UNIVERSITY OF ALBERTA

SPACETIME AND FREE QUANTUM FIELD THEORY

BY TOMÁŠ KOPF

A thesis submitted to the Faculty of Graduate Studies and Research in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

DEPARTMENT OF PHYSICS

Edmonton, Alberta
Fall 1996

ISBN 0-612-18058-1

UNIVERSITY OF ALBERTA

LIBRARY RELEASE FORM

NAME OF AUTHOR: Tomáš Kopf

TITLE OF THESIS: Spacetime and Free Quantum Fields

DEGREE: Doctor of Philosophy

YEAR THIS DEGREE GRANTED: 1996

Tomáš Kopf

Tomáš Kopf
RH 505 Michener Park
Edmonton T6H 4M5
Canada

Date: October 2, 1996

UNIVERSITY OF ALBERTA

FACULTY OF GRADUATE STUDIES AND RESEARCH

The undersigned cerify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled **Spacetime and Free Quantum Field Theory** submitted by **Tomáš Kopf** in partial fulfillment of the requirements for the degree of **Doctor of Pilosophy**.

Don N Page

Don N. Page

W. Israel

Werner Israel

B.A. Campbell

Bruce A. Campbell

Hans P. Künzle

Georg Peschke

Rafael Sorkin

ABSTRACT

For a physical interpretation of a theory of quantum gravity, it is necessary to recover classical spacetime, at least approximately. However, quantum gravity may eventually provide classical spacetimes by giving spectral data similar to those appearing in noncommutative geometry, rather than by giving directly a spacetime manifold. It is shown in this thesis that a globally hyperbolic Lorentzian manifold can be given by spectral data. A new phenomenon in the context of spectral geometry is observed: causal relationships. The employment of the causal relationships of spectral data is shown to lead to a highly efficient description of Lorentzian manifolds, indicating the possible usefulness of this approach.

Connections to free quantum field theory are discussed for both motivation and physical interpretation. It is conjectured that the necessary spectral data can be generically obtained from an effective field theory having the fundamental structures of generalized quantum mechanics: a decoherence functional and a choice of histories.

Acknowledgement

The author is thankful to his supervisor, Professor Don Nelson Page, for the, sometimes amazing, kinds of help and freedom that made this thesis possible.

The author would also like to thank Pavel Krtouš and G[illegible]ke for a number of invaluable discussions.

To Jana. Matúš and Michal

List of Figures

Contents

Introduction

Two experimentally verified theories describe at present the physical world: quantum field theory and general relativity. Both have been extremely successful in their tested ranges of applicability.

Quantum field theory, particularly implemented in the so-called standard model, describes the types and behavior of elementary particles as measured in accelerator experiments and as experienced by everyday contact with matter.

General relativity is concerned with the classical spacetime in which quantum field theory takes place. This spacetime, a four-dimensional Lorentzian manifold with events being its points, can have a complicated structure both locally and globally and can be influenced by the presence of classically understood matter.

Both quantum field theory as applied in the standard model and general relativity indicate intrinsically that they cannot be valid under very extreme circumstances. But moreover they are not fully compatible even under rather usual conditions with the problem being that matter is described by a quantum theory whereas spacetime interacting with the quantum matter is described classically.

For all these reasons it is believed that it should be possible to find a more advanced theory in which also gravity is quantized and in which both the compatibility problem for general relativity and quantum field theory and their internal problems are resolved.

Such theories have already been proposed, most notably string theory. While the internal consistency of such a theory turns out to be a difficult problem, another issue arises once the theory is formulated. How can one relate it to the physical world? The interpretational side of a physical theory has at certain points of history not been trivial but here the question stands with a new urgency. Practically all measurements that are performed in experimental physics use implicitly the notion of classical spacetime. The measurements of positions and times play a dominant role, and there is a clear practical understanding of them. But in a theory where gravity is quantized, there is no classical spacetime in its postulates. The obvious conclusion is that unless one is able to recover from such a theory classical spacetime, at least in an approximative sense, the theory may be a nice piece of mathematics but does not make contact with the physical world and is as a physical theory rather useless.

This work is concerned with providing a tool for recovering classical spacetime from an advanced theory and is thus aimed at the interpretation of a quantum theory of gravity. It is assumed here that such a theory can first be simplified to an effective low energy theory which will look like a usual quantum field theory but without having specified spacetime yet. In such a situation no Lorentzian manifold is present, but there are many structures that contain what one can call spectral information. It comes from the structure of the algebra of observables of the effective theory and eventually from structures like the decoherence functional of generalized quantum mechanics. The problem is thus to describe classical spacetime by spectral data.

There is a theory doing just that for Riemannian spaces: A. Connes' noncommutative geometry. Noncommutative geometry describes classical spaces by commutative algebras of functions on them together with some additional structures on them. It is actually more powerful than is needed here: Noncommutative geometry is able to deal even with noncommutative algebras not corresponding to any classical space. In an indirect way this fact is actually useful even in the present situation where only a classical space is wished for: The understanding of the general noncommutative case is more direct in separating out which concepts are of fundamental importance and which are from a broader perspective just particularities. One structure recognized in this way as being important, the spectral triple, will be especially useful in the considerations presented.

So in a more specific view the problem is to discuss how noncommutative geometry can be used to describe spacetime in the particular commutative case. Unfortunately, the present mathematical framework is able to deal only with spaces of Riemannian type, having a nonnegative distance between any two points. There it is very efficient in using spectral data: Practically all the geometric information is contained in just a few relatively simple structures. The question is whether the same is possible in the Lorentzian case.

The answer to this question is the main topic and result of this work. Compared to Riemannian spectral geometry there is a new phenomenon recognized: causal relationships. Inspired by the thorough discussion of causality in quantum field theory, its place in the framework of noncommutative geometry is found. With this understanding it is possible to show that again, as in Riemannian geometry, the spectral data exhibit a beautiful efficiency in the description of Lorentzian spaces, at least if they are globally hyperbolic which will be assumed throughout.

This gives hope that the adopted approach may turn out to be actually useful in the way it is wished to be useful from the physical context. Several remarks and conjectures on applications in physical interpretations are put forward. Many technical questions are left open for further considerations but have now a clearer formulation and context and can thus be attacked gradually.

The work is organized in the following way:

Chapter 1 contains a review of classical and quantum field theory with special emphasis on a covariant phase space and algebraic approach and especially on the local and causal structure of field theory in the inspiring view of the work of U. Yurtsever.

Chapter 2 shows, after a brief discussion of Connes' spectral triple, first a naïve spectral description of Lorentzian globally hyperbolic manifolds. Then the information contained in causal relationships is discussed and used to obtain a rather compact description of spacetime. The view obtained is the main result of this thesis.

This is summarized in the conclusion.

There are two appendices:

The first appendix contains a review of spinors. This is useful for three reasons. First, a (quantized) Weyl spinor field is part of most of the considerations. Second, spin manifolds play an important role in commutative spectral geometry. Third, a thorough understanding of spinors should be by now, but is not, general knowledge, and thus this appendix serves the author well.

The second appendix gives a brief exposition of some parts of noncommutative geometry in an attempt to support to some degree the spectral triple used in Chapter 2.

CHAPTER 1

Quantization of free fields and local algebras of observables

1. The action principle and the phase space of classical fields.

In this section, classical field theory is reviewed. The treatment adopts to a large degree a functional point of view [1, 2].

The starting point of classical mechanics is kinematics. It is the description of the rather general space of all conceivable histories $\mathcal{F}$ of the considered system. In the case of fields $\mathcal{F}$ may be taken to be the set of all field configurations on spacetime, endowed usually with the structure of an infinite dimensional manifold [2].

The problem of dynamics is then to determine which of the histories in $\mathcal{F}$ actually obey the rules of nature. Those that eventually do are selected by satisfying an equation of motion, forming the space of solutions $\mathcal{S}$. The space of solutions, also called phase space, is assumed to be a submanifold of the space of histories $\mathcal{F}$. The situation is somewhat symbolically expressed in Figure 1.1, a 2-dimensional picture of the infinite dimensional case.

The specific dynamics of a system can be conveniently determined by an action functional $S[\varphi]$, i.e. a map $S : \mathcal{F} \to \mathbb{R}$. The idea is then to determine the elements of the space of solutions $\mathcal{S}$ as the critical points of the action $S[\varphi]$, i.e. $\varphi \in \mathcal{F}$

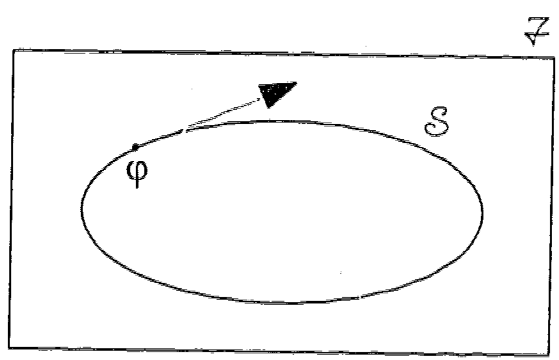

FIGURE 1.1. Kinematics and dynamics. Inside the space of histories $\mathcal{F}$ there is the submanifold of solutions (the phase space) $\mathcal{S}$ determined by dynamics. A point φ on it is a solution of the dynamical equation of motion corresponding to a classical field history. The shown vector tangent to $\mathcal{S}$ at ϕ is to be understood as a functional (Gateaux) derivative in φ.

should be a classical solution, if

(1) $$d_p S[\varphi] = 0.$$

Here d_p is the functional differential and the index p expresses its covariant entry, inspired by the supercondensed notation of B. DeWitt [1,2] and, in another, pictorial notation, by the simplifying power of Feynman diagrams in field theory.

EXAMPLE 1. The action functional of a free scalar field ϕ is

(2) $$S[\phi] = -\frac{1}{2}\int_\Omega (\nabla_\mu \phi \nabla^\mu \phi + m^2\phi^2) d\Omega.$$

Here, the integral is taken over an arbitrary region of spacetime Ω with the measure given by the volume element $d\Omega$, ∇_μ is a covariant derivative and m is a constant (mass). Assuming that the covariant derivative ∇_μ is the one canonically determined by the metric structure of spacetime, the functional differential of the action is

(3) $$d_p S[\phi]\, \delta\phi^p = -\int_\Omega (\nabla_\mu \phi \nabla^\mu \delta\phi + m^2 \phi\delta\phi) d\Omega =$$
$$= \int_\Omega (\nabla_\mu \nabla^\mu \phi - m^2\phi)\delta\phi d\Omega - \int_{\partial\Omega} \delta\phi \nabla_\mu \phi d\Sigma^\mu,$$

with $\partial\Omega$ being the boundary of Ω with the outside-directed hypersurface element $d\Sigma^\mu$.

EXAMPLE 2. The action functional of a free Weyl spinor field ψ is

(4) $$S[\psi] = Re \int_\Omega \tilde{\psi} D\psi d\Omega.$$

The integral is over an arbitrary region of spacetime Ω with the measure given by the volume element $d\Omega$. D is the Dirac operator and $\tilde{\psi}$ is the Dirac adjoint of the Weyl spinor ψ. The field may be taken to be graded commutative (i.e. supercommutative, see [2]). The functional differential of the action is

(5) $$\delta\tilde{\psi}^p \vec{d}_p S[\psi] = Re \int_\Omega (\widetilde{\delta\psi} D\psi + \tilde{\psi} D\delta\psi) d\Omega =$$

(6) $$= 2Re \int_\Omega (\widetilde{\delta\psi} D\psi) d\Omega + Re \int_{\partial\Omega} \tilde{\psi}\gamma_\mu \delta\psi d\Sigma^\mu,$$

with $\partial\Omega$ being the boundary of Ω with the outside-directed hypersurface element $d\Sigma^\mu$. In the graded commutative case, the differential d_p has to be replaced by the left differential $\vec{d}_p$ (or by its right counterpart together with the appropriately reordered expressions).

The differentials calculated in Examples 1, 2 are the sum of an integral over a region of spacetime and of a boundary term B_q, i.e. an integral over the boundary $\partial\Omega$ of the spacetime region Ω. The boundary terms are

(7) $$\delta\phi^q B_q = -\int_\Omega \delta\phi \nabla_\mu \phi d\Sigma^\mu \qquad \text{for a free scalar field.}$$

(8) $$\delta\psi^q B_q = Re \int_{\partial\Omega} \tilde{\psi}\gamma_\mu \delta\psi d\Sigma^\mu \qquad \text{for a free Weyl spinor field.}$$

The correct equation of motion is, however, only obtained if the boundary term is absent, and thus the variational principle 1 has to be modified. This can be achieved in two ways. One is to restrict the variations of the field to to those which do not change the values of the field on the boundary:

DEFINITION 1. The classical solutions for a given action $S[\varphi]$, having the form of an integral of a local function over a spacetime region Ω, are the critical points of $S[\varphi]$ with respect to variations in φ vanishing at the boundary $\partial\Omega$ of Ω.

There is an alternative. One can leave leave the variations of the field arbitrary and subtract the unwanted boundary term from the equation of motion.

DEFINITION 2. The classical solutions for a given action $S[\varphi]$, having the form of an integral of a local function over a spacetime region, are the solutions of the equation

$$d_p S[\varphi] - B_p[\varphi] = 0, \tag{9}$$

where $B_p[\varphi]$ is chosen in such a way that no boundary terms appear on the left side of Equation (9).

The definitions 1, 2 result into the same equation of motion and both give no boundary terms for the values of the field on the boundary $\partial\Omega$ of the spacetime region Ω. In this sense they are equivalent. They give the correct space of solutions $\mathcal{S}$.

EXAMPLE 3. The equation of motion of the free scalar field derived from the action (2) is

$$\nabla_\mu \nabla^\mu \phi - m^2 \phi = 0. \tag{10}$$

EXAMPLE 4. The equation of motion of the free spinor field derived from the action (4) is

$$D\psi = 0. \tag{11}$$

Unlike the case in Figure 1.1, the phase spaces $\mathcal{S}$ of the free scalar field and of the free Weyl spinor field are linear. The linearity results from their quadratic actions (see examples 1, 2). The corresponding symplectic forms have then a global rather than local meaning on the space of solutions $\mathcal{S}$, since a linear space can be identified with its tangent space at zero. This will later be helpful for quantization and is the reason for considering only examples of free fields here. However, so far nothing prevents one to consider at this stage also cases with interaction as long as $\mathcal{S}$ is a manifold.

Given a solution $\varphi \in \mathcal{S}$ (see figure 1.1) one can characterize the functional tangent vectors $\delta\varphi$ at the point φ as solutions of the linearized equation of motion

$$\delta\varphi^p \nabla_p (d_q S - B_q) = 0. \tag{12}$$

Here ∇_p is some functional connection. Since $d_q S - B_q = 0$ in φ, the equation is independent of the particular choice of ∇_p.

Even though the Definitions 1, 2 lead to the same space of solutions $\mathcal{S}$, the definition 2 has the advantage of allowing a simple examination of the boundary. The equation 9 is a functional equation for a form. It can be restricted to the space of solutions $\mathcal{S}$, since forms have, unlike vectors, unique restrictions to subspaces. On $\mathcal{S}$ the equation of motion is identically satisfied:

$$d_{\mathcal{S}} S[\varphi] - B_{\mathcal{S}}[\varphi] = 0 \text{ for } \varphi \in \mathcal{S}. \tag{13}$$

The index z belongs now to the tangent structure of the phase space $\mathcal{S}$. Taking the functional differential of Equation (13) one obtains

$$d_{z_1} d_{z_2} S[\varphi] - d_{z_1} B_z[\varphi] = 0 \text{ for } \varphi \in \mathcal{S}. \tag{14}$$

The differential of a differential is identically zero. Thus $d_{z_1} d_{z_2} S$ vanishes and one has

$$d_{z_1} B_{z_2} = 0 \text{ on the space of solutions } \mathcal{S}. \tag{15}$$

Since B is a boundary term, this equation sets the integral of a local quantity on the boundary of an arbitrary region of spacetime equal to zero for any solution φ from the phase space $\mathcal{S}$.

At this point the treatment will be restricted to the case in which the spacetime M is globally hyperbolic (i.e., is topologically $\Sigma \times \mathbb{R}$, sliced by spacelike Cauchy surfaces [3]) with compact space slices and the spacetime regions considered Ω have as their boundary two Cauchy hypersurfaces $\partial\Omega_1, \partial\Omega_2$.

The decomposition of the boundary $\partial\Omega$ has as a consequence the decomposition of the boundary term $d_{z_1} B_{z_2}(\partial\Omega)$ into two integrals:

$$d_{z_1} B_{z_2}(\partial\Omega) = d_{z_1} B_{z_2}(\partial\Omega_2) - d_{z_1} B_{z_2}(\partial\Omega_1). \tag{16}$$

The parts $d_{z_1} B_{z_2}(\partial\Omega_1)$, $d_{z_1} B_{z_2}(\partial\Omega_2)$ have the same form. The opposite signs in front of them in (16) is caused by the use of the same, future orientation of the hypersurface elements (see Figure 1.2). The importance of the form of one of the parts, e.g. $d_{z_1} B_{z_2}(\partial\Omega_1)$, is that it is a 2-form on the phase space $\mathcal{S}$ which is by Equation (16) independent of the chosen Cauchy hypersurface $\partial\Omega_1$. This means that the space of solutions $\mathcal{S}$ has a canonical 2-form, denoted σ and determined entirely by the action $S[\varphi]$. The 2-form $\sigma_{z_1 z_2}$ is called the symplectic form.

EXAMPLE 5. The symplectic form of the free scalar field follows from the boundary term (7) by functional differentiation and restriction to one Cauchy hypersurface $\partial\Omega_1$:

$$\delta\phi_1{}^{z_1} \sigma^{KG}{}_{z_1 z_2} \delta\phi_2{}^{z_2} = \int_{\partial\Omega_1} (\delta\phi_1 \nabla_\mu \delta\phi_2 - \delta\phi_2 \nabla_\mu \delta\phi_1) d\Sigma^\mu. \tag{17}$$

The symplectic form σ^{KG} is is also called the Klein-Gordon product and is obviously antisymmetric.

EXAMPLE 6. The symplectic form for the free spinor field follows from the boundary term (8) by functional differentiation:

$$\delta\psi_1{}^{z_1} \sigma^D{}_{z_1 z_2} \delta\psi_2{}^{z_2} = Re \int_{\partial\Omega_1} \left(\widetilde{\psi_1}\gamma_\mu\psi_2 - \widetilde{\psi_2}\gamma_\mu\psi_1\right) d\Sigma^\mu. \tag{18}$$

Due to the requirements of quantum field theory, in particular the spin statistics theorem [4], it is necessary to find the symplectic form in the modified case of anticommuting fields. One possibility is to modify the approach above [2]. An alternative is to observe that there is a boundary term connected with the integration by parts for the Dirac operator which vanishes on solutions:

$$\int_{\partial\Omega} \tilde{\phi}\gamma_\mu\psi d\Sigma^\mu = \int_\Omega \tilde{\phi}\gamma_\mu D\psi d\Omega^\mu - \int_\Omega \widetilde{D\phi}\gamma_\mu\psi d\Omega^\mu \tag{19}$$

Again one can consider the case of spacetime regions Ω in a globally hyperbolic spacetime M that have as their boundary two spacelike hypersurfaces $\partial\Omega_1$, $\partial\Omega_2$. The vanishing of the boundary term on solutions means, as in (16), that the integral

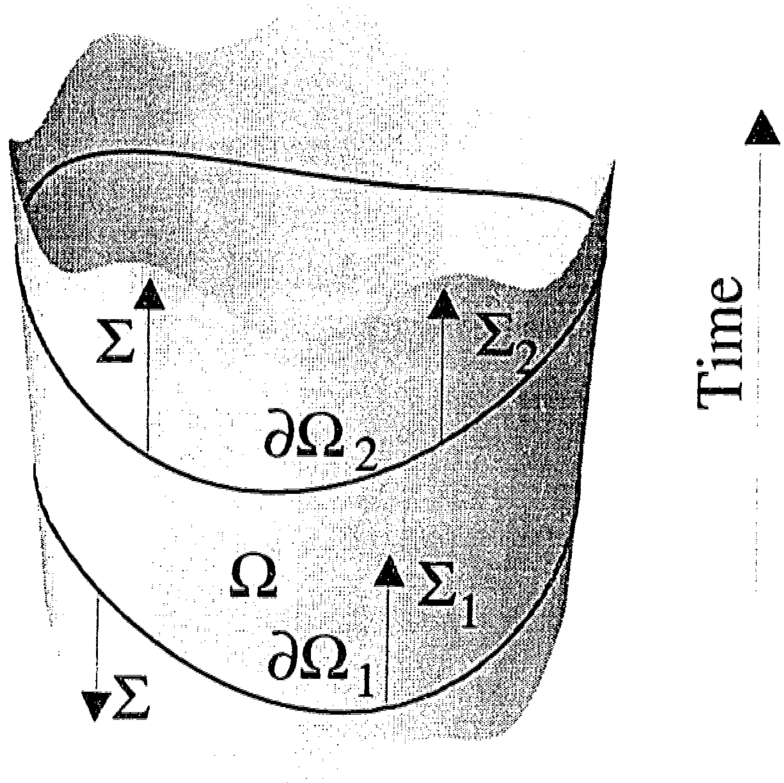

FIGURE 1.2. Orientation of hypersurfaces. A region Ω of a globally hyperbolic spacetime M is enclosed by two Cauchy surfaces $\partial\Omega_1$, $\partial\Omega_2$ forming its boundary $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$. The hypersurface element $d\Sigma$ with its outward orientation as well as the hypersurface elements $d\Sigma_1$, $d\Sigma_2$ oriented into future time direction are shown as normal vectors

over one of the hypersurfaces $\partial\Omega_1$ is independent of the chosen hypersurface and thus a structure solely on the space of solutions. This means that the Weyl spinor phase space is equipped with a hermitean product

(20) $$\int_{\partial\Omega_1} \tilde{\phi}\gamma_\mu \psi \, d\Sigma_1^\mu .$$

2. Bosonic quantization

Once the classical description of a system (e.g. a field) is known, one can make an educated guess of what the correct quantum theory is. This inductive procedure is called quantization. In principle there are two rather different ways to do that, namely quantization by path integrals [5] and canonical quantization (see, e.g. [1,2,6]). In simple examples, as the ones discussed here, they give eventually the same results, but here the latter, canonical quantization, is chosen, since it leads more directly to an algebraic setting used in noncommutative geometry.

In canonical quantization one starts with the classical phase space $\mathcal{S}$ equipped with the antisymmetric symplectic form σ. The functions on the classical phase space $\mathcal{S}$, the classical observables, are then replaced by elements in a noncommutative algebra, the algebra of observables following some rules which turned out to be useful in particular cases. The rules are as follows:

First, a special set $F(\mathcal{S})$ of function on the phase space has to be selected. The set $F(\mathcal{S})$ of chosen classical observables should be closed under taking the Poisson bracket $\{\bullet,\bullet\}$, i.e.,

$$\{a,b\} \in F(\mathcal{S}) \qquad \text{for } a,b \in F(\mathcal{S}), \tag{21}$$

with the Poisson bracket given by the inverse $\widetilde{G}_B$ of the symplectic form σ:

$$\{a,b\} = da \circ \widetilde{G}_B \circ db \qquad \text{for } a,b \in F(\mathcal{S}). \tag{22}$$

Second, a linear map $\hat{\varphi}$ into a complex associative algebra $\mathbf{A}$ should be given.

$$\hat{\varphi} : F(\mathcal{S}) \to \mathbf{A}. \tag{23}$$

The map $\hat{\varphi}$ should satisfy a commutation relation replacing the Poisson bracket by a commutator:

$$\hat{\varphi}(a)\hat{\varphi}(b) - \hat{\varphi}(b)\hat{\varphi}(a) = i\hat{\varphi}(\{a,b\}) \text{ for all } a,b \in F(\mathcal{S}). \tag{24}$$

and its image $\hat{\varphi}(F(\mathcal{S}))$ should generate the algebra $\mathbf{A}$.

NOTE 1. If $F(\mathcal{S})$ contains the constant functions on $\mathcal{S}$ (which have vanishing Poisson brackets with all other functions on $\mathcal{S}$), then their image under the mapping $\hat{\varphi}$ must be in the centre of the algebra $\mathbf{A}$, and if $\mathbf{A}$ is central then the image of constant functions is proportional to the unit $\mathbf{1}$ in the algebra. A not very surprising addition to the quantization rules then usually is the requirement

$$\hat{\varphi}(k) = k\mathbf{1} \qquad \text{for all constant functions } k \text{ on } \mathcal{S}. \tag{25}$$

NOTE 2. The inverse $\widetilde{G}_B$ of the symplectic form σ is well defined only if σ is nondegenerate. If that is not the case (e.g. in the presence of a local gauge symmetry, see [1]), a modification is necessary.

NOTE 3. The Poisson bracket satisfies a Jacobi identity,

$$\{a,\{b,c\}\} + \{b,\{c,a\}\} + \{c,\{a,b\}\} = 0 \qquad \text{for all functions } a,b,c \text{ on } \mathcal{S}, \tag{26}$$

which can be traced back to the closedness of the symplectic form σ. This is helpful, since the mapping of this identity leads to a Jacobi identity in the algebra $\mathbf{A}$,

$$[\hat{\varphi}(a),[\hat{\varphi}(b),\hat{\varphi}(c)]] + [\hat{\varphi}(b),[\hat{\varphi}(c),\hat{\varphi}(a)]] + [\hat{\varphi}(c),[\hat{\varphi}(a),\hat{\varphi}(b)]] = 0 \quad \text{for all } a,b,c \in F(\mathcal{S}), \tag{27}$$

which is a consequence of the associativity of the algebra $\mathbf{A}$. If the Poisson bracket did not satisfy the Jacobi identity, the algebra $\mathbf{A}$ would have to be nonassociative.

One of the difficulties of these rules is the potentially complicated commutation relation (24), and another is the choice of $F(\mathcal{S})$. Obvious choices, like the space of all continuous functions on $\mathcal{S}$, are plagued by inconsistencies or by giving an algebra that is far too big compared with the one that gives a quantum theory in agreement experiment. To deal with this situation, additional information is usually necessary (see e.g. [6]), and even then it is a difficult problem.

The situation radically simplifies if a free system (i.e. one with a quadratic action and thus with a linear phase space $\mathcal{S}$) is considered. The symplectic form σ and thus also the Poisson bracket $\{\bullet,\bullet\}$ are thus global rather than local. The obvious choice of $F(\mathcal{S})$ is then the space $\mathcal{S}^*$ of linear observables. The commutation relation (24) can then, using functional tensor indices x, y, be written as

$$\hat{\varphi}^x a_x \hat{\varphi}^y b_y - \hat{\varphi}^y b_y \hat{\varphi}^x a_x = i a_x \tilde{G}_B^{xy} b_y \mathbf{1} \text{ for all } a_x, b_y \in \mathcal{S}^*, \tag{28}$$

or simply

$$\hat{\varphi}^x \hat{\varphi}^y - \hat{\varphi}^y \hat{\varphi}^x = i\tilde{G}_B^{xy}\mathbf{1} \tag{29}$$

Yet another form of the commutation relation can be obtained by introducing the field operator

$$\Phi(f) = \hat{\varphi}^p \sigma_{pq} f^q \qquad \text{for } f^q \in \mathcal{S}. \tag{30}$$

which leads to

$$\Phi(f)\Phi(g) - \Phi(g)\Phi(f) = i(f \circ \sigma \circ g)\mathbf{1} \qquad \text{for } f, g \in \mathcal{S}. \tag{31}$$

where $f \circ \sigma \circ g = f^p \sigma_{pq} g^q$.

Example 7. A finite dimensional linear phase space can be quantized according to the above scheme. There is, however, still a slight difficulty: It can be shown that the algebra of quantum observables cannot be realized by bounded operators on a Hilbert space and in particular not by a C^*-algebra. This can be improved by exponentiating the commutation relation (31) formally:

$$W(f) = e^{i\Phi(f)}, \tag{32}$$

$$W(f)W(g) = e^{\frac{i}{2} f\circ\sigma\circ g} W(f+g) \qquad \text{for } f, g \in \mathcal{S}. \tag{33}$$

This is called the Weyl form of the commutation relation. It has by the Stone-von Neumann theorem [7] a unique regular irreducible representation (up to unitary isomorphism) by bounded operators in a Hilbert space. So if the Weyl form of the commutation relations is accepted, there is essentially a unique algebra of quantum observables for the system, and quantization has then an unambiguous result.

Note 4. In the unique representation of the Weyl commutation relations, the field operator $\Phi(f)$ can be obtained by differentiation of (32). The result is an unbounded operator. It is not true that this is the unique representation of the commutation relation (31) by possibly unbounded operators on a Hilbert space. For an example see [8].

Example 8. The phase space of a free scalar field can be quantized, accepting again the Weyl form of the commutation relations as in Example 7. In this case the inverse $\tilde{G}_B$ of the symplectic form is the causal Green's function [1] defined as one half of the difference of the advanced and retarded Green.s function. The situation here is however complicated by the exhibition of functional analytic phenomena.

The Stone-von Neumann theorem does not hold in infinite dimensions, and only a unique C^*-algebra may be obtained, generated from smooth field configurations with compact support [7]. There are many unitarily inequivalent representations of this C^*-algebra. It seems that the smooth structure of spacetime, necessary to determine which solutions are smooth, is leaving a fingerprint in the quantization. It is interesting to note that following the cause of this fingerprint it is in the case of spacetimes with compact spacelike Cauchy hypersurfaces, possible to arrive at a unique preferred representation containing all physically acceptable states [9]. This is however possible only if one keeps the information about the origin of the algebra of observables. On the other hand it is known that the double commutant of the algebra of observables in the preferred representation is a von Neumann algebra classified as a factor of type III_1 [10]. This is in some ways a very elementary von Neumann algebra and cannot contain any information on the spacetime on which the field exists.

3. Fermionic quantization

The quantization rules of the previous section cannot be applied in the case of a free spinor field, since its symplectic structure vanishes (see Example 6). In this case, however, another structure, namely the hermitean product (20), is available and can be used for quantization. The imaginary part of this product could be used to replace the symplectic form needed to define a Poisson bracket and to formulate the commutation relation (24). Another possibility is to use the real part μ of the product (20)

$$\mu = Re \int_{\partial\Omega_1} \tilde{\phi}\gamma_\mu \psi d\Sigma_1^\mu. \tag{34}$$

which is symmetric, and to change the commutation relation (24) to

$$\hat{\psi}(a)\hat{\psi}(b) + \hat{\psi}(b)\hat{\psi}(a) = i\hat{\psi}(a \circ \tilde{G}_F \circ b) \qquad \text{for all } a, b \in F(\mathcal{S}). \tag{35}$$

and leaving the rest of the quantization procedure virtually unchanged. Here $\tilde{G}_F$ is the inverse of μ, and $\circ$ denotes a the contraction of indices of two tensors. This way of quantization, using a symmetric rather than an antisymmetric form on the phase space, is called Fermi quantization and is to be distinguished from the previous, called Bose quantization. It is in particular forced onto the quantization of spinors by the spin-statistics theorem [4]

In analogy with (30), one can define the field operator $\Psi(f)$ for a classical solution $f \in \mathcal{S}$

$$\Psi(f) = \hat{\psi}(\mu \circ f), \tag{36}$$

and write the commutation relation (35) in the form

$$\Psi(f)\Psi(g) + \Psi(g)\Psi(f) = i(f \circ \mu \circ g)\mathbf{1} \qquad \text{for } f, g \in \mathcal{S}. \tag{37}$$

EXAMPLE 9. The free Weyl spinor field can be quantized by Fermi quantization. Here the inverse $\tilde{G}_F$ of the symmetric form μ is the causal Green's function of the fermionic field. In contrast to Examples 7, 8, there is no obstruction to finding the unique C^*-algebra of observables which in turn has a unique minimal enveloping von Neumann algebra [11] having, up to isomorphism, a unique regular irreducible representation by bounded operators in a Hilbert space. There is no information whatsoever in this algebra about the smooth structure of spacetime.

4. Local algebras of observables

If the C^*-algebra of observables is considered by itself, without reference to its origin, then it is sufficient to express the evolution of the field by automorphisms and the space of states by normed positive linear functionals (see [12]), but then the physical interpretation is completely lost.

A somewhat similar loss of interpretation can be observed if a classical system is judged on the basis of its phase space only, where canonical transformations can rather arbitrarily change the meaning of coordinates and momenta. It is possible to argue that, e.g., the topology of the phase space is specific to the system, but this is by no means sufficient to give a complete description if there actually is a fundamental distinction between coordinates and momenta.

It is to be expected that the information in the C^*-algebra of observables of a free scalar field, discussed in Example 8, is in the same way of rather weak nature and will be here considered as insignificant, even though there is no fully conclusive evidence for that presented here. In this direction there is, however, no doubt in the case of the free spinor field: As mentioned in Example 9, the algebra of observables does in this case not contain any information about spacetime.

Some structure has thus to be given to the algebra of observables of a quantum field in order to enableone to give its physical interpretation. One could, of course, just remember the whole construction of the algebra of observables, starting with the classical field. In a path integral approach this would not be so bad, since classical histories are part of that framework, but in an algebraic approach to quantum field theory, where the classical field has just the position of an effective approximation, this is definitely not what one would wish to do. The widely accepted solution is to give the algebra of observables the structure of a local algebra [10,12]. The idea is to associate with each region of spacetime Ω a subalgebra $\mathbf{A}(\Omega)$ of the algebra $\mathbf{A}$ of observables. Thus one obtains a set of subalgebras indexed (not necessarily unambiguously) by the set I of open subsets of spacetime.

For many technical purposes it is not necessary to keep the reference to spacetime, and only some properties of the index set I are extracted and required. This is the case of the definition of a quasi-local algebra [10,12]. However, since here interpretation is the main concern, the full link to spacetime will be required [13,14].

Definition 3. A C^*-algebra $\mathbf{A}$ together with a spacetime manifold M is local if the following three conditions all hold:

1. For each open subset Ω of M there is a central (see Note 1) C^*-algebra $\mathbf{A}(\Omega)$, with $\mathbf{A}(\emptyset) = \mathbb{C}$, and $\mathbf{A}(M) = \mathbf{A}$.
2. For any collection $\{\Omega_i\}$ of open subsets of M one has
$$\mathbf{A}\left(\cup_i \Omega_i\right) = \overline{\langle \cup_i \mathbf{A}\left(\Omega_i\right)\rangle}$$
(On the right hand side of this equation is the closure of the algebraic envelope $\langle \cup_i \mathbf{A}(\Omega_i)\rangle$ of $\cup_i \mathbf{A}(\Omega_i)$.)
3. If the regions Ω_1, Ω_2 are not in causal contact, then the corresponding algebras $\mathbf{A}(\Omega_1)$, $\mathbf{A}(\Omega_2)$ commute in the Bose case and graded-commute in the Fermi case.

Example 10. The quantized scalar field can be given the structure of a local algebra. The Green's function $\widetilde{G}_B$ of the field can be used to produce from any smooth density ν with compact support on the spacetime manifold M a solution

f:

(38)
$$f^p = \widetilde{G}_B^{pq} \nu_q,$$

and to each solution f one can by (30) and (32) associate a quantum observable $W(f)$. Given a subset Ω of spacetime, the algebra $\mathbf{A}(\Omega)$ can be then generated by densities with support in Ω. If the supports of two measures ν_1, ν_2 are not causally connected, then the corresponding classical solutions f_1, f_2 can be checked to have vanishing Poisson bracket, and the corresponding quantum observables $W(f_1)$, $W(f_2)$ thus commute.

EXAMPLE 11. The quantized Weyl spinor field can be given the structure of a local algebra. The Green's function $\widetilde{G}_F$ of the field can be used to produce from any smooth density ν on the spacetime manifold M a solution f:

(39)
$$f^p = (\widetilde{G}_F)^{pq} \nu_q$$

and to each solution f one can by (36) associate a quantum observable $\Psi(f)$. Given a subset Ω of spacetime, the algebra $\mathbf{A}(\Omega)$ can be then generated by densities with support in Ω. If the supports of two measures ν_1, ν_2 are not causally connected, then the corresponding classical solutions f_1, f_2 can be checked to have a vanishing product $f_1 \circ \mu \circ f_2$, and the corresponding quantum observables $\Psi(f_1)$, $\Psi(f_2)$ thus anticommute.

5. Reconstruction of spacetime from the lattice of local subalgebras

A very pleasant feature of the local algebra structure is that the C^*-subalgebras $\mathbf{A}(\Omega)$ (with $\Omega \subset M$) of $\mathbf{A}$ are actually sufficient to reconstruct the spacetime M as a topological space and to determine its causal structure, as observed by U.Yurtsever [13, 14]. This will be explained now in some detail.

The local C^*-subalgebras $\mathbf{A}(\Omega)$ of the local algebra $\mathbf{A}$ correspond to open sets in spacetime but in a rather nonunique way. Given a region Ω_1 of spacetime, the C^*-subalgebra $\mathbf{A}(\Omega_1)$ is (by (38) or (39)) generated from measures with support in Ω_1. If now Ω_2 is a second region of spacetime with Ω_1 causally dependent on it, then for each measure with support in Ω_1 there is a measure with support in Ω_2 generating the same element of the quantum field algebra $\mathbf{A}$. Indeed: that Ω_1 is causally dependent on Ω_2 means that each inextendible nonspacelike curve through Ω_1 passes also through Ω_2, and the rest follows from the fact that the classical equation of motion of the field respects causality, so that the fields generated from sources in Ω_1 can also be produced from sources in Ω_2 (see Figure 1.3).

It is possible that two regions are mutually in causal dependence (see Figure 1.4). The maximal region that is with the region Ω mutually causally dependent is called the diamond [14] or the domain of dependence [15] $D(\Omega)$ of Ω.

DEFINITION 4. The diamond (domain of dependence) $D(\Omega)$ of a spacetime region Ω is the set of all points causally dependent on Ω. A point p is causally dependent on the region Ω if all inextendible nonspacelike curves through p pass through Ω.

NOTE 5. One can think of the diamond $D(\Omega)$ of an open spacetime region Ω as of the set of points in which the value of any causality respecting field is fully determined by its value on Ω.

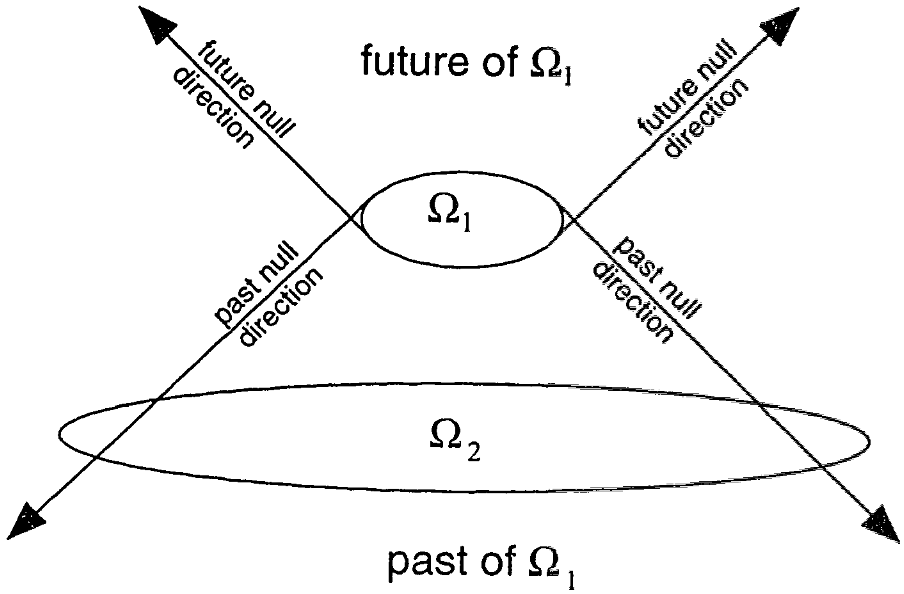

FIGURE 1.3. An example of causal dependence. The region Ω_1 of spacetime is causally dependent on the region Ω_2.

It is easy to check that in a globally hyperbolic spacetime two distinct diamonds give distinct sets of solutions generated by (38) or (39) from measures supported in them.

All this means that diamonds are in one-to-one correspondence with the distinguished family of C^*-subalgebras of the local algebra $\mathbf{A}$.

Both the set L_D of all diamonds in spacetime and the set $L_{\mathbf{A}}$ of C^*-subalgebras can be equipped with partial orders $\leq_D$, $\leq_{\mathbf{A}}$ given by inclusion of subsets or by inclusion of C^*-subalgebras. These partial orders can be promoted to lattices, and the two lattices L_D, $L_{\mathbf{A}}$ are isomorphic by the above mentioned one-to-one correspondence of their elements. Since the lattice structure is at this point important and will be exploited to reconstruct from the local algebra $\mathbf{A}$ spacetime as a topological space, it is useful to review first some of the relevant definitions and to mention some simple and important examples and facts.

DEFINITION 5. A *lattice* is a set L with a partial order $\leq$ such that for every two of elements $a, b \in L$ there is a unique least upper bound $a \vee b$, the *join* of a, b and a unique greatest lower bound $a \wedge b$, the *meet* of a, b. A lattice L is *complete* if the meet and the join of every subset of L exists. A lattice homomorphism from one lattice to another is a map preserving the meet and the join. A complete lattice L has a *least element* $\mathbf{0} = \wedge L$ and a *largest element* $\mathbf{1} = \vee L$.

DEFINITION 6. A *frame* is a complete lattice F satisfying the distributive identity

(40) $$a \wedge (\vee_{i \in I} b_i) = \vee_{i \in I}(a \wedge b_i) \qquad \text{for all } a, b_i \in F$$
and I a possibly infinite index set

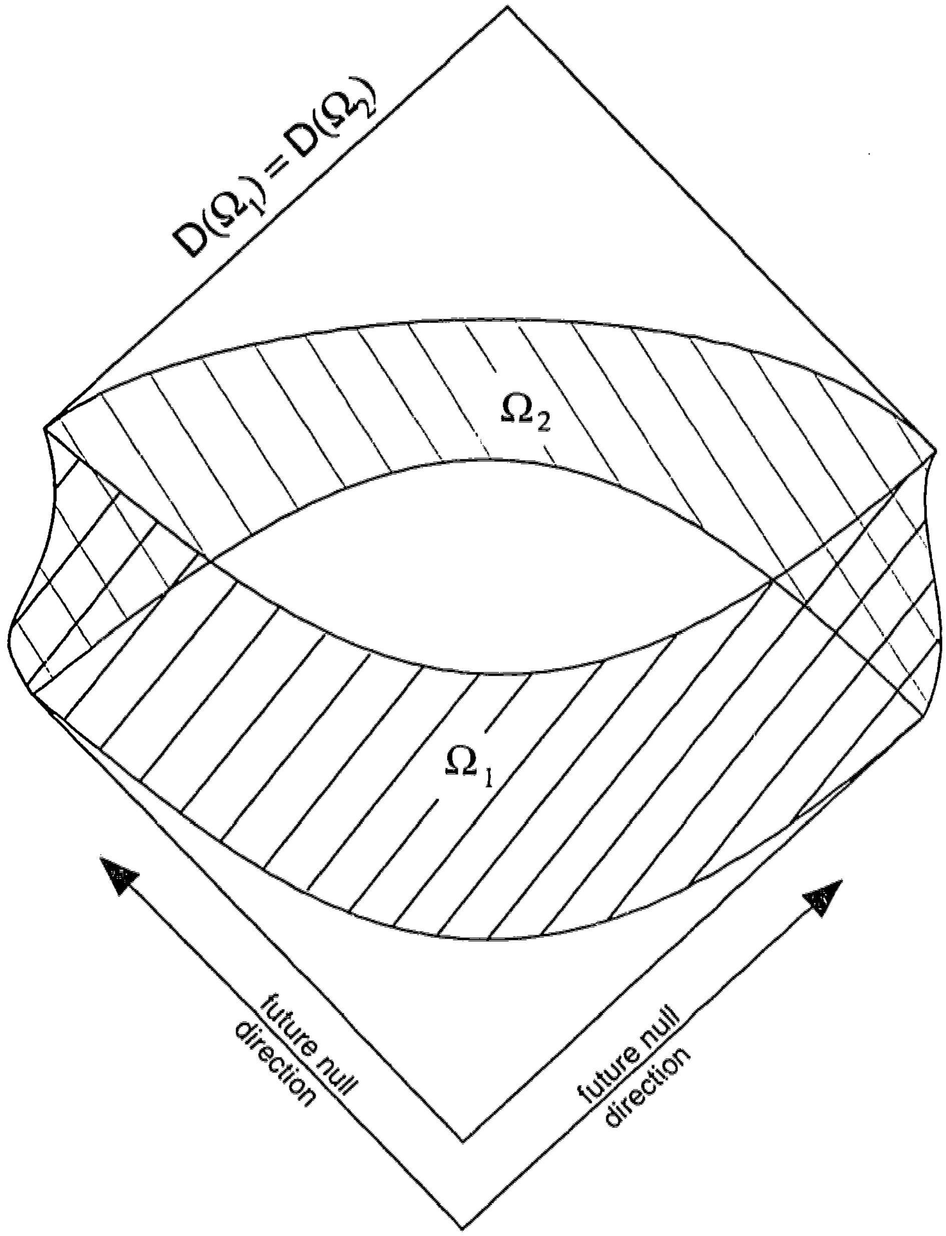

FIGURE 1.4. Mutually causally dependent regions. The regions Ω_1 and Ω_2, as well as their common diamond $D(\Omega_1) = D(\Omega_2)$, are mutually causally dependent.

A *frame map* from one frame to another is a lattice homomorphism preserving the least element $\mathbf{0}$ and the largest element $\mathbf{1}$.

EXAMPLE 12. There is a lattice $L_{\{0,1\}}$ with only two elements $\mathbf{0} \leq \mathbf{1}$. It is trivially also a complete lattice and a frame.

EXAMPLE 13. The set $\Omega(M)$ of open sets of a topological space M is a frame with the partial order given by inclusion and the join $\vee$ and the meet $\wedge$ defined by the union $\cup$ and the interior of the intersection $\cap$ of open sets. It is necessary to take as the meet the interior of the intersection and not just the intersection since an infinite intersection of open sets needs not to be be open.

Example 13 is of fundamental importance since under favorable conditions one can from the lattice $\Omega(M)$ reconstruct the underlying topological space M. The necessary notions for this are contained in the following

DEFINITION 7. The *points* of a frame F are the frame maps from the frame F to the frame $L_{\{0,1\}}$. The set $pt(F)$ of all points of F is understood to be a topological space with the open sets $\mathcal{U}(a)$ given by elements of the frame:

$$\mathcal{U}(a) = \{p \in pt(F) \mid p(a) = 1\} \tag{41}$$

NOTE 6. For an intuitive understanding of Definition 7 it is helpful to insert as the frame in question the frame $\Omega(M)$ of Example 13 and to realize that by choosing a point $p \in M$ one can determine a frame map by assigning to each open set $\mathcal{U}$ the value

$$p(\mathcal{U}) = \begin{cases} 1 & \text{if } p \in \mathcal{U} \\ 0 & \text{if } p \notin \mathcal{U} \end{cases}$$

It is now in question whether one has

$$pt(\Omega(M)) = M \tag{42}$$

as topological spaces. If this is correct one can (at least at the topological level) consider the frame $\Omega(M)$ as a fully respectable substitute of the topological space M.

THEOREM 1. *Every Tychonov space satisfies (42)*

NOTE 7. For the proof of the theorem, see [16]. A Tychonov space is defined by the property that for each closed set A and each point $x \notin A$ there is a continuous function f such that $f(A) = a$, $f(x) = b$ and $a \neq b$. Here it is sufficient to note that any compact Hausdorff space and any metrizable space is Tychonov, and that it is thus a quite broad class of spaces. Concerning connections with noncommutative geometry, it is also useful to note that a commutative C^*-algebra is separable if and only if its Gel'fand transform is metrizable. The Gel'fand transform of a separable commutative C^*-algebra is thus automatically Tychonov.

EXAMPLE 14. The set L_R of open rectangles in $\mathbb{R}^2$ parallel to the axes (including the empty rectangle) is a lattice with the partial order given by inclusion. The meet and the join of two rectangles are shown in Figure 14a. The meet is defined as the intersection, and if infinite meets are to be considered it can be defined as the interior of the intersection, just as in Example 13. With this the lattice L_R is a complete lattice. However, it is not a frame, as is shown in Figure 1.5b.

The open rectangles in Example 14 generate by infinite unions all open sets, i.e. the underlying set of the frame $\Omega(M)$ of Example 13. There may therefore be the possibility to enlarge the lattice L_R to the frame $\Omega(M)$. In fact there is a general construction which does just that for an arbitrary complete lattice L.

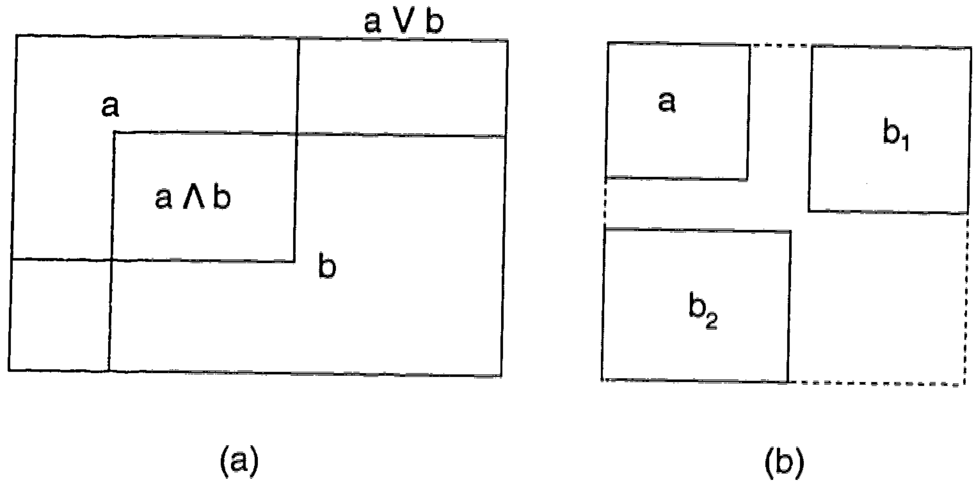

FIGURE 1.5. The lattice L_R of open rectangles. In (a) the join $a \vee b$ and the meet $a \wedge b$ of $a, b \in L_R$ is shown. In (b) a situation is presented where the distributivity condition (40) is not satisfied. Indeed, one has $a \wedge (b_1 \vee b_2) = a$ whereas $(a \wedge b_1) \vee (a \wedge b_2) = 0$

DEFINITION 8. The frame $F(L)$ associated with a complete lattice L is defined as the quotient

$$F(L) = \frac{2^L}{\sim}$$

of the power set 2^L of L (the set of all subsets of L) by the equivalence relation $\sim$ defined as follows: Two subsets K_1, K_2 of L are equivalent, $K_1 \sim K_2$ if and only if

$$\vee_{k \in K_1}(x \wedge k) = \vee_{k \in K_2}(x \wedge k) \text{for all } x \in L.$$

The partial order $\leq$ on $F(L)$ is given by

$$[K_1] \leq [K_2] \Leftrightarrow \vee_{k \in K_1}(x \wedge k) \leq \vee_{k \in K_2}(x \wedge k) \text{for all } x \in L.$$

Here $[K_1], [K_2]$ denote the equivalence classes of K_1, K_2 under $\sim$. The join, meet, least element and largest element in $F(L)$ are given by

$$\begin{aligned} \vee_i[K_i] &= [\cup_i K_i] \\ \wedge_i[K_i] &= [\cap_i K_i] \\ 0 &= [\emptyset] \\ 1 &= [L] \end{aligned}$$

NOTE 8. It is not difficult to show that the structures introduced in Definition 8 indeed turn $F(L)$ into a frame. In particular, the required distributivity (40) can be checked

$$\begin{aligned} [K] \wedge (\vee[K_i]) &= [K \wedge \cup_i K_i] = [\cup_i K \wedge K_i] = \\ &= \vee_i[K \wedge K_i] = \vee_i[K] \wedge [K_i] \end{aligned}$$

for $K, K_i \in L$ and i an index from a possibly infinite index set.

In full analogy with the lattice of rectangles L_R, one can take the complete lattice of diamonds L_D on a manifold M and produce from it by the general construction of Definition 8 the lattice $F(L_D) = \Omega(M)$, and then by Theorem 1 one can obtain back the spacetime manifold as a topological space. The same of course works if one starts instead of the lattice L_D of diamonds with the isomorphic lattice $L_{\mathbf{A}}$ of the local quantum field algebra **A**.

Any two elements of the lattice $L_{\mathbf{A}}$ can be tested for graded commutativity and this encodes causal contact between the two corresponding regions of spacetime. In this way spacetime can be reconstructed from the local algebra **A** as a topological space with its causal structure as long as the knowledge of light cones is sufficient for that (This may be true, e.g. for 4-dimensional Minkowsky space but 1-dimensional Minkowski space is a counter example [17]). This is the main result of [14]

CHAPTER 2

Free quantum fields and spectral data for Lorentzian spaces

1. Overview and motivation

This chapter contains the main new results of this thesis. After a review of Connes' spectral triple in Section 2, two major problems are discussed:

First, it is shown how spectral data can be used to describe spacetime in a 3+1 split form. An important role is played by a one parameter family of spectral triples coexisting on the Hilbert space $\mathcal{S}$. In particular, a set of Dirac triples describing a Weyl spinor field on a globally hyperbolic spacetime manifold is discussed. The treatment of the quantized field turns out to be rather trivial. Besides showing the formulation of the theory, a conjecture is stated suggesting that the structures introduced could eventually be recovered from the fundamental principles of generalized quantum mechanics [18–21].

Second, the causal structure of the spacetime encoded by the spectral data on the Hilbert space $\mathcal{S}$ is examined. Inspired by the understanding of the structure of spacetime of U. Yurtsever [14], it is realized that information about causal relationships in the above description is already automatically encoded by the relationship between the spectral triples coexisting on the same Hilbert space $\mathcal{S}$ and by the inner product on $\mathcal{S}$. This rather simple observation is the main point of this work and has far-reaching consequences. At once a large amount of information becomes available. The proposed spectral data that have not taken into account this fact become considerably redundant. A part of the previous structure which appeared as being a matter of arbitrary choice (the lapse functions N) can be completely dropped since it is now uniquely specified. And even then there is a significant amount of redundancy, and thus in the general case possible overdeterminacy, left. The appearing picture is that noncommutative geometry, which initially appeared to be maybe not as well suited to the description of spacetime as it is to the description of Riemannian spaces, is in the Lorentzian case actually very efficient, even to such a degree that further work will be necessary to control this compact structure in more general situations.

Before approaching the main topics, it is useful to review and extend the motivations for which it is attempted to use a description of spacetime and field theories on spacetime using noncommutative geometry. The physical motivations that have been put forward, are that noncommutative geometry allows a fine structure of spacetime that may be useful for renormalization [22, 23] and that spacetime geometry may be unified with field theory [24–26]. At this point of the mathematical developments [27, 28], it is actually fully justified to use the spectral triple in order only to see how old facts look in the new framework. This is to some degree the case of [29].

Some additional hopes and motivations will be put forward here, and these will actually be particularly important in what follows, even though it is not claimed that they will be completely met by the results obtained.

1. It is clear from the analysis of quantum field measurements affecting macroscopic matter distributions, and thus spacetime geometry [30], that there apparently is not a single classical spacetime underlying a quantum field beyond a semiclassical approximation. It is hoped that a suitable formulation of quantum field theory will be able to accommodate these situations.
2. It is hoped that spacetime and the causal structure of spacetime can be viewed as consequences of structures on the algebra of observables of matter and its noncommutativity and can thus be left out on a more fundamental level, simplifying thus the structure of the theory. Connections between time flow and noncommutativity of an algebra of quantum observables have surfaced in a number of contexts [31, 32], supporting this hope to some degree.
3. It is assumed that the primary output to be interpreted is matter. The observer is assumed to be described as part of the state of the matter field. This leads to the view that classical spacetime has with respect to interpretation a second class position as a sometimes useful book-keeping device for the reality of matter.

NOTE 9. While the first two motivations are in a rather precise way concerned with the range of applicability and with structural issues of the physical theory, the last motivation is of a somewhat vague nature. One may understand it, if one wishes, rather as a psychological statement intended to ease the obvious movement in the direction of abandoning spacetime as a fundamental concept.

2. Connes' spectral triple

A geometric space may be described by its set of points with some additional structures, or, alternatively, by the algebra of functions on it, again with some additional structures. The first point of view is the one of classical geometry. The second may be taken as a starting point for a far more general and powerful theory, A. Connes' noncommutative geometry [27], and is adopted here. In particular, a space can be encoded in the form of a spectral triple [28].

DEFINITION 9. A *spectral triple* $(\mathbf{A}, \mathcal{H}, D)$ is given by an involutive algebra of bounded operators $\mathbf{A}$ in a Hilbert space $\mathcal{H}$ and a selfadjoint operator $D = D^*$ in $\mathcal{H}$ such that

1. The resolvent $(D-\lambda)^{-1}$, $\lambda \notin \mathbb{R}$, of D is compact
2. The commutators $[D, a] = Da - aD$ are bounded, for any $a \in \mathbf{A}$

The triple is said to be *even* if there is a hermitean grading operator Γ on the Hilbert space $\mathcal{H}$ (i.e. $\Gamma^* = \Gamma$, $\Gamma^2 = 1$) such that

$$\Gamma a = a\Gamma \qquad \text{for all } a \in \mathbf{A} \tag{43}$$

$$\Gamma D = -D\Gamma \tag{44}$$

Otherwise the triple is called *odd*

NOTE 10. This section is only concerned with introducing the spectral triple and mentioning its properties to be used in the applications. From that it is not fully clear why one should be interested in exactly this kind of structure, so some motivation is clearly missing here. This is dealt with in Appendix B, which has as its single purpose to show the deep and solid structure of noncommutative geometry that is supporting the spectral triple.

The following example is of great importance.

EXAMPLE 15. On a compact Riemannian spin manifold M (see Appendix A) there is canonically the following spectral triple $(C^\infty(M), L^2(M,S), D)$, the Dirac triple [27], [28]. Here $C^\infty(M)$ is the commutative algebra of smooth complex functions on M, $L^2(M,S)$ is the Hilbert space of square integrable sections of the complex spinor bundle S over M and D is the Dirac operator. The algebra of functions $C^\infty(M)$ acts on the Hilbert space $L^2(M,S)$ by pointwise multiplication

$$(f\psi)(p) = f(p)\psi(p) \qquad \text{for all } f \in C^\infty(M), \psi \in L^2(M,S), p \in M \tag{45}$$

and the commutator with the Dirac operator D with a function f is

$$[D,f] = \gamma df \qquad \text{for } f \in C^\infty(M). \tag{46}$$

γ is the Clifford map from the cotangent bundle into operators on $L^2(M,S)$.

In Example 15 the algebra was taken to be $C^\infty(M)$. Such a choice contains a lot of information and is actually not necessary. In the definition of the Dirac spectral triple it is sufficient to take instead of $C^\infty(M)$ any algebra $\mathbf{A}$ that has the same weak closure (double commutant) $\mathbf{A}''$ as has $C^\infty(M)$. Such an algebra does not necessarily contain any information about the topology or differential structure of M whatsoever. From $\mathbf{A}$ alone only M as a set of points can be obtained as the spectrum of $\mathbf{A}$ (see Section 2 of appendix B). The rest, however, can then be recovered from the structure of the spectral triple including the notion of smooth functions and Lipschitz functions. Lipschitz functions with Lipschitz constant 1 can then be used to define a distance function d on M. This means that a Riemannian spin manifold can be replaced by a spectral triple without the loss of any information about it. The facts are summarized in Proposition 2 (see [28]).

PROPOSITION 2. *Let $(\mathbf{A}, L^2(M,S), D)$ be a Dirac spectral triple associated to a closed Riemannian spin manifold M. Then the compact space M is the spectrum of the commutative C^*-algebra norm closure of*

$$\mathbf{A}_B = \{a \in \mathbf{A}'' \mid [D,a] \text{ bounded}\} \tag{47}$$

while the geodesic distance d on M is given by

$$d(p,q) = \sup\{\mid f(p) - f(q) \mid; f \in \mathbf{A}_B, \| [D,f] \| \leq 1\} \tag{48}$$

It is now in question whether one can reconstruct from a spectral triple a manifold if one is not assured that the spectral triple actually comes from a manifold. With some additional conditions it will certainly be possible to prove in the future a theorem in this direction. One helpful tool for this purpose is a real structure J on the spectral triple [28], [33], even though it presupposes that the space in consideration can be caracterized by a fixed dimension.

EXAMPLE 16. In the case of the Dirac spectral triple of Example 15 a real structure is given by the charge conjugation composed with complex conjugation (see Appendix A).

Before giving its general definition it should be mentioned that for simply connected spaces the real structure ensures that the spectrum of a spectral triple will have the homotopy type of a closed manifold [28], [33]. In addition to that, its dimension is governed by the spectrum of the Dirac operator [34]. So a theorem examining which commutative spectral triples are classical Riemanian manifolds is not out of sight. The considerations of the next sections would be best motivated by such a theorem but making use of it as in [29] is at this point probably premature.

Definition 10. A real structure J on the spectral triple $(\mathbf{A}, \mathcal{H}, D)$ is an antilinear isometry J

(49) $$J : \mathcal{H} \to \mathcal{H}$$

such that

(50) $$JaJ^{-1} = a^* \qquad \text{for all } a \in \mathbf{A}$$

(51) $$J^2 = \epsilon$$

(52) $$JD = \epsilon' DJ$$

(53) $$J\Gamma = \epsilon'' \Gamma J$$

where the signs $\epsilon, \epsilon', \epsilon'' \in \{-1, +1\}$ are given by the following table with ν being the dimension of the space $mod\, 8$:

(54)

ν	0	1	2	3	4	5	6	7
ϵ	1	1	−1	−1	−1	−1	1	1
ϵ'	1	−1	1	1	1	−1	1	1
ϵ''	1		−1		1		−1	

Note 11. The sign ϵ'' in Table (54) is shown for even dimensions only, since for Riemannian spin manifolds only in that case the grading (helicity) operator Γ preserves the irreducible spin representation and has thus a good meaning in it. In the odd case it is assumed that only one of the two irreducible representations is chosen and since Γ switches between the two irreducible repressentations it has no meaning just in one of them. More details on spinors can be found in Appendix A. Also the periodicity $mod\, 8$ of Table (54) is explained there. It is just one of the manifestations of the spinorial chessboard, in this case restricted to its Euclidean row.

3. Spacetime in noncommutative geometry

Here a Lorentzian globally hyperbolic spacetime manifold will be characterized by spectral data. This cannot be done directly by Connes' spectral triple (see Definition 9) since it is well suited for the description of generalized Riemannian spaces only. This is obvious, e.g., from the distance function (48), which cannot be negative. A simple idea to avoid this difficulty is to foliate the spacetime M by a family of spacelike Cauchy slices Σ_t with $t \in \mathbb{R}$ a coordinate time (see Figure 2.1). Each hypersurface Σ_t is then Riemannian and can be characterized by a family of Dirac spectral triples $(L^\infty(\Sigma_t), L^2(\Sigma_t, S), D_t)$ (see Example 15 and Proposition 2) together with some additional information on how the spacelike slices Σ_t are related to each other. In particular, the normal distance between two infinitesimally close Cauchy surfaces Σ_t is encoded by the lapse function N (see [35] and Figure 2.1).

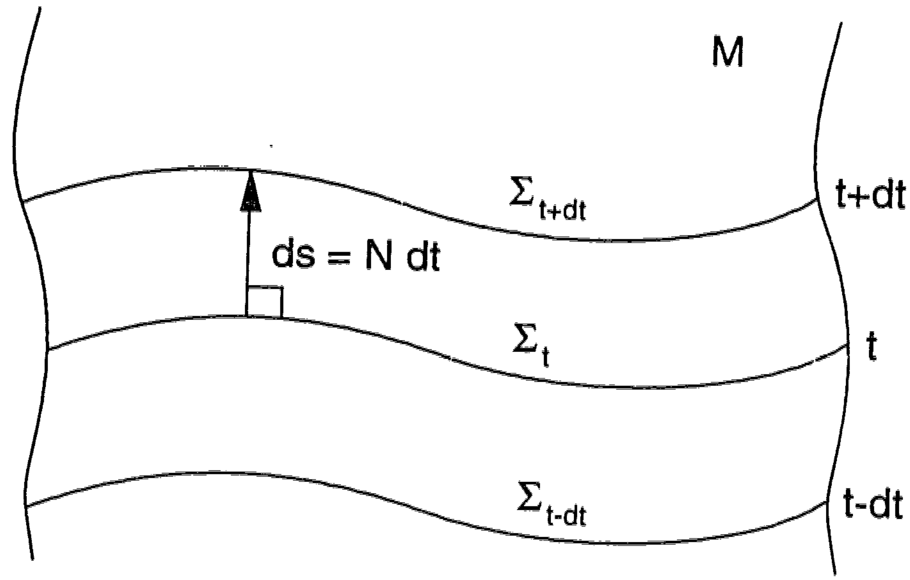

FIGURE 2.1. A Cauchy foliation. The globally hyperbolic manifold M can be sliced by spacelike Cauchy surfaces Σ_t. Each of them can be characterized by a Dirac spectral triple $(L^\infty(\Sigma_t), L^2(\Sigma_t, S), D_t)$ with $L^\infty(\Sigma_t)$ being the algebra of essentially bounded functions on Σ_t, $L^2(\Sigma_t, S)$ being the spinor bundle over Σ_t and D_t being the Dirac operator on Σ_t. The normal distance between infinitesimally close Cauchy surfaces Σ_t, Σ_{t+dt} is characterized by the lapse function N on Σ_t. N can be thought of as an element in the algebra $L^\infty(\Sigma_t) = (C^\infty(\Sigma_t))''$, the double commutant of the algebra of smooth functions.

The only further information needed is the identification $i_t : \Sigma_t \to \Sigma_0$ of points which lie on the same curve normal to the hypersurfaces. This can be established in the spectral data by specifying an automorphism $i_t^* : L^\infty(\Sigma_0) \to L^\infty(\Sigma_t)$

Since the square integrable sections of the spin bundles over the Cauchy surfaces Σ_t, $t \in \mathbb{R}$ are valid Cauchy data for weak solutions of the equation of motion of a Weyl spinor field on M, there is a preferred isomorphism between the spin bundles $L^2(\Sigma_t, S)$ and the space of solutions $\mathcal{S}$ of Weyl spinors. This means that all spectral triples can be understood to share the same Hilbert space $\mathcal{S}$.

Summarizing, a globally hyperbolic spacetime can be described using spectral data by

- a family of spectral triples $(\mathbf{C}_t, \mathcal{S}, D_t)$ with $\mathbf{C}_t$ a commutative von Neumann algebra of bounded operators on $\mathcal{S}$ and D_t Hermitean (possibly unbounded) on $\mathcal{S}$
- a family of lapse functions $N_t \in \mathbf{C}_t$
- an automorphism i^* between any two of the commutative algebras $\mathbf{C}_t$

NOTE 12. Usually it is not required that the identification of Cauchy surfaces has to be done along normal lines. Then the deviation of of the direction of identification from the normal one has to be characterized by a shift vector field $\vec{N}$ on the

Cauchy surfaces [35]. The restriction to the case $\vec{N} = 0$ here avoids the necessity of a replacement of vector fields by spectral concepts.

The above description agrees with [29] except that there the automorphism i^* is omitted. That omission seems to make the spectral data appear incomplete from the point of view presented there.

It is now possible to describe the quantum field theory for Weyl spinors on the spacetime specified by the spectral data. Since the Hilbert space $\mathcal{S}$ in the spectral data is taken to be the space of classical solutions equipped with the canonical Hermitean product (20), this is entirely trivial: The quantum field algebra of observables is just the Clifford algebra generated from $\mathcal{S}$ by the anticommutation relation (37).

This completes the discussion of quantum field theory on spacetime using a spectral approach but not taking in account the causal structure information present in the problem. This is a natural place to reflect on the above with a few comments.

From the point of view of the motivations, one would wish to start from an algebra of quantum observables, to specify the spectral data, and then to construct, if possible, classical spacetime. Such an approach will however bring rather difficult problems: At least in the cases where one hopes to obtain a spacetime that is a topological or smooth manifold, one would wish to have the one-parameter family $\mathbf{C}_t$ in some sense continuous or smooth. (It may be viewed as a continuous or smooth algebra bundle over $\mathbb{R}$). This is an important, but on the other hand technical, issue. Instead of discussing it satisfactorily, the treatment will rely on the case studied here starting with a classical spacetime, producing the space of solutions $\mathcal{S}$ of the Weyl spinor field on it and obtaining by quantization the field algebra $\mathbf{A}$. Then all the facts can be viewed backwards, starting with the field algebra $\mathbf{A}$. This is clearly dishonest to the motivations in using as its input what should be abandoned in the first place: classical spacetime. On the other hand this allows one to go through all the way from the quantum algebra to spacetime avoiding some, in general difficult, arguments bridged by the particular features of this not-so-elegant example. The result is then an understanding of what is important, and with this, one can then gradually face the technically difficult points. This approach has worked so far extremely well in noncommutative geometry. In this context, the aim here is to gain an understanding only, thus considering the example as a valid approach.

For a view starting from the quantum field algebra according to the above motivations, it would also be desirable to have a deeper justification of the introduced structures, particularly for the family of operator algebras $\mathbf{C}_t$ and the family of operators D_t on the space $\mathcal{S}$ generating the algebra of observables $\mathbf{A}$. It will be suggested here in the form of two conjectures that this may eventually be possible.

CONJECTURE 1. Another way to look at the family of commutative algebras $\mathbf{C}_t$ will be offered now. For a given value of the parameter $t = t_0$, the algebra $\mathbf{C}_{t_0}$ splits the space $\mathcal{S}$ into orthogonal subspaces by spectral projections. On the quantum level this means that the field algebra $\mathbf{A}$ is given preferred mutually commuting subspaces. In the case in which the Hilbert space is finite dimensional, these spaces are complex one dimensional. It is conjectured that this structure is sufficient to determine a preferred complete set of commuting projectors in the algebra of observables $\mathbf{A}$ or eventually in its (unique) minimal enveloping von Neumann algebra. If that is the case, then the choice of $\mathbf{C}_{t_0}$ may be understood as the choice of a

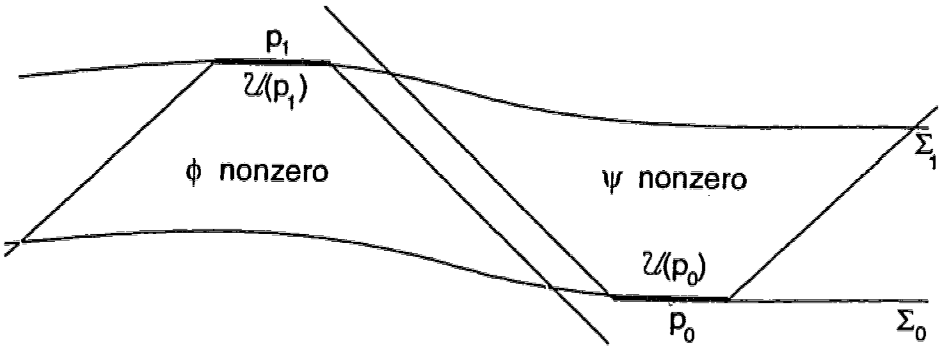

FIGURE 2.2. Causal contact. Any solution ψ with Cauchy data on Σ_0 supported in $\mathcal{U}(p_0)$ has a vanishing inner product with any solution ϕ with Cauchy data on Σ_1 supported in $\mathcal{U}(p_1)$. The points p_0, p_1 are not causally connected.

set of histories in generalized quantum mechanics [18–21]. This would to a large degree justify the introduced structures from a very fundamental point of view.

CONJECTURE 2. If Conjecture 1 is in some way correct, then the family D_t of Hermitean operators on $\mathcal{S}$ can be for a suitable coarse graining recovered from the decoherence functional of generalized quantum mechanics on histories of the quantum field **A**.

These conjectures are a topic of future research. They are stated here only to show that what was reached so far is really following the call of the motivations put forward in Section 1, which would not be so easy to see otherwise.

4. Spectral data and the causal structure of spacetime.

The spectral data describing spacetime as presented in the previous section are sufficient. But they did not take into account the fact that causal structure information is also stored in the family of spectral triples in a way that was not yet exploited. It is here that that insight of the work of U.Yurtsever [13, 14] revieved in Section 5 of Chapter 1 is useful.

To understand that, consider two spacelike Cauchy surfaces Σ_0, Σ_1 on the spacetime manifold (see Figure 2.2). They are described by the spectral triples $(\mathbf{C}_0, \mathcal{S}, D_0)$, $(\mathbf{C}_1, \mathcal{S}, D_1)$. Given two points p_0, p_1 on these Cauchy surfaces ($p_0 \in \Sigma_0$, $p_1 \in \Sigma_1$) it is now possible just to decide whether they are in causal contact or not. If and only if the points p_0, p_1 are not in causal contact, the value of the Weyl spinor field at the point p_0 cannot influence the value of the field at the point p_1. In more precise terms on can say that there exist open neighborhoods $\mathcal{U}(p_0)$, $\mathcal{U}(p_1)$ of the points p_0, p_1 in Σ_0, Σ_1 such that any solution ψ of the equation of motion of the Weyl spinor field with Cauchy data on Σ_0 supported in $\mathcal{U}(p_0)$ has a vanishing inner product with any solution ϕ with Cauchy data on Σ_1 supported in $\mathcal{U}(p_1)$. To identify solutions in $\mathcal{S}$ which have Cauchy data on Σ_i supported in a certain region $\mathcal{U}(p_i) \subset \Sigma_i$ from the spectral data is easy: they are just given as elements of the ranges of the spectral projection corresponding to $\mathcal{U}(p_i)$.

NOTE 13. If one is willing to use generalized eigenvectors (δ-functions) then causal contact can be expressed in the following way. A (generalized) solution with

Cauchy data on Σ_0 supported in the point p_0 is a generalized eigenvector of the algebra $\mathbf{C}_0$ satisfying

$$a\psi = a(p_0)\psi \qquad \text{for } a \in \mathbf{C}_0, \tag{55}$$

with $a(p_0)$ being the value of the function a at the point p_0. The vector ψ can then be for briefness called an eigenvector of point p_0. Then two points are not in causal contact if and only if all their eigenvectors are orthogonal.

One can now summarize:

OBSERVATION 1. Using the family $\mathbf{C}_t$ of commutative algebras represented on the Hilbert space $\mathcal{S}$ of solutions, one can recover spacetime as a set of points and find by the above procedure which points are in causal contact, using the Hermitean inner product on $\mathcal{S}$.

This observation is of central importance. Before using it to reduce the spectral data necessary to describe a Lorentzian spacetime, a two connections will be made.

First, from the point of view of differential equations it is not surprising that the Hermitean inner product on $\mathcal{S}$ contains information on the causal structure, since as mentioned in Example 9 the real part of it is the inverse of the causal Green's function.

Second, from the point of view of quantum field theory the orthogonality of classical solutions with Cauchy data locally supported around two points p_0, p_1 has as its consequence (or, if one wishes, as its origin) the graded commutativity of the corresponding C^*-subalgebras of the local algebra $\mathbf{A}$ of observables generated from $\mathcal{S}$. This is the point where the notion of causality makes contact with Section 5 and with some of the motivations for this work given in Section 1

Now the consequences of Observation 1 will be discussed. First of all, the family of spectral triples $(\mathbf{C}_t, \mathcal{S}, D_t)$ of Section 2 contains already all necessary information about spacetime and no automorphism i^* between the algebras $\mathbf{C}_t$ and no lapse function N need to be specified. Indeed, by knowing the geometry of the Cauchy surfaces Σ_t corresponding to the spectral triples $(\mathbf{C}_t, \mathcal{S}, D_t)$ and the causal structure one can find the normal identifications of points and the normal distances between infinitesimally close Cauchy surfaces (see Figure 2.3).

Thus a large part of the spectral data can be just left out, and the remaining family of spectral triples gives now a quite efficient description. But it is still considerably redundant. To see this is not difficult: If the metric information contained in the operators D_t is omitted, then the conformal structure of spacetime is still rigidly fixed. But not all metrics are conformally related, and thus the D_t determining the metric on the Cauchy surfaces cannot be chosen at will but have to agree with the conformal structure. This means that the spectral data of spacetime can be further reduced. How this has to be done in a useful way will be left for consideration in the future. But even without that a conceptual result is appearing: The spectral data describing a Lorentzian manifold do so in a very efficient way. This result based on Observation 1 is the main claim of this work.

NOTE 14. There is a way of giving less redundant spectral data, if one is willing to lose metric information and keep just the conformal structure of spacetime. It is shown in [27] that for building just differential geometry without metric information, it is sufficient to take, instead of the spectral triple with an unbounded operator D, the same spectral triple but with D replaced by $F = sgn\ D$, the sign of the

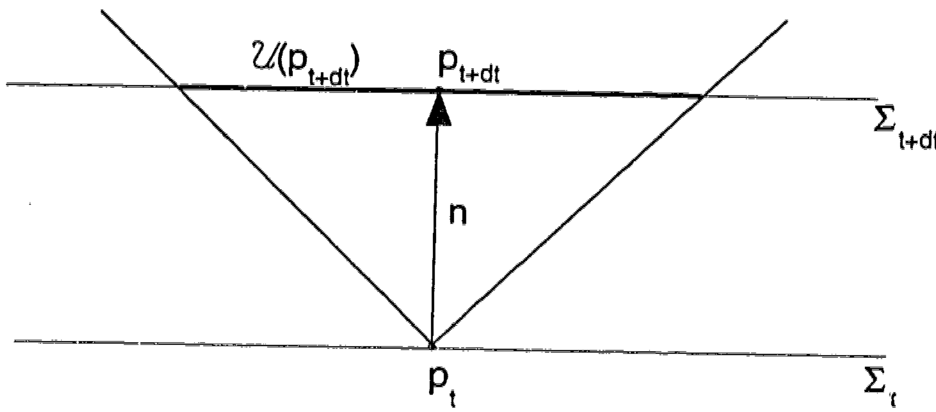

FIGURE 2.3. The geometry of Cauchy surfaces, causal contact and the geometry of spacetime. The point p_t on Σ_t has as its region of causal contact on Σ_{t+dt} the disk $\mathcal{U}(p_{t+dt})$ (including its bounding sphere). The square of the radius of the sphere is the negative of the square of the normal spacetime distance between the Cauchy surfaces Σ_t, Σ_{t+dt}, and the center p_{t+dt} of the sphere $\mathcal{U}(p_{t+dt})$ is the point reached by the normal vector n based in p_t.

operator D (see Appendix B). This is actually a grading operator on $\mathcal{S}$ since $F^2 = \mathbf{1}$. Thus the spectral triple $(\mathbf{C}_t, \mathcal{S}, F_t)$ with a family of grading operators contains the topological and causal as well as differential geometric information on spacetime.

NOTE 15. One may wonder where the efficiency of the spectral data in the presented description comes from. In the case of the spectral triple A. Connes argued [27, 28] that most of the information is not in the algebra of the triple, giving basically just a set of points, nor in the chosen Hermitean operator, fully described by its spectrum, but in the relationship between them. This explanation can be used here again: Most of the information in the spectral data is not in the commutative algebras $\mathbf{C}_t$ represented on $\mathcal{S}$ but in the relationships between them. Indeed, the strong causal structure is purely a result of this.

Conclusion

Motivated by the need to recover classical spacetime from a theory of quantum gravity in order to achieve the theory's physical interpretation, the thesis examines the possibility of describing classical Lorentzian spacetime manifolds by spectral data.

Following in Section 3 a naïve Hamiltonian approach, the spectral data for a Lorentzian manifold are specified as a family of A. Connes' spectral triples with a common Hilbert space and additional structures known from Hamiltonian general relativity: a family of lapse functions and an identification of Cauchy surfaces implemented by isomorphisms of the algebras in the spectral triples. This gives a complete description of spacetime, trivially extended to a free quantum field theory on spacetime.

However, in Section 4 it is realized that the spectral description of spacetime automatically contains unused information on causal relationships. The use of this information leads to a significant reduction of the spectral data. The family of lapse functions and the identification of Cauchy surfaces can be completely left out, and still there is considerable redundancy in the data present. The discovery of the place of causal relationships in spectral geometry thus leads to a very efficient spectral description of spacetime. This is the main result of this thesis.

With the result attained here, there are now two well motivated problems of conceptual importance:

1. The remaining redundancy in the spectral data should be removed and the result put into a useful form to be recognized as standard.
2. The way in which the result may fit into an interpretation of quantum gravity should be clarified, possibly along the lines of Conjectures 1 and 2

Moreover, there are also many further points of technical nature, to be worked out. To suggest just one of them as an example, it would be desirable to have a usefully formulated expression for spacetime distances.

With the insight obtained here, these questions are now open to future investigations.

APPENDIX A

Spinors

This appendix reviews some aspects of the structure of finite dimensional Clifford algebras and their representations [36, 11, 37–40]. Unlike the case of tensors, the structure of spinors is considerably dependent on the dimension and signature of the manifold considered. For the intended use here, only complex representations of the real Clifford algebra connected with a Lorentzian metric in four dimensions are relevant. However, such a restricted view will not provide good insight into the significance of separate concepts in the theory. Therefore the treatment here will start out more generally, attempting to trace the origins of the structures used.

1. Overview

Spinors are vectors in the representation space S of the irreducible representation of a Clifford algebra $\mathbf{Cl}(V, g)$. Clifford algebras $\mathbf{Cl}(V, g)$ in turn are generated from vector spaces V with scalar products g (see Section 2 for details).

Taking as the vector space V the tangent space at a point of a (pseudo-) Riemannian manifold equipped with the metric g as the scalar product, spinors gain geometric significance. Spin bundles with the space of spinors S as fibres may be constructed on vector bundles with scalar products.

This is not entirely trivial since there may be topological obstructions to the existence of a spin bundle over a vector bundle with a scalar product. And once a spin bundle exists, it is not necessarily unique.The situation is to some extent dealt with in the following theorem [37]:

THEOREM 3. *Let E be an oriented vector bundle over a manifold X. Then there exists on E a spin bundle if and only if the second Stiefel-Whitney class $w_2(E)$ vanishes. If $w_2(E) = 0$, then there is a one-to-one correspondence between the spin bundles existing on X and the elements of $H^1(X; \mathbb{Z}_2)$, the first cohomology group of X with values in the group $\mathbb{Z}_2$.*

The point of view so far presented was that spinors are the result of providing a vector space V with a specific algebraic structure, namely extending it to the Clifford algebra $\mathbf{Cl}(V, g)$. There is, however, another, group theoretical. approach: The presence of a metric on the vector space V determines the symmetry groups $\mathbf{O}(V, g)$, $\mathbf{SO}(V, g)$. None of these groups is simply connected, and they have nontrivial (simply connected) covers denoted $\mathbf{Pin}(V, g)$, $\mathbf{Spin}(V, g)$. Now these covering groups are easily realized as subgroups of the inner automorphisms of the Clifford algebra $\mathbf{Cl}(V, g)$. Just for a more definite idea, the definitions are given and supplemented by a theorem [37] describing the covering homomorphisms:

DEFINITION 11. The subgroups $\mathbf{Pin}(V, g)$ and $\mathbf{Spin}(V, g)$ of the inner automorphism group of the Clifford algebra $\mathbf{Cl}(V, g)$ are defined by

$$\mathbf{Pin}(V,g) = \{v_1 v_2 ... v_r \in \mathbf{Cl}(V,g) \mid v_1, v_2 ..., v_r \in V, g(v_k, v_k) = \pm 1\}$$
$$\mathbf{Spin}(V,g) = \{v_1 v_2 ... v_r \in \mathbf{Cl}(V,g) \mid v_1, v_2 ..., v_r \in V \text{ with } r \text{ even}, g(v_k, v_k) = \pm 1\}$$

THEOREM 4. *Let g be a nondegenerate scalar product on the vector space V. Then the following sequences are exact:*

$$(56) \qquad 1 \longrightarrow F \longrightarrow \mathbf{Pin}(V,g) \xrightarrow{\widetilde{Ad}} \mathbf{O}(V,g) \longrightarrow 1$$

$$1 \longrightarrow F \longrightarrow \mathbf{Spin}(V,g) \xrightarrow{\widetilde{Ad}} \mathbf{SO}(V,g) \longrightarrow 1$$

where $F = \begin{cases} \mathbb{Z}_2 = \{-1, +1\} & \textit{if } V \textit{ is a real vector space.} \\ \mathbb{Z}_4 = \{\pm 1, \pm i\} & \textit{if } V \textit{ is a complex vector space.} \end{cases}$

and the homomorphism $\widetilde{Ad}$ is given by

$$(57) \qquad \widetilde{Ad}_\phi(v) = \alpha(\phi) v \phi^{-1} \ \textit{with} \ \begin{cases} \phi & \in \mathbf{Pin}(V,g) \\ v & \in V \\ \alpha(v_1 ... v_r) & = (-1)^r v_1 ... v_r \end{cases}$$

This means that spinors can be regarded as elements of the representation space S of the representations of the covers $\mathbf{Pin}(V,g)$, $\mathbf{Spin}(V,g)$ of $\mathbf{O}(V,g)$, $\mathbf{SO}(V,g)$. This approach is actually more general, since it works also for vector spaces without a metric: The general linear group $\mathbf{GL}(V)$ is not simply connected and has a cover $\widetilde{\mathbf{GL}}(V)$ [41]. For spinor bundles of this type only a manifold structure is necessary and there is no need for a metric. Its disadvantage is that, while $\mathbf{Spin}(V,g)$ and $\mathbf{Pin}(V,g)$ have for finite dimensional vector spaces V finite dimensional representations, the covering group $\widetilde{\mathbf{GL}}(V)$ of $\mathbf{GL}(V)$ has only infinite dimensional ones. Despite this difficulty it would be interesting to try to see an explicit replacement of usual spinors by these ones.

The need for a fixed metric in the definition of spinors causes trouble in theories including a variational principle for gravity, since variations of the metric have consequences that are difficult to control. This is usually improved by choosing a vector bundle isomorphic to the tangent bundle with a fixed metric which is used to construct spinors and transported to the tangent bundle by a particular isomorphism. This isomorphism is often called a soldering form or (in the 4-dimensional case) tetrad and replaces the metric as a dynamical variable [39, 42]. It can then be varied without disturbing the spin structure.

In the cases considered here a fixed metric will be assumed, and therefore it will be sufficient to use the first approach considered in which Clifford algebras and their representations play a central role.

The next section defines Clifford algebras and examines their basic structures, especially the main automorphism, and the main antiautomorphism, the main conjugation. Classification results for complex and real Clifford algebras are given.

The third section deals with the irreducible representations of Clifford algebras, the spin representations. In general, irreducible representations of simple algebras equipped with an antiautomorphism provide their representation space with an inner product that is symmetric or antisymmetric if the antiautomorphism is involutive. The obvious choice in the case of a Clifford algebra is the main antiautomorphism.

Moreover, if complex representations of real Clifford algebras are considered. there is a canonical antiautomorphism on the representation space S called the charge conjugation. Since for a real Clifford algebra a complex representation gives more expressive power than is necessary. the representation can be viewed in a simplified way using the charge conjugation. The charge conjugation either breaks the complex representation down to two copies of a real representation by providing a real structure or forces the complex-linear endomorphisms of the representation to be actually quaternionic-linear.

A combination of the canonical inner product and of the charge conjugation gives then the Hermitean Dirac product appearing in the Lagrangian for free fermions.

2. Clifford Algebras

The Clifford algebra $\mathbf{Cl}(V, g)$ is obtained by providing the vector space V with a multiplication. the Clifford multiplication, produced from the scalar product g by an anticommutation relation. The Clifford multiplication does not preserve V and is thus defined in the form of a map from V into a unital associative algebra:

THEOREM 5. *Let f be a linear map from the vector space V with a scalar product g to a unital associative algebra $\mathbf{A}$,*

$$f : V \to \mathbf{A}.$$

Then f is called a Clifford map if it satisfies:

$$f(v)^2 = g(v, v)\mathbf{1} \text{ for all } v \in V. \tag{58}$$

NOTE 16. It follows by linearity that the anticommutation relation

$$f(u)f(v) + f(v)f(u) = 2g(u, v)\mathbf{1} \tag{59}$$

is satisfied for all $u, v \in V$. As long as the scalar product is nondegenerate. the Clifford map is necessarily injective, and one can identify the vector space V with its image in the algebra. Assuming from now on the nondegeneracy of the scalar product one can write the anticommutation relation (59) in the form:

$$uv + vu = 2g(u, v)\mathbf{1} \tag{60}$$

Given V, g, there are many Clifford maps into different algebras. However. there is a special one:

DEFINITION 12. An algebra together with a Clifford map into it from the vector space V is the Clifford algebra of V and denoted by $\mathbf{Cl}(V, g)$ with the Clifford map γ if it satisfies the following universal mapping property:

For each Clifford map $f : V \to \mathbf{A}$ there exists a unique algebra homomorphism h such that the diagram

(61)
$$\begin{array}{ccc} V & \xrightarrow{\gamma} & \mathbf{Cl}(V,g) \\ & \searrow^{f} & \downarrow h \\ & & \mathbf{A} \end{array}$$

is commutative.

The existence of the Clifford algebra can be shown by a tensor algebra construction [11]. It follows from the universal mapping property that the Clifford algebra is unique up to isomorphisms (for the uniqueness of universals see [43]).

NOTE 17. It will later turn out that the Clifford algebras $\mathbf{Cl}(V,g)$ by themselves are mostly rather boring. It is the Clifford map γ that makes them interesting. It is an important part of the definition of this universal.

The universality of the Clifford algebra ensures immediately the existence of several important structures:

The main automorphism. Each linear isometric map h from vector space V to V',

$$h : V \to V'$$

induces a unique algebra map

$$\theta_h : \mathbf{Cl}(V,g) \to \mathbf{Cl}(V',g'),$$

since $\gamma' \circ h$ is a Clifford map, and by the universal mapping property there exists a unique algebra homomorphism θ_h making the following diagram commutative:

(62)
$$\begin{array}{ccc} V & \xrightarrow{\gamma} & \mathbf{Cl}(V,g) \\ h \downarrow & \searrow^{\gamma' \circ h} & \downarrow \theta_h \\ V' & \xrightarrow[\gamma']{} & \mathbf{Cl}(V',g') \end{array}$$

In particular this is true for all isometries of the vector space V, where now θ_h is an algebra automorphism of the Clifford algebra $\mathbf{Cl}(V,g)$ for each $h \in \mathbf{O}(V,g)$. These automorphisms are called Bogoljubov automorphisms. The map θ is a homomorphism of groups:

(63)
$$\theta : \mathbf{O}(V,g) \to \mathbf{Aut}(\mathbf{Cl}(V,g)),$$

where $\mathbf{Aut}(\mathbf{Cl}(V,g))$ is the automorphism group of $\mathbf{Cl}(V,g)$.

The most important of them, induced by the isometry $-id : v \to -v$ for all $v \in V$ is the main automorphism, denoted by α:

(64)
$$\alpha = \theta_{-id}$$

Since $-id \circ -id = id$, the main automorphism satisfies:

(65)
$$\alpha \circ \alpha = \mathbf{1},$$

and determines thus a grading on $\mathbf{Cl}(V,g)$ which will play an important role later in the analysis of the structure of Clifford algebras.

The main antiautomorphism. Consider the canonical antiautomorphism $\widetilde{id}$ from $\mathbf{Cl}(V,g)$ to its opposite algebra $\mathbf{Cl}(V,g)^\circ$. $\mathbf{Cl}(V,g)^\circ$ is identical to $\mathbf{Cl}(V,g)$ as a vector space, only the multiplication is reversed. Due to the symmetry of the anti-commutation relations (60) the composition $\widetilde{id}\circ\gamma : V \to \mathbf{Cl}(V,g)^\circ$ is a Clifford map, and thus by the universal mapping property there exists a unique automorphism $\beta : \mathbf{Cl}(V,g) \to \mathbf{Cl}(V,g)^\circ$ making the following diagram commutative:

(66)
$$\begin{array}{ccc} V & \xrightarrow{\gamma} & \mathbf{Cl}(V,g) \\ & \searrow^{\widetilde{id}\circ\gamma} & {\scriptstyle \widetilde{id}}\downarrow \ \downarrow{\scriptstyle\beta} \\ & & \mathbf{Cl}(V,g)^\circ \end{array}$$

One may interpret β as an antiautomorphism of $\mathbf{Cl}(V,g)$. This is the main anti-automorphism. It will play a key role later in providing an inner product on the space of spinors.

The main conjugation. If V is a real vector space with a scalar product g and $\mathbf{Cl}(V,g)$ its complex Clifford algebra, then the conjugate algebra, $\overline{\mathbf{Cl}}(V,g)$ is identical with $\mathbf{Cl}(V,g)$ as a ring but conjugate with respect to scalar multiplication. If $\overline{id}$ is the identification of $\mathbf{Cl}(V,g)$ and $\overline{\mathbf{Cl}}(V,g)$ as rings, then $\overline{id}\circ\gamma$ is a Clifford map. and by the universal mapping property there exists a unique algebra homomorphism κ from $\mathbf{Cl}(V,g)$ onto $\overline{\mathbf{Cl}}(V,g)$ making the following diagram commutative:

(67)
$$\begin{array}{ccc} V & \xrightarrow{\gamma} & \mathbf{Cl}(V,g) \\ & \searrow^{\overline{id}\circ\gamma} & {\scriptstyle \overline{id}}\downarrow \ \downarrow{\scriptstyle\kappa} \\ & & \overline{\mathbf{Cl}}(V,g) \end{array}$$

The map κ can be interpreted as an antilinear ring automorphism. This is the main conjugation. It will later be implemented in the spin representation by the charge conjugation.

The main conjugation κ and the main antiautomorphism β commute and their composition

$$* = \kappa\circ\beta = \beta\circ\kappa \tag{68}$$

is the main involution.

This completes the collection of important structures on a Clifford algebra. It was obtained relatively easily from the universal mapping property.

The structure of Clifford algebras. The main automorphism α provides a canonical grading on $\mathbf{Cl}(V,g)$. The Clifford algebra can then be decomposed into its even and odd parts $\mathbf{Cl}_0(V,g)$ and $\mathbf{Cl}_1(V,g)$:

$$\mathbf{Cl}(V,g) = \mathbf{Cl}_0(V,g) \oplus \mathbf{Cl}_1(V,g) \tag{69}$$

with $\mathbf{Cl}_0(V,g)$ being an algebra. The projections P_0. P_1 for the even and odd subspace are

$$P_0 = \frac{1}{2}(1+\alpha). \qquad\qquad P_1 = \frac{1}{2}(1-\alpha). \tag{70}$$

V is an odd subspace of Cl(V,g) since on it $\alpha|_V = -id|_V$.

It is easily seen [11, 38] that the Clifford algebra $\mathbf{Cl}(V, g)$ of an n-dimensional space V is as a vector space isomorphic to the Grassmanian ΛV of V and thus 2^n-dimensional. A basis in $\mathbf{Cl}(V, g)$ is easily given: Let $v_1, v_2, ..., v_n$ be an orthonormal system in V with respect to the scalar product g. Then a basis is the set

(71) $\left\{ v_{\{i_1, i_2, ..., i_k\}} = v_{i_1} v_{i_2} ... v_{i_k} \mid i_1 < i_2 < ... < i_k, \{i_1, i_2, ..., i_k\} \subset \{1, 2, ..., n\} \right\}.$

This is supposed to include

$$v_\emptyset = \mathbf{1}$$

Of particular importance is the volume element

(72) $$\eta = v_{\{1,2,...,n\}} = v_1 v_2 ... v_n$$

which is, up to its sign depending on an orientation, independent of the basis v_i. Depending on whether the vector space V is even or odd dimensional, the volume element η is even or odd with respect to the grading of $\mathbf{Cl}(V, g)$ and leads to substantial structural differences in the even and odd dimensional case.

Complex Clifford algebras. If V is a complex vector space, the corresponding Clifford algebra is isomorphic to a matrix algebra [11]. Denoting by $\mathbf{M}_n$ the algebra of complex $n \times n$-dimensional matrices one has:

(73) $$\mathbf{Cl}(V, g) \cong \begin{cases} \mathbf{M}_{2^m}(\mathbb{C}) & \text{for } dim V = 2m, \\ \mathbf{M}_{2^m}(\mathbb{C}) \oplus \mathbf{M}_{2^m}(\mathbb{C}) & \text{for } dim V = 2m+1. \end{cases}$$

In the even case the Clifford algebras are simple (i.e., have no nontrivial two-sided ideal) and central (i.e., with the centre consisting of the scalars $\lambda\mathbf{1}$, with $\lambda \in \mathbb{C}$). In the odd case the centre is spanned by 1 and η, and there are two central projections $\frac{1}{2}(\mathbf{1} \pm \eta)$ (assuming that the phase of η has been chosen so that $\eta^2 = 1$, as can in the complex case always be done). These central projections split $\mathbf{Cl}(V, g)$ into two simple central pieces.

Real Clifford algebras. The real case is a bit more complicated. Besides the dimension n of V causing differences, the scalar product g can now have different signatures. This is expressed in the form of two nonnegative integer indices k, l for g:

(74) $$g_{k,l} = v^1 v^1 + ... + v^k v^k - v^{k+1} v^{k+1} - ... - v^{k+l} v^{k+l}$$
$$\text{for a dual basis } v^1, ..., v^{k+l} \in V^*.$$

Not only can the signature of $g_{k,l}$ not be changed, as in the complex case by a suitable choice of phase, but also other signs appear with complicated relations to k, l.

The strategy to handle this situation is the following: A case study is necessary for low dimensional Clifford algebras, and higher dimensional Clifford algebras are then built from smaller ones.

The fundamental theorem for combining Clifford algebras is the following [36, 38]:

THEOREM 6. *Let V, W be two real vector spaces with scalar products g, h. Then there is a natural grading-preserving isomorphism of algebras with unity.*

$$\tilde{f} : \mathbf{Cl}(V \oplus W, g \oplus h) \to \mathbf{Cl}(V, g) \hat{\otimes} \mathbf{Cl}(W, h).$$

resulting from the Clifford map

$$f : V \oplus W \to \mathbf{Cl}(V, g) \hat{\otimes} \mathbf{Cl}(W, h) \tag{75}$$
$$(v, w) \to v \hat{\otimes} \mathbf{1} + \mathbf{1} \hat{\otimes} w$$

NOTE 18. In the theorem, $\hat{\otimes}$ is the graded tensor product. The algebra structure of a graded tensor product of two algebras $\mathbf{A}$, $\mathbf{B}$ is given by

$$(a_1 \hat{\otimes} b_1)(a_2 \hat{\otimes} b_2) = (-1)^{\partial b_1 \partial a_2} a_1 a_2 \hat{\otimes} b_1 b_2, \text{with } a_1, a_2 \in \mathbf{A} \text{ and } b_1, b_2 \in \mathbf{B}. \tag{76}$$

Here ∂b_1, ∂a_2 are the $\mathbb{Z}_2$-gradings of b_1, a_2. The proof of Theorem 6 is quite simple: One has only to check that f is a Clifford map, i.e. that $f(v, w)^2 = (g(v, v) + h(w, w))\,\mathbf{1}$, using the fact that V, W are odd. The rest follows from the universal mapping theorem and inspection of surjectivity for the generators.

A variation on this theorem is less general but very useful for avoiding graded tensor products by taking care of the grading using the volume element η defined in (72):

THEOREM 7. *Let $\lambda = \eta^2$ be the the square of the volume element η in an even dimensional real vector space W with a scalar product h and V a real vector space with scalar product g. Then there is a graded isomorphism of algebras with unity,*

$$\tilde{f} : \mathbf{Cl}(V \oplus W, g \oplus h) \to \mathbf{Cl}(V, g) \odot \mathbf{Cl}(W, h),$$

resulting from the Clifford map

$$f : V \oplus W \to \mathbf{Cl}(V, g) \odot \mathbf{Cl}(W, h)$$
$$(v, w) \to v \odot \eta + \mathbf{1} \odot w$$

NOTE 19. Again one has for a proof to check that f is a Clifford map and use the universal mapping property.

The case study of small real Clifford algebras and the combinatorial work of building larger ones can be found in [38]. Here, only the final results will be stated, using the shorthand $\mathbf{Cl}(k, l) = \mathbf{Cl}(V, g_{k,l})$.

The real Clifford algebras satisfy the periodicity property:

$$\mathbf{Cl}(k + 8, l) = \mathbf{Cl}(k, l + 8) = \mathbf{Cl}(k, l) \odot \mathbf{M}_{16}(\mathbb{R}) \tag{77}$$

It is therefore sufficient to find the Clifford algebras for $k, l \in \{0, 1, \ldots, 7\}$. This set of 8×8 algebras is called the spinorial chessboard [38] and is particularly important in the representation theory of real Clifford algebra which also exhibits a periodicity of *mod* 8.

A further reduction of the spinorial chessboard is possible:

$$\mathbf{Cl}(k + m, l + m) = \mathbf{Cl}(k, l) \odot \mathbf{M}_{2^m}(\mathbb{R}) \text{ for } k + l \geq 1. \tag{78}$$

Thus it is necessary to know only $\mathbf{Cl}(1, 1)$ and the zeroth line and the zeroth row on the spinorial chessboard. This can be tabulated as follows [38]:

$$\mathbf{Cl}(1, 1) = \mathbf{A}_0. \tag{79}$$

$$\mathbf{Cl}(\nu, 0) = A_\nu \odot \mathbf{M}_{2^{\frac{1}{2}(\nu - n_\nu)}}(\mathbb{R}). \tag{80}$$

$$\mathbf{Cl}(0, 8 - \nu) = \mathbf{A}_\nu \odot \mathbf{M}_{2^{4 - \frac{1}{2}(\nu + n_\nu)}}(\mathbb{R}), \tag{81}$$

where

(82)

ν	0	1	2	3	4	5	6	7
$\mathbf{A}_\nu$	$\mathbf{M}_2(\mathbb{R})$	$\mathbb{R}\oplus\mathbb{R}$	$\mathbf{M}_2(\mathbb{R})$	$\mathbf{M}_2(\mathbb{C})$	$\mathbf{M}_2(\mathbb{H})$	$\mathbb{H}\oplus\mathbb{H}$	$\mathbb{H}$	$\mathbb{C}$
n_ν	2	1	2	3	4	3	2	1

NOTE 20. The algebra $\mathbb{H}$ is the algebra of quaternions generated as a real unital algebra by two elements i, j satisfying

(83) $$ij + ji = 0, \qquad i^2 = -1, \qquad j^2 = -1.$$

This completes the classification of finite Clifford algebras.

3. Representations of Clifford algebras

Algebras and their representations. The important features of representations of Clifford algebras have a deeper root in the general representation theory of algebras. Therefore, some of its simple but fundamental definitions and theorems are recalled here first.

DEFINITION 13. The centre of an algebra $\mathbf{A}$ over a field K is the subalgebra $\mathbf{Z}(\mathbf{A})$ of the elements in $\mathbf{A}$ commuting with all elements of $\mathbf{A}$. An algebra $\mathbf{A}$ is central if $\mathbf{Z}(\mathbf{A}) = K$.

NOTE 21. The fields of interest here are $K = \mathbb{R}$ and $K = \mathbb{C}$.

DEFINITION 14. An algebra $\mathbf{A}$ is simple if it has no two-sided ideal except $\mathbf{A}$ and $\{0\}$.

EXAMPLE 17. The matrix algebras of real numbers or quaternions, $\mathbf{M}_n(\mathbb{R})$ and $\mathbf{M}_n(\mathbb{H})$ are central and simple as $\mathbb{R}$-algebras. The complex matrix algebra $\mathbf{M}_n(\mathbb{C})$ is central and simple as a $\mathbb{C}$-algebra but fails to be so as an $\mathbb{R}$-algebra since its centre is 2-dimensional in $\mathbb{R}$, namely $\mathbb{C}$. In this connection the following isomorphisms of algebras

(84) $$\mathbf{M}_m(\mathbb{R}) \otimes \mathbf{M}_n(\mathbb{R}) = \mathbf{M}_{mn}(\mathbb{R})$$

(85) $$\mathbf{M}_n(L) = L \otimes \mathbf{M}_n(\mathbb{R}) \qquad \text{for } L = \mathbb{R}, \mathbb{C}, \mathbb{H}.$$

are quite useful as well as the fact [44] that the tensor product of a simple algebra and a central simple algebra is simple.

DEFINITION 15. A *representation* of an algebra $\mathbf{A}$ over the field K is a homomorphism γ from $\mathbf{A}$ into the algebra of operators on a vector space S, the *representation space*.

$$\gamma : \mathbf{A} \to \mathbf{End}_K S$$

The representation is *faithful* if it is injective. A subspace of the representation space S is *invariant* if it is closed under the action of the representation. The representation is *irreducible* if there is no invariant subspace of S apart from S and $\{0\}$.

REMARK 1. The restriction of a representation to an invariant subspace is a representation.

Each algebra $\mathbf{A}$ has the *regular* representation on itself, given by left multiplication. In this case $S = \mathbf{A}$:

(86) $$\gamma(a)b = ab \qquad \text{for } a \in \mathbf{A}, b \in S = \mathbf{A}.$$

If $\mathbf{A}$ is unital then this representation is faithful. If $\mathbf{A}$ is simple, finite dimensional and nonzero then the regular representation must contain a nonzero minimal invariant subspace. The restriction to this subspace is clearly a nontrivial irreducible representation.

Two representations $\gamma_1 : \mathbf{A} \to \mathbf{End}_K S_1$, $\gamma_2 : \mathbf{A} \to \mathbf{End}_K S_2$ of an algebra $\mathbf{A}$ may be connected by an *intertwining transformation*, i.e. a linear map $F : S_1 \to S_2$ satisfying

$$F\gamma_1(a) = \gamma_2(a)F \qquad \text{for all } a \in \mathbf{A}. \tag{87}$$

If F is bijective, then the representations γ_1, γ_2 are said to be *equivalent*.

The subspaces $\ker F \subset S_1$, $F(S_1) \subset S_2$ are invariant spaces of γ_1, γ_2 and thus one has

SCHUR'S LEMMA. *Let $F : S_1 \to S_2$ be a nonzero intertwining transformation between two representations γ_1, γ_2 of the algebra $\mathbf{A}$ on S_1, S_2. Then the following holds:*

If γ_1 is irreducible, then F is injective.
If γ_2 is irreducible, then F is surjective.
If γ_1, γ_2 are irreducible, then F is bijective.

COROLLARY 8. *In particular, if γ $(= \gamma_1 = \gamma_2)$ is irreducible then all operators F commuting with $\gamma(\mathbf{A})$ (i.e. all elements of the* commutant $\mathbf{A}'$ *of $\mathbf{A}$) and nonzero are intertwining and thus invertible.*

If the representation is complex and finite, then each of these operators F has at least one nonzero eigenvalue λ and must then be automatically equal to $\lambda\mathbf{1}$ since $F - \lambda\mathbf{1}$ is not invertible and thus necessarily zero.

COROLLARY 9. *Two nonzero intertwining transformations F, G between two irreducible representations γ_1, γ_2 must be proportional by an invertible operator in the corresponding commutant. Indeed, $G^{-1}F$ and FG^{-1} satisfy:*

$$G^{-1}F\gamma_1 = \gamma_1 G^{-1}F \qquad FG^{-1}\gamma_2 = \gamma_2 FG^{-1} \tag{88}$$

If the representations γ_1, γ_2 are complex, then one has from Corollary 8 that the nonzero intertwining transformation is unique up to multiplication by a complex number:

$$F = \lambda G \qquad \text{for some nonzero } \lambda \in \mathbb{C} \tag{89}$$

THEOREM 10. *Let $\mathbf{A}$ be a finite dimensional and simple algebra. Then all its faithful irreducible representations are equivalent.*

PROOF. Unless $\mathbf{A} = 0$, there exists a nonzero minimal invariant subspace $\mathbf{B}$ in the regular representation of $\mathbf{A}$ providing an irreducible representation $\gamma_{\mathbf{B}}$ of $\mathbf{A}$. It is now easy to show that any faithful irreducible representation $\gamma : \mathbf{A} \to S$ must be equivalent to it. Since γ is faithful, there exists a vector $\psi \in S$ and $b \in \mathbf{B} \subset \mathbf{A}$ such that $b\psi \neq 0$ and an intertwining operator can then be given by

$$F : \mathbf{B} \to S \tag{90}$$

$$b \to \gamma(b)\psi \tag{91}$$

The equivalence then follows from Schur's lemma. □

Theorem 10 is of great importance to Clifford algebras, since they are either simple or a sum of two simple parts, as can be verified with the help of Example 17 by checking the classification results on Clifford algebras from the previous section.

There are two consequences.

First, it is not strictly necessary to distinguish between the faithful irreducible representations of a simple part of a Clifford algebra since all of them are isomorphic. This partially justifies the historically determined use of the same symbol γ for both the Clifford map of a Clifford algebra and for its representation. In the cases in which the Clifford algebras have two inequivalent representations, they will be distinguished as γ_L, γ_R, with L, R standing for *left-handed* and *right-handed*.

Second, Theorem 10 allows one to play a game similar to the one played with the universal mapping property in Section 2: If one can from an irreducible representation, using some of its particular features, construct a second one, then Theorem 10 automatically ensures an intertwining transformation and Corollary 9 shows its uniqueness up to an element in the commutant of the representation. All important structures, including the Dirac products and the charge conjugation, are produced in this way.

It is now appropriate to consider the particular constructions. Since only complex representations of real Clifford algebras have a charge conjugation and only those will be of physical interest, the discussion will be from now on restricted to the case of complex representations of real algebras.

Inner products on spinors. Let $\mathbf{A}$ be finite simple algebra with a faithful irreducible representation $\gamma : \mathbf{A} \to \mathbf{End}S$. Then an antiautomorphism β on $\mathbf{A}$ determines an irreducible representation $\tilde{\gamma}$ on the dual S^* of S by

$$\tilde{\gamma} : \mathbf{A} \to \mathbf{End}S^* \tag{92}$$

$$a \to \gamma(\beta(a))^T \tag{93}$$

with T denoting the transposition, i.e., the action induced on the dual S^*. The representations γ, $\tilde{\gamma}$ are intertwined by an isomorphism B (determined up to the multiplication by a nonvanishing complex number):

$$B : S \to S^* \tag{94}$$

$$\tilde{\gamma}(a)B = B\gamma(a) \tag{95}$$

That is,

$$\gamma(\beta(a))^T B = B\gamma(a) \qquad \text{for all } a \in \mathbf{A}. \tag{96}$$

B is a nondegenerate inner product (2-form on S). The intertwining relation for its inverse is

$$B^{-1}\gamma(\beta(a))^T = \gamma(a)B^{-1} \qquad \text{for all } a \in \mathbf{A}. \tag{97}$$

The following short calculation shows, using the intertwining relations, that if β is involutive, i.e., if $\beta^2 = \mathbf{1}$, then $B^{-1}B^T$ is in the commutant of the representation γ:

$$\gamma(a)B^{-1}B^T = B^{-1}\gamma(\beta(a))^T B^T = B^{-1}(B\gamma(\beta(a)))^T = \tag{98}$$

$$= B^{-1}B^T\gamma(\beta^2 a) \tag{99}$$

Assuming the representations to be complex, Corollary 8 ensures that

(100) $B^{-1}B^T = \lambda \mathbf{1}$ for some nonzero $\lambda \in \mathbb{C}$.

Together with the identity $(B^T)^T = B$ this results in:

(101) $B^T = \pm B$

The inner product B is thus either symmetric or antisymmmetric. Which of these possibility occurs depends on the particular algebra.

All of the above applies strictly to Clifford algebras $\mathrm{Cl}(k,l)$ with $k + l = 2m$ even only, since those are central and simple. In this case it is relatively easy to decide the type of symmetry of B.

First, a complex irreducible representation of a central simple Clifford algebra extends in an obvious way to its complexification

$$\mathrm{Cl}(2m) = \mathbb{C} \otimes \mathrm{Cl}(k,l).$$

The symmetry type of B cannot be affected by that and depends thus on the total dimension $2m = k + l$ only.

Second, by transposition and insertion of $\mathbf{1} = B^{-1}B$ one obtains from (95):

(102) $(B^T B^{-1})\quad B\gamma(\beta(a)) = (B\gamma(a))^T$ for all $a \in \mathrm{Cl}(2m)$.

Taking the basis v_I in $\mathrm{Cl}(2m)$ with $I \subset \{1, 2, ..., 2m\}$ (see (71)) . the action of the main antiautomorphism β on it is, using the the defining property of the Clifford map, given by:

(103) $\beta(v_{i_1} v_{i_2} ... v_{i_q}) = (-1)^{\frac{q(q-1)}{2}} v_{i_1} v_{i_2} ... v_{i_q}$

So $\mathrm{Cl}(2m)$ decomposes as a vector space into

(104) $\mathrm{Cl}(2m) = \mathrm{Cl}^+(2m) \dotplus \mathrm{Cl}^-(2m)$.

$$\text{with} \begin{cases} \mathrm{Cl}^+(2m) \text{spanned by } \{v_I \mid \beta(v_I) = v_I\}. \\ \mathrm{Cl}^-(2m) \text{spanned by } \{v_I \mid \beta(v_I) = -v_I\}. \end{cases}$$

If it is now assumed that B is symmetric, then $B^T B^{-1} = \mathbf{1}$, and (102) gives

(105) B symmetric $\Rightarrow$ $B\gamma(a)$ is $\begin{cases} \text{symmetric for } a \in \mathrm{Cl}^+(2m). \\ \text{antisymmetric for } a \in \mathrm{Cl}^-(2m). \end{cases}$

If it is now assumed that B is antisymmetric, then $B^T B^{-1} = -\mathbf{1}$, and (102) gives

(106) B antisymmetric $\Rightarrow$ $B\gamma(a)$ is $\begin{cases} \text{symmetric for } a \in \mathrm{Cl}^-(2m). \\ \text{antisymmetric for } a \in \mathrm{Cl}^+(2m). \end{cases}$

Due to the faithfulness of the representation γ, the nondegeneracy of B and the same dimension of $\mathrm{Cl}(2m) \cong M_{2^m}$ and the space $S^* \otimes S^*$, the map

(107) $B\gamma : \mathrm{Cl}(2m) \to S^* \otimes S^*$

is an isomorphism. But the space of symmetric forms is of higher dimension than the space of antisymmetric forms. In view of (105) and (106), it is then sufficient to know the dimensions of $\mathbf{Cl}^+(2m)$ and $\mathbf{Cl}^-(2m)$ in order to find the symmetry type of B. That reduces to a purely combinatorial calculation:

$$B^T B^{-1} = sgn\,(\dim \mathbf{Cl}^+ - \dim \mathbf{Cl}^-) = (-1)^{\frac{m(m-1)}{2}}. \tag{108}$$

So the inner product B on spinors for the signature (k,l) satisfies

$$B^T = (-1)^{\frac{(k+l)(k+l-2)}{8}} B. \tag{109}$$

Charge conjugation. Given a complex representation γ of a real algebra $\mathbf{A}$ on S, one can obtain a new representation $\bar{\gamma}$ on the complex conjugate space $\bar{S}$ of S just by complex conjugation. Similar to the previous, an intertwining transformation C, the charge conjugation, can be found. More precisely, the following theorem holds [38]:

THEOREM 11. *Let $\gamma : \mathbf{A} \to \mathbf{End}S$ be a complex faithful and irreducible representation of a central simple real algebra. Then there exists a complex-linear isomorphism $C : S \to \bar{S}$ which intertwines the representations γ, $\bar{\gamma}$, giving*

$$\bar{\gamma}(a)C = C\gamma(a). \tag{110}$$

and is such that

$$\text{either } \bar{C}C = id \text{ or } \bar{C}C = -id. \tag{111}$$

Moreover, if $\mathbf{A}$ has an involutive antiautomorphism β, then the isomorphism $B : S \to S^*$ intertwining γ, $\bar{\gamma}$ can be chosen so that the linear map

$$A = \bar{B}C : S \to \bar{S}^* \tag{112}$$

is Hermitean.

NOTE 22. The centrality of $\mathbf{A}$ is required in order to ensure the simplicity of the complexification $\mathbb{C} \otimes \mathbf{A}$ of $\mathbf{A}$. The conditions (111) are obtained from (110) and its complex conjugate by showing that $\bar{C}C$ is in the commutant of γ and by a proper scaling of C. The hermiticity of A is ensured by a proper choice of a phase: B by itself is determined uniquely up to multiplication by a complex number.

If $\bar{C}C = +1$, then the complex representation can be split into two isomorphic real ones on spaces S^+, S^-:

$$S^{\pm} = \{v \in S \mid \bar{v} = \pm Cv\}. \tag{113}$$

The isomorphism between S^+ and S^- is given by multiplication by i.

If $\bar{C}C = -1$, then there is a quaternionic structure on S (see Note 20) generated from the imaginary unit i and from j defined as

$$jv = \overline{Cv} \qquad \text{for all } v \in S. \tag{114}$$

It follows from the classification of Clifford algebras (see Table 82) that

$$\bar{C}C = \begin{cases} +1 & \text{for } k-l=0 \mod 8 \text{ or } k-l=2 \mod 8 \\ -1 & \text{for } k-l=4 \mod 8 \text{ or } k-l=6 \mod 8 \end{cases} \tag{115}$$

Helicity (Chirality). The two above constructions apply in particular to complex representations of Clifford algebras $\mathbf{Cl}(k,l)$ with $k+l$ even, but are of quite general nature. There is, in the same spirit, a third construction using the main automorphism α specific to Clifford algebras. Composing an irreducible representation of a simple Clifford algebra with the main automorphism α, a new representation γ° is obtained:

$$\begin{aligned} \gamma^\circ : \mathbf{Cl}(k,l) &\to \mathbf{End}S \\ a &\to \gamma(\alpha(a)) \end{aligned} \tag{116}$$

This representation is intertwined with the initial one by the operator Γ, the helicity (chirality) operator:

$$\gamma^\circ(a) = \Gamma\gamma(a)\Gamma^{-1} \qquad \text{for all } a \in \mathbf{Cl}(k,l). \tag{117}$$

Composing the representation γ with the main automorphism α twice does not give a new representation, since $\alpha^2 = id$. However, it has the important consequence that Γ can be chosen so that

$$\Gamma^2 = 1, \tag{118}$$

hence $\Gamma^{-1} = \Gamma$. It also follows from (118) that the helicity operator Γ provides a grading on the representation space S, the space of spinors. The splitting of spinors with respect to the grading Γ into the direct sum of two subspaces S_+, S_- is given by the projectors

$$P_+ = \frac{1}{2}(1+\Gamma), \qquad P_- = \frac{1}{2}(1-\Gamma) \tag{119}$$

For simple Clifford algebras $\mathbf{Cl}(k,l)$, i.e., those with $k+l$ even, the representation of the helicity operator is proportional to the representation of the volume element (72):

$$\Gamma = \pm\lambda^{-\frac{1}{2}}\eta \qquad \text{with } \lambda = \eta^2 \tag{120}$$

This is indeed correct, since η anticommutes with all $\gamma(v)$ for v in the vector space V generating the Clifford algebra:

$$\gamma(v)\gamma(\eta) = -\gamma(\eta)\gamma(v) \quad \text{for all } v \in V, \tag{121}$$

and therefore

$$\begin{aligned} \Gamma\gamma(v)\Gamma &= \pm\lambda^{-\frac{1}{2}}\gamma(\eta)\gamma(v)(\pm\lambda^{-\frac{1}{2}})\gamma(\eta) = \frac{1}{\lambda}\gamma(\eta)\gamma(v)\gamma(\eta) = \\ &= -\frac{1}{\lambda}\gamma(\eta^2)\gamma(v) = \gamma(-v) = \\ &= \gamma(\alpha(v)) \qquad \text{for all } v \in V \end{aligned} \tag{122}$$

The result follows by extension from the space V, since it generates $\mathbf{Cl}(k,l)$.

From (121) one has that the helicity operator Γ which is proportional to η also anticommutes with $\gamma(a)$ for odd $a \in \mathbf{Cl}(k,l)$. This means that the grading of the Clifford algebra given by the main automorphism α is in agreement with the grading given by the helicity operator Γ which turns the representation γ into a graded representation. This can be established in detail using equation (119).

In order to distinguish general spinors in S from the ones in the even and odd subspaces S_+, S_-, the former are called Dirac spinors and later are called Weyl spinors (or reduced spinors) of positive or negative helicity (or of right or left chirality, respectively).

For the Clifford algebra $\mathrm{Cl}(k,l)$ the square of the volume element η is

$$\eta^2 = (-1)^{\frac{(k-l)(k-l-1)}{2}}. \tag{123}$$

The helicity operator Γ is then

$$\Gamma = \begin{cases} \pm\gamma(\eta) & \text{for } k-l=0 \mod 8 \text{ or } k-l=4 \mod 8. \\ \pm i\gamma(\eta) & \text{for } k-l=2 \mod 8 \text{ or } k-l=6 \mod 8. \end{cases} \tag{124}$$

Depending on which case occurs, the helicity operator satisfies either

$$\bar{\Gamma}C = C\Gamma \qquad \text{for } \eta^2 = 1 \tag{125}$$

or

$$-\bar{\Gamma}C = C\Gamma \qquad \text{for } \eta^2 = -1. \tag{126}$$

since equation (110) in the special case of the volume element $\eta \in \mathrm{Cl}(k,l)$ gives

$$\bar{\gamma}(\eta)C = C\gamma(\eta). \tag{127}$$

and η is proportional to the helicity operator Γ.

Assuming that the grading in the space $\bar{S}$ of complex conjugate spinors is given by the complex conjugate $\bar{\Gamma}$, Equations (125) and (126) give the isomorphisms

$$C : S_\pm \to \bar{S}_\pm \qquad \text{for } \eta^2 = 1 \tag{128}$$

$$C : S_\pm \to \bar{S}_\mp \qquad \text{for } \eta^2 = -1 \tag{129}$$

Since, as already mentioned in connection with equation (121), the representation γ of $\mathrm{Cl}(k,l)$ is a graded representation on S, the even and odd subspaces S_+ and S_- of Weyl spinors are preserved under the action of the even subalgebra $\mathrm{Cl}_0(k,l)$. Thus the restriction of the representation γ decomposes into two (irreducible) representations, γ_+ and γ_-, of $\mathrm{Cl}_0(k,l)$. It follows then from equations (125), (126) that the isomorphisms (128) and (129) intertwine γ_+ and γ_- with the complex conjugate representations $\bar{\gamma}_+$ and $\bar{\gamma}_-$:

$$\gamma_\pm = \bar{C}\bar{\gamma}_\pm C \qquad \text{for } \eta^2 = 1. \tag{130}$$

$$\gamma_\mp = \bar{C}\bar{\gamma}_\pm C \qquad \text{for } \eta^2 = -1. \tag{131}$$

Also the commutation relation of the helicity operator Γ with the canonical inner product can be easily established, noting that the volume element η (with $\gamma(\eta)$ proportional to Γ) satisfies

$$\beta(\eta) = (-1)^{\frac{k+l}{2}}\eta. \tag{132}$$

From equation (96) one has then:

$$\Gamma^T B = (-1)^{\frac{k+l}{2}} B\Gamma \tag{133}$$

From all this one can see that the helicity (chirality) operator plays a very important role in the representation theory of Clifford algebras, underpinned later by the use of one of its important products, the Weyl spinors.

Other structures on spinors. The Dirac products. The strategy for finding structures on spinors was so far to construct in a new way another irreducible representation and to obtain an intertwining operator. The ways used to construct new representations were:

- transposition composed with the main antiautomorphism β.

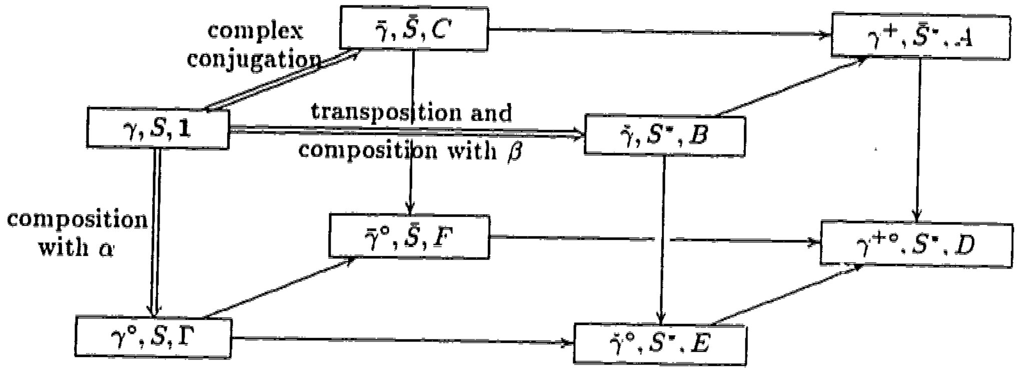

FIGURE A.1
Related spinor representations. The diagram shows by double arrows three ways of producing out of a representation γ of a simple Clifford algebra new ones and is then filled in with their combinations. Each representation is characterized by its name, representation space, and intertwining operator. The definitions of the symbols not mentioned in the text are obvious from the diagram.

- complex conjugation of the representation.
- composition with the main automorphism.

It is now possible to combine these constructions. A first example of this was the construction of the Hermitean product A on spinors, see Equation 112. It was a combination of complex conjugation and transposition. The variety of possibilities is summarized in Figure 3

Two of the possibilities will now be considered in detail:

First, the main automorphism α and the main antiautomorphism β commute, and their composition is an involutive antiautomorphism producing by transposition a new irreducible representation from the irreducible representation γ. The intertwining operator E,

(134) $$E : S \to S^*,$$

satisfies

(135) $$\gamma(\beta(\alpha(a))^T E = E\gamma(a) \qquad \text{for all } a \in \mathrm{Cl}(k,l)$$

and is either a symmetric or an antisymmetric inner product by the same reasoning that led to equation (101) in the case of the inner product B. From the way it was constructed, it follows that with suitable choices of scalar factors

(136) $$E = B\Gamma$$

Second, using all three ways of constructing a new representation at once, i.e. transposition, complex conjugation and composition with the main automorphism, one gets an intertwining operator D

(137) $$D : S \to \bar{S}^*$$

satisfying

(138) $$\gamma(\beta(\alpha(a))^+ D = D\gamma(a) \qquad \text{for all } a \in \mathrm{Cl}(k,l)$$

where + denotes the composition of transposition and conjugation. The operators E and D are by Corollary 9 not determined uniquely, but only up to the multiplication by a nonzero complex number. Using this freedom, one can, in analogy with Theorem 11 achieve that D is Hermitean and related to E by:

$$D = \bar{E}C \tag{139}$$

The Hermitean products A, D are called Dirac products.

Summary of structures on spinors. Given a complex representation of a real simple Clifford algebra with Dirac spinors as the representation space, the following structures can be found:

There are two inner products B, E on the Dirac spinors. Each of them may be symmetric or antisymmetric.

There is a grading operator Γ, the helicity operator, splitting Dirac spinors into Weyl spinors. It may be real or purely imaginary and may commute or anticommute with B and E.

There is a complex linear charge conjugation map C from Dirac spinors to the complex conjugated Dirac spinors. It satisfies $\bar{C}C = \mathbf{1}$ or $\bar{C}C = -\mathbf{1}$.

There are two (Hermitean) Dirac products A and D on the Dirac spinors. The Dirac conjugation $D\psi$ of a spinor ψ given by the Dirac product D is in the physical applications denoted by

$$\tilde{\psi} = D\psi. \tag{140}$$

All the two-valued choices in the properties of B, C, E and Γ above depend on the signature of the scalar product used to generate the Clifford algebra $\mathrm{Cl}(k, l)$. They exhibit in their dependence on the signature a periodicity [38] in each of the nonnegative integer parameters k, l with period 8, enforcing thus again the picture of a spinorial chessboard as it already appeared in the classification of Clifford algebras in Section 2.

APPENDIX B

Noncommutative Geometry

1. Introduction

Classical geometry begins with the concept of a point [45]. The points are the indivisible building blocks of spaces and parts of spaces, the basic objects of interest. The idea of a space consisting of points has to be given first. It may then be equipped with further structures, such as coordinates, topology, measure or smooth structure - they simply come second.

But it is actually these secondary structures that makes a space an object worth investigating, and thus it is somewhat pleasing to note that these concepts can exist also on their own. This comes about in the following way: The points of a space can be organized and described by coordinates, and the coordinates may be given the structure of a commutative algebra. Once we are given this algebra of coordinates, we can completely forget about our space and its points and build up all the structures of interest on the algebra.

Even the space can now be reconstructed from the algebra if one wishes to do so, and each commutative algebra gives in this way a space. The statement of this fact in the case of topological spaces is the fundamental Gel'fand-Naimark theorem [12]. But then it is not inevitable that the space and its points be the fundamental objects anymore. The position of the basic object is now taken by the algebra of coordinates.

Now one can leap from the above change in our point of view of geometry to a changed paradigm: The concepts of classical geometry can be translated to algebraic concepts, and with the right translation they make sense regardless of whether the algebra of coordinates is commutative or noncommutative. With topology, measure theory and differential geometry extended to the vast territory of noncommutative algebras, it is fair to say that almost all the central concepts of classical geometry are just special manifestations of general algebraic concepts. These are now expressed in their natural environment and are the subject of *noncommutative geometry*.

The manifestations of a paradigm shift are clearly present [46]: The old basic notions become derived objects, the understanding of what is important changes, some concepts split up, with the results recognized as separate and coinciding in special cases only. Some problems set entirely in the context of the old framework become at once easy to solve [27].

An important example of this is the concept of a point. In the context of the Gel'fand-Naimark theorem, points are understood as the irreducible representations of the algebra in question. But many noncommutative algebras, among them the matrix algebras of any dimension, have (up to isomorphism) only one irreducible representation. They are all just one-point spaces, but their differential calculus is far from trivial. The idea of a point is in these cases somewhat useless.

The splitting of concepts is what makes noncommutative differential geometry to some degree tricky: The exterior calculus detaches somewhat from ideas around vector fields and diffeomorphisms, and tangent spaces are no more vector bundles.

In order not to lose orientation, it is useful to set up a dictionary between classical and noncommutative geometry. From what was said above it is clear that the dictionary cannot be one to one, and it is not possible to understand fully the noncommutative entries by looking at their classical counterparts, since the latter work only for commutative algebras. But that is not the point of the dictionary. Whoever has worked with the subdictionary between classical and quantum mechanics, found by physicists, should be aware of that. The dictionary is useful, in one direction, for guessing and remembering the structure of the theory once the classical counterparts are clear and learned and, in the other direction, in providing examples and classical interpretations for suitable cases.

The dictionary between classical and noncommutative geometry may look as follows:

	Topology:	
topological space	⟷	commutative C^*-algebra
continuous map	⟷	algebra homomorphism
homeomorphism	⟷	algebra automorphism
open subset	⟷	ideal
point	⟷	irreducible representation
closed subset	⟷	quotient
open dense subset	⟷	essential ideal
one-point compactification	⟷	unitization
connected	⟷	without (central) projections
metrizable	⟷	separable
vector bundle	⟷	finite projective module
K-theory	⟷	algebraic K-theory
	Measure theory:	
measure	⟷	positive linear functional
algebra of L^∞-functions	⟷	von Neumann algebra
Radon derivative	⟷	modular operator
equivalence of measures	⟷	equivalence of GNS-representations
	Differential geometry:	
forms	⟷	Hochschild homology
currents	⟷	Hochschild cohomology
de Rham complex	⟷	Cyclic cohomology
elliptic operator	⟷	Fredholm module
	Other:	
Semigroup	⟷	Bialgebra
Group	⟷	Hopf algebra (Quantum group)

It should be noted that, while the dictionary is straightforward, it also may be a bit confusing. E.g., in the first entry of our dictionary, we have a topological space as being the counterpart of an algebra. But the algebra does not correspond to the space directly, but to the space of functions on that (non-existent, in the noncommutative case) space. So the first entry should rather be:

continuous functions on a topological space $\longleftrightarrow$ commutative C^*-algebra

However that would force one into clumsy terminology or into the definition of new names for dual concepts (as in the case of quantum groups).

2. C*-Algebras

This part is a short introduction to C*-algebras (For more detail see e.g. [47, 12]). Their importance is that C*-algebras play in noncommutative geometry the role of topological spaces and thus one can view this section to some degree as an introduction into the basics of general topology.

A C*-algebra **A** is a complex associative algebra with an involution $(\bullet)^*$ and with a special norm $\| \bullet \|$ in which **A** is complete.

That **A** is a complex algebra means that it is a complex vector space together with a multiplication $\mathbf{m} : \mathbf{A} \times \mathbf{A} \to \mathbf{A}$ between vectors:

$$\mathbf{ab} := \mathbf{m}(\mathbf{a}, \mathbf{b}) \qquad \text{for } \mathbf{a}, \mathbf{b} \in \mathbf{A}. \tag{141}$$

The multiplication is required to be distributive:

$$(\mu \mathbf{a} + \nu \mathbf{b})\mathbf{c} = \mu(\mathbf{ac}) + \nu(\mathbf{bc}), \tag{142}$$

$$\mathbf{c}(\mu \mathbf{a} + \nu \mathbf{b}) = \mu(\mathbf{ca}) + \nu(\mathbf{cb}), \tag{143}$$

for $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbf{A}$ and $\mu, \nu \in \mathbb{C}$.

An associative algebra satisfies moreover

$$(\mathbf{ab})\mathbf{c} = \mathbf{a}(\mathbf{bc}) \tag{144}$$

for $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbf{A}$.

The involution $(\bullet)^*$ is defined by the following properties:

$$(\alpha \mathbf{a})^* = \alpha^* \mathbf{a}^*, \tag{145}$$

$$(\mathbf{a} + \mathbf{b})^* = \mathbf{a}^* + \mathbf{b}^*, \tag{146}$$

$$(\mathbf{ab})^* = \mathbf{b}^* \mathbf{a}^*, \tag{147}$$

$$(\mathbf{a}^*)^* = \mathbf{a}, \tag{148}$$

for $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbf{A}$ and $\alpha, \alpha^* \in \mathbb{C}$

A norm $\| \bullet \|$ on an algebra **A** is a norm on **A** as a vector space satisfying moreover

$$\| \mathbf{ab} \| \leq \| \mathbf{a} \| \| \mathbf{b} \|, \tag{149}$$

$$\| \mathbf{a}^* \| = \| \mathbf{a} \| . \tag{150}$$

A norm complete algebra with all the above structures is called a Banach algebra. Among them the C*-algebras are distinguished by the C*-equation:

$$\| \mathbf{a}^* \mathbf{a} \| = \| \mathbf{a} \|^2 \tag{151}$$

This completes the definition of a C*-algebra.

The C*-equation is very restrictive and has several important consequence:

The topology of the C*-algebra is determined uniquely by its algebraic structure and the involution.

$*$-homomorphisms, i.e., mappings from one C*-algebra to another preserving the algebraic structure and the involution, are necessarily norm decreasing. $*$-isomorphisms are necessarily norm preserving.

There is a particularly simple representation theory, and one can show (using the Gel'fand-Naimark-Segal or GNS construction [12]) that each C*-algebra ($\mathcal{A}$) is isomorphic to a subalgebra **A** of the bounded operators $\mathbb{B}(\ )$ on a Hilbert space . The norm $\| \bullet \|$ is then just the operator norm, and the star operation is given by the adjoint operation on operators.

EXAMPLE 18. Each subalgebra of operators $\mathbb{B}(\ )$ on a Hilbert space , complete in the norm topology and closed under taking adjoints, is a C*-algebra.

This could actually be taken as an alternative and more concrete definition of C*-algebras, since, as mentioned above, each C*-algebra is isomorphic to an operator algebra of this kind. However, from a structural point of view it is more convenient to adopt the more abstract approach.

EXAMPLE 19. The continuous complex functions on a locally compact Hausdorff space **X**, vanishing at infinity and equipped with point by point addition, multiplication, multiplication by scalars and involution and with the supremum topology form a commutative C*-algebra.

Example 19 gives actually the general commutative case as stated by the Gel'fand-Naimark theorem [48, 12]. In order to give a formulation of this fundamental theorem which justifies the claim that C^*-algebras play in noncommutative geometry the role of topological spaces, the following definition is given:

DEFINITION 16. Let **A** be a commutative C^*-algebra. A character ω of **A** is a nonzero linear map $\omega : \mathbf{A} \to \mathbb{C}$ such that

$$\omega(ab) = \omega(a)\omega(b) \tag{152}$$

GEL'FAND-NAIMARK THEOREM. *Let* **A** *be a commutative C^*-algebra and X the set of characters of* **A** *equipped with the weak* topology inherited from the dual* $\mathbf{A}^*$ *of* **A**. *It follows that X is a locally compact Hausdorff space which is compact if and only if* **A** *contains the identity. Moreover,* **A** *is isomorphic to the algebra $C(X)$ of continuous functions over X which vanish at infinity.*

The relationship between C^*-algebras and topological Hausdorff spaces can be used to formulate noncommutative algebraic topology, in particular algebraic K-theory (see [49, 50]), which fits very well with the classical theory and, as will be mentioned later, with noncommutative differential geometry.

3. Hochschild homology, cyclic homology and differential forms

The structure of algebras can in general be analysed in terms of homological algebra. This leads one to Hochschild homology [51], known for a long time.

DEFINITION 17. The Hochschild homology HH_* of an algebra $\mathbf{A}$ is obtained from the complex

(153)
$$\dots \xrightarrow{b} C_n \xrightarrow{b} C_{n-1} \xrightarrow{b} \dots \xrightarrow{b} C_1 \xrightarrow{b} C_0 \xrightarrow{b} 0$$

with the chains $C_n = \mathbf{A} \otimes \mathbf{A}^{\otimes n}$,

$$\dots \xrightarrow{b} \mathbf{A} \otimes \mathbf{A}^{\otimes n} \xrightarrow{b} \mathbf{A} \otimes \mathbf{A}^{\otimes n-1} \xrightarrow{b} \dots \xrightarrow{b} \mathbf{A} \otimes \mathbf{A} \xrightarrow{b} \mathbf{A} \xrightarrow{b} 0$$

and with b being the Hochschild boundary:

(154) $$b(a_0 \otimes \dots \otimes a_n) = \sum_{i=0}^{n-1} a_0 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_n + (-1)^n a_n a_0 \otimes \dots \otimes a_{n-1}.$$

A variant of this is the subcomplex of (153) consisting of the subspaces C_*^λ of cyclicly antisymmetrized chains:

(155) $$\dots \xrightarrow{b} C_n^\lambda \xrightarrow{b} C_{n-1}^\lambda \xrightarrow{b} \dots \xrightarrow{b} C_1^\lambda \xrightarrow{b} C_0^\lambda \xrightarrow{b} 0$$

The homology obtained in this case is the cyclic homology $HC_*(a)$.

One of the main insights of noncommutative geometry now is that the introduced homologies give the right noncommutative version of differential geometry: Hochschild homology modules should be understood as the modules of differential forms. and cyclic homology takes the place of deRham homology.

In order to make this precise it is necessary to introduce the modules $\Omega^n(\mathbf{A})$ of differential forms of a complex commutative unital algebra:

DEFINITION 18. The symmetric module $\Omega^1(\mathbf{A})$ of differential forms is generated by the $\mathbb{C}$-linear symbols

$$adb \qquad \text{with } a, b \in \mathbf{A}$$

with the relations

(156) $$d(ab) = a(db) + b(da)$$

The symmetric modules $\Omega^n(\mathbf{A})$ are then given as

(157) $$\Omega^n(\mathbf{A}) = \Lambda^n_{\mathbf{A}} \Omega^1(\mathbf{A}).$$

Here $\Lambda^n_{\mathbf{A}}$ is the n-th exterior power of $\Omega^1(\mathbf{A})$ over $\mathbf{A}$.

$\Omega^*(\mathbf{A})$ can be given the structure of a graded differential algebra. In the case of $\mathbf{A}$ being the algebra of smooth functions on a manifold M, i.e., $\mathbf{A} = C^\infty(M)$. the spaces $\Omega^n(\mathbf{A})$ can be identified with the spaces of forms on the manifold.

There are now two rather nontrivial results relating Hochschild homology to differential forms and cyclic homology to deRham cohomology contained in the following Theorems (see, e.g., [52]):

THEOREM(HOCHSCHILD-KONSTANT-ROSENBERG). *For any smooth commutative complex algebra* $\mathbf{A}$*, the antisymmetrization map*

$$\epsilon_* : \Omega^* \to HH_*$$

given by

$$\epsilon_n(a_0 da_1 da_2 ... da_n) = \sum_\sigma sgn\, \sigma \quad a_0 \otimes a_{\sigma_1} \otimes a_{\sigma_2} \otimes ... \otimes a_{\sigma_n}$$

(with σ *the permutations of* $(1, 2, ..., n)$*) is an isomorphism of graded algebras.*

THEOREM 12. *If* $\mathbf{A}$ *is a smooth complex commutative algebra, then there is a canonical isomorphism*

$$HC_n(\mathbf{A}) \cong \frac{\Omega^n(\mathbf{A})}{d\Omega^{n-1}(\mathbf{A})} \oplus H^{n-2}_{DR}(\mathbf{A}) \oplus H^{n-4}_{DR}(\mathbf{A}) \oplus ...$$

with $H^*_{DR}(\mathbf{A})$ *being the deRham cohomology of* $\mathbf{A}$*. The last summand is* $H^0_{DR}(\mathbf{A})$ *or* $H^1_{DR}(\mathbf{A})$ *depending on* n *being even or odd.*

NOTE 23. The proofs of these theorems can be found, e.g., in [52]. Smoothness and the deRham cohomology can be defined algebraically (see [52]), but it is sufficient here to know that these notions agree for an algebra of smooth functions $C^\infty(M)$ on a manifold M with the ones used in classical differential geometry.

The above theorems are a clear sign of the suggested relationship between homological algebra and differential geometry. They are, however, not the only one but supplemented by various other results indicating the same. Particularly, A. Connes showed [53, 27] that it is possible to extend the Chern character to noncommutative geometry as a map from (algebraic) K-theory to cyclic homology.

This means that at least in some aspects algebraic topology and differential geometry are found to be compatible which enhances both of them.

Detailed accounts on the relationship between differential geometry and Hochschild and cyclic homology can be found in [52, 54] and to some extent in [55]. A dual formulation in terms of Hochschild cohomology and cyclic cohomology is given in [53, 27, 56]. There are also shorter accounts. e.g. [57–59]

4. Noncommutative differential calculus.

The noncommutative differential calculus of Section 3 can be provided with an integration implemented by the choice of a cyclic cocycle of the algebra $\mathbf{A}$ [53, 27]. It turns out that the structure thus obtained can be particularly given in a quite compact form: The algebra $\mathbf{A}$ is to be represented on a Hilbert space, the differentials are in agreement with insight from quantum mechanics given by commutators with a Hermitean operator F, and the integration is determined by a variation of the trace, the Dixmier trace [60]. A substantial part of this is summarized in the notion of a Fredholm module [53, 27] originating in the works of M. Atiyah [61], A. S. Mishchenko [62], L. G. Brown, R. G. Douglas and P. A. Fillmore [63] and G. Kasparov [64].

DEFINITION 19. Let $\mathbf{A}$ be an involutive complex algebra. Then a Fredholm module over $\mathbf{A}$ is given by

1. an involutive representation π of $\mathbf{A}$ in a Hilbert space $\mathcal{H}$;

2. an operator $F = F^*$, $F^2 = \mathbf{1}$ on $\mathcal{H}$ such that

$$[F, \pi(a)] \text{ is a compact operator for any } a \in \mathbf{A}$$

A Fredholm module is called even if it is given together with a $\mathbb{Z}_2$-grading operator γ, $\gamma = \gamma^*$, $\gamma^2 = \mathbf{1}$ on the Hilbert space $\mathcal{H}$ such that

1. $\gamma\pi(a) = \pi(a)\gamma$ for all $a \in \mathbf{A}$
2. $\gamma F = -F\gamma$

Otherwise the Fredholm module is called odd.

For the definition and properties of the Dixmier trace which supplements Definition 19, see [53, 65, 27]. All that takes the place of the differential calculus in the noncommutative case with infinitesimals given by compact operators with their order determined by the asymptotic behavior of their spectrum (see [27]).

It was realized by A. Connes that the differential geometric information in a Fredholm module can be nicely extended to contain also metric information by replacing the grading operator F by an unbounded Hermitean operator D with $F = sgn\ D$ being the (unitary) sign of D in its polar decomposition and with the metric information contained in $\mid D \mid = (D^*D)^{\frac{1}{2}}$. The resulting structure is the spectral triple of Section 2 which now can be seen to be rooted in the fundamentals of noncommutative geometry.

Bibliography

[1] B. S. DeWitt, *Dynamical theory of groups and fields* (Gordon and Brench, New York, 1965).

[2] B. S. DeWitt, *The space-time approach to quantum field theory* (North Holland, Amsterdam, 1984).

[3] R. M. Wald, *Quantum field theory in curved spacetime and black hole thermodynamics* (The University of Chicago Press, Chicago and London, 1994).

[4] R. F. Streater and A. S. Wightman, *PCT, spin and statistics, and all that* (W. A. Benjamin, Reading, Massachusetts, 1964).

[5] R. P. Feynman and A. R. Hibbs, *Quantum mechanics and path integrals* (McGraw-Hill, New York, 1965).

[6] N. M. J. Woodhouse, *Geometric quantization*, 2nd ed. (Oxford University Press, New York, 1991).

[7] J. C. Baez, I. E. Segal, and Z. Zhou, *Introduction to algebraic and constructive quantum field theory* (Princeton University Press, Princeton, 1992).

[8] N. N. Bogoljubov, A. A. Logunov, A. I. Oksak, and I. T. Todorov, *Obshchie printsipy kvantovoi teorii polia.* (Nauka, Moscow, 1987).

[9] R. Verch, Commun. Math. Phys. **160**, 507 (1994).

[10] R. Haag, *Local quantum physics: fields, particles, algebras* (Springer-Verlag, Berlin Heidelberg, 1992).

[11] R. Plymen and P. Robinson, *Spinors in Hilbert space* (Cambridge University Press, Cambridge, 1994).

[12] O. Bratteli and D. W. Robinson, *Operator algebras and quantum statistical mechanics* (Springer-Verlag, New York, 1979), Vol. I.

[13] U. Yurtsever, Class. Quant. Grav. **11**, 999 (1994).

[14] U. Yurtsever, Class. Quant. Grav. **11**, 1013 (1994).

[15] S. W. Hawking and G. F. R. Ellis, *The large scale structure of space-time* (Cambridge University Press, Cambridge, 1973).

[16] S. Willard, *General topology* (Addison-Wesley, Reading, MA, 1970).

[17] R. Sorkin, private communication.

[18] J. B. Hartle, *Space-time quantum mechanics and the quantum mechanics of spacetime, in Gravitation and quantizations, Les Houches Summer Proceedings, Vol.LVII* (North Holland, Amsterdam, 1992).

[19] C. J. Isham, J. Math. Phys **35**, 2157 (1994).

[20] C. J. Isham and N. Linden, J. Math. Phys **35**, 5452 (1994).

[21] C. J. Isham, N. Linden, and S. Schreckenberg, J. Math. Phys **35**, 6360 (1994).

[22] H. Grosse, C. Klimčík, and P. Prešnajder, Towards finite quantum field theory in non-commutative geometry, e-print: hep-th/9505175.

[23] H. Grosse, C. Klimčík, and P. Prešnajder, Simple field theoretical models on noncommutative manifolds, e-print: hep-th/9510177.

[24] A. Connes and J. Lott, Class. Quant. Grav. **18B**, 29 (1991).

[25] A. H. Chamseddine and A. Connes, A universal action formula, e-print: hep-th/9606056.

[26] A. H. Chamseddine and A. Connes, The spectral action principle, e-print: hep-th/9606001.

[27] A. Connes, *Noncommutative geometry* (Academic Press, San Diego, 1994).

[28] A. Connes, Journal of Math. Physics **36**, 11 (1995).

[29] E. Hawkins, Hamiltonian gravity and noncommutative geometry, e-print: qc-qc/9605068.

[30] D. N. Page and C. D. Geilker, Phys. Rev. Lett. **47**, 979 (1981).

[31] A. Connes and C. Rovelli, Class. Quant. Grav. **11**, 2899 (1994).

[32] D. N. Page, Phys. Rev. Lett. **70**, 4034 (1993).
[33] A. Connes, Gravity coupled with matter and the foundation of non-commutative geometry. e-print: hep-th/9603053.
[34] P. B. Gilkey, *Invariance theory, the heat equation, and the Atiyah-Singer index theorem* (Publish or Perish, Wilmington, Delaware, 1984).
[35] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).
[36] C. Chevalley, *The algebraic theory of spinors* (Columbia University Press, New York, 1954).
[37] H. B. Lawson and M.-L. Michelsohn, *Spin geometry*, Vol. 38 of *Princeton mathematical series* (Princeton University Press, Princeton, 1989).
[38] P. Budinich and A. Trautman, *The spinorial chessboard* (Springer-Verlag, Berlin Heidelberg, 1988).
[39] R. Penrose and W. Rindler, *Spinors and space-time* (Cambridge University Press, Cambridge, 1984).
[40] J. Javůrek, Basic notions of noncommutative geometry, 1995, diploma Thesis.
[41] Y. Ne'eman, Ann. Inst. H. Poincaré **A 28**, 369 (1978).
[42] A. Ashtekar, *Lectures on nonperturbative canonical gravity* (World Scientific, Singapore, 1991).
[43] S. MacLane, *Categories for the working mathematician* (Springer-Verlag, New York, 1971).
[44] J. P. Serre, *Applications algébriques de la cohomologie des groupes. Théorie des groupes simples.* (Ecole Normale Supérieure, Paris, 1950/51).
[45] Euclid, *Elements of geometry* (University Press, Cambridge, 1900).
[46] T. Kuhn, *The structure of scientific revolutions* (University of Chicago Press, Chicago, 1962).
[47] M. Takesaki, *Theory of operator algebras* (Springer-Verlag, New York, 1979), Vol. I.
[48] I. M. Gel'fand and M. A. Naimark, Mat. Sbornik **12**, 197 (1943).
[49] N. E. Wegge-Olsen, *K-theory and C^*-algebras: a friendly approach* (Oxford University Press, New York, 1993).
[50] B. Blackadar, *K-theory for operator algebras* (Springer-Verlag, New York, 1986).
[51] H. Cartan and S. Eilenberg, *Homological algebra* (Princeton University Press, Princeton, 1956).
[52] J.-L. Loday, *Cyclic homology* (Berlin Heidelberg, Springer-Verlag, 1992).
[53] A. Connes, Inst. Hautes Études Sci. Math. Publ. **62**, 257 (1985).
[54] C. A. Weibel, *An introduction to homological algebra* (Cambridge University Press, Cambridge, 1994).
[55] P. Seibt, *Cyclic homology of algebras* (World Scientific, Singapore, 1987).
[56] D. Kastler, *Cyclic cohomology within the differential envelope : an introduction to Alain Connes' non-commutative differential geometry* (Hermann, Paris, 1988).
[57] J. Madore, *An introduction to noncommutative geometry and its physical applications* (London Mathematical Society, Cambridge University Press, Cambridge, 1995).
[58] J. C. Várilly and J. M. Gracia-Bondía, J. Math. Phys. **12**, 3340 (1994).
[59] R. Coquereaux, J. Geom. Phys. **11**, 307 (1993).
[60] J. Dixmier, C. R. Acad. Sci. Paris Ser. A-B **262**, A1107 (1966).
[61] M. F. Atiyah, Global theory of elliptic operators, 1970, proc. Internat. Conf. on Functional Analysis and Related Topics (Tokio 1969).
[62] A. S. Mishchenko, Izv. Akad. Nauk SSSR Ser. Mat. **8**, 85 (1974).
[63] L. G. Brown, R. G. Douglas, and P. A. Fillmore, Ann. of Math. **105**, 265 (1977).
[64] G. Kasparov, Izv. Akad. Nauk SSSR Ser. Mat. **9**, 751 (1975).
[65] A. Connes, *Geometrie non commutative* (InterEditions, Paris, 1990).