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Well-covered graphs: some new sub-classes and complexity results

by

Ramesh S. Sankaranarayana

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Abstract

A graph is said to be well covered if every maximal independent set has the same size, and very well covered if every maximal independent set contains exactly half the vertices in the graph. Well-covered graphs are of interest because while the problem of finding the size of a maximum independent set is NP-complete for graphs in general, it is in P for well-covered graphs.

Many of the existing results in this area deal with characterizations of families of well-covered graphs. This thesis focuses on the algorithmic properties of this family. The first part of this thesis looks at the algorithmic complexities of the following problems for the families of well-covered and very well covered graphs: chromatic number, clique cover, clique partition, dominating cycle, dominating set, Hamiltonian cycle, Hamiltonian path, independent set, independent dominating set, maximum cut, minimum fill-in, recognition, Steiner tree, and vertex cover. While most of the above problems prove to be as difficult for well-covered graphs as for graphs in general, a number of them become tractable when restricted to the family of very well covered graphs.

In the second part of this thesis, an alternative characterization is given for the family of well-covered graphs. This leads to the concept of a maximal intersection of independent sets. Based on this, a hierarchy of four new sub-classes of well-covered graphs is defined. The families are characterized and the algorithmic complexities of the above mentioned problems are studied for these families. It is also shown that the last class in the hierarchy is exactly the family of very well covered graphs without isolated vertices. A generalization of Favaron's theorem for very well covered graphs is also proved.

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Chapter 1

Introduction

The concept of a *well-covered* graph was introduced by Plummer [24] in 1970. He defined a graph as being well covered if every maximal independent set has the same size. These graphs are of interest because while the problem of determining the size of a maximum independent set for a general graph is NP-complete [15], in the case of well-covered graphs, this can be done by determining the size of any maximal independent set.

This thesis looks at the family of well-covered graphs from an algorithmic complexity point of view. The main objectives are: to examine the recognition problem for this family; to study the complexities of some fundamental graph problems for this family; and, in case some of the problems prove intractable, to find non-trivial sub-classes for which such problems can be solved efficiently. These results could form a basis for characterizing classes of tractable, or provably intractable, problems on well-covered graphs. This thesis asks questions of the following nature: Given that the maximum independent set problem is in P, are there other problems that are tractable for this family? Are there problems that are intractable? What is the complexity of the recognition problem? Are there non-trivial sub-classes for which recognition is in P? What are the algorithmic properties of such sub-classes? Can such sub-classes be distinguished algorithmically?

A graph is said to be *very well covered* if every maximal independent set contains exactly half the vertices in the graph. The first part of this thesis looks at the algorithmic complexities of the following problems for the families of well-covered and very well covered graphs: chromatic number, clique cover, clique partition, dominating cycle, dominating set, Hamiltonian cycle, Hamiltonian path, independent set, independent dominating set, maximum cut, minimum fill-in, recognition, Steiner tree, and vertex cover. The recognition problem turns out to be co-NP-complete for well-covered graphs, and in P for very well covered graphs without isolated vertices. While most of the other problems prove to be as difficult for well-covered graphs as graphs in general, a number of them become tractable when restricted to the family of very well covered graphs.

In the second part of this thesis, an alternative characterization is given for the family of well-covered graphs. This leads to the concept of a *maximal* intersection of independent sets. Based on this, a hierarchy of four new sub-classes of well-covered graphs is defined. The families are characterized and the algorithmic complexities of the above mentioned problems are studied for these families. While the first family in the hierarchy has recognition co-NP-complete, the remaining families have recognition in P. All of them have the clique partition problem in P, a problem that is NP-complete for well-covered graphs. The smallest family in the hierarchy is exactly the family of very well covered graphs without isolated vertices.

The last part of this thesis deals with a generalization of Favaron's [8] characterization of very well covered graphs without isolated vertices. She showed that all such graphs have a perfect matching which obeys a certain property. In this thesis, a characterization of the second class in the hierarchy is provided in terms of a clique partition which obeys certain properties. This is shown to reduce to Favaron's characterization when the clique partition considered is a perfect matching. While this result does not immediately fall into the framework of the thesis, it is interesting because it generalizes the structure of very well covered graphs without isolated vertices, while preserving the property of recognition being in P.

1.1 Definitions

A graph is a pair $G = (V, E)$, where V is a finite set of *vertices* and E is a set of unordered pairs (u, v) of distinct vertices of V; each such pair is called an *edge*. $V(G)$ and $E(G)$ are also used to denote the vertex and edge sets, respectively, of a graph G. The *order* of a graph is given by the number of vertices, and the *size* by the number of edges, in it. In what follows, G denotes a simple, undirected, finite graph of order $n =| V |$, with size $m =| E |$ edges. Two vertices u and v are *adjacent*, denoted by $u \sim v$, if $(u, v) \in E$; u and v are called the *end points* or *ends* of the edge (u, v). Two vertices u and v are $non-adjacent$, denoted by $u \not\sim v$, if $(u, v) \notin E$. The *degree* $d(v)$ of a vertex v is the number of vertices adjacent to v. Two edges are adjacent if they have a vertex in common. An edge is said to be *incident* with a vertex v if v is one of its end points. A vertex of degree one is called a *leaf*. An edge that is incident with a leaf is called a *pendant edge*. A graph $H = (V_1, E_1)$ is said to be a *subgraph* of G if $V_1 \subseteq V$ and $E_1 \subseteq E$. Given a vertex set $A \subseteq V$, the subgraph *induced* by A has the vertex set A and the edge set $E(A) = \{(u, v) \in E | u, v \in A\}$, and is denoted by $< A >$.

A set of vertices is *independent* if no two vertices in the set are adjacent. A set of vertices in G forms a *vertex cover* for G if every edge in G is incident with at least one vertex in the set. A set of vertices I_1 is said to *cover* a set of vertices I_2 if every vertex in I_2 is adjacent to some vertex in I_1. A subset of E is a *matching* if no two edges in the set are adjacent. We say that there is a matching from $A \subseteq V$ to $B \subseteq V - A$, if there exists a matching M of G such that every edge in M has one end point in A and the other in B; we can also say that there is a matching from B to A, or there is a matching between A and B. A *perfect matching* is one in which every vertex in G is an end point of some edge in the matching. A set S is a *maximal set* satisfying a certain property P if there is no other set properly containing S that satisfies property P. Set S is *maximum* if there exists no set of greater cardinality that satisfies property P. A similar distinction is made between *minimal* and *minimum*. The size of a maximum independent set in a graph is referred to as $\alpha(G)$. A graph G is a *bipartite* graph if V can be partitioned into two independent sets X and Y. We write the bipartite graph as (X, Y, E). If u is adjacent to v, then u is said to be a *neighbour* of v. $N(v)$ denotes the open neighbourhood of $v \in V$, that is, $N(v) = \{x | x \in V \text{ and } (x, v) \in E\}$. $N[v]$ denotes the closed neighbourhood of v and is given by $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, $N(S) = \cup N(v)\ \forall v \in S$, and $N[S] = N(S) \cup S$. A vertex is *isolated* if it has no neighbours, and *simplicial* if its closed neighbourhood induces a clique. A graph is said to be *chordal* if it does not contain an induced cycle of order greater than three. The *clique cover number* of a graph G is the smallest number of complete subgraphs needed to cover the vertices of G; it is denoted by $\kappa(G)$. A graph G is said to be perfect if $\alpha(< A >) = \kappa(< A >)$ for all $A \subseteq V$. For any additional terms, see [2].

An algorithm is said to run in order $f(n)$ time if its running time is bounded by $cf(n)$ for all possible instances of input of size n, where c is a positive constant. It is called a *polynomial time* algorithm if $f(n)$ is a polynomial in n. An algorithm is said to be *deterministic* if each stage in the execution of the algorithm leads to a unique next stage, and *nondeterministic* if there could be many possible next stages. A problem is said to belong to the class P if there exists a deterministic polynomial time algorithm, and to the class NP if there exists a nondeterministic polynomial time algorithm, which solves it. A problem is said to be NP-hard if the existence of a deterministic polynomial time algorithm for its solution implies the existence of a deterministic polynomial time algorithm for every problem in NP. A problem is said to be NP-complete if it is both NP-hard and is in the class NP. For additional details on complexity and NP-completeness, see [15].

1.2 Organization of this thesis

Chapter 2 looks at related work and gives some examples of well-covered graphs. Chapter 3 studies the complexities of some fundamental graph problems for the families of well-covered and very well covered graphs. Chapter 4 gives an alternative characterization for well-covered and very well covered graphs, and also looks at the nature of the intersections of pairs of maximal independent sets of a well-covered graph. It establishes the conditions under which such intersections are maximal, and under which all such intersections have the same size. Chapter 5 defines and characterizes a hierarchy of four new sub-classes of well-covered graphs. It also shows that the last sub-class in the hierarchy is exactly the family of very well covered graphs without isolated vertices. Chapter 6 studies the complexities of some standard problems for these sub-classes. Chapter 7 provides a generalization of Favaron's theorem for very well covered graphs. Conclusions and future work make up Chapter 8.

Chapter 2

Related work

2.1 Introduction

The concept of a *well−covered* graph was introduced by Plummer [24] who defined a graph to be well covered if every *minimal* vertex cover is also a *minimum* vertex cover. If $V_c \subseteq V$ is a vertex cover for a graph G, then the graph induced by $V - V_c$ cannot contain an edge, as this would contradict the fact that V_c is a vertex cover. That is, $I = V - V_c$ is an independent set. This independent set is a maximal independent set since V_c is a minimal vertex cover. Therefore, an equivalent definition for a well-covered graph is: A graph is well covered if every *maximal* independent set is *maximum*. Note that for a well-covered graph, every independent set is contained in a maximum independent set. Well-covered graphs are interesting because a greedy algorithm can be used to find a maximum independent set, a problem that is intractable for general graphs. Another well known structure for which the greedy algorithm gives an optimal solution is a matroid. For a comprehensive treatment of matroids, see [30].

We first present a few results that have applications in this thesis and then give a summary of some other known results. We then present a few examples of well-covered graphs.

2.2 Results related to the thesis

A graph is said to be *quasi-regularizable* if one can obtain a regular multigraph of non-zero degree from it, by deleting some of the edges if necessary, and replacing the others with several parallel edges. Berge [1] showed that any well-covered graph without isolated vertices is *quasi-regularizable*, and that any quasi-regularizable graph G has the property that for every independent set S of G, $\mid N(S) \mid \geq \mid S \mid$. From this, it is clear that

Corollary 2.1 *For any well-covered graph G without isolated vertices, the following are true:*

a) $\mid N(S) \mid \geq \mid S \mid$ for every independent set S of G.

b) The size of a maximal independent set of G is $\leq \mid V \mid /2$.

A graph is said to be very well covered if every maximal independent set has cardinality $\mid V \mid /2$. Staples [28] was the first to study this family. Favaron [8] gave the following characterization for this family:

Theorem 2.1 (Favaron) *For a graph G, the following are equivalent:*
a) G is very well covered.
b) There exists a perfect matching in G that satisfies P.
c) There exists at least one perfect matching in G and every perfect matching of G satisfies P.

where property P is defined as follows:

Property P: *A matching M in a graph G satisfies property P if for any edge $(u, v) \in M$, $N(u) \cap N(v) = \phi$, and $N(u) - \{v\}$ is adjacent to all of $N(v) - \{u\}$.*

Chvátal and Slater [6] showed that well-covered graph recognition is co-NP-complete, that is, recognizing a graph as being not well covered is NP-complete. This result was arrived at independently by the author and Stewart [27].

2.3 Other results

Ravindra [25] studied well covered bipartite graphs. Let G be a graph and for $e = (u, v) \in E$, let G_e be the subgraph induced by $N(u) \cup N(v)$ in G. Then

Theorem 2.2 (Ravindra) *A bipartite graph G without isolated vertices is well covered if and only if G has a perfect matching M and for every $e \in M$, G_e is a complete bipartite graph.*

If G is a well-covered bipartite graph, then every maximal independent set of G has $\mid V \mid /2$ vertices. That is, any such bipartite graph is very well covered. He also characterized all well-covered trees.

Lewin [19] implicitly characterized well-covered line graphs by characterizing what he called *matching-perfect* graphs. A graph is said to be $matching-perfect$ if every maximal matching is a maximum matching. A line graph G_l of a graph G has a vertex for every edge in G, and two vertices in G_l are joined by an edge if the corresponding two edges in G are adjacent. Hence, a maximal matching in G yields a maximal independent set in G_l, and vice versa. That is, G is matching-perfect if and only if G_l is well covered. It was later proved by Lesk et al. [18] that well-covered line graphs can be recognized in polynomial time.

Staples [29] gave the following classification scheme for well-covered graphs:

Definition 2.1 *Let n be a positive integer. A graph G belongs to class W_n if $\mid V(G) \mid \geq n$ and every n disjoint independent sets in G are contained in n disjoint maximum independent sets.*

W_1 is the class of well-covered graphs, and the W_n classes form a descending chain: $W_1 \supseteq W_2 \supseteq \cdots$. No W_n class is empty, since the complete graph on n vertices belongs to W_n. She also described some ways of constructing W_n graphs, and proved some structural results for these graphs.

The *girth* of a graph is the length of the smallest cycle in it. Finbow and Hartnell [10] characterized well-covered graphs of girth ≥ 8. Finbow et al. later characterized well-covered graphs of girth ≥ 5 [14], and well-covered graphs containing neither 4- nor 5-cycles [13]. They showed that all such graphs can be recognized in polynomial time. A concept similar to that of a graph being well covered is that of a graph being well-dominated. A graph is said to be *well-dominated* if all minimal dominating sets are of the same cardinality. A set $D \subseteq V$ of G is said to be *dominating* if every vertex in G is either in the set or is adjacent to some vertex in it. Finbow et al. [12] showed that well-dominated graphs are also well covered. They gave a characterization of well-dominated graphs having no 3- nor 4-cycles, and of well-dominated bipartite graphs. A dominating set D is said to be *locating* if for every pair of vertices u, v not in D, we have $N(u) \cap D \neq N(v) \cap D$. Finbow and Hartnell [11] showed that graphs in which every independent dominating set is locating form a sub-class of well covered graphs. They also showed that for graphs of girth 5 or more, the two families are identical. Gasquoine, Hartnell, Nowakowski, and Whitehead [16] described techniques for constructing a family of well-covered graphs containing no 4-cycles.

A graph is said to be claw-free if it has no induced subgraph isomorphic to $K_{1,3}$. Whitehead [31] showed how a claw-free well-covered graph containing no 4-cycle, with any given independence number α, can be constructed by linking together α subgraphs, each isomorphic to either K_2 or K_3.

A set S of vertices of a graph is k-independent if each vertex in S is adjacent to at most $k-1$ other vertices in S. Favaron and Hartnell [9] defined a well-k-covered graph as one in which every maximal k-independent set of vertices is maximum. Thus, well-1-covered is the same as well covered. They characterized the well-k-covered trees, and all well-2-covered graphs of girth ≥ 8.

A graph is said to be cubic if the degree of every vertex in it is exactly 3. Campbell [3] characterized the well-covered cubic graphs of connectivity 1 or 2. Campbell and Plummer [5] found all 3-connected cubic planar graphs which are well covered; they showed that there are only four such graphs. Campbell et al. [4] characterized all well-covered cubic graphs and showed that these can be recognized in polynomial time.

A graph is called k-extendable if every independent set of size k is contained in a maximum independent set. Dean and Zito [7] gave the following characterization of well-covered graphs:

Theorem 2.3 *Let C be a clique cover consisting of t cliques of a graph G with independence number $\alpha(G) = t - d$, for some non-negative integer d. Then the following are equivalent:*

a) G is well covered.

b) *G is k-extendable for all $k \in \{1, 2, \ldots, h\}$, where h is the sum of the orders of the $d+1$ largest cliques in C.*

c) *For every $d+1$ cliques $C_1, C_2, \ldots, C_{d+1}$ of the clique cover C with vertex set $W = \cup_{i=1}^{d+1} V(C_i)$, there is no independent set S of $G - W$ such that $\mid W \mid \geq \mid S \mid$ and $W \subseteq N(S)$.*

They also showed that for two classes of perfect graphs, those with bounded clique size and those with no induced 4-cycles, it can be determined whether the graph is well covered in polynomial time.

Moon [21] obtained some results on the number of well-covered trees in various families of trees.

A well-covered graph is defined to be in the *strongly well-covered* class if and only if the deletion of any edge leaves a well-covered graph. Pinter [22] studied the class W_2 as defined by Staples, and the class of strongly well-covered graphs. He showed that these are two different classes, and that there is only one graph common to both classes.

For a more detailed analysis of the work done so far, see Plummer's survey on well-covered graphs [23].

2.4 Examples

We now give some examples of well-covered and very well covered graphs. The only induced paths which are well covered are P_1, P_2, and P_4, the paths on 1, 2, and 4 vertices, respectively. For any other path, one can easily get two maximal independent sets of different sizes by choosing vertices appropriately. For example, if the vertices are numbered $1, 2, 3, \ldots$, choose vertices $1, 3, 5, \ldots$ to form one set and vertices $1, 4, 7, \ldots$ to form another.

The only induced cycles which are well covered are C_3, C_4, C_5, and C_7. For any other cycle, one can find two maximal independent sets of different sizes by using a sequence similar to that given above for paths.

The complete bipartite graph $K_{n,n}$ is very well covered with a maximal independent set size of n. Consider bipartite graphs having no isolated vertices. The complement of any such graph is well covered, as any maximal independent set will contain exactly two vertices.

The complements of k-trees form another family of well-covered graphs. A k-tree is defined recursively as follows: a k-tree on k vertices is a clique on k vertices (k-clique); given a k-tree T_n on n vertices, a k-tree on $n+1$ vertices is obtained by adding a new vertex v_{n+1} to T_n, and making it adjacent to each vertex of some k-clique of T_n, and non-adjacent to the remaining $n-k$ vertices. The complement of a k-tree on k vertices is obviously well covered. To prove this for k-trees on more than k vertices, we need only show that in every such k-tree, every maximal clique is a maximum clique, since a maximal clique in a k-tree corresponds to a maximal independent set in its complement. Rose [26] showed that any k-tree G has a k-clique but no $k+2$-clique. Hence, the size of a maximal clique for a k-tree is bounded by $k+1$. From the definition of a k-tree, any k-tree with more than k

vertices has at least one clique of size $k+1$ and every vertex in it belongs to at least one clique of this size. Hence, every maximal clique in a k-tree with more than k vertices has size $k+1$. Therefore, the complements of k-trees are well covered.

Chapter 3

Complexity results

3.1 Introduction

From the previous chapter, we see that most of the work done so far on well-covered graphs deals with characterizations of this family and of specific subclasses. We now examine the algorithmic properties of such classes by exploring the complexities of some fundamental graph problems like *recognition*, *dominating set*, *Hamiltonian cycle and path*, and *clique cover* for the families of well-covered and very well covered graphs. The very well covered graphs looked at here are those without isolated vertices. We show that recognition is co-NP-complete and that several other problems are NP-complete for well-covered graphs. A number of these problems remain NP-complete, while some of them become tractable, for very well covered graphs. For both families, the isomorphism problem is as hard as the general graph isomorphism. These results are shown in the Table 3.1.

3.2 Recognition

An important question for any family of graphs is that of recognition, that is, given a graph G, can one say whether or not G is well covered? We prove that this problem is co-NP-complete by showing that the complementary problem of deciding whether G is *not well covered* is NP-complete. This result was arrived at independently by Chvátal and Slater [6].

Theorem 3.1 *The recognition problem is co-NP-complete for well-covered graphs.*

Proof:
The decision problem that we are dealing with here is the following: is a given graph G *not* well covered? We first show that this problem is in NP. A graph G is well covered if and only if every maximal independent set is a maximum independent set. To show that G is not well covered, a nondeterministic algorithm only needs to guess two subsets of V and check that they are maximal independent sets of different sizes.

Problem	Well covered		Very well covered	
Member	co-NP-c	$(K_n - K_2)$	P	(Favaron)
Chromatic number	$\Longleftarrow$		NP-c	(leaf vertex)
Clique	$\Longleftarrow$		NP-c	(leaf vertex)
Dominating cycle	$\Longleftarrow$		NP-c	(leaf vertex)
Isomorphism	$\Longleftarrow$		iso-c	(leaf vertex)
Maximum cut	$\Longleftarrow$		NP-c	(leaf vertex)
Minimum fill-in	$\Longleftarrow$		NP-c	(leaf vertex)
Steiner tree	$\Longleftarrow$		NP-c	(leaf vertex)
Independent set	P	(trivial)	$\Longrightarrow$	
Independent dominating set	P	(trivial)	$\Longrightarrow$	
Vertex cover	P	(trivial)	$\Longrightarrow$	
Clique partition	NP-c	$(C_5 - C_5)$	P	(Favaron)
Dominating set	NP-c	$(H - K_3)$	P	(Favaron)
Hamiltonian cycle	NP-c	(K_3)	P	$(K_{n,n})$
Hamiltonian path	NP-c	$(\Downarrow)$	P	(chain graph)

Table 3.1: Complexity results for well-covered and very well covered graphs

$\Longleftarrow$	Result implied from result on right.
$\Longrightarrow$	Result implied from result on left.
$\Downarrow$	Similar transformation to one just above.
$(\ldots)$	Nature of transformation/result or reference.

We transform from a known NP-complete problem, the SATISFIABILITY problem, or SAT. This problem is specified as follows: Given a set U of variables and a collection C of clauses over U, is there a satisfying truth assignment for C? For any instance of SAT with clauses $C = \{c_1, c_2, \ldots, c_m\}$ and variables $U = \{u_1, u_2, \ldots, u_n\}$, we construct a graph $G = (V, E)$, where

$$\begin{aligned}
V &= V_C \cup V_L \text{ where} \\
V_C &= \{c_1, c_2, \ldots, c_m\} \text{ and} \\
V_L &= \{u_1, \overline{u_1}, u_2, \overline{u_2} \ldots, u_n, \overline{u_n}\} \\
E &= \{(c_i, c_j) | 1 \leq i, j \leq m, i \neq j\} \\
&\quad \cup \{(u_i, \overline{u_i}) | 1 \leq i \leq n\} \\
&\quad \cup \{(c_i, u_j) | u_j \text{ is a literal in clause } c_i\} \\
&\quad \cup \{(c_i, \overline{u_j}) | \overline{u_j} \text{ is a literal in clause } c_i\}
\end{aligned}$$

See Figure 3.1. We assume that no clause contains a variable and its negation, as such a clause could be satisfied by any truth assignment and therefore eliminated. G has $2n + m$ vertices. The number of edges in V_C is $m(m-1)/2$, and in V_L is n. The number of edges between V_C and V_L is $\leq mn$, considering the worst

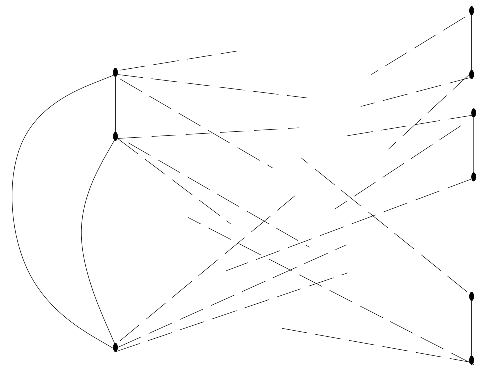

Figure 3.1: Recognition

case of each clause having n literals. Therefore, the number of edges in G is $\leq n + nm + m(m-1)/2$. Thus, G can be constructed in polynomial time.

Consider the graph G. Any independent set can have at most one vertex from V_C and one vertex from each K_2 in V_L. Therefore, the size of a maximum independent set for G is $n+1$. There are many maximal independent sets of this size as one can choose a vertex from V_C, and still pick one vertex from each K_2 in V_L, as no vertex in V_C is adjacent to both the vertices of a K_2 in V_L. In fact, one *must* pick a vertex from each K_2 in V_L, since there are no edges between the K_2's.

Claim 3.1 *C is satisfiable if and only if G is not well covered.*

Proof:
only if:
C is satisfiable. Then we can find a maximal independent set of size n consisting of vertices of V_L corresponding to *true* literals in a satisfying truth assignment for G. Since we have already shown that there are maximal independent sets of size $n+1$ in G, this means that G is not well covered.
if:
G is not well covered. Then there exists a maximal independent set in G of size less than $n+1$. Any independent set can contain at most one vertex from V_C. We have already shown that any independent set containing a vertex from V_C must have one vertex from every K_2 in V_L, giving a maximal independent set of size $n+1$. Thus, any maximal independent set containing fewer than $n+1$ vertices contains only vertices of V_L. Since any such independent set must have one vertex from each K_2, all such maximal independent sets will have size n. For such a set to be maximal, each vertex of V_C should be adjacent to at least one vertex the in the set. No two vertices corresponding to a literal and its negation will be in an independent set, since they are adjacent to each other. Hence, if we assign the value *true* to the literals corresponding to the vertices of V_L in any such maximal

independent set of size n, we will have a satisfying truth assignment for C. This completes the proof of the claim.

Therefore, recognizing a graph to be not well covered is NP-complete.

$\square$

A graph is said to be weakly chordal if neither it, nor its complement, contains a chordless cycle with more than four vertices; see [17]. As mentioned earlier, Dean and Zito [7] showed that for perfect graphs with no induced 4-cycles, the problem of determining whether the graph is well covered is tractable. Since chordal graphs are perfect graphs and have no induced 4-cycles, this means that determining whether a chordal graph is well covered can be done in polynomial time. We now show that the problem of recognizing a graph as being not well covered is NP-complete for weakly chordal and therefore perfect graphs.

Corollary 3.1 *The problem of recognizing a graph as being not well covered is NP-complete for weakly chordal and therefore perfect graphs.*

Proof:
We show that the graph G obtained in the proof of the theorem 3.1 is a weakly chordal graph. We do this by showing that neither G nor its complement contains a chordless cycle with more than four vertices. The proof makes uses of the following observation: Any such cycle in G contains exactly two vertices from V_C, and in G^c exactly two vertices from V_L.

Consider a chordless cycle of length ≥ 5 in G. Any such cycle can have at most two vertices from V_C since $< V_C >$ is a clique; hence, it has at least three vertices from V_L. Since the K_2's in $< V_L >$ are mutually non-adjacent, there have to be at least two vertices from V_C in the cycle. Thus, any such cycle has exactly two vertices from V_C, which are adjacent, and at least three vertices from V_L. Consider the vertices from V_L in such a cycle. If both vertices from a K_2 are in the cycle, they will induce a C_4 as a subgraph with the two vertices from V_C, thus creating a chord. If only one vertex from a K_2 is in the cycle, it will form a K_3 with the vertices from V_C, again creating a chord.

Consider a chordless cycle in G^c of length ≥ 5. $< V_L >$ consists of independent sets of size two with each vertex of any such independent set I being adjacent to every vertex in $V_L - I$. Any four vertices from V_L will induce a 4-cycle as a subgraph, thereby creating a chord in the cycle. Thus, there can be at most three vertices from V_L in the cycle. This means that there are at least two vertices from V_C in the cycle. Since $< V_C >$ is an independent set, the vertices in it are connected to each other only through the vertices in V_L. If there are three vertices from V_L in the cycle, then these will have a P_3 as a subgraph. These vertices, along with any one of the vertices from V_C in the cycle, will have a C_4 or a C_3 as an induced subgraph, thereby creating a chord. Therefore, there are exactly two vertices from V_L in any such cycle, which will then have at least three vertices from V_C. Any two such vertices from V_C, along with the two from V_L, will have a C_4 as a subgraph, thus creating a chord.

Hence, neither G nor its complement has a chordless cycle with more than four vertices. Therefore, G is weakly chordal; that is, the graph obtained in proving theorem 3.1 is a weakly chordal graph. Hence, recognizing non-well-covered weakly chordal graphs is NP-complete. Since it has been proven by Hayward [17] that weakly chordal graphs are also perfect graphs, we conclude that recognizing non-well-covered perfect graphs is also NP-complete.

$\square$

In contrast to this, very well covered graphs can be recognized in polynomial time. From Favaron's characterization, we see that in order to recognize a graph as being very well covered, we just need to show that it has a perfect matching that obeys property P. A maximum matching can be found in polynomial time using the algorithm devised by Micali and Vazirani [20]. Their algorithm runs in order $\sqrt{n}\,m$. Checking if this is perfect requires finding out if the number of edges in the matching is equal to $n/2$. Checking if the neighbours of a pair of vertices that form an edge in the matching are completely connected to each other can be done in order $n + m$ time. Since there are exactly $n/2$ such pairs of vertices, property P can be checked in order $n^2 + nm$ time. Therefore, very well covered graph recognition is in P.

3.3 Independent set and related problems

Some problems are easily solved for the class of well-covered graphs as a result of the definition of this family of graphs. The maximum independent set problem, which is to find an independent set whose size is maximum, is easily seen to be polynomial as we only need to find a maximal independent set. Since a minimum vertex cover is simply the vertex set minus a maximum independent set, the minimum vertex cover problem is also in P for this class of graphs. We also observe that the minimum independent dominating set problem is in P for well-covered graphs because this is equivalent to the problem of finding a minimum cardinality maximal independent set.

3.4 Dominating cycle and other problems

We first show that the dominating cycle problem is NP-complete, and then use the same transformation to show that a number of other problems are also NP-complete for this class of graphs.

Problem: *Given a graph G, does G have a simple cycle such that every vertex in G is in the cycle or is adjacent to some vertex in the cycle?*

Theorem 3.2 *The dominating cycle problem is NP-complete for well-covered graphs.*

Proof:
The problem is known to be in NP. We transform from the Hamiltonian cycle problem for general graphs. Consider any graph G of order n. Construct G_D by adding a leaf vertex to every vertex in G. Thus, G_D has $2n$ vertices and $m + n$ edges, and can be constructed in polynomial time. For an example, see Figure 3.2. Consider the graph thus obtained. The edges with the leaf vertices

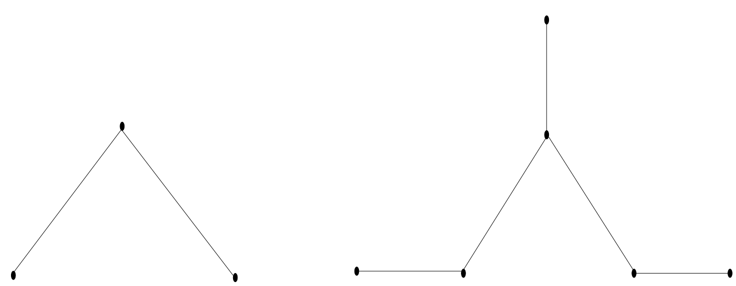

Figure 3.2: Dominating cycle

form a perfect matching for G_D. Therefore, any maximal independent set will have to contain exactly one vertex from every edge in the matching; that is, any maximal independent set will have exactly $\mid V(G_D) \mid /2$ vertices. Therefore, G_D is a very well covered graph.

Claim 3.2 *G has a Hamiltonian cycle if and only if G_D has a dominating cycle.*

Proof:
only if:
G has a Hamiltonian cycle. Therefore, there is a simple cycle in G_D that involves n vertices. All the other vertices in G_D are adjacent to some vertex in the cycle. That is, there is a dominating cycle in G_D.
if:
A dominating cycle in G_D will contain only those vertices that have corresponding ones in G, as the other vertices are of degree 1 and, hence, cannot be part of a cycle. It will also have to contain all such vertices as each one is adjacent to a leaf vertex. Hence, the dominating cycle will contain n vertices and these are the same as the vertices of G. Also, the edges in such a cycle will only be those that have corresponding ones in G, as the new edges added are all pendant edges and, hence, cannot be part of a cycle. Hence, we can find a simple cycle in G that covers all the vertices in it, that is, a Hamiltonian cycle in G. This proves the claim.

Now, the Hamiltonian cycle problem is known to be NP-complete for general graphs; hence, the dominating cycle problem is NP-complete for very well covered graphs, and therefore, for well-covered graphs.

□

The reduction used in the dominating cycle proof yields a number of other results. Consider a graph G which has at least one edge; let G_D be the transformed graph.

The maximum clique size of G_D will be the same as the maximum clique size of G as the pendant edges do not change the maximum clique size. The same holds for chromatic number, which is defined as the minimum number of colours needed to colour the vertices of a graph G such that no two adjacent vertices have the same colour.

Given a graph G and a set of target vertices T, a minimum Steiner tree in G, that is, a sub-tree of G with the minimum number of edges that includes all the vertices of T, will be the same as a minimum Steiner tree in G_D, since the pendant edges and vertices play no role in the minimum Steiner tree.

A minimum fill-in for a graph G is defined as the minimum number of edges required to be added to G to make it chordal. A minimum fill-in for G would be same as a minimum fill-in for G_D as the pendant edges do not play a part in a minimum fill-in.

The unweighted maximum cut problem is defined as follows: given a graph G and an integer k, is there a partitioning of V into disjoint sets V_1 and V_2 such that the number of edges of G with one end-point in V_1 and the other in V_2 is at least k? Clearly, a pendant vertex would be in the opposite partition from its neighbour; otherwise, the size of the cut could be increased by moving one or more pendant vertices. Thus, the size of a maximum cut for G_D is equal to the size of a maximum cut for G plus $\mid V(G) \mid$ (each pendant edge contributes one edge to a maximum cut).

We conclude that the maximum clique size, chromatic number, Steiner tree, minimum fill-in, and maximum cut problems are all NP-complete for very well covered graphs, and hence for well-covered graphs.

We use the same reduction to show that the isomorphism problem is isomorphism complete for very well covered graphs. For arbitrary connected graphs G_1 and G_2, $G_{1D} \cong G_{2D}$ if and only if $G_1 \cong G_2$. Clearly, if two graphs, each with half of the vertices having degree 1 and distinct neighbourhoods, are isomorphic, then the graphs resulting from the removal of the pendant vertices must be isomorphic, and vice versa. Therefore, an algorithm for very well covered graph isomorphism could solve the general graph isomorphism problem, and we conclude that very well covered graph isomorphism is isomorphism complete.

3.5 Hamiltonian cycle

Problem: *Given a graph G, does G contain a simple cycle such that every vertex in G is in the cycle?*

Theorem 3.3 *The Hamiltonian cycle problem is NP-complete for well-covered graphs.*

Proof:
We transform from the Hamiltonian cycle problem for general graphs. Given a graph G of order n ($n > 2$), we construct a graph G_H as follows. For each vertex v_i in G, we construct a K_3 in G_H. One of the vertices of the K_3, say v_{i1}, corresponds to v_i in G; another one, say v_{i2}, forms its image. The third vertex v_{i3} is a simplicial vertex. For every two adjacent vertices v_i and v_j in G, there are three edges between the corresponding K_3's in G_H; these are between v_{i1} and v_{j1}, v_{i1} and v_{j2}, and v_{i2} and v_{j1}. Therefore, G_H has $3n$ vertices and $3m + 3n$ edges. Clearly, this transformation can be done in polynomial time. For an example, see Figure 3.3.

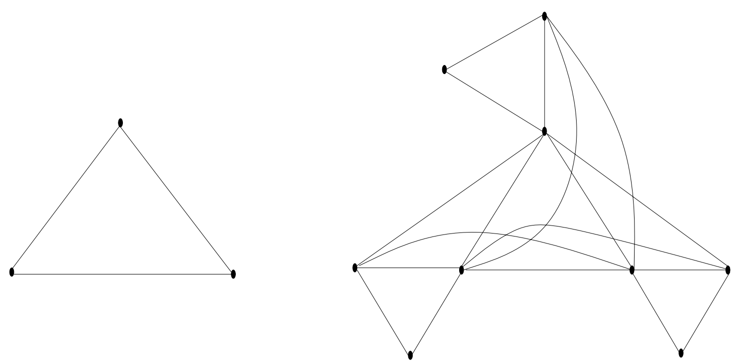

Figure 3.3: Hamiltonian cycle

Consider the graph thus obtained. It has n mutually disjoint K_3's, with each one having a simplicial vertex. Any maximal independent set will have to contain exactly one vertex from each K_3. Hence, this graph is well covered.

Claim 3.3 *G has a Hamiltonian cycle if and only if G_H has a Hamiltonian cycle.*

Proof:
only if:
G has a Hamiltonian cycle. For every vertex in G, there is a corresponding K_3 in G_H. For every edge in G, there are edges connecting two K_3's. Hence, if there is an edge (v_1, v_2) in G, we can always find a path $v_{11}, v_{13}, v_{12}, v_{21}$ in G_H. Therefore, if

G has a Hamiltonian cycle, we can always find a corresponding Hamiltonian cycle for G_H.
if:
G_H has a Hamiltonian cycle. Consider a K_3 which consists of vertices v_{i1}, v_{i2}, and v_{i3} which corresponds to a vertex v_i in G. Since v_{i3} is a simplicial vertex, the path v_{i1}, v_{i3}, v_{i2} will have to be part of any Hamiltonian cycle. Therefore, the part of a Hamiltonian cycle in G_H through a K_3 can be contracted to a single corresponding vertex in G. Of the three edges that connect two K_3's, only one can be part of a Hamiltonian cycle. Any such edge will have a corresponding edge in G. Hence, if G_H has a Hamiltonian cycle, we can always find a corresponding Hamiltonian cycle in G. This proves the claim.

Since the Hamiltonian cycle problem is NP-complete for general graphs, from the above, it is NP-complete for well-covered graphs as well.

□

We now examine the Hamiltonian cycle problem on very well covered graphs. Recall Favaron's characterization of very well covered graphs and the definition of property P from section 2.2. It is clear from this characterization that any edge in a perfect matching cannot be part of a K_3, as this would contradict property P. An edge (u, v) is said to satisfy property P if the neighbour sets of u and v satisfy the conditions of property P.

Theorem 3.4 *A very well covered graph has a Hamiltonian cycle if and only if it is a complete bipartite graph.*

Proof:
Let $G = (V, E)$ be a very well covered graph.
only if:
Suppose G has a Hamiltonian cycle, $C_H = \{v_1, v_2, \ldots, v_n, v_1\}$. Then, $M_1 = \{(v_1, v_2), (v_3, v_4), \ldots, (v_{n-1}, v_n)\}$ and $M_2 = \{(v_2, v_3), (v_4, v_5), \ldots, (v_n, v_1)\}$ are both perfect matchings, and hence each edge in C_H satisfies P. Let us define the C_H-distance between two vertices v_i and v_j of G to be the distance from v_i to v_j in a clockwise traversal of C_H, that is,

$$C_H - distance(v_i, v_j) = \begin{cases} j - i & \text{if } j > i \\ n + j - i & \text{if } j < i \end{cases}$$

Claim 3.4 *For all $1 \leq i, j \leq n$; if i is odd and j is even, then $(v_i, v_j) \in E$.*

Proof:
Suppose not. Let i be odd and j be even such that $(v_i, v_j) \notin E$ and such that no other such pair of nonadjacent vertices has a smaller C_H-distance. Since i is odd and j is even and $(v_i, v_j) \notin E$, we know that $i + 2(modn) \neq j$ and $i + 1(modn) \neq j$. Thus, C_H-distance$(v_i, v_j) > C_H$-distance$(v_{i+2(modn)}, v_j)$, and, hence, $(v_{i+2(modn)}, v_j) \in E$ by our choice of i and j. Therefore, since $(v_{i+1(modn)}, v_{i+2(modn)})$

is a matching edge, we must have $(v_i, v_j) \in E$ by property P. This is a contradiction, which proves the claim.

Finally, if there is an edge between any pair of vertices that are both even or both odd, then we have a matching edge (one of the vertices with one of its neighbours on C_H) in a triangle, which is impossible.

if:

G is a complete bipartite graph. From Favaron's characterization, it has a perfect matching; hence, it must have the same number of vertices in each partition, and is therefore a $K_{n,n}$. Since a $K_{n,n}$ has a Hamiltonian cycle, G has a Hamiltonian cycle.

$\square$

As complete bipartite graphs can be recognized in polynomial time, the Hamiltonian cycle problem is in P for very well covered graphs.

3.6 Hamiltonian path

Problem: *Given a graph G, does G contain a simple path such that every vertex in G is in the path?*

Theorem 3.5 *The Hamiltonian path problem is NP-complete for well-covered graphs.*

Proof:

We transform from the Hamiltonian cycle problem for general graphs. Given a graph G of order n, we construct a graph G_H in the same way as for the Hamiltonian cycle problem, with the following change. Take one of the simplicial vertices, say v_{n3}, and replace it with two vertices v_{n31} and v_{n32}. Replace the edges (v_{n1}, v_{n3}) and (v_{n2}, v_{n3}) with the edges (v_{n1}, v_{n31}) and (v_{n2}, v_{n32}) respectively. Remove the edge (v_{n1}, v_{n2}). The graph G_H will now have $3(n-1)+4$ vertices and $3m + 3(n-1) + 2$ edges. Clearly, this transformation can be done in polynomial time. For an example, see Figure 3.4.

Consider the graph thus obtained. It has $(n-1)$ mutually disjoint K_3's and two pendant edges. Any maximal independent set has to contain exactly one vertex of every K_3 and of every pendant edge. Hence, G is well covered.

Claim 3.5 *G has a Hamiltonian cycle if and only if G_H has a Hamiltonian path.*

Proof:

The proof is similar to the one given for the Hamiltonian cycle problem, except for the following observations.

only if:

G has a Hamiltonian cycle. It is easy to see that we can find a simple path in G_H that starts at one of the leaf vertices, say v_{n31}, ends at the other, and covers all the vertices in G_H, that is, a Hamiltonian path for G_H.

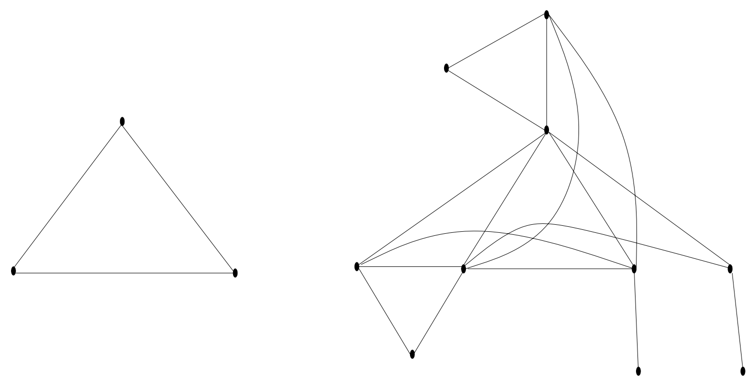

Figure 3.4: Hamiltonian path

if:
G_H has a Hamiltonian path. Since G_H has two leaf vertices in the K_2's, any Hamiltonian path has to start at one of the leaf vertices and end at the other. The four vertices that make up the two K_2's can be contracted to a single vertex in G. Hence, if G_H has a Hamiltonian path, we can always find a simple cycle in G that includes all the vertices in G, that is, a Hamiltonian cycle for G. This proves the claim.

Since the Hamiltonian cycle problem is NP-complete for general graphs, from the above, the Hamiltonian path problem is NP-complete for the family of well-covered graphs.

$\square$

Let us now examine the Hamiltonian path problem with respect to very well covered graphs. The result is similar to the Hamiltonian cycle result.

A bipartite graph $G = (X, Y, E)$ is called a *chain* graph if the vertices of X can be ordered $\{x_1, x_2, \ldots, x_{|X|}\}$ such that $N(x_1) \subseteq N(x_2) \subseteq \cdots \subseteq N(x_{|X|})$. This definition was given by Yannakakis [32]. Note that this implies the existence of an ordering $\{y_1, y_2, \ldots, y_{|Y|}\}$ of the vertices of Y such that $N(y_1) \supseteq N(y_2) \supseteq \cdots \supseteq N(y_{|Y|})$.

Theorem 3.6 *A very well covered graph has a Hamiltonian path if and only if it is a connected chain graph.*

Proof:
only if:
Let $G = (V, E)$, $\mid V \mid = n$, be a very well covered graph, with a Hamiltonian path $P_H = \{v_1, v_2, \ldots, v_n\}$. The edges $M = \{(v_1, v_2), (v_3, v_4), \ldots, (v_{n-1}, v_n)\}$ form a perfect matching, and must therefore satisfy property P.

Claim 3.6 *For all $1 \leq i, j \leq n$; if i is odd and j is even, and $j \leq i+1$,then $(v_i, v_j) \in E$.*

Proof:
Suppose not. Let i be the smallest odd index for which $\exists j \leq i+1$, j even, such that $(v_i, v_j) \notin E$. Let j be the largest index satisfying this. We know that $j < i-2$ because i is odd, j is even, and (v_{i-1}, v_i) and (v_i, v_{i+1}) are edges in P_H. Now since $(v_{i-2}, v_{i-1}) \in M$, it must satisfy property P. Furthermore, since i is as small as possible, it must be that $(v_j, v_{i-2}) \in E$. But then P implies that $(v_i, v_j) \in E$, which contradicts our assumption. This proves the claim.

Finally, if there is an edge between any pair of vertices v_i and v_k, where i and k are both odd, then both vertices are adjacent to $v_{min[i,k]+1}$ by the claim, and thus the matching edge $(v_{min[i,k]}, v_{min[i,k]+1})$ is in a triangle, which is impossible. A similar argument can be used to show that there can be no edges amongst the even vertices. Thus, $v_1, v_3, \ldots, v_{n-1}$ and $v_2, v_4, \ldots, v_n$ are orderings of two independent sets that demonstrate that G is a chain graph.
if:
Let $G = (X, Y, E)$ be a very well covered connected chain graph, with X and Y ordered as in the definition. We know that $\mid X \mid = \mid Y \mid$ because G is very well covered; let $n = \mid X \mid = \mid Y \mid$. Then, $\{x_1, y_1, x_2, y_2, \ldots, x_n, y_n\}$ is a Hamiltonian path.

□

3.7 Clique partition

Problem: *Given a graph G and an integer k, is there a set of k cliques such that every vertex of G is contained in one of the cliques?*

Theorem 3.7 *The clique partition problem is NP-complete for well-covered graphs.*

Proof:
We transform from a known NP-complete problem, the SAT. For any instance of SAT with clauses $C = \{c_1, c_2, \ldots, c_m\}$ and variables $U = \{u_1, u_2, \ldots, u_n\}$, we construct a graph $G = (V, E)$ as follows: G consists of $n + m$ C_5's — one for each clause and one for each variable. The C_5 associated with clause c_i has a distinguished *connector* vertex that we will refer to as c_i. The C_5 associated with variable u_i has two non-adjacent connector vertices, corresponding to the variable and its negation; we will refer to these vertices as u_i and $\overline{u_i}$, respectively. All vertices that are not connectors are adjacent only to other vertices in the same C_5. See Figure 3.5.

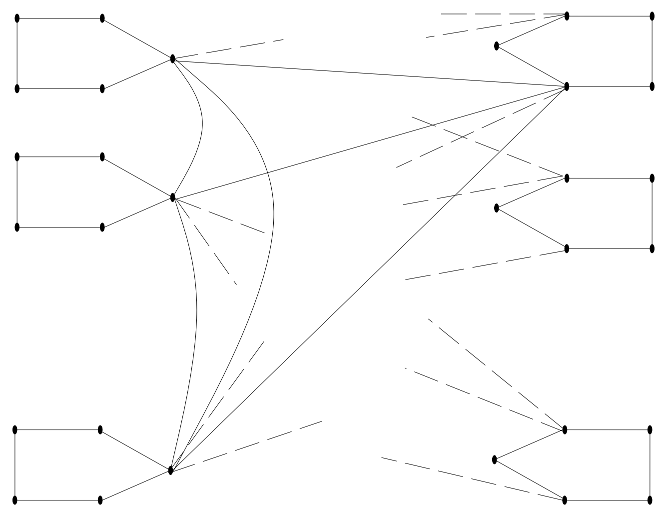

Figure 3.5: Clique partition

The edges of G are

$$
\begin{array}{ll}
 & \text{edges internal to all the } C_5\text{'s} \\
\cup & \{(c_i, u_j) | u_j \text{ is a literal in clause } c_i\} \\
\cup & \{(c_i, \overline{u_j}) | \overline{u_j} \text{ is a literal in clause } c_i\} \\
\cup & \{(c_i, c_j) | \exists \text{ a literal that is in both } c_i \text{ and } c_j\}
\end{array}
$$

The number of edges between clauses and their literals is $\leq mn$, considering the worst case of n literals per clause. The number of edges between the c_i's is $\leq m(m-1)/2$. Therefore, the number of edges in G is $\leq 5(m+n)+mn+m(m-1)/2$; the number of vertices is $5(m+n)$. Hence, G can be constructed in polynomial time.

To see that G is well covered, notice that it can be partitioned into $(n+m)$ disjoint C_5's. There can be at most two vertices from each C_5 in any maximal independent set. Let us see if there can be fewer vertices. This is possible only if some of the vertices in a C_5 have neighbours outside the cycle. Only the connector vertices have neighbours outside the cycle. Consider the C_5's corresponding to the clauses. They have one connector vertex each. Let one of the neighbours of a connector vertex c_i be in a maximal independent set. This leaves a P_4 from the corresponding C_5, and exactly two vertices from it are required in the maximal independent set to cover all its vertices. We still need two vertices from each C_5 that corresponds to a clause to be in any maximal independent set. Now consider the C_5's associated with the literals. They have two connector vertices each which are non-adjacent. If one of them has a neighbour in a maximal independent set, then an argument similar to the one given above holds. If both of them have neighbours in a maximal independent set, then we are left with a K_1 and a K_2, and we need one vertex from each to be in the set in order that the vertices in them may be covered. Therefore, any maximal independent set has to have exactly two vertices from each of the C_5's. Therefore, the graph is well covered.

Claim 3.7 *C is satisfiable if and only if G has a clique partition consisting of $2m + 3n$ cliques.*

Proof:
only if:
Consider a satisfying truth assignment for C. Take the vertices corresponding to the *true* literals along with their neighbours in the C_5's corresponding to the clauses as n cliques of the partition. These cliques cover one connector from each of the variable C_5's and all of the clause connectors. The remaining vertices can be covered with two cliques for each C_5.
if:
There exists a clique partition of size $2m + 3n$. Each of the C_5's must contain at least two cliques of the partition, by the structure of G. At most four vertices of each C_5 can be covered by these $\geq 2m + 2n$ cliques that are internal to the C_5's. Therefore, the remaining $\leq n$ cliques of the partition must cover at least one vertex from each C_5. In fact, since the variable C_5's have no edges amongst themselves, there should be exactly n cliques remaining, each of which covers exactly one vertex from each variable C_5. Also, these n cliques have to cover one vertex from each clause C_5. Since each of these n cliques contains a vertex from a variable C_5, the clause vertices covered must be the connector vertices. Assigning the value *true* to each of the literals corresponding to the variable connectors in the last n cliques yields a satisfying truth assignment for C, since each clause connector is adjacent to one of these variable connectors. This completes the proof of the claim.

We conclude that the clique partition problem is NP-complete for well-covered graphs.

□

The clique partition problem for very well covered graphs is not difficult to solve. From Favaron's characterization, we know that a very well covered graph has a perfect matching. For any graph G, the minimum number of cliques needed to partition the graph is greater than or equal to the size of a maximum independent set in the graph. If G is very well covered, then this size is equal to $\mid V \mid /2$, and any perfect matching is a clique partition of this size.

3.8 Dominating set

Problem: *Given a graph G and integer k, is there a set of k vertices of G such that every vertex not in the set is adjacent to at least one vertex in it?*

Theorem 3.8 *The dominating set problem is NP-complete for well-covered graphs.*

Proof:
We transform from SAT. The reduction is similar to that used for the clique partition proof in the previous section. For an instance of SAT with clauses $C = \{c_1, c_2, \ldots, c_m\}$ and variables $U = \{u_1, u_2, \ldots, u_n\}$, we construct a graph $G = (V, E)$ as follows: We first define the graph H to be a cycle on seven vertices with exactly one chord, which bisects the C_7 into a C_4 and a C_5. The distinguished *connector* vertex is the unique vertex of the C_5 that is not adjacent to any vertex of the C_4. G consists of m H's — one for each clause, and n C_3's — one for each variable. We will denote by c_i the connector vertex that the H associated with the clause c_i has. The C_3 associated with variable u_i has two connector vertices, corresponding to the variable and its negation, u_i and $\overline{u_i}$, respectively. As before, all the vertices which are not connectors are adjacent only to vertices in the same cycle. See Figure 3.6.

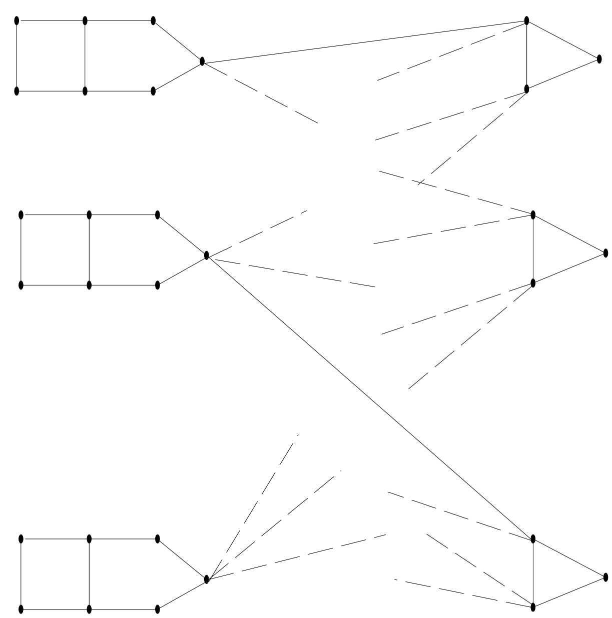

Figure 3.6: Dominating set

The edges of G are

$$\begin{array}{ll} & \text{edges internal to all the clause and variable cycles} \\ \cup & \{(c_i, u_j) | u_j \text{ is a literal in clause } c_i\} \\ \cup & \{(c_i, \overline{u_j}) | \overline{u_j} \text{ is a literal in clause } c_i\} \end{array}$$

The number of edges corresponding to clauses and their literals is $\leq mn$, assuming the worst case of n literals per clause. There are eight edges per H and three per C_3. Therefore, the number of edges in G is $\leq 8m+3n+mn$; the number of vertices is $7m+3n$. Hence, G can be constructed in polynomial time.

Consider the graph G. Any maximal independent set for G can have at most three vertices from each H. Can it have fewer? Only the connector has any neighbours outside H. If there is a connector c_i in some H, say H_i, with at least one of its neighbours in some maximal independent set I, then $H_{ir} = H_i \setminus \{c_i\}$ consists of a C_4 with two pendant edges attached to it. The pendant edges along with the remaining K_2 form a perfect matching for H_{ir}, one that can be easily seen to obey property P. Therefore, H_{ir} is very well covered and any maximal independent set for it has exactly three vertices. Hence, I will still contain three vertices from this H. The C_3's each have a simplicial vertex whose neighbour set is in the C_3, and therefore any maximal independent set for G will have to have exactly one vertex from each triangle. Therefore, any maximal independent set for G will have exactly $3m+n$ vertices; hence, G is well covered.

Claim 3.8 *C is satisfiable if and only if G has a dominating set of size $2m+n$.*

Proof:
only if:
C is satisfiable; therefore, it has a satisfying truth assignment. Vertices corresponding to true literals in this assignment will dominate all the C_3's and one vertex of each H. The remaining vertices can be dominated by choosing two additional vertices from each H. Thus, a dominating set of size $2m+n$ is obtained.
if:
G has a dominating set of size $2m+n$. Any dominating set for G must contain at least one vertex from each of the C_3's, since each C_3 has a simplicial vertex whose neighbour set is in the C_3. In addition, every dominating set has to contain at least two vertices from each H. Therefore, a dominating set of size $2m+n$ contains exactly one vertex from each C_3 and two vertices from each H. In such a dominating set, the connector vertices of all the H's must be dominated by vertices from the C_3's; else, more than two vertices would be required from some H. Thus, the variable connectors that are in the dominating set correspond to the *true* literals of a satisfying truth assignment. This proves the claim.

We conclude that the dominating set problem is NP-complete for well-covered graphs.

□

The dominating set problem can be solved efficiently for very well covered graphs. The result follows from proposition 5.2 of [8], in which it is shown that a minimum dominating set in a very well covered graph consists of one neighbour of each degree 1 vertex in a corresponding irreducible graph.

3.9 Conclusions

We have looked at the families of well-covered and very well graphs from an algorithmic complexity point of view. We have studied the complexities of some standard problems like recognition, Hamiltonian cycle and path, and dominating set, for these families. We conclude that many graph problems are as difficult to solve for well-covered graphs as for graphs in general. Therefore, from an algorithmic complexity point of view, there is little to be gained by restricting our attention to this family. However, some of these problems are tractable for very well covered graphs.

Chapter 4

An alternative characterization

4.1 Introduction

In the previous chapter, we showed that well-covered graph recognition is co-NP-complete, and that the complexity of some standard graph theory problems like the Hamiltonian cycle and path, clique cover, and dominating set are NP-complete for this family. We also observed that very well covered graph recognition is in P, and that some of the above problems such as Hamiltonian cycle are tractable for this sub-class. This points to the possible existence of intermediate classes of well-covered graphs for which some of the problems could prove tractable.

We now provide an *alternative characterization* for well-covered graphs, a characterization based on the interaction between pairs of maximal independent sets of such a graph. We establish the conditions under which the intersection of a pair of maximal independent sets of a well-covered graph is *maximal*, and under which all such intersections have the same size. These two results help define some new sub-classes. The maximal intersection result is the main result of this chapter and is used extensively in decomposing graphs belonging to tbe new sub-classes. The alternative characterization for well-covered graphs also leads to a new characterization for very well covered graphs.

4.2 Definitions

Let $G = (V, E)$ be a simple graph, where $\mid E \mid \neq 0$. Let I_1 and I_2 be maximal independent sets of G. We use R, S, $I_1^{'}$ and $I_2^{'}$ to denote the following:

$$R = I_1 \cap I_2$$

$$S = V - \{I_1 \cup I_2\}$$

$$I_1^{'} = I_1 - R$$

$$I_2^{'} = I_2 - R$$

See Figure 4.1.

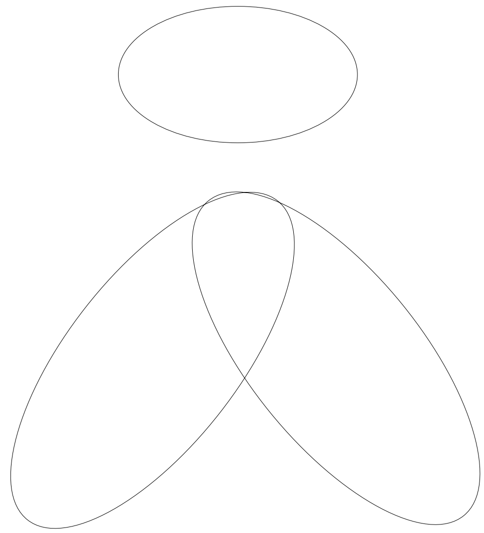

Figure 4.1: Definitions

The intersection R of a pair of maximal independent sets of G is said to be *maximal* if for every pair of maximal independent sets I_a and I_b that contain R, $I_a \cap I_b = R$.

A graph G is said to be *complete k-partite* if its vertex set can be partitioned into k disjoint independent sets, $V = P_1 \cup P_2 \cup \cdots \cup P_k$, for some positive integer $k \leq\mid V \mid$, such that $N(v) = V - P_i$ for each vertex $v \in P_i$, $1 \leq i \leq k$. Each such partition is called a *part*. A graph is said to be *complete k_n-partite* if it is complete k-partite with each part having n vertices.

We state Hall's theorem as it is made use of in this chapter:

Hall's theorem(see [2]): *Let G be a bipartite graph with bipartition (X, Y). Then Y contains a matching that saturates every vertex in X if and only if $\mid N(X_1) \mid \geq \mid X_1 \mid$ for all $X_1 \subseteq X$.*

4.3 An alternative characterization

We give an alternative characterization for well-covered graphs:

Theorem 4.1 (alternative characterization) *A graph G is well covered if and only if for every pair of maximal independent sets I_1 and I_2 of G, $< I_1^{'} \cup I_2^{'} >$ has a perfect matching.*

Proof:
only if:
Let G be well covered. Assume that the statement is not true; that is, there exist maximal independent sets I_1 and I_2 of G such that $< I_1^{'} \cup I_2^{'} >$ does not have a perfect matching. Since G is well covered, $I_1^{'}$ and $I_2^{'}$ are of the same size. Consider

the bipartite graph $< I_1' \cup I_2' >$. Since there does not exist a perfect matching between I_1' and I_2', from Hall's theorem (section 4.2), there exists an independent set $S_1 \subseteq I_1'$ such that $\mid S_1 \mid > \mid N(S_1) \mid$. See Figure 4.2. Let $I_3 = I_2 - N(S_1) \cup S_1$.

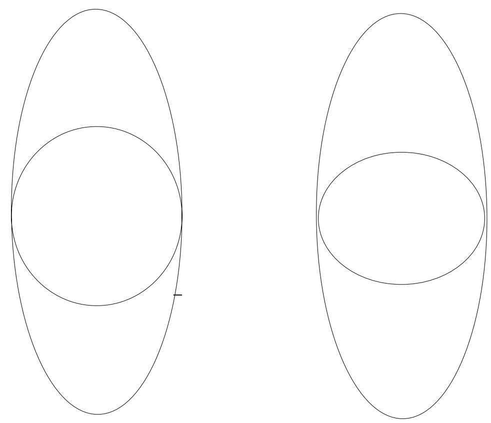

Figure 4.2: S_1 and its neighbour set $N(S_1)$

Since $\mid S_1 \mid > \mid N(S_1) \mid$, $\mid I_3 \mid > \mid I_2 \mid$. This is not possible as I_3 is an independent set and G is well covered.

if:

Let G be a graph such that for any two maximal independent sets I_1 and I_2, $< I_1' \cup I_2' >$ has a perfect matching. Let G not be well covered. Then, there exist maximal independent sets I_1 and I_2 such that $\mid I_1 \mid \neq \mid I_2 \mid$. Let

$$R = I_1 \cap I_2$$

Now,

$$\mid I_1 \mid \neq \mid I_2 \mid$$

Therefore,

$$\mid I_1 - R \mid \neq \mid I_2 - R \mid$$

That is,

$$\mid I_1' \mid \neq \mid I_2' \mid$$

Therefore, $< I_1' \cup I_2' >$ does not have a perfect matching, which is a contradiction.

□

4.4 The maximal intersection theorem

The alternative characterization was based on the interaction between the non-intersecting portions of pairs of maximal independent sets of a well-covered graph. We now take a closer look at such intersections. We first state the conditions under which such an intersection is *maximal*; that is, the conditions under which the intersection R of a pair of maximal independent sets of a well-covered graph G has the property that for every pair of maximal independent sets I_1 and I_2 that it belongs to, $I_1 \cap I_2 = R$. This result, called the *maximal intersection theorem*, is the main result of this chapter and is *the* result used in decomposing graphs belonging to the new sub-classes.

Theorem 4.2 (maximal intersection) *The intersection R of a pair of maximal independent sets I_1 and I_2 of a well-covered graph G is maximal if and only if $< V - N[R] >$ is complete k_n-partite.*

In order to prove this theorem, we need the following two propositions. We first state the conditions under which a graph G is *complete k-partite.*

Proposition 4.1 *A graph G is complete k-partite if and only if for every non-adjacent pair of vertices $u, v \in V$, $N(u) = N(v)$.*

Proof:
only if:
Since the graph is complete k-partite, any non-adjacent pair of vertices must belong to the same part. Therefore, they must have the same neighbour set.
if:
For all $u, v \in V$, $u \not\sim v$, $N(u) = N(v)$. We say that u and v are equivalent if $u \not\sim v$. This relation is an equivalence relation since if $u \not\sim v$ and $v \not\sim w$, $u, v, w \in V$, then $N(u) = N(v)$ and hence $u \not\sim w$. This equivalence relation divides the vertex set V into m equivalence classes, $1 \leq m \leq \mid V \mid$. Let the classes be denoted by $K_1, K_2, \ldots, K_m$. We prove the following claim.

Claim 4.1 *For all $u \in K_i$, $1 \leq i \leq m$, $N(u) = V - K_i$.*

Proof:
Consider a vertex $u \in K_i$. Any other vertex $v \in K_i$ is not adjacent to u because of the equivalence relation. Therefore, the K_i's are mutually disjoint independent sets. Now assume that there exists $w \in K_j$, $i \neq j$, such that $w \notin N(u)$. That is, $u \not\sim w$. This means that $w \in K_i$, from the equivalence relation, which is a contradiction. This proves the claim.

Therefore, the equivalence classes form a partition of the vertex set into mutually disjoint independent sets with the property that a vertex from an independent set is adjacent to all the vertices outside the set; that is, G is complete k-partite.

□

The second proposition states the conditions under which the intersection R of a pair of maximal independent sets of a graph G is maximal. Note that G can be any simple graph, and need not necessarily be well covered.

Proposition 4.2 *Let G be a graph and R be the intersection of a pair of maximal independent sets of G. Then R is a maximal intersection if and only if $< V - N[R] >$ is complete k-partite.*

Proof:
only if:
Assume that $< V - N[R] >$ is not complete k-partite. From proposition 4.1, there exist $u, v \in V - N[R]$, $u \not\sim v$, such that $N(u) \neq N(v)$. That is, there exists $w \in V - N[R]$ such that u(say)$\not\sim w$ and $v \sim w$. See Figure 4.3.

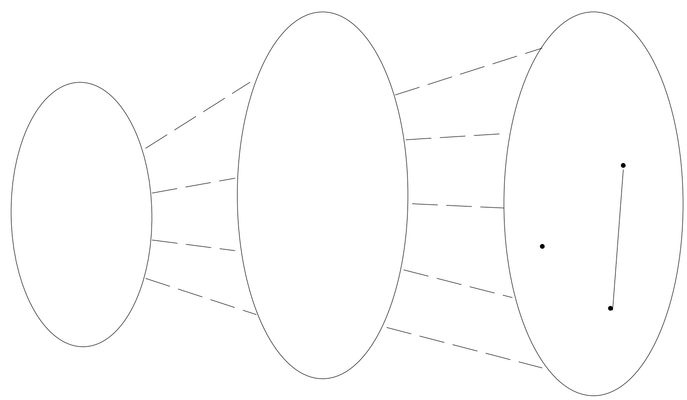

Figure 4.3: $< V - N[R] >$ is not complete k_n-partite

Let

$$I_1 = R \cup \{u\} \cup \{v\}$$

$$I_2 = R \cup \{u\} \cup \{w\}$$

I_1 and I_2 are independent sets. Extend them to form maximal independent sets of G. $I_1 \neq I_2$ as $v \sim w$, $v \in I_1$, $w \in I_2$. Now,

$$I_1 \cap I_2 \supseteq R \cup \{u\} \supset R$$

which is a contradiction.
if:
Since $< V - N[R] >$ is complete k-partite, from proposition 4.1, we have $N(u) = N(v)$ in $< V - N[R] >$, for all $u, v \in V - N[R]$, $u \not\sim v$. Assume that R is not a maximal intersection. Therefore, there exist maximal independent sets I_1 and I_2 of G such that R belongs to both I_1 and I_2, and is properly contained in the intersection R_1 of I_1 and I_2. Let u be an element of R_1, $u \notin R$. Let

$$I_1' = I_1 - R_1$$

and

$$I_2' = I_2 - R_1$$

I_1 and I_2 are maximal independent sets and their intersection is R_1. Hence, every vertex in I_1' has at least one neighbour in I_2' and vice versa. Therefore, there exist $v \in I_1'$ and $w \in I_2'$ such that $v \sim w$. Now, u, v, and w are in $V - N[R]$, $v \sim w$, and $u \not\sim v$. Therefore, $N(u) = N(v)$. This means that $u \sim w$, which is not possible as both u and w are in the maximal independent set I_2.

$\square$

We now prove the maximal intersection theorem, theorem 4.2.

Proof (*of the maximal intersection theorem*):
only if:
G is well covered, and R is maximal. From proposition 4.2, $< V - N[R] >$ is complete k-partite. Now, let two parts of $< V - N[R] >$ be of different sizes. Combining each of these with R would give maximal independent sets of different sizes for G.
if:
$< V - N[R] >$ is complete k_n-partite; therefore, it is also complete k-partite. From proposition 4.2, R is maximal.

$\square$

Thus, we can decompose a well-covered graph into a complete k-partite graph which is well covered, and the graph $< N[R] >$. In the next chapter, we will see that restricting $< N[R] >$ to be well covered leads to the creation of the new sub-classes. We now give an example to show that this need not be true in general.

Consider a C_5. It is well covered because any maximal independent set has exactly two vertices. Therefore, any maximal intersection of a pair of maximal independent sets of a C_5 will have exactly one vertex. This means that when R is maximal, $< N[R] >$ is a P_3 which is not well covered.

In fact, even the graph $< N(R) >$ is not always well covered, as the following example shows. Consider the graph G in the Figure 4.4. Every maximal independent set of G has exactly two vertices, making G a well-covered graph. Now consider the maximal independent sets $I_1 = \{v_1, v_5\}$ and $I_2 = \{v_1, v_6\}$. Their intersection $R = \{v_1\}$ and $N(R) = \{v_2, v_3, v_4\}$, a P_3. Now, $< V - N[R] >$ is a K_2 with the vertex set $\{v_5, v_6\}$. That is, it is a complete bipartite graph and hence, using theorem 4.2, R is a maximal intersection. The graph $< N(R) >$ is a P_3 which is not well covered.

4.5 The fixed intersection theorem

We now state the conditions under which every pair of maximal independent sets of a well-covered graph G intersect in exactly k vertices, for some non-negative integer k. This result, along with the maximal intersection theorem, leads to the definition of the new sub-classes.

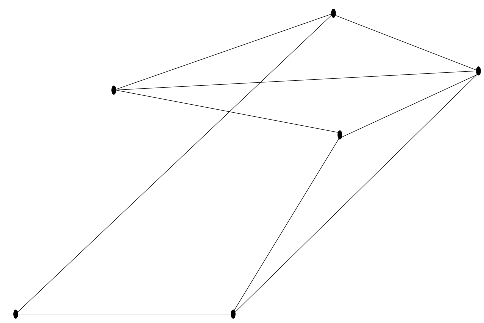

Figure 4.4: $< N(R) >$ is not always well covered

Theorem 4.3 (fixed intersection) *Let G be a well-covered graph and let l be a non-negative integer. Then every pair of maximal independent sets of G intersect in exactly l vertices if and only if G is the union of a complete k_n-partite graph and l isolated vertices.*

Proof:
only if:
Since the intersection of every pair of maximal independent sets of G has a fixed size l, this means that any such intersection has to be maximal, that is, the intersection R of any pair of maximal independent sets I_1 and I_2 of G cannot be a proper subset of an intersection of some other pair of maximal independent sets of G. From theorem 4.2, we see that $< V - N[R] >$ has to be complete k_n-partite.

If $l = 0$, then the theorem is trivially true. Now, assume that $l > 0$, and that the intersection R of some maximal independent sets I_1 and I_2 of G does not consist entirely of isolated vertices. Then, there exist $u \in R$ and $v \in S$ such that $u \sim v$. See Figure 4.5.

Let S_1 be a maximal independent set from S which includes v.

$$\begin{aligned} I_3 &= I_1 \cup S_1 - N(S_1) \\ &= (R - N(S_1)) \cup (I_1^{'} - N(S_1)) \cup S_1 \end{aligned}$$

$\mid R - N(S_1) \mid$ is less than $\mid R \mid$, by the choice of S_1. Let $\mid I_1^{'} - N(S_1) \mid > 0$. Any vertex x in $I_1^{'}$ is adjacent to all of $V - N[R] - I_1^{'}$, by theorem 4.2. Hence, no vertex of $V - N[R] - I_1^{'}$ can be in I_3, that is, I_3 is maximal. But

$$I_3 \cap I_2 = R - N(S_1)$$

and we know that

$$\mid R - N(S_1) \mid < R$$

This is a contradiction. Therefore,

$$\mid I_1^{'} - N(S_1) \mid = 0$$

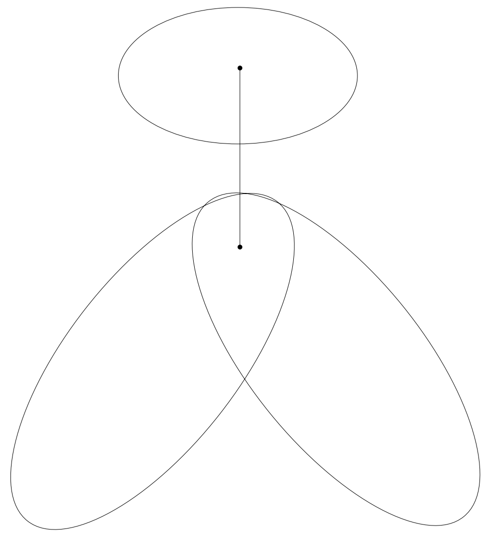

Figure 4.5: Fixed intersection proof

that is,

$$I_3 = (R - N(S_1)) \cup S_1$$

By the same argument,

$$\mid I_2' - N(S_1) \mid = 0$$

Therefore, I_3 is a maximal independent set in G. Then,

$$\mid I_1 \cap I_3 \mid = \mid R - N(S_1) \mid < \mid R \mid$$

Thus, R consists of isolated vertices, whose number equals l.
if:
Follows as G is the union of a complete k_n-partite graph and l isolated vertices.

□

4.6 Very well covered graphs

We now restrict the alternative characterization for well-covered graphs to the family of very well covered graphs. We know from corollary 2.1 that the size of a maximal independent set of a well-covered graph G without isolated vertices is bounded by $\mid V \mid /2$. Hence, any well-covered graph without isolated vertices can be transformed into a very well covered graph by adding an appropriate number of isolated vertices. We therefore turn our attention to very well covered graphs without isolated vertices. The following result is used in the next chapter to show that the smallest sub-class in the hierarchy of new sub-classes is the same as the family of very well covered graphs without isolated vertices.

Theorem 4.4 (very well covered graph characterization) *Let G be a graph without isolated vertices. Then the following are equivalent:*

a) G is very well covered.

b) G is well covered, and for some pair of maximal independent sets I_1 and I_2, $\mid R \mid = \mid S \mid$.

c) G is well covered, and for every pair of maximal independent sets I_1 and I_2, $\mid R \mid = \mid S \mid$.

d) For every pair of maximal independent sets I_1 and I_2 of G, there exists a perfect matching M which satisfies P, in which R matches to S and $I_1^{'}$ matches to $I_2^{'}$.

Property P is the same as the one defined in section 2.2. We recall it here for convenience.

Property P: *A matching M in a graph G satisfies property P if for any edge $(u, v) \in M$, $N(u) \cap N(v) = \phi$, and $N(u) - \{v\}$ is adjacent to all of $N(v) - \{u\}$.*

We need the following proposition in order to prove the theorem.

Proposition 4.3 *Let G be a well-covered graph. Then G is very well covered if and only if there exist maximal independent sets I_1 and I_2 of G such that $\mid R \mid = \mid S \mid$.*

Proof:
only if:
Let I_1 and I_2 be a pair of maximal independent sets of G. Since G is very well covered,

$$\mid I_1 \mid = \mid I_2 \mid = \mid V \mid /2$$

That is,

$$\mid I_1 \mid = \mid V \mid - \mid I_1 \mid$$

This can be written as

$$\mid I_1^{'} \mid + \mid R \mid = \mid I_2^{'} \mid + \mid S \mid$$

Since $\mid I_1 \mid = \mid I_2 \mid$, $\mid I_1^{'} \mid = \mid I_2^{'} \mid$. Therefore,

$$\mid R \mid = \mid S \mid$$

if:
There exist maximal independent sets I_1 and I_2 of G such that $\mid R \mid = \mid S \mid$. Now,

$$\mid I_1 \mid = \mid V \mid - (\mid I_2^{'} \mid + \mid S \mid) \tag{4.1}$$

Since G is well covered, $\mid I_1^{'} \mid = \mid I_2^{'} \mid$. Also, $\mid R \mid = \mid S \mid$. Therefore, equation 4.1 can be rewritten as

$$\mid I_1 \mid = \mid V \mid - (\mid I_1^{'} \mid + \mid R \mid)$$

That is,

$$| I_1 |=| V | - | I_1 |$$

Therefore,

$$| I_1 |=| V | /2$$

As G is well covered, all maximal independent sets in it have the same size, which is $| V | /2$. Therefore, G is very well covered.

$\square$

We now prove the main theorem of this section.

Proof *(of the very well covered graph characterization)* :
a) $\rightarrow$ *b)*
Since G is very well covered, it is also well covered. Statement b) follows from the proposition 4.3.
b) $\rightarrow$ *c)*
From proposition 4.3, if $| R |=| S |$ for some pair of maximal independent sets I_1 and I_2 of a well-covered graph, then G is very well covered. The rest follows.
c) $\rightarrow$ *d)*
Using c), and proposition 4.3, G is very well covered. Let I_1 and I_2 be a pair of maximal independent sets of G. Since G is well covered, using theorem 4.1, we see that $I_1^{'}$ and $I_2^{'}$ have a perfect matching. G does not have any isolated vertices, and R is an independent set in G. Hence, the neighbour set of R is in $V - R$. We observe here that Hall's theorem (section 4.2) should hold for the sets R and $V - R$ as long as R does not contain any isolated vertices, since the theorem only concerns itself with the edges between the two sets. Using corollary 2.1, we see that for any $R_1 \subseteq R$, $| N(R_1) |\geq| R_1 |$. That is, Hall's theorem is satisfied. Hence, there exists a matching from $V - R$ to R, that is, from $N(R)$ to R, that covers all the vertices of R. Now, $N(R) \subseteq S$. Hence, there exists a matching from S to R that covers R. From proposition 4.3, $| R | = | S |$. Therefore, any such matching between R and S is a perfect matching. Since G has no isolated vertices, and since R, S, $I_1^{'}$, and $I_2^{'}$ are mutually disjoint sets, a perfect matching between R and S, and one between $I_1^{'}$ and $I_2^{'}$, together form a perfect matching for G. The fact that this matching satisfies P follows from Favaron's theorem (theorem 2.1).
d) $\rightarrow$ *a)*
Since G has no isolated vertices, it has at least two maximal independent sets. From d), we see that it has a perfect matching which satisfies P. That the graph is very well covered follows from Favaron's theorem (theorem 2.1).

$\square$

4.7 Conclusion

We have given a new characterization for well-covered graphs, and one for very well covered graphs. We have stated the conditions under which the intersection R of a pair of maximal independent sets of a graph G is maximal, and under which every pair of maximal independent sets of G intersect in exactly l vertices, for some non-negative integer l. We have seen that the graph $< N[R] >$ is not necessarily well covered. In the next chapter, we will restrict our attention to those well-covered graphs that have the property that the graph $< N[R] >$ is well covered. We will study four such families of well-covered graphs.

Chapter 5

Some new sub-classes

5.1 Introduction

In the previous chapter, we gave new characterizations for well-covered and very well covered graphs. We defined the conditions under which the intersection R of any pair of maximal independent sets I_1 and I_2 of a well-covered graph G is maximal. We now define a hierarchy of four new sub-classes of well-covered graphs based on this maximal R, each one of which properly contains the one next to it in the hierarchy. We show that the last sub-class in the hierarchy is exactly the family of very well covered graphs without isolated vertices. The objective here it to find new sub-classes of well-covered graphs for which some of the problems that proved to be intractable for well-covered graphs prove to be tractable, and to algorithmically separate the new sub-classes. The algorithmic properties of these sub-classes will be studied in the next chapter.

5.2 Definitions

Let G be a graph whose vertex set V can be partitioned into $1 \leq t \leq \mid V \mid$ disjoint sets $L_1, L_2, \ldots, L_t$, such that for $1 \leq i \leq t$, the subgraph $H_i = < L_i >$ is complete k_n-partite. G is said to be partitioned into complete k_n-partite subgraphs. We call the L_i's *layers*, and the H_i's *lgraphs*. We denote by E_i the edge set, by k_i the number of parts, and by n_i the number of vertices in each part, of H_i, $1 \leq k_i \leq \mid L_i \mid$, $n_i = \mid L_i \mid / k_i$. We denote the parts in H_i by $P_{i1}, P_{i2}, \ldots, P_{ik_i}$. We write H_i as $H_i = (P_{i1}, P_{i2}, \ldots, P_{ik_i}, E_i)$. Since the P_{ij}'s, $1 \leq j \leq k_i$, form a partition of L_i, we say that each L_i consists of, or is made up of, or has, k_i parts; we also talk of a part P_{ij} in the layer L_i. Where there is room for confusion, we shall write P_{ij} as $P_{i,j}$. We say that a part P_a is adjacent to a vertex v if v has a neighbour in P_a. We say that two parts P_a and P_b are adjacent, or *connected*, or are *neighbours*, if there exist $u \in P_a$ and $v \in P_b$ such that $(u, v) \in E$. We say that P_a and P_b are *completely* connected, or that P_a is completely connected to P_b, if $< P_a \cup P_b >$ is complete bipartite. We say that two layers are adjacent if there is a part in one that is adjacent to a part in the other.

5.3 The hierarchy of sub-classes

As stated earlier, we now define a hierarchy of four new sub-classes of well-covered graphs. These sub-classes are named W_{SR}, W_{AR}, W_{ARF}, and W_{AR2}. They have the property that each one is completely contained in the one preceding it, that is, $W_{SR} \supset W_{AR} \supset W_{ARF} \supset W_{AR2}$. The sub-class W_{SR} is properly contained in the family of well-covered graphs, and the family W_{AR2} is the same as the family of very well covered graphs without isolated vertices. These ideas are depicted pictorially in the Figure 5.1.

$$
\begin{array}{c}
\textbf{Well covered} \\
\bigcup \\
\mathbf{W_{SR}} \\
\bigcup \\
\mathbf{W_{AR}} \\
\bigcup \\
\mathbf{W_{ARF}} \\
\bigcup \\
\mathbf{W_{AR2}} = \textbf{Very well covered without isolated vertices}
\end{array}
$$

Figure 5.1: The hierarchy of sub-classes

5.4 The first sub-class W_{SR}

From theorem 4.2, we know that when the intersection R of a pair of maximal independent sets of a well-covered graph G is maximal, $< V - N[R] >$ is complete k_n-partite. We have seen in the last chapter that neither $< N[R] >$ nor $< N(R) >$ is always well covered. We now restrict our attention to the family of well-covered graphs for which we can find a maximal intersection R such that $< N[R] >$ is well covered.

Consider a well-covered graph G_1 with vertex set V_1. Assume that there exists a maximal intersection R_1 in G_1 such that the graph G_2 induced by $V_2 = N[R_1]$ is well covered. Let $H_1 =< V_1 - N[R_1] >$. Thus, G_1 has been decomposed into the graphs H_1 and G_2. We call this the first stage of a decomposition of G_1. Again, assume that there exists a maximal intersection R_2 in G_2 such that the graph G_3 induced by $V_3 = N[R_2]$ is well covered. Let $H_2 =< V_2 - N[R_2] >$. Thus, G_2 has been decomposed into the graphs H_2 and G_3. This is the second stage of a decomposition of G_1. We observe that by restricting $< N[R_i] >$, $i \in \{1, 2\}$, to be well covered, we are able to *recursively* decompose the graph G_1 into the graphs

H_1, H_2 and G_3. We continue the process by assuming that G_3 has a maximal intersection R_3 such that the graph G_4 induced by $V_4 = N[R_3]$ is well covered.

Consider stage j of such a decomposition. We have the graphs H_1 to H_{j-1}, and the well covered graph G_j with vertex set V_j. Assume that there exists a maximal intersection R_j in G_j such that the graph G_{j+1} induced by $V_{j+1} = N[R_j]$ is well covered. Let $H_j =< V_j - N[R_j] >$. That is, the graph G_j has been decomposed into the graphs H_j and G_{j+1}. We again assume that G_{j+1} has a maximal intersection R_{j+1} such that the graph G_{j+2} induced by $V_{j+2} = N[R_{j+1}]$ is well covered.

Since we start with a graph G_1 which has a finite number of vertices, this decomposition stops at some stage, say t. Let the corresponding graph be G_t, with vertex set V_t. Since we cannot decompose the graph any further, this means that we cannot find a pair of maximal independent sets in G_t which have a non-zero intersection. That is, G_t either consists of isolated vertices, or the intersection of every pair of maximal independent sets in G_t is the empty set. From theorem 4.3, the latter will happen when G_t is complete k_n-partite. Therefore, G_t is complete k_n-partite and forms the graph H_t in the decomposition.

Thus, G_1 has been recursively decomposed into the graphs $H_1, H_2, \ldots, H_t$. Let the corresponding vertex sets be given by $L_1, L_2, \ldots, L_t$. As we have seen, H_t is complete k_n-partite, with the number of parts in it being ≥ 1. Consider the graph H_i, $1 \leq i < t$. Now, $H_i =< V_i - N[R_i] > G_i =< V_i >$ and $G_{i+1} =< N[R_i] >$. Since R_i is a maximal intersection in G_i, there exist maximal independent sets $I_{i,1}$ and $I_{i,2}$ in G_i such that their intersection is R_i. From theorem 4.2, H_i is complete k_n-partite. Since $I'_{i,1}$ and $I'_{i,2}$ are maximal independent sets in H_i, H_i has at least two parts. Thus, each of the graphs H_1 to H_{t-1} is complete k_n-partite and has at least two parts. Since the graphs H_i, $1 \leq i \leq t$, are vertex disjoint, the vertex sets L_1 to L_t form a partition of V_1. That is, the H_i's are *lgraphs*, and the V_i's are *layers*, $1 \leq i \leq t$. See Figure 5.2.

We now define a family of graphs which can be recursively decomposed in this manner until we arrive at a $< N[R] >$ which is complete k_n-partite, $k \geq 1$. We call this family W_{SR}.

Definition 5.1 *A graph G is said to belong to the family W_{SR} if*

a) G is complete k_n-partite, or

b) G is well covered and for some maximal R, the intersection of a pair of maximal independent sets of G, $< N[R] >$ belongs to W_{SR}.

From the definition, it is clear that any graph G belonging to W_{SR} can be decomposed as described above into lgraphs H_1 to H_t, $1 \leq t \leq n$, such that each lgraph, except for H_t, has at least two parts; H_t need have only one part. The corresponding layers L_1 to L_t form a partition of the vertex set of G. The recursive definition ensures that at each stage of such a decomposition, we can find a maximal intersection R such that $< N[R] >$ is well covered. At each stage of a decomposition, there can be more than one maximal intersection R with the property that $< N[R] >$ belongs to W_{SR}. Thus, there could be many possible decompositions of G.

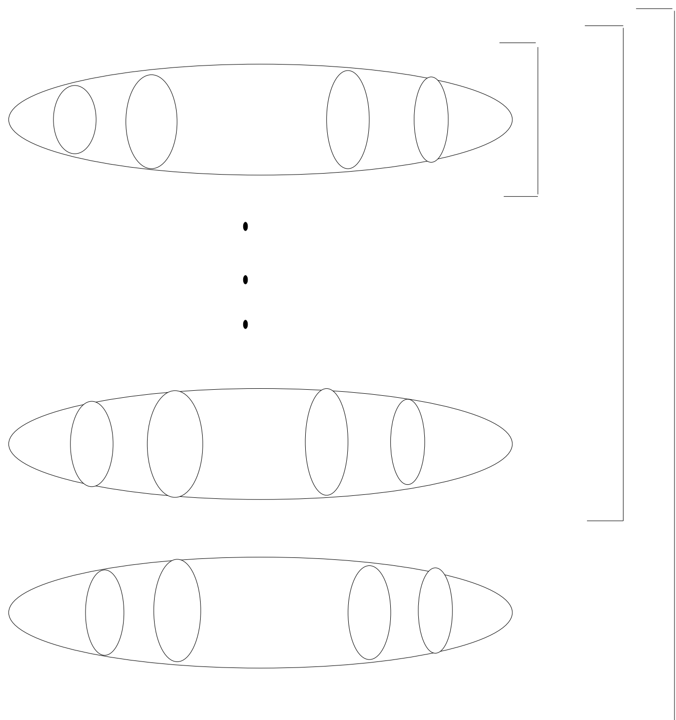

Figure 5.2: The decomposed graph G

We denote by $\mathcal{D}_{SR}(G)$ the set of all decompositions of a graph G belonging to W_{SR}. A decomposition can be represented by an ordered set of lgraphs, or equivalently, by an ordered set of layers. We use the latter representation. That is, each $D_1(G) \in \mathcal{D}_{SR}(G)$ is an ordered set of layers $L_1, L_2, \ldots, L_t$, $1 \leq t \leq n$, with each layer having at least two parts, except for L_t which need have only one.

The above definition leads us to the following characterization of the family W_{SR}:

Theorem 5.1 (W_{SR}) *A graph G belongs to the family W_{SR} if and only if its vertices can be partitioned into layers $L_1, L_2, \ldots, L_t$, $1 \leq t \leq \mid V \mid$, which have the following properties:*

a) The lgraphs $H_i = < L_i >$, $1 \leq i \leq t$, are complete k_n-partite, with every layer except the last one, L_t, having at least two parts. L_t need have just one part.

b) Given a layer L_j, $1 \leq j < t$, there exists at least one part in each of the layers $L_{j+1}, L_{j+2}, \ldots, L_t$, which is not adjacent to any of the vertices in L_j. Furthermore, one set of these parts forms an independent set.

c) Every maximal independent set of G contains exactly one part from each layer.

Proof:

only if:

We assume that G is not complete k_n-partite, since then the statements $a)$ through $c)$ are trivially true. G belongs to W_{SR}. Therefore, G can be decomposed into lgraphs. Let $D_1(G) \in \mathcal{D}_{SR}(G)$ be a decomposition of G into layers L_1 to L_t. We have seen earlier that these layers obey statement *a*), and form a partition of the vertex set of G. In order to prove statements *b*) and *c*), we need the following proposition.

Proposition 5.1 *Let G_j be the graph induced by the layers L_j to L_t, $1 \leq j \leq t$. Then the size of every maximal independent set in G_j is given by $n_j + n_{j+1} + \cdots + n_t$.*

Proof:
The graph G_t is induced by the layer L_t. Since H_t is complete k_n-partite, the size of every maximal independent set in H_t is given by the size of a part in it, namely n_t. Now, consider the graph G_{t-1}. This is induced by the layers L_t and L_{t-1}. From the decomposition, we know that there exists R_{t-1}, the maximal intersection of a pair of maximal independent sets $I_{t-1,1}$ and $I_{t-1,2}$ of G_{t-1}, such that G_t is the graph induced by $V_t = N[R_{t-1}]$. The lgraph H_{t-1} is the graph induced by $L_{t-1} = V_{t-1} - V_t$, and it is complete k_n-partite. We also know that $I'_{t-1,1}$ and $I'_{t-1,2}$ are parts in H_{t-1}. Now, R_{t-1} is a maximal independent set in G_t, because G_t is made up of R_{t-1} and its neighbour set. This means that R_{t-1} is one of the parts of H_t. Since $I_{t-1,1}$ is a maximal independent set in G_{t-1}, its size is given by $\mid R_{t-1} \mid + \mid I'_{t-1,1} \mid$, that is, $n_t + n_{t-1}$. Since G_{t-1} is well covered, the size of every maximal independent set is equal to $n_t + n_{t-1}$, that is, the sum of the sizes of the parts in the lgraphs H_t and H_{t-1}.

Assume that the above is true for graphs G_k to G_t, $t \geq k > 1$. Therefore, the size of every maximal independent set in G_k is given by $n_t + n_{t-1} + \cdots + n_{k-1} + n_k$. Now, consider the graph G_{k-1}, induced by the layers L_t to L_{k-1}. From the decomposition, we know that there exists R_{k-1}, the maximal intersection of a pair of maximal independent sets $I_{k-1,1}$ and $I_{k-1,2}$ of G_{k-1}, such that G_k is the graph induced by $V_k = N[R_{k-1}]$, and the lgraph H_{k-1} is the graph induced by $L_{k-1} = V_{k-1} - V_k$. We know that H_{k-1} is complete k_n-partite, and that $I'_{k-1,1}$ and $I'_{k-1,2}$ are parts in H_{k-1}. The size of $I_{k-1,1}$, a maximal independent set in G_{k-1}, is given by $\mid R_{k-1} \mid + \mid I'_{k-1,1} \mid$. Now, R_{k-1} is a maximal independent set in G_k because G_k is made up of R_{k-1} and its neighbour set. We know that the size of a maximal independent set in G_k is given by the sum of the sizes of the parts in the lgraphs H_k to H_t. Since, $I'_{k-1,1}$ is a part in H_{k-1}, its size is given by n_{k-1}. Therefore, the size of $I_{k-1,1}$ is given by $n_t + n_{t-1} + \cdots + n_k + n_{k-1}$. Since G_{k-1} is well covered, this means that every maximal independent set in it has size equal to the sum of the sizes of the parts in the lgraphs H_{k-1} to H_t. This proves the proposition.

b)
Consider some layer L_j, $1 \leq j < t$. From *a*), this layer is complete k_n-partite, and has at least two parts. From the decomposition, we know that there exists a maximal R_j, the intersection of some pair of maximal independent sets $I_{j,1}$ and $I_{j,2}$ of G_j, such that G_{j+1}, the graph induced by $V_{j+1} = N[R_j]$, is in W_{SR}. We also know that the complete k_n-partite graph H_j is the graph induced by $L_j = V_j - V_{j+1}$. $I'_{j,1}$ and $I'_{j,2}$ are two parts in H_j. From proposition 5.1, we know that the size of

R_j, a maximal independent set in G_{j+1}, is given by the sum of the sizes of the parts in the lgraphs H_{j+1} to H_t, or equivalently, in the layers L_{j+1} to L_t. Since the lgraphs are complete k_n-partite and R_j is an independent set, this means that the number of vertices of R_j that can be present in any given layer is limited by the size of a part in that layer. From the above two statements, we conclude that the vertices of R_j are distributed amongst the layers L_{j+1} to L_t such that they form exactly one part in each layer. Since R_j has no neighbours in L_j, the statement is proved.

c)

G is the graph induced by the layers L_1 to L_t. Since G is in W_{SR}, it is well covered. Using proposition 5.1, we see that the size of every maximal independent set in G is given by the sum of the sizes of the parts in the above layers. Since the lgraphs are complete k_n-partite, every maximal independent set can include at most the vertices of any one part from each layer. From the above two statements, we conclude that every maximal independent set of G contains exactly one part from each layer in the decomposition.

if:

Consider a graph G whose vertex set can be partitioned into layers L_1 to L_t such that the layers obey properties a) through c). Since property c) is obeyed, the graph G is well covered. We prove that G is in W_{SR} by induction. We accomplish this by showing that, for j from t to 1, the subgraph of G induced by the layers L_j to L_t is in W_{SR}. For $j = t$, the subgraph induced by the layers L_j to L_t is H_t, which is complete k_n-partite and is hence in W_{SR}. Suppose that the graph G_{j+1} induced by the layers L_{j+1} to L_t is in W_{SR}, for some $1 \leq j < t$. Consider the subgraph G_j, with vertex set V_j, induced by the layers L_j to L_t. From property c), G_j is well covered. From property b), we can find an independent set I consisting of one part from each of the layers L_{j+1} to L_t such that the set has no neighbours in L_j. From property a), there are at least two parts in L_j. Consider two such parts P_{j1} and P_{j2}. Now, $I \cup P_{j1}$ and $I \cup P_{j2}$ are a pair of maximal independent sets of G_j, since each lgraph is complete k_n-partite and these two sets have one part from each layer in G_j. The intersection of these two sets is $R = I$, which is maximal as $< V_j - N[R] >$ is the lgraph H_j which is complete k_n-partite. Now, the subgraph $< N[R] >$ is the graph $< \cup_{i=j+1}^{t} L_i >$, which is in W_{SR} by induction. Therefore, G_j, and hence G, is in W_{SR}.

□

We now give an example of a graph that is well covered, but does not belong to W_{SR}, a C_5. Any maximal independent set for a C_5 will have two vertices in it. Consider a pair of maximal independent sets I_1 and I_2 of the graph. Assume that they do not have a vertex in common, i.e, R, the intersection of I_1 and I_2 is an empty set. Then, $< V - N[R] >$ is the graph itself. Since a C_5 is not complete k_n-partite, from theorem 4.2, R is not maximal. Therefore, R has to be non-empty for it to be maximal. Since any maximal independent set in a C_5 has two vertices, R will have exactly one vertex. Hence, $< N[R] >$ will be a P_3, since every vertex in a C_5 is of degree two. A P_3 is not well covered and hence we cannot find a maximal R such that $< N[R] >$ is well covered. That is, a C_5 does not belong to W_{SR}. See Figure 5.3. For example, let $I_1 = \{v_1, v_3\}$, and $I_2 = \{v_1, v_4\}$. Then, a

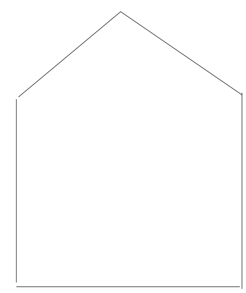

Figure 5.3: A graph not in W_{SR} - C_5

maximal $R = \{v_1\}$. $N[R] = \{v_5, v_1, v_2\}$; hence, $< N[R] >$ is a P_3 which is not well covered.

We give another example to show that not all subgraphs of a graph G in W_{SR} belong to W_{SR}. Let G be partitioned into layers $L_V = \{L_1, L_2, \ldots, L_t\}$ satisfying properties $a)$ to $c)$ of theorem 5.1. From lemma 5.1, any subset of the layers satisfies the above properties and hence the subgraph induced by any such subset is in W_{SR}. Therefore, any subgraph of G induced by whole layers of L_V is in W_{SR}. Hence, we are left with looking at subgraphs which include partial layers of L_V. Consider the graph G in Figure 5.4. It consists of two layers L_1 and L_2 which are

Figure 5.4: A subgraph not in W_{SR}

K_2's, joined by an edge (v_2, v_4) to form a P_4. It is easy to see that the layers obey properties $a)$ to $c)$ of theorem 5.1 and hence the graph is in W_{SR}. Now consider the subgraph induced by the layer L_2 and the vertex v_4 from the layer L_1. This is a P_3 and it is therefore not well covered. Hence, not every subgraph of a graph in W_{SR}, is in W_{SR}.

5.4.1 The uniqueness of the decomposition

We now show that all decompositions of a graph G belonging to W_{SR} yield the same layers, not necessarily in the same order. That is, the layers obtained are unique.

Theorem 5.2 (uniqueness) *Let G be a graph in W_{SR}. Then the following are true:*

- *a) Let $D_a(G) \in \mathcal{D}_{SR}(G)$ be a decomposition of G into layers L_1 to L_t, $1 \leq t \leq\mid V \mid$. Then, any other decomposition $D_b(G) \in \mathcal{D}_{SR}(G)$ will give the same t layers, not necessarily in the same order. Isolated vertices, if any, will always form the layer L_t.*
- *b) Let L_V be a partition of the vertex set V into layers L_1 to L_t, $1 \leq t \leq\mid V \mid$, satisfying properties a) to c) of theorem 5.1. Then any other partition of the vertex set V into layers satisfying properties a) to c) of theorem 5.1. will consist of the same t layers, not necessarily in the same order. Isolated vertices, if any, will always form the layer L_t.*

The above theorem not only says that all decompositions yield the same layers, but also that all partitions of V into layers that satisfy the properties of theorem 5.1 will consist of the same layers. In order to prove this theorem, we need the following results. We first prove the following lemma.

Lemma 5.1 *Let G be a graph in W_{SR}, and let its vertices be partitioned into layers L_1 to L_t satisfying properties a) to c) of theorem 5.1. Then any subset of the layers, with the ordering preserved, will satisfy the same properties.*

Proof:
This is obviously true, as any property that is violated in a subset of the layers is violated in the set of layers L_1 to L_t as well.

$\square$

The next result looks at the nature of a maximal intersection R of a graph $G \in W_{SR}$ which has the property that $< N[R] >$ is also in W_{SR}.

Proposition 5.2 *Let G be a graph in W_{SR} and let its vertex set be partitioned into layers L_1 to L_t satisfying theorem 5.1. Let R be a maximal intersection of a pair of maximal independent sets such that $< N[R] >$ is in W_{SR}. Then, the following are true:*

- *a) R consists of* whole *parts from some of the layers L_1 to L_t.*
- *b) $N[R]$ contains exactly those layers that have a part in R.*
- *c) $V - N[R]$ consists of one complete layer.*

Proof:
a)
From property c) theorem 5.1, we know that any maximal independent set of G has to have exactly one part from each layer. Hence, any intersection R of a pair

of maximal independent sets of G will consist of *whole* partitions from different layers.

b)

Assume not. As the lgraphs are complete k_n-partite, if a part from a layer is in R, then the whole layer is in $N[R]$. Therefore, there exists a layer L_l which has no parts in R, but which has at least one part adjacent to a part in R. Let P_l be the part in L_l which is adjacent to a part P_j in R. Let P_j belong to layer L_j. From a), we know that R consists of *whole* parts from some of the layers. Therefore, the size of R is equal to sum of the sizes of these parts. From theorem 5.1 c), we know that starting with a part from any layer, we can find exactly one part from each of the other layers such that the set so formed is independent. We form an independent set starting with the part P_l. From the above, we should be able to find one part from each of the layers that has a part in R such that the set is still an independent set. This set has size equal to the sum of the sizes of the layers with parts in R plus the size of P_l, that is, $>\mid R \mid$. Since R is a maximal independent set for $< N[R] >$, this means that $< N[R] >$ is not well covered, which contradicts the fact that $< N[R] >$ is in W_{SR}.

c)

Theorem 4.2 says that when the intersection R is maximal, $< V - N[R] >$ is complete k_n-partite. From b), we know that $N[R]$ consists of complete layers. Theorem 5.1 c) tells us that every two layers have to have at least two parts which are non-adjacent. Hence, we conclude that $V - N[R]$ consists of one complete layer.

□

We move on to our next result which shows that every decomposition yields layers which satisfy the properties of theorem 5.1, and that every partition of V into layers which obey the above theorem can be obtained from a decomposition of G.

Proposition 5.3 *Let G be a graph in W_{SR}. Then the following are true:*

a) Any decomposition $D_a(G) \in \mathcal{D}_{SR}(G)$ gives layers $L_1, L_2, \ldots, L_t$, $1 \leq t \leq \mid V \mid$, which form a partition of V and which satisfy properties a) to c) of theorem 5.1.

b) For any partition of V into layers $L_1, L_2, \ldots, L_t$, $1 \leq t \leq\mid V \mid$, which satisfy properties a) to c) of theorem 5.1, there is a decomposition $D_b(G) \in \mathcal{D}_{SR}(G)$ that yields these layers.

Proof:

G is a graph in W_{SR}.

a)

While proving the theorem 5.1, we chose an arbitrary decomposition $D_a(G) \in \mathcal{D}_{SR}(G)$, and showed that the resulting layers obeyed properties a) to c) of the theorem. This proves the statement.

b)

Consider some partition of V into layers $L_1, L_2, \ldots, L_t$ such that the layers obey properties $a)$ to $c)$ of theorem 5.1. We use induction to show that there is a decomposition of G which yields these layers. We assume that $t > 1$ as otherwise there is only one layer and, from property $a)$, we know that the corresponding lgraph is complete k_n-partite, and thus forms a trivial decomposition of G. Let $G_1 = G$. We know that G_1 is the graph induced by the layers L_1 to L_t. From property $b)$, there is a part in each of the layers L_2 to L_t such that the set R_1 formed by these parts is an independent set that has no neighbours in L_1. From property $a)$, L_1 has at least two parts. Consider two such parts P_{11} and P_{12}. Now $R_1 \cup P_{11}$ and $R_1 \cup P_{12}$ are two maximal independent sets in G_1, since each layer induces a complete k_n-partite graph, and these two sets have one part from each layer. Their intersection is R_1, which is maximal from theorem 4.2, as $< V - N[R_1] >$ is the lgraph $H_1 =< L_1 >$ which is complete k_n-partite. From lemma 5.1, the layers L_2 to L_t satisfy the properties of the theorem 5.1. Hence, $G_2 =< N[R_1] >$, which is the graph induced by the layers L_2 to L_t, is in W_{SR}. Thus, we have the first stage of a decomposition of G which yields a layer L_1 and a graph G_2 which is in W_{SR}.

Suppose that the layers L_1 to L_j are the layers obtained in the first j stages of such a decomposition, $1 \leq j < t$. From lemma 5.1, the graph G_{j+1} formed by the layers L_{j+1} to L_t is in W_{SR}. If $j + 1 = t$, we are done. Assume that $j + 1 < t$. From property $b)$, there is a part in each of the layers L_{j+2} to L_t such that the set R_{j+1} formed by these parts is an independent set that has no neighbours in L_{j+1}. From property $a)$, L_{j+1} has at least two parts. Consider two such parts $P_{j+1,1}$ and $P_{j+1,2}$. Now $R_{j+1} \cup P_{j+1,1}$ and $R_{j+1} \cup P_{j+1,2}$ are two maximal independent sets in G_{j+1}, since each layer induces a complete k_n-partite graph, and these two sets have one part from each layer. Their intersection is R_{j+1}, which is maximal from theorem 4.2, since $< V_{j+1} - N[R_{j+1}] >$ is the lgraph $H_{j+1} =< L_{j+1} >$ which is complete k_n-partite. From lemma 5.1, the layers L_{j+2} to L_t obey theorem 5.1. Hence, the graph $G_{j+2} =< N[R_{j+1}] >$ is in W_{SR}. Thus, we have the $j + 1$th stage of a decomposition of G which yields a complete k_n-partite layer L_{j+1} and a graph G_{j+2} which is in W_{SR}. This proves statement $b)$.

$\square$

We are now ready to prove the main theorem of this section.

Proof*(of the uniqueness theorem)*:
a)
$D_a(G)$ is a decomposition of G into layers $L_1, L_2, \ldots, L_t$. We note from proposition 5.3, that the layers obey theorem 5.1. Also, from lemma 5.1, any subset of the layers, with the ordering preserved, will obey properties $a)$ to $c)$ of theorem 5.1. We also note that since G is in W_{SR}, at each stage of a decomposition of G based on definition 5.1, we are guaranteed to find at least one maximal R such that $< N[R] >$ is in W_{SR}. Consider the first stage of some other decomposition $D_b(G)$ of G. Let R_1 be a maximal intersection of a pair of maximal independent sets of G, such that $< N[R_1] >$ is in W_{SR}. From proposition 5.2, $V_2 = N[R_1]$ consists of $t-1$ layers from the layers L_1 to L_t, and $V_{R1} = V - N[R_1]$ consists of the remaining

layer. The graph G_2 induced by V_2 is also in W_{SR}, by definition. Therefore, there exists a maximal intersection R_2 for G_2 such that $< N[R_2] >$ is in W_{SR}. Using proposition 5.2, this will create a $V_3 = N[R_2]$ that consists of $t-2$ layers from the $t-1$ layers in V_2; $V_{R2} = V_2 - N[R_2]$ will consist of the remaining layer. Thus, at stage i of the decomposition, $1 \leq i < t$, we will have a layer L_i which will be one complete layer from the $t-i+1$ layers of V_i, and a graph G_{i+1} whose vertex set V_{i+1} will consist of the remaining $t-i$ layers. At stage t, we will be left with one complete layer which will induce the graph G_t.

Since there are t layers, and each stage of a decomposition yields one layer, there will be t stages in any decomposition of G. Therefore, the layers obtained are independent of the choice of a maximal intersection at each stage of a decomposition; that is, any other decomposition $D_b(G)$ yields the same t layers. Since isolated vertices will be a part of any maximal independent set, they will always form the layer L_t.

That the layers obtained need not be in the same order is easily seen to be true if we consider a graph K consisting of $l > 1$ disjoint K_2's. The vertex pairs forming the edges form the layers of a partition of $V(K)$. These can easily be seen to obey theorem 5.1 and hence K is in W_{SR}. Any set R consisting of one vertex from each of $l-1$ K_2's will be independent as the K_2's are disjoint. It is also maximal since the graph $< V(K) - N[R] >$ is a K_2 which is complete k_n-partite. Choosing different K_2's to form maximal R's will yield different orderings of the layers. Hence the ordering of the layers can be different, as long as the layers obey theorem 5.1.

b)

This follows from proposition 5.3 and *a*).

□

5.4.2 Minimal graphs

We first prove the following result, which is a corollary of theorem 5.1, and then introduce the concept of a minimal graph.

Corollary 5.1 *Let G be a graph in W_{SR} and let its vertex set be partitioned into layers L_1 to L_t satisfying properties a) to c) of theorem 5.1. Then, if parts from different layers are adjacent to each other, then they are completely connected, that is, they induce a complete bipartite subgraph.*

Proof:

Consider any two layers L_j and L_k in such a partition, $j \neq k$. Let P_j and P_k be parts in L_j and L_k respectively. Let $u \in P_k$ such that $\mid P_j - N(u) \mid > 0$, where $\mid P_j \cap N(u) \mid > 0$, that is, u is adjacent to at least one vertex of P_j but not to all the vertices in it. See Figure 5.5.

Construct an independent set I_1, where

$$I_1 = \{u\} \cup (P_j - N(u))$$

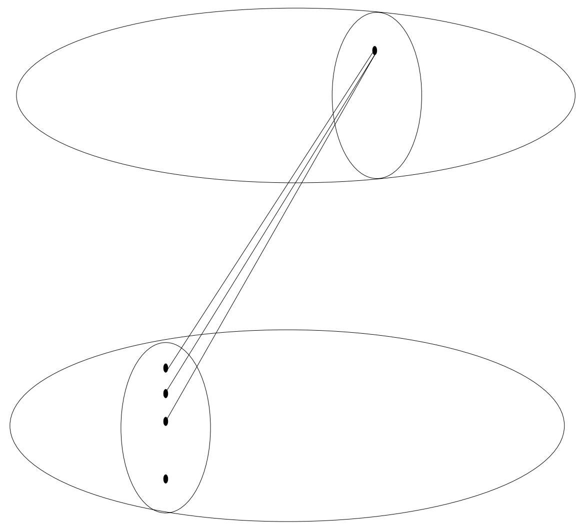

Figure 5.5: u adjacent to some, but not all, vertices of P_j

Extend this to a maximal independent set in G. Now, $P_j - N(u)$ will cover all the vertices of L_j. This will have fewer vertices from L_j than the size of a part in it, thus contradicting theorem 5.1 c). Hence, if a vertex u from a part in one layer is adjacent to some vertices in a part in another layer, then it should be adjacent to all the vertices in that part. Since this works both ways, this means that if parts in different layers are adjacent to one another, then they must be completely connected, that is, the subgraph induced must be complete bipartite.

$\square$

Let G be a graph belonging to W_{SR} and let it be decomposed into layers L_1 to L_t. The above property allows us to replace each part by a single vertex, and the set of edges between two adjacent parts by a single edge. This results in each lgraph in the decomposition being a clique. The resulting graph G_M satisfies theorem 5.1 and hence belongs to W_{SR}. We call such graphs *minimal* graphs. We have seen in the previous section that any partitioning of the vertices of a graph G in W_{SR} into layers that obey the theorem 5.1 leads to the same set of layers. Thus, all partitionings of the vertex set of G leads to the same minimal graph, in the sense that all such graphs are isomorphic to one another. Also, it is easy to see that there could be many graphs in W_{SR} which yield the same minimal graph. We will use minimal graphs to show that graphs belonging to the second sub-class have recognition in P.

As will be shown in the next chapter, the recognition problem for the family W_{SR} is co-NP-complete.

5.5 The second sub-class W_{AR}

While decomposing a graph G belonging to W_{SR}, at each stage, we have to find a maximal intersection R such that $< N[R] >$ is in W_{SR}. This is because we are only guaranteed that there *exists* such a maximal R, and not that any maximal R satisfies this property. We now relax this definition and state that any maximal intersection R at any stage of a decomposition of G yields a graph $< N[R] >$ which is in W_{SR}. This leads to the definition of the second sub-class W_{AR}.

Definition 5.2 *A graph G is said to belong to the family W_{AR} if*
a) G is complete k_n-partite, or
b) G is well covered and for every maximal R, the intersection of a pair of maximal independent sets of G, $< N[R] >$ belongs to W_{AR}.

Clearly, a graph G which belongs to W_{AR} also belongs to W_{SR}. Therefore, the vertices of G can be partitioned into layers $L_1, L_2, \ldots, L_t$ which obey theorem 5.1. We observe that proposition 5.2 holds for every maximal intersection R of a graph G in W_{AR}, since for every such R, $< N[R] >$ is in W_{AR} and hence in W_{SR}.

We now give an example of a graph belonging to W_{SR}, but not to W_{AR}. See Figure 5.6. This graph, call it G, has three layers $\{v_1, v_2, v_3\}, \{v_4, v_5, v_6\}$ and

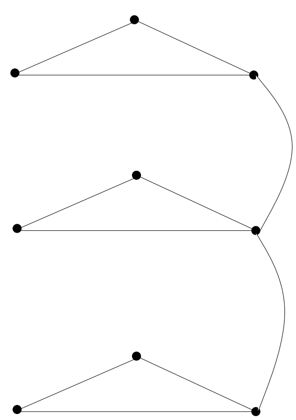

Figure 5.6: Graph in W_{SR} but not in W_{AR}

$\{v_7, v_8, v_9\}$, each of which induces a K_3. G is in W_{SR} as the layers clearly obey theorem 5.1 *a*) to *c*). Choosing maximal independent sets $I_1 = \{v_7, v_6, v_1\}$ and $I_2 = \{v_7, v_6, v_2\}$ gives an intersection $R = \{v_7, v_6\}$. The graph induced by $V - N[R]$ is a bipartite graph with vertices $\{v_1, v_2\}$, and hence, from theorem 4.2, R is a maximal intersection. Now $< N[R] >$ is not well covered as we easily can find two maximal independent sets of different sizes: for example, $\{v_8, v_5, v_3\}$ and $\{v_7, v_6\}$. Hence, not every maximal intersection of G yields a graph which is also well covered. Therefore, using the definition 5.2, G is not in W_{AR}.

We now state the following characterization for graphs belonging to the family W_{AR}.

Theorem 5.3 (W_{AR}) *A graph G belongs to the family W_{AR} if and only if its vertices can be partitioned into layers $L_1, L_2, \ldots, L_t$, $1 \leq t \leq\mid V \mid$, which have the following properties:*

a) The lgraphs $H_i =< L_i >$, $1 \leq i \leq t$, are complete k_n-partite, with every layer except the last one, L_t, having at least two parts. L_t need have just one part.

b) For any two adjacent layers L_j and L_k, there exist parts $P_j \in L_j$ and $P_k \in L_k$ such that $\mid N(P_j) \cap L_k \mid= 0$ and $\mid N(P_k) \cap L_j \mid= 0$, and the parts of $L_j - P_j$ and $L_k - P_k$ are completely connected to each other.

c) The non-common neighbours of any two parts in any layer of the decomposed graph are completely connected to each other.

Proof:

only if:

Since G is in W_{AR}, it is also in W_{SR}. From proposition 5.3, we can decompose G into layers $L_1, L_2, \ldots, L_t$ which satisfy properties a) to c) of theorem 5.1. Statement a) is true since it is the same as the statement a) of theorem 5.1. From corollary 5.1, we see that if a part in a layer is adjacent to a part in another layer, then they are completely connected.

b)

From theorem 5.1 a), we see that the only layer that can have only one part is L_t, in which case it will consist of isolated vertices, since otherwise property b) of the same theorem will be contradicted. Hence, if two layers are adjacent, each of them has at least two parts. Now, assume that statement b) is not true. Let L_m, $t \geq m > 1$, be the layer that contradicts statement b) with some other layer, where m is as large as possible. That is, there exists a layer L_l, $m > l \geq 1$, such that the layers L_m and L_l contradict statement b). We choose L_l to be as close to L_m as possible. Therefore, all the layers between L_m and L_l which are adjacent to L_m, satisfy statement b) with L_m.

The proof lies in showing that the layers L_m and L_l have to satisfy statement b). To do this, we first show that we can form an independent set I_r consisting of one part from each of the layers L_{l+1} to L_t, L_m not included, which has no neighbours in L_m. We then show that some parts in L_l with a certain property can have no neighbours in I_r. We then show that if a part in L_m is adjacent to a part in L_l, it has to be adjacent to all but one part in L_l, and vice versa. This property is then used to show that the layers L_m and L_l satisfy statement b).

We first form the independent set I_r. We observe, from corollary 5.1, that if a part in one layer is adjacent to a part in another layer, then they are completely connected, that is, the subgraph induced is complete bipartite. Consider the graph G_{l+1} induced by the layers L_{l+1} to L_t. Let its vertex set be denoted by V_{l+1}. Consider the layers in G_{l+1} which are adjacent to L_m. Let these layers be $L_{am1}, L_{am2}, \ldots, L_{amq}$, where $t \geq am1, am2, \ldots, amq \geq l+1$. By assumption, each of these layers satisfies statement b) with L_m. Hence, there exist parts

$P_{am1} \in L_{am1}, P_{am2} \in L_{am2}, \ldots, P_{amq} \in L_{amq}$ such that $I_{r1} = \cup_{i=1}^{q} P_{ami}$ has no neighbours in L_m. Let the layers in G_{l+1} which are not adjacent to L_m be given by $L_{rm1}, L_{rm2}, \ldots, L_{rmr}$, $t \geq rm1, rm2, \ldots, rmr \geq l+1$. From theorem 5.1 b), we can find $P_{rm1} \in L_{rm1}, P_{rm2} \in L_{rm2}, \ldots, P_{rmr} \in L_{rmr}$ such that the set I_{r2} formed by these parts is an independent set, and has no neighbours in L_l. Consider the set $I_r = I_{r1} \cup I_{r2}$. We know that this has no neighbours in L_m. We show that I_r is an independent set. Assume not. Then there exist parts P_a and P_b in I_r such that $P_a \sim P_b$. Now, at least one of P_a and P_b has to be from I_{r1} since we know that I_{r2} is an independent set. Let P_a be from I_{r1}. Therefore, there exists a layer L_a, $a \neq m$, $t \geq a > l$, such that $P_a \in L_a$ and L_a is adjacent to L_m. Since L_m and L_a satisfy statement b), there exists P_m in L_m which is adjacent to all but P_a in L_a. Now, P_b has no neighbours in L_m, and P_b is adjacent to P_a. Extending $P_m \cup P_b$ to a maximal independent set for G yields one which has no part from L_a, thus contradicting theorem 5.1 c). Thus, I_r must be an independent set and it has no neighbours in L_m.

We now show that any part in L_l which has a certain property has no neighbours in I_r. Consider the layers L_l and L_m. We prove the following claim.

Claim 5.1 *Let P_{l1} in L_l be non-adjacent to at least two parts P_{m1} and P_{m2} in L_m. Then P_{l1} has no neighbours in I_r.*

Proof:
Assume not. Then there exists $P_a \in I_r$ such that $P_{l1} \sim P_a$. P_a has to be from I_{r1} since I_{r2} has no neighbours in L_l. Therefore, there exists a layer L_a, $a \neq m$, $t \geq a > l$, such that $P_a \in L_a$ and L_a is adjacent to L_m. Since L_m and L_a satisfy statement b), at least one of P_{m1} and P_{m2}, say P_{m1}, is adjacent to all but P_a in L_a. Extending $P_{l1} \cup P_{m1}$ to a maximal independent set for G yields one which has no part from L_a, thus contradicting theorem 5.1 c). This proves the claim.

Next, we show that if a part in L_m is adjacent to a part in L_l, it must be adjacent to all but one part in L_l, and vice versa. We know that a part in L_l can be adjacent to at most all but one part in L_m, and vice versa, since otherwise theorem 5.1 c) will be contradicted. Now, assume that a part in one of these layers is adjacent to at most all but two parts in the other. Consider one such part $P_{j1} \in L_j$, $j \in \{l, m\}$. Let P_{j1} be adjacent to P_{k3}, but not adjacent to P_{k1} or P_{k2}, in the layer L_k, $k \in \{l, m\}, k \neq j$. This leads to two cases.

case a) $j = l, k = m$

We have $P_{j1} \in L_l$ not adjacent to neither of P_{k1} nor P_{k2} in the layer L_m. From claim 5.1, we see that P_{j1} has no neighbours in I_r.

case b) $j = m, k = l$

We have $P_{j1} \in L_m$ which is adjacent to P_{k3}, but not to P_{k1} or P_{k2} in L_l. From theorem 5.1 b), we know that there is a part in L_m which is not adjacent to any part in L_l. This part cannot be P_{j1} as this is adjacent to P_{k3} in L_l. Let this part be P_{j2}. That is, there are two parts P_{j1} and P_{j2} in L_m which are non-adjacent to both P_{k1} and P_{k2}. From claim 5.1, we see that neither P_{k1} nor P_{k2} can have neighbours in I_r. Since I_r has no neighbours in L_m, this means that in both of the above cases, none of P_{j1}, P_{k1} and P_{k2} have neighbours in I_r.

Let $R = I_r \cup P_{j1}$. Let $I_1 = R \cup P_{k1}$ and $I_2 = R \cup P_{k2}$. Consider the graph $G_l =< V_{l+1} \cup L_l >$ with vertex set V_l. I_1 and I_2 are two maximal independent sets of G_l and their intersection is R. $V_l - N[R]$ consists of some, but not all, of the parts of L_k, and hence $< V_l - N[R] >$ is complete k_n-partite. From theorem 4.2, R is maximal. We now argue that G_l is in W_{AR}. At each stage of a decomposition of G yielding layers L_1 to L_t, we obtain a maximal R such that $< N[R] >$ is also in W_{SR}. Since G is in W_{AR}, any maximal R yields a $< N[R] >$ that is in W_{AR}. Therefore, the subgraphs $G_i =< L_i \cup L_{i+1} \cup \cdots \cup L_t >$ obtained at each stage of such a decomposition, $1 \leq i \leq t$, are also in W_{AR}. Hence, G_l is in W_{AR}. Now, $N[R]$ has at least one part P_{k3} from L_k. This contradicts proposition 5.2 *b*). Hence, P_{j1} can have at most one non-neighbour in L_k. We have already seen that P_{j1} has to have at least one non-neighbour in L_k. We conclude that P_{j1} must have exactly one non-neighbour in L_k. Thus, if a part in L_l is adjacent to a part in L_m, it is adjacent to all but one part in L_m, and vice versa.

We now show that the layers L_m and L_l satisfy statement *b*). From theorem 5.1 *b*), there has to be a part P_m in L_m which is not adjacent to any part in L_l. Therefore, any part in L_l that is adjacent to some part in L_m is adjacent to all but P_m in L_m. If every part in L_l had neighbours in L_m, then every part in L_m other than P_m would be adjacent to all of L_l which, as we have already seen, cannot be the case. Hence at most $k_l - 1$ parts of L_l can have neighbours in L_m. Since any part in L_m that has neighbours in L_l has to be adjacent to all but one part in L_l, all but P_m in L_m is adjacent to all but some part P_l in L_l. That is, L_l and L_m satisfy statement *b*), which contradicts our assumption.

This proves the statement.

c)

Suppose that the statement is false. Then there exist parts P_{j1} and P_{j2} in some layer L_j such that they have at least one pair of non-common neighbours $P_l \in L_l$ and $P_m \in L_m$ which are not adjacent to each other, $j \neq l$, $l \neq m$, $m \neq j$, with P_l adjacent to P_{j1} but not to P_{j2}, and P_m adjacent to P_{j2} but not to P_{j1}. Since, from *b*), P_l and P_m are adjacent to $(k_j - 1)$ parts each in L_j, and $P_{j1} \in N(P_l)$ but $\notin N(P_m)$, $N(P_l) \cup N(P_m) \supset L_j$. Now, P_l and P_m can be extended to a maximal independent set which has no part from L_j, thus contradicting theorem 5.1 *c*).

if:

Let $L_V = \{L_1, L_2, \ldots, L_t\}$. Consider some part P_j of layer j, $1 \leq j \leq t$. As the layers are complete k_n-partite, and because of property *b*), the neighbour sets of every vertex in P_j are the same. Hence, if a maximal independent set contains a vertex from P_j, it will contain all the vertices from P_j. Consider the graph G_j with vertex set V_j induced by some k layers of L_V, $1 \leq k \leq t$. From the above, any maximal independent set of G, and hence G_j, contains *whole* parts from the layers in G_j. To prove that G is in W_{AR}, we need the following two results.

We first show that G_j is well covered by proving the following claim.

Claim 5.2 *Every maximal independent set in G_j consists of exactly one part from each layer.*

Proof:
We have already shown that any maximal independent set of G_j has to consist of *whole* parts from the layers. Suppose that the claim is not true. Then there exists at least one layer $L_k \in G_j$ which has no part in some maximal independent set I_1 for G_j. Therefore there exist parts in I_1 which cover the layer L_k. Consider one such part P_{l1} from layer L_l. This means that L_l is adjacent to L_k. Properties *a*), *b*), and *c*), will hold for any subset of the layers L_1 to L_t as otherwise they would not hold for L_V as well. From *b*), we know that there exist parts $P_k \in L_k$ and $P_l \in L_l$ such that $\mid N(P_k) \cap L_l \mid = 0$ and $\mid N(P_l) \cap L_k \mid = 0$, and that the parts of $L_k - P_k$ and $L_l - P_l$ are pairwise complete bipartite. Therefore, P_{l1} has to be a part other than P_l, and it covers all of L_k except for P_k. Therefore, there has to be another part P_L in I_1 which covers P_k. Let this be from the layer L_L. Since L_L is adjacent to L_k, from *b*), there must be a part $P_{k1} \neq P_k \in L_k$ that is not adjacent to any of L_L. Since P_k is adjacent to P_L but not to P_{l1}, and P_{k1} is adjacent to P_{l1} but not to P_L, from *c*), P_{l1} and P_L have to be adjacent. This cannot be true as P_{l1} and P_L belong to I_1 which is an independent set. This proves the claim.

We next prove the following claim about G_j.

Claim 5.3 *Let R be a maximal intersection of a pair of maximal independent sets of G_j. Then the following are true:*
a) $N[R]$ consists of exactly those layers that have a part in R.
b) $V_j - N[R]$ consists of exactly one layer.

Proof:
a)
Any subset of the layers in the partition will also obey properties *a*), *b*), and *c*), as otherwise these properties would be contradicted in the partition also. Suppose that the claim is not true. Then there exists P_i in R such that $N(P_i)$ contains at least one part P_j which belongs to a layer L_j that has no part in R. Let P_i belong to the layer L_i. Since L_i and L_j are adjacent, *b*) applies. Therefore, P_i is adjacent to all but P_{j1} in L_j. Now, P_{j1} cannot be adjacent to anything in R as then property *c*) is contradicted. Therefore, any maximal independent set containing R must contain P_{j1}, which means that the intersection of a pair of such maximal independent sets must contain P_{j1}. Hence, R must contain P_{j1}, which is a contradiction since we assumed that R was maximal. Therefore, $N[R]$ must contain only those layers that have a part in R.
b)
Property *a*) says that every layer induces a complete k_n-partite subgraph. From *a*) of this claim, $V_j - N[R]$ will consist of whole layers. From property *b*), every two such layers will have at least one part in each that has no neighbours in the other. Using theorem 4.2, we see that $< V_j - N[R] >$ has to be complete k_n-partite, for R to be maximal. Statement *b*) follows from the above. This proves the claim.

We now prove that G is in W_{AR} by induction on the layers. All graphs induced by any one layer in the partition are in W_{AR}, since, from a), every layer induces a complete k_n-partite graph. Suppose that all graphs induced by j layers, $1 \leq j < t$, are in W_{AR}. Now, look at graphs induced by $j+1$ layers. From the claim 5.2, all such graphs are well covered. Consider any such graph G_{j+1}, with vertex set V_{j+1}. From the claim 5.3, every maximal intersection R for G_{j+1} has the property that $V_{j+1} - N[R]$ consists of exactly one layer, that is, $N[R]$ consists of j layers. Thus, for graphs induced by $j+1$ layers, every maximal intersection R will result in $< N[R] >$ being a graph induced by some j layers of the partition. By the induction hypothesis, all such graphs are in W_{AR}. Hence, all graphs induced by $j+1$ layers are in W_{AR}. Therefore, G is in W_{AR}.

$\square$

Clearly, all decompositions of a graph G in W_{AR} yield the same layers, and these layers obey the properties of theorem 5.3. That is, the layers are unique.

Property c) of theorem 5.3 states that the non-common neighbours of any two parts in a layer are completely connected to each other. Consider two parts P_{i1} and P_{i2} in a layer L_i. Let $P_j \notin L_i$ be adjacent to both P_{i1} and P_{i2}, and $P_k \notin L_i$, $j \neq k$, be adjacent to P_{i1} but not P_{i2}. Property c) does not say anything about P_j and P_k having to be adjacent to each other, since P_j is a common neighbour. Let P_j be not adjacent to P_k. Using property b) of theorem 5.3, we observe that the set $I = P_j \cup P_k$ will cover the layer L_i. Extending I to a maximal independent set will yield one which has no part from L_i, thus contradicting theorem 5.1 c). This would seem to indicate that property c) of theorem 5.3 is not a strong enough condition. The following corollary shows that properties b) and c) of theorem 5.3 ensure that P_j has to be adjacent to P_k. This result is used in showing that the dominating set problem is in P for the class W_{AR}.

Corollary 5.2 *Let G be a graph in W_{AR}. Let its vertex set be partitioned into layers $L_1, L_2, \ldots, L_t$, $1 \leq t \leq n$, obeying properties a) to c) of theorem 5.3. Let P_{i1} and P_{i2} be any two parts of a layer L_i, $1 \leq i \leq t$. Let $P_j, P_k \notin L_i$, $j \neq k$, be neighbours, at least one non-common, of P_{i1} and P_{i2}. Then P_j is adjacent to P_k.*

Proof:
If P_j and P_k are both non-common neighbours of P_{i1} and P_{i2}, then property c) of theorem 5.3 ensures that P_j is adjacent to P_k. Let P_j be adjacent to both P_{i1} and P_{i2}, and let P_k be adjacent to only P_{j1}. Since P_{i1} and P_{i2} are both adjacent to P_j, from property b) of theorem 5.3, there is a part $P_{i3} \in L_i$ which is not adjacent to P_j. Using the same property b), since P_{i2} is not adjacent to P_k, P_{i3} has to be adjacent to P_k. Now, we have two parts P_{i2} and P_{i3} in a layer L_i which have two non-common neighbours P_j and P_k. From property c) of theorem 5.3, P_j has to be adjacent to P_k.

$\square$

The following proposition states some properties of the layers. It is made use of in proving the generalization of Favaron's theorem.

Proposition 5.4 *Let G be a graph in W_{AR} and let its vertices be partitioned into layers L_1 to L_t satisfying the properties a) to c) of theorem 5.3. Then the following are true:*

- *a) For any layer L_j in G, there exists a part in each of the other layers such that they have no neighbours in L_j and form an independent set, $1 \leq j \leq t$.*
- *b) Every layer is $V - N[R]$ for some R, except for the last layer L_t if it has only one part.*
- *c) Any subset of the layers in any order satisfies the properties a) to c) of theorem 5.3, except if the layer L_t has only one part, in which case, it has to be the last layer in any ordering.*

Proof:
a)
Consider a layer L_j, $1 \leq j \leq t$. Let $L_{j1}, L_{j2}, \ldots, L_{jq}$, $1 \leq q \leq t$, be the layers that are adjacent to L_j. From theorem 5.3 b), we know that there exist parts $P_{j1} \in L_{j1}, P_{j2} \in L_{j2}, \ldots, P_{jq} \in L_{jq}$ which have no neighbours in L_j. We now prove that the set I formed by these parts is an independent set. Assume not. Then there exist parts $P_{jl} \in L_{jl}$ and $P_{jm} \in L_{jm}$ such that P_{jl} is adjacent to P_{jm}, $1 \leq l, t \leq q$. Since L_j is adjacent to L_{jm}, there exists P_j in L_j which is adjacent to all but P_{jm} in L_{jm}. Start an independent set I_1 with P_j and P_{jl} and extend it to a maximal independent set for G. I_1 will not have any part from L_{jm}, thus contradicting theorem 5.1. Extend I to include a part from each of the layers that are non-adjacent to L_j; this is possible since G is in W_{AR}. Thus, I consists of one part from each of the layers other than L_j, and has no neighbours in L_j.
b)
From statement a), we can form an independent set I which consists of a part from each layer of $V - L_j$ such that I has no neighbours in L_j. Since L_j has at least two parts, except if it is the layer L_t and has only one part, we can always find parts $P_{j,1}$ and $P_{j,2}$ in L_j to form maximal independent sets $I_1 = I \cup P_{j,1}$ and $I_2 = I \cup P_{j,2}$. The intersection of these two maximal independent sets is $R = I$, and from proposition 5.2, we know that $N[R]$ consists of all the layers except L_j. Since L_j is complete k_n-partite, using theorem 4.2, we see that R is maximal, and $L_j = V - N[R]$. As G is in W_{AR} and R is a maximal intersection, from the definition 5.2, $N[R]$ is in W_{AR}.
c)
Consider some subset of the layers $L = \{L_{j1}, L_{j2}, \ldots, L_{jq}\}$, $1 \leq j1, j2, \ldots, jq \leq t$, with L_{jq} being the layer L_t if the layer L_t has only one part and is part of the subset. These layers have to obey properties a) through c) of theorem 5.3 as otherwise they would contradict these properties in the partition too. These properties are independent of the ordering of the layers, except for isolated vertices which should form the last layer L_t.

□

As will be seen in the next chapter, the recognition problem for this sub-class is in P.

5.6 The third sub-class W_{ARF}

We now define a sub-class of W_{AR} in which every layer in the decomposed graph has exactly k parts, $k \geq 2$. This rules out graphs with isolated vertices, since isolated vertices form a single layer with one part in the decomposed graph. This sub-class, which we call W_{ARF}, is defined as follows:

Definition 5.3 *A graph G is said to belong to the family W_{ARF} if G belongs to W_{AR} and for some $k \geq 2$, every decomposition of G has exactly k parts in each layer.*

We define a sub-class W_{ARk} of W_{ARF} as follows:

Definition 5.4 *For any $k \geq 2$, a graph G belongs to W_{ARk} if G belongs to W_{ARF} and has exactly k parts in each layer of any decomposition.*

Thus, $W_{ARF} = \cup_{k=2}^{\infty} W_{ARk}$. Since any maximal independent set has exactly one part from each layer in the decomposed graph, the size of any maximal independent set of a graph belonging to W_{ARk} is n/k.

5.7 The fourth sub-class W_{AR2}

From the previous section, we see that W_{AR2} consists of all the graphs in the family W_{ARF} which have exactly two parts in each layer. We now provide the following characterization for this family.

Theorem 5.4 (W_{AR2}) *A graph G belongs to the family W_{AR2} if and only if it is very well covered without isolated vertices.*

Proof:
only if:
G belongs to W_{AR2}. Therefore, it can be decomposed into t layers, with each layer having exactly two parts. As G belongs to W_{AR}, by definition, it obeys theorem 5.3. Since the layers induce subgraphs which are complete 2_n-partite, there are no isolated vertices. As every maximal independent set for G contains exactly one part from each layer, every such set will contain exactly half the number of vertices in the graph. Hence, G is very well covered without isolated vertices.
if:
G is very well covered without isolated vertices. We now state a property of any perfect matching of a very well covered graph without isolated vertices.

Observation 5.1 *Let G be a very well covered graph, and let M be a perfect matching in G. Let $M_1 \subset M$. Then, M_1 satisfies P.*

This can be easily verified.

We partition the vertex set of G into layers $L_1, L_2, \ldots, L_t$, where t is a positive integer, as follows. From theorem 4.4, for any pair of maximal independent sets I_1 and I_2 of G, whenever the intersection R is maximal, there exists a perfect matching between R and S, and I_1' and I_2', which satisfies property P. From theorem 4.2, when R is maximal, the graph induced by $V - N[R]$ is complete k_n-partite. In this case, since $N(R) = S$, the lgraph H_1 induced by $L_1 = V - N[R]$ is complete 2_n- partite. Let G_2 be the graph induced by $N[R]$. From observation 5.1, the matching between R and S satisfies property P, that is, G_2 has a perfect matching which satisfies property P. Using Favaron's theorem 2.1, we see that G_2 is very well covered. Therefore, G_2 can be decomposed as outlined above to give a complete 2_n-partite lgraph $H_2 =< L_2 >$ and a very well covered graph G_3 such that $G_2 =< V_3 \cup L_2 >$. Since we start with a graph G which has a finite number of vertices, this decomposition will stop at some stage, say t. As we cannot decompose the graph any further, this means that we cannot find a pair of maximal independent sets in G_t which have a non-zero intersection. That is, G_t consists of isolated vertices, or is complete k_n-partite. Since G has no isolated vertices, the lgraph $H_t =< L_t >$ is complete 2_n-partite. We now have a decomposition of G into t lgraphs, each of which is complete 2_n-partite, with the corresponding layers being L_1 to L_t. We now show that these layers satisfy properties $a)$ to $c)$ of theorem 5.3. We first prove the following claim.

Claim 5.4 *Every maximal independent set of G contains exactly one part from each of the layers L_1 to L_t.*

Proof:
As the layers induce $K_{n,n}$'s, there can be at most one part from each layer in a maximal independent set. Since G is very well covered, and the parts in each layer have the same size, there has to be exactly one part from each layer in every maximal independent set. This proves the claim.

property a)
Since the lgraphs are complete 2_n-partite, $a)$ is satisfied.
property b)
Let the layers L_j and L_k be adjacent. Let L_j and L_k consist of the parts P_{j1}, P_{j2}, and P_{k1}, P_{k2} respectively. Let P_{j1} be adjacent to P_{k2}. Since the lgraphs are complete 2_n-partite, there exists a perfect matching M for G which consists of n_l edges from the lgraph H_l, $1 \leq l \leq t$. From Favaron's theorem, M obeys property P. Let u_1 be a vertex in P_{j1}. Let it be adjacent to v_1 in P_{k2}. Let (u_1, u_{11}) be in M, where u_{11} is in P_{j2}. Since the lgraphs are complete 2_n-partite, u_{11} is adjacent to all the vertices in P_{j1}. Now, v_1 cannot be adjacent to u_{11} as then we would have the matching edge u_1, u_{11} with a common neighbour, which would contradict property P. That is, (u_1, u_{11}) is a matching edge, with u_1 adjacent to v_1 but not to the rest of the vertices in P_{j1}, and u_{11} is adjacent to all the vertices in $P_{j1} - u_1$, but not to v_1. Therefore, from property P, all the vertices in P_{j1} are adjacent to v_1. That is, if a vertex in P_{j1} is adjacent to a vertex v in P_{k2}, then every vertex in P_{j1} is

adjacent to v. We can use a similar argument to show that this should be true for the vertices in P_{k2} also. Therefore, if parts from different layers are adjacent, then they are complete bipartite. Now, P_{j1} cannot be adjacent to P_{k1} because any maximal independent set of G that contains P_{j1} would have no part from L_k, thus contradicting claim 5.4. For similar reasons, P_{k2} cannot be adjacent to P_{j2}. Therefore, P_{j2} has no neighbours in L_k and P_{k1} has no neighbours in L_j. Thus, if two layers are adjacent then all but one part of one layer is completely connected to all but one part of the other layer, thus satisfying property b).
property c)
While proving b), we have shown that there exists a perfect matching M that contains exactly n_j edges from the lgraph H_j, $1 \leq j \leq t$. Property c) follows from b) and the fact that the perfect matching M satisfies Favaron's theorem.

Therefore, G is in W_{AR}. Since every layer in the decomposition has exactly two parts, G belongs to W_{AR2}.

□

5.8 Conclusions

We have defined and characterized a hierarchy of four sub-families of well-covered graphs. We showed that these sub-classes are recursively decomposable. We showed that for a graph G in W_{SR}, all decompositions yields the same set of layers; that is, the layers are unique. We have also shown that the last sub-family, W_{AR2}, is the same as the family of very well covered graphs without isolated vertices. In the next chapter, we study the algorithmic properties of these new sub-classes.

Chapter 6

Complexity results for the new sub-classes

6.1 Introduction

We have obtained a hierarchy of four new sub-classes of well-covered graphs. We now study the complexity of the same problems that we looked at for well-covered graphs, for these new sub-classes. Since W_{AR2} is the same as the family of very well covered graphs without isolated vertices, we only need to look at those problems that have complexities in P for W_{AR2} and check if these remain the same for the other sub-classes. These problems are recognition, clique partition, Hamiltonian cycle and path, and dominating set. We see that while clique partition remains in P, the rest become intractable as one goes higher up in the hierarchy. These problems separate the classes algorithmically, except for the classes W_{AR} and W_{ARF} which have the same complexities for all the problems. The results are shown in Table 6.1.

Problem	W_{SR}	W_{AR}	W_{ARF}	W_{AR2}
Recognition	co-NP-c	P	$\Longrightarrow$	$\Longrightarrow$
Clique partition	P	$\Longrightarrow$	$\Longrightarrow$	$\Longrightarrow$
Dominating set	NP-c	P	$\Longrightarrow$	$\Longrightarrow$
Hamiltonian cycle	$\Longleftarrow$	$\Longleftarrow$	NP-c	P
Hamiltonian path	$\Longleftarrow$	$\Longleftarrow$	NP-c	P

Table 6.1: Complexity results for the new sub-classes

$\Longleftarrow$ Result implied from result on right.
$\Longrightarrow$ Result implied from result on left.

6.2 The class W_{SR}

6.2.1 Recognition

We first look at the problem of recognition, that is, given a graph G, how difficult is it to determine whether or not G is a member of W_{SR} ? The answer lies in the following theorem.

Theorem 6.1 *The recognition problem is co-NP-complete for the class W_{SR}.*

Proof:
We use the same reduction from SAT as the one used for well-covered graph recognition, with the following addition: Add a new clause c_{m+1} which has a new literal u_{n+1}. Add $(u_{n+1}, \overline{u}_{n+1})$ to the set of literals. This in no way changes the complexity of the problem. See Figure 6.1. Rearrange the above graph in the form of layers

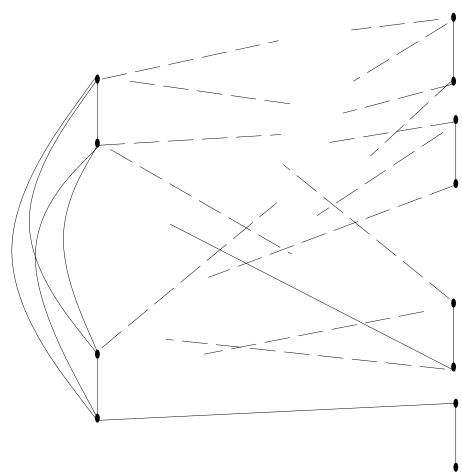

Figure 6.1: W_{SR} recognition - SAT reduction

$L_1, L_2, \ldots, L_t$ as shown in Figure 6.2. Layers L_1 to L_{t-1} induce K_2's and are not connected to each other; hence, they obey statements a) through c) of theorem 5.1. $< L_t >$ is a clique and hence obeys a). L_1 to L_{t-1} also obey b) with respect to L_t as L_1 is not adjacent to any of c_1 to c_m in L_t, and L_2 to L_{t-1} are not adjacent to c_{m+1} in L_t. Therefore, the only property that the above graph can violate is c). L_t cannot cover any other layer as no clause has both a literal and its negation. Therefore, the only possibility is an independent set from L_1 to L_{t-1} covering L_t. As c_{m+1} is covered only by u_{n+1}, this means that a set of independent vertices from L_2 to L_{t-1} covers vertices c_1 to c_m in L_t. Since this is an independent set, it can have at most one vertex from each of the layers L_2 to L_{t-1}.

Claim 6.1 *C is satisfiable if and only if G does not belong to W_{SR}.*

Proof:
only if:

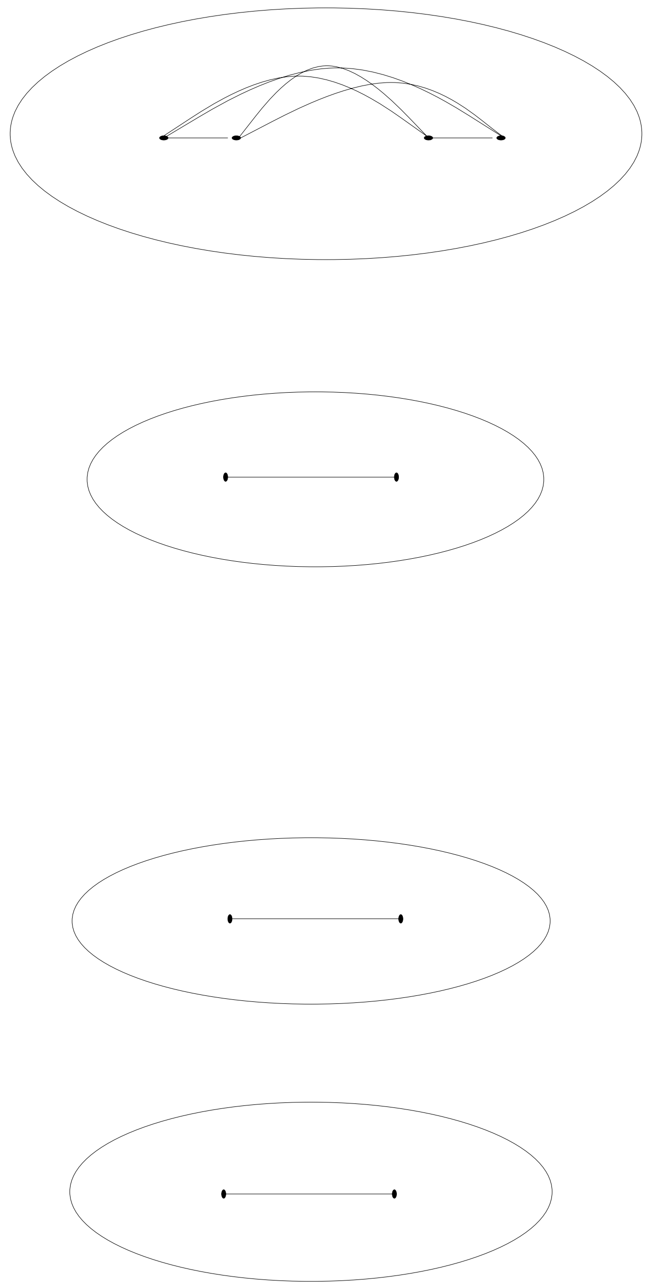

Figure 6.2: W_{SR} recognition - grouping into layers

C is satisfiable. Therefore, there exists a satisfying truth assignation for C, that is, there exists a set of vertices corresponding to *true* literals from layers L_1 to L_{t-1} with there being at most one vertex from each layer. Since the layers are not adjacent to each other, this set is an independent set. This set will include u_{n+1} since this is the only literal in the clause c_{m+1}. Therefore we have an independent set comprised of vertices from the layers L_1 to L_{t-1} which cover all the vertices in L_t. This contradicts c) and hence, G is not in W_{SR}.
if:
G is not in W_{SR}. As we have seen before, only statement c) can be violated. Therefore there exists an independent set of vertices from layers L_1 to L_{t-1} which covers L_t. Since the set is independent, it can have at most one vertex from each of the layers L_1 to L_{t-1}. That is, only a vertex corresponding to a literal, or its negation, will be present in the set. Assigning the value *true* to the literals in the set, we obtain a satisfying truth assignment for C.

□

From the above, it is clear that this group of graphs is as hard to recognize as the family of well-covered graphs. We now look at the complexity of some other problems for this family.

6.2.2 Clique Partition

Problem: *Given a graph G and an integer k, is there a set of k vertex disjoint cliques such that every vertex of G is contained in one of the cliques?*

This problem is not difficult to solve for the class W_{SR}. For any graph G, the minimum number of cliques needed to cover the graph is greater than or equal to the size of a maximum independent set of G. From theorem 5.1 c), we know that the size of a maximal independent set for a graph $G \in W_{SR}$ is equal to the sum of the sizes of the parts of the layers L_1 to L_t. Since each layer is complete k_n-partite, a clique cover of this size exists. Hence the size of a minimum clique cover for G is equal to the size of a maximum independent set in it.

6.2.3 Dominating Set

Problem: *Given a graph G and an integer k, is there a set of k vertices of G such that every vertex not in the set is adjacent to at least one vertex in it?*

Theorem 6.2 *The dominating set problem is NP-complete for the class W_{SR}.*

Proof:
We reduce from the dominating set problem for general graphs. Given a graph G of order n ($n > 2$), we transform it into a graph G_D as follows. For each vertex v_i in G, we have a K_2 whose vertices form the layer L_i in the transformed graph G_D, $1 \leq i \leq n$. There is a vertex $v_{i,1}$ in the K_2 which corresponds to the vertex v_i. Therefore, there are n layers in G_D, each of which induces a K_2. These layers are

numbered L_1 to L_n. There is another layer L_{n+1} which induces a clique of $n+2$ vertices, with a vertex $v_{n+1,i}$ in it for each vertex v_i in G, plus two other vertices. These layers are arranged as shown in Figure 6.3 to form the graph G_D. For each

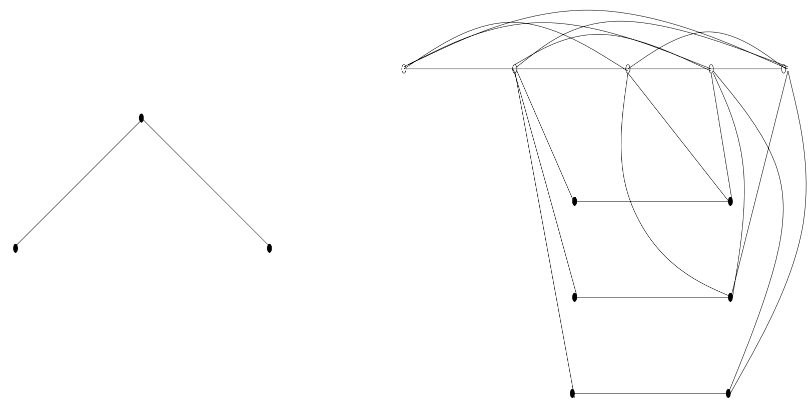

Figure 6.3: W_{SR} - dominating set

edge (v_i, v_j) in G, there is an edge in G_D from the vertex $v_{n+1,i}$ in the layer L_{n+1} to the vertex $v_{j,1}$ in the layer L_j, and from the vertex $v_{n+1,j}$ in the layer L_{n+1} to the vertex $v_{i,1}$ in the layer L_i. For each vertex v_i in G, there is an edge from the vertex $v_{n+1,i}$ to the vertex $v_{i,1}$ in G_D. There is also an edge from the vertex $v_{i,2}$ of each layer, to the vertex $v_{n+1,n+1}$ of the layer L_{n+1}. The vertex $v_{n+1,n+2}$ is a simplicial vertex in the layer L_{n+1}. G_D has $3n+2$ vertices and $2m+3n+(n+2)(n+1)/2$ edges, and can be constructed in polynomial time.

The layers L_1 to L_{n+1} obey statement a) of theorem 5.1 as each one induces a clique containing at least two vertices. Statement b) is obeyed as the layers are non-adjacent to each other except for L_{n+1} which has a simplicial vertex $v_{n+1,n+2}$. Every maximal independent set in G_D will have at most one vertex from each of the layers since the induced subgraphs are all cliques. The layer L_{n+1} is the only layer adjacent to any of the other layers. Since it induces a clique, we can have only one vertex from this layer in an independent set. Since no vertex in this layer is connected to both the vertices in some other layer, there is no possibility that we can exclude all the vertices of any other layer by choosing a vertex from this layer. Also, any independent set will have a vertex from the layer L_{n+1} as $v_{n+1,n+2}$ is a simplicial vertex in this layer. From the above arguments, every maximal independent set has to have exactly one vertex from each of the layers L_1 to L_{n+1}, that is, G_D obeys statement c). Therefore, the graph G_D is in W_{SR}.

Claim 6.2 *G has a dominating set of size $\leq k$ if and only if G_D has a dominating set of size $\leq k+1$, $1 \leq k \leq n$, $n > 2$.*

only if:
G has a dominating set of size $k_1 \leq k$. Let D_G be such a dominating set in G. Choose the corresponding vertices in layer L_{n+1} to form a set D_{GD} in G_D. Since D_G is a dominating set in G, the vertices of D_{GD} will dominate the vertices $v_{j,1}$ of G_D, $1 \leq j \leq n$. Any one vertex in D_{GD} is sufficient to dominate all the vertices in the layer L_{n+1}, since $< L_{n+1} >$ is a clique. Add the vertex $v_{n+1,n+1}$ to the set D_{GD}. This will dominate all the vertices $v_{j,2}$. Therefore, D_{GD} is a dominating set for G_D and has size $k_1 + 1 \leq k + 1$.
if:
G_D has a dominating set of size $k_1 \leq k+1$, $k \leq n$. Let D_{GD} be such a dominating set. Assume that D_{GD} does not contain the vertex $v_{n+1,n+1}$. Consider the vertices $v_{j,2}$, $1 \leq j \leq n$. Each such $v_{j,2}$ is adjacent to the corresponding $v_{j,1}$ and $v_{n+1,n+1}$, and nothing else. Hence, if D_{GD} does not contain $v_{n+1,n+1}$, then it has to contain at least one vertex from each of the K_2's forming the layers L_1 to L_n. Since $v_{n+1,n+2}$ is a simplicial vertex in the layer L_{n+1}, there has to be at least one vertex from this layer in D_{GD}. That is, there has to be at least one vertex from each of the layers L_1 to L_{n+1} in D_{GD}. Since D_{GD} can have at most $n+1$ vertices, this means that there has to be exactly one vertex from each of the layers in G_D in D_{GD}, and each such vertex need only dominate the vertices in the layer that it belongs to. If the vertex from the layer L_{n+1} is not $v_{n+1,n+1}$, replace it with $v_{n+1,n+1}$. Therefore, if D_{GD} does not contain the vertex $v_{n+1,n+1}$, we can always find a vertex v in D_{GD} that can be replaced with the vertex $v_{n+1,n+1}$ such that the new D_{GD} is still a dominating set.

We therefore assume that D_{GD} contains the vertex $v_{n+1,n+1}$. This will dominate all the $v_{j,2}$'s, $1 \leq j \leq n$. Replace each vertex $v_{j,1}$, $1 \leq j \leq n$, in D_{GD} with the corresponding vertex $v_{n+1,j}$ in L_{n+1}. Since there is an edge between $v_{n+1,j}$ and $v_{j,1}$, and the neighbour set of $v_{j,1}$ is contained in $L_{n+1} \cup \{v_{j,2}\}$, this change does not make any difference in the vertices that are dominated by $v_{j,1}$, nor does it change the number of vertices in D_{GD}. Therefore, we now have a new dominating set D_{GD1} of size k_1. The vertices $D_{GD1} \cap L_{n+1} - \{v_{n+1,n+1}, v_{n+1,n+2}\}$, dominate the vertices $\{v_{1,1}, v_{2,1}, \ldots, v_{n,1}\}$ of G_D. Choosing the corresponding vertices in G will yield a dominating set for G of size $\leq k_1 - 1$, that is, $\leq k$.

□

6.3 The class W_{AR}

We now study the complexities of the recognition and dominating set problems for the family W_{AR}.

6.3.1 A Polynomial Recognition Algorithm

The algorithm decomposes a graph G based on the definition 5.2 to obtain the layers L_1 to L_t, and checks if the layers satisfy properties $a)$ to $c)$ of theorem 5.3.

We assume that G is represented by an adjacency list.

Stage A: Decomposition

We decompose the graph G to obtain the layers L_1 to L_t. Since the algorithm also checks if the subgraphs induced are complete k_n-partite, at the end of this stage, property $a)$ is also verified. Let $t = 1$.

1) *while* G is not complete k-partite
2) find two maximal independent sets I_1 and I_2 of G, $I_1 \neq I_2$;
3) find the intersection R of I_1 and I_2;
4) *while* R is not a maximal intersection
5) extend R;
6) *if* $< V - N[R] >$ is complete k_n-partite
7) $L_t = V - N[R]$;
8) $t = t + 1$;
9) $G =< N[R] >$;
else
10) $G = K_{1,2}$;
11) *if* G is complete k_n-partite
12) $L_t = V$;
else
G is not in W_{AR};

Stage B: Reduction

13) *for* all L_i, L_j $i \neq j$
14) *for* all $P_a \in L_i, P_b \in L_j$
15) *if* $P_a \sim P_b$ and $< P_a \cup P_b >$ is not complete bipartite
G is not in W_{AR}.
Construct the minimal graph G_M as follows:
16) create a vertex v_i in G_M for each part P_i in G;
17) $v_i \sim v_j$ in G_M if $P_i \sim P_j$ in G;

Stage C: property b)

18) *for* all L_i, L_j in G_M
19) *if* $v_i \in L_i \sim v_j \in L_j$
20) $S_i = L_i - N(v_j)$;
21) $S_j = L_j - N(v_i)$;
22) *if* $\mid S_i \mid \neq 1$ or $\mid S_j \mid \neq 1$
G is not in W_{AR};
23) *else if* $\mid L_i \cap N(S_j) \mid > 0$ or $\mid L_j \cap N(S_i) \mid > 0$
G is not in W_{AR};
24) *else if* $< (L_i - S_i) \cup (L_j - S_j) >$ is not a clique
G is not in W_{AR};

Stage D: property c)

25) *for* all L_i in G_M
26) *for* all $u, v \in L_i$
27) $S_u = N(u) - N[v]$;
28) $S_v = N(v) - N[u]$;
29) *for* all $w \in S_u$

30) $\quad if\ N(w) \not\supseteq S_v$
G is not in W_{AR};

We now do the correctness and time analysis for the above steps.
Stage A: Decomposition
1)
We observe that if G is complete k-partite, then its complement $\overline{G}$ consists of k disconnected components, each of which is a clique. We can find the complement of a graph in order n^2. The components can be found in order $n+m$ using a depth first search. If the number of vertices in a component is n_1, then the component is a clique if the degree of every vertex in it is equal to $n_1 - 1$. Thus, checking if each component is a clique be done in order $n + m$. Therefore, this step can be done in order n^2.
2)
Finding two maximal independent sets I_1 and I_2 can be done as follows. First include all vertices of degree zero in both I_1 and I_2. Choose a vertex v from any one of the remaining vertices and include it in I_1. Choose one of its neighbours and include it in I_2, thus ensuring that $I_1 \neq I_2$. Extend I_1 and I_2 to maximal independent sets for G using a greedy algorithm. This can be done in order $n+m$.
3)
We find the intersection R of I_1 and I_2 as follows. We mark every vertex in I_1. We go through I_2 and put the marked vertices in a set R. This gives the intersection of the two sets. This can be done in order $n + m$.
4)
From proposition 4.2, we know that R is maximal if and only if $< V - N[R] >$ is complete k-partite. We check if R is maximal by checking if G_R, the graph induced by $V_R = V - N[R]$ is complete k_n-partite. As in 1), this can be done in order n^2.
5)
If R is not maximal, G_R is not complete k_n-partite. Therefore, in $\overline{G_R}$, there is at least one component which is not a clique. Consider such a component H. Let the number of vertices in it be n. Since H is not a clique, there is at least one vertex u in H such that $d(u) < n - 1$. Since H is a connected graph, this implies the existence of a vertex v in H that is not adjacent to u, but is adjacent to a neighbour w of u. These three vertices induce a P_3 in $\overline{G}$. Therefore, in G_R, these three induce a K_2 given by (u, v), and an isolated vertex w. From proposition 4.1, this is the condition for G to be not complete k-partite. These vertices can be found in order $n + m$. Extend R by making $R = R \cup \{w\}$. Thus, this step can be done in order $n + m$.
4) and 5)
Since R can contain at most $n-1$ vertices, the while loop is repeated at most $n-2$ times. Hence, these steps can be done in at most order n^3.
6)
This is the same as 1), except for checking if all the components have the same number of vertices. Like 1), this can be done in order n^2 time.

7)

This can be done in order $n + m$ time.

8)

This can be done in a constant time.

9)

This can be done in order $n + m$ time.

10)

This can be done in a constant time.

1) to 10)

Therefore, steps 1) to 10) take order n^3 time. Since there can be at most $n/2$ layers, considering the case when each layer is a K_2, the whole process can be done in at most order n^4 time.

11)

This, like 6), will take order n^2 time.

12)

This will take order n time.

Therefore, stage A will take at most order n^4 time.

Stage B: Reduction

13 to 15)

If G is in W_{AR}, then it is also in W_{SR}, and hence the layers must obey corollary 5.1. Therefore, if parts from different layers are adjacent to each other, they are complete bipartite. We first check this as it will enable us to reduce the graph. For every two parts P_a and P_b belonging to different layers, we check if they have an edge between them. If so, we check if the parts are complete bipartite. We can do this as follows: Mark each vertex in P_b. For every vertex in P_a, go through its neighbour set and count the number of marked vertices. This can be done in order $n + m$. Since each layer except one, L_t, has to have at least two parts, and each part can consist of just one vertex, the number of layers can be at most $n/2$. As each part can consist of just one vertex, the number of parts can be at most n. Hence, the number of comparisons is of order n^2. Therefore, the above process takes at most order $n^3 + n^2m$ time.

16 to 17)

For properties b) and c), we construct a minimal graph G_M by having a vertex in G_M for each part in G. If two parts are adjacent, we join the corresponding two vertices in G_M by an edge. The graph G_M will have layers which are cliques and parts which are single vertices. This process can be done while doing step 13).

Therefore, this stage would require order $n^3 + n^2m$ time.

We now check if G_M obeys properties b) and c).

Stage C: property b)

18 to 24)

For any two adjacent layers L_a and L_b, we check if there is one vertex in each that has no neighbours in the other, and if the rest of the vertices form a clique. Since the maximum number of layers is $n/2$, the number of layer comparisons would be of order n^2, and the comparison between two layers would be of order $n+m$. Thus, the above process would take at most order $n^3 + n^2m$ time.

Stage D: property c)
25 to 30)
For any two vertices in any layer, we check if the non-common neighbours are completely connected to each other. The number of such vertex pairs is of order n^2, and checking if the non-common neighbours are completely connected will take at most order $n+m$ time. Hence, the whole process will take at most order n^3+n^2m time.
All of the above steps can thus be done in polynomial time, and hence recognition is in P.

6.3.2 Dominating set

Problem: *Given a graph G and an integer k, is there a set of k vertices of G such that every vertex not in the set is adjacent to at least one vertex in it?*

Theorem 6.3 *The dominating set problem is in P for the class W_{AR}.*

Proof:
A part in a layer is said to be a *simplicial part* if its neighbour set is made up of only those vertices that belong to that layer. We first prove the following.

Claim 6.3 *Let G be in W_{AR} and let its vertex set be partitioned into layers $L_1, L_2, \ldots, L_t$ satisfying properties a) to c) of theorem 5.3. Then every layer has a simplicial part or each part of it is adjacent to a layer that has a simplicial part.*

Proof:
Assume not. That is, there exists a layer L_{i1}, $1 \le i1 \le t$, such that L_{i1} does not have a simplicial part, and there exists a part P_{i1} in L_{i1} that is not adjacent to any layer that has a simplicial part. Since it is not a simplicial part, it must have neighbours in some other layer L_{i2}. From theorem 5.3 *b*), we know that it must be adjacent to all but one part P_{i2} in L_{i2}, and P_{i2} must have no neighbours in L_{i1}. By assumption, L_{i2} has no simplicial part; therefore, P_{i2} should be adjacent to all but one part P_{i3} in L_{i3}, and P_{i3} must have no neighbours in L_{i2}. Now P_{i2} is not adjacent to P_{i1} but is adjacent to all but P_{i3} in L_{i3} and all but P_{i2} in L_{i2} are adjacent to P_{i1} in L_{i1}. Therefore, from corollary 5.2, P_{i1} must be adjacent to all but P_{i3} in L_{i3}. Consider the graph G_A formed by the layers L_{i1}, L_{i2}, and L_{i3}. We see that $N(P_{i1}) \supset N(P_{i2}) \supset N(P_{i3})$. Now, P_{i3} cannot be a simplicial part and therefore has to have neighbours in some other layer. This excludes layers L_{i2} and L_{i1} as this would contradict theorem 5.3 *b*). Therefore, P_{i3} must be adjacent to all but one part P_{i4} of L_{i4}, with P_{i4} having no neighbours in L_{i3}. We include this layer in the graph G_A.

We say that layers L_{i1} to L_{ik} obey property P_A if there exist parts $P_{i1} \in L_{i1}, P_{i2} \in L_{i2}, \ldots, P_{ik} \in L_{ik}$ which have the property that $N(P_{i1}) \supset N(P_{i2}) \supset \cdots \supset N(P_{ik})$, $1 \le k \le t$.

Let G_A consist of j layers L_{i1} to L_{ij} of G, $1 \le j < t$. Assume that the layers obey property P_A, that is, $N(P_{i1}) \supset N(P_{i2}) \supset \cdots \supset N(P_{ij})$, where P_{il} is a part in layer L_{il}, $1 \leq l \leq j$. Since P_{ij} is not simplicial, it must be adjacent to all but one part P_{ij+1} of some layer L_{ij+1}, with P_{ij+1} having no neighbours in L_{ij}. In fact, because of property P_A, and theorem 5.3 b), the layer L_{ij+1} cannot be any of L_{i1} to L_{ij-1}. Therefore, L_{ij+1} is a new layer from G. Now, P_{ij} is not adjacent to any of P_{i1} to P_{ij-1} but is adjacent to all but P_{ij+1} in L_{ij+1}, and all but P_{ij} in L_{ij} is adjacent to each of P_{i1} to P_{ij-1} (property P_A). Using corollary 5.2, we have that P_{i1} to P_{ij-1} are adjacent to all but P_{ij+1} in L_{ij+1}. That is, in the new graph G_A formed by $G_A \cup L_{ij+1}$, $N(P_{i1}) \supset N(P_{i2}) \supset \cdots \supset N(P_{ij+1})$. Thus, in order for P_{ij} to be a non-simplicial part, we are forced to extend G_A by adding a new layer P_{ij+1} from G, and these layers obey property P_A.

Thus, in order for P_{i1} to not have neighbours in a layer that has a simplicial part, we are forced to keep extending G_A, and the layers in G_A always obey property P_A. We extend G_A until it is maximal, that is, until no more layers can be added to it; this is bound to happen as G has a finite number t of layers. Let G_A contain k layers, $1 \leq k \leq t$. Since these layers satisfy property P_A, we have $N(P_{i1}) \supset N(P_{i2}) \supset \cdots \supset N(P_{ik})$. Now, there should be some part P_{ik} in L_k that has no neighbours in L_{i1} to L_{ik}. Since this part cannot be simplicial as P_{i1} is adjacent to L_{ik}, there has to another layer in G that P_{ik} is adjacent to. This is not possible as we have assumed that G_A is maximal. Therefore, P_{ik} has to be adjacent to one of the layers L_{i1} to L_{ik-1} which contradicts theorem 5.3 b). Therefore, P_{ik} has to be a simplicial part, in a layer L_{ik} which has parts that are adjacent to P_{i1} in L_{i1}. This contradicts our assumption.

□

Since G is in W_{AR}, from theorem 5.3, we can partition its vertex set into layers that obey properties a) to c) of the theorem. From the recognition algorithm for the class W_{AR}, we know this partitioning can be done in polynomial time. Now, form a set by choosing a non-simplicial part from all the layers that have a simplicial part. This will be a dominating set because of the claim 6.3. This set is a minimum set because a simplicial part can only be dominated by a part from the layer it belongs to. Such a set can be obtained in polynomial time and hence the dominating set problem is in P for the family W_{AR}.

□

6.4 The class W_{ARF}

We now look at the complexities of the Hamiltonian cycle and path problems for this family and prove that these are NP-complete.

6.4.1 Hamiltonian cycle

Problem: *Given a graph G, does G contain a simple cycle such that every vertex in G is in the cycle?*

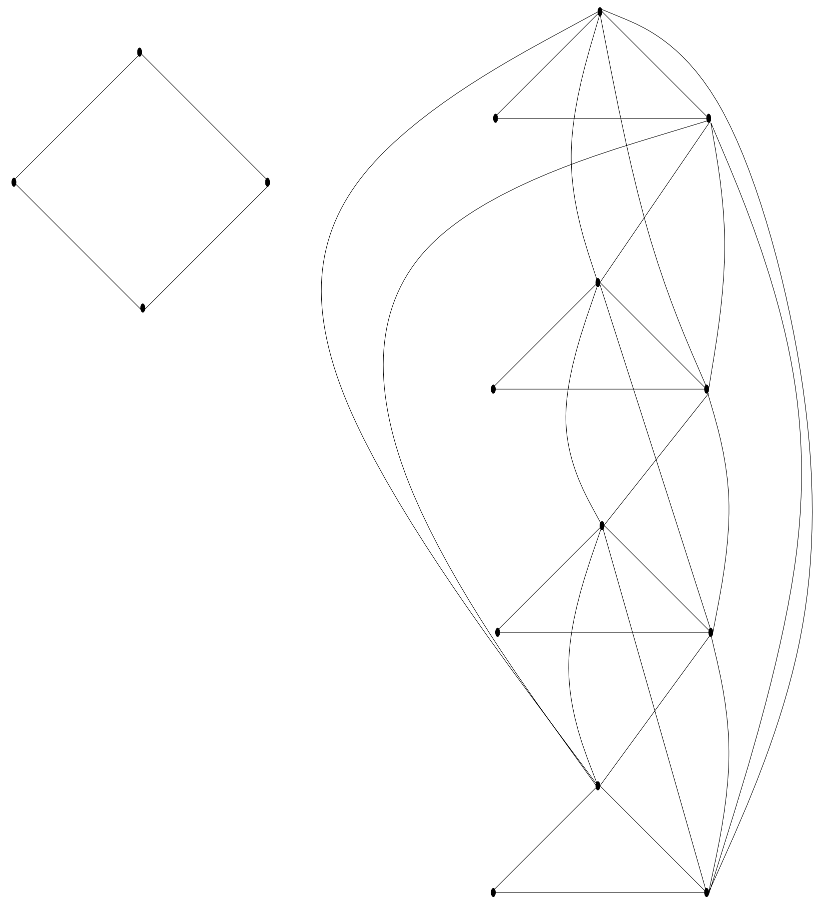

Figure 6.4: Hamiltonian cycle

Theorem 6.4 *The Hamiltonian cycle problem is NP-complete for the family* W_{AR3}.

Proof:
We transform from the Hamiltonian cycle problem for general graphs. Given a graph G of order $n > 2$, we construct a graph G_H as follows. For each vertex v_1 in G, we construct a K_3 in G_H. Two of the vertices of the K_3 correspond to v_1 in G; call them v_{11} and v_{12}. Each of these two vertices are connected to each two vertex pair in G_H that corresponds to a neighbour of v_1 in G. The other vertex v_{13} is a simplicial vertex. Each such K_3 forms an lgraph in G_H. Thus, there are n layers in G_H with $3n$ vertices and $4m + 3n$ edges. Clearly, this transformation can be done in polynomial time. For an example, see Figure 6.4. The layers induce K_3's, each one has a simplicial vertex, and the neighbour sets of the two non-simplicial vertices in each layer are the same; therefore, the layers obey properties a) through c) of theorem 5.3. Therefore, G_H is in W_{AR}. Since each layer has exactly three parts of one vertex each, G_H is in W_{AR3}.

Claim 6.4 *G has a Hamiltonian cycle if and only if G_H has a Hamiltonian cycle.*

Proof:
only if:
G has a Hamiltonian cycle. For every vertex in G, there is a corresponding K_3 in G_H. For every edge in G, there are edges connecting two K_3's. Hence, if there is an edge (v_1, v_2) in G, we can always find a path $v_{11}, v_{13}, v_{12}, v_{21}$ in G_H. Therefore, if G has a Hamiltonian cycle, we can always find a corresponding Hamiltonian cycle for G_H.
if:
G_H has a Hamiltonian cycle. Consider any layer in G_H. It induces a K_3 which corresponds to some vertex in G. Consider one such K_3 which consists of vertices v_{i1}, v_{i2}, and v_{i3} which corresponds to a vertex v_i in G. Since v_{i3} is a simplicial vertex, the path v_{i1}, v_{i3}, v_{i2} has to be part of any Hamiltonian cycle. Therefore, the part of a Hamiltonian cycle in G_H through a K_3 can be collapsed to a single corresponding vertex in G. Of the four edges that connect two K_3's, only one can be part of a Hamiltonian cycle. Any such edge has a corresponding edge in G. Hence, if G_H has a Hamiltonian cycle, we can always find a corresponding Hamiltonian cycle in G. This proves the claim.

Since the Hamiltonian cycle problem is NP-complete for general graphs, from the above, it is NP-complete for the family W_{AR3} and thus for the family W_{ARF} as well.

□

6.4.2 Hamiltonian path

Problem: *Given a graph G, does G contain a simple path such that every vertex in G is in the path?*

Theorem 6.5 *The Hamiltonian path problem is NP-complete for the family W_{AR3}.*

Proof:
We transform from the Hamiltonian cycle problem for general graphs. Given a graph G of order $n > 2$, we construct a graph G_H in the same way as for the Hamiltonian cycle problem, with the following change. We replace one of the layers, say L_n, with two layers as follows. Take the K_3 which forms the lgraph $< L_n >$ and duplicate it to form the lgraph $< L_{n+1} >$. Form two more lgraphs $< L_{n+2} >$ and $< L_{n+3} >$ using K_3's such that L_{n+2} is adjacent only to L_n and L_{n+3} is adjacent only to L_{n+1}; we will call the K_3's forming these lgraphs *leaf* K_3's. Two vertices of L_{n+2} should form a K_4 with the two non-simplicial vertices of L_n; likewise, two vertices of L_{n+3} should form a K_4 with the two non-simplicial vertices of L_{n+1}. The graph G_H has $3(n+3)$ vertices and $4m+4d(v_n)+3(n+3)+8$ edges, where v_n is the vertex in G which forms the layer L_n in G_H. Clearly, this transformation can be done in polynomial time. For an example, see Figure 6.5.

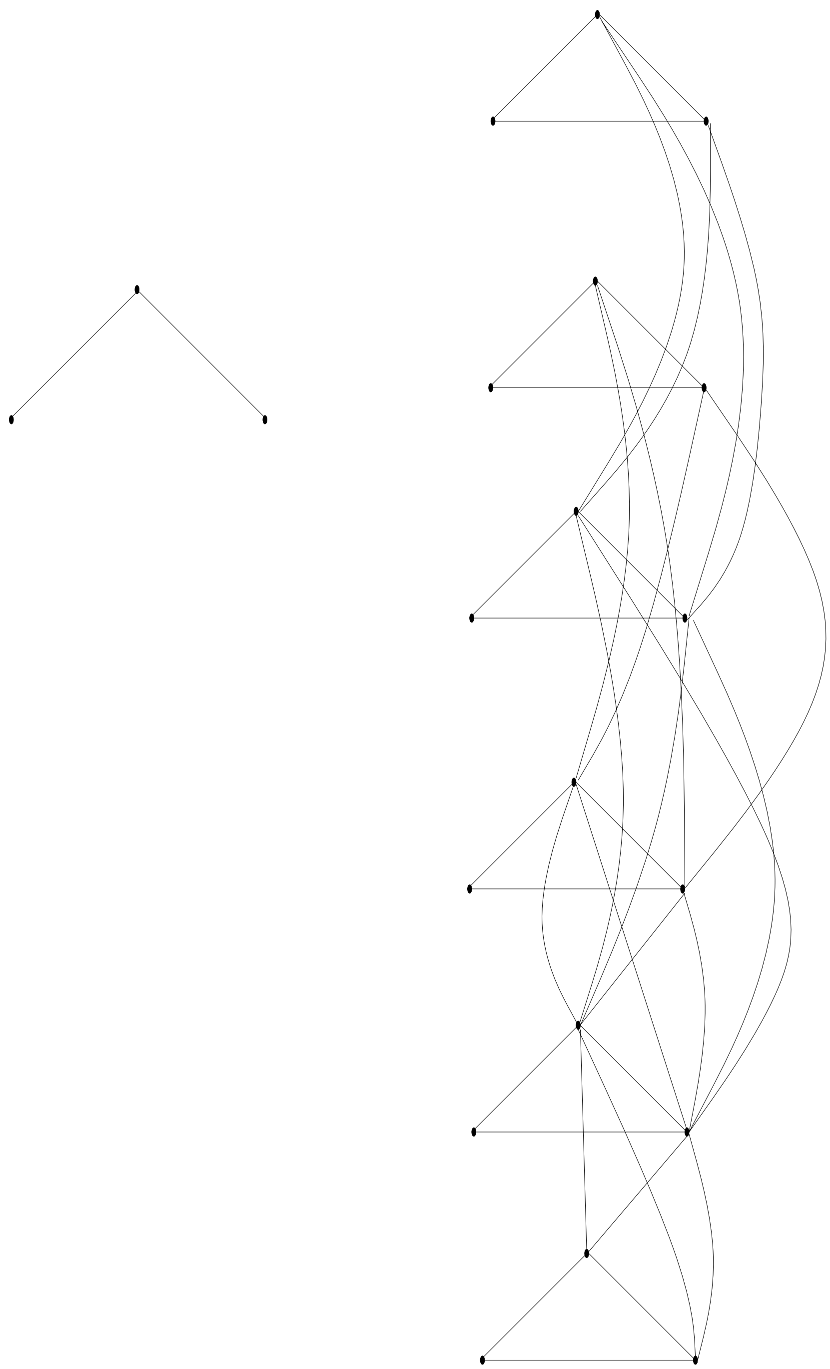

Figure 6.5: Hamiltonian path

It can be easily seen that G_H still obeys properties $a)$ through $c)$ of theorem 5.3, and hence belongs to the family W_{AR}. Since each layer has exactly three parts of one vertex each, G_H is in W_{AR3}.

Claim 6.5 *G has a Hamiltonian cycle if and only if G_H has a Hamiltonian path.*

Proof:
The proof is similar to the one given for Hamiltonian cycle problem, except for the following observations.
only if:
G has a Hamiltonian cycle. It is easy to see that we can find a simple path in G_H that starts at one of the leaf K_3's, say L_{n+2}, ends at the other, and covers all the vertices in G_H, that is, a Hamiltonian path for G_H.
if:
G_H has a Hamiltonian path. As in the case of the Hamiltonian cycle problem, of the four edges that connect two K_3's, only one can be used. Consider the leaf K_3's; once a path enters one of them, there is no way out, since each one is adjacent to exactly one other K_3. Therefore, any Hamiltonian path has to start at one of these K_3's and end at the other. That is, if we ignore the leaf K_3's, the path starts at one of L_n or L_{n+1}, and ends at the other. The two K_3's that make up the lgraphs $< L_n >$ and $< L_{n+1} >$ can be collapsed to a single vertex in G. Hence, if G_H has a Hamiltonian path, we can always find a simple cycle in G that includes all the vertices in G, that is, a Hamiltonian cycle for G. This proves the claim.

Since the Hamiltonian cycle problem is NP-complete for general graphs, from the above, the Hamiltonian path problem is NP-complete for the family W_{AR3} and hence for the family W_{ARF}.

$\square$

From the above, it is seen that the Hamiltonian cycle and path problems are NP-complete for the family W_{ARF}, even when the number of parts per layer is three ($k = 3$).

6.5 The class W_{AR2}

This class is the same as the family of very well covered graphs without isolated vertices. Hence, the results obtained in chapter 3 hold for this family as well.

6.6 Conclusions

We have studied the complexities of the same problems that we looked at for well-covered graphs, for the new sub-classes. Since W_{AR2} is the same as the family of very well covered graphs without isolated vertices, we only needed to look at those problems that had complexities in P for W_{AR2} and check if these remained the same for the other sub-classes. These problems are recognition, clique partition, Hamiltonian cycle and path, and dominating set. We see that while clique partition remains in P, the rest become intractable as one goes higher up in the hierarchy. These problems separate the classes algorithmically, except for the classes W_{AR} and W_{ARF} which have the same complexities for all the problems that we have looked at.

Chapter 7

Generalization of Favaron's theorem

7.1 Introduction

As we have seen earlier, Favaron characterized the class of very well covered graphs without isolated vertices. This characterization showed that all such graphs had a perfect matching which obeyed a certain property P. We have seen that the class W_{AR2} is the same as the family of very well covered graphs without isolated vertices. This is contained in the class W_{AR} which, like W_{AR2}, has the problems of recognition and clique partition in P. A generalization of a matching is a clique partition. We now provide an alternative characterization of the sub-class W_{AR} in terms of a clique partition of size α which obeys a certain property Q. This is shown to be a generalization of Favaron's theorem (theorem 2.1).

7.2 Definitions

Favaron [8] defined the following equivalence relation for very well covered graphs without isolated vertices. We state this here as it is referred to in this chapter.

Definition 7.1 *Let M be a perfect matching of a very well covered graph G. Two vertices x and y are called equivalent if either $x = y$ or if $(x, v), (y, u) \in M$ and $x \in N(u)$ and $y \in N(v)$.*

She showed that the equivalence classes form a partition of the vertex set of G into independent sets with certain properties.

7.3 A generalization of Favaron's theorem

We first state the alternative characterization for the sub-class W_{AR}, which is also a generalization of Favaron's theorem. We say that a clique partition of a graph G is an α-clique partition if the number of cliques in the partition is α, the size of a maximum independent set of G.

Theorem 7.1 (generalization of Favaron's theorem) *The following are equivalent for a graph G.*

a) G belongs to W_{AR}.

b) There exists an α-clique partition in G that satisfies Q.

c) There exists an α-clique partition in G, and every α-clique partition in G satisfies Q.

Hence, if G is in W_{AR}, every clique partition of G satisfies Q. In order to prove this theorem, we need to state some definitions and establish some results. We first define property Q.

Let $C = \{C_1, C_2, \ldots, C_k\}$, $1 \leq k \leq n$, be a clique partition of a graph G, with the corresponding vertex set being $V_1, V_2, \ldots, V_k$. We denote by $C(v)$ the clique, and by $V(v)$ the corresponding vertex set, that $v \in V$ belongs to.

Property Q: We say that a clique partition C satisfies property Q if:

a) $\mid N(v) \cap V_i \mid = 0$ *or* $\mid V_i \mid -1$, $\forall v \in V, 1 \leq i \leq k$.

b) $(w \in V(v), u \in N(v) - V(v), u \notin N(w)) \to (N(u) \supseteq N(w) - N(v))$, $\forall v \in V$.

The first condition states that if a vertex in the graph is adjacent to a vertex in a clique in the clique partition, then it is adjacent to all but one vertex in that clique. The second one states that for every two vertices in a clique, their non-common neighbours are completely adjacent to each other.

Let us see what happens to this property when we restrict the cliques in the partition to be K_2's, that is, a perfect matching for G. Since each C_i is a K_2, from property Q *a*), we have that a vertex in a clique in C can be adjacent to at most one vertex in another clique. This means that the vertices in a clique in C do not have a common neighbour. We can use this fact to rewrite property Q for graphs which have a perfect matching. A perfect matching M is said to satisfy property Q if:

$(w \in V(v), u \in N(v) - V(v)) \to (u \notin N(w))$ and $(N(u) \supseteq N(w) - N(v))$, $\forall v \in V$.

We see that this is the same as the property P defined by Favaron.

In the above theorem (theorem 7.1), if G belongs to W_{AR2}, then an α-clique partition is a perfect matching, and property Q, as we have just seen, reduces to property P. That is, the theorem reduces to Favaron's theorem (theorem 2.1) for very well covered graphs without isolated vertices.

Let C be a clique partition of a graph G, and let C satisfy Q. We define the following equivalence relation:

Definition 7.2 *We say that u and v are equivalent if either $u = v$ or $\mid V(u) \mid = \mid V(v) \mid$ and $x \sim v, y \sim u$, $\forall x \in V(u) - u, y \in V(v) - v$.*

That is, two vertices u and v are said to be equivalent if either they are the same vertex, or if their clique sizes are the same, and every vertex of $V(u) - u$ is adjacent to v, and every vertex of $V(v) - v$ is adjacent to u. Note that two vertices u and v in the same clique cannot be equivalent as this would require each one to be adjacent to itself, which is not permitted.

We need to show that the above is indeed an equivalence relation. We first prove the following proposition.

Proposition 7.1 *Let C be a clique partition that satisfies Q. Then, if u is equivalent to v, $u \neq v$, $C(u) \cup C(v)$ is complete k_n-partite with each part having two vertices, and u and v forming one of the parts.*

Proof:
Since u and v are equivalent, we know that $\mid V(u) \mid = \mid V(v) \mid$. Also, $C(u)$ and $C(v)$ are cliques. Let $V(u) = \{u, u_1, u_2, \ldots, u_j\}$, and $V(v) = \{v, v_1, v_2, \ldots, v_j\}$, $j = \mid V(u) \mid -1$. Since u is adjacent to all of $V(v) - v$, u is not adjacent to v, from Q *a*). Also, v is adjacent to all but u in $V(u)$. Consider some $u_i \in V(u) - u$, $1 \leq i \leq j$. Since u_i is adjacent to v, from property Q *a*), u_i is adjacent to all but some $v_{ii} \in V(v) - v$, $1 \leq ii \leq j$. Likewise, since v_{ii} is adjacent to u, it is adjacent to all but u_i in $V(u)$. Therefore, the vertices of $V(u)$ and $V(v)$ can be paired into disjoint sets of two vertices such that the neighbour set of a vertex in a pair is all but the other vertex in the pair. From the above, u and v forms one such pair. Hence, $C(u) \cup C(v)$ is complete k_n-partite, with each part having two vertices, and u and v forming one of the parts.

□

Now, let u be equivalent to v, $u \neq v$, and v be equivalent to w, $u, v \neq w$. Since $\mid V(u) \mid = \mid V(v) \mid$ and $\mid V(v) \mid = \mid V(w) \mid$, it follows that $\mid V(u) \mid = \mid V(w) \mid$. From proposition 7.1, $C(u) \cup C(v)$ is complete k_n- partite, as is $C(v) \cup C(w)$. Also, $u \not\sim v$ and $v \not\sim w$; with Q *a*), this implies that $C(u) \neq C(w)$. Consider a part $\{x, y\}$ in $C(u) \cup C(v)$, $x \in V(u), y \in V(v)$, $x \neq u, y \neq v$. Now, $v \sim x$ and $w \sim y$. Also, $v \not\sim w$ and $y \not\sim x$. Since v and y are in the same clique $C(v)$ and have non-common neighbours x and w respectively, from property Q *b*), $w \sim x$. Therefore, w is adjacent to all the vertices in $V(u) - u$. In a similar fashion, we can show that u is adjacent to all of $V(w) - w$. That is, u is equivalent to w. Therefore, the relation of Definition 7.2 is an equivalence relation.

Let $E(U)$ denote the equivalence class of u, and let $C(U)$ denote the corresponding clique class, that is, $C(U)$ is made up of the cliques $C(v)$ corresponding to each vertex $v \in E(U)$. Let $V(U)$ represent the vertex set of $C(U)$. We now prove the following proposition.

Proposition 7.2 *The following are true:*
- *a) The equivalence classes partition V into independent sets.*
- *b) The clique classes are complete k_n-partite, with each part forming an equivalence class.*

Proof:
a)
Take any equivalence class $E(U)$ associated with a vertex $u \in V$. From proposition 7.1, the vertices in $E(U)$ are pairwise disjoint, i.e, $E(U)$ is an independent set. As it is an equivalence relation, no vertex can appear in more than one equivalence

class.

b)

Let $v \neq u \in E(U)$, that is, v is equivalent to u. Consider $x \in V(u)$, $x \neq u$. Since v is in $E(U)$, using proposition 7.1, we have that $C(u) \cup C(v)$ is complete k_n-partite, with $\{u, v\}$ forming one of the parts. Therefore, there has to be a $y \in V(v)$, $y \neq v$, that forms a part with x in $C(u) \cup C(v)$. Thus, y is adjacent to all of $V(u) - x$, and x is adjacent to all of $V(v) - y$. Also, $\mid V(u) \mid = \mid V(v) \mid$, since v is equivalent to u. Hence, y is equivalent to x. So for each vertex $v \in E(U)$ that is equivalent to u, we can find a $y \in V(v)$ that is equivalent to x, that is, y is in $E(X)$. Since u is in $C(x)$, as $C(u)$ and $C(x)$ are the same, using a similar argument, we can show that for each $z \in E(X)$, we can find a $w \in V(z)$ that is equivalent to u. Therefore, $\mid E(U) \mid = \mid E(X) \mid$. From *a*), $E(U)$ and $E(X)$ are mutually disjoint independent sets. Thus, each vertex x in $V(u)$ yields an equivalence class $E(X)$, $\mid E(X) \mid = \mid E(U) \mid$, whose vertices are from $V(U)$. Therefore, the $E(X)$'s partition $V(U)$ into mutually disjoint independent sets, all of which have the same size, with each $E(X)$ having exactly one vertex from each clique in $C(U)$. Since every two vertices in $E(U)$ are equivalent to each other, from proposition 7.1, the cliques in $C(U)$ are pairwise complete k_n-partite. From the above, $C(U)$ is complete k_n-partite.

□

Let $C(U)$ be the clique class associated with a vertex u of V. For every other vertex $v \in V(U)$, the clique class $C(V)$ is the same as $C(U)$, since from proposition 7.2, v belongs to either $E(U)$ or an equivalence class $E(X)$ corresponding to a vertex $x \in V(u)$, and $C(U)$ is complete k_n-partite with $E(X)$ forming one of the parts. Hence, when we refer to clique classes, we are referring to the distinct clique classes obtained from the equivalence classes. Clearly, every clique in the clique partition C is a part of some clique class, and no clique can belong to more than one clique class. Hence, the clique classes form a partition of V. We now prove the following.

Proposition 7.3 *Let C be a clique partition of a graph G, and let C satisfy property Q. Then, the corresponding clique classes partition the vertices of G into disjoint sets which satisfy properties a) to c) of theorem 5.3, and hence G is in W_{AR}.*

Proof:

From proposition 7.1, the equivalence classes form a partition of the vertex set of G into independent sets. From the same proposition, each clique class is complete k_n-partite, with each part forming an equivalence class. A clique class is constructed by taking an equivalence class and picking all the cliques in C that contain the vertices of the equivalence class. Since the clique classes are distinct, every equivalence class can be in exactly one clique class. Furthermore, the clique classes contain all the equivalence classes. Hence, the clique classes are vertex disjoint and form a partition of the vertex set of G into disjoint sets, each of which induces a complete k_n-partite subgraph. We show that these disjoint sets satisfy properties a) to c) of theorem 5.3, by showing that the corresponding induced subgraphs, that is, the clique classes, satisfy these properties.

a)

From proposition 7.1, we know that the clique classes are complete k_n-partite. If any isolated vertices are present, they will form one clique class by themselves, as they will all be in the same equivalence class.

b)

Let $C(X)$ and $C(Y)$ be two different clique classes. Let $x \in E(X)$ from the class $C(X)$ be adjacent to $y \in E(Y)$ from the class $C(Y)$. From property Q a), there exists y_1 in $C(y)$ that x is not adjacent to. Since $C(Y)$ is complete k_n-partite, y_1 is adjacent to each $z \in E(Y)$. Since y and y_1 are in the same clique $C(y)$, using property Q b), x is adjacent to all such z. Therefore, x is adjacent to all of $E(Y)$. Now, since every vertex y in $E(Y)$ is adjacent to x in $E(X)$, by a similar argument, y is adjacent to all of $E(X)$. Thus, $E(X)$ and $E(Y)$ are complete bipartite.

Therefore, if parts from different clique classes are adjacent, they are complete bipartite. This enables us to do the following reduction on the clique classes: replace each part in a clique class by a single vertex, thus reducing each clique class to a single clique; replace the set of edges between two adjacent parts by a single edge. Clearly, this transformation preserves the relationship between the clique classes. We say that two cliques are adjacent if there is a part in one that is adjacent to a part in the other. Now, consider any two clique classes $C(X)$ and $C(Y)$ which are adjacent. We consider the classes to be single cliques and call them $C(x)$ and $C(y)$. Let $i = \mid V(x) \mid$ and $j = \mid V(y) \mid$. We have two cases.

case a) $i \neq j$

Assume $i < j$. Since $C(x)$ and $C(y)$ are adjacent, $i, j \geq 2$. Now, if all the vertices of $V(x)$ were to be adjacent to vertices in $V(y)$, from property Q a), the number of edges from $V(x)$ to $V(y)$ would be $i(j-1)$. Now, $i(j-1) > (i-1)j$ which is the maximum number of edges possible from $V(y)$ to $V(x)$. Hence, if every vertex of $V(x)$ were to have neighbours in $V(y)$, then at least one vertex of $V(y)$ would have to be adjacent to all the vertices in $V(x)$, which would contradict Q a). Thus, at most $i-1$ vertices of $V(x)$ can have neighbours in $V(y)$. Since any vertex in $V(y)$ has to be adjacent to exactly $i-1$ vertices in $V(x)$ from property Q a), exactly $i-1$ vertices of $V(x)$ have to have neighbours in $V(y)$.

Can all the vertices of $V(y)$ have neighbours in $V(x)$? We have already determined that there are only $i-1$ vertices in $V(x)$ that have neighbours in $V(y)$. The number of edges from $V(y)$ to these $i-1$ vertices of $V(x)$ is $j(i-1)$. Now, $j(i-1) > (j-1)(i-1)$ which is the maximum number of edges possible from $i-1$ vertices of $V(x)$ to the vertices in $V(y)$. Hence, at least one of the $i-1$ vertices of $V(x)$ should be adjacent to all the vertices of $V(y)$, which contradicts Q a). Thus, at most $j-1$ vertices of $V(y)$ can have neighbours in $V(x)$. Since each vertex of $V(x)$ has to be adjacent to exactly $j-1$ vertices of $V(y)$, exactly $j-1$ vertices of $V(y)$ have to be adjacent to exactly $i-1$ vertices of $V(x)$. To satisfy property Q a), it can be easily seen that these two sets of vertices have to be completely adjacent to each other. Thus, there exists exactly one vertex in $V(x)$ which has

no neighbours in $V(y)$ and vice versa.
case b) $i = j$
Suppose all the vertices of $V(x)$ had neighbours in $V(y)$. The number of edges from $V(x)$ would be $i(i-1)$ which is the same as the maximum possible number of edges from $V(y)$ to $V(x)$. Therefore, all the vertices of $V(y)$ would have neighbours in $V(x)$ too. Consider the equivalence class $E(x)$. This consists of the vertex x. From Q a), there exists a vertex, say w in $C(y)$ which is not adjacent to x. Since x has neighbours in $C(y)$ and w has neighbours in $C(x)$, this means that x is adjacent to all but w in $C(y)$ and w is adjacent to all but x in $C(x)$. That is, w is equivalent to x and therefore $C(x)$ and $C(y)$ belong to the same clique class, which is a contradiction. So at most $i-1$ vertices of $V(x)$ can have neighbours in $V(y)$; from property Q a), we see that exactly $i-1$ vertices of $V(x)$ can have neighbours in $V(y)$. A similar argument can be used to show that exactly $i-1$ vertices of $V(y)$ can have neighbours in $V(x)$. Thus, exactly $i-1$ vertices of $V(x)$ and $V(y)$ are completely adjacent to each other, and there exists exactly one vertex in each class that has no neighbour in the other.

The above two cases prove that the clique classes obey property b) of theorem 5.3.
c)
Consider a clique class $C(X)$. Every part in $C(X)$ has exactly one vertex from each clique $C(v)$, $\forall v \in E(X)$. From b), which we have just proved, if parts from different clique classes are adjacent, they are completely connected. Property c) follows from this and property Q b).

□

We now prove the generalization of Favaron's theorem for the class W_{AR}.

Proof*(of the generalization of Favaron's theorem)*:
a) $\rightarrow$ *c)*
G belongs to W_{AR}. Therefore, the vertices of G can be partitioned into layers L_1 to L_t that satisfy properites a) to c) of theorem 5.3. Since the corresponding lgraphs are complete k_n-partite, from property a) of the theorem, each lgraph can be decomposed into cliques giving an α-clique partition for G. Let one such clique partition be given by $C = C_1, C_2, \ldots, C_k$, $k \leq n$. Since the lgraphs are complete k_n-partite, a vertex v in a clique C_v in a layer is adjacent to exactly $\mid V(v) \mid -1$ vertices in each clique in that layer. As the layers obey property b) of theorem 5.3, if v is adjacent to a clique in another layer, it is adjacent to all but one vertex in that clique. Hence the vertices in C obey property Q a). Since any clique in C has exactly one vertex in each part of a layer that it belongs to, and the parts of the layers obey property c) of theorem 5.3, the vertices in a clique obey Q b). Hence, there exists an α-clique partition in C that satisfies Q.

Since G is in W_{AR}, the number of cliques in any minimum clique partition is equal to α, that is, $k = \alpha$. Consider any α-clique partition $C^1 = C_{11}, C_{12}, \ldots, C_{1k}$. Since $k = \alpha$ and G is in W_{AR}, any maximal independent set for G has to have exactly one vertex from each clique. Consider any decomposition of G, with layers

L_1 to L_t. We show that the cliques in C^1 can be rearranged to form the layers of the decomposition. We ignore isolated vertices as they form one clique each in the clique partition, and the layer L_t in the decomposition. Consider some layer L_i, $1 \leq i \leq t$. From proposition 5.4 *b*), we know that every layer is $V - N[R]$ for some R. Let $R = I_1 \cap I_2$ be a maximal intersection, such that $V - N[R]$ is the layer L_i, where I_1 and I_2 are a pair of maximal independent sets of G. R consists of exactly one vertex from each of $l < k$ cliques of C^1.

We now show that $N[R]$ consists of only vertices from those cliques that have a vertex in R. Assume not. Then there exists $C_j \in C^1$ which has no vertex in R, but which has at least one vertex v in $N[R]$. The graph G_2 induced by $N[R]$ is in W_{AR} from the definition 5.2. Since R is a maximal independent set of G_2, every maximal independent set in G_2 should have size $\mid R \mid$. Since any maximal independent set for G has to have one vertex from every clique in C^1, starting with v, we should be able to find a vertex from every clique that has a vertex in R such that the resulting set is independent. This set has size $>\mid R \mid$ implying that G_2 is not in W_{AR}, which is a contradiction. Hence, $N[R]$ consists of vertices from those cliques that have a vertex in R. That is, the graph induced by $L_i = V - N[R]$ which forms the lgraph H_i, consists of *whole* cliques from C^1 that do not have a vertex in R. Since the number of vertices in R is given by $\alpha - n_i$, there are $\alpha - n_i$ cliques from C^1 in $< N[R] >$. As the number of cliques in C^1 is also α, there are exactly n_i cliques from C^1 forming the layer L_i.

Now, the lgraphs are complete k_n-partite. The size of a part in the layer L_i is n_i, and the maximum possible size of a clique in it is given by k_i, the number of parts in it. Hence, the minimum number of vertex disjoint cliques required to cover the vertices of L_i is n_i, each being of size k_i. Since the cliques in C^1 are vertex disjoint, this means that the n_i cliques forming the lgraph H_i have exactly k_i vertices each, and form a partition of the vertices in H_i. Therefore, each lgraph of the decomposition contains *whole* cliques from C^1 such that the cliques form a clique partition for that lgraph. Since the sum of the n_i's is α, and there are α cliques in C^1, and the lgraphs are vertex disjoint, each clique in C^1 appears in exactly one lgraph in the decomposition. The corresponding layers obey properties *b*) and *c*) of theorem 5.3 and hence, as we have seen earlier, the cliques satisfy property Q.

c) $\rightarrow$ *b)*

Follows.

b) $\rightarrow$ *a)*

There exists a clique partition of G that satisfies Q. From proposition 7.3, G is in W_{AR}.

□

Now let us see what happens to the equivalence relation 7.2 if G is very well covered without isolated vertices. From Favaron's theorem 2.1, there exists a perfect matching for G. Hence, a clique partition for G consists of K_2's. The equivalence relation reduces to:

Definition 7.3 *u and v are equivalent if either $u = v$, or $u \in N(V(v) - v)$ and $v \in N(V(u) - u)$.*

This is the same as the equivalence relation defined by Favaron (definition 7.1), and hence the equivalence classes obtained are the same.

7.4 Conclusion

We have given an alternative characterization for the sub-class W_{AR} in terms of a clique partition of size α which obeys a certain property Q. We have shown that when the cliques in the partition are K_2's, the clique partition reduces to a perfect matching, property Q reduces to property P, and the characterization reduces to Favaron's characterization for very well covered graphs without isolated vertices. This is an interesting result since it generalizes the structure of very well covered graphs as characterized by Favaron.

Chapter 8

Conclusions and future work

8.1 Conclusions

In this thesis, we first studied the algorithmic complexity of the following graph theory problems for well-covered and very well covered graphs: chromatic number, clique cover, clique partition, dominating cycle, dominating set, Hamiltonian cycle, Hamiltonian path, independent set, independent dominating set, maximum cut, minimum fill-in, recognition, Steiner tree, and vertex cover. We saw that many of these problems are as hard for the family of well-covered graphs as for graphs in general. Some of the above problems turn out to be tractable for the family of very well covered graphs without isolated vertices.

We then gave a new characterization for well-covered graphs, and restricted this to a characterization for very well covered graphs. This characterization was based on the interaction between pairs of maximal independent sets I_1 and I_2 of a graph G. Next, we looked at the intersection of a pair of maximal independent sets of G. We defined the conditions under which such an intersection would be maximal, and under which all such intersections would have the same size. Using these two results, we defined and characterized a hierarchy of four new recursively decomposable sub-classes of well-covered graphs. The hierarchy of sub-classes are: W_{SR}, W_{AR}, W_{ARF}, and W_{AR2}. Each one properly contains the one next to it in the hierarchy. We showed that the graphs belonging to these sub-classes can be decomposed into layers which are unique, and which satisfy certain properties. We also proved that the sub-class W_{AR2} is the same as the family of very well covered graphs without isolated vertices.

Next, we looked at the algorithmic complexity of the same problems that we looked at for well-covered graphs, for these new sub-classes. Clearly, the problems that are intractable for very well covered graphs without isolated vertices are also intractable for the new sub-classes. Therefore, we restricted our attention to the following problems: recognition, clique partition, dominating set, and Hamiltonian cycle and path. The clique partition problem proved to be tractable for all the sub-classes. The rest proved tractable as we moved down the hierarchy. We observe that these problems separate the sub-classes algorithmically, except for the sub-classes W_{AR} and W_{ARF}. It is interesting to note that the Hamiltonian cycle and path problems, which are in P for the class W_{AR2}, turn out to be NP-complete for the class W_{AR3}.

Lastly, we generalized Favaron's theorem (theorem 2.1) for very well covered graphs without isolated vertices to the sub-class W_{AR}. Favaron's characterization was based on a perfect matching obeying a property P. We generalized this to an α-clique partition obeying a certain property Q. We showed that this reduces to Favaron's characterization when the α-clique partition considered consists of cliques which are K_2's.

The complexity results are shown in the Table 8.1.

Problem	WC	W_{SR}	W_{AR}	W_{ARF}	W_{AR2}
Member	co-NP-c	co-NP-c	P	$\Longrightarrow$	$\Longrightarrow$
Chromatic number	$\Longleftarrow$	$\Longleftarrow$	$\Longleftarrow$	$\Longleftarrow$	NP-c
Clique	$\Longleftarrow$	$\Longleftarrow$	$\Longleftarrow$	$\Longleftarrow$	NP-c
Dominating cycle	$\Longleftarrow$	$\Longleftarrow$	$\Longleftarrow$	$\Longleftarrow$	NP-c
Isomorphism	$\Longleftarrow$	$\Longleftarrow$	$\Longleftarrow$	$\Longleftarrow$	iso-c
Maximum cut	$\Longleftarrow$	$\Longleftarrow$	$\Longleftarrow$	$\Longleftarrow$	NP-c
Minimum fill-in	$\Longleftarrow$	$\Longleftarrow$	$\Longleftarrow$	$\Longleftarrow$	NP-c
Steiner tree	$\Longleftarrow$	$\Longleftarrow$	$\Longleftarrow$	$\Longleftarrow$	NP-c
Independent set	P	$\Longrightarrow$	$\Longrightarrow$	$\Longrightarrow$	$\Longrightarrow$
Independent dominating set	P	$\Longrightarrow$	$\Longrightarrow$	$\Longrightarrow$	$\Longrightarrow$
Vertex cover	P	$\Longrightarrow$	$\Longrightarrow$	$\Longrightarrow$	$\Longrightarrow$
Clique partition	NP-c	P	$\Longrightarrow$	$\Longrightarrow$	$\Longrightarrow$
Dominating set	$\Longleftarrow$	NP-c	P	$\Longrightarrow$	$\Longrightarrow$
Hamiltonian cycle	$\Longleftarrow$	$\Longleftarrow$	$\Longleftarrow$	NP-c	P
Hamiltonian path	$\Longleftarrow$	$\Longleftarrow$	$\Longleftarrow$	NP-c	P

Table 8.1: Complexity results for well-covered graphs

WC	Well covered.
$\Longleftarrow$	Result implied from result on right.
$\Longrightarrow$	Result implied from result on left.

Let us look at how far we have succeeded in answering the questions that were addressed by this thesis. We have shown that well-covered graph recognition is co-NP-complete. While this does not solve the complexity of the recognition problem for this family, the result indicates that it is highly unlikely that this problem is tractable. We have seen that besides the independent set problem, the independent dominating set and vertex cover problems are also in P for this class. These results follow trivially from the definition of the class. We have also seen that there are many problems that are intractable for this class; for example, the clique partition problem. As we have seen in chapter 2, there are many sub-classes of well-covered graphs which have recognition in P, including the family of very well covered graphs without isolated vertices. We have looked at these two families algorithmically. The results show that many graph theory problems are as hard for the family of well-covered graphs as for graphs in general. Some of them prove tractable for very well covered graphs, thus algorithmically separating the two classes. We have defined and characterized four new sub-classes of well-covered graphs, three of which have recognition in P. The first one, W_{SR} has recognition in co-NP-c. We have shown that the problems studied in this thesis distinguish the sub-classes algorithmically, except for the classes W_{AR} and W_{ARF}.

8.2 Future work

Some questions which remain unanswered in this thesis are: Are there other problems which are in P for well-covered graphs? Is it possible to separate the classes W_{AR} and W_{ARF} algorithmically? What are the algorithmic properties of the other well-covered families which have recognition in P? How do they relate to the sub-classes of this thesis?

All the new sub-classes have been defined based on the concept of a maximal intersection. An obvious question arising from the concept of well-coveredness is: What graphs have the property that every maximal intersection has the same size? Can we characterize such graphs? We call such graphs *well-intersected* graphs. Another question is: What graphs have the property of being both well covered and well intersected? Which of the graphs belonging to the newly defined sub-classes have the property of being well intersected?

Let W_R be the family of well-covered graphs which have the property that there exists a maximal intersection R for which $< N[R] >$ is well covered. This family properly contains the family W_{SR}. Are there other sub-classes of this family for which some of the problems that we have looked at, especially recognition, are tractable?

We have seen how a graph G belonging to W_{SR} can be represented by a corresponding minimal graph G_m. It would be interesting to study the properties of such graphs.

What about the complement of a well-covered graph? What well-covered graphs have the property that the complement is also well covered? Since a maximal independent set in a graph G is a maximal clique in its complement, this would imply that the complement of a well-covered graph has the property that every maximal clique has the same size. We shall call such graphs *well-cliqued* graphs. Hence, our question becomes: What well-covered graphs have the property that they are also well cliqued? Also, what is the nature of well-cliqued graphs?

We have seen that the class W_{AR} has recognition in P. It would be interesting to find an efficient recognition algorithm for this class, and for the class of very well covered graphs without isolated vertices. Also, are there any other sub-classes of W_{SR} that have recognition in P?

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