# Reflected Backward Stochastic Differential Equations for Informational Systems with Applications

by

Safa' Mahmoud Alsheyab

A thesis submitted in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

in

Applied Mathematics

Department of Mathematical and Statistical Sciences
University of Alberta

## Abstract

The core innovation of this thesis lies in studying reflected backward stochastic differential equations (RBSDE hereafter) for informational systems. An informational system is a system where there is discrepancy in the information received by agents over time. In this thesis, we restrict to the case where our system is governed by two flows of informations: The public information $\mathbb{F}$ that is available to all agents and a larger flow of information $\mathbb{G}$ that has additional information about a random time $\tau$. We mathematically formulate our results in a general setting where $\tau$ might not be observable by the flow of information $\mathbb{F}$. This allows our results to be applicable to credit risk theory, to life insurance where mortality and longevity risks are the main challenges, to financial models with arbitrary random horizon, ..., etcetera. Thus, we study RBSDEs that are stopped at $\tau$, and consider $\mathbb{G}$ to be the progressive enlargement of $\mathbb{F}$ with $\tau$, where $\tau$ becomes an observable time when it occurs (a stopping time with respect to $\mathbb{G}$ mathematically speaking). In this setting, we quantify –as explicit as possible– the impact of $\tau$ on the existence, the uniqueness, and the estimate in norm the solution of the RBSDE stopped at $\tau$. We construct an RBSDE under $\mathbb{F}$ that is intimately related to the stopped one, and we single out the exact relationship between their solutions. Importantly, we prove that for any random time, having a positive Azéma supermartingale, there exists a positive discount factor $\widetilde{\mathcal{E}}$, which is a positive and non-increasing $\mathbb{F}$-adapted and RCLL process, that is vital in proving our results without assuming any further assumption on $\tau$. We treat both the linear and general cases of RBSDEs for bounded and unbounded horizon. An application to exponential hedging under random horizon is illustrated in different manners. This gives a clear motivation for our class of stopped RBSDEs that we treat in this thesis.

# Acknowledgements

First and foremost, I would like to express my deep gratitude and special appreciation to my research supervisor and academic role-model Prof. Tahir Choulli for the years of guidance, inspiring discussions, encouragement, and kind support throughout my thesis endeavour. Thank you for introducing me to this project and for constantly following up my work and answering my questions. Thank you for having your doors always open to help me and many other graduate students. I am so proud to have been your student and I am certain that the advices you have conferred onto me will help me advance in my life and future career.

In the meanwhile, I am very grateful to Prof. Thomas Hillen, Prof. Yaozhong Hu, Prof. Alexander Melnikov, Prof. Ivan Mizera, and the external examiner Prof. Nizar Touzi for serving on my committee. Thanks for your invaluable time and patience for reading through this long thesis

No words can express how much I am thankful to Ashraf, my husband, for his endless support at time when I had a tough loads of work to do. He encouraged me every day to go one step closer to my dream. I would like to thank my great sons Hashem, Abdel Rahman, and Kenan who opened his eyes half way through my PhD journey. Thank you for being patients while I am busy and away from you.

A special thanks to my parents, you always supported and encouraged me from past to the present. Thank you for your sacrifices and for your unconditional love and to you I dedicate this work. Finally, I would like to thank my brothers and sisters. Thank you for your prayers, and love.

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# Chapter 1

# Introduction

Since the birth of differential equations in 1671, when Newton worked on the theory of "Flexions", differential equations has known many successful topics. In fact differential equations have been used to calculate the movement or flow of electricity, to formulate many fundamental laws of physics and chemistry, to model behaviours and evolutions of complex systems in biology,.., etcetera.

Stochastic differential equations, in short SDEs, first appeared in 1956, in Einstein's and Smoluchowski's works. SDEs contain a variable which represents random white noise calculated as derivative in some sense of Brownian motion. However, up to our knowledge, Brownian motion was born simultaneously with the modern finance and mathematical finance in 1900 in Bachelier's PhD thesis at the Sorbonne University.

The backward stochastic differential equations, BSDEs hereafter, were introduced in by Bismut (1973) for the linear case, and then extended by Pardoux and Peng (1990) for the general case. Reflected backward stochastic differential equations, in short RBSDEs, were introduced by El Karoui et al. (1997). Today they have become a very powerful tool applied to Mathematics, Physics, Chemistry, Electronics, Biology, Medical science, and almost all sciences.

## 1.1 Reflected backward stochastic differential equations

The BSDEs are roughly speaking inverse problems of the usual Stochastic Differential Equations (SDEs in short). Precisely for BSDEs, the terminal value $\xi$, called the final condition, is given as input and

$$Y_t = \xi + \int_{t\wedge T}^{T} f(s, Y_s, Z_s)ds - \int_{t\wedge T}^{T} Z_s dW_s, \quad t \geq 0. \tag{1.1.1}$$

Here $W$ is the Brownian motion, and $T$ is the terminal time that could be even a random variable, and $f$ is a functional called the *generator*. The RBSDEs are BSDEs as in (1.1.1) with additional condition that the process $Y$ should not get below a barrier process $S$ (also called obstacle). Thus, it is obvious that BSDEs are particular cases of RBDEs by assuming that the barrier process is constant $S = -\infty$. Linear BSDEs are those BSDEs with a linear generator in the variable $(Y, Z)$. These Linear BSDEs with $T$ being a fixed time (not random) were the first BSDEs that appeared in the literature a long time ago, mainly due to stochastic control and in the Black-Scholes formula for pricing options. For this case, we refer the reader to [13], that we consider -in our view– as the first paper about linear BSDEs. Then always in the case of constant $T$, the general case have been introduced by Pardoux and Peng, see [68]. Their aim was to give a probabilistic interpretation of a solution of second order quasi-linear partial differential equation, see [72] and [70]. Since then there has been an upsurging interested in BSDEs due to their tremendous role in mathematical finance, insurance, stochastic control, and partial differential equation,..., etcetera. For more details about these facts and other related topics, we refer the reader to [15, 21, 25, 32, 74, 75, 79, 80, 81, 84].

The RBDEs were first introduced by El Karoui, Kapoudjian, Pardoux, Peng and Quenez in [40]. Besides their role in mathematical finance, RBSDEs play important role providing a probabilistic formula for the viscosity solution of an obstacle problem for a parabolic PDE. In [40], the authors assumed that the underlying filtration is generated by a Brownian motion $W$ and the obstacle $S$ is a continuous process. In [19] , the authors extended these RBDEs to the case where the filtration is not generated by a Brownian motion and the obstacle $S$ may not be continuous.

## 1.2 Informational System

The word information comes from Latin word informatio, which means illumination, exposition, unfolding. In Finance and Economics, information means the acquirement of knowledges about costs, prices, inventory, supply and demand of products, which can be exploited by economic agents to reduce uncertainties in their environment, see Rose [76] further discussions.

From the economic standpoint view, the value of information is huge. It helps to take the better decision and mainly to reduce the risk coming from uncertainty. However, the acquisition of information is subject to a variety of circumstances. In [76], the author stated:

> "Information can be treated and exchanged as an economic commodity, which states that information has some private good features as well. Traders of information can benefit from sale and dissemination of information and therefore undertake the costly process of information acquisition and production".

One major problem is the amount of information available to an information seeker. When one of the parties in a transaction has better information than the other, we call this a market with asymmetric information (see [85, 86]). It was developed as a plausible explanation of market failures (an inefficient distribution of goods and services in a free market). In Akerlof (1970), the author first argued about information asymmetry by asserting that car buyers possess different information that car sellers, giving the seller an incentive to sell goods of poor quality without lowering the price to compensate for the inferiority. For more details about information and uncertainty, we refer the reader to [7, 8, 9, 10, 18, 78] and the references therein to cite a few.

It is unrealistic to assume that all agents know all the prevailing information (e.g. prices). More realistically, agents know some information, hold probabilistic expectation about some other information and totally unconcerned about most. Thus, we consider a market where there are two groups of agents. One group receives through time the public flow of information denoted by $\mathbb{F}$, while the second group (financial managers or insiders only) receives an additional information. Mathematically, this extra information is modelled by the knowledge of some random time $\tau$, when this time occurs. Thus the flow of information received by this agent over time is a bigger flow $\mathbb{G}$ ($\mathbb{F} \subset \mathbb{G}$). This random time can model different situations in various contexts. In fact, in credit risk theory this random time represents a default time, while in insurance market it represents the death time of insured. In (mathematical) finance this random time might represent a random horizon, or the occurrence time of an event that can impact the market somehow.

## 1.3 Our setting in RBSDEs

The BSDEs (or RBSDEs) with a random horizon are among the important and fundamental topics in Finance, Economics, and Mathematical Finance. The most fundamental essential works on BSDE (or RBSDE) with a random horizon started on the paper of Peng [73], where $\tau$ is an $\mathbb{F}$-stopping time. The author describes how the solution to the class of BSDEs with an unbounded random terminal time $\tau$, that is an $\mathbb{F}$-stopping time, is related to semilinear elliptic PDE. It is important to mention that in the case of constant horizon $T$, the solution to the BSDEs are connected to viscosity solutions to a system of semilinear parabolic PDEs, see [69] and the references therein for details. Afterwards, this family of RBSDEs have been extended in various directions in [15, 21, 32, 74, 79], and the references therein to cite a few. For the case second order BSDE under random terminal time, that is an $\mathbb{F}$-stopping time, we refer the reader to the very recent work [59].

The previous results in handling RBSDEs are formulated where the time is deterministic or a random variable that is observable by the public information $\mathbb{F}$. Our framework considers the case where the time is random might not be observable by the public flow of information, that is $\tau$ might not be an $\mathbb{F}$-stopping time with values in $[0,+\infty)$. This allows our results to be applicable to other economic and financial frameworks such as credit risk theory, life insurance where mortality and longevity risks are the main challenges, random horizon problem and so on.

Up to our knowledge, all the existing literature treating this class of RBDEs assumes very strong assumption(s) on $\tau$. The most frequent assumption among these, see [12, 54, 81] and the references therein, we cite the case where $W^\tau$ should remain a martingale under the enlarged filtration (this case is also known in the literature as the immersion assumption). In fact, the Burkholder-Davis-Gundy inequalities for martingales, that are really vital in BSDEs and RBSDEs, fail for martingales stopped at $\tau$ without the immersion assumpion. Without further assumption on $\tau$, we will address three main problems and the challenges induced by these. Our first main problem can be stated as follows.

$$\begin{cases} \text{What are the conditions (the weakest possible) on the data-triplet } (f, S, Y_{T\wedge\tau}) \\ \text{that guarantee the existence and uniqueness of the solution to our RBSDE?} \end{cases} \tag{1.3.1}$$

Our second main problem lies in mimicking the spirit of [3, 27]. In fact, we are interested in finding an RBSDE under $\mathbb{F}$, which will be the counterpart of our RBSDE in $\mathbb{G}$, and determining explicit relationship between their solutions and their data-triplets as well. This task is also highly motivated by its importance in credit risk theory, and we refer the reader to [12, 81] and the references therein to cite a few.

Our third main problem deals with controlling the norm of the solution of the RBSDE with the norm of its data-triplet. For classical RBSDEs (i.e. the case when $\tau$ is a stopping time or a fixed bounded horizon), this control in norm plays important role in studying the stability of RBSDEs. In contrast to the classical case, this problem has numerous challenges in our setting. Among these, we cite the description of the adequate spaces and the norms, for the solutions and/or for the data-triplet, that one should consider.

Inspired by the methods of [59] and [19], we deal with our RBSDE in two steps. In the first step, we consider the case of bounded horizon and we stop at $T \wedge \tau$ for some $T \in (0, +\infty)$ instead fo $\tau$. For this bounded random horizon case, thanks to an interesting probability measure $\widetilde{Q}$ discovered in [29] and intensively used in [30] for portfolio analysis, we fully answer in details the main problems aforementioned and beyond. The second step consists of letting $T$ to go infinity, and derive estimates and results under $P$ instead. This yields to additional serious challenges.

As an application to our RBSDE, we give a preliminary version of the study undertaken intensively in [5], where we show that the RBSDEs consider in this thesis are fully motivated besides they generalize the literature on RBSDEs to a much complex setting.

## 1.4 Summary of the Thesis

This thesis has seven chapters including the current one. The organization of the seven chapters is further detailed, below.

Chapter 2 recalls some stochastic elements and theorems that will be used throughout the thesis. In Section 2.3 we address some vital results on enlargement of filtration $\mathbb{F}$ with $\tau$ and on martingales for the enlarged filtration. In Section 2.4 we define the mathematical model and its preliminaries such as the norms used for the RBSDEs.

Chapter 3 addresses the optimal stopping problem and the Snell envelop under stopping with $\tau$. This is vital as we know the Snell envelope, which is intimately related to linear RBSDEs.

Chapter 4 is devoted to linear RBSDEs depending whether we stop the RBSDE at $\tau \wedge T$ for some fixed planning horizon $T \in (0, +\infty)$, or we stop at $\tau$.

Chapter 5 deals with the general RBSDE, where the generator functional is general but Lipschitz. Here, again, we distinguish the cases depending whether we stop at $\tau \wedge T$ or $\tau$.

Chapter 6 extends Chapters 4 and 5 to the case where the filtration $\mathbb{F}$ is generated by a Brownian motion $W$ and a Poisson process $N$ with intensity $\lambda > 0$, and where the terminal value of the RBSDE $\xi$ is $\mathcal{G}_{T\wedge\tau}$-measurable instead of being $\mathcal{F}_{T\wedge\tau}$-measurable. We consider the case of a fixed and finite deterministic horizon $T \in (0, +\infty)$.

Chapter 7 focuses on giving some applications of RBSDEs in exponential hedging. In the first section we discuss how minimal entropy martingale measures are affected by $\tau$. The second and third sections treat the exponential hedging problem using (R) BSDE for both the primal and dual problems.

# Chapter 2

# Notations and Preliminaries

In this chapter, we introduce some mathematical tools, concepts, and properties on stochastic processes. This chapter contains four sections. The first section recalls stochastic elements and properties that we used in this thesis. In the second and third sections we recall an important theorem about local martingale representation and some other related results in the enlarged filtration. In the last section we define the spaces and norms that we used in our studies. Throughout this thesis, let $(\Omega, \mathcal{H}, \mathbb{H} := (\mathcal{H}_t)_{t\geq 0}, P)$ be a complete

filtered probability space. Where $\mathbb{H} = \{\mathcal{H}_t, 0 \leq t \leq T\}$ is a completed and right-continuous filtration that forms the flow of information.

## 2.1 Notations, Definitions and Properties

In this section, we review some notations, definitions and properties, most of them can be founded in Jacod and Shiryaev [50].

### 2.1.1 Stochastic Basis

A stochastic process is a family of random variables $(X(t))_{t\in[0,T]}$ index by time. The time parameter can be either discrete or continuous, but we will only consider the continuous case.

**Definition 2.1.1** ([50]). *A process $X$ is called RC ( resp. LC, resp. RCLL) if all its paths are right-continuous ( resp. are left-continuous, resp. are right-continuous and admit left-hand limits).*

**Definition 2.1.2** ([50]). *A stopping time is a mapping $T : \Omega \to [0, \infty]$ such that $\{T \le t\} \in \mathcal{H}_t$ for all $t \in [0, \infty)$.*

Let us introduce stochastic intervals as the following: let $S$ and $T$ be two stopping times, then the four kinds of stochastic intervals are the following four random sets:

$$\begin{cases} [\![S,T]\!] = \{(\omega,t) : t \in \mathbb{R}_+, S(\omega) \le t \le T(\omega)\} \\ [\![S,T[\![ = \{(\omega,t) : t \in \mathbb{R}_+, S(\omega) \le t < T(\omega)\} \\ ]\!]S,T]\!] = \{(\omega,t) : t \in \mathbb{R}_+, S(\omega) < t \le T(\omega)\} \\ ]\!]S,T[\![ = \{(\omega,t) : t \in \mathbb{R}_+, S(\omega) < t < T(\omega)\} \end{cases}$$

**Definition 2.1.3** ([83]). *Let $\Omega$ be a nonempty set, and let $\mathcal{B}$ be a collection of subsets of $\Omega$. We say that $\mathcal{B}$ is a $\sigma$-algebra (or $\sigma$-field) provided that:*

(i) *the empty set $\phi$ belongs to $\mathcal{B}$,*

(ii) *whenever a set $A$ belongs to $\mathcal{B}$ , its complement $A^c$ also belongs to $\mathcal{B}$, and*

(iii) *whenever a sequence $(A_n)_{n\geq 1}$ belongs to $\mathcal{B}$, their union $\bigcup_{n=1}^{\infty} A_n$ also belongs to $\mathcal{B}$.*

**Definition 2.1.4** ([50]). (a) *A process $X$ is adapted to the filtration $\mathbb{H}$ if $X_t$ is $\mathcal{H}_t$- measurable for every $t \in \mathbb{R}_+$; A process $X$ is always adapted to its history or natural filtration which is given by*

$$\mathcal{F}_t^X = \sigma(X(s), C \big| s \in [0,t], C \in \mathcal{N})$$

*where $\mathcal{N}$ is the set of all null sets of the state space of the process.*

(b) *The optional $\sigma$-field is the $\sigma$-field $\mathcal{O}(\mathbb{H})$ on $(\Omega, \mathbb{R}_+)$ that is genarated by all RCLL $\mathbb{H}$-adapted processes. Furthermore, a process $X$ that is $\mathcal{O}(\mathbb{H})$-measurable is called optional and it will be denoted by $X \in \mathcal{O}(\mathbb{H})$.*

(c) *The predictable $\sigma$-field is the $\sigma$-field $\mathcal{P}(\mathbb{H})$ on $(\Omega, \mathbb{R}_+)$ that is generated by all LC (left continuous) $\mathbb{H}$-adapted processes. Furthermore, a process $X$ that is $\mathcal{P}(\mathbb{H})$-measurable is called predictable and it will be denoted by $X \in \mathcal{P}(\mathbb{H})$.*

**Proposition 2.1.5** ([50]). *Every process $X$ that is RC and adapted is optional.*

Any predictable process is optional process (i.e. $\mathcal{P}(\mathbb{H}) \subset \mathcal{O}(\mathbb{H})$ ).

**Proposition 2.1.6** ([50]). (a) *If $X$ is RCLL and adapted process, then the two processes $X_-$ and $\Delta X$ are optional.*

(b) *If $X$ is RCLL and adapted process, then $X_-$ is a predictable process; if* *moreover $X$ is predictable, then $\Delta X$ is predictable.*

**Lemma 2.1.7** ([50]). *Let $T$ be a random time (i.e. a nonnegative random variable). $T$ is an $\mathbb{H}$-stopping time if and only if $[\![0, T[\![$, or equivalently $[\![T, \infty[\![$, is an $\mathbb{H}$-optional set.*

**Proposition 2.1.8** ([50]). (a) *If $S$ and $T$ are two stopping times and if $Y$ is $\mathcal{H}_S$-measurable random variable, the four processes $Y1_{[\![S,T]\!]}$, $Y1_{[\![S,T[\![}$, $Y1_{]\!]S,T]\!]}$, $Y1_{]\!]S,T[\![}$ are optional.*

(b) *If $S$ and $T$ are two stopping times and if $Y$ is $\mathcal{H}_S$-measurable random variable, the process $Y1_{]\!]S,T]\!]}$ is predictable.*

**Definition 2.1.9** ([50]). *An $\mathbb{H}$-predictable stopping time is an $\mathbb{H}$-stopping time such that the stochastic interval $[\![0, T[\![$ is $\mathbb{H}$-predictable.*

### 2.1.2 Martingales and Semimartingales

**Definition 2.1.10** ([50]). *A martingale (resp. submartinagle, resp. supermartingale) is an adapted process $X$ on the basis $(\Omega, \mathcal{H}, \mathbb{H}, Q)$ whose $Q$-almost all paths are RCLL, such that every $X_t$ is integrable (i.e. $E|X_t| < +\infty$) and for all $s, t$ such that $s \leq t$ we have:*

$$X_s = E(X_t \big| \mathcal{H}_s) \qquad (\textit{resp. } X_s \leq E(X_t \big| \mathcal{H}_s), \textit{ resp. } X_s \geq E(X_t \big| \mathcal{H}_s)).$$

**Definition 2.1.11** ([50]). *If $X$ is a stochastic process and $T$ is a random time, then $X^T$ is called stopped process and satisfies*

$$X_s^T := X_{s \wedge T}, \qquad s \geq 0.$$

**Definition 2.1.12** ([50]). *An adapted process $X$ on the basis $(\Omega, \mathcal{H}, \mathbb{H}, Q)$ whose $Q$-almost all paths are RCLL is called local martingale process if there exists an increasing sequence of $\mathbb{H}$-stopping times $(T_n)_{n\geq 1} \uparrow +\infty$, such that each stopping process $X^{T_n}$ is an $\mathbb{H}$ martingale.*

**Definition 2.1.13** ([50]). (a) *two local martingales $M$ and $N$ are called orthogonal if their product $MN$ is a local martingale.*
(b) *A local martingale $X$ is called a purely discontinuous $\mathbb{H}$-local martingale (or a pure jump $\mathbb{H}$-local martingale) if $X_0 = 0$ and if it is orthogonal to all continuous local martingales.*

**Theorem 2.1.14** ([50]). *Any local martingale $M$ admits a unique ( up to indistinguishability) decomposition*

$$M = M_0 + M^c + M^d$$

*where $M_0^c = M_0^d = 0$, $M^c$ is a continuous local martingale, and $M^d$ is a purely discontinuous local martingale.*

**Definition 2.1.15** ([50]). (a) *We denote by $\mathcal{M}(\mathbb{H}, Q)$ ( resp. $\mathcal{M}_{loc}(\mathbb{H}, Q)$) the set of all $\mathbb{H}$-martingale ( resp. $\mathbb{H}$-local martingale).*
(b) *If $\mathcal{C}(\mathbb{H})$ is the set of processes that are adapted to $\mathbb{H}$, then $\mathcal{C}_{loc}(\mathbb{H})$ is the set of processes, $X$, for which there exists a sequence of $\mathbb{H}$-stopping times, $(T_n)_{n\geq 1}$, that increases to infinity and $X^{T_n}$ belongs to $\mathcal{C}(\mathbb{H})$, for each $n \geq 1$.*
(c) *We denote by $\mathscr{V}^+$ ( resp. $\mathscr{V}$) the set of all real-valued processes $A$ that are RCLL, adapted, with $A_0 = 0$, and whose each path $t \to A_t(\omega)$ is non-decreasing ( resp. has a finite variation over each finite interval $[0, t]$).*

(d) *We denote by* $\mathcal{A}^+(\mathbb{H}, Q)$ *( resp.* $\mathcal{A}(\mathbb{H}, Q)$*) by the set of all* $A \in \mathscr{V}^+$ *( resp.* $A \in \mathscr{V}$ *) that are integrable:* $E[A_\infty] < \infty$ *( resp. that have integrable variation:* $E[\mathrm{Var}(A)_\infty] < \infty$ *).*

**Lemma 2.1.16** ([50])**.** *Any local martingale that belong to* $\mathscr{V}$ *also belongs to* $\mathcal{A}_{loc}(\mathbb{H}, Q)$*.*

**Definition 2.1.17** ([50])**.** *An* $\mathbb{H}$*-semimartinagle is a process* $X$ *of the form* $X = X_0 + M + A$ *where* $X_0$ *is finite-valued and* $\mathcal{H}_0$*-measurable, where* $M$ *is an* $\mathbb{H}$*-local martingale and* $A \in \mathscr{V}$*. If* $A$ *is predictable, we call* $X$ *a special semimartingale and the decomposition* $X = X_0 + M + A$ *is called the canonical decomposition of* $X$*.*

**Definition 2.1.18** ([50])**.** *We denote by* $L(X, \mathbb{H})$ *the set of* $\mathbb{H}$*-predictable process that is integrable with respect to* $X$ *in the sense of semimartingale.*

**Definition 2.1.19** ([37])**.** *Let* $Y$ *be a uniformly integrable martingale and let* $p \in [1, \infty[$*. We adopt the convention* $Y_{0-} = 0$*. we say that* $Y$ *belongs to* $BMO_p$ *if there exists a constant* $C$ *such that*

$$E[|Y_\infty - Y_{T-}|^p \big| \mathcal{H}_T] \leq C^p \ \text{ a.s. for every stopping time } T.$$

*The smallest constant with this property (or* $\infty$ *if this does not exist) is denoted by* $\|Y\|_{BMO_p}$*.*

### 2.1.3 Predictable and optional (dual) projections

**Theorem 2.1.20** ([50])**.** *Let* $X$ *be a positive or bounded* $\mathcal{B} \otimes \mathcal{H}$*-measurable process. there exists an* $\mathbb{H}$*-optional process* ${}^{o,\mathbb{H}}(X)$ *(called* $\mathbb{H}$*-optional projection* *of $X$ ) and an $\mathbb{H}$-predictable process ${}^{p,\mathbb{H}}(X)$ (called $\mathbb{H}$-predictable projection of $X$ ) such that*

$$E[X_T I_{\{T<\infty\}} \big| \mathcal{H}_T] = {}^{o,\mathbb{H}}(X)_T I_{\{T<\infty\}} \quad a.s. \quad \textit{for any } \mathbb{H}\textit{-stopping time } T,$$

$$E[X_T I_{\{T<\infty\}} \big| \mathcal{H}_{T-}] = {}^{p,\mathbb{H}}(X)_T I_{\{T<\infty\}} \quad a.s. \quad \textit{for any } \mathbb{H}\textit{-predictable time } T.$$

*The two processes ${}^{o,\mathbb{H}}(X)$ and ${}^{p,\mathbb{H}}(X)$ are unique up to evanescent set. A random set $A$ is called evanescent if the set $\{\omega : \exists t \in \mathbb{R}_+ \quad \text{with } (\omega, t) \in A\}$ is $Q$-null .*

**Remark 2.1.21.** *we denote by $X \bullet Y$ by the stochastic integral $\int_0^{\cdot} X_s dY_s$.*

**Theorem 2.1.22** ([50])**.** *Let $A \in \mathcal{A}^+_{loc}(\mathbb{H}, Q)$. There is a process, called the compensator of $A$ and denoted by $A^{p,\mathbb{H}}$, which is unique up to an evanescent set, and which is characterized by being a predictable process in $\mathcal{A}^+_{loc}(\mathbb{H}, Q)$ meeting any one of the following three equivalent statements:*

(i) $A - A^{p,\mathbb{H}}$ *is a local martingale;*

(ii) $E(A^{p,\mathbb{H}}_T) = E(A_T)$ *for all stopping time $T$;*

(iii) $E[(H \bullet A^{p,\mathbb{H}})_\infty] = E[(H \bullet A)_\infty]$ *for all nonnegative predictable processes $H$.*

Sometimes, $A^{p,\mathbb{H}}$ is called "predictable compensator" of $A$, or also "dual predictable projection" of $A$.

We recall an important theorem form martingale inequalities that goes back to Dellacherie and Meyer, see [37, Theorem 99, Chapter VI].

**Theorem 2.1.23.** *Consider a complete filtered probability space given by $(\Omega, \mathcal{F}, \mathbb{H} = (\mathcal{H}_t)_{0 \le t \le T}, P)$. Let $A$ be predictable (optional) increasing process* *whose potential (left potential) $Z$ is bounded above by a martingale $M_t = E[M_\infty|\mathcal{H}_t]$. Then*

$$\|A_\infty\|_\Phi \leq p_\Phi \|M_\infty\|_\Phi, \tag{2.1.1}$$

*where $p_\Phi$ is the constant associated with $\Phi$ and $\Phi$ is increasing convex function defined as the following;*

$$\Phi(t) := \int_0^t \phi(s)ds, \quad p_\Phi := \sup_t \frac{t\phi(t)}{\Phi(t)}. \tag{2.1.2}$$

*for some right continuous increasing function $\phi$ which is positive on $\mathbb{R}^+$.*

Also, we recall the following remark, see [37, Remark 100, Chapter VI].

**Remark 2.1.24.** *Let $B$ be an increasing right continuous locally intergrable process. We denote by $A$ the dual optional or predictable projection of $B$.*

*(i) If $F$ is convex and of moderate growth with exponent $p$, w have*

$$E(F(A_T)) \leq p^p E(F(B_T)). \tag{2.1.3}$$

*(ii) If $F$ is concave, we have*

$$E(F(B_T)) \leq 2E(F(A_T)). \tag{2.1.4}$$

### 2.1.4 Ito's formula and Dolèans-Dade Exponential

**Definition 2.1.25** ([50]). *The quadratic co-variation of the two semimartingales $X$ and $Y$ (the quadratic variation of $X$, when $Y = X$) is the following process:*

$$[X,Y] = XY - X_0Y_0 - X_- \bullet Y - Y_- \bullet X$$

*(it is defined uniquely, up to an evanscent set).*

**Definition 2.1.26** ([50]). *$\langle X, Y\rangle$ denotes to the compensator of $[X,Y]$.*

**Definition 2.1.27** ([37]). *Let $X$ and $Y$ be two local martingales. If the product $XY$ is a special semimartingale, we denote by $\langle X, Y\rangle$ the unique predictable process of finite variation such that $XY - \langle X, Y\rangle$ is a local martingale which is zero at $0$.*

**Theorem 2.1.28** ([50]). *If $X, Y$ are semimartingale, and if $X^c, Y^c$ denote their continuous martingale parts, then*

$$[X,Y]_t = \langle X^c, Y^c\rangle_t + \sum_{s\le t} \Delta X_s \Delta Y_s.$$

**Theorem 2.1.29** (Ito's formula [50]). *Let $X = (X^1, ..., X^d)$ be a $d$-dimensional semimartingale, and $f$ a class $C^2$ function on $\mathbb{R}^d$. Then $f(X)$ is a semimartingale and we have:*

$$f(X_t) = \quad f(X_0) + \sum_{i\le d} D_i f(X_-) \bullet X^i + \frac{1}{2}\sum_{i,j\le d} D_{ij} f(X_-) \bullet \langle X^{i,c}, X^{j,c}\rangle$$

$$+\sum_{s\le t}\left[f(X_s)-f(X_{s-})-\sum_{i\le d}D_if(X_{s-})\Delta X_s^i\right].$$

**Theorem 2.1.30.** *[50] For any $\mathbb{H}$-semimartinagle, $L$, we denote by the semimartingale $\mathcal{E}(L)$ the Doléans-Dade (stochastic) exponential, it is the unique solution to the stochastic differential equation*

$$dX = X_- dL, \quad X_0 = 1,$$

*and is explicitly given by*

$$\mathcal{E}_t(L) = \exp(L_t - L_0 - \frac{1}{2}\langle L^c\rangle_t)\prod_{0<s\le t}(1+\Delta L_s)e^{-\Delta L_s}. \qquad (2.1.5)$$

*Furthermore,*

a) *If $L$ has a finite variation, then $\mathcal{E}(L)$ has a finite variation.*

b) *If $L$ is a local martingale, then $\mathcal{E}(L)$ is a local martingale.*

Here we recall Yor's lemma about the product of two stochastic exponentials.

**Lemma 2.1.31.** *[49] If $X$ and $Y$ are two semimartingales, then*

$$\mathcal{E}(X)\mathcal{E}(Y) = \mathcal{E}(X + Y + [X, Y]).$$

## 2.2 Some useful results from stochastic

**Definition 2.2.1.** *[52] A Brownian motion (or a standard one- dimensional Wiener process) is a continuous, adapted process $(W_t)_{t\ge 0}$, defined on some* *probability space* $(\Omega, \mathcal{F}, P)$, *with the properties* $W_0 = 0$ *a.s. and for* $0 \leq s < t$, *the increment* $W_t - W_s$ *is independent of* $\mathcal{F}_s$ *and is normally distributed with mean zero and variance* $t - s$.

The Poisson process is a discontinuous process that counts the number of random occurrences of some events which happen in a certain time interval. The inter-arrival time between two events occurring is exponentially distributed.

**Definition 2.2.2.** *Let* $(\tau_i)_{i\geq 1}$ *be a sequence of independent exponential random variables with parameter* $\lambda > 0$, *and* $T_n = \sum_{i=1}^{n} \tau_i$. *The process* $(N(t))_{t\geq 0}$ *denoted by* $N(t) = \sum_{n\geq 1} 1_{\{t\geq T_n\}}$ *is called a Poisson process with intensity* $\lambda$.

A Poisson process has piecewise constant sample paths, and it increases by jumps of size 1. Its increments $N(t) - N(s)$ are independent and stationary and have a Poisson distribution with mean $(t-s)\lambda$ for all $t > s \geq 0$.

**Definition 2.2.3.** *A utility function* $U : (0, \infty) \to \mathbb{R}$ *is a strictly increasing and strictly concave* $C^1$ *function that satisfies the Inada conditions*

$$U^{'}(0+) = \lim_{x \downarrow 0} U^{'}(x) = \infty, \textit{ and } \; U^{'}(\infty) = \lim_{x\to\infty} U^{'}(x) = 0.$$

Some examples of utility functions are the so called exponential, power and log utility functions

$$U(x) = 1 - e^{-x}, \quad U(x) = \frac{x^p}{p} \quad \text{and} \quad U(x) = log(x) \quad \text{ for } \quad p \in (-\infty, 1)\backslash\{0\}.$$

**Definition 2.2.4.** *Let* $U$ *be a utility function. The convex conjugate function* *of $U$ is denoted as the function*

$$U^*(y) := sup_{x>0}\{U(x) - xy\}, \quad y > 0.$$

The convex conjugate function of a utility function $U$ is the Legendre-Fenchel transformation of the function $-U(-x)$. Let the inverse of $U^{'}$ be denoted by $I$ so that

$$x = I(U^{'}(x)) = U^{'}(I(x)), \quad \text{for} \quad x > 0.$$

Since $U$ is strictly increasing and strictly concave, $U^{'}$ must be strictly decreasing, and therefore $I$ is also strictly decreasing.
In the following we recall Fatou's Lemma

**Lemma 2.2.5.** *[82] Let $\eta$, $\xi_n$, $n \geq 1$ be random variables.*
(a) *If $\xi_n \geq \eta$ for all $n \geq 1$ and $E\eta > -\infty$, then*

$$E\ \underline{\lim}\ \xi_n \leq \underline{\lim}\ E\ \xi_n.$$

(b) *If $\xi_n \leq \eta$ for all $n \geq 1$ and $E\eta < \infty$, then*

$$\overline{\lim}\ E\ \xi_n \leq E\ \overline{\lim}\ \xi_n.$$

(c) *If $|\xi_n| \leq \eta$ for all $n \geq 1$ and $E\eta < \infty$, then*

$$E\ \underline{\lim}\ \xi_n \leq \underline{\lim}\ E\ \xi_n \leq \overline{\lim}\ E\ \xi_n \leq E\ \overline{\lim}\ \xi_n.$$

In the next theorem we recall a very useful inequality, which is known as

Burkhölder-Davis-Gundy inequality (BDG inequality)

**Theorem 2.2.6.** *[37] Let $\phi$ be a convex function such that $\phi(0) = 0$. For any local martingale $X$,*

$$\frac{1}{4p_\phi}\|\sup_{t\geq 0}|X_t|\|_\phi \leq \|[X,X]_\infty^{\frac{1}{2}}\|_\phi \leq 6p_\phi\|\sup_{t\geq 0}|X_t|\|_\phi.$$

*Here, the constant $p_\Phi$ is defined by*

$$p_\Phi := \sup_t \frac{t\phi(t)}{\Phi(t)} \in [1,+\infty) \quad \textit{where} \quad \Phi(t) := \int_0^t \phi(s)ds. \tag{2.2.1}$$

Also, we give another important inequality, Doob's inequality, in the following theorem

**Theorem 2.2.7.** *[37] Let $X$ be a positive submartingale. Then for all $p > 1$, with $q$ denoting the exponent conjugate to $p$ (i.e. $q := p/(p-1)$)*

$$\|\sup_{t\geq 0} X_t\|_p \leq q \sup_{t\geq 0}\|X_t\|_p.$$

*(The result applies in particular to $|X|$ for every martingale $X$.)*

The following is the Girsanov's Theorem

**Theorem 2.2.8.** *[50] Assume that $P' \overset{loc}{\ll} P$ and let $Z$ be the density process. Let $M$ be a $P$-local martingale such that $M_0 = 0$ and that the $P$-quadratic covariation $[M,Z]$ has $P$-locally integrable variation, and denote by $\langle M,Z\rangle$ its*

*$P$-compensator. Then the process*

$$M^{'} := M - \frac{1}{Z_{-}} \bullet \langle M, Z \rangle$$

*is $P^{'}$-local martingale.*

## 2.3 The random horizon and the progressive enlargement of $\mathbb{F}$

In addition to this initial model $(\Omega, \mathcal{F}, \mathbb{F}, P)$, we consider an arbitrary random time, $\tau$, that might not be an $\mathbb{F}$-stopping time. This random time is parametrized though $\mathbb{F}$ by the pair $(G, \widetilde{G})$, called survival probabilities or Azéma supermartingales, and is given by

$$G_t :=^{o,\mathbb{F}} (I_{[\![0,\tau[\![})_t = P(\tau > t|\mathcal{F}_t) \ \text{ and } \ \widetilde{G}_t :=^{o,\mathbb{F}} (I_{[\![0,\tau]\!]})_t = P(\tau \geq t|\mathcal{F}_t). \tag{2.3.1}$$

Furthermore, the following process

$$m := G + D^{o,\mathbb{F}}, \tag{2.3.2}$$

is a BMO $\mathbb{F}$-martingale and play important role in the analysis of enlargement of filtration. The flow of information that incorporates both $\tau$ and $\mathbb{F}$ is defined using the pair $(D, \mathbb{G})$ given by

$$D := I_{[\![\tau,+\infty[\![}, \ \mathbb{G} := (\mathcal{G}_t)_{t\geq 0}, \ \mathcal{G}_t := \mathcal{G}^0_{t+} \text{ with } \mathcal{G}^0_t := \mathcal{F}_t \vee \sigma\left(D_s, \ s \leq t\right). \tag{2.3.3}$$

Thanks to [2, Theorem 3] and [26, Theorem 2.3 and Theorem 2.11], we recall

**Theorem 2.3.1.** *The following assertions hold.*

(a) *For any $M \in \mathcal{M}_{loc}(\mathbb{F})$, the process*

$$\mathcal{T}(M) := M^\tau - \widetilde{G}^{-1} I_{[\![0,\tau[\![} \bullet [M, m] + I_{[\![0,\tau[\![} \bullet \left( \sum \Delta M I_{\{\widetilde{G}=0<G_-\}} \right)^{p,\mathbb{F}} \tag{2.3.4}$$

*is a $\mathbb{G}$-local martingale.*

(b) *The process*

$$N^{\mathbb{G}} := D - \widetilde{G}^{-1} I_{[\![0,\tau[\![} \bullet D^{o,\mathbb{F}} \tag{2.3.5}$$

*is a $\mathbb{G}$-martingale with integrable variation. Moreover, $H \bullet N^{\mathbb{G}}$ is a $\mathbb{G}$-local martingale with locally integrable variation for any $H$ belonging to*

$$\mathcal{I}^o_{loc}(N^{\mathbb{G}}, \mathbb{G}) := \left\{ K \in \mathcal{O}(\mathbb{F}) \quad \Big| \quad |K| G \widetilde{G}^{-1} I_{\{\widetilde{G}>0\}} \bullet D \in \mathcal{A}_{loc}(\mathbb{G}) \right\}. \tag{2.3.6}$$

For any $q \in [1, +\infty)$ and a $\sigma$-algebra $\mathcal{H}$ on $\Omega \times [0, +\infty)$, we define

$$L^q(\mathcal{H}, P \otimes dD) := \left\{ X \ \mathcal{H}\text{-measurable}: \quad \mathbb{E}[|X_\tau|^q I_{\{\tau<+\infty\}}] < +\infty \right\}. \tag{2.3.7}$$

The following is very useful throughout the thesis, and its proof can be found in [29, Lemma 2.4-(b)].

**Lemma 2.3.2.** *If $G > 0$, then*

$$G = G_0 \widetilde{\mathcal{E}} \mathcal{E}(G_-^{-1} \bullet m), \quad \textit{where} \quad \widetilde{\mathcal{E}} := \mathcal{E}\left( -\frac{1}{\widetilde{G}} \bullet D^{o,\mathbb{F}} \right) =: 1 - V^{\mathbb{F}}. \tag{2.3.8}$$

**Lemma 2.3.3.** *For any nonnegative or integrable process $X$, we always have*

$$E\left[X_t|\mathcal{G}_t\right]I_{\{t\ <\tau\}} = E\left[X_t I_{\{t\ <\tau\}}|\mathcal{F}_t\right]G_t^{-1}I_{\{t\ <\tau\}}. \tag{2.3.9}$$

This lemma, for the case when $X$ is integrable can be found in [34, Chapter XX, Section 37]. For the case of $X$ bering nonnegative can be also deduce from the integrable case using the class monotone theorem. Now, we recall [29, Proposition 4.3] that will be useful throughout the paper.

**Proposition 2.3.4.** *Suppose that $G > 0$ and consider the process*

$$\widetilde{Z} := 1/\mathcal{E}(G_-^{-1} \bullet m). \tag{2.3.10}$$

*Then the following assertions hold.*

(a) *The process $\widetilde{Z}^\tau$ is a $\mathbb{G}$-martingale, and for any $T \in (0, +\infty)$, $\widetilde{Q}_T$ given by*

$$\frac{d\widetilde{Q}_T}{dP} := \widetilde{Z}_{T\wedge\tau}. \tag{2.3.11}$$

*is well defined probability measure on $\mathcal{G}_{\tau\wedge T}$.*

(b) *For any $M \in \mathcal{M}_{loc}(\mathbb{F})$, we have $M^{T\wedge\tau} \in \mathcal{M}_{loc}(\mathbb{G}, \widetilde{Q})$. In particular $W^{T\wedge\tau}$ is a Brownian motion for $(\widetilde{Q}, \mathbb{G})$, for any $T \in (0, +\infty)$.*

**Remark 2.3.5.** *In general, the $\mathbb{G}$-martingale $\widetilde{Z}^\tau$ might not be uniformly integrable, and hence in general $\widetilde{Q}$ might not be extended to $(0, +\infty]$. For these fact, we refer the reader to [29, Proposition 4.3] for details, where conditions for $\widetilde{Z}^\tau$ being uniformly integrable are fully singled out when $G > 0$.*

Now, we recall an important representation theorem of Choulli et al. (2017)

to the case of $\mathbb{G}$-local martingales when the process $G$ never vanishes.

**Theorem 2.3.6.** *Suppose that $G > 0$. Then for any $\mathbb{G}$-local martingale $M^{\mathbb{G}}$, there exists a unique triplet $(M^{\mathbb{F}}, \varphi^{(o)}, \varphi^{(pr)})$ that belongs to $\mathcal{M}_{0,loc}(\mathbb{F}) \times \mathcal{I}^o_{loc}\left(N^{\mathbb{G}}, \mathbb{G}\right) \times L^1_{loc}\left(\widetilde{\Omega}, \mathrm{Prog}(\mathbb{F}), P \otimes D\right)$ and satisfies*

$$\mathbb{E}\left[\varphi^{(pr)}_{\tau} \;\middle|\; \mathcal{F}_{\tau}\right] I_{\{\tau<+\infty\}} = 0, \quad P-a.s., \tag{2.3.12}$$

*and*

$$\left(M^{\mathbb{G}}\right)^{\tau} = M^{\mathbb{G}}_0 + G_{-}^{-2} I_{]\!]0,\tau]\!]} \bullet \mathcal{T}(M^{\mathbb{F}}) + \varphi^{(o)} \bullet N^{\mathbb{G}} + \varphi^{(pr)} \bullet D. \tag{2.3.13}$$

## 2.4 RBSDEs: Definition, spaces and norms

Throughout this section we suppose given a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{H} = (\mathcal{H}_t)_{t\geq 0}, Q)$, where $\mathbb{H} \supseteq \mathbb{F}$ and $Q$ is any probability measure absolutely continuous with respect to $P$. The following definition of RBSDEs is borrowed from [20, Definition 2.1].

**Definition 2.4.1.** *Let $\sigma$ be an $\mathbb{H}$-stopping time, and $(f^{\mathbb{H}}, S^{\mathbb{H}}, \xi^{\mathbb{H}})$ be a triplet such that $f^{\mathbb{H}}$ is Prog$(\mathbb{H}) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$-measurable functional, $S^{\mathbb{H}}$ is a RCLL and $\mathbb{H}$-adapted process, and $\xi^{\mathbb{H}}$ is an $\mathcal{H}_{\sigma}$-measurable random variable.*
(a) *Then an $(\mathbb{H}, Q, \sigma)$-solution to the following RBSDE*

$$\begin{cases} dY_t = -f^{\mathbb{H}}(t, Y_t, Z_t) I_{\{t\leq\sigma\}} dt + Z_t dW_{t\wedge\sigma} - dM_t - dK_t, \quad Y_{\sigma} = \xi^{\mathbb{H}}, \\ Y \geq S^{\mathbb{H}} \text{ on } [\![0, \sigma[\![, \quad \displaystyle\int_0^{\sigma} (Y_{u-} - S^{\mathbb{H}}_{u-}) dK_u = 0 \quad Q\text{-a.s.}. \end{cases} \tag{2.4.1}$$

*is any quadruplet* $(Y^{\mathbb{H}}, Z^{\mathbb{H}}, M^{\mathbb{H}}, K^{\mathbb{H}})$ *satisfying (2.4.1) such that both* $M^{\mathbb{H}}$ *and* $[W^{\sigma}, M^{\mathbb{H}}]$ *belong to* $\mathcal{M}_{0,loc}(Q, \mathbb{H})$*,* $K^{\mathbb{H}}$ *is a RCLL nondecreasing and* $\mathbb{H}$*-predictable,* $M^{\mathbb{H}} = (M^{\mathbb{H}})^{\sigma}$*,* $K^{\mathbb{H}} = (K^{\mathbb{H}})^{\sigma}$*, and*

$$\int_0^{\sigma} \left((Z_t^{\mathbb{H}})^2 + |f^{\mathbb{H}}(t, Y_t^{\mathbb{H}}, Z_t^{\mathbb{H}})|\right) dt < +\infty \quad Q\text{-}a.s. \tag{2.4.2}$$

(b) *We call a class-(D)-*$(\mathbb{H}, Q, \sigma)$*-solution any quadruplet* $(Y^{\mathbb{H}}, Z^{\mathbb{H}}, M^{\mathbb{H}}, K^{\mathbb{H}})$ *which is an* $(\mathbb{H}, Q, \sigma)$*-solution such that* $\left\{Y_{\sigma\wedge\theta}^{\mathbb{H}} : \ \theta \text{ is an } \mathbb{H}\text{-stopping time}\right\}$ *is* $Q$*-uniformly integrable and*

$$E^{Q}\left[\int_0^{\sigma} |f^{\mathbb{H}}(t, Y_t^{\mathbb{H}}, Z_t^{\mathbb{H}})|dt\right] < +\infty.$$

*When* $Q = P$ *we will simply call the quadruplet an* $\mathbb{H}$*-solution, while the filtration is also omitted when there no risk of confusion.*

In this thesis, we are interested in solutions that are integrable somehow. To this end, we recall the following spaces and norms that will be used throughout the paper. We denote by $\mathbb{L}^p(Q)$ is the space of $\mathcal{F}$-measurable random variables $\xi'$, such that

$$\| \ \xi' \ \|_{\mathbb{L}^p(Q)}^p := E^Q\left[|\xi'|^p\right] < \infty.$$

$\mathbb{D}_{\sigma}(Q, p)$ is the space of RCLL and $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}^+)$-measurable processes, $Y$, such that $Y = Y^{\sigma}$ and

$$\|Y\|_{\mathbb{D}_{\sigma}(Q,p)}^p := E^Q\left[\sup_{0\leq t\leq\sigma} |Y_t|^p\right] < \infty.$$

Here $\mathcal{B}(\mathbb{R}^+)$ is the Borel $\sigma$-field of $\mathbb{R}^+$. $\mathbb{S}_{\sigma}(Q, p)$ is the space of Prog$(\mathbb{H})$-

measurable processes $Z$ such that $Z = Z^\sigma$ and

$$\|Z\|^p_{\mathbb{S}_\sigma(Q,p)} := E^Q\left[\left(\int_0^\sigma |Z_t|^2 dt\right)^{p/2}\right] < \infty.$$

For any $M \in \mathcal{M}_{loc}(Q, \mathbb{H})$, we define its $p$-norm by

$$\|M\|^p_{\mathcal{M}^p(Q)} := E^Q\left[[M,M]^{p/2}_\infty\right] < \infty,$$

and the $p$-norm of any $K \in \mathcal{A}_{loc}(Q, \mathbb{H})$ at any random time $\sigma$ is given by

$$\|K\|^p_{\mathcal{A}_\sigma(Q,p)} := E^Q\left[(\mathrm{Var}_\sigma(K))^p\right].$$

Herein and throughout the paper, $\mathrm{Var}(K)$ denotes the total variation process of $K$, and $\mathcal{A}^p_\sigma(Q, \mathbb{H})$ is the set of $K \in \mathcal{A}_{loc}(Q, \mathbb{H})$ such that $\|K\|_{\mathcal{A}_\sigma(Q,p)} < +\infty$.

**Definition 2.4.2.** *Let $p \in (1, +\infty)$. An $L^p(Q, \mathbb{H})$-solution for (2.4.1) is a $(Q, \mathbb{H})$-solution $(Y, Z, M, K)$ that belongs to the set*

$$\mathbb{D}_\sigma(Q,p) \otimes \mathbb{S}_\sigma(Q,p) \otimes \mathcal{M}^p(Q, \mathbb{H}) \otimes \mathcal{A}^p_\sigma(Q, \mathbb{H}).$$

# Chapter 3

# Snell envelope under random horizon

In this chapter we study the Snell envelope theory, the reason behind this theory is the problem of the optimal stopping. To promote the concept of the Snell envelope, we give the following simple example from real life that can be found in [62]: " Let $(Z_n,\ n \in \mathbb{N})$ be a sequence of integrable r.v.'s representing the random sequence of winnings of a gambler at successive time $n$ $(n \in \mathbb{N})$. For every finite stopping time $\nu$, the expectation $E(Z_\nu)$ represents the expected winnings of a gambler who decides to leave the game at the random time $\nu$; let us remark that by supposing $\nu$ a stopping time we are obliging the gambler to be honest, i.e. not to leave the game at time $n$ (event $\{n = \nu\}$) taking into account information other than that available at the moment. The Snell envelope here represents finding the upper bound $\sup_\nu E(Z_\nu)$ and finding the stopping time $\nu^*$ at which we attain this bound ".

To the best of our knowledge, the theory of Snell envelope and optimal stopping origin can be traced back at least to Wald's [87] work in probability and statistics. After that, it becomes very popular in applications in many different areas. The theory of Snell envelope and optimal stopping for a discrete time firstly appearance was in Neveu [62]. For more general treatments and to the case of continuous time, we refer the reader to El Karoui [39], Bismut and Skalli [14], Karatzas and Shreve [53], as well as Dellacherie and Meyer [37].

This chapter contains two sections. The first section presents some preliminaries that will be used in the following section. The second section presents our main results of this chapter. More precisely, in this section, we give an explicit connection between the Snell envelop in both filtrations $\mathbb{F}$ and $\mathbb{G}$. This is interesting by itself, and is useful in proving our results in the next chapters.

## 3.1 Preliminaries on Snell envelope

In this section, we give the definitions and some theories of the Snell envelope. The definition of the Snell envelope in continuous time, which we borrow from [43], is given in the following

**Definition 3.1.1.** *Given a filtered probability space* $(\Omega, \mathcal{F}, \mathbb{H} = (\mathcal{H}_t)_{0\leq t\leq T}, P)$ *and an absolutely continuous probability measure* $Q \ll P$*. Then an adapted process* $U = (U_t)_{t\in[0,T]}$ *is the Snell envelope with respect to* $Q$ *of the process* $X = (X_t)_{t\in[0,T]}$ *if*

*1.* $U$ *is a* $Q$*-supermartingale,*

*2.* $U$ *dominates* $X$*, i.e.* $U_t \geq X_t$ $Q$*-almost surely for all times* $t \in [0, T]$*,* *3. If $V = (V_t)_{t\in[0,T]}$ is a $Q$-supermartingale which dominates $X$, then $V$ dominates $U$.*

**Construction for discrete time (see [43]):**

Given a filtered probability space $\big(\Omega, \mathcal{F}, \mathbb{H} = (\mathcal{H}_n)_{n=0}^N, P\big)$ and an absolutely continuous probability measure $Q \ll P$, then the Snell envelope $(U_n)_{n=0}^N$ with respect to $Q$ of the process $(X_n)_{n=0}^N$ is given by the recursive scheme

$$U_N := X_N,$$

$$U_n := \max(X_n, E^Q[U_{n+1}|\mathcal{H}_n]) \qquad \text{for } n = N-1, \cdots, 0.$$

Here, we give the following set up that can be found in [37]. A process $X$ is said to be of class $D$ if the set of all r.v.s $X_\theta$, where $\theta$ is an arbitrary finite stopping time, is uniformly integrable. Let $(Y_t)_{\{t\geq 0\}} \in D$ denote an optional process; $Y_\infty = 0$. Then $Y_\theta$ is defined for every stopping time $\theta$, and the set $Y_\theta$ ( where $\theta$ runs through the set $I$ for all stopping times) is uniformly integrable. In the following, we give the fundamental existence theorem for the snell envelope

**Theorem 3.1.2** ([37]). *Let $(Y_t)_{\{t\geq 0\}} \in D$ denote an optional process; $Y_\infty = 0$. Then, the following assersions hold*

(a) *There exists a positive optional strong supermartingale $Z$ with the following property:*

*$Z \geq Y$ and for every positive optional strong supermartingale $Z'$ which bounds $Y$ above, $Z' \geq Z$.*

*$Z$ is unique. It is called the snell envelope of $Y$. Moreover, $Z$ belongs to class $D$.*

(b) *For every stopping time $\nu$ we have a.s.*

$$Z_\nu = \operatorname*{ess\,sup}_{\theta \in I, \theta \geq \nu} E[Y_\theta | \mathcal{H}_\nu].$$

And if the paths of $Y$ are continuous. Then, the stopping time $\theta^*$ defined by

$$\theta^* := \inf\{t \geq 0; \quad Z_t = Y_t\}$$

is the optimal one.

## 3.2 The main results

Throughout the thesis, for any two $\mathbb{H}$-stopping times $\sigma_1$ and $\sigma_2$ such that $\sigma_1 \leq \sigma_2$ $P$-a.s., we define $\mathcal{J}_{\sigma_1}^{\sigma_2}(\mathbb{H})$ by

$$\mathcal{J}_{\sigma_1}^{\sigma_2}(\mathbb{H}) := \Big\{ \sigma \ \mathbb{H}\text{-stopping time} \ : \ \sigma_1 \leq \sigma \leq \sigma_2 \ P\text{-a.s.} \Big\}. \tag{3.2.1}$$

Also, throughout the rest of the thesis, we assume the following assumption

$$G > 0 \quad \text{(i.e., } G \text{ is a positive process) and} \quad 0 < \tau < +\infty \quad P\text{-a.s.}. \tag{3.2.2}$$

The condition $\tau > 0$ $P$-a.s. is not restrictive at all due to the fact that throughout the thesis the $\sigma$-algebra $\mathcal{F}_0$ is the trivial, which yields to $G_0$ is number belonging to $(0, 1)$. Thus, the assumption under consideration translates into the condition $G_0 = 1$ only instead of being a real number in $(0, 1)$. The condition $G > 0$ is restrictive in the sense we can not include the $\mathbb{F}$-stopping times in this class of random times, while on the other hand this assumption says that our $\tau$ is truly an unbounded random time.

Throughout the thesis, for any nonnegative or $\mu := P \otimes D$-integrable and $\mathcal{F} \times \mathcal{B}(\mathbb{R}^+)$-measurable process $X$, its $\mathbb{F}$-optional projection with respect to the measure $\mu$ is denoted by $M^P_\mu(X\big|\mathcal{O}(\mathbb{F}))$ and is the unique $\mathbb{F}$-optional process $Y$ satisfying

$$E\left[\int_0^\infty X_s H_s dD_s\right] = E\left[\int_0^\infty Y_s H_s dD_s\right], \quad \text{for any bounded and } \mathbb{F}\text{-optional } H.$$

We give the following useful proposition

**Proposition 3.2.1.** *Suppose (3.2.2) holds, and let $X^{\mathbb{G}}$ be a $\mathbb{G}$-optional process such that $(X^{\mathbb{G}})^\tau = X^{\mathbb{G}}$. Then there exists a unique pair $(X^{\mathbb{F}}, k^{(pr)})$ of processes such that $X^{\mathbb{F}}$ is $\mathbb{F}$-optional, $k^{(pr)}$ is $\mathbb{F}$-progressive,*

$$X^{\mathbb{G}} = X^{\mathbb{F}} I_{[\![0,\tau[\![} + k^{(pr)} \cdot D, \quad \textit{and} \quad X^{\mathbb{F}} = {}^{o,\mathbb{F}}(X^{\mathbb{G}} I_{[\![0,\tau[\![})/G. \tag{3.2.3}$$

*Furthermore, the following assertions hold.*

(a) *$X^{\mathbb{G}}$ is RCLL if and only if $X^{\mathbb{F}}$ is RCLL.*

(b) *$X^{\mathbb{G}}$ is a RCLL $\mathbb{G}$-semimartingale iff $X^{\mathbb{F}}$ is a RCLL $\mathbb{F}$-semimartingale, and*

$$(X^{\mathbb{G}})^\tau = (X^{\mathbb{F}})^\tau + (k^{(pr)} - X^{\mathbb{F}}) \cdot D. \tag{3.2.4}$$

(c) *For any function $f : \mathbb{R} \to \mathbb{R}_+$ such that $f(0) = 0$. $E\left[\sup_{t\geq 0} f(X^{\mathbb{G}}_t)\right] < +\infty$*

*if and only if*

$$f(k^{(pr)}) \in L^1\left(\widetilde{\Omega}, \mathrm{Prog}(\mathbb{F}), P\otimes D\right) \ \ \textit{and} \ E\left[\int_0^{+\infty} \sup_{0\leq s<t} f(X^{\mathbb{F}}_s) dD^{o,\mathbb{F}}_t\right] < +\infty.$$

(d) *Let $\sigma$ be a finite $\mathbb{F}$-stopping time. If* $E\left[\sup_{0\leq t\leq \sigma\wedge\tau} (X^{\mathbb{G}}_t)^+\right] < +\infty$, *then*

$$E\left[G_\sigma \sup_{0\leq t\leq \sigma} (X^{\mathbb{F}}_t)^+\right] + E\left[\int_0^{\sigma} (k^{(op)}_s)^+ dD^{o,\mathbb{F}}_s\right] < +\infty; \quad k^{(op)} = M^P_\mu(k^{(pr)}\big|\mathcal{O}(\mathbb{F})).$$

*Proof.* Consider a $\mathbb{G}$-optional process $X^{\mathbb{G}}$. Then thanks to [3, Lemma B.1 ] (see also [51, Lemma 4.4]), there exists a pair $(X^{\mathbb{F}}, k^{(pr)})$ such that $X^{\mathbb{F}}$ is an $\mathbb{F}$-optional, $k^{(pr)}$ is Prog($\mathbb{F}$)-measurable, and

$$X^{\mathbb{G}} I_{[\![0,\tau[\![} = X^{\mathbb{F}} I_{[\![0,\tau[\![}, \quad \text{and} \quad X^{\mathbb{G}}_\tau = k^{(pr)}_\tau.$$

Furthermore, this pair is unique due to $G>0$. Thus, the condition $X^{\mathbb{G}} = (X^{\mathbb{G}})^\tau$ yields

$$X^{\mathbb{G}} = X^{\mathbb{G}} I_{[\![0,\tau[\![} + X^{\mathbb{G}}_\tau I_{[\![\tau,+\infty[\![} = X^{\mathbb{F}} I_{[\![0,\tau[\![} + k^{(pr)}\cdot D,$$

and the equality (3.2.3) is proved.

a) Thanks to (3.2.3) and the fact that $k^{(pr)}\cdot D$ is a RCLL process, we deduce that $X^{\mathbb{G}}$ is a RCLL process if and only if $X^{\mathbb{F}} I_{[\![0,\tau[\![}$ is a RCLL process. Thus, we consider a RCLL process $X^{\mathbb{G}}$, and let $(T^{\mathbb{G}}_n)$ be the sequence of $\mathbb{G}$-stopping times that increases to infinity, given by

$$T^{\mathbb{G}}_n := \inf\left\{t\geq 0: \quad |X^{\mathbb{G}}_t| > n\right\}, \text{ and satisfies } |X^{\mathbb{G},n}| \leq n, \quad X^{\mathbb{G},n} := X^{\mathbb{G}} I_{[\![0,T^{\mathbb{G}}_n[\![}.$$

By virtue of [3, Proposition B.2-(b)] and $G>0$, there exists a sequence of $\mathbb{F}$-stopping times $(T_n)_n$ which increases to infinity and satisfies $T_n^{\mathbb{G}}\wedge\tau=T_n\wedge\tau$. Furthermore, by applying (3.2.3) to each $X^{\mathbb{G},n}$, on the one hand, we deduce that

$$X^{\mathbb{G},n}I_{[\![0,\tau[\![}=X^{\mathbb{F}}I_{[\![0,\tau[\![}I_{[\![0,T_n[\![}.$$

On the other hand, as $T_n$ increases to infinity, it is clear that $X^{\mathbb{F}}$ is a RCLL if and only if $X^{\mathbb{F}}I_{[\![0,T_n[\![}={}^{o,\mathbb{F}}\ (X^{\mathbb{G},n}I_{[\![0,\tau[\![})G^{-1}$ is RCLL. This latter fact follows directly from combining [33, Théorème 47, pp: 119], the boundedness of $X^{\mathbb{G},n}$, and the right-continuity of $G$. This proves assertion (a).

b) It is clear that $k^{(pr)}\cdot D$ is a RCLL $\mathbb{G}$-semimartingale, and hence $X^{\mathbb{G}}$ is a RCLL $\mathbb{G}$-semimartingale if and only if $X^{\mathbb{F}}I_{[\![0,\tau[\![}$ is a RCLL $\mathbb{G}$-semimartingale. Thus, if $X^{\mathbb{F}}$ is a RCLL $\mathbb{F}$-semimartingale, then $X^{\mathbb{G}}$ is a RCLL $\mathbb{G}$-semimartingale. To prove the converse, we remark that by stopping with $T_n^{\mathbb{G}}$ defined above and by using [33, Théorème 26, Chapter VII, pp: 235], there is no loss of generality in assuming $X^{\mathbb{G}}$ is bounded, which leads to the boundedness of $X^{\mathbb{F}}$, see [3, Lemma B.1] or [51, Lemma 4.4 (b), pp: 63]. Thus, thanks to [33, Théorème 47, pp: 119 and Théorème 59, pp: 268] which implies that the optional projection of a bounded RCLL $\mathbb{G}$-semimartingale is a RCLL $\mathbb{F}$-semimartingale, we deduce that $X^{\mathbb{F}}G={}^{o,\mathbb{F}}\left(X^{\mathbb{G}}I_{[\![0,\tau[\![}\right)$ is a RCLL $\mathbb{F}$-semimartingale. A combination of this with the condition $G>0$ and the fact that $G$ is a RCLL $\mathbb{F}$-semimartingale implies that $X^{\mathbb{F}}$ is a RCLL $\mathbb{F}$-semimartingale. Furthermore, direct calculation yields

$$X^{\mathbb{F}} I_{[\![0,\tau[\![} = (X^{\mathbb{F}})^{\tau} - X^{\mathbb{F}} \cdot D \quad \text{is a } \mathbb{G}\text{-semimartingale,}$$

and (3.2.4) follows from this equality and (3.2.3).

c) Here, we prove assertion (c). To this end, we use (3.2.3) and notice that

$$\begin{aligned} \frac{I}{2} \leq \sup_{t\geq 0} f(X_t^{\mathbb{G}}) &= \max\left(\sup_{0\leq t<\tau} f(X_t^{\mathbb{F}}), f(k_\tau^{(pr)})\right) \leq I, \\ I &:= \int_0^\infty \left(\sup_{0\leq u<t} f(X_u^{\mathbb{F}}) + f(k_t^{(pr)})\right) dD_t. \end{aligned}$$

Hence, we deduce that $E\left[\sup_{t\geq 0} f(X_t^{\mathbb{G}})\right] < +\infty$ iff $E\left[\int_0^\infty f(k_t^{(pr)}) dD_t\right] < +\infty$ and

$$E\left[\int_0^\infty \sup_{0\leq u<t} f(X_u^{\mathbb{F}}) dD_t\right] = E\left[\int_0^\infty \sup_{0\leq u<t} f(X_u^{\mathbb{F}}) dD_t^{o,\mathbb{F}}\right] < +\infty.$$

due to $\sup\limits_{0\leq u<t} f(X_u^{\mathbb{F}})$ being $\mathbb{F}$-optional.

d) Let $\sigma$ be an $\mathbb{F}$-stopping time. By applying assertion (c) to $\left((X_{t\wedge\sigma}^{\mathbb{G}})^+\right)_{t\geq 0}$, we deduce that $E\left[\sup\limits_{0\leq t\leq\sigma\wedge\tau} (X_t^{\mathbb{G}})^+\right] < +\infty$ if and only if

$$E\left[\int_0^\sigma (k_s^{(pr)})^+ dD_s\right] + E\left[\int_0^{+\infty} \sup_{0\leq s<t} (X_{s\wedge\sigma}^{\mathbb{F}})^+ dD_t^{o,\mathbb{F}}\right] < +\infty.$$

On the one hand, a combination of $k^{(op)} = M_\mu^P(k^{(pr)} \big| \mathcal{O}(\mathbb{F}))$,

$$E\left[\int_0^\sigma (k_s^{(pr)})^+ dD_s\right] = M_\mu^P((k^{(pr)})^+ 1_{[\![0,\sigma]\!]}),$$

and Jensen's inequality allows us to derive

$$
\begin{aligned}
&E\left[\int_0^{\sigma}(k_s^{(op)})^+dD_s^{o,\mathbb{F}}\right]=E\left[\int_0^{\sigma}(k_s^{(op)})^+dD_s\right]\\
&=M_\mu^P\Big(\big(M_\mu^P(k^{(pr)}\big|\mathcal{O}(\mathbb{F}))\big)^+1_{[\![0,\sigma]\!]}\Big)\overset{Jensen}{\le}M_\mu^P\Big(M_\mu^P\big((k^{(pr)})^+1_{[\![0,\sigma]\!]}\big|\mathcal{O}(\mathbb{F})\big)\Big)\\
&\overset{Tower}{=}M_\mu^P\big((k^{(pr)})^+1_{[\![0,\sigma]\!]}\big)=E\left[\int_0^{\sigma}(k_s^{(pr)})^+dD_s\right]<\infty.
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
E\left[\sup_{0\le t\le\sigma\wedge\tau}(X_t^{\mathbb{G}})^+\right]&\ge E\left[\sup_{0\le t\le\sigma\wedge\tau}(X_t^{\mathbb{G}})^+1_{\{\tau>\sigma\}}\right]=E\left[\sup_{0\le t\le\sigma}(X_t^{\mathbb{G}})^+1_{\{\tau>\sigma\}}\right]\\
&=E\left[\sup_{0\le t\le\sigma}(X_t^{\mathbb{F}})^+1_{\{\tau>\sigma\}}\right]=E\left[G_\sigma\sup_{0\le t\le\sigma}(X_t^{\mathbb{F}})^+\right].
\end{aligned}
$$

This proves assertion d) and the proof of the proposition is complete. $\square$

**Lemma 3.2.2.** *Let $\sigma_1$ and $\sigma_2$ be two $\mathbb{F}$-stopping times such that $\sigma_1\le\sigma_2$ $P$-a.s.. Then, for any $\mathbb{G}$- stopping time, $\sigma^{\mathbb{G}}$, satisfying*

$$
\sigma_1\wedge\tau\le\sigma^{\mathbb{G}}\le\sigma_2\wedge\tau\quad P\text{-}a.s., \tag{3.2.5}
$$

*there exists an $\mathbb{F}$- stopping time $\sigma^{\mathbb{F}}$ such that*

$$
\sigma_1\le\sigma^{\mathbb{F}}\le\sigma_2\quad and\quad \sigma^{\mathbb{F}}\wedge\tau=\sigma^{\mathbb{G}}\quad P\text{-}a.s. \tag{3.2.6}
$$

*Proof.* Thanks to [34, XX.75 b)] (see also [3, Proposition B.2-(b)]), for the

$\mathbb{G}$-stopping time $\sigma^{\mathbb{G}}$, there exists an $\mathbb{F}$-stopping time $\sigma$ such that

$$\sigma^{\mathbb{G}} = \sigma^{\mathbb{G}} \wedge \tau = \sigma \wedge \tau.$$

Put

$$\sigma^{\mathbb{F}} := \min\left(\max(\sigma, \sigma_1), \sigma_2\right), \tag{3.2.7}$$

and on the one hand remark that $\sigma^{\mathbb{F}}$ is an $\mathbb{F}$- stopping time satisfying the first condition in (3.2.6). On the other hand, it is clear that

$$\min(\tau, \max(\sigma, \sigma_1)) = (\tau \wedge \sigma_1) I_{\{\sigma_1 > \sigma\}} + (\tau \wedge \sigma) I_{\{\sigma_1 \leq \sigma\}} = \max(\sigma \wedge \tau, \sigma_1 \wedge \tau).$$

Thus, by using this equality, we derive

$$\begin{aligned} \sigma^{\mathbb{F}} \wedge \tau \quad &= \tau \wedge \sigma_2 \wedge \max(\sigma, \sigma_1) = (\tau \wedge \sigma_2) \wedge (\tau \wedge \max(\sigma, \sigma_1)) \\ &= (\tau \wedge \sigma_2) \wedge \max(\sigma \wedge \tau, \sigma_1 \wedge \tau) = \sigma \wedge \tau = \sigma^{\mathbb{G}}. \end{aligned}$$

This ends the proof of the lemma. $\square$

The following is our main result of this Chapter, where we write in different manners the Snell envelope of a process under $\mathbb{G}$ as a sum of a transformation of an $\mathbb{F}$-Snell envelope and $\mathbb{G}$-martingales.

**Theorem 3.2.3.** *Suppose $G > 0$, and let $X^{\mathbb{G}}$ be a RCLL and $\mathbb{G}$-adapted process such that $(X^{\mathbb{G}})^{\tau} = X^{\mathbb{G}}$. Then consider the unique pair $(X^{\mathbb{F}}, k^{(pr)})$*

*associated to $X^{\mathbb{G}}$, and denote by*

$$k^{(op)} := M^P_\mu(k^{(pr)}\big|\mathcal{O}(\mathbb{F})) \text{ where } \mu := P\otimes D \text{ and } k^{(\mathbb{F})} := k^{(pr)} - k^{(op)}. \quad (3.2.8)$$

*Then the following assertions hold.*

(a) *If either $X^{\mathbb{G}}$ is nonnegative or $E\left[\sup_{t\geq 0}(X^{\mathbb{G}}_t)^+\right] < +\infty$, then the $(\mathbb{G}, P)$-Snell envelope of $X^{\mathbb{G}}$, denoted $\mathcal{S}(X^{\mathbb{G}}; \mathbb{G}, P)$, is given by*

$$\begin{aligned}\mathcal{S}(X^{\mathbb{G}};\mathbb{G},P) \quad &= \frac{\mathcal{S}(X^{\mathbb{F}}G + k^{(op)}\cdot D^{o,\mathbb{F}};\mathbb{F},P)}{G} I_{[\![0,\tau[\![}\\ &+k^{(\mathbb{F})}\cdot D + \frac{(k^{(op)}\cdot D^{o,\mathbb{F}})_-}{G^2_-}\cdot\mathcal{T}(m) + \left(k^{(op)} + \frac{k^{(op)}\cdot D^{o,\mathbb{F}}}{G}\right)\cdot N^{\mathbb{G}}.\end{aligned} \quad (3.2.9)$$

*Here $\mathcal{S}(X^{\mathbb{F}}G+k^{(op)}\cdot D^{o,\mathbb{F}};\mathbb{F},P)$ denotes the $(\mathbb{F}, P)$-Snell envelope of the process $X^{\mathbb{F}}G + k^{(op)}\cdot D^{o,\mathbb{F}}$.*

(b) *Let $T\in (0, +\infty)$ and $\widetilde{Q}$ be given in (2.3.11). If either $E^{\widetilde{Q}}\left[\sup_{0\leq t\leq T}(X^{\mathbb{G}}_t)^+\right] < +\infty$ or $X^{\mathbb{G}} \geq 0$, then the $(\mathbb{G}, \widetilde{Q})$-Snell envelope of $(X^{\mathbb{G}})^T$, denoted by the process $\mathcal{S}((X^{\mathbb{G}})^T;\mathbb{G},\widetilde{Q})$, is given by*

$$\begin{aligned}\mathcal{S}((X^{\mathbb{G}})^T;\mathbb{G},\widetilde{Q}) \quad &= \frac{\mathcal{S}((X^{\mathbb{F}}\widetilde{\mathcal{E}} - k^{(op)}\cdot\widetilde{\mathcal{E}})^T;\mathbb{F},P)}{\widetilde{\mathcal{E}}} I_{[\![0,\tau[\![} + k^{(\mathbb{F})}\cdot D^T\\ &+\left(k^{(op)} - \frac{k^{(op)}\cdot\widetilde{\mathcal{E}}}{\widetilde{\mathcal{E}}}\right)\cdot (N^{\mathbb{G}})^T.\end{aligned} \quad (3.2.10)$$

*Here the process $\widetilde{\mathcal{E}}$ is given by (2.3.8).*

*Proof.* Let $\theta \in \mathcal{T}^\tau_{t\wedge\tau}(\mathbb{G})$, then thanks to Lemma 3.2.2 there exists $\sigma \in \mathcal{T}^\infty_t(\mathbb{F})$

such that $\theta = \sigma \wedge \tau$. Then notice that

$$\begin{aligned} X^{\mathbb{G}}_{\theta} &= X^{\mathbb{G}}_{\sigma\wedge\tau} I_{\{\sigma<\tau\}} + k^{(pr)}_{\tau} I_{\{\sigma\geq\tau\}} = X^{\mathbb{F}}_{\sigma} I_{\{\sigma<\tau\}} + \int_0^{\sigma} k^{(pr)}_s dD_s \\ &= X^{\mathbb{F}}_{\sigma} I_{\{\sigma<\tau\}} + \left(\frac{k^{(op)}}{\widetilde{G}} \cdot D^{o,\mathbb{F}}\right)_{\sigma\wedge\tau} + (k^{(op)} \cdot N^{\mathbb{G}})_{\sigma} + k^{(\mathbb{F})} \cdot D_{\sigma}. \end{aligned} \tag{3.2.11}$$

The rest of the proof is divided into three parts. The first and second parts prove the assertions (a) and (b) of the theorem under the assumption that $X^{\mathbb{G}}$ is bounded, while the third part relaxes this condition and proves the theorem.

**Part 1.** In this part, we suppose that $X^{\mathbb{G}}$ is bounded, and hence the associated three processes $X^{\mathbb{F}}, k^{(pr)}$ and $k^{(op)}$ are also bounded. As a result, both processes $k^{(op)} \cdot N^{\mathbb{G}}$ and $k^{(\mathbb{F})} \cdot D$ are $\mathbb{G}$-martingales. Thus, by putting

$$L^{\mathbb{G}} := k^{(op)} \cdot N^{\mathbb{G}} + k^{(\mathbb{F})} \cdot D, \tag{3.2.12}$$

and combining the remarks above with Lemma 2.3.3 and taking conditional expectation with respect to $\mathcal{G}_t$ on both sides of (3.2.11), we derive

$$\begin{aligned} Y_t(\theta) := E\left[X^{\mathbb{G}}_{\theta}\big|\mathcal{G}_t\right] &= E\left[X^{\mathbb{F}}_{\sigma} I_{\{\sigma<\tau\}} + \int_0^{\sigma\wedge\tau} \frac{k^{(op)}_s}{\widetilde{G}_s} dD^{o,\mathbb{F}}_s \Big| \mathcal{G}_t\right] + L^{\mathbb{G}}_t \\ &= E\left[X^{\mathbb{F}}_{\sigma} I_{\{\sigma<\tau\}} + \int_{t\wedge\tau}^{\sigma\wedge\tau} \frac{k^{(op)}_s}{\widetilde{G}_s} dD^{o,\mathbb{F}}_s \Big| \mathcal{G}_t\right] + (\frac{k^{(op)}}{\widetilde{G}} \cdot D^{o,\mathbb{F}})_{t\wedge\tau} + L^{\mathbb{G}}_t \\ &\overset{(1)}{=} E\left[X^{\mathbb{F}}_{\sigma} I_{\{\sigma<\tau\}} + \int_{t\wedge\tau}^{\sigma\wedge\tau} \frac{k^{(op)}_s}{\widetilde{G}_s} dD^{o,\mathbb{F}}_s \Big| \mathcal{F}_t\right] \frac{I_{\{\tau>t\}}}{G_t} + (\frac{k^{(op)}}{\widetilde{G}} \cdot D^{o,\mathbb{F}})_{t\wedge\tau} + L^{\mathbb{G}}_t \\ &= E\left[G_{\sigma} X^{\mathbb{F}}_{\sigma} + \int_t^{\sigma} k^{(op)}_s dD^{o,\mathbb{F}}_s \Big| \mathcal{F}_t\right] \frac{I_{\{\tau>t\}}}{G_t} + (\frac{k^{(op)}}{\widetilde{G}} \cdot D^{o,\mathbb{F}})_{t\wedge\tau} + L^{\mathbb{G}}_t \\ &=: \frac{X^{\mathbb{F}}_t(\sigma)}{G_t} I_{\{t<\tau\}} - \frac{(k^{(op)} \cdot D^{o,\mathbb{F}})_t}{G_t} I_{\{t<\tau\}} + (\frac{k^{(op)}}{\widetilde{G}} \cdot D^{o,\mathbb{F}})_{t\wedge\tau} + L^{\mathbb{G}}_t \end{aligned} \tag{3.2.13}$$

The equality (1) in the string of the equalities above is due to $(\sigma<\tau)\subseteq(t<\tau)$ and lemma 2.3.3. Thus, by taking the essential supremum over all $\theta\in\mathcal{T}^{\tau}_{t\wedge\tau}(\mathbb{G})$, we deduce that

$$\begin{aligned}\mathcal{S}(X^{\mathbb{G}};\mathbb{G},P)\quad &= \frac{\mathcal{S}(X^{\mathbb{F}}G+k^{(op)}\cdot D^{o,\mathbb{F}};\mathbb{F},P)}{G}I_{[\![0,\tau[\![}-\frac{(k^{(op)}\cdot D^{o,\mathbb{F}})}{G}I_{[\![0,\tau[\![}\\ &+(\frac{k^{(op)}}{\widetilde{G}}\cdot D^{o,\mathbb{F}})^{\tau}+k^{(op)}\cdot N^{\mathbb{G}}+k^{(\mathbb{F})}\cdot D.\end{aligned}\tag{3.2.14}$$

Furthermore, put $V:=k^{(op)}\cdot D^{o,\mathbb{F}}$ and remark that

$$d(1/G^{\tau})=(G\widetilde{G})^{-1}I_{]\!]0,\tau]\!]}dD^{o,\mathbb{F}}-G_{-}^{-2}d\mathcal{T}(m).$$

Thus, by combining these with Itô, we derive

$$\begin{aligned}d\left(\frac{V^{\tau}}{G^{\tau}}\right)\quad &=V_{-}d\left(\frac{1}{G^{\tau}}\right)+\frac{k^{(op)}}{G}I_{]\!]0,\tau]\!]}dD^{o,\mathbb{F}}\\ &=\frac{V}{G\widetilde{G}}I_{]\!]0,\tau]\!]}dD^{o,\mathbb{F}}-\frac{V_{-}}{G_{-}^{2}}d\mathcal{T}(m)+\frac{k^{(op)}}{\widetilde{G}}I_{]\!]0,\tau]\!]}dD^{o,\mathbb{F}}.\end{aligned}$$

Thus, (3.2.9) follows immediately from combining this equality with (3.2.14) and the easy fact that

$$XI_{[\![0,\tau[\![}=X^{\tau}-X\cdot D,\quad\text{for any}\quad\mathbb{F}\text{-semimartingale } X.\tag{3.2.15}$$

This ends the proof of assertion (a).

**Part 2.** Here, we suppose that $X^{\mathbb{G}}$ is bounded, we fix $T\in(0,+\infty)$ and prove assertion (b). Let $\theta\in\mathcal{T}^{T\wedge\tau}_{t\wedge\tau}(\mathbb{G})$ and $\sigma\in\mathcal{T}^{T}_{t}(\mathbb{F})$ such that $\theta=\sigma\wedge\tau$. Then, similarly as in Part 1, by taking $\widetilde{Q}$-conditional expectation in both sides of

(3.2.11), and using (3.2.12) and the fact that the two processes $k^{(op)}\cdot N^{\mathbb{G}}$ and $k^{(\mathbb{F})}\cdot D$ remain $\mathbb{G}$-martingale under $\widetilde{Q}$ (due the boundedness of $k^{pr)}$ and $k^{(\mathbb{F})}$), we write

$$\begin{aligned}
\widetilde{Y}_t(\theta) &:= E^{\widetilde{Q}}\left[X^{\mathbb{G}}_\theta \big| \mathcal{G}_t\right] = E^{\widetilde{Q}}\left[X^{\mathbb{F}}_\sigma I_{\{\sigma<\tau\}} + \int_0^{\sigma\wedge\tau} \frac{k^{(op)}_s}{\widetilde{G}_s} dD^{o,\mathbb{F}}_s \big| \mathcal{G}_t\right] + L^{\mathbb{G}}_t \\
&= E\left[\frac{\widetilde{Z}_\sigma}{\widetilde{Z}_t} X^{\mathbb{F}}_\sigma I_{\{\sigma<\tau\}} + \int_{t\wedge\tau}^{\sigma\wedge\tau} \frac{k^{(op)}_s \widetilde{Z}_s}{\widetilde{G}_s \widetilde{Z}_t} dD^{o,\mathbb{F}}_s \big| \mathcal{G}_t\right] + \frac{k^{(op)}}{\widetilde{G}} \cdot D^{o,\mathbb{F}}_{t\wedge\tau} + L^{\mathbb{G}}_t \\
&= E\left[\widetilde{Z}_\sigma X^{\mathbb{F}}_\sigma I_{\{\sigma<\tau\}} + \int_{t\wedge\tau}^{\sigma\wedge\tau} G_0 \frac{k^{(op)}_s}{\widetilde{G}_s} dV^{\mathbb{F}}_s \big| \mathcal{F}_t\right] \frac{I_{\{\tau>t\}}}{\widetilde{Z}_t G_t} + \frac{k^{(op)}}{\widetilde{G}} \cdot D^{o,\mathbb{F}}_{t\wedge\tau} + L^{\mathbb{G}}_t \\
&= E\left[\widetilde{\mathcal{E}}_\sigma X^{\mathbb{F}}_\sigma + \int_t^{\sigma} k^{(op)}_s dV^{\mathbb{F}}_s \big| \mathcal{F}_t\right] \frac{I_{\{\tau>t\}}}{\widetilde{\mathcal{E}}_t} + \left(\frac{k^{(op)}}{\widetilde{G}} \cdot D^{o,\mathbb{F}}\right)_{t\wedge\tau} + L^{\mathbb{G}}_t \\
&=: \frac{X^{\mathbb{F}}_t(\sigma)}{\widetilde{\mathcal{E}}_t} I_{\{\tau>t\}} - \frac{(k^{(op)}\cdot V^{\mathbb{F}})_t}{\widetilde{\mathcal{E}}_t} I_{\{\tau>t\}} + \left(\frac{k^{(op)}}{\widetilde{G}} \cdot D^{o,\mathbb{F}}\right)_{t\wedge\tau} + L^{\mathbb{G}}_t \qquad (3.2.16)
\end{aligned}$$

The fifth equality is a consequence of Lemma 2.3.2, where both $\widetilde{\mathcal{E}}$ and $V^{\mathbb{F}}$ are defined. By taking essential supremum over $\theta$ in both sides of (3.2.16), we get

$$\begin{aligned}
\mathcal{S}(X^{\mathbb{G}};\mathbb{G},\widetilde{Q}) \quad &= \frac{\mathcal{S}(X^{\mathbb{F}}\widetilde{\mathcal{E}} + k^{(op)}\cdot V^{\mathbb{F}};\mathbb{F},P)}{\widetilde{\mathcal{E}}} I_{[\![0,\tau[\![} - \frac{(k^{(op)}\cdot V^{\mathbb{F}})}{\widetilde{\mathcal{E}}} I_{[\![0,\tau[\![} \\
&+ \left(\frac{k^{(op)}}{\widetilde{G}} \cdot D^{o,\mathbb{F}}\right)^{\tau} + k^{(op)}\cdot N^{\mathbb{G}} + k^{(\mathbb{F})}\cdot D. \qquad (3.2.17)
\end{aligned}$$

Similar arguments, as in part 1) after equation (3.2.14), applied to $V = k^{(op)}\cdot V^{\mathbb{F}} := -k^{(op)}\cdot\widetilde{\mathcal{E}}$, leads to

$$-(k^{(op)}\cdot V^{\mathbb{F}})\widetilde{\mathcal{E}}^{-1} I_{[\![0,\tau[\![} + (k^{(op)}\widetilde{G}^{-1}\cdot D^{o,\mathbb{F}})^{\tau} = (k^{(op)}\cdot V^{\mathbb{F}})\widetilde{\mathcal{E}}^{-1}\cdot N^{\mathbb{G}}.$$

Thus, (3.2.10) follows from combining this fact with (3.2.17), and the proof of assertion (b) is completed. This ends the second part.

**Part 3:** Herein, we prove the theorem without the boundedness assumption on $X^{\mathbb{G}}$. To this end, by virtue of parts 1 and 2, we remark that the theorem follows immediately as soon as we prove that (3.2.13) and (3.2.16) hold. To this end, we consider the sequence of $\mathbb{G}$-stopping times

$$T_n^{\mathbb{G}} := \inf\left\{t \geq 0 : \ |X^{\mathbb{G}}| > n\right\},$$

that increases to infinity and $X^{\mathbb{G},n} := X^{\mathbb{G}} I_{[\![0,T_n^{\mathbb{G}}[\![}$ is a bounded process. To the sequence $(T_n^{\mathbb{G}})$, we associate a sequence of $\mathbb{F}$-stopping times $(T_n)$, which increases to infinity and $T_n^{\mathbb{G}} \wedge \tau = T_n \wedge \tau$, for all $n \geq 1$. Therefore, thanks to the parts 1 and 2, the bounded $X^{\mathbb{G},n}$ with its associated triplet $(X^{\mathbb{F},n}, k^{(pr,n)}, k^{(op,n)})$ $= (X^{\mathbb{F}}, k^{(pr)}, k^{(op)}) I_{[\![0,T_n[\![}$ fulfills (3.2.13) and (3.2.16).

1) The case when $X^{\mathbb{G}} \geq 0$: In this case we get $X^{\mathbb{G},n} \geq 0$ and it increases to $X^{\mathbb{G}}$, and all components of $(X^{\mathbb{F},n}, k^{(pr,n)}, k^{(op,n)})$ are nonnegative and increase to the corresponding components of $(X^{\mathbb{F}}, k^{(pr)}, k^{(op)})$ respectively. Thus, thanks to the convergence monotone theorem, it is clear that in this case $E[X_\theta^{\mathbb{G},n}|\mathcal{G}_t]$ (respectively $E^{\widetilde{Q}}[X_\theta^{\mathbb{G},n}|\mathcal{G}_t]$) increases to $E[X_\theta^{\mathbb{G}}|\mathcal{G}_t]$ (respectively $E^{\widetilde{Q}}[X_\theta^{\mathbb{G}}|\mathcal{G}_t]$) and $E\left[G_\sigma X_\sigma^{\mathbb{F},n} + \int_t^\sigma k_s^{(op,n)} dD_s^{o,\mathbb{F}} \Big| \mathcal{F}_t\right]$ (respectively $E\left[\widetilde{\mathcal{E}}_\sigma X_\sigma^{\mathbb{F},n} + \int_t^\sigma k_s^{(op,n)} dV_s^{\mathbb{F}} \Big| \mathcal{F}_t\right]$) increases to $E\left[G_\sigma X_\sigma^{\mathbb{F}} + \int_t^\sigma k_s^{(op)} dD_s^{o,\mathbb{F}} \Big| \mathcal{F}_t\right]$ (resp. $E\left[\widetilde{\mathcal{E}}_\sigma X_\sigma^{\mathbb{F}} + \int_t^\sigma k_s^{(op)} dV_s^{\mathbb{F}} \Big| \mathcal{F}_t\right]$). This proves that (3.2.13) and (3.2.16) hold for the case when $X^{\mathbb{G}} \geq 0$, and the theorem is proved in this case.

2) The case of $E\left[\sup_{t\geq 0}(X_t^{\mathbb{G}})^+\right] < +\infty$: Then, on the one hand, Fatou's lemma yields

$$E[X_\theta^{\mathbb{G}}|\mathcal{G}_t] \geq \lim \sup_{n \longrightarrow +\infty} E[X_\theta^{\mathbb{G},n}|\mathcal{G}_t]. \tag{3.2.18}$$

On the other hand, the dominated convergence theorem and $E[X_\theta^{\mathbb{G},n}|\mathcal{G}_t] \geq E[X_\theta^{\mathbb{G}}|\mathcal{G}_t] - E[\sup_{0\leq t\leq T\wedge\tau}(X_t^{\mathbb{G}})^+ I_{\{\theta\geq T_n^{\mathbb{G}}\}}|\mathcal{G}_t]$ imply that

$$\liminf_{n\longrightarrow+\infty} E[X_\theta^{\mathbb{G},n}|\mathcal{G}_t] \geq E[X_\theta^{\mathbb{G}}|\mathcal{G}_t]. \tag{3.2.19}$$

Thus, a combination of (3.2.18) and (3.2.19) implies that $E[X_\theta^{\mathbb{G},n}|\mathcal{G}_t]$ converges to $E[X_\theta^{\mathbb{G}}|\mathcal{G}_t]$ almost surely. These two arguments combined with Proposition (3.2.1)-(d) allow us in the same manner to prove the convergence of $E\left[G_\sigma X_\sigma^{\mathbb{F},n} + \int_t^\sigma k_s^{(op,n)} dD_s^{o,\mathbb{F}} \middle| \mathcal{F}_t\right]$ to $E\left[G_\sigma X_\sigma^{\mathbb{F}} + \int_t^\sigma k_s^{(op)} dD_s^{o,\mathbb{F}} \middle| \mathcal{F}_t\right]$. Thus, assertion (a) follows immediately. The proof of assertion (b), under the assumption $E^{\widetilde{Q}}\left[\sup_{0\leq t\leq T\wedge\tau}(X_t^{\mathbb{G}})^+\right] < +\infty$, mimics exactly the proof of assertion (a), by substituting $\widetilde{Q}$ to $P$, and hence the details will be omitted. This ends the proof of theorem. $\square$

# Chapter 4

# Linear RBSDEs under random horizon

In this Chapter, we consider the linear RBSDE under $\mathbb{G}$ given by

$$\begin{cases} dY_t = -f(t)d(t\wedge\tau) - d(K_{t\wedge\tau} + M_{t\wedge\tau}) + Z_t dW^{\tau}_t, \quad Y_\tau = \xi = Y_T, \\ Y_t \geq S_t, \quad 0 \leq t < T\wedge\tau, \quad \text{and} \quad \mathrm{E}\left[\int_0^{\mathrm{T}\wedge\tau} (\mathrm{Y_{t-}} - \mathrm{S_{t-}})\mathrm{dK_t}\right] = 0. \end{cases} \tag{4.0.1}$$

Here, the barrier process $S$ is an $\mathbb{F}$-adapted and RCLL process, $T \in (0, +\infty]$, and the generator $f$ is an $\mathbb{F}$-progressively measurable process. The terminal value of $Y$, denoted by $\xi$, is an $\mathcal{F}_{T\wedge\tau}$-measurable random variable. This fact, due to the definition of the $\sigma$-algebra $\mathcal{F}_{T\wedge\tau}$, is equivalent to the existence of an $\mathbb{F}$-optional process $h$ such that

$$\xi = h_{T\wedge\tau}. \tag{4.0.2}$$

Here, we study in different aspects the RBSDE under $\mathbb{G}$ having the form of (4.0.1) and those RBSDE under $\mathbb{F}$ that are related to them. In paticular, we derive the explicit relationship between the two solutions, and hence we show how one can pass from an RBSDE under $\mathbb{G}$ to an RBSDE under $\mathbb{F}$ and vice-versa. The key tool and idea behind these results is the optimal stopping problem and the Snell envelope associated to it for models with two flows of information. This Snell concept, which is important in itself besides its application in linear RBSDEs, is treated in Chapter 3.

This chapter has three sections. Section 4.1 handles the case when the horizon time is bounded, or in other word we suppose $T < +\infty$ in this first section. In this setting, we obtain prior estimate results, which are interesting in themselves beyond their role in proving the existence and uniqueness results. In addition to these, we establish an explicit connection between the RBSDE (4.0.1) and its $\mathbb{F}$-RBSDE counterpart. The second section, which is Section 4.2, considers the case of unbounded horizon, or equivalently assumes that $T = +\infty$. Similarly as in the first section, we state our priors estimes, existence and uniqueness of a solution, and the connection between the $\mathbb{G}$-RBSDE and the $\mathbb{F}$-RBSDE counterpart results. In the third section (Section 4.3), we gathered the proofs of the lemmas that are used and vital in the first two sections.

## 4.1 The case of bounded horizon

In this section, we suppose that $T < +\infty$ and the triplet-data $(f, S, \xi)$ satisfies all the conditions described in the paragraph that follows (4.0.1), and $\xi \geq S_{\tau\wedge T}$

$P$-a.s.. Hence, working with the $\mathbb{G}$-triplet data $(f, S, \xi)$ is equivalent to work with the $\mathbb{F}$-triplet data $(f, S, h)$. This section is divided into two subsections. The first subsection elaborates estimates for the solution of the RBSDE (when it exists), while the second subsection addresses the existence and uniqueness of the solution and describes the $\mathbb{F}$-RBSDE counterpart of (4.0.1).

### 4.1.1 Various norm-estimates for the solution

This subsection elaborates estimates for the solution of the RBSDE (4.0.1) when $T < +\infty$. To this end, we start elaborating some useful intermediate results that we summarize in two lemmas.

**Lemma 4.1.1.** *The following assertions hold.*

(a) *For any $T \in (0, +\infty)$, $m^{T\wedge\tau}$ is a BMO $(\widetilde{Q}, \mathbb{G})$-martingale. Furthermore, we have*

$$E^{\widetilde{Q}}\left[[m,m]_{T\wedge\tau} - [m,m]_{t\wedge\tau-}\big|\mathcal{G}_t\right] \leq \|m\|_{BMO(P)}, \quad P\text{-a.s.}. \tag{4.1.1}$$

(b) *For any $t \in (0, T]$, we have*

$$E^{\widetilde{Q}}\left[D^{o,\mathbb{F}}_{T\wedge\tau} - D^{o,\mathbb{F}}_{(t\wedge\tau)-}\big|\mathcal{G}_t\right] \leq \widetilde{G}_t \leq 1, \quad P\text{-a.s.}. \tag{4.1.2}$$

(c) *If $a \in (0, +\infty)$, then* $\max(a, 1)\widetilde{G}^{-1} \cdot D^{o,\mathbb{F}} - \widetilde{V}^{(a)}$ *is nondecreasing, and*

$$E\left[\int_{t\wedge\tau}^{T\wedge\tau} \widetilde{G}_s^{-1} dD^{o,\mathbb{F}}_s \big|\mathcal{G}_t\right] \leq 1, \quad P - a.s., \tag{4.1.3}$$

*where $\widetilde{V}^{(a)}$ is the process defined by*

$$\widetilde{V}^{(a)} := \frac{a}{\widetilde{G}} \cdot D^{o,\mathbb{F}} + \sum \left( -\frac{a\Delta D^{o,\mathbb{F}}}{\widetilde{G}} + 1 - \left(1 - \frac{\Delta D^{o,\mathbb{F}}}{\widetilde{G}}\right)^a \right). \tag{4.1.4}$$

The proof of this lemma is given in Section 4.3. The following lemma, that plays crucial role in our estimations, is interesting in itself and generalizes [28, Lemma 4.8] to more broader cases encountered in our analysis.

**Lemma 4.1.2.** *If $r^{-1} = a^{-1} + b^{-1}$, where $a > 1$ and $b > 1$, then there exists a positive constant $\kappa = \kappa(a, b)$ depending only on $a$ and $b$ such that the following assertion holds.*
*For any triplet $(H, X, M)$ such that $H$ is predictable, $X$ is RCLL and adapted process, $M$ is a martingale, and $|H| \leq |X_-|$, the following inequality holds.*

$$\|\sup_{0\leq t\leq T} |(H \cdot M)_t|\|_r \leq \kappa \|\sup_{0\leq t\leq T} |X_t|\|_a \|[M]_T^{\frac{1}{2}}\|_b.$$

*Proof.* When $H = X_-$, the assertion can be found in [28, Lemma 4.8]. To prove the general case, we remark that, there is no loss of generality in assuming $|X_-| > 0$, and hence the process $H/X_-$ is a well defined process that is predictable and is bounded. Thus, put

$$\overline{M} := \frac{H}{X_-} \cdot M,$$

and remark that $[\overline{M}, \overline{M}] = (H/X_-)^2 \cdot [M, M] \leq [M, M]$. As a result, we derive

$$\|\sup_{0\leq t\leq T} |(H \cdot M)_t|\|_r = \|\sup_{0\leq t\leq T} |(X_- \cdot \overline{M})_t|\|_r \leq \kappa \|\sup_{0\leq t\leq T} |X_t|\|_a \|[\overline{M}]_T^{\frac{1}{2}}\|_b$$

$$\leq \kappa \| \sup_{0\leq t\leq T} |X_t| \|_a \| [M,M]_T^{\frac{1}{2}} \|_b.$$

This ends the proof of the lemma. $\square$

The next lemma connects the solution of (4.0.1) for the case $T < +\infty$ –when it exists– to Snell envelope. Recall that $\mathcal{J}_{t\wedge\tau}^{T\wedge\tau}(\mathbb{G})$ is defined in (3.2.1).

**Lemma 4.1.3.** *Suppose that the triplet $(f, S, \xi)$ satisfies*

$$E^{\widetilde{Q}}\left[|\xi| + \int_0^{T\wedge\tau} |f(s)|ds + \sup_{0\leq u\leq \tau\wedge T} S_u^+\right] < +\infty. \tag{4.1.5}$$

*If $(Y^{\mathbb{G}}, Z^{\mathbb{G}}, M^{\mathbb{G}}, K^{\mathbb{G}})$ is a class-(D)-$(\mathbb{G}, \widetilde{Q}, T\wedge\tau)$-solution to (4.0.1), then*

$$Y_t^{\mathbb{G}} = \operatorname{ess}\sup_{\theta\in\mathcal{J}_{\mathrm{t}\wedge\tau}^{\mathrm{T}\wedge\tau}(\mathbb{G})} \mathrm{E}^{\widetilde{\mathrm{Q}}}\left[\int_{\mathrm{t}\wedge\tau}^{\theta} \mathrm{f(s)ds} + \mathrm{S}_\theta 1_{\{\theta<\mathrm{T}\wedge\tau\}} + \xi 1_{\{\theta=\mathrm{T}\wedge\tau\}} \;\Big|\; \mathcal{G}_{\mathrm{t}}\right]. \tag{4.1.6}$$

The proof of the lemma can be found in Section 4.3, while herein we state our first estimate.

**Theorem 4.1.4.** *Suppose that $T < +\infty$, and let $p \in (1, +\infty)$. Then there exists $C \in (0, +\infty)$, which depends on $p$ only, such that if $(Y^{\mathbb{G}}, Z^{\mathbb{G}}, K^{\mathbb{G}}, M^{\mathbb{G}})$ is a class-(D)-$(\mathbb{G}, \widetilde{Q}, T\wedge\tau)$-solution to (4.0.1), then*

$$\|Y^{\mathbb{G}}\|_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},p)} + \|Z^{\mathbb{G}}\|_{\mathbb{S}_{T\wedge\tau}(\widetilde{Q},p)} + \|M^{\mathbb{G}}\|_{\mathcal{M}^p(\widetilde{Q})} + \|K^{\mathbb{G}}\|_{\mathcal{A}_{T\wedge\tau}(\widetilde{Q},p)} \leq C\Delta_{\widetilde{Q}}(\xi, f, S^+), \tag{4.1.7}$$

*where*

$$\Delta_{\widetilde{Q}}(\xi, f, S^+) := \|\xi\|_{L^p(\widetilde{Q})} + \| \int_0^{T\wedge\tau} |f(s)|ds\|_{L^p(\widetilde{Q})} + \|S^+\|_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},p)}. \quad (4.1.8)$$

*Proof.* This proof is divided into two parts, where we control and estimate, in a way or another, the four terms in the left-hand-side of (4.1.7). To this end, we remark that when the right-hand-side term of (4.1.7) is infinite, then the inequality is trivial. Hence, without loss of generality, for the rest of the proof we assume that $\Delta_{\widetilde{Q}}(\xi, f, S) < +\infty$.

**Part 1.** Remark that, due to $\| \cdot \|_{L^1(\widetilde{Q})} \leq \| \cdot \|_{L^r(\widetilde{Q})}$ for $r \in [1, +\infty]$ which is a direct consequence of Hölder inequality, we get

$$E^{\widetilde{Q}}\left[|\xi| + \int_0^{T\wedge\tau} |f(s)|ds + \sup_{0\leq u\leq \tau\wedge T} S_u^+\right] \leq \Delta_{\widetilde{Q}}(\xi, f, S^+) < +\infty.$$

Then, thanks to Lemma 4.1.3, we conclude that $Y^{\mathbb{G}}$ satisfies (4.1.6), which we recall below for the reader's convenience

$$Y_t^{\mathbb{G}} = \operatorname*{ess\,sup}_{\theta\in\mathcal{J}_{t\wedge\tau}^{T\wedge\tau}(\mathbb{G})} E^{\widetilde{Q}}\left[\int_{t\wedge\tau}^{\theta} f(s)ds + S_\theta 1_{\{\theta\ <T\wedge\tau\}} + \xi 1_{\{\theta=T\wedge\tau\}} \;\Big|\; \mathcal{G}_t\right].$$

Thus, by taking $\theta = T\wedge\tau \in \mathcal{J}_{t\wedge\tau}^{T\wedge\tau}(\mathbb{G})$ and using $\xi \geq -\xi^-$ and $f(s) \geq -(f(s))^-$, we get

$$E^{\widetilde{Q}}\left[-\int_0^{T\wedge\tau} (f(s))^- ds - \xi^- \;\Big|\; \mathcal{G}_t\right] \leq Y_t^{\mathbb{G}},$$

and

$$Y_t^{\mathbb{G}} \leq E^{\widetilde{Q}}\left[\int_0^{T\wedge\tau}(f(s))^+ds + \sup_{0\leq u\leq\tau\wedge T} S_u^+ + \xi^+ \;\Big|\; \mathcal{G}_t\right].$$

This clearly leads to

$$|Y_t^{\mathbb{G}}| \leq \widetilde{N}_t := E^{\widetilde{Q}}\left[\int_0^{T\wedge\tau}|f(s)|ds + \sup_{0\leq u\leq\tau\wedge T} S_u^+ + |\xi| \;\Big|\; \mathcal{G}_t\right]. \tag{4.1.9}$$

Hence, on the one hand, by applying Doob's inequality to $\widetilde{N}$ under $(\widetilde{Q},\mathbb{G})$ we derive

$$\|Y^{\mathbb{G}}\|_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},p)} \leq \|\widetilde{N}\|_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},p)} \leq C_{DB}\Delta_{\widetilde{Q}}(\xi,f,S^+), \tag{4.1.10}$$

where $C_{DB}$ is the universal Doob's constant. On the other hand, by combining $K_{T\wedge\tau}^{\mathbb{G}} = Y_0^{\mathbb{G}} - \xi + \int_0^{T\wedge\tau} f(t)dt - M_{T\wedge\tau}^{\mathbb{G}} + \int_0^{T\wedge\tau} Z_s^{\mathbb{G}}dW_t$, (4.1.9) for $t=0$, and the Burkholder-Davis-Gunndy (BDG for short) inequalities for the $(\widetilde{Q},\mathbb{G})$-martingales $M^{\mathbb{G}}$ and $Z^{\mathbb{G}}\cdot W^\tau$, we get

$$\|K_{T\wedge\tau}^{\mathbb{G}}\|_{L^p(\widetilde{Q})} \leq 2\Delta_{\widetilde{Q}}(\xi,f,S^+) + C_{BDG}\left\{\|M^{\mathbb{G}}\|_{\mathcal{M}^p(\widetilde{Q})} + \|Z^{\mathbb{G}}\|_{\mathbb{S}_{T\wedge\tau}(\widetilde{Q},p)}\right\}. \tag{4.1.11}$$

Here $C_{BDG}$ is the universal BDG constant.

**Part 2.** A combination of Itô applied to $(Y^{\mathbb{G}})^2$ and (4.0.1) implies that

$$\begin{aligned} d(Y^{\mathbb{G}})^2 &= 2Y_-^{\mathbb{G}}dY^{\mathbb{G}} + d[Y^{\mathbb{G}},Y^{\mathbb{G}}] \\ &= -2Y_-^{\mathbb{G}}f(\cdot)d(s\wedge\tau) - 2Y_-^{\mathbb{G}}dK^{\mathbb{G}} + 2Y_-^{\mathbb{G}}Z^{\mathbb{G}}dW^\tau - 2Y_-^{\mathbb{G}}dM^{\mathbb{G}} \end{aligned}$$

$$+\, d[M^{\mathbb{G}}, M^{\mathbb{G}}] + d[K^{\mathbb{G}}, K^{\mathbb{G}}] + (Z^{\mathbb{G}})^2 d(s\wedge\tau) + 2d[K^{\mathbb{G}}, M^{\mathbb{G}}].$$

As $[M^{\mathbb{G}}, K^{\mathbb{G}}] = \Delta K^{\mathbb{G}} \cdot M^{\mathbb{G}}$, the above equality yields

$$\begin{aligned}
&[M^{\mathbb{G}}, M^{\mathbb{G}}]_{\tau\wedge T} + \int_0^{\tau\wedge T} (Z_s^{\mathbb{G}})^2 ds \\
&\leq (1+\frac{1}{\epsilon}) \sup_{0\leq s\leq \tau\wedge T} |Y_s^{\mathbb{G}}|^2 + (\xi)^2 + \left(\int_0^{\tau\wedge T} |f(s)|ds\right)^2 + \epsilon (K_{\tau\wedge T}^{\mathbb{G}})^2 \qquad (4.1.12)\\
&+ 2 \sup_{0\leq s\leq T\wedge\tau} |(\Delta K^{\mathbb{G}} \cdot M^{\mathbb{G}})_s| + 2 \sup_{0\leq s\leq \tau\wedge T} |(Y_-^{\mathbb{G}} \cdot (Z^{\mathbb{G}}\cdot W^\tau - M^{\mathbb{G}}))_s|.
\end{aligned}$$

Furthermore, thanks to (4.0.1) that implies $-\Delta Y^{\mathbb{G}} = \Delta K^{\mathbb{G}} + \Delta M^{\mathbb{G}}$, we remark that

$$|\Delta K^{\mathbb{G}}| \leq 2\overline{M}_-, \quad \text{where} \quad \overline{M}_s := E^{\widetilde{Q}}[\sup_{0\leq t\leq T\wedge\tau} |Y_t^{\mathbb{G}}| \,\big|\mathcal{G}_s].$$

Therefore, by combining this inequality, Lemma 4.1.2 applied to the last two terms in the right-hand-side of (4.1.12) with $a=b=p$, and Doob's inequality applied to $\overline{M}$, we get

$$\begin{aligned}
&\sqrt{2}\|\sqrt{|Y_-^{\mathbb{G}} \cdot (Z^{\mathbb{G}}\cdot W^\tau + M^{\mathbb{G}})|}\|_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},p)} + \sqrt{2}\|\sqrt{|\Delta K^{\mathbb{G}} \cdot M^{\mathbb{G}}|}\|_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},p)} \\
&\leq 2\kappa \|\overline{M}\|^{1/2}_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},p)} \|M^{\mathbb{G}}\|^{1/2}_{\mathcal{M}^p(\widetilde{Q})} \\
&\quad + 2\kappa \|Y^{\mathbb{G}}\|^{1/2}_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},p)} (\|M^{\mathbb{G}}\|_{\mathcal{M}^p(\widetilde{Q})} + \|Z^{\mathbb{G}}\|_{\mathbb{S}_{T\wedge\tau}(\widetilde{Q},p)})^{1/2} \\
&\leq \frac{\kappa^2(1+\sqrt{C_{DB}})^2}{\epsilon} \|Y^{\mathbb{G}}\|_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},p)} + \epsilon(\|M^{\mathbb{G}}\|_{\mathcal{M}^p(\widetilde{Q})} + \|Z^{\mathbb{G}}\|_{\mathbb{S}_{T\wedge\tau}(\widetilde{Q},p)}).
\end{aligned}$$

The last inequality is due to Young's inequality for any $\epsilon \in (0,1)$. Thus, by inserting this latter inequality and (4.1.11) in the resulting inequality from

(4.1.12) after taking square root and the norm in $L^p(\widetilde{Q})$ in both sides, and using the inequality $\|\sqrt{X+Y}\|_{L^p(\widetilde{Q})} \geq (\|\sqrt{X}\|_{L^p(\widetilde{Q})} + \|\sqrt{Y}\|_{L^p(\widetilde{Q})})/2$ for any nonnegative random variables $X$ and $Y$, we obtain

$$
\begin{aligned}
&(\frac{1}{2} - \epsilon - \sqrt{\epsilon} C_{BDG})(\|M^{\mathbb{G}}\|_{\mathcal{M}^p(\widetilde{Q})} + \|Z^{\mathbb{G}}\|_{\mathbb{S}_{T\wedge\tau}(\widetilde{Q},p)}) \\
&\leq \left\{ \frac{\kappa^2(1+\sqrt{C_{DB}})^2}{\epsilon} + \sqrt{1+\frac{1}{\epsilon}} \right\} \|Y^{\mathbb{G}}\|_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},p)} + (1+2\sqrt{\epsilon})\Delta_{\widetilde{Q}}(\xi, f, S^+).
\end{aligned}
$$

Hence, by inserting (4.1.10) in the above inequality and taking $\epsilon$ satisfying $\frac{1}{2} - \epsilon - \sqrt{\epsilon} C_{BDG} > 0$, the inequality (4.1.7) follows immediately from the resulting inequality combined with (4.1.10) and (4.1.11) again. This ends the proof of the theorem. □

The next simple lemma is useful for the proof of our last theorem of this subsection.

**Lemma 4.1.5.** *Let $(X_i)_{i\in I}$ and $(Y_i)_{i\in I}$ be two families of random variables. Then*

$$
|\operatorname*{ess\,sup}_{i\in I} X_i - \operatorname*{ess\,sup}_{i\in I} Y_i| \leq \operatorname*{ess\,sup}_{i\in I} |X_i - Y_i|, \quad P\text{-}a.s.
$$

*Proof.* Remark that, due to the essential supremum definition (see [62] for details), we obtain

$$
\operatorname*{ess\,sup}_{i\in I} X_i = \operatorname*{ess\,sup}_{i\in I}(X_i - Y_i + Y_i) \leq \operatorname*{ess\,sup}_{i\in I}(X_i - Y_i) + \operatorname*{ess\,sup}_{i\in I} Y_i.
$$

Hence, we get

$$\operatorname*{ess\,sup}_{\mathrm{i}\in\mathrm{I}}\mathrm{X_i}-\operatorname*{ess\,sup}_{\mathrm{i}\in\mathrm{I}}\mathrm{Y_i}\leq\operatorname*{ess\,sup}_{\mathrm{i}\in\mathrm{I}}(\mathrm{X_i}-\mathrm{Y_i})\leq\operatorname*{ess\,sup}_{\mathrm{i}\in\mathrm{I}}|\mathrm{X_i}-\mathrm{Y_i}|. \qquad (4.1.13)$$

Then by interchanging the roles of $X_i$ and $Y_i$ we also get

$$\operatorname*{ess\,sup}_{\mathrm{i}\in\mathrm{I}}\mathrm{Y_i}-\operatorname*{ess\,sup}_{\mathrm{i}\in\mathrm{I}}\mathrm{X_i}\leq\operatorname*{ess\,sup}_{\mathrm{i}\in\mathrm{I}}(\mathrm{Y_i}-\mathrm{X_i})\leq\operatorname*{ess\,sup}_{\mathrm{i}\in\mathrm{I}}|\mathrm{X_i}-\mathrm{Y_i}|. \qquad (4.1.14)$$

Therefore, by combining (4.1.13) and (4.1.14) we get our result. $\square$

Now, we elaborate our second main result of this subsection, which gives estimates for the difference of solutions.

**Theorem 4.1.6.** *Suppose $T<+\infty$ and that $(Y^{\mathbb{G},i},Z^{\mathbb{G},i},K^{\mathbb{G},i},M^{\mathbb{G},i})$ is a class-(D)-$(\mathbb{G},\widetilde{Q},T\wedge\tau)$-solution to the RBSDE (4.0.1) associated to $(f^{(i)},S^{(i)},\xi^{(i)})$, for each $i=1,2$. Then for $p>1$, there exist positive $C_1$ and $C_2$ that depend on $p$ only such that*

$$\begin{aligned}&\|\delta Y^{\mathbb{G}}\|_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},p)}+\|\delta Z^{\mathbb{G}}\|_{\mathbb{S}_{T\wedge\tau}(\widetilde{Q},p)}+\|\delta M^{\mathbb{G}}\|_{\mathcal{M}^p(\widetilde{Q})}\\ &\leq C_1\Delta_{\widetilde{Q}}(\delta\xi,\delta f,\delta S)+C_2\sqrt{\|\delta S\|_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},p)}}\sqrt{\sum_{i=1}^2\Delta_{\widetilde{Q}}(\xi^{(i)},f^{(i)},(S^{(i)})^+)},\end{aligned} \qquad (4.1.15)$$

*where $\Delta_{\widetilde{Q}}(\xi^{(i)},f^{(i)},(S^{(i)})^+)$ for $i=1,2$ and $\Delta_{\widetilde{Q}}(\delta\xi,\delta f,\delta S)$ are defined via*

*(4.1.8), and $\delta Y^{\mathbb{G}}, \delta Z^{\mathbb{G}}, \delta K^{\mathbb{G}}, \delta M^{\mathbb{G}}, \delta f, \delta\xi$, and $\delta S$ are given by*

$$\begin{cases} \delta Y^{\mathbb{G}} := Y^{\mathbb{G},1} - Y^{\mathbb{G},2}, \quad \delta Z^{\mathbb{G}} := Z^{\mathbb{G},1} - Z^{\mathbb{G},2}, \\ \delta M^{\mathbb{G}} := M^{\mathbb{G},1} - M^{\mathbb{G},2}, \quad \delta K^{\mathbb{G}} := K^{\mathbb{G},1} - K^{\mathbb{G},2}, \\ \delta f := f^{(1)} - f^{(2)}, \quad \delta\xi := \xi^{(1)} - \xi^{(2)}, \quad \delta S := S^{(1)} - S^{(2)}. \end{cases} \tag{4.1.16}$$

*Proof.* This proof is achieved in two parts, where we control in norm the first term and the remaining terms of the left-hand-side of (4.1.15) respectively.

**Part 1.** By combining Lemma 4.1.3, Lemma 4.1.5 and Jensen's inequality, we get the following

$$|\delta Y_t^{\mathbb{G}}| = |Y_t^{\mathbb{G},1} - Y_t^{\mathbb{G},2}| \leq E^{\widetilde{Q}}\left[\int_0^{T\wedge\tau} |\delta f(s)|ds + \sup_{0\leq s\leq T\wedge\tau} |\delta S_s| + |\delta\xi| \Big| \mathcal{G}_t\right] =: \widetilde{M}_t.$$

By applying Doob's inequality to $\widetilde{M}$ under $(\widetilde{Q}, \mathbb{G})$, we get

$$\|\delta Y^{\mathbb{G}}\|_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},p)} \leq C_{DB}\Delta_{\widetilde{Q}}(\delta\xi, \delta f, \delta S), \tag{4.1.17}$$

where $C_{DB}$ is the universal Doob's constant that depends on $p$ only.

**Part 2.** This part focuses on estimating the term $\int_0^{\cdot} (\delta Z_s^{\mathbb{G}})^2 ds + [\delta M^{\mathbb{G}}, \delta M^{\mathbb{G}}]$. Thus, we put

$$\begin{cases} \mathcal{Q}^{\mathbb{G}} := [\delta M^{\mathbb{G}}, \delta M^{\mathbb{G}}] + \int_0^{\cdot} (\delta Z_s^{\mathbb{G}})^2 d(s\wedge\tau) \\ \Gamma^{\mathbb{G}} := 2\sup_{0\leq t\leq\cdot} |(\delta Y_-^{\mathbb{G}} \cdot (\delta Z^{\mathbb{G}} \cdot W^{\tau} - \delta M^{\mathbb{G}}))_t| + 2\sup_{0\leq t\leq\cdot} |(\Delta(\delta K^{\mathbb{G}}) \cdot \delta M)_t|. \end{cases} \tag{4.1.18}$$

Then we apply Itô to $(\delta Y^{\mathbb{G}})^2$, and we get

$$
\begin{aligned}
d(\delta Y^{\mathbb{G}})^2 &= 2\delta Y^{\mathbb{G}}_- d\delta Y^{\mathbb{G}} + d[\delta Y^{\mathbb{G}}, \delta Y^{\mathbb{G}}] \\
&= -2\delta Y^{\mathbb{G}}_- \delta f(\cdot) d(s\wedge\tau) - 2\delta Y^{\mathbb{G}}_- d\delta K^{\mathbb{G}} + 2\delta Y^{\mathbb{G}}_- \delta Z^{\mathbb{G}} dW^\tau - 2\delta Y^{\mathbb{G}}_- d\delta M^{\mathbb{G}} \\
&\quad + d[\delta M^{\mathbb{G}}, \delta M^{\mathbb{G}}] + d[\delta K^{\mathbb{G}}, \delta K^{\mathbb{G}}] + (\delta Z^{\mathbb{G}})^2 d(s\wedge\tau) + 2d[\delta K^{\mathbb{G}}, \delta M^{\mathbb{G}}].
\end{aligned}
$$

Thanks to this equality and using the notation in (4.1.18), we obtain

$$
\begin{aligned}
\mathcal{Q}^{\mathbb{G}} &\leq (\delta Y^{\mathbb{G}})^2 + 2\int_0^{\cdot} \delta Y^{\mathbb{G}}_{s-}\delta f(s)ds + 2\delta Y^{\mathbb{G}}_- \cdot \delta K^{\mathbb{G}} + \Gamma^{\mathbb{G}}, \\
&\leq 2\sup_{0\leq t\leq \cdot}(\delta Y^{\mathbb{G}}_t)^2 + \left(\int_0^{\cdot} |\delta f(s)|ds\right)^2 + 2\delta S_- \cdot \delta K^{\mathbb{G}} + \Gamma^{\mathbb{G}}.
\end{aligned} \tag{4.1.19}
$$

The last inequality is due to Skorokhod's condition (i.e. it is clear that $(\delta Y^{\mathbb{G}}_- - \delta S_-)\cdot \delta K$ is nonincreasing) and Young's inequality. Furthermore, thanks to (4.0.1), we deduce that

$$
|\Delta(\delta K^{\mathbb{G}})| \leq \widehat{M}_-, \quad \text{where} \quad \widehat{M}_t := 2E^{\widetilde{Q}}[\sup_{0\leq s\leq T\wedge\tau} |\delta Y^{\mathbb{G}}_s| \, \big|\mathcal{G}_t].
$$

Thus, by combining this inequality and Lemma 4.1.2 applied to $\Gamma^{\mathbb{G}}$ with $a = b = p$, and using Doob's inequality for $\widehat{M}$ afterwards, we derive

$$
\begin{aligned}
&\|\sqrt{\Gamma^{\mathbb{G}}_{\tau\wedge T}}\|_{L^p(\widetilde{Q})} \\
&= \|\sqrt{2\sup_{0\leq t\leq\cdot}|(\delta Y^{\mathbb{G}}_-\cdot(\delta Z^{\mathbb{G}}\cdot W^\tau - \delta M^{\mathbb{G}}))_t| + 2\sup_{0\leq t\leq\cdot}|(\Delta(\delta K^{\mathbb{G}})\cdot\delta M)_t|}\|_{L^p(\widetilde{Q})} \\
&\leq \|\sqrt{2\sup_{0\leq t\leq\cdot}|\delta Y^{\mathbb{G}}_-\cdot(\delta Z^{\mathbb{G}}\cdot W^\tau)|}\|_{L^p(\widetilde{Q})} + \|\sqrt{2\sup_{0\leq t\leq\cdot}|\delta Y^{\mathbb{G}}_-\cdot\delta M^{\mathbb{G}}_t|}\|_{L^p(\widetilde{Q})} \\
&\quad + \|\sqrt{2\sup_{0\leq t\leq\cdot}|(\Delta(\delta K^{\mathbb{G}})\cdot\delta M)_t|}\|_{L^p(\widetilde{Q})}
\end{aligned}
$$

$$
\begin{aligned}
&\overset{(1)}{\leq} \kappa \| \sup_{0\leq t\leq T} |\delta Y^{\mathbb{G}}_t| \|_p^{\frac{1}{2}} \| [\delta Z^{\mathbb{G}} \cdot W^{\tau}]_T^{\frac{1}{2}} \|_p^{\frac{1}{2}} + \kappa \| \sup_{0\leq t\leq T} |\delta Y^{\mathbb{G}}_t| \|_p^{\frac{1}{2}} \| [\delta M^{\mathbb{G}}]_T^{\frac{1}{2}} \|_p^{\frac{1}{2}} \\
&\qquad + \kappa \| \sup_{0\leq t\leq T} |\widehat{M}| \|_p^{\frac{1}{2}} \| [\delta M]_T^{\frac{1}{2}} \|_p^{\frac{1}{2}} \\
&\overset{(2)}{\leq} \kappa \| \delta Y^{\mathbb{G}} \|^{\frac{1}{2}}_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},p)} \| \delta Z^{\mathbb{G}} \|^{\frac{1}{2}}_{\mathbb{S}_{T\wedge\tau}(\widetilde{Q},p)} + \kappa \| \delta Y^{\mathbb{G}} \|^{\frac{1}{2}}_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},p)} \| [\delta M^{\mathbb{G}}]_T^{\frac{1}{2}} \|_p^{\frac{1}{2}} \\
&\qquad + \kappa \sqrt{C_{DB}} \| \delta Y^{\mathbb{G}} \|^{\frac{1}{2}}_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},p)} \| [\delta M]_T^{\frac{1}{2}} \|_p^{\frac{1}{2}} \\
&= \kappa \| \delta Y^{\mathbb{G}} \|^{\frac{1}{2}}_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},p)} \| \delta Z^{\mathbb{G}} \|^{\frac{1}{2}}_{\mathbb{S}_{T\wedge\tau}(\widetilde{Q},p)} + \kappa (1 + \sqrt{C_{DB}}) \| \delta Y^{\mathbb{G}} \|^{\frac{1}{2}}_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},p)} \| \delta M^{\mathbb{G}} \|^{\frac{1}{2}}_{\mathcal{M}^p(\widetilde{Q})} \\
&\overset{(3)}{\leq} \frac{\kappa^2 (1 + (1 + \sqrt{C_{DB}})^2)}{\epsilon} \| \delta Y^{\mathbb{G}} \|_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},p)} + \epsilon ( \| \delta Z^{\mathbb{G}} \|_{\mathbb{S}_{T\wedge\tau}(\widetilde{Q},p)} + \| \delta M^{\mathbb{G}} \|_{\mathcal{M}^p(\widetilde{Q})} ).
\end{aligned}
\tag{4.1.20}
$$

Remark that the inequality (1) is due to Lemma 4.1.2 by taking $a = b = p$ which gives $r = \frac{p}{2}$, the inequality (2) is due to Doob's inequality applied to $\widehat{M}$, and the inequality (3) is a consequence of Young's inequality. By using (4.1.19) and the fact that $\|X_1 + X_2\|_{L^p(\widetilde{Q})} \geq \frac{1}{2}\left( \|X_1\|_{L^p(\widetilde{Q})} + \|X_2\|_{L^p(\widetilde{Q})} \right)$ for any random variables $X_1$and $X_2$, we get

$$
\begin{aligned}
&\frac{1}{2} ( \| \delta Z^{\mathbb{G}} \|_{\mathbb{S}_{T\wedge\tau}(\widetilde{Q},p)} + \| \delta M^{\mathbb{G}} \|_{\mathcal{M}^p(\widetilde{Q})} ) \leq \| \sqrt{\mathcal{Q}^{\mathbb{G}}_{T\wedge\tau}} \|_{L^p(\widetilde{Q})} \\
&\leq \| \delta Y^{\mathbb{G}} \|_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},p)} + \Delta_{\widetilde{Q}}(\delta\xi, \delta f, \delta S) + \sqrt{2} \| \delta S \|^{1/2}_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},p)} \| \delta K^{\mathbb{G}} \|^{1/2}_{\mathcal{A}_{T\wedge\tau}(\widetilde{Q},p)} \\
&\quad + \| \sqrt{\Gamma^{\mathbb{G}}_{\tau\wedge T}} \|_{L^p(\widetilde{Q})}.
\end{aligned}
$$

Thus, by combining this latter inequality with (4.1.20), (4.1.17), the fact that $\mathrm{Var}_{\tau\wedge T}(\delta K^{\mathbb{G}}) \leq K^{\mathbb{G},1}_{\tau\wedge T} + K^{\mathbb{G},2}_{\tau\wedge T}$, and Theorem 4.1.4 applied to each $K^{\mathbb{G},i}$, $i = 1, 2$, the proof of the theorem follows with $\epsilon \in (0, 0.5)$, $C_1 = C_{DB} + C_{DB}\left(\epsilon + \kappa^2(1 + (1 + \sqrt{C_{DB}})^2)\right)/(\epsilon(0.5 - \epsilon))$ and $C_2 := \sqrt{2C}(0.5 - \epsilon)^{-1}$ where $C$ is the constant in Theorem 4.1.4. $\square$

### 4.1.2 Existence for the $\mathbb{G}$-RBSDE and its relationship to $\mathbb{F}$-RBSDE

In this subsection, we prove the existence and the uniqueness of the solution to the RBSDE (4.0.1) when $T < +\infty$, we establish an explicit connection between this RBSDE and its $\mathbb{F}$-RBSDE counterpart, and we highlight the explicit relationship between their solutions as well. To this end, we sate the following lemma

**Lemma 4.1.7.** *Let $\widetilde{\mathcal{E}} := \mathcal{E}\left(-\widetilde{G}^{-1} \cdot D^{o,\mathbb{F}}\right)$ and let $L$ be an $\mathbb{F}$-semimartingale. Then we always have*

$$L\widetilde{\mathcal{E}}^{-1} I_{[\![0,\tau[\![} + L\widetilde{\mathcal{E}}^{-1} \cdot N^{\mathbb{G}} = \widetilde{\mathcal{E}}_{-}^{-1} \cdot L^{\tau}.$$

The proof is given in Section 4.3, while below we state our main result.

**Theorem 4.1.8.** *Suppose that $T < +\infty$ and there is $p \in (1, \infty)$ such that*

$$\left\| \int_0^{T\wedge\tau} |f(s)|ds + |\xi| + \sup_{0\leq u\leq \tau\wedge T} S_u^{+} \right\|_{L^p(\widetilde{Q})} < +\infty, \tag{4.1.21}$$

*and consider $(f^{\mathbb{F}}, S^{\mathbb{F}}, \xi^{\mathbb{F}})$ and $V^{\mathbb{F}}$ given by*

$$f^{\mathbb{F}} := \widetilde{\mathcal{E}} f,\ S^{\mathbb{F}} := \widetilde{\mathcal{E}} S,\ \xi^{\mathbb{F}} := \widetilde{\mathcal{E}}_T h_T,\ V^{\mathbb{F}} := 1 - \widetilde{\mathcal{E}},\ \widetilde{\mathcal{E}} := \mathcal{E}\left(-\widetilde{G}^{-1} \cdot D^{o,\mathbb{F}}\right). \tag{4.1.22}$$

*Then the following assertions hold.*

(a) *The following RBSDE under* $\mathbb{F}$*, associated to the triplet* $\left(f^{\mathbb{F}}, S^{\mathbb{F}}, \xi^{\mathbb{F}}\right)$*,*

$$\begin{cases} Y_t = \xi^{\mathbb{F}} + \displaystyle\int_t^T f^{\mathbb{F}}(s)ds + \int_t^T h_s dV_s^{\mathbb{F}} + K_T - K_t - \int_t^T Z_s dW_s, \\ Y_t \geq S_t^{\mathbb{F}} 1_{\{t\ <T\}} + \xi^{\mathbb{F}} 1_{\{t\ =T\}}, \quad \displaystyle\int_0^T (Y_{t-} - S_{t-}^{\mathbb{F}})dK_t = 0, \quad P\text{-}a.s., \end{cases} \tag{4.1.23}$$

*has a unique* $L^p(P, \mathbb{F})$*-solution* $(Y^{\mathbb{F}}, Z^{\mathbb{F}}, K^{\mathbb{F}})$ *satisfying*

$$Y_t^{\mathbb{F}} = \operatorname{ess} \sup_{\sigma \in \mathcal{J}_t^T(\mathbb{F})} \mathrm{E}\left[\int_t^\sigma f^{\mathbb{F}}(s)ds + \int_t^\sigma h_s dV_s^{\mathbb{F}} + S_\sigma^{\mathbb{F}} 1_{\{\sigma<T\}} + \xi^{\mathbb{F}} I_{\{\sigma=T\}} \;\Big|\; \mathcal{F}_t\right]. \tag{4.1.24}$$

(b) *The RBSDE defined in (4.0.1) has a unique* $L^p(\widetilde{Q}, \mathbb{G})$*-solution denoted by* $(Y^{\mathbb{G}}, Z^{\mathbb{G}}, K^{\mathbb{G}}, M^{\mathbb{G}})$*, and*

$$\begin{cases} Y^{\mathbb{G}} = \dfrac{Y^{\mathbb{F}}}{\widetilde{\mathcal{E}}} I_{[\![0,\tau[\![} + \xi I_{[\![\tau,+\infty[\![}, \ Z^{\mathbb{G}} = \dfrac{Z^{\mathbb{F}}}{\widetilde{\mathcal{E}}_-} I_{]\!]0,\tau]\!]}, \\ K^{\mathbb{G}} = \dfrac{1}{\widetilde{\mathcal{E}}_-} \cdot (K^{\mathbb{F}})^\tau \ \textit{and}\ M^{\mathbb{G}} = \left(h - \dfrac{Y^{\mathbb{F}}}{\widetilde{\mathcal{E}}}\right) \cdot N^{\mathbb{G}}. \end{cases} \tag{4.1.25}$$

*Proof.* Assertion (a) is the linear case of a general RBSDE under $\mathbb{F}$ given in Subsection 5.1.2, see (5.1.27). Thus, the proof of the existence and uniqueness of the $L^p(\mathbb{F}, P)$-solution under (4.1.21) of this RBSDE will be omitted here, and we refer the reader to Subsection 5.1.2. Furthermore, the proof of (4.1.24) mimics exactly the proof of (4.1.6). Therefore, the remaining part of this proof deals with assertion (b).

Thanks to the theory of Snell envelop, see [41] for details, there exists a triplet $(Z^{\mathbb{G}}, K^{\mathbb{G}}, M^{\mathbb{G}})$ such that the quadruplet $(Y^{\mathbb{G}}, Z^{\mathbb{G}}, K^{\mathbb{G}}, M^{\mathbb{G}})$ is a solution to the

RBSDE (4.0.1), where

$$\mathrm{Y}_t^{\mathbb{G}} := \operatorname{ess}\sup_{\theta\in\mathcal{J}_{\mathrm{t}\wedge\tau}^{\mathrm{T}\wedge\tau}(\mathbb{G})} \mathrm{E}^{\widetilde{\mathrm{Q}}}\left[\int_{\mathrm{t}\wedge\tau}^{\theta}\mathrm{f(s)ds} + \mathrm{S}_\theta 1_{\{\theta<\mathrm{T}\wedge\tau\}} + \xi 1_{\{\theta=\mathrm{T}\wedge\tau\}} \;\Big|\; \mathcal{G}_{\mathrm{t}}\right]. \quad (4.1.26)$$

Furthermore, it is easy to prove that this solution is a class-(D)-$(\mathbb{G},\widetilde{Q},T\wedge\tau)$-solution. Thus, by combining this with Theorem 4.1.4, we conclude the existence of an $L^p(\widetilde{Q},\mathbb{G})$-solution to (4.0.1). The uniqueness of $L^p(\widetilde{Q},\mathbb{G})$-solution to (4.0.1) follows immediately from Theorem 4.1.6. Hence, the rest of this proof focuses on proving the relationship (4.1.25). To this end, on the one hand, thanks to the Doob-Meyer decomposition under $(\widetilde{Q},\mathbb{G})$, we remark that for any solution $(Y,Z,K,M)$ to (4.0.1) we have $(Y,Z,K,M) = (Y^{\mathbb{G}},Z^{\mathbb{G}},K^{\mathbb{G}},M^{\mathbb{G}})$ if and only if $Y=Y^{\mathbb{G}}$. On the other hand, due to (4.1.26), we have

$$\begin{cases} Y^{\mathbb{G}} + \int_0^{\tau\wedge T\wedge\cdot} f(s)ds = \mathcal{S}(X^{\mathbb{G}};\mathbb{G},\widetilde{Q}) \\ \text{with} \\ X^{\mathbb{G}} := \int_0^{\tau\wedge T\wedge\cdot} f(s)ds + SI_{[\![0,\tau\wedge T[\![} + h_{\tau\wedge T}I_{[\![\tau\wedge T,+\infty[\![}. \end{cases}$$

Therefore, in order to apply Theorem 3.2.3-(b), we need to find the unique pair $(X^{\mathbb{F}},k^{(pr)})$ associated to $X^{\mathbb{G}}$. To this end, we remark that

$$SI_{[\![0,\tau\wedge T[\![} = SI_{[\![0,\tau[\![}I_{[\![0,T[\![} \quad \text{and} \quad h_{\tau\wedge T}I_{[\![\tau\wedge T,+\infty[\![}I_{[\![0,\tau[\![} = h_T I_{[\![0,\tau[\![}I_{[\![T,+\infty[\![},$$

and derive

$$X^{\mathbb{F}} = \int_0^{T\wedge\cdot} f(s)ds + SI_{[\![0,T[\![} + h_T I_{[\![T,+\infty[\![}, \quad k^{(pr)} = \int_0^{T\wedge\cdot} f(s)ds + h_{T\wedge\cdot} = k^{(op)}.$$

Furthermore, it is clear that we have

$$\begin{cases} \widetilde{\mathcal{E}}X^{\mathbb{F}} - k^{(op)} \cdot \widetilde{\mathcal{E}} = \int_0^{T\wedge\cdot} f^{\mathbb{F}}(s)ds + S^{\mathbb{F}} I_{[\![0,T[\![} + (h\cdot V^{\mathbb{F}})^T + \xi^{\mathbb{F}} I_{[\![T,+\infty[\![}, \\ k^{(op)}\widetilde{\mathcal{E}} - k^{(op)} \cdot \widetilde{\mathcal{E}} = \int_0^{T\wedge\cdot} f^{\mathbb{F}}(s)ds + (h\cdot V^{\mathbb{F}})^T + \widetilde{\mathcal{E}}hI_{[\![0,T[\![} + \xi^{\mathbb{F}} I_{[\![T,+\infty[\![}, \end{cases}$$

and

$$Y^{\mathbb{F}} + L^{\mathbb{F}} = \mathcal{S}\left(L^{\mathbb{F}} + \xi^{\mathbb{F}} I_{[\![T,+\infty[\![} + S^{\mathbb{F}} I_{[\![0,T[\![}; \mathbb{F}, P\right),$$

where

$$L^{\mathbb{F}} := \int_0^{T\wedge\cdot} f^{\mathbb{F}}(s)ds + \int_0^{T\wedge\cdot} h_s dV_s^{\mathbb{F}}.$$

Thus, by directly applying Theorem 3.2.3-(b) to $Y^{\mathbb{G}}$, on $[\![0,T]\!]$, we obtain

$$\begin{aligned} Y^{\mathbb{G}} &+ \int_0^{\tau\wedge T\wedge\cdot} f(s)ds = \mathcal{S}(X^{\mathbb{G}}; \mathbb{G}, \widetilde{Q}) \\ &= \frac{\mathcal{S}\left(X^{\mathbb{F}}\widetilde{\mathcal{E}} - k^{(op)}\cdot\widetilde{\mathcal{E}}; \mathbb{F}, P\right)}{\widetilde{\mathcal{E}}} I_{[\![0,\tau[\![} + \frac{L^{\mathbb{F}} + \widetilde{\mathcal{E}}hI_{[\![0,T[\![} + \xi^{\mathbb{F}} I_{[\![T,+\infty[\![}}{\widetilde{\mathcal{E}}} \cdot N^{\mathbb{G}} \\ &= \frac{Y^{\mathbb{F}} + L^{\mathbb{F}}}{\widetilde{\mathcal{E}}} I_{[\![0,\tau[\![} + \frac{L^{\mathbb{F}}}{\widetilde{\mathcal{E}}} \cdot N^{\mathbb{G}} + \left(hI_{[\![0,T[\![} + h_T I_{[\![T,+\infty[\![}\right) \cdot N^{\mathbb{G}} \\ &= \frac{Y^{\mathbb{F}}}{\widetilde{\mathcal{E}}} I_{[\![0,\tau[\![} + \frac{1}{\widetilde{\mathcal{E}}_-} \cdot (L^{\mathbb{F}})^\tau + h_{T\wedge\cdot} \cdot D - \frac{h}{\widetilde{G}} I_{]\!]0,\tau\wedge T]\!]} \cdot D^{o,\mathbb{F}} \\ &= \frac{Y^{\mathbb{F}}}{\widetilde{\mathcal{E}}} I_{[\![0,\tau[\![} + \int_0^{\tau\wedge T\wedge\cdot} f(s)ds + \xi I_{[\![\tau,+\infty[\![}. \end{aligned}$$

The fourth equality follows from Lemma 4.1.7. This proves assertion (b), and the proof of the theorem is complete. $\square$

We end this section by the following important remark that will be useful.

**Remark 4.1.9.** *It worths mentioning (and easy to check) that the main results of this section (especially Theorems 4.1.4,4.1.6 and 4.1.8) remain valid if we replace $T$ with any bounded $\mathbb{F}$-stopping time $\sigma$. In this case, one should use the probability $\widetilde{Q}_\sigma := \widetilde{Z}_{\sigma\wedge\tau} \cdot P$ instead of $\widetilde{Q}$.*

## 4.2 The case of unbounded horizon

In this section, we let $T$ to be infinite in the RBSDE (4.0.1), and get

$$\begin{cases} dY = -f(t)d(t\wedge\tau) - d(K+M) + ZdW^\tau, \quad Y_\tau = \xi = h_\tau, \\ Y_t \geq S_t; \quad 0 \leq t < \tau, \quad \displaystyle\int_0^\tau (Y_{t-} - S_{t-})dK_t = 0, \quad P\text{-a.s..} \end{cases} \tag{4.2.1}$$

It is important to mention that $\widetilde{Q}$ (defined in (2.3.11)) depends heavily on the finite horizon planning $T$, and in general the process $\widetilde{Z}^\tau$ defined in (2.3.10) might not be a $\mathbb{G}$-uniformly integrable martingale, see [29] for details. Thus, the fact of letting $T$ goes to infinity triggers serious challenges in both technical and conceptual sides. In fact, both the condition (4.1.5) and the RBSDE (4.1.23) might not make sense when we take $T$ to infinity, as the limit of $h_T$ when $T$ goes to infinity might not exist even. The rest of this section is divided into two subsections. The first subsection focuses on existence and uniqueness of the solution to (4.2.1), while the second subsection deals with the $\mathbb{F}$-RBSDE counterpart to it.

### 4.2.1 Estimates under $P$ for the solution of (4.0.1)

Our approach to the aforementioned challenges has two steps. The first step relies on the following lemma and the two theorems that follow it, and aims to get rid-off of $\widetilde{Q}$ in the left-hand-sides of the estimates of Theorems 4.1.4 and 4.1.6. The second step addresses the limit of the right-hand-side terms in the estimates of these theorems.

**Lemma 4.2.1.** *Let $T \in (0,+\infty)$, $\widetilde{Q}$ be the probability given in (2.3.11), and $\widetilde{\mathcal{E}}$ be the process defined in (4.1.22). Then the following assertions hold.*

(a) *For any $p \in (1,+\infty)$ and any RCLL $\mathbb{G}$-semimartingale $Y$, we have*

$$E\left[\sup_{0\leq s\leq T\wedge\tau} \widetilde{\mathcal{E}}_s |Y_s|^p\right] \leq G_0^{-1} E^{\widetilde{Q}}\left[\sup_{0\leq s\leq T\wedge\tau} |Y_s|^p\right]. \tag{4.2.2}$$

(b) *For any $a \in (0,+\infty)$, we put $\kappa(a) := 3^{1/a}(5 + (\max(a,a^{-1}))^{1/a})$. Then for any RCLL, nondecreasing and $\mathbb{G}$-optional process $K$ with $K_0 = 0$, we have*

$$E\left[\left(\int_0^{T\wedge\tau} (\widetilde{\mathcal{E}}_{s-})^a dK_s\right)^{1/a}\right] \leq \kappa(a) G_0^{-1} E^{\widetilde{Q}}\left[K_{T\wedge\tau}^{1/a} + \sum_{0<s\leq T\wedge\tau} \widetilde{G}_s (\Delta K_s)^{1/a}\right]. \tag{4.2.3}$$

(c) *For any $p > 1$ and any nonnegative and $\mathbb{G}$-optional process $H$, we have*

$$\begin{aligned} &E\left[(\widetilde{\mathcal{E}}_-^{2/p} H \cdot [N^{\mathbb{G}}, N^{\mathbb{G}}])_{T\wedge\tau}^{p/2}\right] \\ &\leq \kappa(a) G_0^{-1} E^{\widetilde{Q}}\left[(H \cdot [N^{\mathbb{G}}, N^{\mathbb{G}}]_{T\wedge\tau})^{p/2} + (H^{p/2}\widetilde{G} \cdot Var(N^{\mathbb{G}}))_{T\wedge\tau}\right]. \end{aligned} \tag{4.2.4}$$

(d) *For any $p > 1$ and any nonnegative and $\mathbb{F}$-optional process $H$, we have*

$$\begin{aligned} &E\left[(\widetilde{\mathcal{E}}_-^{2/p} H \cdot [N^{\mathbb{G}}, N^{\mathbb{G}}])_{T\wedge\tau}^{p/2}\right] \\ &\leq \kappa(a) G_0^{-1} E^{\widetilde{Q}}\left[(H \cdot [N^{\mathbb{G}}, N^{\mathbb{G}}]_{T\wedge\tau})^{p/2} + 2(H^{p/2} I_{]\!]0,\tau[\![} \cdot D^{o,\mathbb{F}})_T\right]. \end{aligned} \quad (4.2.5)$$

For the sake of simple exposition, we give the proof of this lemma in Section 4.3. In the following, using this lemma, we elaborate estimates for the solution to (4.0.1) under the probability $P$ instead.

**Theorem 4.2.2.** *Suppose $T \in (0, +\infty)$. For any $p > 1$, there exists a positive constant $\widetilde{C}$ that depends on $p$ only such that the unique $L^p(\widetilde{Q}, \mathbb{G})$-solution to (4.0.1), denoted by $(Y^{\mathbb{G}}, Z^{\mathbb{G}}, K^{\mathbb{G}}, M^{\mathbb{G}})$, satisfies*

$$\begin{aligned} &\|\sqrt[p]{\widetilde{\mathcal{E}}} Y^{\mathbb{G}}\|_{\mathbb{D}_{T\wedge\tau}(P,p)} + \|\sqrt[p]{\widetilde{\mathcal{E}}_-} Z^{\mathbb{G}}\|_{\mathbb{S}_{T\wedge\tau}(P,p)} + \|\sqrt[p]{\widetilde{\mathcal{E}}_-} \cdot K^{\mathbb{G}}\|_{\mathcal{A}_{T\wedge\tau}(P,p)} \\ &+ \|\sqrt[p]{\widetilde{\mathcal{E}}_-} \cdot M^{\mathbb{G}}\|_{\mathcal{M}^p(P)} \\ &\leq \widetilde{C}\left\{\|\xi\|_{L^p(\widetilde{Q})} + \|\int_0^{T\wedge\tau} |f(s)| ds\|_{L^p(\widetilde{Q})} + \|S^+\|_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},p)}\right\} =: \widetilde{C}\Delta_{\widetilde{Q}}(\xi, f, S^+). \end{aligned}$$

*where $\widetilde{\mathcal{E}}$ is the process given by (4.1.22).*

*Proof.* By applying Lemma 4.2.1-(b) to $K_t = \int_0^{t\wedge\tau} (Z_s^{\mathbb{G}})^2 ds$ and $a = 2/p$, we get

$$E\left[\left(\int_0^{T\wedge\tau} (\widetilde{\mathcal{E}}_{s-})^{2/p} (Z_s^{\mathbb{G}})^2 ds\right)^{p/2}\right] \leq \kappa(a) G_0^{-1} E^{\widetilde{Q}}\left[\left(\int_0^{T\wedge\tau} (Z_s^{\mathbb{G}})^2 ds\right)^{p/2}\right]. \quad (4.2.6)$$

By applying Lemma 4.2.1-(a) to the process $Y = Y^{\mathbb{G}}$, we obtain

$$E\left[\sup_{0\leq s\leq T\wedge\tau}\widetilde{\mathcal{E}}_s|Y_s^{\mathbb{G}}|^p\right]\leq G_0^{-1}E^{\widetilde{Q}}\left[\sup_{0\leq s\leq T\wedge\tau}|Y_s^{\mathbb{G}}|^p\right]. \tag{4.2.7}$$

By applying Lemma 4.2.1-(b) to the process $K = K^{\mathbb{G}}$ and $a = 1/p$, and using the fact that we always have $\sum_{0<s\leq_{T\wedge\tau}}\widetilde{G}_s(\Delta K_s^{\mathbb{G}})^p \leq (K_{T\wedge\tau}^{\mathbb{G}})^p$, we derive

$$\begin{aligned} E\left[\left(\int_0^{T\wedge\tau}(\widetilde{\mathcal{E}}_{s-})^{1/p}dK_s^{\mathbb{G}}\right)^p\right] &\leq \frac{\kappa(a)}{G_0}E^{\widetilde{Q}}\left[(K_{T\wedge\tau}^{\mathbb{G}})^p+\sum_{0\leq s\leq_{T\wedge\tau}}\widetilde{G}_s(\Delta K_s^{\mathbb{G}})^p\right]\\ &\leq \frac{2\kappa(a)}{G_0}E^{\widetilde{Q}}\left[(K_{T\wedge\tau}^{\mathbb{G}})^p\right]. \end{aligned}\tag{4.2.8}$$

Thanks to Theorem 4.1.8, we have $[M^{\mathbb{G}}, M^{\mathbb{G}}] = H\cdot[N^{\mathbb{G}}, N^{\mathbb{G}}]$ with $H := (h - Y^{\mathbb{F}}\widetilde{\mathcal{E}}^{-1})^2$ being a nonnegative and $\mathbb{F}$-optional process. Thus, a direct application of Lemma 4.2.1-(d) yields

$$\begin{aligned} &E\left[(\widetilde{\mathcal{E}}_-^{2/p}H\cdot[N^{\mathbb{G}}, N^{\mathbb{G}}])_{T\wedge\tau}^{p/2}\right]\\ &\leq \frac{C(p)}{G_0}E^{\widetilde{Q}}\left[(H\cdot[N^{\mathbb{G}}, N^{\mathbb{G}}]_{T\wedge\tau})^{p/2}+2(H^{p/2}I_{]\!]0,\tau[\![}\cdot D^{o,\mathbb{F}})_T\right]. \end{aligned}\tag{4.2.9}$$

To control the second term in the right-hand-side of the above inequality, we remark that $(H^{p/2}I_{]\!]0,\tau[\![}\cdot D^{o,\mathbb{F}}) \leq 2^{p-1}(|h|^p + |Y^{\mathbb{G}}|^p)I_{]\!]0,\tau[\![}\cdot D^{o,\mathbb{F}}$, and we derive

$$2E^{\widetilde{Q}}\left[(H^{p/2}I_{]\!]0,\tau[\![}\cdot D^{o,\mathbb{F}})_T\right]\leq 2^pE^{\widetilde{Q}}\left[|h_\tau|^pI_{\{\tau\leq T\}}\right]+2^pE^{\widetilde{Q}}\left[\sup_{0\leq t\leq\tau\wedge T}|Y_t^{\mathbb{G}}|^p\right].$$

Therefore, by combining this inequality with $h_\tau I_{\{\tau\leq T\}} = \xi I_{\{\tau\leq T\}}$, (4.2.9), (4.2.8), (4.2.7), (4.2.6) and Theorem 4.1.4, the proof of the theorem follows immediately. $\square$

Similarly, the following theorem gives a version of Theorem 4.1.6 where the left-hand-side of its estimate does not involve the probability $\widetilde{Q}$.

**Theorem 4.2.3.** *Suppose that $T < +\infty$, and let $(Y^{\mathbb{G},i}, Z^{\mathbb{G},i}, K^{\mathbb{G},i}, M^{\mathbb{G},i})$ be the $L^p(\widetilde{Q}, \mathbb{G})$-solution to (4.0.1) associated to $(f^{(i)}, S^{(i)}, \xi^{(i)})$, for $i = 1, 2$. Then, there exist positive constants $\widetilde{C}_1$ and $\widetilde{C}_2$ which depend on $p$ only such that*

$$\begin{aligned}
&\|\sqrt[p]{\widetilde{\mathcal{E}}}\delta Y^{\mathbb{G}}\|_{\mathbb{D}_{T\wedge\tau}(P,p)} + \|\sqrt[p]{\widetilde{\mathcal{E}}_-}\delta Z^{\mathbb{G}}\|_{\mathbb{S}_{T\wedge\tau}(P,p)} + \|\sqrt[p]{\widetilde{\mathcal{E}}_-}\cdot\delta M^{\mathbb{G}}\|_{\mathcal{M}^p_{T\wedge\tau}(P)}\\
&\leq \widetilde{C}_1\Delta_{\widetilde{Q}}(\delta\xi, \delta f, \delta S) + \widetilde{C}_2\|\delta S\|^{1/2}_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},p)}\sqrt{\sum_{i=1}^{2}\Delta_{\widetilde{Q}}(\xi^{(i)}, f^{(i)}, (S^{(i)})^+)}. \qquad (4.2.10)
\end{aligned}$$

*Here $\Delta_{\widetilde{Q}}(\xi^{(i)}, f^{(i)}, (S^{(i)})^+)$ for $i = 1, 2$ and $\Delta_{\widetilde{Q}}(\delta\xi, \delta f, \delta S)$ are given via (4.1.8), while $(\delta Y^{\mathbb{G}}, \delta Z^{\mathbb{G}}, \delta M^{\mathbb{G}}, \delta K^{\mathbb{G}})$ and $(\delta\xi, \delta f, \delta S)$ are defined in (4.1.16).*

*Proof.* By applying Lemma 4.2.1-(a) to $Y = \delta Y^{\mathbb{G}}$ and $a = 1/p$, we deduce that

$$E\left[\sup_{0\leq t\leq T}\widetilde{\mathcal{E}}_t|\delta Y^{\mathbb{G}}_t|^p\right] \leq \kappa E^{\widetilde{Q}}\left[\sup_{0\leq t\leq T}|\delta Y^{\mathbb{G}}_t|^p\right]. \qquad (4.2.11)$$

To control the remaining terms in the left-hand-side of (4.2.10), we apply Lemma 4.2.1-(b) to $K = \int_0^{\cdot}(\delta Z^{\mathbb{G}}_s)^2 ds + [\delta M^{\mathbb{G}}, \delta M^{\mathbb{G}}]$ and $a = 2/p$, and get

$$E\left[\left(\int_0^{T\wedge\tau}(\widetilde{\mathcal{E}}_{s-})^{2/p}(\delta Z^{\mathbb{G}}_s)^2 ds + \int_0^{T\wedge\tau}(\widetilde{\mathcal{E}}_{s-})^{2/p} d[\delta M^{\mathbb{G}}, \delta M^{\mathbb{G}}]_s\right)^{p/2}\right]$$

$$\leq \kappa E^{\widetilde{Q}}\left[\left(\int_0^{T\wedge\tau}(\delta Z_s^{\mathbb{G}})^2 ds + [\delta M^{\mathbb{G}}, \delta M^{\mathbb{G}}]_{T\wedge\tau}\right)^{p/2} + \sum_{0\leq t\leq T\wedge\tau}\widetilde{G}_t|\Delta(\delta M^{\mathbb{G}})_t|^p\right]. \tag{4.2.12}$$

Then by using $\Delta(\delta M^{\mathbb{G}}) = (\delta h - \delta Y^{\mathbb{F}}\widetilde{\mathcal{E}}^{-1})\Delta N^{\mathbb{G}} =: H\Delta N^{\mathbb{G}}$ and $\widetilde{\mathcal{E}}^{-1}\delta Y^{\mathbb{F}} = \delta Y^{\mathbb{G}}$ on $\rrbracket 0, \tau\llbracket$ –see Theorem 4.1.8 –, and by mimicking parts 3 and 4 in the proof of Lemma 4.2.1, we derive

$$\begin{aligned}
&E^{\widetilde{Q}}\left[\sum_{0\leq t\leq T\wedge\tau}\widetilde{G}_t|\Delta(\delta M^{\mathbb{G}})_t|^p\right] \\
&\leq E^{\widetilde{Q}}\left[\widetilde{G}|H|^p \cdot \mathrm{Var}(N^{\mathbb{G}})_T\right] = 2E^{\widetilde{Q}}\left[|H|^p I_{\rrbracket 0,\tau\llbracket} \cdot D_T^{o,\mathbb{F}}\right] \\
&\leq 2^p E^{\widetilde{Q}}\left[(|\delta h|^p + |\delta Y^{\mathbb{G}}|^p) I_{\rrbracket 0,\tau\llbracket} \cdot D_T^{o,\mathbb{F}}\right] \leq 2^p E^{\widetilde{Q}}\left[|\delta\xi|^p + \sup_{0\leq t\leq T\wedge\tau}|\delta Y^{\mathbb{G}}|^p\right].
\end{aligned} \tag{4.2.13}$$

Therefore, by combining (4.2.11), (4.2.12), (4.2.13) and Theorem 4.1.6, the proof of the theorem follows immediately. □

Our second step in solving (4.2.1) relies on the following lemma, and focuses on simultaneously letting $T$ to go to infinity and getting rid-off $\widetilde{Q}$ in the norms of the data-triplet.

**Lemma 4.2.4.** *Let $X$ be a non-negative and $\mathbb{F}$-optional process with $X_0 = 0$. Then the following assertions hold.*

(a) *For any $T \in (0, \infty)$, we always have*

$$E^{\widetilde{Q}}[X_{T\wedge\tau}] = E^{\widetilde{Q}}[X_\tau I_{\{\tau\leq T\}}] + G_0 E[X_T \widetilde{\mathcal{E}}_T] = G_0 E\left[\int_0^T X_s dV_s^{\mathbb{F}} + X_T\widetilde{\mathcal{E}}_T\right]. \tag{4.2.14}$$

(b) *If $X/\mathcal{E}(G_-^{-1}\cdot m)$ is bounded, then we get*

$$\lim_{T\to\infty} E^{\widetilde{Q}}[X_{T\wedge\tau}] = G_0\|X\|_{L^1(P\otimes V^{\mathbb{F}})} := G_0 E\left[\int_0^\infty X_s dV_s^{\mathbb{F}}\right]. \qquad (4.2.15)$$

The proof of this lemma can be found in Section 4.3.

### 4.2.2 Existence and uniqueness for (4.2.1): New spaces and norms for the solution and data

It is clear that Lemma 4.2.4 allows us to take the limit of $\widetilde{Q}$-expectations, under some conditions. More importantly, on the one hand, this leads *naturally* to the space $L^p(\Omega\times[0,+\infty),\mathcal{F}\otimes\mathcal{B}(\mathbb{R}^+),P\otimes V^{\mathbb{F}})$ for the data-triplet $(f,h,S)$, endowed with its norm defined –for any $\mathcal{F}\otimes\mathcal{B}(\mathbb{R}^+)$-measurable $X$– by

$$\|X\|^p_{L^p(P\otimes V^{\mathbb{F}})} := E\left[\int_0^\infty |X_t|^p dV_t^{\mathbb{F}}\right]. \qquad (4.2.16)$$

On the other hand, the pair $(Y^{\mathbb{G}},Z^{\mathbb{G}})$ in the solution of (4.0.1) when $T<+\infty$ belongs to $\widetilde{\mathbb{D}}_\sigma(P,p)\times\widetilde{\mathbb{S}}_\sigma(P,p)$, with $\sigma=T\wedge\tau$. The two spaces, which appear *naturally* in our analysis, are given as follows: $(Y,Z)\in\widetilde{\mathbb{D}}_\sigma(P,p)\times\widetilde{\mathbb{S}}_\sigma(P,p)$ if and only if $\sqrt[p]{\widetilde{\mathcal{E}}}(Y,Z)\in\mathbb{D}_\sigma(P,p)\times\mathbb{S}_\sigma(P,p)$. Furthermore the norms of these spaces are defined by

$$\|Y\|_{\widetilde{\mathbb{D}}_\sigma(P,p)} := \|Y\sqrt[p]{\widetilde{\mathcal{E}}}\|_{\mathbb{D}_\sigma(P,p)} \quad \text{and} \quad \|Z\|_{\widetilde{\mathbb{S}}_\sigma(P,p)} := \|Z\sqrt[p]{\widetilde{\mathcal{E}}_-}\|_{\mathbb{S}_\sigma(P,p)}. \qquad (4.2.17)$$

Similarly, for the remaining pair $(K^{\mathbb{G}},M^{\mathbb{G}})$ in the solution of (4.0.1) when $T<+\infty$, we take the norm of the "discounted" pair $(\widetilde{\mathcal{E}}_-^{1/p}\cdot K^{\mathbb{G}},\widetilde{\mathcal{E}}_-^{1/p}\cdot M^{\mathbb{G}})$

under $P$ instead of that of $(K^{\mathbb{G}}, M^{\mathbb{G}})$ under $\widetilde{Q}$.

Below, we elaborate our principal result of this subsection.

**Theorem 4.2.5.** *Let $p \in (1, +\infty)$, and suppose $G > 0$ and $(f, S, h)$ satisfies*

$$\|F + |h| + \sup_{0 \leq u \leq \cdot} |S_u|\|_{L^p(P \otimes V^{\mathbb{F}})} < +\infty, \quad \text{where} \quad F_t := \int_0^t |f(s)| ds. \tag{4.2.18}$$

*Then the following assertions hold.*

(a) *The RBSDE (4.2.1) admits a unique solution $\left(Y^{\mathbb{G}}, Z^{\mathbb{G}}, K^{\mathbb{G}}, M^{\mathbb{G}}\right)$.*

(b) *There exists a positive constant $C$, that depends on $p$ only, such that*

$$\begin{aligned}
&\|Y^{\mathbb{G}}\|_{\widetilde{\mathbb{D}}_\tau(P,p)} + \|Z^{\mathbb{G}}\|_{\widetilde{\mathbb{S}}_\tau(P,p)} + \|\sqrt[p]{\widetilde{\mathcal{E}}_-} \cdot M^{\mathbb{G}}\|_{\mathcal{M}^p(P)} + \|\sqrt[p]{\widetilde{\mathcal{E}}_-} \cdot K^{\mathbb{G}}\|_{\mathcal{A}_\tau(P,p)} \\
&\leq C\|F + |h| + \sup_{0 \leq s \leq \cdot} S_u^+\|_{L^p(P \otimes V^{\mathbb{F}})}.
\end{aligned}$$

(c) *Consider two triplet $(f^{(i)}, S^{(i)}, h^{(i)})$, $i = 1, 2$ satisfying (4.2.18). If the quadruplet $\left(Y^{\mathbb{G},i}, Z^{\mathbb{G},i}, K^{\mathbb{G},i}, M^{\mathbb{G},i}\right)$ denotes the solution to (4.2.1) associated with $(f^{(i)}, S^{(i)}, h^{(i)})$ for each $i = 1, 2$, then there exist positive $C_1$ and $C_2$ that depend on $p$ only such that*

$$\begin{aligned}
&\|\delta Y^{\mathbb{G}}\|_{\widetilde{\mathbb{D}}_\tau(P,p)} + \|\delta Z^{\mathbb{G}}\|_{\widetilde{\mathbb{S}}_\tau(P,p)} + \|\sqrt[p]{\widetilde{\mathcal{E}}_-} \cdot \delta M^{\mathbb{G}}\|_{\mathcal{M}^p(P)} \\
&\leq C_1\||\delta h| + |\delta F| + \sup_{0 \leq u \leq \cdot} |\delta S_u|\|_{L^p(P \otimes V^{\mathbb{F}})} \\
&+ C_2 \sqrt{\|\sup_{0 \leq u \leq \cdot} |\delta S_u|\|_{L^p(P \otimes V^{\mathbb{F}})}} \sqrt{\sum_{i=1}^2 \|F^{(i)} + |h^{(i)}| + \sup_{0 \leq u \leq \cdot} (S_u^{(i)})^+\|_{L^p(P \otimes V^{\mathbb{F}})}}.
\end{aligned}$$

*Here $\delta Y^{\mathbb{G}}, \delta Z^{\mathbb{G}}, \delta M^{\mathbb{G}}, \delta K^{\mathbb{G}}$ and $\delta S$ are given by (4.1.16), $F^{(i)}$ is defined via*

*(4.2.18), and*

$$\delta h := h^{(1)} - h^{(2)},\ \delta F := \int_0^{\cdot} |f_s^{(1)} - f_s^{(2)}| ds. \tag{4.2.19}$$

(d) *Let $\widetilde{V}^{(1/p)}$ be the process defined in (4.1.4). Then, there exists a unique $L^p(P,\mathbb{G})$-solution to*

$$\begin{cases} dY = -Y\left(\frac{\widetilde{G}}{G}\right)^{1/p} I_{]\!]0,\tau]\!]} d\widetilde{V}^{(1/p)} - \widetilde{\mathcal{E}}_-^{1/p} f(t) d(t\wedge\tau) - dK - dM + Z dW^{\tau}, \\ Y_{\tau} = \widetilde{\mathcal{E}}_{\tau}^{1/p}\xi, \quad Y \geq \widetilde{\mathcal{E}}^{1/p} S \quad on \quad [\![0,\tau[\![, \quad \int_0^{\tau} (Y_{u-} - \widetilde{\mathcal{E}}_{u-}^{1/p} S_{u-}) dK_u = 0, \end{cases} \tag{4.2.20}$$

*denoted by $\left(\widetilde{Y}^{\mathbb{G}}, \widetilde{Z}^{\mathbb{G}}, \widetilde{K}^{\mathbb{G}}, \widetilde{M}^{\mathbb{G}}\right)$, and which satisfies*

$$\left(\widetilde{Y}^{\mathbb{G}}, \widetilde{Z}^{\mathbb{G}}, \widetilde{K}^{\mathbb{G}}, \widetilde{M}^{\mathbb{G}}\right) = \left(\widetilde{\mathcal{E}}^{1/p} Y^{\mathbb{G}}, \widetilde{\mathcal{E}}_-^{1/p} Z^{\mathbb{G}}, \widetilde{\mathcal{E}}_-^{1/p} \cdot K^{\mathbb{G}}, \widetilde{\mathcal{E}}_-^{1/p} \cdot M^{\mathbb{G}}\right). \tag{4.2.21}$$

*Proof.* Here, we start by proving that part (d) follows from assertions (a), (b) and (c). Let $\left(Y^{\mathbb{G}}, Z^{\mathbb{G}}, K^{\mathbb{G}}, M^{\mathbb{G}}\right)$ be the unique solution to the RBSDE (4.2.1). Then, thanks to Itô's calculations, we derive

$$\widetilde{\mathcal{E}}^{1/p} Y^{\mathbb{G}} = \widetilde{\mathcal{E}}_-^{1/p} \bullet Y^{\mathbb{G}} + Y^{\mathbb{G}} \bullet \widetilde{\mathcal{E}}^{1/p}, \tag{4.2.22}$$

and

$$\begin{aligned} Y^{\mathbb{G}} \bullet \widetilde{\mathcal{E}}^{1/p} &= Y^{\mathbb{G}} \bullet \left( \frac{-\widetilde{\mathcal{E}}_-^{1/p}}{p\widetilde{G}} \bullet D^{o,\mathbb{F}} + \sum_{0<s\leq\cdot} \left\{ \widetilde{\mathcal{E}}^{1/p} - \widetilde{\mathcal{E}}_-^{1/p} - \frac{1}{p}\widetilde{\mathcal{E}}_-^{1/p-1} \Delta\widetilde{\mathcal{E}} \right\} \right) \\ &= Y^{\mathbb{G}} \bullet \left( \frac{-\widetilde{\mathcal{E}}_-^{1/p}}{p\widetilde{G}} \bullet D^{o,\mathbb{F}} + \sum_{0<s\leq\cdot} \left\{ \widetilde{\mathcal{E}}_-^{1/p} \left(1 - \frac{\Delta D^{o,\mathbb{F}}}{\widetilde{G}}\right)^{1/p} - \widetilde{\mathcal{E}}_-^{1/p} - \frac{1}{p}\widetilde{\mathcal{E}}_-^{1/p-1} \Delta\widetilde{\mathcal{E}} \right\} \right) \end{aligned}$$

$$= Y^{\mathbb{G}} \bullet \left( \frac{-\widetilde{\mathcal{E}}_-^{1/p}}{p\widetilde{G}} \bullet D^{o,\mathbb{F}} + \sum_{0<s\leq\cdot} \widetilde{\mathcal{E}}_-^{1/p} \Big\{ \Big(1 - \frac{\Delta D^{o,\mathbb{F}}}{\widetilde{G}}\Big)^{1/p} - 1 - \frac{1}{p}\widetilde{\mathcal{E}}_-^{-1}\Delta\widetilde{\mathcal{E}} \Big\} \right)$$

$$= -Y^{\mathbb{G}}\widetilde{\mathcal{E}}^{1/p}\Big(\frac{\widetilde{G}}{G}\Big)^{1/p} \bullet \left( \frac{1}{p\widetilde{G}} \bullet D^{o,\mathbb{F}} + \sum_{0<s\leq\cdot} \Big\{ -\Big(1 - \frac{\Delta D^{o,\mathbb{F}}}{\widetilde{G}}\Big)^{1/p} + 1 - \frac{\Delta D^{o,\mathbb{F}}}{p\widetilde{G}} \Big\} \right)$$

$$= -Y^{\mathbb{G}}\widetilde{\mathcal{E}}^{1/p}\left(\frac{\widetilde{G}}{G}\right)^{1/p} \bullet \widetilde{V}^{(1/p)}.$$

By combining this latter inequality with (4.2.22) and then using (4.2.1) afterwards, we get

$$d(\widetilde{\mathcal{E}}^{1/p}Y^{\mathbb{G}}) = -\widetilde{\mathcal{E}}^{1/p}Y^{\mathbb{G}}\left(\frac{\widetilde{G}}{G}\right)^{1/p} I_{]\!]0,\tau]\!]}d\widetilde{V}^{(1/p)} - \widetilde{\mathcal{E}}_-^{1/p}f(t)d(t\wedge\tau) - \widetilde{\mathcal{E}}_-^{1/p}dK^{\mathbb{G}}$$
$$- \widetilde{\mathcal{E}}_-^{1/p}dM^{\mathbb{G}} + \widetilde{\mathcal{E}}_-^{1/p}Z^{\mathbb{G}}dW^{\tau}.$$

This proves that $\big(Y^{\mathbb{G}}, Z^{\mathbb{G}}, K^{\mathbb{G}}, M^{\mathbb{G}}\big)$ is the unique solution to (4.2.1) if and only if $\Big(\widetilde{Y}^{\mathbb{G}}, \widetilde{Z}^{\mathbb{G}}, \widetilde{K}^{\mathbb{G}}, \widetilde{M}^{\mathbb{G}}\Big) := \Big(\widetilde{\mathcal{E}}^{1/p}Y^{\mathbb{G}}, \widetilde{\mathcal{E}}_-^{1/p}Z^{\mathbb{G}}, \widetilde{\mathcal{E}}_-^{1/p}\cdot K^{\mathbb{G}}, \widetilde{\mathcal{E}}_-^{1/p}\cdot M^{\mathbb{G}}\Big)$ is the unique solution to (4.2.20). Furthermore, under (4.2.18), this solution is an $L^p(P,\mathbb{G})$-solution as soon as assertion (b) holds, due to

$$\|\widetilde{Y}^{\mathbb{G}}\|_{\mathbb{D}_\tau(P,p)} = \|Y^{\mathbb{G}}\|_{\widetilde{\mathbb{D}}_\tau(P,p)}, \ \ \|\widetilde{Z}^{\mathbb{G}}\|_{\mathbb{S}_\tau(P,p)} = \|Z^{\mathbb{G}}\|_{\widetilde{\mathbb{S}}_\tau(P,p)}.$$

Thus, on the one hand, the existence of an $L^p(P,\mathbb{G})$-solution to (4.2.20) follows immediately as soon as assertions (a) and (b) hold. On the other hand, the uniqueness of the solution to (4.2.1) is a direct consequence of assertion (c). Therefore, the rest of the proof focuses on proving existence of a solution to (4.2.1) (i.e. the first half of assertion (a)), and assertions (b) and (c) in four parts.

**Part 1.** Here, in this part, we consider an $\mathbb{F}$-stopping time $\sigma$, and we assume that there exists a positive constant $C \in (0, +\infty)$ such that

$$\max\left(|h|^p, \left(\int_0^{\cdot} |f_s|ds\right)^p, \sup_{0\leq u\leq \cdot}(S_u^+)^p\right) \leq C\mathcal{E}(G_-^{-1}\cdot m), \quad \text{on} \quad [\![0,\sigma]\!]. \tag{4.2.23}$$

Our goal, in this part, lies in proving under this assumption that there exists a solution to (4.2.1) and assertion (b) holds for $[\![0, \sigma\wedge\tau]\!]$. To this end, we consider the sequence of data $(f^{(n)}, h^{(n)}, S^{(n)})$ given by

$$f^{(n)} := fI_{[\![0,n\wedge\sigma]\!]},\ h_t^{(n)} := h_{t\wedge\sigma\wedge n},\ S_t^{(n)} := S_{n\wedge t\wedge\sigma},\ \xi^{(n)} := h_{n\wedge\sigma\wedge\tau}, \tag{4.2.24}$$

For any $n \geq 1$, thanks to Theorem 4.1.8-(b) and Remark 4.1.9, the RBSDE (4.2.1) associated to $(f^{(n)}, S^{(n)}, \xi^{(n)})$ has a unique $L^p(\mathbb{G}, \widetilde{Q}_{n\wedge\sigma})$-solution denoted by $(Y^{\mathbb{G},n}, Z^{\mathbb{G},n}, M^{\mathbb{G},n}, K^{\mathbb{G},n})$ for the horizon $n\wedge\sigma\wedge\tau$ (i.e., that corresponds to the case where $T = n\wedge\sigma$). For any $n, m \geq 1$, we apply Theorem 4.2.2 to each $(f^{(n)}, h^{(n)}, S^{(n)})$ and apply Theorem 4.2.3 to the triplet

$$(\delta f, \delta h, \delta S, \delta\xi) := \left(f^{(n)} - f^{(n+m)}, h^{(n)} - h^{(n+m)}, S^{(n)} - S^{(n+m)}, \xi^{(n)} - \xi^{(n+m)}\right),$$

using the bounded horizon $T = (n+m)\wedge\sigma$ for both theorems, and get

$$\begin{aligned}
&E\left[\sup_{0\leq t\leq T}\widetilde{\mathcal{E}}_t|Y_t^{\mathbb{G},n}|^p + \left(\int_0^T (\widetilde{\mathcal{E}}_{s-})^{2/p}|Z_s^{\mathbb{G},n}|^2 ds\right)^{p/2}\right]\\
&+ E\left[\left((\widetilde{\mathcal{E}}_-)^{1/p}\cdot K_T^{\mathbb{G},n}\right)^p + ((\widetilde{\mathcal{E}}_-)^{2/p}\cdot[M^{\mathbb{G},n}, M^{\mathbb{G},n}]_T)^{p/2}\right]\\
&\leq \widetilde{C}E^{\widetilde{Q}}\left[\left(\int_0^{n\wedge\sigma\wedge\tau}|f(s)|ds\right)^p + \sup_{0\leq s\leq n\wedge\sigma\wedge\tau}(S_s^+)^p + |\xi^{(n)}|^p\right],
\end{aligned} \tag{4.2.25}$$

and

$$
\begin{aligned}
&\|Y^{\mathbb{G},n} - Y^{\mathbb{G},n+m}\|_{\widetilde{\mathbb{D}}_\tau(P,p)} + \|Z^{\mathbb{G},n} - Z^{\mathbb{G},n+m}\|_{\widetilde{\mathbb{S}}_\tau(P,p)} \\
&+ \|\sqrt[p]{\widetilde{\mathcal{E}}_-} \cdot (M^{\mathbb{G},n} - M^{\mathbb{G},n+m})^\tau\|_{\mathcal{M}^p(P)} \\
&\leq \widetilde{C}_1 \Delta_{\widetilde{Q}}(\xi^{(n)} - \xi^{(n+m)}, f^{(n)} - f^{(n+m)}, S^{(n)} - S^{(n+m)}) \\
&+ \widetilde{C}_2 \sqrt{\| \sup_{0\leq t\leq \tau\wedge\sigma} |S_{t\wedge n} - S_{t\wedge(n+m)}| \|_{L^p(\widetilde{Q})} \sum_{i\in\{n,n+m\}} \Delta_{\widetilde{Q}}(\xi^{(i)}, f^{(i)}, (S^{(i)})^+)},
\end{aligned}
\tag{4.2.26}
$$

where $\Delta_{\widetilde{Q}}(\xi^{(i)}, f^{(i)}, (S^{(i)})^+)$ is given via (4.1.8), and which we recall below:

$$
\begin{aligned}
&\Delta_{\widetilde{Q}}(\xi^{(i)}, f^{(i)}, (S^{(i)})^+) \\
&:= \|\xi^{(i)}\|_{L^p(\widetilde{Q})} + \|\int_0^{T\wedge\tau} |f^{(i)}(s)|ds\|_{L^p(\widetilde{Q})} + \|(S^{(i)})^+\|_{\mathbb{S}_{T\wedge\tau}(\widetilde{Q},p)}.
\end{aligned}
$$

Next, we calculate the limits, when $n$ and/or $m$ go to infinity, of the right-hand-sides of the inequalities (4.2.25) and (4.2.26). To this end, we start by applying Lemma 4.2.4 to $\left(\int_0^\cdot |f(s)|ds\right)^p$, $\sup_{0\leq s\leq\cdot}(S_s^+)^p$, and $|h|^p$, and get

$$
\begin{cases}
\lim_{n\to\infty} E^{\widetilde{Q}}\left[\left(\int_0^{n\wedge\sigma\wedge\tau} |f(s)|ds\right)^p\right] = G_0 E\left[\int_0^\infty (F_{t\wedge\sigma})^p dV_t^{\mathbb{F}}\right], \\
\lim_{n\to\infty} E^{\widetilde{Q}}\left[\sup_{0\leq s\leq n\wedge\sigma\wedge\tau} (S_s^+)^p\right] = G_0 E\left[\int_0^\infty \sup_{0\leq s\leq t\wedge\sigma} (S_s^+)^p dV_t^{\mathbb{F}}\right], \\
\lim_{n\to\infty} E^{\widetilde{Q}}\left[|\xi^{(n)}|^p\right] = \lim_{n\to\infty} E^{\widetilde{Q}}\left[|h_{n\wedge\sigma\wedge\tau}|^p\right] = G_0 E\left[\int_0^\infty |h_{t\wedge\sigma}|^p dV_t^{\mathbb{F}}\right].
\end{cases}
\tag{4.2.27}
$$

This determines the limits for the right-hand-side terms of (4.2.25). To addresses the limits of the right-hand-side terms of (4.2.26), we remark that

$$\begin{aligned}
&\| \int_0^{T\wedge\tau} |f^{(n)}(s) - f^{(n+m)}(s)|ds\|^p_{L^p(\widetilde{Q})} \\
&= E^{\widetilde{Q}}\left[\left(\int_0^{\tau\wedge(n+m)} I_{[\![0,\sigma]\!]}(s) I_{]\!]n,\infty[\![}(s)|f(s)|ds\right)^p\right].
\end{aligned}$$

increases with $m$. Therefore, by applying Lemma 4.2.4 to $(F^\sigma - F^{n\wedge\sigma})^p$, we derive

$$\begin{aligned}
&\sup_m \| \int_0^{\tau} |f^{(n)}(s) - f^{(n+m)}(s)|ds\|^p_{L^p(\widetilde{Q})} \\
&= G_0 E\left[\int_0^\infty \left(F(s\wedge\sigma) - F(s\wedge n\wedge\sigma)\right)^p dV^{\mathbb{F}}_s\right].
\end{aligned}$$

By combining this equality with (4.2.18) and the dominated convergence theorem, we get

$$\lim_{n\to\infty}\sup_{m\geq 1} \| \int_0^{\tau} |f^{(n)}(s) - f^{(n+m)}(s)|ds\|^p_{L^p(\widetilde{Q})} = 0. \tag{4.2.28}$$

Similar arguments allow us to deduce that

$$\|S^{(n)} - S^{(n+m)}\|^p_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},p)} = E^{\widetilde{Q}}\left[\sup_{n<t\leq(n+m)\wedge\sigma\wedge\tau} |S_n - S_t|^p I_{\{\tau\wedge\sigma>n\}}\right],$$

increases with $m$ to $G_0 E\left[\int_0^\infty \sup_{n<u\leq\sigma\wedge t} |S^+_n - S^+_u|^p I_{\{\sigma\wedge t>n\}} dV^{\mathbb{F}}_t\right]$. Then this yields

$$\lim_{n\to\infty}\sup_{m\geq 1} \|S^{(n)} - S^{(n+m)}\|^p_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},p)} = 0. \tag{4.2.29}$$

Thanks to (4.2.23), we derive

$$E^{\widetilde{Q}}\left[|\xi^{(n)}-\xi^{(n+m)}|^p\right]=E\left[\widetilde{Z}_{(n+m)\wedge\tau}|h_n-h_{\tau\wedge\sigma\wedge(n+m)}|^p I_{\{\tau\wedge\sigma>n\}}\right]$$
$$\leq 2^p CP(\tau\wedge\sigma>n),$$

and hence we obtain

$$\lim_{n\to\infty}\sup_{m\geq 1}E^{\widetilde{Q}}\left[|\xi^{(n)}-\xi^{(n+m)}|^p\right]\leq\lim_{n\to\infty}2CP(\tau\wedge\sigma>n)=0. \tag{4.2.30}$$

Thus, by combining (4.2.28), (4.2.29) and (4.2.30), we conclude that the right-hand-side term of (4.2.26) goes to zero when $n$ goes to infinity uniformly in $m$. This proves that the sequence $(Y^{\mathbb{G},n},Z^{\mathbb{G},n},K^{\mathbb{G},n},M^{\mathbb{G},n})$ is a Cauchy sequence in norm, and hence it converges to $(Y^{\mathbb{G}},Z^{\mathbb{G}},K^{\mathbb{G}},M^{\mathbb{G}})$ in norm and almost surely for a subsequence. On the one hand, we conclude that $(Y^{\mathbb{G}},Z^{\mathbb{G}},K^{\mathbb{G}},M^{\mathbb{G}})$ is in fact a solution to (4.2.1). On the other hand, by taking the limit in (4.2.25) and using Fatou and (4.2.27), the proof of assertion (b) follows immediately. This ends the first part.

**Part 2.** In this part we prove that assertion (c) holds under the assumption (4.2.23) over the interval $[\![0,\sigma\wedge\tau]\!]$, where $\sigma$ is an $\mathbb{F}$-stopping time. To this end, we consider two triplets $(f^{(i)},S^{(i)},h^{(i)})$, $i=1,2$, which satisfy the boundedness assumption (4.2.23), and to which we associate two sequences $(f^{(i,n)},S^{(i,n)},h^{(i,n)})$, $i=1,2$, as in (4.2.24). On the one hand, by virtue of part 1, we deduce that for each $i=1,2$, $(Y^{\mathbb{G},(i,n)},Z^{\mathbb{G},(i,n)},K^{\mathbb{G},(i,n)},M^{\mathbb{G},(i,n)})$ converges in norm and almost surely for a subsequence to the quadruplet $(Y^{\mathbb{G},(i)},Z^{\mathbb{G},(i)},K^{\mathbb{G},(i)},M^{\mathbb{G},(i)})$, which is solution to (4.2.1) for the horizon $\sigma\wedge\tau$.

On the other hand, for each $n \geq 1$, we apply Theorem 4.2.3 for

$$\delta f^{(n)} := f^{(1,n)} - f^{(2,n)}, \quad \delta S^{(n)} := S^{(1,n)} - S^{(2,n)}, \quad \delta\xi^{(n)} := \xi^{(1,n)} - \xi^{(2,n)},$$

and

$$\begin{cases} \delta Y^{\mathbb{G},(n)} := Y^{\mathbb{G},(1,n)} - Y^{\mathbb{G},(2,n)}, \quad \delta Z^{\mathbb{G},(n)} := Z^{\mathbb{G},(1,n)} - Z^{\mathbb{G},(2,n)}, \\ \delta K^{\mathbb{G},(n)} := K^{\mathbb{G},(1,n)} - K^{\mathbb{G},(2,n)}, \quad \delta M^{\mathbb{G},(n)} := M^{\mathbb{G},(1,n)}) - M^{\mathbb{G},(2,n)}, \end{cases}$$

and get

$$\begin{aligned} &\|\delta Y^{\mathbb{G},(n)}\|_{\widetilde{\mathbb{D}}_{T\wedge\tau}(P,p)} + \|\delta Z^{\mathbb{G},(n)}\|_{\widetilde{\mathbb{S}}_{T\wedge\tau}(P,p)} + \|(\widetilde{\mathcal{E}}_-)^{1/p} \cdot \delta M^{\mathbb{G},(n)}\|_{L^p(P)} \\ &\leq \widetilde{C}_1 \Delta_{\widetilde{Q}}(\delta\xi^{(n)}, \delta f^{(n)}, \delta S^{(n)}) + \widetilde{C}_2 \sqrt{\|\delta S^{(n)}\|_{\mathbb{S}(\widetilde{Q},p)} \sum_{i=1}^{2} \Delta_{\widetilde{Q}}\left(\xi^{(i,n)}, f^{(i,n)}, (S^{(i,n)})^+\right)}. \end{aligned} \tag{4.2.31}$$

Similarly, as in the proof of (4.2.27), we use Lemma 4.2.4 and the boundedness assumption (4.2.23) that each triplet $(f^{(i)}, S^{(i)}, h^{(i)})$ $(i = 1, 2)$ satisfies, and get

$$\begin{cases} \lim\limits_{n\to\infty} E^{\widetilde{Q}}\left[\left(\int_0^{n\wedge\sigma\wedge\tau} |\delta f_s^{(n)})| ds\right)^p\right] = G_0 E\left[\int_0^\infty |\delta F_{t\wedge\sigma}|^p dV_t^{\mathbb{F}}\right], \\ \lim\limits_{n\to\infty} E^{\widetilde{Q}}\left[|\delta\xi^{(n)}|^p\right] = G_0 E\left[\int_0^\infty |\delta h_{t\wedge\sigma}|^p dV_t^{\mathbb{F}}\right], \\ \lim\limits_{n\to\infty} E^{\widetilde{Q}}\left[\sup\limits_{0\leq s\leq\sigma\wedge\tau} = |\delta S_s^{(n)}|^p\right] = G_0 E\left[\int_0^\infty \sup\limits_{0\leq s\leq t\wedge\sigma} |\delta S_s|^p dV_t^{\mathbb{F}}\right], \quad i = 1, 2. \end{cases} \tag{4.2.32}$$

Thus, by taking the limit in (4.2.31), using Fatou's lemma for its left-hand-side term, and using (4.2.32) for its right-hand-side term, assertion (c) follows immediately. This ends the second part.

**Part 3.** In this part, we drop the assumption (4.2.23) and prove existence of solution to (4.2.1) and assertion (b). Hence, we consider the following sequence of stopping times

$$T_n := \inf\left\{t \geq 0 \ : \quad \frac{|h_t|^p + |S_t|^p + (\int_0^t |f(s)|ds)^p}{\mathcal{E}_t(G_-^{-1}\cdot m)} > n\right\},$$

and the sequences

$$h^{(n)} := hI_{[\![0,T_n[\![},\ f^{(n)} := fI_{[\![0,T_n]\!]},\ S^{(n)} := SI_{[\![0,T_n[\![},\ \xi^{(n)} := h_\tau I_{\{\tau<T_n\}}. \tag{4.2.33}$$

Thus, for any $n \geq 1$, it is clear that the triplet $(f^{(n)}, h^{(n)}, S^{(n)})$ satisfies (4.2.23) on $[\![0, T_n]\!]$. Thus, thanks to the first and the second parts, we deduce the existence of unique solution to (4.2.1), denoted by $(Y^{\mathbb{G},(n)}, Z^{\mathbb{G},(n)}, K^{\mathbb{G},(n)}, M^{\mathbb{G},(n)})$, associated to $(f^{(n)}, h^{(n)}, S^{(n)})$ with the horizon $T_n\wedge\tau$, which remains a solution for any horizon $T_k \wedge \tau$ with $k \geq n$. Furthermore, we have

$$\begin{aligned}&\left\{\begin{aligned}&\|Y^{\mathbb{G},(n)}\|_{\widetilde{\mathbb{D}}_\tau(P,p)} + \|(\widetilde{\mathcal{E}}_-)^{1/p}\cdot(M^{\mathbb{G},(n)})^\tau\|_{\mathcal{M}^p(P)}\\ &+\|Z^{\mathbb{G},(n)}\|_{\widetilde{\mathbb{S}}_\tau(P,p)} + \|((\widetilde{\mathcal{E}}_-)^{1/p}\cdot K^{\mathbb{G},(n)})_\tau\|_{L^p(P)}\end{aligned}\right\}\\ &\leq C\|F^{(n)} + |h^{(n)}| + \sup_{0\leq u\leq\cdot}(S_u^{(n)})^+\|_{L^p(P\otimes V^{\mathbb{F}})},\end{aligned} \tag{4.2.34}$$

due to assertion (b). Furthermore, for any $n \geq 1$ and $m \geq 1$

$$\begin{aligned}&\|Y^{\mathbb{G},(n)} - Y^{\mathbb{G},(n+m)}\|_{\widetilde{\mathbb{D}}_\tau(P,p)} + \|Z^{\mathbb{G},(n)} - Z^{\mathbb{G},(n+m)}\|_{\widetilde{\mathbb{D}}_\tau(P,p)}\\ &+\|\sqrt[p]{\widetilde{\mathcal{E}}_-}\cdot(M^{\mathbb{G},(n)} - M^{\mathbb{G},(n+m)})^\tau\|_{\mathcal{M}^p(P)}\end{aligned}$$

$$\leq C_1 \| |h^{(n)} - h^{(n+m)}| + |F^{(n)} - F^{(n+m)}| + \sup_{0\leq u\leq \cdot} |S_u^{(n)} - S_u^{(n+m)}| \|_{L^p(P\otimes V^{\mathbb{F}})}$$

$$+ C_2 \sqrt{\| \sup_{0\leq u\leq \cdot} |S_u^{(n)} - S_u^{(n+m)}| \|_{L^p(P\otimes V^{\mathbb{F}})} \Sigma(n,m)}, \qquad (4.2.35)$$

where $\Sigma(n,m)$ is given by

$$\Sigma(n,m) := \sum_{i\in\{n,n+m\}} \|F_t^{(i)} + |h^{(i)}| + \sup_{0\leq u\leq \cdot} (S_u^{(i)})^+ \|_{L^p(P\otimes V^{\mathbb{F}})}.$$

It is important to remark that, thanks to part 2, the latter inequality above follows from assertion (c) applied to

$$\begin{cases} \delta Y^{\mathbb{G}} := Y^{\mathbb{G},(n)} - Y^{\mathbb{G},(n+m)}, \ \delta Z^{\mathbb{G}} := Z^{\mathbb{G},(n)} - Z^{\mathbb{G},(n+m)}, \\ \delta K^{\mathbb{G}} := K^{\mathbb{G},(n)} - K^{\mathbb{G},(n+m)}, \delta M^{\mathbb{G}} := M^{\mathbb{G},(n)} - M^{\mathbb{G},(n+m)}, \delta h := h^{(n)} - h^{(n+m)}, \\ \delta F := \int_0^{\cdot} |f_s^{(n)} - f_s^{(n+m)}| ds, \quad F^{(i)} := \int_0^{\cdot} |f_s^{(i)}| ds, \quad \delta S := S^{(n)} - S^{(n+m)}. \end{cases}$$

Then by virtue of (4.2.18) and the dominated convergence theorem, we derive

$$\lim_{n\to+\infty} \sup_{m\geq 1} \| |h^{(n)} - h^{(n+m)}| + |F^{(n)} - F^{(n+m)}| + \sup_{0\leq u\leq \cdot} |S_u^{(n)} - S_u^{(n+m)}| \|_{L^p(P\otimes V^{\mathbb{F}})}$$

$$\leq \lim_{n\to+\infty} \| I_{[\![T_n,+\infty[\![}(|h| + F + \sup_{0\leq u\leq \cdot} |S_u|) \|_{L^p(P\otimes V^{\mathbb{F}})} = 0.$$

A combination of this with (4.2.35) proves that the sequence of the quadruplet $(Y^{\mathbb{G},(n)}, Z^{\mathbb{G},(n)}, K^{\mathbb{G},(n)}, M^{\mathbb{G},(n)})$ is a Cauchy sequence in norm, and hence it converges to $(Y^{\mathbb{G}}, Z^{\mathbb{G}}, K^{\mathbb{G}}, M^{\mathbb{G}})$ in norm and almost surely for a subsequence. As a result, $(Y^{\mathbb{G}}, Z^{\mathbb{G}}, K^{\mathbb{G}}, M^{\mathbb{G}})$ clearly satisfies (4.2.1), and due to Fatou's lemma and (4.2.34) we conclude that assertion (b) holds. This ends part 3.

**Part 4.** Here we prove assertion (c) under no assumption. Thus, we consider a pair of data $(f^{(i)}, S^{(i)}, h^{(i)}, \xi^{(i)})$, $i = 1, 2$, to which we associate two sequences of $\mathbb{F}$-stopping times $(T_n^{(i)})_n$ for $i = 1, 2$ as in part 3, and two data-sequences $(f^{(n,i)}, h^{(n,i)}, S^{(n,i)})$ which are constructed from $(f^{(i)}, h^{(i)}, S^{(i)})$ and $T_n := \min(T_n^{(1)}, T_n^{(2)})$ via (4.2.33). On the one hand, thanks to part 2, we obtain the existence of $(Y^{\mathbb{G},(n,i)}, Z^{\mathbb{G},(n,i)}, K^{\mathbb{G},(n,i)}, M^{\mathbb{G},(n,i)})_{n\geq 1}$ $(i = 1, 2)$ solution to (4.2.1) for the data $(f^{(n,i)}, h^{(n,i)}, S^{(n,i)})$ with the horizon $T_n \wedge \tau$. Furthermore, the sequence $(Y^{\mathbb{G},(n,i)}, Z^{\mathbb{G},(n,i)}, K^{\mathbb{G},(n,i)}, M^{\mathbb{G},(n,i)})_{n\geq 1}$ converges (in norm and almost surely for a subsequence) to $(Y^{\mathbb{G},i}, Z^{\mathbb{G},i}, K^{\mathbb{G},i}, M^{\mathbb{G},i})$, which is a solution to (4.2.1) for $(f^{(i)}, S^{(i)}, h^{(i)}, \xi^{(i)})$ and the horizon $\tau$. On the other hand, thanks to part 3, we apply assertion (c) to

$$\begin{aligned}&(\delta Y^{\mathbb{G},(n)}, \delta Z^{\mathbb{G},(n)}, \delta K^{\mathbb{G},(n)}, \delta M^{\mathbb{G},(n)})\\&:= (Y^{\mathbb{G},(n,1)}, Z^{\mathbb{G},(n,1)}, K^{\mathbb{G},(n,1)}, M^{\mathbb{G},(n,1)}) - (Y^{\mathbb{G},(n,2)}, Z^{\mathbb{G},(n,2)}, K^{\mathbb{G},(n,2)}, M^{\mathbb{G},(n,2)}),\end{aligned}$$

associated to

$$\begin{aligned}(\delta f^{(n)}, \delta h^{(n)}, \delta S^{(n)}) &:= (f^{(n,1)} - f^{(n,2)}, h^{(n,1)} - h^{(n,2)}, S^{(n,1)} - S^{(n,2)})\\&= (\delta f, \delta h, \delta S) I_{[\![0,T_n[\![},\end{aligned}$$

and get

$$\begin{aligned}&\|\delta Y^{\mathbb{G},(n)}\|_{\widetilde{\mathbb{D}}_\tau(P,p)} + \|\delta Z^{\mathbb{G},(n)}\|_{\widetilde{\mathbb{S}}_\tau(P,p)} + \|\sqrt[p]{\widetilde{\mathcal{E}}_-} \cdot \delta M^{\mathbb{G},(n)}\|_{\mathcal{M}^p(P)}\\&\leq C_1 \| |\delta h^{(n)}| + |\delta F^{(n)}| + \sup_{0\leq u\leq \cdot} |\delta S_u^{(n)}| \|_{L^p(P\otimes V^{\mathbb{F}})}\\&+ C_2 \sqrt{\| \sup_{0\leq u\leq \cdot} |\delta S_u^{(n)}| \|_{L^p(P\otimes V^{\mathbb{F}})}} \sqrt{\sum_{i=1}^{2} \|F^{(n,i)} + |h^{(n,i)}| + \sup_{0\leq u\leq \cdot} (S_u^{(n,i)})^+ \|_{L^p(P\otimes V^{\mathbb{F}})}}.\end{aligned}$$

Then by taking the limits on both sides of this inequality, and using Fatou for the left-hand-side term and the convergence monotone theorem for the right-hand-side term, we conclude that assertion (c) follows immediately for this general case. This ends the fourth part, and the proof of the theorem is complete. $\square$

It is important to mention that our method, in the proof above, requires the assumption $\|\sup_{0\leq u\leq \cdot}|S_u|\|_{L^p(P\otimes V^{\mathbb{F}})}<+\infty$ in (4.2.18) which is stronger than the condition $\|\sup_{0\leq u\leq\cdot}S_u^+\|_{L^p(P\otimes V^{\mathbb{F}})}<+\infty$. Importantly, the assumption (4.2.18) reduces the generality of the theorem and similar results for BSDEs can not be derived from this theorem in contrast to previous theorems. However, our method remains valid and can be easily adapted to the BSDEs setting directly by just ignoring the process $S$ and putting $K^{\mathbb{G}}=0$ throughout the proof, as they are irrelevant. This proves the following

**Theorem 4.2.6.** *Let $p\in(1,+\infty)$ and $F$ be given by (4.2.18). Suppose $G>0$ and $(f,h)$ satisfies $\|F+|h|\|_{L^p(P\otimes V^{\mathbb{F}})}<+\infty$. Then the following assertions hold.*

(a) *The following BSDE*

$$dY=-f(t)d(t\wedge\tau)-dM+ZdW^{\tau},\quad Y_\tau=\xi. \tag{4.2.36}$$

*admits a unique solution $\left(Y^{\mathbb{G}},Z^{\mathbb{G}},M^{\mathbb{G}}\right)$.*

(b) *There exists a positive constant $C$, that depends on p only, such that*

$$\|Y^{\mathbb{G}}\|_{\widetilde{\mathbb{D}}_\tau(P,p)}+\|Z^{\mathbb{G}}\|_{\widetilde{\mathbb{S}}_\tau(P,p)}+\|\sqrt[p]{\widetilde{\mathcal{E}}_-}\cdot M^{\mathbb{G}}\|_{\mathcal{M}^p(P)}\leq C\|F+|h|\|_{L^p(P\otimes V^{\mathbb{F}})}.$$

(c) *Consider two triplet* $(f^{(i)}, h^{(i)})$, $i = 1, 2$, *satisfying* $\|F^{(i)} + |h^{(i)}|\|_{L^p(P\otimes V^{\mathbb{F}})} < +\infty$. *If* $\left(Y^{\mathbb{G},i}, Z^{\mathbb{G},i}, M^{\mathbb{G},i}\right)$ *denotes the solution to (4.2.36) associated with the pair* $(f^{(i)}, h^{(i)})$ *for each* $i = 1, 2$, *then there exist positive* $C_1$ *and* $C_2$ *that depend on* $p$ *only such that*

$$\|\delta Y^{\mathbb{G}}\|_{\widetilde{\mathbb{D}}_\tau(P,p)} + \|\delta Z^{\mathbb{G}}\|_{\widetilde{\mathbb{S}}_\tau(P,p)} + \|\sqrt[p]{\widetilde{\mathcal{E}}_-} \cdot \delta M^{\mathbb{G}}\|_{\mathcal{M}^p(P)} \leq C_1 \||\delta h| + |\delta F|\|_{L^p(P\otimes V^{\mathbb{F}})}.$$

*Here* $\delta Y^{\mathbb{G}}, \delta Z^{\mathbb{G}}$ *and* $\delta M^{\mathbb{G}}$ *are given by (4.1.16),* $F^{(i)}$ *is defined via (4.2.18), and*

$$\delta h := h^{(1)} - h^{(2)},\ \delta F := \int_0^{\cdot} |f_s^{(1)} - f_s^{(2)}| ds.$$

(d) *Let* $\widetilde{V}^{(1/p)}$ *be defined in (4.1.4). Then, there exists a unique* $L^p(P, \mathbb{G})$*-solution to*

$$dY = -Y \left(\frac{\widetilde{G}}{G}\right)^{1/p} I_{]\!]0,\tau]\!]} d\widetilde{V}^{(1/p)} - \widetilde{\mathcal{E}}_-^{1/p} f(t) d(t\wedge\tau) - dM + ZdW^\tau,\ Y_\tau = \widetilde{\mathcal{E}}_\tau^{1/p}\xi,$$

*denoted by* $\left(\widetilde{Y}^{\mathbb{G}}, \widetilde{Z}^{\mathbb{G}}, \widetilde{M}^{\mathbb{G}}\right)$, *and satisfies*

$$\left(\widetilde{Y}^{\mathbb{G}}, \widetilde{Z}^{\mathbb{G}}, \widetilde{M}^{\mathbb{G}}\right) = \left(\widetilde{\mathcal{E}}^{1/p} Y^{\mathbb{G}}, \widetilde{\mathcal{E}}_-^{1/p} Z^{\mathbb{G}}, \widetilde{\mathcal{E}}_-^{1/p} \cdot M^{\mathbb{G}}\right).$$

### 4.2.3 Relationship to RBSDE under $\mathbb{F}$

In this subsection, we establish the RBSDE under $\mathbb{F}$ that is intimately related to (4.2.1), and we connect their solutions as well.

**Theorem 4.2.7.** *Suppose that $G > 0$ and $(f, S, h)$ satisfies (4.2.18) and*

$$E\left[\left(F_\infty \widetilde{\mathcal{E}}_\infty\right)^p\right] < +\infty, \text{ where } \widetilde{\mathcal{E}} := \mathcal{E}(-\widetilde{G}^{-1} \cdot D^{o,\mathbb{F}}) \text{ and } F_\infty := \int_0^\infty |f_s| ds. \tag{4.2.37}$$

*Consider the pair $(f^{\mathbb{F}}, S^{\mathbb{F}})$ given by (4.1.22). Then the following hold.*

(a) *The following RBSDE, under $\mathbb{F}$, generated by the triplet $\left(f^{\mathbb{F}}, S^{\mathbb{F}}, h\right)$*

$$\begin{cases} Y_t = \displaystyle\int_t^\infty f^{\mathbb{F}}(s) ds + \int_t^\infty h_s dV_s^{\mathbb{F}} + K_\infty - K_t - \int_t^\infty Z_s dW_s, \\ Y_t \geq S_t^{\mathbb{F}}, \quad E\left[\displaystyle\int_0^\infty (Y_{t-} - S_{t-}^{\mathbb{F}}) dK_t\right] = 0, \end{cases} \tag{4.2.38}$$

*has a unique $L^p(P, \mathbb{F})$-solution $(Y^{\mathbb{F}}, Z^{\mathbb{F}}, K^{\mathbb{F}})$.*

(b) *The unique $L^p(\widetilde{Q}, \mathbb{G})$-solution to (4.2.1), denoted by $\left(Y^{\mathbb{G}}, Z^{\mathbb{G}}, K^{\mathbb{G}}, M^{\mathbb{G}}\right)$, satisfies*

$$\begin{cases} Y^{\mathbb{G}} = Y^{\mathbb{F}} \widetilde{\mathcal{E}}^{-1} I_{[\![0,\tau[\![} + \xi I_{[\![\tau,+\infty[\![}, \ Z^{\mathbb{G}} = Z^{\mathbb{F}} \widetilde{\mathcal{E}}_-^{-1} I_{]\!]0,\tau]\!]}, \\ K^{\mathbb{G}} = \widetilde{\mathcal{E}}_-^{-1} I_{]\!]0,\tau]\!]} \cdot K^{\mathbb{F}}, \text{ and } M^{\mathbb{G}} = \left(h - Y^{\mathbb{F}} \widetilde{\mathcal{E}}^{-1}\right) \cdot N^{\mathbb{G}}. \end{cases} \tag{4.2.39}$$

*Proof.* On the one hand, remark that, due to the assumptions (4.2.18) and (4.2.37), both random variables $\int_0^\infty |f_s^{\mathbb{F}}| ds$ and $\int_0^\infty |h_s| dV_s^{\mathbb{F}}$ belong to $L^p(P)$. Indeed, in virtue of $(V_\infty^{\mathbb{F}})^{p-1} \leq 1$, this fact follows from the following two inequalities

$$\int_0^\infty |f_s^{\mathbb{F}}| ds = \widetilde{\mathcal{E}}_\infty F_\infty + \int_0^\infty F_s dV_s^{\mathbb{F}}, \quad \text{and} \quad \left(\int_0^\infty |h_s| dV_s^{\mathbb{F}}\right)^p \leq \int_0^\infty |h_s|^p dV_s^{\mathbb{F}}.$$

On the other hand, similar arguments as in the proof of Lemma 4.1.3, one can prove that any $L^p(P,\mathbb{F})$-solution to (4.2.38) $(Y,Z,K)$ satisfies

$$Y_t = \operatorname*{ess\,sup}_{\sigma\in\mathcal{T}_t^\infty(\mathbb{F})} E\left[\int_t^\sigma f_s^{\mathbb{F}} ds + \int_t^\sigma h_s dV_s^{\mathbb{F}} + S_\sigma^{\mathbb{F}} I_{\{\sigma<+\infty\}} \Big| \mathcal{F}_t\right] =: \overline{Y_t}.$$

Furthermore, due to the Snell envelope theory, see [40] for details or see also the proof of Theorem 4.1.8, the Doob-Meyer decomposition of the supermartingale $\overline{Y} + \int_0^\cdot h_s dV_s^{\mathbb{F}} + \int_0^\cdot f_s^{\mathbb{F}} ds = \overline{M} - \overline{K}$. Thanks to the predictable representation theorem, we deduce the existence of $\overline{Z} \in L^2_{loc}(W,\mathbb{F})$ such that $\overline{M} = \overline{Z} \bullet W$. Therefore, the triplet $(\overline{Y},\overline{Z},\overline{K})$ is the solution to (4.2.38). This proves that (4.2.38) has a unique solution, and the proof of assertion (a) is complete.

The rest of the proof deals with assertion (b). Remark that, in virtue of Theorem 4.2.5, the RBSDE (4.2.1) has a unique $L^p(\widetilde{Q},\mathbb{G})$-solution. Therefore, we will prove that $(\widehat{Y},\widehat{Z},\widehat{K},\widehat{M})$ given by

$$\widehat{Y} := \frac{Y^{\mathbb{F}}}{\widetilde{\mathcal{E}}} I_{[\![0,\tau[\![} + \xi I_{[\![\tau,+\infty[\![}, \quad \widehat{Z} := \frac{Z^{\mathbb{F}}}{\widetilde{\mathcal{E}}_-} I_{]\!]0,\tau]\!]}, \quad \widehat{K} := \frac{1}{\widetilde{\mathcal{E}}_-} \cdot (K^{\mathbb{F}})^\tau,$$
$$\text{and } \widehat{M} := \left(h - \frac{Y^{\mathbb{F}}}{\widetilde{\mathcal{E}}}\right) \cdot N^{\mathbb{G}},$$

is a solution to (4.2.1). To this end, we put $\widehat{\Gamma} := Y^{\mathbb{F}}/\widetilde{\mathcal{E}}$, and on the one hand we remark that

$$\widehat{Y} = \widehat{\Gamma} I_{[\![0,\tau[\![} + h_\tau I_{[\![\tau,+\infty[\![} = \widehat{\Gamma}^\tau + (h - \widehat{\Gamma}) \cdot D. \tag{4.2.40}$$

On the other hand, by combining Itô applied to $\widehat{\Gamma}$, (4.2.38) that the triplet $(Y^{\mathbb{F}}, Z^{\mathbb{F}}, K^{\mathbb{F}})$ satisfies, $\widetilde{\mathcal{E}}^{-1} = \mathcal{E}(G^{-1}\cdot D^{o,\mathbb{F}})$, $\widetilde{\mathcal{E}}\widetilde{G} = \widetilde{\mathcal{E}}_- G$, and $dV^{\mathbb{F}} = \widetilde{\mathcal{E}}_- \widetilde{G}^{-1} dD^{o,\mathbb{F}}$

we derive

$$\begin{aligned} d\widehat{\Gamma} &= Y^{\mathbb{F}} d\widetilde{\mathcal{E}}^{-1} + \frac{1}{\widetilde{\mathcal{E}}_-} dY^{\mathbb{F}} = \frac{\widehat{\Gamma}}{\widetilde{G}} dD^{o,\mathbb{F}} + \frac{1}{\widetilde{\mathcal{E}}_-} dY^{\mathbb{F}} \\ &= \frac{\widehat{\Gamma}}{\widetilde{G}} dD^{o,\mathbb{F}} - \frac{f^{\mathbb{F}}}{\widetilde{\mathcal{E}}_-} ds - \frac{h}{\widetilde{\mathcal{E}}_-} dV^{\mathbb{F}} - \frac{1}{\widetilde{\mathcal{E}}_-} dK^{\mathbb{F}} + \frac{Z^{\mathbb{F}}}{\widetilde{\mathcal{E}}_-} dW \\ &= \frac{\widehat{\Gamma} - h}{\widetilde{G}} dD^{o,\mathbb{F}} - f ds - \frac{1}{\widetilde{\mathcal{E}}_-} dK^{\mathbb{F}} + \frac{Z^{\mathbb{F}}}{\widetilde{\mathcal{E}}_-} dW. \end{aligned}$$

Thus, by stopping $\widehat{\Gamma}$ at $\tau$ and inserting the above equality in (4.2.40) and arranging terms we get

$$d\widehat{Y} = -f(t)d(t\wedge\tau) - d\widehat{K} + d\widehat{M} + \widehat{Z} dW^{\tau}, \quad \text{and} \quad \widehat{Y}_{\tau} = \xi. \qquad (4.2.41)$$

This proves that $(\widehat{Y}, \widehat{Z}, \widehat{K}, \widehat{M})$ satisfies the first equation in (4.2.1). Furthermore, it is clear that $Y_t^{\mathbb{F}} \geq S_t^{\mathbb{F}}$ implies the second condition in (4.2.1). To prove the Skorokhod condition (the last condition in (4.2.1)), we combine the Skorokhod condition for the triplet $(Y^{\mathbb{F}}, Z^{\mathbb{F}}, K^{\mathbb{F}})$, the fact that $\widehat{Y}_- \geq S_-$ on $]\!]0,\tau]\!]$, and

$$\begin{aligned} 0 \quad &\leq \int_0^{\tau} (\widehat{Y}_{t-} - S_{t-}) d\widehat{K}_t = \int_0^{\tau} (Y_{t-}^{\mathbb{F}} - S_{t-}^{\mathbb{F}}) \widetilde{\mathcal{E}}_{t-}^{-2} dK_t^{\mathbb{F}} \\ &\leq \int_0^{\infty} (Y_{t-}^{\mathbb{F}} - S_{t-}^{\mathbb{F}}) \widetilde{\mathcal{E}}_{t-}^{-2} dK_t^{\mathbb{F}} = 0, \end{aligned}$$

P-a.s.. This ends the second part, and the proof of theorem is complete. □

**Remark 4.2.8.** (a) *It is clear that, in general, the existence of an $L^p(P,\mathbb{F})$-solution to (4.2.38) exiges stronger assumptions than the existence of $L^p(P,\mathbb{G})$-solution to (4.2.1).*

(b) *It is clear that one can easily derive similar theorem for the BSDE setting. For more details see Theorem 4.2.6 and the discussions before it.*

## 4.3 Proof of Lemmas of Sections 4.1 and 4.2

In this subsection, we give the proof of the intermediate Lemmas 4.1.1, 4.1.3, 4.1.7, 4.2.1 and 4.2.4.

*Proof of Lemma 4.1.1.* Recall that $\Delta m = \widetilde{G} - G_- \leq 1$, and $m$ is a BMO $(\mathbb{F}, P)$-martingale, see Definition 2.1.19. Furthermore, we have

$$\begin{aligned}
&E^{\widetilde{Q}}\left[[m,m]_{T\wedge\tau} - [m,m]_{t\wedge\tau}\big|\mathcal{G}_t\right]\\
&= E\left[\int_{t\wedge\tau}^{T\wedge\tau} \mathcal{E}_s(G_-^{-1}\cdot m)^{-1} d[m,m]_s\big|\mathcal{G}_t\right]\mathcal{E}_{t\wedge\tau}(G_-^{-1}\cdot m)\\
&= E\left[\int_{t\wedge\tau}^{T\wedge\tau} \mathcal{E}_s(G_-^{-1}\cdot m)^{-1} d[m,m]_s\big|\mathcal{F}_t\right]\frac{\mathcal{E}_t(G_-^{-1}\cdot m)}{G_t} I_{\{\tau>t\}}\\
&= E\left[\int_t^T \widetilde{\mathcal{E}}_s d[m,m]_s\big|\mathcal{F}_t\right]\frac{1}{\widetilde{\mathcal{E}}_t} I_{\{\tau>t\}} \leq \|m\|_{BMO(P)}.
\end{aligned}$$

Hence, assertion (a) follows from this latter inequality. Thanks to Lemma 2.3.3, on $(\tau > s)$ we derive

$$\begin{aligned}
&E^{\widetilde{Q}}\left[D^{o,\mathbb{F}}_{T\wedge\tau} - D^{o,\mathbb{F}}_{s-}\big|\mathcal{G}_s\right]\\
&\quad = \Delta D^{o,\mathbb{F}}_s + E\left[\int_{s\wedge\tau}^{T\wedge\tau} \frac{1}{\mathcal{E}_u(G_-^{-1}\cdot m)} dD^{o,\mathbb{F}}_u\big|\mathcal{G}_s\right]\mathcal{E}_{s\wedge\tau}(G^{-1}\cdot m)\\
&\quad = E\left[\int_{s\wedge\tau}^{T\wedge\tau} \frac{1}{\mathcal{E}_u(G_-^{-1}\cdot m)} dD^{o,\mathbb{F}}_u\big|\mathcal{F}_s\right]\frac{\mathcal{E}_{s\wedge\tau}(G^{-1}\cdot m)}{G_s} + \Delta D^{o,\mathbb{F}}_s\\
&\quad = E\left[\int_s^T \frac{\mathcal{E}_u(-\widetilde{G}_-^{-1}\cdot D^{o,\mathbb{F}})}{\mathcal{E}_s(-\widetilde{G}_-^{-1}\cdot D^{o,\mathbb{F}})} dD^{o,\mathbb{F}}_u\big|\mathcal{F}_s\right] + \Delta D^{o,\mathbb{F}}_s \leq \widetilde{G}_s.
\end{aligned}$$

This proves assertion (b). The remaining part of this proof addresses assertion (c). Remark that $1-(1-x)^a \leq \max(a,1)x$, for any $0 \leq x \leq 1$. Thus, in virtue of (4.1.4), we get

$$\Delta\widetilde{V}^{(a)} = 1 - \left(1 - \frac{\Delta D^{o,\mathbb{F}}}{\widetilde{G}}\right)^a \leq \max(1,a)\frac{\Delta D^{o,\mathbb{F}}}{\widetilde{G}}.$$

Hence, by putting

$$W := \frac{\max(1,a)}{\widetilde{G}} \cdot D^{o,\mathbb{F}} - \widetilde{V}^{(a)},$$

we deduce that both

$$I_{\{\Delta D^{o,\mathbb{F}}\neq 0\}} \cdot W = \sum \left\{ \max(1,a)\frac{\Delta D^{o,\mathbb{F}}}{\widetilde{G}} - 1 + \left(1 - \frac{\Delta D^{o,\mathbb{F}}}{\widetilde{G}}\right)^a \right\}$$

and $I_{\{\Delta D^{o,\mathbb{F}}=0\}} \cdot W = \frac{(1-a)^+}{\widetilde{G}} I_{\{\Delta D^{o,\mathbb{F}}=0\}} \cdot D^{o,\mathbb{F}}$ are nondecreasing processes. By combining this with

$$W = I_{\{\Delta D^{o,\mathbb{F}}=0\}} \cdot W + I_{\{\Delta D^{o,\mathbb{F}}\neq 0\}} \cdot W$$

we deduce that assertion (c) holds. This ends the proof of the lemma. $\square$

*Proof of Lemma 4.1.3.* Let $\nu \in \mathcal{J}_{t\wedge\tau}^{T\wedge\tau}(\mathbb{G})$. By using (4.0.1) when $T < \infty$ and by taking the conditional expectation under $\widetilde{Q}$ afterwards, we get

$$\begin{aligned} Y_{t\wedge\tau} &= E^{\widetilde{Q}}\left[\int_{t\wedge\tau}^{\nu} f(s)ds + Y_\nu + K_\nu - K_{t\wedge\tau} \;\Big|\; \mathcal{G}_t\right] \\ &\geq E^{\widetilde{Q}}\left[\int_{t\wedge\tau}^{\nu} f(s)ds + S_\nu 1_{\{\nu\ <\tau\wedge T\}} + \xi 1_{\{\nu\ =\tau\wedge T\}} \;\Big|\; \mathcal{G}_t\right]. \end{aligned}$$

Thus, by letting $\nu$ spans the set $\mathcal{J}_{t\wedge\tau}^{T\wedge\tau}(\mathbb{G})$, we obtain

$$Y_{t\wedge\tau} \geq \operatorname*{ess\,sup}_{\theta\in\mathcal{J}_{t\wedge\tau}^{T\wedge\tau}(\mathbb{G})} E^{\widetilde{Q}}\left[\int_{t\wedge\tau}^{\theta} f(s)ds + S_\theta 1_{\{\theta<\tau\wedge T\}} + \xi I_{\{\theta=\tau\wedge T\}} \;\Big|\; \mathcal{G}_t\right]. \quad (4.3.1)$$

To prove the reverse inequality, we consider the following sequence of stopping times

$$\theta_n := \inf\left\{t\wedge\tau \leq u \leq T\wedge\tau;\quad Y_u < S_u + \frac{1}{n}\right\} \wedge (T\wedge\tau), \quad n \geq 1.$$

Then it is clear that $\theta_n \in \mathcal{J}_{t\wedge\tau}^{T\wedge\tau}(\mathbb{G})$, and

$$Y - S \geq \frac{1}{n} \quad \text{on} \quad [\![ t\wedge\tau, \theta_n [\![, \quad \text{and} \quad Y_- - S_- \geq \frac{1}{n} \quad \text{on} \quad ]\!] t\wedge\tau, \theta_n ]\!].$$

As a result, we get $I_{]\!] t\wedge\tau,\theta_n ]\!]} \cdot K \equiv 0$. Hence, by using (4.0.1) again we get

$$\begin{aligned} Y_{t\wedge\tau} &= Y_{\theta_n} + \int_{t\wedge\tau}^{\theta_n} f(s)ds + \int_{t\wedge\tau}^{\theta_n} d(K+M)_{t\wedge\tau} - \int_{t\wedge\tau}^{\theta_n} Z_s dW_t^\tau \\ &= Y_{\theta_n} + \int_{t\wedge\tau}^{\theta_n} f(s)ds + \int_{t\wedge\tau}^{\theta_n} dM_{t\wedge\tau} - \int_{t\wedge\tau}^{\theta_n} Z_s dW_t^\tau. \end{aligned}$$

By taking conditional expectation under $\widetilde{Q}$, we get

$$Y_{t\wedge\tau} = E^{\widetilde{Q}}[Y_{\theta_n} + \int_{t\wedge\tau}^{\theta_n} f(s)ds|\mathcal{G}_t],$$

which implies

$$\operatorname*{ess\,sup}_{\theta\in\mathcal{J}_{t\wedge\tau}^{T\wedge\tau}(\mathbb{G})} E^{\widetilde{Q}}\left[\int_{t\wedge\tau}^{\theta} f(s)ds + S_\theta 1_{\{\theta\ <\tau\wedge T\}} + \xi I_{\{\theta=\tau\wedge T\}} \;\Big|\; \mathcal{G}_t\right]$$

$$
\begin{aligned}
&\geq E^{\widetilde{Q}}\left[\int_{t\wedge\tau}^{\theta_n} f(s)ds + S_{\theta_n}1_{\{\theta_n<\tau\wedge T\}} + \xi 1_{\{\theta_n=\tau\wedge T\}} \;\Big|\; \mathcal{G}_t\right]\\
&= Y_{t\wedge\tau} + E^{\widetilde{Q}}\left[(S_{\theta_n} - Y_{\theta_n})1_{\{\theta_n<\tau\wedge T\}}\Big|\; \mathcal{G}_t\right]\\
&\geq Y_{t\wedge\tau} - \frac{1}{n}\widetilde{Q}(\theta_n < \tau\wedge T|\mathcal{G}_t).
\end{aligned}
$$

Thus, due to $\widetilde{Q}(\theta_n < \tau\wedge T\big|\mathcal{G}_t) \leq 1$, we take $n$ to infinity and get

$$
\operatorname*{ess\,sup}_{\nu\in\mathcal{J}_{t\wedge\tau}^{T\wedge\tau}(\mathbb{G})} E^{\widetilde{Q}}\left[\int_{t\wedge\tau}^{\nu} f(s)ds + S_\nu 1_{\{\nu\ <\tau\wedge T\}} + \xi 1_{\{\nu=\tau\wedge T\}} \;\Big|\; \mathcal{G}_t\right] \geq Y_{t\wedge\tau}.
$$

By combining this inequality with (4.3.1), we get (4.1.6), and the proof of the lemma is completed. $\square$

*Proof of Lemma 4.1.7.* Let $L$ be an $\mathbb{F}$-semimartingale. Then we derive

$$
\begin{aligned}
\frac{L}{\widetilde{\mathcal{E}}}I_{[\![0,\tau[\![} &= \frac{L^\tau}{\widetilde{\mathcal{E}}^\tau} - \frac{L}{\widetilde{\mathcal{E}}}\cdot D = L\cdot\frac{1}{\widetilde{\mathcal{E}}^\tau} + \frac{1}{\widetilde{\mathcal{E}}_-}\cdot L^\tau - \frac{L}{\widetilde{\mathcal{E}}}\cdot D\\
&= \frac{L}{G\widetilde{\mathcal{E}}_-}I_{]\!]0,\tau]\!]}\cdot D^{o,\mathbb{F}} + \frac{1}{\widetilde{\mathcal{E}}_-}\cdot L^\tau - \frac{L}{\widetilde{\mathcal{E}}}\cdot D\\
&= \frac{L}{\widetilde{G}\widetilde{\mathcal{E}}}I_{]\!]0,\tau]\!]}\cdot D^{o,\mathbb{F}} + \frac{1}{\widetilde{\mathcal{E}}_-}\cdot L^\tau - \frac{L}{\widetilde{\mathcal{E}}}\cdot D\\
&= -\frac{L}{\widetilde{\mathcal{E}}}\cdot N^{\mathbb{G}} + \frac{1}{\widetilde{\mathcal{E}}_-}\cdot L^\tau.
\end{aligned}
$$

The fourth equality follows from the fact that $\widetilde{\mathcal{E}} = \widetilde{\mathcal{E}}_- G/\widetilde{G}$. This ends the proof of the lemma. $\square$

*Proof of Lemma 4.2.1.* This proof has four parts where we prove the four assertions respectively.

**Part 1.** Let $a\in(0,+\infty)$ and $Y$ be a RCLL $\mathbb{G}$-semimartingale, and put

$Y_t^* := \sup_{0\leq s\leq t}|Y_s|$. Then, on the one hand, we remark that

$$\sup_{0\leq t\leq T\wedge\tau}\widetilde{\mathcal{E}}_t|Y_t|^a \leq \sup_{0\leq t\leq T\wedge\tau}\widetilde{\mathcal{E}}_t(Y_t^*)^a. \tag{4.3.2}$$

On the other hand, thanks to Itô, we derive

$$\widetilde{\mathcal{E}}(Y^*)^a = (Y_0^*)^a + \widetilde{\mathcal{E}}\cdot(Y^*)^a + (Y_-^*)^a\cdot\widetilde{\mathcal{E}} \leq (Y_0^*)^a + \widetilde{\mathcal{E}}\cdot(Y^*)^a. \tag{4.3.3}$$

Thus, by combining (4.3.2) and (4.3.3) with $\widetilde{\mathcal{E}} = G/\left(G_0\mathcal{E}(G_-^{-1}\cdot m)\right)$, we get

$$\begin{aligned}
&E\left[\sup_{0\leq t\leq T\wedge\tau}\widetilde{\mathcal{E}}_t|Y_t|^a\right] \leq E\left[(Y_0^*)^a + \int_0^{T\wedge\tau}\widetilde{\mathcal{E}}_s d(Y_s^*)^a\right]\\
&= E[(Y_0^*)^a] + \frac{1}{G_0}E^{\widetilde{Q}}\left[\int_0^{T\wedge\tau}G_s d(Y_s^*)^a\right] \leq G_0^{-1}E^{\widetilde{Q}}\left[(Y_{T\wedge\tau}^*)^a\right].
\end{aligned}$$

This proves assertion (a).

**Part 2.** Let $a\in(0,+\infty)$ and $K$ be a RCLL nondecreasing and $\mathbb{G}$-optional process with $K_0=0$. Then, we remark that

$$\begin{aligned}
\widetilde{\mathcal{E}}_-^a\cdot K &= K\widetilde{\mathcal{E}}^a - K\cdot\widetilde{\mathcal{E}}^a = K\widetilde{\mathcal{E}}^a + K\widetilde{\mathcal{E}}_-^a\cdot\widetilde{V}^{(a)}\\
&= K\widetilde{\mathcal{E}}^a + K_-\widetilde{\mathcal{E}}_-^a\cdot\widetilde{V}^{(a)} + \Delta K\widetilde{\mathcal{E}}_-^a\cdot\widetilde{V}^{(a)},
\end{aligned} \tag{4.3.4}$$

where $\widetilde{V}^{(a)}$ is defined in (4.1.4). As a result, by combining the above equality, the fact that $(\sum_{i=1}^n x_i)^{1/a} \leq n^{1/a}\sum_{i=1}^n x_i^{1/a}$ for any sequence of nonnegative numbers and Lemma 4.1.1, we derive

$$E\left[(\widetilde{\mathcal{E}}_-^a\cdot K_{T\wedge\tau})^{1/a}\right]$$

$$
\begin{aligned}
&\le 3^{1/a} E\left[(K_{T\wedge\tau})^{1/a}\widetilde{\mathcal{E}}_{T\wedge\tau} + (K_-\widetilde{\mathcal{E}}^a_- \cdot \widetilde{V}^{(a)}_{T\wedge\tau})^{1/a} + (\widetilde{\mathcal{E}}^a_-\Delta K \cdot \widetilde{V}^{(a)}_{T\wedge\tau})^{1/a}\right] \\
&\le 3^{1/a} E^{\widetilde{Q}}\left[\sqrt[a]{K_{T\wedge\tau}}\frac{G_{T\wedge\tau}}{G_0}\right] + 4\sqrt[a]{3}E\left[\sup_{0\le t\le T\wedge\tau} K_t^{1/a}\widetilde{\mathcal{E}}_t\right] \\
&\quad + \sqrt[a]{3}E\left[\sqrt[a]{\widetilde{\mathcal{E}}^a_-\Delta K \cdot \widetilde{V}^{(a)}_{T\wedge\tau}}\right].
\end{aligned}
$$

Then, due to $K^{1/a}\widetilde{\mathcal{E}} \le \widetilde{\mathcal{E}} \cdot K^{1/a}$ and $\widetilde{\mathcal{E}} = G/\left(G_0\mathcal{E}(G_-^{-1}\cdot m)\right)$, the above inequality leads to

$$
\begin{aligned}
&E\left[(\widetilde{\mathcal{E}}^a_- \cdot K_{T\wedge\tau})^{1/a}\right] \\
&\le \frac{\sqrt[a]{3}}{G_0}E^{\widetilde{Q}}\left[\sqrt[a]{K_{T\wedge\tau}}\right] + 4\times\sqrt[a]{3}E\left[\int_0^{T\wedge\tau}\widetilde{\mathcal{E}}_t d\sqrt[a]{K_t}\right] + \sqrt[a]{3}E\left[\sqrt[a]{\widetilde{\mathcal{E}}^a_-\Delta K \cdot \widetilde{V}^{(a)}_{T\wedge\tau}}\right] \\
&\le 5\frac{3^{1/a}}{G_0}E^{\widetilde{Q}}\left[(K_{T\wedge\tau})^{1/a}\right] + 3^{1/a}E\left[(\widetilde{\mathcal{E}}^a_-\Delta K \cdot \widetilde{V}^{(a)}_{T\wedge\tau})^{1/a}\right].
\end{aligned} \tag{4.3.5}
$$

Thus, it remains to deal with the last term in the right-hand-side term of the above inequality. To this end, we distinguish the cases whether $a \ge 1$ or $a < 1$.

The case when $a \ge 1$, or equivalently $1/a \le 1$. Then we use the fact that $(\sum x_i)^{1/a} \le \sum x_i^{1/a}$ for any sequence of nonnegative numbers, and get

$$
\begin{aligned}
&E\left[(\Delta K\widetilde{\mathcal{E}}^a_- \cdot \widetilde{V}^{(a)}_{T\wedge\tau})^{1/a}\right] \\
&= E\left[\left(\sum_{0\le t\le T\wedge\tau}\Delta K_t\widetilde{\mathcal{E}}^a_{t-}\Delta\widetilde{V}^{(a)}_t\right)^{1/a}\right] \le E\left[\sum_{0\le t\le T\wedge\tau}(\Delta K_t)^{1/a}\widetilde{\mathcal{E}}_{t-}(\Delta\widetilde{V}^{(a)}_t)^{1/a}\right] \\
&\le a^{1/a}E\left[\sum_{0\le t\le T\wedge\tau}(\Delta K_t)^{1/a}\widetilde{\mathcal{E}}_{t-}\right] = a^{1/a}E\left[\sum_{0\le t\le T\wedge\tau}(\Delta K_t)^{1/a}\frac{\widetilde{G}_t}{G_t}\widetilde{\mathcal{E}}_t\right] \\
&= \frac{a^{1/a}}{G_0}E^{\widetilde{Q}}\left[\sum_{0\le t\le T\wedge\tau}\widetilde{G}_t(\Delta K_t)^{1/a}\right].
\end{aligned}
$$

The last equality follows from $\widetilde{\mathcal{E}}/G = G_0^{-1}/\mathcal{E}(G_-^{-1}\cdot m)$. Thus, by combining this latter inequality with (4.3.5), assertion (b) follows immediately for this case of $a \geq 1$.

For the case of $a \in (0,1)$, or equivalently $1/a > 1$, we use Lemma 4.1.1 and derive

$$
\begin{aligned}
& E\left[(\Delta K\widetilde{\mathcal{E}}^a_-\cdot\widetilde{V}^{(a)})_{T\wedge\tau} - (\Delta K\widetilde{\mathcal{E}}^a_-\cdot\widetilde{V}^{(a)})_{t\wedge\tau-}\,\Big|\,\mathcal{G}_t\right] \\
&= E\left[\int_{t\wedge\tau}^{T\wedge\tau}\Delta K_s\widetilde{\mathcal{E}}^a_{s-}d\widetilde{V}^{(a)}_s + (\Delta K_{t\wedge\tau}\widetilde{\mathcal{E}}^a_{t\wedge\tau-}\Delta\widetilde{V}^{(a)})_{t\wedge\tau}\,\Big|\,\mathcal{G}_t\right] \\
&\leq E\left[\int_{t\wedge\tau}^{T\wedge\tau}\sup_{0\leq u\leq s}\Delta K_u\widetilde{\mathcal{E}}^a_{u-}d\widetilde{V}^{(a)}_s + \sup_{0\leq u\leq t\wedge\tau}\Delta K_u\widetilde{\mathcal{E}}^a_{u-}\,\Big|\,\mathcal{G}_t\right] \\
&= E\left[\int_{t\wedge\tau}^{T\wedge\tau}E[\widetilde{V}^{(a)}_{T\wedge\tau} - \widetilde{V}^{(a)}_{s-}|\mathcal{G}_s]d\sup_{0\leq u\leq s}\Delta K_u\widetilde{\mathcal{E}}^a_{u-} + \sup_{0\leq u\leq t\wedge\tau}\Delta K_u\widetilde{\mathcal{E}}^a_{u-}\,\Big|\,\mathcal{G}_t\right] \\
&\leq E\left[\sup_{0\leq u\leq T\wedge\tau}\Delta K_u\widetilde{\mathcal{E}}^a_{u-}\,\Big|\,\mathcal{G}_t\right].
\end{aligned}
$$

Therefore, a direct application of Theorem 2.1.23, we obtain

$$
\begin{aligned}
E\left[\sqrt[a]{\Delta K\widetilde{\mathcal{E}}^a_-\cdot\widetilde{V}^{(a)}_{T\wedge\tau}}\right] &\leq \frac{1}{\sqrt[a]{a}}E\left[\sup_{0\leq u\leq T\wedge\tau}\widetilde{\mathcal{E}}_{u-}\sqrt[a]{\Delta K_u}\right] \\
&\leq \frac{1}{\sqrt[a]{a}}E\left[\sum_{0\leq u\leq T\wedge\tau}\widetilde{\mathcal{E}}_{u-}\sqrt[a]{\Delta K_u}\right] \\
&= a^{-1/a}G_0^{-1}E^{\widetilde{Q}}\left[\sum_{0\leq u\leq T\wedge\tau}\widetilde{G}_u\sqrt[a]{\Delta K_u}\right].
\end{aligned}
$$

Hence, by combining this inequality with (4.3.5), assertion (b) follows immediately in this case of $a \in (0,1)$, and the proof of assertion (b) is complete.

**Part 3.** Here we prove assertion (c). To this end, we consider $p > 1$, a $\mathbb{G}$-optional process $H$, and we apply assertion (b) to the process $K = H\cdot[N^{\mathbb{G}}, N^{\mathbb{G}}]$

and $a = 2/p$, and get

$$E\left[(\widetilde{\mathcal{E}}_-^{2/p} H \cdot [N^{\mathbb{G}}, N^{\mathbb{G}}])_{T\wedge\tau}^{p/2}\right]$$
$$\leq \frac{C(a)}{G_0} E^{\widetilde{Q}}\left[(H \cdot [N^{\mathbb{G}}, N^{\mathbb{G}}]_{T\wedge\tau})^{p/2} + \sum_{0\leq t\leq T\wedge\tau} \widetilde{G}_t H_t^{p/2} |\Delta N^{\mathbb{G}}|^p\right].$$

Therefore, assertion (c) follows from combining this inequality with

$$\sum_{0\leq t\leq \cdot} \widetilde{G}_t H_t^{p/2} |\Delta N_t^{\mathbb{G}}| \leq \widetilde{G} H^{p/2} \cdot \mathrm{Var}(N^{\mathbb{G}}) \text{ and } |\Delta N^{\mathbb{G}}|^{p-1} \leq 1.$$

**Part 4.** Consider consider $p > 1$ and a nonnegative and $\mathbb{F}$-optional process $H$. Thus, by applying assertion (c), we obtain the inequality (4.2.4). Hence, to get (4.2.5), we remark that $\mathrm{Var}(N^{\mathbb{G}}) = (G/\widetilde{G}) \cdot D + \widetilde{G}^{-1} I_{]\!]0,\tau[\![} \cdot D^{o,\mathbb{F}}$, and due to the $\mathbb{F}$-optinality of $H$ we have

$$\begin{aligned} E^{\widetilde{Q}}\left[\widetilde{G}_t \sqrt{H_t^p} \cdot \mathrm{Var}(N^{\mathbb{G}})_T\right] &= 2E\left[\int_0^T \frac{\sqrt{H_t^p}}{\mathcal{E}_t(G_-^{-1} \cdot m)} I_{]\!]0,\tau[\![}(t) dD_t^{o,\mathbb{F}}\right] \\ &= 2E^{\widetilde{Q}}\left[(\sqrt{H^p} I_{]\!]0,\tau[\![} \cdot D^{o,\mathbb{F}})_T\right]. \end{aligned}$$

Therefore, by combining this with (4.2.4), assertion (d) follows immediately. This ends the proof of the lemma. □

*Proof of Lemma 4.2.4.* Remark that, for any process $H$, we have

$$H_{T\wedge\tau} = H_\tau I_{\{0<\tau\leq T\}} + H_T I_{\{\tau>T\}} + H_0 I_{\{\tau=0\}}.$$

Thus, by applying this to the process $X/\mathcal{E}(G_-^{-1} \cdot m)$, we derive

$$\begin{aligned}
E^{\widetilde{Q}}[X_{T\wedge\tau}] &= E\left[\frac{X_{T\wedge\tau}}{\mathcal{E}_{T\wedge\tau}(G_-^{-1} \cdot m)}\right] \\
&= E\left[\frac{X_\tau}{\mathcal{E}_\tau(G_-^{-1} \cdot m)} I_{\{0<\tau\leq T\}} + \frac{X_T}{\mathcal{E}_T(G_-^{-1} \cdot m)} I_{\{\tau>T\}} + X_0 I_{\{\tau=0\}}\right] \\
&= E\left[\int_0^T \frac{X_s}{\mathcal{E}_s(G_-^{-1} \cdot m)} dD_s^{o,\mathbb{F}} + \frac{X_T}{\mathcal{E}_T(G_-^{-1} \cdot m)} G_T + X_0(1-G_0)\right] \\
&= E\left[G_0 \int_0^T X_s dV_s^{\mathbb{F}} + G_0 X_T \widetilde{\mathcal{E}}_T + X_0(1-G_0)\right].
\end{aligned}$$

Thus, due to $X_0 = 0$, (4.2.14) follows immediately from the latter equality. To prove assertion (b), we take the limit on both sides of (4.2.14) and we use the fact that $G_{\infty-} = \lim_{t\longrightarrow+\infty} G_t = 0$ $P$-a.s. and this ends the proof of the lemma. $\square$

# Chapter 5

# General RBSDEs under random horizon

This chapter extends Chapter 4 to the case where the generator $f$ is a general functional $f(t,\omega,y,z)$ satisfying some Lipschitz' condition on the variables $y$ and $z$. Precisely, we address the following general RBSDE

$$\begin{cases} dY_t = -f(t,Y_t,Z_t)d(t\wedge\tau) - d(K_{t\wedge\tau}+M_{t\wedge\tau}) + Z_t dW^\tau_t, \\ Y_\tau = \xi = Y_T, \\ Y_t \geq S_t, \quad 0\leq t < T\wedge\tau, \quad \text{and} \quad \mathrm{E}\left[\int_0^{\mathrm{T}\wedge\tau}(\mathrm{Y_{t-}}-\mathrm{S_{t-}})\mathrm{dK_t}\right]=0. \end{cases} \tag{5.0.1}$$

Here, the barrier process $S$ is an $\mathbb{F}$-adapted and RCLL process, and the terminal value $\xi$ is an $\mathcal{F}_{T\wedge\tau}$-measurable random variable, or equivalently it takes the form of $\xi = h_{T\wedge\tau}$ for an $\mathbb{F}$-optional process $h$. The deterministic horizon $T\in(0,+\infty]$, while the generator $f$ is a $\mathrm{Prog}(\mathbb{F})\otimes\mathcal{B}(\mathbb{R})\otimes\mathcal{B}(\mathbb{R})$-measurable functional satisfying the following inequality for all $y, y^{'} \in \mathbb{R},\ z, z^{'} \in \mathbb{R}^d$,

$$|f(t,y,z) - f(t,y^{'},z^{'})| \leq C_{Lip}(|y - y^{'}| + \|z - z^{'}\|). \tag{5.0.2}$$

Here $C_{Lip}$ is a positive constant. Throughout our study for this RBSDE, in this chapter, we will distinguish two principal cases depending whether $T \in (0, +\infty)$ or $T = +\infty$. Thus, the rest of this chapter is divided into two sections.

## 5.1 The case of bounded horizon

In this section, we study the RBSDE (5.0.1) when $T < +\infty$. This section has two subsections. The first subsection presents some useful estimate results that are interesting in themselves beyond their role in proving the existence and uniqueness results. These latter results are elaborated in the second subsection besides the explicit connection between the RBSDE (5.0.1) and its $\mathbb{F}$-RBSDE counterpart and their solutions as well.

### 5.1.1 Estimate inequalities for the solution

In this subsection, we derive norm-estimates for the solution of the RBSDE (5.0.1) when it exists. These inequalities play important role in the proof of the existence and uniqueness of the solution of the RBSDE on the one hand. On the other hand, the role of these estimates in studying the stability of RBSDEs is known, for more details we refer the reader to [21] and [19] and the references therein to cite a few.

**Lemma 5.1.1.** *Suppose $T \in (0, +\infty)$, and put $f_0(t) := f(t, 0, 0)$. Then the following assertions hold.*

(a) *If $(Y^{\mathbb{G}}, Z^{\mathbb{G}}, K^{\mathbb{G}}, M^{\mathbb{G}})$ is a class-(D)-$(\mathbb{G}, \widetilde{Q}, \tau \wedge T)$-solution to the RBSDE (5.0.1) that corresponds to $(f, S, \xi)$, then*

$$Y_t^{\mathbb{G}} = \operatorname{ess}\sup_{\theta \in \mathcal{T}_{t\wedge\tau}^{T\wedge\tau}(\mathbb{G})} E^{\widetilde{Q}}\left[\int_{t\wedge\tau}^{\theta} f(s, Y_s^{\mathbb{G}}, Z_s^{\mathbb{G}})ds + S_\theta I_{\{\theta < T\wedge\tau\}} + \xi I_{\{\theta = T\wedge\tau\}} \;\Big|\; \mathcal{G}_t\right] \tag{5.1.1}$$

(b) *If $(Y^{\mathbb{G},i}, Z^{\mathbb{G},i}, K^{\mathbb{G},i}, M^{\mathbb{G},i})$ is a class-(D)-$(\mathbb{G}, \widetilde{Q}, T \wedge \tau)$-solution to the RBSDE (5.0.1) associated to $(f^{(i)}, S^{(i)}, \xi^{(i)})$, $i = 1, 2$, then for any $\alpha > 0$ the following holds*

$$\begin{aligned}
&\exp\left(\frac{\alpha(t\wedge\tau)}{2}\right)|\delta Y_t^{\mathbb{G}}| \\
&\leq \frac{C_{Lip}}{\sqrt{\alpha}} E^{\widetilde{Q}}\left[\sqrt{\int_0^{T\wedge\tau} e^{\alpha s}(\delta Z_s^{\mathbb{G}})^2 ds} + \sqrt{\int_0^{T\wedge\tau} e^{\alpha s}|\delta Y_s^{\mathbb{G}}|^2 ds} \;\Big|\; \mathcal{G}_t\right] \qquad (5.1.2)\\
&+ E^{\widetilde{Q}}\left[\sup_{0<s\leq T\wedge\tau} e^{\alpha s/2}|\delta S_u| + e^{\alpha(T\wedge\tau)/2}|\delta\xi| + \frac{1}{\sqrt{\alpha}}\sqrt{\int_0^{T\wedge\tau} e^{\alpha s}|\delta f_s|^2 ds} \;\Big|\; \mathcal{G}_t\right].
\end{aligned}$$

*Here, $\delta f_t := f^{(1)}(t, Y^{\mathbb{G},1}, Z^{\mathbb{G},1}) - f^{(2)}(t, Y^{\mathbb{G},1}, Z^{\mathbb{G},1})$.*

(c) *If $(Y^{\mathbb{G}}, Z^{\mathbb{G}}, K^{\mathbb{G}}, M^{\mathbb{G}})$ is a class-(D)-$(\mathbb{G}, \widetilde{Q}, T \wedge \tau)$-solution to the RBSDE (5.0.1) that corresponds to $(f, S, \xi)$, then for any $\alpha > 0$, any $\mathbb{F}$-stopping time $\sigma \in \mathcal{T}_0^T(\mathbb{F})$ and any $t \in [0, T]$, the following holds*

$$\begin{aligned}
&\exp\left(\frac{\alpha}{2}(t\wedge\tau)\right)|Y_t^{\mathbb{G}}| I_{\{\sigma\leq\tau\}} I_{\{\sigma\leq t\}} \\
&\leq \frac{C_{Lip}}{\sqrt{\alpha}} E^{\widetilde{Q}}\left[\sqrt{\int_{\sigma\wedge\tau}^{T\wedge\tau} e^{\alpha s}|Y_s^{\mathbb{G}}|^2 ds} + \sqrt{\int_{\sigma\wedge\tau}^{T\wedge\tau} e^{\alpha s}(Z_s^{\mathbb{G}})^2 ds} \;\Big|\; \mathcal{G}_t\right] \qquad (5.1.3)
\end{aligned}$$

$$+ E^{\widetilde{Q}}\left[\sup_{\sigma\wedge\tau\le u\le T\wedge\tau} e^{\alpha\frac{u}{2}}S_u^+ I_{\{\sigma\le\tau\}} + e^{\alpha\frac{T\wedge\tau}{2}}|\xi| I_{\{\sigma\le\tau\}}\ \Bigg|\ \mathcal{G}_t\right]$$
$$+ \frac{1}{\sqrt{\alpha}}E^{\widetilde{Q}}\left[\sqrt{\int_{\sigma\wedge\tau}^{T\wedge\tau} e^{\alpha s}|f_0(s)|^2 ds}\ \Bigg|\ \mathcal{G}_t\right].$$

*Proof.* The proof of assertion (a) mimics the footsteps of the proof of (4.1.6) in Theorem 4.1.8. Thus, the rest of this proof focuses on proving assertions (b) and (c) in two parts.

**Part 1.** This part proves assertion (b). To this end, we start by proving the following

$$\begin{cases} |\delta Y_t^{\mathbb{G}}| \le E^{\widetilde{Q}}\left[\int_{t\wedge\tau}^{T\wedge\tau}|\Delta f_s|ds + \sup_{t\wedge\tau<s\le T\wedge\tau}|\delta S_u| + |\delta\xi|\ \Big|\ \mathcal{G}_t\right], \\ \Delta f_t := f_1(t, Y_t^{\mathbb{G},1}, Z_t^{\mathbb{G},1}) - f_2(t, Y_t^{\mathbb{G},2}, Z_t^{\mathbb{G},2}) \end{cases} \tag{5.1.4}$$

Let $t \in [0, T]$ be arbitrary but fixed. Hence, on the one hand, (5.1.4) follows immediately by applying assertion (a) to each $Y^{\mathbb{G},i}$, $i = 1, 2$, and then using Lemma 4.1.5. On the other hand, due to Hölder's inequality, for any nonnegative and progressively measurable process $h$, and any $\alpha' > 0$, $p_1 > 1$, and $q_1 := p_1/(p_1 - 1)$, we have

$$\int_{t\wedge\tau}^{T\wedge\tau} h_s ds \le \left(\frac{p_1}{\alpha' q_1}\right)^{\frac{1}{q_1}} \exp\left(-\frac{\alpha'(t\wedge\tau)}{p_1}\right)\left(\int_{t\wedge\tau}^{T\wedge\tau} e^{\alpha' s}h_s^{p_1}ds\right)^{\frac{1}{p_1}}. \tag{5.1.5}$$

By using the fact that $|\Delta f_s| \le |\delta f_s| + C_{Lip}|\delta Y_s^{\mathbb{G}}| + C_{Lip}|\delta Z_s^{\mathbb{G}}|$, and applying the above inequality repeatedly, we derive

$$\int_{t\wedge\tau}^{T\wedge\tau}|\Delta f_s|ds$$

$$\leq \frac{1}{\sqrt{\alpha}} \exp\Big(-\frac{\alpha}{2}(t\wedge\tau)\Big) \left\{ \sqrt{\int_{t\wedge\tau}^{T\wedge\tau} e^{\alpha s}|\delta f_s|^2 ds} + C_{Lip}\sqrt{\int_{t\wedge\tau}^{T\wedge\tau} e^{\alpha s}|\delta Y_s^{\mathbb{G}}|^2 ds} \right\}$$
$$+\frac{C_{Lip}}{\sqrt{\alpha}} \exp\Big(-\frac{\alpha}{2}(t\wedge\tau)\Big) \left(\int_{t\wedge\tau}^{T\wedge\tau} e^{\alpha s}(\delta Z_s^{\mathbb{G}})^2 ds\right)^{1/2}.$$

Thus, by combining this inequality with (5.1.4), we derive

$$\exp\left(\frac{\alpha(t\wedge\tau)}{2}\right)|\delta Y_t^{\mathbb{G}}|$$
$$\leq E^{\widetilde{Q}}\left[\sup_{t\wedge\tau<s\leq T\wedge\tau} e^{\alpha s/2}|\delta S_u| + e^{\alpha(T\wedge\tau)/2}|\delta\xi| + \frac{C_{Lip}}{\sqrt{\alpha}}\sqrt{\int_{t\wedge\tau}^{T\wedge\tau} e^{\alpha s}(\delta Z_s^{\mathbb{G}})^2 ds}\ \Bigg|\ \mathcal{G}_t\right]$$
$$+\frac{1}{\sqrt{\alpha}}E^{\widetilde{Q}}\left[C_{Lip}\left(\int_{t\wedge\tau}^{T\wedge\tau} e^{\alpha s}|\delta Y_s^{\mathbb{G}}|^2 ds\right)^{\frac{1}{2}} + \left(\int_{t\wedge\tau}^{T\wedge\tau} e^{\alpha s}|\delta f_s|^2 ds\right)^{\frac{1}{2}}\ \Bigg|\ \mathcal{G}_t\right].$$

Here $C_{Lip}$ is the Lipschitz's constant associated to the driver $f$ defined in (5.0.2). Thus, (5.1.2) follows immediately from the inequality above, and the first part is completed.

**Part 2.** Here we prove assertion (c). Thus, we consider $\alpha > 0$ and an $\mathbb{F}$-stopping time $\sigma$. Similarly as in part 1, for any $t \in [0, T]$, thanks to (5.1.1) we have

$$Y_t^{\mathbb{G}} \geq -E^{\widetilde{Q}}\left[\int_{t\wedge\tau}^{T\wedge\tau} \left(f(s, Y_s^{\mathbb{G}}, Z_s^{\mathbb{G}})\right)^- ds + \xi^-\ \Bigg|\ \mathcal{G}_t\right]$$
$$Y_t^{\mathbb{G}} \leq E^{\widetilde{Q}}\left[\int_{t\wedge\tau}^{T\wedge\tau} \left(f(s, Y_s^{\mathbb{G}}, Z_s^{\mathbb{G}})\right)^+ ds + \sup_{t\wedge\tau\leq\theta\leq T\wedge\tau} S_\theta^+ I_{\{t<\tau\}} + \xi^+\ \Bigg|\ \mathcal{G}_t\right].$$

Thus, by combining these inequalities with $|x| = x^+ + x^-$, we obtain

$$|Y_t^{\mathbb{G}}| \leq E^{\widetilde{Q}}\left[\int_{t\wedge\tau}^{T\wedge\tau} |f(s, Y_s^{\mathbb{G}}, Z_s^{\mathbb{G}})|ds + \sup_{t\wedge\tau\leq\theta\leq T\wedge\tau} S_\theta^+ I_{\{t<\tau\}} + |\xi| \quad \Bigg| \ \mathcal{G}_t\right]. \tag{5.1.6}$$

Then the Lipschitz assumption of $f$ in (5.0.2) implies that

$$\int_{\sigma\wedge\tau}^{T\wedge\tau} |f(s, Y_s^{\mathbb{G}}, Z_s^{\mathbb{G}})|ds \leq \int_{\sigma\wedge\tau}^{T\wedge\tau} |f_0(s)|ds + C_{Lip}\int_{\sigma\wedge\tau}^{T\wedge\tau} |Y_s^{\mathbb{G}}|ds$$
$$+ C_{Lip}\int_{\sigma\wedge\tau}^{T\wedge\tau} |Z_s^{\mathbb{G}}|ds.$$

Hence, by applying (5.1.5) to each term on the right-hand-side above for $p_1 = q_1 = 2$, and inserting the resulting inequality in (5.1.6) afterwards, we obtain for any $t \in [0, T]$

$$e^{\alpha(t\wedge\tau)/2}|Y_t^{\mathbb{G}}|$$
$$\leq \frac{1}{\sqrt{\alpha}} E^{\widetilde{Q}}\left[\left(\int_{t\wedge\tau}^{T\wedge\tau} e^{\alpha s}(f_0(s))^2 ds\right)^{\frac{1}{2}} + C_{Lip}\left(\int_{t\wedge\tau}^{T\wedge\tau} e^{\alpha s}|Y_s^{\mathbb{G}}|^2 ds\right)^{\frac{1}{2}} \ \Bigg| \ \mathcal{G}_t\right]$$
$$+ E^{\widetilde{Q}}\left[\frac{C_{Lip}}{\sqrt{\alpha}}\sqrt{\int_{t\wedge\tau}^{T\wedge\tau} e^{\alpha s}(Z_s^{\mathbb{G}})^2 ds} + \sup_{t\wedge\tau\leq\theta\leq T\wedge\tau} e^{\frac{\theta}{2}} S_\theta^+ I_{\{t<\tau\}} + e^{\alpha(T\wedge\tau)/2}|\xi| \quad \Bigg| \ \mathcal{G}_t\right].$$

Therefore, the inequality (5.1.3) follows immediately from combining the above inequality with $(\sigma \leq t) \cap (\sigma \leq \tau) \in \mathcal{G}_t$. This proves assertion (c) and ends the proof of the lemma. $\square$

Throughout the rest of the thesis, $C_{DB}$ is the Doob's constant, $C_{BDG}$ is the BDG universal constant, $\kappa$ is the positive constant given by Lemma 4.1.2

(the three constants depend on $p \in (1, +\infty)$ only), and $C_{Lip}$ is the Lipschitz constant in (5.0.2).

**Theorem 5.1.2.** *Suppose that $T < +\infty$, and let $p > 1$, $\alpha > \alpha_0(p)$, and $0 < \alpha' < \alpha/2$, where*

$$\alpha_0(p) := \max\left(4C_{Lip} + 4C_{Lip}^2 + 1, 81\left\{1 + \frac{9\sqrt{2}\kappa(1 + C_{DB})}{3 - \sqrt{8}}\right\}^2 C_{DB}^2 C_{Lip}^2\right). \tag{5.1.7}$$

*Then there exists $\widehat{C} > 0$ which depends on $(\alpha, \alpha', p)$ only such that for any $\mathbb{F}$-stopping time $\sigma \in \mathcal{T}_0^T(\mathbb{F})$ and any class-(D)-$(\mathbb{G}, \widetilde{Q}, T \wedge \tau)$-solution to (5.0.1), denoted by the quadruplet $(Y^{\mathbb{G}}, Z^{\mathbb{G}}, M^{\mathbb{G}}, K^{\mathbb{G}})$, we have*

$$\begin{aligned}
&\|e^{\alpha \cdot/2} Y^{\mathbb{G}} I_{\{\tau \geq \sigma\}} I_{[\![\sigma, +\infty[\![}\|_{\mathbb{D}_{\tau \wedge T}(\widetilde{Q}, p)} + \|e^{\alpha \cdot/2} Z^{\mathbb{G}} I_{]\!]\sigma, +\infty[\![}\|_{\mathbb{S}_{\tau \wedge T}(\widetilde{Q}, p)} \\
&+ \|e^{\alpha \cdot/2} Y^{\mathbb{G}} I_{]\!]\sigma, +\infty[\![}\|_{\mathbb{S}_{\tau \wedge T}(\widetilde{Q}, p)} + \|e^{\alpha'(\tau \wedge \cdot)} I_{]\!]\sigma, +\infty[\![} \cdot K^{\mathbb{G}}\|_{\mathcal{A}_{T \wedge \tau}(\widetilde{Q}, p)} \\
&+ \|e^{\alpha(\tau \wedge \cdot)/2} I_{]\!]\sigma, +\infty[\![} \cdot M^{\mathbb{G}}\|_{\mathcal{M}_T^p(\widetilde{Q})} \leq \widehat{C}\left\{\|e^{\alpha(T \wedge \tau)/2} \xi I_{\{\tau \geq \sigma\}}\|_{L^p(\widetilde{Q})}\right\} \\
&+ \widehat{C}\left\{\|e^{(\alpha - \alpha')\cdot} S^+ I_{[\![\sigma, +\infty[\![}\|_{\mathbb{D}_{\tau \wedge T}(\widetilde{Q}, p)} + \|e^{\alpha \cdot/2} f_0(\cdot) I_{]\!]\sigma, +\infty[\![}\|_{\mathbb{S}_{\tau \wedge T}(\widetilde{Q}, p)}\right\}.
\end{aligned}$$

*Proof.* Let $\sigma \in \mathcal{T}_0^T(\mathbb{F})$ be an $\mathbb{F}$-stopping time. Remark that, in virtue of (5.1.3) and Doob's inequality under $(\widetilde{Q}, \mathbb{G})$, on the one hand, we have

$$\begin{aligned}
&\|e^{\alpha \cdot/2} Y^{\mathbb{G}} I_{\{\tau \geq \sigma\}} I_{[\![\sigma, +\infty[\![}\|_{\mathbb{D}_{T \wedge \tau}(\widetilde{Q}, p)} \\
&\leq C_{DB}\left\{\|e^{\alpha \cdot/2} S^+ I_{[\![\sigma, +\infty[\![}\|_{\mathbb{D}_{T \wedge \tau}(\widetilde{Q}, p)} + \|e^{\alpha(T \wedge \tau)/2} \xi I_{\{\tau \geq \sigma\}}\|_{L^p(\widetilde{Q})}\right\} \qquad (5.1.8) \\
&+ \frac{C_{DB} C_{Lip}}{\sqrt{\alpha}}\left\{\|e^{\alpha \cdot/2} Z^{\mathbb{G}} I_{]\!]\sigma, +\infty[\![}\|_{\mathbb{S}_{T \wedge \tau}(\widetilde{Q}, p)} + \|e^{\alpha \cdot/2} Y^{\mathbb{G}} I_{]\!]\sigma, +\infty[\![}\|_{\mathbb{S}_{T \wedge \tau}(\widetilde{Q}, p)}\right\}
\end{aligned}$$

$$+ \frac{C_{DB}}{\sqrt{\alpha}} \| e^{\alpha \cdot /2} f_0 I_{]\!]\sigma,+\infty[\![} \|_{\mathbb{S}_{T\wedge\tau}(\widetilde{Q},p)}.$$

On the other hand, by combining Itô applied to $e^{\alpha t}(Y_t^{\mathbb{G}})^2$, $e^{\alpha(\sigma\wedge\tau)}(Y_{\sigma\wedge\tau}^{\mathbb{G}})^2 \geq 0$, (5.0.1), and Young's inequality (i.e. $2xy \leq \epsilon x^2 + y^2/\epsilon$, $\epsilon > 0$), we put $C := \alpha - 2C_{Lip} - 2C_{Lip}^2 - \epsilon^{-1}$ and derive

$$\begin{aligned}
& C \int_{\sigma\wedge\tau}^{T\wedge\tau} e^{\alpha s}(Y_s^{\mathbb{G}})^2 ds + \frac{1}{2}\int_{\sigma\wedge\tau}^{T\wedge\tau} e^{\alpha s}(Z_s^{\mathbb{G}})^2 ds + \int_{\sigma\wedge\tau}^{T\wedge\tau} e^{\alpha t} d[M^{\mathbb{G}}, M^{\mathbb{G}}]_s \\
& \leq e^{\alpha(T\wedge\tau)}\xi^2 I_{\{\sigma\leq\tau\}} + \epsilon \int_{\sigma\wedge\tau}^{T\wedge\tau} e^{\alpha s}|f_0(s)|^2 ds + 2\int_{\sigma\wedge\tau}^{T\wedge\tau} e^{\alpha s} S_{s-}^{+} dK_s^{\mathbb{G}}, \qquad (5.1.9) \\
& + \sup_{0\leq t\leq T\wedge\tau} |(I_{]\!]\sigma,+\infty[\![} \cdot L^{\mathbb{G},1})_t|.
\end{aligned}$$

In this inequality, we also used the Skorokhod's condition (i.e., $(Y_-^{\mathbb{G}} - S_-)\cdot K^{\mathbb{G}} \equiv 0$), while $L^{\mathbb{G},1} \in \mathcal{M}_{loc}(\mathbb{G})$ is given by

$$L^{\mathbb{G},1} := 2e^{\alpha(\tau\wedge\cdot)}(Y_-^{\mathbb{G}} - \Delta K_s^{\mathbb{G}})) \cdot M^{\mathbb{G}} + 2e^{\alpha(\tau\wedge\cdot)} Y_-^{\mathbb{G}} Z^{\mathbb{G}} \cdot W^{\tau}. \qquad (5.1.10)$$

Throughout this section, for the sake of simplifying notation, we put

$$|||(Z^{\mathbb{G}}, M^{\mathbb{G}})|||_{(\sigma,\alpha,\widetilde{Q})} := \| e^{\frac{\alpha}{2}(\cdot\wedge\tau)} I_{]\!]\sigma,+\infty[\![} \cdot M^{\mathbb{G}} \|_{\mathcal{M}_T^p(\widetilde{Q})} + \| e^{\frac{\alpha \cdot}{2}} Z_s^{\mathbb{G}} I_{]\!]\sigma,+\infty[\![} \|_{\mathbb{S}_{T\wedge\tau}(\widetilde{Q},p)}. \qquad (5.1.11)$$

Thus, by applying Lemma 4.1.2 to $I_{]\!]\sigma,+\infty[\![} \cdot L^{\mathbb{G},1}$ with $a = b = p$, and using Doob's inequality afterwards to the martingale $E^{\widetilde{Q}}[\sup_{0\leq s\leq T\wedge\tau} |Y_s^{\mathbb{G}}| I_{\{\sigma\leq s\wedge\tau\}} \big| \mathcal{G}_t]$, we derive

$$\| \sqrt{|I_{]\!]\sigma,+\infty[\![} \cdot L^{\mathbb{G},1}|} \|_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},p)}$$

$$
\begin{aligned}
&\leq 2\sqrt{\kappa(1+C_{DB})|||(Z^{\mathbb{G}},M^{\mathbb{G}})|||_{(\sigma,\alpha,\widetilde{Q})}\|e^{\frac{\alpha\cdot}{2}}Y^{\mathbb{G}}I_{[\![\sigma,+\infty[\![}\|_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},p)}}\\
&\leq \epsilon_1|||(Z^{\mathbb{G}},M^{\mathbb{G}})|||_{(\sigma,\alpha,\widetilde{Q})}+\frac{\kappa(1+C_{DB})}{\epsilon_1}\|e^{\alpha\cdot/2}Y^{\mathbb{G}}I_{[\![\sigma,+\infty[\![}\|_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},p)}.
\end{aligned} \tag{5.1.12}
$$

Therefore, by combining (5.1.8), (5.1.9), and (5.1.12), and the fact that

$$
\|\sqrt{\sum_{i=1}^{n}X_i}\|_{L^p(\widetilde{Q})}\geq n^{-1}\sum_{i=1}^{n}\|\sqrt{X_i}\|_{L^p(\widetilde{Q})}
$$

for nonnegative random variables $(X_i)_{i=1,...,n}$, we get

$$
\begin{aligned}
&\|e^{\alpha\cdot/2}Y^{\mathbb{G}}I_{[\![\sigma,+\infty[\![}\|_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},p)}+C_1\|e^{\alpha\cdot/2}Y^{\mathbb{G}}I_{]\!]\sigma,+\infty[\![}\|_{\mathbb{S}_{T\wedge\tau}(\widetilde{Q},p)}\\
&+C_2\|e^{\alpha\cdot/2}Z^{\mathbb{G}}I_{]\!]\sigma,+\infty[\![}\|_{\mathbb{S}_{T\wedge\tau}(\widetilde{Q},p)}+C_3\|e^{\alpha\cdot/2}I_{]\!]\sigma,+\infty[\![}\cdot M^{\mathbb{G}}\|_{\mathcal{M}^p_T(\widetilde{Q})}\\
&\leq C_4\|e^{\frac{\alpha}{2}\cdot}f_0(\cdot)I_{]\!]\sigma,+\infty[\![}\|_{\mathbb{S}_{T\wedge\tau}(\widetilde{Q},p)}+C_5\|e^{\frac{\alpha}{2}(T\wedge\tau)}\xi I_{\{\sigma\leq\tau\}}\|_{\mathbb{L}^p(\widetilde{Q})}\\
&+C_6\|e^{\frac{\alpha}{2}\cdot}S^+I_{[\![\sigma,+\infty[\![}\|_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},p)}\\
&+C_7\|e^{(\alpha-\alpha')\cdot}S^+I_{[\![\sigma,+\infty[\![}\|^{1/2}_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},p)}\|e^{\alpha'(\tau\wedge\cdot)}I_{]\!]\sigma,+\infty[\![}\cdot K^{\mathbb{G}}_T\|^{1/2}_{\mathbb{L}^p(\widetilde{Q})},\quad \text{for}\quad \alpha'<\alpha/2,
\end{aligned} \tag{5.1.13}
$$

where $C_i$, $i=1,...,7$ are given by

$$
\begin{cases}
C_1:=\frac{\sqrt{C}}{3}-\left(1+\frac{\kappa(1+C_{DB})}{\epsilon_1}\right)\frac{C_{DB}C_{Lip}}{\sqrt{\alpha}},\\
C_2:=\frac{1}{3\sqrt{2}}-\epsilon_1-\left(1+\frac{\kappa(1+C_{DB})}{\epsilon_1}\right)\frac{C_{DB}C_{Lip}}{\sqrt{\alpha}},\\
C_3:=\frac{1}{3}-\epsilon_1,\quad C_4:=\sqrt{\epsilon}+\frac{C_{DB}}{\sqrt{\alpha}}\left(1+\frac{\kappa(1+C_{DB})}{\epsilon_1}\right),\\
C_5:=1+C_6,\quad C_6:=C_{DB}\left(1+\frac{\kappa(1+C_{DB})}{\epsilon_1}\right),\quad C_7:=\sqrt{2}.
\end{cases} \tag{5.1.14}
$$

Thus, the next step consists of controlling the norm of $K^{\mathbb{G}}$. To this end, we use the RBSDE (5.0.1) and Ito's formula, and derive for any $\alpha' > 0$ and $t > \sigma$

$$\begin{aligned}
&\int_{t\wedge\tau}^{T\wedge\tau} e^{\alpha' s} dK_s^{\mathbb{G}} \\
&= -\int_{t\wedge\tau}^{T\wedge\tau} e^{\alpha' s} dY_s^{\mathbb{G}} - \int_{t\wedge\tau}^{T\wedge\tau} e^{\alpha' s} f(s, Y_s^{\mathbb{G}}, Z_s^{\mathbb{G}}) ds - \int_{t\wedge\tau}^{T\wedge\tau} e^{\alpha' s} dM_t^{\mathbb{G}} \\
&\qquad\qquad + \int_{t\wedge\tau}^{T\wedge\tau} e^{\alpha' s} Z_s^{\mathbb{G}} dW_s^{\tau} \\
&\overset{Ito}{=} -e^{\alpha' T\wedge\tau} Y_{T\wedge\tau}^{\mathbb{G}} + e^{\alpha' t\wedge\tau} Y_{t\wedge\tau}^{\mathbb{G}} + \int_{t\wedge\tau}^{T\wedge\tau} e^{\alpha' s}(\alpha' Y_s^{\mathbb{G}} - f(s, Y_s^{\mathbb{G}}, Z_s^{\mathbb{G}})) ds \\
&\quad - \int_{t\wedge\tau}^{T\wedge\tau} e^{\alpha' s} dM_s^{\mathbb{G}} + \int_{t\wedge\tau}^{T\wedge\tau} e^{\alpha' s} Z_s^{\mathbb{G}} dW_s^{\tau}.
\end{aligned}$$

Therefore, by using this latter equality together with (5.0.2), we derive

$$\begin{aligned}
&E^{\widetilde{Q}}\left[\int_{t\wedge\tau}^{T\wedge\tau} e^{\alpha' s} dK_s^{\mathbb{G}} \,\Big|\mathcal{G}_{t\wedge\tau}\right] \\
&\le E^{\widetilde{Q}}\left[2 \sup_{t\wedge\tau < u \le T\wedge\tau} e^{\alpha' s}|Y_s^{\mathbb{G}}| + \int_{t\wedge\tau}^{T\wedge\tau} e^{\alpha' s}|\alpha' Y_s^{\mathbb{G}} + f(s, Y_s^{\mathbb{G}}, Z_s^{\mathbb{G}})| ds \,\Big|\mathcal{G}_{t\wedge\tau}\right], \\
&\le E^{\widetilde{Q}}\left[2 \sup_{t\wedge\tau < u \le T\wedge\tau} e^{\alpha' s}|Y_s^{\mathbb{G}}| + \int_{t\wedge\tau}^{T\wedge\tau} e^{\alpha' s}(\alpha' + C_{Lip})|Y_s^{\mathbb{G}}| ds \Big|\mathcal{G}_{t\wedge\tau}\right] \\
&\quad + E^{\widetilde{Q}}\left[\int_{t\wedge\tau}^{T\wedge\tau} e^{\alpha' s}|f_0(s)| ds \,\Big|\mathcal{G}_{t\wedge\tau}\right] + C_{Lip} E^{\widetilde{Q}}\left[\int_{t\wedge\tau}^{T\wedge\tau} e^{\alpha' s}|Z_s^{\mathbb{G}}| ds \,\Big|\mathcal{G}_{t\wedge\tau}\right].
\end{aligned}$$

Then by applying (5.1.5) for each term above, and choosing $\alpha' < \alpha/2$, we get for $t > \sigma$

$$\begin{aligned}
&E^{\widetilde{Q}}\left[\int_{t\wedge\tau}^{T\wedge\tau} e^{\alpha' s} dK_s^{\mathbb{G}} \,\Big|\mathcal{G}_{t\wedge\tau}\right] \\
&\le E^{\widetilde{Q}}\left[2 \sup_{\sigma\wedge\tau \le u \le T\wedge\tau} e^{\alpha' s}|Y_s^{\mathbb{G}}| + \frac{\alpha' + C_{Lip}}{\sqrt{\alpha - 2\alpha'}} \sqrt{\int_{\sigma\wedge\tau}^{T\wedge\tau} e^{\alpha s}(Y_s^{\mathbb{G}})^2 ds} \,\Big|\mathcal{G}_{t\wedge\tau}\right]
\end{aligned}$$

$$+\frac{1}{\sqrt{\alpha-2\alpha'}}E^{\widetilde{Q}}\left[C_{Lip}\sqrt{\int_{\sigma\wedge\tau}^{T\wedge\tau}e^{\alpha s}(Z_s^{\mathbb{G}})^2ds}+\sqrt{\int_{\sigma\wedge\tau}^{T\wedge\tau}e^{\alpha s}(f_0(s))^2ds}\ \Big|\mathcal{G}_{t\wedge\tau}\right].$$

Therefore, thanks to Theorem 2.1.23, we deduce that for any $p > 1$ and $\alpha' < \alpha/2$, we have

$$\begin{aligned}
&\|(e^{\alpha'\cdot}I_{]\!]\sigma,+\infty[\![}\cdot K^{\mathbb{G}})_{T\wedge\tau}\|_{L^p(\widetilde{Q})}\\
&\leq C'\left\{\|e^{\frac{\alpha\cdot}{2}}Y^{\mathbb{G}}I_{[\![\sigma,+\infty[\![}\|_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},p)}+\|e^{\frac{\alpha\cdot}{2}}Y^{\mathbb{G}}I_{]\!]\sigma,+\infty[\![}\|_{\mathbb{S}_{T\wedge\tau}(\widetilde{Q},p)}\right\}\\
&+C'\|e^{\frac{\alpha\cdot}{2}}Z^{\mathbb{G}}I_{]\!]\sigma,+\infty[\![}\|_{\mathbb{S}_{T\wedge\tau}(\widetilde{Q},p)}+C'\|e^{\alpha\cdot/2}f_0(\cdot)I_{]\!]\sigma,+\infty[\![}\|_{\mathbb{S}_{T\wedge\tau}(\widetilde{Q},p)},
\end{aligned}\tag{5.1.15}$$

where the constant $C'$ is given by

$$C' := p\max\left(2,\frac{\alpha'+C_{Lip}}{\sqrt{\alpha-2\alpha'}}\right).$$

Remark that for $\alpha > \alpha_0(p)$, and by choosing $\epsilon = 9/5$ and $\epsilon_1 = (3-\sqrt{8})/9\sqrt{2}$, we get $1/9 < C_2 \leq \min(C_1, C_3)$. By inserting (5.1.15) in (5.1.13) and using Young's inequality, we get

$$\begin{aligned}
&\|e^{\frac{\alpha\cdot}{2}}Y^{\mathbb{G}}I_{[\![\sigma,+\infty[\![}\|_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},p)}+\|e^{\frac{\alpha\cdot}{2}}Y^{\mathbb{G}}I_{]\!]\sigma,+\infty[\![}\|_{\mathbb{S}_{T\wedge\tau}(\widetilde{Q},p)}+|||(Z^{\mathbb{G}},M^{\mathbb{G}})|||_{(\sigma,\alpha,\widetilde{Q})}\\
&\leq\overline{C}\left\{\|e^{\frac{\alpha}{2}(\tau\wedge\cdot)}f_0I_{[\![\sigma,+\infty[\![}\|_{\mathbb{S}_T^p(\widetilde{Q})}+\|e^{\frac{\alpha}{2}(T\wedge\tau)}\xi I_{\{\sigma\leq\tau\}}\|_{\mathbb{L}^p(\widetilde{Q})}\right\}\\
&+\overline{C}\|e^{(\alpha-\alpha')\cdot}S^+I_{]\!]\sigma,+\infty[\![}\|_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},p)},
\end{aligned}$$

where $\overline{C} := (20(C')^2 + C_6)/(C_2 - (1/9))$. Therefore, the proof of the theorem follows immediately from combining the above inequality with (5.1.15) and choosing $\widehat{C} = \overline{C}(1 + C') + C'$. This ends the proof of the theorem. $\square$

**Theorem 5.1.3.** *Suppose that $T < +\infty$ and let $p > 1$, $\alpha > \alpha_0(p)$ given in (5.1.7), and $\alpha' \in (0, \alpha/2)$. Then there exist positive $\widehat{C}_j$, $j = 1, 2, 3, 4$, which depend on $(\alpha, \alpha', p)$ only such that $\lim_{\alpha\to\infty} \widehat{C}_1 = 0$ and for $(Y^{\mathbb{G},i}, Z^{\mathbb{G},i}, K^{\mathbb{G},i}, M^{\mathbb{G},i})$, a class-(D)-$(\mathbb{G}, \widetilde{Q}, T \wedge \tau)$-solution to (5.0.1) corresponding to $(f^{(i)}, S^{(i)}, \xi^{(i)})$, $i = 1, 2$, we have*

$$\begin{aligned}
&\|e^{\frac{\alpha}{2}\cdot}\delta Y^{\mathbb{G}}\|_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},p)} + \|e^{\frac{\alpha}{2}\cdot}\delta Y^{\mathbb{G}}\|_{\mathbb{S}_{T\wedge\tau}(\widetilde{Q},p)}\\
&+ \|e^{\frac{\alpha}{2}\cdot}\delta Z^{\mathbb{G}}\|_{\mathbb{S}_{T\wedge\tau}(\widetilde{Q},p)} + \|e^{\frac{\alpha}{2}\cdot}\cdot\delta M^{\mathbb{G}}\|_{\mathcal{M}^p(\widetilde{Q})}\\
&\leq \widehat{C}_1\|e^{\alpha\cdot/2}\delta f\|_{\mathbb{S}_{T\wedge\tau}(\widetilde{Q},p)} + \widehat{C}_2\|e^{\alpha(T\wedge\tau)/2}\delta\xi\|_{\mathbb{L}^p(\widetilde{Q})} + \widehat{C}_3\|e^{\alpha\cdot/2}\delta S\|_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},p)}\\
&+ \widehat{C}_4\sqrt{\|e^{\alpha\cdot/2}\delta S\|_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},p)}\left\{\sum_{i=1}^{2}\Delta(\xi^{(i)}, f^{(i)}, (S^{(i)})^+)\right\}}.
\end{aligned}\tag{5.1.16}$$

*Here $\Delta(\xi^{(i)}, f^{(i)}, (S^{(i)})^+)$ is given by*

$$\begin{aligned}
&\Delta(\xi^{(i)}, f^{(i)}, (S^{(i)})^+)\\
&:= \|e^{\alpha(T\wedge\tau)/2}\xi^{(i)}\|_{L^p(\widetilde{Q})} + \|e^{\alpha(\tau\wedge\cdot)}(S^{(i)})^+\|_{\mathbb{D}_T(\widetilde{Q},p)} + \|e^{\alpha(\tau\wedge\cdot)/2}f^{(i)}(\cdot,0,0)\|_{\mathbb{S}_T(\widetilde{Q},p)},
\end{aligned}\tag{5.1.17}$$

*and $(\delta Y^{\mathbb{G}}, \delta Z^{\mathbb{G}}, \delta M^{\mathbb{G}}, \delta K^{\mathbb{G}})$ and $(\delta f, \delta S, \delta\xi)$ are given by*

$$\begin{aligned}
&\delta Y^{\mathbb{G}} := Y^{\mathbb{G},1} - Y^{\mathbb{G},2},\ \ \delta Z^{\mathbb{G}} := Z^{\mathbb{G},1} - Z^{\mathbb{G},2}, \delta M^{\mathbb{G}} := M^{\mathbb{G},1} - M_2^{\mathbb{G},2},\\
&\delta K^{\mathbb{G}} := K^{\mathbb{G},1} - K^{\mathbb{G},2},\quad \delta S := S^{(1)} - S^{(2)},\quad \delta\xi := \xi^{(1)} - \xi^{(2)},\\
&\delta f_t := f_1(t, Y_t^{\mathbb{G},1}, Z_t^{\mathbb{G},1}) - f_2(t, Y_t^{\mathbb{G},1}, Z_t^{\mathbb{G},1}).
\end{aligned}$$

*Proof.* On the one hand, due to the Lipschitz assumption on $f$, we have

$$|\Delta f_t| := |f_1(t, Y^{\mathbb{G},1}, Z^{\mathbb{G},1}) - f_2(t, Y^{\mathbb{G},2}, Z^{\mathbb{G},2})| \leq |\delta f_t| + C_{Lip}|\delta Y_t^{\mathbb{G}}| + C_{Lip}|\delta Z_t^{\mathbb{G}}|. \tag{5.1.18}$$

On the other hand, in virtue of Lemma 5.1.1-(b) and Doob's inequality, we get

$$\begin{aligned}
&\|e^{\alpha\cdot/2}\delta Y^{\mathbb{G}}\|_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},p)} \\
&\leq C_{DB}\left\{\|e^{\frac{\alpha\cdot}{2}}\delta S_u\|_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},p)} + \|e^{\frac{\alpha}{2}(T\wedge\tau)}\delta\xi\|_{L^p(\widetilde{Q})} + \frac{1}{\sqrt{\alpha}}\|e^{\frac{\alpha\cdot}{2}}\delta f\|_{\mathbb{S}_{T\wedge\tau}(\widetilde{Q},p)}\right\} \qquad (5.1.19)\\
&+ \frac{C_{DB}C_{Lip}}{\sqrt{\alpha}}\left\{\|e^{\alpha\cdot/2}\delta Z^{\mathbb{G}}\|_{\mathbb{S}_{T\wedge\tau}(\widetilde{Q},p)} + \|e^{\alpha\cdot/2}\delta Y^{\mathbb{G}}\|_{\mathbb{S}_{T\wedge\tau}(\widetilde{Q},p)}\right\}.
\end{aligned}$$

By combining Itô applied to $e^{\alpha t}(\delta Y_t^{\mathbb{G}})^2$, $(\delta Y_0^{\mathbb{G}})^2 \geq 0$ and (6.2.10), and putting

$$L^{\mathbb{G}} := e^{\alpha(\tau\wedge\cdot)}(\delta Y_-^{\mathbb{G}} - 2\Delta(\delta K_s^{\mathbb{G}}))\cdot\delta M^{\mathbb{G}} + e^{\alpha(\tau\wedge\cdot)}(\delta Y_-^{\mathbb{G}})\delta Z^{\mathbb{G}}\cdot W^{\tau}, \tag{5.1.20}$$

which belongs to $\mathcal{M}_{loc}(\widetilde{Q}, \mathbb{G})$, we derive

$$\begin{aligned}
&\alpha\int_0^{T\wedge\tau} e^{\alpha s}(\delta Y_s^{\mathbb{G}})^2 ds + \int_0^{T\wedge\tau} e^{\alpha s}(\delta Z_s^{\mathbb{G}})^2 ds + \int_0^{T\wedge\tau} e^{\alpha t} d[\delta M^{\mathbb{G}}, \delta M^{\mathbb{G}}]_s \\
&\leq e^{\alpha(T\wedge\tau)}(\delta\xi)^2 + 2\int_0^{T\wedge\tau} e^{\alpha s}(\delta Y_s^{\mathbb{G}})\Delta f_s ds + 2\int_0^{T\wedge\tau} e^{\alpha s}(\delta Y_{s-}^{\mathbb{G}}) d\delta K_s^{\mathbb{G}} + L_T^{\mathbb{G}}, \\
&\leq e^{\alpha(T\wedge\tau)}(\delta\xi)^2 + 2\int_0^{T\wedge\tau} e^{\alpha s}|\delta Y_s^{\mathbb{G}}|(|\delta f_s| + C_{Lip}(|\delta Y^{\mathbb{G}}| + |\delta Z^{\mathbb{G}}|))ds \\
&\qquad + 2(e^{\alpha\cdot}(\delta Y_-^{\mathbb{G}})\cdot\delta K^{\mathbb{G}})_{T\wedge\tau} + L_T^{\mathbb{G}} \\
&= e^{\alpha(T\wedge\tau)}(\delta\xi)^2 + 2\int_0^{T\wedge\tau} e^{\alpha s}|\delta Y_s^{\mathbb{G}}||\delta f_s| ds + 2\int_0^{T\wedge\tau} e^{\alpha s} C_{Lip}|\delta Y_s^{\mathbb{G}}|^2 ds
\end{aligned}$$

$$
\begin{aligned}
&+ 2\int_0^{T\wedge\tau} e^{\alpha s} C_{Lip}|\delta Y_s^{\mathbb{G}}||\delta Z^{\mathbb{G}}|ds + 2\int_0^{T\wedge\tau} e^{\alpha s}(\delta Y_{s-}^{\mathbb{G}})d\delta K_s^{\mathbb{G}} + L_T^{\mathbb{G}} \\
\le & e^{\alpha(T\wedge\tau)}(\delta\xi)^2 + (\frac{1}{\epsilon} + 2C_{Lip}^2 + 2C_{Lip})\int_0^{T\wedge\tau} e^{\alpha s}|\delta Y_s^{\mathbb{G}}|^2 ds + \epsilon\int_0^{T\wedge\tau} e^{\alpha s}|\delta f_s|^2 ds \\
&+ \frac{1}{2}\int_0^{T\wedge\tau} e^{\alpha s}|\delta Z^{\mathbb{G}}|^2 ds + 2\int_0^{T\wedge\tau} e^{\alpha s}(\delta Y_{s-}^{\mathbb{G}})d\delta K_s^{\mathbb{G}} + L_T^{\mathbb{G}}.
\end{aligned}
$$

Therefore, by arranging terms and putting $C := \alpha - 2C_{Lip} - 2C_{Lip}^2 - \epsilon^{-1}$ , we obtain

$$
\begin{aligned}
& C\int_0^{T\wedge\tau} e^{\alpha s}(\delta Y_s^{\mathbb{G}})^2 ds + \frac{1}{2}\int_0^{T\wedge\tau} e^{\alpha s}(\delta Z_s^{\mathbb{G}})^2 ds + \int_0^{T\wedge\tau} e^{\alpha t} d[\delta M^{\mathbb{G}}, \delta M^{\mathbb{G}}]_s \\
& \le e^{\alpha(T\wedge\tau)}(\delta\xi)^2 + \epsilon\int_0^{T\wedge\tau} e^{\alpha s}|\delta f_s|^2 ds + 2\int_0^{T\wedge\tau} e^{\alpha s}(\delta Y_{s-}^{\mathbb{G}})d\delta K_s^{\mathbb{G}} + L_T^{\mathbb{G}} \\
& \le e^{\alpha(T\wedge\tau)}(\delta\xi)^2 + \epsilon\int_0^{T\wedge\tau} e^{\alpha s}|\delta f_s|^2 ds + 2\int_0^{T\wedge\tau} e^{\alpha s}|\delta S_{s-}|d\mathrm{Var}_s(\delta K^{\mathbb{G}}) + L_T^{\mathbb{G}}.
\end{aligned}
\tag{5.1.21}
$$

The last inequality is a consequence of

$$
e^{\alpha(\tau\wedge\cdot)}(\delta Y_-^{\mathbb{G}}) \cdot \delta K^{\mathbb{G}} \le e^{\alpha(\tau\wedge\cdot)}(\delta S_-^{\mathbb{G}}) \cdot \delta K^{\mathbb{G}} \le e^{\alpha(\tau\wedge\cdot)}|\delta S_-^{\mathbb{G}}| \cdot \mathrm{Var}(\delta K^{\mathbb{G}}),
$$

which is due to Skorokhod's condition. Furthermore, by applying Lemma 4.1.2 to $L^{\mathbb{G}}$ given in (5.1.20) with $a = b = p$ and Doob's inequality afterwards, there exists a constant $\kappa = \kappa(p) > 0$ which depends on $p$ only such that

$$
\begin{aligned}
&\||L^{\mathbb{G}}|^{1/2}\|_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},p)} \\
&\le \sqrt{\kappa(1 + C_{DB})\left\{\|e^{\frac{\alpha}{2}\cdot} \cdot \delta M^{\mathbb{G}}\|_{\mathcal{M}_T^p(\widetilde{Q})} + \|e^{\frac{\alpha}{2}\cdot}\delta Z^{\mathbb{G}}\|_{\mathbb{S}_{T\wedge\tau}(\widetilde{Q},p)}\right\}} \|e^{\frac{\alpha}{2}\cdot}\delta Y^{\mathbb{G}}\|_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},p)}^{1/2}
\end{aligned}
$$

$$\leq \epsilon_1 \|e^{\frac{\alpha}{2}\cdot} \cdot \delta M^{\mathbb{G}}\|_{\mathcal{M}^p_T(\widetilde{Q})} + \epsilon_1 \|e^{\frac{\alpha}{2}\cdot}\delta Z^{\mathbb{G}}\|_{\mathbb{S}_{T\wedge\tau}(\widetilde{Q},p)} + \frac{\kappa(1+C_{DB})}{\epsilon_1}\|e^{\frac{\alpha}{2}\cdot}\delta Y^{\mathbb{G}}\|_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},p)}. \tag{5.1.22}$$

Therefore, by combining (5.1.21), (5.1.22) and (5.1.19) and choosing adequately $\alpha, \epsilon, \epsilon_1$ and using $n^{-1}\sum_{i=1}^n x_i^{p/2} \leq (\sum_{i=1}^n x_i)^{p/2} \leq n^{p/2}\sum_{i=1}^n x_i^{p/2}$ for any positive integer and any sequence of nonnegative number $x_i$, we derive

$$\begin{aligned}
&\frac{1}{3}\left\{\sqrt{C}\|e^{\alpha\cdot/2}\delta Y^{\mathbb{G}}\|_{\mathbb{S}_{T\wedge\tau}(\widetilde{Q},p)} + 2^{-1}\|e^{\alpha\cdot/2}\delta Z^{\mathbb{G}}\|_{\mathbb{S}_{T\wedge\tau}(\widetilde{Q},p)} + \|e^{\alpha(\tau\wedge\cdot)/2}\cdot\delta M^{\mathbb{G}}\|_{\mathcal{M}^p_T(\widetilde{Q})}\right\}\\
&\leq \epsilon\|e^{\frac{\alpha}{2}\cdot}\delta f\|_{\mathbb{S}_{T\wedge\tau}(\widetilde{Q},p)} + \|e^{\frac{\alpha}{2}(T\wedge\tau)}\delta\xi\|_{\mathbb{L}^p(\widetilde{Q})} + \||L^{\mathbb{G}}|^{1/2}\|_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},p)}\\
&+\sqrt{2}\|e^{\frac{\alpha}{2}\cdot}\delta S\|^{1/2}_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},p)}\|e^{\frac{\alpha}{2}\cdot}\cdot\delta K^{\mathbb{G}}\|^{1/2}_{\mathcal{A}_{T\wedge\tau}(\widetilde{Q},p)},\\
&\leq \epsilon\|e^{\alpha(\tau\wedge\cdot)/2}\delta f\|_{\mathbb{S}_{T\wedge\tau}(\widetilde{Q})} + \|e^{\frac{\alpha}{2}(T\wedge\tau)}\delta\xi\|_{\mathbb{L}^p(\widetilde{Q})}\\
&+\sqrt{2}\|e^{\alpha\cdot/2}\delta S\|^{1/2}_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},p)}\|e^{\alpha\cdot/2}\cdot\delta K^{\mathbb{G}})\|^{1/2}_{\mathcal{A}_{T\wedge\tau}(\widetilde{Q},p)} + \epsilon_1\|e^{\alpha(\tau\wedge\cdot)/2}\cdot\delta M^{\mathbb{G}}\|_{\mathcal{M}^p_T(\widetilde{Q})}\\
&+\epsilon_1\|e^{\alpha\cdot/2}\delta Z^{\mathbb{G}}\|_{\mathbb{S}_{T\wedge\tau}(\widetilde{Q},p)} + \frac{\kappa(1+C_{DB})}{\epsilon_1}\|e^{\alpha\cdot/2}\delta Y^{\mathbb{G}}\|_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},p)}.
\end{aligned}$$

Then by combining this equality with (5.1.19) and (5.1.21) we obtain

$$\begin{aligned}
&\|e^{\frac{\alpha}{2}\cdot}\delta Y^{\mathbb{G}}\|_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},\mathbb{G})} + C_1\|e^{\frac{\alpha}{2}\cdot}\delta Y^{\mathbb{G}}\|_{\mathbb{S}_{T\wedge\tau}(\widetilde{Q},p)} + C_2\|e^{\frac{\alpha}{2}\cdot}\delta Z^{\mathbb{G}}\|_{\mathbb{S}_{T\wedge\tau}(\widetilde{Q},p)}\\
&+C_3\|e^{\frac{\alpha}{2}(\tau\wedge\cdot)}\cdot\delta M^{\mathbb{G}}\|_{\mathcal{M}^p(\widetilde{Q})} \leq C_4\|e^{\alpha\cdot/2}\delta f\|_{\mathbb{S}_{T\wedge\tau}(\widetilde{Q},p)} + C_5\|e^{\alpha(T\wedge\tau)/2}\delta\xi\|_{\mathbb{L}^p(\widetilde{Q})}\\
&+C_6\|e^{\alpha\cdot/2}\delta S\|_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},p)} + C_7\sqrt{\|e^{\alpha\cdot/2}\delta S\|_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},p)}\|e^{\alpha\cdot/2}\cdot\delta K^{\mathbb{G}}\|_{\mathcal{A}_{T\wedge\tau}(\widetilde{Q},p)}},
\end{aligned} \tag{5.1.23}$$

where $C_i$, $i = 1, ..., 7$ are given by (5.1.14). Then here we take $\epsilon = 2/\alpha$, $\epsilon_1 = (3-\sqrt{8})/9\sqrt{2}$ and $\alpha > \alpha_1(p)$, and remark that $0 < C_2 \leq \min(C_1, C_3)$.

Furthermore, in virtue of Theorem 5.1.2 with $\sigma = 0$, we get

$$\|e^{\alpha\cdot/2}\cdot\delta K^{\mathbb{G}}\|_{\mathcal{A}_{T\wedge\tau}(\widetilde{Q},p)} \leq \sum_{i=1}^{2}\|e^{\alpha\cdot/2}\cdot K^{\mathbb{G},i}\|_{\mathcal{A}_{T\wedge\tau}(\widetilde{Q},p)} \leq \widehat{C}\sum_{i=1}^{2}\Delta(\xi^{(i)}, f^{(i)}, (S^{(i)})^{+}).$$

Therefore, by inserting this in (5.1.23), the inequality (5.1.16) follows immediately with

$$\widehat{C}_1 = \frac{C_4}{C_2}, \quad \widehat{C}_2 = \frac{C_5}{C_2}, \quad \widehat{C}_3 = \frac{C_6}{C_2}, \quad \widehat{C}_4 = \frac{C_7\sqrt{\widehat{C}}}{C_2}.$$

It is also clear that $\widehat{C}_1$ goes to zero when $\alpha$ goes to infinity. This ends the proof of the theorem. $\square$

### 5.1.2 Existence, uniqueness & connection to $\mathbb{F}$-RBSDEs

In this subsection, we elaborate our results on the existence and uniqueness of the solution to (5.0.1), and describe the form of its $\mathbb{F}$-RBSDE counterpart. To this end, we assume that there exists $\alpha > \alpha_0(p)$ such that

$$E\left[\widetilde{\mathcal{E}}_T\mathcal{K}_T^{(\alpha)}(f,S,h) + \int_0^T \mathcal{K}_s^{(\alpha)}(f,S,h)dV_s^{\mathbb{F}}\right] < +\infty, \qquad (5.1.24)$$

where

$$\mathcal{K}_t^{(\alpha)}(f,S,h) := |e^{\alpha t/2}h_t|^p + \left(\int_0^t e^{\alpha s}|f_0(s)|^2 ds\right)^{p/2} + \sup_{0\leq u\leq t}(e^{\alpha u}S_u^{+})^p, \quad (5.1.25)$$

and $f_0(t) := f(t,0,0)$. One of the main obstacles, herein, lies in guessing the form of the $\mathbb{F}$-RBSDE that corresponds to (5.0.1). To overcome this challenge, we appeal to the linear case and the known method of approximating the solution to the general RBSDE (5.0.1) by the sequence of solutions to linear RBSDEs –as it is adopted in [19] and the references therein–. This is the aim of the following remark.

**Remark 5.1.4.** *Following the footsteps of [19] and the main stream of BSDE literature, we define the sequence of linear RBSDEs under $\mathbb{G}$, whose solutions approximate the solution to the general RBSDE (5.0.1). Thus, we consider the sequence $\left(Y^{\mathbb{G},n}, Z^{\mathbb{G},n}, M^{\mathbb{G},n}, K^{\mathbb{G},n}\right)$ defined recursively as follows.*

$$
\begin{array}{l}
(Y^{\mathbb{G},0}, Z^{\mathbb{G},0}, M^{\mathbb{G},0}, K^{\mathbb{G},0}) := (0,0,0,0),\\
\textit{for any } n \geq 1, \quad \left(Y^{\mathbb{G},n}, Z^{\mathbb{G},n}, M^{\mathbb{G},n}, K^{\mathbb{G},n}\right) \ \textit{is the unique solution to}:\\
\begin{cases}
Y_t = \xi + \displaystyle\int_{t\wedge\tau}^{T\wedge\tau} f(s, Y_s^{\mathbb{G},n-1}, Z_s^{\mathbb{G},n-1})ds + \int_{t\wedge\tau}^{T\wedge\tau} dM_s + \int_{t\wedge\tau}^{T\wedge\tau} dK_s\\
\qquad\qquad\qquad - \int_{t\wedge\tau}^{T\wedge\tau} Z_s dW_s,\\
Y \geq S \quad \textit{on } ]\!]0,\tau[\![, \qquad \displaystyle\int_0^{T\wedge\tau} (Y_{t-} - S_{t-})dK_t = 0.
\end{cases}
\end{array}
$$

*Thus, from this recursive sequence of solutions, and thanks to the linear part fully analyzed in Sections 4.1 and 4.2, we obtain a sequence of RBSDEs under $\mathbb{F}$ and their solutions. This can be achieved by determining $\left(Y^{\mathbb{F},n}, Z^{\mathbb{F},n}, K^{\mathbb{F},n}\right)$ associated to $\left(Y^{\mathbb{G},n}, Z^{\mathbb{G},n}, M^{\mathbb{G},n}, K^{\mathbb{G},n}\right)$ for each $n \geq 0$ as follows.*

1. *As $(Y^{\mathbb{G},0}, Z^{\mathbb{G},0}, M^{\mathbb{G},0}, K^{\mathbb{G},0}) := (0,0,0,0)$, then we get*

$$(Y^{\mathbb{F},0}, Z^{\mathbb{F},0}, K^{\mathbb{F},0}) := (0,0,0).$$

*2. For $n=1$, $\left(Y^{\mathbb{G},1}, Z^{\mathbb{G},1}, M^{\mathbb{G},1}, K^{\mathbb{G},1}\right)$ is the solution to*

$$Y_t = \xi + \int_{t\wedge\tau}^{T\wedge\tau} f(s,0,0)ds + \int_{t\wedge\tau}^{T\wedge\tau} dM_s + \int_{t\wedge\tau}^{T\wedge\tau} dK_s - \int_{t\wedge\tau}^{T\wedge\tau} Z_s dW_s.$$

*Here, the generator/driver is constant in $(Y, Z, M, K)$, and hence in virtue of Theorem 4.1.8 there exists a unique $(Y^{\mathbb{F},1}, Z^{\mathbb{F},1}, K^{\mathbb{F},1})$ solution to the RBSDE (4.1.23) with generator/driver $f^{\mathbb{F},1}(s) := \widetilde{\mathcal{E}}_s f(s,0,0)$ and*

$$\begin{cases} Y^{\mathbb{G},1} = Y^{\mathbb{F},1}\widetilde{\mathcal{E}}^{-1} I_{[\![0,\tau[\![} + \xi 1_{[\![\tau,+\infty[\![}, \ Z^{\mathbb{G},1} = Z^{\mathbb{F},1}\widetilde{\mathcal{E}}_-^{-1}, \\ K^{\mathbb{G},1} = \widetilde{\mathcal{E}}_-^{-1}\cdot K^{\mathbb{F},1}, \quad M^{\mathbb{G},1} = \left(h - Y^{\mathbb{F},1}\widetilde{\mathcal{E}}^{-1}\right)\cdot N^{\mathbb{G}}. \end{cases} \tag{5.1.26}$$

*3. For $n=2$, $\left(Y^{\mathbb{G},2}, Z^{\mathbb{G},2}, M^{\mathbb{G},2}, K^{\mathbb{G},2}\right)$ is the solution to*

$$\begin{aligned} Y_t = \xi &+ \int_{t\wedge\tau}^{T\wedge\tau} f(s, Y_s^{\mathbb{G},1}, Z_s^{\mathbb{G},1})ds + \int_{t\wedge\tau}^{T\wedge\tau} dM_s + \int_{t\wedge\tau}^{T\wedge\tau} dK_s \\ &- \int_{t\wedge\tau}^{T\wedge\tau} Z_s dW_s. \end{aligned}$$

*Thus, by plugging (5.1.26) in this equation, we obtain*

$$\begin{aligned} Y_t = \xi &+ \int_{t\wedge\tau}^{T\wedge\tau} f(s, \frac{Y_s^{\mathbb{F},1}}{\widetilde{\mathcal{E}}_s}, \frac{Z_s^{\mathbb{F},1}}{\widetilde{\mathcal{E}}_{s-}})ds + \int_{t\wedge\tau}^{T\wedge\tau} dM_s + \int_{t\wedge\tau}^{T\wedge\tau} dK_s \\ &- \int_{t\wedge\tau}^{T\wedge\tau} Z_s dW_s. \end{aligned}$$

*The generator here does not depend on $(Y, Z, M, K)$. Hence, again, Theorem 4.1.8 yields the existence of a unique $(Y^{\mathbb{F},2}, Z^{\mathbb{F},2}, K^{\mathbb{F},2})$ solution to* *the RBSDE (4.1.23) under $\mathbb{F}$ with generator/driver*

$$f^{\mathbb{F},2}(s) := \widetilde{\mathcal{E}}_s f\left(s, Y_s^{\mathbb{F},1}/\widetilde{\mathcal{E}}_s, Z_s^{\mathbb{F},1}/\widetilde{\mathcal{E}}_{s-}\right),$$

*and*

$$Y_t^{\mathbb{G},2} = \frac{Y_t^{\mathbb{F},2}}{\widetilde{\mathcal{E}}_t} 1_{\{t\ <\tau\}} + \xi 1_{\{t\geq\tau\}},\ Z^{\mathbb{G},2} = \frac{Z^{\mathbb{F},2}}{\widetilde{\mathcal{E}}_-},\ K^{\mathbb{G},2} = \frac{1}{\widetilde{\mathcal{E}}_-}\cdot K^{\mathbb{F},2},$$
$$\text{and } M^{\mathbb{G},2} = \left(h - \frac{Y^{\mathbb{F},2}}{\widetilde{\mathcal{E}}}\right)\cdot N^{\mathbb{G}}.$$

4. *By iterating this procedure, we get the sequence* $\left(Y^{\mathbb{F},n}, Z^{\mathbb{F},n}, K^{\mathbb{F},n}\right)$ *defined recursively as follows.*

$$\begin{aligned}
&(Y^{\mathbb{F},0}, Z^{\mathbb{F},0}, K^{\mathbb{F},0}) := (0,0,0,0),\\
&Y_t^{\mathbb{F},n} = \xi^{\mathbb{F}} + \int_t^T f^{\mathbb{F}}(s, Y_s^{\mathbb{F},n-1}, Z_s^{\mathbb{F},n-1})ds + \int_t^T h_s dV_s^{\mathbb{F}} + K_T^{\mathbb{F},n} - K_t^{\mathbb{F},n}\\
&\qquad\qquad - \int_t^T Z_s^{\mathbb{F},n} dW_s,\\
&Y_t^{\mathbb{F},n} \geq S_t^{\mathbb{F}} 1_{\{t\ <T\}} + \xi^{\mathbb{F}} 1_{\{t\ =T\}}, \quad \int_0^T (Y_{t-}^{\mathbb{F},n} - S_{t-}^{\mathbb{F}})dK_t^{\mathbb{F},n} = 0,
\end{aligned}$$

*where* $f^{\mathbb{F}}(s,y,z) := \widetilde{\mathcal{E}}_s f\left(s, y(\widetilde{\mathcal{E}}_s)^{-1}, z(\widetilde{\mathcal{E}}_s)^{-1}\right)$. *Thus, thanks to the convergence -in norm and almost surely for a subsequence- of* $\left(Y^{\mathbb{G},n}, Z^{\mathbb{G},n}, M^{\mathbb{G},n}, K^{\mathbb{G},n}\right)$ *and (4.1.25), we deduce that* $\left(Y^{\mathbb{F},n}, Z^{\mathbb{F},n}, K^{\mathbb{F},n}\right)$ *should also converge to* $\left(Y^{\mathbb{F}}, Z^{\mathbb{F}}, K^{\mathbb{F}}\right)$, *and this obtained triplet satisfies*

$$\begin{cases} Y_t = \xi^{\mathbb{F}} + \displaystyle\int_t^T f^{\mathbb{F}}(s, Y_s, Z_s)ds + \int_t^T h_s dV_s^{\mathbb{F}} + K_T - K_t - \int_t^T Z_s dW_s, \\ Y_t \geq S_t^{\mathbb{F}} 1_{\{t\ <T\}} + \xi^{\mathbb{F}} 1_{\{t\ =T\}}, \quad \displaystyle\int_0^T (Y_{t-} - S_{t-}^{\mathbb{F}})dK_t = 0. \end{cases}$$

*This gives us the RBSDE under $\mathbb{F}$ that we are looking for, and this also shows the importance of analyzing the linear case separately besides its own importance.*

Below, we elaborate our main result which connects RBSDE in $\mathbb{G}$ with those in $\mathbb{F}$.

**Theorem 5.1.5.** *Suppose $T < +\infty$, $G > 0$ and both (5.0.2) and (5.1.24) hold. Then the following assertions hold.*
(a) *The following RBSDE under $\mathbb{F}$, associated to the triplet $(S^{\mathbb{F}}, \xi^{\mathbb{F}}, f^{\mathbb{F}})$,*

$$\begin{cases} Y_t = \xi^{\mathbb{F}} + \displaystyle\int_t^T f^{\mathbb{F}}(s, Y_s, Z_s)ds + \int_t^T h_s dV_s^{\mathbb{F}} + K_T - K_t - \int_t^T Z_s dW_s, \\ \\ Y_t \geq S_t^{\mathbb{F}}, \quad t \in [0, T), \quad \displaystyle\int_0^T (Y_{t-} - S_{t-}^{\mathbb{F}})dK_t = 0, \end{cases} \tag{5.1.27}$$

*has a unique $L^p(P, \mathbb{F})$-solution that we denote by $(Y^{\mathbb{F}}, Z^{\mathbb{F}}, K^{\mathbb{F}})$, where*

$$f^{\mathbb{F}}(s, y, z) := \widetilde{\mathcal{E}}_s f\left(s, y\widetilde{\mathcal{E}}_s^{-1}, z\widetilde{\mathcal{E}}_s^{-1}\right), \quad S^{\mathbb{F}} := \widetilde{\mathcal{E}} S, \quad \xi^{\mathbb{F}} := \widetilde{\mathcal{E}}_T h_T. \tag{5.1.28}$$

(b) *There exists a unique $L^p(\widetilde{Q}, \mathbb{G})$-solution to (5.0.1), denoted by the quadru-*

*plet ($Y^{\mathbb{G}}, Z^{\mathbb{G}}, M^{\mathbb{G}}, K^{\mathbb{G}}$), and is given by*

$$\begin{cases} Y^{\mathbb{G}} = Y^{\mathbb{F}}\widetilde{\mathcal{E}}^{-1}I_{[\![0,\tau[\![} + \xi I_{[\![\tau,+\infty[\![}, \quad Z^{\mathbb{G}} = Z^{\mathbb{F}}\widetilde{\mathcal{E}}^{-1}I_{]\!]0,\tau]\!]}, \\ K^{\mathbb{G}} = \widetilde{\mathcal{E}}_-^{-1} \cdot (K^{\mathbb{F}})^{\tau} \quad \textit{and} \quad M^{\mathbb{G}} = \left(h - Y^{\mathbb{F}}\widetilde{\mathcal{E}}^{-1}\right) \cdot N^{\mathbb{G}}. \end{cases} \tag{5.1.29}$$

*Proof.* This proof is divided into two steps, where we prove assertions (a) and (b) respectively.

**Step 1.** On the one hand, put

$$\widetilde{f}^{\mathbb{F}}(t,y,z) := f^{\mathbb{F}}(t, y - (h \cdot V^{\mathbb{F}})_t, z), \quad \widetilde{S}^{\mathbb{F}} := S^{\mathbb{F}} + h \cdot V^{\mathbb{F}},$$
$$\text{and} \quad \widetilde{\xi}^{\mathbb{F}} := \xi^{\mathbb{F}} + (h \cdot V^{\mathbb{F}})_T,$$

and remark that $(\overline{Y}, \overline{Z}, \overline{K})$ is a solution to (5.1.27) iff

$$(Y', Z', K') := (\overline{Y} + h \cdot V^{\mathbb{F}}, \overline{Z}, \overline{K})$$

is a solution to the following RBSDE

$$\begin{cases} Y_t = \widetilde{\xi}^{\mathbb{F}} + \displaystyle\int_t^T \widetilde{f}^{\mathbb{F}}(s, Y_s, Z_s)ds + K_T - K_t - \int_t^T Z_s dW_s, \\ Y_t \geq \widetilde{S}_t^{\mathbb{F}}, \quad t \in [0,T), \quad \displaystyle\int_0^T (Y_{t-} - \widetilde{S}_{t-}^{\mathbb{F}})dK_t = 0. \end{cases} \tag{5.1.30}$$

On the other hand, thanks to (5.1.24), we derive

$$\begin{aligned} \|\widetilde{\xi}^{\mathbb{F}}\|_{L^p(P)} &\leq \|\xi^{\mathbb{F}}\|_{L^p(P)} + \|(|h| \cdot V^{\mathbb{F}})_T\|_{L^p(P)} \\ &\leq \|\xi^{\mathbb{F}}\|_{L^p(P)} + E\left[(|h|^p \cdot V^{\mathbb{F}})_T\right]^{1/p} < +\infty, \end{aligned}$$

$$\|\widetilde{f}^{\mathbb{F}}(\cdot,0,0)\|_{\mathbb{S}_T(P,p)} \leq \|f^{\mathbb{F}}(\cdot,0,0)\|_{\mathbb{S}_T(P,p)} + C_{Lip}\|(|h|\cdot V^{\mathbb{F}})_T\|_{L^p(P)} < +\infty,$$

and

$$\|(\widetilde{S}^{\mathbb{F}})^+\|_{\mathbb{D}_T(P,p)} \leq \|(S^{\mathbb{F}})^+\|_{\mathbb{D}_T(P,p)} + \|(|h|\cdot V^{\mathbb{F}})_T\|_{L^p(P)} < +\infty.$$

Therefore, by combining these inequalities and [19, Theorem 3.1], we conclude that (5.1.30) has a unique $L^p(P,\mathbb{F})$-solution. This ends the first part.

**Step 2.** Here we prove assertion (b). To this end, on the one hand, we remark that due to Theorem 5.1.3 the RBSDE (5.0.1) has at most one $L^p(\widetilde{Q},\mathbb{G})$-solution. On the other hand, when it exists, Theorem 5.1.2 claims that a class-(D)-$(\mathbb{G},\widetilde{Q},T\wedge\tau)$-solution to (5.0.1) is in fact an $L^p(\widetilde{Q},\mathbb{G})$-solution when the triplet $\left(e^{\alpha(T\wedge\tau)/2}\xi, e^{(\alpha-\alpha')\cdot}S^+, e^{\alpha\cdot/2}f_0\right) \in L^p(\widetilde{Q})\otimes\mathbb{D}_{\tau\wedge T}(\widetilde{Q},p)\otimes\mathbb{S}_{\tau\wedge T}(\widetilde{Q},p)$. Furthermore, thanks to Lemma 4.2.4 and its proof, it is not difficult to prove that the latter fact is equivalent to the condition (5.1.24). Thus, the proof of assertion (b) will follows immediately as soon as we prove that the quadruplet $(\overline{Y},\overline{Z},\overline{K},\overline{M})$, give by

$$\overline{Y} := \frac{Y^{\mathbb{F}}}{\widetilde{\mathcal{E}}}I_{[\![0,\tau[\![} + \xi I_{[\![\tau,+\infty[\![},\ \overline{Z} := \frac{Z^{\mathbb{F}}}{\widetilde{\mathcal{E}}}I_{]\!]0,\tau]\!]},\ \overline{K} := \frac{1}{\widetilde{\mathcal{E}}_-}\cdot(K^{\mathbb{F}})^{\tau}$$
$$\text{and } \overline{M} := \left(h - \frac{Y^{\mathbb{F}}}{\widetilde{\mathcal{E}}}\right)\cdot N^{\mathbb{G}},$$

is in fact a class-(D)-$(\mathbb{G},\widetilde{Q},T\wedge\tau)$-solution to (5.0.1). The proof for the fact that $(\overline{Y},\overline{Z},\overline{K},\overline{M})$ is a solution to (5.0.1) mimics exactly Step 2 in the proof of Theorem 4.2.7, and we will omit here. The rest of this proof proves that this solution is a class-(D)-$(\mathbb{G},\widetilde{Q},T\wedge\tau)$-solution as the following. On the one hand, note that due to Lipschitz condition we have

$$
\begin{aligned}
& E^{\widetilde{Q}}\left[\int_0^{T\wedge\tau} |f(s,\overline{Y}_s,\overline{Z}_s)|ds\right] \\
& \leq E^{\widetilde{Q}}\left[\int_0^{T\wedge\tau} |f_0(s)|ds\right] + C_{Lip}E^{\widetilde{Q}}\left[\int_0^{T\wedge\tau} |\overline{Y}_s|ds\right] + C_{Lip}E^{\widetilde{Q}}\left[\int_0^{T\wedge\tau} |\overline{Z}_s|ds\right] \\
& = E^{\widetilde{Q}}\left[\int_0^{T\wedge\tau} |f_0(s)|ds\right] + C_{Lip}E\left[\int_0^{T} |Y^{\mathbb{F}}_s|ds\right] + C_{Lip}E\left[\int_0^{T} |Z^{\mathbb{F}}_s|ds\right] < \infty.
\end{aligned}
$$

On the other hand, thanks to Lemma 2.3.3, we derive

$$
\begin{aligned}
L_t(\sigma) := {} & E\left[\int_t^{\sigma} f^{\mathbb{F}}(s,Y^{\mathbb{F}},Z^{\mathbb{F}})ds + S^{\mathbb{F}}_{\sigma}1_{\{\sigma\ <T\}} + \xi^{\mathbb{F}}1_{\{\sigma\ =T\}}\,\middle|\ \mathcal{F}_t\right]\frac{1_{\{t\ <\tau\}}}{\mathcal{E}_t(-\widetilde{G}^{-1}\bullet D^{o,\mathbb{F}})} \\
& + E\left[\frac{1}{G_0}\int_t^{\sigma}\widetilde{Z}_s h_s dD^{o,\mathbb{F}}_s\ \middle|\ \mathcal{F}_t\right]\frac{1_{\{t\ <\tau\}}}{\mathcal{E}_t(-\widetilde{G}^{-1}\bullet D^{o,\mathbb{F}})} \\
= {} & E\left[\int_t^{\sigma}\widetilde{Z}_s f(s,\frac{Y^{\mathbb{F}}}{\widetilde{\mathcal{E}}},\frac{Z^{\mathbb{F}}}{\widetilde{\mathcal{E}}})G_s ds + \widetilde{Z}_{\sigma}S_{\sigma}G_{\sigma}1_{\{\sigma\ <T\}}\middle|\mathcal{F}_t\right]\frac{1_{\{t\ <\tau\}}}{\widetilde{Z}_tG_t} \\
& + E\left[\widetilde{Z}_T h_T G_T 1_{\{\sigma=T\}} + \int_t^{\sigma}\widetilde{Z}_s h_s dD^{o,\mathbb{F}}_s\middle|\mathcal{F}_t\right]\frac{1_{\{t\ <\tau\}}}{\widetilde{Z}_tG_t} \\
= {} & E\left[\int_t^{\sigma}\widetilde{Z}_s f(s,\frac{Y^{\mathbb{F}}}{\widetilde{\mathcal{E}}},\frac{Z^{\mathbb{F}}}{\widetilde{\mathcal{E}}})1_{\{s<\tau\}}ds + \widetilde{Z}_{\sigma}S_{\sigma}1_{\{\sigma\ <T\wedge\tau\}}\middle|\mathcal{F}_t\right]\frac{1_{\{t\ <\tau\}}}{\widetilde{Z}_tG_t} \\
& + E\left[\widetilde{Z}_T h_T 1_{\{\tau>T=\sigma\}} + \widetilde{Z}_{\tau}h_{\tau}1_{\{\sigma\geq\tau>t\}}\middle|\mathcal{F}_t\right]\frac{1_{\{t\ <\tau\}}}{\widetilde{Z}_tG_t} \\
= {} & E\left[\int_{t\wedge\tau}^{\sigma\wedge\tau}\frac{\widetilde{Z}_s}{\widetilde{Z}_t}f(s,\frac{Y^{\mathbb{F}}}{\widetilde{\mathcal{E}}},\frac{Z^{\mathbb{F}}}{\widetilde{\mathcal{E}}})ds + \frac{\widetilde{Z}_{\sigma\wedge\tau}}{\widetilde{Z}_t}S_{\sigma\wedge\tau}1_{\{\sigma\wedge\tau\ <\tau\wedge T\}}\middle|\mathcal{G}_t\right]1_{\{t\ <\tau\}} \\
& + E\left[\frac{\widetilde{Z}_{\sigma\wedge\tau}}{\widetilde{Z}_t}\xi 1_{\{\sigma\wedge\tau=\tau\wedge T\}}\middle|\mathcal{G}_t\right]1_{\{t\ <\tau\}} \\
= {} & E^{\widetilde{Q}}\left[\int_{t\wedge\tau}^{\theta\wedge\tau} f(s,\overline{Y},\overline{Z})ds + S_{\theta}1_{\{\theta\ <T\wedge\tau\}} + \xi 1_{\{\theta=T\wedge\tau\}}\ \middle|\ \mathcal{G}_t\right]1_{\{t\ <\tau\}}.
\end{aligned}
$$

Thus, by combining the latter equality with the Snell envelope representation of $Y^{\mathbb{F}}$, we get

$$\overline{Y}_t I_{[\![0,\tau[\![} = \frac{Y_t^{\mathbb{F}}}{\widetilde{\mathcal{E}}} I_{[\![0,\tau[\![} = \operatorname{ess}\sup_{\sigma\in\mathcal{J}_t^T(\mathbb{F})} L_t(\sigma)$$
$$= \operatorname{ess}\sup_{\sigma\in\mathcal{J}_t^T(\mathbb{F})} E^{\widetilde{Q}}\left[\int_{t\wedge\tau}^{\theta\wedge\tau} f(s,\overline{Y}_s,\overline{Z}_s)ds + S_\theta 1_{\{\theta\ <T\wedge\tau\}} + \xi 1_{\{\theta=T\wedge\tau\}} \;\Big|\; \mathcal{G}_t\right] 1_{\{t\ <\tau\}}. \tag{5.1.31}$$

Then, by using the analysis as in part 2 of the proof of Theorem 4.1.4, see the inequality (4.1.9), we deduce that (5.1.31) yields

$$|\overline{Y}_t| \leq E^{\widetilde{Q}}\left[|\xi| \;\Big|\; \mathcal{G}_t\right] + |\overline{Y}_t| I_{[\![0,\tau[\![}$$
$$\leq E^{\widetilde{Q}}\left[\int_0^{T\wedge\tau} |f(\overline{Y}_s,\overline{Z}_s)|ds + \sup_{\theta\leq T\wedge\tau} S_\theta^+ + 2|\xi| \;\Big|\; \mathcal{G}_t\right].$$

This proves the solution is of class (D) as defined in Definition 2.4.1-(2). This ends the proof of theorem. □

## 5.2 The case of unbounded horizon

This section considers the case of unbounded horizon, or equivalently $T=+\infty$. Similarly as in the first section, we state our prior estimates, we prove existence and uniqueness of the solution, and we establish the connection between the $\mathbb{G}$-RBSDE and the $\mathbb{F}$-RBSDE counterpart. For the reader convenience, we precisely re-define our RBSDE below as

$$\begin{cases} dY_t = -f(t,Y_t,Z_t)d(t\wedge\tau) - d(K_{t\wedge\tau}+M_{t\wedge\tau}) + Z_t dW_t^\tau, \\ Y_\tau=\xi, \quad Y_t\geq S_t, \quad 0\leq t<\tau, \quad E\left[\int_0^\tau (Y_{t-}-S_{t-})dK_t\right]=0. \end{cases} \tag{5.2.1}$$

Here $(\xi, S, f)$ is such that $S$ is an $\mathbb{F}$-adapted and RCLL process, $f(t,y,z)$ is a $\text{Prog}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$-measurable functional satisfying (5.0.2) and $\xi$ is an $\mathcal{F}_\tau$-measurable random variable, or equivalently there exists an $\mathbb{F}$-optional process $h$ such that $\xi = h_\tau$. This section has three subsections. The first subsection derives estimates and stability inequalities that controls the solutions under $P$ instead of $\widetilde{Q}$. The second subsection introduces the RBSDE under $\mathbb{F}$ and discusses the existence and uniqueness of its solution, while the third subsection solves (5.2.1) and discusses its properties. Recall that the spaces $\widetilde{\mathbb{D}}_\sigma(P,p)$ and $\widetilde{\mathbb{S}}_\sigma(P,p)$ and their norms respective are defined in (4.2.17).

### 5.2.1 Estimate under $P$ for the solution of (5.0.1)

This subsection extends Theorem 4.2.2 and 4.2.3 to the case of general generator $f$. These theorems, that give estimates for the solutions under $P$ instead, are based essentially on Theorems 5.1.2 and 5.1.3 respectively, and represent an important step towards solving (5.2.1).

**Theorem 5.2.1.** *Suppose $T < +\infty$ and let $p > 1$, $\alpha > \alpha_0(p)$ given in (5.1.7) and $\alpha' \in (0, \alpha/2)$. There exists $C > 0$ depending on $(\alpha, \alpha', p)$ only such that, for the class-(D)-$(\mathbb{G}, \widetilde{Q}, T \wedge \tau)$-solution to the RBSDE (5.0.1) associated to $(f, S, \xi)$ denoted by $(Y^{\mathbb{G}}, Z^{\mathbb{G}}, M^{\mathbb{G}}, K^{\mathbb{G}})$, we have*

$$\begin{aligned}
&\|e^{\alpha\cdot/2}Y^{\mathbb{G}}\|_{\widetilde{\mathbb{D}}_{T\wedge\tau}(P,p)} + \|e^{\alpha(\tau\wedge\cdot)/2}Y^{\mathbb{G}}\|_{\widetilde{\mathbb{S}}_{T\wedge\tau}(P,p)} + \|e^{\alpha(\tau\wedge\cdot)/2}Z^{\mathbb{G}}\|_{\widetilde{\mathbb{S}}_{T\wedge\tau}(P,p)}\\
&+\|e^{\alpha\cdot/2}I_{]\!]0,T\wedge\tau]\!]}\sqrt[p]{\widetilde{\mathcal{E}}_-}\cdot M^{\mathbb{G}}\|_{\mathcal{M}^p(P,\mathbb{G})} + \|\int_0^{T\wedge\tau} e^{\alpha' s}\sqrt[p]{\widetilde{\mathcal{E}}_{s-}}dK^{\mathbb{G}}_s\|_{L^p(P)}\\
&\leq C\left\{\|e^{\alpha(T\wedge\tau)/2}\xi\|_{\mathbb{L}^p(\widetilde{Q})} + \|e^{\alpha\cdot/2}f_0\|_{\mathbb{S}_{T\wedge\tau}(\widetilde{Q},p)} + \|\sup_{0\leq t\leq\cdot} e^{(\alpha-\alpha')t}S^+_t\|_{\mathbb{S}_{T\wedge\tau}(\widetilde{Q},p)}\right\},
\end{aligned}$$

*where $\widetilde{\mathcal{E}}$ is defined in (4.1.22), and we recall it herein with $f_0$.*

$$\widetilde{\mathcal{E}}_t := \mathcal{E}_t(-\widetilde{G}^{-1}\cdot D^{o,\mathbb{F}}), \quad \textit{and} \quad f_0(t) := f(t,0,0). \tag{5.2.2}$$

*Proof.* The proof relies essentially on Lemma 4.2.1 and Theorem 5.1.2.
In fact, a direct application of Lemma 4.2.1-(a) to $Y := e^{\alpha(\cdot\wedge T\wedge\tau)/2}Y^{\mathbb{G}}$ yields

$$E\left[\sup_{0\leq s\leq T\wedge\tau} e^{p\alpha s/2}\widetilde{\mathcal{E}}_s|Y_s^{\mathbb{G}}|^p\right] \leq G_0^{-1}E^{\widetilde{Q}}\left[\sup_{0\leq s\leq T\wedge\tau} e^{p\alpha s/2}|Y_s^{\mathbb{G}}|^p\right]. \tag{5.2.3}$$

By applying Lemma 4.2.1-(b) to both cases when $K=\int_0^{\cdot} e^{\alpha s}|Z_s^{\mathbb{G}}|^2 ds$ and when $K=\int_0^{\cdot} e^{\alpha s}|Y_s^{\mathbb{G}}|^2 ds$ afterwards with $a=2/p$, we get

$$\begin{cases} E\left[\left(\int_0^{T\wedge\tau} e^{\alpha s}(\widetilde{\mathcal{E}}_s)^{2/p}|Z_s^{\mathbb{G}}|^2 ds\right)^{p/2}\right] \leq \frac{\kappa}{G_0}E^{\widetilde{Q}}\left[\left(\int_0^{T\wedge\tau} e^{\alpha s}|Z_s^{\mathbb{G}}|^2 ds\right)^{p/2}\right], \\ \\ E\left[\left(\int_0^{T\wedge\tau} e^{\alpha s}(\widetilde{\mathcal{E}}_s)^{2/p}|Y_s^{\mathbb{G}}|^2 ds\right)^{p/2}\right] \leq \frac{\kappa}{G_0}E^{\widetilde{Q}}\left[\left(\int_0^{T\wedge\tau} e^{\alpha s}|Y_s^{\mathbb{G}}|^2 ds\right)^{p/2}\right]. \end{cases} \tag{5.2.4}$$

Similarly, we apply Lemma 4.2.1-(b) to $K=e^{\alpha\cdot/2}\cdot K^{\mathbb{G}}$ with $a=1/p$, we get

$$\begin{aligned} &E\left[\left(\int_0^{T\wedge\tau} e^{\alpha' s}(\widetilde{\mathcal{E}}_{s-})^{1/p}dK_s^{\mathbb{G}}\right)^p\right] \\ &\leq \frac{\kappa}{G_0}E^{\widetilde{Q}}\left[\left(\int_0^{T\wedge\tau} e^{\alpha' s}dK_s^{\mathbb{G}}\right)^p + \sum_{0<s\leq T\wedge\tau}\widetilde{G}_s(e^{\alpha' s}\Delta K_s^{\mathbb{G}})^p\right] \\ &\leq \frac{2\kappa}{G_0}E^{\widetilde{Q}}\left[\left(\int_0^{T\wedge\tau} e^{\alpha' s}dK_s^{\mathbb{G}}\right)^p\right]. \end{aligned} \tag{5.2.5}$$

The last inequality follows from the easy facts that $\widetilde{G}\leq 1$ and $\sum_{0<s\leq T}(\Delta V_s)^p \leq V_T^p$ for any nondecreasing process $V$ with $V_0=0$ and any $p\geq 1$.

The rest of the proof will address the term that involves the $\mathbb{G}$-martingale $M^{\mathbb{G}}$. Thus, thanks to Theorem 5.1.5, we know that $[M^{\mathbb{G}}, M^{\mathbb{G}}] = H' \cdot [N^{\mathbb{G}}, N^{\mathbb{G}}]$ where $H' := (h - Y^{\mathbb{F}}/\widetilde{\mathcal{E}})^2$, which is $\mathbb{F}$-optional. Thus, an application of Lemma 4.2.1-(d) to $H := e^{\alpha\cdot}(h - Y^{\mathbb{F}}/\widetilde{\mathcal{E}})^2$ that is $\mathbb{F}$-optional, we get

$$\begin{aligned}
&E\left[\left((\widetilde{\mathcal{E}})^{2/p} H \cdot [N^{\mathbb{G}}, N^{\mathbb{G}}]_{T\wedge\tau}\right)^{p/2}\right] \\
&\leq \frac{\kappa}{G_0} E^{\widetilde{Q}}\left[\left(H \cdot [N^{\mathbb{G}}, N^{\mathbb{G}}]_T\right)^{p/2} + 2(H^{p/2} I_{]\!]0,\tau[\![} \cdot D^{o,\mathbb{F}})_T\right] \\
&= \frac{\kappa}{G_0} E^{\widetilde{Q}}\left[\left(e^{\alpha\cdot} \cdot [M^{\mathbb{G}}, M^{\mathbb{G}}]_T\right)^{p/2} + 2(H^{p/2} I_{]\!]0,\tau[\![} \cdot D^{o,\mathbb{F}})_T\right]. \qquad (5.2.6)
\end{aligned}$$

Thus, we need to control the second term in the right-hand-side of this inequality. To this end, we remark that $(H^{p/2} I_{]\!]0,\tau[\![} \cdot D^{o,\mathbb{F}}) \leq 2^{p-1}(|he^{\alpha\cdot/2}|^p + |Y^{\mathbb{G}} e^{\alpha\cdot/2}|^p I_{]\!]0,\tau[\![}) \cdot D^{o,\mathbb{F}}$. Thus, by using this, we derive

$$\begin{aligned}
2E^{\widetilde{Q}}\left[(H^{p/2} I_{]\!]0,\tau[\![} \cdot D^{o,\mathbb{F}})_T\right] &\leq 2^p E^{\widetilde{Q}}\left[e^{p\alpha\tau/2}|h_\tau|^p I_{\{\tau\leq T\}}\right] \\
&\quad + 2^p E^{\widetilde{Q}}\left[\sup_{0\leq t\leq\tau\wedge T} e^{p\alpha s/2}|Y_t^{\mathbb{G}}|^p\right]. \qquad (5.2.7)
\end{aligned}$$

Therefore, by combining this inequality with $h_\tau I_{\{\tau\leq T\}} = \xi I_{\{\tau\leq T\}}$, (5.2.6), (5.2.5), (5.2.4), (5.2.3) and Theorem 5.1.2 with $\sigma = 0$, the proof of the theorem follows immediately. $\square$

**Theorem 5.2.2.** *Suppose that $T < +\infty$ and let $p > 1$, $\alpha > \alpha_0(p)$ defined in (5.1.7) and $\alpha' \in (0, \alpha/2)$. Then there exist positive $C_i$, $i = 1, 2, 3$, that depend on $(\alpha, \alpha', p)$ only such that $\lim_{\alpha\to\infty} C_1 = 0$ and for $(Y^{\mathbb{G},i}, Z^{\mathbb{G},i}, K^{\mathbb{G},i}, M^{\mathbb{G},i})$ being a class-(D)-$(\mathbb{G}, \widetilde{Q}, T\wedge\tau)$-solution to the RBSDE (5.0.1) that correspond* *to* $(f^{(i)}, S^{(i)}, \xi^{(i)})$, $i = 1, 2$ *respectively, we have*

$$\begin{aligned}
&\|e^{\frac{\alpha}{2}\cdot}\delta Y^{\mathbb{G}}\|_{\widetilde{\mathbb{D}}_{T\wedge\tau}(P,p)} + \|e^{\frac{\alpha}{2}\cdot}\delta Y^{\mathbb{G}}\|_{\widetilde{\mathbb{S}}_{T\wedge\tau}(P,p)} + \|e^{\frac{\alpha}{2}\cdot}\delta Z^{\mathbb{G}}\|_{\widetilde{\mathbb{S}}_{T\wedge\tau}(P,p)} \\
&\quad + \|e^{\frac{\alpha}{2}\cdot}\sqrt[p]{\widetilde{\mathcal{E}}_-}\cdot\delta M^{\mathbb{G}}\|_{\mathcal{M}^p(P,\mathbb{G})} \leq C_1\|e^{\frac{\alpha}{2}\cdot}\delta f\|_{\mathbb{S}_{T\wedge\tau}(\widetilde{Q},p)} + C_2\|e^{\frac{\alpha}{2}(T\wedge\tau)}\delta\xi\|_{\mathbb{L}^p(\widetilde{Q})} \\
&\qquad\qquad + C_3\sqrt{\|e^{\frac{\alpha}{2}\cdot}\delta S\|_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},p)}\sum_{i=1}^{2}\Delta(\xi^{(i)}, f^{(i)}, (S^{(i)})^+)}.
\end{aligned}$$

*Here* $\Delta(\xi^{(i)}, f^{(i)}, (S^{(i)})^+)$ *is*

$$\begin{aligned}
&\Delta(\xi^{(i)}, f^{(i)}, (S^{(i)})^+) \\
&:= \|e^{\alpha(T\wedge\tau)/2}\xi^{(i)}\|_{\mathbb{L}^p(\widetilde{Q})} + \|e^{\alpha\cdot/2}f_0^{(i)}\|_{\mathbb{S}_{T\wedge\tau}(\widetilde{Q},p)} + \|\sup_{0\leq t\leq\cdot} e^{(\alpha-\alpha')t}(S_t^{(i)})^+\|_{\mathbb{S}_{T\wedge\tau}(\widetilde{Q},p)}
\end{aligned} \tag{5.2.8}$$

*and* $(\delta Y^{\mathbb{G}}, \delta Z^{\mathbb{G}}, \delta M^{\mathbb{G}}, \delta K^{\mathbb{G}})$ *and* $(\delta f, \delta S, \delta\xi)$ *are given by*

$$\begin{cases}
\delta Y^{\mathbb{G}} := Y^{\mathbb{G},1} - Y^{\mathbb{G},2},\ \delta Z^{\mathbb{G}} := Z^{\mathbb{G},1} - Z^{\mathbb{G},2}, \delta M^{\mathbb{G}} := M^{\mathbb{G},1} - M^{\mathbb{G},2}, \\
\delta K^{\mathbb{G}} := K^{\mathbb{G},1} - K^{\mathbb{G},2}, \quad \delta S := S^{(1)} - S^{(2)}, \quad \delta\xi := \xi^{(1)} - \xi^{(2)}, \\
\delta f_t := f^{(1)}(t, Y_t^{\mathbb{G},1}, Z_t^{\mathbb{G},1}) - f^{(2)}(t, Y_t^{\mathbb{G},1}, Z_t^{\mathbb{G},1}).
\end{cases} \tag{5.2.9}$$

*Proof.* By applying Lemma 4.2.1-(a) to $Y_s = e^{\alpha s/2}\delta Y_s^{\mathbb{G}}$ and $a = p$, we obtain

$$E\left[\sup_{0\leq s\leq T\wedge\tau} e^{p\alpha s/2}\widetilde{\mathcal{E}}_s|\delta Y_s^{\mathbb{G}}|^p\right] \leq G_0^{-1}E^{\widetilde{Q}}\left[\sup_{0\leq s\leq T\wedge\tau} e^{p\alpha s/2}|\delta Y_s^{\mathbb{G}}|^p\right]. \tag{5.2.10}$$

By applying Lemma 4.2.1-(b) to both cases when $K = \int_0^\cdot e^{\alpha s}|\delta Z_s^{\mathbb{G}}|^2 ds$ and when $K = \int_0^\cdot e^{\alpha s}|\delta Y^{\mathbb{G}}_s|^2 ds$ afterwards with $a = 2/p$, we get

$$\begin{cases} E\left[\left(\int_0^{T\wedge\tau} e^{\alpha s}(\widetilde{\mathcal{E}}_s)^{2/p}|\delta Z^{\mathbb{G}}_s|^2 ds\right)^{p/2}\right] \leq \frac{\kappa}{G_0} E^{\widetilde{Q}}\left[\left(\int_0^{T\wedge\tau} e^{\alpha s}|\delta Z^{\mathbb{G}}_s|^2 ds\right)^{p/2}\right], \\ \\ E\left[\left(\int_0^{T\wedge\tau} e^{\alpha s}(\widetilde{\mathcal{E}}_s)^{2/p}|\delta Y^{\mathbb{G}}_s|^2 ds\right)^{p/2}\right] \leq \frac{\kappa}{G_0} E^{\widetilde{Q}}\left[\left(\int_0^{T\wedge\tau} e^{\alpha s}|\delta Y^{\mathbb{G}}_s|^2 ds\right)^{p/2}\right]. \end{cases} \tag{5.2.11}$$

Thanks to Theorem 5.1.5, we know that $[\delta M^{\mathbb{G}}, \delta M^{\mathbb{G}}] = H' \cdot [N^{\mathbb{G}}, N^{\mathbb{G}}]$ where $H' := (\delta h - \delta Y^{\mathbb{F}}/\widetilde{\mathcal{E}})^2$, which is $\mathbb{F}$-optional. Thus, an application of Lemma 4.2.1-(d) to $H_s := e^{\alpha s}H'_s$ that is $\mathbb{F}$-optional, and similar argument as in (5.2.7), we get

$$\begin{aligned} & E\left[\left((\widetilde{\mathcal{E}})^{2/p} H \cdot [N^{\mathbb{G}}, N^{\mathbb{G}}]_T\right)^{p/2}\right] \\ & \leq \frac{\kappa}{G_0} E^{\widetilde{Q}}\left[\left(H \cdot [N^{\mathbb{G}}, N^{\mathbb{G}}]_T\right)^{p/2} + 2(H^{p/2} I_{]\!]0,\tau[\![} \cdot D^{o,\mathbb{F}})_T\right] \\ & = \frac{\kappa}{G_0} E^{\widetilde{Q}}\left[\left(e^{\alpha\cdot} \cdot [\delta M^{\mathbb{G}}, \delta M^{\mathbb{G}}]_T\right)^{p/2} + 2(H^{p/2} I_{]\!]0,\tau[\![} \cdot D^{o,\mathbb{F}})_T\right] \\ & \leq \frac{\kappa}{G_0} E^{\widetilde{Q}}\left[\left(e^{\alpha\cdot} \cdot [\delta M^{\mathbb{G}}, \delta M^{\mathbb{G}}]_T\right)^{p/2}\right] \\ & + \frac{2^p\kappa}{G_0}\left\{E^{\widetilde{Q}}\left[|\delta h_\tau|^p e^{p\alpha\tau/2} I_{\{\tau\leq T\}}\right] + E^{\widetilde{Q}}\left[\sup_{0\leq t\leq\tau\wedge T} e^{p\alpha s/2}|\delta Y^{\mathbb{G}}_t|^p\right]\right\} \end{aligned} \tag{5.2.12}$$

Hence, by combining (5.2.10), (5.2.11), (5.2.12) and Theorem 5.1.3, the proof of the theorem follows. $\square$

### 5.2.2 Existence and uniqueness of the solution to (5.2.1)

This subsection elaborates our first main result of this section, which proves the existence and uniqueness of the solution to (5.2.1), and gives estimates for it.

**Theorem 5.2.3.** *Let $p \in (1, +\infty)$, $\alpha > \alpha_0(p)$ defined in (5.1.7) and $\alpha' \in (0, \alpha/2)$. Suppose $G > 0$ and*

$$\|F^{(\alpha)} + e^{\alpha \cdot /2}|h| + \sup_{0 \le u \le \cdot} e^{\alpha u}|S_u|\|_{L^p(P \otimes V^{\mathbb{F}})} < +\infty, \tag{5.2.13}$$

*where $F_t^{(\alpha)} := \sqrt{\int_0^t e^{\alpha s}|f_0(s)|^2 ds}$. Then the following assertions hold.*

(a) *There exists a unique solution $(Y^{\mathbb{G}}, Z^{\mathbb{G}}, M^{\mathbb{G}}, K^{\mathbb{G}})$ to the RBSDE (5.2.1).*

(b) *There exists positive $C$, which depends on $(\alpha, \alpha', p)$ only, such that*

$$\begin{aligned}
&\|e^{\alpha \cdot /2} Y^{\mathbb{G}}\|_{\widetilde{\mathbb{D}}_\tau(P,p)} + \|e^{\alpha \cdot /2} Z^{\mathbb{G}}\|_{\widetilde{\mathbb{S}}_\tau(P,p)} + \|e^{\alpha \cdot /2} Y^{\mathbb{G}}\|_{\widetilde{\mathbb{S}}_\tau(P,p)} \\
&+ \|e^{\alpha' \cdot}(\widetilde{\mathcal{E}}_-)^{1/p} \cdot K_\tau^{\mathbb{G}}\|_{L^p(P)} + \|e^{\alpha \cdot /2}(\widetilde{\mathcal{E}}_-)^{1/p} \cdot (M^{\mathbb{G}})^\tau\|_{\mathcal{M}^p(P)} \\
&\le C\|e^{\alpha \cdot /2}|h| + F^{(\alpha)} + \sup_{0 \le u \le \cdot} e^{\alpha s}(S_u^+)\|_{L^p(P \otimes V^{\mathbb{F}})}.
\end{aligned}$$

(c) *Let $(f, h^{(i)}, S^{(i)})$, $i = 1, 2$, be two triplets satisfying (5.2.13), and the quadruplet $(Y^{\mathbb{G},i}, Z^{\mathbb{G},i}, K^{\mathbb{G},i}, M^{\mathbb{G},i})$ be the solutions to their corresponding RBSDE (5.2.1). There exist $C_1$ and $C_2$ that depend on $\alpha$ and $p$ only such that*

$$\begin{aligned}
&\|e^{\alpha \cdot /2}\delta Y^{\mathbb{G}}\|_{\widetilde{\mathbb{D}}_\tau(P,p)} + \|e^{\alpha \cdot /2}\delta Z^{\mathbb{G}}\|_{\widetilde{\mathbb{S}}_\tau(P,p)} + \|(e^{\alpha \cdot /2}(\widetilde{\mathcal{E}}_-)^{1/p} \cdot \delta M^{\mathbb{G}})^\tau\|_{\mathcal{M}^p(P)} \\
&\le C_1\|e^{\alpha \cdot /2}|\delta h| + \sup_{0 \le u \le \cdot} e^{\alpha u/2}|\delta S_u|\|_{L^p(P \otimes V^{\mathbb{F}})} \\
&\qquad + C_2\sqrt{\|\sup_{0 \le u \le \cdot} e^{\frac{\alpha}{2}u}|\delta S_u|\|_{L^p(P \otimes V^{\mathbb{F}})}}\sqrt{\sum_{i=1}^2 \bar{\Delta}_i}.
\end{aligned}$$

*Here* $(\delta Y^{\mathbb{G}}, \delta Z^{\mathbb{G}}, \delta K^{\mathbb{G}}, \delta M^{\mathbb{G}})$ *and* $(\delta f, \delta S, \delta h)$ *are given by (5.2.9). And,*

$$\bar{\Delta}_i := \|e^{\frac{\alpha}{2}\cdot}|h_i| + F^{(\alpha)} + \sup_{0\leq u\leq \cdot} e^{\alpha u}(S_i(u))^+\|_{L^p(P\otimes V^{\mathbb{F}})}.$$

*Proof.* On the one hand, in virtue of assertion (c), it is clear that (5.2.1) has at most one solution. Thus, the rest of this proof focuses on proving the existence of the solution and assertion (b) and (c). To this end, we divide the rest of the proof into four parts.

**Part 1.** In this part, we consider an $\mathbb{F}$-stoppoing time $\sigma$, and we suppose that there exists a positive constant $C$ such that

$$\max\left(e^{p\alpha\cdot/2}|h|^p, (F^{(\alpha)})^p, \sup_{0\leq t\leq\cdot} e^{p\alpha\cdot}(S_t^+)^p\right) \leq C\mathcal{E}(G_-^{-1}\cdot m) \text{ on } [\![0,\sigma]\!]. \quad (5.2.14)$$

Our goal, in this part, lies in proving under this assumption that there exists a solution to (5.2.1) and assertion (b). To the triplet $(f, S, h)$ satisfying (5.2.14), we associate $(\overline{f}^{(n)}, \overline{S}^{(n)}, \overline{h}^{(n)})$ given by

$$\overline{f}^{(n)} := fI_{[\![0,n\wedge\sigma]\!]},\ \overline{S}_t^{(n)} := S_{n\wedge\sigma\wedge t},\ \overline{h}_t^{(n)} := h_{n\wedge\sigma\wedge t},\ \overline{\xi}^{(n)} := h_{n\wedge\sigma\wedge\tau}. \quad (5.2.15)$$

Then thanks to Theorem 5.1.5, we deduce that for each triplet $(\overline{f}^{(n)}, \overline{S}^{(n)}, \overline{\xi}^{(n)})$, the RBSDE (5.2.1) has a unique solution $(\overline{Y}^{(n)}, \overline{Z}^{(n)}, \overline{M}^{(n)}, \overline{K}^{(n)})$. Then by applying Theorem 5.2.1 to $(\overline{Y}^{(n)}, \overline{Z}^{(n)}, \overline{M}^{(n)}, \overline{K}^{(n)})$ and applying Theorem 5.2.2 to the difference of solutions

$$\begin{aligned}&(\delta Y, \delta Z, \delta M, \delta K)\\ &:= (\overline{Y}^{(n+m)}, \overline{Z}^{(n+m)}, \overline{M}^{(n+m)}, \overline{K}^{(n+m)}) - (\overline{Y}^{(n)}, \overline{Z}^{(n)}, \overline{M}^{(n)}, \overline{K}^{(n)}),\end{aligned}$$

and the horizon $T = (n+m) \wedge \sigma$, we get

$$\begin{aligned}
&\|e^{\alpha\cdot/2}\overline{Y}^{(n)}\|_{\widetilde{\mathbb{D}}_{T\wedge\tau}(P,p)} + \|e^{\alpha(\tau\wedge\cdot)/2}\overline{Y}^{(n)}\|_{\widetilde{\mathbb{S}}_{T\wedge\tau}(P,p)} + \|e^{\alpha(\tau\wedge\cdot)/2}\overline{Z}^{(n)}\|_{\widetilde{\mathbb{S}}_{T\wedge\tau}(P,p)} \\
&+ \|e^{\alpha\cdot/2}I_{]\!]0,T\wedge\tau]\!]}\sqrt[p]{\widetilde{\mathcal{E}}_-}\cdot\overline{M}^{(n)}\|_{\mathcal{M}^p(P,\mathbb{G})} + \|\int_0^{T\wedge\tau} e^{\frac{\alpha}{2}s}\sqrt[p]{\widetilde{\mathcal{E}}_{s-}}d\overline{K}_s^{(n)}\|_{L^p(P)} \qquad (5.2.16)\\
&\leq C\left\{\|e^{\alpha(T\wedge\tau)/2}\overline{\xi}^{(n)}\|_{\mathbb{L}^p(\widetilde{Q})} + \|e^{\alpha\cdot/2}\overline{f}_0^{(n)}\|_{\mathbb{S}_{T\wedge\tau}(\widetilde{Q},p)}\right\}\\
&+ C\|\sup_{0\leq t\leq\cdot} e^{\alpha t}(\overline{S}_t^{(n)})^+\|_{\mathbb{S}_{T\wedge\tau}(\widetilde{Q},p)},
\end{aligned}$$

and

$$\begin{aligned}
&\|e^{\alpha\cdot/2}\widetilde{\mathcal{E}}^{1/p}\delta Y\|_{\mathbb{D}_T(\widetilde{P},p)} + \|e^{\alpha\cdot/2}(\widetilde{\mathcal{E}}_-)^{1/p}|\delta Z\|_{\mathbb{S}_T^p(\widetilde{P},p)} + \|e^{\alpha\cdot/2}(\widetilde{\mathcal{E}}_-)^{1/p}\cdot\delta M\|_{\mathbb{M}_T^p(\widetilde{P})}\\
&\leq C_1\|e^{\alpha(\tau\wedge\cdot)/2}\delta f\|_{\mathbb{S}_T^p(\widetilde{Q})} + C_2\|e^{\alpha(T\wedge\tau)/2}\delta\xi\|_{\mathbb{L}^p(\widetilde{Q})} \qquad (5.2.17)\\
&\quad + C_3\|e^{\alpha(\tau\wedge\cdot)/2}\delta S\|_{\mathbb{D}_T(\widetilde{Q})}^{1/2}\sup_{k\geq n}\sqrt{\Delta(\xi^{(k)}, f^{(k)}, (S^{(k)})^+)}.
\end{aligned}$$

Here we put $\overline{f}_0^{(k)}(t) := \overline{f}^{(k)}(t,0,0)$ and $\Delta(\xi^{(k)}, f^{(k)}, (S^{(k)})^+)$ is given by

$$\begin{aligned}
&\Delta(\xi^{(k)}, f^{(k)}, (S^{(k)})^+)\\
&:= \|e^{\alpha(T\wedge\tau)/2}\overline{\xi}^{(k)}\|_{\mathbb{L}^p(\widetilde{Q})} + \|e^{\alpha(\tau\wedge\cdot)/2}\overline{f}_0^{(k)}\|_{\mathbb{S}_T(\widetilde{Q},p)} + \|e^{\alpha(\tau\wedge\cdot)}(\overline{S}^{(k)})^+\|_{\mathbb{D}_T(\widetilde{Q},p)},
\end{aligned} \qquad (5.2.18)$$

Next, we calculate the limits, when $n$ and/or $m$ go to infinity, of the right-hand-sides of the inequalities (5.2.16) and (5.2.17). It is clear that, in virtue of Lemma 4.2.4, we have

$$\begin{cases} \lim\limits_{n\to\infty} \|e^{\alpha(n\wedge\tau)/2}\overline{\xi}^{(n)}\|^p_{\mathbb{L}^p(\widetilde{Q})} = E\left[\int_0^\infty e^{\alpha sp/2}|h_s|^p dV_s^{\mathbb{F}}\right] \\ \lim\limits_{n\to\infty} \|e^{\alpha(\tau\wedge\cdot)}(\overline{S}^{(n)})^+\|^p_{\mathbb{D}_T(\widetilde{Q},p)} = E\left[\int_0^\infty \sup\limits_{0\le s\le t} e^{\alpha sp}(S_s^+)^p dV_s^{\mathbb{F}}\right] \\ \lim\limits_{n\to\infty} \|e^{\alpha(\tau\wedge\cdot)/2}\overline{f}^{(n)}(\cdot,0,0)\|^p_{\mathbb{S}_T(\widetilde{Q},p)} = E\left[\int_0^\infty (F_t^{(\alpha)})^p dV_t^{\mathbb{F}}\right]. \end{cases} \tag{5.2.19}$$

This determines the limits for the right-hand-side terms of (5.2.16). To addresses the limits of the right-hand-side terms of (5.2.17), we remark that due to the assumption 5.2.14 and in virtue of Lemma 4.2.4, we have

$$\begin{cases} \lim\limits_{n\to\infty}\sup\limits_{m\ge 0} \|e^{\frac{\alpha}{2}(\tau\wedge\cdot)}\delta S\|_{\mathbb{D}_T(\widetilde{Q},p)} \le 2\lim\limits_{n\to\infty} \|e^{\frac{\alpha}{2}(\tau\wedge\cdot)}S^+ I_{[\![n,\infty[\![}\|_{\mathbb{D}_T(\widetilde{Q},p)} = 0 \\ \lim\limits_{n\to\infty}\sup\limits_{m\ge 0} \|e^{\frac{\alpha}{2}(T\wedge\tau)}\delta\xi\|_{\mathbb{L}^p(\widetilde{Q})} = \lim\limits_{n\to\infty}\sup\limits_{m\ge 0} \|e^{\frac{\alpha}{2}(T\wedge\tau)}(h_{(n+m)\wedge\tau} - h_{n\wedge\tau})\|_{\mathbb{L}^p(\widetilde{Q})} = 0, \\ \lim\limits_{n\to\infty}\sup\limits_{m\ge 0} \|e^{\alpha(\tau\wedge\cdot)/2}\delta f(\cdot,0,0)\|_{\mathbb{D}_T(\widetilde{Q},p)} = 0, \\ \sup_{k\ge 0}\sqrt{\Delta(\xi^{(k)}, f^{(k)}, (S^{(k)})^+)} < +\infty, \end{cases} \tag{5.2.20}$$

now we deal with the first term in the right-hand-side of (5.2.17). To this end, on the one hand, we remark that

$$\begin{aligned} |\delta f_t| &= |(\overline{f}^{(n+m)} - \overline{f}^{(n)})(t, \overline{Y}_t^{(n+m)}, \overline{Z}_t^{(n+m)})| = |f(t, \overline{Y}_t^{(n+m)}, \overline{Z}_t^{(n+m)})| I_{\{n<t\le n+m\}} \\ &\le |f(t,0,0)| I_{\{n<t\le n+m\}} + C_{lip}(|\overline{Y}_t^{(n+m)}| + \overline{Z}_t^{(n+m)}|) I_{\{n<t\le n+m\}}. \end{aligned} \tag{5.2.21}$$

On the other hand, thanks to Theorem 5.1.2, applied to the quadruplet

$$(\overline{Y}^{(n+m)}, \overline{Z}^{(n+m)}, \overline{M}^{(n+m)}, \overline{K}^{(n+m)})$$

and $\sigma = n$, we deduce that

$$\begin{aligned}
&\|e^{\alpha(\tau\wedge\cdot)/2}\overline{Z}^{(n+m)}I_{]\!]n,n+m]\!]}\|_{\mathbb{S}_T(\widetilde{Q},p)} + \|e^{\alpha\cdot/2}\overline{Y}^{(n+m)}I_{]\!]n,n+m]\!]}\|_{\mathbb{S}_T(\widetilde{Q},p)}\\
&\leq \widehat{C}\left\{\|e^{\alpha(T\wedge\tau)/2}\overline{\xi}^{(n+m)}I_{\{\tau>n\}}\|_{L^p(\widetilde{Q})} + \|e^{\alpha(\tau\wedge\cdot)}S^+I_{]\!]n,n+m]\!]}\|_{\mathbb{D}_T(\widetilde{Q},p)}\right\}\\
&+\widehat{C}\|e^{\alpha(\tau\wedge\cdot)/2}f(\cdot,0,0)I_{]\!]n,n+m]\!]}\|_{\mathbb{S}_T(\widetilde{Q},p)}
\end{aligned}$$

Therefore, by combining this inequality with (5.2.20) and (5.2.21), we deduce that

$$\lim_{n\to\infty}\sup_{m\geq 0}\|e^{\alpha(\tau\wedge\cdot)/p}\delta f\|_{\mathbb{S}_{(n+m)\wedge\tau}(\widetilde{Q},p)} = 0. \tag{5.2.22}$$

Thus, by combining (5.2.17), (5.2.20) and (5.2.22), we conclude that the sequence $(\overline{Y}^{(n)}, \overline{Z}^{(n)}, \overline{M}^{(n)}, \overline{K}^{(n)})$ is a Cauchy sequence in norm, and hence it converges in norm and almost surely for a subsequence, and its limite is a solution to (5.2.1). This proves assertion (a) of the theorem provided that assertion (c) is true. Furthermore, by taking the limit in (5.2.16) and using Fatou and (5.2.19), the proof of assertion (b) follows immediately. This ends the first part.

**Part 2.** Here we prove assertion (c) under the assumption (5.2.14). Consider two triplets $(f, S^{(i)}, h^{(i)})$, $i = 1, 2$ satisfying (5.2.14). Then for each triplet we associate to it a sequence $(\overline{f}^{(n)}, \overline{S}^{(n,i)}, \overline{h}^{(n,i)})$ defined via (5.2.15). Thus, there exists $(\overline{Y}^{(n,i)}, \overline{Z}^{(n,i)}, \overline{M}^{(n,i)}, \overline{K}^{(n,i)})$, for each $i = 1, 2$, that converges in norm and almost surely for a subsequence to $(\overline{Y}^{\mathbb{G},i}, \overline{Z}^{\mathbb{G},i}, \overline{M}^{\mathbb{G},i}, \overline{K}^{\mathbb{G},i})$ which is solution to (5.2.1) associated to $(f, S^{(i)}, h^{(i)})$. Then by applying Theorem 5.2.2 to the difference of solutions

$$\begin{aligned}&(\delta Y, \delta Z, \delta M, \delta K)\\&:= (\overline{Y}^{(n,1)}, \overline{Z}^{(n,1)}, \overline{M}^{(n,1)}, \overline{K}^{(n,1)}) - (\overline{Y}^{(n,2)}, \overline{Z}^{(n,2)}, \overline{M}^{(n,2)}, \overline{K}^{(n,2)}),\end{aligned}$$

and the horizon $T = n$, we get

$$\begin{aligned}&\|e^{\alpha\cdot/2}\widetilde{\mathcal{E}}^{1/p}\delta Y\|_{\mathbb{D}_T(\widetilde{P},p)} + \|e^{\alpha\cdot/2}(\widetilde{\mathcal{E}}_-)^{1/p}|\delta Z\|_{\mathbb{S}_T^p(\widetilde{P},p)} + \|e^{\alpha\cdot/2}(\widetilde{\mathcal{E}}_-)^{1/p}\cdot\delta M\|_{\mathcal{M}^p(P,\mathbb{G})}\\&\leq C_2\|e^{\alpha(T\wedge\tau)/2}\delta\overline{\xi}^{(n)}\|_{\mathbb{L}^p(\widetilde{Q})} + C_3\|e^{\alpha(\tau\wedge\cdot)/2}\delta\overline{S}^{(n)}\|_{\mathbb{D}_T(\widetilde{Q})}^{1/2}\sqrt{\sum_{k=1}^{2}\Delta(\xi^{(k)}, f^{(k)}, S^{(k)})}.\end{aligned} \tag{5.2.23}$$

Here $\Delta(\xi^{(k)}, f^{(k)}, S^{(k)})$ is given by (5.2.18). Similarly, as in the proof of (5.2.19), we use Lemma 4.2.4 and the boundedness assumption (5.2.14) that each triplet $(f, S^{(i)}, h^{(i)})$ $(i = 1, 2)$ satisfies, we get

$$\begin{cases}\lim\limits_{n\to\infty}\|e^{\alpha(n\wedge\tau)/2}\delta\overline{\xi}^{(n)}\|_{\mathbb{L}^p(\widetilde{Q})}^p = E\left[\int_0^\infty e^{\alpha sp/2}|\delta h_s|^p dV_s^{\mathbb{F}}\right]\\ \lim\limits_{n\to\infty}\|e^{\alpha(\tau\wedge\cdot)/2}(\delta\overline{S}^{(n)})\|_{\mathbb{D}_T(\widetilde{Q},p)}^p = E\left[\int_0^\infty \sup\limits_{0\leq s\leq t} e^{\alpha sp/2}(\delta S_s)^p dV_s^{\mathbb{F}}\right]\end{cases} \tag{5.2.24}$$

Thus, by taking the limit in (5.2.23), using Fatou's lemma for its left-hand-side term, and using (5.2.24) and (5.2.19) for its right-hand-side term, assertion (c) follows immediately. This ends the second part.

**Part 3.** In this part, we drop the assumption (5.2.14) and prove existence of solution to (5.2.1) and assertion (b). Hence, we consider the following sequence of stopping times

$$T_n := \inf\left\{t \geq 0 \ : \ \max(e^{\alpha t p/2}|h_t|^p, (F_t^{(\alpha)})^p, \sup_{0\leq u\leq t} e^{\alpha u p}|S_u|^p) > n\mathcal{E}(G_-^{-1}\cdot m)\right\}. \tag{5.2.25}$$

It is clear that $T_n$ is an $\mathbb{F}$-stopping time that converges to infinity almost surely. Then we associate a sequence, to the triplet $(f, S, h)$, denoted by $(f^{(n)}, S^{(n)}, h^{(n)})$, given by

$$f^{(n)} := fI_{[\![0,T_n[\![}, \ S^{(n)} := SI_{[\![0,T_n[\![}, \ h^{(n)} := hI_{[\![0,T_n[\![}. \tag{5.2.26}$$

Thus, for any $n \geq 1$, it is clear that the triplet $(f^{(n)}, h^{(n)}, S^{(n)})$ satisfies (5.2.14) on $[\![0, T_n]\!]$. Thus, thanks to the first and the second parts, we deduce the existence of unique solution to (5.2.1), denoted by $(Y^{\mathbb{G},(n)}, Z^{\mathbb{G},(n)}, K^{\mathbb{G},(n)}, M^{\mathbb{G},(n)})$, associated to $(f^{(n)}, h^{(n)}, S^{(n)})$ with the horizon $T_n\wedge\tau$, which remains a solution for any horizon $T_k\wedge\tau$ with $k\geq n$. Furthermore, we have

$$\begin{aligned}
&\|e^{\alpha\cdot/2}Y^{\mathbb{G},(n)}\|_{\widetilde{\mathbb{D}}_\tau(P,p)} + \|e^{\alpha\cdot/2}Y^{\mathbb{G},(n)}\|_{\widetilde{\mathbb{D}}_\tau(S,p)} + \|e^{\alpha\cdot/2}(\widetilde{\mathcal{E}}_-)^{1/p}\cdot(M^{\mathbb{G},(n)})^\tau\|_{\mathcal{M}^p(P)}\\
&+\|e^{\alpha\cdot/2}Z^{\mathbb{G},(n)}\|_{\widetilde{\mathbb{S}}_\tau(P,p)} + \|e^{\alpha\cdot/2}(\widetilde{\mathcal{E}}_-)^{1/p}\cdot K_\tau^{\mathbb{G},(n)}\|_{L^p(P)}\\
&\leq C\|F^{(n)} + e^{\alpha\cdot/2}|h^{(n)}| + \sup_{0\leq u\leq\cdot} e^{\alpha s}(S_u^{(n)})^+\|_{L^p(P\otimes V^{\mathbb{F}})},
\end{aligned}\tag{5.2.27}$$

due to assertion (b). Furthermore, for any $n\geq 1$ and $m\geq 1$

$$\|e^{\alpha\cdot/2}(Y^{\mathbb{G},(n)} - Y^{\mathbb{G},(n+m)})\|_{\widetilde{\mathbb{D}}_\tau(P,p)} + \|e^{\alpha\cdot/2}(Z^{\mathbb{G},(n)} - Z^{\mathbb{G},(n+m)})\|_{\widetilde{\mathbb{D}}_\tau(P,p)}$$

$$
\begin{aligned}
&+ \|e^{\alpha\cdot/2}\sqrt[p]{\widetilde{\mathcal{E}}_-}\cdot(M^{\mathbb{G},(n)}-M^{\mathbb{G},(n+m)})^\tau\|_{\mathcal{M}^p(P)} \\
&\le C_1\|e^{\alpha\cdot/2}|h^{(n)}-h^{(n+m)}|+|F^{(\alpha,n)}-F^{(\alpha,n+m)}| \\
&+\sup_{0\le u\le\cdot} e^{\alpha u/2}|S_u^{(n)}-S_u^{(n+m)}|\|_{L^p(P\otimes V^{\mathbb{F}})} \\
&+C_2\sqrt{\|\sup_{0\le u\le\cdot} e^{\alpha u/2}|S_u^{(n)}-S_u^{(n+m)}|\|_{L^p(P\otimes V^{\mathbb{F}})}\sum_{i\in\{n,n+m\}}\Delta_i},
\end{aligned}
\tag{5.2.28}
$$

where

$$
\Delta_i := \|F_t^{(\alpha,i)}+e^{\frac{\alpha\cdot}{2}}|h^{(i)}|+\sup_{0\le u\le\cdot} e^{\alpha u}(S_u^{(i)})^+\|_{L^p(P\otimes V^{\mathbb{F}})}.
$$

Then by virtue of (5.2.13) and the dominated convergence theorem, we put

$$
\begin{aligned}
&\Delta(n,m):= \\
&\|e^{\frac{\alpha\cdot}{2}}|h^{(n)}-h^{(n+m)}|+|F^{(\alpha,n)}-F^{(\alpha,n+m)}|+\sup_{0\le u\le\cdot} e^{\frac{\alpha u}{2}}|S_u^{(n)}-S_u^{(n+m)}|\|_{L^p(P\otimes V^{\mathbb{F}})},
\end{aligned}
$$

and we derive

$$
\begin{aligned}
&\lim_{n\to+\infty}\sup_{m\ge1}\Delta(n,m) \\
&\le \lim_{n\to+\infty}\|I_{[\![T_n,+\infty[\![}(e^{\alpha\cdot/2}|h|+F^{(\alpha)}+\sup_{0\le u\le\cdot} e^{\alpha u/2}|S_u|)\|_{L^p(P\otimes V^{\mathbb{F}})}=0.
\end{aligned}
$$

A combination of this with (5.2.28) proves that the sequence of the quadruplet $(Y^{\mathbb{G},(n)}, Z^{\mathbb{G},(n)}, K^{\mathbb{G},(n)}, M^{\mathbb{G},(n)})$ is a Cauchy sequence in norm, and hence it converges to $(Y^{\mathbb{G}}, Z^{\mathbb{G}}, K^{\mathbb{G}}, M^{\mathbb{G}})$ in norm and almost surely for a subsequence. As a result, $(Y^{\mathbb{G}}, Z^{\mathbb{G}}, K^{\mathbb{G}}, M^{\mathbb{G}})$ clearly satisfies (5.2.1), and due to Fatou's lemma and (5.2.27) we conclude that assertion (b) holds. This ends part 3.

**Part 4.** Here we prove assertion (c) under no assumption( i.e. we drop the assumption (5.2.14)). Let $(f, S, h)$ be a triplet and consider In fact, we consider two triplets $(f, S_i, h_i)$, $i = 1, 2$, to which we associate two sequences of $\mathbb{F}$-stopping times $(T_n^{(i)})_n$ for $i = 1, 2$ as in (5.2.25), and two data-sequences $(f^{(n)}, S^{(n,i)}, h^{(n,i)})$ which are constructed from $(f^{(i)}, h^{(i)}, S^{(i)})$ and $T_n := \min(T_n^{(1)}, T_n^{(2)})$ via (5.2.26). Therefore, for each $i = 1, 2$ and any $n \geq 1$, the triplet $(f^{(n)}, S^{(n,i)}, h^{(n,i)})$ with the horizon $T_n \wedge \tau$ fulfills (5.2.14), and hence due to Part 1, there exists a unique solution $(\overline{Y}^{(n,i)}, \overline{Z}^{(n,i)}, \overline{M}^{(n,i)}, \overline{K}^{(n,i)})$ that converges in norm and almost surely for a subsequence to the quadruplet $(Y^{\mathbb{G},i}, Z^{\mathbb{G},i}, M^{\mathbb{G},i}, K^{\mathbb{G},i})$. Furthermore, we apply assertion (c) to the difference of solutions

$$
\begin{aligned}
&\big(\delta Y^{\mathbb{G},n}, \delta Z^{\mathbb{G},n}, \delta M^{\mathbb{G},n}\big)\\
&= (\overline{Y}^{(n,1)}, \overline{Z}^{(n,1)}, \overline{M}^{(n,1)}, \overline{K}^{(n,1)}) - (\overline{Y}^{(n,2)}, \overline{Z}^{(n,2)}, \overline{M}^{(n,2)}, \overline{K}^{(n,2)}),
\end{aligned}
$$

and get

$$
\begin{aligned}
&\|e^{\alpha\cdot/2}\delta Y^{\mathbb{G},n}\|_{\widetilde{\mathbb{D}}_\tau(P,p)} + \|e^{\alpha\cdot/2}\delta Z^{\mathbb{G},n}\|_{\widetilde{\mathbb{S}}_\tau(P,p)} + \|e^{\alpha\cdot/2}(\widetilde{\mathcal{E}}_-)^{1/p}\cdot(\delta M^{\mathbb{G},n})^\tau\|_{\mathcal{M}^p(P)}\\
&\leq C_1\|e^{\alpha\cdot/2}|\delta h| + \sup_{0\leq u\leq\cdot} e^{\alpha u/2}|\delta S_u|\|_{L^p(P\otimes V^{\mathbb{F}})}\\
&\qquad\qquad + C_2\sqrt{\|I_{[\![0,T_n]\!]}\sup_{0\leq u\leq\cdot} e^{\frac{\alpha}{2}u}|\delta S_u|\|_{L^p(P\otimes V^{\mathbb{F}})}}\sqrt{\sum_{i=1}^2 \bar{\Delta}_i^{T_n}},
\end{aligned}
$$

with

$$
\bar{\Delta}_i^{T_n} := \|I_{[\![0,T_n]\!]}\{e^{\frac{\alpha}{2}\cdot}|h_i| + F^{(\alpha)} + \sup_{0\leq u\leq\cdot} e^{\alpha u}(S_i(u))^+\}\|_{L^p(P\otimes V^{\mathbb{F}})}.
$$

Therefore, by taking the limit on both sides of this inequality, and using Fatou for the left-hand-side term and the convergence monotone theorem for the right-hand-side term, we deduce that assertion (c) holds. $\square$

Herein we state its BSDE version as we did is Subsection 4.2.2.

**Theorem 5.2.4.** *Let $p \in (1, +\infty)$, $\alpha > \alpha_0(p)$ defined in (5.1.7). Suppose $G > 0$ and*

$$\|F^{(\alpha)} + e^{\alpha\cdot/2}|h|\|_{L^p(P\otimes V^{\mathbb{F}})} < +\infty, \quad \textit{where } F_t^{(\alpha)} := \sqrt{\int_0^t e^{\alpha s}|f_0(s)|^2 ds}. \tag{5.2.29}$$

*Then the following assertions hold.*

(a) *There exists a unique solution $(Y^{\mathbb{G}}, Z^{\mathbb{G}}, M^{\mathbb{G}})$ to the following BSDE*

$$dY_t = -f(t, Y_t, Z_t)d(t\wedge\tau) - dM_{t\wedge\tau} + Z_t dW_t^{\tau}, \quad Y_\tau = \xi = h_\tau. \tag{5.2.30}$$

(b) *There exists positive $C$, which depends on $(\alpha, \alpha', p)$ only, such that*

$$\|e^{\frac{\alpha}{2}\cdot}Y^{\mathbb{G}}\|_{\widetilde{\mathbb{D}}_\tau(P,p)} + \|e^{\frac{\alpha}{2}\cdot}Z^{\mathbb{G}}\|_{\widetilde{\mathbb{S}}_\tau(P,p)} + \|e^{\frac{\alpha}{2}\cdot}Y^{\mathbb{G}}\|_{\widetilde{\mathbb{S}}_\tau(P,p)} + \|e^{\alpha\cdot/2}\sqrt[p]{\widetilde{\mathcal{E}}_-}\cdot(M^{\mathbb{G}})^\tau\|_{\mathcal{M}^p(P)}$$
$$\leq C\|e^{\alpha\cdot/2}|h| + F^{(\alpha)}\|_{L^p(P\otimes V^{\mathbb{F}})}.$$

(c) *Let $(f, h^{(i)})$, $i = 1, 2$, be two pairs satisfying (5.2.13), and $(Y^{\mathbb{G},i}, Z^{\mathbb{G},i}, M^{\mathbb{G},i})$ be the solutions to their corresponding BSDE (5.2.30). There exists $C_1$ which depends on $\alpha$ and $p$ only such that*

$$\|e^{\frac{\alpha}{2}\cdot}\delta Y^{\mathbb{G}}\|_{\widetilde{\mathbb{D}}_\tau(P,p)} + \|e^{\frac{\alpha}{2}\cdot}\delta Z^{\mathbb{G}}\|_{\widetilde{\mathbb{S}}_\tau(P,p)} + \|e^{\frac{\alpha}{2}\cdot}\sqrt[p]{\widetilde{\mathcal{E}}_-}\cdot(\delta M^{\mathbb{G}})^\tau\|_{\mathcal{M}^p(P)}$$
$$\leq C_1\|e^{\alpha\cdot/2}|\delta h|\|_{L^p(P\otimes V^{\mathbb{F}})}.$$

*Here* $(\delta Y^{\mathbb{G}}, \delta Z^{\mathbb{G}}, \delta M^{\mathbb{G}})$ *and* $(\delta f, \delta h)$ *are given by (5.2.9).*

*Proof.* The proof of this theorem follows the same footsteps as the proof of Theorem 5.2.3 by just ignoring the process $S$ and putting $K^{\mathbb{G}} = 0$ throughout the proof, as they are irrelevant. $\square$

### 5.2.3 An RBSDE under $\mathbb{F}$ with infinite horizon and its relationship to (5.2.1)

In this subsection, we derive our second main result of this section that addresses the RBSDE under $\mathbb{F}$ given below, and connects it to (5.2.1).

$$\begin{cases} Y_t = \int_t^\infty f^{\mathbb{F}}(s, Y_s, Z_s)ds + \int_t^\infty h_s dV_s^{\mathbb{F}} + K_\infty - K_t - \int_t^\infty Z_s dW_s, \\ Y_t \geq S_t^{\mathbb{F}}, \quad t \geq 0, \quad E\left[\int_0^\infty (Y_{t-} - S_{t-}^{\mathbb{F}})dK_t\right] = 0. \end{cases} \tag{5.2.31}$$

Here $(f^{\mathbb{F}}, S^{\mathbb{F}}, \widetilde{\mathcal{E}})$ denote the functionals defined via (5.1.28). First of all, remark that a solution to this RBSDE is any triplet $(Y, Z, K)$ such that $\lim_{t\to\infty} Y_t$ exists almost surely and is null, and

$$\begin{cases} dY_t = f^{\mathbb{F}}(t, Y_t, Z_t)dt - h_t dV_t^{\mathbb{F}} - dK_t + Z_t dW_t, \\ Y_t \geq S_t^{\mathbb{F}}, \quad t \geq 0, \quad E\left[\int_0^\infty (Y_{t-} - S_{t-}^{\mathbb{F}})dK_t\right] = 0. \end{cases}$$

This latter RBSDE generalizes Hamadène et al . [46] in many aspects. First of all, our obstacle process $S^{\mathbb{F}}$ is arbitrary RCLL and might not be continuous at all. Furthermore, we do not exige that the part $(Y, K)$ of the solution to be continuous. Besides these, our RBSDE has an additional term, $\int_0^\cdot h_s dV_s^{\mathbb{F}}$ that might not be absolutely continuous with respect to the Lebesgue measure.

**Theorem 5.2.5.** *Let $p \in (1,+\infty)$, $(h,S)$ be a pair of $\mathbb{F}$-optional processes, $f$ is a functional satisfying (5.0.2), and $(f^{\mathbb{F}}, S^{\mathbb{F}}, \widetilde{\mathcal{E}})$ is given by (5.1.28). Suppose that $G > 0$ and there exists $\alpha > \alpha_0(p)$ such that $C_1 C_{Lip} < 1$ –where $C_1$ is given by Theorem 5.2.1 –,*

$$\text{(5.2.13) holds and}\quad E\left[\left(\widetilde{\mathcal{E}}_\infty F_\infty^{(\alpha)}\right)^p\right] < +\infty. \tag{5.2.32}$$

*Then the RBSDE (5.2.31) has a unique $L^p(P,\mathbb{F})$-solution $(Y^{\mathbb{F}}, Z^{\mathbb{F}}, K^{\mathbb{F}})$.*

The proof of this theorem is based on the following lemma

**Lemma 5.2.6.** *For $\alpha > \alpha_0(p)$ and $\alpha' \in [0,\alpha/2)$, there exist $C_i$, $i = 1,2,3,4$ that depend on $(\alpha,\alpha',p)$ only such that $\lim_{\alpha\to+\infty} C_1 = 0$ and the following assertions hold.*

(a) *If $(Y^i, Z^i, K^i)$ is an $L^p(\mathbb{F},P)$-solution to the RBSDE (5.2.33) associated to $(f^i, S^i, \xi^i)$, $i = 1,2$, then*

$$\begin{aligned}&\|e^{\frac{\alpha}{2}\cdot}\delta Y\|_{\mathbb{D}(P,p)} + \|e^{\frac{\alpha}{2}\cdot}\delta Z\|_{\mathbb{S}(P,p)}\\ &\leq C_1\|e^{\frac{\alpha}{2}\cdot}\delta f\|_{\mathbb{S}(P,p)} + C_2\|e^{\frac{\alpha}{2}T}\delta\xi\|_{L^p(P)} + C_3\sqrt{\|e^{\frac{\alpha}{2}\cdot}\delta S\|_{\mathbb{S}(P,p)}\|e^{\frac{\alpha}{2}\cdot}\delta K\|_{\mathcal{A}_T(P,p)}}.\end{aligned}$$

(b) *If $(Y,Z,K)$ is a class-(D)-$(\mathbb{F},P,\infty)$-solution to the RBSDE (5.2.33), then*

$$\begin{aligned}&\|e^{\frac{\alpha}{2}\cdot}Y\|_{\mathbb{D}(P,p)} + \|e^{\frac{\alpha}{2}\cdot}Z\|_{\mathbb{S}(P,p)} + \|e^{\frac{\alpha}{2}\cdot}Y\|_{\mathbb{S}(P,p)} + \|e^{\frac{\alpha}{2}\cdot}K_T\|_{L^p(P)}\\ &\qquad\leq C_4\left\{\|e^{\frac{\alpha}{2}\cdot}f_0(\cdot)\|_{\mathbb{S}(P,p)} + \|e^{\frac{\alpha}{2}T}\xi\|_{L^p(P)} + \|e^{(\alpha-\alpha')\cdot}S^+\|_{\mathbb{S}(P,p)}\right\}.\end{aligned}$$

The proof of this lemma mimics those of Theorems 5.1.2 and 5.1.3, and will be omitted here.

*Proof of Theorem 5.2.5.* Remark that due to the assumption (5.2.13), the non-decreasing process $U := \int_0^{\cdot} h_s dV_s^{\mathbb{F}}$ has a limit at infinity. Put

$$\widetilde{f}^{\mathbb{F}}(s,y,z) = f^{\mathbb{F}}(s, y - U_s, z), \quad \widetilde{S}^{\mathbb{F}} := S^{\mathbb{F}} + U, \quad \text{and} \quad \widehat{\xi} := U_\infty = \int_0^\infty h_s dV_s^{\mathbb{F}}.$$

Then $(\overline{Y}, \overline{Z}, \overline{K})$ solves (5.2.31) if and only if $(Y', Z', K') := (\overline{Y} + U, \overline{Z}, \overline{K})$ is a solution to

$$\begin{cases} Y_t = \widehat{\xi} + \int_t^\infty \widetilde{f}^{\mathbb{F}}(s, Y_s, Z_s) ds + K_\infty - K_t - \int_t^\infty Z_s dW_s, \\ \\ Y_t \geq \widetilde{S}_t^{\mathbb{F}}, \quad t \geq 0, \quad E\left[\int_0^\infty (Y_{t-} - \widetilde{S}_{t-}^{\mathbb{F}}) dK_t\right] = 0, \end{cases} \tag{5.2.33}$$

Now, we define the sequence $(Y^{(n)}, Z^{(n)}, K^{(n)})$ as follows: $(Y^{(0)}, Z^{(0)}, K^{(0)}) := (0,0,0)$, and $(Y^{(n)}, Z^{(n)}, K^{(n)})$ is the unique solution to

$$Y_t^{(n)} = \xi + \int_t^\infty \widetilde{f}^{\mathbb{F}}(s, Y_s^{(n-1)}, Z_s^{(n-1)}) ds + \int_t^\infty Z_s^{(n)} dW_s + K_\infty^{(n)} - K_t^{(n)}. \tag{5.2.34}$$

The existence and uniqueness of this $L^p(\mathbb{F}, P)$-solution follow from Theorem 4.2.7 and that

$$\int_t^\infty |\widetilde{f}^{\mathbb{F}}(s, Y_s^{(n-1)}, Z_s^{(n-1)})| ds \leq C_{Lip} \int_t^\infty |\widetilde{f}^{\mathbb{F}}(s)| ds$$
$$+ C_{Lip} \int_t^\infty e^{-\frac{\alpha}{2}s} e^{\frac{\alpha}{2}s} |Y_s^{(n-1)}| ds + C_{Lip} \int_t^\infty e^{-\frac{\alpha}{2}s} e^{\frac{\alpha}{2}s} |Z_s^{(n-1)}| ds,$$

$$\leq C_{Lip}\int_t^\infty |\widetilde{f}^{\mathbb{F}}(s)|ds + C_{Lip}\sqrt{\int_t^\infty e^{-\alpha s}ds}\sqrt{\int_t^\infty e^{\alpha s}|Y_s^{(n-1)}|^2 ds}$$
$$+ C_{Lip}\sqrt{\int_t^\infty e^{-\alpha s}ds}\sqrt{\int_t^\infty e^{\alpha s}|Z_s^{(n-1)}|^2 ds} < \infty.$$

The last inequality above is due to Lemma 5.2.6-(b). Thus, by applying Lemma 5.2.6 to $(Y^{(i)}, Z^{(i)}, K^{(i)})$ and

$$(\delta Y, \delta Z, \delta K) := \left(Y^{(n+m)} - Y^{(n)}, Z^{(n+m)} - Z^{(n)}, K^{(n+m)} - K^{(n)}\right),$$

we define the norm

$$|||(Y, Z, K)||| := \|Y\|_{\mathbb{D}(P,p)} + \|Z\|_{\mathbb{S}(P,p)} + \|K_T\|_{L^p(P)},$$

for any triplet $(Y, Z, K) \in \mathbb{D}(P, p) \times \mathbb{S}(P, p) \times \mathcal{A}^+_{loc}$, and deduce that

$$\sup_{i\geq 0}\left\{|||(Y^{(i)}, Z^{(i)}, K^{(i)})||| + \|Y^{(i)}\|_{\mathbb{S}(P,p)}\right\} < +\infty, \tag{5.2.35}$$

and

$$|||(\delta Y, \delta Z, 0)||| \leq C_1\|\delta f\| \leq C_1 C_{Lip}|||(Y^{(n+m-1)}-Y^{(n-1)}, Z^{(n+m-1)}-Z^{(n-1)}, 0)|||.$$

Thus, by iterating this latter inequality, we get

$$\begin{aligned} |||(Y^{(n+m)} - Y^{(n)}, Z^{(n+m)} - Z^{(n)}, 0)||| &\leq (C_1 C_{Lip})^n|||(Y^{(m)}, Z^{(m)}, 0)||| \\ &\leq (C_1 C_{Lip})^n \sup_{i\geq 0}|||(Y^{(i)}, Z^{(i)}, 0)|||. \end{aligned}$$

By combining this with (5.2.35), we conclude that the sequence $(Y^{(n)}, Z^{(n)})$ is a Cauchy, and hence it converges in norm and almost surely for a subsequence to $(Y, Z)$. Then the convergence of $K^{(i)}$ to some $K$ follows immediately from the RBSDE (5.2.33), and therefore the triplet $(Y, Z, K)$ is a solution to (5.2.33). The uniqueness of the solution to (5.2.33) is a direct consequence of Lemma 5.2.6-(a). This ends the proof of the theorem. $\square$

Below, we establish the relationship between the solution of (5.2.1) and that of (5.2.31).

**Theorem 5.2.7.** *Suppose that the assumptions of Theorem 5.2.5 hold. Then both RBSDEs (5.2.31) and (5.2.1) have unique $L^p(\mathbb{F}, P)$-solutions, denoted by the triplet $(Y^{\mathbb{F}}, Z^{\mathbb{F}}, K^{\mathbb{F}})$ and the quadruplet $(Y^{\mathbb{G}}, Z^{\mathbb{G}}, K^{\mathbb{G}}, M^{\mathbb{G}})$ respectively, and they satisfy*

$$\begin{cases} Y^{\mathbb{G}} = Y^{\mathbb{F}}\widetilde{\mathcal{E}}^{-1} I_{[\![0,\tau[\![} + \xi I_{[\![\tau,+\infty[\![}, \ Z^{\mathbb{G}} = Z^{\mathbb{F}}\widetilde{\mathcal{E}}^{-1} I_{]\!]0,\tau]\!]}, \\ K^{\mathbb{G}} = \widetilde{\mathcal{E}}_{-}^{-1} \cdot (K^{\mathbb{F}})^{\tau} \quad \textit{and} \quad M^{\mathbb{G}} = \left(h - Y^{\mathbb{F}}\widetilde{\mathcal{E}}^{-1}\right) \cdot N^{\mathbb{G}}. \end{cases} \tag{5.2.36}$$

*Proof.* Thanks to Theorems 5.2.5 and 5.2.3, it is clear that both RBSDEs (5.2.31) and (5.2.1) have unique solutions. This proves the first claim of the theorem, while the proof of (5.2.36) follows immediately as soon as we prove that $(\overline{Y}, \overline{Z}, \overline{M}, \overline{K})$ given by

$$\overline{Y} := \frac{Y^{\mathbb{F}}}{\widetilde{\mathcal{E}}} I_{[\![0,\tau[\![} + \xi I_{[\![\tau,+\infty[\![}, \ \overline{Z} := \frac{Z^{\mathbb{F}}}{\widetilde{\mathcal{E}}} I_{]\!]0,\tau]\!]}, \ \overline{K} := \frac{1}{\widetilde{\mathcal{E}}_{-}} \cdot (K^{\mathbb{F}})^{\tau}$$
$$\text{and} \quad \overline{M} := \left(h - \frac{Y^{\mathbb{F}}}{\widetilde{\mathcal{E}}}\right) \cdot N^{\mathbb{G}}.$$

is a solution to (5.2.1). This latter fact can be proved by following exactly the footsteps of Step 2 in the proof of Theorem 4.2.7. This ends the proof of theorem. □

# Chapter 6

# Extension to models with jumps and general terminal value

Throughout this chapter we consider a fixed and finite deterministic horizon $T \in (0, +\infty)$. The main contribution of this chapter lies in extending Chapters 4 and 5 to the case where the filtration $\mathbb{F}$ is generated by a Brownian motion $W$ and a Poisson process $N$ with intensity $\lambda > 0$, and where the terminal value of the RBSDE $\xi$ is $\mathcal{G}_{T\wedge\tau}$-measurable instead of being $\mathcal{F}_{T\wedge\tau}$-measurable. However, we restrict this chapter to the square integrability instead of $p$-integrability of Chapters 4 and 5.

In [11] Barles et al. considered the BSDE when the noise is driven by both Brownian motion and a Poisson random measure. They proved under appropriate assumptions that their BSDE has a unique solution. Also under an appropriate framework, they found that their BSDE's solution is the unique viscosity solution of a system of parabolic integro-partial differential equations. The extension to the case of RBSDE with jumps, which is driven by a Brownian motion and an independent Poisson point process, have been established by Hamadène and Ouknine [48], the authors have shown the existence and uniqueness of a solution when the terminal value is square integrable, the driver is uniformly Lipschitz and the barrier is *rcll* whose jumping times are inaccessible stopping time and hence the reflecting process $K$ is continuous. While, Essaky [44] extends the case of Hamadène et al. when the barrier $S$ is allowed to have predictable jumps then the process $Y$ is so and then the reflecting process $K$ are no longer continuous but just rcll.

We start with a stochastic basis $(\Omega, \mathcal{F}, P)$, with a filtration $\mathbb{F}^{(N,W)} := (\mathcal{F}_t)_{t\geq 0}$ that satisfies the usual conditions of right continuity and completeness. For simplicity we assume that $\mathcal{F}_0$ is trivial and $\mathcal{F}_T = \mathcal{F}$, We give an auxiliary measurable space $(E, \mathcal{B}(E), \lambda)$, where $\lambda$ is a non-nagative $\sigma$-finite measure on $(E, \mathcal{B}(E))$ satisfying $\int_E 1 \wedge |e|^2 \lambda(de) < \infty$. Let $E := \mathbb{R}^l \setminus \{0\}$, then we suppose that the filtration $\mathbb{F} := \mathbb{F}^{(N,W)}$ is generated by the two following mutually independent processes:

- a standard Brownian motion $W = (W_t)_{t\geq 0}$.
- a poisson process $N$ with intensity $\lambda > 0$. Then, $N^{\mathbb{F}}$; $N_t^{\mathbb{F}} := N_t - \lambda t$ is a $(\mathbb{F}^{(N,W)}, P)$- martingale process.

Define the following:

$$(\widetilde{\Omega}, \widetilde{\mathcal{F}}) := (\Omega \times [0,T] \times E, \mathcal{F} \otimes \mathcal{B}([0,T]) \otimes \mathcal{B}(E))$$

Also define

$$\widetilde{\mathcal{P}} := \mathcal{P} \otimes \mathcal{B}(E),$$

where $\mathcal{P}$ is the predictable $\sigma$-field on $\Omega \times [0,T]$. A function on $\widetilde{\Omega}$ that is $\widetilde{\mathcal{P}}$-measurable is called predictable. For more details about random measures we recommend you to read chapter II in [50]. In addition to the spaces that are defined in section 2.4, we define the desired space in this chapter as the following

- $\mathbb{L}^2_T(N,Q)$ is the space of $\mathbb{F}^{(N,W)}$-predictable processes $V$ such that

$$\|V\|^2_{\mathbb{L}^2_T(N,Q)} := E^Q\left[\int_0^T |V_t|^2 dN_t\right] = E^Q\left[\int_0^T |V_t|^2 \lambda dt\right] < \infty.$$

**Remark 6.0.1.** *Note that due to Proposition 2.3.4-b we have that $W^\tau$ and $(N^{\mathbb{F}})^\tau$ are $(\mathbb{G},\widetilde{Q})$-local martingales.*

Suppose that there exists an $\mathbb{F}$-progressively measurable process $h^{(pr)}$ such that

$$\xi = h^{(pr)}_{T\wedge\tau} \quad \text{and} \quad S_{\tau\wedge T} \le \xi \quad P-a.s. \tag{6.0.1}$$

Throughout this chapter define $\mathcal{J}^{\sigma_2}_{\sigma_1}(\mathbb{H})$ as the set of all stopping times in $\mathbb{H}$ with values in $[\sigma_1,\sigma_2]$, also $\widetilde{Q}$ be the probability defined by (2.3.11). For ease, until the end of this section we write $\mathbb{F} := \mathbb{F}^{(N,W)}$.

This chapter consists of two sections. The first section presents the case of $\mathbb{G}$-RBSDE for linear driver, in this section we give some useful estimates and the existence and uniqueness results for the solution of this RBSDE also it presents an explicit connection to its $\mathbb{F}$-RBSDE counterpart. In the second section, we handle the case of non linear driver $\mathbb{G}$-RBSDE. We give some useful estimates and the existence and uniqueness results for the solution of this RBSDE, also we present an explicit connection to its $\mathbb{F}$-RBSDE counterpart.

## 6.1 The linear case

In this section we consider the following linear RBSDEs,

$$\begin{cases} dY = -f(t)d(t\wedge\tau) - d(K+M) + ZdW^{\tau} + Z'd(N^{\mathbb{F}})^{\tau}, \\ Y_{\tau} = Y_T = \xi, \quad Y_t \geq S_t \text{ on } [0, T\wedge\tau), \\ \langle M, W + N^{\mathbb{F}}\rangle^{\mathbb{G}} \equiv 0, \text{ and } \quad E\left[\int_0^{\tau\wedge T}(Y_{t-} - S_{t-})dK_t\right] = 0. \end{cases} \tag{6.1.1}$$

Here $K$ is an increasing process such that $K_0 = 0$, the barrier process $S$ is an $\mathbb{F}$-adapted and RCLL process, $\xi = h^{(pr)}_{T\wedge\tau}$ with $h^{(pr)}$ being $\mathbb{F}$-progressively measurable and its $\mathbb{F}$-optional projection is $h^{(op)} := M^P_{\mu}(h^{(pr)}\big|\mathcal{O}(\mathbb{F})), \quad \mu := P\otimes D$. $T \in (0, +\infty)$, and the generator $f$ is $\mathbb{F}$-progressively measurable function.

### 6.1.1 Estimates for the solution of the BSDE (6.1.1)

This subsection elaborates estimates for the solution of the BSDE (6.1.1).

**Lemma 6.1.1.** *Suppose that the triplet $(f, \xi, S)$ satisfies*

$$E^{\widetilde{Q}}\left[|\xi| + \int_0^{T\wedge\tau}|f(s)|ds + \sup_{0\leq u\leq\tau\wedge T}(S^+_u)\right] < +\infty, \tag{6.1.2}$$

*$\widetilde{Q}$ be the probability defined by (2.3.11). Suppose that $(Y^{\mathbb{G}}, Z^{\mathbb{G}}, Z'^{\mathbb{G}}, M^{\mathbb{G}}, K^{\mathbb{G}})$ is a class-(D)-$(\mathbb{G}, \widetilde{Q}, T\wedge\tau)$-solution to the RBSDE (6.1.1) then for $t \in [0, T]$*

*we have*

$$Y_t^{\mathbb{G}} := \operatorname*{ess\,sup}_{v\in\mathcal{J}_{t\wedge\tau}^{T\wedge\tau}(\mathbb{G})} E^{\widetilde{Q}}\left[\int_{t\wedge\tau}^{v\wedge\tau} f(s)ds + S_v 1_{\{v\ <T\wedge\tau\}} + \xi 1_{\{v=T\wedge\tau\}}\ \Big|\ \mathcal{G}_t\right] \quad (6.1.3)$$

*Proof.* The proof can be obtained by mimicking exactly the footsteps of the proof of Lemma 4.1.3 and using $(Z^{'}\bullet N^{\mathbb{F}})^{\tau}$ is a $(\mathbb{G},\widetilde{Q})$-martingale. □

**Remark 6.1.2.** *For any $\mathbb{F}$-optional process $X$ that belongs to $\mathcal{I}^o(N^{\mathbb{G}},\mathbb{G})$ we have that $X\bullet[N^{\mathbb{G}},N^{\mathbb{F}}] = X\Delta N\bullet N^{\mathbb{G}}$ is a $\mathbb{G}$-martingale. This is due to the fact that $|X\Delta N| \le |X|$. This implies that $X\Delta N \in \mathcal{I}^o(N^{\mathbb{G}},\mathbb{G})$, and hence $X\bullet[N^{\mathbb{G}},N^{\mathbb{F}}] = X\Delta N\bullet N^{\mathbb{G}}$ is a $\mathbb{G}$-martingale.*

**Theorem 6.1.3.** *There exists a positive constant $C$ such that if the quintuplet $(Y^{\mathbb{G}},Z^{\mathbb{G}},Z^{'\mathbb{G}},K^{\mathbb{G}},M^{\mathbb{G}})$ is a class-(D)-$(\mathbb{G},\widetilde{Q},T\wedge\tau)$-solution to (6.1.1), then*

$$\begin{aligned}&E^{\widetilde{Q}}\left[\sup_{0\le t\le T\wedge\tau}(Y_t^{\mathbb{G}})^2 + \int_0^{T\wedge\tau}|Z_s^{\mathbb{G}}|^2ds + \int_0^{T\wedge\tau}|Z_s^{'\mathbb{G}}|^2dN + [M^{\mathbb{G}},M^{\mathbb{G}}]_{T\wedge\tau}\right]\\ &+ E^{\widetilde{Q}}\left[(K_{T\wedge\tau}^{\mathbb{G}})^2\right]\ \le C\left\{\|\xi\|_{L^2(\widetilde{Q})}^2 + \|\int_0^{T\wedge\tau}|f(s)|ds\|_{L^2(\widetilde{Q})}^2 + \|S^+\|_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},2)}^2\right\},\end{aligned} \quad (6.1.4)$$

*Proof.* The proof mimicks the same footsteps of the proof of Theorem 4.1.4, and uses $(Z^{'}\bullet N^{\mathbb{F}})^{\tau}$ is a $(\mathbb{G},\widetilde{Q})$-martingale together with using Remark 6.1.2. □

**Theorem 6.1.4.** *Let $(Y^{\mathbb{G},i},Z^{\mathbb{G},i},Z^{'\mathbb{G},i},K^{\mathbb{G},i},M^{\mathbb{G},i})$ be a class-(D)-$(\mathbb{G},\widetilde{Q},T\wedge\tau)$-solution to the RBSDE (6.1.1) that correspond to $(f_i,S_i,\xi^i)$, $i=1,2$ respec-*

*tively. Then there exist a positive constants $C_1$ and $C_2$ such that*

$$E^{\widetilde{Q}}\left[\sup_{0\leq t\leq T\wedge\tau}(\delta Y_t^{\mathbb{G}})^2+\int_0^{T\wedge\tau}|\delta Z_s^{\mathbb{G}}|^2ds+\int_0^{T\wedge\tau}|\delta Z_s'^{\mathbb{G}}|^2dN+[\delta M^{\mathbb{G}},\delta M^{\mathbb{G}}]_{T\wedge\tau}\right]$$
$$\leq C_1\Delta_{\widetilde{Q}}(\delta\xi,\delta f,\delta S)+C_2\|\delta S_s^{\mathbb{G}}\|_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},2)}\sqrt{\sum_{i=1}^{2}\Delta_{\widetilde{Q}}(\xi^{(i)},f^{(i)},(S^{(i)})^+)}. \quad (6.1.5)$$

*where $\Delta_{\widetilde{Q}}(\xi^{(i)},f^{(i)},(S^{(i)})^+)$ for $i=1,2$ and $\Delta_{\widetilde{Q}}(\delta\xi,\delta f,\delta S)$ are defined via (4.1.8).*

The proof of this theorem mimicks the same footsteps of the proof of Theorem 4.1.6 by using the fact that $(Z'\bullet N^{\mathbb{F}})^\tau$ is $(\mathbb{G},\widetilde{Q})$-martingale together with using Remark 6.1.2.

### 6.1.2 Existence and uniqueness for the $\mathbb{G}$-RBSDE solution and its relationship to $\mathbb{F}$-RBSDE

In this subsection, we prove the existence and uniqueness of the solution to the RBSDE (6.1.1), and we establish explicit connection between this RBSDE and its $\mathbb{F}$-RBSDE counterpart.

**Theorem 6.1.5.** *Suppose that $T<\infty$ and there is $p\in(1,\infty)$ such that*

$$\|\int_0^{T\wedge\tau}|f(s)|ds+|\xi|+\sup_{0\leq u\leq\tau\wedge T}S_u^+\|_{L^2(\widetilde{Q})}<+\infty. \quad (6.1.6)$$

*and consider $(f^{\mathbb{F}},S^{\mathbb{F}},\xi^{\mathbb{F}})$ and $V^{\mathbb{F}}$ given by*

$$f^{\mathbb{F}}:=\widetilde{\mathcal{E}}f,\ S^{\mathbb{F}}:=\widetilde{\mathcal{E}}S,\ \xi^{\mathbb{F}}:=\widetilde{\mathcal{E}}_T h_T^{(op)},\ V^{\mathbb{F}}:=1-\widetilde{\mathcal{E}},\ \widetilde{\mathcal{E}}:=\mathcal{E}\left(-\frac{1}{\widetilde{G}}\cdot D^{o,\mathbb{F}}\right) (6.1.7)$$

*Then the following assertions hold.*

(a) *The following RBSDE under $\mathbb{F}$, associated to the triplet $\left(f^{\mathbb{F}}, S^{\mathbb{F}}, \xi^{\mathbb{F}}\right)$,*

$$\begin{cases} dY_s = -f^{\mathbb{F}}(s)ds - h^{(op)}_s dV^{\mathbb{F}}_s - dK_s + Z_s dW_s + Z'_s dN^{\mathbb{F}}_s, \\ \\ Y_T = \xi,\ Y_t \geq S^{\mathbb{F}}_t 1_{\{t\ <T\}} + \xi^{\mathbb{F}} 1_{\{t\ =T\}},\ \displaystyle\int_0^T (Y_{t-} - S^{\mathbb{F}}_{t-})dK_t = 0,\ P\text{-a.s.}, \end{cases} \tag{6.1.8}$$

*has a unique $L^2(P,\mathbb{F})$-solution $(Y^{\mathbb{F}}, Z^{\mathbb{F}}, Z'^{\mathbb{F}}, K^{\mathbb{F}})$ that satisfies*

$$Y^{\mathbb{F}}_t = \operatorname{ess}\sup_{\sigma\in\mathcal{J}^T_t(\mathbb{F})} E\left[\int_t^\sigma f^{\mathbb{F}}(s)ds + \int_t^\sigma h^{(op)}_s dV^{\mathbb{F}}_s + S^{\mathbb{F}}_\sigma 1_{\{\sigma<T\}} + \xi^{\mathbb{F}} I_{\{\sigma=T\}}\ \Big|\ \mathcal{F}_t\right]. \tag{6.1.9}$$

(b) *The RBSDE defined in (6.1.1) has a unique $L^2(\widetilde{Q},\mathbb{G})$-solution, denoted by $(Y^{\mathbb{G}}, Z^{\mathbb{G}}, Z'^{\mathbb{G}}, K^{\mathbb{G}}, M^{\mathbb{G}})$ given by*

$$\begin{cases} Y^{\mathbb{G}} = \dfrac{Y^{\mathbb{F}}}{\widetilde{\mathcal{E}}} I_{[\![0,\tau[\![} + \xi I_{[\![\tau,\infty[\![}, \quad Z^{\mathbb{G}} = \dfrac{Z^{\mathbb{F}}}{\widetilde{\mathcal{E}}_-} I_{]\!]0,\tau]\!]}, \quad Z'^{\mathbb{G}} = \dfrac{Z'^{\mathbb{F}}}{\widetilde{\mathcal{E}}_-} I_{]\!]0,\tau]\!]} \\ K^{\mathbb{G}} = \dfrac{1}{\widetilde{\mathcal{E}}_-} \bullet (K^{\mathbb{F}})^\tau,\ \text{and } M^{\mathbb{G}} = (\dfrac{Y^{\mathbb{F}}}{\widetilde{\mathcal{E}}} - h^{(op)}) \bullet N^{\mathbb{G}} - (h^{(pr)} - h^{(op)}) \bullet D. \end{cases} \tag{6.1.10}$$

*Proof.* Assertion (a) is the linear case of a general RBSDE under $\mathbb{F}$ given in Subsection 6.2.2, see (6.2.17). Thus, the proof of the existence and uniqueness of the $L^2(\mathbb{F},P)$-solution under (6.1.6) (respectively the solution under (6.1.2)) of this RBSDE will be omitted here, and we refer the reader to Subsection 6.2.2. Furthermore, on the one hand, the proof of (6.1.9) mimics exactly the proof of (6.1.3). On the other hand, under the assumption (6.1.6), the existence and uniqueness of the $L^2(\widetilde{Q},\mathbb{G})$-solution to (6.1.1) satisfying (6.1.10) follows immediately from Theorems 6.1.3 and 6.1.4 and assertion (b). Hence, the remaining proof focuses on proving assertion (b).

Under the assumption (6.1.2), the existence and uniqueness of the solution to (6.1.1) follows from combining Lemma 6.1.1 and the theory of Snell envelop, see [41] for details. Thus, assertion (b) follows as soon as we prove that the solution $(Y^{\mathbb{G}}, Z^{\mathbb{G}}, Z'^{\mathbb{G}}, K^{\mathbb{G}}, M^{\mathbb{G}})$ described by Lemma 6.1.1 satisfies (6.1.10). To this end, on the one hand, thanks to the Doob-Meyer decomposition under $(\widetilde{Q}, \mathbb{G})$, we remark that for any solution $(Y, Z, Z', K, M)$ to (6.1.1) we have $(Y, Z, Z', K, M) = (Y^{\mathbb{G}}, Z^{\mathbb{G}}, Z'^{\mathbb{G}}, K^{\mathbb{G}}, M^{\mathbb{G}})$ if and only if $Y = Y^{\mathbb{G}}$. On the other hand, due to (6.1.3), we have

**Step 1.**

$$\begin{cases} Y^{\mathbb{G}} + \int_0^{\tau\wedge T\wedge\cdot} f(s)ds = \mathcal{S}(X^{\mathbb{G}}; \mathbb{G}, \widetilde{Q}) \\ \text{with} \\ X^{\mathbb{G}} := \int_0^{\tau\wedge T\wedge\cdot} f(s)ds + SI_{[\![0,\tau\wedge T[\![} + h^{(pr)}_{\tau\wedge T} I_{[\![\tau\wedge T,+\infty[\![}. \end{cases}$$

Therefore, in order to apply Theorem 3.2.3-(b), we need to find the unique pair $(X^{\mathbb{F}}, k^{(pr)})$ associated to $X^{\mathbb{G}}$. To this end, we remark that

$$SI_{[\![0,\tau\wedge T[\![} = SI_{[\![0,\tau[\![} I_{[\![0,T[\![} \quad \text{and} \quad h^{(pr)}_{\tau\wedge T} I_{[\![\tau\wedge T,+\infty[\![} I_{[\![0,\tau[\![} = h^{(pr)}_{T} I_{[\![0,\tau[\![} I_{[\![T,+\infty[\![},$$

and derive

$$X^{\mathbb{F}} = \int_0^{T\wedge\cdot} f(s)ds + SI_{[\![0,T[\![} + h^{(pr)}_{T} I_{[\![T,+\infty[\![},$$

and

$$k^{(pr)} = \int_0^{T\wedge\cdot} f(s)ds + h^{(pr)}_{T\wedge\cdot}$$

where its $\mathbb{F}$-optional projection is

$$\begin{aligned} k^{(op)} \quad &:= M^P_\mu(k^{(pr)}\big|\mathcal{O}(\mathbb{F})), \quad \mu := P\otimes D,\\ &= \int_0^{T\wedge\cdot} f(s)ds + h^{(op)}_{T\wedge\cdot}, \end{aligned}$$

where

$$h^{(op)} \quad := M^P_\mu(h^{(pr)}\big|\mathcal{O}(\mathbb{F})), \quad \mu := P\otimes D.$$

Furthermore, we have the following

$$\begin{aligned} \widetilde{\mathcal{E}}X^{\mathbb{F}} - k^{(op)}\cdot\widetilde{\mathcal{E}} \quad &= \int_0^{T\wedge\cdot} f^{\mathbb{F}}(s)ds + S^{\mathbb{F}}I_{[\![0,T[\![} + \xi^{\mathbb{F}}I_{[\![T,+\infty[\![} - h^{(op)}\cdot\widetilde{\mathcal{E}},\\ &= \int_0^{T\wedge\cdot} f^{\mathbb{F}}(s)ds + S^{\mathbb{F}}I_{[\![0,T[\![} + \xi^{\mathbb{F}}I_{[\![T,+\infty[\![} + (h^{(op)}\cdot V^{\mathbb{F}})^T, \end{aligned}$$

$$k^{(op)}\widetilde{\mathcal{E}} - k^{(op)}\cdot\widetilde{\mathcal{E}} = \int_0^{T\wedge\cdot} f^{\mathbb{F}}(s)ds + (h^{(op)}\cdot V^{\mathbb{F}})^T + \widetilde{\mathcal{E}}h^{(op)}I_{[\![0,T[\![} + \xi^{\mathbb{F}}I_{[\![T,+\infty[\![},$$

and

$$Y^{\mathbb{F}} + L^{\mathbb{F}} \quad = \mathcal{S}\left(L^{\mathbb{F}} + \xi^{\mathbb{F}}I_{[\![T,+\infty[\![} + S^{\mathbb{F}}I_{[\![0,T[\![}; \mathbb{F}, P\right),$$

$$L^{\mathbb{F}} := \int_0^{T\wedge\cdot} f^{\mathbb{F}}(s)ds + \int_0^{T\wedge\cdot} h^{(op)}_s dV^{\mathbb{F}}_s.$$

Thus, by directly applying Theorem 3.2.3-(b) to $Y^{\mathbb{G}}$, on $[\![0,T]\!]$, we obtain

$$
\begin{aligned}
&Y^{\mathbb{G}} + \int_0^{\tau\wedge T\wedge\cdot} f(s)ds = \mathcal{S}(X^{\mathbb{G}};\mathbb{G},\widetilde{Q})\\
&= \frac{\mathcal{S}\left(X^{\mathbb{F}}\widetilde{\mathcal{E}} - k^{(op)}\cdot\widetilde{\mathcal{E}};\mathbb{F},P\right)}{\widetilde{\mathcal{E}}} I_{[\![0,\tau[\![} + (k^{(pr)} - k^{(op)})\cdot D^T\\
&+ \left(k^{(op)} - \frac{k^{(op)}\cdot\widetilde{\mathcal{E}}}{\widetilde{\mathcal{E}}}\right)\cdot(N^{\mathbb{G}})^T\\
&= \frac{\mathcal{S}\left(X^{\mathbb{F}}\widetilde{\mathcal{E}} - k^{(op)}\cdot\widetilde{\mathcal{E}};\mathbb{F},P\right)}{\widetilde{\mathcal{E}}} I_{[\![0,\tau[\![} + (h^{(pr)} - h^{(op)})\cdot D^T\\
&+ \left(k^{(op)} - \frac{k^{(op)}\cdot\widetilde{\mathcal{E}}}{\widetilde{\mathcal{E}}}\right)\cdot(N^{\mathbb{G}})^T\\
&= \frac{\mathcal{S}\left(X^{\mathbb{F}}\widetilde{\mathcal{E}} - k^{(op)}\cdot\widetilde{\mathcal{E}};\mathbb{F},P\right)}{\widetilde{\mathcal{E}}} I_{[\![0,\tau[\![} + (h^{(pr)} - h^{(op)})\cdot D^T\\
&+ \frac{L^{\mathbb{F}} + \widetilde{\mathcal{E}}h^{(op)}I_{[\![0,T[\![} + \xi^{\mathbb{F}}I_{[\![T,+\infty[\![}}{\widetilde{\mathcal{E}}}\cdot(N^{\mathbb{G}})^T\\
&= \frac{Y^{\mathbb{F}} + L^{\mathbb{F}}}{\widetilde{\mathcal{E}}} I_{[\![0,\tau[\![} + \frac{L^{\mathbb{F}}}{\widetilde{\mathcal{E}}}\cdot N^{\mathbb{G}} + \left(h^{(op)}I_{[\![0,T[\![} + h^{(op)}_T I_{[\![T,+\infty[\![}\right)\cdot(N^{\mathbb{G}})^T\\
&+ (h^{(pr)} - h^{(op)})\cdot D^T\\
&= \frac{Y^{\mathbb{F}}}{\widetilde{\mathcal{E}}} I_{[\![0,\tau[\![} + \frac{1}{\widetilde{\mathcal{E}}_-}\cdot(L^{\mathbb{F}})^\tau + h^{(op)}_{T\wedge\cdot}\cdot D - \frac{h^{(op)}}{\widetilde{G}} I_{]\!]0,\tau\wedge T]\!]}\cdot D^{o,\mathbb{F}} + (h^{(pr)} - h^{(op)})\cdot D^T\\
&= \frac{Y^{\mathbb{F}}}{\widetilde{\mathcal{E}}} I_{[\![0,\tau[\![} + \int_0^{\tau\wedge T\wedge\cdot} f(s)ds + \xi I_{[\![\tau,+\infty[\![}.
\end{aligned}
$$

The sixth equality follows from Lemma 4.1.7. This proves assertion (b).

**Step 2.** Consider $(Y^{\mathbb{G}}, Z^{\mathbb{G}}, Z'^{\mathbb{G}}, K^{\mathbb{G}}, M^{\mathbb{G}})$ given by (6.1.10), and define the process $\Gamma$ as follows.

$$
\Gamma := \frac{Y^{\mathbb{F}}}{\mathcal{E}(-\widetilde{G}^{-1}\bullet D^{o,\mathbb{F}})} = Y^{\mathbb{F}}\mathcal{E}(G^{-1}\bullet D^{o,\mathbb{F}}), \tag{6.1.11}
$$

and remark that, in virtue of the first equality in (6.1.10), we have

$$Y^{\mathbb{G}} = \Gamma^{\tau} + (h^{(pr)} - \Gamma) \bullet D. \tag{6.1.12}$$

Thanks to direct Ito's calculations, and (6.1.8), we derive

$$\begin{aligned} d\Gamma &= \frac{\Gamma}{\widetilde{G}} dD^{o,\mathbb{F}} + \mathcal{E}_-(G^{-1} \bullet D^{o,\mathbb{F}}) dY^{\mathbb{F}} \\ &= \frac{\Gamma - h^{(op)}}{\widetilde{G}} dD^{o,\mathbb{F}} - f(t)dt - \mathcal{E}_-(G^{-1} \bullet D^{o,\mathbb{F}}) dK^{\mathbb{F}} + Z^{\mathbb{F}} \mathcal{E}_-(G^{-1} \bullet D^{o,\mathbb{F}}) dW \\ &\quad + Z'^{\mathbb{F}} \mathcal{E}_-(G^{-1} \bullet D^{o,\mathbb{F}}) dN^{\mathbb{F}}. \end{aligned} \tag{6.1.13}$$

Then by inserting this latter equation in (6.1.12) and arranging terms we get

$$\begin{aligned} & dY^{\mathbb{G}} \\ &= -f(t)d(t \wedge \tau) - \mathcal{E}_-(G^{-1} \bullet D^{o,\mathbb{F}}) d(K^{\mathbb{F}})^{\tau} - \Gamma dN^{\mathbb{G}} + h^{(pr)} dD \\ &- \frac{h^{(op)}}{\widetilde{G}} dD^{o,\mathbb{F}} + \mathcal{E}_-(G^{-1} \bullet D^{o,\mathbb{F}}) Z^{\mathbb{F}} dW^{\tau} + \mathcal{E}_-(G^{-1} \bullet D^{o,\mathbb{F}}) Z'^{\mathbb{F}} d(N^{\mathbb{F}})^{\tau} \\ =& - f(t)d(t \wedge \tau) - \mathcal{E}_-(G^{-1} \bullet D^{o,\mathbb{F}}) d(K^{\mathbb{F}})^{\tau} - (\Gamma - h^{(op)}) dN^{\mathbb{G}} \\ &+ (h^{(pr)} - h^{(op)}) dD + \mathcal{E}_-(G^{-1} \bullet D^{o,\mathbb{F}}) Z^{\mathbb{F}} dW^{\tau} + \mathcal{E}_-(G^{-1} \bullet D^{o,\mathbb{F}}) Z'^{\mathbb{F}} d(N^{\mathbb{F}})^{\tau} \\ =& - f(t)d(t \wedge \tau) - dK^{\mathbb{G}} - dM^{\mathbb{G}} + Z^{\mathbb{G}} dW^{\tau} + Z'^{\mathbb{G}} d(N^{\mathbb{F}})^{\tau}. \end{aligned} \tag{6.1.14}$$

This proves that the processes defined in (6.1.10) satisfy the first equation in (6.1.1). To prove the second condition in (6.1.1), it is enough to remark that we have

$$Y_t^{\mathbb{F}} \geq S_t^{\mathbb{F}} I_{\{t\ <T\}} + \xi^{\mathbb{F}} I_{\{t\ =T\}},$$

which implies that for any $t \in [0, T)$

$$\frac{Y_t^{\mathbb{F}}}{\mathcal{E}_t(-\widetilde{G}^{-1} \bullet D^{o,\mathbb{F}})} I_{\{t\ <\tau\}} \geq S_t I_{\{t\ <\tau\}}.$$

This is obviously equivalent to the second condition of (6.1.1)). To prove the Skorokhod condition (the last condition in (6.1.1)), we use the Skorokhod condition for the quadruple $(Y^{\mathbb{F}}, Z^{\mathbb{F}}, Z'^{\mathbb{F}}, K^{\mathbb{F}})$ (as it is the solution to the RBSDE (6.1.8) with the data triplet $(f^{\mathbb{F}}, S^{\mathbb{F}}, \xi^{\mathbb{F}})$ defined in (6.1.7)) given by

$$\int_0^T (Y_{t-}^{\mathbb{F}} - S_{t-}^{\mathbb{F}}) dK_t^{\mathbb{F}} = 0, \quad P\text{-a.s.}. \tag{6.1.15}$$

As $Y_-^{\mathbb{G}} - S_- \geq 0$ on $\rrbracket 0, \tau \rrbracket$ and $K^{\mathbb{G}}$ is an increasing process, we get

$$\begin{aligned}
\int_0^T (Y_{t-}^{\mathbb{G}} - S_{t-}) dK_t^{\mathbb{G}} \quad &= \int_0^{T\wedge\tau} (Y_{t-}^{\mathbb{F}} - S_{t-}^{\mathbb{F}}) \mathcal{E}_{t-}(G^{-1} \bullet D^{o,\mathbb{F}})^2 dK_t^{\mathbb{F}} \\
&\leq \int_0^T (Y_{t-}^{\mathbb{F}} - S_{t-}^{\mathbb{F}}) \mathcal{E}_{t-}(G^{-1} \bullet D^{o,\mathbb{F}})^2 dK_t^{\mathbb{F}} = 0, \quad P\text{-a.s.}.
\end{aligned}$$

It is clear that the last equality is equivalent to (6.1.15) due the fact that $K^{\mathbb{F}}$ is nondecreasing and $Y_-^{\mathbb{F}} - S_-^{\mathbb{F}} \geq 0$ . This ends the second step. $\square$

## 6.2 The case of general generator $f$

We finally prove the existence and uniqueness to the solution of RBSDE with general generator and a filtration $\mathbb{F} = \mathbb{F}^{(N,W)}$ is generated by the brownian motion $W$ and the poisson process $N$. In addition to the settings that is given by Section 6.1, we assume that there exists a positive constant $C_{Lip}$ such that

$$|f(t,y_1,z_1,z_1^{'})-f(t,y_2,z_2,z_2^{'})|\leq C_{Lip}(|y_1-y_2|+\|z_1-z_2\|+\sqrt{\lambda}\|z_1^{'}-z_2^{'}\|), \tag{6.2.1}$$

for all $y_1, y_2 \in \mathbb{R}$ ,$z_1, z_2 \in \mathbb{R}^d$ and $z_1^{'}, z_2^{'} \in \mathbb{R}^l$.

In this section, we are interested in the following RBDES,

$$\begin{cases} dY_t = -f(t,Y_t,Z_t,Z_t^{'})d(t\wedge\tau) - d(K_{t\wedge\tau}+M_{t\wedge\tau}) \\ \qquad\qquad +Z_t dW_t^{\tau} + Z_t^{'} d(N_t^{\mathbb{F}})^{\tau}, \\ Y_{\tau}=\xi=Y_T, \quad Y_t\geq S_t \quad ;0\leq t<T\wedge\tau, \\ E\left[\int_0^{T\wedge\tau}(Y_{t-}-S_{t-})dK_t\right]=0, \end{cases} \tag{6.2.2}$$

where $(\xi, S, f)$ is such that $S$ is $\mathbb{F}^{(N,W)}$-adapted and RCLL process, $f(t,y,z,z^{'})$ is a $\text{Prog}(\mathbb{F})\times\mathcal{B}(\mathbb{R})\times\mathcal{B}(\mathbb{R}^d)\times\mathcal{B}(\mathbb{R}^l\setminus\{0\})$-measurable that satisfies the assumption (6.2.1), and $\xi\in L^2(\mathcal{G}_{T\wedge\tau})$.

### 6.2.1 Important estimates for the solution

This subsection derives a number of norm-estimates for the solution of the RBSDEs when this exists. These inequalities play important role in the proof of the existence of uniqueness of the solution of this RBSDE on the one hand. On the other hand, the role of these estimates in studying the stability of RBSDEs is without reproach.

Recall that as $\lambda\bullet(t\wedge\tau)$ is the compensator for $N^{\tau}$ in $(\mathbb{G},\widetilde{Q})$, then for any predictable process $H$ that is bounded or nonnegative, we have

$$E^{\widetilde{Q}}\left[\int_0^{T\wedge\tau} HdN\right] = E^{\widetilde{Q}}\left[\int_0^{T\wedge\tau} H\lambda ds\right].$$

**Lemma 6.2.1.** *The following assertions hold.*

(a) *Let ($Y^{\mathbb{G}}, Z^{\mathbb{G}}, Z'^{\mathbb{G}}, K^{\mathbb{G}}, M^{\mathbb{G}}$) be a class-(D)-($\mathbb{G}, \widetilde{Q}, T\wedge\tau$)-solution to the RBSDEs (6.2.2) that corresponds to $(f, S, \xi)$. Then for any $t \in [0, T]$, we have*

$$Y_t^{\mathbb{G}} = \operatorname*{ess\,sup}_{\theta\in\mathcal{T}_{t\wedge\tau}^{T\wedge\tau}} \mathrm{E}^{\widetilde{Q}}\left[\int_{t\wedge\tau}^{\theta\wedge\tau} f(s, Y_s^{\mathbb{G}}, Z_s^{\mathbb{G}}, Z_s'^{\mathbb{G}})ds + S_\theta I_{\{\theta<T\wedge\tau\}} + \xi I_{\{\theta=T\wedge\tau\}} \,\middle|\, \mathcal{G}_t\right]. \tag{6.2.3}$$

(b) *Let ($Y_i^{\mathbb{G}}, Z_i^{\mathbb{G}}, Z_i'^{\mathbb{G}}, K_i^{\mathbb{G}}, M_i^{\mathbb{G}}$) be a class-(D)-($\mathbb{G}, \widetilde{Q}, T\wedge\tau$)-solution to the RBSDEs (6.2.2) that corresponds to $(f_i, S_i, \xi^i)$, $i = 1, 2$ respectively. Then for any $\alpha > 0$, the following holds*

$$\begin{aligned}
&\exp\left(\frac{\alpha(t\wedge\tau)}{2}\right)|\delta Y_t^{\mathbb{G}}| \\
&\leq E^{\widetilde{Q}}\left[\sup_{0<s\leq T\wedge\tau} e^{\alpha s/2}|\delta S_s| + e^{\alpha(T\wedge\tau)/2}|\delta\xi| + \frac{C_{Lip}}{\sqrt{\alpha}}\sqrt{\int_0^{T\wedge\tau} e^{\alpha s}(\delta Z_s^{\mathbb{G}})^2 ds} \,\middle|\, \mathcal{G}_t\right] \\
&\quad + E^{\widetilde{Q}}\left[\frac{C_{Lip}}{\sqrt{\alpha}}\sqrt{\int_0^{T\wedge\tau} e^{\alpha s}(\delta Z_s'^{\mathbb{G}})^2 dN} + \frac{C_{Lip}}{\sqrt{\alpha}}\sqrt{\int_0^{T\wedge\tau} e^{\alpha s}|\delta Y_s^{\mathbb{G}}|^2 ds} \,\middle|\, \mathcal{G}_t\right] \\
&\quad + E^{\widetilde{Q}}\left[\sqrt{\int_0^{T\wedge\tau} e^{\alpha s}|\delta f_s|^2 ds} \,\middle|\, \mathcal{G}_t\right].
\end{aligned} \tag{6.2.4}$$

*Proof.* The proof of this lemma follows the same footsteps of proving Lemma

5.1.1, by choosing $p = q = 2$ and observing that

$$\Delta f_t := f_1(t, Y_t^{\mathbb{G},1}, Z_t^{\mathbb{G},1}, Z_t'^{\mathbb{G},1}) - f_2(t, Y_t^{\mathbb{G},2}, Z_t^{\mathbb{G},2}, Z_t'^{\mathbb{G},2}),$$

and that $|\Delta f_s| \leq |\delta f_s| + C_{Lip}|\delta Y_s^{\mathbb{G}}| + C_{Lip}|\delta Z_s^{\mathbb{G}}| + C_{Lip}\sqrt{\lambda}|\delta Z'^{\mathbb{G}}|_s$, and noting that

$$\left(\int_{t\wedge\tau}^{T\wedge\tau} e^{\alpha s/2}\delta Z_s'^{\mathbb{G}} dN^{\mathbb{F}}\right)^2 = \left(\int_{t\wedge\tau}^{T\wedge\tau} e^{\alpha s}(\delta Z_s'^{\mathbb{G}})^2 dN\right).$$

□

**Theorem 6.2.2.** *Let $(Y^{\mathbb{G}}, Z^{\mathbb{G}}, Z'^{\mathbb{G}}, M^{\mathbb{G}}, K^{\mathbb{G}})$ be a class-(D)-$(\mathbb{G}, \widetilde{Q}, T\wedge\tau)$-solution to the RBSDE (6.2.2) for the triplet $(f, S, \xi)$. Then for $\alpha$ big enough, there exists a positive constant $C := C(\alpha, T)$ that depends on $(\alpha, T)$ only such that*

$$\begin{aligned}
&E^{\widetilde{Q}}\left[\sup_{0\leq s\leq T\wedge\tau} e^{\alpha s}|Y_s^{\mathbb{G}}|^2 + \left(\int_0^{T\wedge\tau} e^{\alpha t/2} dK_t^{\mathbb{G}}\right)^2 + \int_0^{T\wedge\tau} e^{\alpha s}|Z_s^{\mathbb{G}}|^2 ds\right] \\
&+ E^{\widetilde{Q}}\left[\int_0^{T\wedge\tau} e^{\alpha s}(Z_s'^{\mathbb{G}})^2 dN + \int_0^{T\wedge\tau} e^{\alpha s} d[M^{\mathbb{G}}, M^{\mathbb{G}}]_s\right] \\
&\leq C E^{\widetilde{Q}}\left[\xi^2 + \int_0^{T\wedge\tau} e^{\alpha s}|f(s,0,0,0)|^2 ds + \sup_{0\leq s\leq T\wedge\tau} e^{\alpha s}(S_s^+)^2\right].
\end{aligned}$$

*Proof.* Thanks to the BSDE (6.2.2) and Ito's formula, we get

$$\begin{aligned}
&\int_0^{T\wedge\tau} e^{\alpha\frac{t}{2}} dK_t^{\mathbb{G}} \\
&= -\int_0^{T\wedge\tau} e^{\alpha\frac{t}{2}} dY_t^{\mathbb{G}} - \int_0^{T\wedge\tau} e^{\alpha\frac{t}{2}} f(t, Y_t^{\mathbb{G}}, Z_t^{\mathbb{G}}, Z_t'^{\mathbb{G}})dt - \int_0^{T\wedge\tau} e^{\alpha\frac{t}{2}} dM_t^{\mathbb{G}} \\
&\quad + \int_0^{T\wedge\tau} e^{\alpha\frac{t}{2}} Z_t^{\mathbb{G}} dW_t^{\tau} + \int_0^{T\wedge\tau} e^{\alpha\frac{t}{2}} Z_t'^{\mathbb{G}} dN_t^{\mathbb{F}}
\end{aligned}$$

$$\overset{Ito}{=} -e^{\alpha\frac{T}{2}}Y_T^{\mathbb{G}} + Y_0^{\mathbb{G}} + \int_0^{T\wedge\tau} e^{\alpha\frac{t}{2}}(\frac{\alpha}{2}Y_t^{\mathbb{G}} - f(t, Y_t^{\mathbb{G}}, Z_t^{\mathbb{G}}, Z_t^{'\mathbb{G}}))dt - \int_0^{T\wedge\tau} e^{\alpha\frac{t}{2}}dM_t^{\mathbb{G}}$$
$$+ \int_0^{T\wedge\tau} e^{\alpha\frac{t}{2}}Z_t^{\mathbb{G}}dW_t + \int_0^{T\wedge\tau} e^{\alpha\frac{t}{2}}Z_t^{'\mathbb{G}}dN_t^{\mathbb{F}}$$

then

$$\begin{aligned}
&E^{\widetilde{Q}}\left[\int_0^{T\wedge\tau} e^{\alpha\frac{t}{2}}dK_t^{\mathbb{G}}\big|\mathcal{G}_{t\wedge\tau}\right]\\
&\leq E^{\widetilde{Q}}\left[(2+T)\sup_{0\leq s\leq T\wedge\tau} e^{\alpha s/2}(Y_s^{\mathbb{G}}) + \int_0^{T\wedge\tau} e^{\alpha\frac{t}{2}}|f(t, Y_t^{\mathbb{G}}, Z_t^{\mathbb{G}}, Z_t^{'\mathbb{G}})|dt\big|\mathcal{G}_{t\wedge\tau}\right]\\
&\leq E^{\widetilde{Q}}\left[(2+T)\sup_{0\leq s\leq T\wedge\tau} e^{\alpha s/2}(Y_s^{\mathbb{G}})\big|\mathcal{G}_{t\wedge\tau}\right]\\
&\qquad + C_{Lip}E^{\widetilde{Q}}\left[\int_0^{T\wedge\tau} e^{\alpha\frac{t}{2}}(|f(t,0,0,0)| + |Y_t^{\mathbb{G}}| + |Z_t^{\mathbb{G}}| + \sqrt{\lambda}|Z_t^{'\mathbb{G}}|)dt\big|\mathcal{G}_{t\wedge\tau}\right]\\
&\leq E^{\widetilde{Q}}\left[(2+2T)\sup_{0\leq s\leq T\wedge\tau} e^{\alpha s/2}(Y_s^{\mathbb{G}})\big|\mathcal{G}_{t\wedge\tau}\right]\\
&\qquad + C_{Lip}E^{\widetilde{Q}}\left[\int_0^{T\wedge\tau} e^{\alpha\frac{t}{2}}(|f(t,0,0,0)| + |Z_t^{\mathbb{G}}| + \sqrt{\lambda}|Z_t^{'\mathbb{G}}|)dt\big|\mathcal{G}_{t\wedge\tau}\right].
\end{aligned}$$

Therefore, by using this latter equality together with Theorem 2.1.23 and $(\sum_{i=1}^n x_i)^2 \leq n\sum_{i=1}^n x_i^2$, we derive

$$\begin{aligned}
E^{\widetilde{Q}}\left[\left(\int_0^{T\wedge\tau} e^{\alpha\frac{t}{2}}dK_t^{\mathbb{G}}\right)^2\right] &\leq C_T\left(\|e^{\alpha\frac{t}{2}}Y^{\mathbb{G}}\|^2_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},2)} + \|e^{\alpha\frac{t}{2}}f(t,0,0,0)\|^2_{\mathbb{S}_{T\wedge\tau}(\widetilde{Q},2)}\right)\\
&\quad + C_T\left(\|e^{\alpha\frac{t}{2}}Z^{\mathbb{G}}\|^2_{\mathbb{S}_{T\wedge\tau}(\widetilde{Q},2)} + \|e^{\alpha\frac{t}{2}}Z^{'\mathbb{G}}\|^2_{\mathbb{L}^2_{T\wedge\tau}(N,\widetilde{Q})}\right).
\end{aligned} \tag{6.2.5}$$

On the one hand, applying (6.2.4) by putting

$$(f_1, S_1, \xi^1) \equiv (0,0,0) \quad \text{and} \quad (f_2, S_2, \xi^2) \equiv (f, S, \xi),$$

and then using Doob's inequality under $(\widetilde{Q}, \mathbb{G})$, we have that

$$\begin{aligned}
&\|e^{\alpha\cdot/2}Y^{\mathbb{G}}\|_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},2)}\\
&\leq C_{DB}\left\{\|e^{\alpha\cdot/2}S^{+}\|_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},2)} + \|e^{\alpha(T\wedge\tau)/2}\xi\|_{L^2(\widetilde{Q})}\right\}\\
&+ \frac{C_{DB}}{\sqrt{\alpha}}\|e^{\alpha\cdot/2}f(\cdot,0,0,0)\|_{\mathbb{S}_{T\wedge\tau}(\widetilde{Q},2)}\\
&+ \frac{C_{DB}C_{Lip}}{\sqrt{\alpha}}\left\{\|e^{\alpha\cdot/2}Z^{\mathbb{G}}\|_{\mathbb{S}_{T\wedge\tau}(\widetilde{Q},2)} + \|e^{\alpha\cdot/2}Z'^{\mathbb{G}}\|_{\mathbb{L}^2_{T\wedge\tau}(N,\widetilde{Q})} + \|e^{\alpha\cdot/2}Y^{\mathbb{G}}\|_{\mathbb{S}_{T\wedge\tau}(\widetilde{Q},2)}\right\}.
\end{aligned} \tag{6.2.6}$$

On the other hand, by combining Itô applied to $e^{\alpha t}(Y^{\mathbb{G}}_t)^2$, (6.2.2), and Young's inequality (i.e. $2xy \leq \epsilon x^2 + y^2/\epsilon$ for any $\epsilon > 0$), we derive

$$\begin{aligned}
&\overbrace{(\alpha - 2C_{Lip} - 2C^2_{Lip} - \epsilon^{-1})}^{C}\int_{t\wedge\tau}^{T\wedge\tau} e^{\alpha s}(Y^{\mathbb{G}}_s)^2 ds + \frac{1}{2}\int_{t\wedge\tau}^{T\wedge\tau} e^{\alpha s}(Z^{\mathbb{G}}_s)^2 ds\\
&+ \frac{1}{2}\int_{t\wedge\tau}^{T\wedge\tau} e^{\alpha s}|Z'^{\mathbb{G}}_s|^2 dN + \int_{t\wedge\tau}^{T\wedge\tau} e^{\alpha t} d[M^{\mathbb{G}}, M^{\mathbb{G}}]_s
\end{aligned}$$

$$\begin{aligned}
&\leq e^{\alpha(T\wedge\tau)}\xi^2 + \epsilon\int_{t\wedge\tau}^{T\wedge\tau} e^{\alpha s}|f(s,0,0,0)|^2 ds + 2\int_{t\wedge\tau}^{T\wedge\tau} e^{\alpha s}Y^{\mathbb{G}}_{s-} dK^{\mathbb{G}}_s\\
&+ 2\int_{t\wedge\tau}^{T\wedge\tau} e^{\alpha s}(Y^{\mathbb{G}}_{s-} - \Delta K^{\mathbb{G}}_s)) dM^{\mathbb{G}} + 2\int_{t\wedge\tau}^{T\wedge\tau} e^{\alpha s}(Y^{\mathbb{G}}_{s-}) Z^{\mathbb{G}}_s dW\\
&+ 2\int_{t\wedge\tau}^{T\wedge\tau} e^{\alpha s}(Y^{\mathbb{G}}_{s-}) Z'^{\mathbb{G}}_s dN^{\mathbb{F}}
\end{aligned}$$

$$\begin{aligned}
&\leq e^{\alpha(T\wedge\tau)}\xi^2 + \epsilon\int_{t\wedge\tau}^{T\wedge\tau} e^{\alpha s}|f(s,0,0,0)|^2 ds + 2\int_{t\wedge\tau}^{T\wedge\tau} e^{\alpha s}S^{+}_{s-} dK^{\mathbb{G}}_s\\
&+ 2\int_{t\wedge\tau}^{T\wedge\tau} e^{\alpha s}(Y^{\mathbb{G}}_{s-} - \Delta K^{\mathbb{G}}_s)) dM^{\mathbb{G}} + 2\int_{t\wedge\tau}^{T\wedge\tau} e^{\alpha s}(Y^{\mathbb{G}}_{s-}) Z^{\mathbb{G}}_s dW
\end{aligned}$$

$$+2\int_{t\wedge\tau}^{T\wedge\tau} e^{\alpha s}(Y^{\mathbb{G}}_{s-})Z^{'\mathbb{G}}_s dN^{\mathbb{F}}, \tag{6.2.7}$$

where the last equality is due to the Skorokhod's condition. Thus, by taking the conditional expectation for both sides of the above inequality, and then by combining the result with (6.2.6) and (6.2.5), and noting that $\alpha$ is big we derive the following inequality

$$\begin{aligned}
&\|e^{\alpha\cdot/2}Y^{\mathbb{G}}\|_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},2)} + C_1\|e^{\alpha\cdot/2}Y^{\mathbb{G}}\|_{\mathbb{S}_{T\wedge\tau}(\widetilde{Q},2)}\\
&+C_2\|e^{\alpha\cdot/2}Z^{\mathbb{G}}\|_{\mathbb{S}_{T\wedge\tau}(\widetilde{Q},2)} + C_2^{'}\|e^{\alpha\cdot/2}Z^{'\mathbb{G}}\|_{\mathbb{L}^2_{T\wedge\tau}(N,\widetilde{Q})} + C_3\|e^{\alpha\cdot/2}\bullet M^{\mathbb{G}}\|_{\mathcal{M}^2_T(\widetilde{Q})}\\
&\le C_4\|e^{\alpha\cdot/2}f(\cdot,0,0,0)\|_{\mathbb{S}_{T\wedge\tau}(\widetilde{Q},2)} + C_5\|\xi\|_{\mathbb{L}^2(\widetilde{Q})} + C_6\|e^{\alpha\cdot/2}S^+\|_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},2)}\\
&+C_7\|e^{\alpha/2\cdot}S^+\|^{1/2}_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},2)}\|e^{\alpha/2}\bullet K^{\mathbb{G}}_T\|^{1/2}_{\mathbb{L}^2(\widetilde{Q})},\\
&\le C_4\|e^{\alpha\cdot/2}f(\cdot,0,0,0)\|_{\mathbb{S}_{T\wedge\tau}(\widetilde{Q},2)} + C_5\|\xi\|_{\mathbb{L}^2(\widetilde{Q})} + C_6\|e^{\alpha\cdot/2}S^+\|_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},2)}\\
&+\epsilon_2 C_7\|e^{\alpha/2\cdot}S^+\|_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},2)} + \frac{C_7}{\epsilon_2}\|e^{\alpha/2}\bullet K^{\mathbb{G}}_T\|_{\mathbb{L}^2(\widetilde{Q})},\\
&\le C_4\|e^{\frac{\alpha}{2}\cdot}f(\cdot,0,0,0)\|_{\mathbb{S}_{T\wedge\tau}(\widetilde{Q},2)} + C_5\|\xi\|_{\mathbb{L}^2(\widetilde{Q})} + (C_6+\epsilon_2C_7)\|e^{\frac{\alpha}{2}\cdot}S^+\|_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},2)}\\
&+\frac{C_7C_T}{\epsilon_2}\left(\|e^{\alpha\frac{t}{2}}Y^{\mathbb{G}}\|^2_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},2)} + \|e^{\alpha\frac{t}{2}}f(t,0,0,0)\|^2_{\mathbb{S}_{T\wedge\tau}(\widetilde{Q},2)} + \|e^{\alpha\frac{t}{2}}Z^{\mathbb{G}}\|^2_{\mathbb{S}_{T\wedge\tau}(\widetilde{Q},2)}\right)\\
&+\frac{C_7C_T}{\epsilon_2}\left(\|e^{\alpha\frac{t}{2}}Z^{'\mathbb{G}}\|^2_{\mathbb{L}^2_{T\wedge\tau}(N,\widetilde{Q})}\right)
\end{aligned} \tag{6.2.8}$$

The proof is done by choosing appropriate $\epsilon_2$. $\square$

**Theorem 6.2.3.** *Let $(Y^{\mathbb{G},i}, Z^{\mathbb{G},i}, Z^{'\mathbb{G},i}, K^{\mathbb{G},i}, M^{\mathbb{G},i})$ be a class-(D)-$(\mathbb{G},\widetilde{Q},T\wedge\tau)$-solution to the RBSDE (6.2.2) that corresponds to $(f_i,S_i,\xi^i)$, $i=1,2$ respectively. Then for any $\alpha$ such that $\alpha\ge 1+4C_{Lip}+8C^2_{Lip}$, there exist positive constants $\widehat{C}_i := \widehat{C}_i(\alpha)$, $i=1,..,4$ that depend on $(\alpha)$ only such that $\lim_{\alpha\to\infty}\widehat{C}_1 = 0$*

*and*

$$\begin{aligned}
&\|e^{\alpha\cdot/2}\delta Y^{\mathbb{G}}\|_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},2)}+\|e^{\alpha\cdot/2}\delta Y^{\mathbb{G}}\|_{\mathbb{S}_{T\wedge\tau}(\widetilde{Q},2)}+\|e^{\alpha\cdot/2}\delta Z^{\mathbb{G}}\|_{\mathbb{S}_{T\wedge\tau}(\widetilde{Q},2)}\\
&+\|\delta Z'^{\mathbb{G}}\|_{\mathbb{L}^2_{T\wedge\tau}(N,\widetilde{Q})}+\|e^{\alpha(\tau\wedge\cdot)/2}\bullet\delta M^{\mathbb{G}}\|_{\mathcal{M}^2(\widetilde{Q})}\\
&\leq \widehat{C}_1\|e^{\alpha\cdot/2}\delta f\|_{\mathbb{S}_{T\wedge\tau}(\widetilde{Q},2)}+\widehat{C}_2\|e^{\alpha(T\wedge\tau)/2}\delta\xi\|_{\mathbb{L}^2(\widetilde{Q})}\\
&+\widehat{C}_3\|e^{\alpha\cdot/2}\delta S\|_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},2)}+\widehat{C}_4\sqrt{\|e^{\alpha\cdot/2}\delta S\|_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},2)}\left\{\sum_{i=1}^{2}\Delta_i\right\}}. \qquad (6.2.9)
\end{aligned}$$

*Here* $(\delta Y^{\mathbb{G}},\delta Z^{\mathbb{G}},\delta Z'^{\mathbb{G}},\delta M^{\mathbb{G}},\delta K^{\mathbb{G}})$ *and* $(\delta f,\delta S,\delta\xi)$ *are given by*

$$\begin{aligned}
&\delta Y^{\mathbb{G}}:=Y^{\mathbb{G},1}-Y^{\mathbb{G},2},\ \delta Z^{\mathbb{G}}:=Z^{\mathbb{G},1}-Z^{\mathbb{G},2},\delta Z'^{\mathbb{G}}:=Z'^{\mathbb{G},1}-Z'^{\mathbb{G},2},\\
&\delta M^{\mathbb{G}}:=M^{\mathbb{G},1}-M_2^{\mathbb{G},2},\delta K^{\mathbb{G}}:=K^{\mathbb{G},1}-K^{\mathbb{G},2},\delta S:=S^1-S^2,\\
&\delta\xi:=\xi^1-\xi^2,\quad \delta f_t:=f_1(t,Y_t^{\mathbb{G},1},Z_t^{\mathbb{G},1},Z_t'^{\mathbb{G},1})-f_2(t,Y_t^{\mathbb{G},1},Z_t^{\mathbb{G},1},Z_t'^{\mathbb{G},1}).
\end{aligned}$$

*and*

$$\begin{aligned}
&\Delta_i\\
&:=\|e^{\alpha(T\wedge\tau)/2}\xi^{(i)}\|_{L^2(\widetilde{Q})}+\|e^{\alpha(\tau\wedge\cdot)}(S^{(i)})^+\|_{\mathbb{S}_T(\widetilde{Q},2)}+\|e^{\alpha(\tau\wedge\cdot)/2}f^{(i)}(\cdot,0,0,0)\|_{\mathbb{S}_T(\widetilde{Q},2)}.
\end{aligned}$$

*Proof.* We start by the following simple remark that is due to the Lipschitz assumption on $f$

$$\begin{aligned}
|\Delta f_t|&:=|f_1(t,Y^{\mathbb{G},1},Z^{\mathbb{G},1},Z'^{\mathbb{G},1})-f_2(t,Y^{\mathbb{G},2},Z^{\mathbb{G},2},Z'^{\mathbb{G},2})|\\
&\leq|\delta f_t|+C_{Lip}|\delta Y_t^{\mathbb{G}}|+C_{Lip}|\delta Z_t^{\mathbb{G}}|+C_{Lip}\sqrt{\lambda}|\delta Z_t'^{\mathbb{G}}|. \qquad (6.2.10)
\end{aligned}$$

On the other hand, by using Lemma 6.2.1-(b) and Doob's inequality, we get

$$\begin{aligned}
&\|e^{\alpha\cdot/2}\delta Y^{\mathbb{G}}\|_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},2)}\\
&\leq C_{DB}\left\{\|e^{\frac{\alpha\cdot}{2}}\delta S\|_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},2)}+\|e^{\frac{\alpha}{2}(T\wedge\tau)}\delta\xi\|_{\mathbb{L}^2(\widetilde{Q})}+\frac{1}{\sqrt{\alpha}}\|e^{\frac{\alpha\cdot}{2}}\delta f\|_{\mathbb{S}_{T\wedge\tau}(\widetilde{Q},2)}\right\}\\
&+\frac{C_{DB}C_{Lip}}{\sqrt{\alpha}}\left\{\|e^{\frac{\alpha}{2}\cdot}\delta Z^{\mathbb{G}}\|_{\mathbb{S}_{T\wedge\tau}(\widetilde{Q},2)}+\|e^{\frac{\alpha}{2}\cdot}\delta Z'^{\mathbb{G}}\|_{\mathbb{L}^2_{T\wedge\tau}(N,\widetilde{Q})}+\|e^{\alpha\cdot/2}\delta Y^{\mathbb{G}}\|_{\mathbb{S}_{T\wedge\tau}(\widetilde{Q},2)}\right\}.
\end{aligned}\tag{6.2.11}$$

By combining Itô applied to $e^{\alpha t}(\delta Y_t^{\mathbb{G}})^2$, $(\delta Y_0^{\mathbb{G}})^2\geq 0$, and putting

$$\begin{aligned}
L^{\mathbb{G}}:\ &= e^{\alpha(\tau\wedge\cdot)}(\delta Y_-^{\mathbb{G}}-2\Delta(\delta K_s^{\mathbb{G}}))\cdot\delta M^{\mathbb{G}}+e^{\alpha(\tau\wedge\cdot)}(\delta Y_-^{\mathbb{G}})\delta Z^{\mathbb{G}}\cdot W^{\tau}\\
&\qquad +e^{\alpha(\tau\wedge\cdot)}(\delta Y_-^{\mathbb{G}})\delta Z'^{\mathbb{G}}\cdot N^{\mathbb{F}},
\end{aligned}\tag{6.2.12}$$

which belongs to $\mathcal{M}_{loc}(\widetilde{Q},\mathbb{G})$, we derive

$$\begin{aligned}
&e^{\alpha t}(\delta Y_{t\wedge\tau}^{\mathbb{G}})^2\\
&=(\delta Y_0^{\mathbb{G}})^2+\alpha e^{\alpha t}(\delta Y_t^{\mathbb{G}})^2\bullet(t\wedge\tau)+2e^{\alpha t}\delta Y_{t-}^{\mathbb{G}}\bullet\delta Y_{t\wedge\tau}^{\mathbb{G}}+e^{\alpha t}\bullet[\delta Y^{\mathbb{G}},\delta Y^{\mathbb{G}}]_{t\wedge\tau}\\
&=(\delta Y_0^{\mathbb{G}})^2+\alpha\int_0^{t\wedge\tau}e^{\alpha s}(\delta Y_s^{\mathbb{G}})^2ds+2\int_0^{t\wedge\tau}e^{\alpha s}(\delta Y_{s-}^{\mathbb{G}})d(\delta Y_s^{\mathbb{G}})\\
&\qquad+\int_0^{t\wedge\tau}e^{\alpha s}(\delta Z_s^{\mathbb{G}})^2ds+\int_0^{t\wedge\tau}e^{\alpha s}(\delta Z_s'^{\mathbb{G}})^2dN\\
&\qquad+\int_0^{t\wedge\tau}e^{\alpha t}d[\delta(M^{\mathbb{G}}+K^{\mathbb{G}}),\delta(M^{\mathbb{G}}+K^{\mathbb{G}})]_s\\
&=(\delta Y_0^{\mathbb{G}})^2+\alpha\int_0^{t\wedge\tau}e^{\alpha s}(\delta Y_s^{\mathbb{G}})^2ds-2\int_0^{t\wedge\tau}e^{\alpha s}(\delta Y_s^{\mathbb{G}})\Delta f_s ds\\
&-2\int_0^{t\wedge\tau}e^{\alpha s}(\delta Y_{s-}^{\mathbb{G}})d\delta K_s^{\mathbb{G}}-2\int_0^{t\wedge\tau}e^{\alpha s}(\delta Y_{s-}^{\mathbb{G}})d\delta M_s^{\mathbb{G}}\\
&+2\int_0^{t\wedge\tau}e^{\alpha s}(\delta Y_{s-}^{\mathbb{G}})\delta Z'^{\mathbb{G}}dN_s^{\mathbb{F}}+\int_0^{t\wedge\tau}e^{\alpha s}(\delta Z_s^{\mathbb{G}})^2ds+\int_0^{t\wedge\tau}e^{\alpha s}(\delta Z_s'^{\mathbb{G}})^2dN
\end{aligned}$$

$$+ 2\int_0^{t\wedge\tau} e^{\alpha s}(\delta Y^{\mathbb{G}}_{s-})\delta Z^{\mathbb{G}} dW_s + \int_0^{t\wedge\tau} e^{\alpha t} d[\delta(M^{\mathbb{G}} + K^{\mathbb{G}}), \delta(M^{\mathbb{G}} + K^{\mathbb{G}})]_s$$

Now, by taking $t = T$ we have

$$e^{\alpha T}(\delta Y^{\mathbb{G}}_{T\wedge\tau})^2 = (\delta Y^{\mathbb{G}}_0)^2 + \alpha\int_0^{T\wedge\tau} e^{\alpha s}(\delta Y^{\mathbb{G}}_s)^2 ds - 2\int_0^{T\wedge\tau} e^{\alpha s}(\delta Y^{\mathbb{G}}_s)\Delta f ds$$
$$- 2\int_0^{T\wedge\tau} e^{\alpha s}(\delta Y^{\mathbb{G}}_{s-}) d\delta K^{\mathbb{G}}_s - 2\int_0^{T\wedge\tau} e^{\alpha s}(\delta Y^{\mathbb{G}}_{s-}) d\delta M^{\mathbb{G}}_s$$
$$+ 2\int_0^{T\wedge\tau} e^{\alpha s}(\delta Y^{\mathbb{G}}_{s-})\delta Z^{\mathbb{G}} dW_s + 2\int_0^{T\wedge\tau} e^{\alpha s}(\delta Y^{\mathbb{G}}_{s-})\delta Z'^{\mathbb{G}} dN^{\mathbb{F}}_s + \int_0^{T\wedge\tau} e^{\alpha s}(\delta Z^{\mathbb{G}}_s)^2 ds$$
$$+ \int_0^{T\wedge\tau} e^{\alpha s}(\delta Z'^{\mathbb{G}}_s)^2 dN + \int_0^{T\wedge\tau} e^{\alpha t} d[\delta(M^{\mathbb{G}} + K^{\mathbb{G}}), \delta(M^{\mathbb{G}} + K^{\mathbb{G}})]_s$$

therefore,

$$(\delta Y^{\mathbb{G}}_0)^2 + \alpha\int_0^{T\wedge\tau} e^{\alpha s}(\delta Y^{\mathbb{G}}_s)^2 ds + \int_0^{T\wedge\tau} e^{\alpha s}(\delta Z^{\mathbb{G}}_s)^2 ds + \int_0^{T\wedge\tau} e^{\alpha s}(\delta Z'^{\mathbb{G}}_s)^2 dN$$
$$+ \int_0^{T\wedge\tau} e^{\alpha t} d[\delta M^{\mathbb{G}}, \delta M^{\mathbb{G}}]_s = e^{\alpha T}(\delta Y^{\mathbb{G}}_{T\wedge\tau})^2 + 2\int_0^{T\wedge\tau} e^{\alpha s}(\delta Y^{\mathbb{G}}_s)\Delta f ds$$
$$+ 2\int_0^{T\wedge\tau} e^{\alpha s}(\delta Y^{\mathbb{G}}_{s-}) d\delta K^{\mathbb{G}}_s - [\delta K^{\mathbb{G}}, \delta K^{\mathbb{G}}]_{T\wedge\tau} + L^{\mathbb{G}}$$
$$\leq e^{\alpha T}(\delta Y^{\mathbb{G}}_{T\wedge\tau})^2 + 2\int_0^{T\wedge\tau} e^{\alpha s}|\delta Y^{\mathbb{G}}_s||\Delta f| ds + 2\int_0^{T\wedge\tau} e^{\alpha s}(\delta Y^{\mathbb{G}}_{s-}) d\delta K^{\mathbb{G}}_s + L^{\mathbb{G}}$$

hence,

$$(\delta Y^{\mathbb{G}}_0)^2 + \alpha\int_0^{T\wedge\tau} e^{\alpha s}(\delta Y^{\mathbb{G}}_s)^2 ds + \int_0^{T\wedge\tau} e^{\alpha s}(\delta Z^{\mathbb{G}}_s)^2 ds + \int_0^{T\wedge\tau} e^{\alpha s}(\delta Z'^{\mathbb{G}}_s)^2 dN$$
$$+ \int_0^{T\wedge\tau} e^{\alpha t} d[\delta M^{\mathbb{G}}, \delta M^{\mathbb{G}}]_s$$
$$\leq 2\int_0^{T\wedge\tau} e^{\alpha s}|\delta Y^{\mathbb{G}}_s|(|\delta f(s, Y^{\mathbb{G}}_1, Z^{\mathbb{G}}_1, Z'^{\mathbb{G}}_1)| + C_{Lip}|\delta Y^{\mathbb{G}}_s| + C_{Lip}|\delta Z^{\mathbb{G}}_s|) ds + L^{\mathbb{G}}$$

$$+ e^{\alpha T}(\delta Y^{\mathbb{G}}_{T\wedge\tau})^2 + 2\int_0^{T\wedge\tau} e^{\alpha s}|\delta Y^{\mathbb{G}}_s| C_{Lip}\sqrt{\lambda}|\delta Z^{'\mathbb{G}}_s|ds + 2\int_0^{T\wedge\tau} e^{\alpha s}(\delta Y^{\mathbb{G}}_{s-})d\delta K^{\mathbb{G}}_s$$

$$= e^{\alpha T}(\delta Y^{\mathbb{G}}_{T\wedge\tau})^2 + 2\int_0^{T\wedge\tau} e^{\alpha s}|\delta Y^{\mathbb{G}}_s||\delta f(s, Y^{\mathbb{G}}_1, Z^{\mathbb{G}}_1, Z^{'\mathbb{G}}_1)|ds$$

$$+ 2\int_0^{T\wedge\tau} e^{\alpha s}C_{Lip}|\delta Y^{\mathbb{G}}_s|^2 ds + 2\int_0^{T\wedge\tau} e^{\alpha s}C_{Lip}|\delta Y^{\mathbb{G}}_s||\delta Z^{\mathbb{G}}_s|ds +$$

$$2\int_0^{T\wedge\tau} e^{\alpha s}C_{Lip}|\delta Y^{\mathbb{G}}_s||\delta Z^{'\mathbb{G}}_s|\sqrt{\lambda}ds + 2\int_0^{T\wedge\tau} e^{\alpha s}(\delta Y^{\mathbb{G}}_{s-})d\delta K^{\mathbb{G}}_s + L^{\mathbb{G}}$$

$$\overset{Young}{\leq} e^{\alpha T}(\delta Y^{\mathbb{G}}_{T\wedge\tau})^2 + \frac{1}{\epsilon}\int_0^{T\wedge\tau} e^{\alpha s}|\delta Y^{\mathbb{G}}_s|^2 ds + \epsilon\int_0^{T\wedge\tau} e^{\alpha s}|\delta f(s, Y^{\mathbb{G}}_1, Z^{\mathbb{G}}_1, Z^{'\mathbb{G}}_1)|^2 ds$$

$$+ 2\int_0^{T\wedge\tau} e^{\alpha s}C_{Lip}|\delta Y^{\mathbb{G}}_s|^2 ds + 2C^2_{Lip}\int_0^{T\wedge\tau} e^{\alpha s}|\delta Y^{\mathbb{G}}_s|^2 ds + \frac{1}{2}\int_0^{T\wedge\tau} e^{\alpha s}|\delta Z^{\mathbb{G}}_s|^2 ds$$

$$+ 2C^2_{Lip}\int_0^{T\wedge\tau} e^{\alpha s}|\delta Y^{\mathbb{G}}_s|^2 ds + \frac{1}{2}\int_0^{T\wedge\tau} e^{\alpha s}|\delta Z^{'\mathbb{G}}_s|^2\lambda ds + 2\int_0^{T\wedge\tau} e^{\alpha s}(\delta Y^{\mathbb{G}}_{s-})d\delta K^{\mathbb{G}}_s$$

$$+ L^{\mathbb{G}}$$

therefore

$$(\delta Y^{\mathbb{G}}_0)^2 + \overbrace{(\alpha - 2C_{Lip} - 4C^2_{Lip} - \frac{1}{\epsilon})}^{C}\int_0^{T\wedge\tau} e^{\alpha s}(\delta Y^{\mathbb{G}}_s)^2 ds + \frac{1}{2}\int_0^{T\wedge\tau} e^{\alpha s}(\delta Z^{\mathbb{G}}_s)^2 ds$$

$$+ \int_0^{T\wedge\tau} e^{\alpha s}(\delta Z^{'\mathbb{G}}_s)^2 dN + \frac{1}{2}\int_0^{T\wedge\tau} e^{\alpha s}|\delta Z^{'\mathbb{G}}_s|^2\lambda ds + \int_0^{T\wedge\tau} e^{\alpha t}d[\delta M^{\mathbb{G}}, \delta M^{\mathbb{G}}]_s$$

$$\leq e^{\alpha T}(\delta Y^{\mathbb{G}}_{T\wedge\tau})^2 + \int_0^{T\wedge\tau} e^{\alpha s}\epsilon|\delta f(s, Y^{\mathbb{G}}_1, Z^{\mathbb{G}}_1, Z^{'\mathbb{G}}_1)|^2 ds + 2\int_0^{T\wedge\tau} e^{\alpha s}(\delta Y^{\mathbb{G}}_{s-})d\delta K^{\mathbb{G}}_s$$

$$+ L^{\mathbb{G}}$$

$$\overset{\text{Skorohod}}{\leq} e^{\alpha T}(\delta Y^{\mathbb{G}}_{T\wedge\tau})^2 + \int_0^{T\wedge\tau} e^{\alpha s}\epsilon|\delta f(s, Y^{\mathbb{G}}_1, Z^{\mathbb{G}}_1, Z^{'\mathbb{G}}_1)|^2 ds$$

$$+ 2\int_0^{T\wedge\tau} e^{\alpha s}|\delta S_{s-}|d\operatorname*{Var}_s(\delta K^{\mathbb{G}}_s) + L^{\mathbb{G}}$$

$$= e^{\alpha T}(\delta\xi)^2 + \int_0^{T\wedge\tau} e^{\alpha s}\epsilon|\delta f(s, Y^{\mathbb{G}}_1, Z^{\mathbb{G}}_1, Z^{'\mathbb{G}}_1)|^2 ds + 2\int_0^{T\wedge\tau} e^{\alpha s}|\delta S_{s-}|d\operatorname*{Var}_s(\delta K^{\mathbb{G}}_s)$$

$$+ L^{\mathbb{G}}, \tag{6.2.13}$$

Therefore, by taking the expected value for both sides of 6.2.13), we derive

$$\begin{aligned}
&C\|e^{\alpha\cdot/2}\delta Y^{\mathbb{G}}\|_{\mathbb{S}_{T\wedge\tau}(\widetilde{Q},p)} + 2^{-1}\|e^{\alpha\cdot/2}\delta Z^{\mathbb{G}}\|_{\mathbb{S}_{T\wedge\tau}(\widetilde{Q},p)} + 2^{-1}\|e^{\alpha\cdot/2}\delta Z'^{\mathbb{G}}\|_{\mathbb{E}_{T\wedge\tau}(\widetilde{Q},p)}\\
&+ \|e^{\alpha(\tau\wedge\cdot)/2}\cdot\delta M^{\mathbb{G}}\|_{\mathcal{M}^p_T(\widetilde{Q})} \leq \epsilon\|e^{\alpha\cdot/2}\delta f\|_{\mathbb{S}_{T\wedge\tau}(\widetilde{Q},p)} + \|e^{\alpha(T\wedge\tau)/2}\delta\xi\|_{\mathbb{L}^p(\widetilde{Q})}\\
&+ 2\|e^{\alpha\cdot/2}\delta S\|^{1/2}_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},p)}\|\mathrm{Var_T}(\mathrm{e}^{\alpha\cdot/2}\cdot\delta\mathrm{K}^{\mathbb{G}})\|^{1/2}_{\mathbb{L}^p(\widetilde{\mathrm{Q}})}
\end{aligned}$$

Then by combining this equality with (6.2.11) we obtain

$$\begin{aligned}
&\|e^{\alpha\cdot/2}\delta Y^{\mathbb{G}}\|_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},p)} + \left(C - \frac{C_{DB}C_{Lip}}{\sqrt{\alpha}}\right)\|e^{\alpha\cdot/2}\delta Y^{\mathbb{G}}\|_{\mathbb{S}_{T\wedge\tau}(\widetilde{Q},p)}\\
&+ \left(\frac{1}{2} - \frac{C_{DB}C_{Lip}}{\sqrt{\alpha}}\right)\|e^{\alpha\cdot/2}\delta Z^{\mathbb{G}}\|_{\mathbb{S}_{T\wedge\tau}(\widetilde{Q},p)} + \|e^{\alpha(\tau\wedge\cdot)/2}\cdot\delta M^{\mathbb{G}}\|_{\mathcal{M}^p_T(\widetilde{Q})}\\
&+ \left(\frac{1}{2} - \frac{C_{DB}C_{Lip}}{\sqrt{\alpha}}\right)\|e^{\alpha\cdot/2}\delta Z'^{\mathbb{G}}\|_{\mathbb{E}_{T\wedge\tau}(\widetilde{Q},p)}\\
&\leq \left(\epsilon + \frac{C_{DB}}{\sqrt{\alpha}}\right)\|e^{\alpha\cdot/2}\delta f\|_{\mathbb{S}_{T\wedge\tau}(\widetilde{Q},p)} + (1 + C_{DB})\|e^{\alpha(T\wedge\tau)/2}\delta\xi\|_{\mathbb{L}^p(\widetilde{Q})}\\
&+ C_{DB}\|e^{\frac{\alpha\cdot}{2}}\delta S\|_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},p)} + 2\|e^{\alpha\cdot/2}\delta S\|^{1/2}_{\mathbb{D}_{T\wedge\tau}(\widetilde{Q},p)}\|\mathrm{Var_T}(\mathrm{e}^{\alpha\cdot/2}\cdot\delta\mathrm{K}^{\mathbb{G}})\|^{1/2}_{\mathbb{L}^p(\widetilde{\mathrm{Q}})}
\end{aligned} \tag{6.2.14}$$

Then here we take $\epsilon = 2/\alpha$, then $C = \frac{\alpha}{2} - 2C_{Lip} - 4C^2_{Lip}$, and take $\alpha$ large such that $\frac{1}{2} \leq \frac{\alpha}{2} - 2C_{Lip} - 4C^2_{Lip}$, and remark that for this $\alpha$ we have

$$0 < \frac{1}{2} - \frac{C_{DB}C_{Lip}}{\sqrt{\alpha}} \leq \min\left\{1, C - \frac{C_{DB}C_{Lip}}{\sqrt{\alpha}}\right\}.$$

Furthermore, by Theorem 6.2.2, we get

$$
\begin{aligned}
&\|\mathrm{Var_T}(\mathrm{e}^{\alpha\cdot/2}\bullet\delta\mathrm{K}^{\mathbb{G}})\|_{\mathbb{L}^2(\widetilde{\mathrm{Q}})}\leq\|(\mathrm{e}^{\alpha\cdot/2}\bullet\mathrm{K}^{\mathbb{G},1})_{\mathrm{T}}\|_{\mathbb{L}^2(\widetilde{\mathrm{Q}})}+\|(\mathrm{e}^{\alpha\cdot/2}\bullet\mathrm{K}^{\mathbb{G},2})_{\mathrm{T}}\|_{\mathbb{L}^2(\widetilde{\mathrm{Q}})}\\
&\leq\widehat{C}\sum_{i=1}^{2}\|e^{\frac{\alpha}{2}(T\wedge\tau)}\xi^{(i)}\|_{L^2(\widetilde{Q})}+\|e^{\alpha(\tau\wedge\cdot)}(S^{(i)})^{+}\|_{\mathbb{S}_T(\widetilde{Q},2)}+\|e^{\frac{\alpha}{2}(\tau\wedge\cdot)}f^{(i)}(\cdot,0,0)\|_{\mathbb{S}_T(\widetilde{Q},2)}.
\end{aligned}
$$

Therefore, by plugging this inequality in (6.2.14), the inequality (6.2.9) follows immediately with

$$
\widehat{C}_1=\frac{\frac{2}{\alpha}+\frac{C_{DB}}{\sqrt{\alpha}}}{\frac{1}{2}-\frac{C_{DB}C_{Lip}}{\sqrt{\alpha}}},\ \widehat{C}_2=\frac{1+C_{DB}}{\frac{1}{2}-\frac{C_{DB}C_{Lip}}{\sqrt{\alpha}}},\ \widehat{C}_3=\frac{C_{DB}}{\frac{1}{2}-\frac{C_{DB}C_{Lip}}{\sqrt{\alpha}}},\ \widehat{C}_4=\frac{2\sqrt{\widehat{C}}}{\frac{1}{2}-\frac{C_{DB}C_{Lip}}{\sqrt{\alpha}}}.
$$

It is also clear that $\widehat{C}_1$ goes to zero when $\alpha$ goes to infinity. This ends the proof of the theorem. □

### 6.2.2 Existence, uniqueness & connection to $\mathbb{F}$-RBSDEs

In this section we find the RBSDE under $\mathbb{F}$ that is connected to the RBSDE (6.2.2) under $\mathbb{G}$. Also, we elaborate our results on existence and uniqueness of the solution to (6.2.17). We will assume that

$$
E\left[\widetilde{\mathcal{E}}_T\mathcal{K}_T(f,S,h^{(op)})+\int_0^T\mathcal{K}_s(f,S,h^{(op)})dV_s^{\mathbb{F}}\right]<+\infty,\quad(6.2.15)
$$

where

$$
\mathcal{K}_t(f,S,h^{(op)}):=|h_t^{(op)}|^2+\int_0^t|f(s,0,0,0)|^2ds+\sup_{0\leq u\leq t}(S_u^{+})^2,\quad t\geq0.\ (6.2.16)
$$

**Remark 6.2.4.** *Following the same footsteps of remark 5.1.4 we find the RB-*

*SDE under the flow $\mathbb{F}$ as the following*

$$\begin{cases} dY_t^{\mathbb{F}} = -f^{\mathbb{F}}(s, Y_s^{\mathbb{F}}, Z_s^{\mathbb{F}}, Z_s'^{\mathbb{F}})ds - h_s^{(op)} dV_s^{\mathbb{F}} - dK_t^{\mathbb{F}} + Z_s^{\mathbb{F}} dW_s + Z_s'^{\mathbb{F}} dN_s^{\mathbb{F}}, \\ \\ Y_T^{\mathbb{F}} = \xi^{\mathbb{F}}, \quad Y_t^{\mathbb{F}} \geq S_t^{\mathbb{F}}, \quad t \in [0,T) \quad \int_0^T (Y_{t-}^{\mathbb{F}} - S_{t-}^{\mathbb{F}}) dK_t^{\mathbb{F}} = 0, \quad P\text{-}a.s., \end{cases} \tag{6.2.17}$$

*where $S^{\mathbb{F}}, \xi^{\mathbb{F}}$, and $f^{\mathbb{F}}$ are given by*

$$f^{\mathbb{F}}(s, y, z, z') := \widetilde{\mathcal{E}}_s f\left(s, y\widetilde{\mathcal{E}}_s^{-1}, z\widetilde{\mathcal{E}}_s^{-1}, z'\widetilde{\mathcal{E}}_s^{-1}\right), \quad S_t^{\mathbb{F}} := \widetilde{\mathcal{E}}_t S_t, \quad \xi^{\mathbb{F}} := \widetilde{\mathcal{E}}_T h_T^{(op)}. \tag{6.2.18}$$

In the following theorem we give our main result of this section as the following

**Theorem 6.2.5.** *Let $G > 0$ and $\widetilde{\mathcal{E}}$ be the process defined in (5.2.2). Then the following assertions hold.*
(a) *The RBSDE (6.2.17)-(6.2.18) has a unique $L^2(P, \mathbb{F})$-solution that we denote by $(Y^{\mathbb{F}}, Z^{\mathbb{F}}, Z'^{\mathbb{F}}, K^{\mathbb{F}})$.*
(b) *There exists a unique $L^2(\widetilde{Q}, \mathbb{G})$-solution to (6.2.2), denoted by the quintuplet $(Y^{\mathbb{G}}, Z^{\mathbb{G}}, Z'^{\mathbb{G}}, K^{\mathbb{G}}, M^{\mathbb{G}})$, and is given by*

$$\begin{cases} Y_t^{\mathbb{G}} = \dfrac{Y_t^{\mathbb{F}}}{\widetilde{\mathcal{E}}_t} 1_{\{t\ <\tau\}} + \xi 1_{\{t \geq \tau\}}, \quad Z^{\mathbb{G}} = \dfrac{Z^{\mathbb{F}}}{\widetilde{\mathcal{E}}_t} I_{]\!]0,\tau]\!]}, \quad Z'^{\mathbb{G}} = \dfrac{Z'^{\mathbb{F}}}{\widetilde{\mathcal{E}}_t} I_{]\!]0,\tau]\!]}, \\ K^{\mathbb{G}} = \dfrac{1}{\widetilde{\mathcal{E}}_-} \bullet (K^{\mathbb{F}})^{\tau} \text{ and } M^{\mathbb{G}} = (\dfrac{Y^{\mathbb{F}}}{\widetilde{\mathcal{E}}} - h^{(op)}) \bullet N^{\mathbb{G}} - (h^{(pr)} - h^{(op)}) \bullet D. \end{cases} \tag{6.2.19}$$

*Proof.* The proof of this theorem can be achieved in two steps. Step 1 focuses in proving (a). While step 2 focuses in proving (b).

**Step 1.** The proof of existence and uniqueness of an $L^2(P, \mathbb{F})$- soluoion to the RBSDE (6.2.17)-(6.2.18)) mimics exactly Step 1 in the proof of Theorem 5.1.27 with using [48] instead of [19], and will be omitted.

**Step 2.** Due to Theorem 6.2.3 the RBSDE (6.2.2) has at most one solution. Thus, the proof of assertion (b) will follows immediately as soon as we prove that $(\overline{Y}, \overline{Z}, \overline{Z'}, \overline{K}, \overline{M})$, give by

$$\begin{aligned}
&\overline{Y} := \frac{Y^{\mathbb{F}}}{\widetilde{\mathcal{E}}} I_{[\![0,\tau[\![} + \xi I_{[\![\tau,+\infty[\![}, \quad \overline{Z} := \frac{Z^{\mathbb{F}}}{\widetilde{\mathcal{E}}} I_{]\!]0,\tau]\!]}, \quad \overline{K} := \frac{1}{\widetilde{\mathcal{E}}_-} \bullet (K^{\mathbb{F}})^{\tau}, \\
&\overline{Z'} := \frac{Z'^{\mathbb{F}}}{\widetilde{\mathcal{E}}_t} I_{]\!]0,\tau]\!]}, \quad \text{and} \quad \overline{M} := (\frac{Y^{\mathbb{F}}}{\widetilde{\mathcal{E}}} - h^{(op)}) \bullet N^{\mathbb{G}} - (h^{(pr)} - h^{(op)}) \cdot D,
\end{aligned}$$

is in fact a solution to (6.2.2). The proof of (6.2.19) mimics exactly Step 2 in the proof of Theorem 5.1.27 with using [44] instead of [19], and will be omitted. This ends the proof the theorem. $\square$

# Chapter 7

# Exponential hedging under random horizon

This chapter focuses on giving some applications to RBSDEs in exponential hedging. This shows a clear motivation to our studies of RBSDEs of Chapters 4, 5, and 6. This chapter has three sections. In the first section we give some developments on minimal entropy martingale measures under random horizon $\tau$. The second and third sections address the exponential hedging problem using its BSDEs and RBSDEs for the primal and the dual problems resoectively. It is worth mentioning that this chapter is a preliminary version of paper [5].

## 7.1 Minimal entropy martingale measure for $(S^\tau, \mathbb{G})$

This section addresses the minimal entropy martingale measure/density for the model $(S^\tau \mathbb{G})$. To this end, we recall a useful lemma from Choulli and Stricker (2005,2006).

**Lemma 7.1.1.** *Let $N$ be an $\mathbb{H}$-semimartingale such that $Z := \mathcal{E}(N) > 0$ Then the following holds.*

$$Z \ln(Z) = Z_-(1 + \ln Z_-) \bullet N + Z_- \bullet H^E(N, P). \tag{7.1.1}$$

*Here,*

$$H^E(N, P) = \frac{1}{2}\langle N^c\rangle + \sum_{0<s\leq t} \left((1 + \Delta N)\ln(1 + \Delta N) - \Delta N\right).$$

For the proof, we refer the reader to Choulli and Stricker (2005, 2006).

**Lemma 7.1.2.** *Suppose $G > 0$ and let $\varphi \in \mathcal{I}^o_{loc}(N^{\mathbb{G}}, \mathbb{G})$ satsifying $-\widetilde{G} < G\varphi$ and $\varphi\Delta D^{o,\mathbb{F}} < \widetilde{G}$. Then the following assertions hold.*

(a) *The following process*

$$V(\varphi) := \frac{\varphi}{\widetilde{G}} 1_{\{\widetilde{G}=G\}} \bullet D^{o,\mathbb{F}} - \sum_{0\leq s<\cdot} \Big(\ln(1 - \frac{\varphi_s \Delta D^{o,\mathbb{F}}_s}{\widetilde{G}_s})\Big) \tag{7.1.2}$$

*is a RCLL and $\mathbb{F}$-adapted process with locally integrable variation.*

(b) *If furthermore $0 \leq \varphi$, then $V(\varphi)$ is nondecreasing.*

(c) *We always have we have*

$$\mathcal{E}(\varphi \bullet N^{\mathbb{G}}) = \Big(1 + \frac{\varphi G}{\widetilde{G}} \bullet D\Big) \exp\left(-I_{]\!]0,\tau[\![} \bullet V(\varphi)\right). \tag{7.1.3}$$

*Proof.* The assertions (a) and (b) are obvious and will be omitted here. The rest of the proof proves assertion (c). First remark that

$$\begin{aligned}
\Delta N^{\mathbb{G}} &= \Delta D - \widetilde{G}^{-1} I_{]\!]0,\tau]\!]} \Delta D^{o,\mathbb{F}} = \left(1 - \frac{\Delta D^{o,\mathbb{F}}}{\widetilde{G}}\right) \Delta D - \widetilde{G}^{-1} I_{]\!]0,\tau[\![} \Delta D^{o,\mathbb{F}} \\
&= \frac{G}{\widetilde{G}} \Delta D - \widetilde{G}^{-1} I_{]\!]0,\tau[\![} \Delta D^{o,\mathbb{F}}.
\end{aligned}$$

Therefore, this equality implies that

$$\begin{aligned}
&\sum_{0<s\leq t} \left(\ln(1 + \varphi_s \Delta N_s^{\mathbb{G}}) - \varphi_s \Delta N_s^{\mathbb{G}}\right) \\
&= \left(\ln(1 + \frac{\varphi G}{\widetilde{G}}) - \frac{\varphi G}{\widetilde{G}}\right) \bullet D_t + \sum_{0<s\leq t} \left(\ln(1 - \frac{\varphi_s \Delta D_s^{o,\mathbb{F}}}{\widetilde{G}_s}) + \frac{\varphi_s \Delta D_s^{o,\mathbb{F}}}{\widetilde{G}_s}\right) I_{]\!]0,\tau[\![}(s)
\end{aligned}$$

Thus, a combination of this equality and the explicit form of the stochastic exponential (see 2.1.5)) yields

$$\begin{aligned}
\mathcal{E}_t(\varphi \bullet N^{\mathbb{G}}) &= \exp\left\{(\varphi \bullet N^{\mathbb{G}})_t + \sum_{0\leq s<t} \left(\ln(1 + \varphi_s \Delta N_s^{\mathbb{G}}) - \varphi_s \Delta N_s^{\mathbb{G}}\right)\right\} \\
&= \exp\left\{\begin{array}{l} -\frac{\varphi}{\widetilde{G}} I_{]\!]0,\tau[\![} \bullet D_t^{o,\mathbb{F}} + \ln(1 + \frac{\varphi G}{\widetilde{G}_s}) \bullet D_t \\ + \sum_{0<s\leq t} \left(\ln(1 - \frac{\varphi_s \Delta D_s^{o,\mathbb{F}}}{\widetilde{G}_s}) + \frac{\varphi_s \Delta D_s^{o,\mathbb{F}}}{\widetilde{G}_s}\right) I_{]\!]0,\tau[\![}(s) \end{array}\right\} \\
&= \Big(1 + \frac{\varphi G}{\widetilde{G}} \bullet D\Big) \exp\left\{\begin{array}{l} -\frac{\varphi}{\widetilde{G}} 1_{\{\widetilde{G}=G\}} I_{]\!]0,\tau[\![} \bullet D_t^{o,\mathbb{F}} \\ + \sum_{0<s\leq t} \left(\ln(1 - \frac{\varphi_s \Delta D_s^{o,\mathbb{F}}}{\widetilde{G}_s})\right) I_{]\!]0,\tau[\![}(s) \end{array}\right\}.
\end{aligned}$$

This proves (7.1.3), and the proof of the lemma is complete. $\square$

Below, we elaborate our main results of this section that requires the following notation

$$\mathcal{Z}_{loc}(S^\tau, \mathbb{G}) := \left\{ \frac{Z^\tau \mathcal{E}(\varphi \bullet N^{\mathbb{G}})}{\mathcal{E}(G_-^{-1} \bullet m)^\tau} \mathcal{E}(\psi \bullet D) : \begin{array}{l} (Z, \varphi) \in \mathcal{Z}_{loc}(S, \mathbb{F}) \times \mathcal{I}^o_{loc}(N^{\mathbb{G}}, \mathbb{G}), \\ \psi \in L^1_{loc}(\widetilde{\Omega}, \mathrm{Prog}(\mathbb{F}), P \otimes D), \\ E[\psi_\tau | \mathcal{F}_\tau] = 0 \\ \max\left(-\frac{G}{\tilde{G}}\varphi, \varphi - \varphi\frac{G}{\tilde{G}}\right) < 1, \\ \psi > -1 \ P \otimes D - a.e. \end{array} \right\} \tag{7.1.4}$$

Here $\mathcal{Z}_{loc}(X, \mathbb{H})$ denotes the set of all $\mathbb{H}$-local martingale deflators for the model $(X, \mathbb{H})$. Its definition can be found in [29], and which we recall below.

**Definition 7.1.3.** *Let $X$ be an $\mathbb{H}$-semimartingale and $Z$ be a process. We call $Z$ is a $\mathbb{H}$-local martingale deflator for $X$ (or called local martingale density) if $Z > 0$, $Z_0 = 1$, and there exists an $\mathbb{H}$-predictable process $\varphi$ such that $0 < \varphi \le 1$ and both processes $Z$ and $Z(\varphi \bullet X)$ are $\mathbb{H}$-local martingales. Throughout the thesis, the set of all local martingale deflator for $(X, \mathbb{H})$ will be denoted by $\mathcal{Z}_{loc}(X, \mathbb{H})$.*

Thus, our main theorem in this section is the following

**Theorem 7.1.4.** *Suppose that $G > 0$ and $\Delta D^{o,\mathbb{F}} = 0$. Then the following assertions hold.*

(a) *The following equality holds.*

$$\inf_{Z\in\mathcal{Z}_{loc}(S^\tau,\mathbb{G})} E[Z_T\ln(Z_T)] = \inf_{Z\in\mathcal{Z}^+_{loc}(S^\tau,\mathbb{G})} E\left[Z_T\ln(Z_T)\right], \tag{7.1.5}$$

*where*

$$\mathcal{Z}^+_{loc}(S^\tau,\mathbb{G}) := \left\{\frac{Z^\tau\mathcal{E}(\varphi\bullet N^{\mathbb{G}})}{\mathcal{E}(G_-^{-1}\bullet m)^\tau}\in\mathcal{Z}_{loc}(S^\tau,\mathbb{G}) \;:\; \varphi\geq 0\right\}. \tag{7.1.6}$$

(b) *Let* $\widetilde{Z}$ *be defined in (2.3.10) and put* $Z^\varphi := \mathcal{E}(\varphi\bullet N^{\mathbb{G}})$ *for any* $\varphi\in\mathcal{I}^o_{loc}(N^{\mathbb{G}},\mathbb{G})$. *Then*

$$\inf_{Z\in\mathcal{Z}_{loc}(S,\mathbb{F})} E\left[Z_{T\wedge\tau}\widetilde{Z}_T\ln(Z_{T\wedge\tau}\widetilde{Z}_T)\right]<\infty \tag{7.1.7}$$

*iff*

$$\inf_{\substack{Z\in\mathcal{Z}_{loc}(S,\mathbb{F})\\ \varphi\in\Phi^+_b}} E\left[Z_{T\wedge\tau}\widetilde{Z}_T Z^\varphi_T\ln(Z_{T\wedge\tau}\widetilde{Z}_T Z^\varphi_T)\right]<\infty, \tag{7.1.8}$$

*where*

$$\Phi^+_b := \left\{\varphi\in\mathcal{I}^o_{loc}(N^{\mathbb{G}},\mathbb{G}) \;:\; \varphi\geq 0, \quad \textit{and } \varphi+V(\varphi) \textit{ is a bounded process}\right\}. \tag{7.1.9}$$

(c) *We always have*

$$\inf_{Z\in\mathcal{Z}^+_{loc}(S^\tau,\mathbb{G})} E\left[Z_T\ln(Z_T)\right] \leq \inf_{\substack{Z\in\mathcal{Z}_{loc}(S,\mathbb{F})\\ \varphi\in\Phi^+_b}} E\left[(Z_{T\wedge\tau}\widetilde{Z}_T Z^\varphi_T\ln(Z_{T\wedge\tau}\widetilde{Z}_T Z^\varphi_T)\right].$$

*Proof.* Remark that

$$\left\{Z^\tau\widetilde{Z}Z^\varphi \;\middle|\; Z\in\mathcal{Z}_{loc}(S,\mathbb{F}) \text{ and } \varphi\in\Phi^+_b\right\}\subset \mathcal{Z}^+_{loc}(S^\tau,\mathbb{G}).$$

Thus, assertion (c) follows immediately. The rest of this proof deals with assertions (a) and (b) in two parts.

**Part 1.** Here we prove assertion (a). Remark that $\mathcal{Z}^+_{loc}(S^\tau,\mathbb{G})\subseteq\mathcal{Z}_{loc}(S^\tau,\mathbb{G})$, and hence

$$\inf_{Z\in\mathcal{Z}_{loc}(S^\tau,\mathbb{G})} E(Z_T\ln(Z_T)) \leq \inf_{\mathcal{Z}^+_{loc}(S^\tau,\mathbb{G})} E(Z_T\ln(Z_T)).$$

To prove the reverse inequality, we denote by

$$\varphi+ := \max(\varphi,0) \quad \text{and} \quad \varphi^- := \max(-\varphi,0),$$

and consider $Z^{\mathbb{G}} \in \mathcal{Z}_{loc}(S^\tau,\mathbb{G})$ This implies the existence of $(K^{\mathbb{F}},\varphi,\varphi^{(pr)})$ which belongs to $\mathcal{M}_{0,loc}(\mathbb{F})\times I^o_{loc}(N^{\mathbb{G}},\mathbb{G})\times L^1_{loc}(\tilde{\Omega},\mathrm{Prog}(\mathbb{F}),P\otimes D)$ such that

$$\varphi^{(pr)} > -1\;\; P\otimes D,\quad E\left[\varphi^{(pr)}_\tau \;\middle|\; \mathcal{F}_\tau\right]I_{\{\tau<+\infty\}}=0,\quad P-a.s.,$$
$$\frac{-\tilde{G}}{G}<\varphi<\frac{\tilde{G}}{\tilde{G}-G}.$$

$$Z^{\mathbb{G}} = \mathcal{E}(\mathcal{T}(K^{\mathbb{F}}) - G_-^{-1} \bullet \mathcal{T}(m))\mathcal{E}(\varphi \bullet N^{\mathbb{G}})\mathcal{E}(\varphi^{(pr)} \bullet D).$$

Throughout the rest of the proof, we put

$$Z^1 := \mathcal{E}(\mathcal{T}(K^{\mathbb{F}}) - G_-^{-1} \bullet \mathcal{T}(m)), \quad Z^2 := \mathcal{E}(\varphi \bullet N^{\mathbb{G}}), \quad Z^3 := \mathcal{E}(\varphi^{(pr)} \bullet D).$$

Then we derive

$$\begin{aligned} Z^{\mathbb{G}} \ln Z^{\mathbb{G}} &= Z^{\mathbb{G}} \ln(Z^1 Z^2 Z^3) = Z^{\mathbb{G}} \ln Z^1 + Z^{\mathbb{G}} \ln Z^2 + Z^{\mathbb{G}} \ln Z^3 \\ &= Z^2 Z^3 (Z^1 \ln Z^1) + Z^1 Z^3 (Z^2 \ln Z^2) + Z^1 Z^2 (Z^3 \ln Z^3). \end{aligned}$$

Also we put

$$L^1 := Z^1 \ln Z^1, \quad L^2 := Z^2 \ln Z^2, \quad L^3 := Z^3 \ln Z^3.$$

$Z^1 Z^2 L^3 = (Z^1 Z^2)_- \bullet L^3 + (L^3)_- \bullet (Z^1 Z^2) + [Z^1 Z^2, L^3]$

note that $L_-^3 = \mathcal{E}_-(\varphi^{(pr)} \bullet D) \ln(\mathcal{E}_-(\varphi^{(pr)} \bullet D)) = 1 \ln 1 = 0$, also note that

$$\begin{aligned} &(Z^1 Z^2)_- \bullet L^3 \\ &= (Z^1 Z^2)_- \bullet [(1 + \ln(\mathcal{E}_-(\varphi^{pr} \bullet D)) Z_-^3 \varphi^{pr} \bullet D + \mathcal{E}_-(\varphi^{pr} \bullet D) \bullet H^E(\varphi^{pr} \bullet D)] \\ &= (Z^1 Z^2)_- \bullet [(\varphi^{pr} \bullet D) + H^E(\varphi^{pr} \bullet D)] \\ &= (Z^1 Z^2)_- \bullet (\varphi^{pr} \bullet D) + (Z^1 Z^2)_- \bullet H^E(\varphi^{pr} \bullet D) \\ &= \mathbb{G} - \textit{local martingale} + (Z^1 Z^2)_- \bullet H^E(\varphi^{pr} \bullet D). \end{aligned}$$

As $H^E(\varphi^{pr} \bullet D)$ is non-decreasing and non-negative and $(Z^1 Z^2)_- > 0$, then we deduce that $(Z^1Z^2)_- \bullet H^E(\varphi^{pr} \bullet D) \geq 0$. Hence, $(Z^1Z^2)_- \bullet H^E(\varphi^{pr} \bullet D) = 0$ iff $H^E(\varphi^{pr} \bullet D) = 0$ iff $\varphi^{pr} \bullet D = 0$ iff $\varphi^{pr} = 0$ . Therefore $(Z^1Z^2)_- \bullet H^E(\varphi^{pr} \bullet D)$ takes its infimum value when $\varphi^{pr} = 0$. Hence,

$$(Z^1Z^2)_- \bullet L^3 \geq \mathbb{G}\text{-local martingale.}$$

Finally, note that

$$[Z^1Z^2, L^3] = [\mathcal{E}(\mathcal{T}(K^{\mathbb{F}}) - G_-^{-1} \bullet \mathcal{T}(m))\mathcal{E}(\varphi \bullet N^{\mathbb{G}}), \mathcal{E}(\varphi^{(pr)} \bullet D)\ln(\mathcal{E}(\varphi^{(pr)} \bullet D))]$$

is in $\mathcal{M}_{loc}(\mathbb{G})$. From above we see that $Z^1Z^2L^3 \geq \mathbb{G}$-local martingale. Now, note that

$$Z^2Z^3L^1 = Z^3_- \bullet (Z^2L^1) + (Z^2L^1)_- \bullet Z^3 + [Z^2L^1, Z^3] = Z^2L^1 + \mathbb{G}-\text{local martingale.}$$

And

$$Z^1Z^3L^2 = Z^3_- \bullet (Z^1L^2) + (Z^1L^2)_- \bullet Z^3 + [Z^1L^2, Z^3] = Z^1L^2 + \mathbb{G}\text{-local martingale.}$$

So, we get that

$$E(Z^{\mathbb{G}} \ln Z^{\mathbb{G}}) \geq E(Z^1Z^2 \ln(Z^1Z^2)) + E(\mathbb{G} - \text{local martingale}).$$

But, $Z^1Z^2\ln(Z^1Z^2) = Z^2(Z^1\ln(Z^1)) + Z^1(Z^2\ln(Z^2))$

On the other hand, thanks to a combination of Integration by parts and

Lemma(7.1.1), we deduce that

$$
\begin{aligned}
Z^2\ln(Z^2) &= \mathcal{E}(\varphi\bullet N^{\mathbb{G}})\ln(\varphi\bullet N^{\mathbb{G}})\\
&= \mathcal{E}(\varphi^+\bullet N^{\mathbb{G}})\mathcal{E}(-\varphi^-\bullet N^{\mathbb{G}})\ln(\mathcal{E}(\varphi^+\bullet N^{\mathbb{G}})\mathcal{E}(-\varphi^-\bullet N^{\mathbb{G}}))
\end{aligned}
$$

$$
\begin{aligned}
&= \mathcal{E}(-\varphi^-\bullet N^{\mathbb{G}})\mathcal{E}(\varphi^+\bullet N^{\mathbb{G}})\ln(\mathcal{E}(\varphi^+\bullet N^{\mathbb{G}}))\\
&+ \mathcal{E}(\varphi^+\bullet N^{\mathbb{G}})\mathcal{E}(-\varphi^-\bullet N^{\mathbb{G}})\ln(\mathcal{E}(-\varphi^-\bullet N^{\mathbb{G}}))
\end{aligned}
$$

$$
\begin{aligned}
&= \mathcal{E}(-\varphi^-\bullet N^{\mathbb{G}})\Big((1+\ln(\mathcal{E}_-(\varphi^+\bullet N^{\mathbb{G}}))\mathcal{E}_-(\varphi^+\bullet N^{\mathbb{G}})\bullet\varphi^+\bullet N^{\mathbb{G}}\Big)\\
&+ \mathcal{E}(-\varphi^-\bullet N^{\mathbb{G}})\Big(\mathcal{E}_-(\varphi^+\bullet N^{\mathbb{G}})\bullet H^E(\varphi^+\bullet N^{\mathbb{G}})\Big)\\
&+ \mathcal{E}(\varphi^+\bullet N^{\mathbb{G}})\Big((1+\ln(\mathcal{E}_-(-\varphi^-\bullet N^{\mathbb{G}}))\mathcal{E}_-(-\varphi^-\bullet N^{\mathbb{G}})\bullet(-\varphi^-)\bullet N^{\mathbb{G}}\Big)\\
&+ \mathcal{E}(\varphi^+\bullet N^{\mathbb{G}})\Big(\mathcal{E}_-(-\varphi^-\bullet N^{\mathbb{G}})\bullet H^E(-\varphi^-\bullet N^{\mathbb{G}})\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\overset{Ito}{=} \mathcal{E}(-\varphi^-\bullet N^{\mathbb{G}})\Big((1+\ln(\mathcal{E}_-(\varphi^+\bullet N^{\mathbb{G}}))\mathcal{E}_-(\varphi^+\bullet N^{\mathbb{G}})\bullet\varphi^+\bullet N^{\mathbb{G}}\Big)\\
&+ \mathcal{E}_-(-\varphi^-\bullet N^{\mathbb{G}})\mathcal{E}_-(\varphi^+\bullet N^{\mathbb{G}})\bullet H^E(\varphi^+\bullet N^{\mathbb{G}})\\
&+ \mathcal{E}_-(\varphi^+\bullet N^{\mathbb{G}})H^E_-(\varphi^+\bullet N^{\mathbb{G}})\bullet\mathcal{E}(-\varphi^-\bullet N^{\mathbb{G}})\\
&+ [\mathcal{E}(-\varphi^-\bullet N^{\mathbb{G}}),\mathcal{E}_-(\varphi^+\bullet N^{\mathbb{G}})\bullet H^E(\varphi^+\bullet N^{\mathbb{G}})]\\
&+ \mathcal{E}(\varphi^+\bullet N^{\mathbb{G}})\Big((1+\ln(\mathcal{E}_-(-\varphi^-\bullet N^{\mathbb{G}}))\mathcal{E}_-(-\varphi^-\bullet N^{\mathbb{G}})\bullet(-\varphi^-)\bullet N^{\mathbb{G}}\Big)\\
&+ \mathcal{E}_-(\varphi^+\bullet N^{\mathbb{G}})\mathcal{E}_-(-\varphi^-\bullet N^{\mathbb{G}})\bullet H^E(-\varphi^-\bullet N^{\mathbb{G}})\\
&+ \Big(\mathcal{E}_-(-\varphi^-\bullet N^{\mathbb{G}})\bullet H^E(-\varphi^-\bullet N^{\mathbb{G}})\Big)_-\bullet\mathcal{E}(\varphi^+\bullet N^{\mathbb{G}})\\
&+ [\mathcal{E}(\varphi^+\bullet N^{\mathbb{G}}),\mathcal{E}_-(-\varphi^-\bullet N^{\mathbb{G}})\bullet H^E(-\varphi^-\bullet N^{\mathbb{G}})]
\end{aligned}
$$

$$= \mathbb{G}\text{-local martingale} + \mathcal{E}_-(-\varphi^- \bullet N^{\mathbb{G}})\mathcal{E}_-(\varphi^+ \bullet N^{\mathbb{G}}) \bullet H^E(\varphi^+ \bullet N^{\mathbb{G}})$$
$$+ \mathcal{E}_-(\varphi^+ \bullet N^{\mathbb{G}})\mathcal{E}_-(-\varphi^- \bullet N^{\mathbb{G}}) \bullet H^E(-\varphi^- \bullet N^{\mathbb{G}})$$

$$\geq \mathbb{G}\text{-local martingale}$$
$$+ \mathcal{E}_-(\varphi^+ \bullet N^{\mathbb{G}}) \bullet H^E(\varphi^+ \bullet N^{\mathbb{G}}) + \mathcal{E}_-(\varphi^+ \bullet N^{\mathbb{G}})\mathcal{E}_-(-\varphi^- \bullet N^{\mathbb{G}}) \bullet H^E(-\varphi^- \bullet N^{\mathbb{G}}).$$

However, due to $H^E(-\varphi^- \bullet N^{\mathbb{G}}, \mathbb{G})$ is non-decreasing and non-negative and that $\mathcal{E}_-(\varphi^+ \bullet N^{\mathbb{G}})\mathcal{E}_-(-\varphi^- \bullet N^{\mathbb{G}}) > 0$, we deduce that

$$\mathcal{E}_-(\varphi^+ \bullet N^{\mathbb{G}})\mathcal{E}_-(-\varphi^- \bullet N^{\mathbb{G}}) \bullet H^E(-\varphi^- \bullet N^{\mathbb{G}}) \geq 0.$$

Hence $\mathcal{E}_-(\varphi^+ \bullet N^{\mathbb{G}})\mathcal{E}_-(-\varphi^- \bullet N^{\mathbb{G}}) \bullet H^E(-\varphi^- \bullet N^{\mathbb{G}}, P) = 0$ iff $H^E(-\varphi^- \bullet N^{\mathbb{G}}, P) = 0$ iff $\varphi^- \bullet N^{\mathbb{G}} = 0$ iff $\varphi^- = 0$ . Therefore $\mathcal{E}_-(\varphi^+ \bullet N^{\mathbb{G}})\mathcal{E}_-(-\varphi^- \bullet N^{\mathbb{G}}) \bullet H^E(-\varphi^- \bullet N^{\mathbb{G}})$ takes its infimum value when $\varphi^- = 0$.
Therefore,

$$Z^2 \ln(Z^2) \geq \mathbb{G}\text{-local martingale} + \mathcal{E}_-(\varphi^+ \bullet N^{\mathbb{G}}) \bullet H^E(\varphi^+ \bullet N^{\mathbb{G}}, P).$$

This implies that

$$Z^1 Z^2 \ln(Z^2) \geq \mathbb{G}\text{-local martingale} + Z^1\{\mathcal{E}_-(\varphi^+ \bullet N^{\mathbb{G}}) \bullet H^E(\varphi^+ \bullet N^{\mathbb{G}}, P)\}. \tag{7.1.10}$$

Now, note that

$$Z^2 Z^1 \ln Z^1 = \mathcal{E}(\varphi \bullet N^{\mathbb{G}}) Z^1 \ln Z^1 = \mathcal{E}(\varphi^+ \bullet N^{\mathbb{G}})\mathcal{E}(-\varphi^- \bullet N^{\mathbb{G}}) Z^1 \ln Z^1$$

$$
\begin{aligned}
&= \mathcal{E}_-(\varphi^+ \bullet N^{\mathbb{G}})\mathcal{E}_-(-\varphi^- \bullet N^{\mathbb{G}}) \bullet Z^1 \ln Z^1 + (Z^1 \ln Z^1)_- \bullet \mathcal{E}(\varphi \bullet N^{\mathbb{G}}) \\
&\quad + [\mathcal{E}(\varphi \bullet N^{\mathbb{G}}), Z^1 \ln Z^1] \\
&= \mathbb{G}\text{-local martingale} + \mathcal{E}_-(\varphi^+ \bullet N^{\mathbb{G}})\mathcal{E}_-(-\varphi^- \bullet N^{\mathbb{G}}) \bullet Z^1 \ln Z^1 \\
&\quad + [\mathcal{E}(\varphi \bullet N^{\mathbb{G}}), Z^1 \ln Z^1] \\
&= \mathcal{E}_-(\varphi^+ \bullet N^{\mathbb{G}})\mathcal{E}_-(-\varphi^- \bullet N^{\mathbb{G}}) \bullet \{(1 + \ln Z^1_-)Z^1_- \bullet (\mathcal{T}(K^{\mathbb{F}}) - G_-^{-1} \bullet \mathcal{T}(m)) \\
&+ Z^1_- \bullet H^E(\mathcal{T}(K^{\mathbb{F}}) - G_-^{-1} \bullet \mathcal{T}(m), P)\} \\
&+ [\mathcal{E}(\varphi \bullet N^{\mathbb{G}}), \{(1 + \ln Z^1_-)Z^1_- \bullet (\mathcal{T}(K^{\mathbb{F}}) - G_-^{-1} \bullet \mathcal{T}(m)) \\
&+ Z^1_- \bullet H^E(\mathcal{T}(K^{\mathbb{F}}) - G_-^{-1} \bullet \mathcal{T}(m), P)\}] + \mathbb{G}\text{-local martingale} \\
&= \mathcal{E}_-(\varphi^+ \bullet N^{\mathbb{G}})\mathcal{E}_-(-\varphi^- \bullet N^{\mathbb{G}}) \bullet Z^1_- \bullet H^E(\mathcal{T}(K^{\mathbb{F}}) - G_-^{-1} \bullet \mathcal{T}(m), P) \\
&+ [\mathcal{E}(\varphi \bullet N^{\mathbb{G}}), (1 + \ln Z^1_-)Z^1_- \bullet (\mathcal{T}(K^{\mathbb{F}}) - G_-^{-1} \bullet \mathcal{T}(m))] + \mathbb{G}\text{-local martingale} \\
&= \mathbb{G}\text{-local martingale} \\
&\qquad + \mathcal{E}_-(\varphi^+ \bullet N^{\mathbb{G}})\mathcal{E}_-(-\varphi^- \bullet N^{\mathbb{G}}) \bullet Z^1_- \bullet H^E(\mathcal{T}(K^{\mathbb{F}}) - G_-^{-1} \bullet \mathcal{T}(m), P) \\
&\geq \mathbb{G}\text{-local martingale} + \mathcal{E}_-(\varphi^+ \bullet N^{\mathbb{G}}) \bullet Z^1_- \bullet H^E(\mathcal{T}(K^{\mathbb{F}}) - G_-^{-1} \bullet \mathcal{T}(m), P).
\end{aligned}
\tag{7.1.11}
$$

By combining (7.1.10) and (7.1.11) we get

$$
\begin{aligned}
Z^{\mathbb{G}} \ln Z^{\mathbb{G}} &\geq Z^1 Z^2 \ln Z^2 + Z^2 Z^1 \ln Z^1 + \mathbb{G}\text{-local martingale} \\
&\geq \mathbb{G}\text{-local martingale} + Z^1\{\mathcal{E}_-(\varphi^+ \bullet N^{\mathbb{G}}) \bullet H^E(\varphi^+ \bullet N^{\mathbb{G}}, P)\} \\
&\quad + \mathcal{E}_-(\varphi^+ \bullet N^{\mathbb{G}}) \bullet Z^1_- \bullet H^E(\mathcal{T}(K^{\mathbb{F}}) - G_-^{-1} \bullet \mathcal{T}(m), P)
\end{aligned}
$$

and hence,

$$
E(Z^{\mathbb{G}} \ln Z^{\mathbb{G}}) \geq E[(Z^1 Z^2 \ln Z^2) + (Z^2 Z^1 \ln Z^1)]
$$

$$
\begin{aligned}
&\geq E[Z^1\{\mathcal{E}_-(\varphi^+ \bullet N^{\mathbb{G}}) \bullet H^E(\varphi^+ \bullet N^{\mathbb{G}}, P)\}] \qquad (7.1.12)\\
&\quad + E[\mathcal{E}_-(\varphi^+ \bullet N^{\mathbb{G}}) \bullet Z^1_- \bullet H^E(\mathcal{T}(K^{\mathbb{F}}, P) - G_-^{-1} \bullet \mathcal{T}(m), P)]\\
&= E[Z^{(1)}_T \ln Z^{(1)}_T] \geq \inf_{\mathcal{Z}^+_{loc}(S^\tau, \mathbb{G})} E[Z^{\mathbb{G}} \ln Z^{\mathbb{G}}],
\end{aligned}
$$

where $Z^{(1)} := Z^1 \mathcal{E}(\varphi^+ \bullet N^{\mathbb{G}}) \in \mathcal{Z}^+_{loc}(S^\tau, \mathbb{G})$. This proves assertion (a).

**Part 2.** Here we prove assertion (b). Remark that, it is clear that

$$
\begin{aligned}
&\inf_{Z \in \mathcal{Z}_{loc}(S,\mathbb{F})} E\left[Z_{T\wedge\tau} \widetilde{Z}_T \ln(Z_{T\wedge\tau} \widetilde{Z}_T)\right]\\
&\geq \inf_{\substack{Z \in \mathcal{Z}_{loc}(S,\mathbb{F})\\ \varphi \in \Phi_b^+}} E\left[Z_{T\wedge\tau} \widetilde{Z}_T Z^\varphi_T \ln(Z_{T\wedge\tau} \widetilde{Z}_T Z^\varphi_T)\right].
\end{aligned}
$$

This prove that (7.1.7) implies (7.1.8). To prove the reverse implication, we consider $Z^{\mathbb{G}} \in \mathcal{Z}^+_{loc}(S^\tau, \mathbb{G})$. Then there exists $Z \in \mathcal{Z}_{loc}(S, \mathbb{F})$ and $\varphi \in \mathcal{I}^o_{loc}(N^{\mathbb{G}}, \mathbb{G})$ such that

$$
0 \leq \varphi, \quad \varphi(\widetilde{G} - G) < \widetilde{G}, \quad \text{and } Z^{\mathbb{G}} = \frac{Z^\tau \mathcal{E}(\varphi \bullet N^{\mathbb{G}})}{\mathcal{E}(G_-^{-1} \bullet m)^\tau}.
$$

Put $\widehat{Z}$ as

$$
\widehat{Z} := \frac{Z^\tau}{\mathcal{E}(G_-^{-1} \bullet m)^\tau}.
$$

Therefore,

$$
\begin{aligned}
Z^{\mathbb{G}} \ln Z^{\mathbb{G}} &= \mathcal{E}(\varphi \bullet N^{\mathbb{G}}) \widehat{Z} \ln\Big(\mathcal{E}(\varphi \bullet N^{\mathbb{G}}) \widehat{Z}\Big)\\
&= \mathcal{E}(\varphi \bullet N^{\mathbb{G}}) \widehat{Z} \ln \widehat{Z} + \widehat{Z} \mathcal{E}(\varphi \bullet N^{\mathbb{G}}) \ln \mathcal{E}(\varphi \bullet N^{\mathbb{G}})\\
&= \mathcal{E}(\varphi \bullet N^{\mathbb{G}}) \Big\{ \widehat{Z}_- \Big(1 + \ln(\widehat{Z}_-)\Big) \bullet \widehat{N} + \widehat{Z}_- \bullet H^E(\widehat{N}, P) \Big\}
\end{aligned}
$$

$$
\begin{aligned}
&+ \widehat{Z}\Big\{\mathcal{E}_-(\varphi\bullet N^{\mathbb{G}})\Big(1+\ln(\mathcal{E}_-(\varphi\bullet N^{\mathbb{G}}))\varphi\bullet N^{\mathbb{G}}\Big) + Z^2_-\bullet H^E(\varphi\bullet N^{\mathbb{G}},P)\Big\}\\
&\overset{Ito}{=} \mathcal{E}_-(\varphi\bullet N^{\mathbb{G}})\bullet\Big\{\widehat{Z}_-\Big(1+\ln(\widehat{Z}_-)\Big)\bullet\widehat{N} + \widehat{Z}_-\bullet H^E(\widehat{N},P)\Big\}\\
&+ \Big\{\widehat{Z}_-\Big(1+\ln(\widehat{Z}_-)\Big)\bullet\widehat{N} + \widehat{Z}_-\bullet H^E(\widehat{N},P)\Big\}\bullet(\varphi\bullet N^{\mathbb{G}})\\
&+ \widehat{Z}\bullet\Big\{\mathcal{E}_-(\varphi\bullet N^{\mathbb{G}})\Big(1+\ln(\mathcal{E}_-(\varphi\bullet N^{\mathbb{G}}))\varphi\bullet N^{\mathbb{G}}\Big) + Z^2_-\bullet H^E(\varphi\bullet N^{\mathbb{G}},P)\Big\}\\
&+ \Big\{\mathcal{E}_-(\varphi\bullet N^{\mathbb{G}})\Big(1+\ln(\mathcal{E}_-(\varphi\bullet N^{\mathbb{G}}))\varphi\bullet N^{\mathbb{G}}\Big) + Z^2_-\bullet H^E(\varphi\bullet N^{\mathbb{G}},P)\Big\}_-\bullet\widehat{N}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
E[Z^{\mathbb{G}}\ln Z^{\mathbb{G}}] &= E[\mathcal{E}_-(\varphi\bullet N^{\mathbb{G}})\widehat{Z}_-\bullet H^E_{T\wedge\tau}(\widehat{N},P)]\\
&\qquad + E[\widehat{Z}\mathcal{E}_-(\varphi\bullet N^{\mathbb{G}})\bullet H^E_T(\varphi\bullet N^{\mathbb{G}},P)].
\end{aligned}
$$

Note that for any $\varphi\in\Phi_b^+$, there exists $C>0$ such that

$$
C\leq \mathcal{E}_-(\varphi\bullet N^{\mathbb{G}}) = e^{-(V_-(\varphi))^\tau}\leq 1,
$$

where $V(\varphi) := \frac{\varphi}{\widetilde{G}}\bullet D^{o,\mathbb{F}}$ in this case of $\Delta D^{o,\mathbb{F}}=0$. Hence,

$$
CE[\widehat{Z}_-\bullet H^E_{T\wedge\tau}(\widehat{N},P)] \leq E[\mathcal{E}_-(\varphi\bullet N^{\mathbb{G}})\widehat{Z}_-\bullet H^E_{T\wedge\tau}(\widehat{N},P)] \leq E[\widehat{Z}_-\bullet H^E_{T\wedge\tau}(\widehat{N},P)]. \tag{7.1.13}
$$

By combining this latter inequalities with the fact that $E\widehat{Z}_T\ln(\widehat{Z}_T)] = E[\widehat{Z}_-\bullet H^E_{T\wedge\tau}(\widehat{N},P)]$ whose proof can be found in Choulli and Stricker (2005, 2006), the proof of assertion (b) follows immediately. This ends the proof of the theorem. $\square$

**Theorem 7.1.5.** *If there exists $Z \in \mathcal{Z}_{loc}(S, \mathbb{F})$ such that*

$$E\left[Z_{T\wedge\tau}\widetilde{Z}_T \ln\left(Z_{T\wedge\tau}\widetilde{Z}_T\right)\right] < \infty,$$

*then the following equivalent assertions hold.*

(a) *The minimal entropy martingale measure for $(S^\tau, \mathbb{G})$ exists.*

(b) $\mathcal{Z}_{exp}(S^\tau, \mathbb{G}) := \left\{Z \in \mathcal{Z}_{loc}(S^\tau, \mathbb{G}) \ : \ E[Z_T \ln(Z_T)] < \infty\right\} \neq \phi$.

*Proof.* The proof is trivial by a combination of Theorem 7.1.4 and Theorem 2.1 in Frittelli (2000). □

**Corollary 7.1.6.** *Suppose that $(S, \mathbb{F}, P)$ is complete market model that is arbitrage free, and denote by $\widehat{Z}$ its unique martingale density. Then the minimal entropy martingale measure for $(S^\tau, \mathbb{G})$ exists if*

$$E\left[\widehat{Z}_{T\wedge\tau}\widetilde{Z}_T \ln\left(\widehat{Z}_{T\wedge\tau}\widetilde{Z}_T\right)\right] < \infty.$$

## 7.2 BSDE formulation for the primal problem

Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a complete probability space that is generated by the brownian motion $W$. Throughout this chapter, we consider a financial market which consists of one risk-free asset, whose price process is assumed to be equal to 1 at any date, and one risky asset with price process $S$. throughout this chapter we consider the price process $S$ evolves according to the equation

$$dS_t = S_t(\mu_t dt + \sigma_t dW_t), \tag{7.2.1}$$

with the classical assumptions:

(i) $\mu$, $\sigma$ are predictable processes.

(iii) $\sigma_t > 0$ for any $0 \leq t \leq T$, and there exists a non-negative constant $C$ such that

$$|\mu| + \sigma + \frac{1}{\sigma} \leq C.$$

Let $\lambda$ be the risk premium process. That is $\mu_t := \sigma_t \lambda_t$. A process $\pi$ is called a trading process if it is a predictable process and if $\int_0^T \frac{\pi_t}{S_{t-}} dS_t$ is well defined. Under the assumption that the trading strategy is self-financing, the wealth process $X_t^{x,\pi}$ associated with the trading strategy $\pi$ and the initial capital $x$ satisfies

$$\begin{cases} dX_t^{x,\pi} = \pi(\mu_t dt + \sigma_t dW_t) \\ X_0^{x,\pi} = x. \end{cases}$$

Here, until the rest of this chapter we consider the contingent claim

$$B := g 1_{\tau > T}, \tag{7.2.2}$$

where $g$ is a bounded $F_T$ measurable random variable. Our aim is to study the classical optimization problem

$$V(x, B) = \sup_{\pi \in \mathcal{A}} E[U(X_{T \wedge \tau}^{x,\pi} + B)] \tag{7.2.3}$$

where $\mathcal{A}$ is the set of all admissible strategies.

Let

$$L\left(\int_0^{\cdot\wedge\tau}\frac{dS}{S}\right) \text{ be the set of all processes } \pi \text{ which is } \mathbb{G}- \text{ predictable,}$$
$$\int_0^{\cdot\wedge\tau}\frac{dS}{S} - \text{ integrable and } \mathbb{R}- \text{ valued process.}$$

For the problem (7.2.3) set

$$\mathcal{A} = L\left(\int_0^{\cdot\wedge\tau}\frac{dS}{S}\right),$$

and $U$ is the exponential utility function

$$U(x) = -e^{-\gamma x} \;\; ; \; x \in \mathbb{R} \text{ and } \gamma \text{ is a given constant.}$$

The optimization problem (7.2.3) can be written as

$$V(x,B) = e^{-\gamma x}V(0,B) = -e^{-\gamma x}\inf_{\pi\in\mathcal{A}}E[e^{-\gamma(X^{0,\pi}_{T\wedge\tau}+B)}]$$

Hence, it is enough to study

$$V(0,B) = -\inf_{\pi\in\mathcal{A}}E[e^{-\gamma(X^{0,\pi}_{T\wedge\tau}+B)}]. \tag{7.2.4}$$

Let us Define $\tilde{V}(0,\widetilde{B})$ as the following

$$\tilde{V}(0,\widetilde{B}) := -\inf_{\pi\in\mathcal{A}}E^{\widetilde{Q}}[e^{-\gamma(X^{0,\pi}_{T\wedge\tau}+\widetilde{B})}], \tag{7.2.5}$$

where $-\gamma\widetilde{B} := -\gamma B - \ln(\tilde{Z}_T)$. Note that $V(0,B) = \tilde{V}(0,\widetilde{B})$, for ease we will work on $\tilde{V}(0, \widetilde{B})$.

From now on, let us use the notation $X^{s\wedge\tau,\pi}_{t\wedge\tau}$; $\quad 0 \leq s \leq t \leq T$ to refer to the wealth process restricted to $[s \wedge \tau, t \wedge \tau]$ and started from $x = 0$.

We give the dynamic extension of the initial problem (7.2.5). For any initial time $0 \leq t \leq T$ define

$$\mathcal{L}_t := \operatorname{essinf}_{\pi\in\mathcal{A}} E^{\widetilde{Q}}[e^{-\gamma(X^{t\wedge\tau,\pi}_{T\wedge\tau}+\widetilde{B})}|\mathcal{G}_{t\wedge\tau}]. \tag{7.2.6}$$

Note that $\tilde{V}(0, \widetilde{B}) = -\mathcal{L}(0)$. For any $\pi \in \mathcal{A}$ we define $\mathcal{L}^\pi$ as follows

$$\mathcal{L}^\pi_t := E^{\widetilde{Q}}[e^{-\gamma(X^{t\wedge\tau,\pi}_{T\wedge\tau}+\widetilde{B})}|\mathcal{G}_t], \quad \forall t \in [0, T]. \tag{7.2.7}$$

**Proposition 7.2.1.** *For any $\pi \in \mathcal{A}$, we consider the following function*

$$f^\pi(t, y, z) := y\{-\frac{\gamma^2\pi^2\sigma^2}{2} + \gamma\pi\mu\} + \gamma\pi\sigma z, \ \forall \ (y, z), \tag{7.2.8}$$

*and the associated BSDE given by*

$$\begin{cases} dY_t = f^\pi(t\, Y_t, Z_t)d(t\wedge\tau) + Z_t dW^\tau - U_t dN^{\mathbb{G}} \\ Y_T = e^{-\gamma\tilde{B}}. \end{cases} \tag{7.2.9}$$

*Then this BSDE has a unique solution denoted by $(Y^\pi, Z^\pi, U^\pi)$ satisfying $Y^\pi = \mathcal{L}^\pi$.*

*Proof.* From the definition of the process $\mathcal{L}^\pi$, we deduce that

$$M_t^{\mathbb{G}} := \widetilde{Z}_{t\wedge\tau} e^{-\gamma X_{t\wedge\tau}^{0,\pi}} \mathcal{L}_t^\pi = E[\widetilde{Z}_{T\wedge\tau} e^{-\gamma(X_{T\wedge\tau}^{0,\pi}+\widetilde{B})} | \mathcal{G}_t]$$

is a $\mathbb{G}$-martingale such that its terminal value $M_T^{\mathbb{G}} = h_\tau$ for an $\mathbb{F}$-optional process $h$. Hence, thanks to Theorem 2.3.6, we conclude the existence of $(M^{\mathbb{F}}, \varphi) \in \mathcal{M}_{0,loc}(\mathbb{F}) \times \mathcal{I}_{loc}^o\left(N^{\mathbb{G}}, \mathbb{G}\right)$ such that

$$\tilde{Z}_{t\wedge\tau} e^{-\gamma X_{t\wedge\tau}^{0,\pi}} \mathcal{L}_t^\pi = e^{-\gamma x} \mathcal{L}_0^\pi + \frac{1}{G_-^2} I_{\rrbracket 0,\tau \rrbracket} \bullet \mathcal{T}(M^{\mathbb{F}}) + \varphi \bullet N^{\mathbb{G}}. \tag{7.2.10}$$

Here, $M^{\mathbb{F}}$ is an $\mathbb{F}$-martingale, and due to the martingale representation theorem for the Brownian filtration, we obtain an $\mathbb{F}$-predictable processes $\psi$ such that $\mathcal{T}(M^{\mathbb{F}}) = \psi \bullet \mathcal{T}(W)$. Thus by inserting this in (7.2.10), we get

$$\mathcal{L}_t^\pi = \frac{e^{\gamma X_{t\wedge\tau}^{0,\pi}}}{\tilde{Z}_{t\wedge\tau}} \left( e^{-\gamma x} \mathcal{L}_0^\pi + \frac{1}{G_-^2} \psi \bullet \mathcal{T}(W) + \varphi \bullet N^{\mathbb{G}} \right).$$

Now, by applying Itô and the integration by-part formula repeatedly for the terms in the right-hand-side of the equation above, we get

$$\begin{aligned}
\mathcal{L}_t^\pi = & \frac{e^{\gamma (X_{t\wedge\tau}^{0,\pi})_-}}{(\tilde{Z}_{t\wedge\tau})_-} \bullet (e^{-\gamma x} \mathcal{L}_0^\pi + \frac{1}{G_-^2} \psi \bullet \mathcal{T}(W) + \varphi \bullet N^{\mathbb{G}}) \\
& + (e^{-\gamma x} \mathcal{L}_0^\pi + \frac{1}{G_-^2} \psi \bullet \mathcal{T}(W) + \varphi \bullet N^{\mathbb{G}})_- \bullet \frac{e^{\gamma X_{t\wedge\tau}^{0,\pi}}}{\tilde{Z}_{t\wedge\tau}} \\
& + [\frac{e^{\gamma X_{t\wedge\tau}^{0,\pi}}}{\tilde{Z}_{t\wedge\tau}}, e^{-\hat{\gamma} x} \mathcal{L}_0^\pi + \frac{1}{G_-^2} \psi \bullet \mathcal{T}(W) + \varphi \bullet N^{\mathbb{G}}]
\end{aligned}$$

$$= \frac{e^{\gamma (X_{t\wedge\tau}^{0,\pi})_-}}{(\tilde{Z}_{t\wedge\tau})_-} \bullet (e^{-\gamma x} \mathcal{L}_0^\pi + \frac{1}{G_-^2} \psi \bullet \mathcal{T}(W) + \varphi \bullet N^{\mathbb{G}})$$

$$
\begin{aligned}
&+ (\tilde{Z}_{t\wedge\tau})_- e^{-\gamma(X^{0,\pi}_{t\wedge\tau})_-}\mathcal{L}^\pi_{t-} \bullet \frac{e^{\gamma X^{0,\pi}_{t\wedge\tau}}}{\tilde{Z}_{t\wedge\tau}} \\
&+ \left[\frac{e^{\gamma X^{0,\pi}_{t\wedge\tau}}}{\tilde{Z}_{t\wedge\tau}}, e^{-\gamma x}\mathcal{L}^\pi_0 + \frac{1}{G^2_-}\psi \bullet \mathcal{T}(W) + \varphi \bullet N^{\mathbb{G}}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{e^{\gamma(X^{0,\pi}_{t\wedge\tau})_-}}{(\tilde{Z}_{t\wedge\tau})_-} \bullet (e^{-\gamma x}\mathcal{L}^\pi_0 + \frac{1}{G^2_-}\psi \bullet \mathcal{T}(W) + \varphi \bullet N^{\mathbb{G}}) \\
&\quad + (\tilde{Z}_{t\wedge\tau})_- e^{-\gamma(X^{0,\pi}_{t\wedge\tau})_-}\mathcal{L}^\pi_{t-} \bullet \frac{e^{\gamma(X^{0,\pi}_{t\wedge\tau})_-}}{(\tilde{Z}_{t\wedge\tau})_-} \bullet \big((\beta^{(m)} + \gamma\pi\sigma) \bullet \mathcal{T}(W) \\
&\quad + \{\gamma\pi\mu + \frac{\hat{\gamma}^2\pi^2\sigma^2}{2} + 2\hat{\gamma}\pi\sigma\beta^{(m)} + (\beta^{(m)})^2\} \bullet (t\wedge\tau)\big) \\
&\quad + [\frac{e^{\gamma(X^{0,\pi}_{t\wedge\tau})_-}}{(\tilde{Z}_{t\wedge\tau})_-} \bullet \big((\beta^{(m)} + \gamma\pi\sigma) \bullet \mathcal{T}(W) \\
&+ \{\gamma\pi\mu + \frac{\hat{\gamma}^2\pi^2\sigma^2}{2} + 2\gamma\pi\sigma\beta^{(m)} + (\beta^{(m)})^2\} \bullet (t\wedge\tau)\big)\big), \\
&\quad e^{-\gamma x}\mathcal{L}^\pi_0 + \frac{1}{G^2_-}\psi \bullet \mathcal{T}(W) + \varphi \bullet N^{\mathbb{G}}] \\
&= \frac{e^{\gamma(X^{0,\pi}_{t\wedge\tau})_-}}{(\tilde{Z}_{t\wedge\tau})_-} \bullet (\frac{1}{G^2_-}\psi \bullet \mathcal{T}(W) + \varphi \bullet N^{\mathbb{G}}) \\
&\quad + \mathcal{L}^\pi_{t-} \bullet \Big((\beta^{(m)} + \gamma\pi\sigma) \bullet \mathcal{T}(W)\Big) \\
&\quad + \mathcal{L}^\pi_{t-} \bullet \Big(\{\gamma\pi\mu + \frac{\gamma^2\pi^2\sigma^2}{2} + 2\gamma\pi\sigma\beta^{(m)} + (\beta^{(m)})^2\} \bullet (t\wedge\tau)\Big) \\
&\quad + \frac{e^{\gamma(X^{0,\pi}_{t\wedge\tau})_-}}{(\tilde{Z}_{t\wedge\tau})_-}\frac{\psi}{G^2_-}(\gamma\pi\sigma + \beta^{(m)}) \bullet (t\wedge\tau)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{e^{\gamma(X^{0,\pi}_{t\wedge\tau})_-}}{(\tilde{Z}_{t\wedge\tau})_-} \bullet (\frac{1}{G^2_-}\psi \bullet \mathcal{T}(W) + \varphi \bullet N^{\mathbb{G}}) + \mathcal{L}^\pi_{t-} \bullet \big((\beta^{(m)} + \hat{\gamma}\pi\sigma) \bullet \mathcal{T}(W) \\
&\quad + \Big(\mathcal{L}^\pi_{t-}\{\gamma\pi\mu + \frac{\gamma^2\pi^2\sigma^2}{2} + 2\gamma\pi\sigma\beta^{(m)} + (\beta^{(m)})^2\}\Big) \bullet (t\wedge\tau) \\
&\quad + \Big(\frac{e^{\gamma(X^{0,\pi}_{t\wedge\tau})_-}}{(\tilde{Z}_{t\wedge\tau})_-}\frac{\psi}{G^2_-}(\gamma\pi\sigma + \beta^{(m)})\Big) \bullet (t\wedge\tau)
\end{aligned}
$$

$$
\begin{aligned}
&= \left( \frac{e^{\gamma(X^{0,\pi}_{t\wedge\tau})_-}}{(\tilde{Z}_{t\wedge\tau})_-} \frac{\psi}{G^2_-} + \mathcal{L}^\pi_{t-}(\beta^{(m)} + \hat{\gamma}\pi\sigma) \right) \bullet \mathcal{T}(W) + \frac{e^{\gamma(X^{0,\pi}_{t\wedge\tau})_-}}{(\tilde{Z}_{t\wedge\tau})_-} \varphi \bullet N^{\mathbb{G}} \\
&\quad + \Big( \mathcal{L}^\pi_{t-}\{\gamma\pi\mu + \frac{\gamma^2\pi^2\sigma^2}{2} + 2\gamma\pi\sigma\beta^{(m)} + (\beta^{(m)})^2\} \Big) \bullet (t \wedge \tau) \\
&\quad + \Big( \frac{e^{\gamma(X^{0,\pi}_{t\wedge\tau})_-}}{(\tilde{Z}_{t\wedge\tau})_-} \frac{\psi}{G^2_-} (\gamma\pi\sigma + \beta^{(m)}) \Big) \bullet (t \wedge \tau)
\end{aligned}
$$

$$
\begin{aligned}
&= \left( \frac{e^{\gamma(X^{0,\pi}_{t\wedge\tau})_-}}{(\tilde{Z}_{t\wedge\tau})_-} \frac{\psi}{G^2_-} + \mathcal{L}^\pi_{t-}(\beta^{(m)} + \gamma\pi\sigma) \right) \bullet W^\tau + \frac{e^{\gamma(X^{0,\pi}_{t\wedge\tau})_-}}{(\tilde{Z}_{t\wedge\tau})_-} \varphi \bullet N^{\mathbb{G}} \\
&\quad + \Big( \mathcal{L}^\pi_{t-}\{\gamma\pi\mu + \frac{\gamma^2\pi^2\sigma^2}{2} + 2\gamma\pi\sigma\beta^{(m)} + (\beta^{(m)})^2\} \\
&+ \frac{e^{\gamma(X^{0,\pi}_{t\wedge\tau})_-}}{(\tilde{Z}_{t\wedge\tau})_-} \frac{\psi}{G^2_-} (\gamma\pi\sigma + \beta^{(m)}) - \beta^{(m)} \frac{e^{\gamma(X^{0,\pi}_{t\wedge\tau})_-}}{(\tilde{Z}_{t\wedge\tau})_-} \frac{\psi}{G^2_-} \\
&\quad - \beta^{(m)} \mathcal{L}^\pi_{t-}(\beta^{(m)} + \gamma\pi\sigma) \Big) \bullet (t \wedge \tau)
\end{aligned}
$$

$$
\begin{aligned}
&= \left( \frac{e^{\gamma(X^{0,\pi}_{t\wedge\tau})_-}}{(\tilde{Z}_{t\wedge\tau})_-} \frac{\psi}{G^2_-} + \mathcal{L}^\pi_{t-}(\beta^{(m)} + \gamma\pi\sigma) \right) \bullet W^\tau + \frac{e^{\gamma(X^{0,\pi}_{t\wedge\tau})_-}}{(\tilde{Z}_{t\wedge\tau})_-} \varphi \bullet N^{\mathbb{G}} \\
&\quad + \Big( \mathcal{L}^\pi_{t-}\{\gamma\pi\mu + \frac{\hat{\gamma}^2\pi^2\sigma^2}{2} + \gamma\pi\sigma\beta^{(m)}\} + \frac{e^{\gamma(X^{0,\pi}_{t\wedge\tau})_-}}{(\tilde{Z}_{t\wedge\tau})_-} \frac{\psi}{G^2_-} \gamma\pi\sigma \Big) \bullet (t \wedge \tau)
\end{aligned}
$$

This implies that

$$
\begin{aligned}
d\mathcal{L}^\pi_t &= \left( \frac{e^{\gamma(X^{0,\pi}_{t\wedge\tau})_-}}{(\tilde{Z}_{t\wedge\tau})_-} \frac{\psi}{G^2_-} + \mathcal{L}^\pi_{t-}(\beta^{(m)} + \gamma\pi\sigma) \right) dW^\tau + \frac{e^{\gamma(X^{0,\pi}_{t\wedge\tau})_-}}{(\tilde{Z}_{t\wedge\tau})_-} \varphi dN^{\mathbb{G}} \qquad (7.2.11) \\
&\quad + \Big( \mathcal{L}^\pi_{t-}\{\gamma\pi\mu + \frac{\hat{\gamma}^2\pi^2\sigma^2}{2} + \gamma\pi\sigma\beta^{(m)}\} + \frac{e^{\gamma(X^{0,\pi}_{t\wedge\tau})_-}}{(\tilde{Z}_{t\wedge\tau})_-} \frac{\psi}{G^2_-} \gamma\pi\sigma \Big) d(t \wedge \tau)
\end{aligned}
$$

Put

$$Z^{\pi}_t := \frac{e^{\gamma(X^{0,\pi}_{t\wedge\tau})_-}}{(\tilde{Z}_{t\wedge\tau})_-}\frac{\psi}{G^2_-} + \mathcal{L}^{\pi}_{t-}(\beta^{(m)}I_{]\!]0,\tau]\!]} + \gamma\pi\sigma)I_{]\!]0,\tau]\!]}, \quad U^{\pi}_t := -\frac{e^{\gamma X^{0,\pi}_{t-}}}{\widetilde{Z}_{t-}}\varphi_t. \quad (7.2.12)$$

Then, by using these notations that gives us

$$\psi = G^2_-(\tilde{Z}_{t\wedge\tau})_- e^{-\gamma(X^{0,\pi}_{t\wedge\tau})_-}Z^{\pi}_{t\wedge\tau} - G^2_-(\tilde{Z}_{t\wedge\tau})_- e^{-\gamma(X^{0,\pi}_{t\wedge\tau})_-}\mathcal{L}^{\pi}_{t-}(\gamma\pi\sigma + \beta^{(m)}),$$

we obtain

$$\begin{aligned}
&\mathcal{L}^{\pi}_{t-}\{\gamma\pi\mu + \frac{\gamma^2\pi^2\sigma^2}{2} + \gamma\pi\sigma\beta^{(m)}\} + \frac{e^{\gamma(X^{0,\pi}_{t\wedge\tau})_-}}{(\tilde{Z}_{t\wedge\tau})_-}\frac{\psi}{G^2_-}\gamma\pi\sigma \\
&= \mathcal{L}^{\pi}_-\{-\frac{\gamma^2\pi^2\sigma^2}{2} + \gamma\pi\mu\} + \gamma\pi\sigma Z^{\pi}_{t\wedge\tau}.
\end{aligned}$$

By combining this equation with (7.2.12) and (7.2.11), we get the following dynamics

$$d\mathcal{L}^{\pi}_t = f^{\pi}(t, \mathcal{L}^{\pi}_t, Z^{\pi}_t, U^{\pi}_t)d(t\wedge\tau) + Z^{\pi}_t d\, W^{\tau} - U^{\pi}_t dN^{\mathbb{G}}.$$

This proves the proposition. $\square$

In the following we present our main result in this subsection.

**Theorem 7.2.2.** *Let $\lambda := \mu/\sigma$, and consider the following BSDE*

$$\begin{cases} -dY_t = -\frac{Y_t}{2}\left(Z_t Y_t^{-1} + \lambda\right)^2 d(t\wedge\tau) - Z_t dW^{\tau} + U_t dN^{\mathbb{G}} \\ Y_T = e^{-\hat{\gamma}\widetilde{B}}. \end{cases} \quad (7.2.13)$$

*Then if the solution of this BSDE exists and is denoted by $(Y^{(P)}, Z^{(P)}, U^{(P)})$, then we have that $Y^{(P)} = \mathcal{L}$, where $\mathcal{L}$ is defined in (7.2.6).*

*Proof.* First recall that, by proposition 7.2.1, for any $\pi \in \mathcal{A}$ we have

$$\begin{aligned}\mathcal{L}_t^\pi =& e^{-\gamma\tilde{B}} - \int_{t\wedge\tau}^{T\wedge\tau} \left(\mathcal{L}_s^\pi\{-\frac{\gamma^2\pi^2\sigma^2}{2} + \gamma\pi\mu\} + \gamma\pi\sigma Z_s^\pi\right) ds - \int_{t\wedge\tau}^{T\wedge\tau} Z_s^\pi dW^\tau \\ &+ \int_{t\wedge\tau}^{T\wedge\tau} U_s^\pi dN^{\mathbb{G}}.\end{aligned}$$

and hence,

$$\mathcal{L}_t^\pi = E^{\widetilde{Q}}\left[e^{-\gamma\tilde{B}} - \int_{t\wedge\tau}^{T\wedge\tau} \left(\mathcal{L}_s^\pi\{-\frac{\gamma^2\pi^2\sigma^2}{2} + \gamma\pi\mu\} + \gamma\pi\sigma Z_s^\pi\right) ds|\mathcal{G}_t\right]$$

By using the fact that $\mathcal{L} := \text{essinf}_{\pi\in\mathcal{A}}\mathcal{L}_t^\pi$ for any $0 \le t \le T$, we have

$$\begin{aligned}\mathcal{L}_t &= \text{essinf}_{\pi\in\mathcal{A}_{t\wedge\tau}} E^{\widetilde{Q}}\left[e^{-\gamma\tilde{B}} - \int_{t\wedge\tau}^{T\wedge\tau} \left(\mathcal{L}_s^\pi\{-\frac{\gamma^2\pi^2\sigma^2}{2} + \gamma\pi\mu\} + \gamma\pi\sigma Z_s^\pi\right) ds|\mathcal{G}_t\right] \\ &= E^{\widetilde{Q}}\left[e^{-\gamma\tilde{B}} + \int_{t\wedge\tau}^{T\wedge\tau} \text{essinf}_{\pi\in\mathcal{A}} - \left(\mathcal{L}_s^\pi\{-\frac{\gamma^2\pi^2\sigma^2}{2} + \gamma\pi\mu\} + \gamma\pi\sigma Z_s^\pi\right) ds|\mathcal{G}_t\right]\end{aligned}$$

therefore the process $\mathcal{L}$ corresponds to the solution of a BSDE, whose driver is the essential infimum over $\pi$ of the drivers of $(\mathcal{L}^\pi)_{\pi\in\mathcal{A}}$. And as

$$\operatorname*{essinf}_{\pi\in\mathcal{A}} \; - \left(y\{-\frac{\gamma^2\pi^2\sigma^2}{2} + \gamma\pi\mu\} + \gamma\pi\sigma z\right) = -\frac{y}{2}\left(\frac{z}{y} + \frac{\mu}{\sigma}\right)^2 = -\frac{y}{2}\left(\frac{z}{y} + \lambda\right)^2,$$

where $\lambda := \frac{\mu}{\sigma}$. The optimal $\pi^*$ is given by $\sigma\pi^* = \frac{1}{\hat{\gamma}}\left(\frac{\mu}{\sigma} + \frac{z}{y}\right) = \frac{1}{\hat{\gamma}}\left(\lambda + \frac{z}{y}\right)$ we get our BSDE in (7.2.13). $\square$

## 7.3 BSDE formulation for the dual problem

In this section, we derive the BSDE associated to the dual problem to the primal problem (7.2.5) . To this end, we start deriving the duality and hence defining this dual problem. For the reader's convenience, we recall the definition of relative entropy $\mathcal{H}(Q_1|Q_2)$ for two probabilities $Q_i$, $i=1,2$ as follows.

$$\mathcal{H}(Q_1|Q_2) := \begin{cases} E\left[\frac{dQ_1}{dQ_2}\ln\frac{dQ_1}{dQ_2}\right] & \text{if } Q_1 \ll Q_2 \\ +\infty & \text{otherwise.} \end{cases}$$

Recall that $\lambda$ be the risk premium process. That is $\mu_t := \sigma_t\lambda_t$. For each bounded process $\varphi \in \mathcal{I}^o(N^{\mathbb{G}},\mathbb{G})$, define the probability measure $Q^{\mathbb{G},\varphi}$ as the following

$$Z^{\mathbb{G},\varphi} := \mathcal{E}\left(-\lambda \bullet W^{\tau}\right)\mathcal{E}(\varphi \bullet N^{\mathbb{G}}) \overset{Yor}{=} \mathcal{E}\left(-\lambda \bullet W^{\tau} + \varphi \bullet N^{\mathbb{G}}\right)$$

define the $(\mathbb{G}, Q^{\mathbb{G},\varphi})$-Brownian motion $\widehat{W}$ by the formula:

$$\widehat{W}_t := W_t^{\tau} + \lambda \bullet (t \wedge \tau).$$

The following two probabilities defined on $\mathcal{F}_T$ and $\mathcal{G}_{T\wedge\tau}$ respectively will play important roles in the rest of this chapter.

$$\widehat{Q}^{\mathbb{F}} := \mathcal{E}_T\left(-\lambda \bullet W^{\tau}\right)\cdot P, \quad \widehat{Q}^{\mathbb{G}} := \mathcal{E}_{T\wedge\tau}\left(-\lambda \bullet W^{\tau}\right)\widetilde{Z}_T \cdot P. \tag{7.3.1}$$

Then we consider the dual set to the dual problem that we will define afterwards.

$$\Phi_f := \left\{ \varphi \in \mathcal{I}^o(N^{\mathbb{G}}, \mathbb{G}) \ : -1 < \varphi \text{ and } E^{\widehat{Q}^{\mathbb{G}}}\left[ Z_T^{\varphi} \ln(Z_T^{\varphi}) \right] < \infty \right\}, \ Z^{\varphi} := \mathcal{E}(\varphi \bullet N^{\mathbb{G}}). \tag{7.3.2}$$

This for any $\varphi \in \Phi_f$, the process $Z^{\varphi}$ is a true martinagle under $\widehat{Q}^{\mathbb{G}}$, and hence the following probability that will use later on is well defined

$$Q^{\mathbb{G},\varphi} := Z_T^{\mathbb{G},\varphi} \cdot \widetilde{Q} := Z^{\varphi} \mathcal{E}_{T\wedge\tau}(-\lambda \bullet W) \cdot \widetilde{Q}, \tag{7.3.3}$$

Until the rest of this chapter we assume that $\Delta D^{o,\mathbb{F}} = 0$.

**Lemma 7.3.1.** *For any $\varphi \in \Phi_f$, if $\Delta D^{o,\mathbb{F}} = 0$, then the following two processes*

$$N_t^{\mathbb{G},\varphi} = N_t^{\mathbb{G}} - \frac{\varphi}{1+\varphi} \bullet D \quad \textit{and} \quad \widehat{N}_t^{\mathbb{G},\varphi} = N_t^{\mathbb{G}} - \frac{\varphi}{\widetilde{G}} I_{[\![0,\tau]\!]} \bullet D^{o,\mathbb{F}} \tag{7.3.4}$$

*belong to $\mathcal{M}_{loc}(Q^{\mathbb{G},\varphi})$.*

*Proof.* **Step 1.** This step focuses in proving the first part of (7.3.4). Due to a combination of $N^{\mathbb{G}} := D - \widetilde{G}^{-1} I_{[\![0,\tau[\![} \bullet D^{o,\mathbb{F}}$, $\Delta D^{o,\mathbb{F}} = 0$, and Girsanov's theorem we get that

$$N_t^{\mathbb{G},\varphi} : = N_t^{\mathbb{G}} - \frac{\varphi}{1+\varphi} \bullet D.$$

belongs to $\mathcal{M}_{loc}(Q^{\mathbb{G},\varphi})$.

**Step 2.** This step focuses in proving the second part of (7.3.4). Due to step (1), we have that

$$(1+\varphi) \bullet N^{\mathbb{G},\varphi} \in \mathcal{M}_{loc}(Q^{\mathbb{G},\varphi}).$$

However,

$$\begin{aligned}
(1+\varphi)\bullet N^{\mathbb{G},\varphi} &= (1+\varphi)\bullet (N^{\mathbb{G}} - \frac{\varphi}{1+\varphi}\bullet D)\\
&= (1+\varphi)\bullet \Big(N^{\mathbb{G}} - \frac{\varphi}{1+\varphi}\bullet (N^{\mathbb{G}} + \widetilde{G}^{-1} I_{[\![0,\tau[\![}\bullet D^{o,\mathbb{F}})\Big)\\
&= (1+\varphi)\bullet \Big(\frac{1}{1+\varphi}\bullet N^{\mathbb{G}} - \frac{\varphi}{1+\varphi}\widetilde{G}^{-1} I_{[\![0,\tau[\![}\bullet D^{o,\mathbb{F}}\Big)\\
&= N_t^{\mathbb{G}} - \frac{\varphi}{\widetilde{G}} I_{[\![0,\tau]\!]}\bullet D^{o,\mathbb{F}} = \widehat{N}^{\mathbb{G},\varphi}.
\end{aligned}$$

□

By Theorem 2.1 of Frittelli (2000) there exists a unique equivalent local martingale measure ( ELMM) $\widetilde{Q}^E$ that minimize $\mathcal{H}(Q|\widetilde{Q})$. This $\widetilde{Q}^E$ is called the minimal entropy martingale measure or minimal $\widetilde{Q}$-entropy martingale measure. The density of $\widetilde{Q}^E$ with respect to $\widetilde{Q}$ has the form

$$Z_T^E := \frac{d\widetilde{Q}^E}{d\widetilde{Q}} = c_E e^{\hat{\gamma} X^{0,\pi^E}_{T\wedge\tau}}$$

for some constant $c_E > 0$ and some $\pi^E \in L(\int_0^{\cdot\wedge\tau} \frac{dS}{S})$ such that $X^{0,\pi^E}$ is $\widetilde{Q}^E$-martingale, and

$$E^{\widetilde{Q}^E}\left[\ln\left(\frac{Z_T^E}{Z_t^E}\right)\Big|\mathcal{G}_t\right] = \ln c_E + \hat{\gamma} X^{0,\pi^E}_{t\wedge\tau} - \ln Z_t^E, \quad 0\le t\le T. \tag{7.3.5}$$

Following exactly the same footsteps as above, define the probability measure $\widetilde{Q}_{\widetilde{B}}$ equivalent to $\widetilde{Q}$ by

$$\frac{d\widetilde{Q}_{\widetilde{B}}}{d\widetilde{Q}} := c_{\widetilde{B}} e^{-\hat{\gamma}\widetilde{B}} \qquad \text{with } c_{\widetilde{B}}^{-1} := E^{\widetilde{Q}}[e^{-\hat{\gamma}\widetilde{B}}] \in (0,\infty).$$

let $Z^{\widetilde{B}}$ denotes its density process with $Z^{\widetilde{B}}_T = \frac{d\widetilde{Q}_{\widetilde{B}}}{d\widetilde{Q}}$, we get $\widetilde{Q}^{E,\widetilde{B}}$ is the unique ELMM in $\mathbb{Q}_f(\widetilde{Q}_{\widetilde{B}})$:= { the set of all ELLM $Q$ with $\mathcal{H}(Q|\widetilde{Q}_{\widetilde{B}}) < \infty$}, that minimize $\mathcal{H}(Q|\widetilde{Q}_{\widetilde{B}})$ over all $Q \in \mathbb{Q}_f(\widetilde{Q}_{\widetilde{B}})$, we obtain

$$Z^{E,\widetilde{B}}_T := \frac{d\widetilde{Q}^{E,\widetilde{B}}}{d\widetilde{Q}_{\widetilde{B}}} = c_{E,\widetilde{B}} e^{\hat{\gamma} X^{0,\pi^{E,\widetilde{B}}}_{T\wedge\tau}}$$

for some constant $c_{E,\widetilde{B}} > 0$ and some $\pi^{E,\widetilde{B}} \in L(\int_0^{\cdot\wedge\tau} \frac{dS}{S})$ such that $X^{0,\pi^{E,\widetilde{B}}}$ is $(\widetilde{Q}^{E,\widetilde{B}}, \mathbb{G})$-martingale .

Let

$$\mathcal{A} := \left\{ \pi \in L\left(\int_0^{\cdot\wedge\tau} \frac{dS}{S}\right) \Big| \begin{array}{l} e^{-\hat{\gamma}\int_0^{T\wedge\tau} \frac{\pi_s}{S_s} dS_s} \in \mathbb{L}^1_T(\widetilde{Q}_{\widetilde{B}}, \mathbb{G}), \\ \text{and } \left(\frac{\pi}{S} \bullet S_t\right)_{\{t \le T\wedge\tau\}} \in \mathcal{M}(\widetilde{Q}^{E,\widetilde{B}}, \mathbb{G}) \end{array} \right\} \tag{7.3.6}$$

be the set of all admissible strategies. The rest of this chapter we consider the following assumption

$$B \in L^\infty(\mathbb{G}_T, P), \quad \text{and } E[\widetilde{Z}_T \ln(\widetilde{Z}_T)] < \infty. \tag{7.3.7}$$

We give the relation between the primal problem (7.2.4) and the dual problem due to the following theorem.

**Theorem 7.3.2.** *Assume (7.3.7) holds. Then*

$$\begin{aligned} &-\operatorname*{\mathit{ess}\,inf}_{\pi\in\mathcal{A}} E^{\widetilde{Q}}[e^{-\gamma(X^{t\wedge\tau,\pi}_{T\wedge\tau} + \widetilde{B})} \big| \mathcal{G}_t] \\ &\qquad = -\exp\Big\{ -\operatorname*{\mathit{ess}\,inf}_{\varphi\in\Phi_f} E^{Q^{\mathbb{G},\varphi}} \left[ \ln\left(\frac{Z^{\mathbb{G},\varphi}_T}{Z^{\mathbb{G},\varphi}_t}\right) + \gamma\widetilde{B} \big| \mathcal{G}_t \right] \Big\}. \end{aligned} \tag{7.3.8}$$

*Proof.* Let us start with the LHS of (7.3.8):

$$
\begin{aligned}
&-\operatorname*{ess\,inf}_{\pi\in\mathcal{A}} E^{\widetilde{Q}}[e^{-\gamma(X^{t\wedge\tau,\pi}_{T\wedge\tau}+\widetilde{B})}\big|\mathcal{G}_t] = \operatorname*{ess\,sup}_{\pi\in\mathcal{A}} E^{\widetilde{Q}}[e^{-\gamma(X^{t\wedge\tau,\pi}_{T\wedge\tau}+\widetilde{B})}\big|\mathcal{G}_t] \\
&= \operatorname*{ess\,sup}_{\pi\in\mathcal{A}} -E^{\widetilde{Q}}\left[E[e^{-\gamma\widetilde{B}}]\frac{e^{-\gamma\widetilde{B}}}{E[e^{-\gamma\widetilde{B}}]}e^{-\gamma X^{t\wedge\tau,\pi}_{T\wedge\tau}}\big|\mathcal{G}_t\right] \\
&= c_{\widetilde{B}}^{-1}\operatorname*{ess\,sup}_{\pi\in\mathcal{A}} -E^{\widetilde{Q}}\left[\frac{Z^{\widetilde{B}}_t}{Z^{\widetilde{B}}_t}Z^{\widetilde{B}}_T e^{-\gamma X^{t\wedge\tau,\pi}_{T\wedge\tau}}\big|\mathcal{G}_t\right] \\
&= c_{\widetilde{B}}^{-1}Z^{\widetilde{B}}_t\operatorname*{ess\,sup}_{\pi\in\mathcal{A}} -E^{\widetilde{Q}_{\widetilde{B}}}\left[e^{-\gamma X^{t\wedge\tau,\pi}_{T\wedge\tau}}\big|\mathcal{G}_t\right] \\
&= c_{\widetilde{B}}^{-1}Z^{\widetilde{B}}_t\operatorname*{ess\,sup}_{\pi\in\mathcal{A}} -E^{\widetilde{Q}_{\widetilde{B}}}\left[\frac{Z^{E,\widetilde{B}}_T}{Z^{E,\widetilde{B}}_t}\frac{Z^{E,\widetilde{B}}_t}{Z^{E,\widetilde{B}}_T}e^{-\gamma X^{t\wedge\tau,\pi}_{T\wedge\tau}}\big|\mathcal{G}_t\right] \\
&= c_{\widetilde{B}}^{-1}Z^{\widetilde{B}}_t\operatorname*{ess\,sup}_{\pi\in\mathcal{A}} -E^{\widetilde{Q}^{E,\widetilde{B}}}\left[\frac{Z^{E,\widetilde{B}}_t}{Z^{E,\widetilde{B}}_T}e^{-\gamma X^{t\wedge\tau,\pi}_{T\wedge\tau}}\big|\mathcal{G}_t\right] \\
&= c_{\widetilde{B}}^{-1}Z^{\widetilde{B}}_t Z^{E,\widetilde{B}}_t\operatorname*{ess\,sup}_{\pi\in\mathcal{A}} -E^{\widetilde{Q}^{E,\widetilde{B}}}\left[\frac{1}{c_{E,\widetilde{B}}e^{\gamma X^{0,\pi^{E,\widetilde{B}}}_{T\wedge\tau}}}e^{-\hat{\gamma} X^{t\wedge\tau,\pi}_{T\wedge\tau}}\big|\mathcal{G}_t\right] \\
&= c_{\widetilde{B}}^{-1}c_{E,\widetilde{B}}^{-1}Z^{\widetilde{B}}_t Z^{E,\widetilde{B}}_t\operatorname*{ess\,sup}_{\pi\in\mathcal{A}} -E^{\widetilde{Q}^{E,\widetilde{B}}}\left[e^{-\gamma(X^{t\wedge\tau,\pi}_{T\wedge\tau}+X^{0,\pi^{E,\widetilde{B}}}_{T\wedge\tau})}\big|\mathcal{G}_t\right] \\
&\overset{Jensen}{\leq} c_{\widetilde{B}}^{-1}c_{E,\widetilde{B}}^{-1}Z^{\widetilde{B}}_t Z^{E,\widetilde{B}}_t\operatorname*{ess\,sup}_{\pi\in\mathcal{A}} -\left[e^{-\gamma E^{\widetilde{Q}^{E,\widetilde{B}}}[(X^{t\wedge\tau,\pi}_{T\wedge\tau}+X^{0,\pi^{E,\widetilde{B}}}_{t\wedge\tau})|\mathcal{G}_t]}\right] \\
&= -c_{\widetilde{B}}^{-1}c_{E,\widetilde{B}}^{-1}Z^{\widetilde{B}}_t Z^{E,\widetilde{B}}_t e^{-\gamma X^{0,\pi^{E,\widetilde{B}}}_{t\wedge\tau}}.
\end{aligned}
\tag{7.3.9}
$$

Now, need to calculate the RHS of (7.3.8):

$$
\begin{aligned}
&\operatorname*{ess\,inf}_{\varphi\in\Phi_f} E^{Q^{\mathbb{G},\varphi}}\left[\ln\left(\frac{Z^{\mathbb{G},\varphi}_T}{Z^{\mathbb{G},\varphi}_t}\right)+\gamma\widetilde{B}\big|\mathcal{G}_t\right] \\
&= \operatorname*{ess\,inf}_{\varphi\in\Phi_f}\left(E^{\widetilde{Q}}\left[\frac{dQ^{\mathbb{G},\varphi}/d\widetilde{Q}}{Z^{\mathbb{G},\varphi}_t}\ln\left(\frac{dQ^{\mathbb{G},\varphi}/d\widetilde{Q}}{Z^{\mathbb{G},\varphi}_t}\right)\big|\mathcal{G}_t\right]+E^{Q^{\mathbb{G},\varphi}}\left[\gamma\widetilde{B}\big|\mathcal{G}_t\right]\right) \\
&= \operatorname*{ess\,inf}_{\varphi\in\Phi_f}\left(\begin{array}{l} E^{\widetilde{Q}}\left[\frac{1}{Z^{\mathbb{G},\varphi}_t}\frac{dQ^{\mathbb{G},\varphi}/d\widetilde{Q}_{\widetilde{B}}}{d\widetilde{Q}/d\widetilde{Q}_{\widetilde{B}}}\ln\left(c_{\widetilde{B}}e^{-\hat{\gamma}\widetilde{B}}\frac{dQ^{\mathbb{G},\varphi}/d\widetilde{Q}_{\widetilde{B}}}{Z^{\mathbb{G},\varphi}_t}\right)\big|\mathcal{G}_t\right] \\ +E^{Q^{\mathbb{G},\varphi}}\left[\gamma\widetilde{B}\big|\mathcal{G}_t\right] \end{array}\right)
\end{aligned}
$$

$$
\begin{aligned}
= \ \operatorname*{ess\,inf}_{\varphi\in\Phi_f} & \left( \begin{array}{l} E^{\widetilde{Q}_{\widetilde{B}}}\left[\frac{Z_t^{\widetilde{B}}}{Z_t^{\mathbb{G},\varphi}}\frac{dQ^{\mathbb{G},\varphi}}{d\widetilde{Q}_{\widetilde{B}}}\ln\left(c_{\widetilde{B}}e^{-\hat{\gamma}\widetilde{B}}\frac{dQ^{\mathbb{G},\varphi}/d\widetilde{Q}_{\widetilde{B}}}{Z_t^{\mathbb{G},\varphi}}\right)\big|\mathcal{G}_t\right] \\ +E^{Q^{\mathbb{G},\varphi}}\left[\gamma\widetilde{B}\big|\mathcal{G}_t\right] \end{array} \right) \\
= \ \operatorname*{ess\,inf}_{\varphi\in\Phi_f} & \left( \begin{array}{l} E^{\widetilde{Q}_{\widetilde{B}}}\left[\frac{Z_t^{\widetilde{B}}}{Z_t^{\mathbb{G},\varphi}}\frac{dQ^{\mathbb{G},\varphi}}{d\widetilde{Q}_{\widetilde{B}}}\left(\ln(c_{\widetilde{B}})-\hat{\gamma}\widetilde{B}+\ln(dQ^{\mathbb{G},\varphi}/d\widetilde{Q}_{\widetilde{B}})\right)\big|\mathcal{G}_t\right] \\ -E^{\widetilde{Q}_{\widetilde{B}}}\left[\frac{Z_t^{\widetilde{B}}}{Z_t^{\mathbb{G},\varphi}}\frac{dQ^{\mathbb{G},\varphi}}{d\widetilde{Q}_{\widetilde{B}}}\ln(Z_t^{\mathbb{G},\varphi})\big|\mathcal{G}_t\right]+E^{Q^{\mathbb{G},\varphi}}\left[\gamma\widetilde{B}\big|\mathcal{G}_t\right] \end{array} \right)
\end{aligned}
\tag{7.3.10}
$$

But

$$
\begin{aligned}
&E^{\widetilde{Q}_{\widetilde{B}}}\left[\frac{Z_t^{\widetilde{B}}}{Z_t^{\mathbb{G},\varphi}}\frac{dQ^{\mathbb{G},\varphi}}{d\widetilde{Q}_{\widetilde{B}}}\ln(c_{\widetilde{B}})\big|\mathcal{G}_t\right] = \ln(c_{\widetilde{B}})E^{\widetilde{Q}_{\widetilde{B}}}\left[\frac{Z_t^{\widetilde{B}}}{Z_t^{\mathbb{G},\varphi}}\frac{dQ^{\mathbb{G},\varphi}/d\widetilde{Q}}{d\widetilde{Q}_{\widetilde{B}}/d\widetilde{Q}}\big|\mathcal{G}_t\right] \\
&= \ln(c_{\widetilde{B}})E^{\widetilde{Q}}\left[\frac{Z_T^{\widetilde{B}}}{Z_t^{\widetilde{B}}}\frac{Z_t^{\widetilde{B}}}{Z_t^{\mathbb{G},\varphi}}\frac{dQ^{\mathbb{G},\varphi}/d\widetilde{Q}}{d\widetilde{Q}_{\widetilde{B}}/d\widetilde{Q}}\big|\mathcal{G}_t\right] = \ln(c_{\widetilde{B}})E^{\widetilde{Q}}\left[\frac{Z_T^{\widetilde{B}}}{Z_t^{\widetilde{B}}}\frac{Z_t^{\widetilde{B}}}{Z_t^{\mathbb{G},\varphi}}\frac{Z_T^{\mathbb{G},\varphi}}{Z_T^{\widetilde{B}}}\big|\mathcal{G}_t\right] \\
&= \ln(c_{\widetilde{B}})E^{\widetilde{Q}}\left[\frac{Z_T^{\mathbb{G},\varphi}}{Z_t^{\mathbb{G},\varphi}}\big|\mathcal{G}_t\right] = \ln(c_{\widetilde{B}})E^{Q^{\mathbb{G},\varphi}}\left[1\big|\mathcal{G}_t\right] = \ln(c_{\widetilde{B}}).
\end{aligned}
$$

And,

$$
\begin{aligned}
&E^{\widetilde{Q}_{\widetilde{B}}}\left[\frac{Z_t^{\widetilde{B}}}{Z_t^{\mathbb{G},\varphi}}\frac{dQ^{\mathbb{G},\varphi}}{d\widetilde{Q}_{\widetilde{B}}}\gamma\widetilde{B}\big|\mathcal{G}_t\right] = E^{\widetilde{Q}}\left[\frac{Z_T^{\widetilde{B}}}{Z_t^{\widetilde{B}}}\frac{Z_t^{\widetilde{B}}}{Z_t^{\mathbb{G},\varphi}}\frac{dQ^{\mathbb{G},\varphi}/d\widetilde{Q}}{d\widetilde{Q}_{\widetilde{B}}/d\widetilde{Q}}\gamma\widetilde{B}\big|\mathcal{G}_t\right] \\
&= E^{\widetilde{Q}}\left[\frac{Z_T^{\widetilde{B}}}{Z_t^{\widetilde{B}}}\frac{Z_t^{\widetilde{B}}}{Z_t^{\mathbb{G},\varphi}}\frac{Z_T^{\mathbb{G},\varphi}}{Z_T^{\widetilde{B}}}\gamma\widetilde{B}\big|\mathcal{G}_t\right] = E^{Q^{\mathbb{G},\varphi}}\left[\gamma\widetilde{B}\big|\mathcal{G}_t\right].
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&-\exp\left\{-\operatorname*{ess\,inf}_{\varphi\in\Phi_f} E^{Q^{\mathbb{G},\varphi}}\left[\ln\left(\frac{Z_T^{\mathbb{G},\varphi}}{Z_t^{\mathbb{G},\varphi}}\right)+\hat{\gamma}\widetilde{B}\big|\mathcal{G}_t\right]\right\} \\
&= -\frac{1}{c_{\widetilde{B}}}\exp\Big\{-\operatorname*{ess\,inf}_{\varphi\in\Phi_f} E^{\widetilde{Q}_{\widetilde{B}}}\Big[\frac{Z_t^{\widetilde{B}}}{Z_t^{\mathbb{G},\varphi}}\frac{dQ^{\mathbb{G},\varphi}}{d\widetilde{Q}_{\widetilde{B}}}\left(\ln(\frac{dQ^{\mathbb{G},\varphi}}{d\widetilde{Q}_{\widetilde{B}}})-\ln(Z_t^{\mathbb{G},\varphi})\right)\big|\mathcal{G}_t\Big]\Big\}
\end{aligned}
\tag{7.3.11}
$$

And finally, in the same way as above we have

$$
\begin{aligned}
&\operatorname*{ess\,inf}_{\varphi\in\Phi_f} E^{\widetilde{Q}_{\widetilde{B}}}\left[\frac{Z_t^{\widetilde{B}}}{Z_t^{\mathbb{G},\varphi}}\frac{dQ^{\mathbb{G},\varphi}}{d\widetilde{Q}_{\widetilde{B}}}\Big(\ln(dQ^{\mathbb{G},\varphi}/d\widetilde{Q}_{\widetilde{B}})-\ln(Z_t^{\mathbb{G},\varphi})\Big)\big|\mathcal{G}_t\right]\\
&=\operatorname*{ess\,inf}_{\varphi\in\Phi_f} E^{\widetilde{Q}_{\widetilde{B}}}\left[\frac{Z_t^{\widetilde{B}}}{Z_t^{\mathbb{G},\varphi}}\frac{dQ^{\mathbb{G},\varphi}}{d\widetilde{Q}_{\widetilde{B}}}\left(\ln\left(\frac{dQ^{\mathbb{G},\varphi}/d\widetilde{Q}_{\widetilde{B}}}{Z_t^{\mathbb{G},\varphi}/Z_t^{\widetilde{B}}}\frac{Z_t^{\mathbb{G},\varphi}}{Z_t^{\widetilde{B}}}\right)-\ln(Z_t^{\mathbb{G},\varphi})\right)\big|\mathcal{G}_t\right]\\
&=\operatorname*{ess\,inf}_{\varphi\in\Phi_f} E^{\widetilde{Q}_{\widetilde{B}}}\left[\frac{Z_t^{\widetilde{B}}}{Z_t^{\mathbb{G},\varphi}}\frac{dQ^{\mathbb{G},\varphi}}{d\widetilde{Q}_{\widetilde{B}}}\left(\ln\left(\frac{dQ^{\mathbb{G},\varphi}/d\widetilde{Q}_{\widetilde{B}}}{Z_t^{\mathbb{G},\varphi}/Z_t^{\widetilde{B}}}\right)-\ln(Z_t^{\widetilde{B}})\right)\big|\mathcal{G}_t\right]\\
&=\operatorname*{ess\,inf}_{\varphi\in\Phi_f}\left\{\begin{array}{l} E^{\widetilde{Q}_{\widetilde{B}}}\left[\frac{Z_t^{\widetilde{B}}}{Z_t^{\mathbb{G},\varphi}}\frac{dQ^{\mathbb{G},\varphi}}{d\widetilde{Q}_{\widetilde{B}}}\ln\left(\frac{dQ^{\mathbb{G},\varphi}/d\widetilde{Q}_{\widetilde{B}}}{Z_t^{\mathbb{G},\varphi}/Z_t^{\widetilde{B}}}\right)\big|\mathcal{G}_t\right]\\ \quad -\ln(Z_t^{\widetilde{B}})E^{\widetilde{Q}_{\widetilde{B}}}\left[\frac{Z_t^{\widetilde{B}}}{Z_t^{\mathbb{G},\varphi}}\frac{dQ^{\mathbb{G},\varphi}}{d\widetilde{Q}_{\widetilde{B}}}\big|\mathcal{G}_t\right]\end{array}\right\}\\
&=E^{\widetilde{Q}^{E,\widetilde{B}}}\left[\ln\left(\frac{Z_T^{E,\widetilde{B}}}{Z_t^{E,\widetilde{B}}}\right)\big|\mathcal{G}_t\right]-\ln(Z_t^{\widetilde{B}})\\
&=E^{\widetilde{Q}^{E,\widetilde{B}}}\left[\ln\left(c_{E,\widetilde{B}}e^{\gamma X_{T\wedge\tau}^{0,\pi^{E,\widetilde{B}}}}\right)\big|\mathcal{G}_t\right]-\ln(Z_t^{E,\widetilde{B}})-\ln(Z_t^{\widetilde{B}})\\
&=\ln(c_{E,\widetilde{B}})+E^{\widetilde{Q}^{E,\widetilde{B}}}\left[\gamma X_{T\wedge\tau}^{0,\pi^{E,\widetilde{B}}}\big|\mathcal{G}_t\right]-\ln(Z_t^{E,\widetilde{B}})-\ln(Z_t^{\widetilde{B}})\\
&=\ln(c_{E,\widetilde{B}})+\gamma X_{t\wedge\tau}^{0,\pi^{E,\widetilde{B}}}-\ln(Z_t^{E,\widetilde{B}})-\ln(Z_t^{\widetilde{B}})
\end{aligned}
$$

by substituting this into (7.3.11) we get

$$
\begin{aligned}
&-\exp\left\{-\operatorname*{ess\,inf}_{\varphi\in\Phi_f} E^{Q^{\mathbb{G},\varphi}}\left[\ln\left(\frac{Z_T^{\mathbb{G},\varphi}}{Z_t^{\mathbb{G},\varphi}}\right)+\gamma\widetilde{B}\big|\mathcal{G}_t\right]\right\}\\
&=-\frac{1}{c_{\widetilde{B}}}\exp\left\{-\ln(c_{E,\widetilde{B}})-\gamma X_{t\wedge\tau}^{0,\pi^{E,\widetilde{B}}}+\ln(Z_t^{E,\widetilde{B}})+\ln(Z_t^{\widetilde{B}})\right\}\\
&=-c_{\widetilde{B}}^{-1}c_{E,\widetilde{B}}^{-1}Z_t^{E,\widetilde{B}}Z_t^{\widetilde{B}}e^{-\gamma X_{t\wedge\tau}^{0,\pi^{E,\widetilde{B}}}}. \qquad (7.3.12)
\end{aligned}
$$

□

**Corollary 7.3.3.**

$$-\inf_{\pi\in\mathcal{A}} E^{\widetilde{Q}}[e^{-\gamma(X^{0,\pi}_{T\wedge\tau}+\widetilde{B})}] = -\exp\left\{-\inf_{\varphi\in\Phi_f}\left(\mathcal{H}(Q^{\mathbb{G},\varphi}|\widetilde{Q}) + E^{Q^{\mathbb{G},\varphi}}(\gamma\widetilde{B})\right)\right\}.$$

*Proof.* The proof is done by taking $t = 0$ in theorem 7.3.2. □

The goal of the rest of this section is to solve the dual problem defined by

$$\sup_{\varphi\in\Phi_f}\left(-\mathcal{H}(Q^{\mathbb{G},\varphi}|\widetilde{Q}) - E^{Q^{\mathbb{G},\varphi}}\gamma\widetilde{B})\right). \tag{7.3.13}$$

The dynamic version of this problem can be written as

$$\widetilde{J}_t := \operatorname*{ess\,sup}_{\varphi\in\Phi_\mathrm{f}} \mathrm{E}^{\mathrm{Q}^{\mathbb{G},\varphi}}\left[-\ln\left(\frac{\mathrm{Z}^{\mathbb{G},\varphi}_\mathrm{T}}{\mathrm{Z}^{\mathbb{G},\varphi}_\mathrm{t}}\right) - \gamma\widetilde{\mathrm{B}}\middle|\mathcal{G}_\mathrm{t}\right] \tag{7.3.14}$$

To this end, we consider

$$\Lambda(x) := \ln(1+x) - \frac{x}{1+x}, \quad x > -1. \tag{7.3.15}$$

**Proposition 7.3.4.** *For any $\varphi\in\Phi_f$, we associate $J^\varphi$ given by*

$$J^\varphi_t := E^{Q^{\mathbb{G},\varphi}}\left[-\ln\left(\frac{Z^{\mathbb{G},\varphi}_T}{Z^{\mathbb{G},\varphi}_t}\right) - \gamma\widetilde{B}\middle|\mathcal{G}_t\right], \quad \forall\ 0\le t\le T. \tag{7.3.16}$$

(a) *It holds that*

$$J^\varphi_t = -E^{Q^{\mathbb{G},\varphi}}\left[\int_{t\wedge\tau}^{T\wedge\tau}\Lambda(\varphi_s)dD_s + \frac{1}{2}\int_{t\wedge\tau}^{T\wedge\tau}\lambda_s^2 ds + \gamma\widetilde{B}\ \middle|\mathcal{G}_t\right]. \tag{7.3.17}$$

(b) *If the following BSDE*

$$\begin{cases} dY = (\frac{1}{2}\lambda^2 + \lambda Z)d(t\wedge\tau) + Z_t dW_t^{\tau} - U\left(dN^{\mathbb{G}} - \varphi \widetilde{G}^{-1} I_{[\![0,\tau]\!]} dD_s^{o,\mathbb{F}}\right) \\ \qquad\qquad + \left((1+\varphi)\ln(1+\varphi) - \varphi\right)\widetilde{G}^{-1} I_{[\![0,\tau]\!]} dD_s^{o,\mathbb{F}}. \\ Y_T = -\gamma\widetilde{B}, \end{cases} \tag{7.3.18}$$

*has an integrable* $(\mathbb{G}, T\wedge\tau)$*-solution* $(Y^{\varphi}, Z^{\varphi}, U^{\varphi})$*, then* $Y^{\varphi} = J^{\varphi}$*.*

*Proof.*

$$\begin{aligned} & E^{Q^{\mathbb{G},\varphi}}\left[\ln\left(\frac{Z_T^{\mathbb{G},\varphi}}{Z_t^{\mathbb{G},\varphi}}\right) + \gamma\widetilde{B}\big|\mathcal{G}_t\right] \\ & = E^{Q^{\mathbb{G},\varphi}}\left[\ln\mathcal{E}\left(-\int_{t\wedge\tau}^{T\wedge\tau}\lambda_s dW_s^{\tau} + \int_{t\wedge\tau}^{T\wedge\tau}\varphi_s dN_s^{\mathbb{G}}\right) + \gamma\widetilde{B}\big|\mathcal{G}_t\right], \end{aligned}$$

$$\begin{aligned} & = E^{Q^{\mathbb{G},\varphi}}\left[-\int_{t\wedge\tau}^{T\wedge\tau}\lambda_s dW_s^{\tau} + \int_{t\wedge\tau}^{T\wedge\tau}\varphi_s dN_s^{\mathbb{G}} - \frac{1}{2}\int_{t\wedge\tau}^{T\wedge\tau}\lambda_s^2 ds\big|\mathcal{G}_t\right] \\ & \quad + E^{Q^{\mathbb{G},\varphi}}\left[\sum_{t\wedge\tau<s\leq T\wedge\tau}\left(\ln(1+\varphi_s\Delta N_s^{\mathbb{G}}) - \varphi_s\Delta N_s^{\mathbb{G}}\right) + \gamma\widetilde{B}\big|\mathcal{G}_t\right], \end{aligned}$$

$$\begin{aligned} & = E^{Q^{\mathbb{G},\varphi}}\left[-\int_{t\wedge\tau}^{T\wedge\tau}\lambda_s dW_s + \int_{t\wedge\tau}^{T\wedge\tau}\varphi_s dN_s^{\mathbb{G}} - \frac{1}{2}\int_{t\wedge\tau}^{T\wedge\tau}\lambda_s^2 ds\big|\mathcal{G}_t\right] \\ & \quad + E^{Q^{\mathbb{G},\varphi}}\left[+\int_{t\wedge\tau}^{T\wedge\tau}\left(\ln(1+\varphi_s) - \varphi_s\right)dD_s + \gamma\widetilde{B}\big|\mathcal{G}_t\right], \end{aligned}$$

$$= E^{Q^{\mathbb{G},\varphi}}\left[-\int_{t\wedge\tau}^{T\wedge\tau}\lambda_s\left(d\widehat{W}_s - \lambda_s ds\right) + \int_{t\wedge\tau}^{T\wedge\tau}\varphi_s d(N^{\mathbb{G},\varphi} + \frac{\varphi}{1+\varphi}dD_s)\big|\mathcal{G}_t\right]$$

$$+ E^{Q^{\mathbb{G},\varphi}}\left[-\frac{1}{2}\int_{t\wedge\tau}^{T\wedge\tau}\lambda_s^2 ds + \int_{t\wedge\tau}^{T\wedge\tau}\left(\ln(1+\varphi_s)-\varphi_s\right)dD_s + \gamma\widetilde{B}\Big|\mathcal{G}_t\right],$$

$$= E^{Q^{\mathbb{G},\varphi}}\left[\frac{1}{2}\int_{t\wedge\tau}^{T\wedge\tau}\lambda_s^2 ds + \int_{t\wedge\tau}^{T\wedge\tau}\left(\ln(1+\varphi_s)-\frac{\varphi_s}{1+\varphi}\right)dD_s + \gamma\widetilde{B}\ \Big|\ \mathcal{G}_t\right].$$

This proves (7.3.17). The remaining proof proves assertion (b). Thus, we remark that

$$\begin{aligned}
&J_t^{\varphi} - \int_0^t\left(\ln(1+\varphi_s)-\frac{\varphi_s}{1+\varphi}\right)dD_s - \frac{1}{2}\int_0^{t\wedge\tau}\lambda_s^2 ds\\
&= E^{Q^{\mathbb{G},\varphi}}\left[-\int_0^{T\wedge\tau}\left(\ln(1+\varphi_s)-\frac{\varphi_s}{1+\varphi}\right)dD_s - \frac{1}{2}\int_0^{T\wedge\tau}\lambda_s^2 ds - \gamma\widetilde{B}\Big|\mathcal{G}_t\right]
\end{aligned}$$

is a $(\mathbb{G}, Q^{\mathbb{G},\varphi})$-martingale process. Hence, by Girsanov's theorem, there exists a $(\mathbb{G}, \widetilde{Q})$-martingale process $M^{\mathbb{G},\widetilde{Q},\varphi}$ such that

$$\begin{aligned}
&J_t^{\varphi} - \int_0^{t\wedge\tau}\left(\ln(1+\varphi_s)-\frac{\varphi_s}{1+\varphi}\right)dN_s^{\mathbb{G}} - \frac{1}{2}\int_0^{t\wedge\tau}\lambda_s^2 ds\\
&- \int_0^{t\wedge\tau}\left(\ln(1+\varphi_s)-\frac{\varphi_s}{1+\varphi}\right)\widetilde{G}^{-1}I_{[\![0,\tau]\!]}dD_s^{o,\mathbb{F}}\\
&= M^{\mathbb{G},\widetilde{Q},\varphi} - \langle M^{\mathbb{G},\widetilde{Q},\varphi}, -\lambda\bullet W^{\tau} + \varphi\bullet N^{\mathbb{G}}\rangle_t^{\widetilde{Q}}
\end{aligned} \tag{7.3.19}$$

Again by applying Girsanov's Theorem again to $M^{\mathbb{G},\widetilde{Q},\varphi}$ is $(\mathbb{G}, \widetilde{Q})$-martingale, we deduce the existence of a $(\mathbb{G}, P)$-martingale process $M^{\mathbb{G},P,\varphi}$ such that the following equality holds

$$M^{\mathbb{G},\widetilde{Q},\varphi} = M^{\mathbb{G},P,\varphi} - \langle M^{\mathbb{G},P,\varphi}, G_-^{-1}\bullet\mathcal{T}(m)\rangle_t \tag{7.3.20}$$

Now, by applying Theorem 2.3.6 to $M^{\mathbb{G},P,\varphi}$, which is a $(\mathbb{G}, P)$-martingale, we get the existence of a unique $(M^{\mathbb{F},\varphi}, \varphi^{(o,\varphi)}) \in \mathcal{M}_{0,loc}(\mathbb{F}) \times \mathcal{I}^o_{loc}\left(N^{\mathbb{G}}, \mathbb{G}\right)$ such that

$$\begin{aligned} M^{\mathbb{G},P,\varphi} &= M_0^{\mathbb{G},P,\varphi} + G_-^{-2} I_{]\!]0,\tau]\!]} \bullet \mathcal{T}(M^{\mathbb{F},\varphi}) + \varphi^{(o,\varphi)} \bullet N^{\mathbb{G}} \\ &= M_0^{\mathbb{G},P,\varphi} + G_-^{-2}\psi^{\varphi} \bullet \mathcal{T}(W) + \varphi^{(o,\varphi)} \bullet N^{\mathbb{G}}. \end{aligned} \tag{7.3.21}$$

The $\mathbb{F}$-predictable process $\psi^{\varphi}$ is a consequence of applying the predictable representation theorem to $M^{\mathbb{F},\varphi}$. Therefore, by substituting the above latter equality in (7.3.20) we obtain

$$\begin{aligned} M^{\mathbb{G},\widetilde{Q},\varphi} &= M_0^{\mathbb{G},P,\varphi} + G_-^{-2}\psi^{\varphi} \bullet \mathcal{T}(W) + \varphi^{(o,\varphi)} \bullet N^{\mathbb{G}} \\ &\qquad + \langle M_0^{\mathbb{G},P,\varphi} + G_-^{-2}\psi^{\varphi} \bullet \mathcal{T}(W) + \varphi^{(o,\varphi)} \bullet N^{\mathbb{G}}, G_-^{-1} \bullet \mathcal{T}(m)\rangle^P_{\cdot\wedge\tau} \\ &= M_0^{\mathbb{G},P,\varphi} + G_-^{-2}\psi^{\varphi} \bullet \mathcal{T}(W) + \varphi^{(o,\varphi)} \bullet N^{\mathbb{G}} + \langle G_-^{-2}\psi^{\varphi} \bullet \mathcal{T}(W), G_-^{-1} \bullet \mathcal{T}(m)\rangle^P_t \\ &= M_0^{\mathbb{G},P,\varphi} + \varphi^{(o,\varphi)} \bullet N^{\mathbb{G}}_t + G_-^{-2}\psi^{\varphi} \bullet W^{\tau}_t \end{aligned}$$

and then find (7.3.19) we get that

$$\begin{aligned} & J^{\varphi}_t - \int_0^{t\wedge\tau} \left(\ln(1+\varphi_s) - \frac{\varphi_s}{1+\varphi}\right) dN^{\mathbb{G}}_s - \frac{1}{2}\int_0^{t\wedge\tau} \lambda_s^2 ds \\ & - \int_0^{t\wedge\tau} \left(\ln(1+\varphi_s) - \frac{\varphi_s}{1+\varphi}\right) \widetilde{G}^{-1} I_{[\![0,\tau]\!]} dD^{o,\mathbb{F}}_s \end{aligned}$$

$$\begin{aligned} &= M_0^{\mathbb{G},P,\varphi} + \varphi^{(o,\varphi)} \bullet N^{\mathbb{G}}_t + G_-^{-2}\psi^{\varphi} \bullet W^{\tau}_t \\ &\qquad - \langle M_0^{\mathbb{G},P,\varphi} + \varphi^{(o,\varphi)} \bullet N^{\mathbb{G}} + G_-^{-2}\psi^{\varphi} \bullet W^{\tau}, -\lambda \bullet W^{\tau} + \varphi \bullet N^{\mathbb{G}}\rangle^{\widetilde{Q}}_t \end{aligned}$$

$$= M_0^{\mathbb{G},P,\varphi} + \varphi^{(o,\varphi)} \bullet N^{\mathbb{G}} + G_-^{-2}\psi^{\varphi} \bullet W^{\tau}_t - \langle G_-^{-2}\psi^{\varphi} \bullet W^{\tau}, -\lambda \bullet W^{\tau})\rangle^{\widetilde{Q}}_t$$

$$- \langle \varphi^{(o,\varphi)} \bullet N^{\mathbb{G}}, \varphi \bullet N^{\mathbb{G}} \rangle_t^{\widetilde{Q}}$$

$$= M_0^{\mathbb{G},P,\varphi} + \varphi^{(o,\varphi)} \bullet N^{\mathbb{G}} + \frac{\psi^{\varphi}}{G_-^2} \bullet W_t^{\tau} + \frac{\psi^{\varphi}}{G_-^2} \lambda \bullet (t \wedge \tau) - \left( \varphi^{(o,\varphi)} \varphi \Delta N^{\mathbb{G}} \bullet N^{\mathbb{G}} \right)^p$$

$$= M_0^{\mathbb{G},P,\varphi} + \varphi^{(o,\varphi)} \bullet N^{\mathbb{G}} + \frac{\psi^{\varphi}}{G_-^2} \bullet W_t^{\tau} + \frac{\psi^{\varphi}}{G_-^2} \lambda \bullet (t \wedge \tau) - \left( \varphi^{(o,\varphi)} \varphi \Delta D \bullet N^{\mathbb{G}} \right)^p$$

$$= M_0^{\mathbb{G},P,\varphi} + \varphi^{(o,\varphi)} \bullet N^{\mathbb{G}} + G_-^{-2} \psi^{\varphi} \bullet W_t^{\tau} + G_-^{-2} \psi^{\varphi} \lambda \bullet (t \wedge \tau) - \left( \varphi^{(o,\varphi)} \varphi \bullet D \right)^p$$

$$= M_0^{\mathbb{G},P,\varphi} + \varphi^{(o,\varphi)} \bullet N^{\mathbb{G}} + G_-^{-2} \psi^{\varphi} \bullet W_t^{\tau} + G_-^{-2} \psi^{\varphi} \lambda \bullet (t \wedge \tau)$$
$$- \varphi^{(o,\varphi)} \varphi \widetilde{G}^{-1} I_{[\![0,\tau]\!]} \bullet D^{o,\mathbb{F}}.$$

Therefore,

$$dJ_t^{\varphi} = \left\{ \ln(1 + \varphi_s) - \frac{\varphi_s}{1 + \varphi} + \varphi^{(o,\varphi)} \right\} dN^{\mathbb{G}} + G_-^{-2} \psi^{\varphi} dW_t^{\tau}$$
$$+ \left\{ \frac{1}{2} \lambda_s^2 + G_-^{-2} \psi^{\varphi} \lambda \right\} d(t \wedge \tau) + \left( \ln(1 + \varphi_s) - \frac{\varphi_s}{1 + \varphi} - \varphi^{(o,\varphi)} \varphi \right) \frac{I_{[\![0,\tau]\!]}}{\widetilde{G}} dD_s^{o,\mathbb{F}}.$$

Define $Z^{\varphi} := G_-^{-2} \psi^{\varphi}$ and $U^{\varphi} := -\left( \ln(1 + \varphi) - \frac{\varphi}{1+\varphi} + \varphi^{(o,\varphi)} \right)$, then

$$\varphi^{(o,\varphi)} = -U^{\varphi} - \ln(1 + \varphi) + \frac{\varphi}{1 + \varphi}$$

and hence.

$$\ln(1+\varphi) - \frac{\varphi}{1+\varphi} - \varphi^{(o,\varphi)}\varphi = (1+\varphi)\ln(1+\varphi) - \varphi + U^{\varphi}\varphi.$$

Therefore,

$$\begin{aligned} dJ_t^{\varphi} &= (\frac{1}{2}\lambda^2 + \lambda Z^{\varphi})d(t\wedge\tau) + Z_t^{\varphi}dW_t^{\tau} - U^{\varphi}dN^{\mathbb{G}} \\ &\quad + ((1+\varphi)\ln(1+\varphi) - \varphi + U^{\varphi}\varphi)\,\widetilde{G}^{-1}I_{[\![0,\tau]\!]}dD_s^{o,\mathbb{F}}. \end{aligned}$$

$\square$

Consider the following BSDE

$$\begin{cases} dY_t = \left(\frac{1}{2}\lambda^2 + \lambda_t Z_t\right)d(t\wedge\tau) + Z_t dW_t^{\tau} \\ \qquad\qquad -U_t dN_t^{\mathbb{G}} - \left(U_t + e^{-U_t} - 1\right)\widetilde{G}_t^{-1}I_{]\!]0,\tau]\!]}(t)dD_t^{o,\mathbb{F}}. \\ Y_{T\wedge\tau} = -\gamma\widetilde{B}. \end{cases} \tag{7.3.22}$$

By changing slightly the notations, we can prove that this BSDE belongs to a generalized family of BSDEs.

**Lemma 7.3.5.** *The RBSDE (7.3.22) can be written as*

$$dY_t = \widehat{f}(t,U_t)dA_t^{\tau} + Z_t d\widehat{W}_t^{\tau} - U_t dN^{\mathbb{G}}, \quad Y_{T\wedge\tau} = -\gamma\widetilde{B}, \quad P\text{-a.s.}. \tag{7.3.23}$$

*where*

$$\begin{cases} \widehat{W}_t := W_t + \int_0^t \lambda_s ds,\ \ A_t := t + D_t^{o,\mathbb{F}}, \\ \widehat{f}(t,u) := \frac{1}{2}\lambda_t^2(1-\Gamma_t) - \frac{\Gamma_t}{\widetilde{G}_t}(e^{-u}+u-1), \quad \Gamma_t := \frac{dD_t^{o,\mathbb{F}}}{dA_t}. \end{cases} \tag{7.3.24}$$

The proof is trivial and will be omitted. The BSDE (7.3.23)-(7.3.24) is a generalized BSDE compared to the existing ones of the literature. In fact our process $A$ might not be absolutely continuous with respect to the Lebesgue measure. This gives a new family of BSDEs that we could not find in the literature. In fact all the literature assumes the driver is governed with the real time $t$. This open a new class of BSDEs that deserves attention and investigation in virtue of its financial importance. Furthermore, the driver $f(t,u)$ is not Lipschitz in the variable $u$ for many reasons.

**Theorem 7.3.6.** *Suppose that the BSDE (7.3.23)-(7.3.24) has a solution* $(\widehat{Y},\widehat{Z},\widehat{U})$ *of class (D) under* $\widehat{Q}^{\mathbb{G}}$. *Then the following assertions hold.*

(a) *The following equality holds*

$$\widehat{Y} = \operatorname*{ess\,sup}_{\varphi\in\Phi_{\mathrm{f}}} \mathrm{J}^{\varphi}. \tag{7.3.25}$$

(b) *The quadruplet*

$$(Y^{(D)},Z^{(D)},M^{(D)},K^{(D)}) := \left(\widehat{Y},\widehat{Z},\widehat{U}\bullet N^{\mathbb{G}},\left(\widehat{U}+e^{-\widehat{U}}-1\right)\widetilde{G}_t^{-1}I_{[\![0,\tau]\!]}\bullet D^{o,\mathbb{F}}\right)$$

*is a solution of class (D) under $\widehat{Q}^{\mathbb{G}}$ for the following RBSDE*

$$\begin{cases} dY_t = \frac{1}{2}\lambda_t^2 d(t\wedge\tau) + Z_t d\widehat{W}_t^{\tau} - dM_t - dK_t, \quad Y_{T\wedge\tau} = -\gamma\widetilde{B} \\ Y \geq \widehat{S} \quad on \quad [\![0, T\wedge\tau[\![, \quad \int_0^{T\wedge\tau}(Y_{t-} - \widehat{S}_{t-})dK_t = 0, \quad P\text{-}a.s.. \end{cases} \tag{7.3.26}$$

*Here $\widehat{W}$ is defined in (7.3.24), and $\widehat{S}$ is given by*

$$\begin{aligned} \widehat{S} := & E^{\widehat{Q}^{\mathbb{F}}}\left[-\frac{1}{2}\int_t^T \frac{\widetilde{\mathcal{E}}_s}{\widetilde{\mathcal{E}}_t}\lambda_s^2 ds - \frac{\widetilde{\mathcal{E}}_T}{\widetilde{\mathcal{E}}_t}\Big(\gamma g + \ln\big(\mathcal{E}_T^{-1}(G_-^{-1}\bullet m)\big)\Big)\Big|\mathcal{F}_t\right] \\ & + E^{\widehat{Q}^{\mathbb{F}}}\left[\int_t^T \frac{1}{\widetilde{\mathcal{E}}_t}\ln\big(\mathcal{E}_s(G_-^{-1}\bullet m)\big)dV_s^{\mathbb{F}}\Big|\mathcal{F}_t\right]. \end{aligned} \tag{7.3.27}$$

This theorem conveys two main ideas. On the one hand, the solution to the dual problem is equivalently given by the solution of the BSDE (7.3.23)-(7.3.24) . On the other hand, this BSDE can be considered as a particular case of the class of linear RBSDEs investigated in Chapter 4 of this thesis. This is our main and real motivations for those RBSDEs. This latter fact is very intriguing, as itself conveys that probably the barrier process plays central role somehow in shaping the form of the solution to the RBSDE. This fact deserves more attention.

*Proof of Theorem 7.3.6.* This proof is divided into two parts. The first part proves (a), while the second part proves (b).

**Part 1.** Here, we prove assertion (a). On the one hand, as $J^{\varphi}$ is the solution to the BSDEs (7.3.18), we have

$$
\begin{aligned}
dJ_t^{\varphi} &= (\frac{1}{2}\lambda^2 + \lambda Z^{\varphi})d(t\wedge\tau) + Z_t^{\varphi}dW_t^{\tau} - U^{\varphi}dN^{\mathbb{G}} - f(\varphi, U^{\varphi})\widetilde{G}^{-1}I_{[\![0,\tau]\!]}dD_t^{o,\mathbb{F}} \\
&= (\frac{1}{2}\lambda^2(1-\Gamma) - f(\varphi, U^{\varphi})\widetilde{G}^{-1}\Gamma)dA^{\tau} + Z_t^{\varphi}d\widehat{W}_t^{\tau} - U^{\varphi}dN^{\mathbb{G}} \qquad (7.3.28)
\end{aligned}
$$

where $f(\varphi, U^{\varphi}) := -\left((1+\varphi)\ln(1+\varphi) - \varphi + U^{\varphi}\varphi\right)$. On the other hand, as $\widehat{Y}$ is the solution to the BSDE (7.3.23)-(7.3.24), we have

$$
d\widehat{Y}_t = \widehat{f}(t, U_t)dA_t^{\tau} + \widehat{Z}_t d\widehat{W}_t^{\tau} - \widehat{U}_t dN^{\mathbb{G}}. \qquad (7.3.29)
$$

Here,

$$
\widehat{f}(t,u) := \frac{1}{2}\lambda_t^2(1-\Gamma_t) - \frac{\Gamma_t}{\widetilde{G}_t}(e^{-u} + u - 1).
$$

Therefore, by substacting the equalities (7.3.28) and (7.3.29) we get

$$
\begin{aligned}
&d(J_t^{\varphi} - \widehat{Y}_t) \\
&= ((e^{-\widehat{U}} + \widehat{U} - 1) - f(\varphi, U^{\varphi}))\widetilde{G}^{-1}\Gamma dA^{\tau} + (Z_t^{\varphi} - \widehat{Z}_t)d\widehat{W}_t^{\tau} - (U^{\varphi} - \widehat{U})dN^{\mathbb{G}} \\
&= ((e^{-\widehat{U}} + \widehat{U} - 1) - f(\varphi, \widehat{U}))\widetilde{G}^{-1}\Gamma dA^{\tau} + (f(\varphi, \widehat{U}) - f(\varphi, U^{\varphi}))\widetilde{G}^{-1}\Gamma dA^{\tau} \\
&+ (Z_t^{\varphi} - \widehat{Z}_t)d\widehat{W}_t^{\tau} - (U^{\varphi} - \widehat{U})dN^{\mathbb{G}} \\
&= ((e^{-\widehat{U}} + \widehat{U} - 1) - f(\varphi, \widehat{U}))\widetilde{G}^{-1}\Gamma dA^{\tau} + (Z_t^{\varphi} - \widehat{Z}_t)d\widehat{W}_t^{\tau} - (U^{\varphi} - \widehat{U})d\widehat{N}^{\mathbb{G},\varphi}.
\end{aligned}
\qquad (7.3.30)
$$

Therefore,

$$
J_t^{\varphi} - \widehat{Y}_t = J_T^{\varphi} - \widehat{Y}_T - \int_{t\wedge\tau}^{T\wedge\tau}((e^{-\widehat{U}} + \widehat{U} - 1) - f(\varphi, \widehat{U}))\widetilde{G}^{-1}\Gamma dA^{\tau}
$$

$$-\int_{t\wedge\tau}^{T\wedge\tau}(Z_t^{\varphi}-\widehat{Z}_t)d\widehat{W}_t^{\tau}+\int_{t\wedge\tau}^{T\wedge\tau}(U^{\varphi}-\widehat{U})d\widehat{N}^{\mathbb{G},\varphi}. \tag{7.3.31}$$

Take the conditional expectation for both sides of (7.3.31) with respect to $Q^{\mathbb{G},\varphi}$, we get

$$J_t^{\varphi}-\widehat{Y}_t=E^{Q^{\mathbb{G},\varphi}}\left[\int_{t\wedge\tau}^{T\wedge\tau}(f(\varphi,\widehat{U})-(e^{-\widehat{U}}+\widehat{U}-1))\widetilde{G}^{-1}I_{[\![0,\tau]\!]}\widetilde{G}^{-1}\Gamma dA|\mathcal{G}_t\right]\leq 0. \tag{7.3.32}$$

this implies that,

$$J_t^{\varphi}\leq\widehat{Y}_t,\quad \text{for all } \varphi\in\Phi_f$$

and hence,

$$\operatorname*{ess\,sup}_{\varphi\in\Phi_f} J_t^{\varphi}\leq\widehat{Y}_t.$$

On the other hand, note that

$$\operatorname*{ess\,sup}_{\varphi\in\Phi_f}\left\{-(1+\varphi)\ln(1+\varphi)+\varphi-U^{(D)}\varphi\right\}=e^{-U^{(D)}}-1+U^{(D)}.$$

where the essential supremum occurs on $\widehat{\varphi}:=e^{-U^{(D)}}-1$, or equivalently $U^{(D)}=\ln(1+\widehat{\varphi})=:U^{\widehat{\varphi}}$. By substituting this $\widehat{\varphi}$ in (7.3.18) to get

$$\begin{cases} dJ_t^{\widehat{\varphi}}=(\frac{1}{2}\lambda^2+\lambda Z^{\widehat{\varphi}})d(t\wedge\tau)+Z_t^{\widehat{\varphi}}dW_t^{\tau}-U^{\widehat{\varphi}}dN^{\mathbb{G}} \\ \qquad\qquad -\left((1+\widehat{\varphi})\ln(1+\widehat{\varphi})-\widehat{\varphi}+U^{\widehat{\varphi}}\widehat{\varphi}\right)\widetilde{G}^{-1}I_{[\![0,\tau]\!]}dD_s^{o,\mathbb{F}}. \\ J_T^{\widehat{\varphi}}=-\gamma\widetilde{B}. \end{cases} \tag{7.3.33}$$

$$
\begin{cases}
dJ_t^{\widehat{\varphi}} = (\frac{1}{2}\lambda^2 + \lambda Z^{\widehat{\varphi}})d(t\wedge\tau) + Z_t^{\widehat{\varphi}} dW_t^\tau - U^{\widehat{\varphi}}\left(dN^{\mathbb{G}} - \widehat{\varphi}\widetilde{G}^{-1} I_{[\![0,\tau]\!]} dD_s^{o,\mathbb{F}}\right) \\
\qquad\qquad ((1+\widehat{\varphi})\ln(1+\widehat{\varphi}) - \widehat{\varphi})\,\widetilde{G}^{-1} I_{[\![0,\tau]\!]} dD_s^{o,\mathbb{F}}. \\
J_T^{\widehat{\varphi}} = -\gamma\widetilde{B}.
\end{cases}
$$

therefore, by the uniqueness of the solution of the RBSDE (7.3.33) and as this RBSDE coincide with the RBSDE (7.3.18) we have that $Y^{(D)} = J_t^{\widehat{\varphi}}$. By combining this with the fact that $\operatorname{ess\,sup}_{\varphi\in\Phi_f} J_t^\varphi \leq Y_t^{(D)}$, which is proven in part(a) of this theorem, we have $\operatorname{ess\,sup}_{\varphi\in\Phi_f} J_t^\varphi = Y_t^{(D)}$.

**Part 2.** Here we prove (b). This part is divided into two steps as the following.

**Step 1.** On the one hand, as $\left(\widehat{Y}, \widehat{Z}, \widehat{U}\bullet N^{\mathbb{G}}\right)$ is a solution to (7.3.23) then we have

$$
d\widehat{Y}_t = \widehat{f}(t,\widehat{U}_t)dA_t^\tau + \widehat{Z}_t d\widehat{W}_t^\tau - \widehat{U}_t dN^{\mathbb{G}}, \quad \widehat{Y}_{T\wedge\tau} = -\gamma\widetilde{B}, \quad P\text{-a.s.}. \tag{7.3.34}
$$

where

$$
\begin{cases}
\widehat{W}_t := W_t + \int_0^t \lambda_s ds,\ A_t := t + D_t^{o,\mathbb{F}}, \\
\widehat{f}(t,u) := \frac{1}{2}\lambda_t^2(1-\Gamma_t) - \frac{\Gamma_t}{\widetilde{G}_t}(e^{-u}+u-1), \quad \Gamma_t := \frac{dD_t^{o,\mathbb{F}}}{dA_t}.
\end{cases} \tag{7.3.35}
$$

Thus, by substituting $\widehat{f}(t,u) := \frac{1}{2}\lambda_t^2(1-\Gamma_t) - \frac{\Gamma_t}{\widetilde{G}_t}(e^{-u}+u-1), \quad \Gamma_t := \frac{dD_t^{o,\mathbb{F}}}{dA_t}$ in the (7.3.34) we get that the quadruplet

$$
(Y^{(D)}, Z^{(D)}, M^{(D)}, K^{(D)}) := \left(\widehat{Y}, \widehat{Z}, \widehat{U}\bullet N^{\mathbb{G}}, \left(\widehat{U} + e^{-\widehat{U}} - 1\right)\widetilde{G}_t^{-1} I_{[\![0,\tau]\!]}\bullet D^{o,\mathbb{F}}\right)
$$

is a solution to the following BSDE

$$dY_t = \frac{1}{2}\lambda_t d(t\wedge\tau) + Z_t d\widehat{W}^{\tau}_t - dM_t - dK_t, \quad Y_{T\wedge\tau} = -\gamma\widetilde{B}.$$

On the other hand, due to $\widehat{Y} = \operatorname{ess\,sup}_{\varphi\in\Phi_{\mathrm{f}}} \mathrm{J}^{\varphi}$ we have that

$$\widehat{Y} \geq J^0 = E^{\widehat{Q}}\left[-\frac{1}{2}\int_{t\wedge\tau}^{T\wedge\tau} \lambda_s^2 ds - \gamma\widetilde{B}\big|\mathcal{G}_t\right] =: S^{\mathbb{G}}_t. \tag{7.3.36}$$

To prove the Skorokhod condition, we derive the following

$$\begin{aligned}
&E\left[\int_0^{T\wedge\tau}(Y^{(D)}_{t-} - S^{\mathbb{G}}_{t-})dK^{(D)}_t\right]\\
&= E\left[\int_0^{T\wedge\tau}(Y^{(D)}_{t-} - S^{\mathbb{G}}_{t-})\left(U^{(D)}_t + e^{-U^{(D)}_t} - 1\right)\widetilde{G}^{-1} I_{[\![0,\tau]\!]} dD^{o,\mathbb{F}}_t\right]\\
&= E\left[\int_0^{T}(Y^{\mathbb{F}}_{t-} - S^{o,\mathbb{F}}_{t-})\left(U^{(D)}_t + e^{-U^{(D)}_t} - 1\right)\widetilde{G}^{-1} I_{[\![0,\tau]\!]} dD^{o,\mathbb{F}}_t\right]\\
&= E\left[\int_0^{T}(Y^{\mathbb{F}}_{t-} - S^{o,\mathbb{F}}_{t-})\left(U^{(D)}_t + e^{-U^{(D)}_t} - 1\right) dD_t\right]\\
&= E\left[(Y^{\mathbb{F}}_{\tau-} - S^{o,\mathbb{F}}_{\tau-})\left(U^{(D)}_\tau + e^{-U^{(D)}_\tau} - 1\right) I_{\{\tau\leq T\}}\right]\\
&= E\left[(Y^{(D)}_{\tau-} - S^{\mathbb{G}}_{\tau-})\left(U^{(D)}_\tau + e^{-U^{(D)}_\tau} - 1\right) I_{\{\tau\leq T\}}\right] = 0.
\end{aligned} \tag{7.3.37}$$

Where $Y^{\mathbb{F}}$ and $S^{o,\mathbb{F}}$ are the optional processes in $\mathbb{F}$ that are equals to $Y^{(D)}$ and $S^{\mathbb{G}}$ on $[\![0,\tau[\![$, and hence $Y^{\mathbb{F}}_- = Y^{(D)}_-$ and $S^{o,\mathbb{F}}_- = S^{\mathbb{G}}_-$ on $]\!]0,\tau]\!]$. The last equality in (7.3.37) is because of the following: First, we put

$$\kappa(x) := \ln(1+x) - \frac{x}{1+x}, \quad \text{for any} \quad x > -1,$$

and we remark that

$$
\begin{aligned}
&Y_t^{(D)}\\
&:= \operatorname*{ess\,sup}_{\varphi\in\Phi_f} \big\{ E^{Q^{\mathbb{G},\varphi}} \big[ -\int_{t\wedge\tau}^{T\wedge\tau} \kappa(\varphi_s) dD_s - \frac{1}{2}\int_{t\wedge\tau}^{T\wedge\tau} \lambda_s^2 ds - \gamma \widetilde{B} \big| \mathcal{G}_t \big] \big\}\\
&= \operatorname*{ess\,sup}_{\varphi\in\Phi_f} \left\{ E^{Q^{\mathbb{G},\varphi}} \left[ -\kappa(\varphi_\tau) 1_{\{t<\tau\leq T\}} - \frac{1}{2}\int_{t\wedge\tau}^{T\wedge\tau} \lambda_s^2 ds - \gamma \widetilde{B} \middle| \mathcal{G}_t \right] \right\}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&Y_{\tau-}^{(D)}\\
&= \operatorname*{ess\,sup}_{\varphi\in\Phi_f} \big\{ E^{Q^{\mathbb{G},\varphi}} \big[ -\kappa(\varphi_\tau) 1_{\{\tau-<\tau\leq T\}} - \frac{1}{2}\int_{\tau\wedge\tau}^{T\wedge\tau} \lambda_s^2 ds - \gamma \widetilde{B} \big| \mathcal{G}_{\tau-} \big] \big\}\\
&= -\gamma\widetilde{B} + \operatorname*{ess\,sup}_{\varphi\in\Phi_f} \left\{ E^{Q^{\mathbb{G},\varphi}} \left[ -\left( \ln(1+\varphi_\tau) - \frac{\varphi_\tau}{1+\varphi_\tau} \right) 1_{\{\tau-<\tau\leq T\}} \middle| \mathcal{G}_{\tau-} \right] \right\}\\
&= -\gamma\widetilde{B}.
\end{aligned}
$$

The last inequality is due to

$$
\operatorname*{ess\,sup}_{\varphi\in\Phi_f} \left\{ E^{Q^{\mathbb{G},\varphi}} \left[ -\Lambda(\varphi_\tau) 1_{\{\tau\leq T\}} \middle| \mathcal{G}_{\tau-} \right] \right\} \leq 0 = E^{Q^{\mathbb{G},0}} \left[ -\Lambda(0) 1_{\{\tau\leq T\}} \middle| \mathcal{G}_{\tau-} \right],
$$

which yields

$$
\operatorname*{ess\,sup}_{\varphi\in\Phi_f} E^{Q^{\mathbb{G},\varphi}} \left[ -\left( \ln(1+\varphi_\tau) - \varphi_\tau + \frac{\varphi_\tau^2}{1+\varphi_\tau} \right) 1_{\{\tau\leq T\}} \middle| \mathcal{G}_{\tau-} \right] = 0.
$$

Also, note that

$$S_{\tau-}^{\mathbb{G}} = E^{Q^{\mathbb{G},0}}\left[-\frac{1}{2}\int_{\tau\wedge\tau}^{T\wedge\tau}\lambda_s^2 ds - \gamma\widetilde{B}\big|\mathcal{G}_{\tau-}\right] = E^{Q^{\mathbb{G},0}}\left[-\gamma\widetilde{B}\big|\mathcal{G}_{\tau-}\right] = -\gamma\widetilde{B}.$$

**Step 2.** In this step, we prove that on $[\![0,\tau[\![$, the processes $\widehat{S}$ and $S^{\mathbb{G}}$ defined in (7.3.36) coincide. It is clear that for any $t\in[0,T)$

$$S_t^{\mathbb{G}} = S_t^{\mathbb{G}} 1_{\{t\ <\tau\}} - \gamma\widetilde{B}1_{\{t\ \geq\tau\}} = S_t^{\mathbb{G}} 1_{\{t\ <\tau\}} - \ln\left(\mathcal{E}_\tau^{-1}(G_-^{-1}\bullet m\right) 1_{\{t\ \geq\tau\}}.$$

Now, we find $\widehat{S}_t 1_{\{t\ <\tau\}}$ as the following

$$S_t^{\mathbb{G}} 1_{\{t\ <\tau\}} = E^{\widehat{Q}^{\mathbb{G}}}\left[-\frac{1}{2}\int_{t\wedge\tau}^{T\wedge\tau}\lambda_s^2 ds - \gamma\widetilde{B}\big|\mathcal{G}_t\right] 1_{\{t\ <\tau\}}$$

$$\begin{aligned}
&= E\left[-\frac{1}{2}\int_{t\wedge\tau}^{T\wedge\tau}\frac{\mathcal{E}_s\left(-\lambda\bullet W\right)}{\mathcal{E}_s(G_-^{-1}\bullet m)}\lambda_s^2 ds\big|\mathcal{G}_t\right]\frac{\mathcal{E}_t(G_-^{-1}\bullet m)}{\mathcal{E}_t\left(-\lambda\bullet W\right)} 1_{\{t\ <\tau\}}\\
&\quad + E\left[-\frac{\mathcal{E}_{T\wedge\tau}\left(-\lambda\bullet W\right)}{\mathcal{E}_{T\wedge\tau}(G_-^{-1}\bullet m)}\gamma\widetilde{B}\big|\mathcal{G}_t\right]\frac{\mathcal{E}_t(G_-^{-1}\bullet m)}{\mathcal{E}_t\left(-\lambda\bullet W\right)} 1_{\{t\ <\tau\}}
\end{aligned}$$

$$\begin{aligned}
&= E\left[-\frac{1}{2}\int_{t\wedge\tau}^{T\wedge\tau}\frac{\mathcal{E}_s\left(-\lambda\bullet W\right)}{\mathcal{E}_s(G_-^{-1}\bullet m)}\lambda_s^2 ds\big|\ \mathcal{G}_t\right]\frac{\mathcal{E}_t(G_-^{-1}\bullet m)}{\mathcal{E}_t\left(-\lambda\bullet W\right)} 1_{\{t\ <\tau\}}\\
&+ E\left[-\frac{\mathcal{E}_T\left(-\lambda\bullet W\right)}{\mathcal{E}_T(G_-^{-1}\bullet m)}\left(\gamma g + \ln\left(\mathcal{E}_T^{-1}(G_-^{-1}\bullet m)\right)\right) 1_{\{T\ <\tau\}}\big|\ \mathcal{G}_t\right]\frac{\mathcal{E}_t(G_-^{-1}\bullet m)}{\mathcal{E}_t\left(-\lambda\bullet W\right)} 1_{\{t\ <\tau\}}\\
&\quad + E\left[\frac{\mathcal{E}_\tau\left(-\lambda\bullet W\right)}{\mathcal{E}_\tau(G_-^{-1}\bullet m)}\ln\left(\mathcal{E}_\tau(G_-^{-1}\bullet m)\right) 1_{\{T\ \geq\tau\}}\big|\mathcal{G}_t\right]\frac{\mathcal{E}_t(G_-^{-1}\bullet m)}{\mathcal{E}_t\left(-\lambda\bullet W\right)} 1_{\{t\ <\tau\}}
\end{aligned}$$

$$\begin{aligned}
&= E\left[-\frac{1}{2}\int_t^T\frac{\mathcal{E}_s\left(-\lambda\bullet W\right)}{\mathcal{E}_s(G_-^{-1}\bullet m)}\lambda_s^2 G_s ds\big|\mathcal{F}_t\right]\frac{\mathcal{E}_t(G_-^{-1}\bullet m)}{G_t\mathcal{E}_t\left(-\lambda\bullet W\right)} 1_{\{t\ <\tau\}}\\
&\quad + E\left[-\frac{\mathcal{E}_T\left(-\lambda\bullet W\right)}{\mathcal{E}_T(G_-^{-1}\bullet m)}\left(\gamma g + \ln\left(\mathcal{E}_T^{-1}(G_-^{-1}\bullet m)\right)\right) G_T\big|\mathcal{F}_t\right]\frac{\mathcal{E}_t(G_-^{-1}\bullet m)}{G_t\mathcal{E}_t\left(-\lambda\bullet W\right)} 1_{\{t\ <\tau\}}
\end{aligned}$$

$$+ E\left[\int_t^T \frac{\mathcal{E}_s\left(-\lambda \bullet W\right)}{\mathcal{E}_s(G_-^{-1}\bullet m)} \ln\left(\mathcal{E}_s(G_-^{-1}\bullet m)\right) dD^{o,\mathbb{F}} \big| \mathcal{F}_t\right] \frac{\mathcal{E}_t(G_-^{-1}\bullet m)}{G_t\mathcal{E}_t\left(-\lambda \bullet W\right)} 1_{\{t\ <\tau\}}$$

$$= E^{\widehat{Q}^{\mathbb{F}}}\left[-\frac{1}{2}\int_t^T \widetilde{\mathcal{E}}_s \lambda_s^2 ds - \widetilde{\mathcal{E}}_T\Big(\gamma g + \ln\left(\mathcal{E}_T^{-1}(G_-^{-1}\bullet m)\right)\Big)\big| \mathcal{F}_t\right] \frac{1_{\{t\ <\tau\}}}{\widetilde{\mathcal{E}}_t}$$
$$+ E^{\widehat{Q}^{\mathbb{F}}}\left[\int_t^T \ln\left(\mathcal{E}_s(G_-^{-1}\bullet m)\right) dV_s^{\mathbb{F}} \big| \mathcal{F}_t\right] \frac{1_{\{t\ <\tau\}}}{\widetilde{\mathcal{E}}_t}$$

$$= \widehat{S}_t 1_{\{t\ <\tau\}}$$

where $\widehat{S}$ is defined by (7.3.27). □

**Proposition 7.3.7.** *The RBSDE (7.3.26) has a unique quadruplet solution* $(Y^{(D)}, Z^{(D)}, M^{(D)}, K^{(D)})$ *that is connected to the unique solution* $(Y^{\mathbb{F}}, Z^{\mathbb{F}}, K^{\mathbb{F}})$ *of the following RBSDE under* $\mathbb{F}$

$$\begin{cases} Y_t = \xi^{\mathbb{F}} + \displaystyle\int_t^T f^{\mathbb{F}}(s, Z_s) ds - \int_t^T \ln\left(\widetilde{Z}_s\right) dV_s^{\mathbb{F}} + K_T - K_t - \int_t^T Z_s dW_s, \\ \\ Y_t \geq S_t^{\mathbb{F}}, \quad t \in [0,T), \quad \displaystyle\int_0^T (Y_{t-} - S_{t-}^{\mathbb{F}}) dK_t = 0, \end{cases} \tag{7.3.38}$$

*where*

$$f^{\mathbb{F}}(s,z) := -\widetilde{\mathcal{E}}_s f\left(s, z\widetilde{\mathcal{E}}_s^{-1}\right),\ S^{\mathbb{F}} := \frac{\widehat{S}_t(g)}{\mathcal{E}_t\left(-\lambda \bullet W\right)},\ \xi^{\mathbb{F}} := -\widetilde{\mathcal{E}}_T \gamma \widetilde{B},$$
$$\text{and } \widetilde{\mathcal{E}} := \mathcal{E}(-\widetilde{G}^{-1}\bullet D^{o,\mathbb{F}}) \tag{7.3.39}$$

*as the following*

$$Y^{(D)} = \frac{Y^{\mathbb{F}}}{\widetilde{\mathcal{E}}} I_{[\![0,\tau[\![} - \gamma \widetilde{B} I_{[\![\tau,+\infty[\![}, \quad Z^{(D)} = \frac{Z^{\mathbb{F}}}{\widetilde{\mathcal{E}}}, \quad K^{(D)} = \frac{1}{\widetilde{\mathcal{E}}_-} \bullet (K^{\mathbb{F}})^{\tau}$$

$$\textit{and } M^{(D)} = \left( -\ln\left(\widetilde{Z}\right) - \frac{Y^{\mathbb{F}}}{\widetilde{\mathcal{E}}} \right) \bullet N^{\mathbb{G}}.$$

*Proof.* The proof is a direct application of theorem 5.1.5. □

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