Schur-Weyl Functors for Super Yangians and Deformed Double Current Algebras

by

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Abstract

This thesis establishes new Schur-Weyl functors involving the super Yangian $Y(\mathfrak{gl}_{m|n})$, its deformed double current algebra, the affine super Yangian $\widehat{Y}(\mathfrak{sl}_{m|n})$, and their respective Cherednik algebras.

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Table of Contents

Chapter 1

Introduction

In the 1980's, theoretical physicists and mathematicians introduced quantum groups. While they are valuable in gauge theory and in quantum field theory, their study has evolved into a major branch of representation theory. Quantum groups are deformations of universal enveloping algebras of finite, affine, or Kac-Moody Lie algebras, as well as of finite dimensional, classical Lie superalgebras and affine Lie superalgebras.

Representations of the general linear Lie algebra and representations of the symmetric group have an interplay referred to as Schur duality. This is one of the most classical results in Lie theory. Generalizing this, to find Schur dualities means finding pairs of Lie (super)algebra-type structures and of symmetric group-type structures whose representations are related as in the classical case. In the affine setting, rather than considering commuting actions, we approach Schur-Weyl duality as the existence of a functor sending modules over one structure to modules over another and producing a certain equivalence of categories. Guay has found [15] duality between the trigonometric Cherednik algebra and the affine Yangian. This extends the duality of finite type due to V. Drinfeld [9]. Chari and Pressley [3] have investigated the quantum affine algebra setting. More recently, Flicker [11] extended their work to quantum affine superalgebras. In contrast, not much is known about the double affine case. What is known is that we have three families of double affine quantum algebras (quantum toroidal algebras, affine Yangians, and deformed double current algebras) with corresponding families of Cherednik algebras (elliptic, trigonometric, and rational) and functors between categories of modules, as in the affine case. Recent work on deformed double current algebras include Kalinov's [17] alternative construction to Guay's, as algebras of endomorphisms in the Deligne category. Kalinov has expanded on this construction with Etingof and Rains [10].

Chapter 2 is a compilation of key definitions and preliminary results on superalgebras, Yangians, and Cherednik algebras that are fundamental to what follows. In Chapter 3, we recall universal central extensions and uncover presentations of the Steinberg Lie superalgebras of types $A(m, n)$ and Q. As their respective deformed double current algebras (DDCA) will be deformations of the enveloping algebras of the Steinberg Lie superalgebras, this is worth establishing. Chapter 4 establishes a functor between the categories of modules over the degenerate affine Hecke algebra and modules over the super Yangian $Y(\mathfrak{gl}_{m|n})$. This is then shown to be an equivalence of categories in Theorem 4.2.1. In Chapter 5, we introduce a new quantum superalgebra of double affine type, the deformed double current algebra for $\mathfrak{sl}_{m|n}$. Computations on elements of left-modules over the DDCA lead to Definition 5.1.1. Previously, in the non-super case, the J presentation had produced a straightforward defining action for the P generators. However, without a J presentation in the superalgebra setting, more computing is required. Using the outcome of Chapter 4 we are able to establish an equivalence of categories between left modules over the rational Cherednik algebra and certain right modules over the double deformed current algebra $\mathcal{D}_{t,\kappa}(\mathfrak{sl}_{m|n})$. Theorem 5.1.2 formalizes this key result. This is similar to what Guay had shown for the $\mathfrak{sl}_n$ case [15].

Chapter 6 concludes this thesis with Schur-Weyl functor for affine super Yangians. Much of this is possible thanks to Ueda's minimalistic presentation of the affine super Yangian [23], making it more manageable to define the action of the current operators.

One further application would be to use these new Schur-Weyl functors to construct Fermionic Fock spaces for the super settings. For $Y(\mathfrak{sl}_n)$, Kodera [18] built the higher level Fock space following ideas of Uglov [24], showing that the actions of the affine Lie algebra and the Yangian on the level L Fock space can be glued and extended to an action of the Yangian of $\mathfrak{sl}_n$. It would be natural to use his work as a roadmap to produce an analogous result for superalgebras. Motivations for this include applications to quantum mechanics. Fock space representations can be built in more settings beyond the affine Yangian. One could do this for DDCAs in particular, since we already have a Schur-Weyl functor between them and rational Cherednik algebras, similar to the Schur-Weyl functor between affine Yangians and trigonometric Cherednik algebras. As well, for DDCAs, there are options for further connections that do not exist for affine Yangians, such as considering the cyclotomic version of the rational Cherednik algebra.

Chapter 2

Definitions and Preliminaries

2.1 Lie superalgebras

The generalization of a Lie algebra to include a $\mathbb{Z}_2$-graded is called a *Lie superalgebra*. Elements x are assigned a degree $|x|$ of either 0 or 1 and it is equipped with a *Lie superbracket* satisfying super-analogues of the defining axioms of the Lie algebra, being super skew-symmetry:

$$[x, y] = -(-1)^{|x||y|}[y, x] \tag{2.1}$$

and the super Jacobi identity:

$$(-1)^{|x||z|}[x, [y, z]] + (-1)^{|y||x|}[y, [z, x]] + (-1)^{|z||y|}[z, [x, y]] = 0. \tag{2.2}$$

This thesis will mainly concern specific Lie superalgebras $\mathfrak{gl}_{m|n}$, $\mathfrak{sl}_{m|n}$, and $\mathfrak{q}_n$. The base field will always be the complex numbers $\mathbb{C}$. Cheng and Wang's book on Lie superalgebras [7] was used as a reference for some of these standard definitions and basic preliminary results.

2.1.1 Finite $\mathfrak{sl}_{m|n}$

If V is a vector superspace over $\mathbb{C}$, then $\mathrm{End}(V)$ is an associative superalgebra; when equipped with the superbracket, we call it the general linear Lie superalgebra. Denote it by $\mathfrak{gl}(V)$. If $V = \mathbb{C}^{m|n}$, we write $\mathfrak{gl}(V) = \mathfrak{gl}_{m|n}$.

An element of $\mathfrak{gl}_{m|n}$ is of the form

$$X = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

with $A \in \mathfrak{gl}_m$, $B \in M_{m,n}(\mathbb{C})$, $C \in M_{n,m}(\mathbb{C})$, and $D \in \mathfrak{gl}_n$. Define the *supertrace* as

$$\mathrm{str}(X) := \mathrm{tr}(A) - \mathrm{tr}(D)$$

where tr denotes the usual trace of a square matrix. If $\mathrm{str}(X) = 0$, then X belongs to the subalgebra of $\mathfrak{gl}_{m|n}$ called the *special linear Lie superalgebra* $\mathfrak{sl}_{m|n}$.

For a block matrix $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, we define its *supertranspose* to be $X^{st} = \begin{pmatrix} A^t & C^t \\ -B^t & D^t \end{pmatrix}$.

Note that when $m \neq n$, $\mathfrak{sl}_{m|n}$ is simple.

The Cartan matrix of $\mathfrak{sl}_{m|n}$, $C = (a_{ij})_{i,j=1}^{m+n-1}$, has entries $a_{ii} = 2$ for all $i < m$, $a_{mm} = 0$, $a_{ii} = -2$ for $i > m$, $a_{i+1,i} = a_{i,i+1} = -1$ for all $i < m$, $a_{i+1,i} = a_{i,i+1} = 1$ for all $i \geq m$, and all other entries are 0.

$$C = \left(\begin{array}{cccc|c|cccc}
2 & -1 & & 0 & 0 & & & & \\
-1 & 2 & \ddots & & \vdots & & & & \\
 & \ddots & \ddots & -1 & 0 & & 0 & & \\
0 & & -1 & 2 & -1 & & & & \\
\hline
0 & \dots & 0 & -1 & 0 & 1 & 0 & \dots & 0 \\
\hline
 & & & & 1 & -2 & 1 & & 0 \\
 & & & & 0 & 1 & \ddots & \ddots & \\
 & & 0 & & \vdots & & \ddots & -2 & 1 \\
 & & & & 0 & 0 & & 1 & -2
\end{array}\right)$$

Let the parity for integers $i = 1, \dots, m+n$ be defined as

$$|i| = \begin{cases} 0, i = 1, \dots, m \\ 1, i = m, \dots, m+n \end{cases} .$$

Recall the lower central series of a Lie algebra, the sequence of subalgebras:

$$\mathfrak{g} \supseteq [\mathfrak{g}, \mathfrak{g}] \supseteq [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] \supseteq [\mathfrak{g}, [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]]] \supseteq \dots$$

where we can denote $\mathfrak{g}_0 = \mathfrak{g}$ and recursively define $\mathfrak{g}_n = [\mathfrak{g}, \mathfrak{g}_{n-1}]$ for all $n > 0$. Then, we say that the Lie algebra $\mathfrak{g}$ is *nilpotent* if the lower central series terminates, i.e. there exists an n for which $\mathfrak{g}_n = 0$.

Definition 2.1.1. A Cartan subalgebra of a Lie algebra $\mathfrak{g}$ is a subalgebra $\mathfrak{h}$ satisfying the following two conditions:

1. $\mathfrak{h}$ is a nilpotent Lie algebra

2. The normalizer of $\mathfrak{h}$ in $\mathfrak{g}$ is $\mathfrak{h}$ itself, i.e. if $[X, Y] \in \mathfrak{h}$ for all $X \in \mathfrak{h}$, then $Y \in \mathfrak{h}$

This definition of the Cartan subalgebra works in the superalgebra setting as well. For instance, the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{gl}_{m|n}$ is made up of all diagonal matrices, just as the Cartan subalgebra of $\mathfrak{gl}_n$ is the subalgebra of all diagonal matrices.

We can use the supertrace to define a non-degenerate supersymmetric bilinear form given by

$$(\cdot, \cdot) : \mathfrak{gl}_{m|n} \times \mathfrak{gl}_{m|n} \to \mathbb{C}$$

$$(A, B) = \operatorname{str}(AB)$$

If we restrict the bilinear form given above to $\mathfrak{h}$, it produces a non-degenerate symmetric bilinear form given by

$$(E_{ii}, E_{jj}) = \begin{cases} 1, & 1 \leq i = j \leq m \\ -1, & m+1 \leq i = j \leq m+n \\ 0, & i \neq j. \end{cases}$$

Denote by $\{\varepsilon_i\}_{i=1,\dots,m+n}$ the basis of $\mathfrak{h}^*$ dual to $\{E_{ii}\}_{i=1,\dots,m+n}$. We can identify ε_i with $(E_{ii},\cdot)$ for $1 \le i \le m$ and with $-(E_{ii},\cdot)$ for $m+1 \le i \le m+n$.

Then, the form $(\cdot,\cdot)$ on $\mathfrak{h}$ induces a non-degenerate bilinear form on $\mathfrak{h}^*$, which we can also denote $(\cdot,\cdot)$, given by

$$(\varepsilon_i, \varepsilon_j) = \begin{cases} 1 & 1 \le i = j \le m, \\ -1 & m+1 \le i = j \le m+n, \\ 0 & i \neq j. \end{cases}$$

The root system of $\mathfrak{gl}_{m|n}$ can be expressed as $\Phi = \Phi_0 \cup \Phi_1$ with

$$\Phi_0 = \{\varepsilon_i - \varepsilon_j | i \neq j, |i| = |j|\}$$

$$\Phi_1 = \{\pm(\varepsilon_i - \varepsilon_j) | i \le m < j\}.$$

$\mathfrak{sl}_{m|n}$ has positive simple roots

$$\alpha_1, \dots, \alpha_{m-1}, \alpha_m, \alpha_{m+1}, \dots, \alpha_{m+n-1}$$

where α_m is odd but the rest are even, where $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$.

2.1.2 Finite $\mathfrak{q}_n$

For some positive integer n, we define the *queer Lie superalgebra* $\mathfrak{q}_n$ to be the subalgebra of $\mathfrak{gl}_{n|n}$ consisting of matrices of the form

$$X = \left(\begin{array}{c|c} A & B \\ \hline B & A \end{array}\right).$$

When working in $\mathfrak{q}_n$, it is convenient to label the rows and columns of $\mathfrak{gl}_{n|n}$ as $-n, \dots, -1$ and $1, \dots, n$.

A linear basis for $\mathfrak{q}_n$ consists of $\underline{E}_{ij} := E_{ij} + E_{-i,-j}$ and $\underline{E}_{i,-j} := E_{i,-j} + E_{-i,j}$ for $i, j = 1, \dots, n$.

The *derived superalgebra* $[\mathfrak{q}_n, \mathfrak{q}_n]$, denoted $\mathfrak{sq}_n$, consists of matrices of the form

$$X = \left(\begin{array}{c|c} A & B \\ \hline B & A \end{array}\right)$$

where $A \in \mathfrak{gl}_n$ and $B \in \mathfrak{sl}_n$. Thus, it contains the identity matrix $I_{n|n}$ and we can consider the quotient superalgebra $[\mathfrak{q}_n, \mathfrak{q}_n]/\mathbb{C}I_{n|n}$. Its even part is isomorphic to $\mathfrak{sl}_n$ and its odd part is isomorphic to the adjoint module. For $n \geq 2$, $\mathfrak{sq}_n/\mathbb{C}I_{n|n}$ is simple.

Consider the subalgebra of $\mathfrak{q}_n$ where A, B are diagonal. This is the standard Cartan subalgebra, denoted $\mathfrak{h}$; note that while it is nilpotent and self-normalizing in $\mathfrak{q}_n$, it is not abelian: $\mathfrak{h} = \mathfrak{h}_{\bar{0}} \oplus \mathfrak{h}_{\bar{1}}$, $[\mathfrak{h}_{\bar{0}}, \mathfrak{h}] = 0$ but $[\mathfrak{h}_{\bar{1}}, \mathfrak{h}_{\bar{1}}] = \mathfrak{h}_{\bar{0}}$.

The vectors

$$H_i = E_{ii} + E_{-i,-i},\ i = 1, \ldots, n$$

form a basis for $\mathfrak{h}_{\bar{0}}$ while the vectors

$$\overline{H}_i = E_{-i,i} + E_{i,-i},\ i = 1, \ldots n$$

form a basis for $\mathfrak{h}_{\bar{1}}$. Denote the corresponding dual basis in $\mathfrak{h}_{\bar{0}}^*$ by $\{\varepsilon_i | i = 1, \ldots, n\}$.

With respect to $\mathfrak{h}_{\bar{0}}$, we have the root space decomposition $\mathfrak{q}_n = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$ with root system $\Phi = \Phi_{\bar{0}} \cup \Phi_{\bar{1}}$, with both components coinciding with the root system for $\mathfrak{gl}_n$:

$$\Phi_{\bar{0}} = \Phi_{\bar{1}} = \{\varepsilon_i - \varepsilon_j | 1 \leq i \neq j \leq n\}.$$

$\dim_{\mathbb{C}} \mathfrak{g}_\alpha = 1$ for each $\alpha \in \Phi$, and $\mathfrak{g}_\alpha \subset \mathfrak{g}_i$ for $\alpha \in \Phi_i$ and $i \in \mathbb{Z}_2$. The Weyl group of $\mathfrak{q}_n$ is identified with the symmetric group S_n.

2.1.3 The affine Lie superalgebra $\widehat{\mathfrak{sl}_{m|n}}$

As a vector super-space, the affine Lie superalgebra is

$$\widehat{\mathfrak{sl}_{m|n}} = \mathfrak{sl}_{m|n} \otimes_{\mathbb{C}} \mathbb{C}[v, v^{-1}] \oplus \mathbb{C}c$$

for a central element c that is then equipped with the superbracket:

$$[X_1 \otimes v^r, X_2 \otimes v^s] = [X_1, X_2] \otimes v^{r+s} + \delta_{r+s,0}(X_1, X_2)c \tag{2.3}$$

The root system for $\widehat{\mathfrak{sl}_{m|n}}$ is $\widehat{\Phi}_0 \cup \widehat{\Phi}_1$, where the even roots consist of:

$$\widehat{\Phi}_0 = \{\epsilon_i - \epsilon_j + k\delta | k \in \mathbb{Z}, i \neq j, |i| = |j| \text{ or } i = j \text{ and } k \neq 0\}$$

The corresponding root vectors are $E_{ij}v^k$ for $i \neq j$ and $k\delta$ corresponds to Hv^k for any H belonging to the Cartan subalgebra $\mathfrak{h}$. Then, the odd roots are:

$$\widehat{\Phi}_1 = \{\epsilon_i - \epsilon_j + k\delta | k \in \mathbb{Z}, i \leq m \text{ and } \ j > m, \text{ or } i > m \text{ and } j \leq m\}$$

These correspond to the root vectors $E_{ij}v^k$.

The root $\alpha_0 = \epsilon_{m+n} - \epsilon_1 + \delta$ is actually odd and $(\alpha_0, \alpha_0) = 0$, so we do not obtain an $\mathfrak{sl}_2$-triple.

The generators of the Cartan subalgebra of $\mathfrak{sl}_{m|n}$ are

$$\{h_1, \ldots, h_{m-1}, h_m, h_{m+1}, \ldots, h_{m+n-1}\},$$

where $h_m = E_{mm} + E_{m+1,m+1}$ and for all $i \neq m$, $h_i = E_{i+1,i+1} - E_{ii}$. Further, we have that

$$[h_m, E_{m,m+1}] = 0, \tag{2.4}$$

$$[h_m, E_{m+1,m+2}] = E_{m+1,m+2}. \tag{2.5}$$

This is compatible with the Cartan matrix. Further, the element $e_m = E_{m,m+1}$ generates the upper right block while $f_m = E_{m+1,m}$ generates the lower left block.

For the affine setting, we take $e_0 = E_{m+n,1} \otimes t$ so that $[h_1, e_0] = -e_0$ and $[h_{m+n-1}, e_0] = e_0$. This explains the upperrightmost 1 in what will be $\widehat{C}$. Similarly, we take $f_0 = E_{1,m+n} \otimes t^{-1}$. Then, it follows that $h_0 = [e_0, f_0] = E_{m+n,m+n} + E_{11} - c$ and $[h_0, e_0] = 0 = [h_0, f_0]$.

Following Zhang's guidelines on how to enlarge the Cartan matrix from the Chevalley presentation of $U_q(\mathcal{L}\mathfrak{sl}_{m|n})$ [25], we obtain

$$\widehat{C} = \left(\begin{array}{c|cccc|c|cccc}
0 & -1 & 0 & \dots & \dots & 0 & \dots & \dots & 0 & 1 \\ \hline
-1 & 2 & -1 & & 0 & \vdots & & & & \\
0 & -1 & 2 & \ddots & & \vdots & & 0 & & \\
\vdots & & \ddots & \ddots & -1 & 0 & & & & \\
\vdots & 0 & & -1 & 2 & -1 & & & & \\ \hline
0 & \dots & \dots & 0 & -1 & 0 & 1 & 0 & \dots & 0 \\ \hline
\vdots & & & & & 1 & -2 & 1 & & 0 \\
\vdots & & 0 & & & 0 & 1 & \ddots & \ddots & \\
0 & & & & & \vdots & & \ddots & -2 & 1 \\
1 & & & & & 0 & 0 & & 1 & -2
\end{array}\right).$$

Ueda [23] offers a convenient presentation of $\widehat{\mathfrak{sl}_{m|n}}$ with generators and relations. First, consider the Lie superalgebra

$$\widetilde{\mathfrak{sl}_{m|n}} = \mathfrak{sl}_{m|n} \otimes_{\mathbb{C}} \mathbb{C}[v, v^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d = \widehat{\mathfrak{sl}_{m|n}} \oplus \mathbb{C}d$$

where $[d, X \otimes v^s] = sX \otimes v^s$ for $X \in \mathfrak{sl}_{m|n}$.

Proposition 2.1.2. *(Proposition 2.4 [23]) Suppose that $m, n \geq 2$, $m \neq n$, and $A = (a_{i,j})_{0 \leq i,j \leq m+n-1}$ is a $(m+n) \times (m+n)$-matrix whose components are*

$$a_{i,j} = \begin{cases} (-1)^{|i|} + (-1)^{|i+1|} & \text{if } i = j, \\ -(-1)^{|i+1|} & \text{if } i = j-1, \\ -(-1)^{|i|} & \text{if } i = j+1, \\ 1 & \text{if } (i,j) = (0, m+n-1), (m+n-1, 0), \\ 0 & \text{otherwise.} \end{cases}$$

Then, $\widetilde{\mathfrak{sl}_{m|n}}$ is isomorphic to the Lie superalgebra over $\mathbb{C}$ defined by the gener-

ators $\{x_i^\pm, h_i, d\}$ *for* $0 \le i \le m+n-1$ *subject to the relations:*

$$[d, h_i] = 0, \quad [d, x_i^+] = \begin{cases} x_i^+ & (i=0), \\ 0 & \textit{else}, \end{cases} \tag{2.6}$$

$$[d, x_i^-] = \begin{cases} -x_i^- & (i=0), \\ 0 & \textit{else}, \end{cases} \tag{2.7}$$

$$[h_i, h_j] = 0, \quad [h_i, x_j^\pm] = \pm a_{i,j} x_j^\pm, \tag{2.8}$$

$$[x_i^+, x_j^-] = \delta_{ij} h_i, \quad \mathrm{ad}(x_i^\pm)^{1+|a_{i,j}|} x_j^\pm = 0, \tag{2.9}$$

$$[x_0^\pm, x_0^\pm] = 0, \quad [x_m^\pm, x_m^\pm] = 0, \tag{2.10}$$

$$[[x_{m-1}^\pm, x_m^\pm], [x_{m+1}^\pm, x_m^\pm]] = 0, \quad [[x_{m+n-1}^\pm, x_0^\pm], [x_1^\pm, x_0^\pm]] = 0 \tag{2.11}$$

where the generators $x_m^\pm$ *and* $x_0^\pm$ *are odd and all other generators are even. The isomorphism* Ξ *to* $\widetilde{\mathfrak{sl}_{m|n}}$ *is given by*

$$\Xi(h_i) = \begin{cases} -E_{11} - E_{m+n,m+n} + c & i = 0 \\ (-1)^{|i|} E_{ii} - (-1)^{|i+1|} E_{i+1,i+1} & 1 \le i \le m+n-1 \end{cases}$$

$$\Xi(x_i^+) = \begin{cases} E_{m+n,1} \otimes v & i = 0 \\ E_{i,i+1} & \textit{else} \end{cases}$$

$$\Xi(x_i^-) = \begin{cases} -E_{1,m+n} \otimes v^{-1} & i = 0 \\ (-1)^{|i|} E_{i+1,i} & \textit{else.} \end{cases}$$

Moreover, $\widehat{\mathfrak{sl}_{m|n}}$ *is isomorphic to the Lie superalgebra over* $\mathbb{C}$ *defined by the generators* $\{x_i^\pm, h_i\}$ *for* $0 \le i \le m+n-1$ *and by the relations (2.8)-(2.11).*

2.2 Super Yangians of $\mathfrak{gl}_{m|n}$ and $\mathfrak{sl}_{m|n}$

Yangians, as introduced by Drinfeld [8], were initially intended to generate R-matrices that solve the quantum Yang-Baxter equation. However, these infinite-dimensional Hopf algebras and quantum groups have interesting theory that has been studied on its own right. They can be formulated as degenerations of the quantum loop algebra. Though the application of super Yangians may not me immediately useful, from the standpoint of studying Yangians, to produce the analogues in a super setting is a natural choice. Depending on context, we pick one of several presentations of the Yangian.

Definition 2.2.1. The super Yangian $Y(\mathfrak{gl}_{m|n})$ is the $\mathbb{Z}_2$-graded associative algebra generated by the elements $t_{ij}^{(r)}$ for $r \in \mathbb{Z}_{\geq 0}$, with generating power series $t_{ij}(u)$, where

$$t_{ij}(u) = \sum_{r\geq 0} t_{ij}^{(r)} u^{-r} \tag{2.12}$$

and satisfying:

$$[t_{ij}(u), t_{kl}(v)] = \frac{(-1)^{\nu(i,j;k,l)}}{(u-v)} \left(t_{kj}(u)t_{il}(v) - t_{kj}(v)t_{il}(u)\right) \tag{2.13}$$

$$\nu(i,j;k,l) \equiv 1 + l + (j+l)(i+l) + (i+j)(k+l) \bmod 2 \tag{2.14}$$

Expanding (2.13) yields useful explicit relations on the generators.

$$[t_{ij}^{(r)}, t_{kl}^{(s)}] = (-1)^{\nu(i,j;k,l)} \sum_{a=0}^{\min(r,s)-1} \left(t_{kj}^{(a)} t_{il}^{(r+s-1-a)} - t_{kj}^{(r+s-1-a)} t_{il}^{(a)} \right) \tag{2.15}$$

$$[t_{ij}^{(r+1)}, t_{kl}^{(s)}] - [t_{ij}^{(r)}, t_{kl}^{(s+1)}] = (-1)^{\nu(i,j;k,l)} \left(t_{kj}^{(r)} t_{il}^{(s)} - t_{kj}^{(s)} t_{il}^{(r)} \right) \tag{2.16}$$

This presentation of the Yangian is called the RTT presentation, due to the generators $t_{ij}^{(r)}$ satisfying the following relation, which is itself a compact form of the defining relations:

$$R_{12}(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u-v) \tag{2.17}$$

in $Y(\mathfrak{gl}_{m|n}) \otimes \mathrm{End}(\mathbb{C}^{m|n}) \otimes \mathrm{End}(\mathbb{C}^{m|n})$ where

$$R(u) = 1 + \frac{P}{u} \tag{2.18}$$

$$P = \sum_{a,b=1}^{m+n} (-1)^{|b|} E_{ab} \otimes E_{ba} \tag{2.19}$$

$$T(u) = \sum_{i,j=1}^{m+n} (-1)^{|j|} t_{ij}(u) \otimes E_{ji}. \tag{2.20}$$

The matrix R satisfies the graded Yang-Baxter equation. In particular, $t_{ij}^{(0)} = (-1)^{|j|+1}$.

We then have two other sets of generators: the generators as constructed by Gow [13] and the current generators, which we will present further below in Definition 2.2.6. Based on work by Gelfand and Retakh [12], Gow shows that the matrix $T(u)$ has the Gauss decomposition

$$T(u) = F(u)D(u)E(u) \tag{2.21}$$

for the unique matrices

$$D(u) = \begin{pmatrix} d_1(u) & 0 & \cdots & 0 \\ 0 & d_2(u) & \cdots & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & d_{m+n}(u) \end{pmatrix} \tag{2.22}$$

$$E(u) = \begin{pmatrix} 1 & e_{12}(u) & \cdots & e_{1,m+n}(u) \\ & \ddots & & e_{2,m+n}(u) \\ & & \ddots & \vdots \\ 0 & & & 1 \end{pmatrix} \tag{2.23}$$

$$F(u) = \begin{pmatrix} 1 & & & 0 \\ f_{21}(u) & & \ddots & \vdots \\ \vdots & & \ddots & \\ f_{m+n,1}(u) & f_{m+n,2}(u) & \cdots & 1 \end{pmatrix} \tag{2.24}$$

The matrix entries are generating series for the Yangian $\mathfrak{gl}_{m|n}$. They themselves are the following quasideterminants:

$$d_i(u) = \begin{vmatrix} t_{11}(u) & \cdots & t_{1,i-1}(u) & t_{1i}(u) \\ \vdots & \ddots & & \vdots \\ t_{i1}(u) & \cdots & t_{i,i-1}(u) & \boxed{t_{ii}(u)} \end{vmatrix} \tag{2.25}$$

$$e_{ij}(u) = d_i(u)^{-1} \begin{vmatrix} t_{11}(u) & \cdots & t_{1,i-1}(u) & t_{1j}(u) \\ \vdots & \ddots & \vdots & \vdots \\ t_{i-1,1}(u) & \cdots & t_{i-1,i-1}(u) & t_{i-1,j}(u) \\ t_{i1}(u) & \cdots & t_{i,i-1}(u) & \boxed{t_{ij}(u)} \end{vmatrix} \tag{2.26}$$

$$e_{ij}(u) = \begin{vmatrix} t_{11}(u) & \cdots & t_{1,i-1}(u) & t_{1i}(u) \\ \vdots & \ddots & \vdots & \vdots \\ t_{i-1,1}(u) & \cdots & t_{i-1,i-1}(u) & t_{i-1,i}(u) \\ t_{j1}(u) & \cdots & t_{j,i-1}(u) & \boxed{t_{ji}(u)} \end{vmatrix} d_i(u)^{-1} \tag{2.27}$$

following Gelfand and Retakh's definition [12].

Proposition 2.2.2. *(Theorem 3 of [13]) The Yangian $Y(\mathfrak{gl}_{m|n})$ is isomorphic as an associative superalgebra to the algebra with even generators $d_i^{(r)}, d_i'^{(r)}, f_j^{(r)}, e_j^{(r)}$ for $i \in \{1, \dots, m+n\}, j \in \{1, \dots, m+n-1\}, j \neq m, r \geq 1$ and odd generators $e_m^{(r)}, f_m^{(r)}$ for $r \geq 1$. These are subject to the following relations, where $r, s, t \geq 1$ and i, j, k range over the appropriate admissible values.*

$$d_i^{(0)} = 1 \tag{2.28}$$

$$\sum_{t=0}^{r} d_i^{(t)} d_i'^{(r-t)} = \delta_{r,0} \tag{2.29}$$

$$\left[d_i^{(r)}, d_l^{(s)}\right] = 0 \tag{2.30}$$

$$\left[d_i^{(r)}, e_j^{(s)}\right] = \begin{cases} (\delta_{i,j} - \delta_{i,j+1}) \sum_{t=0}^{r-1} d_i^{(t)} e_j^{(r+s-1-t)}, & 1 \leq j \leq m-1 \\ (\delta_{i,j} + \delta_{i,j+1}) \sum_{t=0}^{r-1} d_i^{(t)} e_j^{(r+s-1-t)}, & j = m \\ -(\delta_{i,j} - \delta_{i,j+1}) \sum_{t=0}^{r-1} d_i^{(t)} e_j^{(r+s-1-t)}, & m+1 \leq j \leq m+n-1 \end{cases} \tag{2.31}$$

$$\left[d_i^{(r)}, f_j^{(s)}\right] = \begin{cases} -(\delta_{i,j} - \delta_{i,j+1}) \sum_{t=0}^{r-1} f_j^{(r+s-1-t)} d_i^{(t)}, & 1 \le j \le m-1 \\ -(\delta_{i,j} + \delta_{i,j+1}) \sum_{t=0}^{r-1} f_j^{(r+s-1-t)} d_i^{(t)}, & j = m \\ (\delta_{i,j} - \delta_{i,j+1}) \sum_{t=0}^{r-1} f_j^{(r+s-1-t)} d_i^{(t)}, & m+1 \le j \le m+n-1 \end{cases} \tag{2.32}$$

$$\left[e_j^{(r)}, f_k^{(s)}\right] = \begin{cases} -\delta_{j,k} \sum_{t=0}^{r+s-1} d_j'^{(t)} d_{j+1}^{(r+s-1-t)}, & 1 \le j \le m-1 \\ +\delta_{j,k} \sum_{t=0}^{r+s-1} d_j'^{(t)} d_{j+1}^{(r+s-1-t)}, & m \le j \le m+n-1 \end{cases} \tag{2.33}$$

$$[e_m^{(r)}, e_m^{(s)}] = 0 \tag{2.34}$$

$$[f_m^{(r)}, f_m^{(s)}] = 0 \tag{2.35}$$

$$[e_j^{(r)}, e_j^{(s)}] = (-1)^{|j|} \left(\sum_{t=1}^{s-1} e_j^{(t)} e_j^{(r+s-1-t)} - \sum_{t=1}^{r-1} e_j^{(r)} e_j^{(r+s-1-t)} \right), \quad j \neq m \tag{2.36}$$

$$[f_j^{(r)}, f_j^{(s)}] = (-1)^{|j|} \left(\sum_{t=1}^{r-1} f_j^{(t)} f_j^{(r+s-1-t)} - \sum_{t=1}^{s-1} f_j^{(r)} f_j^{(r+s-1-t)} \right), \quad j \neq m \tag{2.37}$$

$$[e_j^{(r)}, e_{j+1}^{(s+1)}] - [e_j^{(r+1)}, e_{j+1}^{(s)}] = -(-1)^{|j|} e_j^{(r)} e_{j+1}^{(s)} \tag{2.38}$$

$$[f_j^{(r+1)}, f_{j+1}^{(s)}] - [f_j^{(r)}, f_{j+1}^{(s+1)}] = -(-1)^{|j|} f_{j+1}^{(r)} f_j^{(s)} \tag{2.39}$$

$$\text{if } |j-k| > 1, \text{ then } [e_j^{(r)}, e_k^{(s)}] = 0 \text{ and } [f_j^{(r)}, f_k^{(s)}] = 0 \tag{2.40}$$

$$\text{if } j \neq k, \text{ then } [[e_j^{(r)}, e_k^{(s)}], e_k^{(t)}] + [[e_j^{(r)}, e_k^{(t)}], e_k^{(s)}] = 0 \tag{2.41}$$

$$\text{if } j \neq k, \text{ then } [[f_j^{(r)}, f_k^{(s)}], f_k^{(t)}] + [[f_j^{(r)}, f_k^{(t)}], f_k^{(s)}] = 0 \tag{2.42}$$

$$[[e_{m-1}^{(r)}, e_m^{(1)}], [e_m^{(1)}, e_{m+1}^{(s)}]] = 0 \tag{2.43}$$

$$[[f_{m-1}^{(r)}, f_m^{(1)}], [f_m^{(1)}, f_{m+1}^{(s)}]] = 0 \tag{2.44}$$

These generators $d_i^{(r)}$, $d_i'^{(r)}$, $e_i^{(r)}$, and $f_i^{(r)}$ are coefficients of the following power series, respectively.

$$d_i(u) = \sum_{r \ge 0} d_i^{(r)} u^{-r}$$

$$(d_i(u))^{-1} = \sum_{r \ge 0} d_i'^{(r)} u^{-r}$$

$$e_i(u) = \sum_{r\geq 1} e_i^{(r)} u^{-r}$$
$$f_i(u) = \sum_{r\geq 1} f_i^{(r)} u^{-r}$$

We can recover the generating series $t_{ij}(u)$ by multiplying together and taking commutators of these series.

One way to see the subalgebra $Y(\mathfrak{sl}_{m|n})$ of $Y(\mathfrak{gl}_{m|n})$ is as the subalgebra fixed by all automorphisms μ_f. Here, for a formal power series $f = 1 + f_1 u^{-1} + f_2 u^{-2} + \cdots \in \mathbb{C}[[u^{-1}]]$, the map μ_f is given by $T(u) \mapsto f(u)T(u)$.

Further, we can use this definition to justify that

$$Y(\mathfrak{gl}_{m|n}) \simeq Z_{m|n} \otimes Y(\mathfrak{sl}_{m|n}) \tag{2.45}$$

where $Z_{m|n}$ is the centre of the Yangian $Y(\mathfrak{gl}_{m|n})$, as one can show the centre is generated by the coeffecients of the quantum Berezinian formal power series.

Definition 2.2.3. The subalgebra $Y(\mathfrak{sl}_{m|n})$ is generated by the coefficients of the series $d_1(u)^{-1}d_{i+1}(u)$, $e_i(u)$, and $f_i(u)$ for $i \in \{m+n-1\}$. If we define $h_i(u) := d_1(u)^{-1}d_{i+1}(u)$, then our generators are $h_i^{(r)}, e_i^{(r)}$, and $f_i^{(r)}$.

Both sets of generators of the Yangian can be used to produce a PBW-type basis as shown by Gow [13].

Theorem 2.2.4. ***PBW Theorem*** *Fix some ordering on the generators $t_{ij}^{(r)}$ of $Y(\mathfrak{gl}_{m|n})$. $Y(\mathfrak{gl}_{m|n})$ is equipped with a PBW basis formed by ordered products of these generators, where the odd generators (where $|i| + |j| \equiv 1 \bmod 2$) do not appear with powers of order greater than 1.*

Theorem 2.2.5. ***PBW Theorem*** *Fix some ordering on the generators $f_{ji}^{(r)}$, $d_i^{(r)}$, $e_{ij}^{(r)}$ of $Y(\mathfrak{gl}_{m|n})$ for $1 \leq i < j \leq m+n$ and $r \geq 1$. This ordering must be such that the f generators come first, then the d generators, and then the e generators. We define the $f_{ji}^{(r)}$ and $e_{ij}^{(r)}$ inductively using:*

$$f_{i+1,i}^{(r)} = f_i^{(r)}$$
$$e_{i,i+1}^{(r)} = e_i^{(r)}$$

$$
\begin{aligned}
f_{j,i}^{(r)} =& [f_{j,j-1}^{(1)}, f_{j-1,i}^{(r)}](-1)^{|j-1|} \\
e_{i,j}^{(r)} =& [e_{i,j-1}^{(r)}, e_{j-1,j}^{(1)}](-1)^{|j-1|}
\end{aligned}
$$

for $j > i+1$. Then, $Y(\mathfrak{gl}_{m|n})$ is equipped with a PBW basis formed by ordered products of these generators.

What follows are the current generators for $Y(\mathfrak{sl}_{m|n})$. These can be derived from the previous generators by setting:

$$h_i(u) = d_i(u + \frac{1}{2}(-1)^{|i|}(m-i))^{-1} d_{i+1}(u + \frac{1}{2}(-1)^{|i|}(m-i)) \tag{2.46}$$

$$x_i^+(u) = f_i(u + \frac{1}{2}(-1)^{|i|}(m-i)) \tag{2.47}$$

$$x_i^-(u) = (-1)^{|i|} e_i(u + \frac{1}{2}(-1)^{|i|}(m-i)) \tag{2.48}$$

for $1 \leq i \leq m+n-1$ and coefficients

$$h_i(u) := 1 + \sum_{s \geq 0} h_{i,s} u^{-s-1} \tag{2.49}$$

$$x_i^+(u) := \sum_{s \geq 0} x_{i,s}^+ u^{-s-1} \tag{2.50}$$

$$x_i^-(u) := \sum_{s \geq 0} x_{i,s}^- u^{-s-1}. \tag{2.51}$$

Definition 2.2.6. The super Yangian $Y(\mathfrak{sl}_{m|n})$ is isomorphic to the associative superalgebra over $\mathbb{C}$ defined by the generators $x_{i,s}^{\pm}$ and $h_{i,s}$ for $1 \leq i \leq m+n-1$ and $s \in \mathbb{Z}_+$, and by the relations

$$[h_{i,r}, h_{j,s}] = 0, \tag{2.52}$$

$$[x_{i,r}^+, x_{j,s}^-] = \delta_{ij} h_{i,r+s} \tag{2.53}$$

$$[h_{i,0}, x_{j,s}^{\pm}] = \pm a_{ij} x_{j,s}^{\pm} \tag{2.54}$$

$$[h_{i,r+1}, x_{j,s}^{\pm}] - [h_{i,r}, x_{j,s+1}^{\pm}] = \frac{\pm a_{ij}}{2} \left(h_{i,r} x_{j,s}^{\pm} + x_{j,s}^{\pm} h_{i,r} \right) \text{ for } i,j \text{ not both } m \tag{2.55}$$

$$[h_{m,r+1}, x_{m,s}^{\pm}] = 0 \tag{2.56}$$

$$[x^{\pm}_{i,r+1}, x^{\pm}_{j,s}] - [x^{\pm}_{i,r}, x^{\pm}_{j,s+1}] = \frac{\pm a_{ij}}{2}\left(x^{\pm}_{i,r}x^{\pm}_{j,s} + x^{\pm}_{j,s}x^{\pm}_{i,r}\right) \text{ for } i,j \text{ not both } m \tag{2.57}$$

$$[x^{\pm}_{m,r}, x^{\pm}_{m,s}] = 0 \tag{2.58}$$

$$[x^{\pm}_{i,r}, x^{\pm}_{j,s}] = 0 \text{ if } |i-j| > 1 \tag{2.59}$$

$$[x^{\pm}_{i,r}, [x^{\pm}_{i,s}, x^{\pm}_{j,t}]] + [x^{\pm}_{i,s}, [x^{\pm}_{i,r}, x^{\pm}_{j,t}]] = 0 \text{ if } |i-j| = 1 \tag{2.60}$$

$$[[x^{\pm}_{m-1,r}, x^{\pm}_{m,0}], [x^{\pm}_{m,0}, x^{\pm}_{m+1,s}]] = 0 \tag{2.61}$$

where r, s, and t are arbitrary nonnegative integers and a_{ij} are the elements of the Cartan matrix $C = (a_{ij})_{i,j=1}^{m+n-1}$ of the Lie superalgebra $\mathfrak{sl}_{m|n}$. The generators $x^{\pm}_{m,r}$ are odd while the rest are even.

2.3 Affine Yangians

For a Lie superalgebra $\mathfrak{g}$, its affine Yangian $\widehat{Y(\mathfrak{g})}$ is a deformation of the enveloping algebra of the universal central extension of $\mathfrak{g}[u^{\pm}, v]$. Universal central extensions will be defined in the next chapter. $\widehat{Y(\mathfrak{g})}$ is built from one copy of $Y(\mathfrak{g})$ and one copy of $U\widehat{\mathfrak{g}}$, where $\widehat{\mathfrak{g}}$ is the universal central extension of $\mathfrak{g} \otimes \mathbb{C}[u^{\pm 1}]$. However, for our purposes, we have explicit presentations for the affine Yangians we need.

Definition 2.3.1. The affine super Yangian $\widehat{Y}(\mathfrak{sl}_{m|n})$ is isomorphic to the associative superalgebra over $\mathbb{C}$ defined by the generators $x^{\pm}_{i,s}$ and $h_{i,s}$ for $0 \le i \le m+n-1$ and $s \in \mathbb{Z}_+$, and by the relations (2.52) to (2.61). The a_{ij} are the elements of the enlarged Cartan matrix $\widehat{C} = (a_{ij})_{i,j=0}^{m+n-1}$.

$$
\widehat{C} = \left(\begin{array}{c|cccc|c|cccc}
0 & -1 & 0 & \dots & \dots & 0 & \dots & \dots & 0 & 1 \\
\hline
-1 & 2 & -1 & & 0 & \vdots & & & & \\
0 & -1 & 2 & \ddots & & \vdots & & 0 & & \\
\vdots & & \ddots & \ddots & -1 & 0 & & & & \\
\vdots & 0 & & -1 & 2 & -1 & & & & \\
\hline
0 & \dots & \dots & 0 & -1 & 0 & 1 & 0 & \dots & 0 \\
\hline
\vdots & & & & & 1 & -2 & 1 & & 0 \\
\vdots & & 0 & & & 0 & 1 & \ddots & \ddots & \\
0 & & & & & \vdots & & \ddots & -2 & 1 \\
1 & & & & & 0 & 0 & & 1 & -2
\end{array}\right).
$$

What follows is the affine super Yangian as defined by Mamoru Ueda [23] with parameters $\varepsilon_1, \varepsilon_2 \in \mathbb{C}$.

Definition 2.3.2. Suppose that $m, n \geq 2$ and $m \neq n$, with $\varepsilon_1, \varepsilon_2 \in \mathbb{C}$. The affine super Yangian $Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ is the associative super algebra over $\mathbb{C}$ generated by $x_{i,r}^{+}$, $x_{i,r}^{-}$, $h_{i,r}$ ($i \in \{0, 1, \cdots, m+n-1\}$, $r \in \mathbb{Z}_{\geq 0}$) subject to the relations:

$$[h_{i,r}, h_{j,s}] = 0, \tag{2.62}$$

$$[x_{i,r}^{+}, x_{j,s}^{-}] = \delta_{ij} h_{i,r+s}, \tag{2.63}$$

$$[h_{i,0}, x_{j,r}^{\pm}] = \pm a_{ij} x_{j,r}^{\pm}, \tag{2.64}$$

$$[h_{i,r+1}, x_{j,s}^{\pm}] - [h_{i,r}, x_{j,s+1}^{\pm}] = \pm a_{ij} \frac{\varepsilon_1 + \varepsilon_2}{2} \left\{h_{i,r}, x_{j,s}^{\pm}\right\} - b_{ij} \frac{\varepsilon_1 - \varepsilon_2}{2} [h_{ir}, x_{js}^{\pm}], \tag{2.65}$$

$$[x_{i,r+1}^{\pm}, x_{j,s}^{\pm}] - [x_{i,r}^{\pm}, x_{j,s+1}^{\pm}] = \pm a_{ij} \frac{\varepsilon_1 + \varepsilon_2}{2} \left\{x_{i,r}^{\pm}, x_{j,s}^{\pm}\right\} - b_{ij} \frac{\varepsilon_1 - \varepsilon_2}{2} [x_{ir}^{\pm}, x_{js}^{\pm}], \tag{2.66}$$

$$\sum_{\sigma \in S_{1+|a_{ij}|}} \left[x_{i,r_{\sigma(1)}}^{\pm}, \left[x_{i,r_{\sigma(2)}}^{\pm}, \dots, \left[x_{i,r_{\sigma(1+|a_{ij}|)}}^{\pm}, x_{js}^{\pm}\right] \cdots\right]\right] = 0, \tag{2.67}$$

$$[x_{ir}^{\pm}, x_{is}^{\pm}] = 0 \ (i = 0, m), \tag{2.68}$$

$$[[x^{\pm}_{i-1,r}, x^{\pm}_{i,0}], [x^{\pm}_{i,0}, x^{\pm}_{i+1,s}]] = 0 \ (i = 0, m) \tag{2.69}$$

where

$$a_{ij} = \begin{cases} (-1)^{|i|} + (-1)^{|i+1|} & \text{if } i = j, \\ -(-1)^{|i+1|} & \text{if } j = i+1, \\ -(-1)^{|i|} & \text{if } j = i-1, \\ 1 & \text{if } (i,j) = (m+n-1, 0), (0, m+n-1), \\ 0 & \text{otherwise} \end{cases} \tag{2.70}$$

$$b_{ij} = \begin{cases} -(-1)^{|i+1|} & \text{if } i = j+1, \\ (-1)^{|i|} & \text{if } i = j-1, \\ -1 & \text{if } (i,j) = (0, m+n-1), \\ 1 & \text{if } (i,j) = (m+n-1, 0), \\ 0 & \text{otherwise} \end{cases} \tag{2.71}$$

The minimalistic presentation of $Y(\widehat{\mathfrak{sl}}(m|n))$ given in Theorem 3.13 in [23] will be useful. Assume that $m \geq 2$.

Theorem 2.3.3. *(Theorem 3.13 [23]) Suppose that $m, n \geq 2$ and $m \neq n$. The affine super Yangian $Y(\widehat{\mathfrak{sl}}(m|n))$ is isomorphic to the super algebra generated by $x^{+}_{i,r}, x^{-}_{i,r}, h_{i,r}$ $(i \in \{0, 1, \cdots, m+n-1\}, r = 0, 1)$ subject to the defining relations:*

$$[h_{i,r}, h_{js,}] = 0, \tag{2.72}$$

$$[x^{+}_{i,0}, x^{-}_{j,0}] = \delta_{ij} h_{i,0}, \tag{2.73}$$

$$[x^{+}_{i,1}, x^{-}_{j,0}] = \delta_{ij} h_{i,1} = [x^{+}_{i,0}, x^{-}_{j,1}], \tag{2.74}$$

$$[h_{i,0}, x^{\pm}_{j,r}] = \pm a_{ij} x^{\pm}_{j,r}, \tag{2.75}$$

$$[\widetilde{h}_{i,1}, x^{\pm}_{j,0}] = \pm a_{ij} \left(x^{\pm}_{j,1} - b_{ij} \frac{\varepsilon_1 - \varepsilon_2}{2} x^{\pm}_{j,0} \right), \tag{2.76}$$

$$[x^{\pm}_{i,1}, x^{\pm}_{j,0}] - [x^{\pm}_{i,0}, x^{\pm}_{j,1}] = \pm a_{ij}\frac{\varepsilon_1+\varepsilon_2}{2}\{x^{\pm}_{i,0}, x^{\pm}_{j,0}\} - b_{ij}\frac{\varepsilon_1-\varepsilon_2}{2}[x^{\pm}_{i,0}, x^{\pm}_{j,0}], \tag{2.77}$$

$$(\mathrm{ad}x^{\pm}_{i,0})^{1+|a_{ij}|}(x^{\pm}_{j,0}) = 0 (i \neq j), \tag{2.78}$$

$$[x^{\pm}_{i,0}, x^{\pm}_{i,0}] = 0 (i = 0, m), \tag{2.79}$$

$$[[x^{\pm}_{i-1,0}, x^{\pm}_{i,0}], [x^{\pm}_{i,0}, x^{\pm}_{i+1,0}]] = 0 (i = 0, m), \tag{2.80}$$

where the generators $x^{\pm}_{m,r}$ *and* $x^{\pm}_{0,r}$ *are odd and all other generators are even. We set* $\widetilde{h}_{i,1} = h_{i,1} - \frac{\varepsilon_1+\varepsilon_2}{2}h^2_{i,0}$.

Note that there is a way to simplify this by reducing the second parameter. This will be useful for Chapter 6. The goal is to remove all the relations with $\varepsilon_1 - \varepsilon_2$ that do not include a generator with $i = j = 0$. We do this using the isomorphism τ_a for a chosen $a \in \mathbb{C}$ where $\tau_a(X) = X$ for $X \in \widehat{\mathfrak{sl}}_{m|n}$ and $\tau_a(X^{\pm}_{k,1}) = X^{\pm}_{k1} - (-1)^{|k+1|}a(m-k)X^{\pm}_{k,0}$, $\tau_a(H_{k,1}) = H_{k,1} - (-1)^{|k+1|}a(m-k)H_{k,0}$. Then, in order to remove the desired terms from relations in the minimalistic presentation, we set $a = -\frac{\varepsilon_1-\varepsilon_2}{2}$ and apply τ_a. Set also $\frac{\lambda}{2} = \frac{\varepsilon_1+\varepsilon_2}{2}$.

If $0 \leq i \leq m-1$, then $a_{i,i+1} = -1$, $b_{i,i+1} = 1$. Start with the relation (2.77) and apply τ_a:

$$[x^{\pm}_{i,1}, x^{\pm}_{i+1,0}] - [x^{\pm}_{i,0}, x^{\pm}_{i+1,1}] = \mp\frac{\lambda}{2}\{x^{\pm}_{i,0}, x^{\pm}_{i+1,0}\} - a[x^{\pm}_{i,0}, x^{\pm}_{i+1,0}]$$

$$\begin{aligned}
&[x^{\pm}_{i,1} - a(m-i)x^{\pm}_{i,0}, x^{\pm}_{i+1,0}] - [x^{\pm}_{i,0}, x^{\pm}_{i+1,1} - a(m-i-1)x^{\pm}_{i+1,0}] \\
&\qquad = \mp\frac{\lambda}{2}\{x^{\pm}_{i,0}, x^{\pm}_{i+1,0}\} - a[x^{\pm}_{i,0}, x^{\pm}_{i+1,0}] \\
&[x^{\pm}_{i,1}, x^{\pm}_{i+1,0}] - [x^{\pm}_{i,0}, x^{\pm}_{i+1,1}] - a[x^{\pm}_{i,0}, x^{\pm}_{i+1,0}] \\
&\qquad = \mp\frac{\lambda}{2}\{x^{\pm}_{i,0}, x^{\pm}_{i+1,0}\} - a[x^{\pm}_{i,0}, x^{\pm}_{i+1,0}] \\
&[x^{\pm}_{i,1}, x^{\pm}_{i+1,0}] - [x^{\pm}_{i,0}, x^{\pm}_{i+1,1}] = \mp\frac{\lambda}{2}\{x^{\pm}_{i,0}, x^{\pm}_{i+1,0}\}
\end{aligned}$$

On the other hand, for $m+1 \leq i \leq m+n-1$,

$$[x^{\pm}_{i,1}, x^{\pm}_{i+1,0}] - [x^{\pm}_{i,0}, x^{\pm}_{i+1,1}] = \pm\frac{\lambda}{2}\{x^{\pm}_{i,0}, x^{\pm}_{i+1,0}\} + a[x^{\pm}_{i,0}, x^{\pm}_{i+1,0}]$$

$$[x^{\pm}_{i,1} + a(m-i)x^{\pm}_{i,0}, x^{\pm}_{i+1,0}] - [x^{\pm}_{i,0}, x^{\pm}_{i+1,1} + a(m-i-1)x^{\pm}_{i+1,0}]$$

$$= \pm\frac{\lambda}{2}\{x^{\pm}_{i,0}, x^{\pm}_{i+1,0}\} + a[x^{\pm}_{i,0}, x^{\pm}_{i+1,0}]$$
$$[x^{\pm}_{i,1}, x^{\pm}_{i+1,0}] - [x^{\pm}_{i,0}, x^{\pm}_{i+1,1}] + a[x^{\pm}_{i,0}, x^{\pm}_{i+1,0}]$$
$$= \pm\frac{\lambda}{2}\{x^{\pm}_{i,0}, x^{\pm}_{i+1,0}\} + a[x^{\pm}_{i,0}, x^{\pm}_{i+1,0}]$$
$$[x^{\pm}_{i,1}, x^{\pm}_{i+1,0}] - [x^{\pm}_{i,0}, x^{\pm}_{i+1,1}] = \pm\frac{\lambda}{2}\{x^{\pm}_{i,0}, x^{\pm}_{i+1,0}\}$$

Finally, we consider $i = m$.

$$[x^{\pm}_{m,1}, x^{\pm}_{m+1,0}] - [x^{\pm}_{m,0}, x^{\pm}_{m+1,1}] = \pm\frac{\lambda}{2}\left\{x^{\pm}_{m,0}, x^{\pm}_{m+1,0}\right\} + a[x^{\pm}_{m,0}, x^{\pm}_{m+1,0}]$$
$$[x^{\pm}_{m,1}, x^{\pm}_{m+1,0}] - [x^{\pm}_{m,0}, x^{\pm}_{m+1,1} - ax^{\pm}_{m+1,0}] = \pm\frac{\lambda}{2}\left\{x^{\pm}_{m,0}, x^{\pm}_{m+1,0}\right\} + a[x^{\pm}_{m,0}, x^{\pm}_{m+1,0}]$$
$$[x^{\pm}_{m,1}, x^{\pm}_{m+1,0}] - [x^{\pm}_{m,0}, x^{\pm}_{m+1,1}] + a[x^{\pm}_{m,0}, x^{\pm}_{m+1,0}] = \pm\frac{\lambda}{2}\left\{x^{\pm}_{m,0}, x^{\pm}_{m+1,0}\right\} + a[x^{\pm}_{m,0}, x^{\pm}_{m+1,0}]$$
$$[x^{\pm}_{m,1}, x^{\pm}_{m+1,0}] - [x^{\pm}_{m,0}, x^{\pm}_{m+1,1}] = \pm\frac{\lambda}{2}\left\{x^{\pm}_{m,0}, x^{\pm}_{m+1,0}\right\}$$

i.e. we again produce $[x^{\pm}_{i,1}, x^{\pm}_{i+1,0}] - [x^{\pm}_{i,0}, x^{\pm}_{i+1,1}] = \pm\frac{\lambda}{2}\{x^{\pm}_{i,0}, x^{\pm}_{i+1,0}\}$ for $i = m$.

Next, consider the relation (2.76). With our choice of a, it is equivalent to

$$[\widetilde{h}_{i,1}, x^{\pm}_{i+1,0}] = \pm a_{i,i+1}(x^{\pm}_{i+1,1} + m_{i,i+1}ax^{\pm}_{i+1,0})$$

By definition, $\tau_a(h_{k,1}) = h_{k,1} - (-1)^{|k+1|}a(m-k)h_{k,0}$, so applying τ_a to (2.76) produces

$$[\widetilde{h}_{i,1} - (-1)^{|i+1|}a(m-i)h_{i,0}, x^{\pm}_{i+1,0}]$$
$$= \pm\, a_{i,i+1}(x^{\pm}_{i+1,1} + b_{i,i+1}ax^{\pm}_{i+1,0} - (-1)^{|i+2|}a(m-i-1)x^{\pm}_{i+1,0}).$$

For the cases where $1 \le i \le m-1$ or $m+1 \le i \le m+n-2$, $|i+1| = |i+2|$, $b_{i,i+1} = -(-1)^{|i+1|}$, and this simplifies thusly:

$$[\widetilde{h}_{i,1}, x^{\pm}_{i+1,0}] = \pm a_{i,i+1}(x^{\pm}_{i+1,1} + b_{i,i+1}ax^{\pm}_{i+1,0} + (-1)^{|i+2|}ax^{\pm}_{i+1,0})$$
$$[\widetilde{h}_{i,1}, x^{\pm}_{i+1,0}] = \pm a_{i,i+1}x^{\pm}_{i+1,1}.$$

When $i = m$, $b_{m,m+1} = 1$ and (2.76) becomes

$$[\widetilde{h}_{i,1}, x^{\pm}_{i+1,0}] = \pm a_{i,i+1}(x^{\pm}_{i+1,1} + b_{m,m+1} a x^{\pm}_{i+1,0} - a x^{\pm}_{i+1,0})$$
$$[\widetilde{h}_{i,1}, x^{\pm}_{i+1,0}] = \pm a_{i,i+1} x^{\pm}_{i+1,1}.$$

The upshot of this in the context of the minimalistic presentation of the affine super Yangian is that we can assume that $\varepsilon_1 - \varepsilon_2$ appears only in the relations involving $x^{\pm}_{0,r}$, $x^{\pm}_{m+n-1,s}$ or one of those with an h. As a result, we can replace $\varepsilon_1 - \varepsilon_2$ with another parameter, say β. For instance, one relation is

$$[x^{\pm}_{m+n-1,1}, x^{\pm}_{0,0}] - [x^{\pm}_{m+n-1,0}, x^{\pm}_{0,1}] = \pm\frac{\lambda}{2}\{x^{\pm}_{m+n-1,0}, x^{\pm}_{0,0}\} + \beta[x^{\pm}_{m+n-1,0}, x^{\pm}_{0,0}].$$

2.3.1 Parity sequences

Let $\mathbf{s} = (s_1, \ldots, s_{m+n})$ where $s_i = 0$ or 1 and 1 occurs exactly n times. Then, $\mathbf{s}$ is a *parity sequence*. The standard parity sequence is denoted

$$\mathbf{s}^{stand} = (\underbrace{0, \cdots, 0}_{m}, \underbrace{1 \cdots, 1}_{n}) = \mathbf{s}^{(0)}.$$

Shifts of this parity sequence to the right are denoted

$$\mathbf{s}^{(p)} = (\underbrace{1, \ldots, 1}_{p}, \underbrace{0, \ldots, 0}_{m}, \underbrace{1, \ldots, 1}_{n-p})$$

and for $p \leq n$ this is the cyclical shift of $\mathbf{s}^{(0)}$ by p. Along with these shifts, for a general parity sequence of length $m + n$ $\mathbf{s}$, we assign periodicity via setting $s_i = s_{i+m+n}$ and extending all i to $\mathbb{Z}$. Call the set of all parity sequences of length $m + n$ $S_{m|n}$. Then, we introduce a map $\tau : S_{m|n} \to S_{m|n}$ given by the shift sending $\mathbf{s} = (s_1, \ldots, s_{m+n})$ to $\tau\mathbf{s} = (s_{m+n}, s_1, \ldots, s_{m+n-1})$. We can see that $\mathbf{s}^{(p)} = \tau^p \mathbf{s}$.

Next, we apply this principle of shifted parity sequences to Yangians abiding by these parities. Let $\widehat{Y}^{(0)}_{\varepsilon_1,\varepsilon_2}(m|n) = \widehat{Y}_{\varepsilon_1,\varepsilon_2}(m|n)$ be affine super Yangian as defined above. Define $\widehat{Y}^{(p)}_{\varepsilon_1,\varepsilon_2}(m|n)$ to be the associative superalgebra generated by $x^{\pm}_{i,r}$, $h_{i,r}$ with parities and relations being such that the map

$x_i^{\pm}(u) \mapsto x_{i+1}^{\pm}(u+\sigma)$, $h_i(u) \mapsto h_{i+1}(u+\sigma)$ for a shift σ is an even isomorphism.

2.4 Schur-Weyl Duality

In the classical case, Schur-Weyl duality describes the relationship between representations of the general linear Lie group (or algebra) and representations of the symmetric group.

Both GL_n and S_l act on the tensor space $\mathbb{C}^n \otimes \mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n = (\mathbb{C}^n)^{\otimes l}$ as follows, for $X \in GL_n$, $\sigma \in S_l$:

$$X(v_1 \otimes \cdots \otimes v_l) = Xv_1 \otimes \cdots \otimes Xv_l \tag{2.81}$$

$$(v_1 \otimes \cdots \otimes v_l)\sigma = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(l)} \tag{2.82}$$

Schur-Weyl duality asserts that these commuting actions are centralizers of each other. By the Double Centralizer Theorem, we can decompose $(\mathbb{C}^n)^{\otimes l}$ as a sum of tensor products of simple modules of S_l and simple modules of GL_n. The latter are determined by the former; for a module U of S_l, its pair is $\mathrm{Hom}_{S_l}(U, \mathbb{C}^n)$ which is either an irreducible representation of GL_n or is zero.

This interplay has been generalized to the superalgebra setting, as by Sergeev in 1985 [21]. By Schur-Sergeev duality, the actions of $\mathfrak{gl}_{m|n}$ and the symmetric group S_l commute on $(\mathbb{C}^{m|n})^{\otimes l}$. These actions are described as follows. For homogeneous $X \in \mathfrak{gl}_{m|n}$ and $\sigma_{i,i+1} \in S_l$, and homogeneous elements $v_1, \ldots, v_l \in \mathbb{C}^{m|n}$,

$$\begin{aligned} X.(v_1 \otimes \cdots \otimes v_l) &= X(v_1) \otimes v_2 \otimes \cdots \otimes v_l \\ &\quad + (-1)^{|X||v_1|} v_1 \otimes X(v_2) \otimes v_3 \otimes \cdots \otimes v_l + \cdots \\ &\quad + (-1)^{|X|(|v_1|+\cdots+|v_{l-1}|)} v_1 \otimes \cdots \otimes v_{l-1} \otimes X(v_l) \end{aligned} \tag{2.83}$$

$$\sigma_{i,i+1}.(v_1 \otimes \cdots \otimes v_l) = (-1)^{|v_i||v_{i+1}|} v_1 \otimes \cdots \otimes v_{i+1} \otimes v_i \otimes \cdots \otimes v_l \tag{2.84}$$

Let $\Phi_l : \mathfrak{gl}_{m|n} \to \mathrm{End}((\mathbb{C}^{m|n})^{\otimes l})$ and $\Psi_l : S_l \to \mathrm{End}((\mathbb{C}^{m|n})^{\otimes l})$ be the maps defining the actions of $\mathfrak{gl}_{m|n}$ and S_l respectively.

Lemma 2.4.1. *[7] The actions of* $\mathfrak{gl}_{m|n}$ *and* S_l *on* $(\mathbb{C}^{m|n})^{\otimes l}$ *commute with each other.*

Berele and Regev [2] presented a generalization of Weyl's works into the superalgebra realm, introducing what they call "hook analogues" of classic objects. They define a right action of S_l on $(\mathbb{C}^{m|n})^{\otimes l}$ through a map $\psi : S_l \to \mathrm{End}((\mathbb{C}^{m|n})^{\otimes l})$. Weyl's theorem provides a parametrization of irreducible representations of $GL_n(\mathbb{C})$ by partitions whose Young diagrams lie inside the strip of a certain height k. The main result of Berele and Regev's paper is to provide a hook analogue of this theorem, where they connect semistandard tableaux in two sets of variables to representations of $\mathfrak{gl}_{m|n}(\mathbb{C})$. The upshot of this is a super-analogue of Schur's double centralizing theorem.

Theorem 2.4.2. *(Schur-Sergeev duality, part I) [7] The images of* Φ_l *and* Ψ_l*, being* $\Phi_l(U(\mathfrak{gl}_{m|n}))$ *and* $\Psi_l(\mathbb{C}S_l)$*, satisfy the double centralizer property, i.e.*

$$\Phi_l(U(\mathfrak{gl}_{m|n})) = \mathrm{End}_{S_l}((\mathbb{C}^{m|n})^{\otimes l}) \tag{2.85}$$

$$\Psi_l(\mathbb{C}S_l) = \mathrm{End}_{U(\mathfrak{gl}_{m|n})}((\mathbb{C}^{m|n})^{\otimes l}) \tag{2.86}$$

Throughout this thesis, more analogues of Schur-Weyl duality for Yangians and deformed double current algebras of superalgebras are presented.

2.5 Degenerate Affine Hecke Algebra and Cherednik Algebras

Definition 2.5.1. Let $\kappa \in \mathbb{C}$. The degenerate affine Hecke algebra $H_\kappa^{\mathrm{deg}}(S_l)$ is the associative algebra with generators $\sigma_1^{\pm 1}, \sigma_2^{\pm 1}, \ldots, \sigma_{l-1}^{\pm 1}, \mathsf{x}_1, \mathsf{x}_2, \ldots, \mathsf{x}_l$, and the following defining relations:

$$\sigma_i \sigma_i^{-1} = \sigma_i^{-1} \sigma_i = 1, \tag{2.87}$$

$$\sigma_i^2 = 1, \tag{2.88}$$

$$\sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i-j| > 1, \tag{2.89}$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \tag{2.90}$$

$$\mathsf{x}_i\sigma_i = \sigma_i\mathsf{x}_{i+1} - \kappa, \tag{2.91}$$

$$\mathsf{x}_i\mathsf{x}_j = \mathsf{x}_j\mathsf{x}_i \tag{2.92}$$

Often we see $\kappa = -1$, making the relation (2.91) the familiar

$$\mathsf{x}_i\sigma_i = \sigma_i\mathsf{x}_{i+1} + 1. \tag{2.93}$$

Define elements $\mathsf{y}_k \in H_\kappa^{\deg}(S_l)$ $(l = 1, \ldots, l)$ by

$$\mathsf{y}_k := \sigma_{1k} \cdot \mathsf{x}_1 \cdot \sigma_{1k} = \mathsf{x}_k + \sum_{j<k} \sigma_{jk}. \tag{2.94}$$

We can prove by induction that indeed $\sigma_{1k} \cdot \mathsf{x}_1 \cdot \sigma_{1k} = \mathsf{x}_k + \sum_{j<k} \sigma_{jk}$.

Proof. $k = 1$ holds trivially. Let $k = 2$. Then, by defining relations

$$\mathsf{x}_1\sigma_{12} = \sigma_{12}\mathsf{x}_2 + 1$$

$$\sigma_{12}\mathsf{x}_1\sigma_{12} = \mathsf{x}_2 + \sigma_{12}$$

Assume $\sigma_{1k} \cdot \mathsf{x}_1 \cdot \sigma_{1k} = \mathsf{x}_k + \sum_{j<k} \sigma_{jk}$. Then,

$$\begin{aligned}
\sigma_{1,k+1}\mathsf{x}_1\sigma_{1,k+1} &= \sigma_{k,k+1}\sigma_{1k}\sigma_{k,k+1}\mathsf{x}_1\sigma_{k,k+1}\sigma_{1k}\sigma_{k,k+1} \\
&= \sigma_{k,k+1}\sigma_{1k}\mathsf{x}_1\sigma_{1k}\sigma_{k,k+1} \\
&= \sigma_{k,k+1}\left(\mathsf{x}_k + \sum_{j<k}\sigma_{jk}\right)\sigma_{k,k+1} \\
&= \sigma_{k,k+1}\mathsf{x}_k\sigma_{k,k+1} + \sigma_{k,k+1}\left(\sum_{j<k}\sigma_{jk}\right)\sigma_{k,k+1} \\
&= \mathsf{x}_{k+1} + \sum_{j<k}\sigma_{j,k+1}
\end{aligned}$$

□

The elements y_i satisfy

$$\sigma\mathsf{y}_i = \mathsf{y}_{\sigma(i)}\sigma$$

$$[\mathsf{y}_i, \mathsf{y}_j] = -(\mathsf{y}_i - \mathsf{y}_j)\sigma_{ij}$$

for all $\sigma \in S_l$, $i, j = 1, \ldots, l$.

Proof. We check only the second equality. First, we verify this for $i = 1, j = 2$.

$$\begin{aligned}
[\mathsf{y}_1, \mathsf{y}_2] &= \mathsf{y}_1\mathsf{y}_2 - \mathsf{y}_2\mathsf{y}_1 \\
&= \mathsf{x}_1\sigma_{12}\mathsf{x}_1\sigma_{12} - \sigma_{12}\mathsf{x}_1\sigma_{12}\mathsf{x}_1 \\
&= \mathsf{x}_1\sigma_{12}\left(\sigma_{12}\mathsf{x}_2 + \kappa\right) - \sigma_{12}\left(\sigma_{12}\mathsf{x}_2 + \kappa\right)\mathsf{x}_1 \\
&= \mathsf{x}_1\mathsf{x}_2 + \mathsf{x}_1\sigma_{12}\kappa - \mathsf{x}_2\mathsf{x}_1 - \sigma_{12}\kappa\mathsf{x}_1 \\
&= \left(\mathsf{x}_1\sigma_{12} - \sigma_{12}\mathsf{x}_1\right)\kappa
\end{aligned}$$

If we set $\kappa = -1$,

$$\begin{aligned}
\left(\mathsf{x}_1\sigma_{12} - \sigma_{12}\mathsf{x}_1\right)\kappa &= -\left(\mathsf{x}_1 - \sigma_{12}\mathsf{x}_1\sigma_{12}\right)\sigma_{12} \\
&= -\left(\mathsf{y}_1 - \mathsf{y}_2\right)\sigma_{12}
\end{aligned}$$

Conjugating this equality by σ_{2j} and σ_{1i} gives

$$\begin{aligned}
\sigma_{1i}\sigma_{2j}[\mathsf{y}_1, \mathsf{y}_2]\sigma_{2j}\sigma_{1i} =& -\sigma_{1i}\sigma_{2j}(\mathsf{y}_i - \mathsf{y}_j)\sigma_{ij}\sigma_{2j}\sigma_{1i} \\
\sigma_{1i}\sigma_{2j}[\mathsf{y}_1, \mathsf{y}_2]\sigma_{2j}\sigma_{1i} =& \sigma_{1i}\sigma_{2j}\mathsf{y}_1\mathsf{y}_2\sigma_{2j}\sigma_{1i} - \sigma_{1i}\sigma_{2j}\mathsf{y}_2\mathsf{y}_1\sigma_{2j}\sigma_{1i} \\
=& \sigma_{1i}\mathsf{y}_1\sigma_{2j}\mathsf{y}_2\sigma_{2j}\sigma_{1i} - \sigma_{1i}\sigma_{2j}\mathsf{y}_2\sigma_{2j}\mathsf{y}_1\sigma_{1i} \\
=& \sigma_{1i}\mathsf{y}_1\mathsf{y}_j\sigma_{1i} - \sigma_{1i}\mathsf{y}_j\mathsf{y}_1\sigma_{1i} \\
=& \sigma_{1i}\mathsf{y}_1\sigma_{1i}\mathsf{y}_j - \mathsf{y}_i\sigma_{1i}\mathsf{y}_1\sigma_{1i} \\
=& \mathsf{y}_i\mathsf{y}_j - \mathsf{y}_j\mathsf{y}_i \\
=& [\mathsf{y}_i, \mathsf{y}_j] \\
-\sigma_{1i}\sigma_{2j}(\mathsf{y}_1 - \mathsf{y}_2)\sigma_{12}\sigma_{2j}\sigma_{1i} =& -\sigma_{1i}\sigma_{2j}\mathsf{y}_1\sigma_{12}\sigma_{2j}\sigma_{1i} \\
&+ \sigma_{1i}\sigma_{2j}\mathsf{y}_2\sigma_{12}\sigma_{2j}\sigma_{1i} \\
=& -\sigma_{2j}\sigma_{1i}\mathsf{y}_1\sigma_{12}\sigma_{2j}\sigma_{1i} \\
&+ \sigma_{1i}\mathsf{y}_j\sigma_{2j}\sigma_{12}\sigma_{2j}\sigma_{1i} \\
=& -\sigma_{2j}y_i\sigma_{1i}\sigma_{12}\sigma_{2j}\sigma_{1i} \\
&+ \mathsf{y}_j\sigma_{1i}\sigma_{2j}\sigma_{12}\sigma_{2j}\sigma_{1i}
\end{aligned}$$

$$
\begin{aligned}
&= - \mathsf{y}_i \sigma_{2j}\sigma_{1i}\sigma_{12}\sigma_{2j}\sigma_{1i} + \mathsf{y}_j \sigma_{ij} \\
&= - \mathsf{y}_i \sigma_{ij} + \mathsf{y}_j \sigma_{ij} \\
&= - (\mathsf{y}_i - \mathsf{y}_j)\sigma_{ij}
\end{aligned}
$$

□

We use the notation $\overline{\sigma}_{ij}$ for $i < j$ to denote $\sigma_i \sigma_{i+1} \cdots \sigma_{j-1}$ where σ_i is the transposition $(i\ i+1)$, so $\overline{\sigma}_{ij}$ is the cycle $(i\ i+1\ \cdots\ j-1\ j)$. $\overline{\sigma}_{ji}$ with $j > i$ will denote the cycle $\sigma_{j-1}\sigma_{j-2}\cdots\sigma_i = (j\ j-1\ j-2\ \cdots\ i+1\ i)$.

Lemma 2.5.2. *Suppose $1 \le a < b \le l$. The following identities hold in $H_\kappa^{\deg}(S_\ell)$:*

1. $\overline{\sigma}_{ba}\mathsf{x}_a - \mathsf{x}_b\overline{\sigma}_{ba} = \kappa \sum_{k=a+1}^{b} \overline{\sigma}_{bk}\overline{\sigma}_{k-1,a}$
2. *If $a \le b < e \le l$, then $\overline{\sigma}_{ba}\mathsf{x}_e = \mathsf{x}_e\overline{\sigma}_{ba}$.*
3. *If $a < b \le l$, then $\overline{\sigma}_{ba}\mathsf{x}_b = \mathsf{x}_{b-1}\overline{\sigma}_{ba} - \kappa\overline{\sigma}_{b-1,a}$.*
4. *If $a < e \le b \le l$, then $\overline{\sigma}_{ba}\mathsf{x}_e = \mathsf{x}_{e-1}\overline{\sigma}_{ba} - \kappa\overline{\sigma}_{be}\overline{\sigma}_{e-1,a}$.*

Proof. 1. This is shown by induction on b. Suppose $b = a + 1$. Then

$$\sigma_a x_a - x_{a+1}\sigma_a = \kappa \tag{2.95}$$

holds by defining relations. Note that if $\overline{\sigma}_{ba} = (b\ b-1\ \cdots a)$, then $\overline{\sigma}_{b+1,a} = (b+1\ b \cdots a) = \sigma_{b+1,b}\overline{\sigma}_{ba}$. Multiplying (2.95) on the left by $\sigma_{b,b+1}$ produces

$$
\begin{aligned}
&\sigma_{b,b+1}\overline{\sigma}_{ba}\mathsf{x}_a - \sigma_{b,b+1}\mathsf{x}_b\overline{\sigma}_{ba} = \kappa \sum_{k=a+1}^{b} \sigma_{b,b+1}\overline{\sigma}_{bk}\overline{\sigma}_{k-1,a} \\
\iff\quad & \overline{\sigma}_{b+1,a}\mathsf{x}_a - (x_{b+1}\sigma_{b,b+1} + \kappa)\overline{\sigma}_{ba} = \kappa \sum_{k=a+1}^{b} \overline{\sigma}_{b+1,k}\overline{\sigma}_{k-1,a} \\
\iff\quad & \overline{\sigma}_{b+1,a}\mathsf{x}_a - x_{b+1}\overline{\sigma}_{b+1,a} = \kappa\overline{\sigma}_{ba} + \kappa \sum_{k=a+1}^{b} \overline{\sigma}_{b+1,k}\overline{\sigma}_{k-1,a}
\end{aligned}
$$

$$= \kappa \sum_{k=a+1}^{b+1} \overline{\sigma}_{b+1,k} \overline{\sigma}_{k-1,a}$$

2. This is trivial for $a \leq b < e \leq l$.

3. We prove by induction. If $b = a+1$, by defining relations

$$\sigma_{a,a+1} \mathsf{x}_a - \mathsf{x}_{a+1} \sigma_{a,a+1} = \kappa. \tag{2.96}$$

   Multiplying by $\sigma_{a,a+1}$ on both sides produces

$$\begin{aligned} \mathsf{x}_a \sigma_{a,a+1} - \sigma_{a,a+1} \mathsf{x}_{a+1} &= \kappa \\ -\sigma_{a,a+1} \mathsf{x}_{a+1} &= -\mathsf{x}_a \sigma_{a,a+1} + \kappa \\ \sigma_{a,a+1} \mathsf{x}_{a+1} &= \mathsf{x}_a \sigma_{a,a+1} - \kappa \end{aligned}$$

4. We start with (3) and prove by induction on b. We start with $b = e$:

$$\overline{\sigma}_{ea} x_e = x_{e-1} \overline{\sigma}_{ea} - \kappa \overline{\sigma}_{e-1,a} \tag{2.97}$$

   as desired. Multiplying (4) by $\sigma_{b,b+1}$ produces

$$\begin{aligned} \sigma_{b,b+1} \overline{\sigma}_{ba} \mathsf{x}_e &= \sigma_{b+1,b} \mathsf{x}_{e-1} \overline{\sigma}_{ba} - \kappa \sigma_{b+1,b} \overline{\sigma}_{be} \overline{\sigma}_{e-1,a} \\ \overline{\sigma}_{b+1,a} \mathsf{x}_e &= \mathsf{x}_{e-1} \overline{\sigma}_{b+1,a} - \kappa \overline{\sigma}_{b+1,e} \overline{\sigma}_{e-1,a} \end{aligned}$$

□

Definition 2.5.3. [22] Let $\widetilde{\mathcal{F}}$ be the algebra generated freely by the algebras $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \rtimes \mathbb{C}S_n$, which is $\mathbb{C}\widehat{S_n}$, and $\mathbb{C}[u_1, \ldots, u_n]$ such that the natural inclusion maps $\mathbb{C}\widehat{S_n} \to \widetilde{\mathcal{F}}$ and $\mathbb{C}[u_1, \ldots, u_n] \to \widetilde{\mathcal{F}}$ are algebra homomorphisms.

Then, the *degenerate double affine Hecke algebra* $\mathbb{H}_{\kappa,c}(S_\ell)$ (or *trigonometric Cherednik algebra*) of type A is the unital associative algebra defined as the quotient algebra of $\widetilde{\mathcal{F}}$ by the following relations:

$$s_i u_i = u_{i+1} s_i - 1 \tag{2.98}$$

$$s_i u_{i+1} = u_i s_i + 1 \tag{2.99}$$

$$s_i u_j = u_j s_i \text{ for } 1 \le j \le n,\ j \ne i, i+1 \tag{2.100}$$

$$[u_i, x_j] = \begin{cases} \kappa x_i + \sum_{1\le k<i} x_k s_{ki} + \sum_{i<k\le n} x_i s_{ik} & i = j \\ -x_j s_{ji} & i > j \\ -x_i s_{ij} & i < j \end{cases} \tag{2.101}$$

for $1 \le i \le n-1$.

Definition 2.5.4. Let $t, c \in \mathbb{C}$. The rational Cherednik algebra $\mathsf{H}_{t,c}(S_\ell)$ of type $\mathfrak{gl}_\ell$ is the algebra generated by commuting elements $x_1, \dots, x_\ell$, commuting elements $y_1, \dots, y_\ell$, and $\mathbb{C}[S_\ell]$, subject to the following relations:

$$\sigma x_i \sigma^{-1} = x_{\sigma(i)} \qquad \sigma y_j \sigma^{-1} = y_{\sigma(j)}$$

$$\text{If } i \ne j, \text{ then } [y_j, x_i] = -c\sigma_{ij},\ \ [y_i, x_i] = t + c\sum_{k\ne i} \sigma_{ki}.$$

We previously defined the trigonometric Cherednik algebra $\mathbb{H}_{\kappa,c}(S_\ell)$ using s_i and u_i generators in Definition 2.5.3, but we can frame it in terms of the rational Cherednik algebra as well.

Proposition 2.5.5. *The degenerate double affine Hecke algebra $\mathbb{H}_{\kappa,c}(S_\ell)$ (which is also called the trigonometric Cherednik algebra) is the localization of $\mathsf{H}_{t,c}$ at $x_1 \cdots x_\ell$; in other words, $\mathbb{H}_{\kappa,c}(S_\ell)$ is obtained from $\mathsf{H}_{t,c}(S_\ell)$ by adding the inverses $x_1^{-1}, x_2^{-1}, \dots, x_\ell^{-1}$.*

Let $\mathsf{y}_i = x_i y_i$. It can be checked that the subalgebra of $\mathbb{H}_{\kappa,c}(S_\ell)$ generated by $\mathsf{y}_1, \dots, \mathsf{y}_\ell$ and by S_ℓ is isomorphic to the degenerate affine Hecke algebra of S_ℓ. These elements satisfy $[\mathsf{y}_i, \mathsf{y}_j] = -(\mathsf{y}_i - \mathsf{y}_j)\sigma_{ij}$.

The following algebra will be used in describing the deformed double current algebra of type Q.

Definition 2.5.6. Let $\kappa \in \mathbb{C}$. The *rational double affine Hecke-Clifford algebra* (rational DaHCa) $\ddot{\mathfrak{H}}^{\mathfrak{c}}$ is the $\mathbb{C}$-algebra generated by x_i, y_i, c_i $(1 \le i \le l)$ and S_l, subject to the following relations:

$$x_i x_j = x_j x_i, \qquad y_i y_j = y_j y_i, \qquad (\forall i, j) \tag{2.102}$$

$$\sigma x_i = x_{\sigma(i)}\sigma, \qquad \sigma y_i = y_{\sigma(i)}\sigma, \qquad (\sigma \in S_l) \tag{2.103}$$

$$c_i x_i = -x_i c_i, \qquad c_i y_i = y_i c_i, \tag{2.104}$$

$$c_j x_i = x_i c_j, \qquad c_j y_i = y_i c_j, \qquad (i \neq j) \tag{2.105}$$

$$\sigma c_i = c_{\sigma(i)}\sigma, \qquad (\sigma \in S_l) \tag{2.106}$$

$$c_i^2 = 1 \tag{2.107}$$

$$c_i c_j = -c_j c_i \qquad (i \neq j) \tag{2.108}$$

$$[y_j, x_i] = \kappa(1 + c_j c_i)s_{ij} \qquad (i \neq j) \tag{2.109}$$

$$[y_i, x_i] = -\kappa \sum_{k \neq i}(1 + c_k c_i)s_{ki} \tag{2.110}$$

Denote by $\mathcal{C}_l$ or by $\mathcal{C}(c_1, \ldots, c_l)$ the Clifford algebra generated by $c_1, \ldots, c_l$. The rational DaHCa $\ddot{\mathfrak{H}}^{\mathfrak{c}}$ is a super (i.e. $\mathbb{Z}_2$-graded) algebra with $|c_i| = 1$ and $|x_i| = |y_i| = |s_{ij}| = 0$.

2.6 Universal Central Extensions

Definition 2.6.1. An *extension* of a Lie superalgebra L is a short exact sequence of Lie superalgebras:

$$0 \to I \xrightarrow{e} K \xrightarrow{f} L \to 0.$$

We can simply refer to K as the extension.

As such, an extension of L is the same as an epimorphism $f : K \to L$. A *homomorphism* from an extension $f : K \to L$ to another extension $f' : K' \to L$ is a Lie superalgebra homomorphism $g : K \to K'$ satisfying $f = f' \circ g$. Equivalently, the following diagram commutes:

$$\begin{array}{ccc} K & \xrightarrow{g} & K' \\ & \searrow^{f} & \downarrow{\scriptstyle f'} \\ & & L \end{array}$$

Thus, in particular, $\ker g \subset g^{-1}(\ker f') = \ker f$ and $K' = g(K) + \ker f'$.

We call an extension $f : K \to L$ *split* if there exists a Lie superalgebra homomorphism $s : L \to K$ such that $f \circ s = \mathrm{Id}_L$. Call s a *splitting homomorphism.* If we have such an s, $K = I \oplus s(L)$ where $I = \ker f$ and $s : L \to s(L)$ is an isomorphism with inverse $s^{-1} = f|_{s(L)}$.

A *central extension* of L is an extension $f : K \to L$ such that $\ker f \subset Z(K)$, where $Z(K)$ denotes the centre of K. For a split central extension with splitting homomorphism s, the Lie superalgebra K is a direct product $K = \ker f \times s(L)$.

A central extension $\mathfrak{u} : \mathfrak{K} \to L$ is called a *universal central extension* if there exists a unique homomorphism from $\mathfrak{K}$ to any other central extension K of L. It follows from the universal property that any two universal central extensions of L are isomorphic as extensions and their underlying Lie superalgebras are isomorphic.

Lemma 2.6.2. *(Lemma 1.4 "Central Trick" of [20]) Assume that $f : K \to L$ is a central extension.*

1. *If $f(x) = f(x')$ and $f(y) = f(y')$ then $[x, y] = [x', y']$.*
2. *If g and g' are homomorphisms from some Lie superalgebra P to K such that $f \circ g = f \circ g'$, then $g|_{[P,P]} = g'|_{[P,P]}$. In particular, there exists at most one homomorphism from a covering $p \to L$ to the central extension $f : K \to L$.*

Proof. 1. Since this is a central extension, i.e. $\ker f \subset Z(K)$, $f(x) = f(x')$ gives $x - x' \in \ker f$, so $x' = x + z$ for some $z \in \ker f$. Similarly, $y' = y + z'$ for some $z' \in \ker f$. Then, $[x', y'] = [x+z, y+z'] = [x, y+z'] + [z, y+z'] = [x, y]$.

2. $f \circ g = f \circ g'$ gives that $f(g(x)) = f(g'(x))$, $f(g(y)) = f(g'(y))$ for all $x, y \in P$, so $[g(x), g(y)] = [g'(x), g'(y)]$ by the first statement of this lemma. Then, $g([x, y]) = [g(x), g(y)] = [g'(x), g'(y)] = g'([x, y])$.

□

Lemma 2.6.3. *(Lemma 1.5 of [20]) Let $f : K \to L$ be a central extension of a perfect Lie superalgebra L.*

1. $K = [K, K] + kerf$ *and* $f : [K, K] \to L$ *is a covering.*

2. $Z(K) = f^{-1}(Z(L))$ *and* $f(Z(K)) = Z(L)$.

3. *If* $g : L \to M$ *is a central extension, then so is* $g \circ f : K \xrightarrow{f} L \xrightarrow{g} M$.

4. *If* $f' : K' \to L$ *is a covering and* $g : K \to K'$ *a homomorphism from the extension* $f : K \to L$ *to the extension* $f' : K' \to L$*, then* $g : K \to K'$ *is a central extension and is surjective.*

Proof. 1. L is perfect, so $f([K, K]) = L$ and thus $f|_{[K,K]}$ is a central extension of L. Recall that ker $f \subset Z(K)$, so along with $f([K, K]) = L$ we get that $K = [K, K] + \ker f$. From this, it follows that $[K, K]$ is perfect and thus $f : [K, K] \to L$ is a covering.

2. Take some $z \in K$. $z \in f^{-1}(Z(L)) \Leftrightarrow \forall x \in L, [f(z), x] = 0 \Leftrightarrow \forall y \in K, [f(z), f(y)] = 0 \Leftrightarrow [z, K] \subset \ker f$. In particular, $Z(K) \subset f^{-1}(Z(L))$. To prove the other inclusion, let $z \in f^{-1}(Z(L))$. Then, $[z, K] \subset Z(K)$, and so from the first part of the lemma $[z, K] = [z, [K, K] + \ker f] = [z, [K, K]] = [[z, K], K] + [K, [z, K]] = 0$, i.e., $z \in Z(K)$. $f(Z(K)) = Z(L)$ follows from the surjectivity of f.

3. This follows from $\ker(g \circ f) = f^{-1}(\ker g) \subset f^{-1}(Z(L)) = Z(K)$.

4. If $x \in \ker g$, then $x \in g^{-1}(\ker f') = \ker f \subset Z(K)$. As well, $K' = g(K) + \ker f'$. We get this by taking some $x' \in K'$. The element $f'(x') = l \in L$ has some preimage in K, say $f(x) = l$. Then $g(x) \in K'$ with $f'(g(x)) = f'(x') = l$. Thus, $g(x) - x' \in \ker f'$ and we can conclude that any element of K' can be written as the sum of some element in the image of g and some element in the kernel of f', i.e. $K' = g(K) + \ker f'$. Finally, using that K' is perfect, it follows that $K' = [K', K'] = [g(K), g(K)] = g([K, K])$ and $\ker g \subset \ker f$ is central.

□

Corollary 2.6.4. *(Corollary 1.6 of [20]) Let L be an arbitrary Lie superalgebra. If* $L/Z(L)$ *is perfect, then* $Z(L/Z(L)) = 0$.

Proof. Apply Lemma 2.6.3 (2) to the canonical extension $L \to L/Z(L)$. $\square$

Lemma 2.6.5. *(Lemma 1.7 of [20])* ***(Pullback Lemma)*** *Let $f : L \to M$ be a homomorphism of Lie superalgebras, and suppose $g : N \to M$ is a central extension. Then $P = \{(l, n) \in L \times N : f(l) = g(n)\}$ is a Lie superalgebra (a subalgebra of the direct product $L \times N$), and $pr_1 : P \to L : (l, n) \mapsto l$ is a central extension. The extension $pr_1 : P \to L$ splits (uniquely) if and only if there exists a (unique) Lie superalgebra homomorphism $h : L \to N$ such that $g \circ h = f$.*

$$\begin{array}{ccc} P & \xrightarrow{pr_2} & N \\ {\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle g} \\ L & \xrightarrow{f} & M \end{array}$$

Proof. It is easy to see that P is a Lie superalgebra. $[(l_1, n_1), (l_2, n_2)] = ([l_1, l_2], [n_1, n_2])$ and $f([l_1, l_2]) = [f(l_1), f(l_2)] = [g(n_1), g(n_2)] = g([n_1, n_2])$. As well, $\mathrm{pr}_1 : P \to L$ is clearly a central extension. $\ker \mathrm{pr}_1 = \{(0, n)\}$ and $[(0, n), (l', n')] = ([0, l'], [n, n']) = 0$. Thus, $\ker \mathrm{pr}_1 \subset Z(P)$.

For the "if and only if" statement in the lemma, we first recall that a map $s : L \to P$ *splits* the extension pr_1 if and only if there exists a Lie superalgebra homomorphism $h : L \to N$ such that $s(l) = (l, h(l)) \in P$ for all $l \in L$. Equivalently, $g \circ h = f$. Uniqueness of s is clearly equivalent to uniqueness of h. $\square$

Theorem 2.6.6. *(Theorem 1.8 of [20])* ***(Characterization and properties of universal central extensions)*** *For a Lie superalgebra L, the following are equivalent:*

1. *L is simply connected, i.e. every central extension $L' \to L$ splits uniquely.*
2. *L is centrally closed, i.e., $Id : L \to L$ is a universal central extension.*
3. *Suppose $\mathfrak{u} : L \to M$ is a universal central extension of M. In this case,*
   - *(a) both L and M are perfect, and*
   - *(b) $Z(L) = \mathfrak{u}^{-1}(Z(M))$, $\mathfrak{u}(Z(L)) = Z(M)$.*

Proof.

- (1) $\Leftrightarrow$ (2): (1) holds if and only if for every central extension $f : L' \to L$ there exists a unique homomorphism $g : L \to L'$ such that $f \circ g = \mathrm{Id}_L$. By the definition of a universal central extension, this is equivalent to (2).

- (1) $\Rightarrow$ (3): Let $g : N \to M$ be a central extension and let $\mathrm{pr}_1 : P \to L$ be the central extension constructed in Lemma 2.6.5. By assumption, pr_1 splits uniquely. Thus, again by Lemma 2.6.5, there exists a unique homomorphism $h : L \to N$ such that $g \circ h = \mathfrak{u}$.

- (3) $\Rightarrow$ (a): By Lemma 2.6.3, we know that $\mathfrak{u} : [L, L] \to M$ is a covering. Thus, by the universal property of $\mathfrak{u}$, there exists a unique homomorphism $f : L \to [L, L]$ such that $\mathfrak{u} \circ f = \mathfrak{u}$. Now, let $\iota : [L, L] \to L$ be the canonical injection. Then, $\iota \circ f : L \to L$ is a homomorphism with $\mathfrak{u} \circ (\iota \circ f) = \mathfrak{u}$. Apply the universal property of $\mathfrak{u}$ to see that $\iota \circ f = \mathrm{Id}_L$. This gives $L = \iota(f(L)) \subset [L, L]$, so $L = [L, L]$ and L is perfect. By surjectivity of $\mathfrak{u}$, M is also perfect.

- (3) $\Rightarrow$ (b): This is a special case of Lemma 2.6.3(2).

- (3) $\Rightarrow$ (1): If $f' : L' \to L$ is a central extension, Lemma 2.6.3(3) implies that $\mathfrak{u} \circ f'$ is a central extension of M. Since $\mathfrak{u}$ is a universal central extension, there exists a unique homomorphism $g : L \to L'$ such that $\mathfrak{u} = \mathfrak{u} \circ f' \circ g$. By Lemma 2.6.2(2), this implies $f' \circ g = \mathrm{Id}_L$.

$\square$

Corollary 2.6.7. *(Corollary 1.9 of [20]) Let $f : K \to L$ and $g : L \to M$ be central extensions. Then, $g \circ f : K \to M$ is a universal central extension if and only if $f : K \to L$ is a universal central extension.*

Proof. The conditions (1) and (2) of Theorem 2.6.6 are independent of the maps f or $g \circ f$ and involve only the Lie superalgebra K. Hence, if $g \circ f$ is a central extension, $g \circ f$ is universal if and only if f is. However, if f is universal, K is perfect by 2.6.6(3a) and so $g \circ f$ is indeed a central extension by Lemma 2.6.3(3). $\square$

2.6.1 Universal Central Extensions of Lie superalgebras

Iohara and Koga [16] construct the universal central extension of the Lie superalgebra $\mathfrak{g} \otimes A$, for $\mathfrak{g}$ a basic classical Lie superalgebra over a commutative ring k and A a commutative algebra over k.While the representation theories and Serre relations of affine Lie superalgebras have been studied, Iohara and Koga realize affine Lie superalgebras instead as the UCE of the Lie superalgebra $\mathfrak{g} \otimes \mathbb{C}[t^{\pm 1}]$. The following is their main result of that paper, when $\mathfrak{g}$ is of type $A(m,n)$ and we let $k = \mathbb{C}$.

Theorem 2.6.8. *Let $HC_1(A)$ denote the first cyclic homology group of A. The UCE $\mathfrak{g}(A)$ of $\mathfrak{g} \otimes A$ is given by*

$$\mathfrak{g}(A) \simeq \begin{cases} \mathfrak{g} \otimes A \oplus HC_1(A) & \text{if } \mathfrak{g} \text{ is not of type } A(n,n), \\ \mathfrak{sl}_{n+1|n+1} \otimes A \oplus HC_1(A) & \text{if } \mathfrak{g} \text{ is of type } A(n,n),\ n > 1\ . \end{cases}$$

I will apply this theorem for $\mathfrak{g}$ of type $A(m,n)$ and when A is one of the algebras $\mathbb{C}[u,v]$.

For example, $\mathfrak{g} \otimes \mathbb{C}[u]$ is its own UCE. Instead, we can consider the generalizations $\mathfrak{g} \otimes \mathbb{C}[u^{\pm 1}, v^{\pm 1}]$, $\mathfrak{g} \otimes \mathbb{C}[u^{\pm 1}, v]$, and $\mathfrak{g} \otimes \mathbb{C}[u,v]$ and their universal central extensions. We call these *double affine Lie algebras*. The quantizations of these UCEs are called quantum toroidal algebras, affine Yangians, and deformed double current algebras (DDCA) respectively.

2.7 Deformed Double Current Algebras

For a finite dimensional simple complex Lie algebra $\mathfrak{g}$, its deformed double current algebra (DDCA) $\mathcal{D}(\mathfrak{g})$ is a deformation of the universal enveloping algebra of the universal central extension of the double current algebra $\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[u,v]$. We can see DDCAs as quantum analogues of the rational Cherednik algebra for two reasons. First, when $\mathfrak{g} = \mathfrak{sl}_n$, there exists Schur-Weyl duality between the rational Cherednik algebra of type A and $\mathcal{D}(\mathfrak{sl}_n)$. Second, we can obtain $\mathcal{D}(\mathfrak{g})$ by degenerating twice the quantum toroidal algebra of $\mathfrak{g}$. This parallels how rational Cherednik algebras can be viewed as twice-degenerated elliptic Cherednik algebras.

In Guay's article on deformed double current algebras in type A [15], he provides three equivalent definitions of the DDCA. The one I am most interested in is the one involving the Schur-Weyl functor and that is closely related to the J-presentation of the Yangian.

The deformed double current algebra $\mathcal{D}(\mathfrak{sl}_n)$ is generated by two copies of the current Lie algebra $\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[u]$ and $\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[v]$ with commutation relation between them given by

$$\begin{aligned}[\mathsf{K}(E_{ab}), \mathsf{Q}(E_{cd})] =& \mathsf{P}([E_{ab}, E_{cd}]) + \left(\frac{t}{2} - \frac{n\kappa}{4}\right)(\delta_{bc}E_{ad} + \delta_{ad}E_{cb}) \\ &+ \frac{\kappa}{4}(\delta_{ad} + \delta_{bc})S(E_{ab}, E_{cd}) + \frac{\kappa}{4}\sum_{1\leq i\neq j\leq n} S\left([E_{ab}, E_{ij}], [E_{ji}, E_{cd}]\right)\end{aligned} \tag{2.111}$$

where $\mathsf{K}(X) = X \otimes u$, $\mathsf{Q}(X) = X \otimes v$, and $\mathsf{P}(X) \in Y(\mathfrak{sl}_n)$. $t, \kappa \in \mathbb{C}$ are the two deformation parameters.

Chapter 3

Steinberg Lie Superalgebras

3.1 The Steinberg Lie superalgebra for the type $A(m, n)$ setting

One goal is to find a reasonable definition of the DDCA of $\mathfrak{sl}_{m|n}$, given that it should be a deformation of the UCE of $\mathfrak{sl}_{m|n} \otimes \mathbb{C}[u, v]$. By Theorem 2.6.8 of Iohara and Koga [16], this UCE is $\mathfrak{sl}_{m|n} \otimes \mathbb{C}[u, v] \oplus HC_1(\mathbb{C}[u, v])$, where $HC_1(\mathbb{C}[u, v]) = \Omega^1(\mathbb{C}[u, v])/d(\mathbb{C}[u, v])$ and $m \neq n$, where $\Omega^1(\mathbb{C}[u, v]) = \mathbb{C}[u, v]du \oplus \mathbb{C}[u, v]dv$. The bracket of $\mathfrak{sl}_{m|n} \otimes \mathbb{C}[u, v] \oplus HC_1(\mathbb{C}[u, v])$ is given by

$$[X \otimes p_1(u, v), Y \otimes p_2(u, v)] = [X, Y] \otimes p_1(u, v)p_2(u, v) + (X, Y)\overline{p_2(u, v)dp_1(u, v)}$$

for $p_1(u, v), p_2(u, v) \in \mathbb{C}[u, v]$, $X, Y \in \mathfrak{sl}_{m|n}$ and where $(\cdot, \cdot)$ is the Killing form on $\mathfrak{sl}_{m|n}$, d is the differential, and $\bar{\cdot}$ is the projection from Ω^1 to $\Omega^1/d(\mathbb{C}[u, v])$. However, we know also that another description of this universal central extension is given by the Steinberg Lie superalgebra, defined below.

Definition 3.1.1. [6] Let A be an associative $\mathbb{C}$-superalgebra. For $m+n \geq 3$, the Steinberg Lie superalgebra $\mathfrak{st}_{m|n}(A)$ is defined to be the Lie superalgebra over $\mathbb{C}$, generated by the homogeneous elements $F_{ij}(a)$, $a \in A$ homogeneous, $1 \leq i \neq j \leq m + n$ and $\deg(F_{ij}(a)) = |i| + |j| + |a| \in \mathbb{Z}_2$, subject to the following relations for $a, b \in A$:

$$a \mapsto F_{ij}(a) \text{ is a } \mathbb{C}\text{-linear map,} \tag{3.1}$$

$$[F_{ij}(a), F_{jk}(b)] = F_{ik}(ab), \text{ for distinct } i,\ j,\ k, \tag{3.2}$$

$$[F_{ij}(a), F_{kl}(b)] = 0 \text{ for } i \neq j \neq k \neq l \neq i, \text{ i.e. when } [E_{ij}, E_{kl}] = 0 \tag{3.3}$$

Proposition 3.1.2. *Let*

$$\varphi : \mathfrak{st}_{m|n}(A) \to \mathfrak{sl}_{m|n}(A)$$

be the map given by $F_{ij}(a) \mapsto E_{ij}(a)$. For $m+n \geq 5$, $(\mathfrak{st}_{m|n}(A), \varphi)$ is a central extension of $\mathfrak{sl}_{m|n}(A)$.

Proof. See Proposition 4.1 [6]. □

This is the motivation for finding another definition for the Steinberg Lie superalgebra in the first place. Further, Chen and Sun [6] prove the following result on $\mathfrak{st}_{m|n}(A)$:

Theorem 3.1.3. *If $m+n \geq 3$, the kernel of the central extension $(\mathfrak{st}_{m|n}(A), \varphi)$ of the Lie superalgebra $\mathfrak{sl}_{m|n}(A)$ is isomorphic to $HC_1(A)$ as $\mathbb{C}$-modules. Further, if $m+n \geq 5$, this is a universal central extension.*

I show that $\mathfrak{st}_{m|n}(A)$ is isomorphic to the Lie superalgebra $\widetilde{\mathfrak{st}}_{m|n}(\mathbb{C}[u,v])$ defined below when $A = \mathbb{C}[u,v]$. The motivation for using this second set of generators and relations is to have to consider only finitely many generators rather than infinitely many. This is necessary for finding the DDCA.

Proposition 3.1.4. *Let $\widetilde{\mathfrak{st}}_{m|n}(\mathbb{C}[u,v])$ be the Lie superalgebra generated by elements $\widetilde{F}_{ab}(1)$, $\widetilde{F}_{ab}(u)$, and $\widetilde{F}_{ab}(v)$ for $1 \leq a \neq b \leq m+n$ and $\deg(\widetilde{F}_{ab}(1)) = \deg(\widetilde{F}_{ab}(u)) = \deg(\widetilde{F}_{ab}(v)) = |a| + |b| \in \mathbb{Z}_2$. These generators are subject to the following relations for $a \neq b \neq c \neq d \neq a$:*

$$[\widetilde{F}_{ab}(u), \widetilde{F}_{cd}(u)] = 0, \qquad [\widetilde{F}_{ab}(u), \widetilde{F}_{cd}(v)] = 0,$$
$$[\widetilde{F}_{ab}(v), \widetilde{F}_{cd}(v)] = 0, \qquad [\widetilde{F}_{ab}(u), \widetilde{F}_{cd}(1)] = 0,$$
$$[\widetilde{F}_{ab}(v), \widetilde{F}_{cd}(1)] = 0, \qquad [\widetilde{F}_{ab}(1), \widetilde{F}_{cd}(1)] = 0.$$

For a, b, d and a, c, d all distinct,

$$[\widetilde{F}_{ab}(u), \widetilde{F}_{bc}(u)] = [\widetilde{F}_{ad}(u), \widetilde{F}_{dc}(u)], \qquad [\widetilde{F}_{ab}(u), \widetilde{F}_{bc}(v)] = [\widetilde{F}_{ad}(v), \widetilde{F}_{dc}(u)],$$
$$[\widetilde{F}_{ab}(v), \widetilde{F}_{bc}(v)] = [\widetilde{F}_{ad}(v), \widetilde{F}_{dc}(v)], \qquad [\widetilde{F}_{ab}(u^i), \widetilde{F}_{bc}(u^j)] = \widetilde{F}_{ac}(u^{i+j}),$$
$$[\widetilde{F}_{ab}(v^i), \widetilde{F}_{bc}(v^j)] = \widetilde{F}_{ac}(v^{i+j}),$$

where $i+j = 0, 1$ and $m+n \geq 5$. Then, we have an isomorphism $\widetilde{\mathfrak{st}}_{m|n}(\mathbb{C}[u, v]) \simeq \mathfrak{st}_{m|n}(\mathbb{C}[u, v])$.

Specifically, we will consider $\mathfrak{st}_{m|n}(\mathbb{C}[u, v])$. In $\mathbb{C}[u, v]$, all elements are even, so $\deg(F_{ij}(a)) = |i| + |j|$.

Rewriting the super Jacobi identity for $\mathbb{Z}_2$-homogeneous elements X, Y, Z yields

$$[X, [Y, Z]] = [[X, Y], Z] + (-1)^{|X||Y|}[Y, [X, Z]]$$

which will be a useful relation in the proofs to follow. Recall also that $[X, Y] = -(-1)^{|X||Y|}[Y, X]$.

There is a clear surjective homomorphism of Lie superalgebras

$$\widetilde{\mathfrak{st}}_{m|n}(\mathbb{C}[u, v]) \twoheadrightarrow \mathfrak{st}_{m|n}(\mathbb{C}[u, v])$$

given by

$$\widetilde{F}_{ab}(X) \mapsto F_{ab}(X)$$

for $X = 1, u, v$. To show that this is an isomorphism, we can find an inverse map. This is equivalent to constructing elements $\widetilde{F}_{ab}(u^r v^s)$ in $\widetilde{\mathfrak{st}}_{m|n}(\mathbb{C}[u, v])$, where these elements are to be the image of $F_{ab}(u^r v^s)$ under the map

$$\mathfrak{st}_{m|n}(\mathbb{C}[u, v]) \to \widetilde{\mathfrak{st}}_{m|n}(\mathbb{C}[u, v]).$$

The goal is to construct inductively these elements $\widetilde{F}_{ab}(u^r v^s)$ (where $a \neq b$, $r, s \geq 0$) satisfying

$$[\widetilde{F}_{ab}(u^{r_1} v^{s_1}), \widetilde{F}_{cd}(u^{r_2} v^{s_2})] = \delta_{bc}\widetilde{F}_{ad}(u^{r_1+r_2} v^{s_1+s_2})$$

for $a \neq b$, $c \neq d \neq a$, $r_1, r_2, s_1, s_2 \geq 0$.

Lemma 3.1.5. *We can define inductively elements $\widetilde{F}_{ab}(u^k)$ such that for $a \neq b$, $c \neq d \neq a$, $k, l \geq 0$, the equality*

$$[\widetilde{F}_{ab}(u^k), \widetilde{F}_{cd}(u^l)] = \delta_{bc}\widetilde{F}_{ad}(u^{k+l}) \tag{3.4}$$

is satisfied.

Proof. From the defining relations, we already have that

$$\widetilde{F}_{ac}(u^m) = [\widetilde{F}_{ab}(u), \widetilde{F}_{bc}(u^{m-1})]$$

for $m = 1$ and this is true for any $b \neq a, c$. Indeed, if $d \neq a, b, c$,

$$\begin{aligned}
\widetilde{F}_{ac}(u^1) =& [\widetilde{F}_{ab}(u), \widetilde{F}_{bc}(1)] \\
=& [\widetilde{F}_{ab}(u), [\widetilde{F}_{bd}(1), \widetilde{F}_{dc}(1)]] \\
=& [[\widetilde{F}_{ab}(u), \widetilde{F}_{bd}(1)], \widetilde{F}_{dc}(1)] + (-1)^{(|a|+|b|)(|b|+|d|)}[\widetilde{F}_{bd}(1), [\widetilde{F}_{ab}(u), \widetilde{F}_{dc}(1)]] \\
=& [\widetilde{F}_{ad}(u), \widetilde{F}_{dc}(1)]
\end{aligned}$$

Pick $b \neq a, c$ and set $\widetilde{F}_{ac}(u^2) = [\widetilde{F}_{ab}(u), \widetilde{F}_{bc}(u)]$. This is well-defined and does not depend on the choice of b since, if $d \neq a, b, c$,

$$\begin{aligned}
\widetilde{F}_{ac}(u^2) =& [\widetilde{F}_{ab}(u), \widetilde{F}_{bc}(u)] \\
=& [\widetilde{F}_{ab}(u), [\widetilde{F}_{bd}(1), \widetilde{F}_{dc}(u)]] \\
=& [[\widetilde{F}_{ab}(u), \widetilde{F}_{bd}(1)], \widetilde{F}_{dc}(u)] + (-1)^{(|a|+|b|)(|b|+|d|)}[\widetilde{F}_{bd}(1), [\widetilde{F}_{ab}(u), \widetilde{F}_{dc}(u)]] \\
=& [\widetilde{F}_{ad}(u), \widetilde{F}_{dc}(u)]
\end{aligned}$$

When $b \neq c$, from defining relations we have (3.4) when $k = 0, 1$. If $k = 2, l = 0$, pick some $e \neq a, b, c, d$:

$$\begin{aligned}
[\widetilde{F}_{ab}(u^2), & \widetilde{F}_{cd}(1)] = [[\widetilde{F}_{ae}(u), \widetilde{F}_{eb}(u)], \widetilde{F}_{cd}(1)] \\
&= [\widetilde{F}_{ae}(u), [\widetilde{F}_{eb}(u), \widetilde{F}_{cd}(1)]] - (-1)^{(|a|+|e|)(|e|+|b|)}[\widetilde{F}_{eb}(u), [\widetilde{F}_{ae}(u), \widetilde{F}_{cd}(1)]] \\
&= \delta_{bc}[\widetilde{F}_{ae}(u), \widetilde{F}_{ed}(u)] \\
&= \delta_{bc}\widetilde{F}_{ad}(u^2)
\end{aligned}$$

Similarly, it can be shown that $[\widetilde{F}_{ab}(1), \widetilde{F}_{cd}(u^2)] = \delta_{bc}\widetilde{F}_{ad}(u^2)$.

Thus, we have $[\widetilde{F}_{ab}(u^k), \widetilde{F}_{cd}(u^l)] = \delta_{bc}\widetilde{F}_{ad}(u^{k+l})$ for $0 \leq k+l \leq 2$.

Then, assume we have constructed elements $\widetilde{F}_{ac}(u^k)$ for $0 \leq k \leq M$, $M \geq 2$ satisfying relation (3.4) when $0 \leq k+l \leq M$. Suppose that $a \neq d \neq b \neq c \neq a$ and pick $e \neq a, b, c, d$. Then,

$$\begin{aligned}
&[\widetilde{F}_{ad}(u), \widetilde{F}_{bc}(u^M)] = [\widetilde{F}_{ad}(u), [\widetilde{F}_{be}(u), \widetilde{F}_{ec}(u^{M-1})]] \\
&=[[\widetilde{F}_{ad}(u), \widetilde{F}_{be}(u)], \widetilde{F}_{ec}(u^{M-1})] + (-1)^{(|a|+|d|)(|b|+|e|)}[\widetilde{F}_{be}(u), [\widetilde{F}_{ad}(u), \widetilde{F}_{ec}(u^{M-1})]] \\
&=0
\end{aligned}$$

by induction.

For $a \neq c$, pick $b \neq a, c$ and set $\widetilde{F}_{ac}(u^{M+1}) = [\widetilde{F}_{ab}(u), \widetilde{F}_{bc}(u^M)]$. This does not depend on the choice of b; for instance, take some $d \neq a, b, c$:

$$\begin{aligned}
\widetilde{F}_{ac}(u^{M+1}) &= [\widetilde{F}_{ab}(u), \widetilde{F}_{bc}(u^M)] \\
&= [[\widetilde{F}_{ad}(u), \widetilde{F}_{db}(1)], \widetilde{F}_{bc}(u^M)] \\
&= [\widetilde{F}_{ad}(u), [\widetilde{F}_{db}(1), \widetilde{F}_{bc}(u^M)]] - (-1)^{(|a|+|d|)(|d|+|b|)}[\widetilde{F}_{db}(1), [\widetilde{F}_{ad}(u), \widetilde{F}_{bc}(u^M)]] \\
&= [\widetilde{F}_{ad}(u), \widetilde{F}_{dc}(u^M)]
\end{aligned}$$

We thus have well-defined elements $\widetilde{F}_{ac}(u^{M+1})$. If $a \neq b$, $c \neq d \neq a$, pick $e \neq a, b, c, d$. Then, we have

$$\begin{aligned}
[\widetilde{F}_{ab}(1),\widetilde{F}_{cd}(u^{M+1})] &= [\widetilde{F}_{ab}(1), [\widetilde{F}_{ce}(u), \widetilde{F}_{ed}(u^M)]] \\
&= [[\widetilde{F}_{ab}(1), \widetilde{F}_{ce}(u)], \widetilde{F}_{ed}(u^M)] + (-1)^{(|a|+|b|)(|c|+|e|)}[\widetilde{F}_{ce}(u), [\widetilde{F}_{ab}(1), \widetilde{F}_{ed}(u^M)]] \\
&= \delta_{bc}[\widetilde{F}_{ae}(u), \widetilde{F}_{ed}(u^M)] \\
&= \delta_{bc}\widetilde{F}_{ad}(u^{M+1})
\end{aligned}$$

Assume that for $0 \leq j \leq k-1$, with $1 \leq k \leq M+1$,

$$[[\widetilde{F}_{ab}(u^j), \widetilde{F}_{cd}(u^{M+1-j})] = \delta_{bc}\widetilde{F}_{ad}(u^{M+1}).$$

Now, let $l = M+1-k$. Then, by induction,

$$\begin{aligned}
&[[\widetilde{F}_{ab}(u^k), \widetilde{F}_{cd}(u^l)] = [[\widetilde{F}_{ae}(u), \widetilde{F}_{eb}(u^{k-1})], \widetilde{F}_{cd}(u^l)] \\
&\quad = [\widetilde{F}_{ae}(u), [\widetilde{F}_{eb}(u^{k-1}), \widetilde{F}_{cd}(u^l)]] - (-1)^{(|a|+|e|)(|e|+|b|)}[\widetilde{F}_{eb}(u^{k-1}), [\widetilde{F}_{ae}(u), \widetilde{F}_{cd}(u^l)]] \\
&\quad = \delta_{bc}[\widetilde{F}_{ae}(u), \widetilde{F}_{ed}(u^{k+l-1})] \\
&\quad = \delta_{bc}\widetilde{F}_{ad}(u^{M+1})
\end{aligned}$$

Thus, (3.4) holds when $k+l = M+1$. By induction, (3.4) holds for all $k, l \geq 0$. □

An analogue of Lemma 3.1.5 holds for elements $\widetilde{F}_{ab}(v^s)$.

Finally, we introduce elements $\widetilde{F}_{ab}(u^r v^s)$ while proving Proposition 3.1.6 below.

Proposition 3.1.6. *We can define inductively elements $\widetilde{F}_{ab}(u^k v^l)$ such that for $a \neq b$, $c \neq d \neq a$, and $k_1, k_2, l_1, l_2 \geq 0$, the equality*

$$[\widetilde{F}_{ab}(u^{k_1} v^{l_1}), \widetilde{F}_{cd}(u^{k_2} v^{l_2})] = \delta_{bc}\widetilde{F}_{ad}(u^{k_1+k_2} v^{l_1+l_2}) \tag{3.5}$$

is satisfied.

Proof. We have already shown (3.5) for the cases when $k_1 = k_2 = 0$ or $l_1 = l_2 = 0$.

Take $m \geq 0$. Assume we have defined elements $\widetilde{F}_{ad}(u^k v^l)$ satisfying (3.5) for $0 \leq l \leq 2$ and $0 \leq k \leq m$. Suppose $a \neq d \neq b \neq c \neq a$. Take $e \neq a, b, c, d$ and assume $k \geq 1$. Note that the $k=0, l=2$ case can be dealt with similarly by writing $\widetilde{F}_{cd}(v^2)$ as $[\widetilde{F}_{cd}(v), \widetilde{F}_{ed}(v)]$. Then,

$$\begin{aligned}
&[\widetilde{F}_{ab}(u), \widetilde{F}_{cd}(u^k v^l)] = [\widetilde{F}_{ab}(u), [\widetilde{F}_{ce}(u), \widetilde{F}_{ed}(u^{k-1} v^l)]] \\
&\quad = [[\widetilde{F}_{ab}(u), \widetilde{F}_{ce}(u)], \widetilde{F}_{ed}(u^{k-1} v^l)] + (-1)^{(|a|+|b|)(|c|+|e|)}[\widetilde{F}_{ce}(u), [\widetilde{F}_{ab}(u), \widetilde{F}_{ed}(u^{k-1} v^l)]] \\
&\quad = 0
\end{aligned}$$

by induction. For $a \neq c$, pick $b \neq a, c$ and let $\widetilde{F}_{ac}(u^{m+1} v^l) = [\widetilde{F}_{ab}(u), \widetilde{F}_{bc}(u^m v^l)]$.

This is well-defined (let $d \neq a, b, c$):

$$\begin{aligned}
&[\widetilde{F}_{ab}(u), \widetilde{F}_{bc}(u^m v^l)] = [[\widetilde{F}_{ad}(u), \widetilde{F}_{db}(1)], \widetilde{F}_{bc}(u^m v^l)] \\
&\quad =[\widetilde{F}_{ad}(u), [\widetilde{F}_{db}(1), \widetilde{F}_{bc}(u^m v^l)]] - (-1)^{(|a|+|d|)(|d|+|b|)}[\widetilde{F}_{db}(1), [\widetilde{F}_{ad}(u), \widetilde{F}_{bc}(u^m v^l)]] \\
&\quad =[\widetilde{F}_{ad}(u), \widetilde{F}_{dc}(u^m v^l)]
\end{aligned}$$

Thus, we have well defined elements $\widetilde{F}_{ad}(u^k v^l)$ for $k \geq 0$, $l = 0, 1, 2$.

Assume we have defined elements $\widetilde{F}_{ad}(u^k v^l)$ for $0 \leq k \leq m+1$, $l = 0, 1, 2$ satisfying (3.5) for $k_1 + k_2 \leq m$ and $l_1 + l_2 \leq 2$. We show that (3.5) holds when $k_1 + k_2 = m + 1$

Let $a \neq b$, $c \neq d \neq a$. Pick $e \neq a, b, c, d$. Then,

$$\begin{aligned}
[\widetilde{F}_{ab}(1), \widetilde{F}_{cd}(u^{m+1} v^l)] &= [\widetilde{F}_{ab}(1), [\widetilde{F}_{ce}(u), \widetilde{F}_{ed}(u^m v^l)]] \\
&= [[\widetilde{F}_{ab}(1), \widetilde{F}_{ce}(u)], \widetilde{F}_{ed}(u^m v^l)] + (-1)^{(|a|+|b|)(|c|+|e|)}[\widetilde{F}_{ce}(u), [\widetilde{F}_{ab}(1), \widetilde{F}_{ed}(u^m v^l)]] \\
&= \delta_{bc}[\widetilde{F}_{ae}(u), \widetilde{F}_{ed}(u^m v^l)] \\
&= \delta_{bc}\widetilde{F}_{ad}(u^{m+1} v^l)
\end{aligned}$$

If $k_1 \geq 1$ and $k_1 + k_2 = m + 1$ then, by induction,

$$\begin{aligned}
[\widetilde{F}_{ab}(u^{k_1}), \widetilde{F}_{cd}(u^{k_2} v^l)] &= [[\widetilde{F}_{ae}(u), \widetilde{F}_{eb}(u^{k_1-1})], \widetilde{F}_{cd}(u^{k_2} v^l)] \\
&=[\widetilde{F}_{ae}(u), [\widetilde{F}_{eb}(u^{k_1-1}), \widetilde{F}_{cd}(u^{k_2} v^l)]] \\
&\quad - (-1)^{(|a|+|e|)(|e|+|b|)}[\widetilde{F}_{eb}(u^{k_1-1}), [\widetilde{F}_{ae}(u), \widetilde{F}_{cd}(u^{k_2} v^l)]] \\
&=\delta_{bc}[\widetilde{F}_{ae}(u), \widetilde{F}_{ed}(u^{k_1+k_2-1} v^l)] \\
&=\delta_{bc}\widetilde{F}_{ad}(u^{k_1+k_2} v^l) \\
&=\delta_{bc}\widetilde{F}_{ad}(u^{m+1} v^l).
\end{aligned}$$

By induction on k, we have constructed well-defined elements $\widetilde{F}_{ab}(u^k)$, $\widetilde{F}_{ab}(u^k v)$, and $\widetilde{F}_{ab}(u^k v^2)$ for $k \geq 0$ and shown that they satisfy (3.5). Next, we define elements $\widetilde{F}_{ab}(u^k v^l)$ for all $k, l \geq 0$. Take $m \geq 0$. Assume we have defined elements $\widetilde{F}_{ab}(u^k v^l)$ for all $0 \leq l \leq m$, $k \geq 0$, satisfying (3.5) for $l_1 + l_2 \leq m$

and any $k_1 + k_2 = k$. Suppose $a \neq d \neq b \neq c \neq a$ and take $e \neq a, b, c, d$. Then,

$$\begin{aligned}
[\widetilde{F}_{ab}(u^{k_1}v), &\widetilde{F}_{cd}(u^{k_2}v^m)] = [\widetilde{F}_{ab}(u^{k_1}v), [\widetilde{F}_{ce}(u^{k_2}v), \widetilde{F}_{ed}(u^{k_2}v^{m-1})]] \\
&= [[\widetilde{F}_{ab}(u^{k_1}v), \widetilde{F}_{ce}(u^{k_2}v)], \widetilde{F}_{ed}(u^{k_2}v^{m-1})] \\
&\quad + (-1)^{(|a|+|b|)(|c|+|e|)}[\widetilde{F}_{ce}(u^{k_2}v), [\widetilde{F}_{ab}(u^{k_1}v), \widetilde{F}_{ed}(u^{k_2}v^{m-1})]] \\
&= 0
\end{aligned}$$

by induction.

For $a \neq c$, pick $b \neq a, c$ and let $\widetilde{F}_{ac}(u^k v^{m+1}) = [\widetilde{F}_{ab}(v), \widetilde{F}_{bc}(u^k v^m)]$. This is well-defined, i.e. is not affected by the choice of b (where $d \neq a, b, c$):

$$\begin{aligned}
[\widetilde{F}_{ab}(v), \widetilde{F}_{bc}(u^k v^m)] &= [[\widetilde{F}_{ad}(v), \widetilde{F}_{db}(1)], \widetilde{F}_{bc}(u^k v^m)] \\
&= [\widetilde{F}_{ad}(v), [\widetilde{F}_{db}(1), \widetilde{F}_{bc}(u^k v^m)]] \\
&\quad - (-1)^{(|a|+|d|)(|d|+|b|)}[\widetilde{F}_{db}(1), [\widetilde{F}_{ad}(v), \widetilde{F}_{bc}(u^k v^m)]] \\
&= [\widetilde{F}_{ad}(v), \widetilde{F}_{dc}(u^k v^m)]
\end{aligned}$$

Thus, we have well-defined elements $\widetilde{F}_{ac}(u^k v^l)$ for $k \geq 0$ and $0 \leq l \leq m+1$.

It remains to prove (3.5) when $l_1 + l_2 = m + 1$, assuming it holds when $l_1 + l_2 \leq m$, for all k_1, k_2.

Assume $a \neq b$, $c \neq d \neq a$. Pick $e \neq a, b, c, d$. Then,

$$\begin{aligned}
[\widetilde{F}_{ab}(1), \widetilde{F}_{cd}(u^{k_2}v^{m+1})] &= [\widetilde{F}_{ab}(1), [\widetilde{F}_{ce}(v), \widetilde{F}_{ed}(u^{k_2}v^m)]] \\
&= [[\widetilde{F}_{ab}(1), \widetilde{F}_{ce}(v)], \widetilde{F}_{ed}(u^{k_2}v^m)] \\
&\quad + (-1)^{(|a|+|b|)(|c|+|e|)}[\widetilde{F}_{ce}(v), [\widetilde{F}_{ab}(1), \widetilde{F}_{ed}(u^{k_2}v^m)]] \\
&= \delta_{bc}[\widetilde{F}_{ae}(v), \widetilde{F}_{ed}(u^{k_2}v^m)] \\
&= \delta_{bc}\widetilde{F}_{ad}(u^{k_2}v^{m+1}).
\end{aligned}$$

Further,

$$\begin{aligned}
[\widetilde{F}_{ab}(u^{k_1}), &\widetilde{F}_{cd}(u^{k_2}v^{m+1})] = [\widetilde{F}_{ab}(u^{k_1}), [\widetilde{F}_{ce}(u^{k_2}v^m), \widetilde{F}_{ed}(v)]] \\
&= [[\widetilde{F}_{ab}(u^{k_1}), \widetilde{F}_{ce}(u^{k_2}v^m)], \widetilde{F}_{ed}(v)] \\
&\quad + (-1)^{(|a|+|b|)(|c|+|e|)}[\widetilde{F}_{ce}(u^{k_2}v^m), [\widetilde{F}_{ab}(u^{k_1}), \widetilde{F}_{ed}(v)]
\end{aligned}$$

$$
\begin{aligned}
&=[\delta_{bc}\widetilde{F}_{ae}(u^{k_1+k_2}v^m), \widetilde{F}_{ed}(v)] \\
&=\delta_{bc}[[\widetilde{F}_{ab}(v), \widetilde{F}_{be}(u^{k_1+k_2}v^{m-1})], \widetilde{F}_{ed}(v)] \\
&=\delta_{bc}[\widetilde{F}_{ab}(v), [\widetilde{F}_{be}(u^{k_1+k_2}v^{m-1}), \widetilde{F}_{ed}(v)]] \\
&=\delta_{bc}[\widetilde{F}_{ab}(v), \widetilde{F}_{bd}(u^{k_1+k_2}v^m)] \\
&=\delta_{bc}\widetilde{F}_{ad}(u^{k_1+k_2}v^{m+1})
\end{aligned}
$$

If $l_1 \geq 1$ and $l_1 + l_2 = m + 1$ then, by induction,

$$
\begin{aligned}
[\widetilde{F}_{ab}(u^{k_1}v^{l_1}),\widetilde{F}_{cd}(u^{k_2}v^{l_2})] &= [[\widetilde{F}_{ae}(v), \widetilde{F}_{eb}(u^{k_1}v^{l_1-1})], \widetilde{F}_{cd}(u^{k_2}v^{l_2})] \\
&=[\widetilde{F}_{ae}(v), [\widetilde{F}_{eb}(u^{k_1}v^{l_1-1}), \widetilde{F}_{cd}(u^{k_2}v^{l_2})] \\
&\quad - (-1)^{(|a|+|e|)(|e|+|b|)}[\widetilde{F}_{eb}(u^{k-1}v^{l_1-1}), [\widetilde{F}_{ae}(v), \widetilde{F}_{cd}(u^{k_2}v^{l_2})] \\
&=\delta_{bc}[\widetilde{F}_{ae}(v), \widetilde{F}_{ed}(u^{k_1+k_2}v^{l_1+l_2-1})] \\
&=\delta_{bc}\widetilde{F}_{ad}(u^{k_1+k_2}v^{l_1+l_2})
\end{aligned}
$$

□

3.2 Type Q Steinberg Lie Algebra

Definition 3.2.1. Let $\mathcal{C}$ be the Clifford algebra of rank one: $\mathcal{C} = \text{span}\{1, \underline{c}\}$ with $\underline{c}^2 = 1$.

Definition 3.2.2. Let $A = \mathbb{C}[u, v] \rtimes \mathcal{C} \simeq \mathbb{C} < u, v, \underline{c} > /(\underline{c}^2 - 1, \underline{c}u + u\underline{c}, \underline{c}v - v\underline{c}, uv - vu)$.

The goal in this subsection will be to give a presentation of $\mathfrak{st}_n(A)$ in terms of finitely many generators and relations. We assume $n \geq 5$; similar results should hold for $n = 2, 3, 4$, but the arguments will be more complicated or more relations may required.

We proceed by defining another algebra that should be isomorphic to $\mathfrak{st}_n(A)$:

Definition 3.2.3. Let $\widetilde{\mathfrak{st}_n}(A)$ be Lie superalgebra generated by elements $\widetilde{F}_{ab}(1)$ (even), $\widetilde{F}_{ab}(u)$ (even), $\widetilde{F}_{ab}(\underline{c})$ (odd), $\widetilde{F}_{ab}(v)$ (even), $1 \leq a \neq b \leq n$. They satisfy the following relations when $a \neq b \neq c \neq d \neq a$, $i, j = 0, 1$:

$$[\widetilde{F}_{ab}(u), \widetilde{F}_{cd}(u)] = 0 \tag{3.6}$$
$$[\widetilde{F}_{ab}(u), \widetilde{F}_{cd}(v)] = 0 \tag{3.7}$$
$$[\widetilde{F}_{ab}(v), \widetilde{F}_{cd}(v)] = 0 \tag{3.8}$$
$$[\widetilde{F}_{ab}(u), \widetilde{F}_{cd}(\underline{c}^i)] = 0 \tag{3.9}$$
$$[\widetilde{F}_{ab}(v), \widetilde{F}_{cd}(\underline{c}^i)] = 0 \tag{3.10}$$
$$[\widetilde{F}_{ab}(\underline{c}^i), \widetilde{F}_{cd}(\underline{c}^j)] = 0 \tag{3.11}$$
$$[\widetilde{F}_{ab}(\underline{c}^i), \widetilde{F}_{bc}(\underline{c}^j)] = \widetilde{F}_{ac}(\underline{c}^{i+j}) \tag{3.12}$$
$$[\widetilde{F}_{ab}(u), \widetilde{F}_{bc}(u)] = [\widetilde{F}_{ad}(u), \widetilde{F}_{dc}(u)] \tag{3.13}$$
$$[\widetilde{F}_{ab}(u), \widetilde{F}_{bc}(v)] = [\widetilde{F}_{ad}(v), \widetilde{F}_{dc}(u)] \tag{3.14}$$
$$[\widetilde{F}_{ab}(v), \widetilde{F}_{bc}(v)] = [\widetilde{F}_{ad}(v), \widetilde{F}_{dc}(v)] \tag{3.15}$$
$$[\widetilde{F}_{ab}(u), \widetilde{F}_{bc}(1)] = [\widetilde{F}_{ad}(1), \widetilde{F}_{dc}(u)] \tag{3.16}$$
$$[\widetilde{F}_{ab}(v), \widetilde{F}_{bc}(1)] = [\widetilde{F}_{ad}(1), \widetilde{F}_{dc}(v)] \tag{3.17}$$
$$[\widetilde{F}_{ab}(v), \widetilde{F}_{bc}(\underline{c})] = [\widetilde{F}_{ad}(\underline{c}), \widetilde{F}_{dc}(v)] \tag{3.18}$$
$$[\widetilde{F}_{ab}(u), \widetilde{F}_{bc}(\underline{c})] = -[\widetilde{F}_{ad}(\underline{c}), \widetilde{F}_{dc}(u)] \tag{3.19}$$

One can see through induction using $F_{ab}(u^r v^s \underline{c}^i) = [F_{ac}(u), F_{cb}(u^{r-1} v^s \underline{c}^i)] = [[F_{ac}(u), [F_{cd}(u), F_{db}(u^{r-2} v^s \underline{c}^i)]] = \dots$ and similar identities that $\mathfrak{st}_n(A)$ is generated by these four types of elements.

There is a surjective homomorphism of Lie superalgebras

$$\widetilde{\mathfrak{st}_n}(A) \twoheadrightarrow \mathfrak{st}_n(A)$$

sending

$$\widetilde{F}_{ab}(X) \mapsto F_{ab}(X)$$

for $X = u, v, \underline{c}, 1 \in A$. Surjectivity can be established by induction using

$$F_{ab}(u^r v^s \underline{c}^i) = [F_{ac}(u), F_{cb}(u^{r-1} v^s \underline{c}^i)], \tag{3.20}$$
$$F_{ab}(u^r v^s \underline{c}^i) = [F_{ac}(v), F_{cb}(u^r v^{s-1} \underline{c}^i)], \tag{3.21}$$

$$F_{ab}(u^r v^s \underline{c}) = [F_{ac}(u^r v^s), F_{cb}(\underline{c})]. \tag{3.22}$$

in that we know $F_{ac}(u), F_{ac}(v)$, and $F_{cb}(\underline{c})$ are in the image of the homomorphism and one can show by induction that $F_{cb}(u^{r-1}v^s\underline{c}^i)$, $F_{cb}(u^r v^{s-1}\underline{c}^i)$, and $F_{ac}(u^r v^s)$ are.

We use that

$$\begin{aligned}
[\widetilde{F}_{ab}(u), \widetilde{F}_{bc}(v)] &\mapsto [F_{ab}(u), F_{bc}(v)] = F_{ac}(uv) = F_{ac}(vu) \\
[\widetilde{F}_{ad}(v), \widetilde{F}_{dc}(u)] &\mapsto [F_{ad}(v), F_{dc}(u)] = F_{ac}(vu)
\end{aligned}$$

to show that this surjective homomorphism is in fact an isomorphism of Lie superalgebras. To show this is an isomorphism, which it should be, we can find an inverse homomorphism $\mathfrak{st}_n(A) \to \widetilde{\mathfrak{st}_n}(A)$. Finding this homomorphism is equivalent to determining the image of $F_{ab}(u^r v^s \underline{c}^i)$ and checking the relations inside $\mathfrak{st}_n(A)$.

Thus, we construct by induction elements $\widetilde{F}_{ab}(u^r v^s \underline{c}^i)$ in $\widetilde{\mathfrak{st}_n}(A)$. This is similar to what is done in Section 3.1.

As in the previous case, we assume

$$[\widetilde{F}_{ab}(\underline{c}^i), \widetilde{F}_{bc}(\underline{c}^j)] = \widetilde{F}_{ac}(\underline{c}^{i+j}) \tag{3.23}$$

$$[\widetilde{F}_{ab}(u^i), \widetilde{F}_{bc}(u^j)] = \widetilde{F}_{ac}(u^{i+j}) \tag{3.24}$$

$$[\widetilde{F}_{ab}(v^i), \widetilde{F}_{bc}(v^j)] = \widetilde{F}_{ac}(v^{i+j}) \tag{3.25}$$

for $i, j \geq 0$ and $a \neq b \neq c$.

The goal is to construct elements $\widetilde{F}_{ab}(u^r v^s \underline{c}^i)$ for $r \geq 0$, $s \geq 0$, $i = 0, 1$, $a \neq b$ satisfying

$$[\widetilde{F}_{ab}(u^{r_1} v^{s_1} \underline{c}^{i_1}), \widetilde{F}_{cd}(u^{r_2} v^{s_2} \underline{c}^{i_2})] = (-1)^{i_1 r_2} \delta_{bc} \widetilde{F}_{ad}(u^{r_1+r_2} v^{s_1+s_2} \underline{c}^{i_1+i_2})$$

for $a \neq b$, $c \neq d \neq a$.

Lemma 3.2.4. *We can define inductively elements $\widetilde{F}_{ad}(u^k \underline{c}^i)$ such that, for*

$a \neq b$, $c \neq d \neq a$, $k, l \geq 0$, and $i, j = 0, 1$ the equality

$$[\widetilde{F}_{ab}(u^k \underline{c}^i), \widetilde{F}_{cd}(u^l \underline{c}^j)] = \delta_{bc}(-1)^{il} \widetilde{F}_{ad}(u^{k+l} \underline{c}^{i+j}) \tag{3.26}$$

is satisfied.

Proof. Suppose $i = 0$, $j = 1$. Then, defining relations yield that the elements

$$\widetilde{F}_{ac}(u^m \underline{c}) = [\widetilde{F}_{ab}(u), \widetilde{F}_{bc}(u^{m-1} \underline{c})]$$

for $b \neq a, c$, $1 \leq m \leq 2$ are well-defined. Pick some $d \neq a, b, c$.

$$\begin{aligned}
\widetilde{F}_{ac}(u^m \underline{c}) &= [\widetilde{F}_{ab}(u), \widetilde{F}_{bc}(u^{m-1} \underline{c})] \\
&= [[\widetilde{F}_{ad}(u), \widetilde{F}_{db}(1)], \widetilde{F}_{bc}(u^{m-1} \underline{c})] \\
&= -[[\widetilde{F}_{db}(1), \widetilde{F}_{bc}(u^{m-1} \underline{c})], \widetilde{F}_{ad}(u)] - [[\widetilde{F}_{bc}(u^{m-1} \underline{c}), \widetilde{F}_{ad}(u)], \widetilde{F}_{db}(1)] \\
&= -[\widetilde{F}_{dc}(u^{m-1} \underline{c}), \widetilde{F}_{ad}(u)] \\
&= [\widetilde{F}_{ad}(u), \widetilde{F}_{dc}(u^{m-1} \underline{c})]
\end{aligned}$$

Now, assume we have constructed elements $\widetilde{F}_{ac}(u^k \underline{c}^j)$ for $0 \leq k \leq m$, $m \geq 2$, $j = 0, 1$ satisfying relation (3.26) when $0 \leq k + l \leq m$, $i = 0$, $j = 0, 1$. Suppose $a \neq d \neq b \neq c \neq a$ and pick $e \neq a, b, c, d$. Then,

$$\begin{aligned}
[\widetilde{F}_{ad}(u), &\widetilde{F}_{bc}(u^m \underline{c}^j)] = [\widetilde{F}_{ad}(u), [\widetilde{F}_{be}(u), \widetilde{F}_{ec}(u^{m-1} \underline{c}^j)]] \\
&= -[\widetilde{F}_{be}(u), [\widetilde{F}_{ec}(u^{m-1} \underline{c}^j), \widetilde{F}_{ad}(u)]] - [\widetilde{F}_{ec}(u^{m-1} \underline{c}^j), [\widetilde{F}_{ad}(u), \widetilde{F}_{be}(u)]] \\
&= 0
\end{aligned}$$

by induction. For $a \neq c$, pick $b \neq a, c$ and set $\widetilde{F}_{ac}(u^{m+1} \underline{c}^j) = [\widetilde{F}_{ab}(u), \widetilde{F}_{bc}(u^m \underline{c}^j)]$. This does not depend on the choice of b; let $d \neq b$:

$$\begin{aligned}
[\widetilde{F}_{ab}(u), \widetilde{F}_{bc}(u^m \underline{c}^j)] &= [[\widetilde{F}_{ad}(u), \widetilde{F}_{db}(1)], \widetilde{F}_{bc}(u^m \underline{c}^j)] \\
&= -[[\widetilde{F}_{db}(1), \widetilde{F}_{bc}(u^m \underline{c}^j)], \widetilde{F}_{ad}(u)] - [[\widetilde{F}_{bc}(u^m \underline{c}^j), \widetilde{F}_{ad}(u)], \widetilde{F}_{db}(1)] \\
&= -[\widetilde{F}_{dc}(u^m \underline{c}^j), \widetilde{F}_{ad}(u)] - 0 \\
&= [\widetilde{F}_{ad}(u), \widetilde{F}_{dc}(u^m \underline{c}^j)]
\end{aligned}$$

using (3.26) and that we just showed $[\widetilde{F}_{ad}(u), \widetilde{F}_{bc}(u^m \underline{c}^j)] = 0$.

Thus, we have well-defined elements $\widetilde{F}_{ac}(u^{m+1}\underline{c}^j)$, $j = 0, 1$. Now, assume $a \neq b$, $c \neq d \neq a$. Pick $e \neq a, b, c, d$. Then,

$$\begin{aligned}
[\widetilde{F}_{ab}(1), \widetilde{F}_{cd}(u^{m+1}\underline{c}^j)] &= [\widetilde{F}_{ab}(1), [\widetilde{F}_{ce}(u), \widetilde{F}_{ed}(u^m \underline{c}^j)]] \\
&= -[\widetilde{F}_{ce}(u), [\widetilde{F}_{ed}(u^m \underline{c}^j), \widetilde{F}_{ab}(1)]] - [\widetilde{F}_{ed}(u^m \underline{c}^j), [\widetilde{F}_{ab}(1), \widetilde{F}_{ce}(u)]] \\
&= 0 - [\widetilde{F}_{ed}(u^m \underline{c}^j), \delta_{bc}\widetilde{F}_{ae}(u)] \\
&= \delta_{bc}\widetilde{F}_{ad}(u^{m+1}\underline{c}^j)
\end{aligned}$$

If $k \geq 2$ and $k + l = m + 1$, then, by induction,

$$\begin{aligned}
[\widetilde{F}_{ab}(u^k), \widetilde{F}_{cd}(u^l \underline{c}^j)] &= [[\widetilde{F}_{ae}(u), \widetilde{F}_{eb}(u^{k-1})], \widetilde{F}_{cd}(u^l \underline{c}^j)] \\
&= \delta_{bc}[\widetilde{F}_{ae}(u), \widetilde{F}_{ed}(u^{k-1+l}\underline{c}^j)] \\
&= \delta_{bc}\widetilde{F}_{ad}(u^{k+l}\underline{c}^j) = \delta_{bc}\widetilde{F}_{ad}(u^{m+1}\underline{c}^j)
\end{aligned}$$

By induction on m, this shows (3.2.4) when $i = 0, j = 1$, $\forall k, l \geq 0$.

Finally, we need to show (3.26) holds for $i = 1$, $j = 0, 1$, $k, l \geq 0$. Recall that $[\widetilde{F}_{ab}(u), \widetilde{F}_{bc}(\underline{c})] = -[\widetilde{F}_{ad}(\underline{c}), \widetilde{F}_{dc}(u)]$. The case $i = 1, j = 0$ is similar to the previous case, so assume that $j = 1$.

First, let $k = 0$. Take $a \neq b$, $c \neq d \neq a$. Pick some $e \neq a, b, c, d$.

We use induction on $l \geq 0$ to show that

$$[\widetilde{F}_{ab}(\underline{c}), \widetilde{F}_{cd}(u^l \underline{c}] = (-1)^l \delta_{bc}\widetilde{F}_{ad}(u^l).$$

If $l = 0$,

$$[\widetilde{F}_{ab}(\underline{c}), \widetilde{F}_{cd}(\underline{c})] = \delta_{bc}\widetilde{F}_{ad}(1)$$

Take $l = 1$. Then,

$$\begin{aligned}
[\widetilde{F}_{ab}(\underline{c}), \widetilde{F}_{cd}(u\underline{c})] &= [\widetilde{F}_{ab}(\underline{c}), [\widetilde{F}_{ce}(u), \widetilde{F}_{ed}(\underline{c}]] \\
&= -(-1)[\widetilde{F}_{ce}(u), [\widetilde{F}_{ed}(\underline{c}), \widetilde{F}_{ab}(\underline{c})]] - (-1)^j[\widetilde{F}_{ed}(\underline{c}), [\widetilde{F}_{ab}(\underline{c}), \widetilde{F}_{ce}(u)]] \\
&= -(-1)[\widetilde{F}_{ed}(\underline{c}), -[\widetilde{F}_{ab}(u), \widetilde{F}_{ce}(\underline{c})]]
\end{aligned}$$

$$
\begin{aligned}
&= (-1)[\widetilde{F}_{ed}(\underline{c}), \delta_{bc}\widetilde{F}_{ae}(u\underline{c})] \\
&= -\delta_{bc}[\widetilde{F}_{ae}(u\underline{c}), \widetilde{F}_{ed}(\underline{c})] \\
&= -\delta_{bc}\widetilde{F}_{ad}(u).
\end{aligned}
$$

Now, assume $[\widetilde{F}_{ab}(\underline{c}), \widetilde{F}_{cd}(u^{l-1}\underline{c})] = (-1)^{l-1}\delta_{bc}\widetilde{F}_{ad}(u^{l-1})$. Then,

$$
\begin{aligned}
[\widetilde{F}_{ab}(\underline{c}), \widetilde{F}_{cd}(u^{l}\underline{c})] &= [\widetilde{F}_{ab}(\underline{c}), [\widetilde{F}_{ce}(u), \widetilde{F}_{ed}(u^{l-1}\underline{c})]] \\
&= -(-1)[\widetilde{F}_{ce}(u), [\widetilde{F}_{ed}(u^{l-1}\underline{c}), \widetilde{F}_{ab}(\underline{c})]] - (-1)[\widetilde{F}_{ed}(u^{l-1}\underline{c}), [\widetilde{F}_{ab}(\underline{c}), \widetilde{F}_{ce}(u)]] \\
&= 0 - (-1)[\widetilde{F}_{ed}(u^{l-1}\underline{c}), -[\widetilde{F}_{ab}(u), \widetilde{F}_{ce}(\underline{c})]] \\
&= (-1)[\widetilde{F}_{ed}(u^{l-1}\underline{c}), [\widetilde{F}_{ab}(u), \widetilde{F}_{ce}(\underline{c})]] \\
&= -[\widetilde{F}_{ab}(u), [\widetilde{F}_{ce}(\underline{c}), \widetilde{F}_{ed}(u^{l-1}\underline{c})]] - [\widetilde{F}_{ce}(\underline{c}), [\widetilde{F}_{ed}(u^{l-1}\underline{c}), \widetilde{F}_{ab}(u)]] \\
&= -[\widetilde{F}_{ab}(u), (-1)^{l-1}\widetilde{F}_{cd}(u^{l-1})] \\
&= (-1)^{l}\delta_{bc}\widetilde{F}_{ad}(u^{l})
\end{aligned}
$$

Take arbitrary $k \geq 0$. Then, we want to show that

$$[\widetilde{F}_{ab}(u^{k}\underline{c}), \widetilde{F}_{cd}(u^{l}\underline{c}^{j})] = -\delta_{bc}(-1)^{l}\widetilde{F}_{ad}(u^{k+l}\underline{c}^{j+1}).$$

Again, this is proven using induction. We have shown this to be true for $k = 0$ above. Assume that $[\widetilde{F}_{ab}(u^{m}\underline{c}), \widetilde{F}_{cd}(u^{l}\underline{c}^{j})] = (-1)^{l}\delta_{bc}\widetilde{F}_{ad}(u^{m+l}\underline{c}^{j+1})$ holds for $0 \leq m \leq k$, $l \geq 1$, $j = 0, 1$, $a \neq b$, $c \neq d \neq a$. Pick $e \neq a, b, c, d$. Then,

$$
\begin{aligned}
[\widetilde{F}_{ab}(u^{k+1}\underline{c}), \widetilde{F}_{cd}(u^{l}\underline{c}^{j})] &= [[\widetilde{F}_{ae}(u), \widetilde{F}_{eb}(u^{k}\underline{c})], \widetilde{F}_{cd}(u^{l}\underline{c}^{j})] \\
&= -[[\widetilde{F}_{eb}(u^{k}\underline{c}), \widetilde{F}_{cd}(u^{l}\underline{c}^{j})], \widetilde{F}_{ae}(u)] - (-1)^{j}[[\widetilde{F}_{cd}(u^{l}\underline{c}^{j}), \widetilde{F}_{ae}(u)], \widetilde{F}_{eb}(u^{k}\underline{c})] \\
&= -[(-1)^{l}\delta_{bc}\widetilde{F}_{ed}(u^{k+l}\underline{c}^{j+1}), \widetilde{F}_{ae}(u)] \\
&= (-1)^{l}\delta_{bc}[\widetilde{F}_{ae}(u), \widetilde{F}_{ed}(u^{k+l}\underline{c}^{j+1})] \\
&= (-1)^{l}\delta_{bc}\widetilde{F}_{ad}(u^{k+l+1}\underline{c}^{j+1})
\end{aligned}
$$

□

Similarly, for $a \neq b$, $c \neq d \neq a$, and $k, l, i, j \geq 0$,

$$[\widetilde{F}_{ab}(v^{k}\underline{c}^{i}), \widetilde{F}_{cd}(v^{l}\underline{c}^{j})] = \delta_{bc}\widetilde{F}_{ad}(v^{k+l}\underline{c}^{i+j}). \tag{3.27}$$

Proposition 3.2.5. *We can define inductively elements* $\widetilde{F}_{ad}(u^k v^l \underline{c}^i)$ *such that, for* $a \neq b$, $c \neq d \neq a$, *and* $r_1, r_2, s_1, s_2 \geq 0$, $i_1, i_2 = 0, 1$, *the equality*

$$[\widetilde{F}_{ab}(u^{r_1} v^{s_1} \underline{c}^{i_1}), \widetilde{F}_{cd}(u^{r_2} v^{s_2} \underline{c}^{i_2})] = \delta_{bc}(-1)^{i_1 r_2} \widetilde{F}_{ad}(u^{r_1+r_2} v^{s_1+s_2} \underline{c}^{i_1+i_2}) \tag{3.28}$$

is satisfied

$\widetilde{F}_{ac}(u^r v^s)$ can be defined as in the previous section.

Lemma 3.2.6. *For* $a \neq b$, $c \neq a, b$ *and* $d \neq a, b, c$,

$$[\widetilde{F}_{ac}(u^r v^s), \widetilde{F}_{cb}(\underline{c})] = (-1)^r [\widetilde{F}_{ad}(\underline{c}), \widetilde{F}_{db}(u^r v^s)].$$

Proof. This is a generalization of defining relations (3.19) and (3.18). First, note that when $r = 0$ or $s = 0$, this reduces to Lemma 3.2.4 or its analogue with v instead of u.

It follows that

$$\begin{aligned}
[\widetilde{F}_{ac}(u^r v^s), \widetilde{F}_{cb}(\underline{c})] &= [[\widetilde{F}_{ad}(v^s), \widetilde{F}_{dc}(u^r)], \widetilde{F}_{cb}(\underline{c})] \\
&= [\widetilde{F}_{ad}(v^s), [\widetilde{F}_{dc}(u^r), \widetilde{F}_{cb}(\underline{c})]] - [\widetilde{F}_{dc}(u^r), [\widetilde{F}_{ad}(v), \widetilde{F}_{cb}(\underline{c})]] \\
&= [\widetilde{F}_{ad}(v^s), (-1)^r [\widetilde{F}_{dc}(\underline{c}), \widetilde{F}_{cb}(u^r)]] \\
&= (-1)^r [[\widetilde{F}_{ad}(v^s), \widetilde{F}_{dc}(\underline{c})], \widetilde{F}_{cb}(u^r)] + (-1)^r [\widetilde{F}_{dc}(\underline{c}), [\widetilde{F}_{ad}(v^s), \widetilde{F}_{cb}(u^r)]] \\
&= (-1)^r [[\widetilde{F}_{ad}(\underline{c}), \widetilde{F}_{dc}(v^s)], \widetilde{F}_{cb}(u^r)] \\
&= (-1)^r [[\widetilde{F}_{ad}(\underline{\mathbf{v}}), [\widetilde{F}_{dc}(v^s), \widetilde{F}_{cb}(u^r)]] - (-1)^r [\widetilde{F}_{dc}(v^s), [\widetilde{F}_{ad}(\underline{c}), \widetilde{F}_{cb}(u^r)]] \\
&= (-1)^r [\widetilde{F}_{ad}(\underline{c}), \widetilde{F}_{db}(u^r v^s)]
\end{aligned}$$

□

Proof. We now prove Proposition 3.2.5 We use the result (3.26) from Lemma 3.2.4, (3.27), and induction.

Finally, we consider elements using $\underline{c}$.

By Lemma 3.2.4, and its analogue for v, we already have that

$$[\widetilde{F}_{ab}(u^{r_1} \underline{c}^{i_1}), \widetilde{F}_{cd}(u^{r_2} \underline{c}^{i_2})] = \delta_{bc}(-1)^{r_2 i_1} \widetilde{F}_{ad}(u^{r_1+r_2} \underline{c}^{i_1+i_2}) \tag{3.29}$$

$$[\widetilde{F}_{ab}(v^{s_1} \underline{c}^{i_1}), \widetilde{F}_{cd}(v^{s_2} \underline{c}^{i_2})] = \delta_{bc} \widetilde{F}_{ad}(v^{s_1+s_2} \underline{c}^{i_1+i_2}) \tag{3.30}$$

First, let $i_1 = 0$, $i_2 = 1$. The elements $\widetilde{F}_{ab}(u^r v^s \underline{c}^i)$ given by

$$\widetilde{F}_{ab}(u^r v^s \underline{c}) = [\widetilde{F}_{ac}(u^r v^s), \widetilde{F}_{cb}(\underline{c})]$$

for some $c \neq a, b$ are well-defined, using $e \neq a, b, c$ and $d \neq a, b, c, e$:

$$\begin{aligned}
[\widetilde{F}_{ac}(u^r v^s), &\widetilde{F}_{cb}(\underline{c})] = [[\widetilde{F}_{ae}(u^r v^s), \widetilde{F}_{ec}(1)], \widetilde{F}_{cb}(\underline{c})] \\
&= [\widetilde{F}_{ae}(u^r v^s), [\widetilde{F}_{ec}(1), \widetilde{F}_{cb}(\underline{c})]] + [[\widetilde{F}_{ae}(u^r v^s), \widetilde{F}_{cb}(\underline{c})], \widetilde{F}_{ec}(1)] \\
&= [\widetilde{F}_{ae}(u^r v^s), \widetilde{F}_{eb}(\underline{c})] + [[[\widetilde{F}_{ad}(u^r), \widetilde{F}_{de}(v^s)], \widetilde{F}_{cb}(\underline{c})], \widetilde{F}_{ec}(1)] \\
&= [\widetilde{F}_{ae}(u^r v^s), \widetilde{F}_{eb}(\underline{c})] + [[\widetilde{F}_{ad}(u^r), [\widetilde{F}_{de}(v^s), \widetilde{F}_{cb}(\underline{c})]], \widetilde{F}_{ec}(1)] \\
&\quad + [[[\widetilde{F}_{ad}(u^r), \widetilde{F}_{cb}(\underline{c})], \widetilde{F}_{de}(v^s)], \widetilde{F}_{ec}(1)] \\
&= [\widetilde{F}_{ae}(u^r v^s), \widetilde{F}_{eb}(\underline{c})]
\end{aligned}$$

We also have that

$$\begin{aligned}
[\widetilde{F}_{ac}(u^{r-1} v^{s-1}), &\widetilde{F}_{cb}(uv\underline{c})] = [\widetilde{F}_{ac}(u^{r-1} v^{s-1}, [\widetilde{F}_{ce}(uv), \widetilde{F}_{eb}(\underline{c})]] \\
&= [[\widetilde{F}_{ac}(u^{r-1} v^{s-1}), \widetilde{F}_{ce}(uv)], \widetilde{F}_{eb}(\underline{c})] + [\widetilde{F}_{ce}(uv), [\widetilde{F}_{ac}(u^{r-1} v^{s-1}), \widetilde{F}_{eb}(\underline{c})]] \\
&= [\widetilde{F}_{ae}(u^r v^s), \widetilde{F}_{eb}(\underline{c})] \\
&= \widetilde{F}_{ab}(u^r v^s \underline{c})
\end{aligned}$$

Further, as $\widetilde{F}_{ac}(u^r v^s) = [\widetilde{F}_{ab}(u^{r_1} v^{s_1}), \widetilde{F}_{bc}(u^{r_2} v^{s_2})]$ for any $r_1 + r_2 = r$, $s_1 + s_2 = s$, we can show by induction that

$$\widetilde{F}_{ac}(u^r v^s \underline{c}) = [\widetilde{F}_{ab}(u^{r_1} v^{s_1}), \widetilde{F}_{bc}(u^{r_2} v^{s_2} \underline{c})] \tag{3.31}$$

for any $r_1 + r_2 = r$, $s_1 + s_2 = s$. Assume we have shown that

$$[\widetilde{F}_{ab}(u^{r-k'} v^{s-l'}), \widetilde{F}_{bc}(u^{k'} v^{l'} \underline{c})] = \widetilde{F}_{ac}(u^r v^s \underline{c})$$

for $k' \leq k$, $l' \leq l$ for some $0 \leq k < r$, $0 \leq l < s$. Then, for $e \neq a, b, c$,

$$\begin{aligned}
[\widetilde{F}_{ab}(&u^{r-(k+1)} v^{s-(l+1)}), \widetilde{F}_{bc}(u^{k+1} v^{l+1} \underline{c})] \\
&= [\widetilde{F}_{ab}(u^{r-(k+1)} v^{s-(l+1)}), [\widetilde{F}_{be}(u^{k+1} v^{l+1}), \widetilde{F}_{ec}(\underline{c})]]
\end{aligned}$$

$$
\begin{aligned}
=&[[\widetilde{F}_{ab}(u^{r-(k+1)}v^{s-(l+1)}), \widetilde{F}_{be}(u^{k+1}v^{l+1})], \widetilde{F}_{ec}(\underline{c})] \\
&+ [\widetilde{F}_{be}(u^{k+1}v^{l+1}), [\widetilde{F}_{ab}(u^{r-(k+1)}v^{s-(l+1)}), \widetilde{F}_{ec}(\underline{c})]] \\
=&[\widetilde{F}_{ae}(u^r v^s), \widetilde{F}_{ec}(\underline{c})] \\
=&\widetilde{F}_{ac}(u^r v^s \underline{c})
\end{aligned}
$$

This proves (3.31) by induction. Finally, let $i_1 = 1$ and $i_2 = 0, 1$.

By Lemma 3.2.4, and its analogue for v, we already have that

$$
[\widetilde{F}_{ab}(u^{r_1}\underline{c}), \widetilde{F}_{cd}(u^{r_2}\underline{c}^{i_2})] = \delta_{bc}(-1)^{r_2}\widetilde{F}_{ad}(u^{r_1+r_2}\underline{c}^{1+i_2}) \tag{3.32}
$$

$$
[\widetilde{F}_{ab}(v^{s_1}\underline{c}), \widetilde{F}_{cd}(v^{s_2}\underline{c}^{i_2})] = \delta_{bc}\widetilde{F}_{ad}(v^{s_1+s_2}\underline{c}^{1+i_2}) \tag{3.33}
$$

Our goal is to show that

$$
[\widetilde{F}_{ab}(u^{r_1}v^{s_1}\underline{c}), \widetilde{F}_{cd}(u^{r_2}v^{s_2}\underline{c}^{i_2})] = \delta_{bc}(-1)^{r_2}\widetilde{F}_{ad}(u^{r_1+r_2}v^{s_1+s_2}\underline{c}^{1+i_2}). \tag{3.34}
$$

By Lemma 3.2.6, $[\widetilde{F}_{ac}(u^r v^s), \widetilde{F}_{cb}(\underline{c})] = (-1)^r[\widetilde{F}_{ac}(\underline{c}), \widetilde{F}_{cb}(u^r v^s)]$.

Then, with $e \neq a, b, c, d$, we can write

$$
\begin{aligned}
[\widetilde{F}_{ab}(u^{r_1}v^{s_1}\underline{c}), \widetilde{F}_{cd}(u^{r_2}v^{s_2}\underline{c}^{i_2})] &= [[\widetilde{F}_{ae}(u^{r_1}v^{s_1}), \widetilde{F}_{eb}(\underline{c})], \widetilde{F}_{cd}(u^{r_2}v^{s_2}\underline{c}^{i_2})] \\
=&(-1)^{i_2}[\widetilde{F}_{ae}(u^{r_1}v^{s_1}), [\widetilde{F}_{eb}(\underline{c}), \widetilde{F}_{cd}(u^{r_2}v^{s_2}\underline{c}^{i_2})]] \\
&+ (-1)^{i_2}[[\widetilde{F}_{ae}(u^{r_1}v^{s_1}), \widetilde{F}_{cd}(u^{r_2}v^{s_2}\underline{c}^{i_2})], \widetilde{F}_{eb}(\underline{c})] \\
=&(-1)^{i_2}[\widetilde{F}_{ae}(u^{r_1}v^{s_1}), [\widetilde{F}_{eb}(\underline{c}), \widetilde{F}_{cd}(u^{r_2}v^{s_2}\underline{c}^{i_2})]]
\end{aligned}
$$

If $i_2 = 0$, this becomes

$$
[\widetilde{F}_{ae}(u^{r_1}v^{s_1}), \delta_{bc}(-1)^{r_2}\widetilde{F}_{ed}(u^{r_2}v^{s_2}\underline{c})] = \delta_{bc}(-1)^{r_2}\widetilde{F}_{ac}(u^{r_1+r_2}v^{s_1+s_2}\underline{c}).
$$

If $i_2 = 1$, this becomes

$$
\begin{aligned}
-[\widetilde{F}_{ae}(u^{r_1}v^{s_1}),&[\widetilde{F}_{eb}(\underline{c}), \widetilde{F}_{cd}(u^{r_2}v^{s_2}\underline{c})]] \\
&= - [\widetilde{F}_{ae}(u^{r_1}v^{s_1}), [\widetilde{F}_{eb}(\underline{c}), [\widetilde{F}_{ca}(u^{r_2}v^{s_2}), \widetilde{F}_{ad}(\underline{c})]]] \\
&=[\widetilde{F}_{ae}(u^{1}v^{s_1}), [[\widetilde{F}_{eb}(\underline{c}), \widetilde{F}_{ca}(u^{r_2}v^{s_2})], \widetilde{F}_{ad}(\underline{c})] \\
&\quad + [\widetilde{F}_{ca}(u^{r_2}v^{s_2}), [\widetilde{F}_{eb}(\underline{c}), \widetilde{F}_{ad}(\underline{c})]]]
\end{aligned}
$$

$$
\begin{aligned}
=&[\widetilde{F}_{ae}(u^{r_1}v^{s_1}), \delta_{bc}(-1)^{r_2}[\widetilde{F}_{ea}(u^{r_2}v^{s_2}\underline{c}), \widetilde{F}_{ad}(\underline{c})]] \\
=&[\widetilde{F}_{ae}(u^{r_1}v^{s_1}), \delta_{bc}(-1)^{r_2}\widetilde{F}_{ed}(u^{r_2}v^{s_2})] \\
=&\delta_{bc}(-1)^{r_2}\widetilde{F}_{ad}(u^{r_1+r_2}v^{s_1+s_2}).
\end{aligned}
$$

□

Thus, we have constructed the desired elements $\widetilde{F}_{ab}(u^r v^s \underline{c}^i)$, equivalent to finding an inverse map of the surjection $\widetilde{\mathfrak{st}_n}(A) \twoheadrightarrow \mathfrak{st}_n(A)$, thus showing that $\mathfrak{st}_n(A)$ is isomorphic to $\widetilde{\mathfrak{st}_n}(A)$. This leaves us with a presentation with finitely many generators.

3.2.1 Type Q first cyclic homology group

Recall that $\mathfrak{st}_n(A)$ is the universal central extension of $\mathfrak{sl}_n(A)$ for $n \geq 5$ when A is a $\mathbb{Z}/2\mathbb{Z}$- graded superalgebra over a commutative ring K, from [5]. In [4], $A = \mathbb{C} < u, u^{-1}, \underline{c} > /(\underline{c}^2 - 1, \underline{c}u - u^{-1}\underline{c})$.

Thus, we know that $\mathfrak{st}_n(A)$ for $A = \mathbb{C}[u,v] \rtimes \mathcal{C}$ is the universal central extension of $\mathfrak{sl}_n(A)$. Further, as in the case of the algebra $\mathbb{C}\langle u, u^{-1}, \underline{c}\rangle/(\underline{c}^2 - 1, \underline{c}u - u^{-1}\underline{c})$, the first cyclic homology group of A is isomorphic to the kernel of the epimorphism $\mathfrak{st}_n(A) \twoheadrightarrow \mathfrak{sl}_n(A)$.

To compute $HC_1(A)$, we must first find a minimal spanning set, then check for linear independence. Denote by $\langle A, A\rangle = A \otimes A/I$ for the ideal I generated by $(-1)^{|a_2||a_1|}a_2a_3 \otimes a_1 + (-1)^{|a_3||a_2|}a_3a_1 \otimes a_2 + (-1)^{|a_1||a_3|}a_1a_2 \otimes a_3$ and $a_1 \otimes a_2 + (-1)^{|a_1||a_2|}a_2 \otimes a_1$. Then, $HC_1(A) = \ker(\langle A, A\rangle \to [A, A])$, where this map sends $a_1 \otimes a_2$ to $[a_1, a_2]$.

Recall that when looking at $HC_1(\mathbb{C}[u,v]) = \frac{\Omega^1(\mathbb{C}[u,v])}{d(\mathbb{C}[u,v])}$, we get that

$$
u^r v^s dv = d\left(\frac{u^r v^{s+1}}{s+1}\right) - \frac{r u^{r-1} v^{s+1}}{s+1} du
$$

so we can expect

$$
\overline{u^r v^s \otimes v} = \overline{\frac{-r}{s+1} u^{r-1} v^{s+1} \otimes u}
$$

in $< \mathbb{C}[u,v], \mathbb{C}[u,v] >$. This should also work in $< \mathbb{C}[u,v] \rtimes \mathcal{C}, \mathbb{C}[u,v] \rtimes \mathcal{C} >$ when no $\underline{c}$ elements are involved.

Recall also that as $udv+vdu = d(uv) = 0$, we have $u\otimes v+v\otimes u = 1\otimes uv = 0$.

For readability, overline notation will be suppressed, but these computations do occur in $A \otimes A/I$. Overall, we must consider elements of the forms $a \otimes u$, $a \otimes v$, $a \otimes \underline{c}$ for $a = u^r v^s$ or $a = u^r v^s \underline{c}$ and check for a linear dependence relation in order to find a basis of $HC_1(A)$.

Note also that $X \otimes 1 = 0$ for any $X \in A$:

$$\begin{aligned} X \otimes 1 &= X \cdot 1 \otimes 1 \\ &= -X \otimes 1 - 1 \otimes X \\ &= 1 \otimes X - 1 \otimes X \\ &= 0. \end{aligned}$$

Elementary tensors of the form $u^r v^s \otimes v$

The following two lemmas are used to prove a general statement about these elements.

Lemma 3.2.7.

$$k \cdot u^r v^s \otimes v = u^{r-1} v^{s+1-k} \otimes uv^k - u^{r-1} v^{s+1} \otimes u$$

for $k = 1, \ldots, s+1$.

Proof. Proof by induction. When $k = 1$, we have

$$\begin{aligned} u^r v^s \otimes v &= u \cdot u^{r-1} v^s \otimes v \\ &= -u^{r-1} v^{s+1} \otimes u - uv \otimes u^{r-1} v^s \\ &= u^{r-1} v^s \otimes uv - u^{r-1} v^{s+1} \otimes u. \end{aligned}$$

Next, suppose the claim holds for $k = 1, \ldots, k'$ for some $k' \geq 1$. Then, it holds for $k'+1$ as well:

$$k' u^r v^s \otimes v = u^{r-1} v^{s+1-k'} \otimes uv^{k'} - u^{r-1} v^{s+1} \otimes u$$

$$
\begin{aligned}
&= u^{r-1}v^{s-k'} \cdot v \otimes uv^{k'} - u^{r-1}v^{s+1} \otimes u \\
&= -u^r v^s \otimes v - uv^{k'+1} \otimes u^{r-1}v^{s-k'} - u^{r-1}v^{s+1} \otimes u \\
(k'+1)u^r v^s \otimes v &= -uv^{k'+1} \otimes u^{r-1}v^{s-k'} - u^{r-1}v^{s+1} \otimes u \\
&= u^{r-1}v^{s+1-(k'+1)} \otimes uv^{k'+1} - u^{r-1}v^{s+1} \otimes u
\end{aligned}
$$

□

Lemma 3.2.8.

$$(s+1)u^r v^s \otimes v = -u^l v^{s+1} \otimes u^{r-l} - lu^{r-1}v^{s+1} \otimes u$$

for $l = 1, \dots, r-1$.

Proof. Proof by induction. The base case for $l = 1$ follows from Lemma 3.2.7 when $k = s+1$. Next, suppose the claim holds for $l = 1, \dots, l'$ for some $l' \geq 1$. Then, it holds for $l'+1$ as well:

$$
\begin{aligned}
(s+1)u^r v^s \otimes v &= -u^{l'} v^{s+1} \otimes u^{r-l'} - l'u^{r-1}v^{s+1} \otimes u \\
&= u^{r-l'} \otimes u^{l'} v^{s+1} - l'u^{r-1}v^{s+1} \otimes u \\
&= u^{r-l'-1} \cdot u \otimes u^{l'} v^{s+1} - l'u^{r-1}v^{s+1} \otimes u \\
&= -u^{r-1}v^{s+1} \otimes u - u^{l'+1}v^{s+1} \otimes u^{r-l'-1} - l'u^{r-1}v^{s+1} \otimes u \\
&= -u^{l'+1}v^{s+1} \otimes u^{r-(l'+1)} - (l'+1)u^{r-1}v^{s+1} \otimes u
\end{aligned}
$$

□

Proposition 3.2.9. *For* $r \geq 1,\ s \geq 0$,

$$u^r v^s \otimes v = \frac{-r}{s+1} u^{r-1}v^{s+1} \otimes u$$

Proof. Using Lemma 3.2.8 with $l = r-1$, we get

$$
\begin{aligned}
(s+1)u^r v^s \otimes v &= -u^{r-1}v^{s+1} \otimes u - (r-1)u^{r-1}v^{s+1} \otimes u \\
&= -ru^{r-1}v^{s+1} \otimes u
\end{aligned}
$$

$$\Rightarrow u^r v^s \otimes v = \frac{-r}{s+1} u^{r-1} v^{s+1} \otimes u.$$

□

Elementary tensors of the form $u^r v^s \otimes u$

Initially, these elements were kept as they were; in the previous case, the goal was to produce a u in the second tensor factor. However, the lemma below shows we only take r to be even.

Lemma 3.2.10.

$$u^{2n+1} v^s \otimes u = 0$$

for $n \geq 1$.

Proof.

$$\begin{aligned}
u^{2n+2} v^s \underline{c} \otimes \underline{c} &= u(u^{2n+1} v^s \underline{c}) \otimes \underline{c} \\
&= -u^{2n+1} v^s \otimes u + \underline{c} u \otimes u^{2n+1} v^s \underline{c} \\
&= -u^{2n+1} v^s \otimes u - u\underline{c} \otimes u^{2n+1} v^s \underline{c} \\
&= -u^{2n+1} v^s \otimes u + \underline{c} u^{2n+1} v^s \underline{c} \otimes u - u^{2n+1} v^s \underline{c} u \otimes \underline{c} \\
&= -u^{2n+1} v^s \otimes u - u^{2n+1} v^s \otimes u + u^{2n+2} v^s \underline{c} \otimes \underline{c} \\
\Rightarrow 0 &= -2u^{2n+1} v^s \otimes u
\end{aligned}$$

Thus, $u^{2n+1} v^s \otimes u = 0$. □

Elementary tensors of the form $u^r v^s \otimes \underline{c}$

If $s = 0$ and r is even, these elements disappear:

Lemma 3.2.11.

$$u^{2n} \otimes \underline{c} = 0$$

for $n \geq 1$.

Proof. On the one hand,

$$u^{2n-1} \underline{c} \otimes u = -\underline{c} u^{2n-1} \otimes u = u^{2n} \otimes \underline{c} + u\underline{c} \otimes u^{2n-1}$$

On the other hand,

$$u^{2n-1}\underline{c} \otimes u = -\underline{c}u \otimes u^{2n-1} - u^{2n} \otimes \underline{c} = u\underline{c} \otimes u^{2n-1} - u^{2n} \otimes \underline{c}$$

which means that $u^{2n} \otimes \underline{c} = -u^{2n} \otimes \underline{c}$, thus $u^{2n} \otimes \underline{c} = 0$. □

Note that $[\underline{c}, u] = \underline{c}u - u\underline{c} = -2u\underline{c}$, i.e. $\underline{c} \otimes u$ is not in the kernel of the map so thus not in $HC_1(A)$, so does not need to be addressed. This extends to all odd powers of u tensored with $\underline{c}$:

$$\begin{aligned} [u^{2n+1}, \underline{c}] &= u^{2n+1}\underline{c} - \underline{c}u^{2n+1} \\ &= 2u^{2n+1}\underline{c} \end{aligned}$$

which is not in the kernel of the map $< A, A > \to [A, A]$ for $A = \mathbb{C}[u, v] \rtimes \mathcal{C}$.

Lemma 3.2.12.

$$u^{2n}v^s \otimes \underline{c} = 0$$

Proof. This very closely mimics the proof for Lemma 3.2.11. On the one hand,

$$u^{2n-1}v^s\underline{c} \otimes u = -\underline{c}u^{2n-1}v^s \otimes u = u^{2n}v^s \otimes \underline{c} + u\underline{c} \otimes u^{2n-1}v^s$$

On the other hand,

$$u^{2n-1}v^s\underline{c} \otimes u = -\underline{c}u \otimes u^{2n-1}v^s - u^{2n}v^s \otimes \underline{c} = u\underline{c} \otimes u^{2n-1}v^s - u^{2n}v^s \otimes \underline{c}$$

which means that $u^{2n}v^s \otimes \underline{c} = -u^{2n}v^s \otimes \underline{c}$, thus $u^{2n}v^s \otimes \underline{c} = 0$. □

Elementary tensors of the form $u^r v^s \underline{c} \otimes v$

Lemma 3.2.13.

$$u^{2n+1}v^s\underline{c} \otimes v = 0$$

Proof.

$$\begin{aligned} u^{2n+1}v^s\underline{c} \otimes v &= -\underline{c}u^{2n+1}v^s \otimes v \\ &= v\underline{c} \otimes u^{2n+1}v^s + u^{2n+1}v^{s+1} \otimes \underline{c} \end{aligned}$$

$$= \underline{c}v \otimes u^{2n+1}v^2 + u^{2n+1}v^{s+1} \otimes \underline{c}$$
$$= -u^{2n+1}v^s\underline{c} \otimes v - u^{2n+1}v^{s+1} \otimes \underline{c} + u^{2n+1}v^{s+1} \otimes \underline{c}$$
$$u^{2n+1}v^s\underline{c} \otimes v = 0$$

□

Thus, we can conclude that we only have elements $u^{2n}v^s\underline{c} \otimes v$ of the form $u^r v^s \underline{c} \otimes v$ to consider.

Elementary tensors of the form $u^r v^s \underline{c} \otimes u$

Lemma 3.2.14.

$$u^{2n}\underline{c} \otimes u = -u^{2n+1} \otimes \underline{c}$$

for $n \geq 0$.

Proof. First, note that

$$u^{2n} \otimes u\underline{c} = u \cdot u^{2n-1} \otimes u\underline{c}$$
$$= -u\underline{c}u \otimes u^{2n-1} - u^{2n-1} \cdot u\underline{c} \otimes u$$
$$= u^2\underline{c} \otimes u^{2n-1} - u^{2n}\underline{c} \otimes u$$
$$= u \cdot u\underline{c} \otimes u^{2n-1} - u^{2n}\underline{c} \otimes u$$
$$= -u^{2n} \otimes u\underline{c} - u\underline{c}u^{2n-1} \otimes u - u^{2n}\underline{c} \otimes u$$
$$= -u^{2n} \otimes u\underline{c} + u^{2n}\underline{c} \otimes u - u^{2n}\underline{c} \otimes u$$
$$\therefore u^{2n} \otimes u\underline{c} = -u^{2n} \otimes u\underline{c}$$
$$\therefore u^{2n} \otimes u\underline{c} = 0.$$

Thus,

$$u^{2n}\underline{c} \otimes u = -u^{2n+1} \otimes \underline{c} - u^{2n} \otimes u\underline{c} = -u^{2n+1} \otimes \underline{c}.$$

□

Next, suppose $r = 0$. In general,

$$v^s\underline{c} \otimes u = -uv^s \otimes \underline{c} - \underline{c}u \otimes v^s = -uv^s \otimes \underline{c} + u\underline{c} \otimes v^s.$$

When $s = 1$,

$$\begin{aligned} v\underline{c} \otimes u &= -uv \otimes \underline{c} + u\underline{c} \otimes v \\ &= -uv \otimes \underline{c} + (-vu \otimes \underline{c} - \underline{c}v \otimes u) \\ &= -uv \otimes \underline{c} - uv \otimes \underline{c} - v\underline{c} \otimes u \\ 2v\underline{c} \otimes u &= -2uv \otimes \underline{c} \\ v\underline{c} \otimes u &= -uv \otimes \underline{c}. \end{aligned}$$

Similarly, when $s = 2$, we get $v^2\underline{c} \otimes u = -uv^2 \otimes \underline{c}$. This pattern suggests the following lemma.

Lemma 3.2.15. *For $s \geq 1$,*

$$v^s\underline{c} \otimes u = -uv^s \otimes \underline{c}.$$

Proof.

$$\begin{aligned} v^s\underline{c} \otimes u &= -uv^s \otimes \underline{c} - \underline{c}u \otimes v^s \\ &= -uv^s \otimes \underline{c} + u\underline{c} \otimes v^s \\ &= -uv^s \otimes \underline{c} + (-v^s u \otimes \underline{c} - \underline{c}v^s \otimes u) \\ &= -uv^s \otimes \underline{c} - uv^s \otimes \underline{c} - v^s\underline{c} \otimes u \\ 2v^s\underline{c} \otimes u &= -2uv^s \otimes \underline{c} \\ v^s\underline{c} \otimes u &= -uv^s \otimes \underline{c} \end{aligned}$$

□

Finally, consider the case when $rs \neq 0$.

Lemma 3.2.16.

$$u^{2n}v^s\underline{c} \otimes u = -u^{2n+1}v^s \otimes \underline{c}$$

Proof. This is very similar to the proof of Lemma 3.2.14. First, note that

$$u^{2n}v^s\underline{c} \otimes u = -u^{2n+1}v^s \otimes \underline{c} - \underline{c}u \otimes u^{2n}v^s. \tag{3.35}$$

where $-\underline{c}u \otimes u^{2n}v^s = -u^{2n}v^s \otimes u\underline{c}$. However,

$$\begin{aligned}
u^{2n}v^s \otimes u\underline{c} &= u \cdot u^{2n-1}v^s \otimes u\underline{c} \\
&= -u\underline{c}u \otimes u^{2n-1}v^s - u^{2n-1}v^s \cdot u\underline{c} \otimes u \\
&= u^2\underline{c} \otimes u^{2n-1}v^s - u^{2n}v^s\underline{c} \otimes u \\
&= u \cdot u\underline{c} \otimes u^{2n-1}v^s - u^{2n}v^s\underline{c} \otimes u \\
&= -u^{2n}v^s \otimes u\underline{c} - u\underline{c}u^{2n-1}v^s \otimes u - u^{2n}v^s\underline{c} \otimes u \\
&= -u^{2n}v^s \otimes u\underline{c} + u^{2n}v^s\underline{c} \otimes u - u^{2n}v^s\underline{c} \otimes u \\
\therefore u^{2n}v^s \otimes u\underline{c} &= -u^{2n}v^s \otimes u\underline{c} \\
\therefore u^{2n}v^s \otimes u\underline{c} &= 0. \qquad (3.36)
\end{aligned}$$

Together, (3.35) and (3.36) imply the lemma. □

Elementary tensors of the form $u^r v^s \underline{c} \otimes \underline{c}$

$$\begin{aligned}
u^r v^s \underline{c} \otimes \underline{c} + \underline{c} \cdot \underline{c} \otimes u^r v^s - \underline{c}u^r v^s \otimes \underline{c} &= 0 \\
u^r v^s \underline{c} \otimes \underline{c} - (-1)^r u^r v^s \underline{c} \otimes \underline{c} &= 0 \\
(1 - (-1)^r) u^r v^s \underline{c} \otimes \underline{c} &= 0
\end{aligned}$$

Thus, if r is odd, then $u^r v^s \underline{c} \otimes \underline{c} = 0$. None of the tensors $u^{2n}v^s\underline{c} \otimes \underline{c}$ appear in the kernel and can be disregarded, as $[u^{2n}v^s\underline{c}, \underline{c}] = 2u^{2n}v^s$.

Conclusion

Recall $A = \mathbb{C}[u, v] \rtimes \mathcal{C}$. From the cases considered above, it appears that a spanning set for the kernel of the map $\langle A, A \rangle \to [A, A]$ is

$$\begin{aligned}
&u^{2n}v^s \otimes u,\ n \geq 0, s > 0 \\
&u^{2n}v^s\underline{c} \otimes v,\ n, s \geq 0
\end{aligned}$$

Next, one needs to verify that they are linearly independent. One tool for this would be to introduce another grading on A. It is a $\mathbb{Z}_2$-graded algebra, but it is also graded by $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_2$. Let $\deg(u^r v^s \underline{c}^i) = (r, s, i)$. Then,

$$A = \bigoplus_{(r,s,i) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_2} A[r, s, i]$$

where $A[r, s, i] = \mathbb{C} u^r v^s \underline{c}^i$. Then, the total grading on $A \otimes A$ is given by

$$(A \otimes A)[r, s, i] = \bigoplus_{r_1 + r_2 = r, s_1 + s_2 = s, i_1 + i_2 = i} \mathbb{C} u^{r_1} v^{s_1} \underline{c}^{i_1} \otimes u^{r_2} v^{s_2} \underline{c}^{i_2}$$

and I is also a graded ideal. Thus, $\langle A, A \rangle = A \otimes A / I$ is also graded by $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_2$:

$$\langle A, A \rangle = \bigoplus_{(r,s,i)} \langle A, A \rangle [r, s, i].$$

By definition, two elements of different degrees must be linearly independent. Thus, the elements of the form $u^r v^s \underline{c} \otimes v$ are linearly independent from the others as none of $u^{2n} v^s \otimes u$ can have degree $(r, s + 1, 1)$. Each element of the form $u^r v^s \underline{c} \otimes v$ is also linearly independent from any other element of that form with different values for r or s.

Chapter 4

Schur-Weyl Duality for Super Yangians

4.1 Schur-Weyl Functor

The simplest way to find the Schur-Weyl functor in the $Y(\mathfrak{gl}_{m|n})$ case will be to use the RTT presentation as defined by Gow [13].

Recall the R-matrix

$$R(u) = 1 - \frac{1}{u} P_{12}$$

where P_{12} is the permutation matrix

$$P_{12} = \sum_{i,j=1}^{m+n} (-1)^{|j|} E_{ij} \otimes E_{ji}$$

Let $H^{\deg}(S_l)$ be the degenerate affine Hecke algebra of S_l as defined in Definition 2.5.1. Then, the goal of this chapter is to produce the functor SW:

$$\begin{aligned} SW : H^{\deg}(S_l) - \mathrm{mod}_R &\to Y(\mathfrak{gl}_{m|n}) - \mathrm{mod}_L \\ M &\mapsto M \otimes_{\mathbb{C}[S_l]} \mathbb{C}(m|n)^{\otimes l} \end{aligned}$$

One can obtain a formula for the action of generators of the Yangian using the R-matrix; this is explained below.

The eponymous equation of the RTT presentation gives

$$R_{12}(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u-v)$$

where $T(u) = (t_{ij}(u))$, a matrix with entries power series in u^{-1}. We need a map $Y(\mathfrak{gl}_{m|n}) \to \mathrm{End}_{\mathbb{C}}(SW(M))$ sending $T(u) \in Y(\mathfrak{gl}_{m|n}) \otimes M_{m|n}(\mathbb{C})[[u^{-1}]]$ to some element of $\mathrm{End}_{\mathbb{C}}(SW(M)) \otimes M_{m|n}(\mathbb{C})[[u^{-1}]]$. To do this, one compares the coefficients of power series in the two matrices. SW must preserve RTT relations. This follows from the fact that R is a solution to the quantum Yang-Baxter equation.

Recall that $H^{\deg}(S_l) \simeq \mathbb{C}[\mathsf{x}_1, \ldots, \mathsf{x}_l] \otimes_{\mathbb{C}} \mathbb{C}[S_l]$ as vector spaces.

$R(u - \mathsf{x}_i) = 1 - \frac{P}{u-\mathsf{x}_i} = 1 - \frac{u^{-1}P}{1-\mathsf{x}_i u^{-1}}$ (and use that $\frac{1}{1-t} = 1 + t + t^2 + \cdots$) belongs to $\mathrm{End}(M) \otimes_{\mathbb{C}} \mathrm{End}(\mathbb{C}(m|n))^{\otimes 2}[[u^{-1}]]$

So, if M is a right module over $H^{\deg}(S_l)$, x_i can be seen as an operator on M. Consider the mapping

$$T(u) \mapsto R^{t_1}_{1,l+1}(\mathsf{x}_1 - u)R^{t_2}_{2,l+1}(\mathsf{x}_2 - u) \cdots R^{t_l}_{l,l+1}(\mathsf{x}_l - u) \tag{4.1}$$

where each $R_{i,l+1}(u - \mathsf{x}_i) = 1 - \frac{P_{i,l+1}}{u-\mathsf{x}_i}$.

$$\frac{1}{u - \mathsf{x}_i} = \frac{u^{-1}}{1 - \mathsf{x}_i u^{-1}} \in \mathrm{End}_{\mathbb{C}}(M) \otimes \mathrm{id}^{\otimes l}[[u^{-1}]].$$

Recall that $SW(M) = M \otimes_{\mathbb{C}[S_l]} \mathbb{C}(m|n)^{\otimes l}$.

$$P = \sum_{i,j=1}^{m+n} (-1)^{|j|} E_{ij} \otimes E_{ji}$$

So,

$$P_{k,l+1} = \sum_{i,j=1}^{m+n} (-1)^{|j|} \mathrm{id}_M \otimes \mathrm{id}^{\otimes k-1} \otimes E_{ij} \otimes \mathrm{id}^{\otimes l-k} \otimes E_{ji}$$

Here, the first tensor factor is in $\mathrm{End}(M)$, the last is in $M_{m|n}(\mathbb{C})$, and everything in between is in $\mathrm{End}(\mathbb{C}(m|n))^{\otimes l}$. In other words, $l+1$ is the position of the matrix space.

We need a representation of the super Yangian $Y(\mathfrak{gl}_{m|n})$ on $M \otimes_{\mathbb{C}} \mathbb{C}(m|n)^{\otimes l}$.

We use the mapping (4.1) above. What is to be verified is that the mapping chosen satisfies the RTT relation.

We know

$$R_{12}(v-u)R_{13}(w-u)R_{23}(w-v) = R_{23}(w-v)R_{13}(w-u)R_{12}(v-u)$$

so, applying t_1 to both sides of this equation gives

$$R_{13}^{t_1}(w-u)R_{12}^{t_1}(v-u)R_{23}(w-v) = R_{23}(w-v)R_{12}^{t_1}(w-u)R_{13}^{t_1}(v-u). \quad (4.2)$$

This follows from

$$\begin{aligned}((E_{ab}\otimes E_{ba}\otimes 1)\ (E_{cd}\otimes 1\otimes E_{dc}))^{t_1} &= (-1)^{|a|+|b|)(|c|+|d|)}(E_{ab}E_{cd})^{t_1}\otimes E_{ba}\otimes E_{dc}\\ &= E_{cd}^{t_1}E_{ab}^{t_1}\otimes E_{ba}\otimes E_{dc}\\ &= (E_{cd}\otimes 1\otimes E_{dc})^{t_1}(E_{ab}\otimes E_{ba}\otimes 1)^{t_1}\end{aligned}$$

Note that

$$\begin{aligned}R_{23}(w-v)R_{23}(v-w) &= \left(1-\frac{P}{w-v}\right)\left(1+\frac{P}{w-v}\right)\\ &= 1-\frac{1}{(w-v)^2}\end{aligned}$$

Since $R_{23}(w-v)^{-1} = R_{23}(v-w)\left(1-\frac{1}{(w-v)^2}\right)^{-1}$, multiplying (4.2) by $R_{23}(v-w)$ on both sides produces

$$R_{23}(v-w)R_{13}^{t_1}(w-u)R_{12}^{t_1}(v-u) = R_{12}^{t_1}(v-u)R_{13}^{t_1}(w-u)R_{23}(v-w) \quad (4.3)$$

In the tensor product

$$\mathrm{End}M\otimes_{\mathbb{C}}\mathrm{End}_{\mathbb{C}}(\mathbb{C}^{m|n})^{\otimes l}\otimes M_{m|n}\otimes M_{m|n}$$

we refer to the first factor as the 0th position, the next as the 1st to lth positions, and the last two in $l+1$, $l+2$ are necessary due to the RTT relations. As the $M_{m|n}$ are now in positions $l+1$ and $l+2$, R_{23} has to be replaced by $R_{l+1,l+2}$. Using (4.2),

$$\begin{aligned}
&R_{l+1,l+2}(u-v)R^{t_1}_{1,l+1}(\mathsf{x}_1-u)\cdots R^{t_l}_{l,l+1}(\mathsf{x}_l-u)R^{t_1}_{1,l+2}(\mathsf{x}_1-v)\cdots R^{t_l}_{l,l+2}(\mathsf{x}_l-v)\\
=&R_{l+1,l+2}(u-v)R^{t_1}_{1,l+1}(\mathsf{x}_1-u)R^{t_1}_{1,l+2}(\mathsf{x}_1-v)R^{t_2}_{2,l+1}(\mathsf{x}_2-v)\cdots R^{t_l}_{l,l+2}(\mathsf{x}_l-v)\\
=&R^{t_1}_{1,l+2}(\mathsf{x}_1-v)R^{t_1}_{1,l+1}(\mathsf{x}_1-u)R_{l+1,l+2}(u-v)R^{t_2}_{2,l+1}(\mathsf{x}_2-u)\cdots R^{t_l}_{l,l+2}(\mathsf{x}_l-v)\\
=&R^{t_1}_{1,l+2}(\mathsf{x}_1-v)R^{t_1}_{1,l+1}(\mathsf{x}_1-u)R^{t_2}_{2,l+2}(\mathsf{x}_2-v)\\
&R^{t_2}_{2,l+1}(\mathsf{x}_2-u)\cdots R^{t_l}_{l,l+1}(\mathsf{x}_l-u)R_{l+1,l+2}(u-v)\\
=&R^{t_1}_{1,l+2}(\mathsf{x}_1-v)\cdots R^{t_l}_{l,l+2}(\mathsf{x}_l-v)R^{t_1}_{1,l+1}(\mathsf{x}_1-u)\cdots R^{t_l}_{l,l+1}(\mathsf{x}_2-u)R_{l+1,l+2}(u-v)
\end{aligned}$$

Thus, we do have an action of the super Yangian on $M\otimes_{\mathbb{C}}\mathbb{C}(m|n)^{\otimes l}$ coming from (4.1) as the RTT relation is satisfied. We will eventually want an action over $M\otimes_{\mathbb{C}[S_l]}\mathbb{C}(m|n)^{\otimes l}$

$$R^{t_j}_{j,l+1}(\mathsf{x}_j-u)=\mathrm{Id}-\frac{P^t_{j,l+1}}{\mathsf{x}_j-u} \tag{4.4}$$

$$P=\sum_{a,b=1}^{m+n}(-1)^{|b|}E_{ab}\otimes E_{ba}=\sum_{a,b=1}^{m+n}(-1)^{|a|}E_{ba}\otimes E_{ab} \tag{4.5}$$

$$P^{t_1}=\sum_{a,b=1}^{m+n}(-1)^{|a|}(-1)^{|a|(|a|+|b|)}E_{ab}\otimes E_{ab}=\sum_{a,b=1}^{m+n}(-1)^{|a||b|}E_{ab}\otimes E_{ab} \tag{4.6}$$

Specifically, we use supertranspose here. This means $E^t_{ba}=(-1)^{|a|(|a|+|b|)}E_{ab}=(-1)^{|a|(1+|b|)}E_{ab}$.

Thus, we need to simplify (4.1) as in Proposition 5 in Arakawa [1] or Proposition 5.2 in Nazarov [19]. We verify those identities: $I_i{\cdot}I_k=P_{ik}{\cdot}I_k$ where $I_k=P^{t_k}_{k,l+1}$ for $P_{k,l+1}=\sum(-1)^{|j|}E^{(k)}_{ij}\otimes E^{(l+1)}_{ji}$ and P_{ik} is same as previously defined. The aim is to show that

$$P^{t_i}_{i,l+1}P^{t_k}_{k,l+1}=P_{ik}P^{t_k}_{k,l+1}. \tag{4.7}$$

It is enough to check this for three different indices, e.g. $i = 1, k = 2, l+1 = 3$.

$$\begin{aligned}
P_{13}^{t_1} P_{23}^{t_2} &= \sum_{a,b,c,d} \left((-1)^{|a||b|} E_{ba}^{(1)} \otimes 1 \otimes E_{ba}^{(3)} \right) \left((-1)^{|c||d|} 1 \otimes E_{dc}^{(2)} \otimes E_{dc}^{(3)} \right) \\
&= \sum_{a,b,c,d} (-1)^{|a||b|+|c||d|+(|a|+|b|)(|c|+|d|)} \delta_{ad} E_{ba}^{(1)} \otimes E_{dc}^{(2)} \otimes E_{bc}^{(3)} \\
&= \sum_{a,b,c} (-1)^{|a|+|b||c|} E_{ba}^{(1)} \otimes E_{ac}^{(2)} \otimes E_{bc}^{(3)}
\end{aligned}$$

On the other hand,

$$\begin{aligned}
P_{12} P_{23}^{t_2} &= \left(\sum_{a,d} (-1)^{|a|} E_{da}^{(1)} \otimes E_{ad}^{(3)} \otimes 1 \right) \left(\sum_{b,c} (-1)^{|b||c|} 1 \otimes E_{bc}^{(2)} \otimes E_{bc}^{(3)} \right) \\
&= \sum_{a,b,c,d} (-1)^{|a|+|b||c|} \delta_{bd} E_{da}^{(1)} \otimes E_{ac}^{(2)} \otimes E_{bc}^{(3)} \\
&= P_{13}^{t_1} P_{23}^{t_2}.
\end{aligned}$$

Proposition 4.1.1. *The difference between (4.1) and* $I + \sum_{k=1}^{l} \frac{1}{u-\mathsf{y}_k} P_{k,l+1}^{t_k}$ *as operators on* $M \otimes_{\mathbb{C}} \mathbb{C}(m|n)^{\otimes l}$*, when applied to* $m \otimes \underline{\mathbf{v}}$ *produces an element in* V*, where* V *is the subspace of* $M \otimes_{\mathbb{C}} \mathbb{C}(m|n)^{\otimes l}$ *spanned by* $\widetilde{m}\sigma \otimes \underline{\widetilde{\mathbf{v}}} - \kappa \widetilde{m} \otimes \sigma \underline{\widetilde{\mathbf{v}}}$ *for all* $\sigma \in S_l$*,* $\widetilde{m} \in M$*,* $\underline{\widetilde{\mathbf{v}}} \in \mathbb{C}(m|n)^{\otimes l}$*.*

$$T(u) - \left(I + \sum_{k=1}^{l} \frac{1}{u - \mathsf{y}_k} P_{k,l+1}^{t_k} \right) \tag{4.8}$$

is an operator on $M \otimes_{\mathbb{C}} \mathbb{C}(m|n)^{\otimes l}$ that, when composed with the quotient map $M \otimes_{\mathbb{C}} \mathbb{C}(m|n) \to SW(M)$, becomes 0, where $SW(M) := (M \otimes_{\mathbb{C}} \mathbb{C}(m|n)^{\otimes l})/V$. It thus descends to the zero operator $SW(M) \to SW(M)$. Moreover, $\sum_{k=1}^{l} \frac{1}{u-\mathsf{y}_k} P_{k,l+1}^{t_k}$ is a well-defined operator on $SW(M)$ since $\sigma_{ik} \mathsf{y}_k \sigma_{ik} = \mathsf{y}_i$, so we can say that $T(u)$ descends to an operator on $SW(M)$ equal to $I + \sum_{k=1}^{l} \frac{1}{u-\mathsf{y}_k} P_{k,l+1}^{t_k}$.

The goal is to show for all $k = 1, 2, \ldots, l$ that

$$\left(\mathrm{Id} + \frac{P_{1,l+1}^{t_1}}{u - \mathsf{x}_1} \right) \left(\mathrm{Id} + \frac{P_{2,l+1}^{t_2}}{u - \mathsf{x}_2} \right) \cdots \left(\mathrm{Id} + \frac{P_{k,l+1}^{t_k}}{u - \mathsf{x}_k} \right) \equiv \mathrm{Id} + \sum_{i=1}^{k} \frac{1}{u - \mathsf{y}_i} P_{i,l+1}^{t_i} \tag{4.9}$$

on $M \otimes_{\mathbb{C}} \mathbb{C}(m|n)^{\otimes l}$, where $\equiv$ represents congruence modulo V.

Proof of Proposition 4.1.1. We show (4.9) by induction. For $k = 1$, the two expressions are equal. Let $k > 1$. By induction hypothesis,

$$\left(\mathrm{Id} + \frac{P^{t_1}_{1,l+1}}{u - \mathsf{x}_1}\right)\left(\mathrm{Id} + \frac{P^{t_2}_{2,l+1}}{u - \mathsf{x}_2}\right)\cdots\left(\mathrm{Id} + \frac{P^{t_k}_{k,l+1}}{u - \mathsf{x}_k}\right) \tag{4.10}$$

$$\equiv \left(\mathrm{Id} + \sum_{i=1}^{k-1} \frac{1}{u - \mathsf{y}_i} P^{t_i}_{i,l+1}\right)\left(\mathrm{Id} + \frac{P^{t_k}_{k,l+1}}{u - \mathsf{x}_k}\right) \tag{4.11}$$

$$\equiv \mathrm{Id} + \sum_{i=1}^{k-1} \frac{1}{u - \mathsf{y}_i} P^{t_i}_{i,l+1} + \frac{P^{t_k}_{k,l+1}}{u - \mathsf{x}_k} + \sum_{i=1}^{k-1} \frac{1}{u - \mathsf{y}_i}\frac{1}{u - \mathsf{x}_k} P_{ik} P^{t_k}_{k,l+1} \tag{4.12}$$

$$\equiv \mathrm{Id} + \sum_{i=1}^{k-1} \frac{1}{u - \mathsf{y}_i} P^{t_i}_{i,l+1} + \left(\mathrm{Id} + \sum_{i=1}^{k-1} \frac{1}{u - \mathsf{y}_i}\sigma_{ik}\right)\frac{1}{u - \mathsf{x}_k} P^{t_k}_{k,l+1} \tag{4.13}$$

$$\equiv \mathrm{Id} + \sum_{i=1}^{k-1} \frac{1}{u - \mathsf{y}_i} P^{t_i}_{i,l+1} + \frac{1}{u - \mathsf{y}_k}\left(u - \mathsf{y}_k + \sum_{i=1}^{k-1} \sigma_{ik}\right)\frac{1}{u - \mathsf{x}_k} P^{t_k}_{k,l+1} \tag{4.14}$$

This applied the identity (4.7).

Note that because $\mathsf{y}_i\sigma_{ik} = \sigma_{ik}\mathsf{y}_k$, $\frac{1}{u-\mathsf{y}_i}\sigma_{ik} = \sigma_{ik}\frac{1}{u-\mathsf{y}_k}$.

$$\equiv \mathrm{Id} + \sum_{i=1}^{k-1} \frac{1}{u - \mathsf{y}_i} P^{t_i}_{i,l+1} + \frac{1}{u - \mathsf{y}_k}\left(u - \mathsf{x}_k\right)\frac{1}{u - \mathsf{x}_k} P^{t_k}_{k,l+1} \tag{4.15}$$

$$\equiv \mathrm{Id} + \sum_{i=1}^{k-1} \frac{1}{u - \mathsf{y}_i} P^{t_i}_{i,l+1} + \frac{1}{u - \mathsf{y}_k} P^{t_k}_{k,l+1} \tag{4.16}$$

□

So, as an operator on $SW(M)$, we can write that

$$T(u) = \sum_{k=1}^{l} \frac{1}{u - \mathsf{y}_k} P^{t_k}_{k,l+1}$$

(as an alternative to a product of R-matrices) which gives

$$t_{ab}^{(2)} = \sum_{k=1}^{l} (-1)^{|a||b|} \mathsf{y}_k \otimes E_{ab}^{(k)}$$

(coefficient for E_{ab})

To obtain a functor SW from right modules over $H^{deg}(S_l)$ to left modules over the super Yangian, we need to define $SW(\varphi)$ for any homomorphism $\varphi : M_1 \to M_2$ between two right $H^{deg}(S_l)$-modules: we can set $SW(\varphi) = \varphi \otimes id$ and this is a homomorphism $SW(M_1) \to SW(M_2)$.

4.2 Comparisons to the work of Chari-Pressley in showing an equivalence of categories

The goal is to extend Drinfeld's result [9] to the super setting, i.e. if we let $\tilde{M} \in \mathrm{mod}_L^l - Y(\mathfrak{sl}_{m|n})$, the goal is to show that there exists $M \in \mathrm{mod}_R - H_{\mathrm{aff}}^{\mathrm{deg}}(S_l)$ such that $\tilde{M} = SW(M)$.

Consider the quantum loop setting as considered by Chari and Pressley [3]. In this article, they describe a functor from the category of finite-dimensional representations of the affine Hecke algebra of S_l to the category of finite-dimensional representations of the quantum loop algebra $U_q(\mathcal{L}\mathfrak{sl}_n)$. In particular, if $l < n$, this functor is an equivalence of categories between all finite-dimensional representations of the affine Hecke algebra of S_l and a certain subcategory of representations of $U_q(\mathcal{L}\mathfrak{sl}_n)$. A goal will be to produce equivalents of Theorem 4.2 and Lemma 4.5 of Chari and Pressley [3] for the superalgebra case.

We will use the degenerate affine Hecke algebra case of Lemma 4.3 as well, which follows from Schur-Weyl duality for the Lie superalgebra $\mathfrak{gl}_{m|n}$.

The purpose of Lemma 4.5 is to extend a right module M over $\mathbb{C}[S_l]$ to a right module over $H_{\mathrm{aff}}^{\mathrm{deg}}(S_l)$. Take $\mathbf{m} \in M$, $\mathsf{x}_1, \ldots, \mathsf{x}_l$ polynomial generators of $H_{\mathrm{aff}}^{\mathrm{deg}}(S_l)$. The first goal is to determine what $\mathbf{m}\mathsf{x}_i$ is, as in Section 4.6 of Chari-Pressley [3].

Thus, suppose N is a left module over $Y(\mathfrak{sl}_{m|n})$, integrable of level l, in the sense that N is the direct sum of N_i, where each N_i is an irreducible representation of $\mathfrak{sl}_{m|n}$ contained in $\mathbb{C}(m|n)^{\otimes l}$. Note that we will use this condition for the DDCA case, due to the possibility of infinite dimensional modules there as well.

Theorem 4.2.1. *Let N be a $Y(\mathfrak{sl}_{m|n})$-module that is integrable of level l. Suppose that $m+n > l+1$. Then, there exists a module M over the degenerate affine Hecke algebra $H_\kappa^{\deg}(S_l)$ such that $N \cong M \otimes_{\mathbb{C}[S_l]} \mathbb{C}(m|n)^{\otimes l}$.*

By Schur-Weyl duality for $\mathfrak{sl}_{m|n}$ and the symmetric group S_l,

$$N = SW(M) = M \otimes_{\mathbb{C}[S_l]} \mathbb{C}(m|n)^{\otimes l}$$

where M is a right S_l-module.

The goal is to show that M is a right module over $H_{\text{aff}}^{\deg}(S_l)$.

Let $\{e_1, \dots, e_{m+n}\}$ be the natural basis of $\mathbb{C}(m|n)$. Let $1 \le k \le l$ and $1 \le a, b \le m+n$. Then, as in [3], we introduce the elements:

$$\underline{\mathbf{v}}^{(k)} = e_1 \otimes \cdots \otimes e_b \otimes \cdots \otimes e_l$$

$$\underline{\mathbf{w}}^{(k)} = e_1 \otimes \cdots \otimes e_a \otimes \cdots \otimes e_l$$

where no e_a appears in $\underline{\mathbf{v}}^{(k)}$ and the e_b is in the k-th tensor, and vice versa for $\underline{\mathbf{w}}^{(k)}$. Let $\mathfrak{h}$ be the usual Cartan subalgebra made up of diagonal matrices in $\mathfrak{sl}_{m|n}$. The $\mathfrak{h}$-weight of $\underline{\mathbf{v}}^{(k)}$ is $\varepsilon_1 + \cdots + \varepsilon_b + \cdots + \varepsilon_l$ while the $\mathfrak{h}$-weight of $t_{ab}^{(r)}(\mathbf{m} \otimes \underline{\mathbf{v}}^{(k)})$ (for $\mathbf{m} \in M$, thus $\mathbf{m} \otimes \underline{\mathbf{v}}^{(k)} \in N$) is $\varepsilon_a - \varepsilon_b + \varepsilon_1 + \cdots + \varepsilon_b + \cdots + \varepsilon_l = \varepsilon_1 + \cdots + \varepsilon_a + \cdots + \varepsilon_l$, which is the $\mathfrak{h}$-weight of $\underline{\mathbf{w}}^{(k)}$. We can associate ε_i in the summand of the weight with the appearance of e_i in the tensor product. Thus,

$$t_{ab}^{(r)}(\mathbf{m} \otimes \underline{\mathbf{v}}^{(k)}) = \sum \widetilde{\mathbf{m}} \otimes \widetilde{\underline{\mathbf{w}}}$$

for some $\widetilde{\mathbf{m}} \in M$ and where $\widetilde{\underline{\mathbf{w}}}$ has weight $\varepsilon_1 + \cdots + \varepsilon_a + \cdots + \varepsilon_l$. So, since permuting tensor factors does not affect weight, $\widetilde{\underline{\mathbf{w}}} = \tau(\underline{\mathbf{w}}^{(k)})$ for some $\tau \in S_l$. The action of τ is given by

$$\tau(\underline{\mathbf{w}}^{(k)}) = e_{\tau^{-1}(1)} \otimes \cdots \otimes e_{\tau^{-1}(a)} \otimes \cdots \otimes e_{\tau^{-1}(l)}.$$

Thus,

$$
\begin{aligned}
t_{ab}^{(r)}(\mathbf{m}\otimes\underline{\mathbf{v}}^{(k)}) &= \sum \widetilde{\mathbf{m}}\otimes\widetilde{\underline{\mathbf{w}}} \\
&= \sum_{\tau\in S_l} \widetilde{\mathbf{m}}_\tau \otimes \tau(\underline{\mathbf{w}}^{(k)}) \\
&= \sum_{\tau\in S_l} \underbrace{\widetilde{\mathbf{m}}_\tau \tau}_{m_\tau} \otimes \underline{\mathbf{w}}^{(k)} \\
&= \underbrace{\sum_{\tau\in S_l} \mathbf{m}_\tau}_{\mathbf{m}'} \otimes \underline{\mathbf{w}}^{(k)} \\
&= \mathbf{m}'\otimes\underline{\mathbf{w}}^{(k)}
\end{aligned}
$$

It follows from Lemma 4.3(a) [3] that we obtain a linear map $\mathbf{m}\mapsto\mathbf{m}'$; denote this map by $\chi_{k,r}^{ab}$. The next goal is to show that $\chi_{k,2}^{ab}$ does not depend on a, b, thus can be instead denoted $\chi_{k,2}$.

If $r = 2$, then $t_{ab}^{(2)}(\mathbf{m}\otimes\underline{\mathbf{v}}^{(k)}) = \chi_k^{ab}(\mathbf{m})\otimes\underline{\mathbf{w}}^{(k)}$, setting $\chi_k^{ab} = \chi_{k,2}^{ab}$.

Lemma 4.2.2. *(Equivalent of Lemma 4.3(b) [3])* $\underline{\mathbf{w}} = e_{i_1}\otimes\cdots\otimes e_{i_l}$ *for* i_j *all distinct generate all of* $\mathbb{C}(m|n)^{\otimes l}$*, i.e.* $\mathbb{C}(m|n)^{\otimes l} = U(\mathfrak{sl}_{m|n})\underline{\mathbf{w}}$.

Lemma 4.2.3. *(Equivalent of Lemma 4.5 [3]) For all* $\underline{\mathbf{v}}\in\mathbb{C}(m|n)^{\otimes l}$*,*

$$
t_{ab}^{(2)}(\mathbf{m}\otimes\underline{\mathbf{v}}) = \sum_{j=1}^{l} \chi_j^{ab}(\mathbf{m})\otimes E_{ab}^{(j)}(\underline{\mathbf{v}}). \tag{4.17}
$$

Proof. We have that $t_{ab}^{(2)}(\mathbf{m}\otimes\underline{\mathbf{v}}^{(k)}) = \chi_k^{ab}(\mathbf{m})\otimes\underline{\mathbf{w}}^{(k)} = \chi_k^{ab}(\mathbf{m})\otimes E_{ab}^{(k)}(\underline{\mathbf{v}}^{(k)})$. If $\underline{\mathbf{v}} = e_{i_1}\otimes\cdots\otimes e_{i_l}$, and none of the indices are b (no e_b appears), then $t_{ab}^{(2)}(m\otimes\underline{\mathbf{v}}) = 0$.

One way to see this would be to recognize that the weight of $t_{ab}^{(2)}(m\otimes\underline{\mathbf{v}})$ is the sum of the weights of $t_{ab}^{(2)}$ and $\underline{\mathbf{v}}$, so $\varepsilon_a - \varepsilon_b + \varepsilon_{i_1} + \cdots + \varepsilon_{i_l}$ (recall that the weight of $t_{ab}^{(2)}$ is $\varepsilon_a - \varepsilon_b$). However, the weight of $e_{j_1}\otimes\cdots\otimes e_{j_l}$ is $\varepsilon_{j_1} + \cdots + \varepsilon_{j_l}$; note that all coefficients are ≥ 0. The coefficient of ε_b in the weight of $t_{ab}^{(2)}(\mathbf{m}\otimes\underline{\mathbf{v}})$ is -1, since we have assumed that none of $i_1,\ldots,i_l$ are equal to b. So $\varepsilon_a - \varepsilon_b + \varepsilon_{i_1} + \cdots + \varepsilon_{i_l}$ is not a weight of $\mathbb{C}(m|n)^{\otimes l}$, so

$t^{(2)}_{ab}(\mathbf{m}\otimes\underline{\mathbf{v}}) = 0.$

Next, let $\underline{\mathbf{v}} = e_{i_1}\otimes\cdots\otimes e_{i_l}$ where at least one of $\{i_2,\ldots,i_l\}$ is b. Say

$$\underline{\mathbf{v}} = e_1\otimes\cdots\otimes\overbrace{e_a}^{j_1}\otimes\cdots\cdots\cdots\cdots\otimes\overbrace{e_a}^{j_2}\otimes\cdots\otimes\overbrace{e_a}^{j_3}\otimes\cdots\cdots\cdots$$
$$\underline{\mathbf{v}} = e_1\otimes\cdots\cdots\cdots\cdots\otimes\underbrace{e_b}_{j'_1}\otimes\cdots\cdots\cdots\cdots\cdots\cdots\cdots\otimes\underbrace{e_b}_{j'_2}\otimes\cdots\otimes\underbrace{e_b}_{j'_3}\otimes\cdots$$

Entries other than the $j_1,\ldots,j_r$ and $j'_1,\ldots,j'_s$ are any basis vectors other than $e_a$and e_b.

By Lemma 4.2.2, it is enough to consider the case when all the other vectors in $\underline{\mathbf{v}}$ (not in positions $j_1,\ldots,j_r$ and $j'_1,\ldots,j'_s$) are all distinct. We can prove the formula (4.17) by double induction on r and s.

If $r = 0$, $s = 1$, no e_a appear and there is one e_b in position j'_1. (4.17) follows from the definition of χ.

Assume the formula holds for $\underline{\mathbf{v}}$ with $s = 1$ and up to $r-1$ of e_a. Choose $c \neq a, b$. Let

$$\underline{\mathbf{v}}' = e_1\otimes\cdots\otimes\underbrace{e_a}_{j_1}\otimes\cdots\otimes\underbrace{e_b}_{j'_1}\otimes\cdots\otimes\underbrace{e_a}_{j_2}\otimes\cdots\otimes\underbrace{e_a}_{j_{r-1}}\otimes\cdots\otimes\underbrace{e_c}_{j_r}\otimes\cdots$$

and suppose all other vectors in the tensor factors are from the set $\{e_1,\ldots,e_{m+n}\}-\{e_a,e_b,e_c\}$. Denote by $|\underline{\mathbf{v}}' < j_r|$ the parity of the part of $\underline{\mathbf{v}}'$ up to (and not including) the j_r-th position. Then, $\underline{\mathbf{v}} = (-1)^{(|a|+|c|)|\underline{\mathbf{v}}'<j_r|}E_{ac}\underline{\mathbf{v}}'$.

Since $E_{ac} = t^{(1)}_{ac}$ and $[t^{(2)}_{ab}, E_{ac}] = 0$ by the relations of the Yangian,

$$\begin{aligned}
t^{(2)}_{ab}(\mathbf{m}\otimes\underline{\mathbf{v}}) &= (-1)^{(|a|+|c|)|\underline{\mathbf{v}}'<j_r|}t^{(2)}_{ab}E_{ac}(\mathbf{m}\otimes\underline{\mathbf{v}}') \\
&= (-1)^{(|a|+|c|)|\underline{\mathbf{v}}'<j_r|}(-1)^{(|a|+|b|)(|a|+|c|)}E_{ac}t^{(2)}_{ab}(\mathbf{m}\otimes\underline{\mathbf{v}}') \\
&= (-1)^{(|a|+|c|)(|a|+|b|+|\underline{\mathbf{v}}'<j_r|)}E_{ac}t^{(2)}_{ab}(\mathbf{m}\otimes\underline{\mathbf{v}}').
\end{aligned}$$

By induction on r,

$$\begin{aligned}
E_{ac}t^{(2)}_{ab}(\mathbf{m}\otimes\underline{\mathbf{v}}') &= E_{ac}(\chi^{ab}_{j'_1}(\mathbf{m})\otimes E^{(j'_1)}_{ab}(\underline{\mathbf{v}}')) \\
&= (-1)^{(|a|+|b|)|\underline{\mathbf{v}}'<j'_1|}E_{ac}(\chi^{ab}_{j'_1}(\mathbf{m})\otimes\underline{\mathbf{v}}'')
\end{aligned}$$

$$= (-1)^{(|a|+|b|)|\underline{\mathbf{v}}'<j_1'|}\chi^{ab}_{j_1'}(\mathbf{m}) \otimes E_{ac}(\underline{\mathbf{v}}'')$$
$$= (-1)^{(|a|+|b|)|\underline{\mathbf{v}}'<j_1'|}(-1)^{(|a|+|c|)|\underline{\mathbf{v}}''<j_r|}\chi^{ab}_{j_1'}(\mathbf{m}) \otimes \underline{\mathbf{v}}'''$$
$$= (-1)^{(|a|+|c|)|\underline{\mathbf{v}}''<j_r|}\chi^{ab}_{j_1'}(\mathbf{m}) \otimes E^{(j_1')}_{ab}(\underline{\mathbf{v}})$$

where

$$\underline{\mathbf{v}}'' = \cdots \otimes \cdots \otimes e_a \otimes \cdots \otimes \underbrace{e_a}_{j_1'} \otimes \cdots \otimes e_a \otimes \cdots \otimes e_c$$

(e_b in j_1' position replaced with e_a, so $\underline{\mathbf{v}} = (-1)^{(|a|+|b|)|\underline{\mathbf{v}}''<j_1'|}E_{ab}\underline{\mathbf{v}}''$) and

$$\underline{\mathbf{v}}''' = \cdots \otimes \cdots \otimes e_a \otimes \cdots \otimes e_a \otimes \cdots \otimes e_a \otimes \cdots \otimes e_a$$

(i.e. $\underline{\mathbf{v}}''' = (-1)^{(|a|+|c|)|\underline{\mathbf{v}}''<j_r|}E_{ac}(\underline{\mathbf{v}}'')$ and $\underline{\mathbf{v}}''' = (-1)^{(|a|+|b|)|\underline{\mathbf{v}}<j_1'|}E_{ab}(\underline{\mathbf{v}})$). The signs cancel out to yield

$$t_{ab}(\mathbf{m} \otimes \underline{\mathbf{v}}) = \chi^{ab}_{j_1'}(\mathbf{m}) \otimes E^{(j_1')}_{ab}(\underline{\mathbf{v}}).$$

Now, we proceed to the proof by induction on s. The case for $s = 1$ is addressed above.

Assume that the result holds for all $\underline{\mathbf{v}}$ with fewer than s e_b tensor factors. Pick $c \neq a, b$. Fix the other entries of $\underline{\mathbf{v}}$ as distinct from $\{e_1, \ldots, e_{m+n}\} - \{e_a, e_b, e_c\}$, $m + n \geq l + 1$.

$$\underline{\mathbf{v}}' = \cdots \otimes \underbrace{e_a}_{j_1} \otimes \cdots \otimes \underbrace{e_a}_{j_2} \otimes \cdots \otimes \underbrace{e_c}_{j_1'} \otimes \cdots \otimes \underbrace{e_a}_{j_3} \otimes \cdots \otimes \underbrace{e_c}_{j_2'} \otimes \cdots \otimes \underbrace{e_b}_{j_3'} \otimes \cdots$$

Then, $\underline{\mathbf{v}} = (-1)^{(|b|+|c|)|j_1' \leq \underline{\mathbf{v}}' < j_2'|}\frac{1}{2}E^2_{bc}(\underline{\mathbf{v}}')$; that coefficient of $\frac{1}{2}$ is necessary as E_{bc} can be applied in two tensor factors. Recall that $E_{bc} = t^{(1)}_{bc}$. When a, b, c all distinct, $[E_{bc}, [E_{bc}, E_{ab}]] = 0 = [t^{(1)}_{bc}, [t^{(1)}_{bc}, t^{(1)}_{ab}]]$. Further, $[t^{(1)}_{bc}, [t^{(1)}_{bc}, t^{(2)}_{ab}]] = 0$. This expands to

$$0 = (t^{(1)}_{bc})^2 t^{(2)}_{ab} - ((-1)^{(|b|+|c|)(|a|+|b|)} + (-1)^{(|b|+|c|)(|a|+|b|+|b|+|c|)})t^{(1)}_{bc}t^{(2)}_{ab}t^{(1)}_{bc}$$
$$+ (-1)^{|b|+|c|)}t^{(2)}_{ab}(t^{(1)}_{bc})^2$$
$$= (t^{(1)}_{bc})^2 t^{(2)}_{ab} - ((-1)^{(|b|+|c|)(|a|+|b|)} + (-1)^{(|b|+|c|)(|a|+|c|)})t^{(1)}_{bc}t^{(2)}_{ab}t^{(1)}_{bc}$$

$$+ (-1)^{|b|+|c|)} t_{ab}^{(2)} (t_{bc}^{(1)})^2$$

If we pick b, c to have the same parity, requiring $m, n \geq l+1$, then the coefficient of the middle term will be $-((-1)^0 + (-1)^0) = -2$. Moreover, $|b| + |c|$ is now even, so the signs above involving $(-1)^{|b|+|c|)}$ all become 1.

Then

$$\begin{aligned} t_{ab}^{(2)}(\mathbf{m} \otimes \underline{\mathbf{v}}) &= \frac{1}{2} t_{ab}^{(2)} (t_{bc}^{(1)})^2 (m \otimes \underline{\mathbf{v}}') \\ &= \left(t_{bc}^{(1)} t_{ab}^{(2)} t_{bc}^{(1)} - \frac{1}{2} (t_{bc}^{(1)})^2 t_{ab}^{(2)} \right) (m \otimes \underline{\mathbf{v}}') \end{aligned}$$

$\underline{\mathbf{v}}'$ has fewer e_b, as does $t_{bc}^{(1)}(\underline{\mathbf{v}}')$, so we can apply induction.

Thus,

$$\begin{aligned} (t_{bc}^{(1)})^2 t_{ab}^{(2)} (\mathbf{m} \otimes \underline{\mathbf{v}}') &= (t_{bc}^{(1)})^2 \sum_{k=1}^{l} \chi_k^{ab}(\mathbf{m}) \otimes E_{ab}^{(k)}(\underline{\mathbf{v}}') \\ &= 2 \sum_{t \geq 3} \chi_{j'_t}^{ab}(\mathbf{m}) \otimes E_{ab}^{(j'_t)}(\underline{\mathbf{v}}). \end{aligned}$$

Let $\underline{\mathbf{v}}''$ be $\underline{\mathbf{v}}'$ but with the e_c in the j'_1 position replaced with e_b. Let $\underline{\mathbf{v}}'''$ be the same as $\underline{\mathbf{v}}'$ but with the e_c in the j'_2 position replaced with e_b.

$t_{bc}^{(1)}(\underline{\mathbf{v}}') = \underline{\mathbf{v}}'' + \underline{\mathbf{v}}'''$.By induction on s,

$$\begin{aligned} &t_{ab}^{(2)}(\underline{\mathbf{v}}'' + \underline{\mathbf{v}}''') \\ &\quad = (-1)^{|b|+|c|} \left(\sum_{t \geq 1, t \neq 2} \chi_{j'_t}^{ab}(\mathbf{m}) \otimes E_{ab}^{(j'_t)}(\underline{\mathbf{v}}'') + \sum_{t \geq 2} \chi_{j'_t}^{ab}(\mathbf{m}) \otimes E_{ab}^{(j'_t)}(\underline{\mathbf{v}}''') \right). \end{aligned}$$

Apply $t_{bc}^{(1)} = E_{bc}$ again:

$$t_{bc}^{(1)} \left(\sum_{t \geq 1, t \neq 2} \chi_{j'_t}^{ab}(\mathbf{m}) \otimes E_{ab}^{(j'_t)}(\underline{\mathbf{v}}'') \right) = \sum_{t \geq 1, t \neq 2} \chi_{j'_t}^{ab}(\mathbf{m}) \otimes E_{ab}^{(j'_t)}(\underline{\mathbf{v}})$$

$$t_{bc}^{(1)}\left(\sum_{t\geq 2}\chi_{j'_t}^{ab}(\mathbf{m})\otimes E_{ab}^{(j'_t)}(\underline{\mathbf{v}}''')\right)=\sum_{t\geq 2}\chi_{j'_t}^{ab}(\mathbf{m})\otimes E_{ab}^{(j'_t)}(\underline{\mathbf{v}})$$

Thus,

$$t_{ab}^{(2)}(\mathbf{m}\otimes\underline{\mathbf{v}})=\sum_{t\geq 1,t\neq 2}\chi_{j'_t}^{ab}(\mathbf{m})\otimes E_{ab}^{(j'_t)}(\underline{\mathbf{v}})+\sum_{t\geq 2}\chi_{j'_t}^{ab}(\mathbf{m})\otimes E_{ab}^{(j'_t)}(\underline{\mathbf{v}})-\sum_{t\geq 3}\chi_{j'_t}^{ab}(\mathbf{m})\otimes E_{ab}^{(j'_t)}(\underline{\mathbf{v}})$$
$$=\sum_{t\geq 1}\chi_{j'_t}^{ab}(\mathbf{m})\otimes E_{ab}^{(j'_t)}(\underline{\mathbf{v}})$$

□

We show next that $\chi_k^{ab}=\chi_k^{cd}$ for all a,b,c,d. To do this, first suppose $c\neq a,b$. Then, $E_{cb}=[E_{ca},E_{ab}]$ and $t_{cb}^{(2)}=[t_{ca}^{(1)},t_{ab}^{(2)}]$.

$$\begin{aligned}t_{cb}^{(2)}(\mathbf{m}\otimes\underline{\mathbf{v}})&=\sum_{k=1}^{l}\chi_k^{cb}(\mathbf{m})\otimes E_{cb}^{(k)}(\underline{\mathbf{v}})\\&=[t_{ca}^{(1)},t_{ab}^{(2)}](\mathbf{m}\otimes\underline{\mathbf{v}})\\&=\sum_{k=1}^{l}\chi_k^{ab}(\mathbf{m})\otimes[E_{ca},E_{ab}^{(k)}](\underline{\mathbf{v}})\\&=\sum_{k=1}^{l}\chi_k^{ab}(\mathbf{m})\otimes E_{cb}^{(k)}(\underline{\mathbf{v}})\end{aligned}$$

This applies over all $\mathbf{m}$ and $\underline{\mathbf{v}}$. Thus, $\chi_k^{ab}=\chi_k^{cb}$. Similarly, $\chi_k^{cb}=\chi_k^{cd}$. So, $\chi_k^{ab}=\chi_k^{cd}$. Call the map simply χ_k now.

Next, we show that M is a right $H_\kappa^{\deg}(S_l)$-module.

To define the action of the generators y_i on M, set $\mathbf{m}\mathsf{y}_i=\chi_i(\mathbf{m})$. We use that $t_{ab}^{(2)}(\mathbf{m}\otimes\underline{\mathbf{v}})=\sum_{k=1}^{l}(-1)^{|a||b|}\mathbf{m}.\mathsf{y}_k\otimes E_{ab}^{(k)}(\underline{\mathbf{v}})$ to produce the relations on the y_i generators.

Let's compute $[t_{ab}^{(2)},t_{cd}^{(2)}].(\mathbf{m}\otimes\underline{\mathbf{v}})$ by using first the RTT relation in $Y(\mathfrak{gl}_{m|n})$. Assume that $b\neq c$ and $a\neq d$.

We use that $t_{ab}^{(2)}(\mathbf{m}\otimes\underline{\mathbf{v}})=\sum_{k=1}^{l}\mathbf{m}.\mathsf{y}_k\otimes E_{ab}^{(k)}(\underline{\mathbf{v}})$.

$$
\begin{aligned}
&[t_{ab}^{(2)},t_{cd}^{(2)}].(\mathbf{m}\otimes\underline{\mathbf{v}})\\
&\quad=t_{ab}^{(2)}\left(\sum_{j=1}^{l}\mathbf{m}.y_j\otimes E_{cd}^{(j)}(\underline{\mathbf{v}})\right)-(-1)^{(|a|+|b|)(|c|+|d|)}t_{cd}^{(2)}\left(\sum_{i=1}^{l}\mathbf{m}.\mathsf{y}_i\otimes E_{ab}^{(i)}(\underline{\mathbf{v}})\right)\\
&\quad=\sum_{i,j=1}^{l}\mathbf{m}.\mathsf{y}_j\mathsf{y}_i\otimes E_{ab}^{(i)}E_{cd}^{(j)}(\underline{\mathbf{v}})\\
&\qquad-(-1)^{(|a|+|b|)(|c|+|d|)}\sum_{i,j=1}^{l}\mathbf{m}.\mathsf{y}_i\mathsf{y}_j\otimes E_{cd}^{(j)}E_{ab}^{(i)}(\underline{\mathbf{v}})
\end{aligned}
\tag{4.18}
$$

$$
\begin{aligned}
&[t_{ab}^{(2)},t_{cd}^{(2)}].(\mathbf{m}\otimes\underline{\mathbf{v}})\\
&\quad=(-1)^{|a||b|+|a||c|+|b||c|}\left(t_{cb}^{(0)}t_{ad}^{(3)}-t_{cb}^{(3)}t_{ad}^{(0)}+t_{cb}^{(1)}t_{ad}^{(2)}-t_{cb}^{(2)}t_{ad}^{(1)}\right).(\mathbf{m}\otimes\underline{\mathbf{v}})\\
&\quad=(-1)^{|a||b|+|a||c|+|b||c|}\left(t_{cb}^{(1)}t_{ad}^{(2)}-t_{cb}^{(2)}t_{ad}^{(1)}\right).(\mathbf{m}\otimes\underline{\mathbf{v}})\\
&\quad=(-1)^{|a||b|+|a||c|+|b||c|}\left(t_{cb}^{(1)}\sum_{j=1}^{l}\mathbf{m}.\mathsf{y}_j\otimes E_{ad}^{(j)}(\underline{\mathbf{v}})-t_{cb}^{(2)}\sum_{i=1}^{l}\mathbf{m}\otimes E_{ad}^{(i)}(\underline{\mathbf{v}})\right)\\
&\quad=(-1)^{|a||b|+|a||c|+|b||c|}\left(\sum_{i,j=1}^{l}\mathbf{m}.\mathsf{y}_j\otimes E_{cb}^{(i)}E_{ad}^{(j)}(\underline{\mathbf{v}})\right.\\
&\qquad\left.-\sum_{i,j=1}^{l}\mathbf{m}.\mathsf{y}_j\otimes E_{cb}^{(j)}E_{ad}^{(i)}s(\underline{\mathbf{v}})\right)
\end{aligned}
\tag{4.19}
$$

Recall that $E_{ab}^{(k)}(\underline{\mathbf{v}})=(-1)^{(|a|+|b|)(|i_1|+\cdots+|i_{k-1}|)}(e_{i_1}\otimes\cdots\otimes e_{i_{k-1}}\otimes E_{ab}(e_{i_k})\otimes\cdots\otimes e_{i_l})$.

We choose $\underline{\mathbf{v}}$ to be

$$\underline{\mathbf{v}}=e_{i_1}\otimes\cdots\otimes e_d\otimes\cdots\otimes e_b\otimes\cdots\otimes e_{i_l}$$

with e_d in k_1th position, e_b in k_2th position, and with the other $i_1,\ldots,i_l\neq b,d$, which is possible since $m+n>l+2$.

The goal of the Chari-Pressley proof was to show that the relations of the affine Hecke algebra hold when applied to the module M, extending its Hecke algebra module structure.

In my case, I need to check the relations of the degenerate affine Hecke algebra.

We see that (4.18) is nonzero if $i = k_2$ and $j = k_1$. We get:

$$\begin{aligned}
&[t^{(2)}_{ab}, t^{(2)}_{cd}].(\mathbf{m} \otimes \underline{\mathbf{v}}) \\
&= \sum_{i,j=1}^{l} \mathbf{m}.y_j y_i \otimes E^{(i)}_{ab} E^{(j)}_{cd}(\underline{\mathbf{v}}) \\
&\quad - (-1)^{(|a|+|b|)(|c|+|d|)} \sum_{i,j=1}^{l} \mathbf{m}.\mathsf{y}_i \mathsf{y}_j \otimes E^{(j)}_{cd} E^{(i)}_{ab}(\underline{\mathbf{v}}) \\
&= \mathbf{m}.\mathsf{y}_{k_1} \mathsf{y}_{k_2} \otimes E^{(k_2)}_{ab} E^{(k_1)}_{cd}(\underline{\mathbf{v}}) \\
&\quad - (-1)^{(|a|+|b|)(|c|+|d|)} \mathbf{m}.\mathsf{y}_{k_2} \mathsf{y}_{k_1} \otimes E^{(k_1)}_{cd} E^{(k_2)}_{ab}(\underline{\mathbf{v}}) \\
&= \mathbf{m}.\mathsf{y}_{k_1} \mathsf{y}_{k_2} \otimes E^{(k_2)}_{ab} E^{(k_1)}_{cd}(\underline{\mathbf{v}}) \\
&\quad - \mathbf{m}.\mathsf{y}_{k_2} \mathsf{y}_{k_1} \otimes E^{(k_2)}_{ab} E^{(k_1)}_{cd}(\underline{\mathbf{v}}) \\
&= \mathbf{m}.(\mathsf{y}_{k_1} \mathsf{y}_{k_2} - \mathsf{y}_{k_2} \mathsf{y}_{k_1}) \otimes E^{(k_2)}_{ab} E^{(k_1)}_{cd}(\underline{\mathbf{v}}) \\
&= \mathbf{m}.[\mathsf{y}_{k_1}, \mathsf{y}_{k_2}] \otimes E^{(k_2)}_{ab} E^{(k_1)}_{cd}(\underline{\mathbf{v}}) \qquad (4.20)
\end{aligned}$$

For (4.19), first swap i and j indices in the second sum then set $i = k_2$ and $j = k_1$.

$$\begin{aligned}
&[t^{(2)}_{ab}, t^{(2)}_{cd}].(\mathbf{m} \otimes \underline{\mathbf{v}}) \\
&= (-1)^{|a||b|+|a||c|+|b||c|} \left(\sum_{i,j=1}^{l} \mathbf{m}.\mathsf{y}_j \otimes E^{(i)}_{cb} E^{(j)}_{ad}(\underline{\mathbf{v}}) - \sum_{i,j=1}^{l} \mathbf{m}.\mathsf{y}_i \otimes E^{(i)}_{cb} E^{(j)}_{ad}(\underline{\mathbf{v}}) \right) \\
&= (-1)^{|a||b|+|a||c|+|b||c|} \left(\mathbf{m}.\mathsf{y}_{k_1} \otimes E^{(k_2)}_{cb} E^{(k_1)}_{ad}(\underline{\mathbf{v}}) - \mathbf{m}.\mathsf{y}_{k_2} \otimes E^{(k_2)}_{cb} E^{(k_1)}_{ad}(\underline{\mathbf{v}}) \right) \\
&= (-1)^{|a||b|+|a||c|+|b||c|} \left((\mathbf{m}.\mathsf{y}_{k_1} - \mathbf{m}.\mathsf{y}_{k_2}) \otimes E^{(k_2)}_{cb} E^{(k_1)}_{ad}(\underline{\mathbf{v}}) \right)
\end{aligned}$$

Note that

$$\begin{aligned}
\sigma_{k_1,k_2} E^{(k_2)}_{ab} E^{(k_1)}_{cd}(\underline{\mathbf{v}}) &= (-1)^{|a|} E^{(k_2)}_{ca} E^{(k_1)}_{ac} E^{(k_2)}_{ab} E^{(k_1)}_{cd}(\underline{\mathbf{v}}) \\
&= (-1)^{|a||b|+|b||c|+|a||c|} E^{(k_2)}_{cb} E^{(k_1)}_{ad}(\underline{\mathbf{v}})
\end{aligned}$$

(one can simply look at the indices of the resulting tensor product). Thus,

$$\begin{aligned}(-1)^{|a||b|+|a||c|+|b||c|}\left((\mathbf{m}.\mathsf{y}_{k_1}-\mathbf{m}.\mathsf{y}_{k_2})\otimes E_{cb}^{(k_2)}E_{ad}^{(k_1)}(\underline{\mathbf{v}})\right)\\ =\mathbf{m}.\left(\mathsf{y}_{k_1}-\mathsf{y}_{k_2}\right)\sigma_{k_1,k_2}\otimes E_{ab}^{(k_2)}E_{cd}^{(k_1)}(\underline{\mathbf{v}})\end{aligned} \tag{4.21}$$

Together with Lemma 4.2.2, (4.20) and (4.21) imply the relation $[y_{k_1}, y_{k_2}] = (y_{k_1} - y_{k_2})\sigma_{k_1,k_2}$ as desired.

Another relation to check is that $\sigma\mathsf{y}_i = \mathsf{y}_{\sigma(i)}\sigma$ for all $\sigma \in S_l$.

To do this, we need some $\underline{\mathbf{v}}$ satisfying Lemma 4.2.2 to show that $\mathbf{m}.\sigma\mathsf{y}_i = \mathbf{m}.\mathsf{y}_{\sigma(i)}\sigma$ for all $\mathbf{m} \in M$ and all $\sigma \in S_l$.

Let $\underline{\mathbf{v}}$ have e_b in the ith position with no other tensor factors containing e_b. $\widetilde{\underline{\mathbf{v}}}$ is the same with e_a in ith position. $\underline{\mathbf{v}}'$ is $\underline{\mathbf{v}}$ with e_b in the $\sigma(i)$th position and all other tensor factors manipulated as given by σ. $\widetilde{\underline{\mathbf{v}}}'$ is $\widetilde{\underline{\mathbf{v}}}$ with e_a in the $\sigma(i)$th position and all other tensor factors manipulated as given by σ. Thus, $\widetilde{\underline{\mathbf{v}}} = (-1)^{(|a|+|b|)|\underline{\mathbf{v}}[1,-1]|}$ where $|\underline{\mathbf{v}}[1, i-1]| = |v_1| + \cdots + |v_{i-1}|$ for clarity if $\underline{\mathbf{v}} = v_1 \otimes v_2 \otimes \cdots$.

$$\begin{aligned}\mathbf{m}.\sigma\mathsf{y}_i \otimes \widetilde{\underline{\mathbf{v}}} &= (-1)^{|a||b|}t_{ab}^{(2)}(\mathbf{m}.\sigma \otimes \underline{\mathbf{v}})\\ &= \sum_{k=1}^{l}(-1)^{|a||b+(|a|+|b|)|\underline{\mathbf{v}}[1,i-1]|}\mathbf{m}.\sigma\mathsf{y}_k \otimes E_{ab}^{(k)}(\underline{\mathbf{v}})\end{aligned} \tag{4.22}$$

This is tensored over $\mathbb{C}[S_l]$, so

$$\begin{aligned}t_{ab}^{(2)}(\mathbf{m}.\sigma \otimes \underline{\mathbf{v}}) &= t_{ab}^{(2)}(\mathbf{m} \otimes \sigma\underline{\mathbf{v}})\\ &= t_{ab}^{(2)}((-1)^{(|a|+|b|)|\underline{\mathbf{v}}[i+1.\sigma(i)-1]|}\mathbf{m} \otimes \underline{\mathbf{v}}`)\\ &= \sum_{k=1}^{l}((-1)^{|a||b|+(|a|+|b|)|\underline{\mathbf{v}}[i+1.\sigma(i)-1]|}\mathbf{m}.\mathsf{y}_k \otimes E_{ab}^{(k)}(\underline{\mathbf{v}}`)\\ &= ((-1)^{|a||b|+(|a|+|b|)|\underline{\mathbf{v}}[i+1.\sigma(i)-1]|}\mathbf{m}.\mathsf{y}_{\sigma(i)} \otimes \widetilde{\underline{\mathbf{v}}}`\\ &= \mathbf{m}.\mathsf{y}_{\sigma(i)} \otimes \sigma\widetilde{\underline{\mathbf{v}}}\\ &= \mathbf{m}.\mathsf{y}_{\sigma(i)}\sigma \otimes \widetilde{\underline{\mathbf{v}}}\end{aligned} \tag{4.23}$$

Together, (4.22) and (4.23) imply $\sigma \mathsf{y}_i = \mathsf{y}_{\sigma(i)}\sigma$. Then, the proof that $N \simeq SW(M)$ as $Y(\mathfrak{gl}_{m|n})$-modules and where M is a right module over the degenerate affine Hecke algebra of S_l is now complete.

Let $f : M_1 \to M_2$ be a homomorphism of right modules over $H_\kappa^{\deg}(S_l)$. Then, $SW(f) = f \otimes 1 : M_1 \otimes_{\mathbb{C}[S_l]} \mathbb{C}(m|n)^{\otimes l} \to M_2 \otimes_{\mathbb{C}[S_l]} \mathbb{C}(m|n)^{\otimes l}$ is a homomorphism of left $Y(\mathfrak{gl}_{m|n})$-modules.

$$SW : \mathrm{Hom}_{H_\kappa^{\deg}(S_l)}(M_1, M_2) \to \mathrm{Hom}_{Y(\mathfrak{gl}_{m|n})}(SW(M_1), SW(M_2))$$

We show this is a bijection. First, injectivity follows from that if $SW(f) = 0$, then $f \otimes 1 = 0$ so $f = 0$. Second, for surjectivity, we take $\varphi \in \mathrm{Hom}_{Y(\mathfrak{gl}_{m|n})}(SW(M_1), SW(M_2))$. Then, φ is a homomorphism of $\mathfrak{gl}_{m|n}$-modules as well. So, $\varphi = f \otimes 1$ where f is a homomorphism of right S_l-modules by classical SW-duality for S_l.

Further, it is a homomorphism over the degenerate affine Hecke algebra. This is equivalent to taking $\mathsf{y}_i \in H_\kappa^{\deg}(S_l)$ and showing that $f(\mathbf{m}\mathsf{y}_i) = f(\mathbf{m})\mathsf{y}_i$ for any i.

$l < m + n$, so take $e_1 \otimes \cdots \otimes e_{i-1} \otimes e_{m+n} \otimes e_{i+1} \otimes \cdots \otimes e_l$ where e_{m+n} only appears in the e_i tensor factor.

$$\begin{aligned} &\mathbf{m}\mathsf{y}_i \otimes e_1 \otimes \cdots \otimes e_{i-1} \otimes e_{m+n} \otimes e_{i+1} \otimes \cdots \otimes e_l \\ &\quad = t^{(2)}_{m+n,i}(\mathbf{m} \otimes e_1 \otimes \cdots \otimes e_{i-1} \otimes e_i \otimes e_{i+1} \otimes e_l \end{aligned}$$

So, we can write

$$\begin{aligned} f(\mathbf{m}\mathsf{y}_i) &\otimes e_1 \otimes \cdots \otimes e_{i-1} \otimes e_{m+n} \otimes e_{i+1} \otimes \cdots \otimes e_l \\ &= (f \otimes 1)m\mathsf{y}_i \otimes e_1 \otimes \cdots \otimes e_{i-1} \otimes e_{m+n} \otimes e_{i+1} \otimes \cdots \otimes e_l \\ &= \varphi(m\mathsf{y}_i \otimes e_1 \otimes \cdots \otimes e_{i-1} \otimes e_{m+n} \otimes e_{i+1} \otimes \cdots \otimes e_l) \\ &= \varphi(t^{(2)}_{m+n,i}(m \otimes e_1 \otimes \cdots \otimes e_{i-1} \otimes e_i \otimes e_{i+1} \otimes \cdots \otimes e_l)) \\ &= t^{(2)}_{m+n,i}(\varphi(m \otimes e_1 \otimes \cdots \otimes e_{i-1} \otimes e_i \otimes e_{i+1} \otimes \cdots \otimes e_l)) \\ &= t^{(2)}_{m+n,i}(f \otimes 1)(m \otimes e_1 \otimes \cdots \otimes e_{i-1} \otimes e_i \otimes e_{i+1} \otimes \cdots \otimes e_l) \end{aligned}$$

$$= t^{(2)}_{m+n,i}(f(m) \otimes e_1 \otimes \cdots \otimes e_l)$$
$$= f(m)\mathsf{y}_i \otimes e_1 \otimes \cdots \otimes e_{i-1} \otimes e_{m+n} \otimes e_{i+1} \otimes \cdots \otimes e_l$$

So, $f(\mathbf{m}\mathsf{y}_i)\otimes\underline{\mathbf{v}} = f(m)\mathsf{y}_i\otimes\underline{\mathbf{v}}$. Due to the choice of $\underline{\mathbf{v}}$, we have $f(\mathbf{m}\mathsf{y}_i) = f(\mathbf{m})\mathsf{y}_i$ as desired.

Chapter 5

Schur-Weyl Duality for the Deformed Double Current Superalgebra

The goal of this chapter is to introduce new quantum superalgebras $\mathcal{D}_{t,\kappa}(\mathfrak{sl}_{m|n})$ of double affine type. This is done by deforming the defining relations of the UCE of $\mathfrak{sl}_{m|n}(\mathbb{C}[u,v])$ given in Proposition 3.1.4 in order to produce a Schur-Weyl functor from $\mathsf{H}_{t,\kappa}(S_l)-\mathrm{mod}$ to $\mathcal{D}_{t,\kappa}(\mathfrak{sl}_{m|n})-\mathrm{mod}$. The generators $\widetilde{F}_{ab}(u)$ and $\widetilde{F}_{ab}(v)$ of $\mathfrak{st}_{m|n}$ in Proposition 3.1.4 should be equal to the generators $\mathsf{K}(E_{ab})$ and $\mathsf{Q}(E_{ab})$ of $\mathcal{D}_{t,\kappa}(\mathfrak{sl}_{m|n})$.

Guay constructed a Schur-Weyl functor between deformed double current algebras and rational Cherednik algebras [15]. The DDCA $\mathcal{D}_{\lambda,\beta}(\mathfrak{sl}_n)$ is generated by elements $z, \mathsf{K}(z), \mathsf{Q}(z), \mathsf{P}(z)$ for $z \in \mathfrak{sl}_n$. Suppose $\lambda = c, \beta = \frac{t}{2} - \frac{nc}{4} + \frac{c}{2}$ as described in Section 2.7. Then, Guay introduces a functor $\mathsf{F} : M \to M \otimes_{\mathbb{C}[S_l]} V^{\otimes l}$ sending a right $\mathsf{H}_{t,c}$-module to an integrable left $\mathcal{D}_{\lambda,\beta}(\mathfrak{sl}_n)$-module of level l (as an $\mathfrak{sl}_n$-module). Further, if $l + 2 < n$, this functor is an equivalence. In this chapter, we produce an analogue of that result for $\mathfrak{sl}_{m|n}$ as Theorem 5.1.2.

5.1 $\mathfrak{sl}_{m|n}$ case

One expected distinction to the approach of these computations in the super case is the inclusion of signs in the action of S_l. If σ_i is the transposition $(i\ \ i+1) \in S_l$, $1 \leq i \leq l-1$, S_l is defined to act on $\left(\mathbb{C}^{m|n}\right)^{\otimes l}$ by the formula

$$\begin{aligned}&\sigma_i(v_1 \otimes v_2 \otimes \cdots \otimes v_{i-1} \otimes v_i \otimes v_{i+1} \otimes \cdots \otimes v_l)\\&\quad = (-1)^{|v_i||v_{i+1}|} v_1 \otimes v_2 \otimes \cdots \otimes v_{i-1} \otimes v_{i+1} \otimes v_i \otimes \cdots \otimes v_l.\end{aligned}$$

In practise, it is convenient to use that $\sigma_{kq} = \sum_{i,j=1}^{l} (-1)^{|j|} E_{ij}^{(k)} E_{ji}^{(q)}$. As such, signs also appear in the actions of $X \in \mathfrak{sl}_{m|n}$:

$$E_{ij}^{(k)}(v_{e_1} \otimes \cdots \otimes v_{e_l}) = (-1)^{(|i|+|j|)(|e_1|+\cdots+|e_{k-1}|)} v_{e_1} \otimes \cdots \otimes e_{k-1} \otimes E_{ij}(v_{e_k}) \otimes \cdots \otimes v_{e_l}$$

The goal here is calculate $[\mathsf{K}(E_{ab}), \mathsf{Q}(E_{cd})](\mathbf{m} \otimes \underline{\mathbf{v}})$ and $\mathsf{P}([E_{ab}, E_{cd}])(\mathbf{m} \otimes \underline{\mathbf{v}})$ for $\mathbf{m} \in \mathsf{H}_{t,c}$ and $\underline{\mathbf{v}} \in \left(\mathbb{C}^{m|n}\right)^{\otimes l}$, $E_{ab}, E_{cd} \in \mathfrak{sl}_{m|n}$. The difference between the two will yield what is needed to produce the relation for $[\mathsf{K}(E_{ab}), \mathsf{Q}(E_{cd})]$ and $\mathsf{P}([E_{ab}, E_{cd}])$. We assume that the way that $\mathsf{K}(E_{ab})$, $\mathsf{Q}(E_{ab})$, $\mathsf{P}(E_{ab})$ act is similar to the non-super case, i.e.:

$$\mathsf{K}(E_{ab})(\mathbf{m} \otimes \underline{\mathbf{v}}) = \sum_{k=1}^{l} \mathbf{m} x_k \otimes E_{ab}^{(k)}(\underline{\mathbf{v}})$$

$$\mathsf{Q}(E_{ab})(\mathbf{m} \otimes \underline{\mathbf{v}}) = \sum_{k=1}^{l} \mathbf{m} y_k \otimes E_{ab}^{(k)}(\underline{\mathbf{v}})$$

$$\mathsf{P}(E_{ab})(\mathbf{m} \otimes \underline{\mathbf{v}}) = \sum_{k=1}^{l} \mathbf{m} \frac{y_k x_k + x_k y_k}{2} \otimes E_{ab}^{(k)}(\underline{\mathbf{v}}).$$

Then, for any $z \in \mathfrak{sl}_{m|n}$, $\mathsf{K}(z), \mathsf{Q}(z)$, and $\mathsf{P}(z)$ are defined by linearity.

For convenience, we introduce the notation $|e_1, e_j| := |e_1| + \cdots + |e_j|$, i.e. the sum of the parities of factors 1 through j of the tensor of basis vectors $\underline{\mathbf{v}} = v_{e_1} \otimes \cdots \otimes v_{e_j} \otimes \cdots \otimes v_{e_l}$. These computations will also use the simplified version of two of the relations for the rational Cherednik algebra, being:

$$[y_q, x_k] = -\kappa \sigma_{kq}, \quad k \neq q$$

$$[y_k, x_k] = t + \kappa \sum_{i=1}^{k-1} \sigma_{ik} + \kappa \sum_{j=k+1}^{l} \sigma_{jk}.$$

$$
\begin{aligned}
&[\mathsf{K}(E_{ab}), \mathsf{Q}(E_{cd})](\mathbf{m} \otimes \underline{\mathbf{v}}) = \mathsf{K}(E_{ab}) \left(\sum_{q=1}^{l} \mathbf{m} y_q \otimes E_{cd}^{(q)}(\underline{\mathbf{v}}) \right) \\
&\quad - (-1)^{(|a|+|b|)(|c|+|d|)} \mathsf{Q}(E_{cd}) \left(\sum_{k=1}^{l} \mathbf{m} x_k \otimes E_{ab}^{(k)}(\underline{\mathbf{v}}) \right) \\
&= \sum_{k,q} \mathbf{m} y_q x_k \otimes E_{ab}^{(k)} E_{cd}^{(q)}(\underline{\mathbf{v}}) - (-1)^{(|a|+|b|)(|c|+|d|)} \sum_{k,q} \mathbf{m} x_k y_q \otimes E_{cd}^{(q)} E_{ab}^{(k)}(\underline{\mathbf{v}}) \\
&= \sum_{k,q} \mathbf{m} y_q x_k \otimes (-1)^{(|c|+|d|)(|e_1, e_{q-1}|)} E_{ab}^{(k)}(v_{e_1} \otimes \cdots \otimes E_{cd}(v_{e_q}) \otimes \cdots \otimes v_{e_l}) \\
&\quad - (-1)^{(|a|+|b|)(|c|+|d|)} \sum_{k,q} \mathbf{m} x_k y_q \otimes (-1)^{(|a|+|b|)(|e_1, e_{k-1}|)} E_{cd}^{(q)}(v_{e_1} \otimes \cdots \otimes E_{ab}(v_{e_k}) \otimes \cdots \otimes v_{e_l}) \\
&= \sum_{k<q} \Big(\mathbf{m} y_q x_k \otimes (-1)^{(|c|+|d|)(|e_1, e_{q-1}|)} E_{ab}^{(k)}(v_{e_1} \otimes \cdots \otimes E_{cd}(v_{e_q}) \otimes \cdots \otimes v_{e_l}) \\
&\quad - (-1)^{(|a|+|b|)(|c|+|d|)} \mathbf{m} x_k y_q \otimes (-1)^{(|a|+|b|)(|e_1, e_{k-1}|)} E_{cd}^{(q)}(v_{e_1} \otimes \cdots \otimes E_{ab}(v_{e_k}) \otimes \cdots \otimes v_{e_l}) \Big) \\
&\quad + \sum_{k>q} \Big(\mathbf{m} y_q x_k \otimes (-1)^{(|c|+|d|)(|e_1, e_{q-1}|)} E_{ab}^{(k)}(v_{e_1} \otimes \cdots \otimes E_{cd}(v_{e_q}) \otimes \cdots \otimes v_{e_l}) \\
&\quad - (-1)^{(|a|+|b|)(|c|+|d|)} \mathbf{m} x_k y_q \otimes (-1)^{(|a|+|b|)(|e_1, e_{k-1}|)} E_{cd}^{(q)}(v_{e_1} \otimes \cdots \otimes E_{ab}(v_{e_k}) \otimes \cdots \otimes v_{e_l}) \Big) \\
&\quad + \sum_{k=1}^{l} \Big(\mathbf{m} y_k x_k \otimes (-1)^{(|c|+|d|)(|e_1, e_{k-1}|)} E_{ab}^{(k)}(v_{e_1} \otimes \cdots \otimes E_{cd}(v_{e_k}) \otimes \cdots \otimes v_{e_l}) \\
&\quad - (-1)^{(|a|+|b|)(|c|+|d|)} \mathbf{m} x_k y_k \otimes (-1)^{(|a|+|b|)(|e_1, e_{k-1}|)} E_{cd}^{(k)}(v_{e_1} \otimes \cdots \otimes E_{ab}(v_{e_k}) \otimes \cdots \otimes v_{e_l}) \Big) \\
&= \sum_{k<q} \big(\mathbf{m} y_q x_k \otimes (-1)^{(|c|+|d|)(|e_1, e_{q-1}|)} (-1)^{(|a|+|b|)(|e_1, e_{k-1}|)} \\
&\quad (v_{e_1} \otimes \cdots \otimes E_{ab}(v_{e_k}) \otimes \cdots \otimes E_{cd}(v_{e_q}) \otimes \cdots \otimes v_{e_l}) \\
&\quad - \mathbf{m} x_k y_q \otimes (-1)^{(|a|+|b|)(|e_1, e_{k-1}|)} (-1)^{(|c|+|d|)(|e_1, e_{q-1}|)} \\
&\quad (v_{e_1} \otimes \cdots \otimes E_{ab}(v_{e_k}) \otimes \cdots \otimes E_{cd}(v_{e_q}) \otimes \cdots \otimes v_{e_l}) \big) \\
&\quad + \sum_{k>q} \big(\mathbf{m} y_q x_k \otimes (-1)^{(|c|+|d|)(|e_1, e_{q-1}|)} (-1)^{(|a|+|b|)(|e_1, e_{k-1}|)} (-1)^{(|a|+|b|)(|c|+|d|)} \\
&\quad (v_{e_1} \otimes \cdots \otimes E_{cd}(v_{e_q}) \otimes \cdots \otimes E_{ab}(v_{e_k}) \otimes \cdots \otimes v_{e_l})
\end{aligned}
$$

$$- (-1)^{(|a|+|b|)(|c|+|d|)}\mathbf{m}x_k y_q \otimes (-1)^{(|a|+|b|)(|e_1,e_{k-1}|)}(-1)^{(|c|+|d|)(|e_1,e_{q-1}|)}$$

$$\big(v_{e_1} \otimes \cdots \otimes E_{cd}(v_{e_q}) \otimes \cdots \otimes E_{ab}(v_{e_k}) \otimes \cdots \otimes v_{e_l})\big)$$

$$+ \sum_{k=1}^{l} \big(\mathbf{m}y_k x_k \otimes (-1)^{(|c|+|d|)(|e_1,e_{k-1}|)}(-1)^{(|a|+|b|)(|e_1,e_{k-1}|)}$$

$$(v_{e_1} \otimes \cdots \otimes E_{ab}E_{cd}(v_{e_k}) \otimes \cdots \otimes v_{e_l})$$

$$- (-1)^{(|a|+|b|)(|c|+|d|)}\mathbf{m}x_k y_k \otimes (-1)^{(|a|+|b|)(|e_1,e_{k-1}|)}(-1)^{(|c|+|d|)(|e_1,e_{k-1}|)}$$

$$(v_{e_1} \otimes \cdots \otimes E_{cd}E_{ab}(v_{e_k}) \otimes \cdots \otimes v_{e_l}))$$

$$= \sum_{k<q} (-1)^{(|a|+|b|)(|e_1,e_{k-1}|)+(|c|+|d|)(|e_1,e_{q-1}|)}$$

$$\big(\mathbf{m}(y_q x_k - x_k y_q) \otimes (v_{e_1} \otimes \cdots \otimes E_{ab}(v_{e_k}) \otimes \cdots \otimes E_{cd}(v_{e_q}) \otimes \cdots \otimes v_{e_l})\big)$$

$$+ (-1)^{(|a|+|b|)(|c|+|d|)} \sum_{k>q} (-1)^{(|c|+|d|)(|e_1,e_{q-1}|)+(|a|+|b|)(|e_1,e_{k-1}|)}$$

$$\big(\mathbf{m}(y_q x_k - x_k y_q) \otimes (v_{e_1} \otimes \cdots \otimes E_{cd}(v_{e_q}) \otimes \cdots \otimes E_{ab}(v_{e_k}) \otimes \cdots \otimes v_{e_l})\big)$$

$$+ \delta_{bc} \sum_{k=1}^{l} (-1)^{(|a|+|d|)(|e_1,e_{k-1}|)} (\mathbf{m}y_k x_k \otimes v_{e_1} \otimes \cdots \otimes E_{ad}(v_{e_k}) \otimes \cdots \otimes v_{e_l})$$

$$- \delta_{ad}(-1)^{(|a|+|b|)(|c|+|a|)} \sum_{k=1}^{l} (-1)^{(|b|+|c|)(|e_1,e_{k-1}|)}$$

$$(\mathbf{m}x_k y_k \otimes v_{e_1} \otimes \cdots \otimes E_{cb}(v_{e_k}) \otimes \cdots \otimes v_{e_l})$$

$$= \sum_{k<q} (-1)^{(|a|+|b|)(|e_1,e_{k-1}|)+(|c|+|d|)(|e_1,e_{q-1}|)}$$

$$\mathbf{m}(t\langle y_q, x_k\rangle + \kappa \sum_{\alpha\in\Delta^+} \langle \alpha, y_q\rangle\langle x_k, \alpha^\vee\rangle\sigma_\alpha)\otimes$$

$$(v_{e_1} \otimes \cdots \otimes E_{ab}(v_{e_k}) \otimes \cdots \otimes E_{cd}(v_{e_q}) \otimes \cdots \otimes v_{e_l})$$

$$+ (-1)^{(|a|+|b|)(|c|+|d|)} \sum_{k>q} (-1)^{(|c|+|d|)(|e_1,e_{q-1}|)+(|a|+|b|)(|e_1,e_{k-1}|)}$$

$$\mathbf{m}(t\langle y_q, x_k\rangle + \kappa \sum_{\alpha\in\Delta^+} \langle \alpha, y_q\rangle\langle x_k, \alpha^\vee\rangle\sigma_\alpha)\otimes$$

$$(v_{e_1} \otimes \cdots \otimes E_{cd}(v_{e_q}) \otimes \cdots \otimes E_{ab}(v_{e_k}) \otimes \cdots \otimes v_{e_l})$$

$$+ \delta_{bc} \sum_{k=1}^{l} (-1)^{(|b|+|d|)(|e_1,e_{k-1}|)+(|a|+|b|)(|e_1,e_{k-1}|)}$$

$$(\mathbf{m}y_kx_k \otimes v_{e_1} \otimes \cdots \otimes E_{ad}(v_{e_k}) \otimes \cdots \otimes v_{e_l})$$

$$- \delta_{ad}(-1)^{(|a|+|b|)(|c|+|a|)} \sum_{k=1}^{l} (-1)^{(|a|+|b|)(|e_1,e_{k-1}|)+(|c|+|a|)(|e_1,e_{k-1}|)}$$

$$(\mathbf{m}x_ky_k \otimes v_{e_1} \otimes \cdots \otimes E_{cb}(v_{e_k}) \otimes \cdots \otimes v_{e_l})$$

$$= \sum_{k<q} (-1)^{(|a|+|b|)(|e_1,e_{k-1}|)+(|c|+|d|)(|e_1,e_{q-1}|)}$$

$$\mathbf{m}(\kappa \sum_{i<j} \langle x_i - x_j, y_q \rangle \langle x_k, y_i - y_j \rangle \sigma_{ij}) \otimes$$

$$(v_{e_1} \otimes \cdots \otimes E_{ab}(v_{e_k}) \otimes \cdots \otimes E_{cd}(v_{e_q}) \otimes \cdots \otimes v_{e_l})$$

$$+ (-1)^{(|a|+|b|)(|c|+|d|)} \sum_{k>q} (-1)^{(|c|+|d|)(|e_1,e_{q-1}|)+(|a|+|b|)(|e_1,e_{k-1}|)}$$

$$\mathbf{m}(\kappa \sum_{i<j} \langle x_i - x_j, y_q \rangle \langle x_k, y_i - y_j \rangle \sigma_{ij}) \otimes$$

$$(v_{e_1} \otimes \cdots \otimes E_{cd}(v_{e_q}) \otimes \cdots \otimes E_{ab}(v_{e_k}) \otimes \cdots \otimes v_{e_l})$$

$$+ \delta_{bc} \sum_{k=1}^{l} (-1)^{(|b|+|d|)(|e_1,e_{k-1}|)+(|a|+|b|)(|e_1,e_{k-1}|)}$$

$$(\mathbf{m}y_kx_k \otimes v_{e_1} \otimes \cdots \otimes E_{ad}(v_{e_k}) \otimes \cdots \otimes v_{e_l})$$

$$+ \frac{1}{2}\delta_{bc} \sum_{k=1}^{l} (-1)^{(|b|+|d|)(|e_1,e_{k-1}|)+(|a|+|b|)(|e_1,e_{k-1}|)}$$

$$(\mathbf{m}x_ky_k \otimes v_{e_1} \otimes \cdots \otimes E_{ad}(v_{e_k}) \otimes \cdots \otimes v_{e_l})$$

$$- \frac{1}{2}\delta_{bc} \sum_{k=1}^{l} (-1)^{(|b|+|d|)(|e_1,e_{k-1}|)+(|a|+|b|)(|e_1,e_{k-1}|)}$$

$$(\mathbf{m}x_ky_k \otimes v_{e_1} \otimes \cdots \otimes E_{ad}(v_{e_k}) \otimes \cdots \otimes v_{e_l})$$

$$- \delta_{ad}(-1)^{(|a|+|b|)(|c|+|a|)} \sum_{k=1}^{l} (-1)^{(|a|+|b|)(|e_1,e_{k-1}|)+(|c|+|a|)(|e_1,e_{k-1}|)}$$

$$(\mathbf{m}x_ky_k \otimes v_{e_1} \otimes \cdots \otimes E_{cb}(v_{e_k}) \otimes \cdots v_{e_l})$$

$$- \frac{1}{2}\delta_{ad}(-1)^{(|a|+|b|)(|c|+|a|)} \sum_{k=1}^{l} (-1)^{(|a|+|b|)(|e_1,e_{k-1}|)+(|c|+|a|)(|e_1,e_{k-1}|)}$$

$$(\mathbf{m}y_kx_k \otimes v_{e_1} \otimes \cdots \otimes E_{cb}(v_{e_k}) \otimes \cdots \otimes v_{e_l})$$

$$+\frac{1}{2}\delta_{ad}(-1)^{(|a|+|b|)(|c|+|a|)}\sum_{k=1}^{l}(-1)^{(|a|+|b|)(|e_1,e_{k-1}|)+(|c|+|a|)(|e_1,e_{k-1}|)}$$

$$(\mathbf{m}y_kx_k\otimes v_{e_1}\otimes\cdots\otimes E_{cb}(v_{e_k})\otimes\cdots\otimes v_{e_l})$$

$$=\sum_{k<q}(-1)^{(|a|+|b|)(|e_1,e_{k-1}|)+(|c|+|d|)(|e_1,e_{q-1}|)}$$

$$\mathbf{m}(\kappa\langle x_k-x_q,y_q\rangle\langle x_k,y_k-y_q\rangle\sigma_{kq})\otimes$$

$$(v_{e_1}\otimes\cdots\otimes E_{ab}(v_{e_k})\otimes\cdots\otimes E_{cd}(v_{e_q})\otimes\cdots\otimes v_{e_l})$$

$$+(-1)^{(|a|+|b|)(|c|+|d|)}\sum_{k>q}(-1)^{(|c|+|d|)(|e_1,e_{q-1}|)+(|a|+|b|)(|e_1,e_{k-1}|)}$$

$$\mathbf{m}(\kappa\langle x_q-x_k,y_q\rangle\langle x_k,y_q-y_k\rangle\sigma_{kq})\otimes$$

$$(v_{e_1}\otimes\cdots\otimes E_{cd}(v_{e_q})\otimes\cdots\otimes E_{ab}(v_{e_k})\otimes\cdots\otimes v_{e_l})$$

$$+\frac{1}{2}\delta_{bc}\sum_{k=1}^{l}(-1)^{(|a|+|d|)(|e_1,e_{k-1}|)}$$

$$(\mathbf{m}y_kx_k\otimes v_{e_1}\otimes\cdots\otimes E_{ad}(v_{e_k})\otimes\cdots\otimes v_{e_l})$$

$$+\frac{1}{2}\delta_{bc}\sum_{k=1}^{l}(-1)^{(|a|+|d|)(|e_1,e_{k-1}|)}$$

$$(\mathbf{m}x_ky_k\otimes v_{e_1}\otimes\cdots\otimes E_{ad}(v_{e_k})\otimes\cdots\otimes v_{e_l})$$

$$+\frac{1}{2}\delta_{bc}\sum_{k=1}^{l}(-1)^{(|a|+|d|)(|e_1,e_{k-1}|)}$$

$$(\mathbf{m}y_kx_k\otimes v_{e_1}\otimes\cdots\otimes E_{ad}(v_{e_k})\otimes\cdots\otimes v_{e_l})$$

$$-\frac{1}{2}\delta_{bc}\sum_{k=1}^{l}(-1)^{(|a|+|d|)(|e_1,e_{k-1}|)}$$

$$(\mathbf{m}x_ky_k\otimes v_{e_1}\otimes\cdots\otimes E_{ad}(v_{e_k})\otimes\cdots\otimes v_{e_l})$$

$$-\frac{1}{2}\delta_{ad}(-1)^{(|a|+|b|)(|c|+|a|)}\sum_{k=1}^{l}(-1)^{(|b|+|c|)(|e_1,e_{k-1}|)}$$

$$(\mathbf{m}x_ky_k\otimes v_{e_1}\otimes\cdots\otimes E_{cb}(v_{e_k})\otimes\cdots\otimes v_{e_l})$$

$$-\frac{1}{2}\delta_{ad}(-1)^{(|a|+|b|)(|c|+|a|)}\sum_{k=1}^{l}(-1)^{(|b|+|c|)(|e_1,e_{k-1}|)}$$

$$(\mathbf{m}y_kx_k\otimes v_{e_1}\otimes\cdots\otimes E_{cb}(v_{e_k})\otimes\cdots\otimes v_{e_l})$$

$$
\begin{aligned}
&- \frac{1}{2}\delta_{ad}(-1)^{(|a|+|b|)(|c|+|a|)}\sum_{k=1}^{l}(-1)^{(|b|+|c|)(|e_1,e_{k-1}|)}\\
&(\mathbf{m}x_ky_k \otimes v_{e_1} \otimes \cdots \otimes E_{cb}(v_{e_k}) \otimes \cdots \otimes v_{e_l})\\
&+ \frac{1}{2}\delta_{ad}(-1)^{(|a|+|b|)(|c|+|a|)}\sum_{k=1}^{l}(-1)^{(|b|+|c|)(|e_1,e_{k-1}|)}\\
&(\mathbf{m}y_kx_k \otimes v_{e_1} \otimes \cdots \otimes E_{cb}(v_{e_k}) \otimes \cdots \otimes v_{e_l})\\
=&- \kappa\sum_{k<q}(-1)^{(|a|+|b|)(|e_1,e_{k-1}|)+(|c|+|d|)(|e_1,e_{q-1}|)}\\
&\mathbf{m} \otimes \sigma_{kq}(v_{e_1} \otimes \cdots \otimes E_{ab}(v_{e_k}) \otimes \cdots \otimes E_{cd}(v_{e_q}) \otimes \cdots \otimes v_{e_l})\\
&- \kappa(-1)^{(|a|+|b|)(|c|+|d|)}\sum_{k>q}(-1)^{(|c|+|d|)(|e_1,e_{q-1}|)+(|a|+|b|)(|e_1,e_{k-1}|)}\\
&\mathbf{m} \otimes \sigma_{kq}(v_{e_1} \otimes \cdots \otimes E_{cd}(v_{e_q}) \otimes \cdots \otimes E_{ab}(v_{e_k}) \otimes \cdots \otimes v_{e_l})\\
&+ \frac{1}{2}\delta_{bc}\sum_{k=1}^{l}(-1)^{(|a|+|d|)(|e_1,e_{k-1}|)}\\
&(\mathbf{m}(y_kx_k + x_ky_k) \otimes v_{e_1} \otimes \cdots \otimes E_{ad}(v_{e_k}) \otimes \cdots \otimes v_{e_l})\\
&- (-1)^{(|a|+|b|)(|c|+|a|)}\frac{1}{2}\delta_{ad}\sum_{k=1}^{l}(-1)^{(|b|+|c|)(|e_1,e_{k-1}|)}\\
&(\mathbf{m}(y_kx_k + x_ky_k) \otimes v_{e_1} \otimes \cdots \otimes E_{cb}(v_{e_k}) \otimes \cdots \otimes v_{e_l})\\
&+ \frac{1}{2}\delta_{bc}\sum_{k=1}^{l}(-1)^{(|a|+|d|)(|e_1,e_{k-1}|)}\\
&(\mathbf{m}(y_kx_k - x_ky_k) \otimes v_{e_1} \otimes \cdots \otimes E_{ad}(v_{e_k}) \otimes \cdots \otimes v_{e_l})\\
&+ \frac{1}{2}\delta_{ad}(-1)^{(|a|+|b|)(|c|+|a|)}\sum_{k=1}^{l}(-1)^{(|b|+|c|)(|e_1,e_{k-1}|)}\\
&(\mathbf{m}(y_kx_k - x_ky_k) \otimes v_{e_1} \otimes \cdots \otimes E_{cb}(v_{e_k}) \otimes \cdots \otimes v_{e_l})\\
=&- \kappa\sum_{k<q}(-1)^{(|a|+|b|)(|e_1,e_{k-1}|)+(|c|+|d|)(|e_1,e_{q-1}|)}\\
&\mathbf{m} \otimes \sum_{i,j=1}^{m+n}(-1)^{|j|}E_{ij}^{(k)}E_{ji}^{(q)}(v_{e_1} \otimes \cdots \otimes E_{ab}(v_{e_k}) \otimes \cdots \otimes E_{cd}(v_{e_q}) \otimes \cdots \otimes v_{e_l})\\
&- \kappa(-1)^{(|a|+|b|)(|c|+|d|)}\sum_{k>q}(-1)^{(|c|+|d|)(|e_1,e_{q-1}|)+(|a|+|b|)(|e_1,e_{k-1}|)}
\end{aligned}
$$

$$\mathbf{m}\otimes\sum_{i,j=1}^{m+n}(-1)^{|j|}E_{ij}^{(k)}E_{ji}^{(q)}(v_{e_1}\otimes\cdots\otimes E_{cd}(v_{e_q})\otimes\cdots\otimes E_{ab}(v_{e_k})\otimes\cdots\otimes v_{e_l})$$

$$+\mathsf{P}([E_{ab},E_{cd}])(\mathbf{m}\otimes\underline{\mathbf{v}})+\frac{1}{2}\delta_{bc}\sum_{k=1}^{l}(-1)^{(|a|+|d|)(|e_1,e_{k-1}|)}$$

$$\left(\mathbf{m}(t+\kappa\sum_{i<j}\langle x_i-x_j,y_k\rangle\langle x_k,y_i-y_j\rangle\sigma_{ij})\otimes v_{e_1}\otimes\cdots\otimes E_{ad}(v_{e_k})\otimes\cdots\otimes v_{e_l}\right)$$

$$+\frac{1}{2}\delta_{ad}(-1)^{(|a|+|b|)(|c|+|a|)}\sum_{k=1}^{l}(-1)^{(|b|+|c|)(|e_1,e_{k-1}|)}$$

$$\left(\mathbf{m}(t+\kappa\sum_{i<j}\langle x_i-x_j,y_k\rangle\langle x_k,y_i-y_j\rangle\sigma_{ij})\otimes v_{e_1}\otimes\cdots\otimes E_{cb}(v_{e_k})\otimes\cdots\otimes v_{e_l}\right)$$

$$=-\kappa\sum_{k<q}(-1)^{(|a|+|b|)(|e_1,e_{k-1}|)+(|c|+|d|)(|e_1,e_{q-1}|)}\mathbf{m}\otimes\sum_{i,j=1}^{m+n}(-1)^{|j|}$$

$$E_{ij}^{(k)}(-1)^{(|j|+|i|)(e_1+\cdots+|e_k|+\cdots+|e_{q-1}|)}(-1)^{(|a|+|b|)(|j|+|i|)}$$

$$(v_{e_1}\otimes\cdots\otimes E_{ab}(v_{e_k})\otimes\cdots\otimes E_{ji}E_{cd}(v_{e_q})\otimes\cdots\otimes v_{e_l})$$

$$-\kappa(-1)^{(|a|+|b|)(|c|+|d|)}\sum_{k>q}(-1)^{(|c|+|d|)(|e_1,e_{q-1}|)+(|a|+|b|)(|e_1,e_{k-1}|)}\mathbf{m}\otimes\sum_{i,j=1}^{m+n}(-1)^{|j|}$$

$$E_{ij}^{(k)}(-1)^{(|j|+|i|)(|e_1,e_{q-1}|)}$$

$$(v_{e_1}\otimes\cdots\otimes E_{ji}E_{cd}(v_{e_q})\otimes\cdots\otimes E_{ab}(v_{e_k})\otimes\cdots\otimes v_{e_l})$$

$$+\mathsf{P}([E_{ab},E_{cd}])(\mathbf{m}\otimes\underline{\mathbf{v}})+\frac{1}{2}\delta_{bc}\sum_{k=1}^{l}(-1)^{(|a|+|d|)(|e_1,e_{k-1}|)}$$

$$(\mathbf{m}t\otimes v_{e_1}\otimes\cdots\otimes E_{ad}(v_{e_k})\otimes\cdots\otimes v_{e_l}$$

$$+\mathbf{m}\kappa\left(\sum_{j=k+1}^{l}\sigma_{kj}+\sum_{i=1}^{k-1}\sigma_{ik}\right)\otimes v_{e_1}\otimes\cdots\otimes E_{ad}(v_{e_k})\otimes\cdots\otimes v_{e_l}\Bigg)$$

$$+\frac{1}{2}\delta_{ad}(-1)^{(|a|+|b|)(|c|+|a|)}\sum_{k=1}^{l}(-1)^{(|b|+|c|)(|e_1,e_{k-1}|)}$$

$$(\mathbf{m}t\otimes v_{e_1}\otimes\cdots\otimes E_{cb}(v_{e_k})\otimes\cdots\otimes v_{e_l}$$

$$+\mathbf{m}\kappa\left(\sum_{j=k+1}^{l}\sigma_{kj}+\sum_{i=1}^{k-1}\sigma_{ik}\right)\otimes v_{e_1}\otimes\cdots\otimes E_{cb}(v_{e_k})\otimes\cdots\otimes v_{e_l}\Bigg)$$

$$=\mathsf{P}([E_{ab},E_{cd}])(\mathbf{m}\otimes\underline{\mathbf{v}})-\kappa\sum_{k<q}(-1)^{(|a|+|b|)(|e_1,e_{k-1}|)+(|c|+|d|)(|e_1,e_{q-1}|)}(-1)^{(|a|+|b|)(|j|+|c|)}$$

$$\mathbf{m}\otimes\sum_{j=1}^{m+n}(-1)^{|j|}(-1)^{(|j|+|c|)(e_1+\cdots+|e_k|+\cdots+|e_{q-1}|)}(-1)^{(|c|+|j|)(e_1+\cdots|e_{k-1}|)}$$

$$(v_{e_1}\otimes\cdots\otimes E_{cj}E_{ab}(v_{e_k})\otimes\cdots\otimes E_{jd}(v_{e_q})\otimes\cdots\otimes v_{e_l})$$

$$-\kappa(-1)^{(|a|+|b|)(|c|+|d|)}\sum_{k>q}(-1)^{(|c|+|d|)(|e_1,e_{q-1}|)+(|a|+|b|)(|e_1,e_{k-1}|)}\mathbf{m}\otimes\sum_{j=1}^{m+n}(-1)^{|j|}$$

$$(-1)^{(|j|+|c|)(|e_1,e_{q-1}|)}(-1)^{(|c|+|j|)(|e_1,e_q|+\cdots|e_{k-1}|)}(-1)^{(|c|+|j|)(|j|+|d|)}$$

$$(v_{e_1}\otimes\cdots\otimes E_{jd}(v_{e_q})\otimes\cdots\otimes E_{cj}E_{ab}(v_{e_k})\otimes\cdots\otimes v_{e_l})$$

$$+\delta_{bc}\frac{\kappa}{2}\sum_{k=1}^{l}(-1)^{(|a|+|d|)(|e_1,e_{k-1}|)}\mathbf{m}\otimes\sum_{j=k+1}^{m+n}\sigma_{kj}\,(v_{e_1}\otimes\cdots\otimes E_{ad}(v_{e_k})\otimes\cdots\otimes v_{e_l})$$

$$+\delta_{bc}\frac{\kappa}{2}\sum_{k=1}^{l}(-1)^{(|a|+|d|)(|e_1,e_{k-1}|)}\mathbf{m}\sum_{i=1}^{k-1}\sigma_{ik}\,(v_{e_1}\otimes\cdots\otimes E_{ad}(v_{e_k})\otimes\cdots\otimes v_{e_l})$$

$$+\delta_{ad}(-1)^{(|a|+|b|)(|c|+|a|)}\frac{\kappa}{2}\sum_{k=1}^{l}(-1)^{(|b|+|c|)(|e_1,e_{k-1}|)}$$

$$\mathbf{m}\otimes\sum_{j=k+1}^{l}\sigma_{kj}\,(v_{e_1}\otimes\cdots\otimes E_{cb}(v_{e_k})\otimes\cdots\otimes v_{e_l})$$

$$+\delta_{ad}(-1)^{(|a|+|b|)(|c|+|a|)}\frac{\kappa}{2}\sum_{k=1}^{l}(-1)^{(|b|+|c|)(|e_1,e_{k-1}|)}$$

$$\mathbf{m}\otimes\sum_{i=1}^{k-1}\sigma_{ik}\,(v_{e_1}\otimes\cdots\otimes E_{cb}(v_{e_k})\otimes\cdots\otimes v_{e_l})$$

$$+\frac{1}{2}\delta_{bc}\sum_{k=1}^{l}(-1)^{(|a|+|d|)(|e_1,e_{k-1}|)}\mathbf{m}t\otimes v_{e_1}\otimes\cdots\otimes E_{ad}(v_{e_k})\otimes\cdots\otimes v_{e_l}$$

$$+\frac{1}{2}\delta_{ad}(-1)^{(|a|+|b|)(|c|+|a|)}\sum_{k=1}^{l}(-1)^{(|b|+|c|)(|e_1,e_{k-1}|)}$$

$$\mathbf{m}t\otimes v_{e_1}\otimes\cdots\otimes E_{cb}(v_{e_k})\otimes\cdots\otimes v_{e_l}$$

$$=\mathsf{P}([E_{ab},E_{cd}])(m\otimes\underline{\mathbf{v}})-\kappa\sum_{k<q}(-1)^{(|a|+|b|)(|e_1,e_{k-1}|)+(|c|+|d|)(|e_1,e_{q-1}|)}$$

$$\mathbf{m}\otimes(-1)^{|a|}(-1)^{(|a|+|c|)(e_1+\cdots+|e_k|+\cdots+|e_{q-1}|)}(-1)^{(|c|+|a|)(e_1+\cdots|e_{k-1}|)}(-1)^{(|a|+|b|)(|a|+|c|)}$$

$$(v_{e_1} \otimes \cdots \otimes E_{cb}(v_{e_k}) \otimes \cdots \otimes E_{ad}(v_{e_q}) \otimes \cdots \otimes v_{e_l})$$

$$- \kappa(-1)^{(|a|+|b|)(|c|+|d|)} \sum_{k>q} (-1)^{(|c|+|d|)(|e_1, e_{q-1}|)+(|a|+|b|)(|e_1, e_{k-1}|)}$$

$$\mathbf{m} \otimes (-1)^{|a|}(-1)^{(|a|+|c|)(|e_1, e_{q-1}|)}(-1)^{(|c|+|a|)(|e_1, e_q|+\cdots|e_{k-1}|)}(-1)^{(|a|+|d|)(|c|+|a|)}$$

$$(v_{e_1} \otimes \cdots \otimes E_{ad}(v_{e_q}) \otimes \cdots \otimes E_{cb}(v_{e_k}) \otimes \cdots \otimes v_{e_l})$$

$$+ \delta_{bc} \frac{\kappa}{2} \sum_{k=1}^{l} (-1)^{(|a|+|d|)(|e_1, e_{k-1}|)} \mathbf{m} \otimes$$

$$\sum_{j=k+1}^{m+n} \sum_{r,s=1}^{m+n} (-1)^{|s|} E_{rs}^{(k)} E_{sr}^{(j)} \, (v_{e_1} \otimes \cdots \otimes E_{ad}(v_{e_k}) \otimes \cdots \otimes v_{e_l})$$

$$+ \delta_{bc} \frac{\kappa}{2} \sum_{k=1}^{l} (-1)^{(|a|+|d|)(|e_1, e_{k-1}|)} \mathbf{m} \otimes$$

$$\sum_{i=1}^{k-1} \sum_{r,s=1}^{m+n} (-1)^{|s|} E_{rs}^{(i)} E_{sr}^{(k)} \, (v_{e_1} \otimes \cdots \otimes E_{ad}(v_{e_k}) \otimes \cdots \otimes v_{e_l})$$

$$+ \delta_{ad}(-1)^{(|a|+|b|)(|c|+|a|)} \frac{\kappa}{2} \sum_{k=1}^{l} (-1)^{(|b|+|c|)(|e_1, e_{k-1}|)}$$

$$\mathbf{m} \otimes \sum_{j=k+1}^{m+n} \sum_{r,s=1}^{m+n} (-1)^{|s|} E_{rs}^{(k)} E_{sr}^{(j)} \, (v_{e_1} \otimes \cdots \otimes E_{cb}(v_{e_k}) \otimes \cdots \otimes v_{e_l})$$

$$+ \delta_{ad}(-1)^{(|a|+|b|)(|c|+|a|)} \frac{\kappa}{2} \sum_{k=1}^{l} (-1)^{(|b|+|c|)(|e_1, e_{k-1}|)}$$

$$\mathbf{m} \otimes \sum_{i=1}^{k-1} \sum_{r,s=1}^{m+n} (-1)^{|s|} E_{rs}^{(i)} E_{sr}^{(k)} \, (v_{e_1} \otimes \cdots \otimes E_{cb}(v_{e_k}) \otimes \cdots \otimes v_{e_l})$$

$$+ \frac{t}{2}(\delta_{bc} E_{ad} + (-1)^{(|a|+|b|)(|c|+|d|)} \delta_{ad} E_{cb})(m \otimes \underline{\mathbf{v}})$$

$$= \mathsf{P}([E_{ab}, E_{cd}])(\mathbf{m} \otimes \underline{\mathbf{v}}) - \kappa \sum_{k<q} (-1)^{(|b|+|c|)(|e_1, e_{k-1}|)+(|a|+|d|)(|e_1, e_{q-1}|)+|a||b|+|a||c|+|b||c|}$$

$$\mathbf{m} \otimes (v_{e_1} \otimes \cdots \otimes E_{cb}(v_{e_k}) \otimes \cdots \otimes E_{ad}(v_{e_q}) \otimes \cdots \otimes v_{e_l})$$

$$- \kappa(-1)^{(|a|+|b|)(|c|+|d|)} \sum_{k>q} (-1)^{(|a|+|d|)(|e_1, e_{q-1}|)+(|b|+|c|)(|e_1, e_{k-1}|)+|a||d|+|a||c|+|c||d|}$$

$$\mathbf{m} \otimes (v_{e_1} \otimes \cdots \otimes E_{ad}(v_{e_q}) \otimes \cdots \otimes E_{cb}(v_{e_k}) \otimes \cdots \otimes v_{e_l})$$

$$+ \delta_{bc}\frac{\kappa}{2}\sum_{k=1}^{l}(-1)^{(|a|+|d|)(|e_1,e_{k-1}|)}\mathbf{m}\otimes\sum_{j=k+1}^{m+n}\sum_{r,s=1}^{m+n}(-1)^{|s|}(-1)^{(|s|+|r|)(|e_1,e_{j-1}|)}$$

$$(-1)^{(|s|+|r|)(|a|+|d|)}E_{rs}^{(k)}\left(v_{e_1}\otimes\cdots\otimes E_{ad}(v_{e_k})\otimes\cdots\otimes E_{sr}(v_{e_j})\otimes\cdots\otimes v_{e_l}\right)$$

$$+ \delta_{bc}\frac{\kappa}{2}\sum_{k=1}^{l}(-1)^{(|a|+|d|)(|e_1,e_{k-1}|)}\mathbf{m}\otimes\sum_{i=1}^{k-1}\sum_{r,s=1}^{m+n}(-1)^{|s|}(-1)^{(|s|+|r|)(|e_1|+\cdots+|e_{k-1}|)}$$

$$E_{rs}^{(i)}\left(v_{e_1}\otimes\cdots\otimes E_{sr}E_{ad}(v_{e_k})\otimes\cdots\otimes v_{e_l}\right)$$

$$+ \delta_{ad}(-1)^{(|a|+|b|)(|c|+|a|)}\frac{\kappa}{2}\sum_{k=1}^{l}(-1)^{(|b|+|c|)(|e_1,e_{k-1}|)}$$

$$\mathbf{m}\otimes\sum_{j=k+1}^{m+n}\sum_{r,s=1}^{m+n}(-1)^{|s|}(-1)^{(|s|+|r|)(|e_1,e_{j-1}|)}(-1)^{(|s|+|r|)(|c|+|b|)}$$

$$E_{rs}^{(k)}\left(v_{e_1}\otimes\cdots\otimes E_{cb}(v_{e_k})\otimes\cdots\otimes E_{sr}(v_{e_j})\otimes\cdots\otimes v_{e_l}\right)$$

$$+ \delta_{ad}(-1)^{(|a|+|b|)(|c|+|a|)}\frac{\kappa}{2}\sum_{k=1}^{l}(-1)^{(|b|+|c|)(|e_1,e_{k-1}|)}$$

$$\mathbf{m}\otimes\sum_{i=1}^{k-1}\sum_{r,s=1}^{m+n}(-1)^{|s|}(-1)^{(|s|+|r|)(|e_1,e_{k-1}|)}E_{rs}^{(i)}\left(v_{e_1}\otimes\cdots\otimes E_{sr}E_{cb}(v_{e_k})\otimes\cdots\otimes v_{e_l}\right)$$

$$+ \frac{t}{2}(\delta_{bc}E_{ad}+(-1)^{(|a|+|b|)(|c|+|d|)}\delta_{ad}E_{cb})(\mathbf{m}\otimes\underline{\mathbf{v}})$$

$$=\mathsf{P}([E_{ab},E_{cd}])(m\otimes\underline{\mathsf{v}}) - \kappa\sum_{k<q}(-1)^{(|b|+|c|)(|e_1,e_{k-1}|)+(|a|+|d|)(|e_1,e_{q-1}|)+|a||b|+|a||c|+|b||c|}$$

$$\mathbf{m}\otimes\left(v_{e_1}\otimes\cdots\otimes E_{cb}(v_{e_k})\otimes\cdots\otimes E_{ad}(v_{e_q})\otimes\cdots\otimes v_{e_l}\right)$$

$$- \kappa(-1)^{(|a|+|b|)(|c|+|d|)}\sum_{k>q}(-1)^{(|a|+|d|)(|e_1,e_{q-1}|)+(|b|+|c|)(|e_1,e_{k-1}|)+|a||d|+|a||c|+|c||d|}$$

$$\mathbf{m}\otimes\left(v_{e_1}\otimes\cdots\otimes E_{ad}(v_{e_q})\otimes\cdots\otimes E_{cb}(v_{e_k})\otimes\cdots\otimes v_{e_l}\right)$$

$$+ \delta_{bc}\frac{\kappa}{2}\sum_{k=1}^{l}(-1)^{(|a|+|d|)(|e_1,e_{k-1}|)}\mathbf{m}\otimes\sum_{j=k+1}^{m+n}\sum_{r,s=1}^{m+n}(-1)^{|s|}(-1)^{(|s|+|r|)(|e_1,e_{j-1}|)}$$

$$(-1)^{(|s|+|r|)(|a|+|d|)}(-1)^{(|r|+|s|)(|e_1,e_{k-1}|)}\left(v_{e_1}\otimes\cdots\otimes E_{rs}E_{ad}(v_{e_k})\otimes\cdots\otimes E_{sr}(v_{e_j})\otimes\cdots\otimes v_{e_l}\right)$$

$$+ \delta_{bc}\frac{\kappa}{2}\sum_{k=1}^{l}(-1)^{(|a|+|d|)(|e_1,e_{k-1}|)}\mathbf{m}\otimes\sum_{i=1}^{k-1}\sum_{s=1}^{m+n}(-1)^{|s|}(-1)^{(|s|+|a|)(|e_1,e_{k-1}|)}$$

$$(-1)^{(|a|+|s|)(|e_1,e_{i-1}|)}\left(v_{e_1}\otimes\cdots\otimes E_{as}(v_{e_i})\otimes\cdots\otimes E_{sd}(v_{e_k})\otimes\cdots\otimes v_{e_l}\right)$$

$$
\begin{aligned}
&+\delta_{ad}(-1)^{(|a|+|b|)(|c|+|a|)}\frac{\kappa}{2}\sum_{k=1}^{l}(-1)^{(|b|+|c|)(|e_1,e_{k-1}|)}\\
&\mathbf{m}\otimes\sum_{j=k+1}^{m+n}\sum_{r,s=1}^{m+n}(-1)^{|s|}(-1)^{(|s|+|r|)(|e_1,e_{j-1}|)}(-1)^{(|s|+|r|)(|c|+|b|)}\\
&(-1)^{(|r|+|s|)(|e_1,e_{k-1}|)}\left(v_{e_1}\otimes\cdots\otimes E_{rs}E_{cb}(v_{e_k})\otimes\cdots\otimes E_{sr}(v_{e_j})\otimes\cdots\otimes v_{e_l}\right)\\
&+\delta_{ad}(-1)^{(|a|+|b|)(|c|+|a|)}\frac{\kappa}{2}\sum_{k=1}^{l}(-1)^{(|b|+|c|)(|e_1,e_{k-1}|)}\\
&\mathbf{m}\otimes\sum_{i=1}^{k-1}\sum_{s=1}^{m+n}(-1)^{|s|}(-1)^{(|s|+|c|)(|e_1,e_{k-1}|)}(-1)^{(|s|+|c|)(|e_1,e_{i-1}|)}\\
&(v_{e_1}\otimes\cdots\otimes E_{cs}(v_{e_i})\otimes\cdots\otimes E_{sb}(v_{e_k})\otimes\cdots\otimes v_{e_l})\\
&+\frac{t}{2}(\delta_{bc}E_{ad}+(-1)^{(|a|+|b|)(|c|+|d|)}\delta_{ad}E_{cb})(\mathbf{m}\otimes\underline{\mathbf{v}})\\
=&\mathsf{P}([E_{ab},E_{cd}])(\mathbf{m}\otimes\underline{\mathbf{v}})+\frac{t}{2}(\delta_{bc}E_{ad}+(-1)^{(|a|+|b|)(|c|+|d|)}\delta_{ad}E_{cb})(\mathbf{m}\otimes\underline{\mathbf{v}})\\
&-\kappa\sum_{k<q}(-1)^{|a||b|+|a||c|+|b||c|}\mathbf{m}\otimes E_{cb}^{(k)}E_{ad}^{(q)}(\underline{\mathbf{v}})\\
&-\kappa(-1)^{(|a|+|b|)(|c|+|d|)}\sum_{k>q}(-1)^{|a||d|+|a||c|+|c||d|}\mathbf{m}\otimes(-1)^{(|a|+|b|)(|c|+|d|)}E_{cb}^{(k)}E_{ad}^{(q)}(\underline{\mathbf{v}})\\
&+\delta_{bc}\frac{\kappa}{2}\sum_{k=1}^{l}(-1)^{(|a|+|d|)(|e_1,e_{k-1}|)}\mathbf{m}\otimes\sum_{i=k+1}^{m+n}\sum_{r=1}^{m+n}(-1)^{|a|}(-1)^{(|a|+|r|)(|e_1,e_{i-1}|)}\\
&(-1)^{(|a|+|r|)(|a|+|d|)}(-1)^{(|r|+|a|)(|e_1,e_{k-1}|)}\left(v_{e_1}\otimes\cdots\otimes E_{rd}(v_{e_k})\otimes\cdots\otimes E_{ar}(v_{e_i})\otimes\cdots\otimes v_{e_l}\right)\\
&+\delta_{bc}\frac{\kappa}{2}\sum_{k=1}^{l}(-1)^{(|a|+|d|)(|e_1,e_{k-1}|)}\mathbf{m}\otimes\sum_{i=1}^{k-1}\sum_{r=1}^{m+n}(-1)^{|r|}(-1)^{(|r|+|a|)(|e_1,e_{k-1}|)}\\
&(-1)^{(|a|+|r|)(|e_1,e_{i-1}|)}\left(v_{e_1}\otimes\cdots\otimes E_{ar}(v_{e_i})\otimes\cdots\otimes E_{rd}(v_{e_k})\otimes\cdots\otimes v_{e_l}\right)\\
&+\delta_{ad}(-1)^{(|a|+|b|)(|c|+|a|)}\frac{\kappa}{2}\sum_{k=1}^{l}(-1)^{(|b|+|c|)(|e_1,e_{k-1}|)}\\
&\mathbf{m}\otimes\sum_{i=k+1}^{m+n}\sum_{r=1}^{m+n}(-1)^{|c|}(-1)^{(|c|+|r|)(|e_1,e_{i-1}|)}(-1)^{(|c|+|r|)(|c|+|b|)}\\
&(-1)^{(|r|+|c|)(|e_1,e_{k-1}|)}\left(v_{e_1}\otimes\cdots\otimes E_{rb}(v_{e_k})\otimes\cdots\otimes E_{cr}(v_{e_i})\otimes\cdots\otimes v_{e_l}\right)
\end{aligned}
$$

$$+\delta_{ad}(-1)^{(|a|+|b|)(|c|+|a|)}\frac{\kappa}{2}\sum_{k=1}^{l}(-1)^{(|b|+|c|)(|e_1,e_{k-1}|)}$$

$$\mathbf{m}\otimes\sum_{i=1}^{k-1}\sum_{r=1}^{m+n}(-1)^{|r|}(-1)^{(|r|+|c|)(|e_1|+\cdots+|e_{k-1}|)}(-1)^{(|r|+|c|)(|e_1,e_{i-1}|)}$$

$$(v_{e_1}\otimes\cdots\otimes E_{cr}(v_{e_i})\otimes\cdots\otimes E_{rb}(v_{e_k})\otimes\cdots\otimes v_{e_l})$$

$$=\mathsf{P}([E_{ab},E_{cd}])(m\otimes\underline{\mathbf{v}})+\frac{t}{2}(\delta_{bc}E_{ad}+(-1)^{(|a|+|b|)(|c|+|d|)}\delta_{ad}E_{cb})(\mathbf{m}\otimes\underline{\mathbf{v}})$$

$$-\kappa\left(\sum_{k<q}(-1)^{|a||b|+|a||c|+|b||c|}\mathbf{m}\otimes E_{cb}^{(k)}E_{ad}^{(q)}(\underline{\mathbf{v}})\right.$$

$$\left.+(-1)^{|a||d|+|a||c|+|c||d|}(-1)^{(|a|+|b|)(|c|+|d|)}\sum_{k>q}\mathbf{m}\otimes E_{cb}^{(k)}E_{ad}^{(q)}(\underline{\mathbf{v}})\right)$$

$$+\delta_{bc}\frac{\kappa}{2}\sum_{k=1}^{l}(-1)^{(|a|+|d|)(|e_1,e_{k-1}|)}\mathbf{m}\otimes\sum_{i=k+1}^{m+n}\sum_{r=1}^{m+n}(-1)^{|a|}(-1)^{(|a|+|r|)(|e_1,e_{i-1}|)}$$

$$(-1)^{(|a|+|r|)(|a|+|d|)}(-1)^{(|r|+|a|)(|e_1,e_{k-1}|)}(-1)^{(|a|+|r|)(|e_1,e_{i-1}|)}(-1)^{(|a|+|r|)(|r|+|d|)}$$

$$E_{ar}^{(i)}(v_{e_1}\otimes\cdots\otimes E_{rd}(v_{e_k})\otimes\cdots\otimes v_{e_l})$$

$$+\delta_{bc}\frac{\kappa}{2}\sum_{k=1}^{l}(-1)^{(|a|+|d|)(|e_1,e_{k-1}|)}\mathbf{m}\otimes\sum_{i=1}^{k-1}\sum_{r=1}^{m+n}(-1)^{|r|}(-1)^{(|r|+|a|)(|e_1,e_{k-1}|)}$$

$$E_{ar}^{(i)}(v_{e_1}\otimes\cdots\otimes E_{rd}(v_{e_k})\otimes\cdots\otimes v_{e_l})$$

$$+\delta_{ad}(-1)^{(|a|+|b|)(|c|+|a|)}\frac{\kappa}{2}\sum_{k=1}^{l}(-1)^{(|b|+|c|)(|e_1,e_{k-1}|)}$$

$$\mathbf{m}\otimes\sum_{i=k+1}^{m+n}\sum_{r=1}^{m+n}(-1)^{|c|}(-1)^{(|c|+|r|)(|c|+|b|)}(-1)^{(|r|+|c|)(|e_1,e_{k-1}|)}(-1)^{(|c|+|r|)(|r|+|b|)}$$

$$E_{cr}^{(i)}(v_{e_1}\otimes\cdots\otimes E_{rb}(v_{e_k})\otimes\cdots\otimes v_{e_l})$$

$$+\delta_{ad}(-1)^{(|a|+|b|)(|c|+|a|)}\frac{\kappa}{2}\sum_{k=1}^{l}(-1)^{(|b|+|c|)(|e_1,e_{k-1}|)}$$

$$\mathbf{m}\otimes\sum_{i=1}^{k-1}\sum_{r=1}^{m+n}(-1)^{|r|}(-1)^{(|r|+|c|)(|e_1,e_{k-1}|)}E_{cr}^{(i)}(v_{e_1}\otimes\cdots\otimes E_{rb}(v_{e_k})\otimes\cdots\otimes v_{e_l})$$

Above, we used that

$$
\begin{aligned}
\mathsf{P}([E_{ab},E_{cd}])(\mathbf{m}\otimes\underline{\mathbf{v}}) = &\sum_{k=1}^{l}\mathbf{m}\frac{y_kx_k+x_ky_k}{2}\otimes([E_{ab},E_{cd}])^{(k)}(\underline{\mathbf{v}})\\
=&\frac{1}{2}\sum_{k=1}^{l}\mathbf{m}y_kx_k\otimes([E_{ab},E_{cd}])^{(k)}(\underline{\mathbf{v}})+\frac{1}{2}\sum_{k=1}^{l}\mathbf{m}x_ky_k\otimes([E_{ab},E_{cd}])^{(k)}(\underline{\mathbf{v}})\\
=&\frac{1}{2}\sum_{k=1}^{l}\mathbf{m}y_kx_k\otimes\left(\delta_{bc}E_{ad}-(-1)^{(|a|+|b|)(|c|+|d|)}\delta_{ad}E_{cb}\right)^{(k)}(\underline{\mathbf{v}})\\
&+\frac{1}{2}\sum_{k=1}^{l}\mathbf{m}x_ky_k\otimes\left(\delta_{bc}E_{ad}-(-1)^{(|a|+|b|)(|c|+|d|)}\delta_{ad}E_{cb}\right)^{(k)}(\underline{\mathbf{v}})\\
=&\frac{1}{2}\delta_{bc}\sum_{k=1}^{l}\mathbf{m}y_kx_k\otimes E_{ad}^{(k)}(\underline{\mathbf{v}})-(-1)^{(|a|+|b|)(|c|+|d|)}\delta_{ad}\frac{1}{2}\sum_{k=1}^{l}\mathbf{m}y_kx_k\otimes E_{cb}^{(k)}(\underline{\mathbf{v}})\\
&+\frac{1}{2}\delta_{bc}\sum_{k=1}^{l}\mathbf{m}x_ky_k\otimes E_{ad}^{(k)}(\underline{\mathbf{v}})-\frac{1}{2}(-1)^{(|a|+|b|)(|c|+|d|)}\delta_{ad}\sum_{k=1}^{l}\mathbf{m}x_ky_k\otimes E_{cb}^{(k)}(\underline{\mathbf{v}})\\
=&\frac{1}{2}\delta_{bc}\sum_{k=1}^{l}\mathbf{m}(y_kx_k+x_ky_k)\otimes E_{ad}^{(k)}(\underline{\mathbf{v}})\\
&-\frac{1}{2}(-1)^{(|a|+|b|)(|c|+|d|)}\delta_{ad}\sum_{k=1}^{l}\mathbf{m}(y_kx_k+x_ky_k)\otimes E_{cb}^{(k)}(\underline{\mathbf{v}})\\
=&\frac{1}{2}\delta_{bc}\sum_{k=1}^{l}(-1)^{(|a|+|d|)(|e_1,e_{k-1}|)}\mathbf{m}(y_kx_k+x_ky_k)\otimes v_{e_1}\otimes\cdots\otimes E_{ad}(v_{e_k})\otimes\cdots\otimes v_{e_l}\\
&-(-1)^{(|a|+|b|)(|c|+|d|)}\frac{1}{2}\delta_{ad}\sum_{k=1}^{l}(-1)^{(|c|+|b|)(|e_1|+\cdots+|e_{k-1}|)}\\
&\mathbf{m}(y_kx_k+x_ky_k)\otimes v_{e_1}\otimes\cdots\otimes E_{cb}(v_{e_k})\otimes\cdots\otimes v_{e_l}
\end{aligned}
$$

Note that

$$
\begin{aligned}
(-1)^{(|a|+|b|)(|c|+|d|)}(-1)^{|a||d|+|a||c|+|c||d|} =&(-1)^{|a||c|+|b||c|+|a||d|+|b||d|+|a||d|+|a||c|+|c||d|}\\
=&(-1)^{|b||c|+|b||d|+|c||d|}.
\end{aligned}
$$

We want this to be equal to $(-1)^{|a||b|+|a||c|+|b||c|}$. As well, either $b=c$ or $a=d$. If $b=c$, then $(-1)^{|b||c|+|b||d|+|c||d|}=(-1)^{|b|}$ and $(-1)^{|a||b|+|a||c|+|b||c|}=(-1)^{|b|}$

If $a=d$, $(-1)^{|b||c|+|b||d|+|c||d|} = (-1)^{|b||c|+|a||b|+|a||c|} = (-1)^{|a||b|+|a||c|+|b||c|}$. So, in either case, the signs inside the first pair of round brackets below are the same.

So,

$$
\begin{aligned}
&\mathsf{P}([E_{ab},E_{cd}])(\mathbf{m}\otimes\underline{\mathbf{v}}) + \frac{t}{2}(\delta_{bc}E_{ad} + (-1)^{(|a|+|b|)(|c|+|d|)}\delta_{ad}E_{cb})(\mathbf{m}\otimes\underline{\mathbf{v}})\\
&\quad -\kappa\left(\sum_{k<q}(-1)^{|a||b|+|a||c|+|b||c|}\mathbf{m}\otimes E_{cb}^{(k)}E_{ad}^{(q)}(\underline{\mathbf{v}})\right.\\
&\quad \left.+(-1)^{|a||d|+|a||c|+|c||d|}(-1)^{(|a|+|b|)(|c|+|d|)}\sum_{k>q}\mathbf{m}\otimes E_{cb}^{(k)}E_{ad}^{(q)}(\underline{\mathbf{v}})\right)\\
&\quad +\delta_{bc}\frac{\kappa}{2}\sum_{k=1}^{l}(-1)^{(|a|+|d|)(|e_1,e_{k-1}|)}\mathbf{m}\otimes\sum_{i=k+1}^{m+n}\sum_{r=1}^{m+n}(-1)^{|a|}(-1)^{(|a|+|r|)(|e_1,e_{i-1}|)}\\
&\quad (-1)^{(|a|+|r|)(|a|+|d|)}(-1)^{(|r|+|a|)(|e_1,e_{k-1}|)}(-1)^{(|a|+|r|)(|e_1,e_{i-1}|)}(-1)^{(|a|+|r|)(|r|+|d|)}\\
&\quad E_{ar}^{(i)}(v_{e_1}\otimes\cdots\otimes E_{rd}(v_{e_k})\otimes\cdots\otimes v_{e_l})\\
&\quad +\delta_{bc}\frac{\kappa}{2}\sum_{k=1}^{l}(-1)^{(|a|+|d|)(|e_1,e_{k-1}|)}\mathbf{m}\otimes\sum_{i=1}^{k-1}\sum_{r=1}^{m+n}(-1)^{|r|}(-1)^{(|r|+|a|)(|e_1,e_{k-1}|)}\\
&\quad (-1)^{(|a|+|r|)(|e_1,e_{i-1}|)}(-1)^{(|a|+|r|)(|e_1,e_{i-1}|)}E_{ar}^{(i)}(v_{e_1}\otimes\cdots\otimes E_{rd}(v_{e_k})\otimes\cdots\otimes v_{e_l})\\
&\quad +\delta_{ad}(-1)^{(|a|+|b|)(|c|+|a|)}\frac{\kappa}{2}\sum_{k=1}^{l}(-1)^{(|b|+|c|)(|e_1,e_{k-1}|)}\\
&\quad \mathbf{m}\otimes\sum_{i=k+1}^{m+n}\sum_{r=1}^{m+n}(-1)^{|c|}(-1)^{(|c|+|r|)(|e_1,e_{i-1}|)}(-1)^{(|c|+|r|)(|c|+|b|)}\\
&\quad (-1)^{(|r|+|c|)(|e_1,e_{k-1}|)}(-1)^{(|c|+|r|)(|e_1,e_{i-1}|)}(-1)^{(|c|+|r|)(|r|+|b|)}\\
&\quad E_{cr}^{(i)}(v_{e_1}\otimes\cdots\otimes E_{rb}(v_{e_k})\otimes\cdots\otimes v_{e_l})\\
&\quad +\delta_{ad}(-1)^{(|a|+|b|)(|c|+|a|)}\frac{\kappa}{2}\sum_{k=1}^{l}(-1)^{(|b|+|c|)(|e_1,e_{k-1}|)}\\
&\quad \mathbf{m}\otimes\sum_{i=1}^{k-1}\sum_{r=1}^{m+n}(-1)^{|r|}(-1)^{(|r|+|c|)(|e_1,e_{k-1}|)}(-1)^{(|r|+|c|)(|e_1,e_{i-1}|)}\\
&\quad (-1)^{(|c|+|r|)(|e_1,e_{i-1}|)}E_{cr}^{(i)}(v_{e_1}\otimes\cdots\otimes E_{rb}(v_{e_k})\otimes\cdots v_{e_l})\\
&=\mathsf{P}([E_{ab},E_{cd}])(\mathbf{m}\otimes\underline{\mathbf{v}}) + \frac{t}{2}(\delta_{bc}E_{ad} + (-1)^{(|a|+|b|)(|c|+|d|)}\delta_{ad}E_{cb})(m\otimes\underline{\mathbf{v}})
\end{aligned}
$$

$$
\begin{aligned}
&- \kappa(-1)^{|a||b|+|a||c|+|b||c|} \left(\sum_{k \neq q} \mathbf{m} \otimes E_{cb}^{(k)} E_{ad}^{(q)} (\underline{\mathbf{v}}) \right) \\
&+ \delta_{bc} \frac{\kappa}{2} \sum_{k=1}^{l} \mathbf{m} \otimes \sum_{i=k+1}^{m+n} \sum_{r=1}^{m+n} (-1)^{|r|} E_{ar}^{(i)} E_{rd}^{(k)} \left(v_{e_1} \otimes \cdots \otimes v_{e_l} \right) \\
&+ \delta_{bc} \frac{\kappa}{2} \sum_{k=1}^{l} \mathbf{m} \otimes \sum_{i=1}^{k-1} \sum_{r=1}^{m+n} (-1)^{|r|} E_{ar}^{(i)} E_{rd}^{(k)} \left(v_{e_1} \otimes \cdots \otimes v_{e_l} \right) \\
&+ \delta_{ad} (-1)^{(|a|+|b|)(|c|+|a|)} \frac{\kappa}{2} \sum_{k=1}^{l} \mathbf{m} \otimes \sum_{i=k+1}^{m+n} \sum_{r=1}^{m+n} (-1)^{|r|} E_{cr}^{(i)} E_{rb}^{(k)} \left(v_{e_1} \otimes \cdots \otimes v_{e_l} \right) \\
&+ \delta_{ad} (-1)^{(|a|+|b|)(|c|+|a|)} \frac{\kappa}{2} \sum_{k=1}^{l} \mathbf{m} \otimes \sum_{i=1}^{k-1} \sum_{r=1}^{m+n} (-1)^{|r|} E_{cr}^{(i)} E_{rb}^{(k)} \left(v_{e_1} \otimes \cdots \otimes v_{e_l} \right) \\
=& \mathsf{P}([E_{ab}, E_{cd}])(\mathbf{m} \otimes \underline{\mathbf{v}}) + \frac{t}{2} (\delta_{bc} E_{ad} + (-1)^{(|a|+|b|)(|c|+|d|)} \delta_{ad} E_{cb})(\mathbf{m} \otimes \underline{\mathbf{v}}) \\
&- \kappa(-1)^{|a||b|+|a||c|+|b||c|} \left(\sum_{k \neq q} \mathbf{m} \otimes E_{cb}^{(k)} E_{ad}^{(q)} (\underline{\mathbf{v}}) \right) \\
&+ \delta_{bc} \frac{\kappa}{2} \sum_{i \neq k} \sum_{r=1}^{m+n} (-1)^{|r|} \mathbf{m} \otimes E_{ar}^{(i)} E_{rd}^{(k)} (\underline{\mathbf{v}}) \\
&+ \delta_{ad} (-1)^{(|a|+|b|)(|c|+|a|)} \frac{\kappa}{2} \sum_{i \neq k} \sum_{r=1}^{m+n} (-1)^{|r|} \mathbf{m} \otimes E_{cr}^{(i)} E_{rb}^{(k)} (\underline{\mathbf{v}}) \qquad (5.1)
\end{aligned}
$$

Here, let $S(E_{ij}, E_{kl}) = E_{ij} E_{kl} + (-1)^{(|i|+|j|)(|k|+|l|)} E_{kl} E_{ij}$. Note that $\delta_{ad} \delta_{bc} = 0$ since either $a = d$ or $b = c$ but not both. Also, $\sum_{r=1}^{m+n} (-1)^{|r|} = m - n$.

$$
\begin{aligned}
&\delta_{bc} \frac{\kappa}{4} \sum_{r=1}^{m+n} (-1)^{|r|} S(E_{ar}, E_{rd})(m \otimes \underline{\mathbf{v}}) + \delta_{ad} \frac{c}{4} (-1)^{(|a|+|b|)(|c|+|d|)} \sum_{r=1}^{m+n} (-1)^{|r|} S(E_{cr}, E_{rb})(\mathbf{m} \otimes \underline{\mathbf{v}}) \\
=& \delta_{bc} \frac{\kappa}{4} \sum_{r=1}^{m+n} \sum_{j,k} (-1)^{|r|} \mathbf{m} \otimes E_{ar}^{(j)} E_{rd}^{(k)} (\underline{\mathbf{v}}) \\
&+ \delta_{bc} \frac{\kappa}{4} \sum_{r=1}^{m+n} \sum_{j,k} (-1)^{|r|} (-1)^{(|r|+|d|)(|a|+|r|)} \mathbf{m} \otimes E_{rd}^{(j)} E_{ar}^{(k)} (\underline{\mathbf{v}}) \\
&+ \delta_{ad} \frac{\kappa}{4} (-1)^{(|a|+|b|)(|c|+|d|)} \sum_{r=1}^{m+n} \sum_{j,k} (-1)^{|r|} \mathbf{m} \otimes E_{cr}^{(j)} E_{rb}^{(k)} (\underline{\mathbf{v}})
\end{aligned}
$$

$$+\delta_{ad}\frac{\kappa}{4}(-1)^{(|a|+|b|)(|c|+|d|)}\sum_{r=1}^{m+n}\sum_{j,k}(-1)^{|r|}(-1)^{(|c|+|r|)(|r|+|b|)}\mathbf{m}\otimes E_{rb}^{(j)}E_{cr}^{(k)}(\underline{\mathbf{v}})$$

$$=\delta_{bc}\frac{\kappa}{4}\sum_{r=1}^{m+n}\sum_{j\neq k}(-1)^{|r|}\mathbf{m}\otimes E_{ar}^{(j)}E_{rd}^{(k)}(\underline{\mathbf{v}})$$

$$+\delta_{bc}\frac{\kappa}{4}\sum_{r=1}^{m+n}\sum_{j\neq k}(-1)^{|r|}(-1)^{(|r|+|d|)(|a|+|r|)}\mathbf{m}\otimes E_{rd}^{(j)}E_{ar}^{(k)}(\underline{\mathbf{v}})$$

$$+\delta_{ad}\frac{\kappa}{4}(-1)^{(|a|+|b|)(|c|+|d|)}\sum_{r=1}^{m+n}\sum_{j\neq k}(-1)^{|r|}\mathbf{m}\otimes E_{cr}^{(j)}E_{rb}^{(k)}(\underline{\mathbf{v}})$$

$$+\delta_{ad}\frac{\kappa}{4}(-1)^{(|a|+|b|)(|c|+|d|)}\sum_{r=1}^{m+n}\sum_{j\neq k}(-1)^{|r|}(-1)^{(|c|+|r|)(|r|+|b|)}\mathbf{m}\otimes E_{rb}^{(j)}E_{cr}^{(k)}(\underline{\mathbf{v}})$$

$$+\delta_{bc}\frac{\kappa}{4}\sum_{r=1}^{m+n}\sum_{j=1}(-1)^{|r|}\mathbf{m}\otimes E_{ar}^{(j)}E_{rd}^{(j)}(\underline{\mathbf{v}})$$

$$+\delta_{bc}\frac{\kappa}{4}\sum_{r=1}^{m+n}\sum_{j=1}(-1)^{|r|}(-1)^{(|r|+|d|)(|a|+|r|)}\mathbf{m}\otimes E_{rd}^{(j)}E_{ar}^{(j)}(\underline{\mathbf{v}})$$

$$+\delta_{ad}\frac{\kappa}{4}(-1)^{(|a|+|b|)(|c|+|d|)}\sum_{r=1}^{m+n}\sum_{j=1}(-1)^{|r|}\mathbf{m}\otimes E_{cr}^{(j)}E_{rb}^{(j)}(\underline{\mathbf{v}})$$

$$+\delta_{ad}\frac{\kappa}{4}(-1)^{(|a|+|b|)(|c|+|d|)}\sum_{r=1}^{m+n}\sum_{j=1}(-1)^{|r|}(-1)^{(|c|+|r|)(|r|+|b|)}\mathbf{m}\otimes E_{rb}^{(j)}E_{cr}^{(j)}(\underline{\mathbf{v}})$$

$$=\delta_{bc}\frac{\kappa}{4}\sum_{r=1}^{m+n}\sum_{j\neq k}(-1)^{|r|}\mathbf{m}\otimes E_{ar}^{(j)}E_{rd}^{(k)}(\underline{\mathbf{v}})$$

$$+\delta_{bc}\frac{\kappa}{4}\sum_{r=1}^{m+n}\sum_{j\neq k}(-1)^{|r|}(-1)^{(|r|+|d|)(|a|+|r|)}(-1)^{(|r|+|d|)(|a|+|r|)}\mathbf{m}\otimes E_{ar}^{(k)}E_{rd}^{(j)}(\underline{\mathbf{v}})$$

$$+\delta_{ad}\frac{\kappa}{4}(-1)^{(|a|+|b|)(|c|+|d|)}\sum_{r=1}^{m+n}\sum_{j\neq k}(-1)^{|r|}\mathbf{m}\otimes E_{cr}^{(j)}E_{rb}^{(k)}(\underline{\mathbf{v}})$$

$$+\delta_{ad}\frac{\kappa}{4}(-1)^{(|a|+|b|)(|c|+|d|)}\sum_{r=1}^{m+n}\sum_{j\neq k}(-1)^{|r|}\mathbf{m}\otimes E_{cr}^{(k)}E_{rb}^{(j)}(\underline{\mathbf{v}})$$

$$+\delta_{bc}\frac{\kappa}{4}\sum_{r=1}^{m+n}\sum_{j=1}^{l}(-1)^{|r|}\mathbf{m}\otimes E_{ad}^{(j)}(\underline{\mathbf{v}})$$

$$
\begin{aligned}
&+ \delta_{ad}\delta_{bc}\frac{\kappa}{4}\sum_{r=1}^{m+n}\sum_{j=1}^{l}(-1)^{|r|}(-1)^{(|r|+|d|)(|a|+|r|)}\mathbf{m}\otimes E_{rr}^{(j)}(\underline{\mathbf{v}}) \\
&+ \delta_{ad}\frac{\kappa}{4}(-1)^{(|a|+|b|)(|c|+|d|)}\sum_{r=1}^{m+n}\sum_{j=1}^{l}(-1)^{|r|}\mathbf{m}\otimes E_{cb}^{(j)}(\underline{\mathbf{v}}) \\
&+ \delta_{ad}\delta_{bc}\frac{\kappa}{4}(-1)^{(|a|+|b|)(|c|+|d|)}\sum_{r=1}^{m+n}\sum_{j=1}^{l}(-1)^{|r|}(-1)^{(|c|+|r|)(|r|+|b|)}\mathbf{m}\otimes E_{rr}^{(j)}(\underline{\mathbf{v}}) \\
=&\delta_{bc}\frac{\kappa}{4}\sum_{r=1}^{m+n}\sum_{j\neq k}(-1)^{|r|}\mathbf{m}\otimes E_{ar}^{(j)}E_{rd}^{(k)}(\underline{\mathbf{v}}) \\
&+ \delta_{bc}\frac{\kappa}{4}\sum_{r=1}^{m+n}\sum_{j\neq k}(-1)^{|r|}\mathbf{m}\otimes E_{ar}^{(k)}E_{rd}^{(j)}(\underline{\mathbf{v}}) \\
&+ \delta_{ad}\frac{\kappa}{4}(-1)^{(|a|+|b|)(|c|+|d|)}\sum_{r=1}^{m+n}\sum_{j\neq k}(-1)^{|r|}\mathbf{m}\otimes E_{cr}^{(j)}E_{rb}^{(k)}(\underline{\mathbf{v}}) \\
&+ \delta_{ad}\frac{\kappa}{4}(-1)^{(|a|+|b|)(|c|+|d|)}\sum_{r=1}^{m+n}\sum_{j\neq k}(-1)^{|r|}\mathbf{m}\otimes E_{cr}^{(k)}E_{rb}^{(j)}(\underline{\mathbf{v}}) \\
&+ \delta_{bc}\frac{\kappa}{4}\sum_{r=1}^{m+n}(-1)^{|r|}E_{ad}(\mathbf{m}\otimes\underline{\mathbf{v}}) \\
&+ \delta_{ad}\frac{\kappa}{4}(-1)^{(|a|+|b|)(|c|+|d|)}\sum_{r=1}^{m+n}(-1)^{|r|}E_{cb}(\mathbf{m}\otimes\underline{\mathbf{v}}) \\
=&\delta_{bc}\frac{\kappa}{2}\sum_{r=1}^{m+n}\sum_{j\neq k}(-1)^{|r|}\mathbf{m}\otimes E_{ar}^{(j)}E_{rd}^{(k)}(\underline{\mathbf{v}}) \\
&+ \delta_{ad}\frac{\kappa}{2}(-1)^{(|a|+|b|)(|c|+|d|)}\sum_{r=1}^{m+n}\sum_{j\neq k}(-1)^{|r|}\mathbf{m}\otimes E_{cr}^{(j)}E_{rb}^{(k)}(\underline{\mathbf{v}}) \\
&+ \delta_{bc}\frac{\kappa}{4}(m-n)E_{ad}(\mathbf{m}\otimes\underline{\mathbf{v}}) \\
&+ \delta_{ad}\frac{\kappa}{4}(-1)^{(|a|+|b|)(|c|+|d|)}(m-n)E_{cb}(\mathbf{m}\otimes\underline{\mathbf{v}})
\end{aligned}
$$

So,

$$
\delta_{bc}\frac{\kappa}{2}\sum_{i\neq k}\sum_{r=1}^{m+n}(-1)^{|r|}\mathbf{m}\otimes E_{ar}^{(i)}E_{rd}^{(k)}(\underline{\mathbf{v}})
$$

$$+ \delta_{ad}(-1)^{(|a|+|b|)(|c|+|a|)}\frac{\kappa}{2}\sum_{i\neq k}\sum_{r=1}^{m+n}(-1)^{|r|}\mathbf{m}\otimes E_{cr}^{(i)}E_{rb}^{(k)}(\underline{\mathbf{v}})$$

$$=\delta_{bc}\frac{\kappa}{4}\sum_{r=1}^{m+n}(-1)^{|r|}S(E_{ar},E_{rd})(\mathbf{m}\otimes\underline{\mathbf{v}})$$

$$+ \delta_{ad}\frac{\kappa}{4}(-1)^{(|a|+|b|)(|c|+|d|)}\sum_{r=1}^{m+n}(-1)^{|r|}S(E_{cr},E_{rb})(\mathbf{m}\otimes\underline{\mathbf{v}})$$

$$- \frac{\kappa}{4}(m-n)(\delta_{bc}E_{ad}+(-1)^{(|a|+|b|)(|c|+|d|)}\delta_{ad}E_{cb})(\mathbf{m}\otimes\underline{\mathbf{v}})$$

We return now to the main computations from (5.1).

$$[\mathsf{K}(E_{ab}),\mathsf{Q}(E_{cd})](\mathbf{m}\otimes\underline{\mathbf{v}}) \tag{5.2}$$

$$=\mathsf{P}([E_{ab},E_{cd}])(\mathbf{m}\otimes\underline{\mathbf{v}})+\frac{t}{2}(\delta_{bc}E_{ad}+(-1)^{(|a|+|b|)(|c|+|d|)}\delta_{ad}E_{cb})(\mathbf{m}\otimes\underline{\mathbf{v}})$$

$$-\kappa(-1)^{|a||b|+|a||c|+|b||c|}\left(\sum_{k\neq q}\mathbf{m}\otimes E_{cb}^{(k)}E_{ad}^{(q)}(\underline{\mathbf{v}})\right)$$

$$+\delta_{bc}\frac{\kappa}{2}\sum_{i\neq k}\sum_{r=1}^{m+n}(-1)^{|r|}\mathbf{m}\otimes E_{ar}^{(i)}E_{rd}^{(k)}(\underline{\mathbf{v}})$$

$$+\delta_{ad}(-1)^{(|a|+|b|)(|c|+|a|)}\frac{\kappa}{2}\sum_{i\neq k}\sum_{r=1}^{m+n}(-1)^{|r|}\mathbf{m}\otimes E_{cr}^{(i)}E_{rb}^{(k)}(\underline{\mathbf{v}})$$

$$=\mathsf{P}([E_{ab},E_{cd}])(m\otimes\underline{\mathbf{v}})+\frac{t}{2}(\delta_{bc}E_{ad}+(-1)^{(|a|+|b|)(|c|+|d|)}\delta_{ad}E_{cb})(\mathbf{m}\otimes\underline{\mathbf{v}})$$

$$-\kappa(-1)^{|a||b|+|a||c|+|b||c|}\left(\sum_{k\neq q}\mathbf{m}\otimes E_{cb}^{(k)}E_{ad}^{(q)}(\underline{\mathbf{v}})\right)$$

$$+\delta_{bc}\frac{\kappa}{4}\sum_{r=1}^{m+n}(-1)^{|r|}S(E_{ar},E_{rd})(\mathbf{m}\otimes\underline{\mathbf{v}})$$

$$+\delta_{ad}\frac{\kappa}{4}(-1)^{(|a|+|b|)(|c|+|d|)}\sum_{r=1}^{m+n}(-1)^{|r|}S(E_{cr},E_{rb})(\mathbf{m}\otimes\underline{\mathbf{v}})$$

$$-\frac{\kappa}{4}(m-n)(\delta_{bc}E_{ad}+(-1)^{(|a|+|b|)(|c|+|d|)}\delta_{ad}E_{cb})(\mathbf{m}\otimes\underline{\mathbf{v}})$$

For the last step, look at what expanding $S([E_{ab},E_{ij}],[E_{ji},E_{cd}])$ yields as in the non-super case, but using $\varepsilon(i,j)$ as a placeholder for a potential sign.

$$
\begin{aligned}
&\frac{\kappa}{4}(\delta_{bc}+\delta_{ad}) \sum_{1\le i\neq j\le m+n} \varepsilon(i,j) S([E_{ab},E_{ij}],[E_{ji},E_{cd}]) \\
=&\frac{\kappa}{4}(\delta_{bc}+\delta_{ad}) \sum_{1\le i\neq j\le m+n} \varepsilon(i,j) S(\delta_{ib}E_{aj}-(-1)^{(|a|+|b|)(|i|+|j|)}\delta_{ja}E_{ib}, \\
&\delta_{ic}E_{jd}-(-1)^{(|j|+|i|)(|c|+|d|)}\delta_{jd}E_{ci}) \\
=&\frac{\kappa}{4}\delta_{bc} \sum_{1\le i\neq j\le m+n} \varepsilon(i,j) S(\delta_{ib}E_{aj}-(-1)^{(|a|+|b|)(|i|+|j|)}\delta_{ja}E_{ib}, \\
&\delta_{ib}E_{jd}-(-1)^{(|j|+|i|)(|b|+|d|)}\delta_{jd}E_{bi}) \\
&+\frac{\kappa}{4}\delta_{ad} \sum_{1\le i\neq j\le m+n} \varepsilon(i,j) S(\delta_{ib}E_{aj}-(-1)^{(|a|+|b|)(|i|+|j|)}\delta_{ja}E_{ib}, \\
&\delta_{ic}E_{ja}-(-1)^{(|j|+|i|)(|c|+|a|)}\delta_{ja}E_{ci}) \\
=&\frac{\kappa}{4}\delta_{bc} \sum_{\substack{1\le j\le m+n \\ j\neq b}} \varepsilon(b,j) S(E_{aj}-(-1)^{(|a|+|b|)(|b|+|j|)}\delta_{ja}E_{bb}, E_{jd}-(-1)^{(|j|+|b|)(|b|+|d|)}\delta_{jd}E_{bb}) \\
&+\frac{\kappa}{4}\delta_{bc} \sum_{\substack{1\le i\neq j\le m+n \\ i\neq b}} \varepsilon(i,j) S(-(-1)^{(|a|+|b|)(|i|+|j|)}\delta_{ja}E_{ib}, -(-1)^{(|j|+|i|)(|b|+|d|)}\delta_{jd}E_{bi}) \\
&+\frac{\kappa}{4}\delta_{ad} \sum_{\substack{1\le i\le m+n \\ i\neq a}} \varepsilon(i,a) S(\delta_{ib}E_{aa}-(-1)^{(|a|+|b|)(|i|+|a|)}E_{ib}, \delta_{ic}E_{aa}-(-1)^{(|a|+|i|)(|c|+|a|)}E_{ci}) \\
&+\frac{\kappa}{4}\delta_{ad} \sum_{\substack{1\le i\neq j\le m+n \\ j\neq a}} \varepsilon(i,j) S(\delta_{ib}E_{aj}, \delta_{ic}E_{ja}) \\
=&\frac{\kappa}{4}\delta_{bc} \sum_{\substack{1\le j\le m+n \\ j\neq a,b,d}} \varepsilon(b,j) S(E_{aj},E_{jd}) + \varepsilon(b,a)\frac{\kappa}{4}\delta_{bc} S(E_{aa}-(-1)^{|a|+|b|}E_{bb}, E_{ad}) \\
&+\varepsilon(b,d)\frac{\kappa}{4}\delta_{bd} S(E_{ad}, E_{dd}-(-1)^{|d|+|b|}E_{bb}) \\
&+\frac{\kappa}{4}\delta_{ad} \sum_{\substack{1\le i\le m+n \\ i\neq a,b,c}} \varepsilon(i,a) S(-(-1)^{(|a|+|b|)(|i|+|a|)}E_{ib}, -(-1)^{(|a|+|i|)(|c|+|a|)}E_{ci}) \\
&+\varepsilon(b,a)\frac{\kappa}{4}\delta_{ad} S(E_{aa}-(-1)^{|a|+|b|}E_{bb}, -(-1)^{(|b|+|a|)(|c|+|a|)}E_{cb}) \\
&+\varepsilon(c,a)\frac{\kappa}{4}\delta_{ad} S(-(-1)^{(|a|+|b|)(|c|+|a|)}E_{cb}, E_{aa}-(-1)^{|c|+|a|}E_{cc})
\end{aligned}
$$

$$
\begin{aligned}
=&\frac{\kappa}{4}\delta_{bc}\sum_{1\leq j\leq m+n}\varepsilon(b,j)S(E_{aj},E_{jd})-\varepsilon(b,a)\frac{\kappa}{4}\delta_{bc}S(E_{aa},E_{ad})-\varepsilon(b,b)\frac{\kappa}{4}\delta_{bc}S(E_{ab},E_{bd})\\
&-\varepsilon(b,d)\frac{\kappa}{4}\delta_{bc}S(E_{ad},E_{dd})+\varepsilon(b,a)\frac{\kappa}{4}\delta_{bc}S(E_{aa},E_{ad})-(-1)^{|a|+|b|}\varepsilon(b,a)\frac{\kappa}{4}\delta_{bc}S(E_{bb},E_{ad})\\
&+\varepsilon(b,d)\frac{\kappa}{4}\delta_{bd}S(E_{ad},E_{dd})-(-1)^{|d|+|b|}\varepsilon(b,d)\frac{\kappa}{4}\delta_{bc}S(E_{ad},E_{bb})\\
&+\frac{\kappa}{4}\delta_{ad}\sum_{1\leq i\leq m+n}\varepsilon(i,a)(-1)^{(|i|+|a|)(|b|+|c|)}S(E_{ib},E_{ci})\\
&-\frac{\kappa}{4}\delta_{ad}\varepsilon(a,a)(-1)^{(|a|+|a|)(|b|+|c|)}S(E_{ab},E_{ca})-\frac{\kappa}{4}\delta_{ad}\varepsilon(a,b)(-1)^{(|b|+|a|)(|b|+|c|)}S(E_{bb},E_{cb})\\
&-\frac{\kappa}{4}\delta_{ad}\varepsilon(a,c)(-1)^{(|c|+|a|)(|b|+|c|)}S(E_{cb},E_{cc})\\
&-(-1)^{(|b|+|a|)(|c|+|a|)}\varepsilon(b,a)\frac{\kappa}{4}\delta_{ad}S(E_{aa},E_{cb})\\
&+(-1)^{|a|+|b|}(-1)^{(|b|+|a|)(|c|+|a|)}\varepsilon(b,a)\frac{\kappa}{4}\delta_{ad}S(E_{bb},E_{cb})\\
&-(-1)^{(|a|+|b|)(|c|+|a|)}\varepsilon(c,a)\frac{\kappa}{4}\delta_{ad}S(E_{cb},E_{aa})\\
&+(-1)^{(|a|+|b|)(|c|+|a|)}(-1)^{|c|+|a|}\varepsilon(c,a)\frac{\kappa}{4}\delta_{ad}S(E_{cb},E_{cc})\\
=&\frac{\kappa}{4}\delta_{bc}\sum_{1\leq j\leq m+n}\varepsilon(b,j)S(E_{aj},E_{jd})-\varepsilon(b,b)\frac{\kappa}{4}\delta_{bc}S(E_{ab},E_{cd})\\
&-((-1)^{|a|+|b|}\varepsilon(b,a)+(-1)^{|d|+|b|}\varepsilon(b,d))\frac{\kappa}{4}\delta_{bc}S(E_{bb},E_{ad})\\
&+\frac{\kappa}{4}\delta_{ad}\sum_{1\leq i\leq m+n}\varepsilon(i,a)(-1)^{(|i|+|a|)(|b|+|c|)}S(E_{ib},E_{ci})-\frac{\kappa}{4}\delta_{ad}\varepsilon(a,a)S(E_{ab},E_{cd})\\
&-(-1)^{(|b|+|a|)(|c|+|a|)}(\varepsilon(b,a)+\varepsilon(c,a))\frac{\kappa}{4}\delta_{ad}S(E_{cb},E_{ad})\\
=&\frac{\kappa}{4}\delta_{bc}\sum_{1\leq j\leq m+n}\varepsilon(b,j)S(E_{aj},E_{jd})+\frac{\kappa}{4}\delta_{ad}\sum_{1\leq i\leq m+n}\varepsilon(i,a)(-1)^{(|i|+|a|)(|b|+|c|)}S(E_{ib},E_{ci})\\
&-\frac{\kappa}{4}(\delta_{bc}\varepsilon(b,b)+\delta_{ad}\varepsilon(a,a))S(E_{ab},E_{cd})\\
&-((-1)^{|a|+|b|}\varepsilon(b,a)+(-1)^{|d|+|b|}\varepsilon(b,d))\frac{\kappa}{4}\delta_{bc}S(E_{cb},E_{ad})\\
&-(-1)^{(|b|+|a|)(|c|+|a|)}(\varepsilon(b,a)+\varepsilon(c,a))\frac{\kappa}{4}\delta_{ad}S(E_{cb},E_{ad})\\
=&\frac{\kappa}{4}\delta_{bc}\sum_{1\leq j\leq m+n}\varepsilon(b,j)S(E_{aj},E_{jd})\\
&+\frac{\kappa}{4}\delta_{ad}\sum_{1\leq i\leq m+n}\varepsilon(i,a)(-1)^{(|i|+|a|)(|b|+|c|)}(-1)^{(|i|+|b|)(|c|+|i|)}S(E_{ci},E_{ib})\\
&-\frac{\kappa}{4}(\delta_{bc}\varepsilon(b,b)+\delta_{ad}\varepsilon(a,a))S(E_{ab},E_{cd})
\end{aligned}
$$

$$
\begin{aligned}
&- ((-1)^{|a|+|b|}\varepsilon(b,a) + (-1)^{|d|+|b|}\varepsilon(b,d))\frac{\kappa}{4}\delta_{bc}S(E_{cb},E_{ad})\\
&- (-1)^{(|b|+|a|)(|c|+|a|)}(\varepsilon(b,a)+\varepsilon(c,a))\frac{\kappa}{4}\delta_{ad}S(E_{cb},E_{ad})
\end{aligned}
$$

We need $\varepsilon(i,a)(-1)^{(|i|+|a|)(|b|+|c|)}(-1)^{(|i|+|b|)(|c|+|i|)} = (-1)^{|i|}(-1)^{(|a|+|b|)(|c|+|a|)}$, which gives $\varepsilon(i,a) = (-1)^{|a|}$. So, $\varepsilon(a,a) = \varepsilon(b,a) = \varepsilon(c,a) = (-1)^{|a|}$. Letting $\varepsilon(i,j) = (-1)^{|j|}$ gives $\varepsilon(b,j) = (-1)^{|j|}$, $\varepsilon(b,b) = (-1)^{|b|}$, $\varepsilon(b,d) = (-1)^{|d|}$. So,

$$
\begin{aligned}
&\frac{\kappa}{4}(\delta_{bc}+\delta_{ad})\sum_{1\le i\ne j\le m+n}\varepsilon(i,j)S([E_{ab},E_{ij}],[E_{ji},E_{cd}])\\
=&\frac{\kappa}{4}\delta_{bc}\sum_{1\le j\le m+n}(-1)^{|j|}S(E_{aj},E_{jd}) + \frac{\kappa}{4}\delta_{ad}(-1)^{(|a|+|b|)(|c|+|d|)}\sum_{1\le i\le m+n}(-1)^{|i|}S(E_{ci},E_{ib})\\
&- \frac{\kappa}{4}(\delta_{bc}(-1)^{|b|}+\delta_{ad}(-1)^{|a|})S(E_{ab},E_{cd})\\
&- (2(-1)^{|b|})\frac{\kappa}{4}\delta_{bc}S(E_{cb},E_{ad}) - (-1)^{(|b|+|a|)(|c|+|a|)}(2(-1)^{|a|})\frac{\kappa}{4}\delta_{ad}S(E_{cb},E_{ad})\\
=&\frac{\kappa}{4}\delta_{bc}\sum_{1\le j\le m+n}(-1)^{|j|}S(E_{aj},E_{jd}) + \frac{\kappa}{4}\delta_{ad}(-1)^{(|a|+|b|)(|c|+|d|)}\sum_{1\le i\le m+n}(-1)^{|i|}S(E_{ci},E_{ib})\\
&- \frac{\kappa}{4}((-1)^{|b|}\delta_{bc}+(-1)^{|a|}\delta_{ad})S(E_{ab},E_{cd})\\
&- \frac{\kappa}{2}\left((-1)^{|b|}\delta_{bc}+(-1)^{|b||c|+|a||b|+|a||c|}\delta_{ad}\right)S(E_{cb},E_{ad})
\end{aligned}
$$

Note that if $b=c$, $(-1)^{|b|} = (-1)^{|b|+(|b|+|c|)(|a|+|b|)} = (-1)^{|b||c|+|a||c|+|a||b|}$.

Thus, continuing from (5.2),

$$
\begin{aligned}
&[\mathsf{K}(E_{ab}),\mathsf{Q}(E_{cd})](\mathbf{m}\otimes\underline{\mathbf{v}})\\
&\quad=\mathsf{P}([E_{ab},E_{cd}])(\mathbf{m}\otimes\underline{\mathbf{v}}) + \frac{t}{2}(\delta_{bc}E_{ad}+(-1)^{(|a|+|b|)(|c|+|d|)}\delta_{ad}E_{cb})(\mathbf{m}\otimes\underline{\mathbf{v}})\\
&\qquad- \kappa(-1)^{|a||b|+|a||c|+|b||c|}\left(\sum_{k\ne q}\mathbf{m}\otimes E_{cb}^{(k)}E_{ad}^{(q)}(\underline{\mathbf{v}})\right)\\
&\qquad+ \frac{\kappa}{4}(\delta_{bc}+\delta_{ad})\sum_{1\le i\ne j\le m+n}(-1)^{|j|+(|b|+|i|)(|c|+|i|)}S([E_{ab},E_{ij}],[E_{ji},E_{cd}])(\mathbf{m}\otimes\underline{\mathbf{v}})\\
&\qquad- \frac{\kappa}{4}(m-n)(\delta_{bc}E_{ad}+(-1)^{(|a|+|b|)(|c|+|d|)}\delta_{ad}E_{cb})(\mathbf{m}\otimes\underline{\mathbf{v}})\\
&\qquad+ \frac{\kappa}{4}((-1)^{|b|}\delta_{bc}+(-1)^{|a|}\delta_{ad})S(E_{ab},E_{cd})(\mathbf{m}\otimes\underline{\mathbf{v}})
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\kappa}{2}\left((-1)^{|b|}\delta_{bc}+(-1)^{|b||c|+|a||b|+|a||c|}\delta_{ad}\right)S(E_{cb},E_{ad})(\mathbf{m}\otimes\underline{\mathbf{v}})\\
=&\mathsf{P}([E_{ab},E_{cd}])(\mathbf{m}\otimes\underline{\mathbf{v}})+\frac{t}{2}(\delta_{bc}E_{ad}+(-1)^{(|a|+|b|)(|c|+|d|)}\delta_{ad}E_{cb})(\mathbf{m}\otimes\underline{\mathbf{v}})\\
&-\kappa(-1)^{|a||b|+|a||c|+|b||c|}\left(\sum_{k\neq q}\mathbf{m}\otimes E_{cb}^{(k)}E_{ad}^{(q)}(\underline{\mathbf{v}})\right)\\
&+\frac{\kappa}{4}(\delta_{bc}+\delta_{ad})\sum_{1\leq i\neq j\leq m+n}(-1)^{|j|+(|b|+|i|)(|c|+|i|)}S([E_{ab},E_{ij}],[E_{ji},E_{cd}])(\mathbf{m}\otimes\underline{\mathbf{v}})\\
&-\frac{\kappa}{4}(m-n)(\delta_{bc}E_{ad}+(-1)^{(|a|+|b|)(|c|+|d|)}\delta_{ad}E_{cb})(\mathbf{m}\otimes\underline{\mathbf{v}})\\
&+\frac{\kappa}{4}((-1)^{|b|}\delta_{bc}+(-1)^{|a|}\delta_{ad})S(E_{ab},E_{cd})(\mathbf{m}\otimes\underline{\mathbf{v}})\\
&+\frac{\kappa}{2}\left((-1)^{|b||c|+|a||b|+|a||c|}\delta_{bc}+(-1)^{|b||c|+|a||b|+|a||c|}\delta_{ad}\right)S(E_{cb},E_{ad})(\mathbf{m}\otimes\underline{\mathbf{v}})\\
=&\mathsf{P}([E_{ab},E_{cd}])(\mathbf{m}\otimes\underline{\mathbf{v}})+\frac{t}{2}(\delta_{bc}E_{ad}+(-1)^{(|a|+|b|)(|c|+|d|)}\delta_{ad}E_{cb})(\mathbf{m}\otimes\underline{\mathbf{v}})\\
&-\kappa(-1)^{|a||b|+|a||c|+|b||c|}\left(\sum_{k\neq q}\mathbf{m}\otimes E_{cb}^{(k)}E_{ad}^{(q)}(\underline{\mathbf{v}})\right)\\
&+\frac{\kappa}{4}(\delta_{bc}+\delta_{ad})\sum_{1\leq i\neq j\leq m+n}(-1)^{|j|+(|b|+|i|)(|c|+|i|)}S([E_{ab},E_{ij}],[E_{ji},E_{cd}])(\mathbf{m}\otimes\underline{\mathbf{v}})\\
&-\frac{\kappa}{4}(m-n)(\delta_{bc}E_{ad}+(-1)^{(|a|+|b|)(|c|+|d|)}\delta_{ad}E_{cb})(\mathbf{m}\otimes\underline{\mathbf{v}})\\
&+\frac{\kappa}{4}((-1)^{|b|}\delta_{bc}+(-1)^{|a|}\delta_{ad})S(E_{ab},E_{cd})(\mathbf{m}\otimes\underline{\mathbf{v}})\\
&+\frac{\kappa}{2}(-1)^{|b||c|+|a||b|+|a||c|}S(E_{cb},E_{ad})(\mathbf{m}\otimes\underline{\mathbf{v}})
\end{aligned}
$$

Since $a = d$ or $b = c$, $S(E_{cb}, E_{ad}) = E_{cb}E_{ad} + (-1)^{(|c|+|b|)(|a|+|d|)}E_{ad}E_{cb} = E_{cb}E_{ad} + E_{ad}E_{cb}$. Then,

$$
\begin{aligned}
&\frac{\kappa}{2}(-1)^{|b||c|+|a||b|+|a||c|}S(E_{cb},E_{ad})(\mathbf{m}\otimes\underline{\mathbf{v}})\\
=&\frac{\kappa}{2}(-1)^{|b||c|+|a||b|+|a||c|}(E_{cb}E_{ad}+(-1)^{(|a|+|d|)(|c|+|b|)}E_{ad}E_{cb})(\mathbf{m}\otimes\underline{\mathbf{v}})\\
=&\frac{\kappa}{2}(-1)^{|b||c|+|a||b|+|a||c|}\sum_{i=1}^{m+n}E_{cb}(\mathbf{m}\otimes E_{ad}^{(i)}(\underline{\mathbf{v}}))\\
&+\frac{\kappa}{2}(-1)^{|b||c|+|a||b|+|a||c|}(-1)^{(|a|+|d|)(|c|+|b|)}\sum_{i=1}^{m+n}E_{ad}(\mathbf{m}\otimes E_{cb}^{(i)}(\underline{\mathbf{v}}))
\end{aligned}
$$

$$=\frac{\kappa}{2}(-1)^{|b||c|+|a||b|+|a||c|}\sum_{i,j=1}^{m+n}\mathbf{m}\otimes E_{cb}^{(j)}E_{ad}^{(i)}(\underline{\mathbf{v}})$$
$$+\frac{\kappa}{2}(-1)^{|b||c|+|a||b|+|a||c|}(-1)^{(|a|+|d|)(|c|+|b|)}\sum_{i,j=1}^{m+n}\mathbf{m}\otimes E_{ad}^{(j)}E_{cb}^{(i)}(\underline{\mathbf{v}})$$
$$=\frac{\kappa}{2}(-1)^{|b||c|+|a||b|+|a||c|}\left(\sum_{i\neq j}\mathbf{m}\otimes E_{cb}^{(j)}E_{ad}^{(i)}(\underline{\mathbf{v}})\right)$$
$$+\frac{\kappa}{2}(-1)^{|b||c|+|a||b|+|a||c|}\left(\sum_{i\neq j}\mathbf{m}\otimes E_{cb}^{(i)}E_{ad}^{(j)}(\underline{\mathbf{v}})\right)$$
$$=\kappa(-1)^{|b||c|+|a||b|+|a||c|}\sum_{i\neq j}\mathbf{m}\otimes E_{cb}^{(j)}E_{ad}^{(i)}(\underline{\mathbf{v}})$$

Thus, we get

$$[\mathsf{K}(E_{ab}),\mathsf{Q}(E_{cd})](\mathbf{m}\otimes\underline{\mathbf{v}})$$
$$=\mathsf{P}([E_{ab},E_{cd}])(m\otimes\underline{\mathbf{v}})+\frac{t}{2}(\delta_{bc}E_{ad}+(-1)^{(|a|+|b|)(|c|+|d|)}\delta_{ad}E_{cb})(\mathbf{m}\otimes\underline{\mathbf{v}})$$
$$-\kappa(-1)^{|a||b|+|a||c|+|b||c|}\left(\sum_{k\neq q}\mathbf{m}\otimes E_{cb}^{(k)}E_{ad}^{(q)}(\underline{\mathbf{v}})\right)$$
$$+\frac{\kappa}{4}(\delta_{bc}+\delta_{ad})\sum_{1\leq i\neq j\leq m+n}(-1)^{|j|+(|b|+|i|)(|c|+|i|)}S([E_{ab},E_{ij}],[E_{ji},E_{cd}])(\mathbf{m}\otimes\underline{\mathbf{v}})$$
$$-\frac{\kappa}{4}(m-n)(\delta_{bc}E_{ad}+(-1)^{(|a|+|b|)(|c|+|d|)}\delta_{ad}E_{cb})(\mathbf{m}\otimes\underline{\mathbf{v}})$$
$$+\frac{\kappa}{4}((-1)^{|b|}\delta_{bc}+(-1)^{|a|}\delta_{ad})S(E_{ab},E_{cd})(\mathbf{m}\otimes\underline{\mathbf{v}})$$
$$+\frac{\kappa}{2}(-1)^{|b||c|+|a||b|+|a||c|}S(E_{cb},E_{ad})(\mathbf{m}\otimes\underline{\mathbf{v}})$$
$$=\mathsf{P}([E_{ab},E_{cd}])(\mathbf{m}\otimes\underline{\mathbf{v}})+\frac{t}{2}(\delta_{bc}E_{ad}+(-1)^{(|a|+|b|)(|c|+|d|)}\delta_{ad}E_{cb})(\mathbf{m}\otimes\underline{\mathbf{v}})$$
$$-\kappa(-1)^{|a||b|+|a||c|+|b||c|}\left(\sum_{k\neq q}\mathbf{m}\otimes E_{cb}^{(k)}E_{ad}^{(q)}(\underline{\mathbf{v}})\right)$$
$$+\frac{\kappa}{4}(\delta_{bc}+\delta_{ad})\sum_{1\leq i\neq j\leq m+n}(-1)^{|j|+(|b|+|i|)(|c|+|i|)}S([E_{ab},E_{ij}],[E_{ji},E_{cd}])(\mathbf{m}\otimes\underline{\mathbf{v}})$$
$$-\frac{\kappa}{4}(m-n)(\delta_{bc}E_{ad}+(-1)^{(|a|+|b|)(|c|+|d|)}\delta_{ad}E_{cb})(\mathbf{m}\otimes\underline{\mathbf{v}})$$
$$+\frac{\kappa}{4}((-1)^{|b|}\delta_{bc}+(-1)^{|a|}\delta_{ad})S(E_{ab},E_{cd})(\mathbf{m}\otimes\underline{\mathbf{v}})$$

$$+ \kappa(-1)^{|b||c|+|a||b|+|a||c|} \sum_{i\neq j} \mathbf{m} \otimes E_{cb}^{(j)} E_{ad}^{(i)}(\underline{\mathbf{v}})$$

$$= \mathsf{P}([E_{ab}, E_{cd}])(\mathbf{m} \otimes \underline{\mathbf{v}}) + \frac{t}{2}(\delta_{bc}E_{ad} + (-1)^{(|a|+|b|)(|c|+|d|)}\delta_{ad}E_{cb})(\mathbf{m} \otimes \underline{\mathbf{v}})$$

$$+ \frac{\kappa}{4}(\delta_{bc} + \delta_{ad}) \sum_{1\leq i\neq j\leq m+n} (-1)^{|j|+(|b|+|i|)(|c|+|i|)} S([E_{ab}, E_{ij}], [E_{ji}, E_{cd}])(\mathbf{m} \otimes \underline{\mathbf{v}})$$

$$- \frac{\kappa}{4}(m-n)(\delta_{bc}E_{ad} + (-1)^{(|a|+|b|)(|c|+|d|)}\delta_{ad}E_{cb})(\mathbf{m} \otimes \underline{\mathbf{v}})$$

$$+ \frac{\kappa}{4}((-1)^{|b|}\delta_{bc} + (-1)^{|a|}\delta_{ad})S(E_{ab}, E_{cd})(\mathbf{m} \otimes \underline{\mathbf{v}})$$

From the above computations, we derive that the following relation should hold in $\mathcal{D}_{t,\kappa}(\mathfrak{sl}_{m|n})$, although the element $\mathsf{P}([E_{ab}, E_{cd}])$ has not yet been defined.

$$\begin{aligned}
[\mathsf{K}(E_{ab}),&\mathsf{Q}(E_{cd})] \qquad (5.3)\\
&= \mathsf{P}([E_{ab}, E_{cd}]) + \frac{t}{2}(\delta_{bc}E_{ad} + (-1)^{(|a|+|b|)(|c|+|d|)}\delta_{ad}E_{cb})\\
&+ \frac{\kappa}{4}(\delta_{bc} + \delta_{ad}) \sum_{1\leq i\neq j\leq m+n} (-1)^{|j|+(|b|+|i|)(|c|+|i|)} S([E_{ab}, E_{ij}], [E_{ji}, E_{cd}])\\
&- \frac{\kappa}{4}(m-n)(\delta_{bc}E_{ad} + (-1)^{(|a|+|b|)(|c|+|d|)}\delta_{ad}E_{cb})\\
&+ \frac{\kappa}{4}((-1)^{|b|}\delta_{bc} + (-1)^{|a|}\delta_{ad})S(E_{ab}, E_{cd})
\end{aligned}$$

The relation $[F_{ab}(u), F_{bc}(v)] = [F_{ad}(v), F_{dc}(u)]$ from Proposition 3.1.4 suggests that we should compute the following difference:

$$\begin{aligned}
[\mathsf{K}(E_{ab}),&\mathsf{Q}(E_{bc})] - [\mathsf{Q}(E_{ad}), \mathsf{K}(E_{dc})] \qquad (5.4)\\
&= [\mathsf{K}(E_{ab}), \mathsf{Q}(E_{bc})] + (-1)^{(|a|+|d|)(|c|+|d|)}[\mathsf{K}(E_{dc}), \mathsf{Q}(E_{ad})] \qquad (5.5)\\
&= \mathsf{P}([E_{ab}, E_{bc}]) + \frac{t}{2}E_{ad}\\
&+ \frac{\kappa}{4} \sum_{1\leq i\neq j\leq m+n} (-1)^{|j|+(|b|+|i|)(|b|+|i|)} S([E_{ab}, E_{ij}], [E_{ji}, E_{bc}])\\
&- \frac{\kappa}{4}(m-n)(E_{ac})\\
&+ \frac{\kappa}{4}(-1)^{|b|}S(E_{ab}, E_{bc}) + (-1)^{(|a|+|d|)(|c|+|d|)} \Bigg(\mathsf{P}([E_{dc}, E_{ad}]) + \frac{t}{2}((-1)^{(|d|+|c|)(|a|+|d|)}E_{cb})
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\kappa}{4}\sum_{1\le i\neq j\le m+n}(-1)^{|j|+(|c|+|i|)(|a|+|i|)}S([E_{dc},E_{ij}],[E_{ji},E_{ad}])\\
&-\frac{\kappa}{4}(m-n)((-1)^{(|d|+|c|)(|a|+|d|)}E_{ca})\\
&+\frac{\kappa}{4}(-1)^{|d|}S(E_{dc},E_{ad})\Big)\\
=&\mathsf{P}([E_{ab},E_{bc}])+(-1)^{(|a|+|d|)(|c|+|d|)}\mathsf{P}([E_{dc},E_{ad}]) \qquad (5.6)
\end{aligned}
$$

After cancelling out coordinating terms, what is left belongs in $U(\mathfrak{sl}_{m|n})$.

Next, we consider a,b,c,d such that $[E_{ab},E_{cd}]=0$.

$$
\begin{aligned}
[\mathsf{K}(E_{ab}),&\mathsf{Q}(E_{cd})](\mathbf{m}\otimes\underline{\mathbf{v}})=\mathsf{K}(E_{ab})\left(\sum_{q=1}^{l}\mathbf{m}y_q\otimes E_{cd}^{(q)}(\underline{\mathbf{v}})\right)\\
&-(-1)^{(|a|+|b|)(|c|+|d|)}\mathsf{Q}(E_{cd})\left(\sum_{k=1}^{l}\mathbf{m}x_k\otimes E_{ab}^{(k)}(\underline{\mathbf{v}})\right)\\
&=\sum_{k,q=1}^{l}\mathbf{m}y_qx_k\otimes E_{ab}^{(k)}E_{cd}^{(q)}(\underline{\mathbf{v}})-(-1)^{(|a|+|b|)(|c|+|d|)}\sum_{k,q}\mathbf{m}x_ky_q\otimes E_{cd}^{(q)}E_{ab}^{(k)}(\underline{\mathbf{v}})\\
&=\sum_{k,q=1,k\neq q}^{l}\mathbf{m}(y_qx_k-x_ky_q)\otimes E_{ab}^{(k)}E_{cd}^{(q)}(\underline{\mathbf{v}})\\
&=-\kappa\sum_{k,q=1,k\neq q}^{l}\mathbf{m}\sigma_{kq}\otimes E_{ab}^{(k)}E_{cd}^{(q)}(\underline{\mathbf{v}})\\
&=-\kappa\sum_{k,q=1,k\neq q}^{l}\mathbf{m}\otimes\sigma_{kq}E_{ab}^{(k)}E_{cd}^{(q)}(\underline{\mathbf{v}})\\
&=-\kappa\sum_{k,q=1,k\neq q}^{l}\sum_{e,f=1}^{m+n}\mathbf{m}\otimes(-1)^{|f|}E_{ef}^{(k)}E_{fe}^{(q)}E_{ab}^{(k)}E_{cd}^{(q)}(\underline{\mathbf{v}})\\
&=-\kappa\sum_{k,q=1,k\neq q}^{l}(-1)^{|a|}\mathbf{m}\otimes(-1)^{(|c|+|a|)(|a|+|b|)}E_{ca}^{(k)}E_{ab}^{(k)}E_{ac}^{(q)}E_{cd}^{(q)}(\underline{\mathbf{v}})\\
&=-\kappa\sum_{k,q=1,k\neq q}^{l}(-1)^{|a|+(|c|+|a|)(|a|+|b|)}\mathbf{m}\otimes E_{cb}^{(k)}E_{ad}^{(q)}\\
&=-\kappa(-1)^{|a||b|+|a||c|+|b||c|}E_{cb}E_{ad}(\mathbf{m}\otimes\underline{\mathbf{v}})
\end{aligned}
$$

Thus,

$$[\mathsf{K}(E_{ab}), \mathsf{Q}(E_{cd})](\mathbf{m} \otimes \underline{\mathbf{v}}) = -\kappa(-1)^{|a||b|+|a||c|+|b||c|} E_{ad}E_{cb}(\mathbf{m} \otimes \underline{\mathbf{v}}). \tag{5.7}$$

From this, we can derive a relation in $\mathcal{D}_{t,\kappa}(\mathfrak{sl}_{m|n})$.

Once $\mathcal{D}_{\lambda,\beta}(\mathfrak{sl}_{m|n})$ is defined in terms of $E_{ab}, \mathsf{K}(E_{ab}), \mathsf{Q}(E_{ab})$, elements $\mathsf{P}(E_{ab})$ of $\mathcal{D}_{\lambda,\beta}(\mathfrak{sl}_{m|n})$ could be introduced using the relation (5.3). These $z, \mathsf{P}(z)$ for $z \in \mathfrak{sl}_{m|n}$ should generate a subalgebra of $\mathcal{D}_{\lambda,\beta}(\mathfrak{sl}_{m|n})$ isomorphic to $Y_\lambda(\mathfrak{sl}_{m|n})$. We would use the J-presentation of the Yangian as defined by Drinfeld [8], with $J(z)$ replaced by $\mathsf{P}(z)$.

Definition 5.1.1. The DDCA $\mathcal{D}_{t,\kappa}(\mathfrak{sl}_{m|n})$ is the algebra generated by $\mathfrak{sl}_{m|n}$ and elements $\mathsf{K}(E_{ab}), \mathsf{Q}(E_{ab})$ for $a \neq b$ such that $\mathsf{K}(E_{ab}), E_{cd}$ for all $E_{ab}, E_{cd} \in \mathfrak{sl}_{m|n}$ generate a subalgebra isomorphic to $U(\mathfrak{sl}_{m|n}[u])$. Similarly, $\mathsf{Q}(E_{ab}), E_{cd}$ for all $E_{ab}, E_{cd} \in \mathfrak{sl}_{m|n}$ generate a subalgebra isomorphic to $U(\mathfrak{sl}_{m|n}[v])$. Together, these elements satisfy the relations

$$\begin{aligned}
&[\mathsf{K}(E_{ab}),\mathsf{Q}(E_{bc})] - [\mathsf{Q}(E_{ad}), \mathsf{K}(E_{dc})] \\
&\quad = \frac{t}{2}(E_{ad} + E_{cb}) - \frac{\kappa}{4}(m-n)(E_{ac}) \\
&\qquad + (-1)^{(|a|+|d|)(|c|+|d|)}\Big(-\frac{\kappa}{4}(m-n)((-1)^{(|d|+|c|)(|a|+|d|)}E_{ca})\Big) \\
&\qquad + \frac{\kappa}{4}\sum_{1\le i\neq j\le m+n}(-1)^{|j|+(|b|+|i|)(|b|+|i|)}S([E_{ab}, E_{ij}], [E_{ji}, E_{bc}]) \\
&\qquad + (-1)^{(|a|+|d|)(|c|+|d|)}\frac{\kappa}{4}\sum_{1\le i\neq j\le m+n}(-1)^{|j|+(|c|+|i|)(|a|+|i|)}S([E_{dc}, E_{ij}], [E_{ji}, E_{ad}]) \\
&\qquad + \frac{\kappa}{4}(-1)^{|b|}S(E_{ab}, E_{bc}) + (-1)^{(|a|+|d|)(|c|+|d|)}\frac{\kappa}{4}(-1)^{|d|}S(E_{dc}, E_{ad})\Big)
\end{aligned} \tag{5.8}$$

and for a, b, c, d such that $[E_{ab}, E_{cd}] = 0$,

$$[\mathsf{K}(E_{ab}), \mathsf{Q}(E_{cd})] = -\kappa(-1)^{|a||b|+|a||c|+|b||c|}E_{ad}E_{cb}. \tag{5.9}$$

Theorem 5.1.2. *Let M be a right module over the rational Cherednik algebra $\mathsf{H}_{t,\kappa}(S_l)$). Then $M \otimes_{\mathbb{C}[S_l]} \left(\mathbb{C}^{m|n}\right)^{\otimes l}$ is a left module over $\mathcal{D}_{t,\kappa}(\mathfrak{sl}_{m|n})$, so we have a functor $SW : M \mapsto M \otimes_{\mathbb{C}[S_l]} \left(\mathbb{C}^{m|n}\right)^{\otimes l}$. Moreover, SW provides an equivalence between the category of right $\mathsf{H}_{t,\kappa}(S_l)$-modules and integrable left*

$\mathcal{D}_{t,\kappa}(\mathfrak{sl}_{m|n})$*-modules of level* l *if* $m, n > l$.

Proof. Take $m \geq l, n \geq l$. Let N be an integrable left-module over the DDCA $\mathcal{D}_{t,\kappa}(\mathfrak{sl}_{m|n})$ of level l. By Schur-Weyl duality for S_l and $\mathfrak{sl}_{m|n}$ as seen in Chapter 4, $N = M \otimes_{\mathbb{C}[S_l]} (\mathbb{C}^{m|n})^{\otimes l}$ where M is a right S_l-module.

There exists a homomorphism $U(\mathfrak{sl}_{m|n} \otimes \mathbb{C}[u]) \to \mathcal{D}_{t,\kappa}(\mathfrak{sl}_{m|n})$ sending $E_{ab} \otimes u \mapsto \mathsf{K}(E_{ab})$. Thus, N is also an integrable module over $U(\mathfrak{sl}_{m|n} \otimes \mathbb{C}[u])$ of level l. So, by the results of Chapter 4, $N = M \otimes_{\mathbb{C}[S_l]} (\mathbb{C}^{m|n})^{\otimes l}$ where M is a right module over $\mathbb{C}[x_1, \ldots, x_l] \rtimes S_l$. The analogous result holds replacing u with v and instead sending $E_{ab} \otimes v \mapsto \mathsf{Q}(E_{ab})$: $N = M \otimes_{\mathbb{C}[S_l]} (\mathbb{C}^{m|n})^{\otimes l}$ where M is a right module over $\mathbb{C}[y_1, \ldots, y_l] \rtimes S_l$.

Let $i < j$ and set $\underline{\mathbf{v}} = e_1 \otimes \cdots \otimes e_{i-1} \otimes e_i \otimes e_{i+1} \otimes \cdots \otimes e_{j-1} \otimes e_j \otimes e_{j+1} \otimes \cdots \otimes e_l$. We denote by $[\underline{\mathbf{v}}, r, s]$ the sum of signs $|e_r| + |e_{r+1}| + \cdots + |e_{s-1}| + |e_s|$. Set also:

$$\underline{\mathbf{u}} = e_1 \otimes \cdots \otimes e_{i-1} \otimes e_{l+1} \otimes e_{i+1} \otimes \cdots \otimes e_{j-1} \otimes e_{l+2} \otimes e_{j+1} \otimes \cdots \otimes e_l \quad (5.10)$$

$$\underline{\mathbf{w}} = e_1 \otimes \cdots \otimes e_{i-1} \otimes e_{l+2} \otimes e_{i+1} \otimes \cdots \otimes e_{j-1} \otimes e_{l+1} \otimes e_{j+1} \otimes \cdots \otimes e_l \quad (5.11)$$

Then, by the relation (5.7) derived above, we know that

$$[\mathsf{K}(E_{l+1,i}), \mathsf{Q}(E_{l+2,j})] = -\kappa(-1)^{|l+1||i|+|l+1||l+2|+|i||l+2|} E_{l+2,i} E_{l+1,j} \quad (5.12)$$

Take $\mathbf{m} \in M$. Apply both sides of (5.12) to $\mathbf{m} \otimes \underline{\mathbf{v}}$. This produces:

$$\begin{aligned}
&\sum_{k,q=1}^{l} \mathbf{m} \left(y_q x_k \otimes E^{(k)}_{l+1,i} E^{(q)}_{l+2,j}(\underline{\mathbf{v}}) - (-1)^{(|l+1|+|i|)(|l+2|+|j|)} x_k y_q \otimes E^{(q)}_{l+2,j} E^{(k)}_{l+1,i}(\underline{\mathbf{v}}) \right) \\
&\qquad = -\kappa(-1)^{|l+1||i|+|l+1||l+2|+|i||l+2|} \mathbf{m} \otimes E_{l+2,i} E_{l+1,j}(\underline{\mathbf{v}}) \\
&\Leftrightarrow \\
&\mathbf{m}(y_j x_i (-1)^{(|l+2|+|j|)|\underline{\mathbf{v}},1,j-1|+(|l+1|+|i|)[\underline{\mathbf{v}},1,i-1]} \\
&\qquad - x_i y_j (-1)^{(|l+1|+|j|)|\underline{\mathbf{v}},1,j-1|]+(|l+2|+|i|)[\underline{\mathbf{v}},1,i-1]}) \otimes \underline{\mathbf{u}} \qquad (5.13) \\
&\qquad = -\kappa(-1)^{|l+1||i|+|l+1||l+2|+|i||l+2|+(|l+1|+|j|)[\underline{\mathbf{v}},1,j-1]+(|i|+|l+2|)[\underline{\mathbf{v}},1,i-1]} \mathbf{m} \otimes \underline{\mathbf{w}}
\end{aligned}$$

Note that

$$\underline{\mathbf{w}} = (-1)^{|l+1||l+2|+|l+1|[\underline{\mathbf{v}},i+1,j-1]+|l+2|[\underline{\mathbf{v}},i+1,j-1]}\sigma_{ij}(\underline{\mathbf{u}}) \tag{5.14}$$

Thus, (5.13) is equivalent to

$$\mathbf{m}(y_jx_i - x_iy_j) \otimes \underline{\mathbf{u}} = -\kappa\mathbf{m} \otimes \sigma_{ij}(\underline{\mathbf{u}}) \tag{5.15}$$

$$\Leftrightarrow \mathbf{m}(y_jx_i - x_iy_j) \otimes \underline{\mathbf{u}} = -\kappa\mathbf{m}\sigma_{ij} \otimes \underline{\mathbf{u}} \tag{5.16}$$

$$\Leftrightarrow \mathbf{m}(y_jx_i - x_iy_j + \kappa\sigma_{ij}) \otimes \underline{\mathbf{u}} = 0 \tag{5.17}$$

As the entries of $\underline{\mathbf{u}}$ are distinct, by Lemma 4.2.2, $\underline{\mathbf{u}}$ generates $(\mathbb{C}^{m|n})^{\otimes l}$ as a module over $U(\mathfrak{sl}_{m|n})$.

Here, we see why we keep the conservative assumptions that $m \geq l, n \geq l, l \geq 2$. This implies that any partition of l is an (m,n)-hook partition. Thus (5.17) implies that $\mathbf{m}(y_jx_i - x_iy_j + \kappa\sigma_{ij}) = 0$. We conclude that M is a right module over

$$\mathbb{C} < x_1, \ldots, x_l, y_1, \ldots, y_l > \rtimes S_l/(x_ix_j - x_jx_i, y_iy_j - y_jy_i, y_jx_i - x_iy_j + \kappa\sigma_{ij}). \tag{5.18}$$

The other relations of the rational Cherednik algebra can be produced similarly. $\square$

For instance, we can reproduce the relation $[y_k, x_k] = t + \kappa\sum_{i=1}^{k-1}\sigma_{ik} + \kappa\sum_{j=k+1}^{l}\sigma_{jk}$. To do this, we compute

$$[\mathsf{K}(E_{m+n,1}), \mathsf{Q}(E_{1,m+n-1})] - [\mathsf{K}(E_{1,m+n-1}), \mathsf{Q}(E_{m+n,1})]$$

on $\mathbf{m} \otimes \underline{\mathbf{v}}$ where

$$\underline{\mathbf{v}} = e_2 \otimes e_3 \otimes \cdots \otimes e_i \otimes e_{m+n-1} \otimes e_{i+2} \otimes \cdots \otimes e_{l+1}.$$

$$[\mathsf{K}(E_{m+n,1}),\mathsf{Q}(E_{1,m+n-1})] \tag{5.19}$$

$$
\begin{aligned}
=&\mathsf{P}([E_{m+n,1}, E_{1,m+n-1}]) + \frac{t}{2}E_{m+n,m+n-1} \\
&+ \frac{\kappa}{4}\sum_{1\le i\ne j\le m+n}(-1)^{|j|+|i|}S([E_{m+n,1}, E_{ij}], [E_{ji}, E_{1,m+n-1}]) \\
&- \frac{\kappa}{4}(m-n)E_{m+n,m+n-1} \\
&+ \frac{\kappa}{4}S(E_{m+n,1}, E_{1,m+n-1}) \\
=&\mathsf{P}(E_{m+n,m+n-1}) + \left(\frac{t}{2} - \frac{\kappa}{4}(m-n)\right)E_{m+n,m+n-1} \\
&+ \frac{\kappa}{4}\sum_{1\le i\ne j\le m+n}(-1)^{|j|+|i|}S([E_{m+n,1}, E_{ij}], [E_{ji}, E_{1,m+n-1}]) \\
&+ \frac{\kappa}{4}S(E_{m+n,1}, E_{1,m+n-1})
\end{aligned}
$$

$$
\begin{aligned}
&[\mathsf{K}(E_{1,m+n-1}),\mathsf{Q}(E_{m+n,1})] \\
&=\mathsf{P}([E_{1,m+n-1}, E_{m+n,1}]) + \frac{t}{2}((-1)E_{m+n,m+n-1}) \\
&\quad+ \frac{\kappa}{4}\sum_{1\le i\ne j\le m+n}(-1)^{|j|+(1+|i|)}S([E_{1,m+n-1}, E_{ij}], [E_{ji}, E_{m+n,1}]) \\
&\quad- \frac{\kappa}{4}(m-n)((-1)E_{m+n,m+n-1}) \\
&\quad+ \frac{\kappa}{4}S(E_{1,m+n-1}, E_{m+n,1}) \\
&=\mathsf{P}(E_{m+n,m+n-1}) - \left(\frac{t}{2} - \frac{\kappa}{4}(m-n)\right)(E_{m+n,m+n-1}) \\
&\quad- \frac{\kappa}{4}\sum_{1\le i\ne j\le m+n}(-1)^{|j|+|i|}S([E_{1,m+n-1}, E_{ij}], [E_{ji}, E_{m+n,1}]) \\
&\quad- \frac{\kappa}{4}S(E_{m+n,1}, E_{1,m+n-1})
\end{aligned} \tag{5.20}
$$

Taking the difference (5.19)-(5.20) produces

$$
\begin{aligned}
&[\mathsf{K}(E_{m+n,1}), \mathsf{Q}(E_{1,m+n-1})] - [\mathsf{K}(E_{1,m+n-1}), \mathsf{Q}(E_{m+n,1})] \\
&\quad= \left(t - \frac{\kappa}{2}(m-n)\right)E_{m+n,m+n-1} \\
&\quad+ \frac{\kappa}{2}\sum_{1\le i\ne j\le m+n}(-1)^{|j|+|i|}S([E_{m+n,1}, E_{ij}], [E_{ji}, E_{1,m+n-1}])
\end{aligned} \tag{5.21}
$$

$$+ \frac{\kappa}{2} S(E_{m+n,1}, E_{1,m+n-1})$$

After applying the left-hand side of (5.21) to $\mathbf{m} \otimes \underline{\mathbf{v}}$, we have

$$\begin{aligned}
&\sum_{k,q=1}^{l} \mathbf{m} \left(y_q x_k \otimes E^{(k)}_{m+n,1} E^{(q)}_{1,m+n-1}(\underline{\mathbf{v}}) + x_k y_q \otimes E^{(q)}_{1,m+n-1} E^{(k)}_{m+n,1}(\underline{\mathbf{v}}) \right) \\
&\qquad - \sum_{k,q=1}^{l} \mathbf{m} \left(y_q x_k \otimes E^{(k)}_{1,m+n-1} E^{(q)}_{m+n,1}(\underline{\mathbf{v}}) + x_k y_q \otimes E^{(q)}_{m+n,1} E^{(k)}_{1,m+n-1}(\underline{\mathbf{v}}) \right) \\
&\quad = \sum_{k,q=1}^{l} \mathbf{m} y_q x_k \otimes E^{(k)}_{m+n,1} E^{(q)}_{1,m+n-1}(\underline{\mathbf{v}}) - \sum_{k,q=1}^{l} \mathbf{m} x_k y_q \otimes E^{(q)}_{m+n,1} E^{(k)}_{1,m+n-1}(\underline{\mathbf{v}}) \\
&\quad = \mathbf{m} y_i x_i \otimes \underline{\mathbf{u}} - x_i y_i \otimes \underline{\mathbf{u}} \\
&\quad = \mathbf{m}(y_i x_i - x_i y_i) \otimes \underline{\mathbf{u}} \qquad (5.22)
\end{aligned}$$

Here, $\underline{\mathbf{u}}$ is $\underline{\mathbf{v}}$ with e_{m+n-1} in ith position replaced by e_{m+n}.

We next apply the right-hand side of (5.21) to $\mathbf{m} \otimes \underline{\mathbf{v}}$.

$$\begin{aligned}
&\left(t - \frac{\kappa}{2}(m-n)\right) E_{m+n,m+n-1}(\underline{\mathbf{v}}) \\
&\qquad + \frac{\kappa}{2} \sum_{1 \le k \ne j \le m+n} (-1)^{|j|+|i|} S([E_{m+n,1}, E_{kj}], [E_{jk}, E_{1,m+n-1}])(\underline{\mathbf{v}}) \\
&\qquad + \frac{\kappa}{2} S(E_{m+n,1}, E_{1,m+n-1})(\underline{\mathbf{v}}) \\
&\quad = \left(t - \frac{\kappa}{2}(m-n)\right) \underline{\mathbf{u}} \\
&\qquad + \frac{\kappa}{2} \sum_{1 \ne j \le m+n} (-1)^{|j|} S(E_{m+n,j}, E_{j,m+n-1})(\underline{\mathbf{v}}) \\
&\qquad + \frac{\kappa}{2} S(E_{m+n,1}, E_{1,m+n-1})(\underline{\mathbf{v}}) \\
&\quad = \left(t - \frac{\kappa}{2}(m-n)\right) \underline{\mathbf{u}} \qquad (5.23) \\
&\qquad + \frac{\kappa}{2} \sum_{j=1}^{m+n} (-1)^{|j|} S(E_{m+n,j}, E_{j,m+n-1})(\underline{\mathbf{v}})
\end{aligned}$$

$$
\begin{aligned}
&\sum_{j=1}^{m+n}(-1)^{|j|}S(E_{m+n,j},E_{j,m+n-1})(\underline{\mathbf{v}})\\
&=\sum_{j=1}^{m+n}\sum_{k,q=1}^{l}(-1)^{|j|}(E^{(k)}_{m+n,j}E^{(q)}_{j,m+n-1}(\underline{\mathbf{v}})+E^{(k)}_{j,m+n-1}E^{(q)}_{m+n,j}(\underline{\mathbf{v}}))\\
&=\sum_{j=1}^{m+n}(-1)^{|j|}(\underline{\mathbf{u}})+2\sum_{j=2}^{m+n}\sum_{k\neq q}(-1)^{|j|}E^{(k)}_{m+n,j}E^{(q)}_{j,m+n-1}(\underline{\mathbf{v}})\\
&=(m-n)(\underline{\mathbf{u}})+2\sum_{j=2}^{m+n}\sum_{k\neq q}(-1)^{|j|}E^{(k)}_{m+n,j}E^{(q)}_{j,m+n-1}(\underline{\mathbf{v}})
\end{aligned}
$$

Thus, from (5.22) and (5.23) we deduce

$$
\mathbf{m}(y_ix_i-x_iy_i)\otimes\underline{\mathbf{u}} \tag{5.24}
$$

$$
=\left(t-\frac{\kappa}{2}(m-n)+\frac{\kappa}{2}(m-n)\right)(\mathbf{m}\otimes\underline{\mathbf{u}})+\kappa\sum_{k\neq i}\mathbf{m}\otimes\sigma_{ki}(\underline{\mathbf{u}})
$$

$$
=\mathbf{m}\left(t+\kappa\sum_{k\neq i}\sigma_{ki}\right)\otimes\underline{\mathbf{u}} \tag{5.25}
$$

From this, we can apply Lemma 4.2.2 to conclude that

$$
\mathbf{m}(y_ix_i-x_iy_i)=\mathbf{m}\left(t+\kappa\sum_{k\neq i}\sigma_{ki}\right). \tag{5.26}
$$

Together, with the previous calculations in the proof of this main theorem, we conclude that M is a right module over the rational Cherednik algebra.

5.2 On type Q

We expect to be able to produce an analogous deformed double current algebra for the type Q setting. We know the types of elements that should generate $\mathcal{D}_\kappa(\mathfrak{q}_n)$ as well as that it should contain both $U(\mathfrak{q}_n^{tw}[u])$ and $U(\mathfrak{q}_n[v])$. An alternative approach would be to use the Steinberg Lie superalgebra defined in Section 3.2. This strategy would be to modify the Steinberg's relations to define the DDCA in such a way that when the parameters for the DDCA are set to 0, we obtain the relations for $\widetilde{\mathfrak{st}}_n(A)$. The most complicated of these relations is

$$[\widetilde{F}_{ab}(u), \widetilde{F}_{bc}(v)] = [\widetilde{F}_{ad}(v), \widetilde{F}_{dc}(u)].$$

When determining how to modify this, one must look at the difference $[\widetilde{F}_{ab}(u), \widetilde{F}_{bc}(v)] - [\widetilde{F}_{ad}(v), \widetilde{F}_{dc}(u)]$ with the knowledge that it should lie in $U(\mathfrak{q}_n)$. This uses that $[\widetilde{F}_{ab}(u), \widetilde{F}_{bc}(v)]$ should equal the sum of $\mathsf{P}(E_{ac})$ and an element of $U(\mathfrak{q}_n)$. Similarly, we expect that $[\widetilde{F}_{ad}(v), \widetilde{F}_{dc}(u)]$ is the sum of $\mathsf{P}(E_{ac})$ and something in $U(\mathfrak{q}_n)$, so the difference of the two brackets should yield some element of $U(\mathfrak{q}_n)$. We also expect to have $Y(\mathfrak{q}_n) \subset \mathcal{D}(\mathfrak{q}_n)$, but this remains to be shown.

Chapter 6

Schur-Weyl Duality for Affine Yangians

In this chapter, we establish an equivalence of categories between right modules over the degenerate double affine Hecke algebra $\mathbb{H}_{\kappa,c}(S_\ell)$ and left modules over the affine super Yangian. This is a result Drinfeld has produced for the J presentation [8], Arakawa for the RTT presentation [1], and is now introduced here for the current presentation in the affine case.

6.1 Schur-Weyl Duality in the context of the current presentation

Let M be a right module over $\mathbb{H}_{\kappa,c}(S_\ell)$. We would like to define the action of the current operators $X^{\pm}_{k,r}, H_{k,r}$, $1 \leq k \leq m+n-1$, $r = 0, 1$ on the space $M \otimes_{\mathbb{C}[S_l]} \mathbb{C}(m|n)^{\otimes l}$.

Recall the notation $\overline{\sigma}_{ij} = (i\; i{+}1 \cdots j{-}1\; j)$ for $i < j$, $\overline{\sigma}_{ij} = (i\; i{-}1 \cdots j{+}1\; j)$ for $i > j$. If $\underline{\mathbf{v}} = e_{i_1} \otimes \cdots \otimes e_{i_l}$ with $i_1 \leq i_2 \leq \cdots \leq i_l$, denote by j_k the index at which e_k first appears. Denote by $\tilde{j}_k$ the index at which the last e_k appears. We denote

$$\underline{\mathbf{v}}^-_{j_{k+1}} = e_{i_1} \otimes \cdots \otimes \underbrace{e_k}_{\tilde{j}_k} \otimes \underbrace{e_k}_{j_{k+1}} \otimes e_{k+1} \otimes \cdots \otimes e_{i_l} \tag{6.1}$$

meaning that the first e_{k+1} has its index reduced to e_k. Conversely,

$$\underline{\mathbf{v}}^+_{j_k} = e_{i_1} \otimes \cdots \otimes \underbrace{e_k}_{\tilde{j}_k - 1} \otimes \underbrace{e_{k+1}}_{\tilde{j}_k} \otimes \underbrace{e_{k+1}}_{j_{k+1}} \otimes e_{k+1} \otimes \cdots \otimes e_{i_l} \tag{6.2}$$

sees the last e_k incremented to e_{k+1}.

The current generators of the Yangian $Y(\mathfrak{sl}_{m|n})$ act on $M \otimes_{\mathbb{C}[S_l]} \mathbb{C}(m|n)^{\otimes l}$ as follows:

$$X^+_{k,0}(\mathbf{m} \otimes \underline{\mathbf{v}}) = \mathbf{m} \left(\sum_{d=j_{k+1}}^{\tilde{j}_{k+1}} (-1)^{|k+1|(d-j_{k+1})} \overline{\sigma}_{d,j_{k+1}} \right) \otimes \underline{\mathbf{v}}_{j^-_{k+1}} \tag{6.3}$$

$$X^-_{k,0}(\mathbf{m} \otimes \underline{\mathbf{v}}) = (-1)^{|k|} \mathbf{m} \left(\sum_{d=j_k}^{\tilde{j}_k} (-1)^{|k|(\tilde{j}_k - d)} \overline{\sigma}_{d,\tilde{j}_k} \right) \otimes \underline{\mathbf{v}}_{j^+_k} \tag{6.4}$$

$$X^+_{k,1}(\mathbf{m} \otimes \underline{\mathbf{v}}) = \mathbf{m} \left(\sum_{d=j_{k+1}}^{\tilde{j}_{k+1}} (-1)^{|k+1|(d-j_{k+1})} \overline{\sigma}_{d,j_{k+1}} \right) \left(x_{j_{k+1}} - (-1)^{|k+1|} \frac{\lambda}{2}(m-k) \right) \otimes \underline{\mathbf{v}}^-_{j_{k+1}} \tag{6.5}$$

$$X^-_{k,1}(\mathbf{m} \otimes \underline{\mathbf{v}}) = (-1)^{|k|} \mathbf{m} \left(\sum_{d=j_k}^{\tilde{j}_k} (-1)^{|k+1|(\tilde{j}_k - d)} \overline{\sigma}_{d,\tilde{j}_k} \right) \left(x_{\tilde{j}_k} - (-1)^{|k+1|} \frac{\lambda}{2}(m-k) \right) \otimes \underline{\mathbf{v}}^+_{j_k} \tag{6.6}$$

$$H_{k,1}(\mathbf{m} \otimes \underline{\mathbf{v}}) = \mathbf{m}(-1)^{|k|} \sum_{d=j_k}^{\tilde{j}_k} \left(x_d - \frac{\lambda}{2}(-1)^{|k+1|}(m - k + (-1)^{\delta_{\kappa m}}) \right) \otimes \underline{\mathbf{v}} \tag{6.7}$$

$$- (-1)^{|k+1|} \mathbf{m} \sum_{e=j_{k+1}}^{\tilde{j}_{k+1}} \left(x_e - \frac{\lambda}{2}(-1)^{|k+1|}(m-k-1) \right) \otimes \underline{\mathbf{v}} + \frac{\lambda H^2_{k,0}}{2} \mathbf{m} \otimes \underline{\mathbf{v}}$$

where $H_{k,0}$ acts on $\mathbf{m} \otimes \underline{\mathbf{v}}$ using the natural action on $\underline{\mathbf{v}}$. We observe similarities with the established actions of the loop Yangian, in the $\mathfrak{sl}_n$ case, in its Schur-

Weyl duality with the trigonometric Cherednik algebra [14]. Below, we find that $\lambda = -\kappa$.

Note that it is enough to define these formulas only for nondecreasing $\underline{\mathbf{v}}$; otherwise, one could apply permutations to allow for decreasing indices and these permutations can be moved as needed around the tensor product. Suppose we have such a nondecreasing $\underline{\mathbf{u}}$ where there exists σ and nondecreasing $\underline{\mathbf{v}}$ giving

$$\begin{aligned} X^+_{k,1}(\mathbf{m} \otimes \underline{\mathbf{u}}) =& X^+_{k1}(\mathbf{m} \otimes \sigma\underline{\mathbf{v}}) \\ =& X^+_{k1}(\mathbf{m}\sigma \otimes \underline{\mathbf{v}}) \end{aligned}$$

It is possible that σ is not unique and we see $\underline{\mathbf{u}} = \sigma_1(\underline{\mathbf{v}}) = \sigma_2(\underline{\mathbf{v}})$, $\sigma_1 \neq \sigma_2$. For instance, say $\underline{\mathbf{u}} = e_1 \otimes e_2 \otimes e_1$, $\underline{\mathbf{v}} = e_1 \otimes e_1 \otimes e_2$. Then, we could have $\sigma_1 = (2\ 3)$ and $\sigma_2 = (1\ 2)\sigma_1$.

Take $\underline{\mathbf{u}} = e_{j_1} \otimes \cdots \otimes e_{j_l}$ with $j_1, \ldots, j_l$ not necessarily nondecreasing and $\underline{\mathbf{u}} = \sigma\underline{\mathbf{v}}$ with $\underline{\mathbf{v}}$ nondecreasing as previously defined. Then, we set $H_{k1}(\mathbf{m} \otimes \underline{\mathbf{u}}) = H_{k1}(\mathbf{m}\sigma \otimes \underline{\mathbf{v}})$ and $X^{\pm}_{k1}(\mathbf{m} \otimes \underline{\mathbf{u}}) = X^{\pm}_{k1}(\mathbf{m}\sigma \otimes \underline{\mathbf{v}})$. We have to show this does not depend on the choice of σ.

If $\sigma_1(\underline{\mathbf{v}}) = \sigma_2(\underline{\mathbf{v}})$, then $\sigma_1 = \sigma_2\sigma_3$ where $\sigma_3(\underline{\mathbf{v}}) = \underline{\mathbf{v}}$. As well,

$$\begin{aligned} H_{k1}(\mathbf{m}\sigma_1 \otimes \underline{\mathbf{v}}) &= H_{k1}(\mathbf{m}\sigma_2\sigma_3 \otimes \underline{\mathbf{v}}) \\ =& H_{k1}(\mathbf{m}\sigma_2 \otimes \sigma_3\underline{\mathbf{v}}) \\ =& H_{k1}(\mathbf{m}\sigma_2 \otimes \underline{\mathbf{v}}) \end{aligned}$$

This uses that σ_3 must commute with all x_d if it is in the stabilizer of $\underline{\mathbf{v}}$. Specifically, $\sigma_3\left(\sum_{d=j_k}^{\tilde{j}_k} x_d\right) = \left(\sum_{d=j_k}^{\tilde{j}_k} x_d\right)\sigma_3$. Say $\sigma_3 = (a\ \ a+1)$ with $j_k \leq a \leq \tilde{j}_k - 1$. Then $\sigma_3(x_a + x_{a+1}) = (x_a + x_{a+1})\sigma_3$. The same works with $X^{\pm}_{k1}$. Next, it remains to verify that the formulas (6.5), (6.6), and (6.7) define an action of the super Yangian.

Proof. The main relation to check for $1 \leq k \leq m+n-2$ is

$$[X^{\pm}_{k,1}, X^{\pm}_{k+1,0}](\mathbf{m} \otimes \underline{\mathbf{v}}) - [X^{\pm}_{k,0}, X^{\pm}_{k+1,1}](\mathbf{m} \otimes \underline{\mathbf{v}})$$

$$= \mp(-1)^{|k+1|}\frac{\lambda}{2}\{X^{\pm}_{k,0}, X^{\pm}_{k+1,0}\}(\mathbf{m}\otimes\underline{\mathbf{v}}). \qquad (6.8)$$

Let $\underline{\mathbf{v}}^{-}_{k+1,k+2}$ be obtained from $\underline{\mathbf{v}}$ by replacing the first e_{k+1} by e_k and the first e_{k+2} by e_{k+1}. (Here, as before, $\underline{\mathbf{v}} = e_{i_1}\otimes e_{i_2}\otimes\cdots\otimes e_{i_\ell}$ with $i_1 \le i_2 \le \cdots \le i_\ell$.)

$$
\begin{aligned}
&[X^{+}_{k,1}, X^{+}_{k+1,0}](\mathbf{m}\otimes\underline{\mathbf{v}}) - [X^{+}_{k,0}, X^{+}_{k+1,1}](\mathbf{m}\otimes\underline{\mathbf{v}})\\
=&X^{+}_{k,1}\left(\mathbf{m}\left(\sum_{d=j_{k+2}}^{\tilde{j}_{k+2}}(-1)^{|k+2|(d-j_{k+2})}\overline{\sigma}_{d,j_{k+2}}\right)\otimes\underline{\mathbf{v}}_{j^{-}_{k+2}}\right)\\
&-X^{+}_{k+1,0}\left(\mathbf{m}\left(\sum_{d=j_{k+1}}^{\tilde{j}_{k+1}}(-1)^{|k+1|(d-j_{k+1})}\overline{\sigma}_{d,j_{k+1}}\right)\left(x_{j_{k+1}}-(-1)^{|k+1|}\frac{\lambda}{2}(m-k)\right)\otimes\underline{\mathbf{v}}_{j^{-}_{k+1}}\right)\\
&-X^{+}_{k,0}\left(\mathbf{m}\left(\sum_{d=j_{k+2}}^{\tilde{j}_{k+2}}(-1)^{|k+2|(d-j_{k+2})}\overline{\sigma}_{d,j_{k+2}}\right)\left(x_{j_{k+2}}-(-1)^{|k+2|}\frac{\lambda}{2}(m-k-1)\right)\otimes\underline{\mathbf{v}}_{j^{-}_{k+2}}\right)\\
&+X^{+}_{k+1,1}\left(\mathbf{m}\left(\sum_{d=j_{k+1}}^{\tilde{j}_{k+1}}(-1)^{|k+1|(d-j_{k+1})}\overline{\sigma}_{d,j_{k+1}}\right)\otimes\underline{\mathbf{v}}_{j^{-}_{k+1}}\right)\\
=&\mathbf{m}\left(\sum_{d=j_{k+2}}^{\tilde{j}_{k+2}}(-1)^{|k+2|(d-j_{k+2})}\overline{\sigma}_{d,j_{k+2}}\right)\\
&\left(\sum_{e=j_{k+1}}^{\tilde{j}_{k+1}+1}(-1)^{|k+1|(e-j_{k+1})}\overline{\sigma}_{e,j_{k+1}}\right)\\
&\left(x_{j_{k+1}}-(-1)^{|k+1|}\frac{\lambda}{2}(m-k)\right)\otimes\underline{\mathbf{v}}^{-}_{k+1,k+2}\\
&-\mathbf{m}\left(\sum_{d=j_{k+1}}^{\tilde{j}_{k+1}}(-1)^{|k+1|(d-j_{k+1})}\overline{\sigma}_{d,j_{k+1}}\right)\left(x_{j_{k+1}}-(-1)^{|k+1|}\frac{\lambda}{2}(m-k)\right)\\
&\left(\sum_{e=j_{k+2}}^{\tilde{j}_{k+2}}(-1)^{|k+2|(e-j_{k+2})}\overline{\sigma}_{e,j_{k+2}}\right)\otimes\underline{\mathbf{v}}^{-}_{k+1,k+2}\\
&-\mathbf{m}\left(\sum_{d=j_{k+2}}^{\tilde{j}_{k+2}}(-1)^{|k+2|(d-j_{k+2})}\overline{\sigma}_{d,j_{k+2}}\right)\left(x_{j_{k+2}}-(-1)^{|k+2|}\frac{\lambda}{2}(m-k-1)\right)
\end{aligned}
$$

$$\left(\sum_{e=j_{k+1}}^{\tilde{j}_{k+1}+1}(-1)^{|k+1|(e-j_{k+1})}\overline{\sigma}_{e,j_{k+1}}\right)\otimes\underline{\mathbf{v}}^{-}_{k+1,k+2}$$

$$+\,\mathbf{m}\left(\sum_{d=j_{k+1}}^{\tilde{j}_{k+1}}(-1)^{|k+1|(d-j_{k+1})}\overline{\sigma}_{d,j_{k+1}}\right)\left(\sum_{e=j_{k+2}}^{\tilde{j}_{k+2}}(-1)^{|k+2|(e-j_{k+2})}\overline{\sigma}_{e,j_{k+2}}\right)$$

$$\left(x_{j_{k+2}}-(-1)^{|k+2|}\frac{\lambda}{2}(m-k-1)\right)\otimes\underline{\mathbf{v}}^{-}_{k+1,k+2}$$

$$=\mathbf{m}\left(\sum_{d=j_{k+2}}^{\tilde{j}_{k+2}}(-1)^{|k+2|(d-j_{k+2})}\overline{\sigma}_{d,j_{k+2}}\right)$$

$$\left((-1)^{|k+1|(j_{k+2}-j_{k+1})}\overline{\sigma}_{j_{k+2},j_{k+1}}+\sum_{e=j_{k+1}}^{\tilde{j}_{k+1}}(-1)^{|k+1|(e-j_{k+1})}\overline{\sigma}_{e,j_{k+1}}\right)$$

$$\left(x_{j_{k+1}}-(-1)^{|k+1|}\frac{\lambda}{2}(m-k)\right)\otimes\underline{\mathbf{v}}^{-}_{k+1,k+2}$$

$$-\,\mathbf{m}\left(\sum_{d=j_{k+1}}^{\tilde{j}_{k+1}}(-1)^{|k+1|(d-j_{k+1})}\overline{\sigma}_{d,j_{k+1}}\right)\left(\sum_{e=j_{k+2}}^{\tilde{j}_{k+2}}(-1)^{|k+2|(e-j_{k+2})}\overline{\sigma}_{e,j_{k+2}}\right)$$

$$\left(x_{j_{k+1}}-(-1)^{|k+1|}\frac{\lambda}{2}(m-k)\right)\otimes\underline{\mathbf{v}}^{-}_{k+1,k+2}$$

$$-\,\mathbf{m}\left(\sum_{d=j_{k+2}}^{\tilde{j}_{k+2}}(-1)^{|k+2|(d-j_{k+2})}\overline{\sigma}_{d,j_{k+2}}\right)\left(x_{j_{k+2}}-(-1)^{|k+2|}\frac{\lambda}{2}(m-k-1)\right)$$

$$\left((-1)^{|k+1|(j_{k+2}-j_{k+1})}\overline{\sigma}_{j_{k+2},j_{k+1}}+\sum_{e=j_{k+1}}^{\tilde{j}_{k+1}}(-1)^{|k+1|(e-j_{k+})}\overline{\sigma}_{e,j_{k+1}}\right)\otimes\underline{\mathbf{v}}^{-}_{k+1,k+2}$$

$$+\,\mathbf{m}\left(\sum_{e=j_{k+2}}^{\tilde{j}_{k+2}}(-1)^{|k+2|(e-j_{k+2})}\overline{\sigma}_{e,j_{k+2}}\right)\left(\sum_{d=j_{k+1}}^{\tilde{j}_{k+1}}(-1)^{|k+1|(d-j_{k+1})}\overline{\sigma}_{d,j_{k+1}}\right)$$

$$\left(x_{j_{k+2}}-(-1)^{|k+2|}\frac{\lambda}{2}(m-k-1)\right)\otimes\underline{\mathbf{v}}^{-}_{k+1,k+2}$$

$$=\mathbf{m}\left(\sum_{d=j_{k+2}}^{\tilde{j}_{k+2}}(-1)^{|k+2|(d-j_{k+2})}\overline{\sigma}_{d,j_{k+2}}\right)(-1)^{|k+1|(j_{k+2}-j_{k+1})}\overline{\sigma}_{j_{k+2},j_{k+1}}$$

$$\left(x_{j_{k+1}}-(-1)^{|k+1|}\frac{\lambda}{2}(m-k)\right)\otimes\underline{\mathbf{v}}^{-}_{k+1,k+2}$$

$$+\mathbf{m}\left(\sum_{d=j_{k+2}}^{\tilde{j}_{k+2}}(-1)^{|k+2|(d-j_{k+2})}\overline{\sigma}_{d,j_{k+2}}\right)\left(\sum_{e=j_{k+1}}^{\tilde{j}_{k+1}}(-1)^{|k+1|(e-j_{k+1})}\overline{\sigma}_{e,j_{k+1}}\right)$$

$$\left(x_{j_{k+1}}-(-1)^{|k+1|}\frac{\lambda}{2}(m-k)\right)\otimes\underline{\mathbf{v}}^{-}_{k+1,k+2}$$

$$-\mathbf{m}\left(\sum_{d=j_{k+1}}^{\tilde{j}_{k+1}}(-1)^{|k+1|(d-j_{k+1})}\overline{\sigma}_{d,j_{k+1}}\right)\left(\sum_{e=j_{k+2}}^{\tilde{j}_{k+2}}(-1)^{|k+2|(e-j_{k+2})}\overline{\sigma}_{e,j_{k+2}}\right)$$

$$\left(x_{j_{k+1}}-(-1)^{|k+1|}\frac{\lambda}{2}(m-k)\right)\otimes\underline{\mathbf{v}}^{-}_{k+1,k+2}$$

$$-\mathbf{m}\left(\sum_{d=j_{k+2}}^{\tilde{j}_{k+2}}(-1)^{|k+2|(d-j_{k+2})}\overline{\sigma}_{d,j_{k+2}}\right)\left(x_{j_{k+2}}-(-1)^{|k+2|}\frac{\lambda}{2}(m-k-1)\right)$$

$$(-1)^{|k+1|(j_{k+2}-j_{k+1})}\overline{\sigma}_{j_{k+2},j_{k+1}}\otimes\underline{\mathbf{v}}^{-}_{k+1,k+2}$$

$$-\mathbf{m}\left(\sum_{d=j_{k+2}}^{\tilde{j}_{k+2}}(-1)^{|k+2|(d-j_{k+2})}\overline{\sigma}_{d,j_{k+2}}\right)\left(x_{j_{k+2}}-(-1)^{|k+2|}\frac{\lambda}{2}(m-k-1)\right)$$

$$\left(\sum_{e=j_{k+1}}^{\tilde{j}_{k+1}}(-1)^{|k+1|(e-j_{k+1})}\overline{\sigma}_{e,j_{k+1}}\right)\otimes\underline{\mathbf{v}}^{-}_{k+1,k+2}$$

$$+\mathbf{m}\left(\sum_{e=j_{k+2}}^{\tilde{j}_{k+2}}(-1)^{|k+2|(e-j_{k+2})}\overline{\sigma}_{e,j_{k+2}}\right)\left(\sum_{d=j_{k+1}}^{\tilde{j}_{k+1}}(-1)^{|k+1|(d-j_{k+1})}\overline{\sigma}_{d,j_{k+1}}\right)$$

$$\left(x_{j_{k+2}}-(-1)^{|k+2|}\frac{\lambda}{2}(m-k-1)\right)\otimes\underline{\mathbf{v}}^{-}_{k+1,k+2}$$

$$=\mathbf{m}\left(\sum_{d=j_{k+2}}^{\tilde{j}_{k+2}}(-1)^{|k+2|(d-j_{k+2})}\overline{\sigma}_{d,j_{k+2}}\right)(-1)^{|k+1|(j_{k+2}-j_{k+1})}\overline{\sigma}_{j_{k+2},j_{k+1}}$$

$$\left(x_{j_{k+1}}-(-1)^{|k+1|}\frac{\lambda}{2}(m-k)\right)\otimes\underline{\mathbf{v}}^{-}_{k+1,k+2}$$

$$-\mathbf{m}\left(\sum_{d=j_{k+2}}^{\tilde{j}_{k+2}}(-1)^{|k+2|(d-j_{k+2})}\overline{\sigma}_{d,j_{k+2}}\right)$$

$$\left(x_{j_{k+2}}-(-1)^{|k+2|}\frac{\lambda}{2}(m-k-1)\right)(-1)^{|k+1|(j_{k+2}-j_{k+1})}\overline{\sigma}_{j_{k+2},j_{k+1}}\otimes\underline{\mathbf{v}}^{-}_{k+1,k+2}$$

$$
\begin{aligned}
=&\mathbf{m}\left(\sum_{d=j_{k+2}}^{\tilde{j}_{k+2}}(-1)^{|k+2|(d-j_{k+2})}\overline{\sigma}_{d,j_{k+2}}\right)(-1)^{|k+1|(j_{k+2}-j_{k+1})}\\
&\left(\overline{\sigma}_{j_{k+2},j_{k+1}}x_{j_{k+1}}-x_{j_{k+2}}\overline{\sigma}_{j_{k+2},j_{k+1}}\right)\otimes\underline{\mathbf{v}}^{-}_{k+1,k+2}\\
&-(-1)^{|k+1|}\frac{\lambda}{2}(m-k)\mathbf{m}\left(\sum_{d=j_{k+2}}^{\tilde{j}_{k+2}}(-1)^{|k+2|(d-j_{k+2})}\overline{\sigma}_{d,j_{k+2}}\right)\\
&(-1)^{|k+1|(j_{k+2}-j_{k+1})}\overline{\sigma}_{j_{k+2},j_{k+1}}\otimes\underline{\mathbf{v}}^{-}_{k+1,k+2}\\
&+(-1)^{|k+2|}\frac{\lambda}{2}(m-k-1)\mathbf{m}\left(\sum_{d=j_{k+2}}^{\tilde{j}_{k+2}}-1)^{|k+2|(d-j_{k+2})}\overline{\sigma}_{d,j_{k+2}}\right)\\
&(-1)^{|k+1|(j_{k+2}-j_{k+1})}\overline{\sigma}_{j_{k+2},j_{k+1}}\otimes\underline{\mathbf{v}}^{-}_{k+1,k+2}\\
=&\mathbf{m}\left(\sum_{d=j_{k+2}}^{\tilde{j}_{k+2}}(-1)^{|k+2|(d-j_{k+2})}\overline{\sigma}_{d,j_{k+2}}\right)(-1)^{|k+1|}\\
&\left(\kappa\sum_{f=j_{k+1}+1}^{j_{k+2}}(-1)^{|k+1|(j_{k+2}-f)}\overline{\sigma}_{j_{k+2},f}(-1)^{|k+1|(f-1-j_{k+1})}\overline{\sigma}_{f-1,j_{k+1}}\right)\otimes\underline{\mathbf{v}}^{-}_{k+1,k+2}\\
&-(-1)^{|k+1|}\frac{\lambda}{2}\mathbf{m}\left(\sum_{d=j_{k+2}}^{\tilde{j}_{k+2}}(-1)^{|k+2|(d-j_{k+2})}\overline{\sigma}_{d,j_{k+2}}\right)(-1)^{|k+1|(j_{k+2}-j_{k+1})}\overline{\sigma}_{j_{k+2},j_{k+1}}\otimes\underline{\mathbf{v}}^{-}_{k+1,k+2}\\
=&(-1)^{|k+1|}\kappa\mathbf{m}\left(\sum_{d=j_{k+2}}^{\tilde{j}_{k+2}}(-1)^{|k+2|(d-j_{k+2})}\overline{\sigma}_{d,j_{k+2}}\right)\\
&\left(\sum_{f=j_{k+1}+1}^{j_{k+2}}(-1)^{|k+1|(j_{k+2}-f)}\overline{\sigma}_{j_{k+2},f}(-1)^{|k+1|(f-1-j_{k+1})}\overline{\sigma}_{f-1,j_{k+1}}\right)\otimes\underline{\mathbf{v}}^{-}_{k+1,k+2}\\
&-(-1)^{|k+1|}\frac{\lambda}{2}\mathbf{m}\left(\sum_{d=j_{k+2}}^{\tilde{j}_{k+2}}(-1)^{|k+2|(d-j_{k+2})}\overline{\sigma}_{d,j_{k+2}}\right)\\
&(-1)^{|k+1|(j_{k+2}-j_{k+1})}\overline{\sigma}_{j_{k+2},j_{k+1}}\otimes\underline{\mathbf{v}}^{-}_{k+1,k+2}
\end{aligned}
$$

This applies Lemma 2.5.2.

$$
\begin{aligned}
&(-1)^{|k+1|}\kappa\mathbf{m}\left(\sum_{d=j_{k+2}}^{\tilde{j}_{k+2}}(-1)^{|k+2|(d-j_{k+2})}\overline{\sigma}_{d,j_{k+2}}\right)\\
&\qquad\left(\sum_{f=j_{k+1}+1}^{j_{k+2}}(-1)^{|k+1|(j_{k+2}-f)}\overline{\sigma}_{j_{k+2},f}(-1)^{|k+1|(f-1-j_{k+1})}\overline{\sigma}_{f-1,j_{k+1}}\right)\otimes\underline{\mathbf{v}}^{-}_{k+1,k+2}\\
&\qquad-(-1)^{|k+1|}\frac{\lambda}{2}\mathbf{m}\left(\sum_{d=j_{k+2}}^{\tilde{j}_{k+2}}(-1)^{|k+2|(d-j_{k+2})}\overline{\sigma}_{d,j_{k+2}}\right)\\
&\qquad(-1)^{|k+1|(j_{k+2}-j_{k+1})}\overline{\sigma}_{j_{k+2},j_{k+1}}\otimes\underline{\mathbf{v}}^{-}_{k+1,k+2}\\
&\quad=(-1)^{|k+1|}\kappa\sum_{d\neq e}\mathbf{m}\otimes E^{(d)}_{k+1,k+2}E^{(e)}_{k,k+1}(\underline{\mathbf{v}})-(-1)^{|k+1|}\frac{\lambda}{2}\mathbf{m}\otimes E_{k,k+2}(\underline{\mathbf{v}})\\
&\quad=-(-1)^{|k+1|}\frac{\lambda}{2}\left(2\sum_{d\neq e}\mathbf{m}\otimes E^{(d)}_{k+1,k+2}E^{(e)}_{k,k+1}(\underline{\mathbf{v}})+\mathbf{m}\otimes E_{k,k+2}(\underline{\mathbf{v}})\right)\\
&\quad=-(-1)^{|k+1|}\frac{\lambda}{2}\left\{X^{+}_{k,0},X^{+}_{k+1,0}\right\}(\mathbf{m}\otimes\underline{\mathbf{v}})
\end{aligned}
$$

If we set $\kappa=-\lambda$,

$$
\begin{aligned}
&X^{+}_{k,0}\left(\mathbf{m}\left(\sum_{d=j_{k+2}}^{\tilde{j}_{k+2}}\overline{\sigma}_{d,j_{k+2}}\right)\otimes\underline{\mathbf{v}}_{j^{-}_{k+2}}\right)+X_{k+1,0}\left(\mathbf{m}\left(\sum_{d=j_{k+1}}^{\tilde{j}_{k+1}}\overline{\sigma}_{d,j_{k+1}}\right)\otimes\underline{\mathbf{v}}_{j^{-}_{k+1}}\right)\\
&\quad=\mathbf{m}\left(\sum_{d=j_{k+2}}^{\tilde{j}_{k+2}}\overline{\sigma}_{d,j_{k+2}}\right)\left(\sum_{e=j_{k+1}}^{\tilde{j}_{k+1}+1}\overline{\sigma}_{e,j_{k+1}}\right)\otimes\underline{\mathbf{v}}^{-}_{k+1,k+2}\\
&\qquad+\mathbf{m}\left(\sum_{d=j_{k+1}}^{\tilde{j}_{k+1}}\overline{\sigma}_{d,j_{k+1}}\right)\left(\sum_{e=j_{k+2}}^{\tilde{j}_{k+2}}\overline{\sigma}_{e,j_{k+2}}\right)\otimes\underline{\mathbf{v}}^{-}_{k+1,k+2}\\
&\quad=\mathbf{m}\left(\sum_{d=j_{k+2}}^{\tilde{j}_{k+2}}\overline{\sigma}_{d,j_{k+2}}\right)\overline{\sigma}_{j_{k+2},j_{k+1}}\otimes\underline{\mathbf{v}}^{-}_{k+1,k+2}\\
&\qquad+2\mathbf{m}\left(\sum_{d=j_{k+1}}^{\tilde{j}_{k+1}}\overline{\sigma}_{d,j_{k+1}}\right)\left(\sum_{e=j_{k+2}}^{\tilde{j}_{k+2}}\overline{\sigma}_{e,j_{k+2}}\right)\otimes\underline{\mathbf{v}}^{-}_{k+1,k+2}
\end{aligned}
$$

$$
\begin{aligned}
&E_{k,k+1}E_{k+1,k+2}(\mathbf{m}\otimes\underline{\mathbf{v}}) + E_{k+1,k+2}E_{k,k+1}(\mathbf{m}\otimes\underline{\mathbf{v}}) \\
&\qquad = E_{k,k+1}\sum_{d=1}^{l}\mathbf{m}\otimes E^{(d)}_{k+1,k+2}(\underline{\mathbf{v}}) + E_{k+1,k+2}\sum_{e=1}^{l}\mathbf{m}\otimes E^{(e)}_{k,k+1}(\underline{\mathbf{v}}) \\
&\quad = \sum_{e,d=1}^{l}\mathbf{m}\otimes E^{(e)}_{k,k+1}E^{(d)}_{k+1,k+2}(\underline{\mathbf{v}}) + \sum_{e,d=1}^{l}\mathbf{m}\otimes E^{(d)}_{k+1,k+2}E^{(e)}_{k,k+1}(\underline{\mathbf{v}}) \\
&\quad = \sum_{d=1}^{l}\mathbf{m}\otimes E^{(d)}_{k,k+2}(\underline{\mathbf{v}}) + \sum_{e\neq d}\mathbf{m}\otimes E^{(e)}_{k,k+1}E^{(d)}_{k+1,k+2}(\underline{\mathbf{v}}) + \sum_{e\neq d}^{l}\mathbf{m}\otimes E^{(d)}_{k+1,k+2}E^{(e)}_{k,k+1}(\underline{\mathbf{v}}) \\
&\quad = \sum_{d=1}^{l}\mathbf{m}\otimes E^{(d)}_{k,k+2}(\underline{\mathbf{v}}) + 2\sum_{e\neq d}\mathbf{m}\otimes E^{(e)}_{k,k+1}E^{(d)}_{k+1,k+2}(\underline{\mathbf{v}})
\end{aligned}
$$

Similarly, using the analogous elements and for the other values of k, we can show that $[X^-_{k,1}, X^-_{k+1,0}](\mathbf{m}\otimes\underline{\mathbf{v}}) - [X^-_{k,0}, X^-_{k+1,1}](\mathbf{m}\otimes\underline{\mathbf{v}}) = \frac{\lambda}{2}\{X^-_{k,0}, X^-_{k+1,0}\}(\mathbf{m}\otimes\underline{\mathbf{v}})$.

$$
\begin{aligned}
&[X^-_{k,1}, X^-_{k+1,0}](\mathbf{m}\otimes\underline{\mathbf{v}}) - [X^-_{k,0}, X^-_{k+1,1}](\mathbf{m}\otimes\underline{\mathbf{v}}) \\
&\quad = X^-_{k,1}X^-_{k+1,0}(\mathbf{m}\otimes\underline{\mathbf{v}}) - X^-_{k+1,0}X^-_{k,1}(\mathbf{m}\otimes\underline{\mathbf{v}}) \\
&\qquad - X^-_{k,0}X^-_{k+1,1}(\mathbf{m}\otimes\underline{\mathbf{v}}) + X^-_{k+1,1}X^-_{k,0}(\mathbf{m}\otimes\underline{\mathbf{v}}) \\
&\quad = X^-_{k,1}\left(\mathbf{m}\left(\sum_{d=j_{k+1}}^{\tilde{j}_{k+1}}\overline{\sigma}_{d,\tilde{j}_{k+1}}\right)\otimes\underline{\mathbf{v}}_{j^+_{k+1}}\right) \\
&\qquad - X^-_{k+1,0}\left(\mathbf{m}\left(\sum_{d=j_k}^{\tilde{j}_k}\overline{\sigma}_{d,\tilde{j}_k}\right)\left(x_{\tilde{j}_k} - \frac{\lambda}{2}(k-m)\right)\otimes\underline{\mathbf{v}}_{j^+_k}\right) \\
&\qquad - X^-_{k,0}\left(\mathbf{m}\left(\sum_{d=j_{k+1}}^{\tilde{j}_{k+1}}\overline{\sigma}_{d,\tilde{j}_{k+1}}\right)\left(x_{\tilde{j}_{k+1}} - \frac{\lambda}{2}(k+1-m)\right)\otimes\underline{\mathbf{v}}_{j^+_{k+1}}\right) \\
&\qquad + X^-_{k+1,1}\left(\mathbf{m}\left(\sum_{d=j_k}^{\tilde{j}_k}\overline{\sigma}_{d,\tilde{j}_k}\right)\otimes\underline{\mathbf{v}}_{j^+_k}\right) \\
&\quad = \mathbf{m}\sum_{d=j_{k+1}}^{\tilde{j}_{k+1}}\sum_{e=j_k}^{\tilde{j}_k}\overline{\sigma}_{d,\tilde{j}_{k+1}}\overline{\sigma}_{e,\tilde{j}_k}\left(x_{\tilde{j}_k} - \frac{\lambda}{2}(k-m)\right)\otimes\underline{\mathbf{v}}^+_{k,k+1}
\end{aligned}
$$

$$
\begin{aligned}
&-\mathbf{m}\sum_{e=j_k}^{\tilde{j}_k}\sum_{d=\tilde{j}_k}^{\tilde{j}_{k+1}}\overline{\sigma}_{e,\tilde{j}_k}\left(x_{\tilde{j}_k}-\frac{\lambda}{2}(k-m)\right)\overline{\sigma}_{d,\tilde{j}_{k+1}}\otimes\underline{\mathbf{v}}^{+}_{k,k+1}\\
&-\mathbf{m}\left(\sum_{d=j_{k+1}}^{\tilde{j}_{k+1}}\sum_{e=j_k}^{\tilde{j}_k}\overline{\sigma}_{d,\tilde{j}_{k+1}}\right)\left(x_{\tilde{j}_{k+1}}-\frac{\lambda}{2}(k+1-m)\right)\overline{\sigma}_{e,\tilde{j}_k}\otimes\underline{\mathbf{v}}^{+}_{k,k+1}\\
&+\mathbf{m}\sum_{e=j_k}^{\tilde{j}_k}\sum_{d=\tilde{j}_k}^{\tilde{j}_{k+1}}\overline{\sigma}_{e,\tilde{j}_k}\overline{\sigma}_{d,\tilde{j}_{k+1}}\left(x_{\tilde{j}_{k+1}}-\frac{\lambda}{2}(k+1-m)\right)\otimes\underline{\mathbf{v}}^{+}_{k,k+1}\\
=&\mathbf{m}\sum_{d=j_{k+1}}^{\tilde{j}_{k+1}}\sum_{e=j_k}^{\tilde{j}_k}\overline{\sigma}_{d,\tilde{j}_{k+1}}\overline{\sigma}_{e,\tilde{j}_k}\left(x_{\tilde{j}_k}-\frac{\lambda}{2}(k-m)\right)\otimes\underline{\mathbf{v}}^{+}_{k,k+1}\\
&-\mathbf{m}\sum_{e=j_k}^{\tilde{j}_k}\overline{\sigma}_{e,\tilde{j}_k}\left(x_{\tilde{j}_k}-\frac{\lambda}{2}(k-m)\right)\left(\overline{\sigma}_{\tilde{j}_k,\tilde{j}_{k+1}}+\sum_{d=j_{k+1}}^{\tilde{j}_{k+1}}\overline{\sigma}_{d,\tilde{j}_{k+1}}\right)\otimes\underline{\mathbf{v}}^{+}_{k,k+1}\\
&-\mathbf{m}\left(\sum_{d=j_{k+1}}^{\tilde{j}_{k+1}}\sum_{e=j_k}^{\tilde{j}_k}\overline{\sigma}_{d,\tilde{j}_{k+1}}\right)\left(x_{\tilde{j}_{k+1}}-\frac{\lambda}{2}(k+1-m)\right)\overline{\sigma}_{e,\tilde{j}_k}\otimes\underline{\mathbf{v}}^{+}_{k,k+1}\\
&+\mathbf{m}\sum_{e=j_k}^{\tilde{j}_k}\overline{\sigma}_{e,\tilde{j}_k}\left(\overline{\sigma}_{\tilde{j}_k,\tilde{j}_{k+1}}+\sum_{d=j_{k+1}}^{\tilde{j}_{k+1}}\overline{\sigma}_{d,\tilde{j}_{k+1}}\right)\left(x_{\tilde{j}_{k+1}}-\frac{\lambda}{2}(k+1-m)\right)\otimes\underline{\mathbf{v}}^{+}_{k,k+1}\\
=&\mathbf{m}\sum_{d=j_{k+1}}^{\tilde{j}_{k+1}}\sum_{e=j_k}^{\tilde{j}_k}\overline{\sigma}_{d,\tilde{j}_{k+1}}\overline{\sigma}_{e,\tilde{j}_k}\left(x_{\tilde{j}_k}-\frac{\lambda}{2}(k-m)\right)\otimes\underline{\mathbf{v}}^{+}_{k,k+1}\\
&-\mathbf{m}\sum_{e=j_k}^{\tilde{j}_k}\overline{\sigma}_{e,\tilde{j}_k}\left(x_{\tilde{j}_k}-\frac{\lambda}{2}(k-m)\right)\overline{\sigma}_{\tilde{j}_k,\tilde{j}_{k+1}}\otimes\underline{\mathbf{v}}^{+}_{k,k+1}\\
&-\mathbf{m}\sum_{e=j_k}^{\tilde{j}_k}\overline{\sigma}_{e,\tilde{j}_k}\left(x_{\tilde{j}_k}-\frac{\lambda}{2}(k-m)\right)\left(\sum_{d=j_{k+1}}^{\tilde{j}_{k+1}}\overline{\sigma}_{d,\tilde{j}_{k+1}}\right)\otimes\underline{\mathbf{v}}^{+}_{k,k+1}\\
&-\mathbf{m}\left(\sum_{d=j_{k+1}}^{\tilde{j}_{k+1}}\sum_{e=j_k}^{\tilde{j}_k}\overline{\sigma}_{d,\tilde{j}_{k+1}}\right)\left(x_{\tilde{j}_{k+1}}-\frac{\lambda}{2}(k+1-m)\right)\overline{\sigma}_{e,\tilde{j}_k}\otimes\underline{\mathbf{v}}^{+}_{k,k+1}\\
&+\mathbf{m}\sum_{e=j_k}^{\tilde{j}_k}\overline{\sigma}_{e,\tilde{j}_k}\overline{\sigma}_{\tilde{j}_k,\tilde{j}_{k+1}}\left(x_{\tilde{j}_{k+1}}-\frac{\lambda}{2}(k+1-m)\right)\otimes\underline{\mathbf{v}}^{+}_{k,k+1}
\end{aligned}
$$

$$
\begin{aligned}
&+\mathbf{m}\sum_{e=j_k}^{\tilde{j}_k}\overline{\sigma}_{e,\tilde{j}_k}\left(\sum_{d=j_{k+1}}^{\tilde{j}_{k+1}}\overline{\sigma}_{d,\tilde{j}_{k+1}}\right)\left(x_{\tilde{j}_{k+1}}-\frac{\lambda}{2}(k+1-m)\right)\otimes\underline{\mathbf{v}}^{+}_{k,k+1}\\
=&-\mathbf{m}\sum_{e=j_k}^{\tilde{j}_k}\overline{\sigma}_{e,\tilde{j}_k}\left(x_{\tilde{j}_k}-\frac{\lambda}{2}(k-m)\right)\overline{\sigma}_{\tilde{j}_k,\tilde{j}_{k+1}}\otimes\underline{\mathbf{v}}^{+}_{k,k+1}\\
&+\mathbf{m}\sum_{e=j_k}^{\tilde{j}_k}\overline{\sigma}_{e,\tilde{j}_k}\overline{\sigma}_{\tilde{j}_k,\tilde{j}_{k+1}}\left(x_{\tilde{j}_{k+1}}-\frac{\lambda}{2}(k+1-m)\right)\otimes\underline{\mathbf{v}}^{+}_{k,k+1}\\
=&-\mathbf{m}\sum_{e=j_k}^{\tilde{j}_k}\overline{\sigma}_{e,\tilde{j}_k}\left(x_{\tilde{j}_k}\overline{\sigma}_{\tilde{j}_k,\tilde{j}_{k+1}}-\overline{\sigma}_{\tilde{j}_k,\tilde{j}_{k+1}}x_{\tilde{j}_{k+1}}\right)\otimes\underline{\mathbf{v}}^{+}_{k,k+1}\\
&+\frac{\lambda}{2}(k-m)\mathbf{m}\sum_{e=j_k}^{\tilde{j}_k}\overline{\sigma}_{e,\tilde{j}_k}\overline{\sigma}_{\tilde{j}_k,\tilde{j}_{k+1}}\otimes\underline{\mathbf{v}}^{+}_{k,k+1}\\
&-\frac{\lambda}{2}(k+1-m)\mathbf{m}\sum_{e=j_k}^{\tilde{j}_k}\overline{\sigma}_{e,\tilde{j}_k}\overline{\sigma}_{\tilde{j}_k,\tilde{j}_{k+1}}\otimes\underline{\mathbf{v}}^{+}_{k,k+1}\\
=&-\mathbf{m}\sum_{e=j_k}^{\tilde{j}_k}\overline{\sigma}_{e,\tilde{j}_k}\left(\kappa\sum_{f=\tilde{j}_k+1}^{\tilde{j}_{k+1}}\overline{\sigma}_{fb}\overline{\sigma}_{a,f-1}\right)\otimes\underline{\mathbf{v}}^{+}_{k,k+1}\\
&-\frac{\lambda}{2}\mathbf{m}\sum_{e=j_k}^{\tilde{j}_k}\overline{\sigma}_{e,\tilde{j}_k}\overline{\sigma}_{\tilde{j}_k,\tilde{j}_{k+1}}\otimes\underline{\mathbf{v}}^{+}_{k,k+1}
\end{aligned}
$$

□

For $1 \leq k \leq m+n$, define the operators D_k and $\widetilde{D}_k$ on $M \otimes_{\mathbb{C}[S_\ell]} (\mathbb{C}(m|n))^{\otimes \ell}$ by

$$
D_{k,1}(\mathbf{m}\otimes\underline{\mathbf{v}})=\sum_{e=j_k}^{\tilde{j}_k}\mathbf{m}\left(x_e-(-1)^{|k+1|}\frac{\lambda}{2}(m-k+(-1)^{\delta_{\kappa m}})\right)\otimes\underline{\mathbf{v}} \tag{6.9}
$$

and

$$
\widetilde{D}_{k,1}(\mathbf{m}\otimes\underline{\mathbf{v}})=\sum_{e=j_k}^{\tilde{j}_k}\mathbf{m}\left(x_e-(-1)^{|k|}\frac{\lambda}{2}(m-k)\right)\otimes\underline{\mathbf{v}}. \tag{6.10}
$$

We compute $[D_k, X^{+}_{k,0}](\mathbf{m}\otimes\underline{\mathbf{v}})$ and $[X^{+}_{k,0}, \widetilde{D}_{k+1}](\mathbf{m}\otimes\underline{\mathbf{v}})$ to use when defining a new operator $\widetilde{H}_{k,1}$ and producing its relations.

$$
\begin{aligned}
&[D_{k,1}, X^+_{k,0}](\mathbf{m} \otimes \underline{\mathbf{v}}) \\
=&D_{k,1} \left(\sum_{d=j_{k+1}}^{\tilde{j}_{k+1}} (-1)^{|k+1|(d-j_{k+1})} \mathbf{m}\overline{\sigma}_{d,j_{k+1}} \otimes \underline{\mathbf{v}}_{j^-_{k+1}} \right) \\
&- X^+_{k,0} \left(\sum_{e=j_k}^{\tilde{j}_k} \mathbf{m} \left(x_e - (-1)^{|k|} \frac{\lambda(m-k)}{4} \right) \otimes \underline{\mathbf{v}} \right) \\
=& \sum_{d=j_{k+1}}^{\tilde{j}_{k+1}} \sum_{e=j_k}^{j_{k+1}} (-1)^{|k+1|(d-j_{k+1})} \mathbf{m}\overline{\sigma}_{d,j_{k+1}} \left(x_e - (-1)^{|k+1|} \frac{\lambda}{2}(m-k+(-1)^{\delta_{\kappa m}}) \right) \otimes \underline{\mathbf{v}}_{j^-_{k+1}} \\
&- \sum_{e=j_k}^{\tilde{j}_k} \sum_{d=j_{k+1}}^{\tilde{j}_{k+1}} (-1)^{|k+1|(d-j_{k+1})} \mathbf{m} \left(x_e - (-1)^{|k+1|} \frac{\lambda}{2}(m-k+(-1)^{\delta_{\kappa m}}) \right) \overline{\sigma}_{d,j_{k+1}} \otimes \underline{\mathbf{v}}_{j^-_{k+1}} \\
=& \sum_{d=j_{k+1}}^{\tilde{j}_{k+1}} \sum_{e=j_k}^{\tilde{j}_k} (-1)^{|k+1|(d-j_{k+1})} \mathbf{m}\overline{\sigma}_{d,j_{k+1}} \left(x_e - (-1)^{|k+1|} \frac{\lambda}{2}(m-k+(-1)^{\delta_{\kappa m}}) \right) \otimes \underline{\mathbf{v}}_{j^-_{k+1}} \\
&- \sum_{d=j_{k+1}}^{\tilde{j}_{k+1}} \sum_{e=j_k}^{\tilde{j}_k} (-1)^{|k+1|(d-j_{k+1})} \mathbf{m} \left(x_e - (-1)^{|k+1|} \frac{\lambda}{2}(m-k+(-1)^{\delta_{\kappa m}}) \right) \overline{\sigma}_{d,j_{k+1}} \otimes \underline{\mathbf{v}}_{j^-_{k+1}} \\
&+ \sum_{d=j_{k+1}}^{\tilde{j}_{k+1}} (-1)^{|k+1|(d-j_{k+1})} \mathbf{m}\overline{\sigma}_{d,j_{k+1}} \left(x_{j_{k+1}} - (-1)^{|k+1|} \frac{\lambda}{2}(m-k+(-1)^{\delta_{\kappa m}}) \right) \otimes \underline{\mathbf{v}}_{j^-_{k+1}} \\
=& \sum_{d=j_{k+1}}^{\tilde{j}_{k+1}} \sum_{e=j_k}^{\tilde{j}_k} (-1)^{|k+1|(d-j_{k+1})} \mathbf{m}(\overline{\sigma}_{d,j_{k+1}} x_e - x_e \overline{\sigma}_{d,j_{k+1}}) \otimes \underline{\mathbf{v}}_{j^-_{k+1}} \\
&+ \sum_{d=j_{k+1}}^{\tilde{j}_{k+1}} (-1)^{|k+1|(d-j_{k+1})} \mathbf{m}\overline{\sigma}_{d,j_{k+1}} \left(x_{j_{k+1}} - (-1)^{|k+1|} \frac{\lambda}{2}(m-k+(-1)^{\delta_{\kappa m}}) \right) \otimes \underline{\mathbf{v}}_{j^-_{k+1}} \\
=& \sum_{d=j_{k+1}}^{\tilde{j}_{k+1}} (-1)^{|k+1|(d-j_{k+1})} \mathbf{m}\overline{\sigma}_{d,j_{k+1}} \left(x_{j_{k+1}} - (-1)^{|k+1|} \frac{\lambda}{2}(m-k+(-1)^{\delta_{\kappa m}}) \right)) \otimes \underline{\mathbf{v}}_{j^-_{k+1}}
\end{aligned}
$$

$[X^+_{k,0}, \widetilde{D}_{k+1}](\mathbf{m} \otimes \underline{\mathbf{v}})$

$$
\begin{aligned}
=&X_{k,0}^{+}\left(\sum_{e=j_{k+1}}^{\tilde{j}_{k+1}}\mathbf{m}\left(x_e-(-1)^{|k+1|}\frac{\lambda(m-k-1)}{2}\right)\otimes\underline{\mathbf{v}}\right)\\
&-\widetilde{D}_{k+1}\left(\sum_{d=j_{k+1}}^{\tilde{j}_{k+1}}(-1)^{|k+1|(d-j_{k+1})}\mathbf{m}\overline{\sigma}_{d,j_{k+1}}\otimes\underline{\mathbf{v}}_{j_{k+1}^{-}}\right)\\
=&\sum_{d=j_{k+1}}^{\tilde{j}_{k+1}}\sum_{e=j_{k+1}}^{\tilde{j}_{k+1}}\mathbf{m}(-1)^{|k+1|(d-j_{k+1})}\left(x_e-(-1)^{|k+1|}\frac{\lambda(m-k-1)}{2}\right)\overline{\sigma}_{d,j_{k+1}}\otimes\underline{\mathbf{v}}_{j_{k+1}^{-}}\\
&-\sum_{d=j_{k+1}}^{\tilde{j}_{k+1}}\sum_{e=j_{k+1}+1}^{\tilde{j}_{k+1}}(-1)^{|k+1|(d-j_{k+1})}\mathbf{m}\overline{\sigma}_{d,j_{k+1}}\\
&\left(x_e-(-1)^{|k+1|}\frac{\lambda(m-k-1)}{2}\right)\otimes\underline{\mathbf{v}}_{j_{k+1}^{-}}\\
=&\sum_{d=j_{k+1}}^{\tilde{j}_{k+1}}\sum_{e=j_{k+1}+1}^{\tilde{j}_{k+1}}(-1)^{|k+1|(d-j_{k+1})}\mathbf{m}x_e\overline{\sigma}_{d,j_{k+1}}\otimes\underline{\mathbf{v}}_{j_{k+1}^{-}}\\
&+\sum_{d=j_{k+1}}^{\tilde{j}_{k+1}}\mathbf{m}\left(x_{j_{k+1}}-(-1)^{|k+1|}\frac{\lambda(m-k-1)}{2}\right)(-1)^{|k+1|(d-j_{k+1})}\overline{\sigma}_{d,j_{k+1}}\otimes\underline{\mathbf{v}}_{j_{k+1}^{-}}\\
&-\sum_{d=j_{k+1}}^{\tilde{j}_{k+1}}\sum_{e=j_{k+1}+1}^{\tilde{j}_{k+1}}(-1)^{|k+1|(d-j_{k+1})}\mathbf{m}\overline{\sigma}_{d,j_{k+1}}x_e\otimes\underline{\mathbf{v}}_{j_{k+1}^{-}}\\
=&-(-1)^{|k+1|}\frac{\lambda(m-k-1)}{2}\sum_{d=j_{k+1}}^{\tilde{j}_{k+1}}(-1)^{|k+1|(d-j_{k+1})}\mathbf{m}\overline{\sigma}_{d,j_{k+1}}\otimes\underline{\mathbf{v}}_{j_{k+1}^{-}}\\
&+\sum_{d=j_{k+1}}^{\tilde{j}_{k+1}}(-1)^{|k+1|(d-j_{k+1})}\mathbf{m}x_d\overline{\sigma}_{d,j_{k+1}}\otimes\underline{\mathbf{v}}_{j_{k+1}^{-}}\\
&+\sum_{d=j_{k+1}}^{\tilde{j}_{k+1}}\sum_{e=j_{k+1}}^{d-1}(-1)^{|k+1|(d-j_{k+1})}\mathbf{m}x_e\overline{\sigma}_{d,j_{k+1}}\otimes\underline{\mathbf{v}}_{j_{k+1}^{-}}\\
&+\sum_{d=j_{k+1}}^{\tilde{j}_{k+1}}\sum_{e=d+1}^{\tilde{j}_{k+1}}(-1)^{|k+1|(d-j_{k+1})}\mathbf{m}x_e\overline{\sigma}_{d,j_{k+1}}\otimes\underline{\mathbf{v}}_{j_{k+1}^{-}}\\
&-\sum_{d=j_{k+1}}^{\tilde{j}_{k+1}}\sum_{e=j_{k+1}+1}^{\tilde{j}_{k+1}}(-1)^{|k+1|(d-j_{k+1})}\mathbf{m}\overline{\sigma}_{d,j_{k+1}}x_e\otimes\underline{\mathbf{v}}_{j_{k+1}^{-}}
\end{aligned}
$$

$$= -(-1)^{|k+1|}\frac{\lambda(m-k)}{2}\sum_{d=j_{k+1}}^{\tilde{j}_{k+1}}(-1)^{|k+1|(d-j_{k+1})}\mathbf{m}\overline{\sigma}_{d,j_{k+1}}\otimes\underline{\mathbf{v}}_{j_{k+1}^-}$$

$$+\sum_{d=j_{k+1}}^{\tilde{j}_{k+1}}(-1)^{|k+1|(d-j_{k+1})}\mathbf{m}$$

$$\left(\overline{\sigma}_{d,j_{k+1}}x_{j_{k+1}}-\kappa\sum_{f=j_{k+1}+1}^{d}\overline{\sigma}_{d,f}\overline{\sigma}_{f-1,j_{k+1}}\right)\otimes\underline{\mathbf{v}}_{j_{k+1}^-}$$

$$+\sum_{d=j_{k+1}}^{\tilde{j}_{k+1}}\sum_{e=j_{k+1}}^{d-1}(-1)^{|k+1|(d-j_{k+1})}\mathbf{m}\left(\overline{\sigma}_{d,j_{k+1}}x_{e+1}+\kappa\overline{\sigma}_{d,e+1}\overline{\sigma}_{e,j_{k+1}}\right)\otimes\underline{\mathbf{v}}_{j_{k+1}^-}$$

$$+\sum_{d=j_{k+1}}^{\tilde{j}_{k+1}}\sum_{e=d+1}^{\tilde{j}_{k+1}}(-1)^{|k+1|(d-j_{k+1})}\mathbf{m}\overline{\sigma}_{d,j_{k+1}}x_e\otimes\underline{\mathbf{v}}_{j_{k+1}^-}$$

$$-\sum_{d=j_{k+1}}^{\tilde{j}_{k+1}}\sum_{e=j_{k+1}+1}^{\tilde{j}_{k+1}}(-1)^{|k+1|(d-j_{k+1})}\mathbf{m}\overline{\sigma}_{d,j_{k+1}}x_e\otimes\underline{\mathbf{v}}_{j_{k+1}^-}$$

This applied Lemma 2.5.2. After simplifying further, we get

$$-(-1)^{|k+1|}\frac{\lambda(m-k-1)}{2}\sum_{d=j_{k+1}}^{\tilde{j}_{k+1}}(-1)^{|k+1|(d-j_{k+1})}\mathbf{m}\overline{\sigma}_{d,j_{k+1}}\otimes\underline{\mathbf{v}}_{j_{k+1}^-}$$

$$+\sum_{d=j_{k+1}}^{\tilde{j}_{k+1}}(-1)^{|k+1|(d-j_{k+1})}\mathbf{m}\overline{\sigma}_{d,j_{k+1}}x_{j_{k+1}}\otimes\underline{\mathbf{v}}_{j_{k+1}^-}$$

$$-(-1)^{|k+1|}\kappa\sum_{d=j_{k+1}}^{\tilde{j}_{k+1}}\sum_{f=j_{k+1}+1}^{d}\mathbf{m}(-1)^{|k+1|(d-f)}\overline{\sigma}_{d,f}$$

$$(-1)^{|k+1|(f-1-j_{k+1})}\overline{\sigma}_{f-1,j_{k+1}}\otimes\underline{\mathbf{v}}_{j_{k+1}^-}$$

$$+(-1)^{|k+1|}\kappa\sum_{d=j_{k+1}}^{\tilde{j}_{k+1}}\sum_{e=j_{k+1}}^{d-1}\mathbf{m}(-1)^{|k+1|(d-e-1)}\overline{\sigma}_{d,e+1}$$

$$(-1)^{|k+1|(e-j_{k+1})}\overline{\sigma}_{e,j_{k+1}}\otimes\underline{\mathbf{v}}_{j_{k+1}^-}$$

$$+\sum_{d=j_{k+1}}^{\tilde{j}_{k+1}}\sum_{e=j_{k+1}+1}^{d}(-1)^{|k+1|(d-j_{k+1})}\mathbf{m}\overline{\sigma}_{d,j_{k+1}}x_e\otimes\underline{\mathbf{v}}_{j_{k+1}^-}$$

$$+ \sum_{d=j_{k+1}}^{\tilde{j}_{k+1}} \sum_{e=d+1}^{\tilde{j}_{k+1}} (-1)^{|k+1|(d-j_{k+1})} \mathbf{m}\overline{\sigma}_{d,j_{k+1}} x_e \otimes \underline{\mathbf{v}}_{j_{k+1}^-}$$

$$- \sum_{d=j_{k+1}}^{\tilde{j}_{k+1}} \sum_{e=j_{k+1}+1}^{\tilde{j}_{k+1}} (-1)^{|k+1|(d-j_{k+1})} \mathbf{m}\overline{\sigma}_{d,j_{k+1}} x_e \otimes \underline{\mathbf{v}}_{j_{k+1}^-}$$

$$= \sum_{d=j_{k+1}}^{\tilde{j}_{k+1}} (-1)^{|k+1|(d-j_{k+1})} \mathbf{m}\overline{\sigma}_{d,j_{k+1}} \left(x_{j_{k+1}} - (-1)^{|k+1|} \frac{\lambda(m-k-1)}{2} \right) \otimes \underline{\mathbf{v}}_{j_{k+1}^-}$$

Set $\widetilde{H}_{k,1}(\mathbf{m} \otimes \underline{\mathbf{v}}) = (-1)^{|k|} D_k(\mathbf{m} \otimes \underline{\mathbf{v}}) - (-1)^{|k+1|} \widetilde{D}_{k+1}(\mathbf{m} \otimes \underline{\mathbf{v}})$. Then, as operators on $M \otimes_{\mathbb{C}[S_\ell]} (\mathbb{C}(m|n))^{\otimes \ell}$, the following relations are satisfied:

$$[\widetilde{H}_{k,1}, X^+_{k,0}] = ((-1)^{|k|} + (-1)^{|k+1|}) X^+_{k,1}, \tag{6.11}$$

$$[\widetilde{H}_{k,1}, X^+_{k-1,0}] = -(-1)^{|k|} X^+_{k-1,1}, \tag{6.12}$$

$$[\widetilde{H}_{k,1}, X^+_{k+1,0}] = -(-1)^{|k+1|} X^+_{k+1,1}. \tag{6.13}$$

Similar relations hold when $X^+_{k,0}$ is replaced by $X^-_{k,0}$.

Note that $X^\pm_{kr}$ is an odd generator only when $k = m$ for $1 \le k \le m+n-1$. Thus, the sign $(-1)^{\delta_{km}}$ becomes necessary to compensate in computations such as the following defining relation.

$$(-1)^{|k|}[X^+_{k1}, X^-_{k0}](\mathbf{m} \otimes \underline{\mathbf{v}}) = X^+_{k1} \left(\mathbf{m} \sum_{d=j_k}^{\tilde{j}_k} (-1)^{|k|(\tilde{j}_k - d)} \overline{\sigma}_{d,\tilde{j}_k} \otimes \underline{\mathbf{v}}_{j_k^+} \right)$$

$$- (-1)^{\delta_{km}} X^-_{k0} \left(\mathbf{m} \left(\sum_{e=j_{k+1}}^{\tilde{j}_{k+1}} (-1)^{|k+1|(e-j_{k+1})} \overline{\sigma}_{e,j_{k+1}} \right) \right.$$

$$\left. \left(x_{j_{k+1}} - (-1)^{|k+1|} \frac{\lambda(m-k)}{2} \right) \otimes \underline{\mathbf{v}}_{j_{k+1}^-} \right)$$

$$= \sum_{d=j_k}^{\tilde{j}_k} \sum_{e=\tilde{j}_k}^{\tilde{j}_{k+1}} (-1)^{|k|(\tilde{j}_k - d)} (-1)^{|k+1|(e-\tilde{j}_k)} \mathbf{m}\overline{\sigma}_{d,\tilde{j}_k}$$

$$\left(\overline{\sigma}_{e,\tilde{j}_k} \left(x_{\tilde{j}_k} - (-1)^{|k+1|} \frac{\lambda(m-k)}{2} \right) \right) \otimes \underline{\mathbf{v}}$$

$$- (-1)^{\delta_{km}} \sum_{e=j_{k+1}}^{\tilde{j}_{k+1}} \sum_{d=j_k}^{j_{k+1}} (-1)^{|k+1|(e-j_{k+1})} (-1)^{|k|(j_{k+1}-d)}$$

$$\mathbf{m}\overline{\sigma}_{e,j_{k+1}} \left(x_{j_{k+1}} - (-1)^{|k+1|} \frac{\lambda(m-k)}{2} \right) \overline{\sigma}_{d,j_{k+1}} \otimes \underline{\mathbf{v}}$$

$$= \sum_{d=j_k}^{\tilde{j}_k} \sum_{e=\tilde{j}_k+1}^{\tilde{j}_{k+1}} (-1)^{|k|(\tilde{j}_k-d)} (-1)^{|k+1|(e-\tilde{j}_k)} \mathbf{m}\overline{\sigma}_{d,\tilde{j}_k} \left(\overline{\sigma}_{e,\tilde{j}_k} \left(x_{\tilde{j}_k} \right) \right) \otimes \underline{\mathbf{v}}$$

$$+ \sum_{d=j_k}^{\tilde{j}_k} (-1)^{|k|(\tilde{j}_k-d)} \mathbf{m}\overline{\sigma}_{d,\tilde{j}_k} x_{\tilde{j}_k} \otimes \underline{\mathbf{v}}$$

$$- \sum_{d=j_k}^{\tilde{j}_k} \sum_{e=\tilde{j}_k}^{\tilde{j}_{k+1}} (-1)^{|k|(\tilde{j}_k-d)} (-1)^{|k+1|(e-\tilde{j}_k)} \mathbf{m}\overline{\sigma}_{d,\tilde{j}_k}$$

$$\left(\overline{\sigma}_{e,\tilde{j}_k} \left((-1)^{|k+1|} \frac{\lambda(m-k)}{2} \right) \right) \otimes \underline{\mathbf{v}}$$

$$- (-1)^{\delta_{km}} \sum_{e=j_{k+1}+1}^{\tilde{j}_{k+1}} \sum_{d=j_k}^{j_{k+1}} (-1)^{|k+1|(e-j_{k+1})} (-1)^{|k|(j_{k+1}-d)}$$

$$\mathbf{m}\overline{\sigma}_{e,j_{k+1}} \left(x_{j_{k+1}} \right) \overline{\sigma}_{d,j_{k+1}} \otimes \underline{\mathbf{v}}$$

$$- (-1)^{\delta_{km}} \sum_{d=j_k}^{j_{k+1}} (-1)^{|k|(j_{k+1}-d)} \mathbf{m} x_{j_{k+1}} \overline{\sigma}_{d,j_{k+1}} \otimes \underline{\mathbf{v}}$$

$$+ (-1)^{\delta_{km}} \sum_{e=j_{k+1}}^{\tilde{j}_{k+1}} \sum_{d=j_k}^{j_{k+1}} (-1)^{|k+1|(e-j_{k+1})} (-1)^{|k|(j_{k+1}-d)} \mathbf{m}\overline{\sigma}_{e,j_{k+1}}$$

$$\left((-1)^{|k+1|} \frac{\lambda(m-k)}{2} \right) \overline{\sigma}_{d,j_{k+1}} \otimes \underline{\mathbf{v}}$$

$$= \sum_{d=j_k}^{\tilde{j}_k} \sum_{e=\tilde{j}_k+1}^{\tilde{j}_{k+1}} (-1)^{|k|(\tilde{j}_k-d)} (-1)^{|k+1|(e-\tilde{j}_k)} \mathbf{m}\overline{\sigma}_{d,\tilde{j}_k} \overline{\sigma}_{e,\tilde{j}_k} x_{\tilde{j}_k} \otimes \underline{\mathbf{v}}$$

$$+ \sum_{d=j_k}^{\tilde{j}_k} (-1)^{|k|(\tilde{j}_k-d)} \mathbf{m}\overline{\sigma}_{d,\tilde{j}_k} x_{\tilde{j}_k} \otimes \underline{\mathbf{v}}$$

$$- (-1)^{\delta_{km}} \sum_{e=j_{k+1}+1}^{\tilde{j}_{k+1}} \sum_{d=j_k}^{j_{k+1}} (-1)^{|k+1|(e-j_{k+1})} (-1)^{|k|(j_{k+1}-d)}$$

$$\mathbf{m}\overline{\sigma}_{e,j_{k+1}} \left(x_{j_{k+1}} \right) \overline{\sigma}_{d,j_{k+1}} \otimes \underline{\mathbf{v}}$$

$$
\begin{aligned}
&-(-1)^{\delta_{km}}\sum_{d=j_k}^{j_{k+1}}(-1)^{|k|(j_{k+1}-d)}\mathbf{m}x_{j_{k+1}}\overline{\sigma}_{d,j_{k+1}}\otimes\underline{\mathbf{v}}\\
&-\sum_{d=j_k}^{\tilde{j}_k}\sum_{e=j_{k+1}}^{\tilde{j}_{k+1}}(-1)^{|k|(\tilde{j}_k-d)}(-1)^{|k+1|(e-\tilde{j}_k)}\\
&\mathbf{m}\overline{\sigma}_{d,\tilde{j}_k}\left(\overline{\sigma}_{e,\tilde{j}_k}\left((-1)^{|k+1|}\frac{\lambda(m-k)}{2}\right)\right)\otimes\underline{\mathbf{v}}\\
&+(-1)^{\delta_{km}}\sum_{e=j_{k+1}}^{\tilde{j}_{k+1}}\sum_{d=j_k}^{\tilde{j}_k}(-1)^{|k+1|(e-j_{k+1})}(-1)^{|k|(j_{k+1}-d)}\\
&\mathbf{m}\overline{\sigma}_{e,j_{k+1}}\left((-1)^{|k+1|}\frac{\lambda(m-k)}{2}\right)\overline{\sigma}_{d,j_{k+1}}\otimes\underline{\mathbf{v}}\\
&-\sum_{d=j_k}^{\tilde{j}_k}(-1)^{|k|(\tilde{j}_k-d)}\mathbf{m}\overline{\sigma}_{d,\tilde{j}_k}(-1)^{|k+1|}\frac{\lambda(m-k)}{2}\otimes\underline{\mathbf{v}}\\
&+(-1)^{\delta_{km}}\sum_{e=j_{k+1}}^{\tilde{j}_{k+1}}(-1)^{|k+1|(e-j_{k+1})}\mathbf{m}\overline{\sigma}_{e,j_{k+1}}(-1)^{|k+1|}\frac{\lambda(m-k)}{2}\otimes\underline{\mathbf{v}}
\end{aligned}
$$

$$
\begin{aligned}
&(-1)^{|k+1|}((-1)^{\delta_{km}}l_{k+1}-l_k)\frac{\lambda}{2}(m-k)\\
&\quad+\sum_{d=j_k}^{\tilde{j}_k}\sum_{e=\tilde{j}_k+1}^{\tilde{j}_{k+1}}(-1)^{|k|(\tilde{j}_k-d)}(-1)^{|k+1|(e-\tilde{j}_k)}\mathbf{m}\overline{\sigma}_{d,\tilde{j}_k}\overline{\sigma}_{e,\tilde{j}_k}x_{\tilde{j}_k}\otimes\underline{\mathbf{v}}\\
&\quad+\sum_{d=j_k}^{\tilde{j}_k}(-1)^{|k|(\tilde{j}_k-d)}\mathbf{m}\overline{\sigma}_{d,\tilde{j}_k}x_{\tilde{j}_k}\otimes\underline{\mathbf{v}}\\
&\quad-(-1)^{\delta_{km}}\sum_{e=j_{k+1}+1}^{\tilde{j}_{k+1}}\sum_{d=j_k}^{j_{k+1}}(-1)^{|k+1|(e-j_{k+1})}(-1)^{|k|(j_{k+1}-d)}\\
&\quad\mathbf{m}\overline{\sigma}_{e,j_{k+1}}\left(x_{j_{k+1}}\right)\overline{\sigma}_{d,j_{k+1}}\otimes\underline{\mathbf{v}}\\
&\quad-(-1)^{\delta_{km}}\sum_{d=j_k}^{j_{k+1}}(-1)^{|k|(j_{k+1}-d)}\mathbf{m}x_{j_{k+1}}\overline{\sigma}_{d,j_{k+1}}\otimes\underline{\mathbf{v}}\\
&\quad=(-1)^{|k+1|}((-1)^{\delta_{km}}l_{k+1}-l_k)\frac{\lambda}{2}(m-k)
\end{aligned}
$$

$$+\sum_{d=j_k}^{\tilde{j}_k}\sum_{e=\tilde{j}_k+1}^{\tilde{j}_{k+1}}(-1)^{|k|(\tilde{j}_k-d)}(-1)^{|k+1|(e-\tilde{j}_k)}\mathbf{m}\overline{\sigma}_{d,\tilde{j}_k}$$

$$\left(x_e\overline{\sigma}_{e,\tilde{j}_k}+\kappa\sum_{j=\tilde{j}_k+1}^{e}\overline{\sigma}_{ej}\overline{\sigma}_{j-1,\tilde{j}_k}\right)\otimes\underline{\mathbf{v}}$$

$$+\sum_{d=j_k}^{\tilde{j}_k}(-1)^{|k|(\tilde{j}_k-d)}\mathbf{m}\overline{\sigma}_{d,\tilde{j}_k}x_{\tilde{j}_k}\otimes\underline{\mathbf{v}}$$

$$-(-1)^{\delta_{km}}\sum_{e=j_{k+1}+1}^{\tilde{j}_{k+1}}\sum_{d=j_k}^{j_{k+1}}(-1)^{|k+1|(e-j_{k+1})}(-1)^{|k|(j_{k+1}-d)}$$

$$\mathbf{m}\left(x_e\overline{\sigma}_{e,j_{k+1}}+\kappa\sum_{j=j_{k+1}+1}^{e}\overline{\sigma}_{ej}\overline{\sigma}_{j-1,j_{k+1}}\right)\overline{\sigma}_{d,j_{k+1}}\otimes\underline{\mathbf{v}}$$

$$-(-1)^{\delta_{km}}\sum_{d=j_k}^{j_{k+1}}(-1)^{|k|(j_{k+1}-d)}\mathbf{m}x_{j_{k+1}}\overline{\sigma}_{d,j_{k+1}}\otimes\underline{\mathbf{v}}$$

$$=(-1)^{|k+1|}((-1)^{\delta_{km}}l_{k+1}-l_k)\frac{\lambda}{2}(m-k)$$

$$+\sum_{d=j_k}^{\tilde{j}_k}\sum_{e=\tilde{j}_k+1}^{\tilde{j}_{k+1}}(-1)^{|k|(\tilde{j}_k-d)}(-1)^{|k+1|(e-\tilde{j}_k)}\mathbf{m}\overline{\sigma}_{d,\tilde{j}_k}$$

$$\left(x_e\overline{\sigma}_{e,\tilde{j}_k}+\kappa\sum_{j=\tilde{j}_k+1}^{e}\overline{\sigma}_{ej}\overline{\sigma}_{j-1,\tilde{j}_k}\right)\otimes\underline{\mathbf{v}}$$

$$+\sum_{d=j_k}^{\tilde{j}_k}(-1)^{|k|(\tilde{j}_k-d)}\mathbf{m}\overline{\sigma}_{d,\tilde{j}_k}x_{\tilde{j}_k}\otimes\underline{\mathbf{v}}$$

$$-(-1)^{\delta_{km}}\sum_{e=j_{k+1}+1}^{\tilde{j}_{k+1}}\sum_{d=j_k}^{j_{k+1}}(-1)^{|k+1|(e-j_{k+1})}(-1)^{|k|(j_{k+1}-d)}$$

$$\mathbf{m}\left(x_e\overline{\sigma}_{e,j_{k+1}}+\kappa\sum_{j=j_{k+1}+1}^{e}\overline{\sigma}_{ej}\overline{\sigma}_{j-1,j_{k+1}}\right)\overline{\sigma}_{d,j_{k+1}}\otimes\underline{\mathbf{v}}$$

$$-(-1)^{\delta_{km}}\sum_{d=j_k}^{j_{k+1}}(-1)^{|k|(j_{k+1}-d)}\mathbf{m}x_{j_{k+1}}\overline{\sigma}_{d,j_{k+1}}\otimes\underline{\mathbf{v}}$$

$$=(-1)^{|k+1|}((-1)^{\delta_{km}}l_{k+1}-l_k)\frac{\lambda}{2}(m-k)$$
$$+\sum_{d=j_k}^{\tilde{j}_k}\sum_{e=\tilde{j}_k+1}^{\tilde{j}_{k+1}}(-1)^{|k|(\tilde{j}_k-d)}(-1)^{|k+1|(e-\tilde{j}_k)}\mathbf{m}x_e\overline{\sigma}_{d,\tilde{j}_k}\overline{\sigma}_{e,\tilde{j}_k}\otimes\underline{\mathbf{v}}$$
$$+\kappa\sum_{d=j_k}^{\tilde{j}_k}\sum_{e=\tilde{j}_k+1}^{\tilde{j}_{k+1}}\sum_{j=\tilde{j}_k+1}^{e}(-1)^{|k|(\tilde{j}_k-d)}(-1)^{|k+1|(e-\tilde{j}_k)}\mathbf{m}\overline{\sigma}_{d,\tilde{j}_k}\overline{\sigma}_{ej}\overline{\sigma}_{j-1,\tilde{j}_k}\otimes\underline{\mathbf{v}}$$
$$+\sum_{d=j_k}^{\tilde{j}_k}(-1)^{|k|(\tilde{j}_k-d)}\mathbf{m}\left(x_d\overline{\sigma}_{d,\tilde{j}_k}-\kappa\sum_{j=d+1}^{\tilde{j}_k}\overline{\sigma}_{d,j-1}\overline{\sigma}_{j,\tilde{j}_k}\right)\otimes\underline{\mathbf{v}}$$
$$-(-1)^{\delta_{km}}\sum_{e=j_{k+1}+1}^{\tilde{j}_{k+1}}\sum_{d=j_k}^{j_{k+1}}(-1)^{|k+1|(e-j_{k+1})}(-1)^{|k|(j_{k+1}-d)}\mathbf{m}x_e\overline{\sigma}_{e,j_{k+1}}\overline{\sigma}_{d,j_{k+1}}\otimes\underline{\mathbf{v}}$$
$$-(-1)^{\delta_{km}}\kappa\sum_{e=j_{k+1}+1}^{\tilde{j}_{k+1}}\sum_{d=j_k}^{j_{k+1}}\sum_{j=j_{k+1}+1}^{e}(-1)^{|k+1|(e-j_{k+1})}(-1)^{|k|(j_{k+1}-d)}$$
$$\mathbf{m}\overline{\sigma}_{ej}\overline{\sigma}_{j-1,j_{k+1}}\overline{\sigma}_{d,j_{k+1}}\otimes\underline{\mathbf{v}}$$
$$-(-1)^{\delta_{km}}\sum_{d=j_k}^{j_{k+1}}(-1)^{|k|(j_{k+1}-d)}\mathbf{m}x_{j_{k+1}}\overline{\sigma}_{d,j_{k+1}}\otimes\underline{\mathbf{v}}$$
$$=(-1)^{|k+1|}((-1)^{\delta_{km}}l_{k+1}-l_k)\frac{\lambda}{2}(m-k)$$
$$+\sum_{d=j_k}^{\tilde{j}_k}\sum_{e=\tilde{j}_k+1}^{\tilde{j}_{k+1}}(-1)^{|k|(\tilde{j}_k-d)}(-1)^{|k+1|(e-\tilde{j}_k)}\mathbf{m}x_e\overline{\sigma}_{d,\tilde{j}_k}\overline{\sigma}_{e,\tilde{j}_k}\otimes\underline{\mathbf{v}}$$
$$+\kappa\sum_{d=j_k}^{\tilde{j}_k}\sum_{e=\tilde{j}_k+1}^{\tilde{j}_{k+1}}\sum_{j=\tilde{j}_k+1}^{e}(-1)^{|k|(\tilde{j}_k-d)}(-1)^{|k+1|(e-\tilde{j}_k)}$$
$$\mathbf{m}\overline{\sigma}_{d,\tilde{j}_k}\overline{\sigma}_{ej}\overline{\sigma}_{j-1,\tilde{j}_k}\otimes\underline{\mathbf{v}}$$
$$+\sum_{d=j_k}^{\tilde{j}_k}(-1)^{|k|(\tilde{j}_k-d)}\mathbf{m}\left(x_d\overline{\sigma}_{d,\tilde{j}_k}-\kappa\sum_{j=d+1}^{\tilde{j}_k}\overline{\sigma}_{d,j-1}\overline{\sigma}_{j,\tilde{j}_k}\right)\otimes\underline{\mathbf{v}}$$
$$-(-1)^{\delta_{km}}\sum_{e=j_{k+1}}^{\tilde{j}_{k+1}}\sum_{d=j_k}^{j_{k+1}}(-1)^{|k+1|(e-j_{k+1})}(-1)^{|k|(j_{k+1}-d)}\mathbf{m}x_e\overline{\sigma}_{e,j_{k+1}}\overline{\sigma}_{d,j_{k+1}}\otimes\underline{\mathbf{v}}$$

$$+(-1)^{\delta_{km}}\sum_{d=j_k}^{j_{k+1}}(-1)^{|k|(j_{k+1}-d)}\mathbf{m}x_{j_{k+1}}\overline{\sigma}_{d,j_{k+1}}\otimes\underline{\mathbf{v}}$$

$$-(-1)^{\delta_{km}}\kappa\sum_{e=j_{k+1}+1}^{\tilde{j}_{k+1}}\sum_{d=j_k}^{j_{k+1}}\sum_{j=j_{k+1}+1}^{e}(-1)^{|k+1|(e-j_{k+1})}$$

$$(-1)^{|k|(j_{k+1}-d)}\mathbf{m}\overline{\sigma}_{ej}\overline{\sigma}_{j-1,j_{k+1}}\overline{\sigma}_{d,j_{k+1}}\otimes\underline{\mathbf{v}}$$

$$-(-1)^{\delta_{km}}\sum_{d=j_k}^{j_{k+1}}(-1)^{|k|(j_{k+1}-d)}\mathbf{m}x_{j_{k+1}}\overline{\sigma}_{d,j_{k+1}}\otimes\underline{\mathbf{v}}$$

$$=(-1)^{|k+1|}((-1)^{\delta_{km}}l_{k+1}-l_k)\frac{\lambda}{2}(m-k)$$

$$+\sum_{d=j_k}^{\tilde{j}_k}\sum_{e=\tilde{j}_k+1}^{\tilde{j}_{k+1}}(-1)^{|k|(\tilde{j}_k-d)}(-1)^{|k+1|(e-\tilde{j}_k)}\mathbf{m}x_e\overline{\sigma}_{d,\tilde{j}_k}\overline{\sigma}_{e,\tilde{j}_k}\otimes\underline{\mathbf{v}}$$

$$+\kappa\sum_{d=j_k}^{\tilde{j}_k}\sum_{e=\tilde{j}_k+1}^{\tilde{j}_{k+1}}\sum_{j=j_{k+1}+1}^{e}(-1)^{|k|(\tilde{j}_k-d)}(-1)^{|k+1|(e-\tilde{j}_k)}\mathbf{m}\overline{\sigma}_{d,\tilde{j}_k}\overline{\sigma}_{ej}\overline{\sigma}_{j-1,\tilde{j}_k}\otimes\underline{\mathbf{v}}$$

$$+\kappa\sum_{d=j_k}^{\tilde{j}_k}\sum_{e=\tilde{j}_k+1}^{\tilde{j}_{k+1}}(-1)^{|k|(\tilde{j}_k-d)}(-1)^{|k+1|(e-\tilde{j}_k)}\mathbf{m}\overline{\sigma}_{d,\tilde{j}_k}\overline{\sigma}_{e,\tilde{j}_k+1}\otimes\underline{\mathbf{v}}$$

$$+\sum_{d=j_k}^{\tilde{j}_k}(-1)^{|k|(\tilde{j}_k-d)}\mathbf{m}\left(x_d\overline{\sigma}_{d,\tilde{j}_k}\right)\otimes\underline{\mathbf{v}}$$

$$-\kappa\sum_{d=j_k}^{\tilde{j}_k}\sum_{j=d+1}^{\tilde{j}_k}(-1)^{|k|(\tilde{j}_k-d)}\mathbf{m}\overline{\sigma}_{d,j-1}\overline{\sigma}_{j,\tilde{j}_k}\otimes\underline{\mathbf{v}}$$

$$-(-1)^{\delta_{km}}\sum_{e=j_{k+1}}^{\tilde{j}_{k+1}}\sum_{d=j_k}^{\tilde{j}_k}(-1)^{|k+1|(e-j_{k+1})}(-1)^{|k|(j_{k+1}-d)}\mathbf{m}x_e\overline{\sigma}_{e,j_{k+1}}\overline{\sigma}_{d,j_{k+1}}\otimes\underline{\mathbf{v}}$$

$$-(-1)^{\delta_{km}}\sum_{e=j_{k+1}}^{\tilde{j}_{k+1}}(-1)^{|k+1|(e-j_{k+1})}\mathbf{m}x_e\overline{\sigma}_{e,j_{k+1}}\otimes\underline{\mathbf{v}}$$

$$-(-1)^{\delta_{km}}\kappa\sum_{e=j_{k+1}+1}^{\tilde{j}_{k+1}}\sum_{d=j_k}^{\tilde{j}_k}\sum_{j=j_{k+1}+1}^{e}(-1)^{|k+1|(e-j_{k+1})}$$

$$(-1)^{|k|(j_{k+1}-d)}\mathbf{m}\overline{\sigma}_{ej}\overline{\sigma}_{j-1,j_{k+1}}\overline{\sigma}_{d,j_{k+1}}\otimes\underline{\mathbf{v}}$$

$$-(-1)^{\delta_{km}}\kappa\sum_{e=j_{k+1}+1}^{\tilde{j}_{k+1}}\sum_{j=j_{k+1}+1}^{e}(-1)^{|k+1|(e-j_{k+1})}\mathbf{m}\overline{\sigma}_{ej}\overline{\sigma}_{j-1,j_{k+1}}\otimes\underline{\mathbf{v}}$$

$$=(-1)^{|k+1|}((-1)^{\delta_{km}}l_{k+1}-l_k)\frac{\lambda}{2}(m-k)$$

$$+\sum_{d=j_k}^{\tilde{j}_k}\sum_{e=\tilde{j}_k+1}^{\tilde{j}_{k+1}}(-1)^{|k|(\tilde{j}_k-d)}(-1)^{|k+1|(e-\tilde{j}_k)}\mathbf{m}x_e\overline{\sigma}_{d,\tilde{j}_k}\overline{\sigma}_{e,\tilde{j}_k}\otimes\underline{\mathbf{v}}$$

$$-(-1)^{\delta_{km}}\sum_{e=j_{k+1}}^{\tilde{j}_{k+1}}\sum_{d=j_k}^{\tilde{j}_k}(-1)^{|k+1|(e-j_{k+1})}(-1)^{|k|(j_{k+1}-d)}\mathbf{m}x_e\overline{\sigma}_{e,j_{k+1}}\overline{\sigma}_{d,j_{k+1}}\otimes\underline{\mathbf{v}}$$

$$+\sum_{d=j_k}^{\tilde{j}_k}(-1)^{|k|(\tilde{j}_k-d)}\mathbf{m}\left(x_d\overline{\sigma}_{d,\tilde{j}_k}\right)\otimes\underline{\mathbf{v}}$$

$$-(-1)^{\delta_{km}}\sum_{e=j_{k+1}}^{\tilde{j}_{k+1}}(-1)^{|k+1|(e-j_{k+1})}\mathbf{m}x_e\overline{\sigma}_{e,j_{k+1}}\otimes\underline{\mathbf{v}}$$

$$-\kappa\sum_{d=j_k}^{\tilde{j}_k}\sum_{j=d+1}^{\tilde{j}_k}(-1)^{|k|}\mathbf{m}\otimes\underline{\mathbf{v}}$$

$$+\kappa\sum_{d=j_k}^{\tilde{j}_k}\sum_{e=\tilde{j}_k+1}^{\tilde{j}_{k+1}}\sum_{j=j_{k+1}+1}^{e}(-1)^{|k|(\tilde{j}_k-d)}(-1)^{|k+1|(e-\tilde{j}_k)}\mathbf{m}\overline{\sigma}_{d,\tilde{j}_k}\overline{\sigma}_{ej}\overline{\sigma}_{j-1,\tilde{j}_k}\otimes\underline{\mathbf{v}}$$

$$+\kappa\sum_{d=j_k}^{\tilde{j}_k}\sum_{e=\tilde{j}_k+1}^{\tilde{j}_{k+1}}(-1)^{|k|(\tilde{j}_k-d)}(-1)^{|k+1|(e-\tilde{j}_k)}\mathbf{m}\overline{\sigma}_{d,\tilde{j}_k}\overline{\sigma}_{e,\tilde{j}_k+1}\otimes\underline{\mathbf{v}}$$

$$-(-1)^{\delta_{km}}\kappa\sum_{e=j_{k+1}+1}^{\tilde{j}_{k+1}}\sum_{d=j_k}^{\tilde{j}_k}\sum_{j=j_{k+1}+1}^{e}(-1)^{|k+1|(e-j_{k+1})}(-1)^{|k|(j_{k+1}-d)}$$

$$\mathbf{m}\overline{\sigma}_{ej}\overline{\sigma}_{j-1,j_{k+1}}\overline{\sigma}_{d,j_{k+1}}\otimes\underline{\mathbf{v}}$$

$$-(-1)^{\delta_{km}}\kappa\sum_{e=j_{k+1}+1}^{\tilde{j}_{k+1}}\sum_{j=j_{k+1}+1}^{e}(-1)^{|k+1|(e-j_{k+1})}\mathbf{m}\overline{\sigma}_{ej}\overline{\sigma}_{j-1,j_{k+1}}\otimes\underline{\mathbf{v}}$$

$$=(-1)^{|k+1|}((-1)^{\delta_{km}}l_{k+1}-l_k)\frac{\lambda}{2}(m-k)$$

$$+\sum_{d=j_k}^{\tilde{j}_k}\sum_{e=\tilde{j}_k+1}^{\tilde{j}_{k+1}}(-1)^{|k|(\tilde{j}_k-d)}(-1)^{|k+1|(e-\tilde{j}_k)}\mathbf{m}x_e\overline{\sigma}_{d,\tilde{j}_k}\overline{\sigma}_{e,\tilde{j}_k}\otimes\underline{\mathbf{v}}$$

$$- (-1)^{\delta_{km}} \sum_{e=j_{k+1}}^{\tilde{j}_{k+1}} \sum_{d=j_k}^{\tilde{j}_k} (-1)^{|k+1|(e-j_{k+1})}(-1)^{|k|(j_{k+1}-d)} \mathbf{m} x_e \overline{\sigma}_{e,j_{k+1}} \overline{\sigma}_{d,j_{k+1}} \otimes \underline{\mathbf{v}}$$

$$+ \sum_{d=j_k}^{\tilde{j}_k} \mathbf{m} x_d \otimes \underline{\mathbf{v}} - (-1)^{\delta_{km}} \sum_{e=j_{k+1}}^{\tilde{j}_{k+1}} \mathbf{m} x_e \otimes \underline{\mathbf{v}}$$

$$- \kappa \sum_{d=j_k}^{\tilde{j}_k} \sum_{j=d+1}^{\tilde{j}_k} (-1)^{|k|} \mathbf{m} \otimes \underline{\mathbf{v}}$$

$$+ \kappa \sum_{d=j_k}^{\tilde{j}_k} \sum_{e=\tilde{j}_k+1}^{\tilde{j}_{k+1}} \sum_{j=j_{k+1}+1}^{e} (-1)^{|k|(\tilde{j}_k-d)}(-1)^{|k+1|(e-\tilde{j}_k)} \mathbf{m} \overline{\sigma}_{d,\tilde{j}_k} \overline{\sigma}_{ej} \overline{\sigma}_{j-1,\tilde{j}_k} \otimes \underline{\mathbf{v}}$$

$$+ \kappa \sum_{d=j_k}^{\tilde{j}_k} \sum_{e=\tilde{j}_k+1}^{\tilde{j}_{k+1}} (-1)^{|k+1|} \mathbf{m} \otimes \underline{\mathbf{v}}$$

$$- (-1)^{\delta_{km}} \kappa \sum_{e=j_{k+1}+1}^{\tilde{j}_{k+1}} \sum_{d=j_k}^{\tilde{j}_k} \sum_{j=j_{k+1}+1}^{e} (-1)^{|k+1|(e-j_{k+1})}(-1)^{|k|(j_{k+1}-d)}$$

$$\mathbf{m} \overline{\sigma}_{ej} \overline{\sigma}_{j-1,j_{k+1}} \overline{\sigma}_{d,j_{k+1}} \otimes \underline{\mathbf{v}}$$

$$- (-1)^{\delta_{km}} \kappa \sum_{e=j_{k+1}+1}^{\tilde{j}_{k+1}} \sum_{j=j_{k+1}+1}^{e} (-1)^{|k+1|} \mathbf{m} \otimes \underline{\mathbf{v}}$$

$$= (-1)^{|k+1|}((-1)^{\delta_{km}} l_{k+1} - l_k) \frac{\lambda}{2}(m-k)$$

$$+ \sum_{d=j_k}^{\tilde{j}_k} \sum_{e=\tilde{j}_k+1}^{\tilde{j}_{k+1}} (-1)^{|k|(\tilde{j}_k-d)}(-1)^{|k+1|(e-\tilde{j}_k)} \mathbf{m} x_e \overline{\sigma}_{d,\tilde{j}_k} \overline{\sigma}_{e,\tilde{j}_k} \otimes \underline{\mathbf{v}}$$

$$- (-1)^{\delta_{km}} \sum_{e=j_{k+1}}^{\tilde{j}_{k+1}} \sum_{d=j_k}^{\tilde{j}_k} (-1)^{|k+1|(e-j_{k+1})}(-1)^{|k|(j_{k+1}-d)} \mathbf{m} x_e \overline{\sigma}_{e,j_{k+1}} \overline{\sigma}_{d,j_{k+1}} \otimes \underline{\mathbf{v}}$$

$$+ \sum_{d=j_k}^{\tilde{j}_k} \mathbf{m} x_d \otimes \underline{\mathbf{v}} - (-1)^{\delta_{km}} \sum_{e=j_{k+1}}^{\tilde{j}_{k+1}} \mathbf{m} x_e \otimes \underline{\mathbf{v}}$$

$$- \kappa \sum_{d=j_k}^{\tilde{j}_k} \sum_{j=d+1}^{\tilde{j}_k} (-1)^{|k|} \mathbf{m} \otimes \underline{\mathbf{v}}$$

$$
\begin{aligned}
&+\kappa\sum_{d=j_k}^{\tilde{j}_k}\sum_{e=\tilde{j}_k+1}^{\tilde{j}_{k+1}}\sum_{j=j_{k+1}+1}^{e}(-1)^{|k|(\tilde{j}_k-d)}(-1)^{|k+1|(j-\tilde{j}_k)}\mathbf{m}\overline{\sigma}_{d,\tilde{j}_k}\overline{\sigma}_{j-1,\tilde{j}_k}\otimes\underline{\mathbf{v}}\\
&+\kappa\sum_{d=j_k}^{\tilde{j}_k}\sum_{e=\tilde{j}_k+1}^{\tilde{j}_{k+1}}(-1)^{|k+1|}\mathbf{m}\otimes\underline{\mathbf{v}}\\
&-(-1)^{\delta_{km}}\kappa\sum_{e=j_{k+1}+1}^{\tilde{j}_{k+1}}\sum_{d=j_k}^{\tilde{j}_k}\sum_{j=j_{k+1}+1}^{e}(-1)^{|k+1|(j-j_{k+1})}(-1)^{|k|(j_{k+1}-d)}\\
&\mathbf{m}\overline{\sigma}_{j-1,j_{k+1}}\overline{\sigma}_{d,j_{k+1}}\otimes\underline{\mathbf{v}}\\
&-(-1)^{\delta_{km}}\kappa\sum_{e=j_{k+1}+1}^{\tilde{j}_{k+1}}\sum_{j=j_{k+1}+1}^{e}(-1)^{|k+1|}\mathbf{m}\otimes\underline{\mathbf{v}}\\
=&(-1)^{|k+1|}((-1)^{\delta_{km}}l_{k+1})\frac{\lambda}{2}(m-k)-(-1)^{\delta_{km}}\sum_{e=j_{k+1}}^{\tilde{j}_{k+1}}\mathbf{m}x_e\otimes\underline{\mathbf{v}}\\
&-(-1)^{|k+1|}(l_k)\frac{\lambda}{2}(m-k)+\sum_{d=j_k}^{\tilde{j}_k}\mathbf{m}x_d\otimes\underline{\mathbf{v}}\\
&+\sum_{d=j_k}^{\tilde{j}_k}\sum_{e=\tilde{j}_k+1}^{\tilde{j}_{k+1}}\mathbf{m}x_e\Big((-1)^{|k|(\tilde{j}_k-d)}(-1)^{|k+1|(e-\tilde{j}_k)}\overline{\sigma}_{d,\tilde{j}_k}\overline{\sigma}_{e,\tilde{j}_k}\\
&-(-1)^{\delta_{km}}(-1)^{|k+1|(e-j_{k+1})}(-1)^{|k|(j_{k+1}-d)}\overline{\sigma}_{e,j_{k+1}}\overline{\sigma}_{d,j_{k+1}}\Big)\otimes\underline{\mathbf{v}}\\
&-(-1)^{|k|}\kappa\,(0+1+\cdots+l_k-1)\,(\mathbf{m}\otimes\underline{\mathbf{v}})\\
&+(-1)^{|k+1|}\kappa l_k l_{k+1}\mathbf{m}\otimes\underline{\mathbf{v}}\\
&-(-1)^{\delta_{km}}(-1)^{|k+1|}\kappa\,(0+1+\cdots+l_{k+1}-1)\,(\mathbf{m}\otimes\underline{\mathbf{v}})\\
&+\kappa\sum_{d=j_k}^{\tilde{j}_k}\sum_{e=\tilde{j}_k+1}^{\tilde{j}_{k+1}}\sum_{j=j_{k+1}+1}^{e}\mathbf{m}\Big((-1)^{|k|(\tilde{j}_k-d)}(-1)^{|k+1|(j-\tilde{j}_k)}\overline{\sigma}_{d,\tilde{j}_k}\overline{\sigma}_{j-1,\tilde{j}_k}-\\
&(-1)^{\delta_{km}}(-1)^{|k+1|(j-j_{k+1})}(-1)^{|k|(j_{k+1}-d)}\overline{\sigma}_{j-1,j_{k+1}}\overline{\sigma}_{d,j_{k+1}}\Big)\otimes\underline{\mathbf{v}}\\
=&-(-1)^{\delta_{km}}\sum_{e=j_{k+1}}^{\tilde{j}_{k+1}}\mathbf{m}\left(x_e-(-1)^{|k+1|}\frac{\lambda}{2}(m-k)\right)\otimes\underline{\mathbf{v}}\\
&+\sum_{d=j_k}^{\tilde{j}_k}\mathbf{m}\left(x_d-(-1)^{|k+1|}\frac{\lambda}{2}(m-k)\right)\otimes\underline{\mathbf{v}}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{d=j_k}^{\tilde{j}_k}\sum_{e=\tilde{j}_k+1}^{\tilde{j}_{k+1}}\mathbf{m}x_e(-1)^{|k|(\tilde{j}_k-d)}(-1)^{|k+1|(e-\tilde{j}_k)}\\
&\left(1-(-1)^{\delta_{km}}(-1)^{|k+1|}(-1)^{|k|}\right)\otimes\sigma_{ed}\underline{\mathbf{v}}\\
&-(-1)^{|k|}\kappa\frac{l_k(l_k-1)}{2}(\mathbf{m}\otimes\underline{\mathbf{v}})+(-1)^{|k+1|}\kappa l_k l_{k+1}\mathbf{m}\otimes\underline{\mathbf{v}}\\
&-(-1)^{\delta_{km}}(-1)^{|k+1|}\kappa\frac{l_{k+1}(l_{k+1}-1)}{2}(\mathbf{m}\otimes\underline{\mathbf{v}})\\
&+\kappa\sum_{d=j_k}^{\tilde{j}_k}\sum_{e=\tilde{j}_k+1}^{\tilde{j}_{k+1}}\sum_{j=j_{k+1}+1}^{e}\mathbf{m}(-1)^{|k|(\tilde{j}_k-d)}(-1)^{|k+1|(j-\tilde{j}_k)}\\
&\left(1-(-1)^{\delta_{km}}(-1)^{|k+1|}(-1)^{|k|}\right)\otimes\sigma_{ed}\underline{\mathbf{v}}\\
=&-(-1)^{\delta_{km}}\sum_{e=j_{k+1}}^{\tilde{j}_{k+1}}\mathbf{m}\left(x_e-(-1)^{|k+1|}\frac{\lambda}{2}(m-k+(-1)^{\delta_{km}})\right)\otimes\underline{\mathbf{v}}\\
&+\sum_{d=j_k}^{\tilde{j}_k}\mathbf{m}\left(x_d-(-1)^{|k+1|}\frac{\lambda}{2}(m-k-1)\right)\otimes\underline{\mathbf{v}}\\
&-(-1)^{|k|}\frac{\kappa}{2}\left(l_k^2+l_{k+1}^2-2(-1)^{\delta_{km}}l_k l_{k+1}\right)(\mathbf{m}\otimes\underline{\mathbf{v}})\\
=&-(-1)^{\delta_{km}}\sum_{e=j_{k+1}}^{\tilde{j}_{k+1}}\mathbf{m}\left(x_e-(-1)^{|k+1|}\frac{\lambda}{2}(m-k+(-1)^{\delta_{km}})\right)\otimes\underline{\mathbf{v}}\\
&+\sum_{d=j_k}^{\tilde{j}_k}\mathbf{m}\left(x_d-(-1)^{|k+1|}\frac{\lambda}{2}(m-k-1)\right)\otimes\underline{\mathbf{v}}\\
&-(-1)^{|k|}\frac{\kappa}{2}\left(l_k-(-1)^{\delta_{km}}l_{k+1}\right)^2(\mathbf{m}\otimes\underline{\mathbf{v}})\\
=&-(-1)^{\delta_{km}}\sum_{e=j_{k+1}}^{\tilde{j}_{k+1}}\mathbf{m}\left(x_e-(-1)^{|k+1|}\frac{\lambda}{2}(m-k+(-1)^{\delta_{km}})\right)\otimes\underline{\mathbf{v}}\\
&+\sum_{d=j_k}^{\tilde{j}_k}\mathbf{m}\left(x_d-(-1)^{|k+1|}\frac{\lambda}{2}(m-k-1)\right)\otimes\underline{\mathbf{v}}\\
&+(-1)^{|k|}\frac{\lambda H_{k0}^2}{2}(\mathbf{m}\otimes\underline{\mathbf{v}})\\
=&H_{k1}(\mathbf{m}\otimes\underline{\mathbf{v}})
\end{aligned}
$$

6.2 The map ψ and extension of the operators to $X^{\pm}_{01}, H_{0,1}$

We can consider shifted superspaces $\mathbb{C}(1|m|n-1) = \mathbb{C} \oplus \mathbb{C}^m \oplus \mathbb{C}^{n-1}$ and $\mathbb{C}(2|m|n-2)$. In each, the even part is equal to $\mathbb{C}^m$ and the odd parts are $\mathbb{C} \oplus \mathbb{C}^{n-1}$ and $\mathbb{C}^2 \oplus \mathbb{C}^{n-1}$ respectively. Denote by $|k|_1$ the parity of the vector e_k in $\mathbb{C}(1|m|n-1)$ and by $|k|_2$ its parity in $\mathbb{C}(2|m|n-2)$. In practice, this notation applies to any such shifted superspace $\mathbb{C}(t|m|n-t)$ and $|k|_t = |k-t|$, but the motivation to use the subscripted notation is to more clearly indicate the superspace in question.

The change of parity for the vectors corresponds to an analogous change of the Cartan matrix. Indeed, the new matrix is the Cartan matrix of the Lie superalgebra $\mathfrak{sl}(1|m|n-1)$ and can be used to define the super Yangian $Y(\mathfrak{sl}(1|m|n-1))$.

We now introduce the linear map ψ,

$$\psi : (\mathbb{C}(m|n))^{\otimes \ell} \to (\mathbb{C}(1|m|n-1))^{\otimes \ell}, \quad \psi(e_i) = e_{i+1}, \tag{6.14}$$

which extends to a linear isomorphism between $M \otimes_{\mathbb{C}[S_\ell]} (\mathbb{C}(m|n))^{\otimes \ell}$ and $M \otimes_{\mathbb{C}[S_\ell]} (\mathbb{C}(1|m|n-1))^{\otimes \ell}$ by

$$\psi(\mathbf{m} \otimes \underline{\mathbf{v}}) = \mathbf{m} x_1^{-\delta_{j_1,\gamma}} x_2^{-\delta_{j_2,\gamma}} \cdots x_\ell^{-\delta_{j_\ell,\gamma}} \otimes e_{i_1+1} \otimes e_{i_2+1} \otimes \cdots \otimes e_{i_\ell+1} \tag{6.15}$$

where $\gamma = m+n-1$ and indices are modulo $m+n$. Note that ψ^2 maps to $(\mathbb{C}(2|m|n-2))^{\otimes l} = (\mathbb{C}^2 \oplus \mathbb{C}^m \oplus \mathbb{C}^{n-2})^{\otimes \ell}$.

For $a \in \mathbb{C}$, the map τ_a between generators of $\widehat{Y}(\mathfrak{sl}_{m|n})$ is defined here as

$$\begin{aligned}
\tau_a(X^{\pm}_{k,0}) =& X^{\pm}_{k-1,0} \\
\tau_a(H_{k,0}) =& H_{k-1,0} \\
\tau_a(X^{\pm}_{k,1}) =& X^{\pm}_{k-1,1} - (-1)^{|k|} a X^{\pm}_{k-1,0} \\
\tau_a(X^{\pm}_{0,1}) =& X^{\pm}_{m+n-1,1} - (a+\beta) X^{\pm}_{m+n-1,0}
\end{aligned}$$

where $1 \le k \le m+n-1$.

We can define an action of $Y(\mathfrak{sl}(1|m|n-1))$ on $M \otimes_{\mathbb{C}[S_\ell]} (\mathbb{C}(1|m|n-1))^{\otimes \ell}$ using

$$X^+_{k,1}(\mathbf{m}\otimes\underline{\mathbf{v}}) = \mathbf{m}\left(\sum_{d=j_{k+1}}^{\tilde{j}_{k+1}} (-1)^{|k+1|_1(d-j_{k+1})}\overline{\sigma}_{d,j_{k+1}}\right)\left(x_{j_{k+1}}-(-1)^{|k+1|_1}\frac{\lambda}{2}(m-k)\right)\otimes\underline{\mathbf{v}}_{j^-_{k+1}} \tag{6.16}$$

for $2 \leq k \leq m+n-1$ and

$$X^+_{1,1}(\mathbf{m}\otimes\underline{\mathbf{v}}) = \mathbf{m}\left(\sum_{d=j_{k+1}}^{\tilde{j}_{k+1}} \overline{\sigma}_{d,j_{k+1}}\right)\left(x_{j_{k+1}}+\frac{\lambda}{2}(m-2)\right)\otimes\underline{\mathbf{v}}_{j^-_{k+1}}. \tag{6.17}$$

The following lemma is a key result for this chapter.

Lemma 6.2.1. *For $2 \leq k \leq m+n-1$, $r = 0,1$ $a = -\frac{\lambda}{2}, c = \lambda$, and $\beta = -t + \frac{\lambda}{2}(m+n+4)$, we have:*

$$X^{\pm}_{k,r}(\psi(\mathbf{m}\otimes\underline{\mathbf{v}})) = \psi(\tau_a(X^{\pm}_{k,r})(\mathbf{m}\otimes\underline{\mathbf{v}})),\;\; H_{k,r}(\psi(\mathbf{m}\otimes\underline{\mathbf{v}})) = \psi(\tau_a(H_{k,r})(\mathbf{m}\otimes\underline{\mathbf{v}})) \tag{6.18}$$

$$X^{\pm}_{1,r}(\psi^2(\mathbf{m}\otimes\underline{\mathbf{v}})) = \psi^2(\tau_a^2(X^{\pm}_{1,r})(\mathbf{m}\otimes\underline{\mathbf{v}})),\;\; H_{1,r}(\psi^2(\mathbf{m}\otimes\underline{\mathbf{v}})) = \psi^2(\tau_a^2(H_{1,r})(\mathbf{m}\otimes\underline{\mathbf{v}}). \tag{6.19}$$

Proof. It is enough to prove this for $r = 1$ and $D_{k,1}$.

Let $\underline{\mathbf{u}} = \psi(\underline{\mathbf{v}})$ and let p be the number of indices in $\underline{\mathbf{v}}$ that are equal to $m+n$; i.e. $\tilde{j}_{m+n} - j_{m+n} + 1 = p$. This $\underline{\mathbf{u}}$ may not be in standard order, i.e. the indices may no longer be non-decreasing, so we call the reordered tensor $\underline{\mathbf{w}}$. This $\underline{\mathbf{w}}$ can be produced by applying the needed product of transpositions to $\underline{\mathbf{u}}$:

$$\underline{\mathbf{w}} = \overline{\sigma}^{-1}_{lp}\overline{\sigma}^{-1}_{l-1,p-1}\cdots\overline{\sigma}^{-1}_{l-p+1,1}(\underline{\mathbf{u}})$$

essentially moving the last p vectors, all equal to e_1, to the first p positions instead. Set $\rho = \overline{\sigma}_{l-p+1,1}\cdots\overline{\sigma}_{l-1,p-1}\overline{\sigma}_{l,p}$; then $\underline{\mathbf{u}} = \rho(\underline{\mathbf{w}})$. Define h_k and $\tilde{h}_k$ for $\underline{\mathbf{w}}$ as j_k and $\tilde{j}_k$ were defined for $\underline{\mathbf{v}}$.

Suppose $2 \leq k \leq m+n-1$. Then,

$$D_{k,1}(\psi(\mathbf{m}\otimes\underline{\mathbf{v}})) = D_{k,1}(\mathbf{m}x^{-1}_{l-p+1}x^{-1}_{l-p+2}\cdots x^{-1}_{l}\otimes\underline{\mathbf{u}})$$

$$= D_{k,1}(\mathbf{m}x_{l-p+1}^{-1}x_{l-p+2}^{-1}\cdots x_l^{-1}\otimes\rho(\underline{\mathbf{w}}))$$
$$= D_{k,1}(\mathbf{m}x_{l-p+1}^{-1}x_{l-p+2}^{-1}\cdots x_l^{-1}\rho\otimes\underline{\mathbf{w}})$$
$$= \mathbf{m}x_{l-p+1}^{-1}x_{l-p+2}^{-1}\cdots x_l^{-1}\rho$$
$$\left(\sum_{d=h_k}^{\tilde{h}_k}(\mathsf{x}_d-(-1)^{|k+1|_1}\frac{\lambda}{2}(m-k+(-1)^{\delta_{k,m+1}}+(-1)^{\delta_{k1}}))\right)\otimes\underline{\mathbf{w}}$$

$$\psi(D_{k-1,1}(\mathbf{m}\otimes\underline{\mathbf{v}}))=\psi\left(\mathbf{m}\left(\sum_{d=j_{k-1}}^{\tilde{j}_{k-1}}(\mathsf{x}_d-(-1)^{|k|}\frac{\lambda}{2}(m-k+1+(-1)^{\delta_{k-1,m}}))\right)\otimes\underline{\mathbf{v}}\right)$$
$$=\mathbf{m}\left(\sum_{d=j_{k-1}}^{\tilde{j}_{k-1}}(\mathsf{x}_d-(-1)^{|k|}\frac{\lambda}{2}(m-k+1+(-1)^{\delta_{k-1,m}}))x_{l-p+1}^{-1}x_{l-p+2}^{-1}\cdots x_l^{-1}\right)\otimes\rho\underline{\mathbf{w}}$$
$$=\mathbf{m}\left(\sum_{d=j_{k-1}}^{\tilde{j}_{k-1}}(\mathsf{x}_d-(-1)^{|k|}\frac{\lambda}{2}(m-k+1+(-1)^{\delta_{k-1,m}}))x_{l-p+1}^{-1}x_{l-p+2}^{-1}\cdots x_l^{-1}\rho\right)\otimes\underline{\mathbf{w}}$$

Thus, to verify that $D_{k,1}(\psi(\mathbf{m}\otimes\underline{\mathbf{v}}))=\psi(\tau_a(D_{k-1,1}(\mathbf{m}\otimes\underline{\mathbf{v}})))$, it is enough to show that

$$x_{l-p+1}^{-1}x_{l-p+2}^{-1}\cdots x_l^{-1}\rho\left(\sum_{d=h_k}^{\tilde{h}_k}(\mathsf{x}_d-(-1)^{|k+1|_1}\frac{\lambda}{2}(m-k+(-1)^{\delta_{k,m+1}}))\right)\tag{6.20}$$
$$=\sum_{d=j_{k-1}}^{\tilde{j}_{k-1}}(\mathsf{x}_d-(-1)^{|k|}\frac{\lambda}{2}(m-k+1+(-1)^{\delta_{k-1,m}})+(-1)^{|k|}a)x_{\ell-p+1}^{-1}x_{\ell-p+2}^{-1}\cdots x_\ell^{-1}\rho$$

Since we have set $a=-\frac{\lambda}{2}$, $|k+1|_1=|k|$, and by definition $h_k=j_{k-1}+p$, it is enough to prove that

$$x_{l-p+1}^{-1}x_{l-p+2}^{-1}\cdots x_l^{-1}\rho\mathsf{x}_d=\mathsf{x}_{d-p}x_{l-p+1}^{-1}x_{l-p+2}^{-1}\cdots x_l^{-1}\rho\tag{6.21}$$

when $h_k\le d\le\tilde{h}_k$.

In the trigonometric Cherednik algebra, when $k < f$,

$$\begin{aligned} x_f \mathsf{x}_k x_f^{-1} &= \frac{t}{2} + x_k x_f y_k x_f^{-1} + c \sum_{j<k} \sigma_{jk} \\ x_f y_k x_f^{-1} - y_k &= c\sigma_{kf} x_f^{-1} \\ x_f \mathsf{x}_k x_f^{-1} &= \frac{t}{2} + x_k y_k + c x_k \sigma_{kf} x_f^{-1} + c \sum_{j<k} \sigma_{jk} \\ &= \mathsf{x}_k + c\sigma_{kf} \\ x_f^{-1} \mathsf{x}_k x_f &= \mathsf{x}_k - c\sigma_{kf} x_k^{-1} x_f \end{aligned} \tag{6.22}$$

Applying (6.22) to part of ρx_d, we produce

$$\begin{aligned} \overline{\sigma}_{\ell,p} \mathsf{x}_d &= \overline{\sigma}_{\ell,p} \left(\frac{t}{2} + x_d y_d + c \sum_{f=1}^{d-1} \sigma_{fd} \right) \\ &= \left(\frac{t}{2} + x_{d-1} y_{d-1} + c \sum_{f=1}^{d-1} \overline{\sigma}_{\ell,p} \sigma_{fd} \overline{\sigma}_{\ell,p}^{-1} \right) \overline{\sigma}_{\ell,p} \\ &= \left(\frac{t}{2} + x_{d-1} y_{d-1} + c \sum_{f=1}^{p-1} \overline{\sigma}_{\ell,p} \sigma_{fd} \overline{\sigma}_{\ell,p}^{-1} + \overline{\sigma}_{\ell,p} \sigma_{pd} \overline{\sigma}_{\ell,p}^{-1} + c \sum_{f=p+1}^{d-1} \overline{\sigma}_{\ell,p} \sigma_{fd} \overline{\sigma}_{\ell,p}^{-1} \right) \overline{\sigma}_{\ell,p} \\ &= \left(\frac{t}{2} + x_{d-1} y_{d-1} + c \sum_{f=1}^{p-1} \sigma_{f,d-1} + \sigma_{l,d-1} + c \sum_{f=p+1}^{d-1} \sigma_{f-1,d-1} \right) \overline{\sigma}_{\ell-1,p} \\ &= (\mathsf{x}_{d-1} + c\sigma_{\ell,d-1}) \overline{\sigma}_{\ell,p}. \end{aligned}$$

Similarly,

$$\begin{aligned} \overline{\sigma}_{\ell-1,p-1} \overline{\sigma}_{\ell,p} \mathsf{x}_d &= \overline{\sigma}_{\ell-1,p-1} (\mathsf{x}_{d-1} + c\sigma_{\ell,d-1}) \overline{\sigma}_{\ell,p} \\ &= (\mathsf{x}_{d-2} + c\sigma_{\ell-1,d-2} + c\sigma_{\ell,d-2}) \overline{\sigma}_{\ell-1,p-1} \overline{\sigma}_{\ell,p}. \end{aligned}$$

After applying iterations of these transformations making up ρ, we see that

$$\rho \mathsf{x}_d = (\mathsf{x}_{d-p} + c\sigma_{\ell-p+1,d-p} + c\sigma_{\ell-p+2,d-p} + \cdots + c\sigma_{\ell,d-p})\rho. \tag{6.23}$$

As well, using (6.22) and (6.23),

$$\begin{aligned}
x_\ell^{-1}\rho\mathsf{x}_d &= x_\ell^{-1}(\mathsf{x}_{d-p} + c\sigma_{\ell-p+1,d-p} + c\sigma_{\ell-p+2,d-p} + \cdots + c\sigma_{\ell,d-p})\rho \\
&= (x_\ell^{-1}\mathsf{x}_{d-p}x_\ell x_\ell^{-1} + c\sigma_{\ell-p+1,d-p}X_\ell^{-1} + c\sigma_{\ell-p+2,d-p}x_\ell^{-1} + \cdots + c\sigma_{\ell,d-p}x_{d-p}^{-1})\rho \\
&= (\mathsf{x}_{d-p} - c\sigma_{d-p,\ell}x_{d-p}^{-1}x_\ell)x_\ell^{-1}\rho + (c\sigma_{\ell-p+1,d-p}x_\ell^{-1} + c\sigma_{\ell-p+2,d-p}x_\ell^{-1} + \cdots + c\sigma_{\ell,d-p}x_{d-p}^{-1})\rho \\
&= (\mathsf{x}_{d-p})x_\ell^{-1}\rho + (c\sigma_{\ell-p+1,d-p}x_\ell^{-1} + c\sigma_{\ell-p+2,d-p}x_\ell^{-1} + \cdots + c\sigma_{\ell-1,d-p}x_\ell^{-1})\rho \\
&= (\mathsf{x}_{d-p} + (c\sigma_{\ell-p+1,d-p} + c\sigma_{\ell-p+2,d-p} + \cdots + c\sigma_{\ell-1,d-p}))x_\ell^{-1}\rho
\end{aligned}$$

Similarly,

$$x_{\ell-1}^{-1}x_\ell^{-1}\rho\mathsf{x}_d = (\mathsf{x}_{d-p} + c(\sigma_{\ell-p+1,d-p} + \sigma_{\ell-p+2,d-p} + \cdots + \sigma_{\ell-2,d-p}))x_{\ell-1}^{-1}x_\ell^{-1}\rho$$

Eventually, we get

$$x_{\ell-p+1}^{-1}x_{\ell-p+2}^{-1}\cdots x_\ell^{-1}\rho\mathsf{x}_d = \mathsf{x}_{d-p}x_{\ell-p+1}^{-1}x_{\ell-p+2}^{-1}\cdots x_\ell^{-1}\rho$$

which is (6.21) as desired.

Finally, we consider the relations (6.2.1); again checking for $D_{1,1}$ is sufficient. Set $\underline{\mathbf{u}} = \psi^2(\underline{\mathbf{v}})$. Again, denote by $\underline{\mathbf{w}}$ the tensor obtained from $\underline{\mathbf{u}}$ by rearranging the factors into non-descending order. Let p_1 be the numbers of indices in $\underline{\mathbf{v}}$ equal to $m+n$ and let p_2 be the number of those equal to $m+n-1$. Set $p = p_1 + p_2$. Then, $\underline{\mathbf{u}} = \rho(\underline{\mathbf{w}})$ where $\rho = \overline{\sigma}_{\ell-p+1,1}\cdots\overline{\sigma}_{\ell-1,p-1}\overline{\sigma}_{\ell,p}$.

$$\begin{aligned}
D_{1,1}(\psi^2(\mathbf{m}\otimes\underline{\mathbf{v}})) &= D_{1,1}(\mathbf{m}x_{\ell-p+1}^{-1}x_{\ell-p+2}^{-1}\cdots x_\ell^{-1}\otimes\underline{\mathbf{u}}) \\
&= D_{1,1}(\mathbf{m}x_{\ell-p+1}^{-1}x_{\ell-p+2}^{-1}\cdots x_\ell^{-1}\otimes\rho(\underline{\mathbf{w}})) \\
&= D_{1,1}(\mathbf{m}x_{\ell-p+1}^{-1}x_{\ell-p+2}^{-1}\cdots x_\ell^{-1}\rho\otimes\underline{\mathbf{w}}) \\
&= \mathbf{m}x_{\ell-p+1}^{-1}x_{\ell-p+2}^{-1}\cdots x_\ell^{-1}\rho\left(\sum_{d=1}^{p_2}(\mathsf{x}_d + \frac{\lambda}{2}(m-1))\right)\otimes\underline{\mathbf{w}}
\end{aligned} \tag{6.24}$$

$$\begin{aligned}\psi^2(D_{m+n-1,1}(\mathbf{m}\otimes\underline{\mathbf{v}})) &= \psi^2\left(\mathbf{m}\left(\sum_{d=j_{m+n-1}}^{\tilde{j}_{m+n-1}}(\mathsf{x}_d-\frac{\lambda}{2}(n-1))\right)\otimes\underline{\mathbf{v}}\right)\\ &= \mathbf{m}\left(\sum_{d=j_{m+n-1}}^{\tilde{j}_{m+n-1}}(\mathsf{x}_d-\frac{\lambda}{2}(n-1))x_{\ell-p+1}^{-1}x_{\ell-p+2}^{-1}\cdots x_\ell^{-1}\right)\otimes\rho\underline{\mathbf{w}}\\ &= \mathbf{m}\left(\sum_{d=j_{m+n-1}}^{\tilde{j}_{m+n-1}}(\mathsf{x}_d-\frac{\lambda}{2}(n-1))x_{\ell-p+1}^{-1}x_{\ell-p+2}^{-1}\cdots x_\ell^{-1}\rho\right)\otimes\underline{\mathbf{w}}\end{aligned}\tag{6.25}$$

Unlike the first case, we do not need (6.24) and (6.25) to be equal. Rather, their difference should come from τ_a.

We need to find $\overline{\sigma}_{\ell-p+d,d}\mathsf{x}_d$, assuming $1\le d\le p_2$. Using basic relations, we rewrite

$$\begin{aligned}\sigma_{d+1,d}\mathsf{x}_d &= \mathsf{x}_{d+1}\sigma_{d+1,d}+\kappa\\ &= (\mathsf{x}_{d+1}+\kappa\sigma_{d+1,d})\sigma_{d+1,d}\\ \sigma_{d+2,d+1}\sigma_{d+1,d}\mathsf{x}_d &= \mathsf{x}_{d+1}\sigma_{d+1,d}+\kappa\\ &= (\mathsf{x}_{d+2}+\kappa\sigma_{d+2,d+1}+\kappa\sigma_{d+2,d})\sigma_{d+2,d+1}\sigma_{d+1,d}\end{aligned}$$

In general,

$$\overline{\sigma}_{\ell-p+d,d}\mathsf{x}_d = (\mathsf{x}_{\ell-p+d}+\kappa\sum_{j=d}^{\ell-p+d-1}\sigma_{\ell-p+d,j})\overline{\sigma}_{\ell-p+d,d}$$

$$\overline{\sigma}_{\ell-p+d-1,d-1}(\mathsf{x}_{\ell-p+d}+\kappa\sum_{j=d}^{\ell-p+d-1}\sigma_{\ell-p+d,j}) = (\mathsf{x}_{\ell-p+d}+\kappa\sum_{j=d-1}^{\ell-p+d-2}\sigma_{\ell-p+d,j})\overline{\sigma}_{\ell-p+d-1,d-1}$$

By downwards induction,

$$\overline{\sigma}_{\ell-p+1,1}\cdots\overline{\sigma}_{\ell-p+d-1,d-1}(\mathsf{x}_{\ell-p+d}+\kappa\sum_{j=d}^{\ell-p+d-1}\sigma_{\ell-p+d,j})$$

$$=(\mathsf{x}_{\ell-p+d}+\kappa\sum_{j=1}^{\ell-p}\sigma_{\ell-p+d,j})\overline{\sigma}_{\ell-p+1,1}\cdots\overline{\sigma}_{\ell-p+d-1,d-1}$$

As a result,

$$\rho\mathsf{x}_d=(\mathsf{x}_{\ell-p+d}+\kappa\sum_{j=1}^{\ell-p}\sigma_{\ell-p+d,j})\rho. \tag{6.26}$$

Next, we determine $x_{\ell-p+1}^{-1}x_{\ell-p+2}^{-1}\cdots x_{\ell}^{-1}\mathsf{x}_{\ell-p+d}$. When $f<p-d$,

$$\begin{aligned}x_{\ell-f}^{-1}\mathsf{x}_{\ell-p+d}&=\mathsf{x}_{\ell-p+d}x_{\ell-f}^{-1}-c\sigma_{\ell-f,\ell-p+d}x_{\ell-p+d}^{-1}\\&=(\mathsf{x}_{\ell-p+d}-cx_{\ell-p+d}\sigma_{\ell-f,\ell-p+d}x_{\ell-p+d}^{-1})x_{\ell-f}^{-1}.\end{aligned} \tag{6.27}$$

We also need

$$\begin{aligned}x_{\ell-p+d}^{-1}\mathsf{x}_{\ell-p+d}&=x_{\ell-p+d}^{-1}\left(\frac{t}{2}+x_{\ell-p+d}y_{\ell-p+d}+c\sum_{j<\ell-p+d}\sigma_{j,\ell-p+d}\right)\\&=\frac{t}{2}x_{\ell-p+d}^{-1}+y_{\ell-p+d}+c\sum_{j<\ell-p+d}\sigma_{j,\ell-p+d}x_j^{-1}\\&=\left(\frac{t}{2}+y_{\ell-p+d}x_{\ell-p+d}\right)x_{\ell-p+d}^{-1}+c\sum_{j<\ell-p+d}\sigma_{j,\ell-p+d}x_j^{-1}\\&=\left(\frac{t}{2}+t+x_{\ell-p+d}y_{\ell-p+d}+c\sum_{f\neq\ell-p+d}\sigma_{f,\ell-p+d}\right)x_{\ell-p+d}^{-1}\\&+c\sum_{j<\ell-p+d}\sigma_{j,\ell-p+d}x_j^{-1}\\&=\left(\mathsf{x}_{\ell-p+d}+t+c\sum_{f>\ell-p+d}\sigma_{f,\ell-p+d}\right)x_{\ell-p+d}^{-1}+c\sum_{j<\ell-p+d}\sigma_{j,\ell-p+d}x_j^{-1}\end{aligned} \tag{6.28}$$

As well, when $f<k$,

$$x_f^{-1}\mathsf{x}_kx_f=\frac{t}{2}+x_kx_f^{-1}x_kx_f+c\sum_{j<k,j\neq f}\sigma_{jk}+cx_f^{-1}\sigma_{fk}x_f$$

$$
\begin{aligned}
&= \frac{t}{2} + x_k(x_k - c\sigma_{jk}x_k^{-1}) + c \sum_{j<k, j\neq f} \sigma_{jk} + c\sigma_{fk}x_k^{-1}x_f \\
&= \frac{t}{2} + x_k x_k + c \sum_{j<k, j\neq f} \sigma_{jk} \\
&= \mathsf{x}_k - c\sigma_{jk} \qquad (6.29)
\end{aligned}
$$

This implies that, when $p - d < f < \ell$,

$$x_{\ell-f}^{-1}\mathsf{x}_{\ell-p+d} = \mathsf{x}_{\ell-p+d}x_{\ell-f}^{-1} - c\sigma_{\ell-f,\ell-p+d}x_{\ell-f}^{-1}$$

Using (6.27), we obtain

$$
\begin{aligned}
&x_{\ell-p+d+1}^{-1}x_{\ell-p+2}^{-1}\cdots x_\ell^{-1}\mathsf{x}_{\ell-p+d} \\
&\quad =(\mathsf{x}_{\ell-p+d} - c\sum_{f=0}^{p-d-1} x_{\ell-p+d}\sigma_{\ell-f,\ell-p+d}x_{\ell-p+d}^{-1})x_{\ell-p+d+1}^{-1}X_{\ell-p+2}^{-1}\cdots x_\ell^{-1}
\end{aligned}
$$

As a result, using (6.28), we have

$$
\begin{aligned}
&x_{\ell-p+d}^{-1}x_{\ell-p+d+1}^{-1}x_{\ell-p+2}^{-1}\cdots x_\ell^{-1}\mathsf{x}_{\ell-p+d} \\
&\quad =x_{\ell-p+d}^{-1}\mathsf{x}_{\ell-p+d}x_{\ell-p+d+1}^{-1}x_{\ell-p+2}^{-1}\cdots x_\ell^{-1} \\
&\qquad - c\sum_{f=0}^{p-d-1} \sigma_{\ell-f,\ell-p+d}x_{\ell-p+d}^{-1}x_{\ell-p+d+1}^{-1}x_{\ell-p+2}^{-1}\cdots x_\ell^{-1} \\
&\quad =(\mathsf{x}_{\ell-p+d} + t)x_{\ell-p+d}^{-1}x_{\ell-p+d+1}^{-1}x_{\ell-p+2}^{-1}\cdots x_\ell^{-1} \\
&\qquad + c\sum_{f=0}^{p-d-1} \sigma_{\ell-f,\ell-p+d}x_{\ell-p+d}^{-1}x_{\ell-p+d+1}^{-1}X_{\ell-p+2}^{-1}\cdots x_\ell^{-1} \\
&\qquad + c\sum_{j=1}^{\ell-p+d-1} \sigma_{j,\ell-p+d}x_j^{-1}x_{\ell-p+d+1}^{-1}x_{\ell-p+2}^{-1}\cdots x_\ell^{-1} \\
&\qquad - c\sum_{f=0}^{p-d-1} \sigma_{\ell-f,\ell-p+d}x_{\ell-p+d}^{-1}x_{\ell-p+d+1}^{-1}x_{\ell-p+2}^{-1}\cdots x_\ell^{-1} \\
&\quad =(\mathsf{x}_{\ell-p+d} + t)x_{\ell-p+d}^{-1}x_{\ell-p+d+1}^{-1}x_{\ell-p+2}^{-1}\cdots x_\ell^{-1}
\end{aligned}
$$

$$+ c \sum_{j=1}^{\ell-p+d-1} \sigma_{j,\ell-p+d} x_j^{-1} x_{\ell-p+d+1}^{-1} x_{\ell-p+2}^{-1} \cdots x_\ell^{-1}$$

We can apply (6.29) to conclude that:

$$\begin{aligned}
&x_{\ell-p+1}^{-1} x_{\ell-p+2}^{-1} \cdots x_\ell^{-1} \mathsf{x}_{\ell-p+d} \\
&\quad = x_{\ell-p+1}^{-1} x_{\ell-p+2}^{-1} \cdots x_{\ell-p+d-1}^{-1} \mathsf{x}_{\ell-p+d} x_{\ell-p+d}^{-1} x_{\ell-p+d+1}^{-1} x_{\ell-p+2}^{-1} \cdots x_\ell^{-1} \\
&\qquad + c \sum_{j=1}^{\ell-p+d-1} x_{\ell-p+1}^{-1} x_{\ell-p+2}^{-1} \cdots x_{\ell-p+d-1}^{-1} x_{\ell-p+d}^{-1} \sigma_{j,\ell-p+d} x_{\ell-p+d+1}^{-1} x_{\ell-p+2}^{-1} \cdots x_\ell^{-1} \\
&\quad = \left(\mathsf{x}_{\ell-p+1} + t - c \sum_{f=p-1}^{p-d+1} \sigma_{\ell-f,\ell-p+d} \right) x_{\ell-p+1}^{-1} x_{\ell-p+2}^{-1} \cdots \mathsf{x}_{\ell-p+d-1}^{-1} x_{\ell-p+1}^{-1} x_{\ell-p+2}^{-1} \cdots x_{\ell-p+d-1}^{-1} \\
&\qquad + c \sum_{j=1}^{\ell-p+d-1} x_{\ell-p+1}^{-1} x_{\ell-p+2}^{-1} \cdots x_{\ell-p+d-1}^{-1} x_{\ell-p+d}^{-1} \sigma_{j,\ell-p+d} x_{\ell-p+d+1}^{-1} x_{\ell-p+2}^{-1} \cdots x_\ell^{-1} \\
&\quad = (\mathsf{x}_{\ell-p+1} + t) x_{\ell-p+1}^{-1} x_{\ell-p+2}^{-1} \cdots x_{\ell-p+d-1}^{-1} x_{\ell-p+1}^{-1} x_{\ell-p+2}^{-1} \cdots x_{\ell-p+d-1}^{-1} \\
&\qquad + c \sum_{j=1}^{\ell-p} x_{\ell-p+1}^{-1} x_{\ell-p+2}^{-1} \cdots x_{\ell-p+d-1}^{-1} x_{\ell-p+d}^{-1} \sigma_{j,\ell-p+d} x_{\ell-p+d+1}^{-1} x_{\ell-p+2}^{-1} \cdots x_\ell^{-1}
\end{aligned}$$

Returning to (6.26) and multiply both sides by the product $x_{\ell-p+1}^{-1} x_{\ell-p+2}^{-1} \cdots x_\ell^{-1}$. We can then apply the previous calculation to produce:

$$\begin{aligned}
&x_{\ell-p+1}^{-1} x_{\ell-p+2}^{-1} \cdots x_\ell^{-1} \rho \mathsf{x}_d = x_{\ell-p+1}^{-1} x_{\ell-p+2}^{-1} \cdots x_\ell^{-1} \mathsf{x}_{\ell-p+d} \rho \\
&\qquad + \kappa \sum_{j=1}^{\ell-p} x_{\ell-p+1}^{-1} x_{\ell-p+2}^{-1} \cdots x_\ell^{-1} \sigma_{\ell-p+d} \rho \\
&\quad = (\mathsf{x}_{\ell-p+1} + t) x_{\ell-p+1}^{-1} x_{\ell-p+2}^{-1} \cdots x_{\ell-p+d-1}^{-1} x_{\ell-p+1}^{-1} x_{\ell-p+2}^{-1} \cdots x_{\ell-p+d-1}^{-1} \rho \\
&\qquad + c \sum_{j=1}^{\ell-p} x_{\ell-p+1}^{-1} x_{\ell-p+2}^{-1} \cdots x_{\ell-p+d-1}^{-1} x_{\ell-p+d}^{-1} \sigma_{j,\ell-p+d} x_{\ell-p+d+1}^{-1} x_{\ell-p+2}^{-1} \cdots x_\ell^{-1} \rho \\
&\qquad + \kappa \sum_{j=1}^{\ell-p} x_{\ell-p+1}^{-1} x_{\ell-p+2}^{-1} \cdots x_\ell^{-1} \sigma_{\ell-p+d} \rho
\end{aligned}$$

Since $\kappa = -c$, the last two lines cancel out. This gives

$$x_{\ell-p+1}^{-1} x_{\ell-p+2}^{-1} \cdots x_{\ell}^{-1} \rho \mathsf{x}_d = \kappa \sum_{j=1}^{\ell-p-1} x_{\ell-p+1}^{-1} x_{\ell-p+2}^{-1} \cdots x_{\ell}^{-1} \sigma_{\ell-p+d} \rho$$

Note that

$$\tau_a^2(D_{1,1}) = D_{m+n-1,1} - (2a+\beta) D_{m+n-1,0}$$

and, as a result, the last equation in Lemma 6.2.1 follows

$$D_{1,1}(\psi^2(\mathbf{m} \otimes \underline{\mathbf{v}})) = \psi^2((D_{m+n-1,1} - (2a+\beta) D_{m+n-1,0}(\mathbf{m} \otimes \underline{\mathbf{v}})) \qquad (6.30)$$

As we've shown from (6.24), the left-hand side of (6.30) equals

$$\mathbf{m} \sum_{d=1}^{p_2} (x_{\ell-p+d} + t) x_{\ell-p+1}^{-1} x_{\ell-p+2}^{-1} \cdots x_{\ell}^{-1} \rho \otimes \underline{\mathbf{w}} + \mathbf{m} \sum_{d=1}^{p_2} \frac{\lambda}{2}(m-1) x_{\ell-p+1}^{-1} x_{\ell-p+2}^{-1} \cdots x_{\ell}^{-1} \rho \otimes \underline{\mathbf{w}}$$

and from (6.25), the right-hand side equals

$$\mathbf{m} \left(\sum_{d=j_{m+n-1}}^{\tilde{j}_{m+n-1}} (\mathsf{x}_d - \frac{\lambda}{2}(n-1) - 2a - \beta) x_{\ell-p+1}^{-1} x_{\ell-p+2}^{-1} \cdots x_{\ell}^{-1} \rho \right) \otimes \underline{\mathbf{w}}$$

If $j_{m+n-1} \leq d \leq \tilde{j}_{m+n-1}$, then $d = \ell - p + e$ with $1 \leq e \leq p_2$. As a result, (6.30) holds when

$$t + \frac{\lambda}{2}(m-1) = -\frac{\lambda}{2}(n-1) - 2a - \beta$$

Since $a = -\frac{\lambda}{2}$, this is equivalent to

$$t + \frac{\lambda}{2} m = -\frac{\lambda}{2} n + 2\lambda - \beta \qquad (6.31)$$

and thus we need $\beta = -t + \frac{\lambda}{2}(m+n+4)$. □

For this lemma to hold for $0 \leq k \leq m+n-1$, it will have to hold for $k = 1$

in particular:

$$X_{1,1}^{\pm}(\psi(\mathbf{m}\otimes\underline{\mathbf{v}})) = \psi(\tau_a(X_{1,1}^{\pm})(\mathbf{m}\otimes\underline{\mathbf{v}})),$$
$$H_{1,1}(\psi(\mathbf{m}\otimes\underline{\mathbf{v}})) = \psi(\tau_a(H_{1,1})(\mathbf{m}\otimes\underline{\mathbf{v}}))$$

Then, since $\tau_a(X_{1,1}^{\pm}) = X_{0,1}^{\pm} - aX_{0,0}^{\pm}$ and $\tau_a(H_{1,1}) = H_{0,1} - aH_{0,0}$, we set

$$\psi^{-1}(X_{1,1}^{\pm}(\psi(\mathbf{m}\otimes\underline{\mathbf{v}}))) + aX_{0,0}^{\pm}(\mathbf{m}\otimes\underline{\mathbf{v}}) = X_{0,1}^{\pm}(\mathbf{m}\otimes\underline{\mathbf{v}})$$
$$\psi^{-1}(H_{1,1}(\psi(\mathbf{m}\otimes\underline{\mathbf{v}}))) + aH_{0,0}(\mathbf{m}\otimes\underline{\mathbf{v}}) = H_{0,1}(\mathbf{m}\otimes\underline{\mathbf{v}}).$$

With this definition of $X_{0,1}^{\pm}(\mathbf{m}\otimes\underline{\mathbf{v}})$ and $H_{0,1}(\mathbf{m}\otimes\underline{\mathbf{v}})$, the first part of Lemma 6.2.1 holds for $k=1$. We verify that it is also true for $k=0$.

$$\begin{aligned} X_{0,1}^{\pm}(\psi(\mathbf{m}\otimes\underline{\mathbf{v}})) &= \psi^{-1}(X_{1,1}^{\pm}(\psi^2(\mathbf{m}\otimes\underline{\mathbf{v}})) + aX_{0,0}^{\pm}(\psi(\mathbf{m}\otimes\underline{\mathbf{v}})) \\ &= \psi^{-1}(\psi^2(\tau_a^2(X_{1,1}^{\pm})(\mathbf{m}\otimes\underline{\mathbf{v}}))) + aX_{0,0}^{\pm}(\psi(\mathbf{m}\otimes\underline{\mathbf{v}})) \\ &= \psi(\tau_a(X_{0,1}^{\pm} - aX_{0,0}^{\pm})(\mathbf{m}\otimes\underline{\mathbf{v}})) + aX_{0,0}^{\pm}(\psi(\mathbf{m}\otimes\underline{\mathbf{v}})) \\ &= \psi(\tau_a(X_{0,1}^{\pm})(\mathbf{m}\otimes\underline{\mathbf{v}})) \end{aligned}$$

Thanks to Lemma 6.2.1, we now have a guide for extending the action of the super Yangian of $\mathfrak{sl}_{m|n}$ on $M \otimes_{\mathbb{C}[S_l]} (\mathbb{C}(m|n))^{\otimes \ell}$ to the affine super Yangian. Next, we need to verify that these choices of actions of $X_{0,r}^{\pm}$ and $H_{0,r}$ on that space satisfy the defining relations of the affine super Yangian when $r = 0, 1$. For a relation involving $X_{0,r}^{\pm}$ and $X_{m+n-1,r}$, it holds for the corresponding operators on $M \otimes_{\mathbb{C}[S_l]} (\mathbb{C}(m|n))^{\otimes \ell}$ if and only if it holds for $\tau_a(X_{0,1}^{\pm})$ and $\tau_a(X_{m+n-1,r}^{\pm})$. This is because ψ is an isomorphism and $X_{k,1}^{\pm}(\mathbf{m}\otimes\underline{\mathbf{v}}) = \psi(\tau_a(X_{k,1}^{\pm})(\psi^{-1}(\mathbf{m}\otimes\underline{\mathbf{v}}))$ when $k=0$ or $m+n-1$. $\tau_a(X_{0,1}^{\pm})$ and $\tau_a(X_{m+n-1,r}^{\pm})$ do not depend on $X_{0,0}^{\pm}$ or $X_{0,1}^{\pm}$, so those relations are known to hold from the case of the Yangian of $\mathfrak{sl}_{m|n}$. The same argument applies to the relations involving $X_{0,r}^{\pm}$ and $X_{1,r}^{\pm}$ and to the relations involving also $H_{0,1}$ and $H_{0,0}$.

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