Expected Credit Loss Impairment: Early Recognition vs. Income Volatility

by

Logan Ewanchuk

A thesis submitted in partial fulfillment of the requirements for the degree of

Master of Science

in

Statistics

Department of Mathematical and Statistical Sciences
University of Alberta

© Logan Ewanchuk, 2019

Abstract

Recently, a new accounting standard was established, namely the International Financial Reporting Standard 9, requiring banks to build provisions using forward-looking expected loss models. When there is a significant increase in credit risk of a loan, additional provisions must be charged to the income statement. Banks need to set a threshold for each loan, defining what such a significant increase in credit risk constitutes. A low threshold allows banks to recognize credit risk early, but leads to greater income volatility. By introducing a statistical framework, this trade-off between early recognition of credit risk and avoidance of excessive income volatility is modelled. We analyze the resulting optimization problem for various models in both continuous-time and discrete-time settings, relate it to the banking stress test of the European Union, and illustrate it using historical default data by Standard and Poor's.

Preface

Chapter 4 of this thesis is based on Ewanchuk and Frei (2019), published in the journal *Risks* on April 14, 2019, and written jointly by Logan Ewanchuk and Christoph Frei. The two authors contributed equally to Ewanchuk and Frei (2019).

Acknowledgements

To start, I would like to express the appreciation I have for my academic supervisor, Dr. Christoph Frei. His expertise and encouraging nature were invaluable in contributing to the successful writing of this thesis. Additionally, his industrial knowledge further enhanced the overall experience, as the career advice given will benefit me to a great extent, on top of the academic guidance.

I would like to thank Dr. Richard Hou of Canadian Western Bank, who enabled a research collaboration and continues to provide quality supervision during my internship with the bank. Financial support for this research collaboration by the Natural Sciences and Engineering Research Council of Canada under grants EGP 531183-18 and EGP2 536672-18 is gratefully acknowledged.

Furthermore, thank you to Dr. Hou, Dr. Adam Kashlak, and Dr. Michael Maier for serving on my thesis examination committee.

Lastly, I appreciate the support of friends and family, including my sister Laurel, during the busy two academic years of my master's studies.

Contents

List of Tables

List of Figures

List of Abbreviations

EaD	exposure at default
ECL	expected credit loss
EU	European Union
IFRS	International Financial Reporting Standard
LGD	loss given default
PD	probability of default

Chapter 1

Introduction

Credit risk analysis is an integral part of research for financial institutions, as borrowers can default on their debts in many situations. A consequence of credit risk, for instance, is that banks incur a loss if an obligor is unable to pay off a loan at its maturity date. As a result, banks need to have sufficient capital to cover losses that may occur in adverse scenarios from all relevant types of risk. To explain further, capital requirements described by the Basel Committee on Banking Supervision (2005) involve the equity a financial institution must hold, and are directly dependent on risk-weighted assets such as loans. Estimation of credit risk is also important to determine an appropriate interest rate for each loan, as higher interest rates associated with riskier loans acknowledge an increased risk of non-payment. Nonetheless, if the riskiness of a loan increases after it is issued, further provisions need to be built to account for the possibility of greater losses ensuing.

The global financial crisis of 2007–2008 highlighted the delayed recognition of credit losses as a weakness of the accounting standards at that time. As a result, the International Accounting Standards Board (2014) decided to introduce International Financial Reporting Standard (IFRS) 9, which is an accounting standard that requires banks to recognize increased credit risk of loans early and build additional provisions for such loans. Because loss provisions affect banks' income statements, it is critical to have accurate estimations of these values for each reporting period. However, the building of provisions and their subsequent release for reduced credit risk leads to undesired volatility in the income statements of banks. In this thesis, we introduce and analyze a framework to model this trade-off between early recognition of increased credit risk and avoidance of excessive income volatility.

IFRS 9, being mandatory since January 1, 2018, requires banks to consider forward-looking expected loss impairment models. Randall and Thompson (2017) outlined that forward-looking information regarding credit risk includes qualitative information, such as the current economic environment, along with both statistical models and non-statistical quantitative information. On the contrary, the previously used backward-looking approach required a trigger event to occur prior to any credit losses being reported. Regardless of the specific information considered, there is still a subjective aspect involved in a financial institution's assessment of credit risk for loans, but it is crucial for each bank to have a sound underlying modelling framework.

Under IFRS 9, banks are required to estimate the expected credit loss (ECL) for each loan and build corresponding provisions. Compared to charging only highly likely or even realized credit losses, this procedure allows for early recognition of credit risk, well before an actual default occurs. The new standards require various ECL measures for loans classified into three different buckets of progressively higher default risk, compared to the initial default risk when the loan is issued; see Cohen and Edwards (2017); Maggi et al. (2017). To avoid misunderstandings, these IFRS 9 buckets are not to be confused with the typical rating buckets in which banks group obligors with similar credit ratings, within their diversified loan portfolio. Table 1.1 provides a summary of the structure of the IFRS 9 buckets. ECL is estimated over a one-year period for a loan in bucket 1, when the obligor generally possesses "good credit" relative to their initial credit quality, so it is considered to be a performing loan in this case.

IFRS 9	**loan type**	**ECL**
bucket 1	performing	one-year
bucket 2	underperforming	lifetime
bucket 3	impaired	lifetime

Table 1.1: ECL calculation criteria for loans, relative to their IFRS 9 classification.

The reclassification of a loan takes place when deteriorating credit quality of an obligor is observed, based on predefined warning signs. These warning signs involve an increased probability of default, in accordance with the current forecast of the credit environment. Therefore, IFRS 9 bucket 2 represents underperforming loans that have experienced a significant increase in credit risk since initial recognition. Furthermore, impaired loans, which result in the bank actually incurring credit losses, are classified in bucket 3.

An important feature of this classification is that, for underperforming loans (bucket 2), lifetime ECL is estimated over the remaining term to maturity, rather than one-year ECL. Clearly, the ECL of a loan in bucket 2, especially in the early stages of the period of the loan, would typically be much greater than the ECL of a loan in bucket 1. It must be noted that for loans in bucket 3, ECL is estimated based on the bank's exposure along with the estimated recovery values. The subjective nature of this IFRS 9 classification is critical when analyzing, from the perspective of a financial institution, what a significant increase in credit risk actually is, as this is not specified in IFRS 9.

It is quite challenging to decide on the threshold for precisely defining a significant increase in credit risk, for numerous reasons. If an extremely conservative framework is implemented by a bank, its obligors would only need to experience minor indicators of credit downturn to warrant a reclassification from bucket 1 to bucket 2. In this instance, the bank's ECL calculations are prone to volatility, as loans could transfer from one-year ECL to lifetime ECL frequently. However, setting a threshold that is very lenient towards obligors' credit quality would indeed create more stable ECL calculations for the bank, but at the same time lead to late recognition of credit risk with respect to these loans. Finding a balance between early recognition of credit risk and income volatility, in this sense, is an interesting problem that is at the heart of this thesis and directly linked to the new loss impairment standards of IFRS 9. Indeed, a survey by the European Banking Authority (2017) found that "72% of the banks included in the survey anticipate that IFRS 9 impairment requirements will increase volatility in profit or loss. Respondents mentioned that this was mainly due to the 'cliff effect' when moving exposures from stage 1 to stage 2 (from 12-month ECL to lifetime ECL), and to the inclusion of forward-looking information, which will need to be reassessed at each reporting period, in the ECL estimation."

Our setting is based on the structural model of default by Merton (1974). It essentially states that an obligor can only meet its financial obligations if the value of their assets exceeds that of the liabilities at the time of maturity, for a single debt obligation. While the actual nature of an individual obligor's debt is considerably more complex, with default possible at several times, the preceding assumptions do provide us with a quality, widely used starting point for credit risk modelling. There have been extensions in terms of asset value modelling (see Bluhm et al. (2010); Bohn and Stein (2009); McNeil et al. (2015) for an overview), but the Merton model remains the "prototype" of many credit risk models, such as Bluhm and Overbeck (2003); Frei and Wunsch (2018); Gordy (2000). In particular, the Merton model is at the basis of the capital requirement described by the Basel Committee on Banking Supervision (2005), whose framework Miu and Ozdemir (2017) suggest to employ for IFRS 9 purposes. Moreover, the Basel Committee on Banking Supervision (2015) provides supervisory guidance on ECL accounting. Such models have also been applied for stress testing; see, for example, Miu and Ozdemir (2009); Simons and Rolwes (2009); Yang and Du (2015).

Since Merton (1974) models the underlying asset value of the obligor as a stochastic process over time, we can formulate an optimization problem that considers the impact on the income statement at different reporting moments. The optimization problem consists of two penalization terms: (1) a term to penalize for failing to early recognize an eventual default, and (2) a term that penalizes for income volatility. We weight the two terms by a tuning parameter, which determines the relative importance of low income volatility compared to early credit risk recognition. While IFRS 9 does not define exactly what a significant increase in credit risk is, and thus leaves flexibility in choosing our tuning parameter, we apply the stress test framework of the European Banking Authority (2018) to obtain a suitable estimation for this parameter. Overall, two problem formulations are considered, namely, a continuous-time framework that is consistent with typical quantitative finance modelling, as well as a discrete-time formulation relevant to the realistic periodic reporting dates for obligor specific information. Our problem formulation in a continuous-time framework leads to an optimization problem that we analyze numerically and extend to a situation with multiple obligors. In this discrete-time setting, we solve the optimization problem analytically for certain distributions of the asset value process. For the classical setting of Merton (1974), where the asset value process is modelled as a Brownian motion observed at discrete moments in time, we recast it as an optimization problem that we solve efficiently by numerical routines, and provide an illustration using default data from Standard and Poor's (2018).

The content of this thesis is organized as follows. Chapter 2 introduces a credit risk model that captures the trade-off between early recognition of credit risk and income volatility in a continuous-time setting, and compares the model with the stress test framework of the European Banking Authority (2018). Chapter 3 extends this analysis to consider many obligors grouped in a particular credit rating bucket, taking into account default correlations in the modelling structure. The optimization problem is analyzed in a discrete-time formulation in Chapter 4, where the latent asset return has either a continuous distribution or discrete distribution, in which the process is modelled by alternatives to Brownian motion. Ultimately, our model is applied to credit rating migration and default data from Standard and Poor's (2018). Chapter 5 concludes.

Chapter 2

Continuous-time formulation

2.1 Theoretical examination of the problem

When considering an obligor being issued a loan from a bank, we can model the trade-off between income volatility and the early recognition of credit risk. The net asset value of the obligor at time $t \in [0,T]$ can be represented by A_t for a stochastic process $(A_t)_{0 \leq t \leq T}$, akin to the Merton model's basic framework, and we assume that this stochastic process has stationary and independent increments.[1] In particular, we specify that $(A_t - A_0)_{0 \leq t \leq T}$ is a Brownian motion, and measure time t in years. Recall from Section I.1 in Revuz and Yor (1999) that Brownian motion is a stochastic process with continuous paths and stationary, independent, and normally distributed increments. The starting point $A_0 = k$ is the initial distance to default of an obligor, based on their initial default probability. We then let c denote the threshold in which the obligor's credit quality drops from bucket 1 to bucket 2 in

[1] A stochastic process $(A_t)_{0 \leq t \leq T}$ has independent increments if for any $0 \leq t_1 < t_2 < \cdots < t_N \leq T$, the increments $A_{t_2} - A_{t_1}, A_{t_3} - A_{t_2}, \ldots, A_{t_N} - A_{t_{N-1}}$, are independent. It has stationary increments if for any $0 \leq s < t \leq T$, the increment $A_t - A_s$ has the same distribution as A_{t-s}.

terms of the IFRS 9 classification, for some constants $c \in [0,k]$. Figure 2.1 provides an example of the net asset value of an obligor with a loan over $T = 10$ years. To

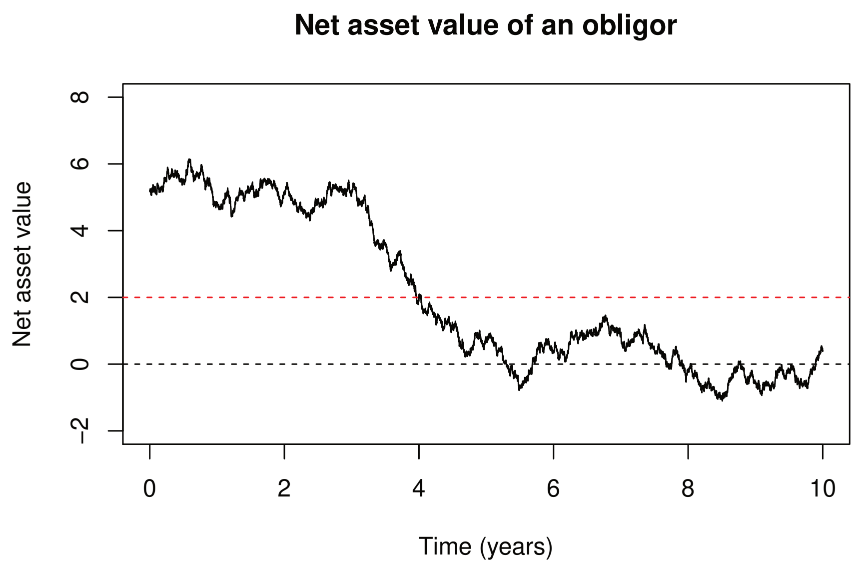

Figure 2.1: Simulation of the net asset value of an obligor for a ten-year loan. The chosen parameters are $k = 5.2$ and $c = 2$.

explain further, if $A_t > c$, the obligor's loan is in bucket 1, signifying "good credit" relative to their initial credit quality at the time the loan was issued. We define that a default at time T occurs if $A_T < 0$, and consequently, the initial distance to default of the obligor is equal to k. Furthermore, when $A_t \leq c$, the loan is in bucket 2, which means the bank's provisions for this obligor's particular loan are calculated based on lifetime ECL rather than one-year ECL. We assume that there needs to be enough assets to pay back the loan only at the maturity time T, so that we consider default events only at T, consistent with the modelling of default in the Merton model. Hence, in Figure 2.1, although the obligor's loan experiences a significant increase in credit risk at approximately $T = 4$, a default does not occur, since its asset value evidently exceeds that of its liabilities at maturity, $T = 10$.

Prior to formally defining the actual optimization problem, we examine the two portions of the prospective objective function, which will model the trade-off between income volatility and the early recognition of credit risk. The *recognition portion* is

$$\int_0^{T-1} (T-1-t)\, P(A_t > c \mid A_T < 0)\, dt\,, \tag{2.1}$$

while the *volatility portion* is

$$\int_0^{T-1} (T-1-t)\, \mathrm{Var}(\mathbb{1}_{A_t > c})\, dt. \tag{2.2}$$

As we subsequently explain in more detail, the recognition portion penalizes for failing to early recognize a default, while the volatility portion measures fluctuations resulting from reclassifications between buckets 1 and 2. We first examine the probability, $P(A_t > c \mid A_T < 0)$, which is part of (2.1), representing the conditional probability that the net asset value of the obligor at time t exceeds c, given a default occurs at time T. Hence, if this probability is low, the early recognition of credit risk is likely, since experiencing the warning sign of a loan being classified in bucket 2 before a default occurs is probable. Intuitively, this probability being low would correspond to a high threshold value c, near $A_0 = k$. In (2.2), the variance term $\mathrm{Var}(\mathbb{1}_{A_t > c})$ describes the stability, or lack thereof, of the process in terms of changing from being above the threshold c (when the indicator $\mathbb{1}_{A_t > c}$ is one) to below the threshold c (when the indicator $\mathbb{1}_{A_t > c}$ is zero), or vice versa. As we would expect, if c is high, the net asset value of an obligor is more prone to fluctuation between classification buckets 1 and 2. This would lead to frequent adjustments in the calculation of ECL, changing from one-year ECL to lifetime ECL, and vice versa, ultimately contributing to substantial earnings volatility from the perspective of the bank that issues the loan. We recall here that ECL equals the product of probability of default (PD), loss given default (LGD), and exposure at default (EaD). In practice, LGD and EaD are often assumed to be constant so that the PD, which we capture in this term, models the default risk. Together, the integrals in both (2.1) and (2.2) capture the net asset value throughout time, up until the final year of the loan. There is no impact on provisions when switching above and below the threshold c after $T-1$, as one-year ECL equals lifetime ECL during the final year of the loan, hence the upper limit of integration being $T-1$. The coefficient $(T-1-t)$ in both portions conveys that the early recognition of credit risk, along with the management of income volatility, is more critical earlier in the term of the loan. Thus, each portion has a negative, approximately linear relationship with time. First of all, the earlier the risk of non-payment is identified, the sooner loan loss provisions can be built proactively. Relative to income volatility, the earlier a significant increase in credit risk occurs, the larger the impact on a bank's income relative to ECL estimation, hence the factor $(T-1-t)$.

We model the inherent balance between early recognition of credit risk and income volatility through the optimization problem

$$\min_{c \leq k} \left(\int_0^{T-1} (T-1-t) \, P(A_t > c \mid A_T < 0) \, dt + \lambda \int_0^{T-1} (T-1-t) \, \mathrm{Var}(\mathbb{1}_{A_t > c}) \, dt \right), \tag{2.3}$$

where $\lambda > 0$ is a tuning parameter that, in practical terms, determines how much importance is placed on minimizing income volatility from the viewpoint of a bank. Moreover, each portion of (2.3) can be simplified, confirming mathematically that (2.1) and (2.2) are decreasing and increasing, respectively, in the threshold value c.

Proposition 1. *The optimization problem* (2.3) *can be written as* $\min_{c \le k} f(c)$*, where*

$$f(c) = \int_0^{T-1} (T-1-t) \left(\frac{\int_{\frac{c-k}{\sqrt{t}}}^{\infty} \Phi\left(\frac{-k-x\sqrt{t}}{\sqrt{T-t}}\right) \varphi(x)\,dx}{\Phi\left(\frac{-k}{\sqrt{T}}\right)} \right) dt$$
$$+ \lambda \int_0^{T-1} (T-1-t) \left(\Phi\left(\frac{c-k}{\sqrt{t}}\right) - \Phi^2\left(\frac{c-k}{\sqrt{t}}\right) \right) dt. \tag{2.4}$$

Proof. We first examine the conditional probability $P(A_t > c \mid A_T < 0)$, which is part of (2.1). This term from the recognition portion can be written as

$$\begin{aligned} P(A_t > c \mid A_T < 0) &= \frac{P(A_t > c, A_T < 0)}{P(A_T < 0)} \\ &= \frac{P(A_t > c, A_T - A_t < -A_t)}{P(A_T < 0)} \\ &= \frac{E[\mathbb{1}_{A_t > c}\, \mathbb{1}_{A_T - A_t < -A_t}]}{P(A_T < 0)} \\ &= \frac{E\left[\, E[\mathbb{1}_{A_t > c}\, \mathbb{1}_{A_T - A_t < -A_t} \mid A_t]\right]}{P(A_T < 0)} \\ &= \frac{E\left[\mathbb{1}_{A_t > c}\, E[\mathbb{1}_{A_T - A_t < -A_t} \mid A_t]\right]}{P(A_T < 0)}. \end{aligned} \tag{2.5}$$

Within the numerator in (2.5), we further simplify

$$\begin{aligned} E[\mathbb{1}_{A_T - A_t < -A_t} \mid A_t] &= E[\mathbb{1}_{A_T - A_t < -a}]\,|_{a = A_t} \\ &= P\left(\frac{A_T - A_t}{\sqrt{T-t}} < \frac{-a}{\sqrt{T-t}}\right)\Bigg|_{a = A_t} \\ &= \Phi\left(\frac{-a}{\sqrt{T-t}}\right)\Bigg|_{a = A_t} \\ &= \Phi\left(\frac{-A_t}{\sqrt{T-t}}\right), \end{aligned} \tag{2.6}$$

where Φ is the cumulative distribution function of a standard normally distributed random variable. In the first equality of (2.6), we used the well-known result that $E[f(X,Y)|Y] = E[f(X,y)|_{y=Y}$ for independent random variables X and Y, and a measurable function f with $E[|f(X,Y)|] < \infty$; see for example Theorem 6.4 of Kallenberg (2002). Subsequently, substituting our expression in (2.6) into (2.5), we have that

$$\begin{aligned}
\frac{E\left[\mathbb{1}_{A_t>c}\, E[\mathbb{1}_{A_T-A_t<-A_t} \mid A_t]\right]}{P(A_T<0)} &= \frac{E\left[\mathbb{1}_{A_t>c}\,\Phi\left(\frac{-A_t}{\sqrt{T-t}}\right)\right]}{\Phi\left(\frac{-k}{\sqrt{T}}\right)} \\
&= \frac{E\left[\mathbb{1}_{N>\frac{c-k}{\sqrt{t}}}\ \Phi\left(\frac{-k-N\sqrt{t}}{\sqrt{T-t}}\right)\right]}{\Phi\left(\frac{-k}{\sqrt{T}}\right)} \\
&= \frac{\int_{-\infty}^{\infty} \mathbb{1}_{x>\frac{c-k}{\sqrt{t}}}\ \Phi\left(\frac{-k-x\sqrt{t}}{\sqrt{T-t}}\right)\varphi(x)\,dx}{\Phi\left(\frac{-k}{\sqrt{T}}\right)} \\
&= \frac{\int_{\frac{c-k}{\sqrt{t}}}^{\infty} \Phi\left(\frac{-k-x\sqrt{t}}{\sqrt{T-t}}\right)\varphi(x)\,dx}{\Phi\left(\frac{-k}{\sqrt{T}}\right)}\,, \qquad (2.7)
\end{aligned}$$

where $N \sim \mathcal{N}(0,1)$, and φ is the probability density function of a standard normally distributed random variable. Note that since $A_0 = k$, A_t has the same distribution as $k+N\sqrt{t}$. This expression is certainly decreasing in c, due to the fact that this threshold constant is included only in the lower limit of integration in the numerator in (2.7). Similarly, we can analyze $\mathrm{Var}(\mathbb{1}_{A_t>c})$ in (2.2), deriving

$$\begin{aligned}
\mathrm{Var}(\mathbb{1}_{A_t>c}) &= E[\mathbb{1}_{A_t>c}] - (E[\mathbb{1}_{A_t>c}])^2 \\
&= P(A_t>c) - (P(A_t>c))^2 \\
&= (1-P(A_t<c)) - (1-P(A_t<c))^2,
\end{aligned}$$

which we further simplify to

$$
\begin{aligned}
\operatorname{Var}(\mathbb{1}_{A_t>c}) &= (1-P(A_t<c)) - (1-2P(A_t<c)+(P(A_t<c))^2) \\
&= P(A_t<c) - (P(A_t<c))^2 \\
&= P\left(\frac{A_t-k}{\sqrt{t}} < \frac{c-k}{\sqrt{t}}\right) - \left(P\left(\frac{A_t-k}{\sqrt{t}} < \frac{c-k}{\sqrt{t}}\right)\right)^2 \\
&= \Phi\left(\frac{c-k}{\sqrt{t}}\right) - \Phi^2\left(\frac{c-k}{\sqrt{t}}\right),
\end{aligned}
\tag{2.8}
$$

which is a function of the form $f(x) = x - x^2$, increasing for $x < \frac{1}{2}$. Hence, if $\Phi\left(\frac{c}{\sqrt{t}}\right) < \frac{1}{2}$, which is satisfied since $c < k$ and $t > 0$, we make the conclusion from (2.8) that $\operatorname{Var}(\mathbb{1}_{A_t>c})$ is indeed increasing in c. As we would expect, if c is high, or near $A_0 = k$, the obligor's loan would be more likely to alternate between one-year and lifetime ECL calculations repeatedly.

Hence, using the expressions in (2.7) and (2.8), we reach the optimization problem representation in (2.4). $\square$

As preparation for the numerical analysis in the next section, we determine the derivatives of the objective function (2.4). Differentiating twice, using the Leibniz integral rule, we find that

$$
\begin{aligned}
f'(c) = \int_0^{T-1} &-\frac{T-1-t}{\sqrt{t}\,\Phi\left(\frac{-k}{\sqrt{T}}\right)} \left(\Phi\left(\frac{-c}{\sqrt{T-t}}\right)\varphi\left(\frac{c-k}{\sqrt{t}}\right)\right) dt \\
&+ \lambda \int_0^{T-1} \frac{T-1-t}{\sqrt{t}} \left(\varphi\left(\frac{c-k}{\sqrt{t}}\right) - 2\Phi\left(\frac{c-k}{\sqrt{t}}\right)\varphi\left(\frac{c-k}{\sqrt{t}}\right)\right) dt
\end{aligned}
$$

and

$$
\begin{aligned}
f''(c) = \int_0^{T-1} & \frac{T-1-t}{\sqrt{t}\,\Phi\left(\frac{-k}{\sqrt{T}}\right)} \varphi\left(\frac{c-k}{\sqrt{t}}\right) \\
& \times \left(\frac{1}{\sqrt{T-t}} \varphi\left(\frac{-c}{\sqrt{T-t}}\right) + \frac{c-k}{t} \Phi\left(\frac{-c}{\sqrt{T-t}}\right) \right) dt \\
& + \lambda \int_0^{T-1} - \frac{(T-1-t)}{\sqrt{t}} \left(\frac{c-k}{t} \varphi\left(\frac{c-k}{\sqrt{t}}\right) + 2 \frac{\varphi^2\left(\frac{c-k}{\sqrt{t}}\right)}{\sqrt{t}} \right. \\
& \qquad \left. - 2 \frac{c-k}{t} \Phi\left(\frac{c-k}{\sqrt{t}}\right) \varphi\left(\frac{c-k}{\sqrt{t}}\right) \right) dt.
\end{aligned}
$$

The objective function and its first two derivatives, along with the initial assumptions of this section, provide sufficient information to proceed with obtaining a solution for our optimal threshold, the desired minimizer.

2.2 Results and illustration of solutions

Due to the complexity of the preceding expressions in Section 2.1, numerical integration using the programming language R is performed, fixing $T = 10$ (loan issued over ten years) and k such that $P(A_T < 0) = 0.05$, representing a default probability of 5%. Therefore, $P\left(\frac{A_T - A_0}{\sqrt{T}} < \frac{-k}{\sqrt{T}}\right) = 0.05$ and $k = -\sqrt{T}\,\Phi^{-1}(0.05) \approx 5.2$. Letting c^* be the solution to the minimization problem, given some λ, we need to satisfy $f'(c^*) = 0$ and $f''(c^*) > 0$, so we obtain a minimum in the interval $k \geq c^* \geq 0$.

The portions of $f(c)$, with the parameter values used in this example, are shown in Figure 2.2. Evidently, the recognition and volatility portions do not necessarily contribute equally, so we examine how the choice of λ affects the solution c^*. From an intuitive standpoint, as λ increases, we place more significance on managing in-

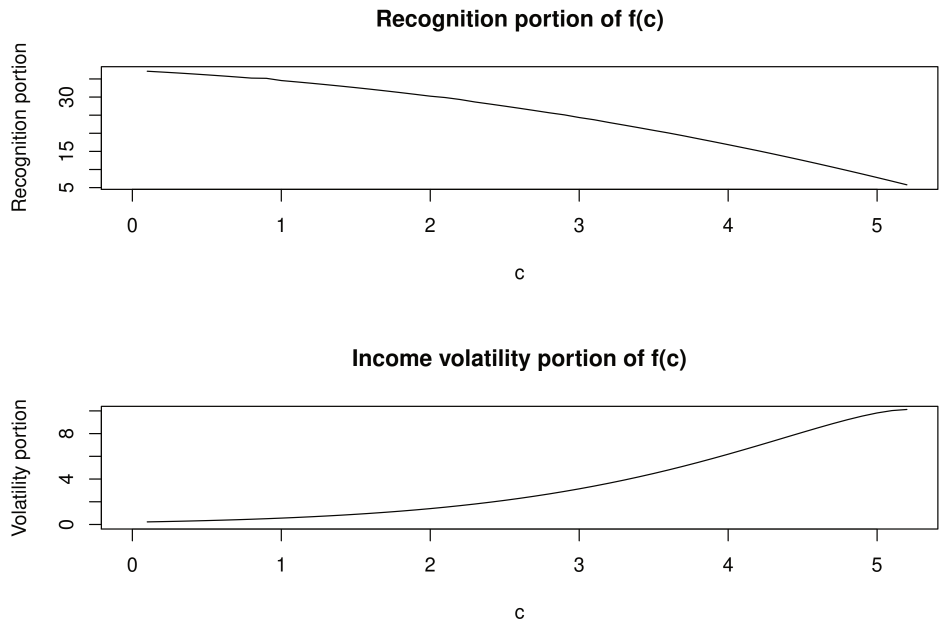

Figure 2.2: Recognition and volatility portions of $f(c)$, with $T = 10$, $k \approx 5.2$ and $\lambda = 1$.

come volatility, therefore recognizing credit risk early becomes less prominent, and the corresponding c^* decreases. Clearly, in the extreme case where λ is sufficiently large such that $c^* \approx 0$, the net assets of the obligor are unlikely to fluctuate frequently between IFRS 9 buckets 1 and 2. This situation would be ideal from the perspective of minimizing income volatility, yet disastrous in terms of recognizing credit risk early. Solutions, for various λ values, are seen in Figure 2.3. We observe that for many λ values, both a maximum and minimum exist in the appropriate interval, $k \geq c^* \geq 0$. Obviously, the solution of interest is the minimum, and the red line approximately indicates, for various λ values, where $f''(c^*) = 0$. It follows that if we select λ "reasonably," we will have a solution to the minimization problem such that $f(c^*)$ is convex, thus $f'(c^*) = 0$ and $f''(c^*) > 0$, as required. Additionally, it must be explained that for very small λ values, a solution is not visible in Figure 2.3. In those cases, $f(c)$ is strictly decreasing in this interval, resulting in a

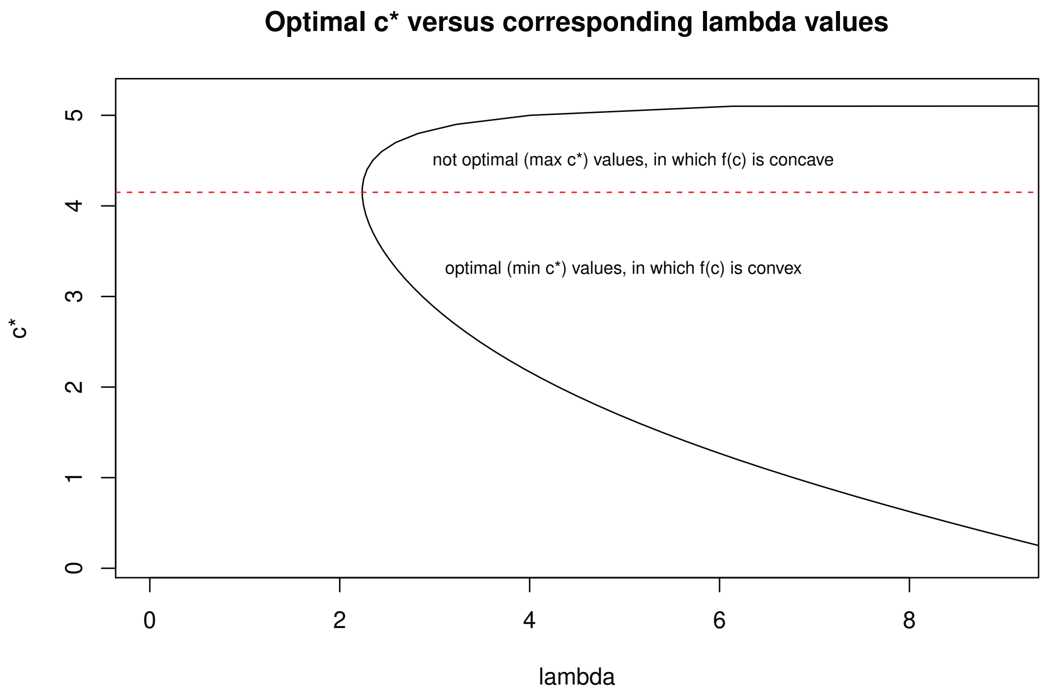

Figure 2.3: Optimal thresholds versus corresponding λ values, with $T = 10$ and $k \approx 5.2$.

solution of $c^* = k \approx 5.2$. This is not feasible from a practical view, as a loan issued would initially be classified in IFRS 9 bucket 2, leading to maximal income volatility for the bank that issues it. We see the specific nature of the objective function for some viable λ values, and respective solutions, in Figure 2.4. The shapes of the three objective functions in this interval are quite similar, although the decreasing relationship between λ and c^* is noticeable, as expected. For this example, we fixed k based on some particular obligor's initial default probability, given the loan was issued over T years. Another natural objective is to study the general relationship between the distance to default of an obligor and the consequent solution c^*.

A basic criticism of the trade-off modelled in our problem formulation is that the tuning parameter, λ, could be chosen in a subjective manner by a financial

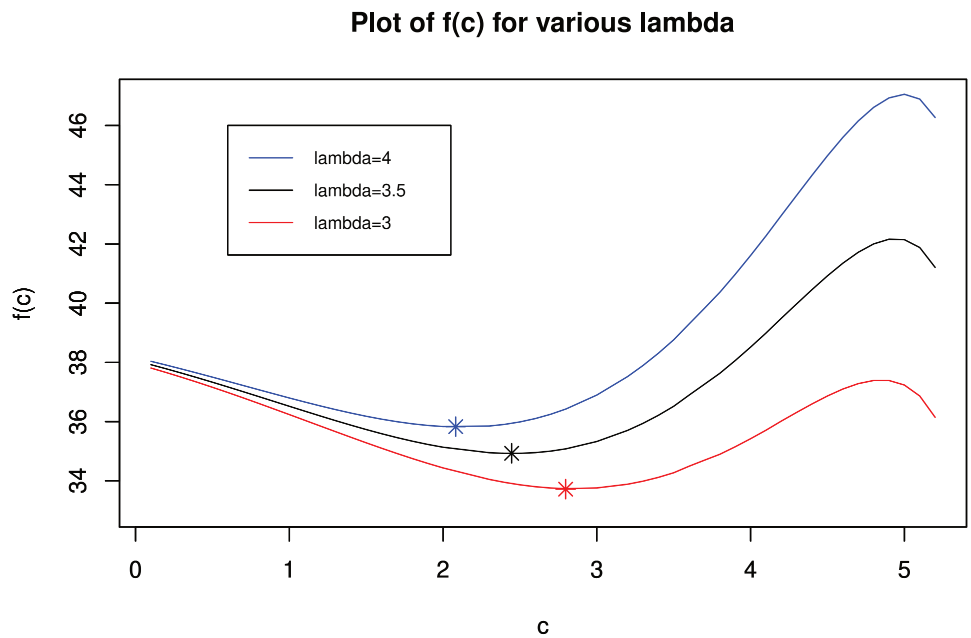

Figure 2.4: Plot of $f(c)$ for various λ, with $T = 10$ and $k \approx 5.2$. The optimal thresholds, determined numerically, are $c^* \approx 2.80$ ($\lambda = 3$), $c^* \approx 2.45$ ($\lambda = 3.5$) and $c^* \approx 2.09$ ($\lambda = 4$).

institution, based on the extent to which they are concerned about early recognition of credit risk compared to income volatility. To justify what reasonable values for λ are in practice, we make a comparison with a European Union (EU) definition. Although IFRS 9 does not specify explicitly what constitutes a significant increase, the EU stress test of the European Banking Authority (2018) defines a significant increase in credit risk as being 200% from the initial credit risk. To convert this definition into our setting, consider a loan with maturity T and current time t, so that the remaining time to maturity is $T - t$. The lifetime default probability at the initial time is $P(A_T < 0) = \Phi\left(\frac{-k}{\sqrt{T}}\right)$. Assuming that the net asset value A_t equals a at time t, the lifetime default probability at time t is

$$P(A_T < 0 \mid A_t = a) = P(A_t + A_T - A_t < 0 \mid A_t = a) = P(a + W_{T-t} < 0)$$

for a Brownian motion W. We use the property that Brownian motion has independent increments, and that $P(W_T - W_t \leq a) = P(W_{T-t} \leq a)$ for any constant a. Since W_{T-t} is normally distributed with mean zero and variance $(T-t)$, we obtain that $P(a + W_{T-t} < 0) = \Phi\left(\frac{-a}{\sqrt{T-t}}\right)$. By the framework of the EU stress test, the risk of a loan has significantly increased if the current lifetime default probability is at least triple the initial lifetime default probability, which means $P(a + W_{T-t} < 0) \geq 3\,P(A_T < 0)$. We now define a^* as the critical value so that $P(a^* + W_{T-t} < 0) = 3\,P(A_T < 0)$. Assuming the loan is in the middle of the lifetime, such that $t = \frac{T}{2}$, we compare $P(a^* + W_{T-t} < 0)$ with the corresponding value $P(c^* + W_{T-t} < 0)$ for our optimizer c^*. Figure 2.5 conveys the relationship between the initial lifetime default probability $P(A_T < 0)$ and the critical lifetime default probability $P\left(c^* + W_{\frac{T}{2}} < 0\right)$, for time $t = \frac{T}{2}$. A positive relationship exists for each λ value shown, since the critical lifetime default probability increases, intuitively, as the initial lifetime default probability increases. Numerically, it is determined that when $P(A_T < 0) \approx 0$, the corresponding $P\left(c^* + W_{\frac{T}{2}} < 0\right) \approx 0$ also, regardless of λ, so the intercept for each line is indeed approximately zero. In comparison to the EU stress test reference line, we see that for each λ, the relationship does not appear to be perfectly linear. However, we can still feasibly suggest that the value $P(a^* + W_{T-t} < 0)$ from the EU stress test corresponds to a parameter choice of λ between 3.5 and 4, for this loan with maturity of $T = 10$. It is not critical for us to choose λ completely analogously to the EU framework. Rather, this comparison is made in order to explain how values of the tuning parameter can be inferred as a guideline for practical applications of our framework.

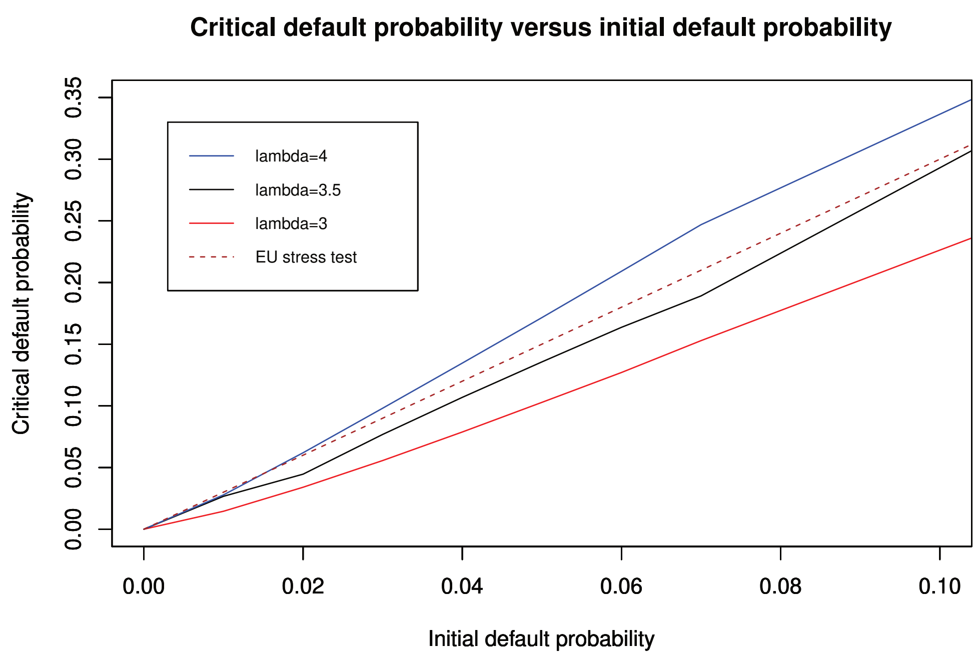

Figure 2.5: Plot of $P\left(c^* + W_{\frac{T}{2}} < 0\right)$ versus $P(A_T < 0)$, for various λ, with $T = 10$. The dashed line shows the relationship outlined by the EU stress test, which is linear by definition.

Chapter 3

Multiple obligors and their latent asset correlation

3.1 Theoretical formulation for this framework

We can extend our problem formulation to consider multiple obligors, which certainly takes into account a bank's full loan portfolio. While obligors are actually grouped into various homogeneous buckets, we will examine in this section obligors in only one particular bucket. It must be acknowledged that these buckets created by banks are not directly associated with the IFRS 9 classification buckets previously discussed. Now, for N obligors, the tradeoff between the early recognition of credit risk and income volatility in (2.3) can be formulated in the new optimization problem

$$\min_{c\le k}\left(\int_0^{T-1}(T-1-t)\sum_{i=1}^{N}\Big(P(A_t^i>c\mid A_T^i<0)\Big)\,dt\right.$$
$$\left.+\lambda\int_0^{T-1}(T-1-t)\,\mathrm{Var}\left(\sum_{i=1}^{N}\mathbb{1}_{A_t^i>c}\right)dt\right), \tag{3.1}$$

where $\lambda>0$ is the tuning parameter, and A_t^i is the net asset value of obligor i at time t. As in the Vasicek (2002) model, the net asset value is defined as

$$A_t^i=k+\sqrt{\rho}\,W_t+\sqrt{1-\rho}\,\beta_t^i$$

with W and β^i representing the systematic and idiosyncratic (individual) components, respectively, and $\rho\in[0,1]$ being a correlation coefficient for this specific bucket. More precisely, $W_t\sim\mathcal{N}(0,t)$ represents the systematic factor of this bucket (equal for all obligors in it) at time t, while $\beta_t^i\sim\mathcal{N}(0,t)$ models the idiosyncratic factor of obligor i at time t. Note that $\beta^1,\ldots,\beta^N$ and W are modelled as independent Brownian motions. Also, observe that in (3.1), we evaluate the variance of the sum of the net asset value indicators for all obligors in this bucket at time t. The net asset values of different obligors in the bucket are not independent, so we have that

$$\mathrm{Var}\left(\sum_{i=1}^{N}\mathbb{1}_{A_t^i>c}\right)\neq\sum_{i=1}^{N}\mathrm{Var}\big(\mathbb{1}_{A_t^i>c}\big).$$

However, from the bank's perspective, the variability of the net asset value indicator functions throughout time for the entire bucket is of interest, rather than summing the individual variances for each individual obligor, which justifies the optimization problem formulation in this regard.

In relation to the average net asset value, we have

$$\begin{aligned}
\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}A_t^i &= \lim_{N\to\infty}\left(\frac{1}{N}(N\cdot k)+\frac{1}{N}(N\sqrt{\rho}\,W_t)+\frac{1}{N}\sqrt{1-\rho}\sum_{i=1}^{N}\beta_t^i\right)\\
&= \lim_{N\to\infty}\left(k+\sqrt{\rho}\,W_t+\sqrt{1-\rho}\left(\frac{1}{N}\sum_{i=1}^{N}\beta_t^i\right)\right)\\
&= k+\sqrt{\rho}\,W_t+\sqrt{1-\rho}\lim_{N\to\infty}\left(\frac{1}{N}\sum_{i=1}^{N}\beta_t^i\right)\\
&= k+\sqrt{\rho}\,W_t \quad \text{almost surely,}
\end{aligned}$$

due to the fact that, for the random variable β_t^i,

$$\lim_{N\to\infty}\left(\frac{1}{N}\sum_{i=1}^{N}\beta_t^i\right)=0 \quad \text{almost surely}$$

by the strong law of large numbers. Evidently, this limit suggests that as the number of obligors increases to very large values, the idiosyncratic factor of each obligor becomes negligible in terms of the average net asset value for a specific bucket. It is intuitive that this average for the rating bucket has a systematic component but not an idiosyncratic one, as we should theoretically have obligors with similar credit quality in a certain bucket. Therefore, negating an obligor's individual randomness from a large-scale perspective, on average, is quite reasonable.

Proposition 2. *The objective function in* (3.1) *can be written as* $\min_{c\leq k} f(c,\rho)$ *for*

$$\begin{aligned}
f(c,\rho) &= \int_0^{T-1}(T-1-t)N\,\frac{\int_{\frac{c-k}{\sqrt{t}}}^{\infty}\Phi\left(\frac{-k-x\sqrt{t}}{\sqrt{T-t}}\right)\varphi(x)\,dx}{\Phi\left(\frac{-k}{\sqrt{T}}\right)}\,dt\\
&\quad+\lambda\int_0^{T-1}(T-1-t)\,\mathrm{Var}\left(\sum_{i=1}^{N}\mathbb{1}_{A_t^i>c}\right)dt\,, \qquad (3.2)
\end{aligned}$$

with

$$\mathrm{Var}\left(\sum_{i=1}^{N} \mathbb{1}_{A_t^i > c}\right) = \left(N^2 - N\right) E\left[\Phi^2\left(\frac{c - k - \sqrt{\rho}\, W_t}{\sqrt{t}\,\sqrt{1-\rho}}\right)\right] + N\,\Phi\left(\frac{c-k}{\sqrt{t}}\right) - N^2\,\Phi^2\left(\frac{c-k}{\sqrt{t}}\right). \tag{3.3}$$

Proof. Since we have the same framework as in the one-obligor case, we can simplify the summation of the conditional probability statement within the recognition portion in (3.1), in the form

$$\begin{aligned} \sum_{i=1}^{N} P(A_t^i > c \mid A_T^i < 0) &= N\,P(A_t^1 > c \mid A_T^1 < 0) \\ &= N\,\frac{\int_{\frac{c-k}{\sqrt{t}}}^{\infty} \Phi\left(\frac{-k-x\sqrt{t}}{\sqrt{T-t}}\right) \varphi(x)\,dx}{P(A_T^1 < 0)} \\ &= N\,\frac{\int_{\frac{c-k}{\sqrt{t}}}^{\infty} \Phi\left(\frac{-k-x\sqrt{t}}{\sqrt{T-t}}\right) \varphi(x)\,dx}{\Phi\left(\frac{-k}{\sqrt{T}}\right)}, \end{aligned}$$

where we use that, for the first equality above, the A_t^i are indeed identically distributed; the second equality follows from (2.7). Essentially, this makes the probability statement equivalent to the expression relative to one specific obligor, with a coefficient of N to account for all obligors in the rating bucket. We also note that this recognition portion of the overall objective function does not depend on ρ, the bucket's correlation coefficient.

For the income volatility portion of (3.1), we examine $\mathrm{Var}\left(\sum_{i=1}^{N} \mathbb{1}_{A_t^i > c}\right)$, applying the law of total variance, while conditioning on the systemic factor of the bucket, W_t, to arrive at the following two terms:

$$\operatorname{Var}\left(\sum_{i=1}^{N} \mathbb{1}_{A_t^i>c}\right) = \underbrace{E\left[\operatorname{Var}\left(\sum_{i=1}^{N} \mathbb{1}_{A_t^i>c} \middle| W_t\right)\right]}_{(3.4.1)} + \underbrace{\operatorname{Var}\left(E\left[\sum_{i=1}^{N} \mathbb{1}_{A_t^i>c} \middle| W_t\right]\right)}_{(3.4.2)}. \tag{3.4}$$

As the A_t^i are conditionally independent, given W_t, (3.4.1) can be written as

$$\begin{aligned}
E\left[\operatorname{Var}\left(\sum_{i=1}^{N} \mathbb{1}_{A_t^i>c} \middle| W_t\right)\right] &= E\left[\sum_{i=1}^{N} \operatorname{Var}\left(\mathbb{1}_{A_t^i>c} \middle| W_t\right)\right] \\
&= E\left[\sum_{i=1}^{N} E\left[\mathbb{1}_{A_t^i>c} \middle| W_t\right] - \left(E\left[\mathbb{1}_{A_t^i>c} \middle| W_t\right]\right)^2\right] \\
&= E\left[\sum_{i=1}^{N} P(A_t^i > c\,|W_t) - \left(P(A_t^i > c\,|W_t)\right)^2\right] \\
&= E\left[\sum_{i=1}^{N} P(A_t^i < c\,|W_t) - \left(P(A_t^i < c\,|W_t)\right)^2\right] \\
&= E\left[\sum_{i=1}^{N} P\left(\frac{\beta_t^i}{\sqrt{t}} < \frac{c-k-\sqrt{\rho}\,W_t}{\sqrt{t}\sqrt{1-\rho}}\right)\right. \\
&\qquad \left. - \left(P\left(\frac{\beta_t^i}{\sqrt{t}} < \frac{c-k-\sqrt{\rho}\,W_t}{\sqrt{t}\sqrt{1-\rho}}\right)\right)^2\right] \\
&= E\left[\sum_{i=1}^{N} \Phi\left(\frac{c-k-\sqrt{\rho}\,W_t}{\sqrt{t}\sqrt{1-\rho}}\right) - \Phi^2\left(\frac{c-k-\sqrt{\rho}\,W_t}{\sqrt{t}\sqrt{1-\rho}}\right)\right] \\
&= N E\left[\Phi\left(\frac{c-k-\sqrt{\rho}\,W_t}{\sqrt{t}\sqrt{1-\rho}}\right) - \Phi^2\left(\frac{c-k-\sqrt{\rho}\,W_t}{\sqrt{t}\sqrt{1-\rho}}\right)\right] \\
&= N\left(\Phi\left(\frac{c-k}{\sqrt{t}}\right) - E\left[\Phi^2\left(\frac{c-k-\sqrt{\rho}\,W_t}{\sqrt{t}\sqrt{1-\rho}}\right)\right]\right).
\end{aligned}$$

For the last equality, we note that

$$
\begin{aligned}
E\left[\Phi\left(\frac{c-k-\sqrt{\rho}\,W_t}{\sqrt{t}\,\sqrt{1-\rho}}\right)\right] &= E\left[P\left(N \le \frac{c-k-\sqrt{\rho}\,W_t}{\sqrt{t}\,\sqrt{1-\rho}}\right)\right] \\
&= E\left[\mathbb{1}_{N \le \frac{c-k-\sqrt{\rho}\,W_t}{\sqrt{t}\sqrt{1-\rho}}}\right] \\
&= E\left[\mathbb{1}_{N\sqrt{1-\rho}+\frac{\sqrt{\rho}}{\sqrt{t}}W_t \le \frac{c-k}{\sqrt{t}}}\right] \\
&= E\left[\mathbb{1}_{\tilde{N} \le \frac{c-k}{\sqrt{t}}}\right] \\
&= E\left[P\left(\tilde{N} \le \frac{c-k}{\sqrt{t}}\right)\right] \\
&= E\left[\Phi\left(\frac{c-k}{\sqrt{t}}\right)\right] \\
&= \Phi\left(\frac{c-k}{\sqrt{t}}\right),
\end{aligned}
$$

where N is a standard normal random variable, independent of W_t, so that $\tilde{N} = N\sqrt{1-\rho} + \frac{\sqrt{\rho}}{\sqrt{t}}W_t$ is also standard normally distributed. Thus, we have a simplified expression for (3.4.1), mainly by incorporating the formula for the identically distributed A_t^i, and the fact that $\beta_t^i \sim \mathcal{N}(0,t)$.

As for (3.4.2), we proceed with the derivation

$$
\begin{aligned}
\mathrm{Var}\left(E\left[\sum_{i=1}^{N}\mathbb{1}_{A_t^i > c}\,\middle|\,W_t\right]\right) &= \mathrm{Var}\left(N\,E\left[\mathbb{1}_{A_t^1 > c}\,\middle|\,W_t\right]\right) \\
&= N^2\,\mathrm{Var}\Big(P\big(A_t^1 > c\,\big|\,W_t\big)\Big) \\
&= N^2\,\mathrm{Var}\Big(1-P\big(A_t^1 < c\,\big|\,W_t\big)\Big) \\
&= N^2\,\mathrm{Var}\left(P\left(\frac{\beta_t^1}{\sqrt{t}} < \frac{c-k-\sqrt{\rho}\,W_t}{\sqrt{t}\,\sqrt{1-\rho}}\,\middle|\,W_t\right)\right) \\
&= N^2\,\mathrm{Var}\left(\Phi\left(\frac{c-k-\sqrt{\rho}\,W_t}{\sqrt{t}\,\sqrt{1-\rho}}\right)\right),
\end{aligned}
$$

which we further simplify to

$$
\begin{aligned}
\mathrm{Var}\left(E\left[\sum_{i=1}^{N}\mathbb{1}_{A_t^i>c}\,\middle|W_t\right]\right) &= N^2\left(E\left[\Phi^2\left(\frac{c-k-\sqrt{\rho}\,W_t}{\sqrt{t}\sqrt{1-\rho}}\right)\right]\right. \\
&\qquad \left.-E\left[\Phi\left(\frac{c-k-\sqrt{\rho}\,W_t}{\sqrt{t}\sqrt{1-\rho}}\right)\right]^2\right) \\
&= N^2\left(E\left[\Phi^2\left(\frac{c-k-\sqrt{\rho}\,W_t}{\sqrt{t}\sqrt{1-\rho}}\right)\right]-\Phi^2\left(\frac{c-k}{\sqrt{t}}\right)\right),
\end{aligned}
$$

and combining this term with (3.4.1) so that we deduce (3.3). Through these simplifications, the objective function in (3.1) is indeed equivalent to (3.2). □

A subsequent observation from Proposition 2 is that when $\rho = 0$, we see that

$$
\begin{aligned}
\mathrm{Var}\left(\sum_{i=1}^{N}\mathbb{1}_{A_t^i>c}\right) &= \left(N^2-N\right)\left(\Phi^2\left(\frac{c-k}{\sqrt{t}}\right)\right)+N\,\Phi\left(\frac{c-k}{\sqrt{t}}\right)-N^2\,\Phi^2\left(\frac{c-k}{\sqrt{t}}\right) \\
&= N\left(\Phi\left(\frac{c-k}{\sqrt{t}}\right)-\Phi^2\left(\frac{c-k}{\sqrt{t}}\right)\right),
\end{aligned}
$$

justifying that, for the case with uncorrelated net asset values in a rating bucket, the variance of this sum is equivalent to the sum of the variances. Naturally, we have a similar expression as in the volatility portion derivation in (2.8) for one obligor, but with a coefficient of N. From (3.3), it is also evident that for the one-obligor case, $N = 1$, we predictably arrive at the same expression as (2.8), independently of ρ.

Furthermore, we have that

$$
\begin{aligned}
E\left[\Phi^2\left(\frac{c-k-\sqrt{\rho}\,W_t}{\sqrt{t}\sqrt{1-\rho}}\right)\right] &= E\left[\Phi^2\left(\frac{\frac{c-k}{\sqrt{t}}-\sqrt{\rho}\,N}{\sqrt{1-\rho}}\right)\right] \quad \text{for } N\sim\mathcal{N}(0,1) \\
&= \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\,\Phi^2\left(\frac{\frac{c-k}{\sqrt{t}}-\sqrt{\rho}\,x}{\sqrt{1-\rho}}\right)\mathrm{e}^{-\frac{x^2}{2}}\,dx,
\end{aligned}
$$

which is useful, along with the Leibniz integral rule, in finding the derivative

$$
\begin{aligned}
\frac{d}{dc} f(c,\rho) = \int_0^{T-1} & -N \frac{T-1-t}{\sqrt{t}\,\Phi\left(\frac{-k}{\sqrt{T}}\right)} \Phi\left(\frac{-c}{\sqrt{T-t}}\right) \varphi\left(\frac{c-k}{\sqrt{t}}\right) dt \\
& + \lambda \int_0^{T-1} (T-1-t) \Bigg[\int_{-\infty}^{\infty} \frac{2\left(N^2-N\right)}{\sqrt{2\pi}\sqrt{t}\sqrt{1-\rho}} \Phi\left(\frac{\frac{c-k}{\sqrt{t}} - \sqrt{\rho}\,x}{\sqrt{1-\rho}}\right) \\
& \qquad \times \varphi\left(\frac{\frac{c-k}{\sqrt{t}} - \sqrt{\rho}\,x}{\sqrt{1-\rho}}\right) \mathrm{e}^{-\frac{x^2}{2}}\, dx \\
& \qquad + \frac{N}{\sqrt{t}} \left(\varphi\left(\frac{c-k}{\sqrt{t}}\right) - 2N\Phi\left(\frac{c-k}{\sqrt{t}}\right) \varphi\left(\frac{c-k}{\sqrt{t}}\right) \right) \Bigg] dt,
\end{aligned}
$$

necessary for solving the optimization problem. Once again, numerical integration will be performed, because of the complex nature of these expressions.

3.2 Effect of the various parameters on the minimizer

We now investigate the influence of ρ and N on the objective function, along with the corresponding solutions. For the following examples, $T = 10$ and $k \approx 5.2$ are fixed, consistent with our parameter choices from Section 2.2. To paraphrase, we now have many obligors with similar credit quality, that are all issued loans with the same term and have equivalent initial default probabilities. In the upcoming analysis, we also set $\lambda = 1$ and $\rho = 0.05$, then establish some knowledge regarding the two portions of the objective function. Not surprisingly, the recognition portion is the same as in the one-obligor case seen in (2.1), multiplied by N, therefore it is decreasing in c and its influence is simply magnified by the number of obligors in this bucket. Also, this portion is independent of ρ, thus examining the recognition portion further provides minimal additional knowledge. A much different situation exists for the volatility portion of the objective function, due to the complex nature of the variance term expressed in detail in (3.3). Realistically assuming that $N > 1$ for a rating bucket, we use numerical integration to visualize some results for this advanced expression. For the parameter values previously mentioned, Figure 3.1 displays the effect of the volatility portion for $N = 100$ and $N = 1000$. Clearly, as c increases, the values of the income volatility portion with $N = 1000$ represent considerably more than a tenfold increase, when compared to the same function with $N = 100$.

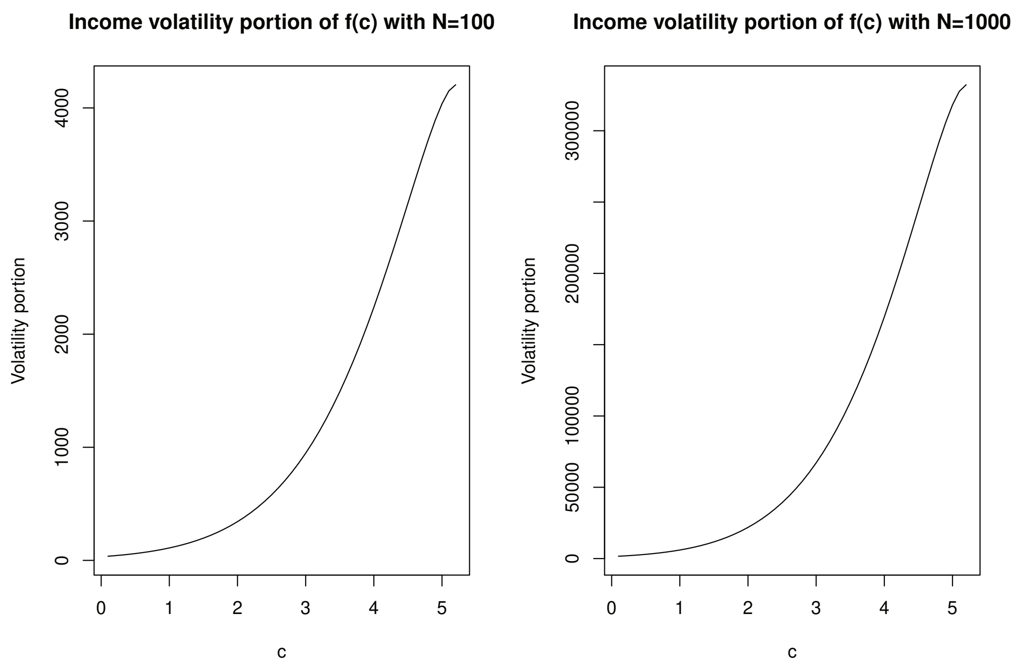

Figure 3.1: Volatility portion of $f(c)$, with $T = 10$, $k \approx 5.2$, $\lambda = 1$ and $\rho = 0.05$. Plots for both $N = 100$ and $N = 1000$ are shown.

It is evident that the variance term in (3.3) can be written as $\mathrm{Var}\left(\sum_{i=1}^{N} \mathbb{1}_{A_t^i > c}\right) = \alpha N^2 + \beta N$, with α and β depending on functions of the values of the standard normal cumulative distribution function. In particular,

$$\alpha = E\left[\Phi^2\left(\frac{c-k-\sqrt{\rho}\,W_t}{\sqrt{t}\,\sqrt{1-\rho}}\right)\right] - \Phi^2\left(\frac{c-k}{\sqrt{t}}\right)$$

and

$$\beta = \Phi\left(\frac{c-k}{\sqrt{t}}\right) - E\left[\Phi^2\left(\frac{c-k-\sqrt{\rho}\,W_t}{\sqrt{t}\,\sqrt{1-\rho}}\right)\right].$$

The derivations of α and β are completely analogous to the expressions in (3.3.2) and (3.3.1), respectively. As a result, both the variance of a conditional expectation, along with the expected value of a conditional variance, are non-negative. Hence, it is clear that $\mathrm{Var}\left(\sum_{i=1}^{N} \mathbb{1}_{A_t^i > c}\right)$ is also non-negative, grows quadratically in N, and diverges to positive infinity as $N \to \infty$. To summarize in practical terms, the influence of the number of obligors is more drastic for the volatility portion than it is for the recognition portion. A consequence is that for larger sized rating buckets, a smaller value of the tuning parameter (for example, $\lambda < 1$ in this case) may be needed so that the volatility portion does not have an excessive contribution to the overall objective function. Recall that, intuitively, as we place more importance on managing income volatility, the optimal threshold c^* gets closer to zero, which disregards the early recognition of credit risk.

A sensible next step is to analyze the relationship between λ and corresponding solutions to the optimization problem, for different numbers of obligors in the rating bucket. With the same parameter values, we observe in Figures 3.2 and 3.3 that the solution "path" is similar to the one-obligor case, as we have a decreasing relationship between c^* and λ, with some local maxima, not useful for our purposes, also existing in the interval $k \geq c^* \geq 0$. Note that the red line in each figure indicates the approximate cutoff point between concavity and convexity of $f(c)$.

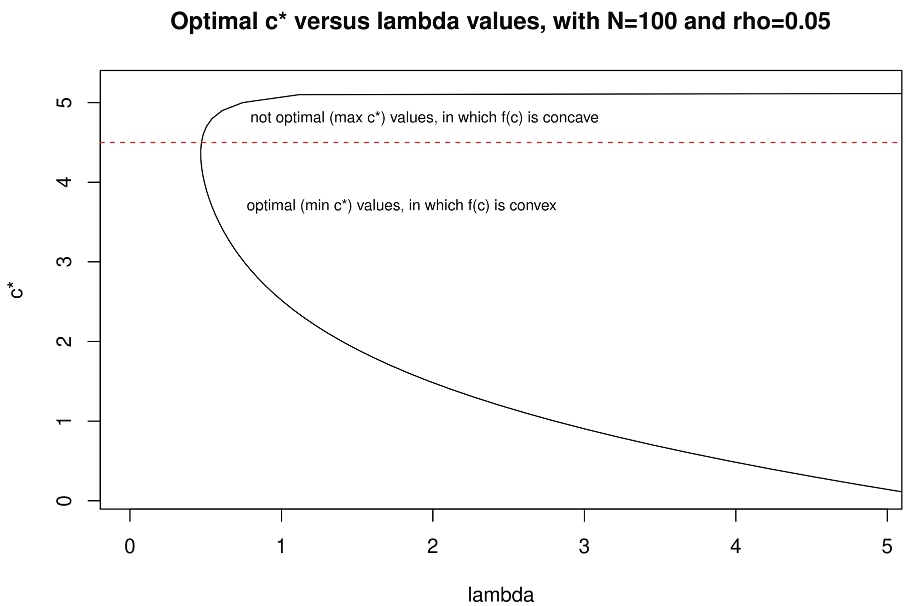

Figure 3.2: Optimal thresholds versus corresponding λ values, with $T = 10$, $k \approx 5.2$ and $\rho = 0.05$, for bucket size of $N = 100$.

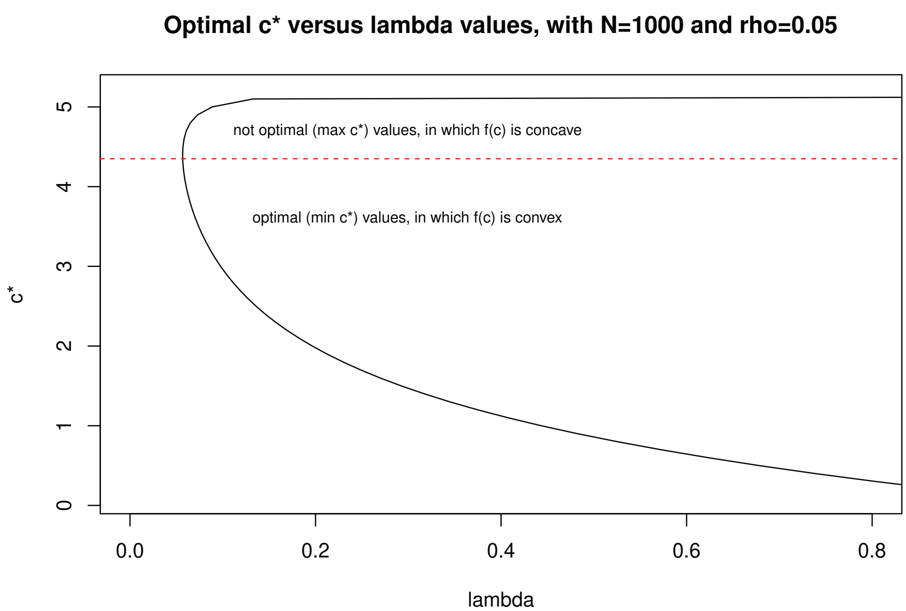

Figure 3.3: Optimal thresholds versus corresponding λ values, with $T = 10$, $k \approx 5.2$ and $\rho = 0.05$, for bucket size of $N = 1000$.

This convexity requirement certainly needs to be satisfied in this multiple-obligor formulation, with the minima being of interest, as expected.

A more insightful conclusion can be made when looking at the plot for $N = 1000$ in Figure 3.3. To avoid having a solution extremely close to zero, $\lambda < 1$ must be selected, reinforcing the notion that our volatility portion of the objective function has a stronger relationship with N. Therefore, for very large bucket sizes, the balance between the two portions can only be maintained if we shrink the considerably larger values of income volatility, to some extent. This is done to ensure the early recognition of credit risk is considered to be sufficiently important relative to our formulation. Moreover, the subjective nature of choosing λ has already been discussed, but in general, to avoid having a solution approaching either endpoint, it is reasonable to claim that $1 \leq \lambda \leq 2$ (for $N = 100$) and $0.1 \leq \lambda \leq 0.3$ (for $N = 1000$) are feasible tuning parameter choices.

Ultimately, we can select $\lambda = 1$ (for $N = 100$) and $\lambda = 0.1$ (for $N = 1000$), then scrutinize the relationship between ρ and related solutions c^*. Note that in the preceeding plots of this section, a common choice of $\rho = 0.05$ was arbitrarily used, reasonable for a qualifying revolving retail portfolio, for example. It is natural to study our objective function for different ρ values, in which Figures 3.4 and 3.5 are prominent. We deduce that, regardless of the number of obligors in the bucket, as ρ increases, c^* decreases. An explanation for this in layman's terms is based on the fact that the income volatility portion of $f(c)$ depends on ρ, unlike the recognition portion. Hence, as the correlation within the rating bucket becomes more significant, and one obligor is reclassified from one-year ECL to lifetime ECL, we would expect many other obligors to also experience a significant increase in credit risk, based on their highly correlated net asset values. As a result, the total ECL for

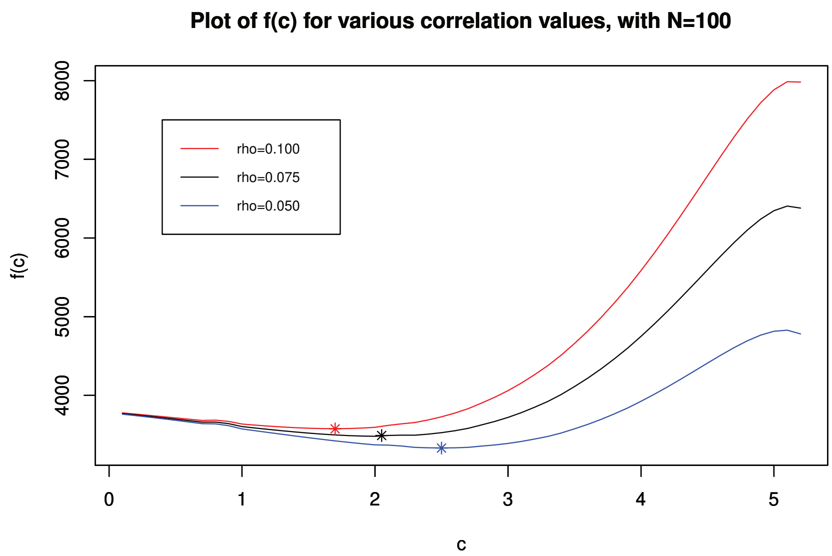

Figure 3.4: Plot of $f(c)$ for various ρ, with $T = 10$, $k \approx 5.2$ and $\lambda = 1$, with a bucket size of $N = 100$. The optimal thresholds, determined numerically, are $c^* \approx 2.49$ ($\rho = 0.05$), $c^* \approx 2.05$ ($\rho = 0.075$) and $c^* \approx 1.70$ ($\rho = 0.1$).

this rating bucket would be unstable to estimate from the perspective of the bank. To counter this, the conservative approach, focusing on the income volatility aspect, is to set the threshold c^* as far away from k as possible. In this extreme case, which was confirmed numerically and does not depend on N, when $\rho \to 1$ ("perfect" correlation within the bucket), $c^* \to 0$ indeed. Another general conclusion is that because only the income volatility portion depends on ρ, making λ larger will increase the influence of ρ on the solution of the optimization problem. As seen in Figure 3.5, for $N = 1000$, our choice of $\lambda = 0.1$ somewhat mitigated any profound effect of ρ. However, numerically, for larger values of N and λ, solutions tend to zero relatively hastily as ρ increases incrementally. Since $c^* \approx 0$ is not practical, these results are not displayed in the plots, although this relationship is still important to acknowledge theoretically.

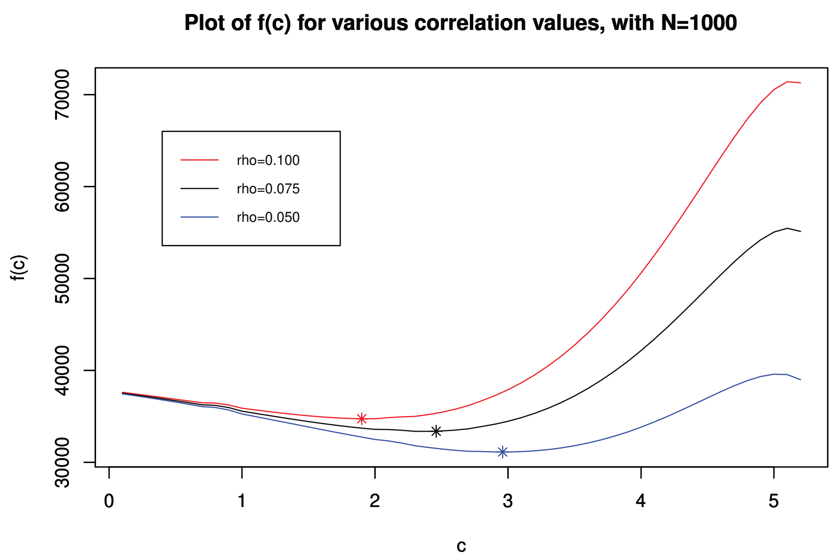

Figure 3.5: Plot of $f(c)$ for various ρ, with $T = 10$, $k \approx 5.2$ and $\lambda = 0.1$, with a bucket size of $N = 1000$. The optimal thresholds, determined numerically, are $c^* \approx 2.96$ ($\rho = 0.05$), $c^* \approx 2.46$ ($\rho = 0.075$) and $c^* \approx 1.92$ ($\rho = 0.1$).

Chapter 4

Discrete-time formulation

4.1 Overview of modelling structure

In this chapter, an alternative formulation involving discrete-time observations is incorporated. This logic is quite realistic, as obligor specific information from a bank is typically observed at monthly or quarterly reporting dates. Here, we consider instances $t_0 = 0 < t_1 < \cdots < t_N = T$ and assume that $t_{N-1} \leq T - 1$. This assumption is without loss of generality, as we could simply disregard the instances after time $T - 1$. There is no impact on provisions when switching above and below the threshold c after $T - 1$, since the loan's one-year ECL equals lifetime ECL when there is less than a year until maturity. We do not impose that $(A_t)_{0 \leq t \leq T}$ is necessarily a Brownian motion, but only assume that this stochastic process has stationary and independent increments. Time continues to be measured in years, and we maintain that the starting point $A_0 = k$ is the initial distance to default of an obligor, depending directly on their initial default probability. From this discrete-time perspective, the *recognition portion* is

$$\sum_{j=1}^{N-1} (T-1-t_j)\, P(A_{t_j} > c \mid A_T < 0)\,, \tag{4.1}$$

and the *volatility portion* is

$$E\left[\sum_{j=1}^{N-1} (T-1-t_j)\Big(\mathbb{1}_{A_{t_j}>c} - \mathbb{1}_{A_{t_{j-1}}>c}\Big)^2\right]. \tag{4.2}$$

It is clear that the recognition portion, along with the time-factor coefficient $(T-1-t_j)$ in both portions, is constructed analogously to the logic of Section 2.1, of course with a summation included rather than an integral. On the contrary, the volatility portion takes here the effective variation between reporting dates into account and does not follow directly from the continuous-time formulation previously discussed. In particular, the expression $\big(\mathbb{1}_{A_{t_j}>c} - \mathbb{1}_{A_{t_{j-1}}>c}\big)^2$ in (4.2) gives the value 1 (and zero otherwise) if the process changes from being above to below (or vice versa) the threshold c from time t_{j-1} to t_j. We can write the volatility portion as

$$\begin{aligned}
&E\left[\sum_{j=1}^{N-1} (T-1-t_j)\Big(\mathbb{1}_{A_{t_j}>c} - \mathbb{1}_{A_{t_{j-1}}>c}\Big)^2\right] \\
&= E\left[\sum_{j=1}^{N-1} (T-1-t_j)\Big(\mathbb{1}_{A_{t_j}>c, A_{t_{j-1}}\le c} + \mathbb{1}_{A_{t_j}\le c, A_{t_{j-1}}>c}\Big)\right] \\
&= \sum_{j=1}^{N-1} (T-1-t_j)\, E\Big[\mathbb{1}_{A_{t_j}>c, A_{t_{j-1}}\le c} + \mathbb{1}_{A_{t_j}\le c, A_{t_{j-1}}>c}\Big] \\
&= \sum_{j=1}^{N-1} (T-1-t_j)\Big(P\big(A_{t_j} > c, A_{t_{j-1}} \le c\big) + P\big(A_{t_j} \le c, A_{t_{j-1}} > c\big)\Big).
\end{aligned}$$

Naturally, we merge the two portions, and model the trade-off in a new optimization problem

$$\min_{c \leq k} f(c),$$

where

$$\begin{aligned} f(c) = &\sum_{j=1}^{N-1} (T-1-t_j) P(A_{t_j} > c \mid A_T < 0) \\ &+ \lambda \sum_{j=1}^{N-1} (T-1-t_j) \Big(P\big(A_{t_j} > c, A_{t_{j-1}} \leq c\big) + P\big(A_{t_j} \leq c, A_{t_{j-1}} > c\big) \Big), \end{aligned} \tag{4.3}$$

with $\lambda > 0$ being our familiar tuning parameter. Various approaches to modelling the net asset value process $(A_t)_{0 \leq t \leq T}$ will be investigated in the following subsections, including both continuous and discrete distributions for the increments of $(A_t)_{0 \leq t \leq T}$.

4.2 Analyzing the optimization problem for continuous asset distribution

In this section, we study the situation where the net asset value process has a continuous distribution. Before analyzing specific distributions, we rephrase our optimization problem in terms of the distribution of A_t. We denote by G_t and g_t the cumulative distribution function and probability density function, respectively, of the net asset value change $A_t - A_0$ for a fixed time t.

Proposition 3. *The function $f(c)$ in* (4.3) *can be written as*

$$
\begin{aligned}
f(c) = \sum_{j=1}^{N-1} (T-1-t_j)\, \frac{\int_{c-k}^{\infty} G_{T-t_j}(-k-x)\, g_{t_j}(x)\, dx}{G_T(-k)} \\
+ \lambda \sum_{j=1}^{N-1} (T-1-t_j) \Bigg(\int_{-\infty}^{c-k} \Big(1 - G_{t_j - t_{j-1}}\big(c-k-x\big)\Big)\, g_{t_{j-1}}(x)\, dx \\
+ \int_{c-k}^{\infty} G_{t_j - t_{j-1}}\big(c-k-x\big)\, g_{t_{j-1}}(x)\, dx \Bigg). \qquad (4.4)
\end{aligned}
$$

Proof. Because the net asset value process has stationary increments, the cumulative distribution function of $A_{t_j} - A_{t_k}$ for $j > k$ is $G_{t_j - t_k}$. For the recognition portion of $f(c)$, we derive the conditional probability

$$
\begin{aligned}
P(A_{t_j} > c \mid A_T < 0) &= \frac{P\big(A_{t_j} > c, A_T - A_{t_j} < -A_{t_j}\big)}{P(A_T < 0)} \\
&= \frac{E\Big[\mathbb{1}_{A_{t_j} > c}\, E\big[\mathbb{1}_{A_T - A_{t_j} < -A_{t_j}} \big| A_{t_j}\big]\Big]}{P\big(A_T - A_0 < -k\big)} \\
&= \frac{E\Big[\mathbb{1}_{A_{t_j} > c}\, E[\mathbb{1}_{A_T - A_{t_j} < -a}]\ |_{a = A_{t_j}}\ \Big]}{G_T(-k)} \\
&= \frac{E\Big[\mathbb{1}_{A_{t_j} > c}\, G_{T-t_j}(-A_{t_j})\Big]}{G_T(-k)} \\
&= \frac{E\Big[\mathbb{1}_{A_{t_j} - A_0 > c-k}\, G_{T-t_j}\big(-k - (A_{t_j} - A_0)\big)\Big]}{G_T(-k)} \\
&= \frac{\int_{c-k}^{\infty} G_{T-t_j}(-k-x)\, g_{t_j}(x)\, dx}{G_T(-k)}, \qquad (4.5)
\end{aligned}
$$

which follows from the procedure in Section 2.1, leading to an expression similar to (2.7). The volatility portion simplification involves comparable methodology, with some slight differences. First, we deduce that

$$
\begin{aligned}
P\big(A_{t_j} > c, A_{t_{j-1}} \leq c\big) &= P\big(A_{t_{j-1}} \leq c, A_{t_j} - A_{t_{j-1}} > c - A_{t_{j-1}}\big) \\
&= E\Big[\mathbb{1}_{A_{t_{j-1}} \leq c} E\big[\mathbb{1}_{A_{t_j} - A_{t_{j-1}} > c - A_{t_{j-1}}} \big| A_{t_{j-1}}\big]\Big] \\
&= E\Big[\mathbb{1}_{A_{t_{j-1}} \leq c} \Big(1 - G_{t_j - t_{j-1}}\big(c - A_{t_{j-1}}\big)\Big)\Big] \\
&= E\Big[\mathbb{1}_{A_{t_{j-1}} - A_0 \leq c-k} \Big(1 - G_{t_j - t_{j-1}}\big(c - k - (A_{t_{j-1}} - A_0)\big)\Big)\Big] \\
&= \int_{-\infty}^{c-k} \Big(1 - G_{t_j - t_{j-1}}\big(c - k - x\big)\Big) g_{t_{j-1}}(x)\, dx.
\end{aligned}
$$

This is a condensed version of the derivation, as we are following the same procedure as in the computation of (4.5). Similarly, this method yields

$$
P\big(A_{t_j} \leq c, A_{t_{j-1}} > c\big) = \int_{c-k}^{\infty} G_{t_j - t_{j-1}}\big(c - k - x\big)\, g_{t_{j-1}}(x)\, dx,
$$

so that the overall volatility portion is

$$
\begin{aligned}
\lambda \sum_{j=1}^{N-1} (T - 1 - t_j) \bigg(& \int_{-\infty}^{c-k} \Big(1 - G_{t_j - t_{j-1}}\big(c - k - x\big)\Big) g_{t_{j-1}}(x)\, dx \\
& + \int_{c-k}^{\infty} G_{t_j - t_{j-1}}\big(c - k - x\big)\, g_{t_{j-1}}(x)\, dx \bigg).
\end{aligned}
$$

Merging the two portions, the objective function (4.3) results in (4.4). □

To extend this analysis, a distribution must be established for the increments of the net asset value, with the purpose of solving the optimization problem analytically when possible. In Subsection 4.2.1, we give a simple example where we can find an explicit formula for the optimal threshold, while in Subsection 4.2.2 we present and numerically analyze the optimization problem in the case of the net asset value given by Brownian motion.

4.2.1 Considering a shifted exponential distribution for modelling the net asset value

In this subsection, we assume the increments in the net asset value of an obligor are of equal time length, such that $t_j = \frac{jT}{N}$, with $t_j - t_{j-1}$ being constant. In addition, the distribution of an increment, $A_{t_j} - A_{t_{j-1}}$, is given by G with corresponding density g. More specifically, we consider the shifted exponential distribution with shift parameter δ and mean parameter θ, such that

$$g(x) = \frac{1}{\theta} \mathrm{e}^{-\frac{x-\delta}{\theta}}$$

and

$$G(x) = 1 - \mathrm{e}^{-\frac{x-\delta}{\theta}}$$

with $x \geq \delta$ and $\theta > 0$. In our situation, we specify $\delta < 0$ to account for the obviously realistic possibility of net losses occurring in any given time interval prior to maturity. Under the assumption of equal time steps, we now have $E(A_{t_j}) = k + j(\theta + \delta)$ and $\mathrm{Var}(A_{t_j}) = j\theta^2$, observing that the net asset value has greater variability over longer periods of time. We further note that $-k > \delta$, so that a default can theoretically occur in any time increment, with the initial default probability depending on k directly.

In this situation, we examine the case when $T = t_N = t_2 = 2t_1$, such that we have only one reporting date of the net asset value of the obligor, exactly halfway to the time of maturity of the loan. We state an additional simple, albeit important assumption, that $T > 2$. Otherwise, our problem would have no relevance to IFRS 9 expected credit loss, as we cannot have early recognition of credit risk if $t_1 = \frac{T}{2} \leq 1$, since there would be no possible impact on provisions at the only reporting date prior to maturity, for $t_1 \geq T-1$. We can find an explicit formula for the optimal threshold, given in the following result.

Proposition 4. *Consider the assumptions of this subsection and suppose that*

$$\frac{1-\mathrm{e}^{\frac{k+\delta}{\theta}}}{1-\mathrm{e}^{\frac{k+2\delta}{\theta}}\left(\frac{\theta-(k+2\delta)}{\theta}\right)} < \lambda < \frac{1-\mathrm{e}^{\frac{\delta}{\theta}}}{1-\mathrm{e}^{\frac{k+2\delta}{\theta}}\left(\frac{\theta-(k+2\delta)}{\theta}\right)} \tag{4.6}$$

then the optimal threshold is given by

$$\begin{aligned} c^* &= \theta \ln\left[1-\lambda\left(1-\mathrm{e}^{\frac{k+2\delta}{\theta}}\left(\frac{\theta-(k+2\delta)}{\theta}\right)\right)\right]-\delta \\ &= \theta \ln\left[1-\lambda\Big(P(A_T<0)\Big)\right]-\delta. \end{aligned} \tag{4.7}$$

Proof. Under the assumptions of this subsection, the objective function (4.3) is simplified as

$$\begin{aligned} f(c) &= \left(T-1-\frac{T}{2}\right)P\big(A_{\frac{T}{2}}>c \mid A_T<0\big) \\ &\quad +\lambda\left(T-1-\frac{T}{2}\right)\Big(P\big(A_{\frac{T}{2}}>c, A_0\leq c\big)+P\big(A_{\frac{T}{2}}\leq c, A_0>c\big)\Big). \end{aligned}$$

Under the assumption that $A_0 = k > c$, we eliminate the unrealistic possibility that the loan is in IFRS 9 bucket 2 at the time it is issued. Consequently, we deduce that $P\big(A_{\frac{T}{2}}>c, A_0\leq c\big)=0$ and $P\big(A_{\frac{T}{2}}\leq c, A_0>c\big)=P\big(A_{\frac{T}{2}}\leq c\big)$, because $P(A_0\leq c)=0$ and $P(A_0>c)=1$. Therefore, for our case where $N=2$, the volatility portion of $f(c)$ from (4.3) simply becomes $\lambda\left(T-1-\frac{T}{2}\right)P\big(A_{\frac{T}{2}}\leq c\big)$. Furthermore,

$$f(c) = \frac{T-2}{2}P\big(A_{\frac{T}{2}}>c \mid A_T<0\big)+\lambda\frac{T-2}{2}P\big(A_{\frac{T}{2}}\leq c\big),$$

which, using (4.4), is expressed in terms of functions G and g as

$$f(c) = \frac{T-2}{2} \frac{\int_{c-k}^{\infty} G(-k-x)\, g(x)\, dx}{P(A_T < 0)} + \lambda \frac{T-2}{2} G(c-k). \tag{4.8}$$

The next step in this problem is to further simplify $f(c)$, take the partial derivative with respect to c, and arrive at an expression for the solution to the minimization problem, c^*, depending on the other parameters involved. For the distribution of A_T, we write $A_T = A_T - A_{\frac{T}{2}} + A_{\frac{T}{2}} - A_0 + A_0$, which shows that $A_T - A_0$ is the sum of two independent and identically distributed random variables, namely, the random variable $A_T - A_{\frac{T}{2}}$ and the random variable $A_{\frac{T}{2}} - A_0$. This means that the distribution of $A_T - A_0$ is a convolution, given by $P(A_T - A_0 < x) = \int_{-\infty}^{\infty} G(x-y)\, g(y)\, dy$. Then the initial default probability $P(A_T < 0)$ is derived from

$$\begin{aligned}
P(A_T - A_0 < -k) &= \int_{-\infty}^{\infty} G(-k-y)\, g(y)\, dy \\
&= \int_{-\infty}^{\infty} \left(1 - \mathrm{e}^{-\frac{-k-y-\delta}{\theta}}\right) \frac{1}{\theta} \mathrm{e}^{-\frac{y-\delta}{\theta}}\, dy \\
&= \int_{\delta}^{-k-\delta} \left(1 - \mathrm{e}^{-\frac{-k-y-\delta}{\theta}}\right) \frac{1}{\theta} \mathrm{e}^{-\frac{y-\delta}{\theta}}\, dy \\
&= 1 - \mathrm{e}^{\frac{k+2\delta}{\theta}} \left(\frac{\theta - (k+2\delta)}{\theta}\right),
\end{aligned}$$

with the limits of integration in the second last equality being a consequence of the fact that $g(y) = 0$ for $y < \delta$, and $G(-k-y) = 0$ for $-k-y < \delta$, or $y > -k-\delta$.

Obtaining $P(A_T < 0)$ now allows us to write our objective function in (4.8) as

$$\begin{aligned} f(c) &= \frac{T-2}{2}\left[\frac{\int_{c-k}^{-k-\delta}\left(1-\mathrm{e}^{-\frac{-k-x-\delta}{\theta}}\right)\frac{1}{\theta}\,\mathrm{e}^{-\frac{x-\delta}{\theta}}\,dx}{1-\mathrm{e}^{\frac{k+2\delta}{\theta}}\left(\frac{\theta-(k+2\delta)}{\theta}\right)} + \lambda\left(1-\mathrm{e}^{-\frac{c-k-\delta}{\theta}}\right)\right] \\ &= \frac{T-2}{2}\left[\frac{\mathrm{e}^{-\frac{c-k-\delta}{\theta}} + \frac{1}{\theta}\,\mathrm{e}^{\frac{k+2\delta}{\theta}}(c+\delta-\theta)}{1-\mathrm{e}^{\frac{k+2\delta}{\theta}}\left(\frac{\theta-(k+2\delta)}{\theta}\right)} + \lambda\left(1-\mathrm{e}^{-\frac{c-k-\delta}{\theta}}\right)\right], \end{aligned}$$

noting that, in the first equality, $G(-k-x) = 0$ for $-k-x < \delta$, or equivalently $x > -k-\delta$, thus the integral

$$\int_{c-k}^{\infty} G(-k-x)\,g(x)\,dx = \int_{c-k}^{-k-\delta} G(-k-x)\,g(x)\,dx.$$

The next natural step is finding the derivative

$$\begin{aligned} f'(c) &= \frac{T-2}{2}\left[\frac{-\frac{1}{\theta}\,\mathrm{e}^{-\frac{c-k-\delta}{\theta}} + \frac{1}{\theta}\,\mathrm{e}^{\frac{k+2\delta}{\theta}}}{1-\mathrm{e}^{\frac{k+2\delta}{\theta}}\left(\frac{\theta-(k+2\delta)}{\theta}\right)} + \lambda\left(\frac{1}{\theta}\,\mathrm{e}^{-\frac{c-k-\delta}{\theta}}\right)\right] \\ &= \frac{T-2}{2\theta\left(1-\mathrm{e}^{\frac{k+2\delta}{\theta}}\left(\frac{\theta-(k+2\delta)}{\theta}\right)\right)}\Bigg[\mathrm{e}^{\frac{k+2\delta}{\theta}} - \mathrm{e}^{\frac{-c+k+\delta}{\theta}} \\ &\qquad + \lambda\left(\mathrm{e}^{\frac{-c+k+\delta}{\theta}} - \mathrm{e}^{\frac{-c+2k+3\delta}{\theta}}\left(\frac{\theta-(k+2\delta)}{\theta}\right)\right)\Bigg], \end{aligned}$$

and in order to equate this to zero, we first acknowledge that $T > 2$, $\theta < \infty$, and the default probability $P(A_T < 0) = 1-\mathrm{e}^{\frac{k+2\delta}{\theta}}\left(\frac{\theta-(k+2\delta)}{\theta}\right)$ is of course finite. Therefore, to satisfy $f'(c) = 0$, we require

$$\mathrm{e}^{\frac{k+2\delta}{\theta}} - \mathrm{e}^{\frac{-c+k+\delta}{\theta}} + \lambda\left(\mathrm{e}^{\frac{-c+k+\delta}{\theta}} - \mathrm{e}^{\frac{-c+2k+3\delta}{\theta}}\left(\frac{\theta-(k+2\delta)}{\theta}\right)\right) = 0,$$

implying

$$e^{\frac{-c+k+\delta}{\theta}}\left[e^{\frac{c+\delta}{\theta}}-1+\lambda\left(1-e^{\frac{k+2\delta}{\theta}}\left(\frac{\theta-(k+2\delta)}{\theta}\right)\right)\right]=0\,.$$

Examining the term $e^{\frac{-c+k+\delta}{\theta}}$, we observe from our initial specifications that $-k>\delta$ or $k+\delta<0$, so that $-c+k+\delta<0$. Since we desire a solution in the interval $0<c^*<k$, with $|\delta|>k>c$, we assume the shift parameter δ is finite, ensuring that $-c+k+\delta>-\infty$. This is reasonable, as we do not realistically consider that infinite net losses can occur in a given time interval. Moreover, since $\theta>0$, and all parameters are finite by assumption, we know $e^{\frac{-c+k+\delta}{\theta}}>0$. All that remains is solving

$$e^{\frac{c+\delta}{\theta}}-1+\lambda\left(1-e^{\frac{k+2\delta}{\theta}}\left(\frac{\theta-(k+2\delta)}{\theta}\right)\right)=0$$

yielding our result for optimal c^* given by (4.7).

However, we must establish an appropriate interval for λ to verify that our solution (minimum) exists in the appropriate interval $0<c^*<k$, along with confirmation that $f''(c^*)>0$, hence $f(c^*)$ is convex. Initially, we analyze the inequality

$$0<c^*<k$$

or

$$0<\theta\ln\left[1-\lambda\left(1-e^{\frac{k+2\delta}{\theta}}\left(\frac{\theta-(k+2\delta)}{\theta}\right)\right)\right]-\delta<k,$$

which is equivalent to the inequalities in (4.6) and gives our applicable interval for the tuning parameter. From (4.7), considering the domain of the natural log function, we see that (4.6) also satisfies

$$1-\mathrm{e}^{\frac{k+2\delta}{\theta}}\left(\frac{\theta-(k+2\delta)}{\theta}\right)>0$$

or

$$\lambda<\frac{1}{1-\mathrm{e}^{\frac{k+2\delta}{\theta}}\left(\frac{\theta-(k+2\delta)}{\theta}\right)}$$

since $\mathrm{e}^{\frac{\delta}{\theta}}>0$. Furthermore, we take the second partial derivative

$$\begin{aligned}f''(c)=\frac{T-2}{2\theta^2\left(1-\mathrm{e}^{\frac{k+2\delta}{\theta}}\left(\frac{\theta-(k+2\delta)}{\theta}\right)\right)}\Bigg[&\mathrm{e}^{\frac{-c+k+\delta}{\theta}}\\ &+\lambda\left(\mathrm{e}^{\frac{-c+2k+3\delta}{\theta}}\left(\frac{\theta-(k+2\delta)}{\theta}\right)-\mathrm{e}^{\frac{-c+k+\delta}{\theta}}\right)\Bigg],\end{aligned}$$

and following the same logic used when analyzing the first partial derivative, for $f''(c)>0$ we require

$$\mathrm{e}^{\frac{-c+k+\delta}{\theta}}\left(1-\lambda\left(1-\mathrm{e}^{\frac{k+2\delta}{\theta}}\left(\frac{\theta-(k+2\delta)}{\theta}\right)\right)\right)>0,$$

meaning

$$1-\lambda\left(1-\mathrm{e}^{\frac{k+2\delta}{\theta}}\left(\frac{\theta-(k+2\delta)}{\theta}\right)\right)>0,$$

or

$$\lambda<\frac{1}{1-\mathrm{e}^{\frac{k+2\delta}{\theta}}\left(\frac{\theta-(k+2\delta)}{\theta}\right)},$$

already satisfied by (4.6), once again due to the fact that $\mathrm{e}^{\frac{\delta}{\theta}}>0$. This provides certainty that $f(c^*)$ is convex, and to summarize, we have a proper solution to the optimization problem in (4.7), given λ is selected in the interval derived in

(4.6). $\square$

Focusing on the solution in detail, it is clear that c^* is decreasing in λ, consistent with the notion from prior chapters that λ is associated with income volatility management, thus an increased weight placed on it results in later recognition of credit risk. It is also intuitive to note that, as k increases, keeping θ and δ constant, the initial default probability decreases, since the distance to default becomes larger at the time of origination of the loan. To confirm, we compute the first partial derivative

$$\begin{aligned}\frac{d}{dk}P(A_T < 0) &= \frac{d}{dk}\left[1 - \mathrm{e}^{\frac{k+2\delta}{\theta}}\left(\frac{\theta - (k+2\delta)}{\theta}\right)\right]\\ &= \underbrace{\frac{\mathrm{e}^{\frac{k+2\delta}{\theta}}}{\theta}}_{>0}\underbrace{\frac{k+2\delta}{\theta}}_{<0} < 0,\end{aligned}$$

leading us to the conclusion that $P(A_T < 0)$ is indeed decreasing in k. From the initial assumptions of this subsection, we use the fact that $\theta > 0$ and $k+2\delta < 0$ to confirm the sign of the derivative. As $P(A_T < 0)$ decreases, we see from (4.7) that c^* increases, thus the overall solution is increasing in k.

The relationships of θ and δ with (4.7) are more complex, as they are interdependent on the values of the other parameters in their effect on c^*. However, using appropriate sets of parameter choices that result in a solution existing in the interval $0 < c^* < k$, some basic trends in these relationships can be observed. To demonstrate this notion visually, we fix $\lambda = 3$ and $k = 3.5$, and examine the effect of θ on c^* for different values of δ such that (4.6) is satisfied. In Figure 4.1, we indeed see that c^* is increasing in θ, regardless of δ, in the applicable interval relative to k, of $0 < c^* < 3.5$. Additionally, we notice that c^* is a concave function of δ, and the

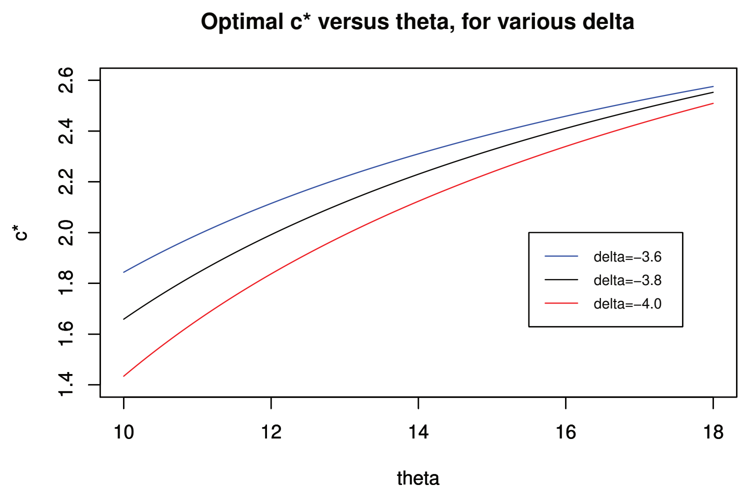

Figure 4.1: Optimal thresholds versus corresponding θ values, with $\lambda = 3$ and $k = 3.5$, for various values of the shift parameter δ.

increasing trend is less pronounced for larger values of θ. A logical explanation is that, in general, $E(A_{t_j}) = k + j(\theta + \delta)$ is increasing in θ, so if the mean net asset value over the time interval is greater, we expect that the obligor is less likely to default. This implies that the consequent initial default probability is lower, consistent with the increasing relationship between c^* and k. As for the association being weaker for higher θ, recall that $\text{Var}(A_{t_j}) = j\theta^2$ is also increasing in θ, meaning the net asset value is more volatile as this parameter value increases, providing simple reasoning as to why the curves "flatten out" in Figure 4.1. Likewise, using the same predetermined λ and k values, the relationship between δ and c^* is investigated, for various values of θ such that (4.6) is satisfied. Conclusively determining the influence of δ on c^* theoretically is also of considerable complexity. Figure 4.2 conveys that, for these particular parameter selections, c^* is also increasing in δ. To explain further, the shift parameter δ represents the worst case scenario, or maximal loss, for an obligor in a given time interval. Therefore, as it increases, it

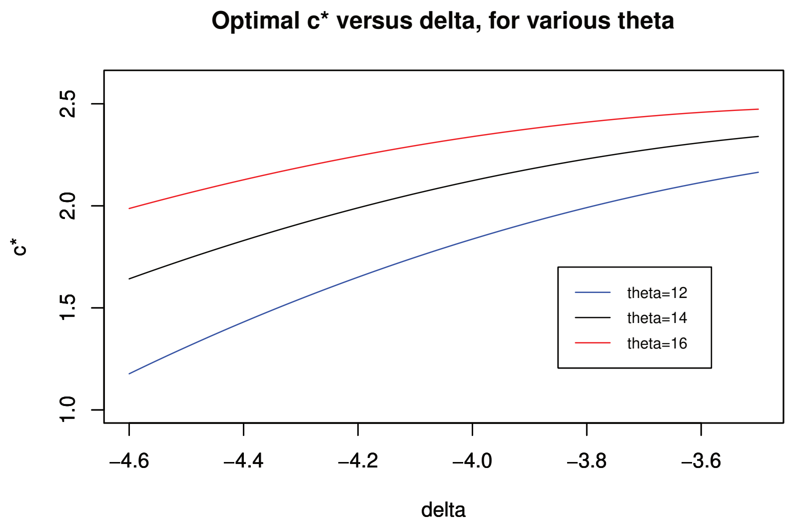

Figure 4.2: Optimal thresholds versus corresponding δ values, with $\lambda = 3$ and $k = 3.5$, for various θ values.

indicates the obligor is of higher credit quality, as they have less potential to have a profound loss. In other words, this obligor would also be "farther from default" in layman's terms, so c^* should also increase, using reasoning from other relationships already confirmed. Nevertheless, it is evident that c^* is a concave function of δ, with the increasing relationship becoming slightly weaker for higher θ values. To provide justifiable evidence that the relationship is not conclusive, we can actually find parameter selections such that (4.6) is satisfied, and c^* is decreasing in δ. Figure 4.3 reveals that in this instance, with $\theta = 100$, we indeed have a decreasing trend with respect to δ, with desired solutions existing in the appropriate interval $0 < c^* < k$. This counterintuitive observation is interesting, yet not overly relevant, as it typically corresponds to very large θ values and extremely low initial default probabilities. In fact, for the example in Figure 4.3, for $\delta = -3.6$, the resulting $P(A_T < 0) \approx 6.68 \cdot 10^{-4}$, a miniscule default risk. Lastly, to provide some context for these numerous parameters, we precisely set $\lambda = 3$, $k = 3.5$, $\theta = 14$,

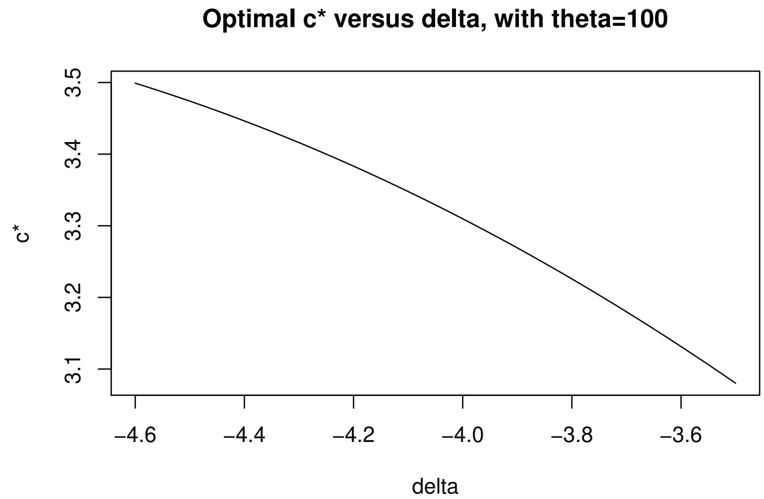

Figure 4.3: Optimal thresholds versus corresponding δ values, with $\lambda = 7$, $k = 3.5$ and $\theta = 100$.

and $\delta = -3.6$, so the initial default probability of the obligor is $P(A_T < 0) \approx 0.03$. In practice, this is a realistic default risk, and the solution for this particular set of parameter values is $c^* \approx 2.31$, which can be seen in Figure 4.4, where the optimal threshold values are plotted for various λ. The decreasing relationship between c^* and λ is evident.

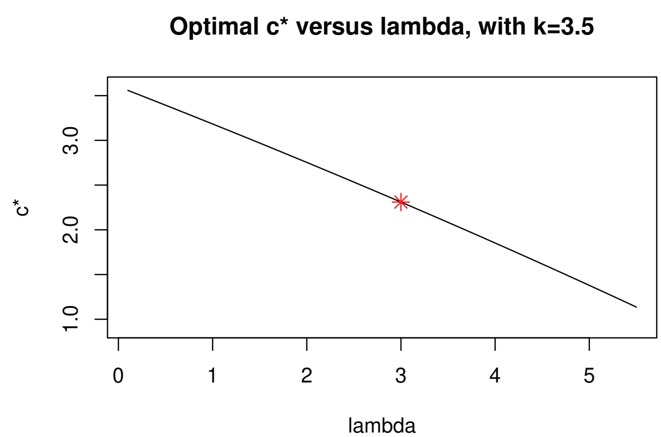

Figure 4.4: Optimal thresholds versus corresponding λ values, with $k = 3.5$, $\theta = 14$ and $\delta = -3.6$. A specific solution, for $\lambda = 3$ ($c^* \approx 2.31$) is labelled in red.

4.2.2 Modelling the net asset value with Brownian motion

We now consider the case of the net asset value process driven by Brownian motion. Concretely, we assume in this subsection that $(A_t - A_0)_{0\le t\le T}$ is a Brownian motion. With this modelling assumption, we can rephrase the objective function. We denote by Φ and φ the cumulative distribution function and probability density function, respectively, of a standard normally distributed random variable. Since $G_t(x) = \Phi\big(\frac{x}{\sqrt{t}}\big)$ in this case of Brownian motion, we immediately obtain from Proposition 3 the following result.

Proposition 5. *If* $(A_t - A_0)_{0\le t\le T}$ *is a Brownian motion, the function* $f(c)$ *in* (4.3) *can be written as*

$$\begin{aligned} f(c) = \sum_{j=1}^{N-1} (T-1-t_j) \frac{\int_{\frac{c-k}{\sqrt{t_j}}}^{\infty} \Phi\left(\frac{-k-x\sqrt{t_j}}{\sqrt{T-t_j}}\right) \varphi(x)\,dx}{\Phi\left(\frac{-k}{\sqrt{T}}\right)} \\ + \lambda \sum_{j=1}^{N-1} (T-1-t_j) \left(\int_{-\infty}^{\frac{c-k}{\sqrt{t_{j-1}}}} \left(1 - \Phi\left(\frac{c-k-x\sqrt{t_{j-1}}}{\sqrt{t_j - t_{j-1}}}\right)\right) \varphi(x)\,dx \right. \\ \left. + \int_{\frac{c-k}{\sqrt{t_{j-1}}}}^{\infty} \Phi\left(\frac{c-k-x\sqrt{t_{j-1}}}{\sqrt{t_j - t_{j-1}}}\right) \varphi(x)\,dx \right). \end{aligned} \tag{4.9}$$

Taking the partial derivative with respect to c in (4.9), we find that

$$\begin{aligned} f'(c) = \sum_{j=1}^{N-1} -\frac{(T-1-t_j)}{\sqrt{t_j}\,\Phi\left(\frac{-k}{\sqrt{T}}\right)} \left(\Phi\left(\frac{-c}{\sqrt{T-t_j}}\right) \varphi\left(\frac{c-k}{\sqrt{t_j}}\right)\right) \\ + \lambda \sum_{j=1}^{N-1} (T-1-t_j) \left[-\int_{-\infty}^{\frac{c-k}{\sqrt{t_{j-1}}}} \frac{\varphi(x)}{\sqrt{t_j - t_{j-1}}} \varphi\left(\frac{c-k-x\sqrt{t_{j-1}}}{\sqrt{t_j - t_{j-1}}}\right) dx \right. \\ \left. + \int_{\frac{c-k}{\sqrt{t_{j-1}}}}^{\infty} \frac{\varphi(x)}{\sqrt{t_j - t_{j-1}}} \varphi\left(\frac{c-k-x\sqrt{t_{j-1}}}{\sqrt{t_j - t_{j-1}}}\right) dx \right]. \end{aligned}$$

As these equations are quite elaborate and complicated, arriving at an analytical solution relative to (4.3) is not feasible, although numerical results are certainly attainable. To demonstrate an example of the objective function and its relevant minimum, consider a loan issued over ten years ($T = 10$) with yearly reporting dates ($N = 10$), assuming equal time increments of $(t_j - t_{j-1}) = 1$ for $j \in \{1, 2, \ldots, 10\}$. We fix the initial distance to default, k, such that $P(A_T < 0) = 0.05$. Accordingly, $P\left(\frac{A_T - A_0}{\sqrt{T}} < \frac{-k}{\sqrt{T}}\right) = 0.05$ and $k = -\sqrt{T}\,\Phi^{-1}(0.05) \approx 5.2$. We observe the shape of the objective function for some reasonably chosen λ values, and the respective solutions to the minimization problem, in Figure 4.5.

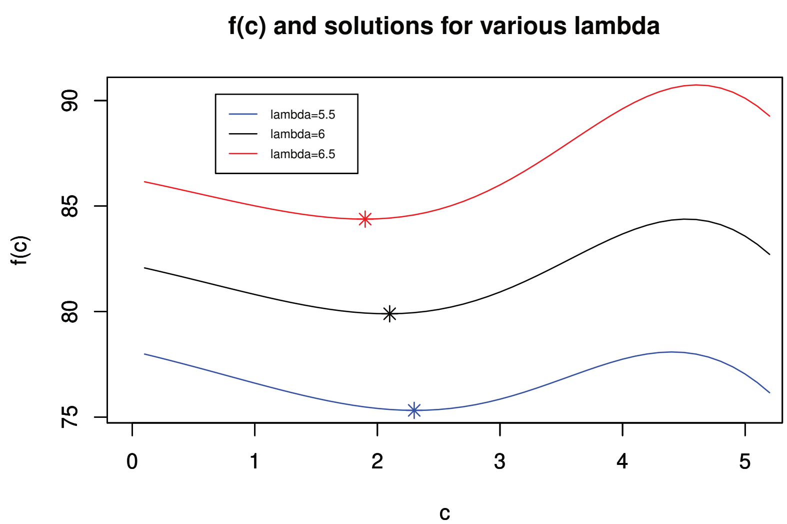

Figure 4.5: Plot of $f(c)$ for various λ, with $T = 10$, $N = 10$, and $k \approx 5.2$. The optimal thresholds, determined numerically, are $c^* \approx 2.32$ ($\lambda = 5.5$), $c^* \approx 2.07$ ($\lambda = 6$) and $c^* \approx 1.92$ ($\lambda = 6.5$).

Note that for this specific example, smaller values of the tuning parameter ($\lambda < 5$) produce solutions of $c^* \approx k \approx 5.2$, indicating the volatility portion is not properly weighted, since the early recognition of credit risk would be considered overwhelmingly important in that situation. Although an "ideal" selection of λ is very subjective in terms of balancing the significance of the two portions of $f(c)$, the choices in this example clearly result in solutions existing in the desired interval of $0 < c^* < k$. In addition, the decreasing relationship between c^* and λ is apparent, identical to the conclusion established in the previous subsection, where the shifted exponential distribution was used to model the net asset value.

Remark. We could study a model extension to a situation with multiple obligors, similar to Chapter 3. However, the analysis would involve an expression of the form

$$E\left[\left(\sum_{\ell=1}^{L}\left(\mathbb{1}_{A_{t_j}^{\ell}>c}-\mathbb{1}_{A_{t_{j-1}}^{\ell}>c}\right)\right)^2\right],$$

compared to $\mathrm{Var}\big(\sum_{\ell=1}^{L}\mathbb{1}_{A_t^{\ell}>c}\big)$ from Chapter 3, where A^{ℓ} denotes the net asset value of obligor $\ell = 1,\dots,L$. Because of this different expression, the study becomes more convoluted without providing new conceptual insights, so that we refrain from spelling out the details, but we note that the resulting optimization problem for this discrete-time setting with multiple obligors could still be analyzed numerically.

4.3 Analyzing the optimization problem for discrete asset distribution

4.3.1 Specific increments

In a discrete-time framework, we can consider a situation where the increments $A_{t_j} - A_{t_{j-1}}$ are explicitly defined in terms of assuming possible values, along with their corresponding transition probabilities. In particular, we again consider the case where $T = t_N = t_2$, letting $t_1 = \frac{T}{2}$ be the midpoint between the time the loan is issued and time of maturity. Consistent with prior formulations, $A_{t_0} = A_0 = k > 0$, and in this instance the optimization problem corresponding to (4.3) is simplified as

$$\min_{c\leq k} f(c),$$

where

$$\begin{aligned} f(c) = {} & \frac{T-2}{2} P\big(A_{\frac{T}{2}} > c \mid A_T < 0\big) \\ & + \lambda \frac{T-2}{2}\Big(P\big(A_{\frac{T}{2}} > c, A_0 \leq c\big) + P\big(A_{\frac{T}{2}} \leq c, A_0 > c\big)\Big), \end{aligned}$$

equivalent to

$$f(c) = \frac{T-2}{2} P\big(A_{\frac{T}{2}} > c \mid A_T < 0\big) + \lambda \frac{T-2}{2}\Big(P\big(A_{\frac{T}{2}} > c, k \leq c\big) + P\big(A_{\frac{T}{2}} \leq c, k > c\big)\Big), \tag{4.10}$$

assuming again that $\lambda > 0$. The increments are independent, with fixed transition probabilities

$$P\big(A_{\frac{T}{2}} - A_0 = b_1\big) = P\big(A_T - A_{\frac{T}{2}} = b_2\big) = p_1 > 0.5,$$

$$P\big(A_{\frac{T}{2}} - A_0 = -b_1\big) = P\big(A_T - A_{\frac{T}{2}} = -b_2\big) = 1 - p_1 < 0.5.$$

Here, we do not require identical distribution of the two increments, but impose $b_2 > k > b_1 > 0$. We also specify $k + b_1 - b_2 < 0 < k - b_1 + b_2$, which implies the initial default probability $P(A_T < 0) = (1-p_1)^2 + p_1(1-p_1) = 1 - p_1 < 0.5$. Therefore, relative to a specific obligor, we select $p_1 = 1 - P(A_T < 0)$. Ultimately, the solution to the optimization problem, in terms of possible intervals for c^*, can be expressed as a piecewise function dependent on λ. We do consider the endpoints of the interval $0 \leq c^* \leq k$ in this instance. The objective function is

$$f(c) = \begin{cases} \frac{T-2}{2} & \text{if } 0 \leq c < k - b_1 \\ \frac{T-2}{2}\big(p_1 + \lambda(1-p_1)\big) & \text{if } k - b_1 \leq c < k \\ \frac{T-2}{2} p_1(1+\lambda) & \text{if } c = k \end{cases}$$

and since the coefficient $\frac{T-2}{2}$ appears in each scenario, we examine all the other coefficients as a means of solving for the minimum of $f(c)$. First, we note that

$$p_1 + \lambda(1-p_1) < p_1(1+\lambda)$$

regardless of the value of λ, since $1 - p_1 < p_1$ for our specified $p_1 > 0.5$. Therefore, we are certain that a solution of $c^* = k$ will not be chosen. Secondly, we see that $p_1 + \lambda(1-p_1) < 1$ if $\lambda < 1$, for any $p_1 > 0.5$, leading to a clear solution to the minimization problem in terms of optimal c^* selection intervals:

$$\text{select} \begin{cases} k - b_1 \leq c^* < k & \text{if } \lambda < 1 \\ \text{any } 0 \leq c^* < k & \text{if } \lambda = 1 \\ 0 \leq c^* < k - b_1 & \text{if } \lambda > 1 \end{cases}$$

However, the instance when $\lambda = 1$ provides us with no information other than $c^* \neq k$, while the case with $0 \leq c^* < k - b_1$ allows no opportunity to observe a possible IFRS 9 significant increase in credit risk prior to an eventual default. To conclude, choosing $0 < \lambda < 1$, resulting in a solution of $k - b_1 \leq c^* < k$, is most feasible in this setting.

4.3.2 General increments

In this subsection, we still consider just two reporting dates, $\frac{T}{2}$ and T, but in each period we have K different possible values for the increments $A_{t_j} - A_{t_{j-1}}$, including $b_1, \ldots, b_K$, comprised of both positive and negative integers, with relevant probabilities $p_1, \ldots, p_K$. A general discrete random variable is modelled in each step, where we still assume that the changes in the two time steps are independent and identically distributed. In particular, we explicitly let X and Y be the independent random variables, in time order, so that $A_{\frac{T}{2}} = k + X$ and $A_T = k + X + Y$. To evaluate the objective function in this instance, starting with the recognition portion of $f(c)$, we compute

$$
\begin{aligned}
P\left[A_{\frac{T}{2}} > c | A_T < 0\right] &= \frac{P\left[A_{\frac{T}{2}} > c, A_T < 0\right]}{P[A_T < 0]} \\
&= \frac{P[X > c - k, X + Y < -k]}{P[X + Y < -k]} \\
&= \frac{P[\bigcup_{n=1}^K \{X = b_n\} \cap \{X > c - k\} \cap \{X + Y < -k\}]}{P[\bigcup_{n=1}^K \{X = b_n\} \cap \{X + Y < -k\}]} \qquad (4.11) \\
&= \frac{\sum_{n=1}^K P[\{X = b_n\} \cap \{X > c - k\} \cap \{X + Y < -k\}]}{\sum_{n=1}^K P[\{X = b_n\} \cap \{X + Y < -k\}]} \\
&= \frac{\sum_{n=1}^K P[\{X = b_n\} \cap \{b_n > c - k\} \cap \{b_n + Y < -k\}]}{\sum_{n=1}^K P[\{X = b_n\} \cap \{b_n + Y < -k\}]} \\
&= \frac{\sum_{\{n: b_n > c - k\}} P[\{X = b_n\} \cap \{b_n + Y < -k\}]}{\sum_{n=1}^K P[\{X = b_n\} \cap \{b_n + Y < -k\}]} \\
&= \frac{\sum_{\{n: b_n > c - k\}} p_n P[Y < -k - b_n]}{\sum_{n=1}^K p_n P[Y < -k - b_n]} \qquad (4.12) \\
&= \frac{\sum_{\{n: b_n > c - k\}} p_n \sum_{\{\ell: b_\ell < -k - b_n\}} p_\ell}{\sum_{n=1}^K p_n \sum_{\{\ell: b_\ell < -k - b_n\}} p_\ell},
\end{aligned}
$$

using that the events in (4.11) are disjoint, and independence of X and Y in (4.12). Under the assumption that $A_0 = k > c$, from the volatility portion of $f(c)$, we have that

$$
\begin{aligned}
P\left(A_{\frac{T}{2}} \leq c\right) &= P[X \leq c-k] \\
&= \sum_{n=1}^{K} P[\{X = b_n\} \cap \{X \leq c-k\}] \\
&= \sum_{\{n: b_n \leq c-k\}} P[\{X = b_n\}] \\
&= \sum_{\{n: b_n \leq c-k\}} p_n \,,
\end{aligned}
$$

resulting in (4.10) being expressed in this setting as

$$
\begin{aligned}
f(c) &= \frac{T-2}{2} \frac{\sum_{\{n: b_n > c-k\}} p_n \sum_{\{\ell: b_\ell < -k-b_n\}} p_\ell}{\sum_{n=1}^{K} p_n \sum_{\{\ell: b_\ell < -k-b_n\}} p_\ell} + \lambda \frac{T-2}{2} \sum_{\{n: b_n \leq c-k\}} p_n \\
&= \frac{T-2}{2} \left[\frac{\sum_{\{n: b_n > c-k\}} p_n \sum_{\{\ell: b_\ell < -k-b_n\}} p_\ell}{\sum_{n=1}^{K} p_n \sum_{\{\ell: b_\ell < -k-b_n\}} p_\ell} + \lambda \sum_{\{n: b_n \leq c-k\}} p_n \right] \\
&= \frac{T-2}{2} \left[\frac{\sum_{\{n: b_n > c-k\}} p_n \sum_{\{\ell: b_\ell < -k-b_n\}} p_\ell}{\sum_{n=1}^{K} p_n \sum_{\{\ell: b_\ell < -k-b_n\}} p_\ell} \right. \\
&\qquad \left. + \frac{\sum_{n=1}^{K} p_n \sum_{\{\ell: b_\ell < -k-b_n\}} p_\ell \left(\lambda \sum_{\{n: b_n \leq c-k\}} p_n\right)}{\sum_{n=1}^{K} p_n \sum_{\{\ell: b_\ell < -k-b_n\}} p_\ell} \right].
\end{aligned}
$$

Notice that the expression for $f(c)$ depends on c only in the index sets in two of the summations. This implies that $f(c)$ is piecewise constant with jumps when $c = b_n + k$ for $n = 1, \ldots, N$. Moreover, the sum $\sum_{\{n: b_n > c-k\}} p_n$ is decreasing in c, as it relates to the early recognition of credit risk. On the contrary, the sum $\sum_{\{n: b_n \leq c-k\}} p_n$ in the second term of the numerator of the objective function, is increasing in c, being part of the volatility portion. As a result, the optimal c^* is chosen in an interval between $b_n + k$ and $b_{n+1} + k$ such that the expression for $f(c)$ on this interval is minimal compared to values on other intervals. Therefore, the solution is similar to that of Subsection 4.3.1, but more intricate and cumbersome. Due to the complexity of this expression, which depends on the exact distance to default along with the incremental and probability values, a computer program is required to select the minimum numerically. For each n, it will select c_n with $b_n + k \leq c_n < b_{n+1} + k$ and compute $f(c_n)$. Because f is constant on such intervals, the specific choice of c_n within the interval does not affect the value of $f(c_n)$. Among all $f(c_n)$ for different n, the program will choose the one with the smallest value, say $f(c_{n^*})$, as the minimum.

4.4 Relating our model to the European Union stress test and Standard and Poor's default data

For this section, we use the assumption from Subsection 4.2.2 regarding the net asset value modelling, such that $(A_t - A_0)_{0 \leq t \leq T}$ is a Brownian motion, and apply it in practical settings.

4.4.1 Selecting λ in comparison to the European Union framework

Similar to the procedure in Section 2.2, to rationalize possible selections of λ, we relate the tuning parameter choice in this discrete-time formulation to the EU stress test definition of a significant increase in credit risk. Recall that the stress test framework of the European Banking Authority (2018) defines a significant increase in credit risk as being 200% from the initial credit risk. We maintain that a^* is the critical value so that $P(a^* + W_{T-t} < 0) = 3\,P(A_T < 0)$, and compare $P(a^* + W_{T-t} < 0)$

with its relative value $P(c^* + W_{T-t} < 0)$ for our optimal threshold c^*. For different initial default probabilities, the minimizers are determined numerically using the objective function in (4.9). Figure 4.6 displays the relationship between the initial lifetime default probability $P(A_T < 0)$ and the critical lifetime default probability $P\left(c^* + W_{\frac{T}{2}} < 0\right)$, at the midpoint $t = \frac{T}{2}$. As expected, a positive relationship between

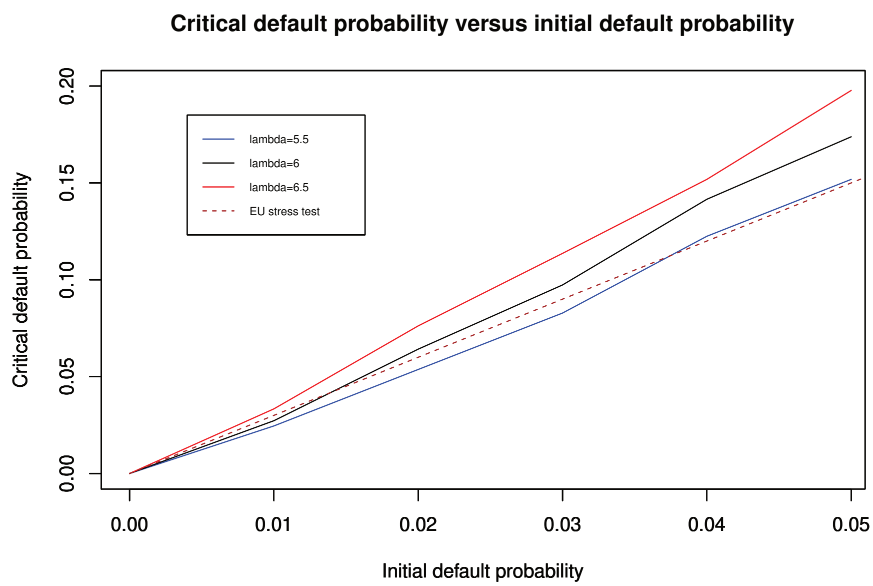

Figure 4.6: Plot of $P\left(c^* + W_{\frac{T}{2}} < 0\right)$ versus $P(A_T < 0)$, for various λ, with $T = 10$. The dashed line shows the relationship outlined by the EU stress test, which is linear by definition.

the critical lifetime default probability and the initial lifetime default probability is observed for each λ value displayed. Evidently, it is reasonable to conclude that, for this ten-year loan, the value $P(a^* + W_{T-t} < 0)$ from the stress test threshold corresponds to a parameter choice of λ between 5.5 and 6. When comparing this result to the λ selections investigated in Section 2.2, slightly lower tuning parameter values are appropriate in the continuous-time setting. As the two volatility portions we introduced, (2.2) and (4.2), differ in terms of their interpretations of ECL instability for a bank, it is not surprising that the two formulations do not produce exactly the same results. Nonetheless, the precise value of λ will always be subjective, so we simply verified that our discrete-time problem formulation is indeed also comparable to a practical framework.

4.4.2 Analyzing the income volatility portion with default data by Standard and Poor's

To directly relate our model to recent financial data, we use Standard and Poor's (2018) global corporate average default rates and yearly credit grouping transition rates. Each defined credit rating group has assigned one-year and lifetime probabilities of default, based on corporate averages from 1981 to 2017, which is relevant for any future length of time up until maturity of a loan. Additionally, yearly transition probabilities to all possible credit rating groups are available. From this information, evaluating what percentage of loans experience a significant increase in credit risk, depending on the threshold c^*, is critical in terms of determining the severity of income volatility for a bank in a particular year.

As far as integrating this data into our problem formulation, we first specify an initial rating group, and subsequently observe the performance at time $t = 1$ year of a loan issued over $T = 10$ years. The transition rates reveal proportions of loans in each respective rating group possessing lower credit quality, one year since origination. Furthermore, under the modelling structure of Subsection 4.4.1, possible threshold solutions are approximated using $P(c^* + W_{T-t} < 0) = P(c^* + W_{10-1} < 0) = P(c^* + W_9 < 0)$, in which nine-year probabilities of default over the remaining time to maturity are used, relative to the current credit rating group at time $t = 1$. For various c^* values defined by the cutoff points described by progressive downgrades

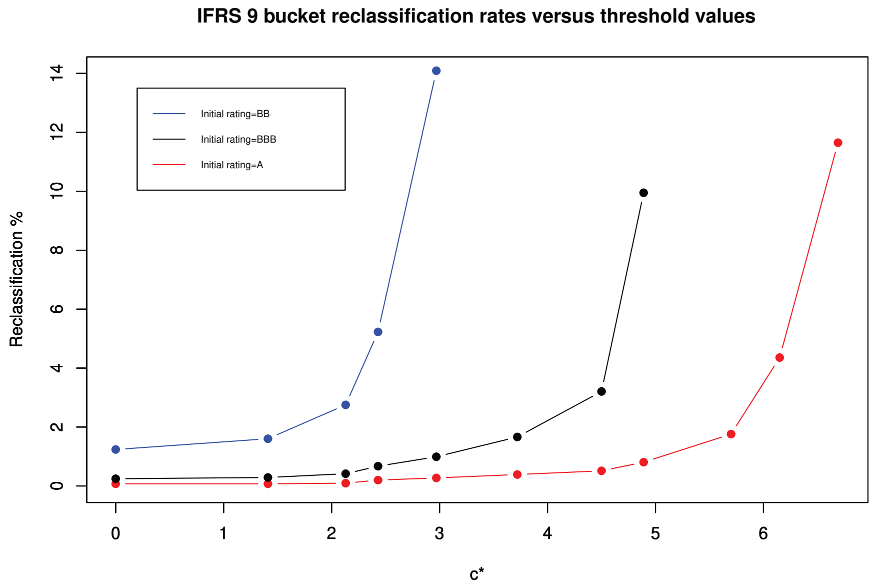

Figure 4.7: Plot of reclassification percentage from IFRS 9 bucket 1 to buckets 2 or 3, after year one of a ten-year loan, for different initial ratings.

in credit rating groups, the yearly transition rates reveal the percentage of loans that would transfer to an IFRS 9 bucket of worsened performance status. Assuming each loan examined originated in bucket 1, and that $c^* < k$, Figure 4.7 displays the influence of the threshold level on the IFRS 9 bucket reclassification rate, for three different initial rating groups. With an initial rating group of A, the initial distance to default for a ten-year loan is $k = -\sqrt{10}\Phi^{-1}(0.0147) \approx 6.89$, and if we attribute, for example, a significant increase in credit risk over a one-year period being a downgrade to rating BBB or worse, then $c^* = -\sqrt{9}\Phi^{-1}(0.0285) \approx 5.71$. From our illustration, it is evident that the loan reclassification rate from one-year ECL to lifetime ECL is more profound for lower initial credit rating groups (such as BB), meaning these obligors that have a higher risk of non-payment also contribute significantly to increased income volatility for the bank that issues their loans.

The actual calculation of ECL involves sophisticated models for each component, including PD, EaD, and LGD for a credit facility. For demonstration purposes, a proxy ECL calculation can be created to exemplify the relationship between ECL and IFRS 9 loan reclassification, incorporating the same default data from Standard and Poor's (2018). We still consider the case when $T = 10$, and assess three different samples of 100 similar loans having the same initial and current rating group, namely ratings A, BBB, and BB for the three samples, as of the midpoint of $T = 5$ years. After year 5, we have three samples of 100 loans each, with one sample experiencing a common migration from rating group A to BBB, the second showing a deterioration from rating group BBB to BB, and the third having a transition from rating group BB to B. In summary, all 100 obligors in a particular sample have undergone a credit quality downgrade of three notches. We then fix EaD = \$500,000 and LGD = 0.1, and use both one-year and lifetime PDs (for the five remaining years to maturity) to calculate proxy ECLs, simply from the product of the three components. Figure 4.8 emphasizes the difference between one-year and lifetime ECL for each sample from an initial rating group, assuming all loans originated in IFRS 9 bucket 1. Essentially, if a downgrade of three rating groups is

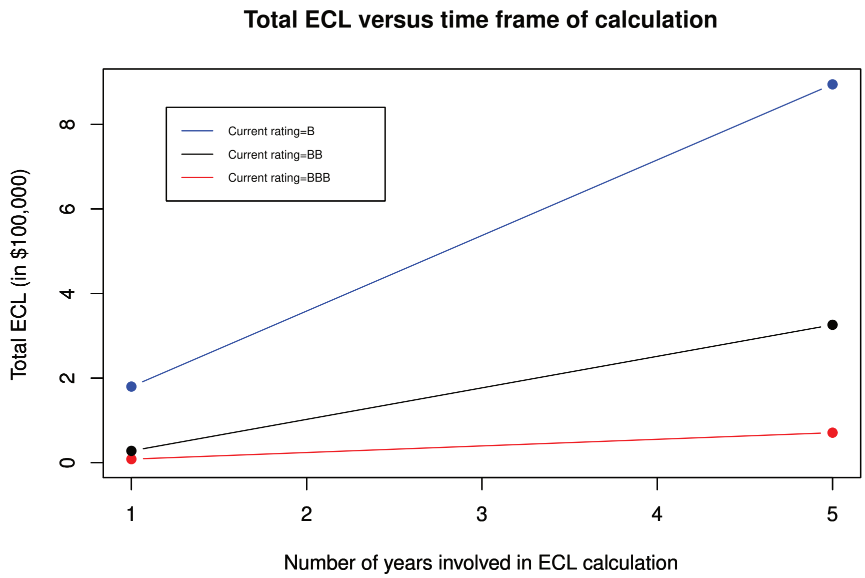

Figure 4.8: Plot of total ECL versus number of years involved in the calculation, for samples of 100 loans grouped in the same rating at the midpoint of a ten-year loan.

deemed sufficient enough to constitute a significant increase in credit risk, lifetime (five-year) ECL is computed. Otherwise, ECL for a one-year horizon is calculated. Consistent with the interpretation of Figure 4.7, we observe that the effect of reclassification from bucket 1 to bucket 2 is considerably more drastic for inferior rating groups. Clearly, the total ECL for a sample of just 100 loans increases sharply, by over $700,000, when current rating group B, corresponding to initial rating group BB, is classified in IFRS 9 bucket 2 rather than bucket 1.

Chapter 5

Conclusion

While the delayed recognition of credit losses, particularly in the United States subprime mortgage market, was heavily scrutinized following the global financial crisis of 2007–2008, the actual solution to this concern involves a complex trade-off involving income volatility. The recent introduction of the accounting standard IFRS 9 specifies that lifetime ECL should be estimated for loans that have experienced a significant increase in credit risk, as compared to one-year ECL. Although additional provisions address the importance of the early recognition of credit risk, the exact thresholds set by banks for IFRS 9 loan reclassification directly affect the stability, or lack thereof, of their ECL calculations.

Modelling this trade-off statistically, and incorporating both continuous-time and discrete-time formulations, allows us to examine the optimal threshold selection and its dependence on various factors. Since each relevant model analyzed is influenced by a predetermined initial default risk, our approach can be applied appropriately to an extensive variety of loans in different portfolios. By considering a problem variation that accounts for multiple obligors in a credit rating group, we determined how other critical factors, such as asset correlation and rating bucket size, impact the optimal threshold.

To supplement the knowledge obtained from the numerous parameters and relative solutions to our optimization problem, the applicability of the model structure is demonstrated through a practical definition as well as recent default data. The evaluation of several optimal threshold values, in comparison to the European Banking Authority (2018) stress test, conveys that our framework is effective in comparing banks' different definitions of what comprises a significant increase in credit risk. The immediate consequence of establishing a precise threshold, in association with this statistical model, is illustrated with Standard and Poor's (2018) historical default data, clearly capturing the relevance of the volatility portion from the inherent objective function. Regardless, our framework not only provides solutions to this intricate trade-off, but reveals the realistic subjectivity involved in balancing the early recognition of credit risk with income volatility in a suitable manner.

Bibliography

Basel Committee on Banking Supervision. An explanatory note on the Basel II IRB risk weight functions. Bank for International Settlements, **2005**, available at www.bis.org.

Basel Committee on Banking Supervision. Guidance on credit risk and accounting for expected credit losses. Bank for International Settlements, **2015**, available at www.bis.org.

Bluhm, C. and Overbeck, L. Systematic risk in homogeneous credit portfolios. In: Bol, G., Nakhaeizadeh, G., Rachev, S., Ridder, T. and Vollmer, K.-H. (eds.): Credit Risk: Measurement, Evaluation and Management (Contributions to Economics), Springer, **2003**, pp. 35–48.

Bluhm, C., Overbeck, L., and Wagner, C. *An Introduction to Credit Risk Modeling*. Chapman & Hall/CRC, 2nd edition, **2010**.

Bohn, J. and Stein, R. *Active Credit Portfolio Management in Practice*. Wiley Finance, **2009**.

Cohen, B.H. and Edwards Jr, G.A. The new era of expected credit loss provisioning. *BIS Quarterly Review*, **2017**, available at www.bis.org.

European Banking Authority. Report on results from the second EBA impact assessment of IFRS 9, **2017**, available at www.eba.europa.eu.

European Banking Authority. 2018 EU-Wide Stress Test. European Union, **2018**, available at www.eba.europa.eu.

Ewanchuk, L. and Frei, C. Recent regulation in credit risk management: a statistical framework. *Risks* **2019**, *7*(2), 40.

Frei, C. and Wunsch, M. Moment estimators for autocorrelated time series and their application to default correlations. *Journal of Credit Risk* **2018**, *14*(1), 1–29.

Gordy, M. A comparative anatomy of credit risk models. *Journal of Banking and Finance* **2000**, *24*(1–2), 119–149.

International Accounting Standards Board. IASB completes reform of financial instruments accounting. **2014**, available at www.ifrs.org

Kallenberg, O. Foundations of Modern Probability, 2nd Edition, Springer, **2002**.

Maggi, F., Natale, A., Pepanides, T., Risso, E., and Schrock, G. IFRS 9: A silent revolution in banks' business models. *McKinsey & Company*, **2017**.

McNeil, A., Frey, R., and Embrechts, P. *Quantitative Risk Management: Concepts, Techniques and Tools*. Revised Edition. Princeton University Press, **2015**.

Merton, R. On the pricing of corporate debt: the risk structure of interest rates. *Journal of Finance* **1974**, *29*(2), 449–470.

Miu, P. and Ozdemir, B. Stress-testing probability of default and migration rate with respect to Basel II requirements. *Journal of Risk Model Validation* **2009**, *3*(4), 1–36.

Miu, P. and Ozdemir, B. Adapting the Basel II advanced internal-ratings-based models for International Financial Reporting Standard 9. *Journal of Credit Risk* **2017**, *13*(2), 53–83.

Randall, M. and Thompson, S. IFRS 9 impairment: significant increase in credit risk. *PwC In depth*, **2017**, available at www.pwc.com.

Revuz, D. and Yor, M. *Continuous Martingales and Brownian Motion*, Springer, Berlin, 3rd edition, **1999**.

Simons, D. and Rolwes, F. Macroeconomic default modeling and stress testing. *International Journal of Central Banking* **2009**, *5*(3), 177–204.

Standard and Poor's. Default, transition, and recover: 2017 annual global corporate default study and rating transitions. **2018**, available at www.spratings.com.

Vasicek, O. The distribution of loan portfolio value. *Risk* **2002**, *15*(12), 160–162.

Yang, B.H. and Du, Z. Stress testing and modeling of rating migration under the Vasicek model framework: empirical approaches and technical implementation. *Journal of Risk Model Validation* **2015**, *9*(2), 33–47.