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Paris' Law Parameters Estimation for Fatigue Crack Prediction of An Aluminum Alloy Plate Under Cyclic Loading

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Abstract—Fatigue-induced crack initiation and growth are common in cyclically loaded structures. Crack reduces the structures' strength and increases the risk of failure. Engineers have been using degradation models like Paris' law to relate crack length to the magnitude of the applied load to schedule repair and maintenance of such structures. The presented study developed a methodology to estimate the Paris' law parameters that can be used to predict the fatigue lifetime of 2024-T3 aluminum alloy plate under cyclic loading. An optimization method is also developed to optimize Paris' law parameters and their standard deviation, making the model more reliable. At the same time, the case of unavailable magnitude of the applied load is considered in this research. The optimized parameters are further updated using Bayesian updating with the help of condition monitoring data to increase the accuracy of crack length estimation. Virkler crack propagation data for an aluminum alloy plate is used to develop and validate the proposed method. The experimental samples' validation results show that our model's average error or structure's lifetime prediction based on crack length is 1.5% when the crack on validation samples reaches 71% of the failure threshold.

Keywords-component; Paris' law, Probabilistic fatigue crack prediction, Genetic Algorithm, Bayesian Updating.

I. INTRODUCTION

Fatigue is a phenomenon of initiation and growth of damage in a structure under cyclic loading [1]. Fatigue-induced crack is the most common damage observed in structures made out of metal and alloy [2]. This type of crack may initiate in the elastic load limit and gradually grow as the number of loading cycles increases [3]. The growth of the crack reduces the load tolerance capacity of the structure and puts the structure at risk of failure. Often the structure with crack does not fail immediately; instead, the crack grows until it reaches a critical limit, and then the structure fails [4]. If the growing crack in the structure is not detected early, its failure can cause damage to life and property. The study of fatigue cracks can help prevent sudden failures in machines and structures; therefore, it is important to study the crack propagation.

The fatigue crack prediction process is usually performed with the help of a crack propagation model. One widely used and frequently studied model is Paris' law [5]. Paris' law crack propagation model is used to estimate the crack length using a mathematical relation between current crack length and future crack length as shown in Eq. (1) where, m and C are the model parameters to be tuned, the term $\frac{da}{dN}$ is the crack growth rate with a being crack length and N being the loading cycle, the parameter S is the load parameter whose value is obtained from the applied load, and the term Y is the geometric correction factor.

$$\frac{da}{dN} = C(SY\sqrt{a\pi})^m. \quad (1)$$

The main challenge with using Paris' law is estimating proper values of its parameters because there are uncertainties in the crack propagation path [6]–[8]. The crack propagation path depends on factors such as manufacturing defects, varying environmental conditions, load uncertainties [9]–[11]. Changes in those factors can alter the propagation path. In this situation, estimating one proper set of Paris law parameters is challenging. If the values of the Paris law parameters are improper, the predicted fatigue crack length will face a significant error.

This study works on fatigue crack prediction of aluminum alloy plates under cyclic loading. The study aims to address the identified shortcomings in existing research: weakness of optimization strategies that might generate a large standard deviation of parameters and lack of investigation in the case of unavailable magnitude of the applied load. This study tries to find the proper standard deviation of Paris' law parameters during optimization to obtain an accurate prior distribution of Paris' law parameters. The Bayesian updating method is further used to improve those parameters. This study also systematically estimates the value of load parameter in Paris' law for the case of unavailable load magnitude.

II. LITERATURE REVIEW

Researchers have presented different methods to calculate the Paris' law parameters. Many of those methods have quantified the randomness in crack propagation path using a probabilistic Paris' law method [9], [10]. The parameters of the probabilistic form of Paris' law are assumed to follow a distribution such as normal and lognormal distribution [11] [12]. Many papers like [7], [9], [10] that made such assumptions have developed two-step methods, which is parameter optimization followed by the Bayesian updating, to estimate Paris' law parameters. Optimization is performed

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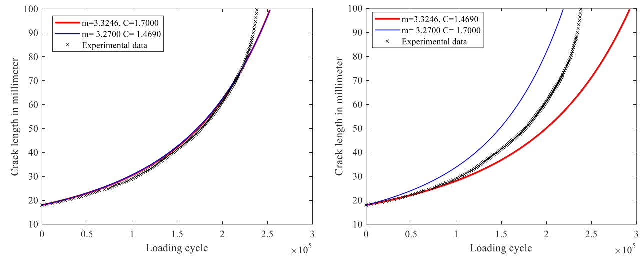
offline in those methods, while Bayesian updating is performed online when new evidence from condition monitoring is available.

A. Review of Parameters Optimization

Different offline optimization methods are available to estimate the probability distribution of Paris' law parameters. Anis Ben Abdessalem[9] minimized the sum of squared error in predicting crack length using the simulated annealing method. Xu Du et al. [7] maximize the log-likelihood of the integrated Paris' law equation using the maximum likelihood method. Similarly, Fuqiong Zhao, Zhigang Tian, Eric Bechhoefer and Yong Zeng[10] used regression fitting to optimize the parameters. A similar strategy is observed in all three papers. They have multiple experimental crack propagation paths available, and they considered each path individually for their optimization. The obtained values of the parameters are further used to form the distribution of the parameters. The methodology presented in the abovementioned papers estimated one set of parameters to represent a crack propagation path. However, some other papers, such as [13], [14], suggest that Paris' law parameters are dependent on each other, and different combinations of those parameters may give the same path.

The condition mentioned above, i.e. different combinations of Paris' law parameters, may give the same path is tested using the crack propagation data from [6]. Two different combinations of parameters are obtained after performing the optimization twice on the same crack propagation path. In Fig. 1, (a)., the red curve is plotted using obtained parameters m_1, C_1 and the blue curve is plotted using obtained parameters m_2, C_2. It is observed that both curves are very close to each other. Since the value of m_1, m_2 and C_1, C_2 are also very close, we form the cross combination m_1, C_2 and m_2, C_1 and plotted red and blue curves with respective parameters in Fig. 1, (b). The two curves obtained in the second figure are distinct enough to show that they do not represent the same crack propagation trend. From the illustration, it can be concluded that the existing methods which didn't consider the possibility of multiple combinations of parameters may generate a large standard deviation of the parameters' distributions. Therefore, further study is needed to obtain the best possible value for the standard deviation of m and C parameters to illustrate the uncertainty of crack length estimation. A better standard deviation of parameters would form a more accurate prior distribution, increasing the efficiency of the online parameters updating process.

In addition, there is no proper investigation to deal with the situation of unavailable magnitude of the applied load. As mentioned earlier, Paris' law has load parameter S in its equation that takes the value of the magnitude of the applied cyclic load. Usually, the magnitude of applied load is available, like in [9], [10]. However, there can be situations when the magnitude of applied load is unknown. Dealing with an unknown magnitude of applied load is not properly investigated in existing methods, and some methods have even neglected the load value in Paris' law equation [10], [19]. Therefore, further investigation is required to identify a proper value of the load parameter for unavailable load magnitude.



(a) (b)

Figure 1. (a). two combinations of parameters give one path. (b). Cross combination of parameters from 'Figure 1. a' give distinct paths

B. Review of Bayesian Updating of Parameters

Bayesian updating updates the probability of a hypothesis whenever new evidence is available. In recent papers, online Bayesian updating methods with the help of condition monitoring data are used to improve the accuracy of estimated parameters. Bayes' theorem is used for Bayesian updating given by Eq. 2 [16]. In Eq.2, prior information is multiplied by likelihood to get an updated posterior. Prior is the distribution obtained from optimization, and likelihood is the conditional probability of new evidence from condition monitoring data given the prior. After updating, the means of the Paris' law parameters move towards an accurate value while their standard deviations decrease.

$$Posterior \propto Likelihood \times Prior \qquad (2)$$

Many papers used Markov Chain Monte Carlo (MCMC) sampling algorithm to estimate posterior distribution [7], [10], [13], [15], [17]–[19]. MCMC is a popular simulation method used in engineering analysis to generate samples from an unknown distribution [20]. One major concern about using MCMC is the computational time[19]. In [19], Fuqiong Zhao tries to reduce the computational time of the MCMC process for Bayesian updating of Paris' law by introducing the Polynomial chaos expansion method to the calculation of likelihood. That method by Fuqiong Zhao [19] can estimate posterior distribution by generating fewer samples than required for MCMC, which reduces the computational time from 43 hours per update to 17 minutes per update.

An alternative to MCMC is proposed by A. A. Ben [9], who uses the Nested Sampling (NS) method instead of MCMC to generate samples from the posterior distribution, and they conclude that NS can be an alternative method to MCMC, but they do not claim improvement in computational time.

Moreover, some papers have estimated posterior using the Bayes' theorem [14], [21], [22]. Bayes' theorem is used to estimate the posterior of one parameter in [21]. In [22], both m and C parameters are individually updated using Bayes' theorem. Similarly,[14] used Bayes' theorem and performed individual updating of parameters and updating the joint distribution of Paris' law parameters. Those papers show that the use of Bayes' theorem for Paris' law parameters updating is a good alternative to computationally heavy algorithms like

MCMC and NS. Thus, the concept of using Bayes' theorem for updating is adopted in this study.

III. PROPOSED METHODOLOGY

This study proposes a method to estimate Paris' law parameters effectively. The proposed methodology is based on the Paris' law parameter estimation framework from [10], [19], [21]. The detailed flowchart of the proposed methodology is illustrated in Fig. 2. This framework consists of two parts. The first part is optimization, an offline method to find the optimum value of Paris's law parameters. The second part is online updating of the estimated parameters with the help of condition monitoring data using the Bayesian Theorem. The left three blocks in the figure represent the optimization method to estimate Paris' law's parameters (*m*, *C*, and *S*). The estimated values of parameters from the optimization are used to form a joint distribution which is used as prior distribution for the Bayesian updating method with condition monitoring data. After updating, the updated parameters are used in lifetime prediction.

The flowchart in Fig. 2 has data inputs in two places, i.e., crack length history data for Paris' law parameters optimization method and condition monitoring data being used for the Bayesian updating method. Both of those data consist of crack length and respective loading cycles. The crack length history data is the history of crack propagation, whereas condition monitoring data denotes new observation. In this study, the proposed method is validated using data of accelerated fatigue test experiments from [6]. This experimental data contains crack length vs respective loading cycles, also called crack propagation path, of 68 samples. Among the 68 crack propagation paths, 63 paths are used for training, and the remaining are reserved for validation. The training data are used in the optimization of Paris' law parameters, and validation data are used for the validation during Bayesian updating.

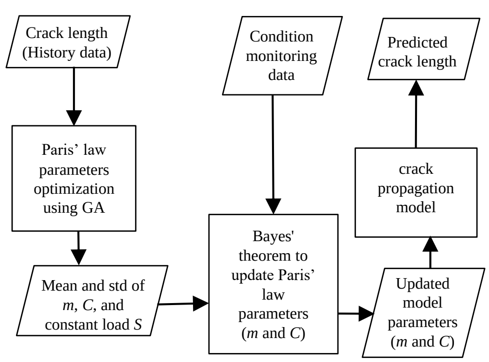


Figure 2. Proposed methodology

A. *Optimization of Paris' Law Using GA*

This study presents an optimization method to estimate three parameters *m*, *C*, and *S* of Paris' law. The optimization method considers multiple crack propagation paths together during optimization and aims to optimize the Paris' law parameters and their standard deviations.

The objective function defined for the optimization is shown in Eq. (3). in which n is the number of training data, p is the number of inspection points in a crack propagation path, a_k is the measured crack length, $\hat{a}_k$ is the predicted crack length and, $\frac{\sigma_m}{\mu_m}$ and $\frac{\sigma_C}{\mu_C}$ are the coefficients of variation of *m* and *C*, respectively. There are two parts in this equation, the first part, which is ($\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{p}\sum_{k=1}^{p}\frac{|a_k-\hat{a}_k|}{a_k}\right)_i$), represents the model error that is the error in the prediction of crack length using Paris' law. The second part, which is ($\frac{1}{4}\left(\frac{\sigma_m}{\mu_m}+\frac{\sigma_C}{\mu_C}\right)$), represents the standard deviation of the parameters *m* and *C*. The constant $\frac{1}{4}$ is to ensure that the algorithm does not emphasize one of the two parts in the objective function while minimizing it. If the constant is too large, then the algorithm emphasizes reducing the standard deviation. In contrast, if the constant is very small, the algorithm emphasizes minimizing the model error. While minimizing this objective function, it is intended to minimize the model error while finding a suitable standard deviation of *m* and *C*, different values for the constant is tried, and the proper result is obtained with the value $\frac{1}{4}$.

$$\text{Obj} = min\left[\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{p}\sum_{k=1}^{p}\frac{|a_k-\hat{a}_k|}{a_k}\right)_i+\frac{1}{4}\left(\frac{\sigma_m}{\mu_m}+\frac{\sigma_C}{\mu_C}\right)\right]. \quad (3)$$

In the first part of the objective function, Paris' law is used to predict the crack length $\hat{a}_k$. The Paris' law is in Eq. (1). This Paris' law equation is an ordinary differential equation, and it is needed to solve the equation before using it for crack length prediction. For the integration, the numerical integration method is followed. The numerical integration of Paris' law is performed using the Euler method

The optimization is performed using GA, which stopped after 100,000 generations with the best objective function value 0.032. The time taken by GA was 22 hours. The optimal results obtained are provided in Table I. The model parameters from Table I are used in Paris' law to plot the obtained model in Fig. 3. It is observed from Fig.3 that the obtained model shows uncertainties in the crack propagation path; however, the standard deviation of the prediction made with the help of this model is still large. For example, the standard deviation of the loading cycle at the crack length of 79.6 mm, (assumed failure threshold) is 16,887 cycles. The predicted mean lifetime is 248,394 cycles. The obtained mean and standard deviation of the predicted lifetime are used to represent the lifetime distribution with a normal distribution. The obtained normal distribution quantifies the uncertainty in the crack propagation path. The normal distribution provides a range of lifetime i.e. [197,733, 299,055] cycles. This range covers 99.7% of paths (±3-sigma range). The width of the ±3-sigma range is 101,322 cycles. The maximum error observed in lifetime prediction is 19.39% for sample number 49, which is the sample farthest from predicted mean lifetime. This observation shows that the obtained model may work for samples lying in the one sigma range (68.27% paths), but the prediction may not be accurate enough for all other samples. Using condition monitoring data with the help of the Bayesian updating process is one of the most popular methods in the literature to improve the crack path prediction accuracy, and we used it to update the obtained model calculated using GA optimization.

TABLE I. RESULT OF OPTIMIZATION

Description	Variable	Value
Mean value of m	μ_m	3.3795
The std of m	σ_m	0.1435
Mean value of C	μ_C	1.9663
The std of C	σ_C	0.8076
load parameter	S	0.3084

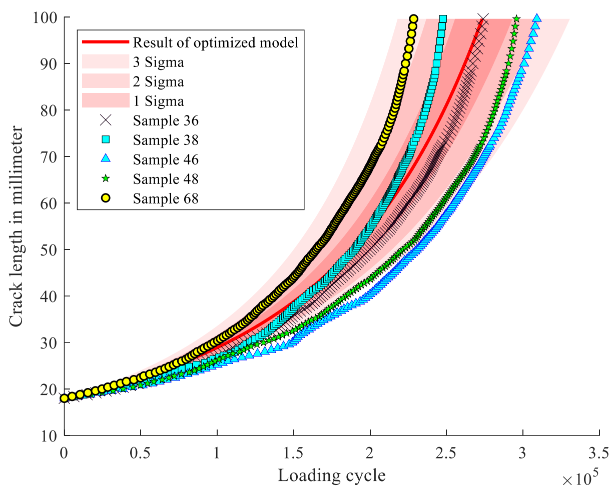


Figure 3. Result of obtained model plotted with some experimental crack propagation paths

B. *Bayesian Updating of Paris' Law Parameters*

The estimated values of parameters from the optimization are used to form a joint distribution called prior distribution, which is updated using the Bayesian updating method with condition monitoring data. The Bayesian updating method used in this study is illustrated in the flowchart in Fig. 4. There are two inputs to the Bayesian updating method shown in the flowchart. The first is the optimization result, and the second is the condition monitoring data. The optimization result consists of 63 values for *m*, 63 values for *C*, and 1 value for *S*. All 63 crack propagation paths are obtained from replicated experiments on identical samples and performed under the same loading condition. Due to this reason, it is assumed that all samples have the same value for load parameter *S*. Therefore, only two parameters of Paris' law, *m* and *C,* are updated. It is observed that *m* follows a normal distribution, and *C* follows a lognormal distribution. Bivariate normal distribution of *m* and *logC* is obtained from the optimization, and it is used as the prior distribution for the Bayesian updating. The flowchart of Fig.4 shows how when new evidence is available; it is used to calculate the conditional probability given the prior distribution, which is the likelihood of the prior distribution. This likelihood is multiplied with the prior to get an updated distribution called the posterior distribution.

The validation data consist of 163 inspection points. The crack length and the respective loading cycle are available at each inspection point. It is initially assumed that the crack propagation path of the considered validation samples is unknown for the validation. Then based on the available prior distribution, the lifetime is predicted. The predicted lifetime consists of the mean and standard deviation of the prediction. After that, the value of crack length and loading cycle for the first inspection point of the considered validation samples is assumed as new evidence from the condition monitoring., Bayesian updating is performed on the prior distribution using the new evidence. After updating, a new updated distribution is obtained called posterior distribution. The lifetime prediction is again performed using the posterior distribution. Finally, the predicted lifetime with prior and posterior is compared. It is expected that after updating, the mean prediction shifts towards the real value while the standard deviation of the prediction reduces. After one updating and prediction cycle is complete, another round of updating is performed considering the posterior as new prior and the crack vs loading cycle data from the second inspection point as the new evidence. This updating cycle can be repeated as long as required. In this study, 120 updates are performed on two different validation data sets.

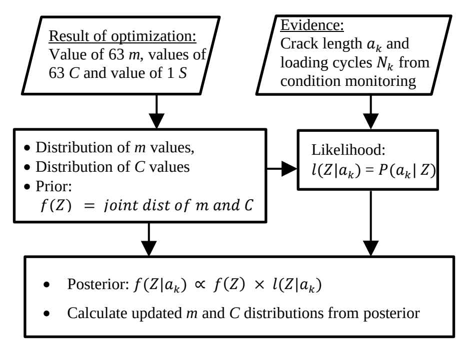


Figure 4. The Bayesian updating process of Paris' law parameters

The model shown in Fig. 4 is validated using five samples. The validation results for two samples among five are illustrated in this paper. In Table II, the change in the mean and standard deviation of the parameters after different numbers of updates are illustrated, whereas, in Table III, the result of lifetime prediction using updated parameters is illustrated. Sample 15 represents the left extreme crack propagation path near sample 68 in Fig. 2, and sample 49 represents the right extreme crack propagation path near sample 48 in Fig. 2. The real failure cycle for sample 15 is 212,237 cycles, and that of sample 49 is 308,158 cycles. For both samples, a total of 120 updates were performed. The pdf of the predicted lifetime is plotted after every 24 updates, which is illustrated in Fig. 4 a and b for samples 15 and 49, respectively. The prediction assumes that the sample has a failure threshold each in the crack length of 79.66mm. This crack length is 80% of the maximum crack length available in crack propagation data.

Table III and Fig. 5 show that the general trend of error in and standard deviation of lifetime prediction decreases with an increase in the number of updates. Before updating, the standard deviation was 16,887 cycles, which decreased to 50 cycles for sample 15 and 98 cycles for sample 49 after 120 updates. Total 120 updates are performed in under 2 minutes. Moreover, the error in lifetime prediction is about 2% for sample 15 and 1.48% for sample 49 after 96 updates.

TABLE II. MEAN AND STD OF PARAMETERS AFTER A DIFFERENT NUMBER OF UPDATES

After update	Sample 15			Sample 49		
	μ_m	σ_m	μ_{logC}	μ_m	σ_m	μ_{logC}
1	3.7070	0.0790	1.1390	3.5783	0.1154	0.9711
24	4.1616	0.0315	2.8995	3.3919	0.0147	0.1532
48	4.1130	0.0063	2.6772	3.4623	0.0073	0.4674
72	4.1103	0.0033	2.6770	3.4752	0.0041	0.5690
96	4.1101	0.0020	2.6794	3.4737	0.0024	0.5821
120	4.1093	0.0013	2.6844	3.4744	0.0012	0.5893

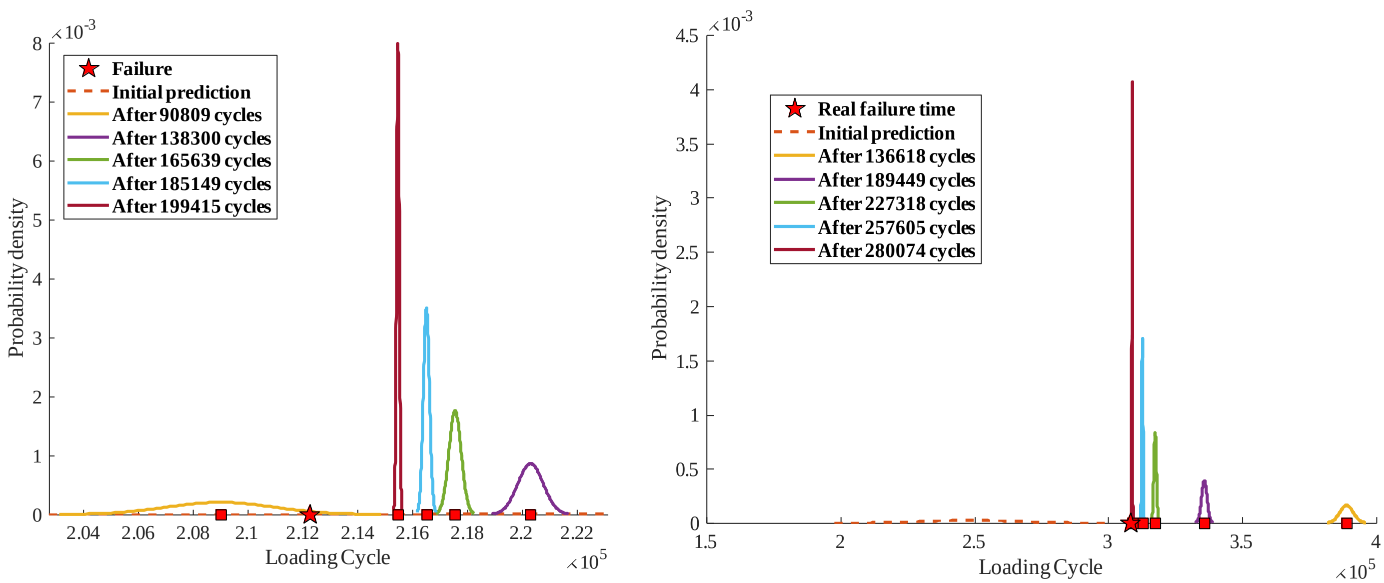
The validation results using five experimental samples (Sample15, 27,42,44, 49) show that the average error of lifetime prediction is 1.5% after the 96th update. The assumed threshold crack length was 79.6 mm. The crack length on the samples during the 96th update was 56 mm. A total of 120 updates were performed for all five samples, during which the crack length on samples was 66 mm. After the 120th update, the standard deviation of the predicted lifetime decreases from 16,887 cycles to an average of 50 cycles. The Bayesian updating algorithm developed in this study took 1 minute and 15 seconds to complete 120 updates.

TABLE III. RESULTS OF BAYESIAN UPDATING

Number of updates	Sample 15				Sample 49			
	Inspection cycle	Mean (cycles)	Std (cycles)	% Error	Inspection cycle	Mean (cycles)	Std (cycles)	% Error
0	0	248,394	16,887	17.036	0	248,394	16,88	19.393
24	90,809	208,988	1,955	1.530	105,465	388,897	2,460	26.200
48	138,30	220,303	463	3.800	153,715	335,824	1,029	8.977
72	165,63	217,549	227	2.502	190,301	317,458	479	3.017
96	185,14	216,506	114	2.011	212,739	312,747	235	1.489
120	199,41	215,466	50	1.521	231,774	308,940	98	0.253

IV. CONCLUSION

This study proposed a method to estimate Paris' law parameter for efficient and accurate fatigue lifetime prediction of an aluminum alloy plate with fatigue crack. The proposed optimization method optimizes the parameters *m*, *C* and *S* (Paris' law parameters) and their standard deviations. Since the magnitude of applied load is not available, the load parameter is optimized during the optimization. The optimized *m* and *C* parameters are used to form a prior distribution for a Bayesian updating framework to increase fatigue lifetime accuracy. As the standard deviation of *m* and *C* parameters are optimized, the prior distribution is more accurate and informative. An accurate prior helps in the performance of Bayesian updating by reducing the computational time and improving the model accuracy. During the validations, it is observed that the proposed method is fast while maintaining the accuracy of prediction.



(a) (b)

Figure 5. Predicted lifetime distribution (lifetime distribution is represented as normal probability distribution) using optimized parameters (dash line, before updating) and updated parameters after a different number of updates for (a) sample 15 and (b) sample 49

References

[1] S. V. Petinov, "Fatigue of material and structures," in *Solid Mechanics and Its Applications*, New York: Springer International Publishing, 2018, vol. 251, ch. 5, pp. 43-70.

[2] H. J. Lim, Y. J. Lee, and H. Sohn, "Continuous fatigue crack length estimation for aluminum 6061-T6 plates with a notch," *Mechanical Systems and Signal Processing*, vol. 120, pp. 356–364, Apr. 2019.

[3] "EN380 Naval materials science and engineering course notes," *U.S. Naval Academy.*

[4] E. Santecchia, A.M.S. Hamouda, F. Musharavati, E. Ealnezhad, M. Cabibbo, M.E. Mehtedi and S. Spigarelli, "A review on fatigue life prediction methods for metals," *Advances in Materials Science and Engineering*, vol. 2016, Aug. 2016.

[5] P. Paris, A. Director, and F. Erdogan, "A critical analysis of crack propagation laws," *J. Basic Eng.*, ASME, vol. 85, no. 4, pp. 528-533, Dec. 1963.

[6] D.A. Virkler, B.M. Hillberry, and P. K. Goel, "The statistical nature of fatigue crack propagation," *J. Eng. Mater. Technol*., ASME, vol. 101, no. 2, pp. 148-153, Apr. 1979.

[7] X. Du, Y. T. He, C. Gao, K. Liu, T. Zhang, and S. Zhang, "MCMC-based fatigue crack growth prediction on 2024-T6 aluminum alloy," *Mathematical Problems in Engineering*, vol. 2017, 2017.

[8] A. A. Ben, R. Azais, M. Tauzet-Cortina, A. Gegout-Petit, and M. Puiggali "Stochastic modelling and prediction of fatigue crack propagation using piecewise-deterministic Markov processes," *Journal of Risk and Reliability*, vol. 230, no. 4, pp. 405–416, Jun. 2016.

[9] A. A. Ben, "Model selection, updating and prediction of fatigue crack propagation using nested sampling algorithm," in *Congress Francais de mechanique*, Lille, France, Sept. 2017.

[10] F. Zhao, E. Bechhoefer, Y. Zeng, and F. Zhao, "An integrated prognostics method under time-varying operating conditions," *IEEE Transactions on Reliability*, vol. 64, no. 2, pp. 673–689, Jun. 2015.

[11] Y. Z. Li, S. P. Zhu, D. Liao, and X. P. Niu, "Probabilistic modeling of fatigue crack growth and experimental verification," *Engineering Failure Analysis*, vol. 118, Dec. 2020.

[12] P. C. Gope and A. Kabdwal, "Probabilistic aspects of fatigue crack growth parameters under single overload in 6061-T6 aluminium alloys," in *IOP Conference Series: Materials Science and Engineering*, Nov. 2019, vol. 653, no. 1.

[13] D. An, J. H. Choi, and N. H. Kim, "Identification of correlated damage parameters under noise and bias using Bayesian inference," *Structural Health Monitoring*, vol. 11, no. 3, pp. 293–303, May. 2012.

[14] A. Coppe, R. T. Haftka, and N.-H. Kim, "Effect of number of Paris Law parameters on Bayesian inference for remaining useful life estimation," in *52nd AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics and Materials Conf*., Denver, Colorado, Jun 2011.

[15] X. Du, Y. He, T. Zhang, C. Gao, T. Zhang, and S. Zhang, "An effective Bayesian method for probability fatigue crack propagation modeling through test data," *Mathematical Problems in Engineering*, vol. 2019, 2019. [16] "Bayesian infeernce wikipedia."

[17] B. A. Zárate, J. M. Caicedo, J. Yu, and P. Ziehl, "Bayesian model updating and prognosis of fatigue crack growth," *Engineering Structures*, vol. 45, pp. 53–61, Dec. 2012.

[18] T. Dong, D. An, and N. H. Kim, "Prognostics 102: efficient Bayesian-based prognostics algorithm in Matlab," in *Fault Detection, Diagnosis and Prognosis*, London: IntechOpen, 2019, ch 2, pp. 5-25.

[19] F. Zhao, "Integrated prognostics for component health management," Ph.D. dissertation, Fac. of Eng., Univ. of Alberta, Edmonton, Canada, 2015.

[20] J. S. Speagle, "A conceptual introduction to Markov chain Monte Carlo methods," Sep. 2019.

[21] F. Zhao, Z. Tian, and Y. Zeng, "Uncertainty quantification in gear remaining useful life prediction through an integrated prognostics method," *IEEE Transactions on Reliability*, vol. 62, no. 1, pp. 146–159, 2013.

[22] J. Y. Yoon, T. H. Lee, K. H. Ryu, Y. J. Kim, S. H. Kim, and J. W. Park, "Bayesian model updating for the corrosion fatigue crack growth rate of Ni-base alloy X-750," *Nuclear Engineering and Technology*, vol. 53, no. 1, pp. 304–313, Jan. 2021.