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# THE STRUCTURE OF THE REGULAR REPRESENTATION OF A LOCALLY COMPACT GROUP

BY

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled THE STRUCTURE OF THE REGULAR REPRESENTATION OF A LOCALLY COMPACT GROUP submitted by KEITH F. TAYLOR in partial fulfilment of the requirements for the degree of Doctor of Philosophy in Mathematics

(Supervisor)

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Date

TO JULIA

The dedication of this thesis is small reward for her patience.

ABSTRACT

For any locally compact group G, let $VN(G)$ denote the von Neumann algebra generated by the left regular representation of G. We show that, if the finite part of $VN(G)$ is nonzero, then it is isomorphic to $VN(G/K)$, where K is a certain compact normal subgroup of G. Groups for which $VN(G)$ has a nonzero Type I, finite part are characterized. For such groups we show that there is a compact normal subgroup K of G such that the Type I, finite part of $VN(G)$ is isomorphic to $VN(G/K)$.

We also investigate the center, Z, of $VN(G)$. We show that, for [SIN] groups, Z is contained in the von Neumann subalgebra of $VN(G)$ that is generated by the elements of G with relatively compact conjugacy classes. An example is given to show that this is not true, in general, even for the class of unimodular groups.

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* * *

# 1. Introduction

Let $G$ be a locally compact group and $VN(G)$ the von Neumann algebra generated by the left regular representation of $G$ on $L^2(G)$. Definitions and the terminologies used in this introduction can be found in Chapter 2.

The structure of $VN(G)$ as a von Neumann algebra is closely tied to the topological group structure of $G$. A trivial example is that $VN(G)$ is an abelian von Neumann algebra if and only if $G$ is an abelian group. The purpose of what follows is to investigate this connection between the structures of $VN(G)$ and $G$. Much of what is already known is outlined below.

In 1950, Segal [28] showed that $VN(G)$ is semi-finite if $G$ is a unimodular group. $VN(G)$ is also semi-finite if $G$ is a connected group as was shown in 1969 by Dixmier [5]. A gap in Dixmier's proof was corrected by Pukanszky [25] in 1972. In [3] (13.10.5), Dixmier proved that $VN(G)$ is a finite von Neumann algebra if and only if $G$ is a [SIN]-group. This result goes back to the work of Godement, in the early 1950's, on the theory of characters of a locally compact group ([10], page 46).

More recently, in [17], Kaniuth gave necessary and sufficient conditions on the structure of a [SIN]-group $G$, for $VN(G)$ to be Type I or Type $II_1$. For discrete groups $G$ the conditions for $VN(G)$ to be Type I or Type $II_1$ were presented in a more elementary fashion by Smith in [30]. This enabled Formanek, in [9], to prove that the Type I part of $VN(G)$ is isomorphic, in a canonical manner, to $VN(G/K)$, where K is a well defined finite normal subgroup of G. Formanek's work prompted this author to investigate whether or not similar characterizations hold for nondiscrete groups. The results of these investigations constitute this thesis.

In Chapter 4, conditions on G are given for $VN(G)$ to have a nonzero finite part. When nonzero, the finite part of $VN(G)$ is isomorphic to $VN(G/K_f)$, where $K_f$ is a certain compact normal subgroup of G.

The intersection of the Type I part and the finite part of $VN(G)$ is referred to as the Type I, finite part of $VN(G)$. Necessary and sufficient conditions are found on G for $VN(G)$ to have a nonzero Type I, finite part. If G satisfies these conditions, then there exists a compact normal subgroup, $K_{I,f}$, such that the Type I, finite part of $VN(G)$ is isomorphic to $VN(G/K_{I,f})$. The theorem which provides conditions on G, for $VN(G)$ to have a nonzero Type I, finite part is a direct generalization if Kaniuth's theorem in [17] which gives conditions on a [SIN]-group G for $VN(G)$ to be Type $II_1$. The proof of this generalization, which appears in Chapter 6 is very different and simpler than Kaniuth's proof of the result for [SIN]-groups.

The identification of the Type I, finite part of $VN(G)$ with $VN(G/K_{I,f})$ is a generalization of Formanek's result mentioned earlier.

On a slightly different theme, the center of $VN(G)$ is investigated in Chapter 5. Results that generalize known results on discrete groups were found for [SIN]-groups. Since these results depend heavily on the finitness of $VN(G)$ when G is a [SIN]-group and find application in the chapter on the Type I, finite part of $VN(G)$, the chapter on the center is wedged between the chapter on the finite part of $VN(G)$ and the one on the Type I, finite part. It is shown by example that the results on the center do not hold in general for non-[SIN]-groups.

Preliminary definitions and necessary facts from the areas of von Neumann algebra theory and abstract harmonic analysis are presented in Chapter 2. Although this makes for tedious reading, it is convenient to have all preliminaries gathered together.

In Chapter 3, several propositions are stated and proven. They provide the tools that will find frequent application in later chapters. In particular, the technique used in associating central projections in $VN(G)$ with compact normal subgroups of $G$ is presented.

Chapters 4, 5, and 6 contain the results discussed above on the finite part, the center and the Type I, finite part. The results on the Type I, finite part can be extended to general representations. This is included in Chapter 6.

The final chapter is a concluding summary of the techniques and results of this thesis.

## 2. Notation and Preliminaries

This chapter consists of three sections. The first section deals with general von Neumann algebras and the facts about their classification scheme that will be used later. The second section lists some notation from the theory of harmonic analysis on locally compact groups that will be used frequently. Also the definition and basic properties of the von Neumann algebra generated by the left regular representation are given.

The third section contains the pertinent definitions involving the topological group properties of locally compact groups. From time to time, in later chapters, these properties will be mentioned as necessary, so they are gathered together here.

Additional details on von Neumann algebras can be found in Dixmier [4] or Sakai [27]. Hewitt and Ross [13] is a basic reference for harmonic analysis. Many of the well known results in either area will be used without explicit reference.

### VON NEUMANN ALGEBRAS

If $H$ is any Hilbert space, let $B(H)$ denote the algebra of all bounded linear operators on $H$. The weak operator topology on $B(H)$ is that topology determined by the family of semi-norms $\{P_{\xi,\eta} : \xi, \eta \in H\}$, where

$$P_{\xi,\eta}(T) = |\langle T\xi, \eta \rangle|, \quad \text{for all } T \in B(H).$$

The words "weak operator topology" will usually be shortened to WOT.

A von Neumann algebra, $\mathcal{M}$, on $\mathcal{H}$ is a WOT-closed, self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$. Von Neumann algebras always have an identity. Let $\mathcal{M}^P$ and $\mathcal{M}^U$ denote, respectively, the lattice of projections and the group of unitary operators in $\mathcal{M}$. Let $\mathcal{M}'$ denote the set of all elements of $\mathcal{B}(\mathcal{H})$ that commute with all the elements of $\mathcal{M}$, the so-called commutant of $\mathcal{M}$. The set $\mathcal{M}'$ also forms a von Neumann algebra and von Neumann's famous double commutant theorem asserts that $\mathcal{M} = (\mathcal{M}')'$.

Let $\mathcal{Z}$ denote the center of $\mathcal{M}$, that is,

$$\mathcal{Z} = \mathcal{M} \cap \mathcal{M}'.$$

For each $P \in \mathcal{M}^P$, let $c(P)$ denote the smallest central projection (an element of $\mathcal{Z} \cap \mathcal{M}^P$) which dominates $P$. The projection $c(P)$ is called the central support of $P$ (see Sakai [27], 1.10.6). The classification scheme for von Neumann algebras involves classifying the projections in these algebras. Two projections $P, Q \in \mathcal{M}^P$ are said to be equivalent if there exists a $W \in \mathcal{M}$, such that $W^*W = P$ and $WW^* = Q$. A projection is said to be finite if it is equivalent to no proper subprojection.

If $P \in \mathcal{M}^P$, then $P\mathcal{M}P$ is a von Neumann algebra on $P\mathcal{H}$. Notice that if $E \in \mathcal{Z} \cap \mathcal{M}^P$, then $E\mathcal{M}E = E\mathcal{M}$. A projection, $P \in \mathcal{M}^P$, is said to be abelian if $P\mathcal{M}P$ is commutative.

Let $E$ be a central projection in $\mathcal{M}$, then:

$E$ is said to be Type I if there exists an abelian $P \in \mathcal{M}^P$ with $E = c(P)$,

$E$ is said to be semi-finite if there exists a finite $P \in \mathcal{M}^P$ with $E = c(P)$,

$E$ is said to be purely infinite, or Type III, if $E$ dominates no nonzero finite projection,

$E$ is said to be properly infinite if $E$ dominates no nonzero finite central projection,

$E$ is said to be Type II if it is semi-finite and dominates no nonzero abelian projection.

$E$ is said to be Type $II_1$, if it is both Type II and finite,

$E$ is said to be Type I, finite if it is both Type I and finite.

The von Neumann algebra $M$ is referred to as being in any of the above classes if its identity element, $I$, belongs to that class.

For any of the above classes there exists a maximal central projection in that class which dominates all other central projections in that class. For convenience these central projections are denoted as follows:

Let $E_I$ denote the maximal Type I central projection,
$E_{II}$ denote the maximal Type II central projection,
$E_{III}$ denote the maximal Type III central projection,
$E_{II_1}$ denote the maximal Type $II_1$ central projection,
$E_f$ denote the maximal finite central projection,
$E_s$ denote the maximal semi-finite central projection,
$E_{I,f}$ denote the maximal Type I, finite central projection.

The following relations hold,

$$E_I E_{II} = E_I E_{III} = E_{II} E_{III} = 0,$$

$$I = E_I + E_{II} + E_{III} ,$$

$$E_s = E_I + E_{II} ,$$

$$E_f \leq E_s ,$$

$$E_{I,f} = E_I E_f ,$$

$$E_{II_1} = E_{II} E_f ,$$

$$E_f = E_{I,f} + E_{II_1} .$$

The equation, $I = E_I + E_{II} + E_{III}$, decomposes $\mathcal{M}$ as the direct sum. $\mathcal{M} = E_I\mathcal{M} \oplus E_{II}\mathcal{M} \oplus E_{III}\mathcal{M}$, of von Neumann algebras of Types I, II and III respectively. Similarily,

$$\mathcal{M} = E_f\mathcal{M} \oplus (I-E_f)\mathcal{M},$$

is a decomposition of $\mathcal{M}$ into a finite von Neumann algebra and a properly infinite von Neumann algebra. Throughout the later chapters $E_f\mathcal{M}$ will be known as the finite part of $\mathcal{M}$ and similarily for the other distinguished central projections of $\mathcal{M}$.

The Type I, finite part of $\mathcal{M}$ can be further decomposed in a useful manner.

For each positive integer $n$, a von Neumann algebra $\mathcal{N}$ is said to be Type $I_n$ if there exists a family of $n$ mutually orthogonal, equivalent, abelian projections which sum as the identity in $\mathcal{N}$. This is equivalent to $\mathcal{N}$ being isomorphic to the $n \times n$-matrices over it's center (Sakai [27], 2.3.3).

In $\mathcal{M}$, there exists a maximal central projection $E_n$ of Type $I_n$ (that is, $E_n\mathcal{M}$ is Type $I_n$), for each positive integer $n$. The elements of $\{E_n : n = 1,2,\cdots\}$ are mutually orthogonal and

$$E_{I,f} = \sum_{n=1}^{\infty} E_n$$

$M$ is said to be Type $I_{\leq n}$ if $I = \sum_{k=1}^{n} E_k$.

2.1. Remark. If $M$ is not of Type $I_{\leq n}$, then there exists a copy of $\mathbb{C}_{n+1}$ (the $(n+1)\times(n+1)$-complex matrices) in $M$.

To see this, note that if $M$ is not of Type $I_{\leq n}$, then there exists a set of $(n+1)$ mutually orthogonal equivalent projections in $M$. As in Lemma 9.3 of Smith [29], a copy of $\mathbb{C}_{n+1}$ can then be constructed in $M$.

For any natural number $k$, let $P_k$ denote the standard polynomial in $k$ non-commuting variables

$$P_k(a_1,a_2,\cdots,a_k) = \sum(-1)^{\sigma}\, a_{\sigma(1)}a_{\sigma(2)}\cdots a_{\sigma(k)},$$

where the sum is over all permutations $\sigma$ of $\{1,\cdots,k\}$ and $(-1)^{\sigma}$ denotes the signature of the permutation.

2.2. Remark. (Amitsur and Levitski [1]). For any commutative algebra $R$, let $R_m$ denote the algebra of $m\times m$-matrices over $R$. Then $R_m$ satisfies $P_{2n}$ (that is, $P_{2n}$ is identically zero on $R_m$) if and only if $m \leq n$.

2.3. Proposition. *Let $M$ be a von Neumann algebra and $n$ a natural number. Then $M$ satisfies $P_{2n}$ if and only if $M$ is of Type $I_{\leq n}$.*

Proof: If $M$ satisfies $P_{2n}$, then Remarks 2.1 and 2.2 imply that $M$ is of Type $I_{\leq n}$.

Conversely, if $M$ is of Type $I_{\leq n}$, then $M$ can be written as the direct sum of algebras $M_k$, $1 \leq k \leq n$, where each $M_k$ is of Type $I_k$. By Remark 2.2, each $M_k$ satisfies $P_{2n}$. Therefore $M$ satisfies $P_{2n}$.

2.4. Proposition. *Let $N$ be a von Neumann subalgebra of $M$ having the same identity element. If $M$ has a nonzero Type I, finite part, then so does $N$.*

Proof: Let $E_n$ be a nonzero Type $I_n$ central projection in $M$. The two-sided ideal of $N$ given by $\{T \in N : TE_n = 0\}$ is of the form $FN$ for some projection $F$, central in $N$. The projection $I-F$ is nonzero and $(I-F)N$ is isomorphic to $E_n N$. Therefore $(I-F)N$ satisfies $P_{2n}$ and by Proposition 2.3 is a Type $I_{\leq n}$ von Neumann algebra. Hence $N$ has a nonzero Type I, finite part.

Proposition 2.4 is essentially Lemma 1 of Kaplansky [19].

If $M$ is any von Neumann algebra, then $M$ has a unique predual. That is, there exists a unique, up to isomorphism, Banach space, $M_*$, such that $M = (M_*)^*$. The topology of the duality, $(M, M_*)$, is referred to as the $\sigma$-topology on $M$.

An element $\phi$ in $M_*$ is said to be positive if,

$$(\phi, T^*T) \geq 0, \quad \text{for all } T \in M.$$

If $\phi$ is positive, then $\|\phi\| = (\phi, I)$. To each positive $\phi$ in $M_*$, there corresponds a projection, $S(\phi)$, in $M^P$, called the support of $\phi$ in $M$, such that for any $P \in M^P$,

$$(\phi,P) = 0 \quad \text{if} \quad S(\phi)P = 0,$$
$$(\phi,P) > 0 \quad \text{if} \quad 0 \neq P \leq S(\phi).$$

Furthermore, for all $T \in M$,

$$(\phi,T) = \big(\phi, S(\phi)T\big) = \big(\phi, TS(\phi)\big).$$

See Sakai [27], page 31 for the proofs of these facts.

A positive $\phi \in M_*$ is called a finite $\sigma$-continuous trace or simply a trace on $M$ if,

$$(\phi, I) = \|\phi\| = 1,$$

and

$$(\phi, U^* TU) = (\phi, T), \quad \text{for all } T \in M,\ U \in M^u.$$

Note that, if $\phi$ is a trace, then $S(\phi)$ is a central projection.

The maximal finite central projection, $E_f$, in $M$ can be characterized as,

$$E_f = \text{l.u.b.}\ \{S(\phi): \phi \text{ is a trace on } M\}.$$

This concludes the listing of basic facts on von Neumann algebras.

## LOCALLY COMPACT GROUPS

Let $G$ denote a locally compact group. Let $CB(G)$ denote the Banach space of bounded continuous complex valued functions on $G$ with the supremum norm. Let $C_o(G)$ and $C_{oo}(G)$ denote the subspaces of $CB(G)$ consisting of functions that vanish at infinity and with compact support, respectively.

Let $M(G)$ denote the Banach algebra of bounded, regular, Borel measures on $G$ with convolution as multiplication and total variation as norm. As a Banach space $M(G)$ is identified with the dual of $C_o(G)$.

Let the left invariant Haar integral on $C_{oo}(G)$ be denoted by

$$\int_G f(x)dx \quad \text{or} \quad \int f(x)dx, \quad \text{for all } f \in C_{oo}(G).$$

Left invariance means,

$$\int f(yx)dx = \int f(x)dx, \quad \text{for all } y \in G,\ f \in C_{oo}(G).$$

If $A$ is a Haar measurable subset of $G$, then the Haar measure of $A$ is denoted $|A|$.

For $1 \leq p \leq \infty$, let $L^p(G)$ denote the usual spaces, where $G$ is equipped with left Haar measure. Under the inner product, $\langle f|g\rangle = \int f(x)\bar{g}(x)dx$, the space $L^2(G)$ is a Hilbert space.

For any complex valued functions $f$ and $g$ on $G$ and any $x \in G$, the following notation is used.

$$\check{f}(x) = f(x^{-1})$$

$$\tilde{f}(x) = \bar{f}(x^{-1})$$

$$\ell_x f(y) = f(x^{-1}y), \quad \text{for all } y \in G,$$

$$r_x f(y) = f(yx), \quad \text{for all } y \in G.$$

$$f*g(x) = \int f(y)g(y^{-1}x)dy, \quad \text{whenever the right side exists.}$$

Notice, that if $f,g \in L^2(G)$ and $x \in G$, then $(f*\tilde{g})^\vee(x) = \langle \ell_x f|g\rangle$ and $(f*\tilde{g})^\vee \in C_o(G)$.

Usually $\{\ell_x: x \in G\}$ will be considered as a set of operators on $L^2(G)$, in which case they are all unitary operators with $\ell_x^* = \ell_{x^{-1}}$. In fact, the map $x \to \ell_x$ is a continuous unitary representation of $G$ on $L^2(G)$, where continuity is with respect to the weak operator topology. The symbol, $\ell$, will also be used for the induced representation of the algebra $M(G)$ in $\mathcal{B}(L^2(G))$. For $\mu \in M(G)$, the operator $\ell(\mu)$ is such that

$$\langle \ell(\mu) f | g \rangle = \int \int \ell_x f(y) \bar{g}(y) \, dy \, d\mu(x)$$
$$= \langle \mu * f | g \rangle, \quad \text{for all } f,g \in L^2(G).$$

Hence, $\ell(\mu)$ is the operator of left convolution by $\mu$ on $L^2(G)$. If $f \in L^1(G)$, then $\ell(f)$ is left convolution by $f$ on $L^2(G)$.

Let $VN(G)$ denote the von Neumann algebra generated by $\{\ell_x : x \in G\}$ in $\mathcal{B}(L^2(G))$. Then,

$$VN(G) = \text{WOT-cl } \ell(M(G)) = \text{WOT-cl } \ell(L^1(G)).$$

Also, if $\{r_x : x \in G\}$ is considered a set of elements of $\mathcal{B}(L^2(G))$, then $VN(G) = \{r_x : x \in G\}'$.

In [8], P. Eymard shows that $VN(G)$ may be identified with the dual of the Fourier algebra, $A(G)$, of $G$. The Fourier algebra may be described as the subspace of $C_o(G)$ consisting of all functions of the form $(f*\tilde{g})^{\vee}$, for $f,g \in L^2(G)$. Each $T \in VN(G)$ acts on $A(G)$ as follows, for $\phi = (f*\tilde{g})^{\vee} \in A(G)$

$$(\phi, T) = \langle Tf | g \rangle.$$

In [8], Eymard defines a norm on $A(G)$ such that $A(G)$ is a Banach space and the above action of $VN(G)$ on $A(G)$ identifies $VN(G)$ with the dual of $A(G)$. Here, the norm on $A(G)$ will be calculated by means of this duality.

An element $\phi$ of $A(G)$ will be considered either as a continuous function on $G$ or as a $\sigma$-continuous linear functional on $VN(G)$, whichever is convenient. For instance,

$$\phi(x) = (\phi, \lambda_x), \quad \text{for all } x \in G.$$

Note that, since the set of maps $T \to \langle Tf|g\rangle$, for $f$ and $g$ in $L^2(G)$, are exactly all the $\sigma$-continuous linear functionals on $VN(G)$, the weak operator topology and the $\sigma$-topology on $VN(G)$ coincide.

Let $P_1(G)$ denote the set of $\phi \in A(G)$ that are positive as linear functionals on $VN(G)$ with $\|\phi\| = 1$. Then,

$$P_1(G) = \{(f*\tilde{f})^{\vee} : \|f\|_2 = 1\}.$$

Let $T_1(G)$ denote the set of all traces on $VN(G)$. Then,

$$T_1(G) = \{\phi \in P_1(G) : (\phi, U^*TU) = (\phi, T), \text{ for all } T \in VN(G),\ U \in VN(G)^u\}.$$

The set $P_1(G)$ is a semi-group of continuous functions on $G$ under pointwise multiplication, It will be seen later that $T_1(G)$ is a subsemi-group of $P_1(G)$.

## CLASSIFICATION OF LOCALLY COMPACT GROUPS

There are many different classes of groups that will be mentioned in later chapters. For the convenience of the reader, their definitions are gathered together here.

There are, naturally, the classes of discrete groups, abelian groups or compact groups, whose definitions are obvious.

Let $G$ be a locally compact group. A set $V$, in $G$, is said to be invariant if $xVx^{-1} = V$, for all $x \in G$.

If there exists a compact invariant neighbourhood of the identity in $G$, then $G$ is said to be an [IN]-group.

If there exists a basic neighbourhood system of the identity consisting of invariant sets, then $G$ is a [SIN]-group.

Discrete, abelian or compact groups are [SIN]-groups and all [SIN]-groups are [IN]-groups.

If the left Haar measure on $G$ is also right invariant, then $G$ is said to be unimodular. As was pointed out in Grosser and Moskowitz [12], 2.4, any [IN]-group is unimodular.

For any $x \in G$, let $\mathcal{O}_x = \{yxy^{-1} : y \in G\}$. Let $G_{FC^-}$ denote the normal subgroup of $G$ consisting of all $x$ such that $\mathcal{O}_x$ is relatively compact. It is shown by example in Tits [34] that $G_{FC^-}$ is not necessarily a closed subgroup of $G$. If $G = G_{FC^-}$, then $G$ is said to be a [FC$^-$]-group.

Let $G'$ denote the subgroup generated by $\{xyx^{-1}y^{-1} : x,y \in G\}$. Then $\overline{G'}$ is known as the topological commutator subgroup of $G$. If $\overline{G'}$ is compact, then $G$ is said to be a [FD$^-$]-group.

A representation $\pi$ of $G$ on a Hilbert space $H_\pi$ means a homomorphism of $G$ into the group of unitary operators on $H_\pi$ that is continuous with respect to the weak operator topology. A representation $\pi$ is said to be irreducible if the only subspaces of $H_\pi$ invariant under $\pi(G)$ are $(0)$ and $H_\pi$. It is said to be finite dimensional if $H_\pi$ is.

If $G$ has sufficiently many finite dimensional representations to separate points, then $G$ is said to be maximally almost periodic or a [MAP]-group.

If every irreducible representation is finite dimensional, then $G$ is said to be a [Type I, finite]-group. This class of groups have sometimes been referred to as [MOORE]-groups.

Since every locally compact group has sufficiently many irreducible representations to separate points, any [Type I, finite]-group is a [MAP]-group.

For additional information on these classes of groups and their inter-relations see Grosser and Moskowitz [12], Robertson [26] and Moore [21].

## 3. Traces, Projections and Subgroups

In this chapter, the structure of the traces on $VN(G)$ is analyzed to the extent that is necessary for applications later. An order preserving coorespondence is developed between the nonzero central projections in $VN(G)$ and the compact normal subgroups of $G$. This correspondence will aid in showing that certain special central projections in $VN(G)$ are canonically associated with certain well described compact normal subgroups. A first application of this technique is provided in the relatively easy case of the maximal abelian central projection in $VN(G)$.

The traces on $VN(G)$ can be identified in terms of their behavior as functions on $G$.

<u>3.1. Proposition</u>. *Let* $\phi \in P_1(G)$. *Then* $\phi \in T_1(G)$ *if and only if* $\phi(x) = \phi(yxy^{-1})$, *for all* $x,y \in G$.

<u>Proof</u>: If $\phi \in T_1(G)$, then for any $x$ and $y$ in $G$ the tracial property of $\phi$ implies $(\phi, \lambda_y \lambda_x \lambda_y^*) = (\phi, \lambda_x)$. Since $\lambda_y \lambda_x \lambda_y^* = \lambda_{yxy^{-1}}$ and $(\phi, \lambda_x) = \phi(x)$, therefore $\phi(x) = \phi(yxy^{-1})$.

Conversely, if $\phi(x) = \phi(yxy^{-1})$, then $(\phi, \lambda_x \lambda_y) = (\phi, \lambda_y \lambda_x)$ for all $x,y \in G$. Therefore, $(\phi,AB) = (\phi,BA)$ for all $A$ and $B$ in the linear span of $\{\lambda_x : x \in G\}$. Hence, $(\phi,AT) = (\phi,TA)$ and $(\phi,ST) = (\phi,TS)$, for all $S$ and $T$ in $VN(G)$, by the ultraweak continuity of multiplication in $VN(G)$.

<u>Corollary</u>. $T_1(G)$ *is a subsemigroup of* $P_1(G)$.

As a result of the above proposition, [SIN] groups are characterized by an abundance of the set $T_1(G)$.

3.2. Proposition. [illegible] $G$ [illegible]. [illegible] $G$ [illegible] [SIN] [illegible] $x \neq e$ [illegible] $G$, [illegible] $T_1(G)$ [illegible] $(x) \neq \phi(e)$.

Proof: Let $\mathcal{U}$ be a neighbourhood system of $e$ consisting of compact invariant sets. For each $V \in \mathcal{U}$ let $f_V \in L^2(G)$ be defined by $f_V(x) = 1/|V|^{1/2}$ if $x \in V$ and $0$ otherwise, where $|V|$ denotes the Haar measure of $V$. Note that $G$ must be unimodular. Then $\phi_V = (f_V * \tilde{f}_V)^{\vee}$ is in $T_1(G)$, since $V$ is invariant. If $V \in \mathcal{U}$ and $x \in G$ is such that $x \notin V^{-1}V$, then $\phi_V(x) = 0$. Since $\phi_V(e) = 1$, then $\phi_V(x) \neq \phi_V(e)$.

Conversely, for each $\phi \in T_1(G)$ and each natural number $n \geq 2$, let $V_{\phi,n} = \{x \in G: |\phi(x)-1| \leq 1/n\}$. Then each $V_{\phi,n}$ is a compact invariant neighbourhood of $e$ and $\{e\} = \cap\{V_{\phi,n}: \phi \in T_1(G)$ and $n = 2,3,\cdots\}$. A simple compactness argument shows that the collection of all finite intersections of such sets $V_{\phi,n}$ forms a basic neighbourhood system of $e$, consisting of invariant sets.

It is clear from the above proof that the following characterization of [IN]-groups also holds.

3.3. Proposition. [illegible] $G$ [illegible]. [illegible] $G$ [illegible] [IN]-[illegible] $T_1(G) \neq \emptyset$.

3.4. Proposition. [illegible] $P_1(G)$ [illegible] $f$ [illegible] $L^2(G)$ [illegible] $= (f * \tilde{f})^{\vee}$. [illegible] $S(\phi)$ [illegible] $\{r_x f: x \in G\}$.

Proof: Let $P$ denote the projection onto the closed linear span of $\{r_x f: x \in G\}$. Since $\langle \phi, S(\phi)\rangle = \langle \phi, I\rangle$, it is clear that $\langle S(\phi)f|f\rangle = \langle f|f\rangle$. Therefore, $S(\phi)f = f$. For each $x$ in $G$, the operator $r_x$ is in $VN(G)'$. Therefore, $S(\phi)r_x f = r_x S(\phi)f = r_x f$. Hence, $S(\phi) \geq P$.

Conversely, since $\langle \phi, P\rangle = \langle Pf|f\rangle = \langle f|f\rangle = \langle \phi, I\rangle$, it is clear that $S(\phi) \leq P$.

3.5. Remark. If $\phi \in T_1(G)$, then $S(\phi)$ must be a central projection. Suppose $\phi = (f * \tilde{f})^{\vee}$ for $f \in L^2(G)$, let $P'_\phi$ denote the projection onto the closed linear span $\{\lambda_x f: x \in G\}$ in $L^2(G)$. Then $P'_\phi \in VN(G)'$ and $S(\phi)$ is the central support of $P'_\phi$. Hence, $P'_\phi VN(G)$ is isomorphic to $S(\phi)VN(G)$. See Dixmier [4], (I 1.4 and I 2.1).

If $H$ and $K$ are Hilbert spaces, let $H \otimes K$ denote their Hilbert space tensor product (see Dixmier [4] I 2.3). Suppose $M$ and $N$ are von Neumann algebras on $H$ and $K$, respectively. The tensor product of $M$ and $N$ is the von Neumann algebra, denoted $M \otimes N$, on $H \otimes K$ that is generated by $\{S \otimes T: S \in M \text{ and } T \in N\}$.

It will be necessary to have the following relation between the support projections of two elements of $T_1(G)$ and the support projection of their pointwise product.

3.6. Proposition. *Let $\phi$ and $\psi$ be in $T_1(G)$, and let $S(\phi\psi)$ be the support of their product $\phi\psi$. Then $S(\phi\psi)VN(G)$ is isomorphically contained in $S(\phi)VN(G) \otimes S(\psi)VN(G)$.*

Proof: Suppose $h \in L^2(G)$ is such that $(\phi\psi)(x) = (h * \tilde{h})^{\vee}(x) = \langle \lambda_x h|h\rangle$ for all $x \in G$. Then, in the notation of Remark 3.5, the representation $x \to \lambda_x P'_{\phi\psi}$ of $G$ on the range of $P'_{\phi\psi}$ is a cyclic representation of $G$ with cyclic vector $h$. The von Neumann algebra generated by this representation is $P'_{\phi\psi}VN(G)$.

Let $f$ and $g$ in $L^2(G)$ be such that $\phi = (f * \tilde{f})^{\vee}$ and $\psi = (g * \tilde{g})^{\vee}$. Then $(\phi\psi)(x) = \phi(x)\psi(x) = \langle \rho_x f | f \rangle \langle \rho_x g | g \rangle$. Therefore, $(\phi\psi)(x) = \langle [\rho_x S(\phi)] \otimes [\rho_x S(\psi)](f \otimes g) | f \otimes g \rangle$, for all $x \in G$.

Let $X$ denote the closed subspace of $S(\phi)L^2(G) \otimes S(\psi)L^2(G)$ generated by $\{[\rho_x S(\phi)] \otimes [\rho_x S(\psi)](f \otimes g) : x \in G\}$ and let $P_X$ be the projection of $S(\phi)L^2(G) \otimes S(\psi)L^2(G)$ onto $X$. Let $\mathfrak{A}$ denote the von Neumann subalgebra of $S(\phi)VN(G) \otimes S(\psi)VN(G)$ generated by $\{[\rho_x S(\phi)] \otimes [\rho_x S(\psi)] : x \in G\}$. Then $P_X \in \mathfrak{A}'$. The von Neumann algebra generated by $\{[\rho_x S(\phi)] \otimes [\rho_x S(\psi)]P_X : x \in G\}$ on $X$ is $P_X\mathfrak{A}$. By Dixmier [3], 13.4.5(iii), the von Neumann algebra $P'_{\phi\psi}VN(G)$ is spacially isomorphic to $P_X\mathfrak{A}$ which is isomorphic to $E\mathfrak{A}$, where $E$ is the central support of $P_X$. Therefore $E\mathfrak{A}$ is an isomorphic image of $S(\phi\psi)VN(G)$, contained in $S(\phi)VN(G) \otimes S(\psi)VN(G)$.

For each $P \in VN(G)^P$, let $N_P = \{x \in G : \rho_x P = P\}$. It is routine to show that $N_P$ is a closed subgroup of $G$ which is normal if $P$ is central.

3.7. Remark. For $\phi \in P_1(G)$, let $x \in N_{S(\phi)}$. Then,

$$\phi(x) = \langle \phi, \rho_x \rangle = \langle \phi, \rho_x S(\phi) \rangle = \langle \phi, S(\phi) \rangle = 1.$$

Conversely, if $\phi(x) = 1$, then $\langle \rho_x f | f \rangle = 1 = \|f\|_2^2$, where $f \in L^2(G)$ is such that $\phi = (f * \tilde{f})^{\vee}$. By the Cauchy-Schwartz theorem,

$$\rho_x f = \frac{\langle \rho_x f | f \rangle}{\langle f | f \rangle} f = f.$$

Hence, $\rho_x r_y f = r_y \rho_x f = r_y f$, for all $y \in G$. By Proposition 3.4, it is clear that $\rho_x S(\phi) = S(\phi)$. Therefore, $N_{S(\phi)} = \{x \in G : \phi(x) = 1\}$.

Compare with Hewitt and Ross [13], 32.6.

3.8. Proposition. *Let* G *be any locally compact group. The map that takes a projection* P *in* $VN(G)$ *to the subgroup* $N_P$ *of* G *has the following properties:*

(i) *if* $P \neq 0$, *then* $N_P$ *is compact,*

(ii) *if* $P_1 \leq P_2$, *then* $N_{P_2} \subseteq N_{P_1}$,

(iii) *if* $E$ *is a non-empty family of projections in* $VN(G)$ *and* $P = \text{l.u.b. } E$, *then* $N_P = \cap\{N_Q: Q \in E\}$.

Proof: Part (ii) is clear. To see part (i) let $f \in L^2(G)$, such that $Pf = f$ and $\|f\|_2 = 1$. If $\phi = (f * \tilde{f})^{\vee}$, then $\phi \in P_1(G)$ and by Proposition 3.4, $S(\phi) \leq P$. From (ii), it follows that $N_{S(\phi)} \supseteq N_P$. By Remark 3.7, $N_{S(\phi)}$ is compact, which implies that $N_P$ is compact. In part (iii) it follows from (ii) that $N_P \subseteq \cap\{N_Q: Q \in E\}$. Conversely, suppose $x \in N_Q$, for each $Q \in E$. Then $\ell_x f = f$, for every f in the range of Q and for each $Q \in E$. Therefore, $\ell_x f = f$, for every f in the closed linear span of the union of the ranges of the projections in $E$. But that closed linear span is the range of P. Therefore, $x \in N_P$.

For each compact normal subgroup K of G, let $\mu_K$ denote the regular Borel measure on G which, when restricted to K, is normalized Haar measure on K and such that $\mu_K(G \sim K) = 0$. Then $\mu_K$ is a central idempotent in $M(G)$ and $\ell(\mu_K)$ is a nonzero central projection in $VN(G)$. Let $E_K = \ell(\mu_K)$.

The following Proposition is analagous to Proposition 2.9 of Ernest [7].

<u>3.9. Proposition</u>. *Let* K *be a compact normal subgroup of* G. *Then* $E_K VN(G)$ *is isomorphic to* $VN(G/K)$.

<u>Proof</u>: Following Eymard [8] (3.23), let j denote the isomorphism of $L^2(G/K)$ into $L^2(G)$ given by

$$jh(x) = h(xK), \quad \text{for each } h \in L^2(G/K) \text{ and } x \in G.$$

If the Haar measure on $L^2(G/K)$ is properly scaled, then j is a Hilbert space isomorphism of $L^2(G/K)$ onto $E_K L^2(G)$. For each T in $VN(G)$, define $\Phi T$ on $L^2(G/K)$ by

$$\Phi T(h) = j^{-1} \circ T \circ j(h), \quad \text{for each } h \in L^2(G/K).$$

This is well defined since $E_K T = TE_K$, for all $T \in VN(G)$. Eymard shows that $\Phi$ is a von Neumann algebra homomorphism of $VN(G)$ onto $VN(G/K)$. It is clear that the kernel of $\Phi$ is $(I-E_K)VN(G)$. Hence $\Phi$, restricted to $E_K VN(G)$, is an isomorphism.

<u>3.10. Proposition</u>. *Let* G *be a locally compact group,* K *a compact normal subgroup of* G, *and* E *a nonzero central projection in* $VN(G)$. *Then*

(*i*) $K = N_{E_K}$

(*ii*) $E \leq E_{N_E}$

<u>Proof</u>: If K is a compact normal subgroup of G, then it is clear that $K \subseteq N_{E_K}$. Conversely, suppose $x \in N_{E_K}$, then

$$\mathcal{L}_x(jh) = jh, \quad \text{for each } h \in L^2(G/K).$$

This implies that $x \in K$. Thus (i) holds.

To see that (ii) holds, let $f \in EL^2(G)$ and let $\phi = (f * \tilde{f})^\vee$. If $x \in N_E$, then $\ell_x f = f$ and therefore $\phi(xy) = \phi(y)$, for any $y \in G$. Hence $\phi$ is constant on cosets of $N_E$. Therefore, by Eymard [8] (3.25), there exists $f_1 \in E_{N_E} L^2(G)$ such that $\phi = (f_1 * \tilde{f}_1)^\vee$. Therefore $S(\phi) \leq E_{N_E}$, which implies that $f \in E_{N_E} L^2(G)$, by Proposition 3.4. Since $f$ is arbitrary, $E \leq E_{N_E}$.

In several of the main theorems that follow the proofs center around showing that for the special central projection under consideration equality actually holds in Proposition 3.10(ii). As an illustration of this technique, a simple case will be presented immediately.

Let $E_1$ denote the maximal Type $I_1$ central projection in $VN(G)$ (that is, the maximal abelian central projection). Recall that, if $N$ is any closed normal subgroup of $G$, then $G/N$ is abelian if and only if $\overline{G'} \subseteq N$.

3.11. Theorem. *(a)* $E_1 \neq 0$ *if and only if* $G$ *is an* $[FD^-]$*-group.*

*(b) If* $E_1 \neq 0$, *then* $E_1 = E_{\overline{G'}}$. *Therefore* $E_1 VN(G)$ *is isomorphic to* $VN(G/\overline{G'})$.

Proof: If $E_1 \neq 0$, then $N_{E_1}$ is a compact normal subgroup of $G$ by Proposition 3.8. Also $N_{E_1}$ is the kernel of the homomorphism $x \to \ell_x E_1$ of $G$ into the abelian group $(E_1 VN(G))^u$. Hence $G/N_{E_1}$ is abelian, which implies that $\overline{G'} \subseteq N_{E_1}$. Therefore, $\overline{G'}$ is compact. Furthermore, $E_1 \leq E_{N_{E_1}} \leq E_{\overline{G'}}$.

Conversely, if $\overline{G'}$ is compact, then $E_{\overline{G'}} VN(G)$ is isomorphic to $VN(G/\overline{G'})$ and so must be abelian since $G/\overline{G'}$ is. Therefore, $0 \neq E_{\overline{G'}} \leq E_1$.

This proves both parts (a) and (b).

The results which have been proven in this chapter are heavily dependent on the work of Eymard in [8]. These propositions find frequent applications in later chapters so the importance of Eymard's work is emphasized.

## 4. The Finite Part of $VN(G)$

Let $E_f$ denote the maximal finite central projection in $VN(G)$. The properties of the finite part of $VN(G)$ when it is nonzero, are such that there exists a compact normal subgroup $K_f$ with $E_f = \ell(\mu_{K_f})$. This result will be established together with necessary and sufficient conditions on $G$ for $VN(G)$ to have a nonzero finite part.

It is well known that $VN(G)$ is finite if and only if $G$ is a [SIN]-group (see Dixmier [3], 13.10.5; actually the result goes back to Theorem 6 of Godement [10]). A short proof is provided here.

**4.1. Proposition.** *Let $G$ be a locally compact group. Then $VN(G)$ is a finite von Neumann algebra if and only if $G$ is a [SIN]-group.*

Proof: If $E_f = I$, then $I = \text{l.u.b.}\{S(\phi): \phi \in T_1(G)\}$. Therefore $\{e\} = N_I = \cap\{N_{S(\phi)}: \phi \in T_1(G)\}$ by Proposition 3.8(iii). Proposition 3.2 and Remark 3.7 imply that $G$ is a [SIN]-group.

Conversely, if $\mathcal{U}$ is a basic neighbourhood system of $e$ consisting of compact invariant sets, then for each $V \in \mathcal{U}$ let $f_V = \chi_V/|V|^{\frac{1}{2}}$. As in the proof of Proposition 3.8, $\phi_V = (f_V * \tilde{f}_V)^{\vee} \in T_1(G)$. Therefore $S(\phi_V) \leq E_f$. Suppose $g \in (I-E_f)L^2(G)$, then $\ell_x g$ is in $(I-E_f)L^2(G)$ for each $x \in G$. Therefore, for each $V \in \mathcal{U}$, since $S(\phi_V)g = 0$, it is clear that $\langle \ell_x g, f_V\rangle = 0$, for every $x \in G$. Hence $g * \tilde{f}_V = 0$. But $\tilde{f}_V = f_V$ and $\{f_V/|V|^{\frac{1}{2}}: V \in \mathcal{U}\}$ forms an approximate identity when acting on $L^2(G)$ by convolution. Therefore $g = 0$, which implies that $E_f = I$.

Recall that $G_{FC}^-$ is the normal subgroup of $G$ consisting of elements with relatively compact conjugacy classes. It is clear that $G_{FC}^{-}$ is an open subgroup if $G$ is an [IN]-group. That openness of $G_{FC}^{-}$ implies that $G$ is an [IN]-group and $VN(G)$ has a nonzero finite part is less obvious.

4.2. Proposition. *Let* $G$ *be a locally compact group. The following are equivalent:*

(i) $VN(G)$ *is not properly infinite,*

(ii) $G$ *is an* [IN]-*group,*

(iii) $G_{FC}^{-}$ *is an open subgroup of* $G$.

Proof: The equivalence of (ii) and (iii) is due to Wu and Yu [38], (Theorem 1).

To see that (i) is equivalent to (ii), note that $T_1(G) \neq \emptyset$ if and only if $VN(G)$ is not properly infinite, then apply Proposition 3.3.

4.3. Theorem. *Let* $G$ *be an* [IN]-*group. There exists a compact normal subgroup* $K_f$ *such that the finite part of* $VN(G)$ *is isomorphic to* $VN(G/K_f)$.

Proof: Let $K_f = N_{E_f}$. By Propositions 3.9 and 3.10(ii), it suffices to show that $E_{K_f} \leq E_f$.

Since $E_f = \text{l.u.b.}\{S(\phi): \phi \in T_1(G)\}$, by Proposition 3.8(iii),

$$K_f = \cap\{N_{S(\phi)}: \phi \in T_1(G)\}.$$

Each $\phi$ in $T_1(G)$ is constant on cosets of $K_f$, so by (3.25) of Eymard [8], each such $\phi$ can be considered as an element of $T_1(G/K_f)$.

Therefore, the identity element of $G/K_f$ is the only element of $\cap\{N_{S(\phi)}: \phi \in T_1(G/K_f)\}$. By Proposition 3.2 and Remark 3.7, the locally compact group $G/K_f$ is a [SIN]-group. Proposition 4.1 implies that $VN(G/K_f)$ is a finite von Neumann algebra. Therefore, $E_{K_f}$ is a finite projection. So $E_{K_f} \leq E_f$.

4.4. Remark. In 1951, Iwasawa [14] proved that if $G$ is an [IN]-group, then there exists a unique minimal compact normal subgroup $K$ of $G$, such that $G/K$ is a [SIN]-group. Clearly this subgroup $K$ is exactly $K_f$.

To illustrate Theorem 4.3, an example of a non-[SIN]-group which is an [IN]-group will be given together with the resulting decomposition of $VN(G)$. This example is related to an example in Hewitt and Ross [13] (7.19(b)).

Example. For each integer $n$ let $D_n = \{-1,1\}$, the two element group. Let $D$ be the direct product group with the product topology,

$$D = \prod_{n=-\infty}^{\infty} D_n.$$

---

For $x = (x_n)_{n=-\infty}^{\infty}$ in $D$, let $\alpha x \in D$ be such that

$$(\alpha x)_n = x_{n+1}, \text{ for all } n.$$

Then $\alpha$ is an automorphism of $D$ and its powers give an action of the integers $Z$ on $D$. Let $G$ be the semi-direct product,

$$G = D \otimes_\alpha Z.$$

That is, $G$ is the topological space $D \times Z$ with multiplication

$$(x,n)(y,m) = \big(x(\alpha^n y),\ n+m\big),$$

for all $x,y \in D$, $n,m \in Z$.

It is clear that $D \times \{0\}$ is an open, compact invariant neighbourhood of the identity. In fact,

$$G/(D \times \{0\}) = Z.$$

By Remark 4.4, the subgroup $K_f$, given by Theorem 4.3, must be contained in $D \times \{0\}$. If $K_f$ is smaller than $D \times \{0\}$, then there exists a $\phi \in T_1(G)$ such that, $D \times \{0\} \not\subseteq N_{S(\phi)}$. Therefore, there exists a $\varepsilon > 0$ such that, $D \times \{0\} \not\subseteq V$, where $V = \{(x,n) \in G: |\phi(x,n) - 1| < \varepsilon\}$. Then $(D \times \{0\}) \cap V$ is an invariant neighbourhood of the identity in $G$ which is properly contained in $D \times \{0\}$. This will be shown to be impossible. Let $U = (D \times \{0\}) \cap V$.

Consider $U$ as a subset of $D$ and for each $n$, let $U_n$ be the projection of $U$ onto $D_n$.

$$U_n = \{x_n : x \in U\}$$

Since $U$ is a neighbourhood of the identity in $D$, with the product topology, there exists as positive integer $n_o$, such that, $U_n = D_n$, for all $|n| \geq n_o$.

Calculating, $(x,n)^{-1}(y,0)(x,n)$ for $(x,n),(y,0) \in G$, yields

$$\begin{aligned}(x,n)^{-1}(y,0)(x,n) &= (\alpha^{-n}x^{-1},-n)(y,0)(x,n)\\ &= (\alpha^{-n}x^{-1}\alpha^{-n}y,-n)(x,n) = (\alpha^{-n}y,0)\end{aligned}$$

fore,

$$\big((x,n)^{-1}\,U(x,n)\big)_m = U_{m-n}, \quad \text{for all } m.$$

Since $U$ is invariant, this implies that,

$$U_k = U_j, \quad \text{for all integers } k \text{ and } j.$$

Therefore, $U_n = D_n$, for all $n$. Hence $U = D$, which is a contradiction. Therefore $K_f = D \times \{0\}$ and $G/K_f = Z$.

It follows from classical harmonic analysis results that $VN(Z) = L^\infty(T)$, where $T$ is the circle group. Thus Theorem 4.3 yields the decomposition of $VN(G)$ as

$$L^\infty(T) \oplus (I-E_f)VN(G),$$

where $(I-E_f)VN(G)$ is a properly infinite von Neumann algebra.

## 5. The Center of $VN(G)$ for [SIN]-Groups

Let $G$ be a [SIN]-group and let $Z$ denote the center of $VN(G)$. The fact that $VN(G)$ is finite exactly when $G$ is a [SIN]-group can be used to obtain some knowledge of $Z$.

If $H$ is a subgroup of $G$, then let $VN(H,G)$ denote the von Neumann subalgebra of $VN(G)$ generated by $\{\ell_x : x \in H\}$. The minimal subgroup $H$ such that $Z \subseteq VN(H,G)$ will be determined in Theorem 5.3. For discrete groups $G$, the center of $VN(G)$ is contained in $VN(G_{FC}^-, G)$. This follows from representing $VN(G)$ as a subset of $\ell^2(G)$, as in Murray and von Neumann [23], section 5. Theorem 5.3 generalizes this result.

A result closely related to the above mentioned theorem is that the center of $M(G)$, when embedded in $VN(G)$ is WOT-dense in $Z$. An example will be given to show that neither this nor Theorem 5.3 is true in general for non-[SIN]-groups.

For a subset $A$ of a linear space, let $\text{co}\, A$ denote the convex hull of $A$.

For each $T \in VN(G)$, let

$$K(T) = \text{WOT-cl-co}\,\{U^*TU : U \in VN(G)^u\}.$$

Since $VN(G)$ is a finite von Neumann algebra, there exists a linear map $\natural: VN(G)$ onto $Z$, which takes $T \to T^\natural$, with the following properties,

i) $T \geq 0$ implies $T^\natural \geq 0$,

ii) $T^\natural \in K(T) \cap Z$, in fact $K(T) \cap Z$ is a singleton,

iii) $(U^*TU)^{\natural} = T^{\natural}$, for all $U \in VN(G)^u$,

iv) $\natural$ is WOT-WOT-continuous on $VN(G)$.

This map is known as the center valued trace. Additional details and properties can be found in Sakai [27] (2.4). Note that (ii) implies $C^{\natural} = C$, for all $C \in Z$.

It is necessary to introduce another concept whose relation to the center of $VN(G)$ is not immediately obvious. For $f \in CB(G)$, let $Of = \{\ell_x f : x \in G\}$, then $f$ is said to be almost periodic if $Of$ is relatively compact in $CB(G)$. Let $AP(G)$ denote the closed subspace of $CB(G)$ consisting of almost periodic functions. It has been a long established fact (see von Neumann [36]) that there exists a unique invariant mean on $AP(G)$. That is, there exists $m \in AP(G)^*$ with the following properties,

(a) $m \geq 0$, $\|m\| = 1$,

(b) $m(f) = m(\ell_x f) = m(r_x f)$, for all $f \in AP(G)$, $x \in G$.

For each $x \in G$, let $\delta_x$ be point evaluation as an element of $CB(G)^*$, $C_o(G)^*$ or $AP(G)^*$. Then the convex hull of the set $\{\delta_x : x \in G\}$ is w*-dense in the set of positive, norm 1 functionals on $CB(G)$, $C_o(G)$, or $AP(G)$. Therefore, there exists a net

$$\{\sum_{i=1}^{n_\alpha} \lambda_i^\alpha \delta_{x_i^\alpha}\} \subseteq co\{\delta_x : x \in G\}$$

which converges w* to $m$ in $AP(G)^*$.

All of the above preliminaries are necessary for the proof of the following proposition from which the desired results on the center of $VN(G)$ follow easily.

5.1. **Proposition.** *Let* $G$ *be a* [SIN]*-group. Then* $T \in VN(G_{FC}^-, G)$ *implies that* $T^\natural \in VN(G_{FC}^-, G)$.

Proof: Since $\natural$ is linear and WOT-continuous, it suffices to show that $\rho_x^\natural \in VN(G_{FC}^-, G)$, for each $x \in G_{FC}^-$. This will be accomplished by modifying a technique used in Sakai [27], page 210.

For each $f$ and $g$ in $L^2(G)$ and $y \in G$, let

$$F_{f,g,y}(a) = \langle \rho_{a^{-1}ya} f | g \rangle, \quad \text{for all } a \in G.$$

It is easy to see that $F_{f,g,y} \in CB(G)$ and the following identities hold for each $b \in G$.

$$(*) \qquad r_b F_{f,g,y} = F_{\ell_b f, \ell_b g, y}$$

$$(**) \qquad \ell_b F_{f,g,y} = F_{f,g,byb^{-1}} .$$

Lemma 1. *If* $x \in G_{FC}^-$, *then* $F_{f,g,x} \in AP(G)$, *for all* $f,g \in L^2(G)$.

Proof of Lemma 1: It is sufficient to show that the map: $y \to F_{f,g,y}$ is continuous from $G$ into $CB(G)$, since then, by (**), $\mathcal{O}F_{f,g,x}$ is the continuous image of a relatively compact set $\mathcal{O}_x$ in $G$.

Let $f$ and $g$ be any fixed elements of $L^2(G)$. Let $\varepsilon > 0$. Since $G$ is a [SIN]-group, there exists an invariant neighbourhood $V$ of $e$, such that for every $v \in V$,

$$\|\ell_v f - f\|_2 < \varepsilon / \|g\|_2 .$$

Let $y$ and $y'$ in $G$ be such that $y'^{-1} y \in V$, then invariance of $V$ implies $a^{-1} y'^{-1} ya \in V$ for every $a \in G$. Hence

$$
\begin{aligned}
|F_{f,g,y}(a) - F_{f,g,y'}(a)| &\le |\langle \ell_{a^{-1}ya} f - \ell_{a^{-1}y'a} f \,|\, g\rangle| \\
&\le \|\ell_{ya} f - \ell_{y'a} f\|_2 \, \|g\|_2 \\
&= \|\ell_{a^{-1}y'^{-1}ya} f - f\|_2 \, \|g\|_2 < \epsilon, \quad \text{for every } a \in G.
\end{aligned}
$$

Therefore, $\|F_{f,g,y} - F_{f,g,y'}\|_\infty < \epsilon$ if $y'^{-1}y \in V$.

This concludes the proof of Lemma 1.

Let $m$ denote the unique two-sided invariant mean on $AP(G)$. Let $x$ be a fixed element of $G_{FC}^-$. For each $f,g \in L^2(G)$ define,

$$\phi(f,g) = m(F_{f,g,x}).$$

This defines a bounded, conjugate bilinear form on $L^2(G)$. Hence, there exists a $T_x \in \mathcal{B}(L^2(G))$, such that,

$$\langle T_x f \,|\, g\rangle = m(F_{f,g,x}), \quad \text{for all } f,g \in L^2(G).$$

<u>Lemma 2</u>. $T_x \in VN(G)'$.

---

<u>Proof of Lemma 2</u>: For each $a \in G$, since $m$ is right invariant,

$$
\begin{aligned}
&\langle \ell_a^* T_x \ell_a f \,|\, g\rangle \\
&= \langle T_x \ell_a f \,|\, \ell_a g\rangle \\
&= m(F_{\ell_a f, \ell_a g, x}) \\
&= m(r_a F_{f,g,x}), \quad \text{by } (*) \\
&= m(F_{f,g,x}) \\
&= \langle T_x f \,|\, g\rangle, \quad \text{for all } f,g \in L^2(G).
\end{aligned}
$$

Hence, $T_x \in \{\rho_a : a \in G\}'' = VN(G)''$.

Lemma 3. $T_x \in \text{WOT-cl-co}\{\rho_{a^{-1}xa} : a \in G\} \subseteq VN(G_{FC}^{-}, G)$.

Proof of Lemma 3: If $\gamma = \sum_{i=1}^{n} \lambda_i \delta_{a_i} \in \text{co}\{\delta_a : a \in G\}$ in $AP(G)^*$, then

$$\gamma(F_{f,g,x}) = \left\langle \left(\sum_{i=1}^{n} \lambda_i \rho_{a_i^{-1}xa_i}\right) f, g \right\rangle,$$

for all $f, g \in L^2(G)$.

Since $m \in w^*\text{-cl-co}\{\delta_a : a \in G\}$, it follows that $T_x \in \text{WOT-cl-co}\{\rho_{a^{-1}xa} : a \in G\}$. □

The second containment is clear, since $x \in G_{FC}^{-}$ implies $\mathcal{O}_x \subseteq G_{FC}^{-}$.

Returning to the proof of the proposition, since

$$\text{WOT-cl-co}\{\rho_{a^{-1}xa} : a \in G\} \subseteq K(\rho_x),$$

by Lemmas 2 and 3,

$$T_x \in K(\rho_x) \cap Z.$$

But $\rho_x^\natural$ is the unique element of this intersection. Hence, $T_x = \rho_x^\natural$ and by Lemma 3, this implies that $\rho_x^\natural \in VN(G_{FC}^{-}, G)$.

This completes the proof of the proposition.

Let $Z(M(G))$ denote the center of $M(G)$, a measure $\mu \in Z(M(G))$ is called a central measure on $G$.

5.2. Corollary. *If $G$ is a [SIN]-group, then for each $x \in G_{FC}^{-}$, there exists a unique positive central measure, $\mu_x$, in*

$$w^*\text{-cl-co}\{\delta_{a^{-1}xa} : a \in G\}.$$

Proof: It follows from the proof of Proposition 5.1, that there exists a net $\{\mu_\alpha\} \subseteq \text{co}\{\delta_{a^{-1}xa} : a \in G\}$ such that,

$$\ell_{\mu_\alpha} \xrightarrow{WOT} \ell_x^{\natural}.$$

But $\{\mu_\alpha\}$ is a bounded net in $M(G)$, so by taking a subnet it may be assumed that there exists a $\mu_x \in M(G)$ such that $\mu_\alpha \xrightarrow{w^*} \mu_x$. Hence, $\ell_{\mu_\alpha} \xrightarrow{WOT} \ell_{\mu_x}$, so $\ell_{\mu_x} = \ell_x^{\natural}$ and $\mu_x$ must be a central measure in $M(G)$. The uniqueness and positivity are obvious.

The machinery is now available to prove the following two theorems locating the center of $VN(G)$ for [SIN]-groups.

5.3. Theorem. *If* $G$ *is a* [SIN]-*group, then* $Z \subseteq VN(G_{FC}^-, G)$.

5.4. Theorem. *If* $G$ *is a* [SIN]-*group, then* $\ell(Z(M(G)))$ *is* WOT-*dense in* $Z$.

A proof is provided for Theorem 5.3. Theorem 5.4 can be proven by a similar application of the Hahn-Banach Theorem.

Proof of Theorem 5.3: Suppose, to the contrary, that there exists a $C \in Z$, such that $C \notin VN(G_{FC}^-, G)$. Since $VN(G_{FC}^-, G)$ is WOT-closed, by the Hahn-Banach Theorem, there exists a $\phi_0 \in A(G)$, such that,

$$(\phi_0, C) = 1,$$

and

$$(\phi_0, T) = 0, \quad \text{for all } T \in VN(G_{FC}^-; G).$$

Let a linear functional $\phi$ on $VN(G)$ be defined by,

$$(\phi,T) = (\phi_o,T^{\natural}), \quad \text{for all } T \in VN(G).$$

Since $\natural$ is WOT-continuous, $\phi$ is WOT-continuous and so must be in $A(G)$.

If $x \in G_{FC}{}^-$, then $\ell_x^{\natural} \in VN(G_{FC}{}^-,G)$ by Proposition 5.1. Therefore,

$$\phi(x) = (\phi,\ell_x) = (\phi_o,\ell_x^{\natural}) = 0.$$

For any $x,y \in G$,

$$\phi(yxy^{-1}) = \left(\phi_o, (\ell_{yxy^{-1}})^{\natural}\right) = (\phi_o,\ell_x^{\natural}) = \phi(x).$$

Hence, $\phi$ is constant on conjugacy classes in $G$. Therefore, $\phi(x) = 0$, for all $x \in G \sim G_{FC}{}^-$; so $\phi = 0$.

But,

$$(\phi,C) = (\phi_o,C^{\natural}) = (\phi_o,C) = 1,$$

which is a contradiction. Therefore, $Z \subseteq VN(G_{FC}{}^-,G)$.

Let $Z(L^1(G))$ denote the center of the convolution algebra $L^1(G)$. It is shown in Mosak [22] that $L^1(G)$ has an approximate identity $\{f_\alpha\}_{\alpha\in\Delta}$ consisting of $Z(L^1(G))$ functions if and only if $G$ is a [SIN]-group. Let $f_\alpha = \chi_{V_\alpha}/|V_\alpha|$, where $\{V_\alpha\}_{\alpha\in\Delta}$ is the directed set of basic invariant neighbourhoods of $e$. Therefore, if $\mu \in Z(M(G))$, then $f_\alpha * \mu \in Z(L^1(G))$, for each $\alpha \in \Delta$ and $f_\alpha * \mu \xrightarrow{w^*} \mu$. Hence, $Z(L^1(G))$ is w*-dense in $Z(M(G))$. Since $\lambda: M(G) \to VN(G)$ is w*-WOT-continuous, the following corollary follows from Theorem 5.3.

5.5. Corollary. *If* G *is a* [SIN]-*group, then* $\lambda(Z(L^1(G)))$ *is* WOT-*dense in* Z.

Neither Theorem 5.3 or Theorem 5.4 can be extended even to unimodular groups as is shown by the following example. The example is a connected, unimodular group G such that $G_{FC}^- = \{e\}$ and $VN(G)$ is not a factor. It is clear that such a group cannot satisfy the conclusion of Theorem 5.3 since $VN(G_{FC}^-, G)$ is just $\{\lambda I: \lambda \in \mathbb{C}\}$ in this case. That the same group cannot satisfy the conclusion of Theorem 5.4 follows from the fact that for connected locally compact groups H, the center of $M(H)$ is supported on $H_{FC}^-$ (see Greenleaf, Moskowitz and Rothschild [11], Theorem 1.5).

The Example: Let $\mathbb{R}^2$ denote the group of ordered pairs of real numbers under addition and $\mathbb{R}_+^*$ the group of positive real numbers under multiplication. For each $t \in \mathbb{R}_+^*$ let t act on $\mathbb{R}^2$ via $\alpha_t(x,y) = (tx, y/t)$, for all $(x,y) \in \mathbb{R}^2$. Let G be the semi-direct product $\mathbb{R}^2 \otimes_\alpha \mathbb{R}_+^*$.

It is clear that G is connected. To see that $G_{FC}^- = \{e\}$ let $[(x,y),t] \in G$. If $[(v,w),s] \in G$, then

$$[(v,w),s]^{-1}[(x,y),t][(v,w),s] = [(\frac{x-v+tv}{s}, sy - sw + \frac{sw}{t}), t].$$

Therefore, $\mathcal{O}_{[(x,y),t]} = \{[(\frac{x-v+tv}{s}, sy - sw + \frac{sw}{t}), t] : [(v,w),s] \in G\}$ which is not relatively compact unless $(x,y) = (0,0)$ and $t = 1$.

It follows from Hewitt and Ross [13], (15.29)(b), that G will be unimodular if

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\alpha_t(x,y))dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy,$$

for all $t = \mathbb{R}_+^*$ and $f \in C_{oo}(\mathbb{R}^2)$. But

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\alpha_t(x,y))\, dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(tx,\, y/t)dxdy$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x',y')\, \frac{dx'}{t}\, t\, dy'$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy.$$

Therefore $G$ is unimodular.

That $VN(G)$ is not a factor is due to the fact that the group $\mathbb{R}_+^*$ does not act ergotically on $\mathbb{R}^2$. The full argument is presented below.

If $t \in \mathbb{R}_+^*$, then the automorphism $\alpha_t$ induces a unitary $U_t$ acting on $L^2(\mathbb{R}^2)$ by,

$$U_t\, f(x,y) = f(\alpha_t^{-1}(x,y)) = f(x/t,\, ty),$$

for all $x,y \in \mathbb{R}^2$, $f \in L^2(\mathbb{R}^2)$. In turn $U_t$ induces an automorphism $\bar{\alpha}_t$ of $VN(\mathbb{R}^2)$ onto itself by

$$\bar{\alpha}_t\, T = U_t\, T\, U_t^*, \quad \text{for all } T \in VN(\mathbb{R}^2).$$

Since $\mathbb{R}^2$ is abelian and its own dual group, there is a unitary map from $L^2(\mathbb{R}^2)$ to $L^2(\mathbb{R}^2)$ which carries each $f \in L^2(\mathbb{R}^2)$ to its Fourier transform $\hat{f}$. This induces a spatial isomorphism $\Phi$ of $L^\infty(\mathbb{R}^2)$ onto $VN(\mathbb{R}^2)$. That is,

$$((\Phi F)f)^\wedge(\gamma,\delta) = F(\gamma,\delta)\hat{f}(\gamma,\delta),$$

for all $(\gamma,\delta) \in \mathbb{R}^2$, $f \in L^2(\mathbb{R}^2)$ and $F \in L^\infty(\mathbb{R}^2)$.

For each $t \in \mathbb{R}_+^*$, let $\tilde{\alpha}_t$ be the automorphism of $L^\infty(\mathbb{R}^2)$ given by, $\tilde{\alpha}_t = \bar{\alpha}_t \circ \Phi$. Then,

$$(\tilde{\alpha}_t F)(\gamma, \delta) = F(t\gamma, \delta/t), \quad \text{for all} \quad (\gamma, \delta) \in \mathbb{R}^2.$$

Sutherland in [32], Proposition 2.2, gives a proof that implies that $VN(\mathbb{R}^2 \otimes_\alpha \mathbb{R}_+^*) = VN(G)$ is isomorphic to the crossed product, $R(VN(\mathbb{R}^2); \bar{\alpha}, \mathbb{R}_+^*)$, of the von Neumann algebra $VN(\mathbb{R}^2)$ by the action $\bar{\alpha}$ of $\mathbb{R}_+^*$. Therefore, $VN(G)$ is isomorphic to $R(L^\infty(\mathbb{R}^2); \tilde{\alpha}, \mathbb{R}_+^*)$. See Takesaki [33], Proposition 3.4.

The construction of crossed product von Neumann algebras is presented in Takesaki [33], Section 3. For the purpose of this example it is sufficient to note that there exist a Hilbert space $H$, an isomorphism $\pi_{\tilde{\alpha}}$ of $L^\infty(\mathbb{R}^2)$ into $B(H)$ and a unitary representation $\lambda$ of $\mathbb{R}_+^*$ on $H$ such that,

$$\lambda(t)\, \pi_{\tilde{\alpha}}(F) \lambda(t)^* = \pi_{\tilde{\alpha}} \circ \tilde{\alpha}_t(F), \quad \text{for all} \quad t \in \mathbb{R}_+^*, \; F \in L^\infty(\mathbb{R}^2),$$

and $R(L^\infty(\mathbb{R}^2); \tilde{\alpha}, \mathbb{R}_+^*)$ is generated by $\pi_{\tilde{\alpha}}(L^\infty(\mathbb{R}^2))$ and $\lambda(\mathbb{R}_+^*)$.

Let $A = \{(\gamma, \delta): 0 \leq \gamma, \; \delta < \infty\}$ and let $F = \chi_A$. Then $\tilde{\alpha}_t(F) = F$, for all $t \in \mathbb{R}_+^*$, so $\pi_{\tilde{\alpha}}(F)$ is a central element in $R(L^\infty(\mathbb{R}^2); \tilde{\alpha}, \mathbb{R}_+^*)$ that is not a constant multiple of the identity. Therefore, $VN(G)$ is not a factor.

## 6. The Type I, Finite Part of $VN(G)$.

Let $E_{I,f}$ denote the maximal Type I, finite central projection in $VN(G)$. The class of groups for which $E_{I,f} \neq 0$ is determined in this chapter and it is shown that, for such groups, there exists a compact normal subgroup $K_{I,f}$ such that $E_{I,f} = E_{K_{I,f}}$.

In [30], Smith provided a simpler proof of the following theorem of Kaniuth [16]:

If $G$ is a discrete group, then $VN(G)$ is Type I if and only if $G$ has an abelian subgroup of finite index.

In that case, $VN(G)$ is actually Type $I_{\leq n}$, for some $n < \infty$, as is pointed out in Formanek [9].

The following is the non-discrete version of this theorem.

6.1. Theorem. *Let $G$ be a locally compact group. There exists a natural number $n$ such that $VN(G)$ is Type $I_{\leq n}$ if and only if $G$ has an abelian subgroup of finite index.*

Proof: If $VN(G)$ is of Type $I_{\leq n}$, then $VN(G)$ satisfies $P_{2n}$. Since $L^1(G)$ is isomorphically contained in $VN(G)$ it is clear that $L^1(G)$ also satisfies $P_{2n}$. Therefore any *-representation of $L^1(G)$ must satisfy $P_{2n}$. Hence, the dimension of any irreducible representation of $G$ must be less than $n$. By Theorem 1 of Moore [21], there is an abelian subgroup of finite index in $G$.

Suppose $G_1$ is an abelian subgroup of finite index in $G$. Let $x_1 = e, x_2, \ldots, x_k$ be a complete set of right coset representatives in $G$. Let $W$ be the linear span of $\{\ell_y : y \in G_1\}$. Let $B = W\ell_{x_1} + W\ell_{x_2} + \cdots + W\ell_{x_k}$, then $VN(G)$ is the weak operator topology closure of $B$. As in Smith [29], page 404, the algebra $B$ can be isomorphically embedded in the $k \times k$-matrices over the abelian algebra $W$. Therefore, both $B$ and $VN(G)$ satisfy $P_{2k}$. Therefore, $VN(G)$ is Type $I_{\leq k}$, by Proposition 2.3.

The following two propositions will be useful in determining those groups $G$ for which $VN(G)$ has a non-zero Type I, finite part.

Suppose $H$ is an open subgroup of a locally compact group $G$. Let $VN(H,G)$ denote the von Neumann subalgebra of $VN(G)$ generated by $\{\ell_x : x \in H\}$ acting on $L^2(G)$. Eymard shows in [8] (3.21,2°) that $VN(H,G)$ is isomorphic to $VN(H)$. The following proposition is Lemma 7 of Kaniuth [17].

<u>6.2. Proposition</u>. *Let $H$ be an open subgroup of $G$. If $VN(G)$ has a non-zero Type I, finite part, then so does $VN(H)$.*

<u>Proof</u>: Apply Proposition 2.4.

<u>6.3 Proposition</u>. *Let $K$ be a compact normal subgroup of $G$. The Type I, finite part of $VN(G)$ is non-zero if and only if the Type I, finite part of $VN(G/K)$ is non-zero.*

<u>Proof</u>: Since $VN(G/K)$ is isomorphic to $E_K VN(G)$ by Proposition 3.9, it is clear that the existence of a non-zero Type I, finite central projection in $VN(G/K)$ implies the existence of one in $VN(G)$. To prove the converse, it is sufficient to show that $E_{I,f}E_K \neq 0$, where $E_{I,f}$ is the maximal Type I, finite projection in $VN(G)$.

If $E_{I,f} \neq 0$, then, for some $n$, there exists a $\psi$ in

$T_1(G)$ such that $S(\psi)$ is Type $I_n$. Suppose that $f \in L^2(G)$ is such that $\psi = (f * \tilde{f})^{\vee}$. Then $\bar{\psi} = (\bar{f} * \tilde{\bar{f}})^{\vee}$.

<u>Claim</u>: $S(\bar{\psi})$ is also Type $I_n$.

To prove this, define $\Gamma$ from $VN(G)$ onto $VN(G)$ by

$$(\Gamma T)(h) = (T\bar{h})^{-}, \text{ for every } T \in VN(G) \text{ and } h \in L^2(G).$$

Then $\Gamma(S+T) = \Gamma S + \Gamma T$ and $\Gamma(ST) = (\Gamma S)(\Gamma T)$, for every $S,T \in VN(G)$. Therefore, if $E$ is any central projection in $VN(G)$ and $P_k$ is the standard polynomial of degree $k$, then $E\,VN(G)$ satisfies $P_k$ if and only if $(\Gamma E)VN(G)$ satisfies $P_k$. Hence $\Gamma(S(\psi))$ is Type $I_n$. But $\Gamma(S(\psi)) = S(\bar{\psi})$.

Since $S(\psi)VN(G)$ and $S(\bar{\psi})VN(G)$ are both Type $I_n$, it follows that $S(\psi)VN(G) \otimes S(\bar{\psi})VN(G)$ is Type $I_{n^2}$ (Sakai [27], 2.6.2). Therefore $S(\psi)VN(G) \otimes S(\bar{\psi})VN(G)$ satisfies $P_{2n^2}$. From Proposition 3.6, it follows that $S(|\psi|^2)VN(G)$ satisfies $P_{2n^2}$. Hence $S(|\psi|^2)$ is Type $I_{\leq n^2}$.

Let $h$ in $L^2(G)$ be such that $|\psi|^2 = (h * \tilde{h})^{\vee}$. Since $|\psi|^2$ is a real-valued function, it is clear that $|\psi|^2 = h * \tilde{h}$.

Suppose that $E_K h = 0$. That is, $\mu_K * h = 0$, which implies that $\mu_K * |\psi|^2 = 0$. This is a contradiction, since $|\psi|^2(x) \geq 0$, for every $x \in G$ and $|\psi|^2(e) = 1$. Therefore $E_K h \neq 0$.

Since $h \in S(|\psi|^2)L^2(G)$ which is contained in $E_{I,f}L^2(G)$, it is clear that $E_K E_{I,f} \neq 0$.

In [31], Smith proves that if $G$ is a unimodular group and $VN(G)$ has a nonzero Type $I_n$ central projection, then the index of $G_{FC}^{-}$ in $G$ is less than or equal to $n^2$. By virtue of the results of Chapter 5, this can be proven in a manner analogous to the treatment for discrete groups as in Smith [29], Theorem 9.4. In light of Proposition 4.2, it is not necessary to assume that $G$ is unimodular.

6.4. Proposition. *Let $G$ be a locally compact group such that $VN(G)$ has a nonzero Type $I_n$ part, then the index of $G^{\circ}_{FC^-}$ in $G$ is less than or equal to $n^2$.*

Proof: Since the index of the $FC^-$-subgroup in the group is not changed by taking the quotient by a compact normal subgroup, without loss of generality, $G$ can be assumed to be a [SIN]-group, by Theorem 4.3.

Let $E_n$ be a nonzero Type $I_n$ central projection in $VN(G)$. Then $E_n VN(G)$ is isomorphic to the algebra of $n \times n$-matrices over its center, $E_n Z$. Suppose $e = x_1, x_2, \cdots, x_{n^2+1}$ are from distinct cosets of $G_{FC^-}$ in $G$. As in Smith [29], Theorem 9.4, there exist $E_n C_1, \cdots, E_n C_{n^2+1} \in E_n Z$, not all zero, such that

$$\sum_{i=1}^{n^2+1} \varepsilon_{x_i} E_n C_i = 0.$$

Suppose $1 \leq j \leq n^2+1$ is such that $E_n C_j \neq 0$. Since $E_n C_j \in Z$, for any open subgroup $H$ of $G$, there exists an $f \in L^2(G)$, supported on $H$, such that $E_n C_j f \neq 0$. This is so, since any $g \in L^2(G)$ can be written,

$$g = \sum_{\alpha \in \Delta} \varepsilon_{x_\alpha} f_\alpha,$$

where $\{x_\alpha\}_{\alpha \in \Delta}$ is a complete set of coset representatives of $H$ in $G$ and each $f_\alpha$ is supported on $H$.

The subgroup $G_{FC^-}$ is open in $G$; so there exists an $f \in L^2(G)$, supported on $G_{FC^-}$ such that $E_n C_j f \neq 0$. But,

$$\sum_{i=1}^{n^2+1} \ell_{x_i} E_n C_i f = 0.$$

By Theorem 5.3, each $E_n C_i \in VN(G_{FC^-}, G)$ and $f$ is supported on $G_{FC^-}$; so each $E_n C_i f$ is supported on $G_{FC^-}$. Then, $\ell_{x_i} E_n C_i f$ is supported on $x_i G_{FC^-}$, for each $i$. Therefore, $\{\ell_{x_i} E_n C_i f\}_{i=1}^{n^2+1}$ is an orthogonal set of elements of $L^2(G)$, not all zero, with sum zero. This is a contradiction. Therefore, $G_{FC^-}$ can have at most $n^2$ cosets in $G$.

While proving Theorem 1 of [30], Smith showed that if $D$ is a discrete $[FC^-]$-group and $VN(D)$ has a nonzero Type I part, then $D$ is a $[FD^-]$-group. A slightly more general result will be required later and this is provided by Lemma 6.6 below. The proofs of Lemmas 6.5 and 6.6 are, to a large degree, extracted from Smith [30], Lemma 2 and Theorem 1.

<u>6.5. Lemma.</u> *Let* $G$ *be a locally compact group and* $H$ *a closed subgroup of* $G$. *If there exists a central projection* $E \in VN(G)$ *such that* $EVN(G)$ *and* $EVN(H,G)$, *are both Type* $I_n$, *then* $C(H)$, *the centralizer of* $H$ *in* $G$, *has a relatively compact commutator subgroup.*

<u>Proof:</u> Since $EVN(G)$ is isomorphic to the $n \times n$-matrices over its center, it has a faithful family of irreducible representations in $\mathbb{C}_n$. Let $\rho$ be one such irreducible representation. Since $\rho(E) \neq 0$,

$$\rho\{EVN(H,G)\} = \mathbb{C}_n, \text{ also.}$$

Suppose there exists $x, y \in C(H)$ such that $E\ell_{x^{-1}y^{-1}xy} \neq E$, then there exists an irreducible representation $\rho$ of $EVN(G)$ in $\mathbb{C}_n$, such that $\rho(E\ell_{x^{-1}y^{-1}xy} - E) \neq 0$. Hence

$$\rho(E\ell_x)\rho(E\ell_y) \neq \rho(E\ell_y)\rho(E\ell_x).$$

But $x \in C(H)$, so, $\rho(E\ell_x)$ commutes with $\rho\{EVN(H,G)\} = \mathbb{C}_n$, which is a contradiction. Therefore,

$$E\ell_{x^{-1}y^{-1}xy} = E, \quad \text{for all} \quad x,y \in C(H).$$

Hence, $C(H)' \subseteq N_E$, which is compact.

6.6. Lemma. *Let* $A$ *be an abelian group and* $D$ *a discrete* $[FC^-]$-*group. If* $VN(A \times D)$ *has a nonzero Type I part, then* $D$ *is a* $[FD^-]$-*group.*

Proof: Since $A \times D$ is a [SIN]-group, $VN(A \times D)$ must have a nonzero Type $I_n$ projection, $E_n$, for some $n \geq 1$. Then $E_n VN(A \times D)$ satisfies the standard polynomial identity $P_{2n}$ but not $P_{2(n-1)}$. Let $e_1$ denote the identity in $A$. By multilinearity of $P_{2(n-1)}$, there exists $d_1, \cdots, d_{2(n-1)} \in D$ such that,

$$P_{2(n-1)}\left(E_n\ell_{(e_1,d_1)}, \cdots, E_n\ell_{(e_1,d_{2(n-1)})}\right) \neq 0.$$

Let $H$ be the normal subgroup of $D$ generated by $\{d_1, \cdots, d_{2(n-1)}\}$ and their finitely many conjugates. Since $H$ is finitely generated, $C(H)$ has finite index in the $[FC^-]$group $D$. If it can be shown that $C(H)$ has a finite commutator subgroup, then Neumann [24], Lemma 4.1, implies that $D$ is a $[FD^-]$-group.

To see that $C(H)'$ is finite, note that $C(A \times H)$ in $A \times D$ is $\{e_1\} \times C(H)$. Since $E_n VN(A \times H, A \times D)$ satisfies $P_{2n}$ but not $P_{2(n-1)}$, then the maximal Type $I_n$ central projection $E$ in $VN(A\cdot H, A\cdot D)$ must be nonzero. Since $A\cdot H$ is normal in $A\cdot D$ and $E$ is maximal, $\ell_{x^{-1}} E \ell_x = E$, for all $x \in A\cdot D$. Therefore $E$ is a central Type $I_n$ projection in $VN(A\cdot D)$ also. An application of Lemma 6.5 completes the proof.

The following theorem is a generalization of Theorem 2 in Kaniuth [17], where he assumes that $G$ is a [SIN]-group. The method of proof given here is entirely different.

<u>6.7. Theorem.</u> *Let* $G$ *be a locally compact group. For* $VN(G)$ *to have a nonzero Type I, finite part, it is necessary and sufficient that the following conditions hold:*

*i) the index of* $G_{FC}^-$ *in* $G$ *is finite,*

*ii) the commutator subgroup of* $G_{FC}^-$ *has compact closure in* $G$.

<u>Proof</u>: Suppose that $VN(G)$ has a nonzero Type $I_n$ part for some $n$. Proposition 6.4 implies that the index of $G_{FC}^-$ in $G$ is less than or equal to $n^2$. Proposition 4.2 implies that $G_{FC}^-$ is open in $G$. So $VN(G_{FC}^-)$ has a non-zero Type I, finite part by Proposition 6.2.

By Theorem 4.3, there exists a compact normal subgroup $K$ of $G_{FC}^-$ such that the finite part of $VN(G_{FC}^-)$ is isomorphic to $VN(G_{FC}^-/K)$. From Proposition 6.3, it follows that $VN(G_{FC}^-/K)$ has a non-zero Type I part. Let $H = G_{FC}^-/K$. Then $H$ is both a [SIN]-group and a [$FC^-$]-group. By Wilcox [37], there exists a compact normal subgroup $N$ of $H$, such that $H/N$ is the direct product of a vector group, $V$, and a discrete [$FC^-$]-group, $D$. By Proposition 6.3, the Type I part of $VN(V\times D)$ is nonzero. Therefore, the commutator subgroup of $D$ is finite by Lemma 6.6. Since $H/N \cong V \cdot D$ and $G_{FC}/K \cong H$, it follows from Hewitt and Ross [13] (5.24) that the commutator subgroup of $G_{FC}^-$ has compact closure. Therefore, part (ii) is established since $G_{FC}^-$ is open in this case.

Suppose that (i) and (ii) hold. Let $K$ denote the closure, in $G$, of the commutator subgroup of $G_{FC}^-$. Let $\sigma: G \to G/K$ denote the canonical homomorphism. From (i), it follows that $\sigma(G_{FC}^-)$ is an abelian subgroup of finite index in $G/K$. Theorem 6.1 implies that $VN(G/K)$ is Type I, finite. Hence, the Type I, finite part of $VN(G)$ is non-zero.

Recall that a locally compact group is called a [Type I, finite]-group if every irreducible representation is finite dimensional. Note that an irreducible representation is finite dimensional if and only if the von Neumann algebra it generates is Type I, finite.

<u>6.8. Remark</u>. If $G$ is a [Type I, finite]-group, then it is clear from Dixmier [3] (4.2.1 and 5.5.2) that $VN(G)$ is Type I. It follows from Moore [21] (Lemma 4.1), that $VN(G)$ is a finite von Neumann algebra. Therefore $VN(G)$ is Type I, finite.

<u>6.9. Remark</u>. In [26], Robertson calls [Type I, finite]-groups, [MOORE]-groups. He gives the following characterization. A proof can be found in Kaniuth [17] (page 234).

For a group $G$ to be a [Type I, finite]-group, it is necessary and sufficient that each of the following is satisfied.

i) the index of $G_{FC}^-$ in $G$ is finite,

ii) the commutator subgroup of $G_{FC}^-$ is relatively compact in $G_{FC}^-$,

iii) the group $G_{FC}$ is maximally almost periodic.

Recall that a group is maximally almost periodic if there exist sufficiently many finite dimensional unitary representations to separate points.

6.10. Lemma. *Let D be a discrete group. If D has a faithful unitary representation in a Type $I_0$, finite von Neumann algebra, M, then D is maximally almost periodic.*

Proof: Since M decomposes into the direct sum of Type $I_n$ von Neumann algebras and each Type $I_n$ von Neumann algebra has sufficiently many *-representations in $\mathbb{C}_n$ to separate points, there are sufficiently many finite dimensional representations of M (hence of D) to separate points.

6.11. Lemma. *Let G be a locally compact group satisfying the following properties:*

*i) there exists a compact normal subgroup [illegible] G, such that G/K is a [Type I, finite]-group [illegible]*

*ii) there exists a faithful representation π of G into the unitary group of a Type I, finite von Neumann algebra M.*

*Then G is a [Type I, finite]-group.*

Proof: From the characterization given in Remark 6.9, it is not hard to see that $K_1 = \cap\{K: K$ compact, $G/K$ is a [Type I, finite]-group$\}$ is such that $G/K_1$ is also a [Type I, finite]-group. So it can be assumed that the subgroup K given in (i) is minimal such that G/K is a [Type I, finite]-group.

If $K \neq \{e\}$, then let $k \in K$ be such that $k \neq e$. For some $n$ there exists a Type $I_n$ central projection $E_n$ in $M$, such that $\pi(k)E_n \neq E_n$.

Let $C^*(G)$ denote the group $C^*$-algebra of $G$ (see Dixmier [3], 13.9). Any representation $\sigma$ of $G$ induces a representation, also denoted $\sigma$, of $C^*(G)$ and conversely.

Let

$$J_\sigma = \{a \in C^*(G): \sigma(a) = 0\},$$

and

$$J = \{a \in C^*(G): \pi(a)E_n = 0\}.$$

If unitarily equivalent representations are identified, then $J$ remains well defined. Let $S$ denote the set of equivalence classes of irreducible representations, $\sigma$, such that $J \subseteq J_\sigma$. By Dixmier [3], 2.9.7,

$$J = \cap\{J_\sigma: \sigma \in S\}.$$

Since $C^*(G)/J$ is algebraically contained in $E_n M$, it must satisfy the polynomial identity $P_{2n}$. Thus, for each $\sigma \in S$, the identity $P_{2n}$ is also satisfied by $C^*(G)/J_\sigma$ and therefore by $\sigma(C^*(G))$. Since $\sigma$ is irreducible, it must be finite dimensional. By minimality of $K$,

$$\sigma(x) = I, \quad \text{for all } x \in K, \sigma \in S.$$

Let $\mu_K$ be the central idempotent in $M(G)$ as defined in Chapter 3. Then $\pi(\mu_K)$ and $I - \pi(\mu_K) = \pi(\delta_e - \mu_K)$ are central projections in $M$. Since $\pi(k)E_n \neq E_n$, for some $k \in K$,

$$E_n(I - \pi(\mu_K)) \neq 0.$$

By considering an approximate identity in $L^1(G)$, an $f \in L^1(G)$ can be found such that,

$$E_n\,\pi\big((\delta_e - \mu_K) * f\big) \neq 0.$$

Let $f_0 = (\delta_e - \mu_K) * f = f - \mu_K * f$. Since, $\sigma(\mu_K) = I$ for all $\sigma \in S$, then $\sigma(f_0) = 0$, for all $\sigma \in S$. Hence

$$f_0 \in \cap\{J_\sigma : \sigma \in S\},$$

while

$$\pi(f_0)E_n \neq 0.$$

Contradicting,

$$J = \cap\{J_\sigma : \sigma \in S\}.$$

Therefore, $K = \{e\}$ and $G$ is a [Type I, finite]-group.

<u>6.12. Theorem</u>. *Let* $G$ *be a locally compact group such that* $VN(G)$ *has a non-zero Type I, finite part. There exists a compact normal subgroup,* $K_{I,f}$ *of* $G$*, such that the Type I, finite part of* $VN(G)$ *is isomorphic to* $VN(G/K_{I,f})$.

<u>Proof</u>: Let $E_{I,f}$ denote the maximal Type I, finite central projection of $VN(G)$. Let $K_{I,f} = N_{E_{I,f}}$. By Propositions 3.9 and 3.10(ii), it suffices to prove that $E_{I,f} \leq E_{K_{I,f}}$. For this, it suffices to prove that $VN(G/K_{I,f})$ is Type I, finite.

Let $H = G/K_{I,f}$. Since $E_{I,f} \leq E_{K_{I,f}}$, by Theorem 6.7, the index of $H_{FC}^-$ in $H$ is finite and the commutator subgroup of $H_{FC}^-$ is relatively compact. If $H$ can be shown to be a [Type I, finite]-group, then by Remarks 6.9 and 6.8, the proof that $VN(G/K_{I,f})$ is

Type I, finite will be complete.

Theorem 4.3 and the definition of $K_{I,f}$ imply $H$ is a [SIN]-group. The theorem in Wilcox [37] implies that there exists a compact normal subgroup $N$ of $H$ such that

$$(H_{FC}^{-}/N) = V \times D,$$

where $V$ is a vector group and $D$ is a discrete group. By Proposition 6.3, the Type I, finite part of $VN(H/N)$ is nonzero. The kernel of the representation of $H/N$ in the Type I, finite part of $VN(H/N)$ is a compact normal subgroup of $H/N$ and, thus, must be a subgroup of $N$. Hence it can be assumed, by possibly increasing $N$, that $H/N$ has a faithful representation in a Type I, finite von Neumann algebra. Therefore $D$ is maximally almost periodic by Lemma 6.10. Since

$$(H/N)_{FC}^{-} = H_{FC}^{-}/N = V \times D,$$

it must be maximally almost periodic. Hence, $H/N$ is a [Type I, finite]-group by Remark 6.9.

Therefore, [illegible] satisfies Lemma 6.11 with $K$ as $N$ and $M$ as the image of $E_{I,f}VN(G)$ in $VN(H)$. Hence, $H$ is a [Type I, finite]-group. This completes the proof of the theorem.

<u>6.13. Corollary.</u> *Let* $G$ *be a locally compact group. If for some compact normal subgroup* $K$ *of* $G$, *the group* $G/K$ *has only finite dimensional irreducible representations, then* $K_{I,f}$ *is the unique minimal compact normal subgroup of* $G$ *such that the quotient group has only finite dimensional irreducible representations.*

If $G$ satisfies the hypothesis of Theorem 6.12, then let

$\mu_{K_{I,f}}$ denote the central idempotent in $M(G)$ defined by $K_{I,f}$. For any representation $\pi$ of $G$, let $M_\pi$ denote the von Neumann algebra generated by $\pi(G)$.

6.14. Corollary. *If $G$ satisfies the hypothesis of Theorem 6.12 and $\pi$ is any representation of $G$, then $\pi(\mu_{K_{I,f}})$ is the maximal Type I, finite central projection in $M_\pi$.*

Proof: Since $K_{I,f}$ is contained in the kernel of the representation that takes $x \in G$ to $\pi(x)\pi(\mu_{K_{I,f}})$ in $\pi(\mu_{K_{I,f}})M_\pi$, it may be considered as a representation of $G/K_{I,f}$. By Lemma 4.1 of Moore [21], every representation of a [Type I, finite]-group generates a Type I, finite von Neumann algebra. Therefore, $\pi(\mu_{K_{I,f}})$ is a Type I, finite projection in $M_\pi$.

Suppose $F \geq \pi(\mu_{K_{I,f}})$ is also a Type I, finite central projection in $M_\pi$. Let $N_F = \{x \in G: \pi(x)F = F\}$, then $N_F \subseteq K_{I,f}$. By Lemma 6.11, the group $G/N_F$ is a [Type I, finite]-group. By minimality of $K_{I,f}$, then $N_F = K_{I,f}$. Therefore,

$$\pi(k)F = F, \quad \text{for all } k \in K_{I,f}.$$

Hence, $\pi(\mu_{K_{I,f}}) = \pi(\mu_{K_{I,f}})F = F$.

6.15. Corollary. *If $G$ satisfies the hypothesis of Theorem 6.12 and $\pi$ is any irreducible unitary representation of $G$, then $\pi$ is finite dimensional if and only if $K_{I,f} \subseteq \ker \pi$.*

6.16. Remark. If $G$ is a discrete group, then Thoma [35] proves that $G$ is a [Type I, finite]-group if and only if $G$ has an abelian subgroup of finite index. In [9], Formanek proves that if $G$ is discrete and

$VN(G)$ has a non-zero Type I part, then there exists a minimal finite normal subgroup, $K$, such that the quotient group has an abelian subgroup of finite index. Furthermore, the Type I part of $VN(G)$ is isomorphic to $VN(G/K)$. This is the discrete group version of Theorem 6.12.

The following theorem was proven by Kaniuth in [17]. Here it follows from the proof of Theorem 6.12 and Remark 6.8.

<u>6.17. Theorem.</u> (Kaniuth). $VN(G)$ *is a Type I, finite von Neumann algebra if and only if* $G$ *is a [Type I, finite]-group.*

An example will now be given to illustrate Theorem 6.12. This example is discussed in Grosser and Moskowitz [12], page 39,e. The calculation of the details below follow as in Section 5.10 of Grosser and Moskowitz [12].

<u>Example</u>. Let

$$H = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} : x,y,z \in \mathbb{R} \right\} .$$

The commutator subgroup, $H'$, of $H$ is given by,

$$H' = \left\{ \begin{pmatrix} 1 & 0 & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : y \in \mathbb{R} \right\} .$$

Let

$$\Gamma = \left\{ \begin{pmatrix} 1 & 0 & n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : n \text{ is an integer} \right\} .$$

Let $G = H/\Gamma$. Then $G' = H'/\Gamma$ and is isomorphic to the circle group $\mathbb{T}$. For this group $K_f = K_{I,f} = G'$ and

$$G/K_{I,f} = \mathbb{R}^2 .$$

Therefore, $VN(G)$ decomposes as $VN(\mathbb{R}^2) \oplus (I-E_{I,f})VN(G)$. But $VN(\mathbb{R}^2)$ is isomorphic to $L^\infty(\mathbb{R}^2)$ and $E_{I,f} = E_f$ for this group. Hence $VN(G)$ is isomorphic to $L^\infty(\mathbb{R}^2) \oplus (I-E_f)VN(G)$.

In the above example, $G$ is connected so the fact that $E_{I,f} = E_f$ is no accident. It follows from the following theorem of Kadison and Singer [15].

*If $G$ is a connected locally compact group, then $VN(G)$ has no Type $II_1$ part.*

In [6] and [7], Ernest introduced another von Neumann algebra which is associated with a locally compact group. This von Neumann algebra will be denoted $W^*(G)$. The following properties of $W^*(G)$ are taken from Ernest [6] and [7].

There exists a Hilbert space $H_\omega$ and a unitary representation $\omega$ of $G$ on $H_\omega$, such that $W^*(G)$ is the von Neumann algebra generated by $\omega(G)$. For any representation $\pi$ of $G$, there exists a $\sigma$-continuous representation $\tilde{\pi}$ of $W^*(G)$ such that $\pi(x) = \tilde{\pi}(\omega(x))$, for all $x \in G$. Furthermore, $M_\pi = \tilde{\pi}(W^*(G))$. In this sense, $\omega$ is a universal representation of $G$ and Corollary 6.14 can be reformulated as follows.

<u>6.19. Theorem.</u> *If $G$ satisfies the hypothesis of Theorem 6.12, then $c(\omega_{K_{I,f}})$ is the maximal Type I, finite central projection in $W^*(G)$.*

## 7. Summary

In this chapter, a summary is made of what has been proven so that the reader may get an overview of the techniques used.

Perhaps the single most important tool which is used is the association, developed in Chapter 3, between the compact normal subgroups in $G$ and the nonzero central projections in $VN(G)$. The order preserving map, which takes a compact normal subgroup $K$ of $G$ to the central projection $E_K$, is not, in general, onto. Often it is far from being onto, as in the case where $G = \mathbb{R}$. There is only one compact normal subgroup, $\{0\}$, of $\mathbb{R}$; but $VN(\mathbb{R})$ is isomorphic to $L^\infty(\mathbb{R})$ and has as many central projections as there are inequivalent Lebesgue measurable sets in $\mathbb{R}$.

Most of the energy expended in the previous chapters has gone towards showing in three different cases, that a certain central projection in $VN(G)$ is in the range of this map, $K \to E_K$. The three cases are discussed below.

If $E_1$ is the maximal abelian central projection in $VN(G)$, then the topological commutator subgroup, $\overline{G'}$, is compact if $E_1 \neq 0$ and $E_1 = E_{\overline{G'}}$. The key to this result is the fact that $VN(H)$ is abelian if and only if $H$ is abelian.

Since it was known that $VN(H)$ is finite if and only if $H$ is a [SIN]-group, it was possible to use this characterization to show that if the maximal finite central projection $E_f$ in $VN(G)$ is nonzero, then $E_f = E_{K_f}$, for a certain compact normal subgroup $K_f$ of $G$. It turns out that $K_f$ is the minimal compact normal subgroup $K$ such that $G/K$ is a [SIN]-group. Note that $\overline{G'}$ is the minimal closed normal subgroup $N$ of $G$ such that $G/N$ is abelian.

The situation in the third case is slightly more complicated but retains much of the above pattern. It was known that if $G$ is a [Type I, finite]-group, then $VN(G)$ is Type I, finite. The class of groups for which the maximal Type I, finite central projection, $E_{I,f}$, is nonzero, was then characterized. Using this characterization a compact normal subgroup $K_{I,f}$ was shown to exist such that $E_{I,f} = E_{K_{I,f}}$. In fact, $K_{I,f}$ is the minimal compact normal subgroup $K$ of $G$ such that $G/K$ is a [Type I, finite]-group. This leads to the corollary that $VN(G)$ is Type I, finite if and only if $G$ is a [Type I, finite]-group.

Since, for any compact normal subgroup $K$ of $G$, the von Neumann algebras $E_K VN(G)$ and $VN(G/K)$ are isomorphic, the above characterizations of the maximal abelian, finite and Type I, finite central projections in $VN(G)$ identify the respective parts of $VN(G)$ with von Neumann algebras generated by the left regular representations of the quotients of $G$ by the respective compact normal subgroups.

One of the steps in characterizing the groups, $G$, for which $E_{I,f} \neq 0$, is to show that if $VN(G)$ has a Type $I_n$ part, then the index of $G_{FC}^-$ in $G$ is less than or equal to $n^2$. For unimodular groups, this was proven by Martha Smith in [31], by making use of the construction of $VN(G)$ as the von Neumann algebra generated by the Hilbert algebra $C_o(G)$ as contained in Dixmier [3], 13.10.2. If $G$ is a discrete group, then there is a simple proof, also due to Smith ([29], Lemma 9.4), of the above result. This latter proof generalizes easily to general groups if it is known that the center of $VN(G)$ is contained in $VN(G_{FC}^-, G)$. That is the proof that is given for Proposition 6.4 after first reducing to the case of a [SIN]-

group and using the fact that for [SIN]-groups, the center of $VN(G)$ is contained in $VN(G_{FC}^{-}, G)$.

Aside from the above discussion, the results on the center of $VN(G)$, for [SIN]-groups $G$, are included as another illustration of the connection between the structure of $VN(G)$ and the topological group structure of $G$.

Consideration of the center of $VN(G)$ for non-[IN]-groups usually involves consideration of the action of some group on some measure space as in the example in Chapter 5. It would be desirable to have a method of studying the center of $VN(G)$ for non-[IN]-groups that avoids these measure theoretic techniques.

Finding necessary and sufficient conditions on general locally compact groups $G$ for $VN(G)$ to be a factor is an interesting but, in the author's opinion, a very difficult problem.

Other questions related to the general theme of this thesis, that remain unanswered, are listed below.

What are necessary and sufficient conditions on $G$ for $VN(G)$ to be a Type I von Neumann algebra? If the answer to this question is found, then it will be possible to decide whether the Type I part of $VN(G)$ is isomorphic to $VN(G/K)$ for some compact normal subgroup $K$ of $G$.

Similarly, what are necessary and sufficient conditions on $G$ for $VN(G)$ to be semi-finite? Again, when the answer is found, the characterization of the semi-finite part may be possible.

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