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University of Alberta

Mathematics Teacher Understanding as an Emergent Phenomenon

by

Florence Anne Glanfield ©

A thesis submitted to the Faculty of Graduate Studies and Research in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled Mathematics Teacher Understanding as an Emergent Phenomenon submitted by Florence Anne Glanfield in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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Dedication

With love and appreciation, this work is dedicated to

my parents

Betty & Oliver Glanfield;

my sister

Brenda;

my aunt

Emmeline;

my grandparents

Katie Florence & Stanley Fraser Wylie;

&

the teachers who influenced

Sister Nativity
Mrs. Olga Mearti
Mr. Ken Saunderson
Dr. Sol Sigurdson
Mr. Alan Ormerod
Dr. Margaret Haughey
Dr. Tom Kieren
Dr. Jean Clandinin
Joyce
Julia
Marilyn

Abstract

This study examined the research question "In what ways do mathematics teachers grow in their understanding of mathematical processes within the context of professional conversation?" An enactivist view of cognition is used as a frame to consider teacher understanding.

This thesis is offered as an explanatory piece that considers mathematics teacher understanding as an emergent phenomenon. Within the professional conversation of four teachers about mathematical processes, individual understanding, collective understanding, and understanding within the body of mathematics was noticed as emerging.

Narrative inquiry is used to describe the emergence of mathematics teacher understanding. Two detailed narrative accounts are included to highlight the complexity and complicity of teachers' conversations. From the two narrative accounts, five moments are selected and interpreted further through the frames of individual understanding, collective understanding, and understanding within the body of mathematics. Pirie and Kieren's (1994) theory of dynamical growth of mathematical understanding is used to interpret emergent individual understanding; Davis and Simmt's (2003) work on collective understanding is used to interpret emergent collective understanding; and Davis's (1996) work around understanding within the body of mathematics is used to interpret emergent understanding within the body of mathematics.

Some of the patterns that emerged in the interpretations of the selected moments are mathematics teacher understanding is intertwined with teachers' lived histories and student understanding; a teacher may not overtly express their understanding to others,

yet changing understanding has occurred; teacher understanding of mathematical processes is affected by the way in which they themselves experienced the processes; changing collective understanding emerges in the collective; developing a shared or distributed understanding within a collective is possible; because conversation itself is an emergent phenomenon, we can see emergent understanding within it; the Pirie-Kieren theory can be used to describe emergent mathematical understanding; and mathematics lives in mathematics teacher conversations.

A series of questions for both in-service and pre-service teacher education are posed in the conclusion. The questions are raised within Maturana's (1988) explanatory path of objectivity-in-parenthesis and are offered as suggestions for action and further research.

Acknowledgements

My life has co-emerged with the lives of many others. My 'quilt' has been implicated by the 'quilts' of others. Not only do we have our own quilts, but we also have quilts together. I have been blessed to have relationships with the following 'fellow quilters.' With love and appreciation, I thank:

My father, Oliver; my sister Brenda and her family, Bernie, Lara, and Valisa; my aunt Emmeline and uncle Ray; and my cousins Juliette, David, Susan, and Andy and their families; for their embracing love.

My husband, Andy, who entered my life after my research quilt had started, and who has become a part of the fabric of my quilt.

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CHAPTER 1

Introduction

In 1990, Virginia Richardson wrote a paper that addressed two questions: "What is involved in bringing about significant and worthwhile change in teaching practices?" and "How can or should research aid in this process" (p. 10)? The paper concluded with implications and suggestions for practice and research. One of Richardson's suggestions was that opportunities be created for "teachers to interact and have conversations around standards, theory, and classroom activities" (p. 16). I read this article when I was working with Alberta Education and was responsible for implementing a new provincial curriculum[1] in high school mathematics. Many of my professional conversations at the time included comments such as "teachers just need to change..." and "if only teachers did ...." As I continued my work with Alberta Education, I began to discover the value of Richardson's suggestion. I began to see that policymakers were not the only individuals who should be involved in conversations about standards, theory, and classroom activities. It was during this time, that I first became interested in my research question.

Research Question

This thesis presents possibilities and occasions for discussing the following research question:

> In what ways do mathematics teachers grow in their understanding of mathematical processes?

I have posed this question within the framework provided by enactivist views and

[1] Throughout this thesis I refer to a curriculum document in mathematics. Technically, the Alberta Department of Education refers to this document as a Program of Studies. The common language used, however, is curriculum.

principles.[2] In developing the discussion for this research question, I would like to share the way in which my path has been laid in coming to the research question. In this chapter I will describe the path towards the research question.

Growing Into my Research Life

For much of my life, I have probably not noticed how the pieces and experiences of my life fit together to describe how I have come to this research question. My first experience in understanding how the pieces fit together was about the time that I began doctoral studies. I went to see the movie "How to Make an American Quilt" (1995). Essentially, the movie was about a group of old friends who were making a wedding quilt. Each individual in the group created a square that in some way depicted their own story about love. Each square was brought together to form this quilt. The movie itself was a kind of quilt, a collection of individual stories brought together to describe the idea of "love" and marriage.

At that time, I saw a connection between this movie and a classroom. If we look at the entire group of individuals in the classroom as a quilt, then each individual in the classroom brings their own story to the class – each individual contributes an individual patch to the quilt. The children and the teacher are quilting together; as head quilter, the teacher brings all of these individuals together to form the quilt - the quilt being symbolic of the class's shared experiences. Since that time, I have come to consider that my own research can also be a quilt and that the pieces, or the fabric, of my research include the experiences in life that form the "quilt" that I am currently living and working with.

The difficulty with this metaphor is that the quilt appears to be static; yet, I have come to understand that the process of quilt-making is a powerful metaphor for a research

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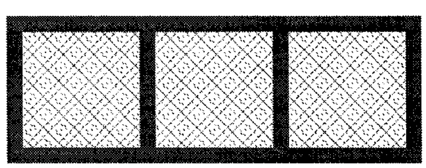
[2] I elaborate on enactivist views and principles in Chapter 2.

life. In my use of this image, the quilt also provides a powerful metaphor. The quilt is one thing; it may appear to be static. Yet it is also a composite of individual choices, memories and experiences. So, too, we might say that we know an individual based on what we see. However, as individuals we each come to be who we are at a particular point in time by living a variety of experiences. So, although any individual—a researcher, a teacher, a student or a policy-maker—may seem to be 'static' and 'unchanged,' this is just not so. In fact we are constantly in a state of change because we are complex beings. The experiences and stories of our experiences are ever-changing. However, as in the movie, we cannot see the whole quilt until we have heard and noted the stories of each experience that is added to the quilt at a particular point in time. Who I am as a researcher can be pieced together through the stories of my experiences leading to this research life.

This dynamic quilt, or a fractal quilt, is in a constant state of transformation. But rather than changing simply by growing larger through the addition of new pieces, this quilt transforms as new pieces are added and existing pieces are transformed. My own quilt of experiences as a policy-maker, as a teacher, as a researcher, and as a daughter in my own family, illustrates how this quilt of experiences comes together.



Piecing Together my Teaching Life

This piece of the quilt has early images: a child running from a school bus, and a blackboard. My most vivid memory of my childhood is of my mother telling the story of my first experience with school, my first day of school. My mother would tell me how I ran from the school bus after my first day of school and announced that when I grew up I was going to be a teacher. I spent my first two years of school racing home each night to "teach" my sister all the "stuff" I had learned in school on any particular day. My Christmas gift that first year of school was a blackboard. Many of my noon hours were spent 'tutoring' my classmates and when my friends and I played together, we often played school. I, of course, was the teacher. If I could not find any 'real' students, my mother said that I would be 'teaching' my toys.

My entire school life involved making decisions about how I could achieve my goal of becoming a teacher:

> My whole school life circulated around that - always making choices as to going to university to become a teacher (I guess that that only happened when I was old enough to know that I had to attend university in order to become a teacher.) It was truly never a question in my life as to whether or not I would attend university - I just would! By the time I was in high school though, clearly English was not a strong part of studies - I did not do very well in that subject area - I guess not doing very well meant working around the 60% mark. I took English 30 in the first semester of grade 12 and received a final mark of 55%. Now this worried me (I wonder now why it didn't bother me before. Perhaps I don't remember it bothering me) - but I thought that I might not 'get into university' with a mark like that in English. Most importantly though I thought that 55% in English 30 meant that I could not write - so when I was looking at the university calendars to find out which faculty I could study in, a big part of my decision was based on the English 30 mark. I wanted to be a mathematics or a music teacher - when I looked at the Faculty of Education, you needed to take an English course; when I looked at music - you needed to take an English course; but when I looked at science with a degree in mathematics, I discovered that (at that time) you could take a science degree without an English course - so guess where I enrolled? Although I still wanted to teach, I completed my science degree, took a year off, worked as a radio operator and typist, and then applied for the Faculty of Education. In an after degree program, you didn't need to take an English course. Well, I was on my way to reaching my goal of being a teacher. I finished the B.Ed. in December and started teaching in January. After about 6 months of teaching I was feeling lost, I had always worked for becoming a teacher and now felt that I didn't have any goals to work towards so I decided to study music again and work at becoming a phenomenal teacher!

Personal Reflection, September 17, 1996

In these first experiences around teaching, teaching was couched in a public school situation. However, my professional life has circulated around many types of experiences since my experiences in a high school mathematics classroom. I began to do some work with Alberta Education when I was close to completing my master's work. I was hired, as a consultant, to write teacher resource manuals to support the implementation of a new high school mathematics curriculum. These resource manuals were intended to provide teachers across Alberta with examples of teaching strategies that might help students to meet the intended goals. The curriculum revision introduced the following ideas into high school mathematics: students communicating mathematically; being able to justify and explain their thinking (reasoning); problem solving; and making connections among mathematical concepts and between mathematics and "the world outside mathematics."

At the time, I believed that my role was to create a resource to teach teachers about how the curriculum might operate in the classroom. The teacher resource manuals were full of activities teachers might use to promote communication, reasoning, problem solving and connections in their classrooms. It was very disappointing to learn that after two years of research and writing, few 'experienced' teachers saw these teacher resource manuals as a resource. I learned a lot from these experiences - and mostly what I started to think about was what it meant for a Ministry of Education to "create a curriculum" and then what it meant to "implement a curriculum." I also started to think about how teachers interpret curriculum documents.

From this experience with curriculum implementation, I moved into managing the development of a provincial assessment program - a program that has students write a final examination at the end of their grade 12 year. The final examination for the course contributed 50% to their overall final mark. My role was to interpret the curriculum and manage the development of the exam so that it would reflect the philosophy of the curriculum. In this work and in working with many different teachers from across Alberta, I was struck with the widely differing ideas about the intent of the curriculum. It occurred to me that if I worked with 700 teachers over those 5 years then there were at least 700 different interpretations of the curriculum when, at the beginning, I had thought there was only one. What fascinated me about working with all of these teachers was that when we came together to talk about a common topic such as performance standards, my understanding of the topic emerged and changed. As we participated in these conversations, I learned so much from other people; at the same time I believe that they learned from me. It was the interaction among us that helped me to strengthen and come to know what I understood at that point in time about a particular topic.

It also seemed to me that, each time I participated in a conversation and a group discussion, I could think about a topic in many different ways - hence my understanding of the topic was emerging and changing. Probably the most significant impact on my thinking though over these years of working with Alberta Education was in my thinking about my role as a teacher. Certainly, in writing teacher resource manuals, my role as a teacher had been very much one of 'telling' or 'suggesting.' Gradually I began to work as a teacher in creating occasions for learning about a particular concept. I had a role in the conversations and discussions about the particular concept, but our collective thinking and understanding were emerging as we were participating.

Piecing Together my Research Life

My first recollection of thinking about 'research' was during my first two years of teaching. I had read an article about interaction patterns between mathematics teachers and their students. The article suggested that mathematics teachers tended to interact more often with boys, and in some way the conclusion of the article was that, because mathematics teachers spent more time talking and interacting with the boys in their classroom, boys generally outperformed girls in mathematics. At the same time, I was reading an article about general interaction patterns that teachers have in classrooms; for example, if you were right-handed then you would interact more with students on the right-hand side of the room. I decided that I would 'collect data' on myself and observe my pattern of interacting with my students. Well, indeed, I did interact more with students on the right side of my room. Not only did I interact with them more often, I always started moving around the classroom from that side of the room. I remember one class in particular where there was a group of boys in that spot as well. However, the gender-related issues did not appear to emerge in my observations, although one class had a group of boys that sat on the right hand side, another class had a group of girls. What this experience did teach me though, was that I was able to reflect on my actions as a teacher and think about the implications of my actions on student's learning.

On making a decision to attend graduate school though, many fears of inadequacy in my writing re-emerged. The following reflection shows how, with support, I was able to begin to grow beyond these fears of inadequacy.

> Teaching was natural for me though - it always just seemed 'to happen.' About 4 years later, I started thinking that I wanted to go to school again - but didn't think I had high enough marks to get into a masters program - and even if I did, what area would I go into? My mind churned over this

> many many times - until one afternoon in April and I was sitting in a meeting with some colleagues and school administrators - everyone of them had a master's degree in education - I think all of them had their degrees in Administration - as I looked around the room, I thought 'well, if these guys could do it, I can.' I knew that I would have to write in a masters program - so decided to apply for both a graduate diploma program and a masters program in educational administration. Well, I got accepted into both programs - completely confusing the Department of Educational Administration - and in the long run decided I would try a course-based masters degree (I then wouldn't have to write a thesis). That first year, I also registered for a learning to write course (this was at the time when the University of Alberta had the English test for undergraduates and then they took sort of 'upgrading' courses.) Well, the due date on my first critique came and went - and I didn't hand one in. The professor asked me why I didn't and I, of course, replied that I hadn't written it yet because I couldn't write and that I had signed up for the course about learning to write - well my professor suggested that I write the critique and she would read it for my writing! The result of this activity was that the professor told me that she thought that I could write as well as or better than most of the people in the program - and I was surprised - with a 55% in English 30! I was still hesitant about writing - but with each paper and assignment it got a little easier - in the second term I even decided to switch into a thesis-based program because I got a chance to see some theses and how they were written.
>
> Personal reflection September 17, 1996

My master's research was very much a quantitative study, one in which a statistical analysis was performed on the data and where I talked about statistically significant results and looked at and interpreted measures of reliability and validity. The study, High School Students' Satisfaction with School Experiences, although highly quantitative, also asked students to identify their most satisfying experiences and most dissatisfying experiences in school. Interestingly enough, for both of these dimensions, experiences with their teachers were most often mentioned. As I was interpreting these data, I was drawn to the stories of the experiences that students were sharing. The prevalent theme to these stories of experiences was teacher actions.

Another experience with a "research project" following my master's thesis was when I was working with Alberta Education. I coordinated Alberta's involvement in an international study of mathematics achievement. I was involved in replicating a research design in Alberta. This meant that I needed to identify students who would participate in the research and collect data from those students. In this study, the instrument and design were already in place; I needed only to put the structure in place to collect the data. The researchers who initiated the project analyzed the data.

What is interesting about my research experiences in my classroom and then the research experiences that I had subsequently, is that my first experience was as a classroom teacher/researcher while in the second two examples, I was very much an observer and researcher "from the outside looking in." Later, I noticed that my view about research was changing from an observer to a participant:

> About a year ago, I decided to apply for a Fellowship from the Alberta Teachers' Association. When I look back on that application, I realize how cold and impersonal my writing was. I was truly describing researching teaching from an outsiders' point of view, although interestingly enough the topic that I described is not currently very far from where I am now, I used language like "I will observe" and "I will interview." When I look back on my SSHRC application this year, I noticed how my research methodology language has changed to language like conversations with and reflecting upon my own teaching being included, I think it demonstrates how my thinking is changing.
>
> Personal Reflection, December 18, 1996

Piecing Together my Narrative Life

The oral tradition of story telling is a part of my history. My family history includes storytellers of the aboriginal, Scottish, French and English histories, of the merging together of these cultures. Sharing stories of experiences was a part of growing up within my family, stories from and about my grandparents', great grandparents', great great grandparents' and parents' lives. However, even thinking that this tradition contributed to my teaching life began to occur to me only a few years ago:

> My whole adventure with thinking about narrative and my area of research interest did begin many years ago when I was working with Alberta Education. Narrative came naturally when I taught (or as some people would prefer to call it, when I 'did' a session at a conference or a professional development day). As I reflected on my teaching, I realized that my most 'successful lessons' were when I was able to relate my own story as a teacher thinking about assessment, for example - the one I use frequently is one that really did occur in my classroom and really did cause me to alter my way of thinking about assessment of student learning. I will share this question with you - the experience happened about 13 years ago - I had read about writing in mathematics classrooms and decided to ask my students the following question on an examination "Explain why $\frac{1}{2}+\frac{1}{3}=\frac{5}{6}$." I expected my students to write out the algorithm in describing how to find a common denominator or to draw and describe a picture showing the addition of these two fractions. One of my students saw this question another way and replied in the following way "Well Miss Glanfield, $\frac{1}{2}+\frac{1}{3}$ does not equal $\frac{5}{6}$ and this is why. Look on this test, your question number one is worth two marks and your question number two is worth three marks. Now if I got one of the two marks on question number one and one of the three marks on question number two then I would have a total of 2 out of 5 marks. So $\frac{1}{2}+\frac{1}{3}=\frac{2}{5}$." Two things immediately hit me when I saw this response, one was 'wow, imagine if I had only asked the students to add one-half and one-third without the explanation, this student would receive the answer wrong and I wouldn't know why - I would only assume that he didn't know how to add the two fractions;' and my second immediate reaction was 'let's look at how I indicate marks on my tests and sure enough I always wrote the marks as a ratio with no numerator as the student's mark would appear as the numerator.' It is amazing the number of teachers who are in awe when I tell this story because most of the people that I know have asked questions without the explaining and have indicated the score for a question in the way that I did. As a teacher in a mathematics classroom my student's response also left me thinking about the mathematics that we teach and why we teach it as well as what do these numbers mean without any context? My student clearly was thinking of fractions as ratios and all the rules that he had learned about adding fractions do not hold when you add ratios - also as a teacher I needed to be clearer in how I asked the question if I wanted my students to tell me about the addition of fractions. This story about my experience with assessment and thinking about assessment helps my 'teacher students' in my 'sessions' to develop a context that they can connect with a story of their own.

My second thought about the role of story was when I was asked to chair a gathering of people who all worked within the Student Programs and Evaluation Division with Alberta Education. There were about 100 or so people, all with teaching experience, meeting at the middle of June - the Assistant Deputy Minister called the meeting. The intent of the meeting was to share successes of the projects that were 'in progress' or 'completed' during the school year. This was also a follow-up to a meeting that occurred in September so that we would all be aware of how our work 'fit' into the work of the Division and of the entire Department of Education. I was feeling a bit apprehensive about chairing this meeting as many people were frustrated that they were asked to attend - it was taking them away from a day of work on their own projects and many of the individuals saw it as an opportunity for certain projects to be 'highlighted' - illustrating that the highlighted projects were more important than all of the individual projects that were continuous. . . . Now you know that I just love doing things like this - participating in these 'whole picture' discussions and making connections among projects - so I readily agreed to attend and act as a chairperson for the meeting. When I opened the meeting I told a short story - a story of how I was thinking about the meeting and why were they all here. I asked people to close their eyes as I told the story so that they could imagine their own experiences. My story related to the anxiety I would feel at the beginning of a school year - walking back into the school after some vacation, anxious to visit with my colleagues, anxious to meet new colleagues, anxious to know what my classes were going to be like, anxious to know what subjects I was going to teach, and generally anxious and optimistic about the new school year - every thing was exciting and I looked forward to it all. Our September meeting was like that - I want you to think about how you felt when you were meeting new colleagues and hearing about new projects in September - it really felt like the beginning of a year in the school. Now it is June and I want you to think about the feelings that you had in June in your school - some were feelings of relief, some were feelings of 'will I finish everything I need to finish', some were joy with regards to the students we had met and knew we had influenced, some were sadness with regards to those students who we had met and felt that we did not have much influence with, some were anxious in terms of what will happen next year - all of these feelings but we always had a chance to think about our past school year. This June meeting is intended for you to have the chances to think about 'where you have been' and 'where should I be going' for the next school year. Well, when I relayed my story about thinking about this meeting you wouldn't believe the change in the 'sense' of the room - one that was quite tension-filled to one that was more relaxed and inviting. But it was through sharing a story that once again helped people create their own context and internal story for this meeting.

> Of course, with both of these cases I initially didn't think of the power of the stories - I only thought about this as being a good teacher - a good teacher also helps students to identify where they are at and provokes students to build their thinking from those experiences.
>
> More directly related to my graduate school experience was when I was first taking the curriculum theory courses last year - I realized myself that I was much more attuned to the class discussion and thinking when I would hear how my classmates were bringing meaning to the readings - often the sharing of those meanings was being done through story - things we thought about when we read and thought about the articles.
>
> During these experiences with story, a good friend of mine was working on her research in the area of how women continue to learn in the work place and each time we would meet she would be probing and asking questions of me - when I reflect on these experiences, I realize that most of my responses would always include a story of some sort so that my friend could understand the context and the experiences from which my responses were coming.
>
> Personal Reflection, December 18, 1996

I also started to think about the role of sharing stories about experiences in a project that I was working with:

> I convinced my colleagues that our 'debriefing' meeting should contain time for each of the lead teachers to tell their story about being involved in our 'process.' However, I just didn't 'let the stories' unfold, I also asked some very pointed questions so that we could get some 'real' sense of the work with, and of course, because my interest is in how teachers grow, their sense of how their teaching changed over the past few months. Of course, we also wanted to know their stories about being involved. It was important that, not only did the lead teachers share their story but that my colleagues and I also share our stories about what we learned and thought about during this project. Of course, one of the things that was most exciting is that everyone wanted to stay and participate for a second year.
>
> Personal Reflection, March 10, 1997

My teaching, research and narrative lives all describe who I am and point to my interest in the research question "In what ways do mathematics teachers grow in their understanding of mathematical processes?" These 'lives' are the threads that form the fabric of my quilt. The threads weave together to describe who I am. At the same time, the threads continue to weave. I know that my perceptions of the ways in which I see my role in the way in which mathematics teachers grow in their understanding of mathematical processes has changed, from a way where 'I as the expert just needs to tell the teacher how to do it' to where 'I can only share what I know, a teacher is a knowing individual, and they will make sense of the ideas in their own way.' The research and this writing about the research can be thought of as a 'quilt-making process.' The quilt, like the research, continues to develop and takes on many forms in its development. As the quilt-maker develops her skills, some of the process becomes easier; as she extends her reach into new areas, it becomes more challenging and more rewarding.

Ultimately, this quilt-making process will tell a story, a story that will illustrate the weaving together of theory and lived experiences. This particular story is one developed by four experienced teachers, three colleagues and myself. My passion for creating this quilt is illustrated in this personal reflection:

The Theory Unfolding in the Story I'd Like to Tell

> This thesis comes from a passion, a passion about learning. I am so very tired and frustrated (I've sat through so many meetings and conversations in my professional life, hearing that teachers need more mathematics courses) that there is a perception both in the teaching community and the non-teaching community that teachers are not professionals and do not continue to learn. I want this thesis to show just in what ways teachers are always in a state of change and learning and that formal (i.e. course bound) education is not the only way in which that learning/changing occurs.
>
> I think that this thesis is really about telling my story in a way. I had all of the credentials when I started my professional life and yet I didn't really know much about teaching. Over the years that I taught high school, however, I realized I was continuing to come to know and it was through the conversations that I was having with my colleagues. It was through sharing the development of teaching units and student assessments and generally meeting to talk about mathematics on a somewhat regular basis (I don't quite remember how often.) It was through these beginning

conversations that I began my development as a professional. These conversations stimulated my thinking about teaching and students learning.

I think that the importance of these conversations really came to light when I worked with Alberta Education. It was during these times that I realized that I was in a state of change, of expanding the possible. I consistently was engaged in conversations with teachers and government officials about tough ideas about learning mathematics and what this means for thinking about teachers teaching.

These events helped me to realize how I learn. For example, it always took me a lot of effort to read an article on my own and to make sense of that article. It was only in conversation, with peers, about the article that I began to make sense about the article. I realized that in my talking I was making sense. I realized that the 'talking through' an idea is the way that I come to make sense of the idea. In the 'talking through' my previous experiences connecting with the idea would emerge. In my talking I realize what I know and then it becomes known to me. It seemed to me that this was how I'd always come to know. I remember 'tutoring' my friends in high school mathematics. It was in my 'talking through' of the mathematical concepts with my friends that I came to realize what I knew about the concepts.

I also began to realize that I couldn't just be making sense of ideas but that I needed to think about the way these ideas were enacted in my life. I think about how my parents always told me that, 'words don't really matter, it's your actions that are remembered.' (I'm not sure if this was the exact phrase, but it is something like that). I think that this idea consistently caused me to think about my actions as I came to see myself as a leader in mathematics education. I needed to not only talk the talk but walk the talk. Hence, my action in my own teaching became important. What does all of this mean? In my talking with colleagues I come to realize what I know, what I know becomes known to me, and in my action I come to realize what I know looks like, and then when I talk about my action I come to realize what I know and how who I am has changed, and the recursiveness continues.

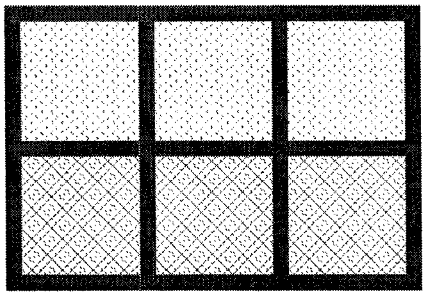
For me then, knowing is being and being is in community. Having said this, I feel uncomfortable with our schools and the way in which we treat knowing. I've never had a sense that I was a good student, in the traditional sense of the word. What I mean by traditional is the notion of obtaining high marks, most classrooms are not interactive in nature and thus many school classrooms do not support learners like myself. When I think about professional learning, most teacher professional development does not support learners like myself. I was even a part of the system until

> I realized what I knew about how I know.
>
> This is why enactivism was so very exciting for me to learn about; it began to describe the way in which I've come to know. So, in this thesis, I want to show in what ways enactivist views and principles provide a way of thinking about teacher learning.

This passion folds back to the research question, "In what ways do mathematics teachers grow in their understanding of mathematical processes?" This question led me to a framework of enactivist views and principles as one way to think about teacher understanding. I will lay the path for discussing this question by first of all discussing the theory, introducing the three other teachers in my community, and looking at how, through conversation about standards, theory and classroom practice (Richardson, 1990) I might be able to describe the ways in which mathematics teachers grow in their understanding of mathematical processes.

CHAPTER 2

Backing for the Quilt



What are the fibers that make up the backing of this quilt? What are the materials that piece together the patches in this quilt? I am thinking of the theoretical structure of my research as the quilt's backing and its connecting material. The backing is literally a background that holds the individual pieces together. What is the nature of this material? Why did I choose this material over other material? From where is this material drawn? The 'backing' for this research quilt is located within the fields of mathematics education, enactivism, narrative inquiry and narrative knowledge.

Within the Mathematics Education Community

Within the mathematics education community, we are coming to realize in new ways that teachers are learners in their practices (Ball, 2003), just as we expect students to be in theirs. I want to consider a means by which we might describe how teachers grow or change in their understanding of mathematical processes. These are the aspects of school mathematics that policy makers and curriculum developers are asking teachers to think about. Usiskin (1997) describes how, thirty years ago, policy makers focused on asking students to become mathematicians by placing a heavy emphasis on mathematical content. Under that view, teachers were considered to be knowers of mathematics and in their practice were expected to impart that mathematical knowledge to their students. In that model, growth in teachers' knowing would require knowing more mathematics.

During the 1980s and 1990s, policy makers focused on the role of teacher practices in the classroom. They sought to introduce new teaching practices. These new aspects of practices were highlighted in the NCTM's *Curriculum and Evaluation Standards for School Mathematics* (1989) and the *Professional Standards for Teaching Mathematics* (1991). *Curriculum and Evaluation Standards* (NCTM, 1989) identified the content that should be taught, provided examples of how it should be taught, and suggested ways that student evaluation of that content should be conducted. *Professional Standards for Teaching Mathematics* (NCTM, 1991) identified standards for teaching mathematics, the evaluation of teaching mathematics, and the professional development of teachers of mathematics. Throughout the early 1990s, NCTM published many supporting documents to help teachers implement the ideas contained in these two seminal documents.

Provincial governments across Canada responded to the NCTM's *Curriculum and Evaluation Standards for School Mathematics* (1989) in particular, by creating two types of curriculum documents. One identified what students were supposed to learn, the curriculum itself; teacher resource manuals identified how teachers might teach students.

During this time, publishers also changed the way textbooks were developed. Prior to the NCTM's *Curriculum and Evaluation Standards for School Mathematics* (1989), most textbook publishers only produced material for students to use, that is, a student textbook. Little attention, if any, was given to the development of teacher support material. The teacher's copy of the student's mathematics textbook was often a duplicate of the student textbook with the answers given or a few other suggestions printed in it.

Textbook development in the 1990s began to look dramatically different. Textbook companies, encouraged by provincial ministries of education, started to pay as much attention to the development of teacher material as they did to the development of student material. Textbook series developed during the 1990s included detailed teacher support material that suggested a variety of ways that the material could be taught and assessed, provided historical perspectives of the content, and outlined a variety of activities that could be used to enhance the material presented in the student textbook. State departments of education and textbook companies responded similarly to the NCTM's *Curriculum and Evaluation Standards for School Mathematics* (1989).

The underlying assumption in the development of all the teacher support material was that classroom teachers were currently not, nor did they have the skills for, implementing the suggested instructional strategies and practices envisioned and described in documents like the Curriculum and Evaluation Standards for School Mathematics (NCTM, 1989) and the provincial curriculum documents.[3]

In all of this work to define and implement provincial curriculum documents, the way in which teachers themselves bring meaning to the curriculum was neglected. Although a lot of time and effort was spent in sharing the intent of the documents, little time was given to providing opportunities for teachers to "examine and revise their assumptions about the nature of mathematics, how it should be taught, and how students learn mathematics; reflect on learning and teaching individually and with colleagues; and participate actively in the professional community of mathematics educators" (NCTM, 1991, p. 160, 168). As Clark and Florio-Ruane write, "the time has come for a radical

[3] This comment is based on my personal involvement in writing and developing teacher support material for a textbook company and for a provincial government.

shift in thought and action in support of sustainable teacher learning and teacher research. This shift is needed to engage teachers as reasoning and responsible professionals in the process of refining their knowledge" (2001, p. 6).

In 2000, the NCTM published the *Principles and Standards for School Mathematics*, which identified teaching itself as one of the six principles of a "high-quality, engaging mathematics program" (p. 11). The teaching principle suggests "effective teaching requires reflection and continual efforts to seek improvement" (p. 17). Several authors have also asserted that teachers' understanding of mathematics is fundamental to improved student learning (Ma, 1999; Aubrey, 1997; Ball,[4] 1988). As I read these texts, the authors imply that teachers need deep understanding or profound understanding of fundamental mathematics (Ma, 1999). Simon and Tzur (1999) would call these deficit studies; studies that focus on what teachers "lack, do not know, or are unable to do" (p. 255). Simon and Tzur (1999) indicate, "deficit studies cannot provide comprehensive understanding of teachers' perspectives. They do not inform teacher education about the teachers' knowledge that might be built upon" (p. 255).

The texts of studies like Ma (1999), Aubrey (1997) and Ball (1988) appear to suggest that once teachers have developed this understanding, or acquired such understanding, that they will then become effective teachers. That is, in their view, understanding is an acquisition, a specific body of knowledge or a set of rules for specific practices.

How would this statement change if we consider that understanding is generative and dynamic? How would this statement change within a view of teacher knowledge that

[4] You will notice that I reference Ball on more than one occasion. Ball's work has shifted focus since 1988. Her recent work focuses on the question "what mathematical knowledge is needed to teach mathematics?"

was more than knowledge of subject matter and static? How might we begin to describe the way in which teachers' understandings grow in this perspective? Based on evidence from work with teachers who are considered sound practitioners, I will attempt to provide ideas that allow us to think about how or in what ways teachers' understandings grow or change. (In this work, the notion of growth and change are used synonymously.) The enactivist view of cognition provides a useful framework for reflecting on how our understanding grows.

Within Enactivism

Enactivism emerges principally from Bateson's (1979, 1987) work in ecology, and Varela, Thompson, and Rosch's (1991) work on embodied knowing. These works can be related to a body of knowledge on the biological roots of human knowing and to a philosophy of thinking about such knowing found in the works of Maturana (e.g., 1988, 1991) and Maturana and Varela (e.g., 1980, 1987), as well as in process views of knowledge (e.g., Maturana & von Foerster, 2000).

Within the theory of enactivism, cognition is seen as "embodied action." Varela, Thompson, and Rosch describe embodied action in this way:

> By using the term *embodied* we mean to highlight two points: first, that cognition depends upon the kinds of experience that come from having a body with various sensorimotor capacities, and second, that these individual sensorimotor capacities are themselves embedded in a more encompassing biological, psychological, and cultural context. By using the term *action* we mean to emphasize once again that sensory and motor processes, perception and action, are fundamentally inseparable in lived cognition. (1991, p. 172-173)

They further describe an enactive approach:

> The enactive approach consists of two points: (1) perception consists in perceptually guided action and (2) cognitive structures emerge from the recurrent sensorimotor patterns that enable action to be perceptually guided. (Varela, Thompson, and Rosch, 1991, p. 173)

Varela (1992) further states that

> cognition cannot be properly understood without common sense, and this is none other than our *bodily and social history*, the inevitable conclusion is that knower and known, subject and object, stand in relation to each other as mutual specification: they arise together. (p. 253)

Davis and Sumara (1997) elaborate this view of knowing:

> [T]he cognizing agent is recast as *part of* the context. As the learner learns, the context changes, simply because one of its components changes. Conversely, as the context changes, so does the very identity of the learner...both the cognizing agent and everything with which it is associated are in constant flux, each adapting to the other in the same way that the environment evolves simultaneously with the species that inhabit it. (p. 111)

Davis (1996) indicates that

> from an enactivist perspective, "understanding" is discussed in terms of action rather than conceptual structure. Words and concepts are interpreted as patterns of acting so that shared understandings are possible. Understandings are not merely dynamic, they are relationally, contextually, and temporally specific. As one moves from a particular situation, one's understandings, as revealed in one's actions, may change dramatically. So, while understandings may be shared during moments of interaction, they inevitably diverge as the participants come back to their "selves". (p. 200)

MacDonald, a science educator who also situates his work in enactivism, indicates that "understanding or meaning making is not about stepping out of and contemplating the world. It involves noticing how we are a part of the world and how we are connected to it" (1996, p. 29).

Maturana and Varela write that

> The phenomenon of knowing cannot be taken as though there were "facts" or objects out there that we grasp and store in our head. The experience of anything out there is validated in a special way by the human structure, which makes possible "the thing" that arises in the description. This circularity, this connection between action and experience, this inseparability between a particular way of being and how the world appears to us, tells us that *every act of knowing brings forth a world*. This feature of knowing will invariably be our problem, our starting point, and the guideline of all that we present in the following pages. All this can

> be summed up in the aphorism *All doing is knowing, and all knowing is doing.* (1987, p. 25-26)

In an enactivist view of the world then, "we cannot stand outside the world in which we find ourselves, to consider how its contents match their representations of it: we are always already immersed in it" (Varela, 1992, p. 252). Hence, "cognitive capacities are inextricably linked to a history that is lived, much like a path that does not exist but is laid down in walking. Consequently, the view of cognition is not that of solving problems through representations, but as a creative bringing forth of a world" [5] (Varela, 1992, p. 255).

In the enactivist view of cognition then, bringing forth of a world is the "burning issue of knowledge" (Maturana & Varela, 1987, p. 27) and that it is "associated with the deepest roots of our cognitive being, however strong our experience may be" (Maturana & Varela, 1987, p. 27). Bringing forth of a world "manifests itself in *all* our actions and all our being" (Maturana & Varela, 1987, p. 27) and "it is the ongoing process of living which has shaped our world in the back-and-forth between what we describe as external constraints from our perceptual perspective and the internally generated activity" (Varela, 1992, p. 253).

Kieren poses enactivism as a view of mathematical knowing which "observes knowing as occurring in the <u>inter-action</u> of the individual and the world which (s)he is shaping and in which (s)he is acting" (1997, p. 16) and "how each student thinks or acts with respect to a mathematical situation is fully determined by his/her structure and his/her lived history of mathematical actions in relevant situations" (Kieren, 1997, p. 17). In my research, teachers are the individuals and are a part of the environment in which they finds themselves acting and inter-acting. Through the occasions in discussing and thinking about mathematical processes with me, with colleagues in the research group, with their students, and on their own reflection (Maturana & Varela, 1987) they continually bring forth a world—that is to say, the world in which they each are living.

Developing understanding then, from an enactivist perspective, is "an ongoing interpretation which cannot be adequately captured as a set of rules and assumptions since it is a matter of action and history, an understanding picked up by imitation and by becoming a member of an understanding which is already there" (Varela, 1992, p. 252). In classrooms, enactivism sees teaching as "in-the-middle which is a way in the middle between a completely teacher centred and a student centred view" (Kieren, 1997, p. 16). A teacher lives out some portion of the understanding, which is already there, and has a "special responsibility to help the students be in touch with the larger world of mathematics and to help students see themselves as part of that larger community" (Kieren, 1997, p. 14). Hence the teacher occasions (Kieren, 1997) students' thought/actions and helps them see that they are responsible for their thinking within the community, or within the understanding which is already there. That is to say, the teacher participates in, but does not determine student learning. The action of the student is not caused by the environment or the teacher, but is "determined by the students' own lived histories of actions" (Kieren, 1997, p. 10).

In my story (at the end of Chapter 1), my individual mathematical understanding in high school grew out of my actions and explanation of the actions when I was tutoring my peers. When I was in class, taking notes did not make much sense to me nor did

[5] Notice that the bringing forth of a world of significance is done through the structures of the person and is co-determined by the environment in which this world building occurs.

reading the articles. It was only in the explanation of those actions when I was tutoring my friends that my individual mathematical understanding developed. Similarly, in discussing the articles I made sense of the ideas presented. Kieren writes that "in an enactivist practice students (and the teacher) are expected to explain their thinking for the community; the teacher deliberately fosters practices that will do that. It is in actions and explanation of those actions that individual understanding grows" (1997, p. 18). In this research project, my role was to be a teacher deliberately fostering a practice where we as teachers[6] were expected to explain our thinking in the community that we were building.

Participants in an enactive community are expected to explain their thinking. As the researcher, I fostered practices that encouraged this act. Similarly, from an enactivist practice, students (and the teacher) are expected to explain their thinking for the community; the teacher deliberately fosters practices that will do that (Kieren, 1997, p. 18). From an enactivist perspective, it is in action and explanation of those actions that individual understanding grows and that the structures of everyone change. Simmt (2000) and Simmt and Kieren (1999, 2002) modeled this idea of bringing forth a world in the following figure.

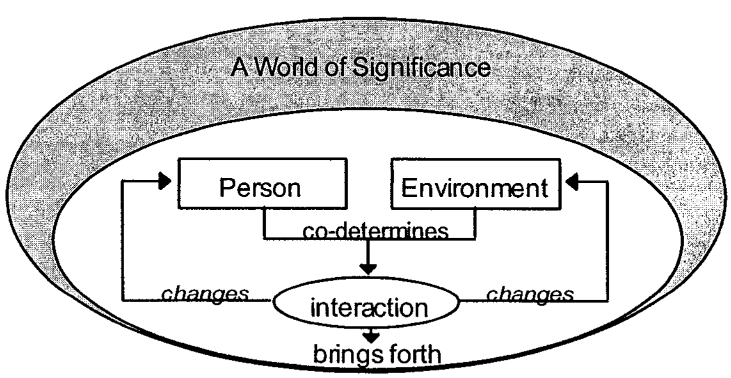


Figure 1. Simmt and Kieren's model for knowing in interaction. Used with permission.

[6] Three teachers, Marilyn, Joyce, and Julia participated in this study with me. A more detailed account of

Kieren and Simmt (2002) write that this

> diagram allows us to observe both the individual determinants and the social/environmental co-determinants of knowing all-at-once. Such knowing and the related understandings are observed then as coemerging phenomena. . . The knowing actions and representations of the individual can become occasions for the understanding acts of others (for example, other students, the teacher or the researcher) and for changes in the environment in which they exist. . . It is in this way that the cognitive domain can be observed as increased. (p. 2)

What might the schemata resemble in terms of this research? In this research, the teachers and I met on several occasions. During our times together we discussed the provincial curriculum, mathematical processes, and classroom activities (standards, theory, and classroom activities as suggested by Richardson, 1990). Each time we came together as a group or as a mathematical community (Davis and Simmt, 2003) the environment for each individual included each of the other individuals in the group. For example, my environment included each of the three teachers and the conversation in which we were engaging. Hence each individual's world of significance was a co-emergent phenomenon, because the environment and the world(s) of significance were being generated and co-determined in the interaction of the individual and others. The group, or mathematical community, was bringing forth a world of significance at the same time that each individual was bringing forth a world of significance. Pirie and Kieren (1994) describe the individual understanding as a dynamical process, one that can be observed in action and inter-action. Kieren and Simmt (2002) then suggest that the collective understanding is also dynamic and that it can be observed in the patterns in the actions and interactions.[7]

Teaching and learning are complex phenomena because they involve human

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each participant and our process is given in Chapters 3 and 4.

beings. Individually and in community, human beings are dynamic, unpredictable and inter-related (Davis & Sumara, 1997). Using narrative inquiry as a form of studying such complex phenomena resonates for me. Connelly and Clandinin describe the study of narrative as "the study of the ways humans experience the world" (1990, p. 2). Connelly and Clandinin write of narrative as both phenomenon and method, that is, experience is a narrative phenomenon that can best be understood by inquiring narratively into it. Bruner (1986), who writes of narrative as a mode of knowing separate from paradigmatic knowing, suggests that it is our "sensitivity to narrative" that provides the "major link between our own sense of self and our sense of others in the social world around us" (p. 69).

Bruner talks about learning in a similar manner when he relates it to an individual's culture: "I have come increasingly to recognize that most learning in most settings is a communal activity, a sharing of the culture. It is not just that the child must make his knowledge his own, but that he must make it his own in a community of those who share his sense of belonging to a culture" (1986, p. 127). Enactivism helps us understand how this happens. As the individual agents make meaning in context the context itself changes—the personal and his/her culture co-emerge. Narrative inquiry helps us inquire into the stories being lived and told and also helps us tell of meaning making/culture making in meaningful ways.

How is change considered within the enactivist theory? Sumara and Davis (1997) offer that growth or change can be thought of as "enlarging the space of the possible…we are collectively moving toward increased complexity" (p. 303). In interacting with one

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[7] Collective understanding is in this way a second or third order phenomenon to an observer. It occurs not simply in action and interaction but is about patterns in those actions and interactions.

another, with the researcher and with their students, it might be expected that teachers generate new interpretations of curriculum and new objects of practice, as well as forming new and more connections among these ways of seeing curriculum. In this study, I looked for patterns in the connected set of ways of knowing; that is, I investigated ways in which increasing complexity is a feature of growing understanding.

Within Narrative

In an enactive view, a person's actions, while coemergent with and co-determined by their environment, are fully affected by his or her lived history, and his or her structure, (Kieren, 1997) which includes schemes but also entails affective responses which can be observed as intentions to act in particular ways. Narrative inquiry resonates with this enactivist view. In narrative inquiry, stories are told by participants to describe their work and explain their actions (Connelly & Clandinin, 1990, p. 6). The study of narrative is the study of the ways humans experience the world ; narrative is a way of characterizing the phenomenon of human experience (Connelly & Clandinin, 1990, p. 2). Varela indicates that cognitive capacities are inextricably linked to a history that is lived, much like a path that does not exist but is laid down in walking (1992, p. 255). I am suggesting that we can understand and describe the history that is lived and the collective path that is laid down in walking narratively. As Connelly and Clandinin write, "when one engages in narrative inquiry the process becomes even more complex, for as researchers, we become part of the process. The two narratives of participant and researcher become, in part, a shared narrative construction and reconstruction through inquiry" (2000, p. 5). In this study, there are narratives of several participants, myself as researcher and participant, and the collective narrative of the participants within our research group.

Clandinin & Connelly write:

> In our work, keeping experience in the foreground comes about by periodic returns to the writings of Dewey (for example, [1916], 1961, 1934, 1938). For Dewey, education, experience, and life are inextricably intertwined. When one asks what it means to study education, the answer - in its most general sense - is to study experience. Following Dewey, the study of education is the study of life - for example, the study of epiphanies, rituals, routines, metaphors, and everyday actions. We learn about education from thinking about life, and we learn about life from thinking about education. This attention to experience and thinking about education *as* experience is part of what educators do in schools. (2000, p. xxiii-xiv)

In this research, a group of teachers and myself came together to talk about ideas related to implementing a new curriculum in the province of Alberta. The curriculum document described the outcomes that students were expected to know but also included the mathematical processes of communication, reasoning, problem solving, and connections.[8] The document suggested that students come to learn the outcomes by participating in these mathematical processes (Western Canadian Protocol, 1996). In a sense, the WCP authors were attempting to describe situations that teachers should create in their classrooms so that students could come to know the outcomes. Because I had often experienced the notion of coming to know within a community, I wanted to come to know how teachers made sense of the mathematical processes of communication, reasoning, problem solving, and connections. Therefore, I invited a group of my colleagues to join me in this conversation. For me, it is through the sharing and explaining in community, in authentic conversation (Clark, 2001), that I could begin to consider the ways in which mathematics teachers grow or change in their understanding of mathematical processes. Conversation, sustained among a group over time, is a

[8] Please see Appendix A for a description of the mathematical processes in the Alberta document.

powerful medium to explore ideas presented in the curriculum documents, because "the reconstitution of experience through personal narrative allows for safe exploration of uncharted territory and imagining the possible" (Clark & Florio-Ruane, 2001, p. 12).

Hopkins (1994) and Clandinin and Connelly (2000) also suggest that through narrative we are able to capture a sense of our changing selves in a changing environment. The stories we tell ourselves, of who we are, our stories to live by, become a way to represent our changing identities (Connelly and Clandinin, 1999). I am hoping that through narrative inquiry, I will be able to describe the enlargement of the space of the possible and the ways in which this might occur for teachers.

The language that we use in our interactions is critical in considering this research question. Maturana and Varela (1987) say that each reflection takes place in language and "every reflection brings forth a world" (p. 26). In our conversations with others, "we can count on constant transactional calibration in language, and we have ways of calling for repairs in one another's utterances to assure such calibration" (Bruner, 1986, p. 62-63). This points once again to an enactivist view. Varela might identify this calibration in language as "co-determination" (1992). Coming to know is co-determined by us as individuals, the environment and others, "all at once" (Maturana & Varela, 1987).

According to Bruner,

> language not only transmits, it creates or constitutes knowledge or "reality." Part of that reality is the stance that the language implies toward knowledge and reflection, and the generalized set of stances one negotiates creates in time a sense of one's self. The language of education is the language of culture creating, not of knowledge consuming or knowledge acquisition alone. (1986, p. 132-333)

In this research, the language that we use in our conversations about the curriculum, mathematical processes and the classroom activities, is what I will take as actions that reflect teacher understanding.

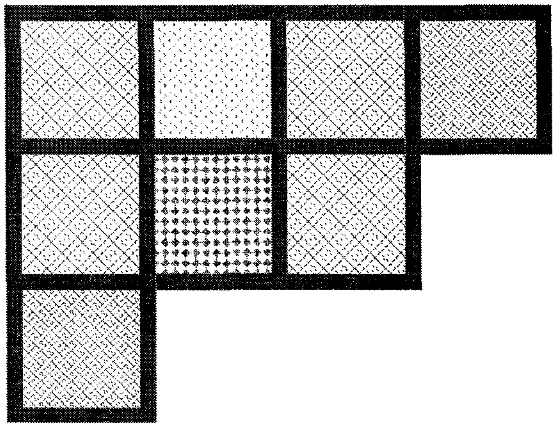
This thesis provides a space for conversation about describing the growth of teacher understanding. Readers of this thesis will remember that the ideas presented are meant to prompt further discussion about this important phenomenon, not to provide a definitive model or final explanation for that growth.

The emerging patterns in this theory, the patterns forming the fibres of the quilt's backing, show the ways in which growing understanding is manifest. The fibres are held together by dynamic theories of personal and collective understanding. These theories and their ideas are backed by enactive, narrative, and embodied views of knowing, views that see knowing as a complex, dynamic phenomenon.

In the next chapter you will meet the three teachers who were a part of the mathematical community that pursued the research question. Subsequently I will use the ideas of collective understanding (Davis & Simmt, 2003; Kieren & Simmt, 2002) and "dynamical personal understanding" (Pirie & Kieren, 1994) to suggest ways in which teachers grow in their understanding of mathematical processes.

CHAPTER 3

Gathering of the Pieces



Our research group—Joyce, Julia, Marilyn and I—became a mathematical community or a collective learning system (Davis and Simmt, 2003) over the course of our interactions together. Our mathematical community, or collective learning system, came together to consider the ways in which we could think about mathematical processes in our high school classrooms. Davis and Simmt describe five "necessary but insufficient conditions that must be met in order for systems to arise and maintain their fitness within dynamic contexts—that is, to learn" (p. 147). These five interdependent conditions are internal diversity, redundancy, decentralized control, organized randomness, and neighbour interactions (p. 147). Davis and Simmt suggest that a system's intelligence, or range of possibilities, is "dependent on, but not necessarily determined by, the variation among and the mutability of its parts" (p. 148).

Internal diversity within a system "must be assumed" (Davis & Simmt, 2003, p. 149) and I offer here the stories of each of the three teachers who joined me in our collective learning system. Each of us, within our collective learning system, brings our own story of lived experiences to the system. You have read my story in Chapter 1. In this chapter I share the stories of Joyce, Julia, and Marilyn, to explore the internal diversity of our mathematics community.

I will also use the stories to explore the condition of redundancy (Davis & Simmt, 2003) within our mathematical community. Davis and Simmt (2003) say, "sameness among agents…is essential in triggering a transition from a collection of *me's* to a collective of *us*" (p. 150). Further, Davis and Simmt (2003) write

> Redundancy is a complement to diversity. Whereas internal diversity is more outward-oriented in that it enables novel actions and possibilities in response to contextual dynamics, internal redundancy is more inward-oriented, enabling the moment-to-moment interactivity of the agents. (p. 150)

Introducing Joyce

It was a beautiful summer day, the sky was an azure blue and there were fluffy white clouds floating by. I was on my way to see Joyce so that we could begin to talk about my research project. As I made my way to the lake just south of Edmonton, a lake where Joyce and her family vacation each year, I thought about how I met Joyce and how we were now at this place in my doctoral research. We first met when Joyce was an undergraduate student in the Faculty of Education about 11 years previously. I was one of the graduate assistants for a secondary mathematics education course in which Joyce was enrolled. I was assigned as Joyce's university faculty liaison for her student teaching experience. This meant that Joyce was a part of a small group that met with me twice a week. It also meant that I was one of Joyce's supervisors during her student teaching experience. I had an opportunity to observe Joyce's beginning teacher practices. Several years after Joyce completed her degree, we met again when she started teaching at a local college and I was working with Alberta Education. Joyce has taught at the college, in the adult upgrading program, for about nine years. She teaches the same mathematics courses that are taught in the public high school system. The only difference is that Joyce's students are students who generally did not complete their high school program and are returning to school to complete it. Joyce's students write the provincial examinations, just as do the students in the public school system.

The scenery was beautiful; I could see farmers in their fields and children playing in their yards as I drove through prairie land dotted with groves of trees. I'd never been to Wizard Lake before and was surprised when Joyce suggested we meet at the lake, as she would still be on vacation. As Joyce was giving me the directions "turn left at the Esso station in Calmar and follow the road signs to Wizard Lake. Then turn left, right, and follow the lake. We are at campsite number 21, look for a grey truck," I wondered about this lake. I didn't know it existed and yet I had driven through Calmar many many times, on my way to visit my family. I arrived at the lake around ten o'clock in the morning, Joyce, her husband, and nephew were sitting at the picnic table, waiting for me. Joyce's husband and nephew decided to go for a walk so that she and I could talk. With a fresh cup of coffee in hand, I turned on my newly-purchased tape recorder and we began to talk. I asked Joyce to tell me about her teaching career and why or how she decided to become a mathematics teacher.

Joyce told me that she became a mathematics teacher because she always liked mathematics, that mathematics was easy for her, and that it made sense to her. She also told the story of her "really good" high school mathematics teacher who made mathematics exciting and relevant. Her teacher would always say things like "well here's where you can use this concept, and if you were doing this type of job, you could use this other concept there." At one point in time, prior to finishing high school, Joyce did consider enrolling in the Faculty of Engineering. However, because she "had experience teaching music and Sunday school lessons and because her grandmother was a teacher," Joyce decided to pursue teaching to become a mathematics teacher, because "if you like something, it's easier to teach it." We were interrupted by a motorboat on the lake and we changed our conversation to discuss the way in which motorboats pollute lakes.

Joyce continued by saying that she really liked teaching and had now been teaching for 11 years. In teaching, Joyce said, you meet different people every year so every year it's new and always exciting. Joyce attempts to make her math class interesting in the way that her high school teacher made it interesting for her, she tries to ask herself, in her planning, "where might my students ever use this?" or she tries to give her students examples of where they might see the mathematics on a daily basis. For example, when Joyce introduces the locus definition for the ellipse, she uses the shape of the A & W sign to illustrate the concept.

Joyce offered me a second cup of coffee and I asked her how she'd answer the question "what is mathematics?" I wanted to know about how she would describe mathematics, because I was thinking that, perhaps how teachers teach students mathematics might be related to how a teacher perceived the topic of mathematics. Joyce said that she sees mathematics as the "background or building block," but was not sure how to describe what she meant by the statement, so she added two examples, "For example, you need mathematics in order to build this trailer and you use mathematics when you are landscaping." Finally, Joyce concluded that mathematics affected a lot of things in the "background of our lives."

I also wanted to know how Joyce would describe herself as a teacher, in her classroom. Joyce described herself as "probably a fairly traditional teacher," where she would try to tell her students about why things are done the way they are done in mathematics, but would often "just give my students the information." Joyce's description continued with her thoughts about the teaching strategies suggested in the new provincially-mandated curriculum. She indicated that she believed that the intent of the new curriculum was for students to "sort of discover things," although she, as a teacher, would still "direct them." So, this would mean that she wouldn't just tell her students, for example, that "these are the rules for factoring," but that somehow, both she and her students would approach the rules for factoring "differently." She also shared that she would also have access to the computer lab at her school this year and that she would ask her students to buy a graphing calculator so that she could begin to integrate technology into her program.

Joyce had already considered the new course she was about to teach by looking at the new textbook she would use. She had examined the first few chapters of the textbook and noticed that there were features that she had not used before. (In the particular book that Joyce would use, there was a feature called "Explore and Inquire." In this feature, the authors of the textbook asked students to "examine this pattern" and "describe what you see happening in the pattern.") On initial examination, Joyce told me that she thought that this type of feature would be times when she could have her students working together in a group to come to a conclusion or a discovery and then there would be the opportunity to discuss their conclusions or discoveries with their peers. Another area that Joyce thought that she might try during the coming year would be to include math journals in her classes, because she believes that students have to be able to communicate mathematics:

"That's where we are heading," she said.

As I was preparing to leave, Joyce reminded me that she believed her role as a teacher would become "sort of like a facilitator" because she really did see that both the curriculum and the textbook as asking students to become more independent in their study of mathematics.

It was around noon when we finished our conversation. Joyce's husband and nephew had returned from their walk and they invited me to stay for an over-the-fire grilled cheese sandwich. The sandwich was tasty and Joyce and I made arrangements to meet with Marilyn and Julia the following week.

As I left Joyce, I marvelled at this remarkable teacher. She was truly interested in engaging in conversations about mathematical processes and implementing curriculum. She was very honest about her classroom practices and I knew that she frequently participated in professional development activities. She was at one of those activities when I asked if she would work with me on my research. I remembered when I was facilitating a workshop in June, Joyce was sitting near the back of the room, and when I saw her, I knew I wanted to ask her to participate in my research group. I walked over and briefly spoke with Joyce at the break and she did not hesitate to say yes to my invitation.

Introducing Julia

I don't remember the first time I met Julia, I only remember how often my supervisor, Tom Kieren, would use her name when he would talk about "amazing teachers" or whenever he would talk about his research. Julia's name always appeared as the teacher who invited him to come to her classroom. I knew that Julia had joined Tom and other colleagues in conference presentations, that she had completed a master's degree in Secondary Education, and that she had invited another doctoral student into her classroom for his research.

I do, though, remember the first time I was in Julia's classroom. I was a graduate assistant and Tom was working with Julia's grade 9 class on the polynomial engineering project. Julia's students always appeared to be ready to try the activities that Tom had prepared and Julia would walk around and observe what her students were doing and were being asked to do.

On a second occasion Tom and I needed a class to try our idea of "number line dancing," an activity designed to have students actually role play as numbers on a number line and then moving, based on a series of mathematical operations, to other places on the number line. (For example, if a student was at +1, and then was asked to 'add positive three,' then that student would move to the right 3 steps on the number line.) Our idea was that if a whole class of students were participating in this number line dance, then students would begin to learn about the geometric relationships that exist to the arithmetic numerical operations. Julia again, invited us to her class, as a place to try our idea.

It was exciting to know that I would now have the chance to work with Julia, this amazing teacher that I had heard so much about, had met on a few occasions, and was somewhat in awe of.

As I drove up to the school, you could definitely tell it was still August and that teachers were still on vacation. The parking lot was deserted. I met school custodians as I entered the school and walked down the hallway. The school secretary was busy with preparing for school opening as I entered the main office. Without hesitation, the secretary looked up from her work and asked how she could help me. I told her that I had an appointment to meet with both Julia and Marilyn. Using the school intercom system, the secretary called their names. I was invited to go up stairs to the mathematics department workroom. We decided to meet at the high school where Julia and Marilyn taught for this first interview so that Julia and Marilyn could continue with some planning when our interview was finished. The school was testing their bell system, so our interview was interrupted several times.

The mathematics department workroom was a small cozy rectangular space with two easy chairs, a bookcase, and a computer desk. There were a couple of office-type chairs and a table, where I placed my tape recorder. There were no windows in the room but three doors. One door led to Marilyn's classroom, one to the hallway, and one to another teacher's classroom. The room reminded me of the workroom that I shared with my mathematics department colleagues some fifteen years earlier. I turned the tape recorder on, recorded a few words, performed a check (to make sure I had the cassette tape on the right side) of the recording, and Julia and I proceeded.

Julia told me that she started teaching about 25 years earlier (the past six years at a high school), and said that teaching was "something that I always wanted to do." Julia told the story of playing school when she was a little girl, lining up her dolls so that they could be her students. Although Julia always liked the idea of teaching, she did go through a period of time, while in high school, of wondering what she "should do." Part of Julia's wondering stemmed from societal expectations of woman at that time. Julia described those expectations as "in those days girls were supposed to be teachers or nurses or secretaries." Julia resented these expectations and shared that she had decided that she was "just not following the norm," so decided to become a doctor, or still questioning, perhaps a social worker, or … a teacher.

When Julia went to university, she enrolled in the Faculty of Education, with a major in science. This meant that she was taking all of the "right courses" in case she really did want to become a doctor. Julia watched her peers in science, met them in labs and observed that they were "all just so competitive to get 9s [top marks in the stanine scale]." Julia described the experience as "such a dog-eat-dog, don't help anybody, and don't talk to anybody cause you've got to get 9s to get in medicine" place and that that experience was not who she was. At the same time, Julia was really enjoying her classes in education, so she decided to stay with teaching, and felt good about the decision because "there were very few females in the science and biology areas." As Julia reminded me, "at least I wasn't doing what everybody else did." She continued with saying that she "had just this real rebel attitude." As a teenager, Julia said that she would say to her mom, "you can't make me into what you think I should be, you know I'm going to be what I'm going to be. I can't do just anything 'cause you think that's what you think I should do, you know." Julia mused that these opinions were "just part of my personality."

Although Julia's major teaching area when she graduated from university was biology, she has never taught it. Julia's first teaching job was to teach mathematics in a junior high school. Julia was hired to teach mathematics and science in her second teaching position, also in a junior high school. Over a five-year period at this second school, Julia realized that she was teaching more mathematics classes than she was science classes and decided that she "should have some math courses." (Julia told me that she was also really enjoying teaching mathematics at that time.) So, Julia decided to enrol in a graduate diploma program in mathematics education. When she was close to completing the course work for the graduate diploma program, Julia applied for and received a sabbatical leave from her school board, transferred the courses from the diploma program to a master's degree program, and took a year to finish course work and her thesis. Since completing her master's degree, Julia has been teaching mostly mathematics and computer-related courses, since her thesis topic was around mathematics and computers.

Because I had heard many stories of Julia's teaching, I asked Julia to tell me how she thought that her 'rebel attitude' might be reflected in her teaching. Julia replied by saying "that's interesting, I'm not sure if I've ever thought that." However, she continued to describe how she's "probably more flexible and more adaptable than most teachers." For example, Julia described that she is quite willing to look at new ideas and say 'Oh well, I'll give that a try,' or, 'I'll see where that goes.' She also continued with saying that she really enjoys when someone, like Tom Kieren, wants to come into her class, because she would be learning something and that she really liked the variety. Essentially, Julia believed that these experiences benefit everybody, her students and herself. Julia told me that she would not describe herself as a 'traditionalist' and that this, rather than the 'rebel attitude,' had probably affected her teaching.

I asked Julia to tell me more about how she saw herself as 'not a traditionalist.' Julia gave me an example from the 1988 curriculum change in junior high mathematics. A significant change in the 1988 junior high mathematics curriculum was incorporating the use of manipulative materials to help students' understanding of mathematical concepts. Julia told me that the 1988 curriculum suggested that teachers should start using the manipulative materials, so she started to use materials when she was teaching fractions. Although many of her colleagues were suggesting that they would never use manipulative materials in their classes, Julia just 'went for it.'

Because Julia embraced changes in her practice, she was often asked by her colleagues to lead "discussions on how they might incorporate or begin to incorporate manipulative materials in their teaching." Julia would share how she started working with the materials and would volunteer to help her colleagues 'getting started.'

I was still fascinated that Julia had said that she had a 'rebel attitude' and wondered if her description of mathematics would be 'somewhat rebel.' I asked Julia how she would describe mathematics. To Julia, mathematics is the "use of numbers and symbols to solve problems." She continued to say that we need to study mathematics "because there are applications in the real world and in order to do certain careers, you need to use mathematics," and that a person needs to know and understand mathematics. Julia told me that she knows a lot of people who are terrified of numbers and therefore "don't do bank statements, income tax, or fractions," and yet this is "everyday life, mathematics you do everyday." Julia also believed that mathematics teaches a person how to "organize thoughts, do things in a systematic manner, and think logically. It gives a person another perception on the world." At the time, I wondered what part of Julia's definition of mathematics was rebel, particularly when Julia said "It gives a person another perception on the world," I realized that this too was part of my definition of mathematics. Did I also have a 'rebel attitude?'

Julia left the room and I waited for Marilyn to arrive. As I was waiting, I remembered how Julia agreed to become involved in my research project. Julia and Marilyn had attended the same professional development experience that Joyce had. Julia and Marilyn were in the same cohort and they were asking me about my research plans. As I was describing my work, Julia and Marilyn asked me questions and wanted clarification. As I was talking with them, it occurred to me that I could invite them to become involved in my study. When I mentioned the possibility, they both wanted more information, so we decided to talk more in depth about the study. I was so pleased that, after our in-depth discussion, Julia and Marilyn both agreed to be a part of the study.

Introducing Marilyn

I have always felt like I've known Marilyn for a long time. It seemed that she had always been a part of my professional life, at least since I started working with Alberta Education. Our professional paths continued to cross, at conferences and through in-service sessions. I remembered the year that Marilyn took a leave from her school board to write student material for the Alberta Distance Learning Centre. During that year, Marilyn and I would periodically see one another at meetings.

I'd come to know Marilyn as a leader in the mathematics education community in her school, school district, and province. I remember how, when I, as an Alberta Education representative, would meet with the department heads of Marilyn's school district (Marilyn was then a department head), Marilyn's comments and questions would always require deep thought to answer. Marilyn was also an experienced textbook author, so I understood her influence as a mathematics educator to stem beyond our province. Marilyn was a mathematics teacher and person that I always admired and I was looking forward to learning more about her as we worked together in this study.

I made sure I had a new tape in the tape recorder and waited for Marilyn to join me in the mathematics department workroom. When Marilyn entered the room, I was ready to begin our conversation.

I asked Marilyn to tell me about how she decided to become a mathematics teacher. Marilyn told me that she's always been a 'math person.' Marilyn's mother likes to tell the story that Marilyn could add, subtract, multiply (by repeatedly adding), and count to 100 when she was four years old. Marilyn also remembered that, when she was in school, she loved learning about the "divisibility rules and multiplying algorithms" because she saw the inherent patterns in the rules and algorithms. Marilyn also added that she could add faster than most calculators, again, because she saw the patterns.

When Marilyn was in high school, she had decided that she would be a nurse. She was a volunteer 'candy-striper' at the local hospital at the time and just 'loved' the experience. This all changed one day in grade 12, though. One day, in math class during Marilyn's grade 12 year, her high school mathematics teacher and principal of the school, asked his students about their plans for the following year. Marilyn responded that she was going to be a nurse. There was no reaction at the time from her teacher but two days later he phoned Marilyn's parents and said "Can I come to visit you on Sunday?" Marilyn continued, "So there he was, on his way out to the farm, and he was bringing the parish priest and my mom is thinking, 'Oh, my God, what's wrong?'" Marilyn's math teacher told her parents that he thought she should go into education and that she should specialize in math. Marilyn told me, "So this is where I am. It was a decision that was made for me. Basically, I wasn't even privy to the conversation. It was him, the parish priest, and my mom and dad." Marilyn found out later that the only reason the parish priest came along was because the priest knew where she and her family lived.

So Marilyn finally registered in the Faculty of Education in February, specializing in mathematics. Still not sure about this mathematics teacher idea though, Marilyn also enrolled in religious education courses. When Marilyn completed her university education she had a Bachelor of Education (major in mathematics) and a Bachelor of Religious Education. Marilyn started teaching twenty years ago.

While in university, Marilyn told me that she "probably fought through every single math course, because I didn't have a lot of time to get used to this." Marilyn questioned the mathematics courses she took in university, "I could never see the reason for all those university math courses. Why was the stuff so hard, and why were we even doing this if I'm not even going to come close to teaching it?" Once Marilyn started teaching though, she knew that "this was it." She also started appreciating the mathematics courses she took while in university because those courses gave her the background that she needed for teaching; they helped her make connections between mathematical concepts.

Marilyn's high school mathematics teacher played a profound role in her teaching life. He died a few years ago. Marilyn attended the prayer service and spoke with his wife. Marilyn was able to tell his wife "he's the reason I'm a math teacher." His wife remembered Marilyn's name as the person whose name "was on the list for all the awards."

Marilyn's friends tell her that "the teacher is in her" and that if she wasn't teaching and was nursing, that she would probably be involved in nursing education. At the time of our conversation, Marilyn was in her twentieth year of teaching and with the exception of one year when she taught language arts and religion, she'd always taught mathematics and some religious education courses. Marilyn's focus has been teaching mathematics.

I was interested to know how Marilyn, as someone who has "always been a math person," viewed mathematics. Marilyn described her view of mathematics as "Mathematics. Of course, it's manipulation of numbers. Then those numbers can be replaced with variables and it becomes a manipulation of abstract numbers, shall we say. It's just looking at replacements. It's the development of patterns. Because you can't do the abstract, you can't do the algebra unless you understand the patterns in numbers. A lot of mathematics is what you do, like you do mathematics. You can talk about mathematics, you can write about mathematics. But you also have to 'do mathematics.' It can be such a concrete thing. And I think that's where we've come in leaps and bounds on where mathematics used to be, where you didn't look at the concrete. So, for me doing it involves the actual, like the manipulation of using bingo chips, or using algebra tiles or, or even just having good real-life examples."

Marilyn then told me about her teaching. She has a tendency "to start with the concrete or the practical application and then move into the mathematics. Like do the actual and then move into the abstract." She chooses to plan her lessons in this way because this gives her students a reason for studying the mathematics and gave me some examples. One example that Marilyn shared with me was from the study of conic sections. When Marilyn begins the lesson about the locus definition in Mathematics 30, she asks her students to imagine a goat tied, with a rope, to a post. The geometric shape that the goat could create if the goat walked around the post would be a circle with the length of the rope as the radius of the circle. If the goat was tied to the corner of a building, then the geometric shape would be three quarters of a circle. Marilyn hoped that these illustrations of the path that the goat could create would be a concrete example of the locus of a set of points and said that she has to "make it real for the kids." Marilyn concluded her example with "but I have to make it real for myself first I think. It has to be real for me."

So, for Marilyn, mathematics has to be "stuff that you do, stuff that you can think about," and "being able to make the connection" between the symbols and numbers on a piece of paper and putting it into "real-life." Marilyn and I finished our conversation with Marilyn telling me about how she had been thinking about the meaning of words in mathematics. Marilyn told me that she had a really hard time "doing problem solving" when she was in high school. In high school, the problems always felt contrived although the numerical solutions always "worked nicely." Problem solving started making sense to Marilyn when she started teaching because that was when she had to "explain it to someone else." She added, "That is when you really know the math."

We then began to start talking about the possibilities of our work together. We heard a knock on the door, I turned off the tape recorder, and it was Julia. Julia, Marilyn, and I chatted for a while about our work together and made plans to meet the following week. Marilyn and Julia also shared with me that they had been thinking about our work together and wondered if we could investigate the second unit of study in the Mathematics 10 Pure Curriculum, polynomials. The first unit of study would happen in the first couple of weeks of September and would give both students and teachers a chance to come to know one another.

Collective Learning System

A review of the narratives of these teachers and their perspectives on their practices reveals the diversity as well as the redundancy in who they are and their practices. For example, we see the diversity and the redundancy within the group when we examine Joyce, Julia, and Marilyn's teaching experiences, their experiences in becoming a mathematics teacher, how they view mathematics, and how they view themselves as teachers.

Diversity

Teaching Experiences. Joyce brings the history of teaching in an adult upgrading program for 11 years, Julia brings the experience of teaching in a junior high school for most of her 25 year career with only recently teaching in a high school (6 years), whereas Marilyn brings a variety of teaching experiences in her 20 years (classroom teacher, distance education teacher, textbook author, and department head) at the senior high school level to our community.

Becoming a Mathematics Teacher. Joyce liked mathematics, found it easy and it made 'sense' to her. She also had experience teaching music and Sunday school. Julia reported that teaching was something that she always wanted to do. Julia came to teaching mathematics via her interest in science. Marilyn, although she describes herself as 'always a math person,' came to teaching mathematics in an entirely different way. Marilyn's parents and her high school mathematics teacher made the decision.

Views of Mathematics. Joyce views mathematics as a "building block" and that mathematics affected a lot of things in the "background of our lives." To Julia, mathematics is the "use of numbers and symbols to solve problems and it teaches a person how to organize thoughts and do things in a systematic manner." Marilyn's view of mathematics is as the "manipulation of numbers, the numbers can be replaced by variables and then it becomes the manipulation of abstract numbers. It's the development of patterns." For Marilyn, mathematics also has to be "done, the actual manipulation of numbers, variables, and bingo chips."

As a Mathematics Teacher. Joyce views herself as a 'fairly traditional teacher' where she tries to tell her students about why things are done the way they are done in mathematics. Julia describes herself as a teacher who is probably more flexible and more adaptable than most and who is willing to give things a try and to see where 'they go.' Marilyn describes herself in a systematic way, where she has a tendency to start with the concrete or the practical application and then move into the abstract.

Redundancies

At the same time that the diversity exists within this mathematical community, similarities or redundancies also exist. We can also point to the redundancy that we notice within each of the experiences of Joyce, Julia, and Marilyn in these four areas.

Teaching Experiences. Although each of Joyce, Julia, and Marilyn has different teaching experiences, their teaching experiences are all within the Alberta system, working with the Alberta curriculum documents. Each of their experiences required them to interpret meanings within these documents.

Becoming a Mathematics Teacher. Joyce, Julia, and Marilyn all have similar educational experiences in becoming a mathematics teacher. Each of them attended the Faculty of Education at the University of Alberta and completed degrees with a major in mathematics education. Although Julia's major teaching area was science when she completed her undergraduate degree, she pursued a graduate degree in mathematics education.

Views of Mathematics. Each of Joyce, Julia, and Marilyn's views of mathematics point to mathematics as a human activity. It is a 'part of your background' (Joyce), you use it (Julia), and you do it (Marilyn).

As a Mathematics Teacher. Although each of Joyce, Julia, and Marilyn describe themselves as teachers in a variety of ways, similarities also resonate. For example, each of these teachers regularly participated in professional development experiences. The experiences are varied; however, the participation in the experiences point to shared interest in professional development. In addition, each of these three teachers had been thinking about the way in which the new provincial curriculum would affect their classroom practices.

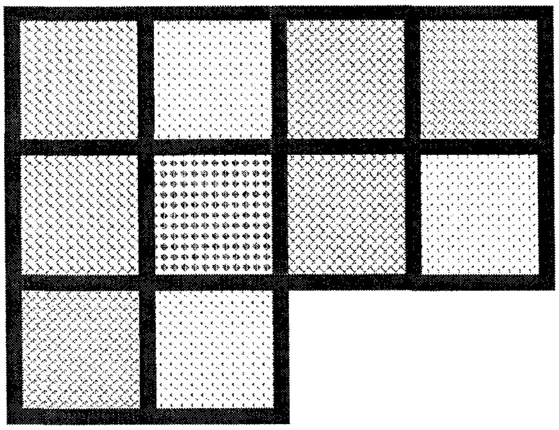
When I met with each of Joyce, Julia, and Marilyn I outlined the question that I wanted to know more about, "in what ways do mathematics teachers grow in their understanding of mathematical processes?" I also shared with each of them that I believed that it would only be when we were working together in a collaborative and collective manner that our thinking and knowing would emerge. I also shared that, at our first meeting as a group, I thought we would begin to outline and determine the type of 'project' that we would work on together. In this sharing of how I saw our work together emerging, my actions were in some way, pointing to a third condition for a collective learning system to exist, that of decentralized control, in the same way that Davis and Simmt (2003) describe this notion in a mathematics classroom: "effective teaching is not maintaining control over ideas and correctness, but the capacity to disperse control" (p. 153). At the same time, there is some inconsistency here as I invited each of Joyce, Julia, and Marilyn to come and investigate the question with me. In that invitation, I had control. However, it is my actions as a researcher, or the teacher in this community, that is critical in this discussion. In the next chapter, I will examine the ways in which our community satisfies the conditions for a collective learning system. In particular, I will elaborate further as to what it is about this context that helps to fulfill the necessary, interdependent conditions for observing the emergence of dynamic collective and personal understandings of mathematical processes and patterns of growth in them.

To do this, I will examine the stories of the times when our emerging mathematical community came together. As I do this I will begin to write in ways that invite you as reader into the conversation about the ways mathematics teachers grow in their understanding of mathematical processes.

CHAPTER 4

Inquiring Narratively: Composing Narratives of Experience

(A Reading of Two Pieces)



Up to this point in my thesis, you have read about how I came to the research question, “in what ways do mathematics teachers grow in their understanding of mathematical processes?” In Chapter 2 I described the recent efforts in mathematics curriculum reform, introduced elements of enactivist thought and described narrative as a mode of inquiry for this research question. In Chapter 3 I introduced you to the three teachers who joined me in pursuing this research question.

Although my research question was about teacher knowledge (Clandinin, 2000) and ways in which we might describe change in teacher knowledge, I have struggled with how I might gain insight into describing and understanding such change. I recalled how much I had learned about teacher knowing during the days when we had ‘item -writing’ committees for the diploma examination, when I worked with Alberta Education. This work history suggested to me the usefulness of studying a ‘committee’ of teachers. I then invited Marilyn, Joyce, and Julia to join me to discuss the implementation of the mathematical processes in the high school mathematics classroom. It would be through our conversations and working together, that I would be able to achieve insight into their knowledge.

In this chapter I describe our collective work. When I met separately with Marilyn, Joyce, and Julia, I described my research interest and asked that we meet collectively for one and a half days prior to the beginning of school. We agreed to do this.

A Brief Account of Our Work Together

On the first day that we met we talked about our individual meanings of the mathematical processes and developed a way in which we might visually represent our meanings. On the second day, we talked about the way in which we might work together and decided that we would meet an additional three times. We met as a group five different times over a period of three months. A description of each of our meetings follows.

First Meeting

Our first meeting[9] was divided into two half-days. We introduced ourselves, described our individual meanings about the mathematical processes, and built a representation of our collective thinking. I treat this first meeting as having two parts and will describe them in two narrative accounts.

Second Meeting

Our conversation during the second meeting was focussed on the ways in which we could understand elements of the new Alberta high school curriculum in our collective thinking and to think about the ways we would focus on the mathematical processes in our work together. We compared our representation of the relationship

[9] You will notice that the description of the first meeting is much shorter than the description of the following meetings. The reason for this will be clarified within the next few pages.

between the mathematical processes with that of the curriculum document and the textbooks that teachers were using. We noticed that, in the curriculum document, the definition for each process included a reference to the remaining three processes. For example, within the definition of connections, problem solving, reasoning and communication were mentioned. Within the definition of problem solving, connections, reasoning, and communication were mentioned, and so on. We concluded that the curriculum writers, like us, had difficulty in thinking about the processes separately. We also noticed that there was no mention of community in the curriculum documents. Although a classroom is mentioned, the curriculum still only refers to individual students. We found that interesting because, in our discussion of the mathematical processes, community was prominent.

During this meeting we decided on a topic that we would work on: the second unit of study in the new Pure Mathematics 10 curriculum, that of polynomials. Marilyn and Julia had also suggested this topic when I first met with them (see Chapter 3). We also decided that we would focus on the process of communication, recognizing that the four processes of problem solving, reasoning, connections, and communication are interconnected. However, the polynomial unit, in our opinion, is one of the most abstract units of study for students. The unit is filled with mathematical symbols; we thought it would be important to focus on students making sense of those symbols. The process of communication emerged because both Marilyn and Joyce were also teaching Mathematics 30. Students need to be able to communicate mathematically on the Mathematics 30 diploma exam.

We decided that we would create a project[10] that we would ask students to complete prior to beginning the unit. The project would be designed to review material that students were to have studied prior to entering Mathematics 10. The project would also give us, as teachers, a sense of what our students remembered. At least one week prior to the beginning of the unit, we would ask students to communicate to us in some format their understanding of the previously studied material. Joyce, Julia, and Marilyn would decide separately whether or not students would have time in class to work on this project. The development of this project then provided a site for considering teacher understanding.

We recorded our discussions. I took the notes home and formatted the project on my computer. I then sent the project to Joyce, Julia, and Marilyn for comments. I made changes and sent out a final copy. I also included possible scoring criteria that we could discuss at our next meeting.

Third Meeting

Our third meeting was held about one and a half weeks into the school year. Julia, Marilyn, and Joyce had each started their new school year. We met after school and our conversation focused on the implementation of the new course. The purpose of our meeting was to talk about the implementation of 'communication' activities and to review the proposed scoring criteria for the project that we had developed.

One point of extensive conversation was the way in which Marilyn, Joyce, and Julia were using the features that focused on communication in their textbooks. Joyce was implementing the journal activities from the textbook she was using. Marilyn and Julia were implementing the "Communicating the Ideas" feature in the textbook they

[10] A copy of the student project we developed can be found in Appendix B.

were using.

We further developed a series of questions[11] that asked students to communicate their understanding of polynomials. Marilyn, Joyce, and Julia wanted to be able to use the questions in a way that made sense for them in their classrooms. We decided that we would discuss how each of them used the questions at our next meeting.

We also clarified that the student project was going to be handed in to each teacher about the day that they would begin to teach the unit of polynomials. Marilyn, Joyce, and Julia would grade their students' projects and bring the projects to our fourth meeting.

We recorded our discussions. I took our notes for the set of questions we developed home after the meeting, typed them, and sent them to Marilyn, Joyce, and Julia.

Fourth Meeting

Our fourth meeting was held about a week after Marilyn, Joyce, and Julia started the study of polynomials. The purpose of this meeting was to use our scoring criteria to grade the student projects. Although Marilyn, Joyce, and Julia had graded their own students' projects, we wanted to collectively talk about our individual standards for student work and we wanted to see examples of the different types of responses students were giving. Once again, we met after school.

During this meeting we decided that, at the end of the unit on polynomials, we would give the students their projects back and ask them to "show how they would change their answers" for questions 2, 6, and 7 on the project because those questions focused on the 'big ideas' in the unit. We decided to ask students for this because we thought we might be able to see growth in student responses after they had studied the unit. We would re-grade those questions. We also thought that this opportunity to show how they would change their answers would give those students who were not happy with the marks that they earned on the project a chance to improve their grade.

Marilyn, Joyce, and Julia each shared what they had learned about what their students remembered from previously studied material when grading the projects. This information helped each of them to be better prepared for teaching the unit.

We then discussed the type of unit assessment that would be used and whether or not it would be important for each class to write the same unit test. After much deliberation, we decided that it was not necessary to have all of the students write the same unit test.

We also discussed the questions we had generated in our third meeting and decided that each individual teacher would decide on the way in which they would use the questions in their classroom.

Fifth Meeting

Our fifth meeting was held about two weeks after Julia, Joyce, and Marilyn had finished teaching the unit on polynomials. We met after school. The focus of this meeting was, once again, to look at the student projects and to examine the ways in which students responded to the task of showing how they would change their answers to questions 2, 6, and 7.

We analyzed student projects, looking at how students communicated their thinking. We spent most of our time looking at individual samples of student work from the project papers and talked about what we noticed about students' mathematical

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[11] A copy of the developed questions can be found in Appendix B.

understanding, our expectations and standards. Now that we were able to closely look at student work, we all discussed the way in which the questions on the project might be worded differently. For example, one of the questions on the project asked the students to describe the "significance of the diagram." We wondered if students understood what we meant by that statement and concluded, after we looked at examples of student responses, that most students did understand our question.

During this meeting we also spent time talking about the use of the 'communication' activities that each teacher had decided to focus on. Joyce told us that she'd been using the journal idea with her Mathematics 10, Mathematics 20, and Mathematics 30 classes. Almost all of her Mathematics 10 students were participating in the activity and handing them in. Only about half of her students in the Mathematics 20 and 30 classes were participating. She suspected that the reason that the students in her Mathematics 20 and 30 classes were not participating was because they had not experienced journal activities prior to this year. Julia and Marilyn indicated that they would continue to work with the "Communicating the Ideas" feature in their textbook as they described in our third meeting.

We also noticed that, in our focus on communication in our teaching, we were helping students make connections. The questions that we asked encouraged students to communicate what they knew and also helped them with problem solving. Through their communication, we gained insight into their reasoning. So, these kinds of questions get at all of the mathematical process ideas of thinking about mathematics.

Through the descriptions of these five meetings, you can see how our work evolved and unfolded over time.

Beginning to Inquire into the Conversations

At this point in your reading of this thesis, you might expect to read about the data that I collected in order to answer my research question about growth in teacher understanding of mathematical processes. My original intent in the next portion of the thesis was to share the detailed inquiry into the transcripts of all of the times that our collective met. Indeed, all of the audiotapes of our meeting times have been transcribed. I started the process of using the transcripts for developing six narrative accounts (two for the first meeting) and began the interpretation of the research texts (Clandinin and Connelly, 2000) created from our first meeting time.

In the writing of these research texts, a byproduct of this work emerged: the idea that a mathematics teacher's understanding arises in conversation. As I worked with the transcripts and began to compose narrative accounts, I could see emergent understandings,[12] not as simply an individual phenomenon, but arising also collectively, within the teacher conversation.[13] I began to notice features of individual teacher understanding, understanding related to the collective, and understandings within the body of mathematics. Such noticing is a feature of narrative inquiry. As Clandinin and Connelly describe it, "narrative inquiry carries more of a sense of a search, a 're-search,' a searching again. Narrative inquiry carries more of a sense of continual reformulation of an inquiry than it does a sense of problem definition and solution" (2000, p. 124). In my re-searching of the transcripts and narrative accounts, I noticed the emerging and

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[12] By "emergent understanding" I mean that understanding is not predictable.

[13] By saying emergent understanding "within the teacher conversation" I mean to point out that such understanding is co-determined by the individual actions but also is occasioned by the inter-action with others. The conversation is also changed by emerging understanding, that is, changing understanding changes the way in which the conversation proceeds. Because understanding is emergent in a conversation it is at once not predictable ahead of time and develops in part to maintain the relationships in the conversations (e.g. Gordon Calvert, 2001; Gadamer, 1992).

changing teacher understandings within the context of conversation. I thought that I was going to write about emerging understandings over a period of three months, but in writing the narrative accounts and studying what I felt were critical moments in them, I found I was writing about emerging and changing understandings within our conversations about mathematical processes and implementing the new mathematics curriculum. I could see the emerging and changing understandings in each conversation. This idea of understanding as emerging and coemerging is consistent with an enactivist view of knowing as occurring in action.

Hence, an elaboration of my research question as developed in my re-searching of the transcripts and narrative accounts might be: "In what ways do mathematics teachers grow in their understanding of mathematical processes within the context of professional conversation?" By "professional conversation" I am pointing to a conversation among teachers that has as its topic an element of practice (be it instructional or curricular). Enactivist views and principles provided me with a way of thinking about this aspect of teacher understanding.

My intention was to interpret each of the six narrative accounts (two for the first meeting and one for each subsequent meeting), to begin to describe the ways teachers' understandings about mathematical processes changed. Using the quilt metaphor, my intent was to provide an interpretation of the entire 'quilt,' that is, all six accounts. Instead, with the reformulation of the question, I ask you to think about looking at the entire quilt but to imagine examining two pieces of the quilt closely. The two pieces of the quilt that I will use for exploring the question of emerging and changing teacher understanding within the context of conversation will be the two narrative accounts of our first meeting day.

Thus, in the research process I began to realize the richness in our conversation and realized that our changing understanding was occurring within the conversation itself—well before we "implemented" our teaching plans in the classroom. A reformulation of my research question, and hence the structure of this thesis, emerged in the "world of significance" (Maturana and Varela, 1987) that I was bringing forth in the interpretive and writing acts. Life emerges and unfolds as we lay the path in walking (Varela, 1992).

Under this view, in this chapter I provide you with two narrative accounts, or research texts, derived from our first meeting day. I could have chosen small pieces or quotes from the research texts to illustrate the features of changing understanding; however, I have maintained the entire research text in the form of two narrative accounts. I made this decision because the two narrative accounts maintain the complexity and complicity (Sumara and Davis, 1997) of the situation, which are an important part of the changes I was studying. By using the terms complexity and complicity, I mean to point to the ways in which our conversation was emerging and evolving and the ways our interactions were changing each of us.

Constructing the Narrative Accounts

I wrote the two narrative accounts, *Coming to Know One Another* and *Building a Representation of our Thinking* based on the audio recordings of our first meeting. Each recording was transcribed. I listened and re-listened to the recordings, read and re-read the transcripts in order to write these narrative accounts. As you read the narrative accounts you'll see some of the narrative in a poetic form. When the poetry is italicized it represents our collective conversation. When the poetry is not italicized it represents my interpretation of my own writing.

The poetry emerged in my reading and re-reading. I began to realize the poetic structure of some of the writing and in the conversation. When the poetry is italicized, it is offered as an alternative representation of the conversation—one that perhaps better foregrounds the subjective character of the representation but that, I hope, also preserves the emergent collective insight.

From the narrative accounts, I hope to portray the complexity of our conversations and of the teachers' thinking and interacting within them.[14]

The First Narrative Account: Coming to Know One Another

On the first day that Marilyn, Julia, Joyce and I met, I was nervous and excited. We had decided to meet at the university, where I was able to book a classroom, room 382. The room had tables in it and I thought that it would be a good space for us to meet and for me to set up the tape recorders. Room 382 was at the end of a hallway and accommodated twenty people at the most, so it had a nice intimacy associated with it. Meeting at the university was also important because it was a central location, about half way between Joyce's school and Julia and Marilyn's school. We also believed that there would be fewer interruptions at the university. If we were in one of the schools, there would be a lot of people around, readying the schools for the beginning of the school year.

We were going to meet in the afternoon only, so I went to the room early to make sure it was ready. I arranged the tables so that four people, their materials, and two tape

[14] It is important to note that the two narrative accounts and all other writings in this thesis has been reviewed by each person in our collective.

recorders could all be accommodated. As I was moving the tables around, I realized that the room itself brought a lot of memories for me. It was the room in which I had taken a class, one of the first classes in my PhD program. The room prompted many memories of being challenged to think about the meaning of curriculum and in learning about the different ways in which individuals have described curriculum.

I was nervous for two reasons. Firstly, because this was the first time that I was engaging in an act where I considered myself a researcher. In all of my other experiences working with teachers, I really did see myself as the teacher of teachers. On this day I was thinking of myself as a teacher of other teachers, as a colleague among teachers, and as a researcher.

The second reason I was a bit nervous was that although I knew each person individually, Joyce had not had the chance to meet Julia and Marilyn, other than perhaps seeing them at a professional development seminar prior to this day. Julia and Marilyn worked together in the same school and were, professionally and personally, good friends. I thought about how important it would be for me, as the group facilitator, to ensure that there were truly spaces for each of us to contribute to the conversation. I was very aware that, because I had had several opportunities to talk about what I believed in many different contexts, that I did not want to dominate the conversation. Yet, I also did not want to be a non-participant. I was also aware that because Julia and Marilyn often had the chance to talk together about ideas, it might be easy for Joyce and me to both feel as if we were not participating. I was also really conscious that I did not want to be the only person who completely defined the project that we were going to develop. I was thinking about this and fumbling with the cassette tapes when Joyce first, and then

Marilyn and Julia arrived.

We introduced ourselves and then I shared, once again, my research interest. I described that our work together would be co-emerging because I really did not have a pre-determined view of what we would develop or a way in which our previous ideas would form a basis for new ideas. I shared that, within the context of investigating the implementation of mathematical processes in the secondary mathematics curriculum, I was interested in teacher knowledge and that such knowledge showed itself in teacher actions and in reflections on such actions. The way in which we chose to implement or consider the implementation of mathematical processes would be defined by our research group. Although I had a sense of what I thought we should do on this first day together, I also explained that I hoped that, if any one felt awkward with the direction, then we should stop and discuss the process. I also explained, that in my proposal, I suggested that we "work on" lesson or unit materials that focus on mathematical processes; however, we would decide together as to how we would work on this problem.

I suggested that we participate in an activity where we individually thought about the way in which we would define the mathematical processes of problem solving, reasoning, communication, and connections and then share those meanings so that we could develop a shared (Davis and Simmt, 2003) meaning of the processes. We stopped the tape recorder and each wrote our own meanings of the four processes. Once I noticed that we had all finished with our writing I asked who would start sharing their meanings first.

Communication.

Julia said, "I'll go first." She read what she had written for the process of communication:

> Communication. (pause) The use of mathematical language and symbols. Being able to describe or explain what you're thinking and why. Using symbols and diagrams to help explain a process. Being able to organize your thought processes and then describe them.

Marilyn added "Being able to talk about mathematics, being able to verbalize it and talk on paper too." Joyce agreed and said "Yeah, that's what I had too." Joyce continued with "Or tell somebody about, that doesn't know the language. And explain it so they can understand." Marilyn agreed with the comments, "Yep. You're right about it, it's not just words that you're using either, it's diagrams and symbols."

Julia then reminded the group that she also had the concept of organizing in her definition, "I think that one of the things is the organizing. The kids have a lot of trouble saying 'this is how I did it, I went here and this is why I went here.'"

I was suddenly aware that Joyce might not have had a chance to share her full definition so I asked "What did you write, Joyce?" Joyce looked at her paper and concluded, "Same kind of thing. I had both verbalizing mathematics, being able to explain things to someone who wasn't familiar with the terminology so they could understand the concept. Sort of verbal." Julia and Marilyn both agreed and together said, "Verbalizing."

I noticed that a 'shared' definition of the process of communication was beginning to emerge because each individual definition either overlapped another or built on another. I then shared the meaning I made of communication:

> I chose two examples, '2x + y = 7' and '3 + 4.'
> These expressions communicate something.
> The symbols communicate something.
>
> Uh-huh

Uh-huh

Writing, talking, and acting.

Uh-huh
Uh-huh

Communication is important,
How we belong together in a group.

Uh-huh

Being involved together in a community.
Don't know what that means yet.
But, it's something.

When I finished reading the meaning I'd written, the group agreed, "Yeah!" "Right!" and "That's right!" Marilyn added, "It's all community and communication. If you can't communicate, then you don't belong to the community. Both have the same root word."

Connections.

Julia started our conversation by reading what she had written for connections, "Connecting the mathematics within the strands and to other school situations, and to occupations outside of school. Knowing that mathematics can be described in many different ways." She looked up and said, "I put down the word 'links.'"

Joyce added, "Like being able to apply math through real situations. I was thinking about career connections in the book [in reference to the text book Joyce would be using] and the same sort of thing you had. Being able to use previously learned skills with new concepts. So linking." "Across grade and subject," offered Marilyn and everyone agreed.

Joyce continued, "Not just in math but in the other things like science or social."

Julia offered an illustration of what she meant, "It's like when we teach students how to solve right triangles. We always start off with Pythagoras. Pythagoras gives us one perspective. When we introduce the trigonometric ratios of sine, cosine, and tangent, then we have to look at the right triangle and ask ourselves 'what information do I know?' Based on the answer to that question, we then decide which trigonometric ratio to use to solve the triangle. But, we might also still be able to use Pythagoras. So, although we've introduced some new ideas, we can still use the old ideas. We can bring back what we've previously learned."

Marilyn made the connection to the Catholic school in which she and Julia teach, "There's always connections. We try to bring the Christian component into our classrooms; not only the way in which we teach, but also in what we teach. I know that at every Catholic education conference I attend we discuss Christianity in mathematics, in social studies, and in science."

I just couldn't resist now making the connection between Pythagoras and religion. (Barrow, 1992) and relayed how fascinated I was when I learned that there were a group of followers of Pythagoras, the Pythagoreans, who were like a religious group. Their 'religion' was based in part on the powers of the Pythagorean numbers, the triads. We all mused about this idea for a brief moment and I then shared my writing about connections. I read:

\- - - - - -

Connecting to something else
Works
From other experiences

\- - - - - -

Who makes the connections?
Teacher?
Or the student?

\- - - - - -

In response to my reading, Julia said, "That's interesting. I know that Marilyn and I have talked about how we've noticed that our exceptionally talented math students appear to be making their own connections all of the time. They don't go on until they know exactly how 'this' fits and then they're quite satisfied. Whereas, other kids don't do that. Then I think that's the teacher's responsibility to help them make those connections and help them see those fits."

I continued with my reading:

\- - - - - -

How do we help
Students
Make connections?

\- - - - - -

I make connections
While
I'm thinking.

\- - - - - -

Everyone agreed and Julia added, "That's exactly what they [students] need." Marilyn agreed, "Yeah! Sometimes you'll say something, look at particular students, and you can just see connections happening. Remember teaching Roman?" At this point Julia and Marilyn recollected a former student, Roman, with a series of looks, nods, and 'Hmms.' Joyce and I looked at one another, clearly not a part of the story of teaching Roman.

I did feel prompted to ask though, “Do you think it’s ‘natural’ for some people and not for others to make connections? Or do we all do it?”

Julia quickly responded, “I’m thinking that maybe we all do it in some ways and in some areas but maybe not all of us do it mathematically.”

This prompted me to ask, “When students learn the concepts of mathematics, do they automatically make connections between the concepts?”

Marilyn answered, “Some, but I think it’s a minority. When you give them an idea they’ll go with it, a lot more will go with it. But, if you didn’t make the connections for them most of them wouldn’t be making it.” Marilyn continued with her story of Roman to illustrate her point “you see there are the students like Roman. Roman would ask ‘Does this mean this?’ or ‘Could you do it this way?’ The questions he asked referred to material he had studied in previous grades. Other students in his class would be looking at him wondering….” Julia finished Marilyn’s sentence “where is this coming from? What planet is he on?” Marilyn finished her statement with “I think all the students would know that stuff, but they would never think of it. Not on their own motivation.”

Julia then commented that as Marilyn described students making connections she was thinking about the relationship between connections, community, and communication. Marilyn, Julia, and I were clearly on the same wave length:

Communication and community
Part of connecting?

Another side of the word ‘connections’

Connect
Connecting with people

It’s a part of community

Being able to communicate with people
Listeners are making

Connections to other things they know

Perhaps not consciously connecting
Con-scious? Con-nect?

Reasoning.

Julia started by reading what she had written, "Reasoning is logical sequence, being able to link a sequence of steps, infer the next step, (pause) apply mathematical principles to infer relationships."

Marilyn then shared what she had written, "Through reasoning we can bring meaning to mathematics of course. Because there 'is the reason for,' we know this algorithm so that we can do this (pause)." Julia finished with "We know this, therefore we can do this. It's a hard idea to explain."

Joyce offered her meaning, "being able to apply common sense and logic to a mathematical problem" and Julia said "I wrote down the word 'precision' as well because some things, like some mathematics..." Marilyn finished with "some mathematics requires an incredible amount of precision."

Julia continued, "Precision, and being able to know and reason when, what's important. How precise?" Marilyn suggested, "Like a multiple-choice test. You can't require any more precision than that."

I really struggled with how to write about the mathematical process of reasoning. I was not surprised by this struggle as I remembered how I would brush over 'reasoning' when I would facilitate workshops about mathematical processes. Even though I had read about reasoning in the NCTM's *Curriculum and Evaluation Standards* (1989) many times, I felt I did not have a real sense of the way in which I could think about reasoning in mathematics. I then shared my writing about reasoning:

> Problem solving -----
> Reasoning?
> Separate? Random?
>
> Thinking?
> Making Meaning?
> In mathematics?

Marilyn offered, "Well maybe there are two sides to that. Like, there is the logic part of it. Cause if you didn't, you can't be logical without reasoning. If you can't follow the steps through with something that's a little bit non-routine. Without being very logical. And you can't get those steps and that logic until you reason them."

Because I was struggling with trying to write a meaning for reasoning, I asked, "Is reasoning developing logic?" We all paused for a moment, each of us deep in thought. Footsteps outside the door brought us back to our conversation and the room we were in.

Marilyn started with her thoughts, "There's probably more than just two sides to this word. We're looking at it, maybe, from the very logical mathematical side, because of the minds that we are. So the word 'reason' to me, means to make meaning. There's that side to 'reason.'

Julia wondered out loud, "Well, reason 'why.' Are we saying 'why?' Or is it 'reason' to do something. So you're, you're asking a 'why' question?"

I then shared that I used to think about reasoning like solving logic puzzles. I shared the story that I used to have logic puzzles readily available in my classroom. When students were finished their homework, I would encourage them to work on a logic puzzle. I did this because I thought my students would be engaged in reasoning if they were in the process of solving a logic puzzle. My teacher in grade 9 science used to encourage her students in the same way. Julia added that she thought that there was "more to it than that even though that's a very common point of view."

I continued with my pondering and wondered aloud, "But then how does reasoning fit in mathematics? Does it fit in the learning of mathematics? Or do we now say that 'students have learned about some mathematical ideas and skills.' Here is a new idea, now select from your previous learnings to 'reason out' the new idea?"

Marilyn suggested, "Well what did they say about 'The Age of Reasoning?' Isn't it about age six or seven, that children learn to reason? Like before that it, it's really hard to talk to a two-year old and give them reasons why. You just say this is the way it is and that's the way it has to be." Julia questioned, "They can't choose? They don't choose because of reasons?"

Marilyn continued, "How does learning change after the age of reasoning? (pause) Like before, and maybe that's part of mathematics, there are some students that just want the rote learning. They say 'just tell me what I have to do. I can do 500 problems, they're all the same.' And that's like a child before the age of 6."

I asked, "Is reasoning about making connections?" I was still struggling. Marilyn added "And understanding."

Again I said, "I'm having a hard time with this one. I don't understand reasoning." Julia asked "Don't know what reasoning means?" I responded, "I don't know what reasoning means. Logic appears to be a part of the definition; all three of you used the term logic. Is that like inductive and deductive logic? Or?" Marilyn agreed, "Yeah, for sure. If you don't have the powers of reasoning there's no way you can make an inference." Marilyn continued with an example to illustrate her point, "For example, you are not going to teach statistics to an elementary student. Not the kind of statistics where you ask students to examine the collected data and draw inferences or predictions about the population, based on the sample where the data was collected. Elementary students don't have those skills."

I wondered if Marilyn's comments about elementary students meant that elementary students might ask 'why' but may not know 'why?' Our dialogue was emerging:

> *Asking 'why' and knowing 'why?'*
> *Asking 'why' and knowing 'why.'*
>
> *Being able to find out 'why?'*
> *Reasoning 'why.'*
>
> *Being able to hypothesize 'why?'*
> *Hypothesize. 'Why?'*

Having our conversation move towards the notion of 'hypothesize' reminded Marilyn of the difficulty that grade 9 students have with hypothesizing in science classes. She believed that it was because the students were having a difficult time in "thinking about what could happen because their experience is with what does happen."

Our conversation reminded me of watching my nieces, when they were young, so I shared the following story:

> My oldest niece, Lara and I would watch Walt Disney movies together. She could watch a movie over and over again. One time, she and I were watching "The Little Mermaid." I knew she had watched this movie several times, so at one point during the movie I asked her to tell me what she thought would happen next. Lara replied with "I don't know." I was amazed and couldn't believe that she could watch the movie as many times as she had and not know what would happen in the next scene. I wondered "how could she not know?" Then, it seemed like just a few months later, we were watching the movie again. Lara turned to me at one

point and said "Be careful Aunty Florence this is the scary part." At another time during the movie Lara said "The next part is my favorite part."

Was not being able to talk about the sequence peculiar to Lara? No. I observed it again when my youngest niece Valisa was about the same age that Lara was. She too would watch a movie repeatedly, not be able to tell 'what happened next,' and then, what seemed like overnight, would be able to describe the next scene. So perhaps reasoning is about sequencing, or in some way understanding consequences.

As our conversation about reasoning was drawing to a close:

Understanding consequences
About hypothesizing

Is reasoning
Generalizing?

That's way up there
Way down the road
Generalizing

Is generalizing
More like problem solving?

Problem Solving.

Again Julia began the sharing process by describing what she had written for problem solving, "I thought to myself, 'Well, it's pulling together all of the other processes. Doing them in many different ways to view, and approach a situation. I thought problem-solving is kind of like (pause) writing an essay. There's not necessarily one correct response, but the response depends on interpretation and the ability to support the idea. That's why I say problem solving is generalizing."

Joyce then shared what she had written, "Being able to apply learned mathematical concepts and skills to new situations. Problem solving is not the same thing as all the other processes. Problem solving is not just a distinct process, it's a bundle of them."

Julia nodded in agreement and added, "You've got to communicate, to make connections, and to reason. You have to know your basic skills and you have to pull all of those things together."

Marilyn then offered, "I think that we have to make problem solving intra-curricular rather than extra-curricular. Problem solving has often been seen as something that we do as 'extra' to the rest of the mathematics."

Julia agreed and shared a conversation that she had with some colleagues to illustrate her agreement:

> The other day we were talking about and remembering the 'problem solving pages' in the old textbooks. In those textbooks, you would study a whole chapter of content, and at the end of the chapter, there would be a page or two pages of problems. The problems though were not related to the content studied in the chapter. I would often hear students say 'I can't do problem solving.' Yet, if we think about problem solving as a process, students were 'doing it all along,' while they were studying the chapter.

> I shared my writing:
>
> What is problem solving?
>
> Reasoning
> Solve Problems
> Thinking
>
> Is the process
> Or the answer
> The purpose?
>
> What is a problem?
>
> Anything or
> Any situation
> I've not seen before?
>
> Is this the same

As making 'sense'
Or making 'meaning'
To arrive at a con-clusion?

After I read what I had written, I asked Julia about generalizing and how she saw generalizing connecting to problem solving. She replied that she thought that that might be what problem solving is "about, coming to a conclusion, seeing an overall picture."

Our conversation reminded me of the many conversations that I'd had with teachers about problem solving. In my conversations with teachers, it seemed to me that two different views of problem solving emerge. One view is where you have a problem to solve, that is where you apply some mathematics to find an answer. A second view was that problem solving is a process of thinking. I shared this thought with the group:

So, we problem solve
Each day

As we go through life
Problem solving

Thinking
Making connections

Communicating
Reasoning

Can't find an answer
Or a conlusion
If you can't reason

Marilyn continued, "Maybe that's what's been wrong with problem-solving. It's only been defined as the kind that, that requires the answer." We then talked about the new textbooks. If we looked at the textbooks that were authorized we could see this dual tension existing, each chapter contained a page, or two, called 'problem solving' yet, there were questions throughout the chapter that could be considered a part of the process of problem solving. We wondered though if a textbook could actually reflect the process of problem solving because of the medium and the role of a textbook.

Julia then suggested that perhaps the only way that we can have students involved in the process of problem solving is if they are engaged in a long-term project, display, or being a part of an activity where students have to explain the project, display, or activity. Marilyn reminded Julia of a project that Julia's IOP[15] students did in grade 9 mathematics. Julia's students were involved in designing and building a cabin. We asked Julia to share what she did with her students:

> My students were great and they really loved this project. The idea was that this class would make plans for a 'cabin at a lake' and then actually build the cabin in the school field. There were 10 students in the class, 5 boys and 5 girls. The boys built a cabin and the girls built a cabin.
>
> The students first created a scale diagram of their cabin. From their scale diagrams we talked about measurement, ratio, and proportion. My students didn't know those ideas. Then we moved to actually building the cabin outside. This is not an easy task to complete. They had to stake out their floor plan and decide how to ensure the corners were right angles. The cabin consisted of a lot of right angles, if the walls were not at right angles to one another, then they would not meet. So, there was a lesson about the Pythagorean Theorem and right angles.
>
> After the cabins were built, a teacher's aide and I went outside after school and measured the walls, to make sure the cabin represented the scale model the students had created. We were going to give a prize for the 'best cabin.' Well, both cabins were excellent, we had to give both groups a prize.
>
> It was a really good experience. Another interesting thing about this class and project was that there was a boy in the class with muscular dystrophy, (That's why I had a teacher's aide that year.) He could do very little of the physical work, but he sat there with the plans and could tell the kids what to do and explain why 'this' should be 'that' long.

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[15] IOP means the Integrated Occupational Program, a program for "students with exceptional needs." (1994, Alberta Education, p. 1) Students in this program are not eligible for high school graduation. The program was developed "for students who continue to experience difficulty in learning." The purpose of the program is to "enable students to: become responsible members of society, develop entry-level vocational abilities, and become aware of the need and opportunity for lifelong learning." (1994, p. 1)

Marilyn responded to Julia's story with "there, that's problem solving. These were IOP kids and they could problem solve. Because problem solving is not just arriving at an answer, it's the process." Joyce and I nodded in agreement.

Marilyn continued verbalizing her thoughts and asked, "Maybe we, as teachers, understand that problem solving is the process, but do we verbalize that very often? Do you think our students have come to think about problem solving in that way, or is it just a finding an answer to a problem in the book? I'm wondering if we (as teachers) even know how to tell our students that."

Julia replied, "It's hard enough for me to just think about it right now, let alone me telling my students." Joyce and I again nodded in agreement. I wondered if we, as teachers, should be helping students become conscious of this idea about problem solving.

Marilyn continued, "Often our best students have shivers when they are asked to solve a problem, because they perceive that it is so hard." I shared how the statement immediately reminded me about what I thought problem solving was when I was in school, word problems. I always found solving word problems hard. I remembered that I was often confused when I had to write a mathematical expression for a word statement like 'a number is 5 less than a second number.' I always wondered, 'where does the minus sign go in relation to the 'x'?'

My story reminded Julia of her experience, "I would read those problems about trains and cars, and think 'I have no idea what this problem is about.'" Marilyn and Joyce both indicated that they also felt similarly this way about problem solving when they were in school.

Marilyn suggested "it must be about comprehension and that links to communication." In a sense, were Julia and I not comprehending the language when we were in school? Marilyn, Julia, and Joyce all suggested that they did not learn how to problem solve until they started to teach about how to problem solve. I too remembered feeling that way after my first year of teaching. When I first started teaching, problem solving was solving the word problems in the textbook. I remembered working hard to develop techniques to help my students read and interpret word problems. So, I had finally 'figured out' how to solve those textbook word problems, when I was teaching in a high school.

Marilyn continued with suggesting that she imagined that most current students would still consider problem solving as solving word problems in the textbook. Julia asked, "Isn't problem-solving in every school subject? Aren't we teaching problem-solving all the time?" This question caused us to think beyond the subject of mathematics.

Marilyn responded, "It's not called problem solving in other school subjects, it's called decision making, but I think it is still problem solving. I think decision making models can be models for problem solving."

Julia suggested that the decision making model could also be a model for hypothesizing in science. She continued, "the reason you do an experiment, is to find the answer to a question, and that's what you are doing when you are problem solving." I had never thought of a science experiment as problem solving so I asked Julia, where, in a science experiment does the thinking occur? Julia offered the following explanation of how she saw the problem solving process related to an experiment in science:

> It's when you ask yourself, "How might I answer a particular question?" One of the ways I might answer this question is by doing an experiment. There might be a variety of ways of answering the question, but you could choose to do an experiment. It's like doing research. What question are we going to ask? How are we going to go about answering that question? That's endless, right?

This lead me to think about generalizing again and how generalizing is related to problem solving. Again our ideas were emerging:

Generalizing
A part of problem solving?

Generalizing
A result of problem solving?

Hypothesizing
A part of problem solving?

Hypothesizing
The beginning of problem solving?

Generalizing is
Hypothesizing?

Hypothesizing is
Generalizing?

"I'm just listening to all this, going 'wow.' I've never thought about this." said Joyce. We all agreed that these ideas were not something that we generally thought about either, but in this conversation, the ideas were emerging.

It was time to take a break.

<u>The Second Narrative Account: Building a Representation of Our Thinking</u>

When we returned from our break, a break that included snacking on the brownies that I'd picked up at a local grocery story, I suggested we think about a way in which our conversation and the meanings that we each brought about the mathematical processes of problem solving, reasoning, connections, and communication could be represented. I thought about our group as participating in this activity because I'd used a similar type of activity in a workshop that I did in Yellowknife.[16]

Julia grabbed a pen, a large piece of flip chart paper. I suggested we use a Venn diagram. Julia acted as our recorder:

How are we going to do this?
Three circles and
Problem-solving in the middle?
Or problem-solving that encompasses the whole thing.

It's where?
In the square. [This refers to the rectangle drawn around the three circles.]
It's the square.
It's the universe.

On the outside?
It's not what it's called, 'the universe?'
Yeah. Is that what it's called?
Oh Yeah!

At this point, Julia asked "Remember when we used to study set theory in grade 7?" I immediately remembered my grade 7 mathematics teacher, Mrs. Mearti and shared my memory:

> I'll never forget the first day of school in grade 7. We had brand new math textbooks and I remember studying set theory. It took me a year of practice to master making the 'squiggly brackets' for the set problems. I believe Mrs. Mearti was learning the mathematics along with us. I believe she was the first teacher that I observed to be learning the mathematical ideas with us and in a direct way. She was as strict as possible and wanted us to be neat in our notebooks. At the same time she was going to university to complete her education degree. She had originally experienced the normal school and was returning to university each summer to complete her degree. I was really impressed with her when I discovered that she was going to school. I think she stands out as a significant teacher for me because she would continue to learn. I believe she was the first teacher that I knew of that was 'continuing her education.'

[16] The workshop in Yellowknife was used in my proposal for this thesis. Please see Appendix C for a description of work that was done in Yellowknife.

This prompted Marilyn to remember that she too had practiced making the 'squiggly brackets' and now her students think 'that she does a nice job.' Julia remembered students who would practice making them when she was teaching grade 7; Joyce said "I still can't make the silly things." We all giggled at the memories and Marilyn reminded us of our task:

I don't think we can just stick problem solving on the universe
I think it's also the nucleus.
The core
The core

Cause there are two perspectives of problem solving?
Yeah Problem-solving is the process
But it's also...finding the answer
Okay.... Uh-huh

So we draw up three circles?
What would be in the other?
What do you mean by three circles then?
Communication, connections, and reasoning.

Okay... sure...yeah.

Problem solving is the intersection of those three?
Plus it's the outward, it's the universe.
Is that what we're saying?
Yeah.

I don't know. Are we saying that?
I don't know,
I think that seems reasonable.
The problem solving is, but, is that the process in there?

Oh, right!
I would say this is the linking of all the processes.
So, that's how problem solving is the linking of all the processes of
Communicating, connecting, and reasoning.

Where is finding an answer then?
That might be the universe.
It's your generalization, it's your answer,
It's your conclusion, it would be a unique.

You need all of these things.
And you need the problem.
So you're saying the universe is problem-solving.
All of the realities of problem-solving?

Julia suddenly stopped listening and said "Somebody listening would probably go 'Oh, my gawd.' This is mind boggling." We chuckled again. Marilyn again brought us back to our task.

Maybe problem solving can be universe.
Maybe we can solve a problem by just...
Reasoning.
Reasoning. Or just...

Making connections.
Or maybe it can be a whole.
Maybe it doesn't have to be.
Can it? Like, like, it's everything.

Maybe it only needs to be in the middle.
Can it be the universe?
Both definitions fall in the middle?
Yeah. Yeah. Both definitions fall in the middle.

If we see problem solving as a process.
It, it's linking two.
It is the questioning.
Two or all three of those processes.

If I put it in the middle,
Aren't I saying that I need all three?
With the true Venn diagram?
I think so.

That's why I'm asking.
When I look at our diagram.
Problem solving does not occur
Without communication, reasoning, and connections occurring.

At this point, it appeared that we were somewhat stumped. Marilyn suggested, "Maybe the Venn diagram is too limiting? Perhaps we need another dimension." Julia jumped in with "Maybe we need spheres." We all laughed. Was there another way we could represent our thinking? As the recorder, Julia asked "Shall we scrap this and think of something else?" Neither Marilyn, Joyce nor I thought we should scrap our work, but that it "could be put aside; it becomes a record of part of our thinking. Chapter 1 of our thinking." I asked Julia about the way that she was thinking about the representation.

Julia offered:

> I think that they are linked. I'm not so sure that the Venn diagram says what we want it to say, but I think the idea of linking them is very valid. Like rectangles that are with connected with lines, just like in the old textbooks when they would describe the 'family of parallelograms.' Where squares are part of rectangles, rectangles are part of parallelograms. You want something to be parts of the whole. Right?

Joyce suggested "What about something we could show, like, interwoven? Like braided together?" Julia clarified, "weaving together, like integrated?" Marilyn responded, "a woggle." Joyce and I were mystified, neither of us had previously heard about a woggle, so we asked Marilyn to tell us about a woggle.[17]

> It's a mathematical thing. Mariko, one of our colleagues, had her Math 10 students make them last year. She used a rectangular piece of felt with two slits in it. The material between the slits was divided into thirds and then the pieces are braided. There is some way of twisting all the felt together so that's it's still in one piece. You make this little braided thing. It looks like a braid. It also looks something like a mobius strip.

Although we all thought that the woggle would be a good representation for our descriptions, no one could draw one, and so we continued to discuss other possible representations. I suggested an orange could be used, with the processes being the segments of the orange. Julia responded, "I can't draw that. Can somebody else draw it?" We all laughed as we responded no. Julia continued though, by saying "that's a good idea though. Maybe we can use a circle with segments that won't look three-dimensional. We could draw semi-circles. There must be some program out there." I asked, "What would the orange peel represent in terms of our conversation?" I continued with sharing how I thought the orange, even though I suggested it, might not be a good representation: "The orange treats the processes separately. I could take a segment, say connections, away and I should be able to talk about that one segment. Whereas, we've been saying how all of the processes are related." Julia continued, "Where the Venn diagram says that there is a relationship." Marilyn, once again, took us back to the diagram Julia had drawn:

Why can't all of these intersections be problem solving?
Where problem solving can be the nucleus of all three of them.
All of that is problem-solving?
Yeah. That would work.

Is problem solving only connected with
Communication and connections?
Without reasoning?
Where is reasoning?

Part of those?
Or is reasoning the universe?
Yeah. Maybe. For the orange peel
Reasoning is what keeps things together.

What's at the centre?
The centre is the problem,
The problem to be solved
The big circle,

That's the problem solving process

Or would the inside one be hypothesizing or generalizing?
It's concluding.
When I come up with an answer to a problem, I've made a conclusion?
There's a variety of ways to answer.

[17] A woggle is a type of knot. Pope (2002) describes the woggle as "Technically, these are knots, not braids, because braids are composed of separate interwoven strands. None-the-less, "braid" seems a most descriptive word for the beauty of this family of knots."

That conclusion can be a hypothesis or a generalization.
It can ask another question
It can make a prediction
Or it can be finite.

What would be in the centre again?
Conclusion
Conclusion, hypothesis, and then
Generalization

Reasoning, it will be then be what we call the universe?
Universe.
Yeah
And problem solving is a process.

Okay, let's try that.
Let's see what it looks like.
And you know what?
This is kind of fun.

We laugh, Julia continued to sketch. We continue to think about the diagram.

On the bottom,
Problem solving process?
In the middle,
Hypothesis, conclusions, generalization...

Uh-huh
Anything else?
We might find some other ones to put in there.
I think that's nice to start with.

What are you thinking about?
Thinking about that diagram.
Just admiring it,
I hadn't thought about it this way before.

We paused to look at and consider the emerging diagram. Marilyn started our pondering with "Sometimes you try diagrams and they are too limiting. But if you don't have a diagram then there's too much information. This will be a picture of our overall thinking. It's like using graphs, tables, and charts in statistics. Without the picture, the words in themselves won't mean anything." I suggested that this act of constructing the diagram was a way to 'practise' the visualization and representation that we've talked about. Joyce added that the diagram was also a form of communication and problem solving, because it shows how we got to this point. Julia tried to refocus our group on the diagram, but I continued, "So we've now drawn a visualization of some sort but it really only makes sense to those of us who were in here, because we've had and participated in all of the conversation." Marilyn replied, "Yes, but it will still give somebody else a picture to look at. It may be kind of interesting to ask that person, 'what does this mean to you?' We would get a sense if the picture communicated our thinking."

Julia started again:

What do we want to say under communication?
I like the word community
Words and phrases
Cause that's what we build, is a community.

Communication
Write, Talk and Act
Write, Talk, and Act
Yeah

The other side of communication is comprehension.
Understanding someone's communication.
Or should it be listening?
Yeah, listening and reading.

Not just reading the written text but the symbolic text.
And reading a diagram, and
Interpreting an action.
So interpretation is a good word under there.

There's something about variables and symbols
Use of mathematical language and symbols.
Symbols should be there.
Okay, symbols.

Belonging to the community
Share a common language,
A mathematical language

A formal language of mathematics

How about this,
Write Talk Act
Links to Listen
Uh-huh Uh-huh

Connections
Linking
Linking and consciousness.
Making meaning

Linking things and experiences.
Linking ideas and experiences?
Ideas like past knowledge.
Cross-curricular, cross-grade?

Links to previous learning?
Careers?
Is that linking to real-life?
Isn't that experiences?

Past experiences, but now careers
Future experiences
Could we call that anticipating?
Oh, anticipating experiences.

You are anticipating the links.
Not just to reinforce the past,
Or to reuse the past
It's also to prepare us for the future

Real-life, what did you mean?
Using real-life examples to
Give meaning to or for the mathematics
Context...or application? Yeah sure.

Problem-solving process
There are strategies like
Making a diagram
Guess and test

Interpretation?
Organizing?
When you communicate?
Connections to the past, linking ideas and experiences.

Should application be in here?
Wouldn't the context be just the connections?
Wouldn't the consciousness
Consciousness would be just a connection

Where does questioning fit?
Communicating
That's part of the problem solving process
It should go in the middle

Being able to apply learned
Mathematical concepts and skills
To new situations
I think that's a good idea.

How do we make those links?
By organizing and interpreting

Community could be in there
Building the community
Being part of the community
Using the community as a connection

Julia and Marilyn both stop to look at the diagram. Julia began, "It's amazing, actually amazing. I've never thought of those before, like this." Marilyn added, "Neither have I."

So what is reasoning?
Reasoning is thinking?
Logic. Different kind of thinking.
What's another kind of thinking?

Intuitive thinking?
It's intuition, not logic.
It's hard to describe
Intuition as knowing

It's hard to describe
Cause when I think of the word
Knowing
I can explain why I know

With intuitive, "I just know"

That's how I feel
I know that's the right way
Can't explain it

You know it has all your whole background
What's brought you to that choice
No one particular thing
That you can put your finger on

Is reasoning being able to say why?
Do you always have to explain why?
I think in math there should be a why
In our context there should be a why

And to be able to explain how
Explain how and why you know this
So how is kind of the logic
Are you saying there's no intuition in math?

Oh no. Oh no. Oh no.
Being able to say how I reached
The hypothesis, conclusion, or generalization
Oh support, supporting evidence.

Maybe supporting arguments.
About why
Or how
And what

Justification?
Some, but there is support
Justify too strong?
Support is better

I know I use the word often
Justify
I like support
Support is better

Yeah, support works
Justify is more like communication
And support is more open
Yeah, yeah.

So when we say intuition in mathematics
Is that when we say "I know it?"

It doesn't get at the 'why.'
Yeah, okay.

I think intuition can be there
May have gotten the original idea
Through intuition
But what do you do with it?

Can you work backwards?
Can you take the process from there?
But to belong to a community...
I have to explain why I know

If not, I'm not a part
I only have the math
I only have the knowledge
You only have the knowledge.

I have inductive
And deductive down here
For reasoning
What type of support do you give?

That's what it is
We want to say logic?
Does deductive reasoning say logic?
Yeah...yeah.

Inference, support
Is inference a form of deductive?
I think so
Or is inference under generalization?

Isn't inferring inductive reasoning?
I don't know.
It's like a hypothesis.
So maybe it goes there. [pointing to the centre of the Venn diagram]

It's different than a hypothesis.
It's different than a generalization.
Support...
This is why I can say this now.

I look at reasoning
Is it the verb or the noun?
And that can be connections.

I suppose that's why it's the universe.

So, it's still support?
It's still support.
You can support the idea
In two ways

What about precision?
When you do deductive reasoning
You are very precise
You are very logical

Or maybe precision belongs
In the problem solving process
Isn't precision a strategy?
It's important to be precise in mathematics.

To know when to be precise
Oh definitely
To know when that's going to matter
Depends on your purpose

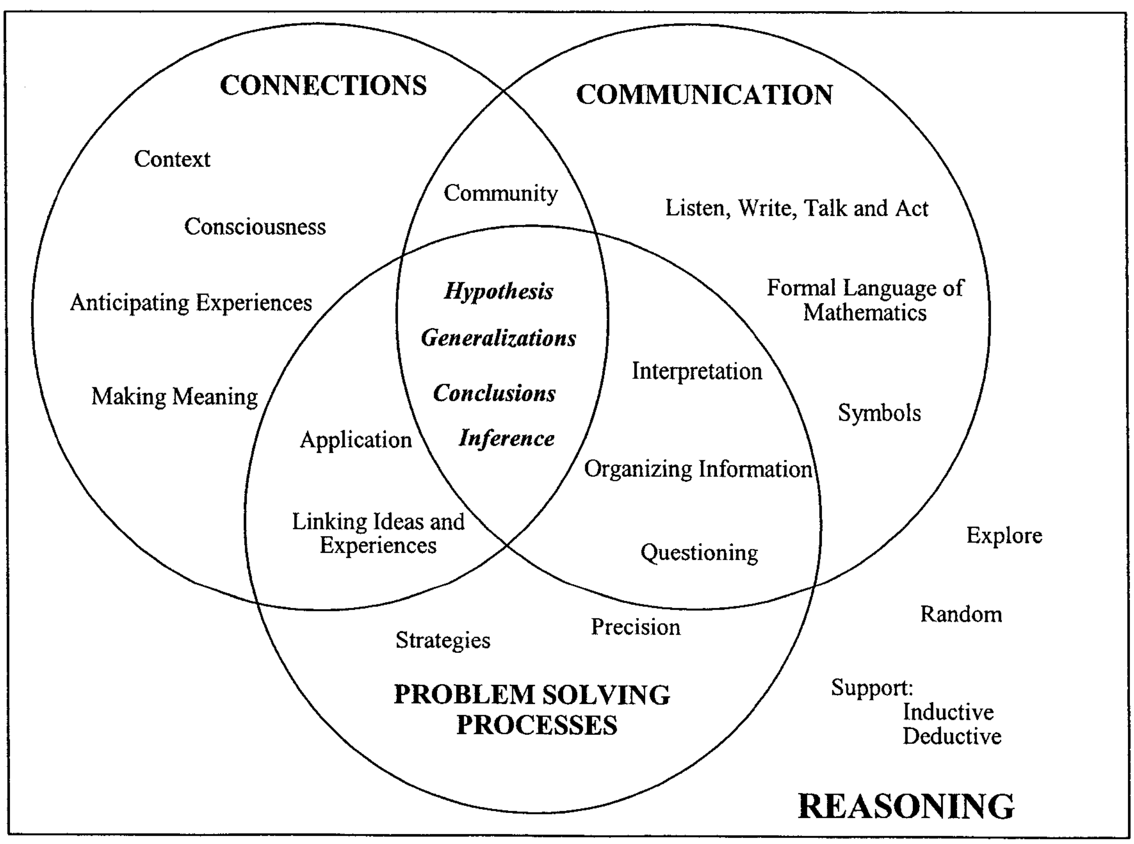
Julia yawned, "Oh, we've done good, good work here." Joyce agreed. I asked if each of us could see our definitions reflected in the diagram. We all agreed we could see our own definitions in the diagram, but Julia remembered that I had used the word random. She asked me to explain again what I thought I meant by that word. I started:

> I said it when I was thinking about reasoning. There are some people that talk about mathematics as being a logical sequential, step process. But if you listen to them solving a problem, their thinking doesn't appear that way. Their thinking appears more, intuitive, random perhaps. In mathematics we start from the specifics and we generalize, right? Or, do we, maybe, bring in different things; it appears random and then comes together, like, an artist. As I say that, and talking about it, I'm not convinced that there is a difference. I've heard artists say that they don't see their work as a logical step process. They are just painting something, for example, and something else emerges within them. Dancers move to how they feel. Writers sometimes just write and ideas emerge. But many people see mathematics as following a set of steps. And it's always these steps. But in mathematics, are there really a set of steps? No. There is an element of randomness. But the perception is that there's always steps.

Marilyn suggested, "Well, if you have no place to start when solving a problem, an intuitive place to start, then what do you do? You go random, don't you?" Julia added "sort of like mulling it over in your mind, like a brain storm." I agreed and built on Julia's idea, "you're brain storming with yourself." "You're driving and you're thinking about. Or falling asleep and you're thinking" continued Julia. Marilyn added "And you wake up in the middle of the night." Joyce added "even your conscious, sometimes you don't realize it that you're mulling it over." We all agreed that this happens.

I continued in my wondering and asked "Is there space in our mathematics curriculum for us to work with students in this area. I called this 'mulling' over random. Do we acknowledge that this is a part of problem solving? Yet generally we have so much content that students have to know, yet this is a part of the process."

That night Julia took her sketch home and created a computer-generated version of our representation:



Preparing for Chapter 5

Now that you have had a chance to read the two narrative accounts, *Coming to Know One Another* and *Building a Representation of our Thinking,* I wouldd like to draw your attention to some features of enactivism that can be seen through these conversations. Davis and Sumara write:

> the conversation winds and wanders, arriving at places that, quite simply, could never have been anticipated...the conversation is something more than the coordinated actions of autonomous agents...the conversation is not subject to predetermined goals, but unfolds within the reciprocal, codetermined actions of the persons involved (1997, p. 110).

They further suggest that "the conversation might be thought of as a process of 'opening' ourselves to others, at the same time opening the possibility of affecting our understandings of the world" (1997, p. 110).

Clandinin and Connelly describe conversations similarly: "[C]onversations are marked by equality among participants and by flexibility to allow participants to establish forms and topics appropriate to their group inquiry...conversation entails listening...the listener's response may constitute a probe into experience that takes the representation of experience far beyond what is possible in an interview" (2000, p. 109).

Gordon Calvert (2001) sees conversation as "voices co-versing in prose, in poetry. Bodies rhythming in language, in gesture" (p. 47) and notes that conversation "carries with it a sense of embodiment, presence, responsiveness, and responsibility. In our conversations with others we explore a topic of mutual interest and at the same time, further our relationship with each other" (p. 47). Gordon Calvert also notes that "conversations proceed on relevant concerns and directions that emerge in the moment and arise from the phenomenological history that each person brings to the conversation"

(p. 47).

These two narrative accounts are examples of representations of conversations, professional conversations, where each of us brought forth our lived histories as teachers as we participated in the conversations. In the narrative accounts I used examples of the poetry of the conversation, the co-determined actions, and the moments that emerged and were prompted by our individual lived histories.

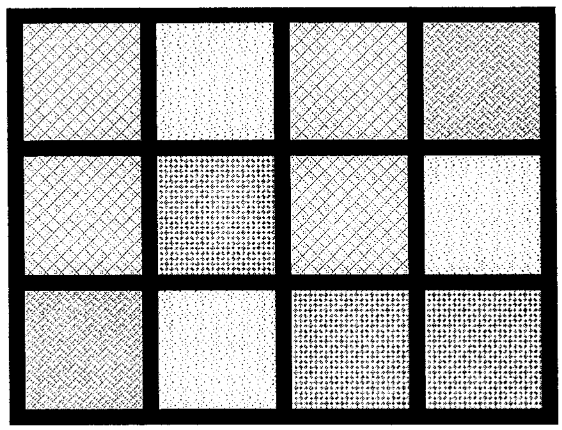
The narrative accounts in this chapter, selected from and constructed based on our first meeting, illustrate the richness of teachers' cognition about practical and curricular ways of understanding mathematical processes. That is, one sees the cognitive domain with respect to these issues and the sphere of possibilities expanded and just how the particular conversational contributions of each maintain the collective conversation and generates, elaborates, or supports developing ideas. Thus, I see this conversation as a site for the emergent understanding of the teachers.

In the next chapter, I will interpret pieces of the narrative accounts to show the ways in which I have come to see emergent teacher understanding of mathematical processes within our conversation.

CHAPTER 5

Recursive Elaborations of the Narrative Accounts: Inquiring into the Narratives of Experience

(Closely Examining the Pieces)



As you read the two narrative accounts, *Coming to Know One Another* and *Building a Representation of our Thinking*, you also brought forth a world, a world that reflects your experience, actions and lived history. What story would you tell of the world that you brought forward as you read the narratives?

As you read in Chapter 4, in the process of re-searching and composing the narrative accounts I noticed emerging and changing teacher understanding within the context of conversation. In writing the narrative accounts and studying what seemed to be critical moments in them, I found I was writing about emerging and changing understandings within that one conversation. I began to reformulate my research question to be “In what ways do mathematics teachers grow in their understanding of mathematical processes within the context of professional conversation?”

As I worked with the narrative accounts, I could see emergent understanding within the teacher conversation. I began to notice features of individual understanding, teacher understanding within the collective, and understanding within the body of mathematics. I began to wonder about these features as the evolutions of three threads in the conversation.

Using the quilt metaphor and the reformulated research question, I asked you to imagine examining two pieces of the quilt closely. The two pieces of the quilt I used for the context of exploring the question of emerging and changing teacher understanding within the context of conversation were the narrative accounts of our first meeting day.

In this chapter, I provide my interpretation of the two pieces of the quilt by selecting several moments within the pieces, or the two narrative accounts. The interpretations of the moments are focused on the evolutions of three 'threads'[18] in the conversation. The threads are related to individual understanding, the social or collective understanding, and understanding within the body of mathematics. The individual teachers' understanding thread is a weave of individual experiences within each person's lived history, which includes his/her life as a teacher within the body of mathematical knowledge.

Returning to the quilt metaphor once again, I could think of the idea of narrative inquiry as forming the backing for the "pieces" (the two narrative accounts) in Chapter 4. In this interpretive chapter, I use ideas from enactivism to both connect and outline the quilt pieces. The three threads, backed by enactivist ideas, can be seen as the elaborative quilting on the quilt. This elaborative quilting adds new features and provides a connecting structure to the quilt. Thus, the interpretations in this chapter, which are based on the complex narrative accounts and backed by ideas from enactivism, provide key

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[18] I use the word threads here to suggest that, as we closely look at the moments, they are once again pieces of the quilt. The two narrative accounts are part of the quilt and the moments are also part of the quilt. Threads, too, are part of the fabric of the quilt.

ideas relating to the elaborated research questions I have posed.

I will re-present moments within the two narratives to explain the way in which I observed teacher understanding as an emergent phenomenon. In no way am I suggesting that my explanation has a universal or compelling character (Maturana, 1987). I offer this possibility of an explanation that both respects and is conditioned by the contingencies of the personal lived histories of the participants, the task and the conversational character, and ask you to reflect on this possibility.

Selecting the Moments

How did I select which moments in the narrative accounts to use? As Maturana and Varela observe, “everything said is said by someone” (1987, p. 26). In this case, I am the someone, and these moments will be significant in terms of my lived experiences. There are hundreds of moments in the two narrative accounts; the moments that I chose were significant to me because as I was reading and re-reading the narratives, these moments emerged as moments that prompted me to remember other stories in my experience. The moments of the conversation I have chosen illustrate the way in which I see emergent teacher understanding because of the experiences I have had. The selection of the moments was also affected by the research question, because my experiences include reflection on and posing the research question. I began to see patterns in the interactions in the moments related to changing understanding and this is also a part of my experiences. These experiences are part of the history that I bring to this collective and the history that I bring to describing the path that our collective has laid.

Maturana and Varela also say “every reflection brings forth a world” (1987, p.

26).[19] Hence, as I wrote the interpretations, reflecting on the narrative accounts, I was bringing forth a world. As you read the narrative accounts, and now as you read the interpretations, you will also bring forth a world, a world that reflects your experience, actions and lived history but is also conditioned by the narratives and interpretations.

The Interpretive Threads

In the interpretive writing, drawing from each narrative account, I have used three 'threads' to re-story moments in the experience (Clandinin and Connelly, 1995) of being together. I am a part of the collective that laid down a path of knowing and understanding acts as teachers in conversation. In engaging in these interpretive acts, I am stepping out of the conversation and act as a researcher to provide explanations[20] upon further reflection. I have composed the interpretive acts as the researcher. I now look back on the narratives and point to moments in the narrative accounts to provide an explanation of those moments.

The three interpretative 'threads' explain the way I have come to think about emergent teacher understanding. The three threads are individual understanding (related to Pirie and Kieren's (1994) "dynamical theory for the growth of mathematical understanding"), social or collective understanding (related to Davis and Simmt's (2003) conditions of complexity for the emergence of a mathematical community), and understanding within the body of mathematics (based on Davis's (1996) work).

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[19] Notice that the "world" is always a co-emerging phenomenon; determined by the structure of the person(s) but co-determined by environment including others. Thus the "world" arises from the environment as well as the reflective acts of the person(s).

[20] I have chosen to use Gordon Calvert's (2001) notion of explanations as re-presentation as a frame to describe, or explain, how I have made sense of teacher understanding of mathematical processes within the context of conversation. Gordon Calvert suggests that "explanations as re-presentation" may be "evoked when a person is asked to summarize his or her thinking to persons outside of the conversation" (2001, p. 82).

Individual Understanding

The first thread is based on Pirie and Kieren's (1994) "dynamical theory for the growth of mathematical understanding." I will be elaborating aspects of the theory throughout the chapter, but offer a few introductory remarks here.

Pirie and Kieren's (1994) theory views mathematical understanding as a "whole, dynamic, leveled but non-linear, transcendently recursive process" (p. 166). Although looking at children's mathematical understanding generated the theory, Pirie and Kieren (1994) indicate that the theory is "not age related" (p. 185). Pirie and Kieren describe eight "potential levels or distinct modes" (1994, p. 170) of understanding. These levels or modes are primitive knowing, image making, image having, property noticing, formalizing, observing, structuring, and inventising (I will provide their definition of this invented word later). Pirie and Kieren (1994) have represented the model as a "sequence of nested circles or layers…emphasizing that each layer contains all previous layers and is embedded in all succeeding layers" (p. 172). They "see growth as represented by back and forth movement between levels" (Pirie and Kieren, 1994, p. 172).

An illustration of the Pirie-Kieren model is shown below:

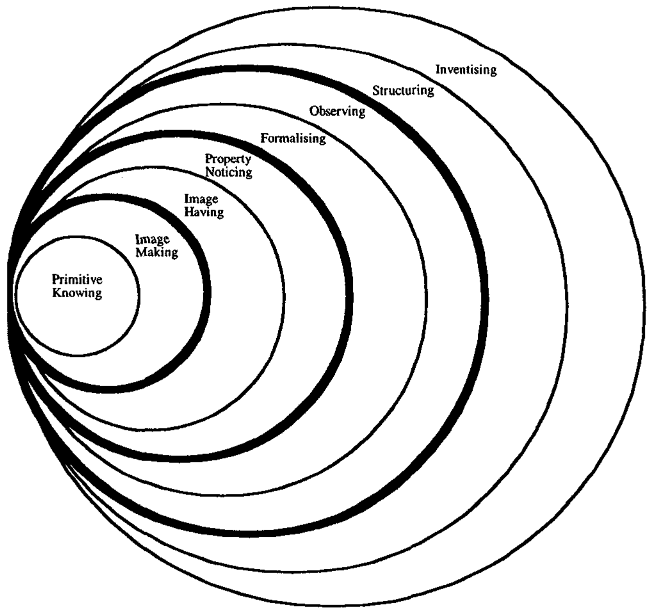


Figure 2. Model for the Pirie-Kieren theory of the dynamical growth of mathematical understanding. Used with permission.

Pirie and Kieren (1994) say that "the process of coming to understand starts at a level called *primitive knowing*...[which] does not imply low level mathematics, but is rather the starting place for the growth of any particular mathematical understanding" (p. 170) and it is "what the observer, the teacher or researcher assumes the person doing the understanding can do initially" (p. 170). Image making is when the "learner is asked to make distinctions in previous knowing and use it in new ways" (Pirie and Kieren, 1994, p. 170). Image having is when a person "can use a mental construct about a topic without having to do the particular activities which brought it about" (Pirie and Kieren, 1994, p. 170). Property noticing occurs when "one can manipulate or combine aspects of ones images to construct context specific, relevant properties" (Pirie and Kieren, 1994, p. 170). Formalizing is when the person "abstracts a method or common quality from the previous image dependent know how which characterized the noticed properties" (Pirie and Kieren, 1994, p. 170). A person at the observing level is one who is in a "position to reflect on and coordinate such formal activity and express such coordinations as theorems" (Pirie and Kieren, 1994, p. 171). Structuring occurs when "one attempts to think about ones formal observations as a theory" (Pirie and Kieren, 1994, p. 171). A person at the inventising level has "a full structured understanding and may therefore be able to break away from the preconceptions which brought about this understanding and create new questions which grow into a totally new concept" (Pirie and Kieren, 1994, p. 171).

An important feature in the Pirie and Kieren (1994) theory is that of "folding back" (p. 173) and they describe this idea as follows

> When faced with a problem or question at any level, which is not immediately solvable, one needs to *fold back* to an inner level in order to extend one's current, inadequate understanding. This returned-to, inner level activity, however, is not identical to the original inner level actions; it is now informed and shaped by outer level interests and understandings...The inner level action is part of a recursive reconstruction of knowledge, necessary to further build outer level knowing (p. 173).

The darker lines on the model represent a second feature of the Pirie-Kieren theory, that of the 'don't need boundaries.' Pirie and Kieren (1994) call these darker rings "the 'don't need' boundaries in order to convey the idea that beyond the boundary one does not need the specific inner understanding that gave rise to the outer knowing" (p. 173).

As Towers and Davis (2002) demonstrate, the Pirie-Kieren theory can also be used to analyze the structure of children's collective mathematical understanding. As they note, "each student is complicit in the unfolding understandings of the other" (p. 326). Hence it seems possible to use ideas from this theory in my interpretation of teachers' growing understanding of mathematical processes in a curricular/instructional context on both individual and collective levels.

An important consideration in the Pirie-Kieren theory is that their "theory attempts to elaborate in detail the constructivist definition of understanding as a continuing process of organizing one's knowledge structures" (1994, p. 166) and that they have used the theory in a "variety of learning environments as a tool to observe the mathematical behaviour of students as they work on a single mathematical task and as they build and organize mathematical knowledge structures over periods of time" (p. 181). I am suggesting that an observer could also use the theory as a tool to observe emergent mathematical understanding when teachers are talking about mathematics, and in this case specifically mathematical processes.

Collective Understanding

The second thread I use in the interpretation of our experiences views our group as a collective learning system (Davis and Simmt, 2003). As with the Pirie-Kieren theory, I elaborate important points throughout the chapter, but provide orienting remarks here.

Davis and Simmt (2003) describe five "necessary but insufficient conditions that must be met in order for [such] systems to arise and maintain their fitness within dynamic contexts—that is, to learn" (p. 147). The five interdependent conditions are internal diversity, redundancy, decentralized control, organized randomness, and neighbour interactions (Davis & Simmt, 2003, p. 147). In Chapter 3, I described the conditions of internal diversity and redundancy and suggested that the lived histories of each of us could be used to describe the ways in which our collective learning system met those conditions. I continue to look for examples of the conditions of redundancy and internal diversity in the moments I interpret in this chapter and also look for examples of the conditions of decentralized control, organized randomness, and neighbor interactions.

Davis and Simmt (2003) describe decentralized control as when "the system itself 'decides' what is and is not acceptable" (p. 153) and that "appropriate action can only be conditioned by external authorities, not imposed" (p. 153). Organized randomness is a "structural condition that helps to determine the balance between redundancy and diversity among agents" (Davis and Simmt, 2003, p. 154). Davis and Simmt (2003) further say that the structures "that define complex systems…maintain a delicate balance between sufficient organization to orient agents' actions and sufficient randomness to allow for flexible and varied response. Such situations are matters of neither 'everyone does the same thing' nor 'everyone does their own thing' but of everyone participating in a joint project" (p. 155). Davis and Simmt (2003) write that neighbour interactions are not the interactions among "physical bodies or social groupings" (p. 156) but "rather…the neighbors that must 'bump' against one another are ideas, hunches, queries, and other manners of representation" (p. 156). I use this second thread to examine the way in which emergent understanding exists within our mathematical community, as described by Davis and Simmt (2003).

Understanding Within the Body of Mathematics

The third thread I use to interpret the narrative accounts is that of the body of mathematics (Davis, 1996; Gordon Calvert, 2001). In what ways do these teachers invoke mathematical ideas into their understanding acts? Davis (1996) asked, "How does the discipline contribute to our perceptions and define our actions?" (p. 80). How does our collective enact mathematics into our understanding acts from cultural practices of the larger mathematical community? To what extent are such enactments personal? To what extent does the body of mathematics provide a source of "redundancy" needed to support a more collective reason or understanding?

In what way do these threads come together as part of my inquiry process? I return to the image of the quilt. The quilt itself is an emergent phenomenon as an image of this research. I take two pieces of the quilt (the narrative accounts), from those pieces of the quilt I take samples of 'moments' and use a recursive process to examine and re-examine the moments in a fractal-like way. The first level of recursivity, or iteration, is selecting the moments; the second iteration is describing why the moment was chosen; the third iteration is the re-interpretation using the thread of the Pirie-Kieren theory of emergent mathematical understanding; the fourth iteration is the re-interpretation of the moment using Davis-Simmt's collective learning system conditions of complexity; and the fifth iteration is the re-interpretation of the moment within the body of mathematics.[21]

In Chapter 6, I will try to give an overall picture featuring the main consequences from these recursive interpretations as well as suggesting how the consequential ideas might be used in research and particularly in the practices of teacher education.

Reading the Interpretations

In the presentation of the interpretations, I have pulled the 'moment' from the narrative account. A key term for narrative inquirers is temporality, in the sense that "experiences taken collectively are temporal" (Clandinin and Connelly, 2000, p. 19), hence a moment could be thought of as a 'mini-narrative.' To describe the moments, I have used samples taken directly from the two narrative accounts in Chapter 4.

I have selected five moments to illustrate my thinking about teacher

[21] Notice how the idea of recursion is used here. Each interpretation extends the input to further interpretations; but later interpretations necessarily extend and re-construct earlier ones. Thus the interpretive process is not simply iterative, but is such that the interpretations impact one another.

understanding as an emergent phenomenon. The moments I have selected connect with my story; however there are many moments within the two narrative accounts. The moments that an observer might select, identify or be interested in will come out of their individual lived history. I share why each moment is significant to me. The way I make sense of my world is through understanding my experience narratively, hence, the reason the moment is significant to me will connect to my experiential history. Because part of my experiential history is thinking about teacher understanding as an emergent phenomenon, the moments are also places that I observed emergent teacher understanding. Narrative inquirers realize the recursivity of their lives; we keep 're-telling' stories. As the stories are told and re-told, we re-live our lives. As complex beings our lives are like fractals and fractal filaments.

How I've Organized the Interpretations

Five moments have been chosen, the 'Connections' moment, the 'Wow' moment, the 'Right! That's Right!' moment, the 'Woggle' moment, and the 'Random' moment. For each moment, there are five levels of interpretation: the description of the moment, why the moment was selected, an interpretation through the Pirie-Kieren thread, an interpretation through the collective dynamic thread, and an interpretation through the body of mathematics thread. Periodically, in describing the features of the Pirie-Kieren theory, the collective dynamic thread (Davis-Simmt), or the body of mathematics thread, I use examples from my lived history to illustrate the interpretation I have made.

As you read the interpretations, the following figure may assist you in helping to make sense of my organization.

Table of Moments and Interpretations
(Recursive Elaborations of the Moments)

The 'Connections' Moment
- Interpretation 1.1 Description of the Moment
  - Interpretation 1.2 Why this Moment?
    - Interpretation 1.3 Individual Understanding Thread – Primitive Knowing
      - Interpretation 1.4 Collective Dynamic Thread – Internal Diversity
        - Interpretation 1.5 Body of Mathematics Thread –Diverse Interpretations

The 'Wow' Moment
- Interpretation 2.1 Description of the Moment
  - Interpretation 2.2 Why this Moment?
    - Interpretation 2.3 Individual Understanding Thread – Image Having/Making
      - Interpretation 2.4 Collective Dynamic Thread – Redundancy
        - Interpretation 2.5 Body of Mathematics Thread – Problem Solving

The 'Right! That's Right!' Moment
- Interpretation 3.1 Description of the Moment
  - Interpretation 3.2 Why this Moment?
    - Interpretation 3.3 Individual Understanding Thread - Dynamical
      - Interpretation 3.4 Collective Dynamic Thread – Organized Randomness
        - Interpretation 3.5 Body of Mathematics Thread – Mathematics in Community

The 'Woggle' Moment
- Interpretation 4.1 Description of the Moment
  - Interpretation 4.2 Why this Moment?
    - Interpretation 4.3 Individual Understanding Thread – Emergent Possibilities
      - Interpretation 4.4 Collective Dynamic Thread – Neighboring Interactions
        - Interpretation 4.5 Body of Mathematics Thread – Embodied Mathematics

The 'Random' Moment
- Interpretation 5.1 Description of the Moment
  - Interpretation 5.2 Why this Moment?
    - Interpretation 5.3 Individual Understanding Thread – Folding Back
      - Interpretation 5.4 Collective Dynamic Thread – Decentralized Control
        - Interpretation 5.5 Body of Mathematics Thread – Mathematics as an Emergent Phenomena

Figure 3. Table of moments and interpretations.

The 'Connections' Moment

Interpretation 1.1 Description of the Moment.

We all mused about this idea for a brief moment and I then shared my writing about connections. I read:

\- - - - - -

Connecting to something else
Works
From other experiences

\- - - - - -

Who makes the connections?
Teacher?
Or the student?

\- - - - - -

In response to my reading, Julia said, "That's interesting. I know that Marilyn and I have talked about how we've noticed that our exceptionally talented math students appear to be making their own connections all of the time. They don't go on until they know exactly how 'this' fits and then they're quite satisfied. Whereas, other kids don't do that. Then I think that's the teacher's responsibility to help them make those connections and help them see those fits."

I continued with my reading:

\- - - - - -

How do we help
Students
Make connections?

\- - - - - -

I make connections
While
I'm thinking.

\- - - - - -

Everyone agreed and Julia added, "That's exactly what they

> [students] need." Marilyn agreed, "Yeah! Sometimes you'll say something, look at particular students, and you can just see connections happening. Remember teaching Roman?" At this point Julia and Marilyn recollected a former student, Roman, with a series of looks, nods, and 'Hmms.' Joyce and I looked at one another, clearly not a part of the story of teaching Roman.
>
> I did feel prompted to ask though, "Do you think it's 'natural' for some people and not for others to make connections? Or do we all do it?"
>
> Julia quickly responded, "I'm thinking that maybe we all do it in some ways and in some areas but maybe not all of us do it mathematically."
>
> This prompted me to ask, "When students learn the concepts of mathematics, do they automatically make connections between the concepts?"
>
> Marilyn answered, "Some, but I think it's a minority. When you give them an idea they'll go with it, a lot more will go with it. But, if you didn't make the connections for them most of them wouldn't be making it." Marilyn continued with her story of Roman to illustrate her point "you see there are the students like Roman. Roman would ask 'Does this mean this?' or 'Could you do it this way?' The questions he asked referred to material he had studied in previous grades. Other students in his class would be looking at him wondering...." Julia finished Marilyn's sentence "where is this coming from? What planet is he on?" Marilyn finished her statement with "I think all the students would know that stuff, but they would never think of it. Not on their own motivation."
>
> Julia then commented that as Marilyn described students making connections she was thinking about the relationship between connections, community, and communication. Marilyn, Julia, and I were clearly on the same wave length.

Interpretation 1.2 Why this Moment?

The 'connections' moment reminded me of a meeting that I once attended. I was the chairperson for a group of people who were writing a book about mathematics assessment in primary classrooms. One of the members of our team had been a mathematics education professor for many years. During one of our conversations we were discussing the way in which we ask questions and why we ask children questions about mathematics. Charles, the mathematics education professor, made the comment "I think I know why. The reason to ask questions is so that we are able to see the connections children make." At the time I remember rolling my eyes and thinking "Of course! Why else?" and wondering, "How could he not have thought of that before?" Had I thought of the idea of the reason that we ask children questions is to discover the connections that they've made before?

Interpretation 1.3 Individual Understanding Thread – Primitive Knowing.

Of interest to me during this moment is the way in which our conversations bring forth our lived histories of working with students. The notion of 'making sense' of the mathematical process of connections was affected by our classroom histories, both in my writing about connections, in the questions I ask about connections, and in our subsequent conversation about 'students who make connections' and 'those students who do not.'

When Julia and Marilyn are talking about the differences between their students, we can consider that they are exploring the idea of the assumptions that teachers make about what students' already know. Julia says that she notices her "exceptionally talented math students appear to be making their own connections all of the time. They don't go on until they know exactly how 'this' fits and then they're quite satisfied." Marilyn provides two examples of the way in which a particular student would ask questions, 'does it mean?' and 'could it be done this way?' referring in each case to his [Roman's] previous experiences. It is through these questions that Marilyn would have a sense of this particular student's primitive knowing.

The Pirie-Kieren theory helps us consider emergent mathematical understanding. Because we are complex beings, we bring forth our embodied lives in our conversations.

In this conversation about connections we see Marilyn and Julia's teaching lives coming forth. When the question, 'Who makes the connections?' is asked, Julia is reminded of a conversation that she and Marilyn have had about the differences between students, those students who are 'exceptionally talented math students' and 'other kids.' When students ask me questions such as 'does it mean?' and 'could it be done this way?' I too am gaining a sense of the connections that those individual students are making and perhaps insight into their primitive knowing about a particular mathematical concept.

Our conversation within the 'connections' moment can also be used to further explore Pirie-Kieren's notion of primitive knowing. Primitive knowing "does not imply low level mathematics, but is rather the starting place for the growth of any particular mathematical understanding. It is what the observer, the teacher or researcher assumes the person doing the understanding can do initially" (Pirie and Kieren, 1994, p. 170). In essence, though, primitive knowing is the knowing that each individual brings to the understanding of a new idea; it is their knowing composed in their lived histories. What can be said of the teachers' primitive knowing about the mathematical process of connections in this moment? My lived history came forth in the writing I did for my description of the mathematical process of connections. My primitive knowing is situated in a teaching context and my writing, "Who makes the connections? Teacher? Or the student?" points to this. When I write "How do we help students make connections?" my primitive knowing points to the teacher as being the facilitator of the 'making connections' and that somehow the teacher should bring forward the connections students are making. In a sense this implies that the teacher would decide which connections students are supposed make. However, I do not want to suggest that a teacher must know all of the connections that a student could make. If my understanding of the mathematical process of connections comes from a history of one where the teacher facilitates by asking students about the connections they are making then, in my classroom, I too would try to understand the connections students are making in action. I could sample students in my classroom to determine the connections that they are making and encourage students to attend to the connections they are making. In this process, I not only become aware of the multiple perspectives that my students bring as their primitive knowing to the topic, but I am also in the process of re-imaging my understanding of the mathematical process of connections. I will have new images from my students that will change my image of connections.

Because mathematics is the study of patterns and relationships, it seems to me that I as a mathematics teacher could perhaps bring forward the idea of the connections that a particular concept has with another concept. The way in which I bring forward those connections and the 'story' that I tell around that concept, will invoke other connections for students. Sometimes the language that I use as a teacher excludes students from making connections. However, when I share the meaning that I've made of a particular concept with students and the way in which I've made meaning of that particular concept, the way in which I share that meaning making will help students make connections not only to other mathematical concepts but to each individual student's lived history. What is important is that I, as teacher, am aware of the language I use in 'bringing forth the possible connections.'

Further in this moment, Marilyn acknowledges that "all the students would know the stuff, but they would never think of it. Not on their own motivation." How do I, as a teacher, come to know an individual student's primitive knowing about a particular concept if the student does not ask the questions? Generally, as a teacher I make assumptions about the primitive knowing of each student in my classroom, based on assuming the common experiences of the students. For example, if I am teaching a grade 8 class mathematics, I generally make the assumption that all of my students have experienced grade 7 mathematics. My assumption of the grade 7 mathematics experience is based on my own experiences as a teacher. Each of my students may have experienced the grade 7 mathematics curriculum, but they will still bring forth a variety of lived histories that will shape the grade 8 experience in diverse ways. It is the assumptions that I make as a teacher that help in my interactions with my students. I will not be aware of the individual lived histories, or the primitive knowing that a child brings to a concept, until they present themselves.

Thus how I, as a teacher, prompt this in my classroom can be shaped by my image of the mathematical process of connections. Marilyn appears to already have this inter-relationship image of the mathematical process of connections but suggests that the teacher must prompt or ask students to re-member such inter-relationships. This is not to say that 'inter-connection' is a necessary feature or pre-determined feature of the objective of 'making mathematical connections.' But having an image of the mathematical process allows me to make my own teaching interpretation rather than simply following a set of pre-given activities.

This is the part of the complexity of teaching. How do we make well-educated assumptions about the primitive knowing of our students while at the same time being aware of the diversities that cannot be known until they present themselves?

Like the children, the talk of these teachers indicate that they have different primitive knowings which they bring to understanding mathematical processes in action, because, like their students they express themselves variably about this. Their primitive knowings and assumptions about them are not always easily observable. Nonetheless, if I as a teacher educator were to lead a group of teachers in a workshop about the meaning of the mathematical process of connections, I specifically might build in ways of interacting which would allow the teachers to reveal such knowing and would build activities which would allow teachers to form useful images of the mathematical process for teaching practices or for classroom curriculum.

Interpretation 1.4 Collective Dynamic Thread – Internal Diversity.

Each person in our complex learning system is a complex being. This means that we bring a multi-faceted lived histories into the conversations in which we are engaged. Each of us brings a variety of teaching experiences to our collective and because of the diversity in these experiences there is the possibility of new shared ideas being developed. Because our lived histories as teachers of mathematics had enough commonality, we could come to see teaching practices or make sense of episodes with which we were otherwise unfamiliar. For example, when Marilyn mentions Roman, a student that she and Julia both taught, Julia and Marilyn remembered Roman with a series of 'looks, nods, and Hmms.' Joyce and I did not know Roman, yet, as the conversation unfolds, Joyce and I can come to share a common understanding about students like Roman because we too recollect those students that ask 'does this mean this?' Our lived teaching experiences, although diverse and at the same time common within our collective, help our collective to develop a shared understanding of students 'like Roman' and students 'not like Roman.'

This example points to the usefulness of internal group diversity in previous experiences to the generation of both new personal understanding of mathematical processes and in the emergence of features of collective understanding. As Joyce, Julia, and I listened to Marilyn's story about Roman, we could imagine the meaning and context of Marilyn's experience. As Marilyn describes the emergent practice to others it could be a possible means of development of more sophisticated understanding of the content and practices for seeing mathematical processes in action.

At the same time that our conversation in this moment was affected by our lived histories, it also points to the internal diversity of classroom collectives. The students in a classroom collective are diverse and bring a variety of histories to the classroom. This conversation points to that diversity. For example, when asked "when students learn the concepts of mathematics, do they automatically make connections between the concepts?" Marilyn offers an answer "some, but I think it's a minority. When you give them an idea they'll go with it, a lot more will go with it. But if you didn't make the connections for them most of them wouldn't be making it." In this statement Marilyn is referring to the diversity of students that she has worked with in her classroom. There are students who will make connections 'automatically' (at least in some personal if unobserved way) and there are students who will require a prompt by a teacher. It is the shape of such prompts that will arise out of a teacher's understanding of the mathematics in the lesson. It is possible and even likely that thinking about and imagining examples of student actions in making connections will be important in developing a useful teaching image of the mathematical processes.

Recognizing the diversity within classroom collectives is an ongoing emergent act for the teachers in our collective. When we look closely at Marilyn's response, we might consider the types of ideas that teachers might give students in order to help students make connections between mathematical concepts. Ball and Bass (2003) call the knowledge that mathematics teachers need to know during these emergent acts the mathematical knowledge needed for teaching. In this understanding, at a particular moment, a teacher must provide some students with an idea to help them make connections. The teacher must know mathematics in a way that can provide students with an idea and, at the same time, ensure that the student can understand the way in which they are providing the idea. At that time, teachers consider the questions or ideas that will engage all of the learners in their complex system.

Davis and Simmt (2003) articulate the importance of diversity within a complex system when they write "the extent of a system's intelligence is linked to its range of possible innovations, which in turn is rooted in the diversity represented among its agents. A system's range of possibilities-its intelligence-is thus dependent on, but not necessarily determined by, the variation among and the mutability of its parts" (p. 148). Our collective understanding about the mathematical process of connections emerged within our conversation and was affected by the diversity of our lived histories of teaching.

Interpretation 1.5 Body of Mathematics Thread – Diverse Interpretations.

In this moment I began to think about the mathematics that teachers need to know in order for them to help their students make connections, particularly when Julia said "I'm thinking that maybe we all do it [make connections] in some ways and in some areas but maybe not all of us do it mathematically." As a mathematics teacher, I want my students to begin to make 'mathematical connections' because mathematics itself is inter-related. For example, when I introduce my students to the features of a cubic function, I would like my students to realize that the features of a cubic function are related to the features of a quadratic function, a linear function, a quartic function, andpolynomial functions more generally. It is this desire in me as mathematics teacher that prompted me to ask 'when students learn the concepts of mathematics, do they automatically make connections between them?' in this moment. This question brings forth my lived history of the body of mathematics, as a series of inter-related concepts. What experiences did I have that affected my image of mathematics as a series of inter-related concepts?

Bass and Ball (2003) ask, in what way must teachers be with mathematics?" This is a profound question that points to, not more high level mathematics courses for teachers, but different courses. Ball (2003) writes, "teachers need to be able to use representations skillfully, choose them appropriately, and map carefully between a given representation, the numbers involved, and the operations or processes being modeled. This requires significant mathematical skill, insight, and understanding, again well beyond the knowledge required to carry out a procedure oneself" (p. 3). Not only must teachers understand a mathematical idea but they must be able to "explain it in ways that are both mathematically valid and accessible" (Ball, 2003, p. 2) to all of their students. Bass and Ball's work has been with elementary school mathematics and leaves me to wonder about the way in which high school mathematics teachers need to be with mathematics. For example, high school mathematics teachers not only have to know how to factor quadratic expressions, but they must be able to explain the way in which factoring a quadratic expression is related to factoring numbers and different representations for factoring quadratic expressions. This is more than knowing how to factor a quadratic expression.

In teacher education programs, pre-service teachers are generally required to take mathematics content courses and mathematics education courses. The mathematics content courses are generally selections from the areas of calculus, linear algebra, geometry, and statistics. The content in these courses are beyond the content in high school mathematics. The mathematics education courses tend to focus on a variety of instructional strategies. Some content is addressed in the mathematics education courses; however, addressing a whole high school mathematics curriculum in a mathematics education course of 39 hours is impossible. Our pre-service teachers do mathematics and learn about instructional strategies that they can use in their classrooms, but do not have the opportunity to talk about the mathematics that they know and the ways in which they know it.

Where do our pre-service teachers begin to talk about the mathematics, in the way that Ball and Bass (2003) describe, that they are going to teach? Where do they begin to realize the diverse primitive knowings that they, as learners, bring to a concept, and hence come to realize the diverse primitive knowings that their students will bring to a concept? Because our lived histories affect our primitive knowing our pre-service teachers need to have the opportunity to talk about the way in which their individual history has affected their primitive knowing of a mathematical concept.

Once teachers are teaching, there are seldom opportunities or occasions for teachers to talk about the mathematics that they know and the ways in which they know it. When new curricula are being introduced, the professional development experiences are generally someone telling teachers about the curriculum. This is a great way to distribute information; however, this does not give teachers the opportunity to explore the meaning that they are making about the new curricula.

The experiences that I brought to this moment compelled me to ask, “When students learn the concepts of mathematics, do they automatically make connections between them?” My history was filled with experiences that required me to talk about the mathematics I know and the way in which I know it. In the times I facilitated workshops across the province and ‘item-writing committees’ I was engaged in conversations about the way in which mathematical ideas are inter-related. In my own high school teaching experience, I used to tell my students that mathematical ideas are all inter-connected; however, I am not convinced that I was explicit about the inter-relationships at the time. I imagine that I shared my view of the inter-relationships. Through the conversations with colleagues, however, my understanding of the inter-relationships became more explicit. As a teacher educator, I need to plan classroom experiences that focus on the connections that my students make between mathematical concepts.

It is also important for my pre-service education students to come to realize that the diversity of students in a classroom collective requires diversity within the teacher in order for shared meanings to develop and in order to enlarge the space of the possible. It is also important for me to remember that, when I am working with practicing teachers, that I should plan experiences where they are asked to talk about the mathematics that they know and the connections that they have come to make within their knowings. In the conversations about how we know mathematics and the connections that we make, we come to realize the diversity of knowing mathematics. We notice this in the 'connections' moment when Julia makes the comment that she was thinking about the relationship between connections, community, and communication when Marilyn was describing how she knew when students were making connections. In this comment, Julia points to her image of the mathematical process of connections being broader than only including the connections between mathematical concepts that I was talking about.

The 'Wow' Moment

Interpretation 2.1 Description of the Moment.

> Marilyn continued, "Often our best students have shivers when they are asked to solve a problem, because they perceive that it is so hard." I shared how the statement immediately reminded me about what I thought problem solving was when I was in school, word problems. I always found solving word problems hard. I remembered that I was often confused when I had to write a mathematical expression for a word statement like 'a number is 5 less than a second number.' I always wondered, 'where does the minus sign go in relation to the 'x'?'
>
> My story reminded Julia of her experience, "I would read those problems about trains and cars, and think 'I have no idea what this problem is about.'" Marilyn and Joyce both indicated that they also felt similarly this way about problem solving when they were in school.
>
> Marilyn suggested "it must be about comprehension and that links to communication." In a sense, were Julia and I not comprehending the language when we were in school? Marilyn, Julia, and Joyce all suggested that they did not learn how to problem solve until they started to teach about how to problem solve. I too remembered feeling that way after my first year of teaching. When I first started teaching, problem solving was solving the word problems in the textbook. I remembered working hard to develop techniques to help my students read and interpret word problems. So, I had finally 'figured out' how to solve those textbook word problems, when I was teaching in a high school.
>
> Marilyn continued with suggesting that she imagined that most current students would still consider problem solving as solving word problems in the textbook. Julia asked, "Isn't problem-solving in every school subject? Aren't we teaching problem-solving all the time?" This question caused us to think beyond the subject of mathematics.

Marilyn responded, "It's not called problem solving in other school subjects, it's called decision making, but I think it is still problem solving. I think decision making models can be models for problem solving."

Julia suggested that the decision making model could also be a model for hypothesizing in science. She continued, "the reason you do an experiment, is to find the answer to a question, and that's what you are doing when you are problem solving." I had never thought of a science experiment as problem solving so I asked Julia, where, in a science experiment does the thinking occur? Julia offered the following explanation of how she saw the problem solving process related to an experiment in science:

> It's when you ask yourself, "How might I answer a particular question?" One of the ways I might answer this question is by doing an experiment. There might be a variety of ways of answering the question, but you could choose to do an experiment. It's like doing research. What question are we going to ask? How are we going to go about answering that question? That's endless, right?

This led me to think about generalizing again and how generalizing is related to problem solving. Again our ideas were emerging:

Generalizing
A part of problem solving?

Generalizing
A result of problem solving?

Hypothesizing
A part of problem solving?

Hypothesizing
The beginning of problem solving?

Generalizing is
Hypothesizing?

Hypothesizing is
Generalizing?

"I'm just listening to all this, going 'wow.' I've never thought about this." said Joyce. We all agreed that these ideas were not something that we generally thought about either, but in this conversation, the ideas were emerging.

Interpretation 2.2 Why this Moment?

This moment emerged for me through a series of emotions. As I listened and re-listened to the audiotape, I could feel the anguish that I used to feel as I was attempting to solve word problems while in school. At the same time I could feel the anxiety I felt when I was teaching and came to the section in the textbook on solving word problems. To this day, when I am working with students in my teacher education classes, I feel the anguish and anxiety when asked a question about a particular word problem. I have learned, as I did when I was teaching high school, to talk through the word problem in an interpretive manner. I use the 'talking through' as a way of teaching my students about the importance of the interpretive act that both teachers and students make when they come to 'word problems.'

Interpretation 2.3 Individual Understanding Thread – (Primitive Knowing +) Image Having/Making.

In this moment, Joyce participates within our conversation of problem solving as solving word problems and then is quiet throughout the conversation of problem solving in all school subjects. One might view this silence as Joyce not participating in the conversation or that her understanding is not changing. However, if we think of the Pirie-Kieren theory, we are prompted to consider that Joyce is acting throughout the 'problem solving in all subject areas' conversation. We can consider that Joyce is acting in silence because she says "I'm just listening to all of this, going 'wow.' I've never thought about this." In this statement, Joyce is saying that her action is that of listening to the 'problem solving'. Her comment indicates that this was her action.[22]

I, as the observer, could hypothesize that Joyce's primitive knowing around problem solving, in this conversation and because she was involved in the conversation, is that of problem solving as the word problems in the textbook. I can now hypothesize that Joyce has grown in her understanding of problem solving because she has acted and made the comment. Joyce, perhaps, has a new image of problem solving or is in the process of making one.

One of the features of the Pirie-Kieren theory is that "each level beyond primitive knowing is composed of a complementarity of *acting* and *expressing* and each of these aspects of the understanding growth is necessary before moving on from any level" (1994, p. 175). They further say, "growth occurs through…first acting then expressing but more often growth occurs through, at least, the to-and-fro movements of these complementary aspects" (Pirie and Kieren, 1994, p. 175). Elaborating on the terms 'acting' and 'expressing,' Pirie and Kieren say that "acting can encompass mental as well as physical activities and expressing is to do with making overt to others or to oneself the nature of those activities" (1994, p. 175). If an observer is using this model, however, the expressing must be made overtly in order for the observer to see.

I, as the observer, cannot suggest which level of understanding that Joyce is enacting with her comment because her actions were mental (Pirie and Kieren, 1994) and not able to be observed by an observer. If I as the teacher (and observer) had asked Joyce to explain what she meant by her comment then we might be able to describe the level at which Joyce's emergent understanding appears to be. In her response to my prompt for

[22] Because both individual and collective understanding is occurring in a conversation, it is useful to think about the role of silence in the conversation. Was it the case that Joyce's silence "made room for" the contributions of others that later impacted them and particular her understanding actions?

explanation, Joyce could be expressing and "articulating *what* was involved in the actions" (Pirie and Kieren, 1994, p. 175). I can hypothesize, however, that when Joyce makes the statement "I'm just listening to all this, going 'wow.' I've never thought about this," she is changing her image of the mathematical process of problem solving. It is clear that she is listening to the conversation and could, if asked, compare the ideas and offer her own. However, she is not asked to in this moment. Joyce could also perhaps see different possibilities for her practice, but her 'wow' statement makes space for us to reconsider what we have just been talking about and see if some of what they have said might change their images as well.

In this moment I also think about classrooms. How do I as a teacher, know that the quiet student is not developing further understanding? An observer cannot suggest that they might know about an individual's level of understanding or an individual's image of a particular concept if that individual has not acted or expressed overtly in some way. This is important because, as a teacher, I have a responsibility to my students to learn how they are coming to understand, and what images they have of, mathematical concepts. What this does suggest for me is that, if I am enacting the mathematical processes in my classroom, then I need to plan for occasions for students to express their understanding and their image of mathematical concepts. Some students will not overtly be expressing their understanding or their images; hence, I must be conscious that these students may be forming mental images and mental understandings.

What is important here is that understanding teacher understanding is wound up in understanding student understanding and is not separate from it. It is also important in teacher education activities, as in the work of our group, to not make assumptions about the people who do not contribute in an overt way. It is one thing for there not to be a place in which that individual can participate. As a facilitator of teacher education activities I need to ensure there is 'space' for each person to express and act, in whatever way they come to do it.

Interpretation 2.4 Collective Dynamic Thread – (Internal Diversity +) Redundancy.

In this moment the members of our group have histories, which have common elements, so that our conversation is shared. For example, in this moment we can point to our experiences with solving word problems while in school and teaching how to solve word problems when we first started teaching.

Davis and Simmt (2003) would call these shared histories redundancies in a collective system. They write that the "redundancies that underlie a system's robustness can be difficult to interpret because they tend to serve as the ground of activity, not the figure" (Davis and Simmt, 2003, p. 160) and it "enables interactions among agents" (Davis and Simmt, 2003, p. 150).

The redundancy among our lived histories allowed us to talk about problem solving in the way we did. At the same time the diversity among our lived histories emerged as the conversation unfolded. Marilyn's comment "it's not called problem solving in other school subjects, it's called decision making" comes forward because of Marilyn's experiences in teaching subjects other than mathematics. Because we share some common experiences, our collective understanding about the relationship between the mathematical acts of problem solving, generalizing, and hypothesizing and the human acts of problem solving, generalizing, and hypothesizing emerges. I could interpret Marilyn's comment "it's not called problem solving in other school subjects, it's called decision making" as the 'figure' in our conversation, then where, in this moment, might we see the redundant 'ground?' In our comments following Marilyn's, Julia and I appear to be able to interpret Marilyn's ideas in our own terms, hence communication between Julia, Marilyn, and myself exists. When Joyce makes her statement "I'm just listening to all this, going 'wow.'...," she too is bringing the diverse ideas into her understanding. Joyce has lived history features in common with the rest of us to realize the importance of these 'new' ideas for her. This inter-play shows just how extended 'group intelligence' is enabled by diverse individual acts and at the same time is potentially generative of new individual modes of understanding.

Our conversation also points to the redundancy that exists within the entire school experience for students. Students study, for example, the decision-making model in social studies and religious studies, the inquiry model in science, and problem solving in mathematics. All of these models are intended to teach students about how they might go about finding a solution to a problem that they face in their 'real life.' How often have students had the chance to come to understand the relationship between these models?

Interpretation 2.5 Body of Mathematics Thread – Problem Solving.

I also explore the mathematical process of problem solving in this moment. This moment begins with Marilyn sharing a piece of her teaching history and talking about how "often our best students have shivers when they are asked to solve a problem, because they perceive that it is so hard." I then share a piece of my history of learning mathematics and tell about how hard I found problem solving when I was in school. Then we all suggest that we did not learn how to problem solve until we started teaching. I wonder what we did 'learn' how to do. As I shared in this moment, I 'finally figured out' how to solve the textbook word problems when I was teaching. Yes, I 'figured out' how to solve the textbook word problems. I suspect now that I 'figured out' the pattern that existed within the textbook that I was using at the time, but did I 'figure out' problem solving?

Our understandings of the mathematical processes are affected by our lived histories. If our lived histories as students learning mathematics are that problem solving is solving textbook word problems, then in what ways do these experiences affect our understanding of problem solving when we come to be teachers? What kinds of experiences must we 'live in' in order for our understanding of problem solving to be something richer than the textbook word problems? In other words, is our embodiment in the body of mathematics going to be localized in curricular or textbook suggestions? Are we able to live out our teaching of problem solving enacting larger mathematical ideas in our teaching or learning context?

In the 1980s, there was a focus on problem solving in mathematics curricula. Many books were published about the importance of problem solving. In many cases the books suggested a model for solving the problems. One of the common models was a four step model: understand the problem; make a plan; implement the plan; and look back. These four steps are not unlike the decision making model in social studies and religious studies and the inquiry model in science. In mathematics, a common expectation in high school problem solving is to write at least one symbolic, and likely, algebraic expression in order to solve the problem. If problem solving is more than solving a word problem and is a mathematical process, then what experiences must I have as a teacher where I come to know the meaning of the mathematical process of problem solving? I need experiences not only in solving problems but also in reflecting on the processes that I used to solve the problem. It will be in the expression of my reflections of my problem solving where I will come to see that problem solving as a process cannot be reduced to a four step model. In a sense I will come to learn, in the expression of my reflections, that problem solving is a meaning-making process.

It is likely that our collective image of the mathematical process of problem solving is deeply intertwined with the four step problem solving model. I suggest this because we were each teaching in the 1980s when this model was prominently introduced into mathematics curricula. The 'look back' step in the model was often reduced to 'looking back' over your work to see if you had made any computational or mathematical errors. However, the look back step also included features such as generalizing to other problems. I would like to hypothesize our collective image of the mathematical process was changing when we began to talk the relationship between problem solving in mathematics and other school subjects and then when we talked about generalizing and hypothesizing. In the act of talking about the mathematical process of problem solving, we were changing our images of this process.

The primitive knowing in our collective conversation about problem solving was that of the textbook word problems that we experienced as students and when we first began teaching. Because of these redundancies in our experience, our collective was able to begin to develop a shared understanding of the possibility of thinking about problem solving as decision making and problem solving as scientific inquiry. We could consider expanding this possibility when we moved towards thinking about how generalizing as a part of problem solving or a result of problem solving and hypothesizing as a part of problem solving or the beginning of problem solving. Our collective was bringing forth a world of problem solving that was different than the previously brought forth world of the individuals.

The 'Right! That's Right!' Moment

Interpretation 3.1 Description of the Moment.

Julia said "I'll go first." She read what she had written for the process of communication:

> Communication. (pause) The use of mathematical language and symbols. Being able to describe or explain what you're thinking and why. Using symbols and diagrams to help explain a process. Being able to organize your thought processes and then describe them.

Marilyn added "Being able to talk about mathematics, being able to verbalize it and talk on paper too." Joyce agreed and said "Yeah, that's what I had too." Joyce continued with "Or tell somebody about, that doesn't know the language. And explain it so they can understand." Marilyn agreed with the comments, "Yep. You're right about it, it's not just words that you're using either, it's diagrams and symbols."

Julia then reminded the group that she also had the concept of organizing in her definition, "I think that one of the things is the organizing. The kids have a lot of trouble saying 'this is how I did it, I went here and this is why I went here.'"

I was suddenly aware that Joyce might not have had a chance to share her full definition so I asked "What did you write, Joyce?" Joyce looked at her paper and concluded, "Same kind of thing. I had both verbalizing mathematics, being able to explain things to someone who wasn't familiar with the terminology so they could understand the concept. Sort of verbal." Julia and Marilyn both agreed and together said, "verbalizing."

I noticed that a 'shared' definition of the process of communication was beginning to emerge because each individual definition either overlapped another or built on another. I then shared the meaning I made of communication:

> I chose two examples, '2x + y = 7' and '3 + 4.'
> These expressions communicate something.
> The symbols communicate something.

Uh-huh
Uh-huh

Writing, talking, and acting.

Uh-huh
Uh-huh

Communication is important,
How we belong together in a group.

Uh-huh

Being involved together in a community.
Don't know what that means yet.
But, it's something.

When I finished reading the meaning I'd written, the group agreed, "Yeah!" "Right!" and "That's right!" Marilyn added, "It's all community and communication. If you can't communicate, then you don't belong to the community. Both have the same root word."

Interpretation 3.2 Why this Moment?

This moment is significant for me because it reminded me of the first time I noticed language and communication in my own teaching. It was within my first six months of teaching and my lesson for the day was about factoring quadratic expressions. I remember being at the blackboard and saying to my students in a Math 13 class, "there are two special kinds of quadratic expressions, a perfect square and a difference of squares." At that point I looked up and noticed the 30 faces in my classroom. It appeared that there was a glaze over the faces; something like the glaze you might see on a fruit pizza. Seeing those glazed looks prompted me to think about the language that I used in my teaching. In that moment in my professional life I was truly aware of the language that I used as a teacher. That one incident has replayed in my mind several times.

Interpretation 3.3 Individual Understanding Thread – ((Primitive Knowing +) Image Having/Making +) Dynamical.

Understanding is not a static state. My lived history in mathematics contains many examples of this. Before addressing the 'Right! That's Right!' moment, I will share an example, from my lived history of learning mathematics, to illustrate the dynamical process of understanding.

This story that I will share with you is my story of coming to understand the division of fractions. When I was teaching high school mathematics in the early 1980s, I remember thinking about the meaning of the division of fractions. I had been using a rectangular area model to teach my students about the multiplication of two binomial expressions. This meant that I used the area model to represent the multiplication of two whole numbers and then generalized to the concept to two binomials. After class, one of my students asked if the area model would also work for the multiplication of fractions. We played around with the idea and discovered that we could use the model for the multiplication of fractions.

That night I wondered about the division of fractions. I remember being stuck in my exploration, partly stuck with the language used to talk about the division of fractions, and eventually I left the task. About 10 years after that exploration a colleague asked me to coauthor a book about fractions. We were going to use an area model to model the addition, subtraction, multiplication and division of fractions. (The area model we used in the book was a different representation than the one I had used previously; however it was still an area model).

In the 10 years, I had had many an opportunity to think about the way in which our language brings forward mathematical meaning, so my primitive knowing about fractions and the arithmetic operations with fractions was now different than it was earlier. During the process of writing the book, we left the division of fractions till the end. When we reached the stage in writing the section about the division of fractions we worked for many hours with the model, talking about our actions, writing the symbols, and writing in text the words that described our meaning of the division of fractions. In working with these multiple representations, I came to understand the division of fractions with the area model. From my primitive understanding, I had redeveloped my image of fractions as area and division was a feature of that image. In addition, in moving back and forth between these different representations, I was moving back and forth between different levels of understanding. I was moving between the image making, image having, and property noticing levels as I was trying to determine the way in which the language and actions were consistent within any one of the representations that I was using.

In our collective during this research, the concept that we discussed was less concrete than the concept of the division of fractions. The Pirie-Kieren theory can be used by an observer, such as a teacher or a researcher, or by an individual such as myself to examine my own emergent mathematical understanding. In this thread I develop a way in which I might talk about the emergent mathematical understanding observed in the definition of communication that I shared with our collective:

> I chose two examples, '$2x + y = 7$' and '$3 + 4$.'
> These expressions communicate something.
> The symbols communicate something.
>
> Uh-huh
> Uh-huh

Writing, talking, and acting.

Uh-huh
Uh-huh

Communication is important,
How we belong together in a group.

Uh-huh

Being involved together in a community.
Don't know what that means yet.
But, it's something.

The expressions, '2x + y = 7' and '3 + 4' could be thought of as expressions related to my primitive knowing of the mathematical process of communication. Much of my school experience was working with mathematical symbols such as these. At the same time these expressions point to an image of the mathematical process of communication, that is the idea of the symbols as communication; hence I also see this beginning representation as being at the 'image having' level of the Pirie-Kieren model. When I said, "these expressions communicate something," I recognize that both sets of symbols share a common quality, the quality of representation. That is, these sets of symbols represent mathematical acts as well as expressed relationships or quantities. In my statement, I express my knowing that because the sets of symbols represent a mathematical act, they are a form of communication. Using the Pirie-Kieren theory I could suggest that the statement "these expressions communicate something" is an expression, at this moment in time, of my image or at least a feature of such an image. If my expression had a 'for all' quality, for example 'any relationship or quantity in mathematics can be represented by an expression,' then my emergent mathematical understanding could be thought of at the formalizing level. At the formalizing level, a person "abstracts a method or common quality from the previous image" (Pirie and Kieren, 1994, p. 170). The formalizing level has a 'justifiable for all' quality.

Now, let me step out and act as an observer in this moment. Of course, determining whether or not the statement "these expressions communicate something" has a 'for all' quality, will depend upon the way in which an observer interprets the statement. If the statement is interpreted locally, that is for the symbols that I used, $2x + y = 7$ and $3 + 4$, then an observer could suggest that the statement does not have a 'for all' quality. If, however, the statement is interpreted to mean that all symbolic mathematical expressions communicate something, then the observer could suggest that the statement could be at the formalizing level. The statement "writing, talking, and acting" points to my changing understanding of communication. The symbolic statements are statements that are used to represent mathematical acts whereas the statement "writing, talking, and acting" points to my own being and that communication is part of my structure. This is an embodied view of communication. The statement "being involved together in a community" points to my making the connection to communication as being related to being "in community." I could hypothesize that I 'folded back' to a different image of the mathematical process of communication, that is, being involved in a community, which of course relates to being embodied in the body of mathematics.

Another way that I might think about using the Pirie-Kieren theory in this moment is thinking about Julia's emergent understanding of the mathematical process of communication. As Julia reads her definition of the mathematical process of communication, we might consider this definition as her primitive knowing in this moment. In listening to the contributions of Marilyn and Joyce, it is likely that Julia's understanding is changing. We see that Julia is making connections between what Joyce and Marilyn are saying and her definition when she reminds us "I think one of the things, is the organizing. The kids have a lot of trouble saying 'this is how I did it, I went here and this is why I went here.'" We see Julia's lived history of teaching emerging in this moment to provide an image for us as to what she means about the idea of organizing in her definition, because it appears that Julia did not see the idea of organizing in Marilyn and Joyce's comments.[23] We also see Julia having another image of her mathematical process of communication when she agreed with Joyce's writing and said, "verbalizing." Julia's action of making the statement "verbalizing" and then again at the end of the moment, when she agrees with my definition of the mathematical process of communication, could be seen as an indication that Julia's understanding of the mathematical process of communication was changing. As a teacher, however, if I were to not only hypothesize that Julia's understanding of the mathematical process of communication was changing, then I would have to ask her to explain her actions.

Interpretation 3.4 Collective Dynamic Thread – ((Internal Diversity +) Redundancy +) Organized Randomness.

Imagine you are sitting at a basketball game, watching two teams play. There are many observable features and rules at play. For example, there is a particular court size for the game; in order for a team to score points the ball must go through the hoop; there are five players on each team; there is a limited time for each quarter in the game; and when the whistle blows the teams stop playing. But the basketball game itself is a whirr of motion, of what may look like randomness; the game itself is an emergent

[23] This is another illustration of understanding both emerging in, being occasioned by, and extending the nature of the topic of conversation.

phenomenon, each team attempting to score points. This is an example of the organized randomness in a complex system. The rules for the game provide the organization, while the playing of the game is an emergent phenomenon that has random features. For example, players move, to an observer, in unpredictable directions. The players are playing to score points; that is to say, the team as a collective has a project that they are trying to complete, that of scoring the most points. The organizers, or the rules, in this particular example, are overt; however, in other complex systems the organizers are not always overt.

Davis and Simmt explain that organized randomness is a "structural condition that helps to determine the balance between redundancy and diversity among agents. Complex systems are rule-bound, but those rules determine only the boundaries of activity, not the limits of possibility" (2003, p. 154). Describing a structural condition is conditioned by the observer.

In our collective, there were 'rules' or 'structural conditions' that determined our 'boundary of activity.' Largely these 'rules' or 'organizers' were social in nature. When you look closely at our conversation in this moment you will see that there were many instances of agreeing with one another to elaborate another's point. This is the way that we, as a group of teachers, engage in professional conversation. Perhaps it is also the way Marilyn, Joyce, Julia and I have each learned how to engage in conversations, both personal and professional. In any case, it was an unspoken 'organizer' for our collective. Because of redundant features of our lived histories, particularly that of engaging in professional development activities, our collective adopted this practice. Another example of a tacit 'organizer' in our collective is that people in our collective had an opportunity to share their definition of the mathematical process of communication. Yet, looking closely at our conversation we see that the conversation is fluid and random. There is a blending of ideas that emerge as a shared understanding of the mathematical process of communication is developed. No one person is 'designated' to share her definition first or second. Once one person shares her definition, it prompted others to join. Developing a shared understanding of communication is the 'game' that our collective is 'playing' during this moment, no one is in charge and the possibilities are not limited. Our shared understanding of the mathematical process emerged from our conversation about using symbols, diagrams, and mathematical language to the notion of 'writing, talking, and acting' to being involved together in a community.

At the same time, though, I do show that I am very much the teacher in this group of teachers. This moment is the first time that we shared our writing about the mathematical processes. Julia starts the moment. I, as the teacher, know that Julia and Marilyn know each other very well professionally and personally. I wanted to make sure that Joyce would know that she, too, was a part of our group. Seeing the part I played as teacher, I wondered what it means if there is a 'leader' or 'teacher' within a complex system. What role does that person play?

In one sense, I, as the teacher, play the role of an organizer of this system in that I needed to 'maintain' the organizers. For example, I ensured each person had the chance to talk about her definition of the mathematical process of communication. At the same time I needed to 'let go' and be a part of the collective system. I did this when I did not define the order in which we would read our definitions. You will also notice that I did not interrupt and add my suggestions to either of Marilyn's, Julia's, or Joyce's work. I listened, and added mine, when it was my turn in the conversation. In this way, I did not limit the possibilities of the system.

Interpretation 3.5 Body of Mathematics Thread – Mathematics in Community.

This moment points to two aspects of mathematics. The first one is the importance of mathematics in our culture and the way in which our society relies on mathematical ideas when we talk about the symbols, mathematical language, and diagrams to explain our thinking. We use symbols to communicate and, as humans, are engaged in mathematical acts regularly. For example, when we read a bus schedule, we are engaging in a mathematical act. The second aspect of mathematics that is addressed here is the idea of the importance of being in community when learning mathematics. This idea is brought forth when Marilyn mentions the connection between the root of community and communication. Much has been written about the importance of social interaction in learning mathematics in recent years (e.g. Cobb, Yackel, et. al., 1992; 1995). When we are in community we bring forth mathematics because we are human. In many ways these two aspects are intertwined.

Throughout this moment we refer to this important idea of being in community as we are learning and this relationship is either implicitly or explicitly stated. For example, when Julia says "Being able to describe or explain what you're thinking and why;" when Marilyn says "being able to talk about mathematics, being able to verbalize it and talk on paper too;" when Joyce says "or tell somebody about, that doesn't know the language. And explain it so they can understand;" and when I say, "being involved together in a community." All of these examples point to the idea that "human beings are not self-contained, self-sufficient subjects contingently and externally related to one another, but beings who are formed, from the very beginning, in and through their social interactions" (Levin, in Davis, 1996, p. 187)

From an enactivist perspective, we interact with the environment and others and bring forth a world of significance. Our interactions with the environment include interactions with other people and in this moment we see each of Joyce, Julia, Marilyn and myself pointing to the importance of social interaction in bringing forth a world of significance.

The 'Woggle' Moment

Interpretation 4.1 Description of the Moment.

At this point, it appeared that we were somewhat stumped. Marilyn suggested, "Maybe the Venn diagram is too limiting? Perhaps we need another dimension." Julia jumped in with "Maybe we need spheres." We all laughed. Was there another way we could represent our thinking? As the recorder, Julia asked "Shall we scrap this and think of something else?" Neither Marilyn, Joyce nor I thought we should scrap our work, but that it "could be put aside; it becomes a record of part of our thinking. Chapter 1 of our thinking." I asked Julia about the way that she was thinking about the representation.

Julia offered:

> I think that they are linked. I'm not so sure that the Venn diagram says what we want it to say, but I think the idea of linking them is very valid. Like rectangles that are with connected with lines, just like in the old textbooks when they would describe the 'family of parallelograms.' Where squares are part of rectangles, rectangles are part of parallelograms. You want something to be parts of the whole. Right?

Joyce suggested "What about something we could show, like, interwoven? Like braided together?" Julia clarified, "weaving together, like integrated?" Marilyn responded, "a woggle." Joyce and I were mystified, neither of us had previously heard about a woggle, so we asked Marilyn to tell us about a woggle.

> It's a mathematical thing. Mariko, one of our colleagues,

> had her Math 10 students make them last year. She used a rectangular piece of felt with two slits in it. The material between the slits was divided into thirds and then the pieces are braided. There is some way of twisting all the felt together so that's it's still in one piece. You make this little braided thing. It looks like a braid. It also looks something like a mobius strip.

Although we all thought that the woggle would be a good representation for our descriptions, no one could draw one, and so we continued to discuss other possible representations. I suggested an orange could be used, with the processes being the segments of the orange. Julia responded, "I can't draw that. Can somebody else draw it?" We all laughed as we responded no. Julia continued though, by saying "that's a good idea though. Maybe we can use a circle with segments that won't look three-dimensional. We could draw semi-circles. There must be some program out there." I asked, "What would the orange peel represent in terms of our conversation?" I continued with sharing how I thought the orange, even though I suggested it, might not be a good representation: "The orange treats the processes separately. I could take a segment, say connections, away and I should be able to talk about that one segment. Whereas, we've been saying how all of the processes are related." Julia continued, "Where the Venn diagram says that there is a relationship." Marilyn, once again, took us back to the diagram Julia had drawn.

Interpretation 4.2 Why this Moment?

This moment is significant for me because it reminded me of my school experiences with geometry. All of the representations of three-dimensional mathematical objects when I was in school were two-dimensional. I was the student in the class who saw two squares with some joining lines when my teachers would draw a cube on the blackboard. I'm also reminded of the number of times I would use the phrase "imagine that this is a sphere, or a cube" when I was teaching and attempting to draw the three dimensional objects on the two dimensional blackboard.

It appears to me that, in this moment, each of us in our collective had similar experiences in school and hence, were limited to only using a two dimensional representation for our thinking. This moment also prompted me to think about the way in which computers have affected our teaching lives when Julia says "I can't draw that. Can somebody else draw it?" and makes the connection to her computer science teaching when she says, "There must be some program out there."

Interpretation 4.3 Individual Understanding Thread – (((Primitive Knowing +) Image Having/Making +) Dynamical +) Emergent Possibilities.

When Marilyn makes the statement "Is the Venn diagram too limiting?" she is questioning our understanding of the diagram we were using to represent our ideas. This question could be thought of as pointing our collective to inventing new ideas, or new mathematics. The question prompts us to think of different ways in which we could represent our ideas, different models that might be used or new mathematics, different than the idea of a Venn diagram. We examine ways in which we can show the interconnectedness of each of the ideas of problem solving, reasoning, connections, and communication through a series of linked rectangles, a sphere (orange), a braid, or a woggle. Our lived histories bring each of these ideas forward. For example, it is Julia's history of teaching that brings forward the example of the linked rectangles. It is Marilyn's history with her colleague that brings forward the woggle. We even begin to talk about why some of the suggested models would not clearly represent the interconnectedness of the ideas (e.g. the orange because you can take away a segment). We are limited in the ways we could explore our new mathematics because we are not able to build a two-dimensional representation of these three-dimensional ideas.

In this collective moment, we are attempting to build a representation for our thinking about the mathematical processes of communication, problem solving, reasoning, and connections. Our primitive knowing, based on our collective experience of mathematics, is to represent our thinking with a Venn diagram. We realize that it might be better represented with a three dimensional representation. We move through the levels of image having and property noticing with our different representations but return to our primitive knowing, that of a Venn diagram. This movement back to our primitive knowing is an example of the "folding back" feature of the Pirie-Kieren theory. We return and use the Venn diagram in a different way now, because we've come to a shared understanding of the meaning of the Venn diagram for our work.

In our conversation we are asking new questions and beginning to create new mathematics, or use mathematical ideas in new ways, to solve a representation and a re-presentational problem. The outer most level of the Pirie-Kieren theory is called inventising, or creating "new questions that might grow into a totally new concept" (1994, p. 171). I might suggest that inventising here occurs in the blending of ideas, that is, our conversation illustrates that we have a fully structured understanding of the Venn diagram as a representation of relationships. In the blend of ideas of the linking rectangles, the woggle, the weaving, and the orange we are exploring the representation of mathematical ideas, using the mathematical idea of representation in new ways. However, the notion of inventising in the Pirie-Kieren theory is related to the idea that the new questions might "grow into a totally new concept." In our conversation, our ideas did blend to illustrate our understanding of the Venn diagram and to explore alternative representations, however, we did not move into a new concept. We remained within the concept of the representation. Our conversation illustrates the possibility of growing to a new concept, however, we "fold back" to our primitive knowing of the Venn diagram.

When we "fold back" to our primitive knowing, we have a new understanding of the Venn diagram.

Interpretation 4.4 Collective Dynamic Thread – (((Internal Diversity +) Redundancy +) Organized Randomness +) Neighboring Interactions.

Our joint project in this moment is to consider a representation, other than a Venn diagram, that would better represent the interconnectedness of our emerging ideas of problem solving, reasoning, connections, and communication. Many ideas are exchanged and explored as a collective. One of the 'organizers' in this moment is that our collective system only had flip chart paper and a felt pen. Hence, although we were talking about different three-dimensional ways that we could represent our ideas, we were limited to the drawing 'skill' that each of us brought to the collective. Davis and Simmt (2003) call the sharing of these ideas neighboring interactions.

"Agents within a complex system must be able to affect one another's activities" (Davis and Simmt, 2003, p. 155). It is not only our interactions that are important here in affecting one another, however, it is the ideas that we are sharing. These ideas are 'bumping' against one another. Davis and Simmt (2003) describe 'neighbors' as "ideas, hunches, queries, and other manners of representation" (p. 156).

Neighboring interactions are lots of ideas that are 'bumping' against one another. In this 'bumping' of ideas in our collective, new mathematical representations are emerging. In this moment, the ideas of the linking rectangles, the 'interwoven braid,' the woggle, and the sphere, or orange, are 'bumping.' Because of the redundancies and diversity within our collective the mathematical idea of re-presentation is emerging. In this moment, the focus was on our interpretation of the diverse and emergent ideas of the different representations for our thinking. Looking at ways to solve this mathematical problem of representation and re-presentation emerged within the collective. The line that distinguishes between the threads of individual knowing and collective knowing is blurry here.

As I examine and re-examine this moment, I also think about what neighbouring interactions might look like in a high school mathematics classroom. Davis and Simmt (2003) suggest that the "idea of neighbour interactions prompts careful consideration of strategies for representation of concepts and understanding." I would like to use an example from the student project that our collective designed to illustrate my thinking. Three of the questions on our student project asked students to (a) tell what you know about what it means to add, subtract, multiply and divide two quantities; (b) choose polynomials from the list above [a list of polynomial expressions were provided on the project] to demonstrate the operations of addition, subtraction, multiplication, and division of polynomials; and (c) how do you see the operations with quantities connecting to operations with polynomials? Our intent in these questions was to discover our student's primitive knowing about the way in which they understood the relationship between the quantities and polynomials. Hence, we examined our students' responses to learn about their primitive knowing. But, what if we had paid attention to the classroom collective? If we had considered the classroom collective, then we could've used our student responses to foster the "bumping" of "ideas, hunches, queries and other manners of representation." What if we not only thought of what we, as teachers, could learn from the student responses but what other students in the classroom collective could learn? If we had our students share their responses to these three questions, they would come to know about a variety of ways of thinking about the relationship between quantities and polynomial expressions. In the sharing of their responses our students would be explaining their thinking and becoming aware of multiple images for the relationships between quantities and polynomial expressions.

Interpretation 4.5 Body of Mathematics Thread – Embodied Mathematics.

We had a collective history of using a Venn diagram, a representation of set relationships from mathematics, for a representation of our thinking. At one point Marilyn asks us if the Venn diagram is too limiting. In the discussion of a different representation for our thinking, new mathematical ideas were emerging and, because we are humans, we brought forth these new mathematical representations of the way in which we might represent our thinking. This notion of being human and bringing forth mathematics is an idea of the theory of embodied mathematics (Lakoff and Nunez, 2000).

Lakoff and Nunez (2000) say that "the only access that human beings have to any mathematics at all, either transcendent or otherwise, is through concepts in our minds that are shaped by our bodies and brains and realized physically in our neural systems. For human beings-or any other embodied beings-mathematics *is* embodied mathematics. The only mathematics we can know is the mathematics that our bodies and brains allow us to know. For the reason, the *theory of embodied mathematics*…as a theory of the only mathematics we know or can know, is a theory of what mathematics *is*-what it really is!" (p. 346) and add "mathematics is a mental creation that evolved to study objects in the world" (p. 350). Within the theory of embodied mathematics, "mathematics is a product of human beings" (p. 351).

According to Lakoff and Nunez (2000), one of the central concepts of mathematics is containment. They write, "the concept of containment is central to much of mathematics. Closed sets of points are conceptualized as containers, as are bounded intervals, geometric figures, and so on." and argue that "the concepts of containment and orientation are not special to mathematics but are used in thought and language generally" (p. 33). Further, Venn diagrams are "typically conceptualized metaphorically as containers and derive their logics from the logic of conceptual container schemas" (p. 45).

In this moment, we examined the validity of the Venn diagram as a representation of the way in which we saw the relationships between the mathematical processes. After exploring some different representations, we return to the idea of the Venn diagram. Partly, I believe, we returned to the Venn diagram because of our collective history with set theory. I could also suggest, however, that we returned to the Venn diagram because it seemed "natural" (Lakoff and Nunez, 2000, p. 45) for us. In other words, we returned to the Venn diagram, because we are human.

Hence, in being human in our collective, we were bringing forth mathematics.

<u>The 'Random' Moment</u>

<u>Interpretation 5.1 Description of the Moment.</u>

Julia yawned, "Oh, we've done good, good work here." Joyce agreed. I asked if each of us could see our definitions reflected in the diagram. We all agreed we could see our own definitions in the diagram, but Julia remembered that I had used the word random. She asked me to explain again what I thought I meant by that word. I started:

> I said it when I was thinking about reasoning. There are some people that talk about mathematics as being a logical sequential, step process. But if you listen to them solving a problem, their thinking doesn't appear that way. Their thinking appears more, intuitive, random perhaps. In

> mathematics we start from the specifics and we generalize, right? Or, do we, maybe, bring in different things; it appears random and then comes together, like, an artist. As I say that, and talking about it, I'm not convinced that there is a difference. I've heard artists say that they don't see their work as a logical step process. They are just painting something, for example, and something else emerges within them. Dancers move to how they feel. Writers sometimes just write and ideas emerge. But many people see mathematics as following a set of steps. And it's always these steps. But in mathematics, are there really a set of steps? No. There is an element of randomness. But the perception is that there's always steps.

Marilyn suggested, "Well, if you have no place to start when solving a problem, an intuitive place to start, then what do you do? You go random, don't you?" Julia added "sort of like mulling it over in your mind, like a brain storm." I agreed and built on Julia's idea, "you're brain storming with yourself." "You're driving and you're thinking about. Or falling asleep and you're thinking" continued Julia. Marilyn added "And you wake up in the middle of the night." Joyce added "even your conscious, sometimes you don't realize it that you're mulling it over." We all agreed that this happens.

I continued in my wondering and asked "Is there space in our mathematics curriculum for us to work with students in this area. I called this 'mulling' over random. Do we acknowledge that this is a part of problem solving? Yet generally we have so much content that students have to know, yet this is a part of the process."

<u>Interpretation 5.2 Why this Moment?</u>

This moment emerged for me because of the number of conversations that I have been a part of where individuals will say things like "well, I tell my son (or daughter) that all you need to do in order to get a good mark in a mathematics class is to figure out the steps. It's all about rules; you just have to memorize the rules." This tension of whether or not solving problems is an emergent process also existed when I was one of the authors on a series of textbooks. In the writing of the book our author team decided that we'd use a particular 'four step problem solving model' when we would write the problem solving features. Not only is there a public perception that mathematics is about following a set of steps, but in my action of writing the textbook I was perpetuating that perception. This moment in our collective conversation reminded me of the personal tension that I felt as a mathematics educator in participating in the writing of the 'problem solving features.'

Interpretation 5.3 Individual Understanding Thread – ((((Primitive Knowing +) Image Having/Making +) Dynamical +) Emergent Possibilities +) Folding Back.

I will return to my illustration of understanding the division of fractions to illustrate the way in which emerging understanding is a process that is dynamic and requires a constant 'rebuilding' on previous understanding. When I was coming to understand the meaning of the division of fractions, I was engaged in working with an area model, speaking, writing, and using the symbolic language. I would look at an expression like $\frac{1}{2} \div \frac{1}{3}$, build an area representation and then say, "one half divided by one third" means "what portion of one third of a whole is in one half of a whole?" By comparing the areas I could 'see' that there were $1\frac{1}{2}$ portions of $\frac{1}{3}$ of a whole in one half of the same whole. In my work since, I have been able to take this image of the division of fractions with an area model and use it to explore what the division of fractions means if I use a number line representation, a set representation, and different area representations. Each time I'm faced with a new representation of a fraction, I return, or fold back, to my primitive knowing of the division of fractions that I made when I was coauthoring the book with a colleague that I wrote of earlier. Of course, that primitive knowing continues to change because I now have experiences with different representations for the meaning of a fraction.

Pirie and Kieren describe 'folding back,' another feature of their theory, as "vital to growth of understanding…which reveals the non-unidirectional nature of coming to understand mathematics" (1994, p. 173). "When one is faced with a problem or question at any level, which is not immediately solvable, one needs to fold back to an inner level in order to extend one's current, inadequate understanding" (Pirie and Kieren, 1994, p. 173). The 'folding back' feature of the Pirie-Kieren theory points to the idea that emergent mathematical understanding is a recursive process, a process where one is continually re-forming previous understandings (even informal) to allow one to extend one's understanding to more formal understanding. The process is 'all at once' and continuous. I began to unpack this moment as an illustration of 'folding back.'

The conversation around 'random' is not only in its self an example of folding back, but points to the notion of 'folding back' in other instances. For example, as we begin the conversation about 'random,' I fold back to images that I have of 'random' or 'intuitive' acts, the acts of a dancer and a painter. In trying to make sense of what the word 'random' means in this context, we see several examples of folding back to our primitive knowings. For example, when Marilyn says "Well, if you have no place to start when solving a problem, an intuitive place to start, then what do you do? You go random don't you?" and when Joyce says "You don't realize that you're mulling it over," we have a sense of 'folding back' to something each of Marilyn and Joyce know, considering how they might begin the solution to a problem.

The notion of 'folding back' here is also illustrated using the idea of solving a problem. I see this idea of connecting to the personal history of the teachers as they share their contributions to thinking about the way in which I meant the word random. Using the Pirie-Kieren theory, I see how when we start to solve a problem, the teachers and I begin with our primitive knowing or understanding about the problem. If we have no 'sense' of where to start in the problem solving, we 'mull it over' and that mulling it over helps to build a 'thicker' understanding of the problem. When we go to solve the problem though, we 'fold back' to the problem and our primitive knowing. However, our primitive knowing will now be different because we 'mulled over' the idea. It has been informed and shaped by our outer level interests and understandings. It is a recursive process.

Interpretation 5.4 Collective Dynamic Thread – ((((Internal Diversity +) Redundancy +) Organized Randomness +) Neighboring Interactions +) Decentralized Control.

Reflecting on the basketball story that I used earlier, a complex system, like the basketball teams playing a game of basketball organizes itself. As you are watching the game in action there appears to be no one team member in charge or who directs where the ball will go. The game, who has the ball, and the direction the ball moves, etc., emerges. When a system organizes itself, as in this basketball game, Davis and Simmt (2003) call it decentralized control. There is no all-powerful central controller determining each move or predetermining the sequence.

In this moment where our collective is talking about the idea of 'random' we can see an example of decentralized control. Although I was responsible for initially gathering this group of people together, you will notice that throughout this moment and in the narrative accounts in Chapter 4, that I was not 'really in charge.' In the fluidness of the conversation, the system itself, organized itself in inter-action. In this moment we see that Julia leads the discussion and asks me to explain what I meant by random. When I finish my explanation, Marilyn, Julia, and Joyce all contribute to making sense of my explanation. They each bring forward their interpretations with comments like "sort of mulling it over in your mind," "brain storming with yourself," and "sometimes you don't realize."

In this process of emergence we, as a complex system, are trying to develop a shared (Davis and Simmt, 2003) understanding of what I meant when I said 'random.' Davis and Simmt (2003) say, "the suggestion that ideas can be shared only makes sense if the observer allows knowing and knowledge to be spread across agents' actions in collective contexts" (p. 154). In this moment our joint project was developing a shared understanding of what I meant when I said the word 'random.' Each of our lived histories came forward in our actions to make sense of the meaning of the word random.

The conversation in this moment also points our collective to considering classroom practise when I wondered out loud and asked, "Is there space in our mathematics curriculum for us to work with students in this area? Do we acknowledge that this 'mulling over' or 'brainstorming with one's self' is a part of problem solving?"

In what ways do we help students to see that the problem solving process emerges? In what ways is mathematics itself an emergent phenomena?

<u>Interpretation 5.5 Body of Mathematics Thread – An Emergent Phenomena.</u>

In this moment we see a conversation around the idea of 'random' in coming to solve a problem. The notion of 'random' in this moment is not taken as a mathematical definition. In the mathematical definition, we would be looking for references to the probability of an event. In this moment the random reference is made with respect to the idea of thinking, 'intuition,' and 'mulling it over.' Emergent acts. The random reference is in contrast to the idea that "mathematics as being a logical sequential, step process," or a predetermined act.

As I wrote earlier, there is a perception that mathematics is about following a set of steps. When you are confronted with a word problem in school mathematics, for example, you follow a set of steps to determine a solution. In this moment I also think about the "four step problem solving model" that I wrote about earlier. The first step in the model was "understand the problem." Essentially, what our collective is describing in this moment is the process that we, as humans, participate in as we come to "understand the problem." We do not immediately "write an algebraic statement" to show our understanding of a problem, our understanding of a problem emerges just as mathematics itself is an emergent phenomenon.

Steen (1990) writes, "mathematics, in the common lay view, is a static discipline based on formulas taught in the school subjects of arithmetic, geometry, algebra, and calculus" (p. 1). Steen continues with "outside public view, mathematics continues to grow at a rapid rate, spreading into new fields and spawning new applications" (1990, p. 1).

In what way is mathematics emergent? I will illustrate with a specific example.

The computer was designed around the mathematical task of computing. Our earliest computers were machines that performed algorithms for adding, subtracting, multiplying, and dividing numbers. The development of the computer has gone beyond the computation of algorithms to include computer graphics. The computer graphics are allowing mathematicians to see patterns now that they were not able to see before. New mathematics is emerging from these new observations. This is an example where new mathematics emerges from the old.

In this moment we think of mathematics as an emerging, not a static, subject. When the reference is made to the dancer dancing, the writer writing, and the painter painting we see the dance emerging, the writing emerging, and the painting emerging. In the 'mulling it over' we can consider mathematics emerging. This is true for the individual but for the larger mathematical communities such as the curriculum makers, test makers, university teachers, or communities of practicing mathematicians as well.

Preparing for Chapter 6

In this chapter I provided an elaboration of five different moments taken from the narrative accounts in Chapter 4 to illustrate my thinking about emergent teacher understanding of mathematical processes. In the recursive elaborations using the interpretive threads of individual understanding, the collective dynamic, and the body of mathematics we have come to see that Julia, Marilyn, Joyce, and I brought forward multiple lived histories.

Not only did I recursively elaborate the moments but I also attempted to recursively elaborate on the interpretive threads. In this chapter I presented the recursive elaborations as if I was reading the following 'quilt' horizontally. I also attempted to provide, within the writing, a recursive elaboration of the threads, or as if I was reading the quilt in Figure 4 vertically.

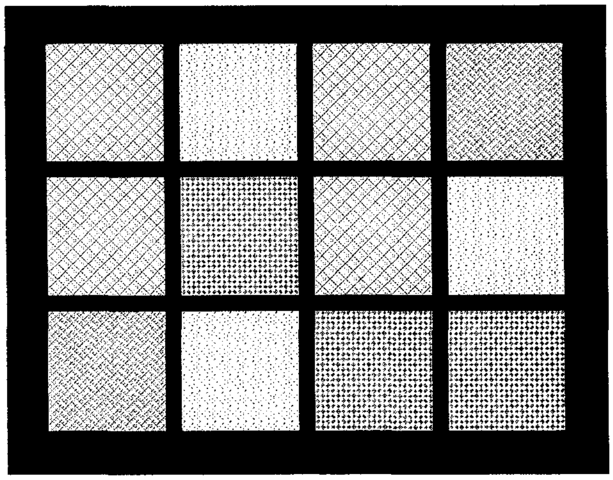
Description of the Moments	Why this Moment?	Individual Understanding Thread	Collective Dynamic Thread	Body of Mathematics Thread
The 'Connections' Moment	Connections	Primitive Knowing	Internal Diversity	Diverse Interpretations
The 'Wow' Moment	Wow	Image Having/Making	Redundancy	Problem Solving
The 'Right! That's Right!' Moment	Right! That's Right!	Dynamical	Organized Randomness	Mathematics in Community
The 'Woggle' Moment	Woggle	Emergent Possibilities	Neighboring Interactions	Embodied Mathematics
The 'Random' Moment	Random	Folding Back	Decentralized Control	Mathematics as an Emergent Phenomenon

<u>Figure 4.</u> A 'quilt' representation of the recursive elaborations.

In this chapter we saw that teacher understanding is wound up in their lived histories, histories that include being a student and being a teacher. The threads of individual understanding, the collective understanding, and understanding within the body of mathematics were woven through conversation. In the next chapter, I elaborate on these ideas to highlight implications of my work for teachers, teacher educators, policy makers, and curriculum writers.

CHAPTER 6

Binding and Using the Quilt



At this point in the writing, a "quilt" has been "completed" and it is time to put on a binding. In this metaphor, the fractal quilt that I have been working with is what quilters call a "story" (Otto, 1991) quilt. It is a quilt that tells a story, that is, a quilt with a voice.

Otto (1991) says that a fairly intricate quilt could contain one thousand pieces. In my work, I have attempted to describe closely a sample of those one thousand pieces. In these descriptions:

1. I have described how I came to think about a research question around the ways I might describe how teachers change in their understanding of the mathematical processes of reasoning, communication, problem solving, and connections by sharing my lived history.
2. I have described the backing for the quilt, that is, the material that forms the backing of the quilt. This material, which forms a theoretical basis for considering the question, comes from the fields of mathematics education, enactivism, narrative inquiry and narrative knowledge.

3. I have shared the stories of three teachers who joined me in helping to make the quilt. At the same time their stories and who they are, just as my story and who I am, form a part of the quilt.
4. I have described how, in the process of closely describing pieces of the quilt, that is in the writing of the narrative accounts of our meeting times and interpreting moments in those accounts, that I began to notice three aspects of changing teacher understanding pertaining to individual teacher understanding, understanding related to the collective, and understanding within the body of mathematics emerging within our conversations.
5. I have described how, just like the quilter as she is arranging the pieces to build a pattern, the pattern that she thinks she's started out with becomes a different pattern, that in the writing and interpretive work an elaboration of my research question emerged and the question became "In what ways do mathematics teachers grow in their understanding of mathematical processes within the context of professional conversation?"
6. I have described why I maintained the entire two narrative accounts in order to show the overall context and the complexity of teacher conversation in action. Once again, something like a quilter, who, when they are arranging the pieces of their quilt need to be aware of the complexity of the acts in paying attention to "harmonies of color, fabric, and form" (Otto, 1991, p. 68).
7. Using the reformulated research question and noticing features of individual teacher understanding, understanding related to the collective, and understandings within the body of mathematics emerging within our

conversations, I described how I would use three interpretive threads to elaborate on the features of teacher understanding. In quilt-making terms, the threads provided the elaborate stitching (or quilting) that attaches the pieces of the quilt to the backing

8. I have described the elaborate stitching on the quilt, that is, the recursive interpretations of five moments from the two narrative accounts to illustrate the way in which I saw emerging and changing understanding in the individual, within the collective, and within the body of mathematics.

Now, I will attempt to describe the binding of this quilt. Otto (1991) says that

> binding a quilt can be achieved a couple of ways. You can turn the back edge forward...Or you can stitch on a separate edge. The separate edge is often recommended since it can be replaced if the quilt suffers from tension, stretching, age, accident. Sometimes, a quilt can benefit from an attached border; can make the fusion whole yet relaxed. (p. 70)

I might have chosen to "turn the back edge forward" for the binding of my quilt if this thesis was of a different nature. Suppose I had written this thesis, or "sewn this quilt," as the description of a research project. The quilt would still be a story quilt; that is, it would tell a story, but it would be a different story. If this thesis was the description of a research project, then the story would have an ending: the results of the research project. The initial pattern that I had for my quilt was a study such as that, one that described the way in which I observed mathematics teachers changing in their understanding of the mathematical processes over a period of time. Such a study would be self-bounding, with the "folding forward" or summarization serving as the binding. The quilt could then be displayed for people to look at. However, if the quilt is on display then it cannot be used in other functional ways.

I am choosing to bind my quilt by attaching a separate edge to it, as I hope I will have to “replace” the binding many times as the quilt is used over and over again. Especially because this research and this thesis is about exploring the idea of mathematics teacher understanding as an emergent phenomenon and I have invited the reader to join me in this conversation. I hope that, as I continue in my research life, many other people will join me in this conversation. That is, I hope that the exploration is never complete and that it continues to emerge. I use Maturana’s (1988) notion of an explanation when I think about quilt, or my research as a quilt; that is, I am attempting to offer an explanation of mathematics teacher understanding as an emergent phenomenon.

Maturana (1988) described two explanatory paths, the path of objectivity-without-parenthesis and the path of objectivity-in-parenthesis. The explanatory path that I have chosen to think about my research is that of objectivity-in-parenthesis. Maturana described the path of objectivity-in-parenthesis as:

> the observer explicitly accepts: a) that he or she is, as a human being, a living system; b) that his or her cognitive abilities as an observer are biological phenomena because they are altered when his or her biology is altered; and c) that if he or she wants to explain his or her cognitive abilities as an observer, he or she must do so showing how they arise as biological phenomena in his or her realization as a living system. (p. 29)

Maturana further described that if an observer has adopted the objectivity-in-parenthesis explanatory path then “the observer has to accept as his or her constitutive features all constitutive features of living systems, particularly their inability to distinguish in experience what we distinguish in daily life as perception and illusion” (1988, p. 29) and that “it follows that an observer has no operational basis to make any statement or claim about objects, entities or relations as if they existed independently of what he or she does” (p. 30).

Maturana (1988) further wrote that

> When one observer accepts this explanatory path, he or she becomes aware that two observers, who bring forth two explanations that exclude each other in front of what, for a third observer, seems to be the same situation, are not giving different explanations for the same situation, but that all three are operating in different yet equally legitimate domains of reality, and are explaining different aspects of their respective praxes of living. The observer that follows this explanatory path realizes that he or she lives in a multiversa, that is, in many different, equally legitimate, but not equally desirable, explanatory realities, and that in it an explanatory disagreement is an invitation to a responsible reflection of coexistence, and not an irresponsible negation of the other. (p. 31-32.)

Hence, in offering an objectivity-in-parenthesis explanation of this research, I invite readers to consider that I was one of the teachers as well as the researcher within our mathematical community. Therefore, I have a "basis to make any statement or claim about objects, entities or relations" (Maturana, 1988, p. 30) because they do not exist independently from me. I offer the explanation within the context of my lived history, as a teacher, policy maker, curriculum writer, and a teacher educator. All of these experiences were interwoven as I wrote the recursive elaborations of the moments in Chapter 5.

My explanatory path used the features from the Pirie-Kieren (1994) theory of the dynamical growth of mathematical understanding to explore emergent individual understanding; the features from the Davis-Simmt (2003) conditions of complexity to explore emergent collective understanding or emergent understanding within the collective; and Davis's (1996) ideas about emergent understanding within the body of mathematics. What were some of the patterns that emerged within the explanatory path that I chose?

Prior to discussing some of the patterns that I noticed emerging throughout my explanation, I would like to draw your attention to the context of the conversations in my work. Our conversations were professional conversations about the mathematical processes. During the time that our mathematical community was meeting, we were involved in implementing a new grade 10 mathematics curriculum. Throughout the narrative accounts, we did not "do" traditional mathematics but we talked about how we came to make sense of the mathematical ideas within the mathematical processes of reasoning, problem solving, connections, and communication. I make this point, because in the experiences I have had, there is a general belief about high school mathematics teachers in the professional development community. This belief can be characterized as follows: "We do not engage high school mathematics teachers in conversation about teaching mathematics or about mathematics." I am not sure where or how this belief has come about; however, it is almost as if professional developers are not certain whether or not they will understand, or appropriately shape, the conversation of high school mathematics teachers. What I have learned, however, is that when I engage high school teachers in conversations about mathematics and teaching mathematics, I come to learn the diversity in their thinking and my own understanding is changed.[24] In addition, many high school teachers also feel that "they have had the best professional development" when they are engaged in such conversations.

This is a comment that I repeatedly heard when I was responsible for organizing the marking sessions for the grade 12 mathematics diploma examinations. During these marking sessions, teachers would work in groups of 6 to 8 to mark students' papers. While marking student papers, teachers would often have conversations in their group about the wording of particular questions on the exam or about the way that students

[24] As evidenced in the complexities of the narrative accounts in Chapter 4 and made explicit, particularly in the 'Wow' moment in Chapter 5. This diversity is evident in the interpretation of teacher actions in my research.

responded to a particular question. The conversations would often begin with phrases like "Oh, I haven't thought about it that way before!" or "Did you see how this student answered this question?" The whole group would look at the question or the student's response and talk about the way each of them thought about the content and then how they themselves would teach the idea. In these conversations, it is likely that an individual's understanding about a particular concept or idea changed.

Now, back to considering the ideas about emergent teacher understanding in this story. As I reflect back on these groups of teachers marking, they were likely what I would now describe as a complex learning system. All five "necessary but insufficient conditions" of internal diversity, redundancy, decentralized control, organized randomness, and neighbor interactions (Davis and Simmt, 2003, p. 147) could be observed in these working groups. Teacher comments about these experiences as being the "best professional development they've participated in" did not refer to the actual marking, but the emergent conversation within the context of marking.

Emergent Patterns

In this section I now look back at the recursive elaborations in Chapter 5 and describe the patterns that I noticed. I offer these noticed patterns as part of the explanatory path that I have chosen, that of the path of objectivity-in-parenthesis. These patterns are my recursive interpretations of the narrative accounts and moments and are invitations or suggestions for action.

Teacher Understanding in Action

What did I notice about teacher understanding in action throughout my exploration?

Mathematics teacher understanding is intertwined with our lived histories. How we come to understand and how our understanding emerges within interaction is intertwined with our experiences in learning mathematics, in becoming and being a teacher, and in living. That is, teacher understanding is intertwined with, and affected, by teachers' lives. Teaching and mathematics are embedded in our lives. I bring forward my history and experiences as I come to make sense of an emergent conversation. An example of this intertwining was evident in the 'Wow' moment. In our community's discussion of making sense of the mathematical process of problem solving in the 'Wow' moment, each of us 'brought forth' experiences in learning about solving textbook word problems while in high school. This 'bringing forth' of solving textbook word problems while in high school then reminded each of us about what it was like to try to make sense of teaching students about solving textbook problems when we first started teaching. In bringing forth the teaching experience, we are bringing forward our teaching lives. From this experience with textbook word problems in our teaching lives, we moved to thinking about problem solving in other school subjects and finally to the idea of generalizing and hypothesizing, acts that we engage in in our lives. In Joyce's comment that she'd "never thought about this," and we all agreed, I could suggest that our collective somehow saw the image of problem solving differently than we had prior to the conversation. Our experiences in learning about solving textbook word problems, our experiences with learning to teach about solving textbook word problems, and our experiences with hypothesizing and generalizing were brought forth in this conversation as a new way of 'making sense' of the mathematical process of problem solving.

Understanding teacher understanding is wound up in understanding student understanding and is not separate from it. Understanding student understanding is part of a teacher's lived experience; hence, often in explaining my own understanding, I will use examples of student understanding, either of my own students' or of my own experience as a student. This means that in conversations teachers might bring forward classroom experiences to provide the rest of the group with an image of their own understanding of the mathematical processes. An example of bringing forward classroom experiences to provide our collective with an image of the mathematical process of connections was in the 'Connections' moment. In this moment, Julia, after listening to what I shared, was reminded of how she has observed students in her own classrooms. She described how she had noticed that "exceptionally talented math students appear to be making their own connections all of the time. They don't go on until they know exactly how 'this' fits and then they're quite satisfied." She continued by contrasting what she has observed about 'other kids.' Julia's observation about what she has noticed about collective student understanding, provided an image for each of the others in our collective about what it means for mathematical connections in action. An image that we could all "picture" because of the redundancies in our teaching experiences.

Another example of understanding teacher understanding being wound up in understanding student understanding is in the 'Wow' moment. When I brought forth my experience as a student learning how to solve textbook word problems and explaining my confusion when I "had to write a mathematical expression for a word statement like 'a number is 5 less than a second number'" it points to ways in which students make sense of the relationship between mathematical expressions and verbal or written expressions.

A teacher may not overtly express their understanding to others, yet changing understanding has occurred. There were times in our conversation when one person or another would be silent. Did this mean that that person was not engaged? I observed this throughout each of the elaborations in Chapter 5, but a particular example of this silence was in the ''Wow' moment when Julia, Marilyn, and I talked about problem solving. It appeared that Joyce was not 'participating' as she was not overt, to others, in her statements or in her participation in the conversation. However, when there was a break, or what appeared to be a conclusion of some sort in our conversation, Joyce said, "I'm just listening to all this, going, 'wow.' I've never thought about this." Joyce's act of making this statement prompted each of Julia, Marilyn, and I to reflect on what we'd just said and agree with Joyce. The ideas that had emerged in our conversation were ideas that each of Julia, Marilyn, and I did not generally think about either. Although I, as an observer, cannot tell, the particular way in which Joyce's understanding has changed; I could suggest that she was "participating" in our conversation by listening.

Collective Understanding

What did I notice about collective understanding?

The role of teacher or teacher educator is as an organizer of the complex system, such as our group or a school class, in that the teacher or teacher educator needs to 'maintain' the organizers. For example, in the 'Right! That's Right!' moment, we see that I was aware Joyce might not have had the chance to fully share her definition of the mathematical process of communication, so I specifically asked Joyce "what did you write?" In this moment, I acted as 'the leader' or 'the teacher' because of what I knew about the lived histories of each of 'my students,' Julia, Marilyn, and Joyce, and myself. I acted in this way to ensure each person had the chance to talk about their definition of the mathematical processes, knowing that because Marilyn, Julia, and I were regularly involved in professional conversations that if I were not conscious, Joyce may not have a chance to share her understandings. In that same moment, however, I as the leader or the teacher, 'let go,' and was a part of the collective system. I did not limit the possibilities of the learning system by interrupting and interspersing my ideas constantly. I participated in the system as both a colleague and as a leader. In other words the teacher educator must realize that "the collective actually makes space for, and supports the development of, individual students' ideas" (Davis and Simmt, 20003, p. 147). Davis and Simmt's hypothesis is that the "individual learner's mathematical understandings might be better supported – not compromised – if the teacher pays more attention to the grander learning system" (p. 164).

<u>Developing a shared or distributed understanding within a collective is possible.</u> Each of the moments elaborated on in Chapter 5 as well as the narrative accounts in Chapter 4 illustrated development of a shared, or distributed, understanding within a collective. An example of this is in the narrative account *Building a Representation of our Thinking*. Throughout this narrative account we see a conversation about the way in which our collective could visually represent our thinking about the relationships in our definitions of the mathematical processes. The conversation reaches a point, in which we are 'stumped,' as considered in the 'Woggle' moment. In the 'Woggle' moment, we debate about the use of the Venn diagram as a representation of the relationships that we saw. Although we explore a variety of three-dimensional representations, we return to the Venn diagram, but now have a different image of it. We continue in our deliberations

'end up' with a representation of the mathematical processes, the Venn diagram at the end of the narrative account.

Within the narrative account, our collective recognized that we were developing a shared understanding of our thinking when Marilyn suggested that "Sometimes you try diagrams and they are too limiting. But if you don't have a diagram then there's too much information. This [referring to our Venn diagram] will be a picture of our overall thinking. It's like using graphs, tables, and charts in statistics. Without the picture, the words in themselves won't mean anything." I then suggested that the act of constructing the diagram was a way to 'practise' the visualization and representation that we had talked about. Joyce added that the diagram was also a form of communication and problem solving, because it showed how we got to this point. I then made the point that the diagram would really only make sense to each of us, because we were involved in the conversation. Marilyn replied to my comment with "Yes, but it will still give somebody else a picture to look at. It may be kind of interesting to ask that person, 'what does this mean to you?' We would get a sense if the picture communicated our thinking." In this moment we see an example of understanding as a property of the collective.

Changing collective understanding emerges in the collective. When I first suggested using a Venn diagram to represent our thinking, in the narrative account *Building a Representation of Our Thinking*, there was no discussion as to whether or not the Venn diagram could represent a way in which our conversation and the meanings that we each brought about the mathematical processes of problem solving, reasoning, connections, and communication. The conversation about constructing the Venn diagram started immediately because of the redundancies within our lived experiences of learning and teaching mathematics. Using the Venn diagram made sense to each of us and hence, that is we had a collective understanding of the Venn diagram. However, in the process of constructing the Venn diagram, we reach a point where we are stumped and question whether the Venn diagram is too limiting. As described in the "Woggle" moment, our collective explored a variety of three-dimensional representations, but returned to the Venn diagram. In returning to the Venn diagram, however, we now have a different image of it and hence our collective understanding about the Venn diagram changed. Our collective understanding of the Venn diagram emerged within our collective's conversation.

A second example of this idea of collective understanding changing in the collective is when our collective was talking about the mathematical process of communication, in the 'Right! That's Right!' moment, we see the notion of changing understanding of the mathematical process of communication. At the beginning of the 'Right! That's Right!' moment, we noticed that each of our definitions were rooted in our mathematical and teaching experiences. For example, Julia refered to "the use of mathematical language and symbols;" Marilyn said "being able to talk about mathematics, being able to verbalize it and talk on paper too;" Joyce said "or tell somebody about, that doesn't know the language [referring to mathematical language]. . .so they can understand;" and I offered two symbolic statements "2x + y = 7 and 3 + 4." At the end of the 'Right! That's Right!' moment, an observer could suggest that our collective understanding of the mathematical process of communication extended outside of our teaching and mathematical experiences to our living experience, that of being in community, when Marilyn added, "It's all community and communication. If you can't communicate, then you don't belong to the community. Both have the same root word." At this point our collective had developed an image of the mathematical process of communication as "being in community."[25]

Understanding Mathematical Processes

In what way or ways is understanding mathematical processes mathematical?

Teacher understanding of the mathematical processes is affected by the way in which they themselves experienced the processes. For example, when our collective was talking about solving word problems in school in the 'Wow' moment, each of us had experienced problem solving in our high school experiences and in our beginning teaching experiences in a particular way. In our conversation, we brought forward our lived history of problem solving as solving the textbook word problems. This pattern points to the importance of teachers and teacher educators being conscious of their actions in a classroom. When I make choices as a teacher about the way in which I plan the occasions for talking about problem solving, or any of the other mathematical processes, I am now conscious that my understanding of the mathematical processes in action will affect my students' understanding of the mathematical processes. Similarly, as a teacher educator I am now conscious that the way in which I enact the mathematical processes of problem solving, reasoning, communications, and connections in my pre-service and in-service education classes will affect the understanding that my students enact of the mathematical processes.

[25] Our collective understanding of the mathematical process of communication as "being in community" arose again in our collective's second meeting, as described in Chapter 4. During the second meeting we noticed that the curriculum document did not mention the idea of being in community and that the writing in the document refers only to individuals.

The mathematical processes exist within the culture of mathematics. Although the mathematical processes can be described as 'human acts,' there are also uniquely mathematical features within the processes. One example of this uniqueness is the use of specialized symbols and symbolic expressions to communicate meaning, as described in the 'Right! That's Right!' moment when I used the expressions '2x + y = 7' and '3 + 4.'

Pirie-Kieren Theory

What does my research contribute to the Pirie-Kieren theory of the "dynamical growth of mathematical understanding?"

The Pirie-Kieren theory is a useful model to use in describing emergent mathematical understanding. The way in which I used the model, however, was not aligned with the way in which Pirie-Kieren have used the model. Pirie and Kieren suggested that their model portrays "theory [which] attempts to elaborate in detail the constructivist definition of understanding as a continuing process of organizing one's knowledge structures" (1994, p. 166) and have used the theory in a "variety of learning environments as a tool to observe the mathematical behaviour of students as they work on a single mathematical task and as they build and organize mathematical knowledge structures over periods of time" (p. 181). That is, Pirie and Kieren have used the theory as they have observed children and adults doing mathematics. I have used the Pirie-Kieren theory, as described in each of the interpretations in Chapter 5, as a tool to describe the emergent mathematical understanding of teachers talking about doing mathematics, and specifically in talking about the mathematical processes. Therefore, the Pirie-Kieren theory cannot only be used to "observe the mathematical behaviour of students as they work on a single mathematical task" but it can also be used as a tool to observe emergent mathematical understanding as students (or teachers) talk about mathematical tasks.

The teacher or teacher educator can also use the idea of multiple forms of understanding as described by the Pirie-Kieren theory. Each of the moments interpreted in Chapter 5, represents teachers expressing their primitive knowing, making images, having images, altering images and folding back to try to find a basis for changing their images. Because these teachers were functioning in a conversation, their own individual actions showing changing understanding both changed the conversations and the possible understanding of others. Each person's understanding was shaped by participating in the conversation and in maintaining professional (and personal) relationships with the others.

An observer such as a teacher or teacher educator cannot suggest that they might know about an individual's level of understanding if that individual has not acted or expressed overtly, to others, in some way. This is important because, as a teacher and a teacher educator, I have a responsibility to my students to learn how they are coming to understand mathematical concepts described in the curriculum and the images that they have about those mathematical ideas.

Emergent Understanding in Conversations

How does being in conversation contribute to all of the above emerging understanding?

Because conversation itself is an emergent phenomenon, we can see emergent understanding within it. As each person contributes to the conversation, they are bringing forth their lived history and understanding of the topic at hand. The fluidness of conversation allows the contributors to bring forth images of their understanding that represent changes in their own structures but also provide possibilities for changing or elaborating the collective understanding of the group. An example of this is evident in the 'Random' moment. In the 'Random' moment, Julia asked me to explain what I thought I meant by the word 'random.' In my description of the word, I brought forth my lived history of my conversations with painters and dancers and my conversations with people about mathematics. In my description, we see images of my understanding of the word 'random.' We see that these images of mine provided the possibilities of developing a collective understanding for our group about the way in which we think about 'random.'

Mathematics lives in mathematics teacher conversations. Engaging mathematics teachers in conversation about the ways in they come to think about ideas and the histories that they bring to the conversation illustrates the way in which mathematics itself does not 'live' in some curriculum document or textbook, but in the actions of teachers and learners. By listening to, and being a part of conversations, we see that the culture of mathematics emerges.

Each of the moments interpreted in Chapter 5 showed evidence of a culture of mathematics emerging. One particular example was in the 'Woggle' moment. Within the culture of mathematics, we look for limits and limiting factors. In the 'Woggle' moment Marilyn suggested "Maybe the Venn diagram is too limiting." This prompted our collective to examine alternative mathematical representations to the Venn diagram, thinking about mathematical ideas in new ways and attempting to re-present the relationships we had seen between the mathematical processes. Although we moved back to the Venn diagram, we had explored within the culture of mathematics. In this moment, the exploring culture of mathematics emerged.

Contributions of Narrative Inquiry

What contributions does narrative inquiry make to understanding mathematics teacher understanding?

Narrative inquiry and, in my research particularly, the narrative accounts in Chapter 4 provide evidence of the complexity and complicity in our professional conversation. Living out our histories in conversation provides the context for emergence of understanding. Our individual understanding, conversation, and narrative lives interweave to reveal the complexity of emergent understanding. A reader could only read the recursive elaborations of the moments in Chapter 5 and suggest that, they might 'know' the whole story. However, without reading the narrative accounts in Chapter 4, a reader would likely be unable to come to see the complexity within the conversation.

This complexity both contributes to and is indeed a feature of emerging teacher understanding of the mathematical processes. Thus, while features of this understanding and its changing nature are highlighted by the interpretations in Chapter 5, the understanding as a dynamical phenomenon, in its individual, collective, or mathematically embodied aspects, is grounded in the continuing conversations reflected by the narratives as wholes.

Using the Quilt

In what ways might this quilt, and the emergent patterns in the quilt, be used by teachers, teacher educators, policy makers, and curriculum writers in teacher education activities? I offer a series of questions as possibilities of emergent events, recognizing that "emergent events cannot be caused, but they might be occasioned. A shift in interpretive focus is implicit here, away from what *must* or *should* happen toward what *might* or *could* happen" (Davis and Simmt, 2003, p. 147). I offer a series of questions within the explanatory path of objectivity-in-parenthesis. These questions have emerged within my recursive interpretations of the narrative accounts, moments, and noticed patterns and can be considered as ongoing research questions. The act of now describing the ways in which I might use this quilt is, once again, recursive.

In-Service Teacher Education

As a person planning the facilitation of an in-service teacher activity, my quilt could be used to answer the following questions about the activities I have planned or are about to plan:

1. In what way or ways could the activities help teachers come to understand that their mathematical understanding:
    a. is intertwined with their lived histories?
    b. is wound up in understanding student understanding and is not separate from it?
    c. may be changing, even if they have not overtly expressed their understanding to others and, when they are teaching, they may notice this with children?
2. In what way or ways could the activities help teachers come to understand their role in a classroom collective learning system and the features of a collective learning system?
3. In what way or ways could the activities help teachers come to understand that developing a shared or distributed understanding within a collective learning system is possible?

4. In what way or ways could the activities help teachers come to understand that changing understanding of a collective emerges within the collective?
5. In what way or ways could the activities help teachers come to understand that their understanding of the mathematical processes they are teaching is affected by the way in which they themselves experienced the processes?
6. In what way or ways could the activities help teachers come to understand that their enactment of the mathematical processes in their classroom will affect the way in which their students come to understand the mathematical processes?
7. In what way or ways could the activities help teachers come to understand that the mathematical processes exist within the culture of mathematics?
8. In what way or ways could the activities help teachers come to understand the features of the Pirie-Kieren theory as a tool they could use to describe the dynamic growth of mathematical understanding?
9. In what way could the activities help teachers come to understand the importance of having professional conversations about mathematics?
10. In what way could the activities help teachers come to understand the importance of their lived history and the sharing of those histories through narrative?
11. In what way could the activities help teachers come to see themselves as complex beings, that their students are complex beings, and that their classrooms could be "an adaptive, self-organizing – complex – unity" (Davis and Simmt, 2003, p. 164)?

Pre-Service Teacher Education

The quilt could be used as frame in thinking about pre-service teacher education programs. For example, I currently teach in a pre-service mathematics teacher education program that requires students to take mathematics content courses and mathematics education courses. The mathematics content courses are generally selections from the areas of calculus, linear algebra, geometry, and statistics and the content within each of these courses is beyond the content in high school mathematics. The mathematics education courses tend to focus on a variety of instructional strategies. In addition, our students must take courses in learning theory, instructional strategies, administration, and diversity in classrooms. My quilt could be used to ask the following questions about our pre-service mathematics teacher education program (and I suspect other pre-service mathematics teacher education programs): [26]

1. In what way or ways could a pre-service mathematics teacher education program help students come to understand that their mathematical understanding:
    a. is intertwined with their lived histories?
    b. is wound up in understanding student understanding and is not separate from it?
    c. may be changing, even if they have not overtly expressed their understanding to others and, when they are teaching, they may notice this with children?

[26] As you read these questions, you will likely ask, "Are these not the same questions posed for in-service teacher education?" They are similar questions and at the same time very different questions. The reasons that they are different is that the lived experiences of teachers will be very different from those experiences of pre-service teachers. Hence, within the context of the collective, the questions are different.

2. In what way or ways could a pre-service mathematics teacher education program help students come to understand their role as a teacher in a collective learning system and the features of a collective learning system?
3. In what way or ways could a pre-service mathematics teacher education program help students come to understand that developing a shared or distributed understanding within a collective learning system is possible?
4. In what way or ways could a pre-service mathematics teacher education program help students come to understand that changing understanding of a collective emerges within the collective?
5. In what way or ways could a pre-service mathematics teacher education program help students come to understand that their understanding of the mathematical processes they are teaching is affected by the way in which they themselves experienced the processes?
6. In what way or ways could a pre-service mathematics teacher education program help students come to understand that the mathematical processes exist within the culture of mathematics?
7. In what way or ways could a pre-service mathematics teacher education program help students come to understand the features of the Pirie-Kieren theory as a tool they could use to describe the dynamic growth of mathematical understanding?
8. In what way or ways could a pre-service mathematics teacher education program help students come to understand the importance of having professional conversations about mathematics?

9. In what way or ways could a pre-service mathematics teacher education program help students come to understand the importance of lived history and the sharing of those histories through narrative?

10. In what way or ways could a pre-service mathematics teacher education program help students come to see themselves as complex beings, that their future students are complex beings, and that their future classrooms could be "an adaptive, self-organizing – complex – unity" (Davis and Simmt, 2003, p. 164)?

The Last Stitch

In composing this thesis, I could be seen as the quilt-maker. Metaphorically speaking, we are all quilters or quilt-makers. Researchers, teachers, and students are all involved in creating a quilt that tells their life story. We are all quilters, but some are not very good. What is the difference? Well, the difference is whether or not we are conscious that we are involved in quilt making in living out our lives.[27] As a teacher I need to be conscious that I am part of a quilt that is being developed by each child in my classroom. As a teacher educator I am conscious that I am a part of the process that will define the quilt that each of my pre-service and in-service teachers are quilting. When teachers are asked to explain their understanding of the mathematical processes I know that:

- their understanding is intertwined with their lived experiences in how they come to understand mathematics (individual understanding) and that this will be connected with their experiences in school mathematics, in university mathematics classes, and in life;

[27] The binding that we choose at the end is almost like the eulogy that will be given at our funeral.

- how they come to understand teaching mathematics (collective understanding) will be connected to their experiences as a student in schools, as a student in mathematics education and mathematics classes, and as a teacher
- their understanding is connected to their entire life within the body of mathematics.

As beings in the world, to be a quilter who can "hold the work between her fingers and examine the stitches" or who "will lay out the quilt and analyze the overall pattern the stitches follow" (Otto, 1991, p. 161), we need to be conscious that we co-determine the quilts of others. All acts by an individual implicate another.

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APPENDIX A

Description of the Mathematical Processes

This appendix describes the mathematical processes from the Western Canadian Protocol and from the National Council of Teachers of Mathematics (1989, 2000).

The Western Canadian Protocol identified seven mathematical processes that "students must encounter in a mathematics program in order to achieve the goals of mathematics education and to encourage lifelong learning in mathematics" (1996, p. 5). These seven processes are communication, connections, estimation and mental mathematics, problem solving, reasoning, technology, and visualization.

The Western Canadian Protocol essentially used the NCTM's 1989 definitions for the processes of communication, connections, problem solving, and reasoning.

The National Council of Teachers of Mathematics (NCTM), a professional organization of mathematics educators, published the *Curriculum and Evaluation Standards for School Mathematics* (1989). Within that document, four mathematical processes were identified as threading throughout K - grade 12 mathematics education. These four processes were problem solving, reasoning, communication, and connections. Descriptions as to what NCTM meant by each of these processes are detailed below.

Problem Solving

> Problem solving is not a distinct topic but a process that should permeate the entire program and provide the context in which concepts and skills can be learned. (p. 23)
>
> Classrooms with a problem-solving orientation are permeated by thought-provoking questions, speculations, investigations, and explorations; in this environment, the teacher's primary goal is to promote a problem-solving approach to the learning of all mathematics content. (p. 23)

Problem solving is the process by which students experience the power and usefulness of mathematics in the world around them. It is also a method of inquiry and application, interwoven throughout the Standards to provide a consistent context for learning and applying mathematics. Problem situations can establish a "need to know" and foster the motivation for the development of concepts. (p. 75)

The nonroutine problem situations envisioned in these standards are much broader in scope and substance than isolated puzzle problems. They are also very different from traditional word problems, which provide contexts for using particular formulas or algorithms but do not offer opportunities for true problem solving. Real-world problems are not ready-made exercises with easily processed procedures and numbers. Situations that allow students to experience problems with "messy" numbers or too much or not enough information or that have multiple solutions, each with different consequences, will better prepare them to solve problems they are likely to encounter in their daily lives. (p. 76)

Problem solving—which includes the ways in which problems are represented, the meanings of the language of mathematics, and the ways in which one conjectures and reasons—must be central to schooling so that students can explore, create, accommodate to changed conditions, and actively create new knowledge over the course of their lives. (p. 4)

<u>Communication</u>

Mathematics can be thought of as a language that must be meaningful if students are to communicate mathematically and apply mathematics productively. Communication plays an important role in helping children construct links between their informal, intuitive notions and the abstract language and symbolism of mathematics; it also plays a key role in helping children make important connections among physical, pictorial, graphic, symbolic, verbal, and mental representations of mathematical ideas. (p. 26)

....Exploring, investigating, describing, and explaining mathematical ideas promote communication. Teachers facilitate this process when they pose probing questions and invite children to explain their thinking. (p. 26)

Communicating helps children to clarify their thinking and sharpen their understandings. Representing, talking, listening, writing, and reading are key communication skills and should be viewed as integral parts of the mathematics curriculum. Probing questions that encourage children to think and explain their thinking orally or in writing help them to understand more clearly the ideas they are expressing. (p. 26 - 27)

As students progress from grade 5 to grade 8, their ability to reason abstractly matures greatly. Concurrent with this enhanced ability to abstract common elements from situations, to conjecture, and to generalize—in short, to do mathematics—should come an increasing sophistication in the ability to communicate mathematics. But this development cannot occur without deliberate and careful acquisition of the language of mathematics. (p. 78)

Communication involves the ability to read and write mathematics and to interpret meanings and ideas. Writing and talking about their thinking clarifies students' ideas and gives the teacher valuable information from which to make instructional decisions. (p. 78 - 79)

Teachers foster communication in mathematics by asking questions or posing problem situations that actively engage students, including situations that encourage students to create problems themselves. (p. 79)

Reasoning

A classroom that values reasoning also values communicating and problem solving, all of which are components of the broad goals of the entire elementary school curriculum. (p. 29)

A climate should be established in the classroom that places critical thinking at the heart of instruction. Both teachers' and children's statements should be open to question, reaction, and elaboration from others in the classroom. Such a climate depends on all members of the class expressing genuine respect and support for one another's ideas. Children need to know that being able to explain and justify their thinking is important and that how a problem is solved is as important as its answer. This minds-set is established when children have opportunities to apply their reasoning skills and when justifying one's thinking is an expected component of problem discussions. (p. 29)

Children should be encouraged to justify their solutions, thinking processes, and conjectures in a variety of ways. Manipulatives and other physical models help children relate processes to their conceptual underpinnings and give them concrete objects to talk about in explaining and justifying their thinking. (p. 29)

Reasoning is fundamental to the knowing and doing of mathematics......Conjecturing and demonstrating the logical validity of conjectures are the essence of the creative act of doing mathematics. To give more students access to mathematics as a powerful way of making sense of the world, it is essential that an emphasis on reasoning pervade all mathematical activity. (p. 81)

The seeds of logical thinking are planted as students learn to describe objects or processes accurately and to elaborate their properties, similarities, differences, and relationships. Students should be encouraged to explain their reasoning in their own words. Listening to their peers and their teacher describe other strategies helps students refine their thoughts and the language they use to express their thoughts. (p. 81 - 82)

Connections

A classroom in which making connections is emphasized exhibits several notable characteristics. Ideas flow naturally from one lesson to another, rather than each lesson being restricted to a narrow objective. Lessons frequently extend over several days so that connections can be explored, discussed, and generalized. Once introduced, a topic is used throughout the mathematics program. Teachers seize opportunities that arise from classroom situations to relate different areas and uses of mathematics. Children are asked to compare and contrast concepts and procedures. They are helped to construct bridges between the concrete and the abstract and between different ways of representing a problem or concept. Learning and using mathematics are important aspects of the entire school curriculum. (p. 32)

When children enter school, they have not segregated their learning into separate school subjects or topics within an academic area. Thus, it is particularly important to build on the wholeness of their perspective of the world and expand it to include more of the world of mathematics. This can be done in many ways, both within and outside the realm of mathematics. (p. 32)

Students should have many opportunities to observe the interaction of mathematics with other school subjects and with everyday society. To accomplish this, mathematics teachers must seek and gain the active participation of teachers of other disciplines in exploring mathematical ideas through problems that arise in their classes. This integration of mathematics into contexts that give its symbols and processes practical meaning is an overarching goal of all the standards. It allows students to see how one mathematical idea can help them understand others, and it illustrates the subject's usefulness in solving problems, describing and modeling real-world phenomena, and communicating complex thoughts and information in a concise and precise manner. (p. 84)

This persistent attention to recognizing and drawing connections among topics will instill in students an expectation that the ideas they learn are useful in solving other problems and exploring other mathematical concepts. .....Curriculum materials can foster an attitude in students that

> will encourage them to look for connections, but teachers must also look for opportunities to help students make mathematical connections. (p. 85)

In 2000, the National Council of Teachers of Mathematics published an up-dated version of the 1989 document, titled *The Principles and Standards for School Mathematics* (PSSM). In the PSSM, the process standards—problem solving, reasoning and proof, communication, connections, and representation—highlight ways of acquiring and using content knowledge (NCTM, 2000, p. 29).

Problem Solving

> Problem solving means engaging in a task for which the solution method is not known in advance. In order to find a solution, students must draw on their knowledge, and through this process, they will often develop new mathematical understandings. Solving problems is not only a goal of learning mathematics but also a major means of doing so. Students should have frequent opportunities to formulate, grapple with, and solve complex problems that require a significant amount of effort and should then be encouraged to reflect on their thinking.
>
> By learning problem solving in mathematics, students should acquire ways of thinking, habits of persistence and curiosity, and confidence in unfamiliar situations that will serve them well outside the mathematics classroom. In everyday life and in the workplace, being a good problem solver can lead to great advantages.
>
> Problem solving is an integral part of all mathematics learning, and so it should not be an isolated part of the mathematics program. Problem solving in mathematics should involve all the five content areas described in these Standards. The contexts of the problems can vary from familiar experiences involving students' lives or the school day to applications involving the sciences or the world of work. Good problems will integrate multiple topics and will involve significant mathematics (2000, p. 51).

Reasoning

> Mathematical reasoning and proof offer powerful ways of developing and expressing insights about a wide range of phenomena. People who reason and think analytically tend to note patterns, structure, or regularities in both real-world situations and symbolic objects; they ask if those patterns are accidental or if they occur for a reason; and they conjecture and prove.

Ultimately, a mathematical proof is a formal way of expressing particular kinds of reasoning and justification.

Being able to reason is essential to understanding mathematics. By developing ideas, exploring phenomena, justifying results, and using mathematical conjectures in all content areas and—with different expectations of sophistication—at all grade levels, students should see and expect that mathematics makes sense. Building on the considerable reasoning skills that children bring to school, teachers can help students learn what mathematical reasoning entails. By the end of secondary school, students should be able to understand and produce mathematical proofs—arguments consisting of logically rigorous deductions of conclusions from hypotheses—and should appreciate the value of such arguments.

Reasoning and proof cannot simply be taught in a single unit on logic, for example, or by "doing proofs" in geometry. Proof is a very difficult area for undergraduate mathematics students. Perhaps students at the postsecondary level find proof so difficult because their only experience in writing proofs has been in a high school geometry course, so they have a limited perspective (Moore 1994). Reasoning and proof should be a consistent part of students' mathematical experience in prekindergarten through grade 12. Reasoning mathematically is a habit of mind, and like all habits, it must be developed through consistent use in many contexts (NCTM, 2000, p. 55).

<u>Communication</u>

Communication is an essential part of mathematics and mathematics education. It is a way of sharing ideas and clarifying understanding. Through communication, ideas become objects of reflection, refinement, discussion, and amendment. The communication process also helps build meaning and permanence for ideas and makes them public. When students are challenged to think and reason about mathematics and to communicate the results of their thinking to others orally or in writing, they learn to be clear and convincing. Listening to others' explanations gives students opportunities to develop their own understandings. Conversations in which mathematical ideas are explored from multiple perspectives help the participants sharpen their thinking and make connections. Students who are involved in discussions in which they justify solutions—especially in the face of disagreement—will gain better mathematical understanding as they work to convince their peers about differing points of view (Hatano and Inagaki 1991). Such activity also helps students develop a language for expressing mathematical ideas and an appreciation of the need for precision in that language. Students who have opportunities, encouragement, and support for speaking, writing, reading, and listening in mathematics classes reap dual benefits: they communicate to learn mathematics, and they learn to communicate mathematically.

Because mathematics is so often conveyed in symbols, oral and written communication about mathematical ideas is not always recognized as an important part of mathematics education. Students do not necessarily talk about mathematics naturally; teachers need to help them learn how to do so (Cobb, Wood, and Yackel 1994). As students progress through the grades, the mathematics about which they communicate should become more complex and abstract. Students' repertoire of tools and ways of communicating, as well as the mathematical reasoning that supports their communication, should become increasingly sophisticated. Support for students is vital. Students whose primary language is not English may need some additional support in order to benefit from communication-rich mathematics classes, but they can participate fully if classroom activities are appropriately structured (Silver, Smith, and Nelson 1995).

Students need to work with mathematical tasks that are worthwhile topics of discussion. Procedural tasks for which students are expected to have well-developed algorithmic approaches are usually not good candidates for such discourse. Interesting problems that "go somewhere" mathematically can often be catalysts for rich conversations. Technology is another good basis for communication. As students generate and examine numbers or objects on the calculator or computer screen, they have a common (and often easily modifiable) referent for their discussion of mathematical ideas (2000, p. 59).

Connections

When students can connect mathematical ideas, their understanding is deeper and more lasting. They can see mathematical connections in the rich interplay among mathematical topics, in contexts that relate mathematics to other subjects, and in their own interests and experience. Through instruction that emphasizes the interrelatedness of mathematical ideas, students not only learn mathematics, they also learn about the utility of mathematics.

Mathematics is not a collection of separate strands or standards, even though it is often partitioned and presented in this manner. Rather, mathematics is an integrated field of study. Viewing mathematics as a whole highlights the need for studying and thinking about the connections within the discipline, as reflected both within the curriculum of a particular grade and between grade levels. To emphasize the connections, teachers must know the needs of their students as well as the mathematics that the students studied in the preceding grades and what they will study in the following grades. As the Learning Principle emphasizes, understanding involves making connections. Teachers should build on students' previous experiences and not repeat what students have already done. This approach requires students to be responsible for what they have learned and for using that knowledge to understand and make sense of new ideas (2000, p. 63).

Representation

The term *representation* refers both to process and to product—in other words, to the act of capturing a mathematical concept or relationship in some form and to the form itself. Some forms of representation—such as diagrams, graphical displays, and symbolic expressions—have long been part of school mathematics. Unfortunately, these representations and others have often been taught and learned as if they were ends in themselves. Representations should be treated as essential elements in supporting students' understanding of mathematical concepts and relationships; in communicating mathematical approaches, arguments, and understandings to one's self and to others; in recognizing connections among related mathematical concepts; and in applying mathematics to realistic problem situations through modeling. New forms of representation associated with electronic technology create a need for even greater instructional attention to representation (2000, p. 66).

APPENDIX B

Student Project and Developed Questions

Student Project

Mathematics 10 Pure

Polynomial Project

Polynomials is a topic that you have examined in your previous math experience. There are a variety of ideas and language associated with this topic. While working through this project you will explore these ideas and this language.

You may work alone, with a partner, or in a group to complete this project. Projects can be presented to me in a variety of ways; for example,

- on paper;
- as a video;
- audio;
- through a poster;
- an oral presentation; or
- any combination.

This project is due September 25, 1998.

Here are 6 polynomial expressions:

$x^2 + 7x + 12$	$3x^4$	$6x^6 - 18x^5 + 24x^4$
$-2x^2 + 6x - 8$	$x + 4$	$3x^2 + 5x - 6$

1. Choose one of the expressions shown above and explain why it can be called a polynomial.

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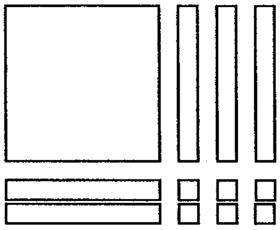
2. A model of the polynomial $x^2 + 5x + 6$ is the following rectangular area:



a. In what ways does this rectangular area represent this polynomial?

b. On the template included with this project, cut the pieces that can be used to form this rectangular area. Use these cut pieces to rearrange the pieces of the rectangle to build another rectangular area that represents this polynomial.

c. Describe the effect that different values of x have on the polynomial and its model.

3. Choose one of the 6 polynomials from above and show a physical model of this polynomial.

4. Tell what you know about what it means to add, subtract, multiply and divide two quantities.

5. Choose polynomials from the list above to demonstrate the operations of addition, subtraction, multiplication, and division of polynomials.

6. How do you see the operations with quantities connecting to operations with polynomials?

7. Factoring is a key idea in working with polynomials. Choose one of the polynomials above to explain what it means to factor.

Draft Scoring Criteria for Polynomial Project

The Polynomial Project will be scored based on the following criteria:

Mathematical Content (4 marks)

4 marks The material demonstrates mathematical understanding. It reflects the research or analysis done.

3 marks The material demonstrates mathematics understanding but there is a mechanical error in the mathematics. The material also reflects some research or analysis.

2 marks The material shows some mathematical understanding, but there are gaps in the ideas, or they are not applied consistently. Research is not shown.

1 mark There are serious mathematical errors, or information has been recopied without analysis. There is no apparent understanding of the mathematics.

Communication (3 marks)

3 marks Concepts are explained clearly and related to the original problem. There is a logical structure to the material. Diagrams, if they are included, are clearly labelled and easy to understand.

2 marks Concepts are explained but not related to the original problem. Some steps may be left out. Diagrams are mislabelled or hard to interpret.

1 mark The material is not well explained. The explanation of the concepts is unclear. The diagrams are not clear or do not match the content.

Presentation (2 marks)

2 marks The presentation of the material is inviting and easy to follow.

1 mark The presentation is messy and hard to read.

Originality (3 marks)

3 marks The project and all supporting work appear to be original. That is the material shows a creative and personal touch.

2 marks Most of the project appears to be original. Some of the sections may have come from other sources.

1 mark It does not appear that the project work is original.

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Developed Questions

Ideas that we want students to think about when studying polynomials.

1. We use the language of monomial, binomial, trinomial and polynomial when working with algebraic expressions. Describe the difference between and the relationship among each of these terms.

2. In what ways is factoring a polynomial expression similar to finding the factors of a number?

3. Explain what it means to factor a polynomial expression.

4. Describe how factoring an expression like $4x^2 - 25$ is similar to factoring an expression like $x^2 - y^2$.

5. Describe how factoring an expression like $x^2 - y^2$ is similar to factoring an expression like $a^2x^2 - b^2y^2$.

6. Explain the steps in multiplying two polynomials. Show how this is related to numbers.

7. How is division related to factoring polynomials? Describe how this thinking is the same as thinking about the division and factoring of numbers.

8. Explain how to factor $(x + t)^2 + 6(x + t) + 8$.

9. You probably have heard about those people who claim they can calculate values very quickly. For example, they can calculate the product of 45×21 very quickly. Show how the procedure that we used to multiply two binomials can be used to calculate the product of 45 and 21.

10. Find a different pair of binomials that could also be used to find the product of 45 and 21.

11. The division algorithm can be expressed in the forms: $\frac{P}{D} = Q + \frac{R}{D}$; $P = DQ + R$; and $P(x) = D(x)Q(x) + R$. Describe how these forms are related.

12. Write 428 as a second-degree polynomial expression in one variable. Identify the value of the variable to show that the polynomial expression that you have written has a value of 428 when it is evaluated at that value.

APPENDIX C

Yellowknife Work

About a year prior to working with my research group, I had the chance to work with a group of teachers from across the Northwest Territories in Yellowknife. Members in this group spanned teachers who taught K and grade 1 to teachers who taught grades 10 - 12. The work was surrounding how these individuals might take a leadership role in working with their colleagues to implement a new mathematics curriculum.

One of the activities that I planned during this work was that of talking about what we, as individuals, mean when we talk about mathematics as problem solving, communications, or reasoning and mathematical connections. Teachers in the room were divided into four groups, one group to discuss each topic. Each individual in the group was then asked to write what they understood the process to mean, and then share what they had written with their group. Once each individual had shared what they had written, the group was to develop a 'shared' statement about their understandings of the process. These 'shared' statements would be what the group would share with their colleagues in the room.

In the transcript that follows, individuals in the group are sharing what they have written as to how they view mathematical connections. Some of the selections include a dialogue between group members as individuals who are sharing their ideas (the individual sharing their writing is indicated in bold print) and these ideas become more clearly understood through the dialogue.

Mathematical Connections

PAM: I'm dealing with kids that are in Grade 8 and 9 for math and I see connecting math with other disciplines and experience to show the relevance of math to the students and it eliminates the "Why do we have to do this? This is stupid." It helps them see that learning is sort of holistic...math isn't just Miss X or Mr. Y, it's language arts, it's English, it's getting paid, it's balancing your cheque book, it's paying bills, and that there is a reality to it and a necessity.

AKIDA: That is the big connection we seem to make, that the works of connecting to the real world as opposed to having math be something you do out of a book or if you ask primary children what math is, most will simply indicate the workbook. . .that's what math is to them.

PAM: Or in high school, "period two and three on Mondays and Wednesdays". I had that learning must be meaningful and relevant and seen as useful to the student. It also means integration.

AKIDA: Well, I also talked about the same kind of thing. I had that math is rather meaningless in isolation. That... connections between the classroom and the real-world are crucial to making math part of the child's world. And that connections between areas of math, between the actual "hands-on" and then for the child to say "Oh, yeah, I see what's happening here!" and those kinds of connections just in terms of building the child's confidence. They can think this and make those connections between what is happening in terms of the concrete things to thinking processes of their own ... so I guess connections in that way too, I suppose, is just between the concrete and the symbolic, and the satisfaction of being able to make those kinds of leaps.

ELAINE: I think 'connections' in math is learning skills and concepts which apply to everyday living. I think it was already mentioned earlier, like addition and subtraction used for shopping or when you get paid you add up all the hours and stuff there's the deduction and stuff, and you have to pay bills and stuff. How much money I have can only get so much money so the stuff they learn connects to the real world. But there's also things like problem solving and I was thinking of those mounds up there and going well even if it was not really geared to where these things that are necessary, I was thinking about the patterns one over there and going, well if I learn about patterns and stuff, or if I like sewing and stuff and get stuck, there so that's what I meant about skills and concepts apply to everyday living.

AKIDA: I don't mind repeating myself. I find the same thing. Working with concrete objects, especially in kindergarten and grade 1, and then having to come to paper work, or something and they don't realize that while they are working with the concrete materials they don't realize that they'll be seeing it as well too, only on paper. When they see it, it's like, oh well, what are we going to do and then once they sit down and actually do the work, wow, we just did this with the concrete materials, so much easier to work with.

JOYCE: I just had 'connections' means to 'join up.' It's a way to find or relate to other parts. And so we talk about connecting in terms of getting to know something better. We talk about connecting in terms of building relationships and seeing them together. And so I had this sort of vision of almost a road map, roads connecting all these different places and much see math the same way. You need all these 'roads' to take the math. I've got a little 4-year old at home and I look at what she can do with math, and I think she's making all kinds of connections with mathematics to her real life and doing all kinds of problem solving.

PAM: How does that happen? I'm just thinking about connections and how it occurs. How can you ensure that that happens, or do you?

JOYCE: I think it's genetic. Got it from her mother.

PAM: It seems to be something that sometimes is difficult in the classroom.

JOYCE: I think it's the questions and the thinking about it. To give children the opportunity to do that. I want to play this game, well, we need 5 die to play it so we have two there, how many more do you need to go and find?

PAM: It's tied to language.

JOYCE: It's tied to language.

What I learned from the "Yellowknife" Experience

Each of these pieces of transcript show that, based on their experiences, each of Joyce, Pam, Akida, and Elaine have an understanding of mathematical connections. In my study, these transcripts would be the initial story. This story that each person told describes their understandings of mathematical connections.

All of the groups in this work session included teachers from different grade levels, as did the group shown here. I believe that, in this case and context, diversity was important because the people in this work session were identified as teacher-leaders for their entire school district and it was intended that each of these individuals would work with teachers at many different grade levels. However, for my own research, the process of telling and retelling stories might be more valid for study participants if they were teachers at similar grade levels. The stories would have more meaning to the individuals involved.

Some groups in this activity referred to the definition of a mathematical process as stated by the National Council of Teachers of Mathematics. For example, one of the people in the group that was working with mathematics as communication, started with the following statement:

> Communications. Students need to communicate mathematical ideas clearly and effectively, orally and in writing. Communications will help the students make connections among different representations mathematical ideas, namely -- pictorial, graphic -- verbal and mental representations. Taken from NCTM. Far enough to arrive at an answer students must be able to communicate effectively how the answer was obtained, in other words visits the opportunity to explore, to investigate, to write, to listen to, to discuss, and to explain ideas in their own language of mathematics. Thus students can create their own language, both informal and their own assumptions in the abstract language and symbolism of mathematics. Basically I guess we are suppose to discuss the concept of communication with --- the NCTM or would we like to see more work or is there criticism of it. I think there is a small criticism possibly. Almost all communication that I do is in my head. And I'm pretty sure that everybody else's communication is done totally in their head sometimes before--. But you always think it. You think in words. You think mathematical. You have mathematical concepts roll around in your head. If you do you are able to communicate those concepts verbally. I don't think that this point about enough of the idea that most communication is actually a one on one with yourself in your head. You have to be able to understand mathematics that way first, mental mathematics without numbers -- mental mathematics you can get mathematical ideas.

What was interesting with this group was that because they began to work with a definition, we never fully see the individual stories of what they believe the process to be. So, in order to develop some sense of teachers' understandings of the processes at the beginning of the study, I need to ensure that we are talking about what the study participants have come to understand the processes to be.

My study intends to describe teachers' growing understanding of mathematical processes. To get at how understandings of these processes grow, we would experience an activity something like the one described above and then ask, if our understandings of mathematical processes are the things that are identified, then what actions do we take as teachers in our classrooms to show these understandings?

If these four teachers, Elaine, Joyce, Akida, and Pam, were in my study, we would work together to try to identify a shared understanding of mathematical connections. Later in the transcript with this group, there is talk about how they are going to represent their conversation:

> So, we want a web? What do we want in the middle? We can put connections in the middle because that's probably what we....things we can connect math to, maybe, pardon, some of the things we connect it to ....you had mentioned a couple, you mentioned the balanced cheque books, the patterns. . .
>
> I like the patterns.
>
> I like that analogy.
>
> I have a wonderful poster on my office door that says "Why do I need this?" There are 500 or 600 titles and things that you need math for. I have students who will ask me "why do I need math to be fisherman, a sports fisherman?" And I said, well you have to be able to read a map, and understand compass headings, and understand weights and, you know, read the time, like all that stuff, and he was, like, oh yeah. It amazes them sometimes when they see what is actually out there.
>
> When Where How even Why Making connections in terms of even the logic thinking that somebody mentioned, you know, just the facts that give them framework. Like, if you are going to do something do this first and this second and this third and you have to be able to think in that order to plan it out. Connections even to other subjects. Oh, definitely. Science is an obvious one. We were doing calculations on the board for insulation

> values and that's multiplication of decimals, that's math. What are you doing?
>
> We were doing a Social Studies research project on statistical analysis unit on Third-World poverty and wealth in the world. It must drive them nuts! Insurrection in the classroom, "You don't even teach math." No, I don't.
>
> What about that scaffolding, maybe we should work that in too in terms of…? Should children take the leaps to join things together, so, what is that, intuitive? It teaches that intuitiveness, that's intuitive isn't it? It's intuitive to be able to make that leap. And so it promotes that it's one of the things, it connects. Develops that. It develops brain connections. Yeah! Thought connections. Develops those kind of brain connections. Actually there's some really interesting brain cognitive research that talks about if you are not using certain patterns, you lose them. Use it or lose it. That's some of the hardest to get back, if you've been out of math for a long long time. . .
>
> "Use it, or lose it."

This piece of the transcript shows that, in developing a shared understanding of mathematical connections, an understanding that this group will share with others, the group's members reconsider their initial discussions. This piece of transcript, and the activity in its entirety, was aimed at how these individuals might begin to initiate conversations about the mathematical processes with their colleagues. In my study, the difference is that we are considering how individuals demonstrate their understandings of the processes with their students.