Nori's Conjecture Revisited

by

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Abstract

Our main interest has to do with a conjecture of Nori, based on the weak Lefschetz theorem for Betti cohomology. We first prove Nori's conjecture under the assumption of the existence of a Bloch-Beilinson filtration. In the second part of my thesis, I re-prove a result of Paranjape on smooth general complete intersections, that supports Nori's conjecture.

I dedicate this thesis to the souls of my beloved aunts Myassar and Muneera,
may their souls rest in peace

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Chapter 1

Introduction

We study the relation between the topology of a smooth projective variety and a general subvariety. One of the measures of topology is a suitable cohomology theory; specifically Betti = singular cohomology.

Another measure is the group of cycles modulo rational equivalence - the Chow group. These two are conjecturally related by a series of conjectures of A. A. Beilinson and S. Bloch (see [6]), and Nori (see [[13]]).

The classical (weak) Lefschetz theorem asserts that if X is a smooth projective variety with dimension n and Y is a smooth hyperplane section of X with inclusion map $j : Y \hookrightarrow X$. Then

$$j^* : H^i(X, \mathbb{Z}) \to H^i(Y, \mathbb{Z})$$

is an isomorphism if $i < n - 1$, and injective for $i = n - 1$.

There is the hard Lefschetz theorem (HLT):

$$L_X^i : H^{n-i}(X, \mathbb{Q}) \to H^{n+i}(X, \mathbb{Q}),$$

(L_X the operation of intersecting with the hyperplane class) is an isomorphism, and Hard Lefschetz conjecture (HLC).

The HLT is equivalent to the statement

$$L_X^{n-i} : H^i(X, \mathbb{Q}) \to H^{2n-i}(X, \mathbb{Q})$$

is an isomorphism, for $i \leq n$.

Then for all i satisfying $0 \leq i \leq n$, the hard Lefschetz conjecture states: The inverse

$$\Lambda_X^{n-i} : H^{2n-i}(X, \mathbb{Q}) \to H^i(X, \mathbb{Q})$$

to L_X^{n-i} is algebraic, namely, induced by an algebraic cycle.
Nori's conjecture is then:

Conjecture 1.0.1. *If X is a smooth projective variety with dimension n and Y is a smooth hyperplane section of X with inclusion map $j : Y \hookrightarrow X$. Then*

$$j^* : CH^r(X, \mathbb{Q}) \xrightarrow{\sim} CH^r(Y, \mathbb{Q})$$

if $2r < \dim Y$.

We prove Nori's conjecture under the assumption of the existence of a Bloch-Beilinson filtration.

If we prefer not to work with any assumptions, then the following result of Paranjape can be proved, using different methods:

Theorem 1.0.2. *Given integers* $1 \leq d_1 \leq d_2 \leq ... \leq d_r$ *and any nonnegative integer* l, *let* $X \subset \mathbb{P}^{n+r}$ *be a smooth complete intersection of multi degree* $(d_1, ..., d_r)$. *If* n *is sufficiently large then*

$$CH^l(X)_{\mathbb{Q}} \simeq \mathbb{Q}$$

More precisely, Kapil H. Paranjape proves this theorem and estimates the values of n in his paper [13].

We prove the theorem for a smooth general (in the sense of the Zariski topology, $t \in \mathbb{P}^{N(d_1)} \times ... \times \mathbb{P}^{N(d_r)}$ general if $t \in$ Zariski open subset of $\mathbb{P}^{N(d_1)} \times ... \times \mathbb{P}^{N(d_r)}$ characterized by generic properties (eg X_t *smooth*, *etc*,...)) complete intersection, using different methods. Our hope is that our proof will provide effective values of n.

1.1 Precise Results

In the second chapter we will give the definition of the algebraic cycles and go over some examples. Then we will give the cycle class map and Hodge conjecture. We will also give the construction of the Abel Jacobi map that is induced from the kernel of the cycle class map, which will take us to the Mumford's famous theorem for 0-cycles on surfaces and Bloch-Beilinson conjecture.

In the third chapter, We will give some cohomological machinery, starting by talking about spectral sequences and double complexes, then giving some hypercohomology to end up by defining Deligne cohomology and giving a simple example.

In the fourth chapter, we will give Lewis filtration and make an approach for it using the material given in the second chapter. At the end of the chapter, we will prove Nori's conjecture based on the conjectural Bloch-Beilinson filtration.

Lastly in the fifth chapter, we start by introducing Fano varieties, then reprove the result of Paranjape for a smooth general complete intersection and providing some examples on finding the minimal value of n.

Chapter 2

Preliminaries

Let us start with the definition of the complete intersection:

Definition 2.0.1. *Let $z = (z_0, ..., z_{n+r})$ and assume given $f_1, ..., f_r$ homogeneous polynomials of degrees $(d_1, ..., d_r)$ in z such that the corresponding ideal $\mu = (f_1, ..., f_r)$ is prime and that $X = V(\mu) \subset \mathbb{P}^{n+r}$ is of dimension n. Then X is called a complete intersection of type $(d_1, ..., d_r)$. X is called smooth if at every point of X, its Jacobian has the full rank r.*

Example 2.0.2. *Fermat's Cubic. $X = V(z_0^3 + z_1^3 + z_2^3 + z_3^3) \subset \mathbb{P}^3$ is a projective algebraic manifold: It is given by the zeros of an irreducible cubic homogeneous polynomial and $X \cap U_i = V(1 + x^3 + y^3 + z^3)$ for $U_i = \{[z_0, ..., z_3] | z_i \neq 0\}$ where variables x, y, z adjusted according charts $U_i \cong \mathbb{C}^3$. Then the Jacobian of the polynomial has rank 1 at everywhere in X hence X is smooth.*

Example 2.0.3. *The elliptic quartic curve in $\mathbb{P}^3$. Let $X = V(z_0^2 - z_0z_2 - z_1z_3, z_1z_2 - z_0z_3 - z_2z_3) \subset \mathbb{P}^3$ is a projective algebraic manifold: It is given by the zeros of an irreducible cubic homogeneous polynomials and for $U_i = \{[z_0, ..., z_3] | z_i \neq 0\}$, $X \cap U_0 = V(1 - y - xz, xy - z - yz), X \cap U_1 = V(x^2 - xy - z, y - xz - yz), X \cap U_2 = V(x^2 - x - yz, y - xz - z), X \cap U_3 = V(x^2 - xz - y, yz - x - z)$ where variables x, y, z are adjusted according charts $U_i \cong \mathbb{C}^3$.*

Then the Jacobian of these all have rank 2 and hence it is a smooth complete intersection.

2.1 Algebraic cycles

Let $k = \overline{k}$ be an algebraically closed field and W/k a quasi-projective variety of dimension d say. Notice that $k = \overline{k}$ implies $W(k) \neq \emptyset$. where $W(k)$ are the k valued points (Nullstellensatz). Note again that $W(k)$ represents the codimension d points in W/k.

Definition 2.1.1. *A codimension r algebraic cycle Z on W is a formal sum (i.e. $\mathbb{Z}$-linear combination) of codimension r irreducible subvarieties on W.*

Example 2.1.2. *A codimension d algebraic cycle $Z = \sum_{j=1}^{m} n_j \cdot p_j$, where $p_j \in W$, $m \geq 1$ and $n_j \in \mathbb{Z}$.*

Definition 2.1.3. *$z^r(W) = z_{d-r}(W)$ is the free abelian group generated by irreducible subvarieties of codimension r $(= d - r)$ in W.*

Example 2.1.4. *The free abelian group generated by $W(k)$, is denoted by $z^d(W) = z_0(W)$, where $z_0(W)$ represents dimension. Any such point in $W(k)$ is the same thing as an irreducible subvariety of codimension d.*

Example 2.1.5. *Let $W = \mathbb{P}^2$. Put $\tilde{z_1} = V(z_0 z_2^2 - z_1^3 - z_0^2 z_1 - z_0^3)$, $\tilde{z}_2 = V(z_0 z_2^2 - z_1^3)$ then $5\tilde{z}_1$ - $2\tilde{z}_2 \in z^1(\mathbb{P}^2) = z_1(\mathbb{P}^2)$.*

Let us state the weak Lefschetz theorem now:

Theorem 2.1.6. *(Weak Lefschetz) Let Y be a smooth hyperplane section of a smooth projective variety X with dimension n with inclusion map $j : Y \hookrightarrow X$. Then*

$$j^* : H^i(X, \mathbb{Z}) \to H^i(Y, \mathbb{Z})$$

is an isomorphism if $i < n-1$, and injective for $i = n-1$.

Assume $0 \leq i \leq n$ and let L_X be the operator of taking cup product with the hyperplane class on X (relative to a projective embedding of X) then

Theorem 2.1.7. *(Hard Lefschetz theorem) (HLT)*

$$L_X^i : H^{n-i}(X, \mathbb{Q}) \to H^{n+i}(X, \mathbb{Q})$$

is given by $A \to A \cap H_1 \cap H_2 ... \cap H_i$ where $H_j \sim_{rat} H_X$, where rational equivalence defined in the next section, is an isomorphism.

Proposition 2.1.8. *Let X, Y be projective algebraic manifolds of dimensions n, m respectively, and let $\xi \in z^k(X \times Y)_{\mathbb{Q}}$ be an algebraic cycle of codimension k. Let $r = k - n$, and $\ell \in \mathbb{Z}$. Then ξ induces $\rho_\xi : H^\ell(X, \mathbb{Z}) \to H^{\ell+2r}(Y, \mathbb{Z})$ a morphism of Hodge structures of type (r, r).*

Definition 2.1.9. *Let X, Y be projective algebraic manifolds of dimensions n, m respectively and p, q be integers ≥ 0 such that $p + q$ is even. A linear map $\lambda : H^p(X, \mathbb{Q}) \to H^q(Y, \mathbb{Q})$ is said to be algebraic if it is induced by algebraic cycle $\xi \in z^{(2n-p+q)/2}(X \times Y)_{\mathbb{Q}}$.*

Conjecture 2.1.10. *(Hard Lefschetz conjecture) (HLC)*

Recall the HLT

$$L_X^{n-i} : H^i(X, \mathbb{Q}) \to H^{2n-i}(X, \mathbb{Q})$$

is an isomorphism. Then for all i satisfying $0 \leq i \leq n$: The inverse

$$\Lambda_X^{n-i} : H^{2n-i}(X, \mathbb{Q}) \to H^i(X, \mathbb{Q})$$

to L_X^{n-i} is algebraic.

Note that Lefschetz's theorems do not hold for singular spaces using singular cohomology, but there is a (co)homology theory where they do hold, namely intersection (co-)homology invented by R. MacPherson and M. Goresky.

2.2 Chow Groups

Definition 2.2.1. *Let's take two codimension r cycles Z_1 and Z_2 on W. We say that Z_1 and Z_2 are rationally equivalent if there exists a codimension r cycle Y in $W \times \mathbb{P}^1$ such that*

$$Z_1 - Z_2 = (\pi_W)_*(W \times \{0\} \bullet Y) - (\pi_W)_*(W \times \{\infty\} \bullet Y).$$

Definition 2.2.2. *Let $V \in z^{r-1}(W)$ be irreducible and $f \in k(V)^*$, then $div(f) = (f)_0 - (f)_\infty$ (zeros minus poles of f on Z, including multiplicities).*

Definition 2.2.3. *(Alternate definition of rational equivalence) The codimension r cycles Z_1 and Z_2 are rationally equivalent if $Z_1 - Z_2 = \sum_{i=1}^{N} div_{V_i}(f_i)$ where $codim(V_i) = r - 1$, $f_i \in \mathbb{C}(V_i)^*$, i.e. f_i is a rational function.*

Example 2.2.4. *Let D_1 and D_2 be codimension one cycles on X (also called divisors) . Then they are rationally equivalent if $D_1 - D_2 = (f)$, where f is a rational function on X.*

Definition 2.2.5. *$z^r_{rat}(W)$ is the subgroup generated by $div_V(f)$ where $V \in z^{r-1}(W)$ is irreducible and $f \in k(V)^*$.*

Definition 2.2.6. *$CH^r(W) = z^r(W)/z^r_{rat}(W)$ is the r-th Chow group of W.*

Example 2.2.7. *$CH^{n-r}(\mathbb{P}^n) = CH_r(\mathbb{P}^n) = \mathbb{Z}\mathbb{P}^r$.*

Proof. Consider the short exact sequence $\mathrm{CH}_r(\mathbb{P}^{n-1}) \to \mathrm{CH}_r(\mathbb{P}^n) \to \mathrm{CH}_r(\mathbb{P}^n - \mathbb{P}^{n-1})$.

For $r = n$ we have $\mathrm{CH}_n(\mathbb{P}^n) = \mathbb{Z}\mathbb{P}^n$.

For $r < n$ we have $\mathbb{P}^n - \mathbb{P}^{n-1} = \mathbb{C}^n$ and $CH_r(\mathbb{C}^n) = 0$.

Now, $\mathrm{CH}_r(\mathbb{P}^{n-1}) \to \mathrm{CH}_r(\mathbb{P}^n)$ is surjective.

Donig injuction on n, assume that $\mathrm{CH}_r(\mathbb{P}^{n-1}) = \mathbb{Z}\mathbb{P}^r$

Claim 2.2.8. *$\mathbb{Z}\mathbb{P}^r \to CH_r(\mathbb{P}^n)$ is injective.*

Proof. As $\mathbb{P}^r \subset \mathbb{P}^n$ we can always find a complimenatry projective space in $\mathbb{P}^n$ such that $\mathbb{P}^{n-r} \cap \mathbb{P}^r = p_0$.

Now, $\mathbb{Z}\mathbb{P}^r \to \mathbb{Z}p_0$, $\mathbb{Z}\mathbb{P}^r$ is not raionally equivalent to 0 since $\mathbb{P}^{n-r} \cap \mathbb{P}^r = p_0$. Hence, $\mathbb{Z}\mathbb{P}^r \to \mathrm{CH}_r(\mathbb{P}^n)$ is injective.

□

And so, $\mathbb{Z}\mathbb{P}^r \to \mathrm{CH}_r(\mathbb{P}^n)$ is an isomorphism. Hence, $\mathrm{CH}_r(\mathbb{P}^n) = \mathbb{Z}\mathbb{P}^r$ and we are done.

□

That the group $\mathrm{CH}^\bullet(W) := \bigoplus_{r=0}^d \mathrm{CH}^r(W)$, has a ring structure under intersection if W is smooth and quasi-projective, is due to the following theorem.

Theorem 2.2.9. *If W is smooth and quasi-projective, then the following holds:*

1. Given $\xi_1 \in z^{r_1}(W), \xi_2 \in z^{r_2}(W)$, there exists $\xi_2' \in z^{r_2}(W)$, $\xi_2' \sim_{rat} \xi_2$ such that ξ_1, ξ_2' meet properly, viz., of codimension $r_1 + r_2$.

2. Given the correct intersection property in 1., we have an appropriate definition of intersection multiplicity inducing the ring structure:

$$CH^{r_1}(W) \otimes CH^{r_2}(W) \xrightarrow{\cap} CH^{r_1+r_2}(W).$$

2.3 Hodge Theory

Definition 2.3.1. *Let $\mathbb{A} \subseteq \mathbb{R}$ be a subring, most commonly, $\mathbb{A} = \mathbb{Z},\ \mathbb{Q}$. An $\mathbb{A}$-Hodge structure (HS) V of weight $N \in \mathbb{Z}$ is given by the following datum:*

1. *A finitely generated $\mathbb{A}$-module V, and either of the two equivalent statements:*

2. $V_{\mathbb{C}} = \bigoplus_{p+q=N} V^{p,q}$,*satisfying* $V^{p,q} = \bar{V}^{q,p}$, *where* $^-$ *is complex conjugation induced from conjugation on the second factor* $\mathbb{C}$ *of* $V_{\mathbb{C}} := V \otimes \mathbb{C}$. *Equivalently,*

2′. *A (finite) descending filtration:*

   $V_{\mathbb{C}} \supset \cdots \supset F^r \supset F^{r+1} \supset \cdots \supset \{0\}$, *satisfying* $V_{\mathbb{C}} = F^r \bigoplus \bar{F}^{N-r+1},\ \forall r \in \mathbb{Z}$.

Remark 2.3.2. *The equivalence of above conditions 2. and 2′. can be seen as follows. For the decomposition in condition 2., put*

$$F^r V_{\mathbb{C}} = \bigoplus_{p+q=N,\ p\geq r} V^{p,q}.$$

Conversely, given $\{F^r\}$ in condition 2′., put $V^{p,q} = F^p \cap \bar{F}^q$.

Example 2.3.3. *We can multiply Hodge structures using tensor product. If H_1 is a Hodge structure of weight i and H_2 is a Hodge structure of weight j then $H_1 \otimes H_2$ is a Hodge structure of weight $i + j$.*

Example 2.3.4. *(Hodge) Let $W/\mathbb{C}$ be smooth, projective. Then $H^i(W,\mathbb{Z})$ is a $\mathbb{Z}$-Hodge structure of weight i.*

Example 2.3.5. *$\mathbb{A}(r) := (2\pi i)^r\mathbb{A}$ is an $\mathbb{A}$-Hodge structure of weight $-2r$ and of pure Hodge type $(-r,-r)$, called the Tate twist.*

Example 2.3.6. *Let $W/\mathbb{C}$ be smooth projective. Then $H^i(W,\mathbb{Q}(r)) := H^i(W,\mathbb{Q})\otimes \mathbb{Q}(r)$ is a $\mathbb{Q}$-Hodge structure of weight $i-2r$.*

Theorem 2.3.7 (Poincaré and Serre duality)**.** *Let $W/\mathbb{C}$ be a smooth projective variety of dimension d . The following pairings*

$H^i(W,\mathbb{C}) \times H^{2d-i}(W,\mathbb{C}) \to \mathbb{C}$.

$H^{p,q}(W,\mathbb{C}) \times H^{d-p,d-q}(W,\mathbb{C}) \to \mathbb{C}$ induced by

$$(w_1,w_2) \mapsto \int_W w_1 \wedge w_2,$$

are non-degenerate.

Hence $H^r(W) \simeq H^{2d-r}(W)^\vee$, $H^{p,q}(W) \simeq H^{d-p,d-q}(W)^\vee$.

Theorem 2.3.8 (Poincaré duality with twists)**.** *Let $W/\mathbb{C}$ be smooth with dimension d and Y is a subvariety then*

1. $H^i(W,\mathbb{Q}(r)) \simeq H_{2d-i}(W,\mathbb{Q}(d-r))$.

More generally,

2. $H^i_Y(W, \mathbb{Q}(r)) \simeq H_{2d-i}(Y, \mathbb{Q}(d-r))$ *(Note that 2. $\Rightarrow$ 1. by setting $Y = W$).*

Corollary 2.3.9.

$$\frac{H^i(X,\mathbb{C})}{F^r H^i(X,\mathbb{C})} \simeq \{F^{d-r+1}H^{2d-i}(X,\mathbb{C})\}^\vee.$$

Definition 2.3.10. *An $\mathbb{A}$-mixed Hodge structure ($\mathbb{A}$-MHS) is given by the following datum:*

- *A finitely generated $\mathbb{A}$-module $V_\mathbb{A}$,*
- *A finite descending "Hodge" filtration on $V_\mathbb{C} := V_\mathbb{A} \otimes \mathbb{C}$,*

$$V_\mathbb{C} \supset \cdots \supset F^r \supset F^{r+1} \supset \cdots \supset \{0\},$$

- *A finite increasing "weight" filtration on $V_\mathbb{A} \otimes \mathbb{Q} := V_\mathbb{A} \otimes_\mathbb{Z} \mathbb{Q}$,*

$$\{0\} \subset \cdots \subset W_{\ell-1} \subset W_\ell \subset \cdots \subset V_\mathbb{A} \otimes \mathbb{Q}$$

such that $\{F^r\}$ induces a (pure) HS of weight ℓ on $Gr^W_\ell := W_\ell/W_{\ell-1}$.

Theorem 2.3.11. *(Deligne [4]) Let Y be a complex variety. Then $H^i(Y,\mathbb{Z})$ has a canonical and functorial $\mathbb{Z}$-MHS, which agrees with the aforementioned (pure) Hodge structure in the case where Y is smooth and projective.*

Example 2.3.12. *Let $\bar{U}$ be a compact Riemann surface, $\emptyset \neq \Sigma \subset \bar{U}$ a finite set of points, and put $U := \bar{U} \setminus \Sigma$. According to Deligne, $H^1(U, \mathbb{Z}(1))$ carries a $\mathbb{Z}$-MHS. The Hodge filtration on $H^1(U,\mathbb{C})$ is defined in terms of a filtered complex of holomorphic differentials on U with logarithmic poles along Σ.*

One can "observe" the MHS via weights as follows. Poincaré duality gives us $H^1_\Sigma(\bar{U},\mathbb{Z}) \simeq H_1(\Sigma,\mathbb{Z}) = 0$ *since* Σ *is finite, and the localization sequence in cohomology below is a sequence of MHS (Deligne, op. cit.).*

$$... \to H^1_\Sigma(\bar{U},\mathbb{Z}(1)) \to H^1(\bar{U},\mathbb{Z}(1)) \to H^1(U,\mathbb{Z}(1)) \to H^2_\Sigma(\bar{U},\mathbb{Z}(1)) \to \; H^2(\bar{U},\mathbb{Z}(1)) \to 0$$

Notice that, $H^2_\Sigma(\bar{U},\mathbb{Z}(1)) \simeq H_0(\Sigma,\mathbb{Z}(0))$ *and* $H^2(\bar{U},\mathbb{Z}(1)) \simeq H_0(\bar{U},\mathbb{Z}(0))$.

But $H^1_\Sigma(\bar{U},\mathbb{Z}(1)) = H_1(\Sigma) = 0$.

And $H^2(\bar{U},\mathbb{Z}(1)) = H_0(\bar{U},\mathbb{Z}(0)) = \mathbb{Z}\{p\}$ *since* $\bar{U}$ *is path connected implies that*

$$H^0(\Sigma,\mathbb{Z}(0))^\circ := \ker\left(H^2_\Sigma(\bar{U},\mathbb{Z}(1)) \to H^2(\bar{U},\mathbb{Z}(1))\right) \simeq \mathbb{Z}(0)^{|\Sigma|-1}.$$

So we get

$$0 \to H^1(\bar{U},\mathbb{Z}(1)) \to H^1(U,\mathbb{Z}(1)) \to H^0(\Sigma,\mathbb{Z}(0))^\circ \to 0,$$

where

$W_0 = H^1(U,\mathbb{Z}(1))$, $W_{-1} = \mathrm{Im}\big(H^1(\bar{U},\mathbb{Z}(1)) \to H^1(U,\mathbb{Z}(1))\big) = H^1(\bar{U},\mathbb{Z}(1)$ *since the map is injective.,* $W_{-2} = 0$.

Then $Gr^W_{-1}H^1(U,\mathbb{Z}(1)) \simeq H^1(\bar{U},\mathbb{Z}(1))$ *has pure weight* -1 *and* $Gr^W_0 H^1(U,\mathbb{Z}(1)) \simeq \mathbb{Z}(0)^{|\Sigma|-1}$ *has pure weight* 0. *As they has pure Hodge structure since* $\bar{U}$ *and* Σ *are both smooth projective.*

Definition 2.3.13. *Let* V *be an* $\mathbb{A}$*-MHS. We put*

$$\Gamma_{\mathbb{A}} V := \hom_{\mathbb{A}-MHS}(\mathbb{A}(0), V),$$

and

$$J_{\mathbb{A}}(V) = Ext^1_{\mathbb{A}-MHS}(\mathbb{A}(0), V).$$

In the case where $\mathbb{A} = \mathbb{Z}$ or $\mathbb{A} = \mathbb{Q}$, we simply put $\Gamma = \Gamma_{\mathbb{A}}$ and $J = J_{\mathbb{A}}$.

Example 2.3.14. *Suppose that $V = V_{\mathbb{Z}}$ is a $\mathbb{Z}$ (pure) HS of weight $2r$. Then $V(r) := V \otimes \mathbb{Z}(r)$ is of weight 0, and (up to the twist) one can identify ΓV with $V_{\mathbb{Z}} \cap F^r V_{\mathbb{C}} = V_{\mathbb{Z}} \cap V^{r,r} := \varepsilon^{-1}(V^{(r,r)}$, where $\varepsilon : V_{\mathbb{Z}} \to V_{\mathbb{C}}$.*

Example 2.3.15. *(see [3], [6]) Suppose that V be an $\mathbb{A}$-MHS. There is the identification due to J. Carlson*

$$J(V) \simeq \frac{W_0 V_{\mathbb{C}}}{F^0 W_0 V_{\mathbb{C}} + W_0 V},$$

where in the denominator term, $V := V_{\mathbb{A}}$ is identified with its image $V_{\mathbb{A}} \to V_{\mathbb{C}}$ (viz., quotient out torsion).

For example, if $\{E\} \in Ext^1_{MHS}(\mathbb{Z}(0), V)$ corresponds to the short exact sequence of MHS:

$$0 \to V \to E \xrightarrow{\alpha} \mathbb{Z}(0) \to 0,$$

then one can find $x \in W_0 E$ and $y \in F^0 W_0 E_{\mathbb{C}}$ such that $\alpha(x) = \alpha(y) = 1$ (see [10]).

Then $x - y \in V_{\mathbb{C}}$ descends to a class in $W_0V_{\mathbb{C}}/\{F^0W_0V_{\mathbb{C}} + W_0V\}$, which defines the map from $Ext^1_{MHS}(\mathbb{Z}(0), V)$ to $W_0V_{\mathbb{C}}/\{F^0W_0V_{\mathbb{C}} + W_0V\}$.

Let us define the cycle class map:

Definition 2.3.16. *The cycle class map for a smooth projective X over $\bar{k} \subseteq \mathbb{C}$ is the Poincaré dual of the fundamental class map,*

$$cl_r : CH^r(X) \to \Gamma\big(H^{2r}(X, \mathbb{Z}(r))\big),$$

where

$H_{2d-2r}(|\xi|, \mathbb{Q}(d-r)) \to H_{2d-2r}(X, \mathbb{Q}(d-r)) \simeq H^{2r}(X, \mathbb{Q}(r))$

$\xi \mapsto (2\pi i)^{r-d}\{\xi\} \mapsto (2\pi i)^r[\xi]$.

This map fails to be surjective in general for $r > 1$.([8], 67).

Conjecture 2.3.17 (Hodge$_{\mathbb{Q}}$)**.** *(HC)*

$$cl_r : \mathrm{CH}^r(X) \otimes \mathbb{Q} \to \Gamma(H^{2r}(X, \mathbb{Q}(r))),$$

is surjective.

2.4 Abel-Jacobi Map

Let us define the Abel-Jacobi map first.

Definition 2.4.1. *Abel-Jacobi map*

$$AJ_X : \mathrm{CH}^r_{\mathrm{hom}}(X) \to J(H^{2r-1}(X, \mathbb{Z}(r))),$$

is defined as follows. Recall that

$$J(H^{2r-1}(X, \mathbb{Z}(r))) \simeq \frac{F^{d-r+1}H^{2d-2r+1}(X, \mathbb{C})^\vee}{H_{2d-2r+1}(X, \mathbb{Z}(d-r))},$$

Prescription for Φ_r: Let $\xi \in \mathrm{CH}^r_{\mathrm{hom}}(X)$. Then $\xi = \partial\zeta$ bounds a $2d - 2r + 1$ real dimensional chain ζ in X. Let $\{w\} \in F^{d-r+1}H^{2d-2r+1}(X, \mathbb{C})$.

Define:

$$\Phi_r(\xi)(\{w\}) = \frac{1}{(2\pi \mathrm{i})^{d-r}} \int_\zeta w \quad \text{(modulo periods)}.$$

That Φ_r is well-defined follows from the fact that $F^\ell H^i(X, \mathbb{C})$ depends only on the complex structure of X, namely

$$F^\ell H^i(X, \mathbb{C}) \simeq \frac{F^\ell E^i_{X,d-\mathrm{closed}}}{d\big(F^\ell E^{i-1}_X\big)},$$

where we recall that E_X^i are the C^∞ complex-valued i-forms on X.

Let $\{w\}$ *and* $\{\tilde{w}\} \in (F^{d-r+1}E^{2d-2r+1})_{d-closed}$ *be given, such that* $[w] = [\tilde{w}] \in F^{d-r+1}H^{2d-2r+1}$.

Then $w - \tilde{w} \in dF^{d-r+1}E_X^{2d-2r}$ so there exists

$$\eta \in F^{d-r+1}E_X^{2d-2r}\ , w - \tilde{w} = d\eta.$$

Now $\int_\zeta w \quad - \int_\zeta \tilde{w} \quad = \int_\zeta d\eta \quad = \int_{\partial\zeta} \eta \quad = \int_\xi \eta \quad = 0$ by the Hodge type, and we're done.

Alternate take for Φ_r: Let $\xi \in \mathrm{CH}^r_{\mathrm{hom}}(X)$.

First observe that $H^{2r-1}_{|\xi|}(X, \mathbb{Z}) \simeq H_{2d-2r+1}(|\xi|, \mathbb{Z}) = 0$ as $\dim_{\mathbb{R}} |\xi| = 2d - 2r$.

Secondly there is a fundamental class map $\xi \mapsto \{\xi\} \in H_{2d-2r}(|\xi|, \mathbb{Z}(d - r)) \simeq H^{2r}_{|\xi|}(X, \mathbb{Z}(r))$ (Poincaré duality).

Further, since ξ is nulhomologous, we have by duality

$$[\xi] \in H^{2r}_{|\xi|}(X, \mathbb{Z}(r))^\circ := \ker\left(H^{2r}_{|\xi|}(X, \mathbb{Z}(r)) \to H^{2r}(X, \mathbb{Z}(r))\right).$$

Hence ξ determines a morphism of MHS, $\mathbb{Z}(0) \to H^{2r}_{|\xi|}(X, \mathbb{Z}(r))^\circ$.

From the short exact sequence of MHS

$$0 \to H^{2r-1}(X,\mathbb{Z}(r)) \to H^{2r-1}(X\backslash|\xi|,\mathbb{Z}(r)) \to H^{2r}_{|\xi|}(X,\mathbb{Z}(r))^{\circ} \to 0,$$

we can pullback via this morphism to obtain another short exact sequence of MHS,

$$0 \to H^{2r-1}(X,\mathbb{Z}(r)) \to E \to \mathbb{Z}(0) \to 0.$$

Then $\Phi_r(\xi) := \{E\} \in \mathrm{Ext}^1_{\mathrm{MHS}}\big(\mathbb{Z}(0), H^{2r-1}(X,\mathbb{Z}(r))\big)$.

This class $\{E\}$ is easy to calculate in $J(H^{2r-1}(X,\mathcal{Z}(r))))$, in terms of a membrane integral.

Note that via duality,

$$E \subset H^{2r-1}(X\backslash|\xi|,\mathbb{Z}(r)) \simeq H_{2d-2r+1}(X,|\xi|,\mathbb{Z}(d-r)),$$

and that if ζ is a real $2d-2r+1$ chain such that $\partial\zeta = \xi$ on X, then $\{\zeta\} \in H_{2d-2r+1}(X,|\xi|,\mathbb{Z})$.

One can show that the class $x \in W_0E$ corresponding to the current

$$\frac{1}{(2\pi\mathrm{i})^{d-r}}\int_{\zeta},$$

maps to $1 \in \mathbb{Z}(0)$.

Now choose $y \in F^0 W_0 E_{\mathbb{C}}$ also mapping to $1 \in \mathbb{Z}(0)$, (see [10]).

By Hodge type alone, the current corresponding to $x - y$ in the Poincaré dual description of $J^r(X)$ is the same as for $x = \frac{1}{(2\pi \mathrm{i})^{d-r}} \int_{\zeta}$,
which is precisely the classical description of the Griffiths Abel-Jacobi map.

2.5 Mumford's famous theorem for 0-cycles on surfaces

Theorem 2.5.1. *(See [8] (Ch. 15)) [Mumford's famous theorem for 0-cycles on surfaces]*

Let X be smooth projective surface over $\mathbb{C}$, with geometric genus := dim$_{\mathbb{C}}$ $H^{2,0}_{\mathbb{C}} \neq 0$, i.e, there exists a nontrivial holomorphic 2-form on X. Then

$$\ker \left(AJ_X : \mathrm{CH}^2_{\mathrm{hom}}(X) \to J(H^3(X, \mathbb{Z}(2))) \right)$$

is enormous.

Note that enormous means that the kernel cannot be represented as an abelian variety.

Outline of proof of the theorem: Consider the $N-$th symmetric product $S^{(N)} := X^N / \{action\ of\ the\ symmetric\ group\ on\ the\ N-letters\}$.

Let $\xi \in S^{(N)}(X)$ be in the form $\xi = p_1 + ... + p_N$, means ignoring the ordering of the $N - tuple$. Where $S^{(N)}$ is the connected component of the Chow variety of effective $0-cycles$ of degree N on X, so its known to be projective algebraic.

Now, the singularities of $S^{(N)}$ are concentrated on $\{p_1 + ... + p_N \mid$ $not\ all\ of\ the\ \{p_1, ..., p_N\}\ are\ distinct\}$.

Let $k_N : S^{(N)} \to CH_0(X)$ given by $A \to [A]$ [and $k_{N,M} : S^{(N)} \times S^{(M)} \to CH_0(X)$ given by $(A, B) \to A - B$].

Lemma 2.5.2. *The fibres of k_N are c-closed. [The fibres of $k_{N,M}$ as well]. [Where c-closed means countable unions of closed subvarieties of S^N)].*

Idea of the proof : Let $\xi_1, \xi_2 \in S^{(N)}(X)$. Then ξ_1 and ξ_2 are rationally equivalent. and as mentioned before that ξ_1 and ξ_2 are rationally equivalent if there exists a map $f : \mathbb{P}^1 \to S^{(N+k)}(X) \times S^{(k)}(X)$
such that $f(0) = (A_0, B_0), f(\infty) = (A_\infty, B_\infty)$, $\xi_1 = A_0 - B_0, \xi_2 = A_\infty - B_\infty)$, for some $k \geq 1$.

Then rational curves in $S^{(N+k)}(X) \times S^{(k)}(X)$ is a Chow variety and represented by a countable union of projective varieties.

Now, just vary $k \in \mathbb{N}$ and relate this to $S^{(N)}(X)$ to get the result.

Now, since the c-closed sets have a unique decomposition into irreducibles, the dimension of a c-closed set make sense.

Thus we can define $\delta_N := dim k_N(S^{(N)}(X)) = 2N - min\{$ dimensions of fibres of $k_N\}$.

Let ω be a nontrivial holomorphic 2-form on X. Thus in local holomorphic coordinates $z = (z_1, z_2)$ on X, $\omega = h(z) dz_1 \wedge dz_2$, where $h(z)$ holomorphic.

Now, $\omega(p) : T_p(X) \times T_p(X) \to \mathbb{C}$ is an alternating map, for $p \in X$, non-degenerate if $\omega(p) \neq 0$, i.e. , if $h(p) \neq 0$.

Let $pr_j : X^N \to X, 1 \leq j \leq N$ be the j-th projection.

Then $\sum_{j=1}^{N} pr_j^*(\omega)$ is a holomorphic 2-form on X^N that is invariant under the action of the symmetric group.

So there is an induced 2-form ζ_N on $S^{(N)}(X)$, which is meromorphic along $S^{(N)}(X)_{sing}$.

Lemma 2.5.3. *Let S be a smooth quasi projective variety, and $f : S \to S^{(N)}(X)$ a morphism.*

Then $f^*(\zeta_N)$ is a holomorphic 2-form on S (even if $f(S) \subset S^{(N)}(X)_{sing}$); moreover if $f(s)_{s\in S}$ are rationally equivalent to each other, then $f^*(\zeta_N) = 0$.

Now back to the proof of the Theorem. It is enough to show that $Pg(X) \neq 0 \implies \{\delta_N\}_{N\in\mathbb{N}}$ is an unbounded sequence.

Let ω be a nonzero holomorphic 2-form on X, Then ζ_N is nonzero over a nonempty Zariski open subset $U_N \subset S^{(N)}(X)\backslash S^{(N)}(X)_{sing}$.

Suppose that $\Sigma_N \to U_N$ by f is a nonsingular quasi-projective variety for which $f(t)_{t\in\Sigma_N}$ are rationally equivalent to each other.

Then $f^*(\zeta_N) = 0$. For $p \in \Sigma_N$, $T_p(\Sigma_N) \subset T_p(U_N) \simeq \mathbb{C}^{2N}$ is an isotropic subspace for $\zeta_N(p)$.

But the maximal isotropic subspace have dimension $= 2N/2 = N$.

Now $dim\Sigma_N \leq N$, so $\delta_N \geq 2N - N = N$. Now just let $N \to \infty$ and were done.

2.6 Bloch-Beilinson conjecture

Mumford's theorem implies that $\mathrm{CH}^{\bullet}(W/\mathbb{C};\mathbb{Q})$ can be highly complicated.

Further, it was shown by C. Schoen, Griffiths-Green, Lewis, [6] that for any subfield $k \subset \mathbb{C}$ of transcendence degree ≥ 1 over $\mathbb{Q}$, that there exists smooth projective W and an r such that

$AJ_W : \mathrm{CH}^r(W) \to J\big(H^{2r-1}(W,\mathbb{Z}(r))\big)$ has non-zero kernel. This leads us to:

Conjecture 2.6.1 (Bloch-Beilinson Conjecture (BBC))**.** *If W/k is smooth and projective over a number field k then the Abel-Jacobi map*

$$AJ_W : CH^r(W/k;\mathbb{Q}) \longrightarrow J(H^{2r-1}(W,\mathbb{Q}(r))),$$

is injective.

Chapter 3

Cohomological machinery

3.1 A primer on spectral sequences

Spectral sequences were invented by Jean Leray.

Let us consider a bounded complex $(K^\bullet, d)$ of abelian groups, where for simplicity $K^{\bullet<0} = 0$. [Note that $K^{\bullet>>1} = 0$].

$$K^0 \xrightarrow{d} K^1 \xrightarrow{d} K^2 \xrightarrow{d} \cdots, \quad d^2 = 0.$$

Thus we have

$$H^p(K^\bullet) := \frac{\ker d : K^p \to K^{p+1}}{dK^{p-1}}.$$

Next, we will assume that this complex has a descending filtration of subcomplexes:

$$K^\bullet = F^0K^\bullet \supset F^1K^\bullet \supset F^2K^\bullet \supset \cdots \supset F^{N+1}K^\bullet = \{0\},$$

where again $F^{\bullet \geq 0}$ is out of convenience, and being a subcomplex means that $dF^\nu K^p \subset F^\nu K^{p+1}$.
This induces a corresponding associated ν-th graded complex $(Gr^\nu_F K^\bullet, d)$. Now put

$$F^\nu H^p(K^\bullet) := \frac{F^\nu K^p_{d-\text{closed}}}{F^\nu \cap (dK^{p-1})}.$$

This gives

$$H^p(K^\bullet) = F^0H^p(K^\bullet) \supset F^1H^p(K^\bullet) \supset \cdots \supset F^\nu H^p(K^\bullet) \supset F^{\nu+1}H^p(K^\bullet) \supset \cdots$$

Definition 3.1.1. A spectral sequence is a sequence $\{E_r, d_r\}$, $(r \geq 0)$, of bi-graded groups

$$E_r = \bigoplus_{p,q} E_r^{p,q},$$

with differentials

$$d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}, \quad d_r^2 = 0,$$

such that $H^*(E_r) = E_{r+1}$.

Proposition 3.1.2. *Given a filtered complex* $(K^\bullet, d, F^\bullet)$*, then there exists a spectral sequence* $\{E_r\}$ *with:*

$$E_0^{p,q} = \frac{F^p K^{p+q}}{F^{p+1} K^{p+q}} =: Gr_F^p K^{p+q}$$

$$E_1^{p,q} = H^{p+q}(Gr_F^p K^\bullet)$$

$$E_\infty^{p,q} = Gr_F^p \big(H^{p+q}(K^\bullet) \big)$$

We say that the spectral sequence abuts to $H^\bullet(K^\bullet)$ *and write*

$$E_r \Rightarrow H^{p+q}(K^\bullet).$$

Proof. The $E_0^{p,q}$ term is already defined. Let d_0 be induced by d:

$$
\begin{array}{ccccc}
E_0^{p,q-1} & \xrightarrow{d_0} & E_0^{p,q} & \xrightarrow{d_0} & E_0^{p,q+1} \\
\Big\| & & \Big\| & & \Big\| \\
\frac{F^pK^{p+q-1}}{F^{p+1}K^{p+q-1}} & \xrightarrow{d} & \frac{F^pK^{p+q}}{F^{p+1}K^{p+q}} & \xrightarrow{d} & \frac{F^pK^{p+q+1}}{F^{p+1}K^{p+q+1}}
\end{array}
$$

Then $E_1^{p,q}$ is by definition the cohomology in the middle part, which is precisely $H^{p+q}(Gr_F^p K^\bullet)$. Next, let us define

$$
E_r^{p,q} := \frac{\left\{\xi \in F^pK^{p+q} \,\middle|\, d\xi \in F^{p+r}K^{p+q+1}\right\}}{\left\{d\left(F^{p-r+1}K^{p+q-1}\right) + F^{p+1}K^{p+q}\right\} \bigcap \text{Numerator}},
$$

which is consistent with $E_0^{p,q}$ and $E_1^{p,q}$. Obviously, for $r >> 1$.

$$
E_r^{p,q} = E_\infty^{p,q} = \frac{\left\{\xi \in F^pK^{p+q} \,\middle|\, d\xi = 0\right\}}{\left\{d\left(K^{p+q-1}\right) + F^{p+1}K^{p+q}\right\} \bigcap \text{Numerator}}
$$

$$
=: Gr_F^p H^{p+q}(K^\bullet).
$$

Therefore it suffices to show that for all $r \geq 0$:

$$E_{r+1}^{p,q} = \frac{\ker d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}}{d_r\left(E_r^{p-r,q+r-1}\right)}.$$

But this follows from the definitions (drop "$\bigcap$ Numerator" for notational convenience):

$$\begin{array}{ccc}
E_r^{p-r,q+r-1} & == & \frac{\left\{\xi \in F^{p-r}K^{p+q-1} \,\middle|\, d\xi \in F^pK^{p+q}\right\}}{d\left(F^{p-2r+1}K^{p+q-2}\right)+F^{p-r+1}K^{p+q-1}} \\
d_r\Big\downarrow & & \Big\downarrow d \\
E_r^{p,q} & == & \frac{\left\{\xi \in F^pK^{p+q} \,\middle|\, d\xi \in F^{p+r}K^{p+q+1}\right\}}{d\left(F^{p-r+1}K^{p+q-1}\right)+F^{p+1}K^{p+q}} \\
d_r\Big\downarrow & & \Big\downarrow d \\
E_r^{p+r,q-r+1} & == & \frac{\left\{\xi \in F^{p+r}K^{p+q+1} \,\middle|\, d\xi \in F^{p+2r}K^{p+q+2}\right\}}{d\left(F^{p+1}K^{p+q}\right)+F^{p+r+1}K^{p+q+1}}
\end{array}$$

where

$$E_{r+1}^{p,q} = \frac{\left\{\xi \in F^pK^{p+q} \,\middle|\, d\xi \in F^{p+r+1}K^{p+q+1}\right\}}{d\left(F^{p-r}K^{p+q-1}\right) + F^{p+1}K^{p+q}}$$

$\square$

3.2 Double complexes and the Grothendieck spectral sequences

Again, for simplicity of notation, we will assume non-negative indices.
Consider a (bounded) double complex

$$K^{\bullet,\bullet} = \bigoplus_{p,q\geq 0} K^{p,q},\ d : K^{p,q} \to K^{p+1,q},\ \delta : K^{p,q} \to K^{p,q+1},$$

with

$$d^2 = \delta^2 = 0,\ d\delta + \delta d = 0.$$

We can form the associated single complex

$$\mathbf{s}K^n := \bigoplus_{p+q=n} K^{p,q},\ D = d + \delta,$$

where we observe that

$$D^2 = d^2 + \delta^2 + d\delta + \delta d = 0.$$

The total complex $(\mathbf{s}K^\bullet, D)$ has two descending filtrations, viz.,

$$'F^\nu \mathbf{s}K^n := \bigoplus_{p+q=n, p\geq\nu} K^{p,q}$$

$$''F^\nu \mathbf{s}K^n := \bigoplus_{p+q=n, q\geq\nu} K^{p,q}.$$

This automatically leads to two spectral sequences:

$$'E_r \Rightarrow H_D^{p+q}(\mathbf{s}K^\bullet)$$

$$''E_r \Rightarrow H_D^{p+q}(\mathbf{s}K^\bullet).$$

Note that

$$'E_1^{p,q} = H_D^{p+q}(Gr^p_{'F}\mathbf{s}K^\bullet) = H_\delta^q(K^{p,\bullet}),$$

$$''E_1^{p,q} = H_D^{p+q}(Gr^p_{''F}\mathbf{s}K^\bullet) = H_d^q(K^{\bullet,p}).$$

Note that

$$D = d + \delta = \begin{cases} d & \text{on } {}'E_1 \\ \delta & \text{on } {}''E_1 \end{cases},$$

hence

$$d_1 = d : H_\delta^q(K^{p,\bullet}) = {}'E_1^{p,q} \to {}'E_1^{p+1,q} = H_\delta^q(K^{p+1,\bullet})$$

$$d_1 = \delta : H_d^q(K^{\bullet,p}) = {}''E_1^{p,q} \to {}''E_1^{p+1,q} = H_d^q(K^{\bullet,p+1}).$$

Therefore

$${}'E_2^{p,q} = H_d^p\big(H_\delta^q(K^{\bullet,\bullet})\big),$$

$${}''E_2^{p,q} = H_\delta^p\big(H_d^q(K^{\bullet,\bullet})\big).$$

3.3 Hypercohomology

Let $(\mathcal{S}^{\bullet \geq 0}, d)$ be a (bounded) complex of sheaves on X. One has a Cech double complex

$$\big(C^\bullet(\mathcal{U}, \mathcal{S}^\bullet),\ d,\ \delta\big),$$

where $\mathcal{U}$ is an open cover of X. The k-th hypercohomology is given by the k-th total cohomology of the associated single complex

$$\big(M^\bullet := \oplus_{i+j=\bullet} C^i(\mathcal{U}, \mathcal{S}^j),\ D = d \pm \delta\big),$$

viz.,

$$\mathbb{H}^k(\mathcal{S}^\bullet) := \lim_{\overrightarrow{\mathcal{U}}} H^k(M^\bullet).$$

Associated to the double complex are two filtered subcomplexes of the associated single complex, with two associated Grothendieck spectral sequences abutting to $\mathbb{H}^k(\mathcal{S}^\bullet)$ (where $p + q = k$):

$${}'E_2^{p,q} := H_\delta^p(X, \mathcal{H}_d^q(\mathcal{S}^\bullet))$$

$${}''E_2^{p,q} := H_d^p(H_\delta^q(X, \mathcal{S}^\bullet)).$$

The first spectral sequence shows that quasi-isomorphic complexes yield the same hypercohomology. The second spectral sequence is generally used for calculations.

Alternate take. Two complexes of sheaves $\mathcal{K}_1^\bullet, \mathcal{K}_2^\bullet$ are said to be quasi-isomorphic if there is a morphism $h : \mathcal{K}_1^\bullet \to \mathcal{K}_2^\bullet$ inducing an isomorphism on cohomology $h_* : \mathcal{H}^\bullet(\mathcal{K}_1^\bullet) \xrightarrow{\sim} \mathcal{H}^\bullet(\mathcal{K}_2^\bullet)$.

Take a complex of acyclic sheaves $(\mathcal{K}^\bullet, d)$ (viz., $H^{i>0}(X, \mathcal{K}^j) = 0$ for all j) quasi-isomorphic to $\mathcal{S}^\bullet$.

Then from the second spectral sequence,

$$\mathbb{H}^i(\mathcal{S}^\bullet) := H^i\big(\Gamma(\mathcal{K}^\bullet)\big),$$

where in this situation we define $\Gamma(\mathcal{K}^\bullet) := \Gamma(X, \mathcal{K}^\bullet) := H^0(X, \mathcal{K}^\bullet)$. For example if $\mathcal{L}^{\bullet,\bullet}$ is an [double complex] acyclic resolution of $\mathcal{S}^\bullet$, then the associated single complex $\mathcal{K}^\bullet = \oplus_{i+j=\bullet}\mathcal{L}^{i,j}$ is acyclic and quasi-isomorphic to $\mathcal{S}^\bullet$.

3.4 Deligne cohomology

Let $\mathbb{A} \subseteq \mathbb{R}$ be a subring and $r \geq 0$ an integer. We recall the Tate twist $\mathbb{A}(r) = (2\pi \mathrm{i})^r \cdot \mathbb{A}$, and declare $\mathbb{A}(r)$ a pure $\mathbb{A}$-Hodge structure of weight $-2r$ and of (pure) Hodge type $(-r, -r)$.

We introduce the Deligne complex $\mathbb{A}_{\mathbb{D}}(r)$:

$$\mathbb{A}(r) \to \mathcal{O}_X \xrightarrow{d} \Omega_X \xrightarrow{d} \cdots \xrightarrow{d} \Omega_X^{r-1}.$$

Definition 3.4.1. Deligne cohomology[1] is given by the hypercohomology:

$$H^i_{\mathbb{D}}(X, \mathbb{A}(r)) := \mathbb{H}^i(\mathbb{A}_{\mathbb{D}}(r)).$$

Example 3.4.2. *When $\mathbb{A} = \mathbb{Z}$, we have a quasi-isomorphism*

$$\mathbb{Z}_{\mathbb{D}}(1) \approx \mathcal{O}_X^{\times}[-1],$$

hence

$$H^2_{\mathbb{D}}(X, \mathbb{Z}(1)) \simeq H^1(X, \mathcal{O}_X^{\times}) =: \mathrm{Pic}(X) \simeq CH^1(X).$$

$$H^1_{\mathbb{D}}(X, \mathbb{Z}(1)) \simeq H^0(X, \mathcal{O}_X^{\times}) \simeq \mathbb{C}^{\times} \simeq H^0_{Zar}(X, \mathcal{K}_{1,X}) =: CH^1(X, 1).$$

[1]This definition applies to *any* complex manifold, not just projective algebraic X. It is the definition of *analytic* Deligne cohomology.

Chapter 4

Lewis Filtration

4.1 Leray Filtration

This is one of the most versatile spectral sequences in the literature. First some business about a push-forward.

Let $f : X \to Y$ be a continuous map of 'nice' spaces, and $\mathcal{F}$ a sheaf on X. The push-forward

$f_*\mathcal{F}$ (or direct image sheaf) is the sheaf on Y given by

$$U \subset Y \text{ open } \mapsto f_*\mathcal{F}(U) := \mathcal{F}(f^{-1}(U)).$$

Assume given a flasque resolution of $\mathcal{F}$, viz.,

$$0 \to \mathcal{F} \to \mathbb{A}^\bullet.$$

Note that $f_*\mathbb{A}$ is flasque for any flasque sheaf $\mathbb{A}$ on X. Furthermore because of flasqueness,

$$\mathbb{H}^i(f_*\mathbb{A}^\bullet) = H^i\big(\Gamma(Y, f_*\mathbb{A}^\bullet)\big) = H^i\big(\Gamma(X, \mathbb{A}^\bullet)\big) \simeq H^i(X, \mathcal{F}).$$

The E_2-term of one of the Grothendieck spectral sequences associated to $\mathbb{H}^i(f_*\mathbb{A}^\bullet)$ is again, via flasqueness:

$$E_2^{p,q} = H^p(X, \mathcal{H}^q(f_*\mathbb{A}^\bullet)) = H^p(X, R^q f_*\mathcal{F}) \Rightarrow H^{p+q}(X, \mathcal{F}).$$

Keep in mind that $R^q f_*\mathcal{F}$, called the Leray cohomology sheaf, is really the sheaf associated to the presheaf:

$$U \subset Y \text{ open } \mapsto H^q(f^{-1}(U), \mathcal{F}).$$

4.2 Lewis Filtration

The complexity of Chow groups is measured in terms of a filtration.

Theorem 4.2.1. *[9] Let $X/\mathbb{C}$ be smooth projective of dimension d. Then for all r, there is a filtration,*

$$CH^r(X;\mathbb{Q}) = F^0 \supset F^1 \supset \cdots \supset F^\nu \supset F^{\nu+1} \supset$$

$$\cdots \supset F^r \supset F^{r+1} = F^{r+2} = \cdots ,$$

which satisfies the following

(i) $F^1 = CH^r_{\text{hom}}(X;\mathbb{Q})$.

(ii) $F^2 \subset \ker AJ_X \otimes \mathbb{Q} : CH^r_{\text{hom}}(X;\mathbb{Q}) \to J\big(H^{2r-1}(X,\mathbb{Q}(r))\big)$.

(iii) $F^{\nu_1} CH^{r_1}(X;\mathbb{Q}) \bullet F^{\nu_2} CH^{r_2}(X;\mathbb{Q}) \subset F^{\nu_1+\nu_2} CH^{r_1+r_2}(X;\mathbb{Q})$, *where $\bullet$ is the intersection product.*

(iv) F^ν *is preserved under the action of correspondences between smooth projective varieties over $\mathbb{C}$.*

(v) *Let $gr_F^\nu := F^\nu / F^{\nu+1}$ and assume that the Künneth components of the diagonal class $[\Delta_X] = \oplus_{p+q=2d}[\Delta_X(p,q)] \in H^{2d}(X \times X, \mathbb{Q}(d))$ are algebraic. Then*

$$\Delta_X(2d-2r+\ell, 2r-\ell)_*\big|_{gr_F^\nu CH^r(X;\mathbb{Q})} = \delta_{\ell,\nu} \cdot \text{Identity}.$$

(vi) *Let* $D^r(X) := \bigcap_\nu F^\nu$*. If BBC+HC holds then* $D^r(X) = 0$*.*

4.2.1 Approach via Lewis

Consider Deligne complex $\mathbb{A}_{\mathbb{D}}(r)$:

$$\mathbb{A}(r) \to \mathcal{O}_X \xrightarrow{d} \Omega_X \xrightarrow{d} \cdots \xrightarrow{d} \Omega_X^{r-1}.$$

So we have a short exact sequence

$$0 \to \Omega_X^{\bullet<r}[-1] \to \mathbb{A}_{\mathbb{D}}(r) \to \mathbb{A}(r) \to 0,$$

$\Omega_X^{\bullet<r} = \Omega_X^\bullet/\Omega_X^{\bullet\geq r}$.

And $\mathbb{H}^i(\Omega_X^\bullet) = H^i(X, \mathbb{C})$.

We have a fact that $H^i(F^p\Omega_X^\bullet) = F^pH^i(X, \mathbb{C})$.

And $H^i(\Omega_X^{\bullet<r}) = H^i(X, \mathbb{C})/F^rH^i(X, \mathbb{C})$.

Now from

$$0 \to \Omega_X^{\bullet<r}[-1] \to \mathbb{A}_{\mathbb{D}}(r) \to \mathbb{A}(r) \to 0,$$

we have

$$\mathbb{H}^{2r}(\Omega_X^{\bullet<r}[-1]) \to \mathbb{H}^{2r}(\mathbb{A}_{\mathbb{D}}(r)) \to \mathbb{H}^{2r}(\mathbb{A}(r)) \to \mathbb{H}^{2r+1}(\Omega_X^{\bullet<r}[-1]),$$

But

$\mathbb{H}^{2r+1}(\Omega_X^{\bullet<r}[-1]) = \mathbb{H}^{2r}(\Omega_X^{\bullet<r}) = \frac{H^{2r}(X,\mathbb{C})}{F^rH^{2r}(X,\mathbb{C})}$.

So

$$\frac{H^{2r-1}(X,\mathbb{C})}{F^rH^{2r-1}(X,\mathbb{C}) + H^{2r-1}(X,\mathbb{A}(r))} \to H^{2r}_{\mathbb{D}}(X,\mathbb{A}(r)) \to H^{2r}(X,\mathbb{A}(r)) \to \frac{H^{2r}(X,\mathbb{C})}{F^rH^{2r}(X,\mathbb{C})}.$$

Putting $\mathbb{Q}$ instead of $\mathbb{A}$ getting that there is a short exact sequence:

$$0 \to J(H^{2r-1}(X,\mathbb{Q}(r))) \to H^{2r}_{\mathbb{D}}(X,\mathbb{Q}(r)) \to \Gamma(H^{2r}(X,\mathbb{Q}(r))) \to 0.$$

Suppose X is a quasi-projective, Beilinson introduces an absolute Hodge cohomology $H^{2r}_{\mathcal{H}}(\mathcal{X},\mathbb{Q}(r))$, very similar to Deligne cohomology, and shows that we have the following short exact sequence:

$$0 \to J(H^{2r-1}(X,\mathbb{Q}(r))) \to H^{2r}_{\mathcal{H}}(X,\mathbb{Q}(r)) \to \Gamma(H^{2r}(X,\mathbb{Q}(r))) \to 0.$$

Consider a $\overline{\mathbb{Q}}$-spread $\rho : \mathcal{X} \to \mathcal{S}$, where ρ is smooth and proper. Let η be the generic point of $\mathcal{S}$, and put $K := \overline{\mathbb{Q}}(\eta)$.

Write $X_K := \mathcal{X}_\eta$ then $X = \mathcal{X}_\eta \times \mathbb{C} = X_K \times \mathbb{C}$. From [9] we introduced a decreasing filtration $\mathcal{F}^\nu \mathrm{CH}^r(\mathcal{X};\mathbb{Q})$, with the property that $Gr^\nu_{\mathcal{F}} \mathrm{CH}^r(\mathcal{X};\mathbb{Q}) \hookrightarrow E_\infty^{\nu,2r-\nu}(\rho)$, where $E_\infty^{\nu,2r-\nu}(\rho)$ is the ν-th graded piece of the Leray filtration on the lowest weight part $\underline{H}^{2r}_{\mathcal{H}}(\mathcal{X},\mathbb{Q}(r))$ of Beilinson's absolute Hodge cohomology $H^{2r}_{\mathcal{H}}(\mathcal{X},\mathbb{Q}(r))$ associated to ρ.

That lowest weight part $\underline{H}^{2r}_{\mathcal{H}}(\mathcal{X},\mathbb{Q}(r)) \subset H^{2r}_{\mathcal{H}}(\mathcal{X},\mathbb{Q}(r))$ is given by the image $H^{2r}_{\mathcal{H}}(\overline{\mathcal{X}},\mathbb{Q}(r)) \to H^{2r}_{\mathcal{H}}(\mathcal{X},\mathbb{Q}(r))$, where $\overline{\mathcal{X}}$ is a smooth compactification of $\mathcal{X}$.

There is a cycle class map $\mathrm{CH}^r(\mathcal{X};\mathbb{Q}) := \mathrm{CH}^r(\mathcal{X}/\overline{\mathbb{Q}};\mathbb{Q}) \to \underline{H}^{2r}_{\mathcal{H}}(\mathcal{X},\mathbb{Q}(r))$, which is conjecturally injective under the Bloch-Beilinson conjecture assumption, and using the fact that there is a short exact sequence

$$0 \to J(H^{2r-1}(\mathcal{X},\mathbb{Q}(r))) \to H^{2r}_{\mathcal{H}}(\mathcal{X},\mathbb{Q}(r)) \to \Gamma(H^{2r}(\mathcal{X},\mathbb{Q}(r))) \to 0.$$

To see this, consider the diagram,

$$\begin{array}{ccccccccc}
0 & \to & \mathrm{CH}^r_{hom}(X,\mathbb{Q}) & \to & \mathrm{CH}^r(X,\mathbb{Q}) & \to & \mathrm{CH}^r(X,\mathbb{Q})/\mathrm{CH}^r_{hom}(X,\mathbb{Q}) & \to & 0 \\
 & & \downarrow AJ & & \downarrow \psi_r & & \downarrow cl_r & & \\
0 & \to & J(H^{2r-1}(\mathcal{X},\mathbb{Q}(r))) & \to & H^{2r}_{\mathcal{H}}(\mathcal{X},\mathbb{Q}(r)) & \to & \Gamma(H^{2r}(\mathcal{X},\mathbb{Q}(r))) & \to & 0
\end{array}$$

(*Let* $\xi \in \mathrm{CH}^r(\mathcal{X}/\overline{\mathbb{Q}};\mathbb{Q})$ goes to $0 \in \underline{H}^{2r}_{\mathcal{H}}(\mathcal{X},\mathbb{Q}(r))$ then it will go to $0 \in \Gamma(H^{2r}(\mathcal{X},\mathbb{Q}(r)))$
and therefore, $\xi \in CH^r_{hom}(\mathcal{X},\mathbb{Q}(r))$ so it will go to $0 \in J(H^{2r-1}(\mathcal{X},\mathbb{Q}(r)))$ by the AJ and hence $\xi = 0$ by BBC).

Regardless of whether or not injectivity holds, the filtration $\mathcal{F}^\nu\mathrm{CH}^r(\mathcal{X};\mathbb{Q})$ is given by the pullback of the Leray filtration on $\underline{H}^{2r}_{\mathcal{H}}(\mathcal{X},\mathbb{Q}(r))$ to $\mathrm{CH}^r(\mathcal{X};\mathbb{Q})$.

Recall that $R^q\rho_*\mathbb{Q}$ is the sheaf associated to the presheaf given by

$U \subset \mathcal{S} \to H^q(\rho^{-1}(U),\mathbb{Q})$.

It is proved in [9] that the term $E^{\nu,2r-\nu}_\infty(\rho)$ fits in a short exact sequence:

$$0 \to \underline{E}^{\nu,2r-\nu}_\infty(\rho) \to E^{\nu,2r-\nu}_\infty(\rho) \to \underline{\underline{E}}^{\nu,2r-\nu}_\infty(\rho) \to 0,$$

where

$$\underline{\underline{E}}^{\nu,2r-\nu}_\infty(\rho) = \Gamma(H^\nu(\mathcal{S},R^{2r-\nu}\rho_*\mathbb{Q}(r))),$$

$$\underline{E}^{\nu,2r-\nu}_\infty(\rho) = \frac{J(W_{-1}H^{\nu-1}(\mathcal{S},R^{2r-\nu}\rho_*\mathbb{Q}(r)))}{\Gamma(Gr^0_W H^{\nu-1}(\mathcal{S},R^{2r-\nu}\rho_*\mathbb{Q}(r)))}$$

$$\subset J(H^{\nu-1}(\mathcal{S},R^{2r-\nu}\rho_*\mathbb{Q}(r))).$$

[Here the latter inclusion is a result of the short exact sequence:

$$W_{-1}H^{\nu-1}(\mathcal{S}, R^{2r-\nu}\rho_*\mathbb{Q}(r)) \hookrightarrow W_0 H^{\nu-1}(\mathcal{S}, R^{2r-\nu}\rho_*\mathbb{Q}(r)) \twoheadrightarrow Gr_W^0 H^{\nu-1}(\mathcal{S}, R^{2r-\nu}\rho_*\mathbb{Q}(r))].$$

One then has (by definition)

$$F^\nu \mathrm{CH}^r(X_K;\mathbb{Q}) = \lim_{\substack{\rightarrow \\ U\subset S/\overline{\mathbb{Q}}}} \mathcal{F}^\nu \mathrm{CH}^r(\mathcal{X}_U;\mathbb{Q}), \quad \mathcal{X}_U := \rho^{-1}(U).$$

$$F^\nu \mathrm{CH}^r(X_{\mathbb{C}};\mathbb{Q}) = \lim_{\substack{\rightarrow \\ K\subset \mathbb{C}}} F^\nu \mathrm{CH}^r(X_K;\mathbb{Q}).$$

Further, since direct limits preserve exactness,

$$Gr_F^\nu \mathrm{CH}^r(X_K;\mathbb{Q}) = \lim_{\substack{\rightarrow \\ U\subset S/\overline{\mathbb{Q}}}} Gr_{\mathcal{F}}^\nu \mathrm{CH}^r(\mathcal{X}_U;\mathbb{Q}),$$

$$Gr_F^\nu \mathrm{CH}^r(X_{\mathbb{C}};\mathbb{Q}) = \lim_{\substack{\rightarrow \\ K\subset \mathbb{C}}} Gr_F^\nu \mathrm{CH}^r(X_K;\mathbb{Q}).$$

4.3 Goals

• We intend to study the complexity of $CH^r(X, \mathbb{Q})/D^r(X)$. This will require the joint work of Lewis-Shuji Saito (See [11]).

• Another filtration $\mathcal{F}^\nu$ naturally occurs. We would like to compare $\mathcal{F}^\nu$ with the Lewis filtration.

• Higher Chow analogues.

One approach to these problems is to look at cases where the "motive" of X degenerates, viz., $\Delta_X \sim_{\text{rat}} \Gamma_0 + \cdots + \Gamma_N$ where the supports of Γ_j are restricted.

• We intend to prove Nori's Conjecture in the sense of Lewis BB filtration.

• We plan to prove Kapil H. Paranjape theorem [13] for the smooth general subvariety by a different method and give some precise information for the minimum values of n.

4.4 Partial results

Let $k \subseteq \mathbb{C}$ be a subfield and denote by $\mathrm{Sm}(k)$ the category of smooth projective (geometrically irreducible) varieties over k.
Now let $X \in \mathrm{Sm}(\mathbb{C})$. The dimension of X will be denoted by d_X. Put $\delta_X(r,\nu) := d_X + \nu - r$. [$F^\nu\mathrm{CH}$ = Lewis filtration, $\mathcal{F}^\nu\mathrm{CH}$ defined below].

4.4.1 A new filtration

Definition 4.4.1. *Let $X \in Sm(\mathbb{C})$ be given. Let us introduce the descending filtration $\{\mathcal{F}^\nu CH^r(X;\mathbb{Q})\}_{\nu\geq 0}$ by the formula*

$$\mathcal{F}^\nu CH^r(X;\mathbb{Q}) = \bigcap_{\substack{Y\in Sm(\mathbb{C}) \\ w\in CH^{\delta_X(r,\nu)-\ell}(X\times Y;\mathbb{Q}) \\ 1\leq\ell\leq\nu}} \big\{\ker w_* : CH^r(X;\mathbb{Q}) \to CH^{\nu-\ell}(Y;\mathbb{Q})\big\}.$$

Claim 4.4.2. $\mathcal{F}^0 CH^r(X;\mathbb{Q}) = CH^r(X;\mathbb{Q})$.

Proof. Obvious. □

Claim 4.4.3. $\mathcal{F}^{r+1} CH^r(X;\mathbb{Q}) = 0$.

Proof. Choose $\ell = 1$, $\nu = r+1$ and $Y = X$. Then the diagonal $\Delta_X \in \mathrm{CH}^{d_X}(X\times X;\mathbb{Q}) = CH^{\delta_X(r,\nu)-\ell}(X\times Y;\mathbb{Q})$, has $\ker\Delta_{X,*} = 0$. □

Thus we have a descending filtration

$$\mathrm{CH}^r(X;\mathbb{Q}) = \mathcal{F}^0 \supset \mathcal{F}^1 \supset \mathcal{F}^2 \supset \cdots \supset \mathcal{F}^r \supset 0.$$

Claim 4.4.4. *$\mathcal{F}^\nu$ is functorial with respect to correspondences between smooth projective varieties.*

Proof. Obvious. □

Definition 4.4.5. *Let's take a codimension r cycle Z on W. We say that Z is numerically equivalent to zero if*

$$< Z, Y >= 0 \ for \ all \ Y > 0 \ and \ Y \in Z_r(W).$$

Claim 4.4.6. $\mathcal{F}^1 CH^r(X;\mathbb{Q}) = CH^r_{\mathrm{num}}(X;\mathbb{Q})$.

Proof. First of all, suppose that $\xi_1 \in \mathrm{CH}^r_{\mathrm{num}}(X)$. Then according to Definition 7.1 , $\nu = \ell = 1$, and for any given pair (Y, w), with $w_* : \mathrm{CH}^r(X;\mathbb{Q}) \to \mathrm{CH}^0(Y;\mathbb{Q})$, we have $w_*(\xi_1) \in \mathrm{CH}^0_{\mathrm{num}}(Y;\mathbb{Q}) = 0$.
Thus $\mathrm{CH}^r_{\mathrm{num}}(X;\mathbb{Q}) \subseteq \mathcal{F}^1\mathrm{CH}^r(X;\mathbb{Q})$.

To arrive at the reverse inclusion, let $\xi_1 \in \mathrm{CH}^r(X;\mathbb{Q})$. If $\xi_1 \notin \mathrm{CH}^r_{\mathrm{num}}(X;\mathbb{Q})$, then by definition of $\mathrm{CH}^r_{\mathrm{num}}(X;\mathbb{Q})$, there exists $\xi_2 \in \mathrm{CH}^{d_X - r}(X;\mathbb{Q})$ such that $\deg(\xi_1 \cap \xi_2) \neq 0$.

Now choose $w = \xi_2 \times X \in \mathrm{CH}^{d_X - r}(X \times X;\mathbb{Q})$ to show that $\xi_1 \notin \mathcal{F}^1\mathrm{CH}^r(X;\mathbb{Q})$. Therefore $\mathcal{F}^1\mathrm{CH}^r(X;\mathbb{Q}) \subseteq \mathrm{CH}^r_{\mathrm{num}}(X;\mathbb{Q})$, and hence $\mathcal{F}^1\mathrm{CH}^r(X;\mathbb{Q}) = \mathrm{CH}^r_{\mathrm{num}}(X;\mathbb{Q})$.

$\square$

Claim 4.4.7. *If $D^r(X) = 0$, then $F^\nu CH^r(X;\mathbb{Q}) \subseteq \mathcal{F}^\nu CH^r(X;\mathbb{Q})$, $\forall r$.*

Proof. Obvious.

$\square$

Claim 4.4.8. *We have*

$$\big\{ \ker AJ_X : CH^r_{\mathrm{hom}}(X;\mathbb{Q}) \to J\big(H^{2r-1}(X,\mathbb{Q}(r))\big)\big\} \subseteq \mathcal{F}^2 CH^r(X;\mathbb{Q}).$$

Proof. Bearing in mind that $\mathcal{F}^2 \subset \mathcal{F}^1$, it makes sense to consider $\ell = 1$, $\nu = 2$, and $w \in \mathrm{CH}^{\delta_X(r,\nu)-\ell}(X \times Y;\mathbb{Q})$.
We end up with a commutative diagram

$$\begin{array}{ccc}
\mathrm{CH}^r_{\mathrm{hom}}(X;\mathbb{Q}) & \xrightarrow{w_*} & \mathrm{CH}^1_{\mathrm{hom}}(Y;\mathbb{Q}) \\
{\scriptstyle AJ_X}\downarrow & & \downarrow{\scriptstyle \wr} \\
J\big(H^{2r-1}(X,\mathbb{Q}(r))\big) & \xrightarrow{[w]_*} & J\big(H^1(Y,\mathbb{Q}(1))\big),
\end{array}$$

where the RHS ($\simeq$) is due to the theory of the Picard variety. The rest is clear.

$\square$

Definition 4.4.9. *Let's take two codimension r cycles Z_1 and Z_2 on W. We say that Z_1 and Z_2 are algebraically equivalent if there exists a curve $\mathcal{E}$ and a codimension r cycle Y in $W \times \mathcal{E}$ such that*

$$Z_1 - Z_2 = (\pi_W)_*(W \times \{c\} \bullet Y) - (\pi_W)_*(W \times \{d\} \bullet Y).$$

where c and d are points on the curve $\mathcal{E}$.

Let $\mathrm{CH}^r_{\mathrm{alg}}(V)$ be the subgroup of $\mathrm{CH}^r(V)$ of those rational equivalence classes which are themselves algebraically equivalent to zero.

Definition 4.4.10. *Let $z \in CH^r_{alg}(V)$ and (Y, S) a couple consisting of a smooth projective variety S, $Y \in CH^{d-r+1}(S \times V)$. Put*

$$Y(z) = pr_S(S \times z) \bullet Y).$$

The cycle class $[z] \in CH^r_{alg}(V)$ is called incidence equivalent to zero if $Y(z) = 0$ for all couples (S, Y).

Corollary 4.4.11. *Under the assumption of the HC,*

$$\mathcal{F}^2 CH^r_{alg}(X;\mathbb{Q}) = CH^r_{alg,\mathrm{inc}}(X;\mathbb{Q}) = CH^r_{alg,AJ}(X;\mathbb{Q}),$$

where the latter equality (requiring the HC) comes from [8].

Corollary 4.4.11 is the original motivation for introducing $\mathcal{F}^\nu$.

There are two conjectures that are of interest:

Conjecture 4.4.12. *The filtration is compatible with products:*

$$\mathcal{F}^{\nu_1} CH^{r_1}(X;\mathbb{Q}) \times \mathcal{F}^{\nu_2} CH^{r_2}(X;\mathbb{Q}) \xrightarrow{\bullet} \mathcal{F}^{\nu_1+\nu_2} CH^{r_1+r_2}(X;\mathbb{Q}),$$

where $\bullet$ is the intersection product.

Conjecture 4.4.13. *(Factoring through the Grothendieck motive.) Let's assume given an algebraic Künneth decomposition of the diagonal class Δ_X, modulo numerical equivalence, viz.,*

$$\Delta_X = \bigoplus_{p+q=2d_X} \Delta_X(p,q).$$

Then

$$\Delta_X(2d_X - 2r + s, 2r - s)_* \Big|_{Gr^{\nu}_{\mathcal{F}} CH^r(X;\mathbb{Q})} = \delta_{s,\nu} \cdot \mathrm{Id}_X,$$

where $\delta_{s,\nu}$ is Kronecker's delta function.

4.5 Nori's conjecture

In this section we prove Nori's conjecture under the assumption of the existence of a Bloch-Beilinson filtration [13].

Consider fields $k \subset K \subset \mathbb{C}$, where K/k is finitely generated. We consider the Bloch-Beilinson filtration constructed in [9] We recall:

Conjecture 4.5.1. *(Hard Lefschetz conjecture) (HLC) Recall the HLT*

$$L_X^{n-i} : H^i(X, \mathbb{Q}) \to H^{2n-i}(X, \mathbb{Q})$$

is an isomorphism. Then for all i satisfying $0 \leq i \leq n$: The inverse

$$\Lambda_X^{n-i} : H^{2n-i}(X, \mathbb{Q}) \to H^i(X, \mathbb{Q})$$

to L_X^{n-i} is algebraic.

Conjecture 4.5.2. *(Nori)*

If X is a smooth projective variety with dimension n and Y is a smooth hyperplane section of X with inclusion map $j : Y \hookrightarrow X$. Then

$$j^* : CH^r(X, \mathbb{Q}) \xrightarrow{\sim} CH^r(Y, \mathbb{Q})$$

if $2r < \dim Y$.

Theorem 4.5.3. *Given the Lewis BB filtration, with all the required assumptions (HC + BBC)), then conjecture (4.5.2) holds.*

Proof. Consider a $\overline{\mathbb{Q}}$-spread $\rho : \mathcal{X} \to \mathcal{S}$, where ρ is smooth and proper, $\mathcal{X}$ and $\mathcal{S}$ are smooth and quasi projective.

Recall that $R^q\rho_*\mathbb{C}$ is the sheaf associated to $\mathcal{S}$ and the operation $U \subset \mathcal{S} \to H^q(\rho^{-1}(U), \mathbb{C})$.
We have short exact sequences:

$$0 \to \underline{E}_\infty^{\nu,2r-\nu}(\rho_X) \to E_\infty^{\nu,2r-\nu}(\rho_X) \to \underline{\underline{E}}_\infty^{\nu,2r-\nu}(\rho_X) \to 0,$$

And

$$0 \to \underline{E}_\infty^{\nu,2r-\nu}(\rho_Y) \to E_\infty^{\nu,2r-\nu}(\rho_Y) \to \underline{\underline{E}}_\infty^{\nu,2r-\nu}(\rho_Y) \to 0,$$

Let $t \in \mathcal{S}$,

Now, we have by HLT

$j^* : H^{2r-\nu}(X_t, \mathbb{Q}) \xrightarrow{\sim} H^{2r-\nu}(Y_t, \mathbb{Q})$, with algebraic inverse $(j^*)^{-1}H^{2r-\nu}(Y_t, \mathbb{Q}) \xrightarrow{\sim} H^{2r-\nu}(X_t, \mathbb{Q})$, using the Hodge conjecture.

But we have by definition

$(R_X^{2r-\nu}\rho_*\mathbb{Q})_t \simeq H^{2r-\nu}(X_t,\mathbb{Q})$.

And

$(R_Y^{2r-\nu}\rho_*\mathbb{Q})_t \simeq H^{2r-\nu}(Y_t,\mathbb{Q})$.

Implies that

$\underline{E}_\infty^{\nu,2r-\nu}(\rho_X) \simeq \underline{E}_\infty^{\nu,2r-\nu}(\rho_Y)$ and
$\underline{\underline{E}}_\infty^{\nu,2r-\nu}(\rho_X) \simeq \underline{\underline{E}}_\infty^{\nu,2r-\nu}(\rho_Y)$.

Then by Five lemma,

$E_\infty^{\nu,2r-\nu}(\rho_X) \simeq E_\infty^{\nu,2r-\nu}(\rho_Y)$.

Now, we have the inclusions $\mathrm{Gr}_F^\nu \mathrm{CH}^r(X,\mathbb{Q}) \subset E_\infty^{\nu,2r-\nu}(\rho_X)$, $G_F^\nu \mathrm{CH}^r(Y,\mathbb{Q}) \subset E_\infty^{\nu,2r-\nu}(\rho_Y)$ and an isomorphism $j^* : E_\infty^{\nu,2r-\nu}(\rho_X) \simeq E_\infty^{\nu,2r-\nu}(\rho_Y)$.

By a commutative diagram

$$\begin{array}{ccc} \mathrm{Gr}_F^\nu \mathrm{CH}^r(X,\mathbb{Q}) & \hookrightarrow & E_\infty^{\nu,2r-\nu}(\rho_X) \\ \downarrow & & \downarrow \wr \\ \mathrm{Gr}_F^\nu \mathrm{CH}^r(Y,\mathbb{Q}) & \hookrightarrow & E_\infty^{\nu,2r-\nu}(\rho_Y) \end{array}$$

, this provides an injection $j^* : G_F^\nu \mathrm{CH}^r(X,\mathbb{Q}) \hookrightarrow G_F^\nu \mathrm{CH}^r(Y,\mathbb{Q})$. But the inverse $(j^*)^{-1}$ is also algebraic by the Hodge conjecture, and determines an injection $(j^*)^{-1} : G^{\nu}_F\mathrm{CH}^r(Y,\mathbb{Q}) \hookrightarrow G^{\nu}_F\mathrm{CH}^r(X,\mathbb{Q})$. This implies the isomorphism: $\mathrm{Gr}^{\nu}_F\mathrm{CH}^r(X,\mathbb{Q}) \simeq Gr^{\nu}_F\mathrm{CH}^r(Y,\mathbb{Q})$.

Consider the short exact sequence,

$$\begin{array}{ccccccccc}
0 & \to & \mathrm{Gr}^{\nu}_F\mathrm{CH}^r(X,\mathbb{Q}) = F^r\mathrm{CH}^r(X,\mathbb{Q}) & \to & F^{r-1}\mathrm{CH}^r(X,\mathbb{Q}) & \to & \mathrm{Gr}^{\nu-1}_F\mathrm{CH}^r(X,\mathbb{Q}) & \to & 0 \\
 & & \downarrow\wr & & \downarrow & & \downarrow\wr & & \\
0 & \to & \mathrm{Gr}^{\nu}_F\mathrm{CH}^r(Y,\mathbb{Q}) = F^r\mathrm{CH}^r(Y,\mathbb{Q}) & \to & F^{r-1}\mathrm{CH}^r(Y,\mathbb{Q}) & \to & \mathrm{Gr}^{\nu-1}_F\mathrm{CH}^r(Y,\mathbb{Q}) & \to & 0
\end{array}$$

By the Five lemma, we have,

$F^{r-1}\mathrm{CH}^r(X,\mathbb{Q}) \simeq F^{r-1}\mathrm{CH}^r(Y,\mathbb{Q})$.

So we get,

$$\begin{array}{ccccccccc}
0 & \to & F^{r-1}\mathrm{CH}^r(X,\mathbb{Q}) & \to & F^{r-2}\mathrm{CH}^r(X,\mathbb{Q}) & \to & \mathrm{Gr}^{\nu-2}_F\mathrm{CH}^r(X,\mathbb{Q}) & \to & 0 \\
 & & \downarrow\wr & & \downarrow & & \downarrow\wr & & \\
0 & \to & F^{r-1}\mathrm{CH}^r(Y,\mathbb{Q}) & \to & F^{r-2}\mathrm{CH}^r(Y,\mathbb{Q}) & \to & \mathrm{Gr}^{\nu-2}_F\mathrm{CH}^r(Y,\mathbb{Q}) & \to & 0
\end{array}$$

By the Five lemma again, we have

$F^{r-2}\mathrm{CH}^r(X,\mathbb{Q}) \simeq F^{r-2}\mathrm{CH}^r(Y,\mathbb{Q}).$

So by induction, we get

$F^{0}\mathrm{CH}^r(X,\mathbb{Q}) \simeq F^{0}\mathrm{CH}^r(Y,\mathbb{Q}).$

Which is equal to

$$\mathrm{CH}^r(X,\mathbb{Q}) \simeq \mathrm{CH}^r(Y,\mathbb{Q}).$$

$\square$

Chapter 5

Fano Varieties

5.1 Introduction to Fano Varieties

Definition 5.1.1. *$G(k+1, N+1) = k+1-$dimensional subspaces $\mathbb{C}^{k+1} \subset \mathbb{C}^{N+1} = k-$dimensional subspaces $\mathbb{P}^k \subset \mathbb{P}^N$ is a Grasmannian space with dimension $(k+1)(N+1-(k+1)) = (k+1)(N-k)$.*

Definition 5.1.2. *Let $X \subset \mathbb{P}^n$ be a variety. Then $\Omega_X(k) = \{\mathbb{P}^k\text{'s} \subset \mathbb{P}^n | \mathbb{P}^k \subset X\} \subset G(k+1, N+1)$ is called the Fano variety of $\mathbb{P}^k$'s in X.*

Theorem 5.1.3. *(Borcea) [2] Let $X \subset \mathbb{P}^{n+r}$ be a generic complete intersection of type $(d_1, ..., d_r)$. Then $\Omega_X(k)$ is non-empty and smooth of pure dimension $\delta = (k+1)(n+r-k) - \sum_{j=1}^{r} \binom{d_j+k}{k}$, provided $\delta \geq 0$ and X is not quadric. In the case X is a quadric, we require $n \geq 2k$. Furthermore, if $\delta > 0$ or if in the case X quadric, $n > 2k$, the $\Omega_X(k)$ is connected (hence irreducible).*

Example 5.1.4. *Let $X \subset \mathbb{P}^3$ be a smooth cubic surface. That is $X = V(f)$, where f is a cubic homogeneous polynomial with a Jacobian of rank 1. Then $dim X = 2$, and $deg X = 3$.*
Lets calculate δ for $k = 1, n = 2, r = 1$ and $d = 3$: $\delta = (1+1)(3-1) - \binom{(3+1)}{1} = 4 - 4 = 0$. Implying $\Omega_X(1)$ consists of points.

In fact, it is a well-known result that there are 27 *lines on a cubic smooth surface on* $\mathbb{P}^3$*. Hence* $\Omega_X(1)$ *consists of* 27 *points.*

Example 5.1.5. *Let* $X \subset \mathbb{P}^5$ *be generic quintic fourfold. That is* $n = dimX = 4$*, and* $d = degX = 5$*,* $r = 1$*.*
Lets calculate δ for $k = 1 : \delta = (1+1)(5-1) - \binom{(5+1)}{1} = 8 - 6 = 2$*. Also* $n - 2k = 4 - 2 > 0$ *so by Borcea Theorem,* $\Omega_X(1)$ *is smooth irreducible of dimension* 2*.*

Now, we will define the cylinder correspondence and cylinder homomorphism.

Definition 5.1.6. *[12]*

$$P(X) = \{(c, x) \in \Omega_X(k) \times X \mid x \in \mathbb{P}^k_c\},$$

is called the cylinder correspondence and the cylinder homomorphism map ϕ_* *is induced by the intersection with* $P(X)$*;* $\phi_* : H^{n-2k}(\Omega_X(k), \mathbb{Q}) \to H^n(X, \mathbb{Q})$ *given by* $\phi_*(\gamma) = Pr_{2,*}((\gamma \times X \cap P(X)))_{\Omega_X(k)\times X}$*.*

It is well-known that via Poincaré duality, ϕ_* defines a cohomological map preserving Hodge structures (see [8], Lecture 7]), and in particular, the image of ϕ_* in $H^n(X, \mathbb{Q})$ defines a subHodge structure of weight $\leq min\{n - 2k, dim\ \Omega_X(k)\}$.

We define the level of the Hodge structure

Definition 5.1.7. *[7]* $Level(H^*(X)) =\ max\{p - q | H^{(p,q)}(X) \neq 0\}$

We conclude that a necessary condition for ϕ_* to be surjective is that:

dim $\Omega_X(k) \geq n - 2k$ where $n - 2k =$ level of $H^n(X, \mathbb{Q})$.

[We can assume that $n-2k \geq 0$, otherwise $H^n(X,\mathbb{Q})=0$].

Definition 5.1.8. *Let X be a projective variety $\subset \mathbb{P}^n$ with dimension d. Then $degX = deg(\mathbb{P}^{n-d} \cap X)$.*

Let $X \subset \mathbb{P}^N$ be a complete intersection of dimension n given by r homogeneous polynomials, this means X is obtained by taking exactly r hypersurface sections of $\mathbb{P}^N$, i.e, $X = V(f_1, ..., f_r)$. By the weak Lefschetz theorem for $i < n$ we have:

$H^i(\mathbb{P}^N, \mathbb{Z}) \overset{\simeq}{\to} H^i(X, \mathbb{Z})$.

The cohomology groups of $\mathbb{P}^N$;

$H^i(\mathbb{P}^N, \mathbb{Z}) = \mathbb{Z}$ if $0 \leq i \leq 2N$ is even and 0 otherwise.

Now by the strong Lefschetz theorem we have : $H^{n-i}(X,\mathbb{Q}) \simeq H^{n+i}(X,\mathbb{Q}), 0 \leq i \leq n$, hence the only nontrivial cohomology of X is $H^n(X,\mathbb{Q})$.

Hence $X \subset \mathbb{P}^{n+r}$ a smooth generic complete intersection of multidegree $(d_1, ..., d_r)$ with $dimX = n$. we have Level $(H^*(X)) = level(H^n(X))$ provided that $H^n(X) \neq 0$.
In fact, we know the value of the level from [1] where the Hodge level of $H^n(X)$ is given by $n-2k$ with

$k = [(n - \sum_{s \neq j} (d_i - 1)/d_s]$

and $d_s = max\{d_1, ..., d_r\}$. Note that this means we have $H^n(X,\mathbb{C}) = F^k H^n(X,\mathbb{C})$.

From now, let $X \subset \mathbb{P}^{n+r}$ be a smooth generic complete intersection of multidegree $(d_1, ..., d_r)$ and Let $Z \subset \mathbb{P}^{n+r+1}$ be a smooth generic complete intersection of multidegree $(d_1, ..., d_r)$ such that $X = Z \cap \mathbb{P}^{n+r}$.

Proposition 5.1.9. *Let $Z \subset \mathbb{P}^{n+r+1}$ be a smooth generic complete intersection of multidegree $(d_1, ..., d_r)$ and assume $l \geq 0$ where $l = k(n+1+r-k) - \sum_{j=1}^{r} \binom{d_j+k}{k}$. Then through every point of Z there passes a $\mathbb{P}^k \subset Z$.*
In particular, there is an l-dimensional family of $\mathbb{P}^k$'s in a general Z passing through a generic point $p \in Z$. Finally , we have $l \geq 0$ if and only if ϕ_* surjectivity condition holds for $X = Z \cap \mathbb{P}^{n+r}$.

Corollary 5.1.10. *Given Z in Proposition 5.1.9. If ϕ_* surjectivity condition holds, then Z is covered by a family of $\mathbb{P}^k$'s; moreover, for a general Z and a generic point $p \in Z$, there passes an l-dimensional family of $\mathbb{P}^k$'s.*

Proof. If ϕ_* surjectivity condition holds then by Proposition 5.1.9 we have $l \geq 0$ and through every point $p \in Z$ there passes a $\mathbb{P}^k_p$ implying $Z = \bigcup_{p \in Z} \mathbb{P}^k_p$. $\square$

We can view X as a hyperplane section of Z with the inclusion map $j : X \hookrightarrow Z$. Then if ϕ_* surjectivity condition holds and by Borcea Theorem and the Corollary 5.1.10 we have:

(i) $\Omega_X(k)$ is non-empty, smooth and of pure dimension $(n-2k)+l = \delta$, where $l \geq 0$ is given in the previous Proposition.

(ii) $\Omega_Z(k)$ is non-empty, smooth and of pure dimension $(n+1-k)+l$.

(iii) Through a generic point $p \in Z$, there passes an ℓ-dimensional family of $\mathbb{P}^k$'s.

Now, consider one of the irreducible components of $\Omega_Z(k)$ which describes a covering family of $\mathbb{P}^k$'s on Z, let us denote it by $\tilde{\Omega_Z}$.
Let Ω_Z be a subvariety cut out by ℓ general hyperplane sections of $\tilde{\Omega_Z}$ and

define $\Omega_X = \Omega_X(k) \cap \Omega_Z$.

For what follows, we need to introduce the dual projective space.

Definition 5.1.11. $\mathbb{P}^{N,*} = \{\mathbb{P}^{N-1}\text{'s} \subset \mathbb{P}^N\} = (G(N, N+1)$, *where* $[a_0, ...a_N] \in \mathbb{P}^{N,*}$ *corresponds to* $\mathbb{P}^{N-1} = V(a_0 z_0 + ... + a_N z_N) \subset \mathbb{P}^N$.

Theorem 5.1.12. *(Bertini's theorem) Let* $X \subset \mathbb{P}^N$ *be a smooth projective variety of dimension* n*. Then there is a non-empty Zariski open subset* $U^* \subset \mathbb{P}^{N,*}$ *such that for any* $t \in U^*$ *:*

(a) ${\mathbb{P}^{N-1}}_t \cap X$ *is smooth (i.e,* ${\mathbb{P}^{N-1}}_t$ *is nowhere tangent to* X.

(b) if $n \geq 2$ *then* ${\mathbb{P}^{N-1}}_t \cap X$ *is irreducible.*

Recall that:

Definition 5.1.13. *Let* $V \in z^{r-1}(W)$ *be irreducible and* $f \in k(V)^*$*, then* $div(f) = (f)_0 - (f)_\infty$ *(zeros minus poles of* f *on* Z*, including multiplicities).*

Let X be a smooth projective variety. Recall:

Definition 5.1.14. *[5] Two divisors* D_1 , D_2 *are linearly (rationally) equivalent if they differ by a principal divisor :* $D_1 - D_2 = (f)$ *for some* $f \in \mathbb{C}(X)^*$.

We recall the standard definition of a linear system.

Definition 5.1.15. *[5] If* $D = \sum_{j=1}^{t} n_j Z_j \in Div(X)$, *then,*

$\mathcal{L}(D) = \{f \in \mathbb{C}(X) | f = 0 \; or \; f \neq 0 \; and \; (f) + D \geq 0, i.e., \; all \; coefficients \; of \; (f) + D \; are \; nonnegative\}$
$= \{f \in \mathbb{C}(X) | ord_{Z_i}(f) \geq -n_i, 1 \leq i \leq t \; and \; ord_Z(f) \geq 0 \; all \; other \; Z\}.$

A complete linear system on a general variety V is defined as the set of all effective divisors linearly equivalent to some given divisor D. It is denoted by $|D|$.

Definition 5.1.16. *[5] A linear system is a subset L of some $|D|$ such that*

$$V = \{f \in \mathbb{C}(X) | f = 0 \; or \; f \neq 0 \; and \; (f) + D \in L\}$$

is a vector space over $\mathbb{C}$. Equivalently, L is a linear subspace of $|D|$ in its structure of projective space. The linear systems $|D|$ themselves are called complete linear systems. The base points of a complete system L are defined by

$$(Base \; pts. \; of \; L) = \cap_{D' \in L}(support \; of \; D').$$

Theorem 5.1.17. *(Second theorem of Bertini) A generic element of a linear system on an algebraic variety X cannot have singular points that are not base points of the linear system or singular points of X.*

By Bertini's theorem we can assume:

(iv) Ω_Z is smooth and irreducible of dimension $n+1-k$.

(v) Ω_X is smooth and of pure dimension $n-2k$.

We have a commutative diagram where the π's and ρ's are projections: We consider the following diagram

$$
\begin{array}{ccccccc}
P(X) & & \xrightarrow{\pi_X} & & X & & \\
& \searrow i & & \pi \nearrow & & \searrow j & \\
& & \tilde{X} & & & & Z \\
& & & \searrow & & \pi_Z \nearrow & \\
\rho_X \Big\downarrow & & \rho \Big\downarrow & & P(Z) & & \\
& & & \swarrow \rho_Z & & & \\
\Omega_X & \hookrightarrow & \Omega_Z & & & &
\end{array} \tag{5.1}
$$

We have:

(a) $\tilde{X} = \pi_Z^{-1}(X)$ is smooth by the second theorem of Bertini.[$\tilde{X}$ is a general member of a linear system on $P(Z)$ with no base points, obtained by the pullback of linear systems on Z to $P(Z)$].

(b) π and π_Z are generically finite to one and onto of degree q say.

(c) $\rho_X : P(X) \to \Omega_X$ and $\rho_Z : P(Z) \to \Omega_Z$ are $\mathbb{P}^k$ bundles. (i.e $\rho_X^{-1}(c) = \mathbb{P}_c^k \ since \ \rho_X^{-1}(c) = \{(c,p) | p \in \mathbb{P}_c^k\} \ and \ P(X) \ is \ a \ manifold.$

(e) $dimX = dim\tilde{X} = n, \ dimZ = dimP(Z) = n+1, \ dimP(X) = n-k, \ dim\Omega_X = n-2k, \ dim\Omega_Z = n-k+1$ [Note that all varieties here are smooth].

Definition 5.1.18. *[7] A hypersurface is general if it corresponds to a point in a non-empty Zariski open subset of* $\mathbb{P}^{\binom{n+1+d}{d}-1}$ *governed by certain generic properties (eg. smoothness of the Fano variety of* $\mathbb{P}^k$*'s on* X*, etc.).*

Example 5.1.19. *To illustrate the general idea, we give a cohomological example. Assume given a general (in the sense of the Zariski topology) projective variety* $X \subset \mathbb{P}^5$ *be generic quintic fourfold. That is* $n = dimX = 4$*, and* $d = degX = 5$*.*
We have $k = [5/5] = 1$ *and a general hypersurface* $Z \subset \mathbb{P}^6$ *with* $dimZ = 5$ *and* $degZ = 5$ *such that* $Z \cap \mathbb{P}^5 = X$*.*

$$\Omega_X = \{\mathbb{P}^1 \subset \mathbb{P}^5 \mid \mathbb{P}^1 \subset X\}.$$

Furthermore, $\dim \Omega_X = (1+1)(5-1) - \binom{(5+1)}{1} = 8 - 6 = 2.$

Recall

$$\Omega_Z = \{\mathbb{P}^1 \subset \mathbb{P}^6 \mid \mathbb{P}^1 \subset Z\}.$$

Then $\dim \Omega_Z = 4$*. Also recall*

$$P(X) = \{(c, x) \in \Omega_X \times X \mid x \in \mathbb{P}^1_c\},$$

and

$$P(Z) = \{(c, z) \in \Omega_Z \times Z \mid z \in \mathbb{P}^1_c\}.$$

$$\pi_X : P(X) \to X, \ \rho_X : P(X) \to \Omega_X.$$

We refer to diagram (5.1).

Let $c \in \Omega_Z$*. Then* $\mathbb{P}^1_c$ *either meets* X *in a single point (hence* $c \in \Omega_Z \backslash \Omega_X$*) or* $\mathbb{P}^1_c$ *lies in* X *(hence* $c \in \Omega_X$*). Recall* $\tilde{X} = \pi_Z^{-1}(X)$*, and diagram:*

$$\begin{array}{ccccc}
\tilde{X}\backslash P(X) & \hookrightarrow & \tilde{X} \simeq B_{\Omega_X}(\Omega_Z) & \hookleftarrow & P(X) \\
\wr\downarrow & & \rho\downarrow & & \downarrow\rho_X \\
\Omega_Z\backslash\Omega_X & \hookrightarrow & \Omega_Z & \hookleftarrow & \Omega_X
\end{array}$$

It follows that from the fact that $\tilde{X} \simeq B_{\Omega_X}(\Omega_Z)$ and from the diagram 5.1 that

$$H_4(\tilde{X}, \mathbb{Q}) \simeq H_4(\Omega_Z, \mathbb{Q}) \oplus H_2(\Omega_X, \mathbb{Q}) \simeq \rho^* H_4(\Omega_Z, \mathbb{Q}) \oplus \rho_X^* H_2(\Omega_X, \mathbb{Q}).$$

Now by 5.1(b), $\pi_* \circ \pi^* = \times q$, and therefore $\pi_* : H_4(\tilde{X}, \mathbb{Q}) \to H_4(X, \mathbb{Q})$ is surjective.

Under the decomposition of $H_4(\tilde{X}, \mathbb{Q})$, the surjective morphisim $\pi_* : H_4(\tilde{X}, \mathbb{Q}) \to H_4(X, \mathbb{Q})$ is a sum of two morphisms:

$$\pi_* : \rho^*(H_4(\Omega_Z, \mathbb{Q})) \to H_4(X, \mathbb{Q})$$

$$\pi_* : i_* \circ \rho_X^*(H_2(\Omega_X, \mathbb{Q})) \to H_4(X, \mathbb{Q}).$$

Now, we have from the diagram 5.1:

$$\pi_* \circ \rho^* = \pi_* \circ (\rho_Z \circ j)^* = \pi_* \circ j^* \circ \rho_Z^* = j^* \circ \pi_{Z,*} \circ \rho_Z^*.$$

Hence $\pi_* \circ \rho^*(H_4(\Omega_Z, \mathbb{Q})) \subset j^*(H_4(Z, \mathbb{Q}))$.

Let H_X be a hyperplane section of X. Then applying the weak Lefschetz theorem to the inclusion map j, we deduce that

$j^*(H_4(Z,\mathbb{Q}))\overset{\mathrm{PD}}{\simeq} j^*(H^4(Z,\mathbb{Q})) = \mathbb{Q}H_X \wedge H_X$. Hence $\pi_* \circ \rho^*(H_4(\Omega_Z,\mathbb{Q})) = \mathbb{Q}H_X \wedge H_X$.

And then since $H_4(H_X^2,\mathbb{Q}) \simeq \mathbb{Q}H_X^2 \simeq \mathbb{Q}$, we conclude that the cylinder homomorphism

$\phi_* : H_2(\Omega_X,\mathbb{Q}) \to H_4(X,\mathbb{Q})/H_4(H_X^2,\mathbb{Q})$ is surjective.

Note that to extend these ideas to Chow groups, Z has to be choosen carefully with regard to a general X.

5.2 Some evidence towards Nori's theorem

Theorem 5.2.1. *Given integers $1 \leq d_1 \leq d_2 \leq ... \leq d_r$ and any nonnegative integer l, Let $X \subset \mathbb{P}^{n+r}$ be a smooth general subvariety of multidegree $(d_1, ..., d_r)$. If n is sufficiently large then*

$$CH^l(X)_{\mathbb{Q}} \simeq \mathbb{Q}.$$

Proof. Let $X \subset \mathbb{P}^{n+r}$ be an n-dimensional smooth complete intersection with multidegree $(d_1, ..., d_r)$, *i.e*, $X = V(F_1, ..., F_r)$ where F_i are homogeneous polynomials of degree d_i, and $Z \subset \mathbb{P}^{n+r+1}$ such that $Z \cap V(z_{n+r+1}) = Z \cap \mathbb{P}^{n+r} = X$, $Z = V(G_1, ..., G_r)$ where $G_i = F_i$ for all $i = 1, ..., r-1$ and $G_r = F_r + z_{n+r+1}^{d_r}$.

Now consider the projection from $[0, ..., 0, 1]$, $\nu : Z \to \mathbb{P}^{n+r}$. Let $W = \nu(Z)$ that is if $p = [p_0, ..., p_{n+r+1}] \in Z$ then $\nu(p) = [p_0, ..., p_{n+r}] \in W$.

Note that, $p = [p_0, ..., p_{n+r+1}] \in Z$ means $F_i(p_0, ..., p_{n+r}) = 0$ for $i = 1, ..., r-1$, and $F_r(p_0, ..., p_{n+r}) + p_{n+r+1}^{d_r} = 0$. There are two possibilities:

1. $p_{n+r+1} = 0 \Rightarrow [p_0, ..., p_{n+r}] \in X$.

2. $p_{n+r+1} \neq 0 \Rightarrow [p_0, ..., p_{n+r+1}] \notin X$.

Hence $X \subset W$ and $W = V(F_1, ..., F_{r-1})$.

Now consider the inclusion maps $j : X \hookrightarrow Z$, $i : X \hookrightarrow W$.

Proposition 5.2.2. *Let X, Z, and W be given as above. Then, as correspondences, the following diagram is commutative*

$$\begin{tikzcd}
X \arrow[rr, "j"] \arrow[dr, "i"'] & & Z \arrow[dl, "\nu"] \\
& W &
\end{tikzcd}$$

Proof. $\nu j = i$: Let $[p_0, ..., p_{n+r}] \in X$, *then* $\nu j([p_0, ..., p_{n+r}]) = \nu([p_0, ..., p_{n+r}, 0]) = [p_0, ..., p_{n+r}] \in W$. Also $i([p_0, ..., p_{n+r}]) = [p_0, ..., p_{n+r}]$. □

Proposition 5.2.3. *Let X, Z, and W be given as above. Then, as correspondences, the following diagram is commutative*

$$\begin{tikzcd}
Z \arrow[rr, "d_r j^*"] \arrow[dr, "\nu_*"'] & & X \\
& W \arrow[ur, "i^*"'] &
\end{tikzcd}$$

Proof. We have to show that $i^*\nu_* = d_r j^*$: Let $p \in W$, we can see that $\nu^{-1}(p) = p$ if $p \in i(X)$ and d_r distinct points if $p \notin i(X)$ so $deg(\nu) = d_r$.

Note that $(graph)^t(i) \circ graph(\nu) = (graph)^t(j)$ as sets in $Z \times X$ and $(graph)^t(j)$ is irreducible since X is.
So as varieties there are multiplicities: $i^* \circ \nu_* = l.j^*$ for some $l \in \mathbb{N}$.

To show that $l = d_r$, let's consider $l.X = < l.j^*(Z), X >_X = < i^* \circ \nu_*(Z), X >_X = < i^*(deg(\nu)).W, X >_X = d_r < i^*W, X > = d_r.X$.

□

Proposition 5.2.4. *Let $X \subset \mathbb{P}^{n+r}$ be a general complete intersection with multidegree $(d_1, ..., d_r)$ satisfying that dim $\Omega_X(k) \geq n - 2k$ and let $Z \subset \mathbb{P}^{n+r+1}$ be given as above. Then $\Omega_Z(k)$ is smooth and of pure dimension $(n+1-k)+\ell$, where ℓ is given in proposition [5.1.9].(see [14] for the proof).*

Claim 5.2.5. *If $X = V(F_1, ..., F_r) \subset \mathbb{P}^{n+r}$ is an n-dimensional smooth complete intersection with multidegree $(d_1, ..., d_r)$ and $W = V(F_1, ..., F_{r-1}) \subset \mathbb{P}^{n+r}$, described as above, is a smooth complete intersection with multidegree $(d_1, ..., d_{r-1})$ then $Z = V(F_1, ..., F_{r-1}, F_r + z_{n+r+1}^{d_r}) \subset \mathbb{P}^{n+r+1}$ is an n+1-dimensional smooth complete intersection with multidegree $(d_1, ..., d_r)$.*

Proof. Note that a variety $Y \subset \mathbb{P}^N$ is smooth if and only if $Cone(Y) - 0 \subset \mathbb{C}^{N+1}$ is smooth.

From our assumptions $Cone(X) - 0 = V(F_1, ..., F_r) \subset \mathbb{C}^{n+r+1}$ and $Cone(W) - 0 = V(F_1, ..., F_{r-1}) \subset \mathbb{C}^{n+r}$ are smooth complete intersections. That means that the Jacobian of both have full rank at every point of theirs.

Lets consider the Jacobian of $Cone(Z) - 0$:

Jacobian of $Cone(Z) - 0 =$

$$\begin{bmatrix} & 0 \\ Jacobian\ of\ Cone(W) - 0 & . \\ & . \\ & 0 \\ & \\ (dF_r/dz_0)...(dF_r/dz_{n+r}) & d_r z_{n+r+1}^{d_r} \end{bmatrix}$$

Now let $p = [p_0, ..., p_{n+r+1}]$ be any point in Z. Then

1. If $p_{n+r+1} = 0$ then from the proof of the last Proposition, $[p_0, ..., p_{n+r}] \in X \cap W$ and

Jacobian of $Cone(Z) - 0 =$

$$\begin{bmatrix} & 0 \\ \textit{Jacobian of } \text{Cone(X)-0} & . \\ & . \\ & \\ & 0 \end{bmatrix}$$

Then in this case the Jacobian of $Cone(Z) - 0$ will have full rank at $[p_0, ..., p_{n+r+1}]$ as the Jacobian of $Cone(X) - 0$ has rank r at $[p_0, ..., p_{n+r}]$.

2. If $p_{n+r+1} \neq 0$ then from the proof of Proposition 5.2.2 and the fact that $[0, ..., 0, 1] \notin Z$, $[p_0, ..., p_{n+r}] \in W - X$.
Then the Jacobian of $Cone(W) - 0$ will have rank r-1 at $[p_0, ..., p_{n+r}]$ implying the Jacobian of $Cone(Z) - 0$ will have full rank at $[p_0, ..., p_{n+r+1}]$.

Hence, in any case the Jacobian of $Cone(Z)-0$ will have full rank at every point of $Cone(Z)-0$, implying $Z = V(F_1, ..., F_{r-1}, F_r + z_{n+r+1}^{d_r}) \subset \mathbb{P}^{n+r+1}$ is an n+1-dimensional smooth complete intersection with multidegree $(d_1, ..., d_r)$ ([8], Lecture 1 , 1.17).

$\square$

By taking hyperplane sections, we can assume that $dim\ \Omega_X = n - 2k$ and $dim\ \Omega_Z = n - k + 1$.

We refer to diagram (5.1), where it is known from [14] that $\tilde{X}$ is still smooth, despite our specially choosen Z for a general X.

$dim\ \pi_X(P(X)) = n - k = \{$ dimension of fibres + dimension of the base = k +n-2k, recall that ρ_X, ρ_Z are $\mathbb{P}^k$-bundles$\}$ implying that π_X is not surjective since $dimP(X) = n - k < dimX = n$.

Now consider the cylinder homomorphism map:

$$\phi_{X,*} : \mathrm{CH}^{l-k}(\Omega_X(k))_{\mathbb{Q}} \xrightarrow{\pi_{X,*} \circ \rho_X^*} \mathrm{CH}^l(X)_{\mathbb{Q}}$$

by $c \in \Omega_X$ goes to $\mathbb{P}_c^k \subset X$.

If we fix l and let $n \to \infty$ then $k \to \infty$ [Note that $k = [(n - \sum_{s \neq j} (d_i - 1)/d_s]$.

Let $d_s = max\{d_1, ..., d_r\}$. Note that this means we have $H^n(X, \mathbb{C}) = F^k H^n(X, \mathbb{C})$.

This implies that $l-k<0$ and so $\mathrm{CH}^{l-k}(\Omega_X(k))_{\mathbb{Q}} = 0$ for $n >> 1$.

But as we mentioned in example 5.1.19 $\tilde{X} = \{\Omega_Z - \Omega_X\} \coprod P(X)$.

So $\mathrm{CH}^l(X)_{\mathbb{Q}} = \pi_*(\mathrm{CH}^l(\tilde{X})_{\mathbb{Q}}) = \phi_{X,*}\mathrm{CH}^{l-k}(\Omega_X(k))_{\mathbb{Q}} + j^*\mathrm{CH}^l(Z)_{\mathbb{Q}}$ [7].

If $r \geq 1$, Take W as described before with $n >> 1$, then by proposition 5.2.3, $\mathrm{CH}^l(X)_{\mathbb{Q}} = j^*\mathrm{CH}^l(Z)_{\mathbb{Q}} = i^*\mathrm{CH}^l(W)_{\mathbb{Q}} \simeq \mathbb{Q}$, i.e., one needs to prove $\mathrm{CH}^l(W)_{\mathbb{Q}}) \simeq \mathbb{Q}$, by induction.

If $r=1$ then $X \subset \mathbb{P}^{n+1}$ so $W = \mathbb{P}^{n+1}$ and therefore $\mathrm{CH}^l(W)_{\mathbb{Q}} \simeq \mathbb{Q}$.

Suppose that our statement is true for r, $X = V(F_1, ..., F_r)$ and $\mathrm{CH}^l(X)_{\mathbb{Q}}$ $= \mathbb{Q}$ then $\mathrm{CH}^l(W)_{\mathbb{Q}} = \mathrm{CH}^l(V(F_1, ..., F_{r-1})) = \mathbb{Q}$, by induction.

Now, for r+1, $X = V(F_1, ..., F_{r+1})$ then $W = V(F_1, ..., F_r)$ and so by the inductive assumption, $\mathrm{CH}^l(W)_{\mathbb{Q}} \simeq \mathbb{Q}$.

□

5.2.1 Some examples on finding the value of n

We will give some examples now on finding the value of n, that is minimal with regard to this process. Whether this process gives effective values of minimal n is unclear.

Example 5.2.6. *Let X be a hypersurface with degree d and dim n then recall that from the proof*

$$\phi_{X,*} : CH^{l-k}(\Omega_X(k))_{\mathbb{Q}} \xrightarrow{\pi_{X,*}\circ\rho_X^*} CH^l(X)_{\mathbb{Q}}.$$

We show that $l-k<0$ ends by

$$CH^l(X)_{\mathbb{Q}} \simeq \mathbb{Q}.$$

So since $k=[(n+1)/d]$, we take $k=l+1$ then $l-k<0$ implies that $l+1-[(n+1)/d]\leq 0$ hence, $n\geq dl+d-1$. Hence a minimum value of n by this process is $dl+d-1$.

Example 5.2.7. *Let X be a complete intersection with degree d and dim n. Then recall that from the proof*

$$\phi_{X,*} : CH^{l-k}(\Omega_X(k))_{\mathbb{Q}} \xrightarrow{\pi_{X,*}\circ\rho_X^*} CH^l(X)_{\mathbb{Q}}.$$

We show that $l-k<0$ ends by

$$CH^l(X)_{\mathbb{Q}} \simeq \mathbb{Q}.$$

So since $k=[(n-\sum_{s\neq j}(d_i-1)/d_s]$

and $d_s = max\{d_1,...,d_r\}$, take $k=l+1$ then $l-k<0$ implies that $l+1-[(n-\sum_{s\neq j}(d_i-1)/d_s]\leq 0$.

hence, $n \geq l + 1 + [\sum_{s \neq j} (d_i - 1)/d_s]$.

Hence $l + 1 + [\sum_{s \neq j} (d_i - 1)/d_s]$ *is the minimum value of* n.

But $\ell = k(n + 1 + r - k) - \sum_{j=1}^{r} \binom{d_j+k}{k} \geq 0$.

So $(l + 1)(n + 1 + r - l - 1) - \sum_{j=1}^{r} \binom{d_j+l+1}{l+1} \geq 0$.

So $n \geq \sum_{j=1}^{r} \binom{d_j+l+1}{l+1} |(l + 1) + l - r$.

Therefore, a minimum value of n *is* $\sum_{j=1}^{r} \binom{d_j+l+1}{l+1} |(l + 1) + l - r$.

Now for W *in the Theorem we have* $W = V(F_1, ..., F_{r-1})$ *so a minimum value of* n *is* $\sum_{j=1}^{r-1} \binom{d_j+l+1}{l+1} |(l + 1) + l - r - 1$.

Consider that $Z_W = V(G_1, ..., G_{r-1})$ *where* $G_i = F_i$ *for all* $i = 1, ..., r - 2$ *and* $G_{r-1} = F_{r-1} + z_{n+r}^{d_{r-1}}$. *Where* Z_W *is subset from* $\mathbb{P}^{n+r}$ *such that* $Z_W \cap V(z_{n+r}) = Z_W \cap \mathbb{P}^{n+r-1} = X$.

And now consider $\nu : Z_W \to \mathbb{P}^{n+r-1}$. *Let* $W_1 = \nu(Z_W)$ *that is if* $p = [p_0, ..., p_{n+r}] \in Z_W$ *then* $\nu(p) = [p_0, ..., p_{n+r-1}] \in W_1$.

Note that, $p = [p_0, ..., p_{n+r}] \in Z_W$ *means* $F_i(p_0, ..., p_{n+r-1}) = 0$ *for* $i = 1, ..., r - 2$, *and* $F_{r-1}(p_0, ..., p_{n+r-1}) + p_{n+r}^{d_{r-1}} = 0$. *There are two possibilities:*

1. $p_{n+r} = 0 \Rightarrow [p_0, ..., p_{n+r-1}] \in W$.

2. $p_{n+r} \neq 0 \Rightarrow [p_0, ..., p_{n+r} \notin W$.

Hence $W \subset W_1$ *and* $W_1 = V(F_1, ..., F_{r-2})$.

In the same way of the proof of Theorem 5.2.1, we have

$$CH^l(W)_{\mathbb{Q}} \simeq \mathbb{Q}$$

.

Hence, in the same way we get a minimum value of n *is* $\sum_{j=1}^{r-2}\binom{d_j+l+1}{l+1}|(l+1)+l-r-2$.

Continuing with the same procedure we end up by $W_{r-1} = V(F_1)$ *and hence a minimum value of* n *is* $\binom{d+l+1}{l+1}/(l+1)+l-1$.

Example 5.2.8. *Let* X *be a general intersection of two cubics* $\subset \mathbb{P}^{n+2}$.

Then X *is a complete intersection and we have* $d_1 = d_2 = 3$.

If we find W *in the same way of the theorem then* $W = V(\mathit{one\ cubic})$.

Let $l = 2$ *and* $k = l + 1 = 3$.
Now, recall that $k = [(n - \sum_{s\neq j}(d_i - 1)/d_s]$

and $d_s = max\{d_1, ..., d_r\}$.

And recall that $\ell = k(n+1+r-k) - \sum_{j=1}^{r}\binom{d_j+k}{k} \geq 0$. *But we have here* $r = 1$ *and* $d_1 = d_s = 3$.

Then $\ell = 3(n+1-3) - \binom{6}{3} \geq 0$.

$n \geq 2 + (20/3)$ *implies that a minimum value of* n *is* 8.

Example 5.2.9. *Let* X *be a general intersection of a cubic and quartic* $\subset \mathbb{P}^{n+2}$.

Then X is a complete intersection and we have $d_1 = 3, d_2 = 4$.

If we find W in the same way of the theorem then $W = V(a\ cubic)$

Let $l = 3$ and $k = l + 1 = 4$.
Now, recall thet $k = [(n - \sum_{s \neq j} (d_i - 1)/d_s]$

and $d_s = max\{d_1, ..., d_r\}$.

And recall that $\ell = k(n + 1 + r - k) - \sum_{j=1}^{r} \binom{d_j+k}{k} \geq 0$. But we have here $r = 1$ and $d_1 = d_s = 3$.

Then $\ell = 4(n + 1 - 4) - \binom{6}{4} \geq 0$.

$n \geq 3 + (15/4)$ implies that a minimum value of n is 6.

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