My heart's in the Highlands, my heart is not here...

Robert Burns (1759 - 1796)

University of Alberta

Applications of a Scalar Field to de Sitter Quantum Gravity and to Hořava-Lifshitz Gravity

by

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Abstract

In the first part of the thesis, we study the minimally-coupled massless scalar field in de Sitter spacetime. Because of the non-linear nature of general relativity, the direct analysis of the graviton is very complicated. So, we use the scalar field as an analogue to the graviton, and shift invariance as an analogue to gauge invariance of the graviton. Physical observables are restricted to those with shift invariance. Starting from a massive scalar field in the Euclidean vacuum, we take the massless limit of the Wightman function in this state. We propose to use this two-point function in the massless limit with the divergent part dropped off as an intermediate tool to calculate two-point functions of physical operators. Examples for the two-point functions of gradients of the field and for the n-point products of the differences of the field values are calculated. We find that as long as one considers only shift-invariant operators, there does exist a well-defined vacuum state, and the correlation functions are free of IR divergences and exhibit the cluster decomposition property. This suggests that there should exist a de Sitter-invariant vacuum for the graviton on de Sitter, as long as one considers only gauge invariant operators.

In the second part, we study vacuum static solutions with spherical symmetry in the IR limit of Hořava-Lifshitz gravity. In this case, the problem can be greatly simplified by using a trick to project the 4D theory into a 3D massless scalar field minimally coupled to 3D Euclidean gravity. Then the solution to Hořava-Lifshitz gravity can be generated from the Schwarzschild solution in general relativity by a constant rescaling of the 3D scalar field, though this is in general not a black hole solution. This solution has a naked singularity and should be regarded as the exterior to some spherical distribution of matter. The nontrivial parameter (i.e. the parameter of the theory, not the integration constant) of the solution is constrained by physical considerations. In particular, using the correspondence between the IR limit of Hořava-Lifshitz and Einstein-aether theory, it is also constrained by conditions arising from the latter.

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Chapter 1

Introduction

In this thesis, we use the minimally-coupled massless scalar field to study de Sitter quantum gravity and Hořava-Lifshitz gravity.

As one application, we study a minimally-coupled massless scalar field in a de Sitter background in the first part of the thesis. We hope to shed some light on perturbative quantum gravity in de Sitter by investigating its massless scalar field counterpart as a poor-man's model for the graviton.

A quantum state may be defined by the expectation values it gives for quantum operators. It is well known that for calculating the expectation values of all products of fields at distinct points and of suitably smeared field operators, there is no de Sitter-invariant vacuum state for a massless minimally-coupled scalar field in de Sitter background [1, 2, 3, 4, 5, 6]. Indeed, taking the massless limit of the two-point function in the de Sitter-invariant Euclidean vacuum of a free massive scalar field leads to a divergence, so that this expectation value cannot be both a well-defined finite value and de Sitter invariant.

In order to obtain an infrared (IR)-finite result for the two-point function of a massless scalar field, one generally has to abandon the de Sitter invariance of the vacuum. For example, the two-point function may have time-dependent terms which break de Sitter invariance [5, 6].

Since the action of the minimally-coupled massless scalar field involves only derivative terms, it has a global symmetry under an arbitrary constant shift in field values, $\phi \to \phi + \text{const}$. We propose that one should impose this symmetry as a physical requirement on the operators whose expectation values define the quantum state, thus excluding the non-shift-invariant operators whose expectation values are infinite or undefined in the massless limit of the de Sitter-invariant Euclidean vacuum. In this thesis we shall call non-shift-invariant operators unphysical, so physical observables shall here be required to be shift invariant.

For example, physical n-point functions should be restricted to those that can be written in terms of differences of fields at different points and/or in terms of field derivatives. Operators like $\phi(x)\phi(y)$, $\{\phi(x), \phi(y)\}$ or $\phi^2(x) - \phi^2(y)$, which are not shift-invariant, do not correspond to physical observables for a massless scalar field with a shift-invariant Lagrangian, but operators like $[\phi(u) - \phi(v)][\phi(x) - \phi(y)]$ do. Then, as we shall see, the above-mentioned IR divergence is only associated with the unphysical correlation functions. For correlation functions made of shift-invariant operators, the IR divergences cancel out. A similar viewpoint has been expressed by Kirsten and Garriga [7], who obtained several results overlapping ours. Related constructions for the massless scalar field have also been carried out using Krein spaces in [8, 9].

Diffeomorphism invariance for the graviton h_{ab} takes the form of gauge invariance of its action under the transformation

$$h_{ab} \to h_{ab} + \nabla_a \xi_b + \nabla_b \xi_a, \tag{1.1}$$

where ξ_a is an arbitrary infinitesimal coordinate transformation. The graviton two-point function is IR divergent in some physical (e.g., transverse, traceless and synchronous) gauges in the spatially flat patch of de Sitter [10]. However, it is not gauge invariant in the sense of (1.1). Indeed, it was shown that one can gauge away the IR divergence in the physical gauge [11] and in some covariant gauges [12]. Moreover, it is shown in [13] that in the open de Sitter patch the two-point function in the physical gauge is free of such IR problems. These results suggest that the IR divergence of the graviton propagator may not contribute to the physical two-point functions, i.e. those constructed from gauge-invariant operators, such as the product of two linearized Weyl tensors [14, 15, 16].

The massless scalar field with shift invariance considered here is an analogue of the graviton with gauge invariance.[1] In general, the dynamics of the graviton in Einstein's general relativity (GR) is complicated due to the high non-linearity of its equations of motion (EOM) and the issue of gauge symmetry. A tractable first step is to study the linear perturbation of gravity. In this case, we can use our free scalar field as the 'guinea pig' to study the graviton. Indeed they share the same EOM when the graviton is in the physical gauge [17]; see details in Appendix A. As we shall see, the IR divergence of the massless scalar field can be removed if the shift invariance is taken into account, so that one only requires the quantum state to give expectation values to physical shift-invariant operators. Then correlation functions of physical operators are IR finite.

As the second application, we use a minimally-coupled massless scalar field to obtain

[1] Of course, this is not a one-to-one correspondence. Indeed, the constant shift of the scalar field is a 1-dimensional group whereas the gauge group of the graviton is infinite-dimensional. It is an "analogue" to the extent that in both cases, physical observables should be constructed from operators constrained by some symmetries.

static solutions in the low-energy limit of Hořava-Lifshitz (HL) gravity [18]. HL gravity was designed as a theory that has a non-relativistic ultraviolet (UV) fixed point, where the relativistic scaling of time and spatial coordinates is replaced by anisotropic scaling. In this way, time and space are treated on a different footing, and the boost invariance of the Lorentz symmetry is broken. The benefit of adopting such Lorentz violation is that the theory becomes power-counting renormalizable (although its renormalizability beyond power-counting is still an open question). In particular, the Lorentz violation is realized by introducing a fixed foliation as an extra structure on the spacetime manifold. Since the foliation structure is essentially characterized by a scalar field, there is an extra scalar degree of freedom, in addition to the two degrees of freedom corresponding to the two polarizations of the graviton in GR. Originally, HL gravity was expected to recover Einstein's GR as the IR fixed point. Nevertheless, it was found later that the extra scalar degree of freedom does not decouple at the IR and thus GR is not recovered [19, 20]. Instead, one obtains some modified gravity theory [21] with the modifications tightly constrained by experiments.

Phenomenologically it is still interesting to study the IR (low-energy) limit of HL gravity. In this thesis we focus on the static solution with spherical symmetry. This kind of solution was first studied in [22] within the original model proposed by Hořava [18] with the condition of detailed balance. We solve the problem in the extended version of HL gravity [23]. In this case, something new we do is that, we transform the problem into a 3D Euclidean one with a massless scalar field minimally coupled to Einstein gravity. This trick makes it very easy to obtain the static solution in the IR limit of HL gravity from the Schwarzschild solution of GR via a constant rescaling of the 3D scalar field. Our solution is essentially equivalent to the one obtained in [24] and [25].

This thesis is organized as follows. Chapter 1 is devoted to the first application of scalar field in de Sitter quantum gravity. We review basics of the de Sitter geometry and quantum field theory (QFT) in de Sitter background in the first two sections. In Section 2.3, we propose that for a massless scalar field, one should restrict the physical observables to those that are shift invariant. By taking the massless limit of the Wightman function in the Euclidean vacuum for the massive case, we obtain an intermediate tool for calculating two-point functions for shift-invariant operators. In Section 2.4, we give some examples of calculating two-point functions of the gradients of the field and the differences of the field values. In Section 2.5, we compare our results with previous works and comment on the implication of our work for understanding the problem of IR divergences in de Sitter quantum gravity. The second part of the thesis, consisting of Chapter 3 and 4, is devoted to the application of a scalar field in HL gravity. We first introduce the motivation and some key features of HL gravity in Section 3.1. Then we discuss the correspondence between Einstein-aether theory and the IR limit of HL gravity and the issue of recovering full general covariance in Section 3.2. After directly solving the EOM of HL gravity in Section 4.1 to obtain the static spherically symmetric solution, in Section 4.2, we use a simple method of projecting the 4D problem into a 3D Euclidean problem of a scalar field coupled with 3D GR. The constraints on the parameter of the solution are discussed and the geometric features of the solution are analyzed in Section 4.3. We summarize our results and comment on directions for future work in the Conclusion.

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Chapter 2

Massless Scalar Field in de Sitter

2.1 Useful Properties of the de Sitter Geometry

The n-dimensional de Sitter spacetime is a solution to the Einstein's equation

$$G_{ab} + \Lambda g_{ab} = 0, \tag{2.1}$$

where the positive cosmological constant Λ is related to the asymptotic Hubble constant via

$$\Lambda = \frac{(n-1)(n-2)}{2} H^2. \tag{2.2}$$

The geometry of n-dimensional de Sitter spacetime can be conveniently described as a hypersurface (in fact, a hyperboloid)

$$\eta_{AB} X^A(x) X^B(x) = H^{-2}, \tag{2.3}$$

embedded in $(n+1)$-dimensional Minkowski spacetime

$$ds_M^2 = \eta_{AB} dX^A dX^B \tag{2.4}$$

where we use X to denote Minkowski coordinates, and we use x for points on de Sitter. See Fig. 2.1. The hypersurface (2.3) explicitly preserves the Lorentz symmetry $SO(n,1)$ of the Minkowski spacetime. As a consequence, the $SO(n,1)$ is also the isometry of the n-dimensional de Sitter spacetime, and there are $n(n+1)/2$ Killing vectors. So de Sitter spacetime is a maximally symmetric space. There are various coordinate charts that cover part or the whole manifold of de Sitter. Details of these charts and their relations to the coordinates of the embedding Minkowski spacetime are summarized in Appendix B. Note also that there is no globally defined timelike Killing vector field; therefore de Sitter is not stationary globally.

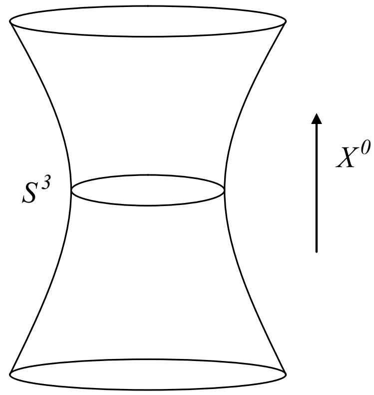


Figure 2.1: de Sitter as a hypersurface embedded in a higher dimensional Minkowski spacetime.

The geodesic distance μ with the affine parameter λ in de Sitter is

$$\mu(x,x') = \int d\lambda \sqrt{g_{ab}\dot{x}^a\dot{x}^b}, \tag{2.5}$$

where an overdot represents a derivative with respect to that affine parameter. Note that, in this definition, μ is imaginary for timelike geodesics. A more convenient quantity equivalently characterizing the geodesic distance is defined by

$$Z(x,x') = H^2\eta_{AB}X^A(x)X^B(x'), \tag{2.6}$$

which is de Sitter invariant, again, due to the fact that a de Sitter isometry corresponds to a Lorentz transformation in the embedding space.[1] Following from its definition, Z has the property that it changes sign when one point is replaced by its antipodal point, i.e.

$$Z(x,y) = -Z(\bar{x},y), \tag{2.7}$$

where $\bar{x}$ denotes the antipodal point to x, simply because $X^A(\bar{x}) = -X^A(x)$.

In addition, for two points x and x' on de Sitter, one can also measure their distance in the embedding space via the Minkowski distance given by

$$\sigma^2(x,x') \equiv \eta_{AB}[X^A(x) - X^A(x')][X^B(x) - X^B(x')]. \tag{2.8}$$

[1] Intuitively, Z is related to some 'angle'. Indeed, in the Euclidean case, the dS_n becomes a sphere S^n (with radius $R = 1/H$), and the embedding space becomes $\mathbb{R}^{n+1}$. Then any point on S^n satisfies $|X| = R$, and $X \cdot X' = |X||X'|\cos\Omega$. Now Z becomes $Z = \cos\Omega = X \cdot X'H^2$.

Now Z is related to the Minkowski distance by

$$\sigma^2 = 2(1-Z)H^{-2} \tag{2.9}$$

while it is related to μ via

$$\mu(x,x') = \cos^{-1} Z, \quad (-1 < Z < 1), \tag{2.10}$$

$$\mu(x,x') = \cos^{-1}(Z - i\epsilon), \quad (Z > 1), \tag{2.11}$$

where μ is real for spacelike geodesics $(-1 < Z < 1)$ and imaginary for timelike ones $(Z > 1)$, where $\mu = i\tau$ in terms of the geodesic proper time separation τ. Note that there is a branch cut along the real axis for $Z > 1$ where the values of μ on both sides are pure imaginary and differ by a sign. Here we choose the convention to pick out the limit from below the real axis. $Z < -1$ corresponds to the case where the separation between the two points is spacelike, but there is no geodesic connecting them. In this case μ can still be defined via analytic continuation as [26, 27]

$$\mu(x,x') = \cos^{-1}(Z + i\epsilon), \quad (Z < -1). \tag{2.12}$$

The range of Z is illustrated in Fig. 2.2.

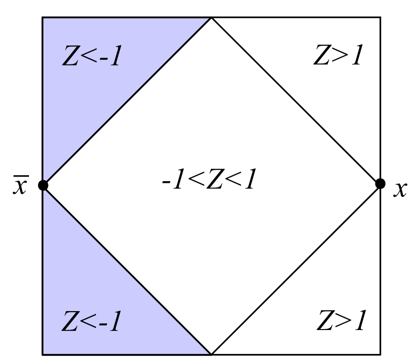


Figure 2.2: Range of Z measured with respect to the point x. Points in the shaded region $Z < -1$ cannot be connected to x by any geodesics. $\bar{x}$ is the antipodal point to x.

There are two useful unit vectors defined by

$$n_a = \nabla_a \mu(x,x'), \quad n_{b'} = \nabla_{b'} \mu(x,x'),$$

which are pointing outward at each end of the geodesic. They should be distinguished from the normally defined tangent vectors of the geodesic, which are always pointing in the direction of increasing λ. In particular, for timelike geodesics, n is imaginary. So $n^a n_a = n^{b'} n_{b'} = 1$ for both spacelike and timelike cases. Note also that

$$g_{ab'} n^{b'} = -n_a, \tag{2.13}$$

$$\nabla_a n_{b'} = -(\csc \mu)(g_{ab'} + n_a n_{b'}), \tag{2.14}$$

where $g_{ab'}$ is the parallel propagator which parallel transports a vector along the geodesic.

In the following we will only consider 4-dimensional de Sitter spacetime with $n = 4$. The generalization to other dimensions is straightforward. We use units in which the asymptotic Hubble constant $H \equiv \sqrt{\Lambda/3}$ is unity for simplicity of notation. In other units, one may use dimensional analysis to restore the correct powers of H. More details about the following properties can be found in [26, 27].

2.2 Quantization of Minimally-Coupled Free Scalar Field

2.2.1 Basics of Quantum Scalar Field Theory on de Sitter

The total action for a scalar field coupled to gravity can be written as

$$S_{tot} = \int d^4x \sqrt{-g}[\frac{1}{2\kappa^2} R - 2\Lambda - (\partial\phi)^2 - (m^2 + \xi R)\phi^2 - V(\phi)]. \tag{2.15}$$

In the following, we will only consider a minimally-coupled free scalar field, i.e. $\xi = 0, V = 0$. But we will keep the mass term. Moreover, we will neglect the back reaction of the scalar field on the spacetime. In other words, the metric is regarded as a non-dynamic background. Thus we are essentially only concerned with the action of the free scalar field

$$S = -\int d^4x \sqrt{-g}[(\partial\phi)^2 + m^2\phi^2], \tag{2.16}$$

where the implicit measure is defined on de Sitter. The resulting equation of motion is given by

$$(\Box - m^2)\phi = 0, \tag{2.17}$$

and one must choose appropriate boundary conditions for the solutions.

In the following, we will use $u_n(x)$ to denote the mode functions, which are solutions to the EOM, and which form a complete basis orthonormal to each other

$$(u_n, u_m) = \delta_{mn}, \tag{2.18}$$

with respect to the Klein-Gordon inner product

$$(\phi, \psi) \equiv -i \int_\Sigma d^3x \sqrt{h} n^\mu (\phi \partial_\mu \psi^* - \partial_\mu \phi \psi^*) \tag{2.19}$$

where Σ is any Cauchy surface with timelike future directed normal n^μ, h stands for the determinant of the induced metric on Σ, and a star denotes complex conjugate. Such an inner product is independent of any particular Σ. Mode functions in global coordinates and conformally flat coordinates are summarized in Appendix E.

As a consequence of quantization, the classical field is promoted to an operator (or more precisely, an operator-valued distribution) in the Hilbert space,

$$\phi(x) \to \hat{\phi}(x) = \sum_n [\hat{a}_n u_n(x) + \hat{a}_n^\dagger u_n^*(x)] \tag{2.20}$$

where $\hat{a}_n$ and $\hat{a}_n^\dagger$ are annihilation and creation operators, respectively, which satisfy the relation

$$[\hat{a}_m, \hat{a}_n^\dagger] = \delta_{mn}. \tag{2.21}$$

The vacuum state $|0\rangle$ is defined by

$$\hat{a}_n |0\rangle = 0 \quad \forall n, \tag{2.22}$$

and the Fock representation of the Hilbert space is obtained by applying the creation operators on the vacuum state. In the following, for notational convenience, we will neglect the hat notation and operators should be distinguished according to their context.

In quantum field theory, there are various two-point functions, which are summarized in Appendix C. To avoid unnecessary complication, it is sufficient for our purpose to analyze only the Wightman two-point function, defined by the vacuum expectation value

$$G^W(x, y) \equiv \langle 0|\phi(x)\phi(y)|0\rangle, \tag{2.23}$$

for a minimally-coupled scalar field of mass m, which obeys the homogeneous equation of motion[2]

$$(\Box_x - m^2) G^W(x, y) = 0. \tag{2.24}$$

Various Green's functions, such as the Hadamard function or Feynman function, can be obtained from G^W (see, e.g. [28]). Given a set of complete mode functions u_n, the

[2]Of course, to obtain a specific G^W as a solution to the EOM, one must specify corresponding boundary conditions. In addition, one also requires that for a physical state, the short-distance singularity structure of the Wightman function should satisfy certain conditions such that the state becomes a so-called Hadamard state [96].

Wightman function can be easily expressed as

$$G^W(x,y) = \sum_n u_n(x)u_n^*(y), \tag{2.25}$$

which depends on the state with respect to which the expectation value is taken. Thus in the following we will use this two-point function as a character of different vacuum states. On the other hand, the commutator, which can be written as

$$iG_c(x,y) = [\phi(x), \phi(y)] = \sum_n [u_n(x)u_n^*(y) - u_n^*(x)u_n(y)], \tag{2.26}$$

is a state-independent two-point function.

Causality requires that field operators at two points with spacelike separations should commutate, i.e.

$$[\phi(x), \phi(y)] = 0 \quad \text{for } (x-y)^2 > 0. \tag{2.27}$$

In this case, we also have $G_c(x,y) = 0$, $G^W(x,y)$ is symmetric about the two points, and $G^W(x,y) = G^{(1)}(x,y)/2$. These properties do not hold in general for non-spacelike separations.

For de Sitter-invariant states, this two-point function can only depend on the de Sitter-invariant distance, so $G^W(x,y) = G^W(Z(x,y))$. If one introduces

$$z(x,x') \equiv \frac{1-Z(x,x')}{2} = \frac{1}{4}\eta_{AB}[X^A(x) - X^A(x')][X^B(x) - X^B(x')] = \sin^2\frac{1}{2}\mu(x,x'), \tag{2.28}$$

which is one-fourth the square of the distance between the points in the 5D Minkowski spacetime and for nearby points is also approximately one-fourth the square of the geodesic distance $\mu(x,x')$ in the 4D de Sitter spacetime itself, then the Wightman two-point function obeys the equation

$$z(1-z)\frac{d^2}{dz^2}G^W + (2-4z)\frac{d}{dz}G^W - m^2 G^W = 0. \tag{2.29}$$

In general, the de Sitter-invariant vacuum is not unique. There are a family of solutions corresponding to different vacua [5]. Moreover, the general solutions have two singularities, at $z = 0$ or $Z = 1$ (when one point is on the lightcone of the other), and at $z = 1$ or $Z = -1$ (when one point is on the lightcone of the point antipodal to the other). Among these solutions, however, there is one two-point function that is the Wightman function for a unique Euclidean vacuum, or Bunch-Davies vacuum, which is only singular when the two points are lightlike related, at $z = 0$, and which can be obtained by analytic continuation from the Euclidean de Sitter space (i.e., a four-sphere), namely [29, 30, 31, 32, 33, 34, 5, 6, 35, 36]

$$G^W(z) = c_m\ {}_2F_1(h_+, h_-; 2; 1-z), \tag{2.30}$$

where $h_\pm = (3/2)[1 \pm \sqrt{1-(4/9)m^2}]$ and ${}_2F_1$ is the Gaussian hypergeometric function. The constant $c_m = \Gamma(h_+)\Gamma(h_-)/(4\pi)^2$ is obtained by requiring the Wightman function to have the same singularity strength as that in Minkowski space. See Appendix D for more details. Note that for x, x' timelike separated, $z < 0$, one should take care of the branch cut of the hypergeometric function by using the $i\epsilon$ prescription (c.f. [26, 37, 38])

$$z(x,x') \to \begin{cases} z(x,x') - i\epsilon & \text{if} \quad t > t' \\ z(x,x') + i\epsilon & \text{if} \quad t' > t \end{cases}. \tag{2.31}$$

In the following, the same prescription is also implicit in the logarithmic function, where, for timelike separations,

$$\ln z(x,x') = \begin{cases} \ln|z| - i\pi & \text{if} \quad t > t' \\ \ln|z| + i\pi & \text{if} \quad t' > t \end{cases}. \tag{2.32}$$

2.2.2 The α-Vacua

In general, there is no uniquely defined vacuum state for QFT on curved background, unlike the case in flat spacetime. Indeed, in Minkowski spacetime, the presence of the Poincare symmetry, and in particular the existence of a timelike Killing vector field, is crucial for the existence of a unique, 'natural' vacuum, i.e. the Minkowski vacuum, defined as the trivial representation of the Poincare group. More particularly, there is a set of preferred complete modes

$$u_{\vec{k}} = \frac{1}{\sqrt{2\omega}} e^{i\vec{k}\cdot\vec{x} - i\omega t} \tag{2.33}$$

in terms of which a free scalar field can be expressed as

$$\phi(\vec{x}, t) = \sum_{\vec{k}} [a_{\vec{k}} u_{\vec{k}} + a^\dagger_{\vec{k}} u^*_{\vec{k}}]. \tag{2.34}$$

Then the Minkowski vacuum is defined as the state annihilated by all $a_{\vec{k}}$.

In curved spacetimes, there is no Poincare symmetry, and in many cases there simply doesn't exist a timelike Killing vector, which is just the case for de Sitter. It is well-known that there is a family of physically inequivalent vacua, the α-vacua, for a free minimally-coupled massive scalar field on de Sitter [5]. These α-vacua can be constructed from the Euclidean vacuum via the Bogolyubov transformation (following the convention in [38])

$$\tilde{u}_n = N_\alpha (u_n + e^\alpha u_n^*) \tag{2.35}$$

where $N_\alpha = 1/\sqrt{1-\exp(\alpha+\alpha^*)}$ and u_n here specifically denotes the modes of the Euclidean vacuum. Such a transformation is uniquely characterized by a constant $\alpha \in \mathbb{C}$. One can also remove a total phase ambiguity by fixing Re $\alpha < 0$ such that $N_\alpha \in \mathbb{R}$. Correspondingly the creation/annihilation operators are related to those associated with the Euclidean vacuum via

$$\tilde{a}_n = N_\alpha(a_n - e^{\alpha^*}a_n^\dagger). \tag{2.36}$$

Then the α-vacuum is defined by

$$\tilde{a}_n|\alpha\rangle = 0 \quad \forall n. \tag{2.37}$$

Now the Wightman function of the α-vacua can be expressed as [38]

$$G_\alpha^W(x,y) = N_\alpha^2[G(x,y) + e^{\alpha+\alpha^*}G(y,x) + e^\alpha G(\bar{x},y) + e^{\alpha^*}G(x,\bar{y})], \tag{2.38}$$

where $G(x,y)$ is the Wightman function of the Euclidean vacuum (2.30), which depends only on the de Sitter-invariant quantity z (up to an appropriate $i\epsilon$ prescription). Thus the Wightman functions of the α-vacua are also de Sitter invariant. $G_\alpha^W(x,y)$ generally has two singularities, at $(x-y)^2 = 0$ and at $(x-\bar{y})^2 = 0$. The Euclidean G^W, as a particular member of the α-vacua with Re $\alpha = -\infty$, is the only one with just one singularity at $(x-y)^2 = 0$.

For example, in global coordinates (B.2)[38], one can obtain the Euclidean vacuum, i.e. the mode functions, by analytic continuing the Lorentzian time $\tau \to -i\tau_E$, with $\tau_E \in (-\pi/2, \pi/2)$, which gives the round metric on S^4. Then one can solve the EOM in the Lorentzian global chart to obtain the modes for the in and out vacua, as two particular examples within the α-vacua. The in vacuum is the state with no incoming particles at past infinity, whereas the out vacuum is the state with no outgoing particles at future infinity. See details in Appendix E. In flat coordinates (B.6), it is argued in [39] that the state with no particles on the horizon $\eta = -\infty$ corresponds to the Euclidean vacuum.

2.3 de Sitter-Invariant Vacuum for Massless Scalar Field with Shift Invariance

In the Euclidean vacuum, as $m \to 0$, G^W in Eq. (2.30) can be expanded as [6, 27] (with a z-independent $\mathcal{O}(m^0)$ term dropped)

$$G^W = \frac{1}{16\pi^2}\frac{6}{m^2} + \frac{1}{16\pi^2}\left(\frac{1}{z} - 2\ln z\right) + \mathcal{O}(m^2). \tag{2.39}$$

As one can see, the Wightman function is divergent in the massless limit. This divergence is, however, eliminated for n-point combinations of the Wightman function that are shift invariant. For such shift-invariant operators of a massless scalar field, we can drop the mass dependence of the massive Wightman function in the massless limit to get

$$G = \frac{1}{16\pi^2}\left(\frac{1}{z} - 2\ln z\right), \tag{2.40}$$

the shift-invariant part of the two-point function of a minimally-coupled massless scalar field in the de Sitter-invariant vacuum state $|0\rangle$, where $z = z(x, x')$ is given by Eq. (2.28) as one-fourth the invariant interval between the two points in the 5D Minkowski spacetime in which the 4D de Sitter may be embedded as a unit hyperboloid.

An equivalent way to get $G(x, x')$ is to start with the spectral representation of the two-point function $G_E(x, x')$ for a massive scalar field on the Euclidean de Sitter space (four-sphere) [40, 9]. Note that the Lorentzian Wightman function of the Euclidean vacuum can be obtained via continuing the Euclidean $z(x, x')$ on the S^4 to its Lorentzian version, and assuming the same $i\epsilon$ prescription as in (2.31). G_E obeys

$$(\Box_x - m^2)G_E(x, x') = -\delta(x, x'), \tag{2.41}$$

with $\Box_x$ being the Laplacian with respect to the x coordinates and an implicit factor $1/\sqrt{g}$ in the covariant delta function. Hence this Green's function may be written as

$$G_E(x, x') = \sum_n \frac{\phi_n(x)\phi_n(x')}{\lambda_n}, \tag{2.42}$$

where the $\phi_n(x)$ are an orthonormal set of real eigenfunctions, obeying the equation

$$(\Box_x - m^2)\phi_n(x) = -\lambda_n\phi_n(x), \tag{2.43}$$

with eigenvalues $\lambda_n = m^2 + l_n(l_n + 3)$ and having $(l + 1)(l + 3/2)(l + 2)/3$ orthonormal eigenfunctions sharing the same nonnegative integer value of $l_n = l$. The lowest eigenvalue, which we shall label as $n = 0$ with $l_0 = 0$, is $\lambda_0 = m^2$, corresponding to the constant eigenfunction $\phi_0(x) = 1/\sqrt{V_4}$, where $V_4 = 8\pi^2/3$ in our units with $H = 1$, so that V_4 is the volume of the unit S^4.

Clearly the $n = 0$ term in this spectral representation of $G_E(x, x')$ diverges when $m = 0$. One can also directly see that when $m = 0$, there is no solution to the equation (2.41) on a compact Euclidean manifold such as the four-sphere, since then the integral over x of the left hand side is identically zero, whereas the integral of minus the covariant delta function on the right hand side gives -1. However, if we omit the zero-eigenvalue term in the sum, we get a result that is finite even when $m = 0$ and then is uniquely defined as [40, 9]

$$G_E^0(x, x') = \sum_{n \neq 0} \frac{\phi_n(x)\phi_n(x')}{\lambda_n}. \tag{2.44}$$

Our Lorentzian $G(x, x')$ can thus be regarded as an analytic continuation of G_E^0.

This shift-invariant part of the two-point function obeys the equation

$$\Box_x G_E^0(x, x') = \Box_{x'} G_E^0(x, x') = -\delta(x, x') + \frac{1}{V_4}, \tag{2.45}$$

so it is not a Green's function for the Laplacian on the four-sphere; such a Green's function does not exist. A similar equation holds for the Lorentzian Wightman two-point function $G(x, x')$ with the δ function absent (since the Wightman function is by definition a solution to the homogeneous EOM). Therefore, the de Sitter-invariant $G(x, x')$ is not the two-point function $\langle\psi|\phi(x)\phi(x')|\psi\rangle$ in any quantum state $|\psi\rangle$ of the original Fock space, consistent with Allen's proof [5] that there exists no de Sitter-invariant Fock vacuum state, in which a de Sitter-invariant two-point function would be defined.

However, in our alternative set of quantum states in which only shift-invariant operators are assigned expectation values, the failure of $G(x, x')$ to be a solution to the homogeneous equation of motion cancels out. The shift-invariant expectation values will never have a single $G(x, x')$ term with the argument x, but terms with argument x will always occur in pairs, such as $G(x, x') - G(x, x'')$, and the Laplacian with respect to x acting on such a combination will always be zero in the Lorentzian spacetime.

2.4 Examples of Shift-Invariant Correlation Functions

We regard the shift invariance as a physical constraint on constructing observables. Then there is a de Sitter-invariant vacuum for a massless minimally-coupled scalar field as long as one considers only shift-invariant operators. In this section, we give some examples of these operators and show that their correlation functions are indeed both de Sitter invariant and free of IR divergences.

2.4.1 Two-Point Functions of Derivatives

The shift-invariant correlation function for the derivatives of a massless scalar field is readily calculated to be

$$\langle 0|\nabla_a\phi(x)\nabla_{b'}\phi(x')|0\rangle = \nabla_a\nabla_{b'}G(x, x') = \frac{(4 + 2z)n_a n_{b'} + (1 + 2z)g_{ab'}}{32\pi^2 z^2}. \tag{2.46}$$

For two points separated by proper time $T = -i\mu(x, x')$, so that $Z = \cosh T$ and

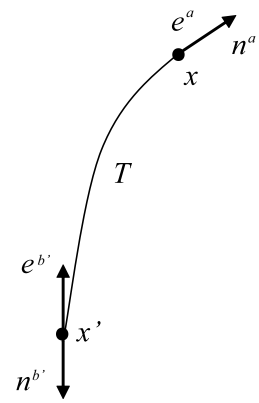


Figure 2.3: Two timelike separated points x and x', with derivatives of the field along two timelike directions.

$z \equiv (1-Z)/2 = -\sinh^2(T/2)$, we have

$$n^a = ie^a, \quad n^{b'} = i(-e^{b'}), \tag{2.47}$$

where e^a and $e^{b'}$ are unit tangent vectors of the geodesic at x and x' satisfying $e^a e_a = e^{b'} e_{b'} = -1$. See Fig. 2.3. Then, using Eq. (2.40), we have

$$\langle 0| e^a e^{b'} \nabla_a \phi(x) \nabla_{b'} \phi(x') |0\rangle = e^a e^{b'} \nabla_a \nabla_{b'} G(\mu) = \frac{3}{32\pi^2 z^2} = \frac{3}{32\pi^2 \sinh^4(T/2)}. \tag{2.48}$$

This correlation function goes to zero exponentially rapidly as $T \to \infty$.

If e^a and $e^{b'}$ are two unit spatial vectors orthogonal to n^a and $n^{b'}$, and $e^{b'}$ is obtained by parallel transporting e^a along the geodesic, as shown in Fig. 2.4, then

$$\langle 0| e^a \nabla_a \phi(x) e^{b'} \nabla_{b'} \phi(x') |0\rangle = \frac{1+2z}{32\pi^2 z^2} = \frac{1-2\sinh^2(T/2)}{32\pi^2 \sinh^4(T/2)}. \tag{2.49}$$

This also vanishes exponentially as $T \to \infty$, though asymptotically at half the rate of that of the previous case.

For two spacelike separated points that are connected by a spacelike geodesic of length $\mu = L$, with $e^a = -n^a$ and $e^{b'} = n^{b'} = -g^{ab'} n_a$ now being spacelike tangent vectors to the

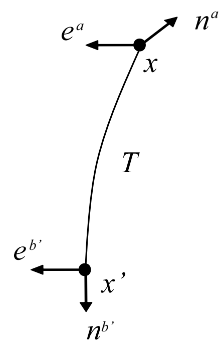


Figure 2.4: Two timelike separated points x and x', with derivatives of the field along two spacelike directions.

geodesic, we have

$$\langle 0|e^a\nabla_a\phi(x)e^{b'}\nabla_{b'}\phi(x')|0\rangle = -\frac{3}{32\pi^2 z^2} = -\frac{3}{32\pi^2\sin^4(L/2)}. \tag{2.50}$$

On the other hand, if e^a and $e^{b'} = e_a g^{ab'}$ are not tangent vectors to the geodesic but are orthogonal to n^a and to $n^{b'}$ respectively, we have

$$\langle e^a\nabla_a\phi(x)e^{b'}\nabla_{b'}\phi(x')\rangle = \frac{1+2z}{32\pi^2 z^2} = \frac{1+2\sin^2(L/2)}{32\pi^2\sin^4(L/2)}. \tag{2.51}$$

2.4.2 Correlation Functions of Differences

The difference between field operators at different points is shift invariant. So, if we calculate the correlation function (cf. [7] Eq. (45) for a similar construction)

$$\begin{aligned} G_{xy,uv} &\equiv \lim_{m\to 0}\langle 0|[\phi(x)-\phi(y)][\phi(u)-\phi(v)]|0\rangle \\ &= \lim_{m\to 0}[G^W(x,u)-G^W(y,u)-G^W(x,v)+G^W(y,v)], \end{aligned} \tag{2.52}$$

then the divergent terms as $m\to 0$ cancel out, leaving only

$$\begin{aligned} G_{xy,uv} &= G(x,u)-G(y,u)-G(x,v)+G(y,v) \\ &= \frac{1}{16\pi^2}\left\{\left[\frac{1}{z(x,u)}-2\ln z(x,u)\right]-\left[\frac{1}{z(y,u)}-2\ln z(y,u)\right]\right. \\ &\qquad \left. -\left[\frac{1}{z(x,v)}-2\ln z(x,v)\right]+\left[\frac{1}{z(y,v)}-2\ln z(y,v)\right]\right\}, \end{aligned} \tag{2.53}$$

which is free of IR divergences. This will be further confirmed in the examples given later.

Note that although $\Box_x$, the Laplacian with respect to x, acting on $G(x, x')$ has a nonzero constant term, the combination that appears in $G_{xy,uv}$ does give

$$\Box_x G_{xy,uv} = 0, \tag{2.54}$$

with the constant term cancelling out. However, the cancellation also implies that one could replace the de Sitter-invariant $G(x, x')$ with a suitable non-de Sitter-invariant $\tilde{G}(x, x')$ and still obtain de Sitter-invariant shift-invariant expectation values such as $G_{xy,uv}$. For example,

$$\tilde{G}(x, x') = G(x, x') + f(x) + f'(x') \tag{2.55}$$

with arbitrary functions f and f' will give the same

$$\tilde{G}(x, u) - \tilde{G}(y, u) - \tilde{G}(x, v) + \tilde{G}(y, v) = G_{xy,uv}. \tag{2.56}$$

In particular, if one chooses f and f' to obey

$$\Box_x f(x) = \Box_{x'} f'(x') = -1/V_4, \tag{2.57}$$

then instead of

$$\Box_x G(x, x') = \Box_{x'} G(x, x') = 1/V_4, \tag{2.58}$$

one gets

$$\Box_x \tilde{G}(x, x') = \Box_{x'} \tilde{G}(x, x') = 0, \tag{2.59}$$

so that $\tilde{G}(x, x')$ obeys the equations of motion of a true Wightman function. One might think (as we did) that $\tilde{G}(x, x')$ would be the Wightman function of a suitable non-de Sitter-invariant Fock state, but Albert Roura [41] has convinced us that this is apparently not the case.

For example, following Allen [5], one can solve the equation of motion of a massless scalar field in the $k = 0$ FLRW coordinate system of de Sitter spacetime (flat spatial slices),

$$ds^2 = -dt^2 + e^{2t} d\vec{x}^2. \tag{2.60}$$

Then it seemed that one could choose a vacuum defining all n-point functions (not just the shift-invariant ones) that is not fully de Sitter invariant but only $E(3)$ invariant, corresponding to the symmetries of rotations and translations on the flat spatial slices. There it appeared that one could choose the two-point function to be [5] (up to an arbitrary additive constant)

$$D(x, x') = \frac{1}{16\pi^2} \left(\frac{1}{z} - 2 \ln z + 2t + 2t' \right). \tag{2.61}$$

Note that this is not de Sitter invariant due to the last two terms depending on the time coordinates, which correspond to $f(x) = t/(3V_4)$ and $f'(x') = t'/(3V_4)$. However, in the similarly defined shift-invariant correlation function $G_{xy,uv}$, the t- and t'-dependent terms cancel out. The remaining part is exactly the same as Eq. (2.53). As we see, although the two-point function $D(x, x')$ is not de Sitter invariant, it is not shift invariant either and thus may be regarded as "unphysical." On the other hand, the shift-invariant correlation function $G_{xy,uv}$ obtained from it is de Sitter invariant.

However, Allen and Folacci [6], following work by Ford and Vilenkin [42], pointed out that the procedure used to obtain the two-point function $D(x, x')$ involved a regulator that dropped zero modes, so it is not the two-point function of a Fock state, but rather of an "unrealizable limit of a continuous family of Fock vacuum states." The two-point functions of actual Fock states have not only the additive functions $f(x)$ and $f'(x')$ but also product terms that do not cancel when one forms the shift-invariant four-point function $G_{xy,uv}$ from them. Indeed, in [6], under the "unrealizable limit" ($\alpha \to 0$, c.f. Eq. (4.13) of that paper), the zero modes of the O(4) invariant vacuum diverge, but the resulting two-point function is finite and essentially corresponds to our $\tilde{G}$. In particular, the product term disappear and the time-dependent terms are exactly the same as our $f(x)$ and $f'(x')$ given below for the $k = +1$ case (after a coordinate transformation).

More generally in the $k = 0$ FLRW coordinate system, one could have

$$f(x) = f(t) = t/(3V_4) + c_1 e^{-3t} + c_2, \tag{2.62}$$

with two arbitrary coefficients c_1 and c_2, and similarly for $f'(x')$, to give $\Box_x \tilde{G}(x, x') = \Box_{x'} \tilde{G}(x, x') = 0$ but the same $G_{xy,uv} = \tilde{G}(x, u) - \tilde{G}(y, u) - \tilde{G}(x, v) + \tilde{G}(y, v)$. However, it appears doubtful that there are Fock states that give these two-point functions either.

In the $k = -1$ FLRW coordinate system with hyperbolic spatial slices and scale factor $a(t) = \sinh t$, one can have

$$f(t) = (\ln \sinh t + 1/\sinh^2 t)/(3V_4) + c_1(\ln \tanh(t/2) + \cosh t/\sinh^2 t) + c_2, \tag{2.63}$$

and similarly for $f'(x')$ to get an analogous two-point function $\tilde{G}(x, x') = G(x, x') + f(x) + f'(x')$. Similar to the case with the $k = 0$ FLRW coordinate system, one can get an idealized limiting state giving this two-point function, but the results of [6] suggest that this is also impossible with a actual Fock state.

In the $k = +1$ FLRW coordinate system with three-sphere spatial slices and scale factor $a(t) = \cosh t$, one can have

$$f(t) = (\ln \cosh t - 1/\cosh^2 t)/(3V_4) + c_1(\arctan \sinh t + \sinh t/\cosh^2 t) + c_2, \tag{2.64}$$

to get yet another non-de Sitter-invariant two-point function $\tilde{G}(x,x') = G(x,x') + f(x) + f'(x')$ obeying $\Box_x \tilde{G}(x,x') = \Box_{x'} \tilde{G}(x,x') = 0$ but the same de Sitter-invariant $G_{xy,uv} = \tilde{G}(x,u) - \tilde{G}(y,u) - \tilde{G}(x,v) + \tilde{G}(y,v)$ as obtained from the de Sitter-invariant $G(x,x')$ that does not obey the equation of motion for a Wightman function. However, again it appears to be impossible to get this $\tilde{G}(x,x')$ as the Wightman function of an actual Fock state [7, 27, 41].

2.4.3 Some Examples

One may consider many configurations. One simple example is four points x, y, u, v all in order along a timelike geodesic, with x-y and u-v proper time separations t, and x-u and y-v separations $T > t$, as shown in Fig. 2.5. Then it is very easy and straightforward to show, using the relation (2.11) and (2.28), that

$$\begin{aligned} G_{xy,uv} &= \frac{1}{16\pi^2}\left\{\frac{1}{\sinh^2(\frac{T-t}{2})} + \frac{1}{\sinh^2(\frac{T+t}{2})} - \frac{2}{\sinh^2(\frac{T}{2})}\right. \\ &+ \left. 2\ln[\sinh^2(\frac{T-t}{2})\sinh^2(\frac{T+t}{2})] - 4\ln[\sinh^2(\frac{T}{2})]\right\} \\ &= \frac{1}{4\pi^2}\left[\frac{(\cosh^2 T + \cosh T - \cosh t - 1)(\cosh t - 1)}{(\cosh T - 1)(\cosh T - \cosh t)^2}\right. \\ &- \left.\ln\frac{\cosh T - 1}{\cosh T - \cosh t}\right], \end{aligned} \tag{2.65}$$

and that for $T \gg t$, $G_{xy,uv} \to 0$. In the following, we will consider some less trivial examples.

In the static coordinate system,

$$ds^2 = -(1-r^2)dt^2 + (1-r^2)^{-1}dr^2 + r^2 d\Omega^2, \tag{2.66}$$

the r-coordinate lines (e.g., lines in which all the other coordinates are fixed) are geodesics, i.e.

$$V^b \nabla_b V^a = 0,$$

where $V^a = (\partial_r)^a/\sqrt{g_{rr}}$ is the normalized unit vector in the r-direction, the derivative with respect to the radial proper distance

$$\mu = \int_0^r \frac{dr'}{\sqrt{1-r'^2}} = \sin^{-1} r \quad (\text{so } r = \sin\mu). \tag{2.67}$$

In general the t-coordinate lines are not geodesic, but an exception is the t-coordinate line that passes through the origin.

The embedding mapping between the 5D bulk space X^α $(\alpha = 0, \ldots, 4)$ and the static

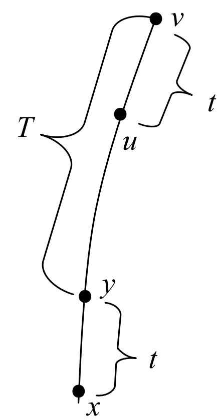


Figure 2.5: Two pairs of points x-y and u-v, with x-y and u-v proper time separations t, and x-u and y-v separations $T > t$.

patch $x^\mu = (t, r, \theta, \phi)$ is given by (B.9). Inserting these transformations in Eq. (2.6) gives

$$Z(x, x') = \sqrt{(1 - r^2)(1 - r'^2)} \cosh (t - t') + rr' \cos \Omega, \tag{2.68}$$

where

$$\cos \Omega \equiv \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\phi - \phi'). \tag{2.69}$$

One can then get $z(x, x')$ from Eq. (2.28).

Configuration 1

For this configuration in Fig. 2.6, the coordinates of the four points are

$$\begin{aligned} x \quad &: \{t = 0, r = r_L, \theta = 0, \phi = 0\}, \quad y : \{t = 0, r = r_L, \theta = \pi, \phi = 0\}, \\ u \quad &: \{t = t_T, r = r_L, \theta = 0, \phi = 0\}, \quad v : \{t = t_T, r = r_L, \theta = \pi, \phi = 0\}, \end{aligned} \tag{2.70}$$

where the geodesic distance between x and y (and between u and v) is L, while that between x and u (and between y and v) is T. t_T can be determined in terms of T via

$$z(x, u) = - \sinh^2(T/2) = -(1 - r_L^2) \sinh^2(t_T/2). \tag{2.71}$$

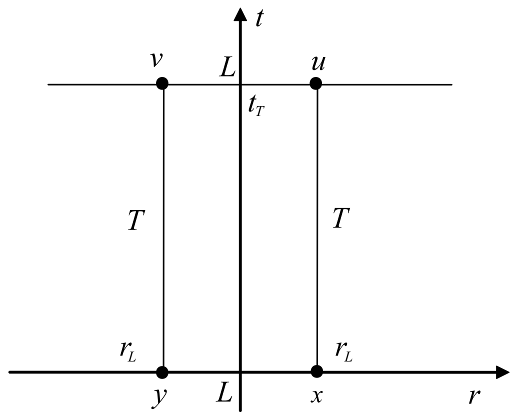


Figure 2.6: Configuration 1. (x, y) and (u, v) are regarded as two clusters with a timelike separation T.

The interval between x and y (and between u and v) is

$$z(x,y) = z(u,v) = \sin^2(L/2) = r_L^2. \tag{2.72}$$

The interval between x and v (and between y and u) is, using Eq. (2.71) and Eq. (2.72),

$$z(x,v) = z(y,u) = \frac{1}{2}[1 + r_L^2 - (1 - r_L^2)\cosh(t_T)] = \sin^2(L/2) - \sinh^2(T/2). \tag{2.73}$$

Then the correlation function is

$$G_{xy,uv} = \frac{1}{8\pi^2}\left[\frac{1}{\sinh^2(T/2) - \sin^2(L/2)} - \frac{1}{\sinh^2(T/2)} + 2\ln\left(1 - \frac{\sin^2(L/2)}{\sinh^2(T/2)} - i\epsilon\right)\right], \tag{2.74}$$

where the $i\epsilon$ picks out the branch of the logarithm (with a branch cut along the negative real axis) with the imaginary term $-\pi i$ when the argument of the logarithm is negative, which is when $\sinh(T/2) < \sin(L/2)$, so that the two separations of (x, v) and (y, u) are spacelike. The shift-invariant correlation function goes to zero for $T \gg L$, i.e. as the two clusters are separated far away in time.

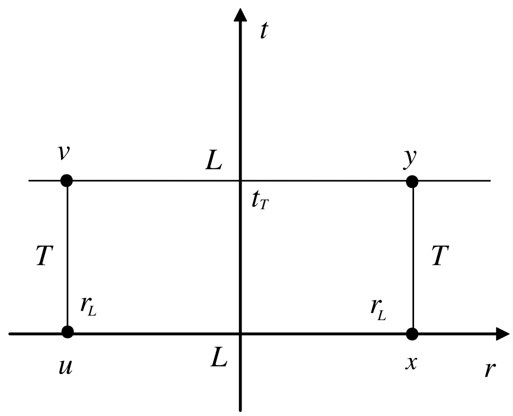


Figure 2.7: Configuration 2. (x, y) and (u, v) are regarded as two clusters with a spacelike separation L.

Configuration 2

In this case we choose four points located at

$$\begin{aligned} x &: \{t=0, r=r_L, \theta=0, \phi=0\}, \quad y : \{t=t_T, r=r_L, \theta=0, \phi=0\}, \\ u &: \{t=0, r=r_L, \theta=\pi, \phi=0\}, \quad v : \{t=t_T, r=r_L, \theta=\pi, \phi=0\}, \end{aligned} \tag{2.75}$$

where the geodesic distance between x and u (and between y and v) is L, while that between x and y (and between v and u) is T, as shown in Fig. 2.7. The relation between t_T and T can be expressed explicitly by calculating $z(x, y)$:

$$z(x, y) = -\sinh^2(T/2) = -(1 - r_L^2)\sinh^2(t_T/2). \tag{2.76}$$

The geodesic distance L between x and u (and between y and v) can be characterized by

$$z(x, u) = z(y, v) = \sin^2(L/2).$$

Moreover, the interval between x and v (and between y and u) is given by

$$z(x, v) = z(y, u) = \sin^2(L/2) - \sinh^2(T/2). \tag{2.77}$$

Note that when the $i\epsilon$ prescription is explicit, due to the opposite time ordering of (x, v) and (y, u), $z(x, v)$ and $z(y, u)$ are not equal but differ by the sign of $i\epsilon$. Therefore, even when these two separations are timelike, their imaginary contributions to the Wightman functions cancel.

Then the correlation function is

$$G_{xy,uv} = \frac{1}{8\pi^2}\left[\frac{1}{\sin^2(L/2)} - \frac{1}{\sin^2(L/2) - \sinh^2(T/2)} + 2\ln\left|1 - \frac{\sinh^2(T/2)}{\sin^2(L/2)}\right|\right], \quad (2.78)$$

which goes to zero as $T \ll L$. The absolute value here is due to the proper treatment of the $i\epsilon$, with one $+\pi i$ canceling another $-\pi i$ when $\sinh(T/2) > \sin(L/2)$. Kirsten and Garriga [7] have previously noted the linear divergence in the time as $T \to \infty$.

2.5 Discussion

We have shown that if one restricts the definition of a quantum state to giving expectation values of shift-invariant quantities (such as products of differences of field values) involving a massless scalar field, then there does exist a perfectly well-behaved de Sitter-invariant vacuum state for a massless scalar field. It just does not give well-defined expectation values to quantities that are not shift invariant, such as the product of two field values. Since the Lagrangian of a massless scalar field is shift invariant, it is natural to restrict the expectation values given by the quantum state to shift-invariant quantities, analogous to the way that one restricts to the expectation values of gauge-invariant quantities for a gauge-invariant Lagrangian.

2.5.1 Relations to Previous Works

Our results were perhaps first partially anticipated by Pathinayake, Vilenkin, and Allen [43], who showed that the related theory of a massless antisymmetric tensor field[3] B does have a de Sitter-invariant quantum state with a well-defined two-point function. In particular, they consider the field given by

$$S_B = -\frac{1}{12}\int H^2, \quad (2.79)$$

where H is a 3-form field as the field strength of the 2-form B field, i.e. $H = dB$. As a peculiarity of 4 dimension, there is always a one-to-one correspondence between 3-form and 1-form fields via the Hodge duality. Thus one can introduce the 1-form $^*H$ as

$$^*H_\alpha = \frac{1}{6}\epsilon_{\alpha\mu\nu\rho}H^{\mu\nu\rho}. \quad (2.80)$$

[3] Such a field resembles the so-called Kalb-Ramond field in string theory, where the generic dimension of the target spacetime is 26 (bosonic) or 10 (supersymmetric). Here in our context, the spacetime is 4 dimensional, which is a key fact for the correspondence between the B field and the gradient of a scalar field.

Then it is not hard to see that once H satisfies the EOM $\nabla_\mu H^{\mu\nu\rho} = 0$, its dual field ${}^*H$ is closed, i.e. $d\,{}^*H = 0$. Thus it is at least locally exact, i.e. ${}^*H = d\phi$ for a scalar field ϕ. Then the EOM of H implies that ϕ satisfies the EOM of a minimally-coupled massless scalar field

$$\Box\phi = 0. \tag{2.81}$$

Therefore, the expectation values of combinations of the dual of the three-form field strength ${}^*H$ can be interpreted as giving the expectation values of gradients of a massless scalar field.

In [43], B is regarded as a fundamental field and is quantized in both a de Sitter-breaking gauge and the Feynman gauge which preserves de Sitter invariance; then, despite the fact that the Wightman function of B may or may not be de Sitter invariant depending on the gauge, the two-point function of (the Hodge dual of) the gauge invariant field strength is always de Sitter invariant. In particular, they obtained (changing a minus sign due to different signature in that paper)

$$\langle\, {}^*H_a(x)\, {}^*H_{b'}(x')\rangle = \frac{1}{32\pi^2}\left[(4z^{-2}+2z^{-1})n_a n_{b'} + (z^{-2}+2z^{-1})g_{ab'}\right], \tag{2.82}$$

which is the same as our result (2.46) for the Wightman function of gradients of the scalar field. In principle one can take combinations of line integrals of ${}^*H$ to give differences of the scalar field at the two endpoints of the lines and hence to get the expectation values of the shift-invariant operators of the massless scalar field considered here, but that was not done explicitly in [43].

Similar results in Euclidean de Sitter, i.e. the S^4, are given in [40] using the BRST quantization associated with the shift invariance of the scalar field. As we mentioned before, the box operator in S^4 has a discrete spectrum $\{\phi_n\}$, and the field can be expanded as

$$\phi(x) = \sum_n a_n \phi_n(x), \tag{2.83}$$

where the coefficients a_n should be regarded as coordinate-independent variables of the path integral. Then the action becomes

$$S = \frac{1}{2}(m^2 a_0^2 + \sum_{n\neq 0} \lambda_n^2 a_n^2). \tag{2.84}$$

The IR divergence in this case is due to the fact that as $m \to 0$, there will be no damping of the zero mode in the partition function which contains

$$\int da_0 e^{-S} \to \int da_0 \sim \infty. \tag{2.85}$$

By introducing the ghost fields associated with the shift invariance, the BRST method introduces, among others, a 'gauge' parameter α and a 'gauge-fixing' term

$$\frac{1}{2\alpha V_4}(\int_{S^4}\phi)^2 = \frac{1}{2\alpha}a_0^2, \tag{2.86}$$

which breaks the shift invariance. It is this term that provides the necessary damping of the contribution from the zero mode a_0 to remove the divergence as $m \to 0$. Then the original IR divergence just corresponds to taking the 'gauge' parameter $\alpha \to \infty$. In other words, the IR divergence is not physical but just a 'gauge' artifact. However, the BRST method cannot be directly applied to Lorentzian de Sitter due to the infinite volume of the non-compact manifold. Moreover, the author did not consider restricting the quantum state to giving only the expectation values of shift-invariant operators on the Lorentzian de Sitter spacetime.

Bros, Epstein, and Moschella [9], following earlier work in two-dimensions by Bertola, Corbetta, and Moschella [8], also used a construction similar to ours of removing the Euclidean zero-eigenvalue eigenfunction (the constant function) and then noting that this gives "a local de Sitter invariant quantization of that field on the space of test functions having zero mean value." As Moschella expresses it [44], our work "is exploring concrete states belonging to the physical subspace we have generally constructed."

Kirsten and Garriga [7] have given the results perhaps most obviously similar to ours, a de Sitter-invariant state that is not normalizable. They note, "This should not be taken as an indication that the state is pathological: it simply means that all values of [the spatial mean of the scalar field] are equally probable." A Euclidean derivation of this de Sitter-invariant state has been given by Tolley and Turok [45]. What is perhaps somewhat new in our work, besides the detailed results, is the explicit description of the quantum state as being defined by giving the expectation values of just the shift-invariant operators.

In our detailed results we have shown, in the case of field derivatives, that the two-point function becomes vanishing as the timelike separation goes to infinity. In the case of products of field differences, the two clusters (x, y) and (u, v) in both Configuration 1 and 2 become uncorrelated as the separation between the two pairs of points becomes large, although there is a divergence in Configuration 2 in which one takes the product of differences of fields between points that are moved apart to arbitrarily great separation.

In other words, the shift-invariant expectation values obey a cluster-decomposition property in that the field derivatives become only weakly correlated at widely separated points. For the product of field differences, if each of the clusters in which the field differences are taken is kept at fixed size but separated widely from the other cluster whose field differences are multiplied by the first difference, then the expectation value goes to zero. That is, the correlation between the differences of fields at two point-pairs that each have fixed separation goes to zero as one pair is widely separated from the other pair. There is no IR divergence for such quantities, but only for products of differences of fields in which the separations of the points whose different field values are used in the product are taken to infinity.

2.5.2 Implication for de Sitter Quantum Gravity

Although the massless scalar field is only a poor-man's toy model for gravity, it does suggest that there is a de Sitter-invariant gravitational quantum state with no IR divergences for the correlations of local gauge-covariant quantities, such as correlations between the Weyl tensor at one location and at another. Mora and Woodard [15], and Mora, Tsamis, and Woodard [16], correcting some minor errors in earlier work by Kouris [14], have recently confirmed this at the linearized level, though, contrary to our own opinion, they conjecture that this lack of IR divergences would not be true at the next order. Cf. similar results by Fröb, Roura, and Verdaguer that include the effect of one-loop corrections from matter fields [46] and apply for the full Riemann tensor [47].

On the other hand, the divergence one gets for the expected product of differences of massless scalar field values when the points giving the differences are pulled apart to infinity may be analogous to the divergences in the gauge-fixed graviton propagator when the gauge fixing is taken over a spacetime separation that is taken to infinity (see, e.g., [10, 48, 49]). For example, in a classical model for de Sitter spacetime with quantum fluctuations that is a classical spacetime with small metric perturbations, one may define the following two-time function along a central timelike geodesic that might well diverge in the limit that the proper time between the two times goes to infinity:

As illustrated by Fig. 2.8, if t is the proper time along the central timelike geodesic, let the two times be $t = 0$ and $t = T$. At point P, the $t = 0$ point along the central timelike geodesic, construct the full set of spatial geodesics orthogonal to the timelike geodesic and form the locus of points at proper distance r from the $t = 0$ point, a two-dimensional 'sphere' S of proper radius r. Parallel-propagate the four-velocity of the central timelike geodesic along each of the spatial geodesics to the 'sphere' and use these timelike vectors as the four-velocities of new timelike geodesics running forward in time from the 'sphere' at $t = 0$. Go forward by a proper time T along each of these new timelike geodesics to get a second approximate sphere S' that would be nearly simultaneous to P', the $t = T$ point along the central timelike geodesic.

Denote $\langle X \rangle$ as the average of a quantity X over the S^2

$$\langle X \rangle = \frac{1}{A} \int X dA. \tag{2.87}$$

Then we can construct the following two dimensionless invariant measures of the distortion

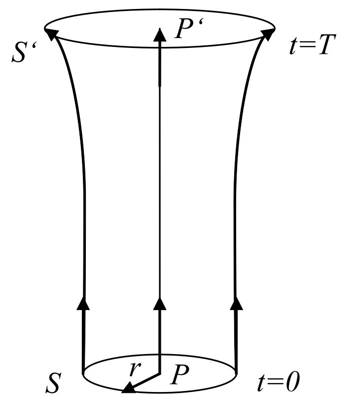


Figure 2.8: Distortion of a small 'sphere' evolving along a central timelike geodesic in perturbed de Sitter spacetime.

of the shape of this second approximate sphere S': a bulk 'ellipticity' E

$$E = \frac{\langle r'^2 \rangle - \langle r' \rangle^2}{\langle r' \rangle^2}, \tag{2.88}$$

i.e. the variance of the proper radial distance from point P' to the sphere, divided by the square of the average radial distance, and a surface 'distortion factor' F

$$F = A^2(S')\langle R^2(S')\rangle - 64\pi^2, \tag{2.89}$$

that is, the square of the area $A(S')$ of S', multiplied by the intrinsic Ricci scalar of S', minus $64\pi^2$ which is the value this product has for a round sphere[4]. For pure de Sitter spacetime, the linear size of the initial sphere (if it is small, $r \ll 1$) will just grow by a factor of $\cosh T$, but E and F will remain zero. For distortions of the spacetime with wavelength much longer than the size of the sphere, and for a small sphere size, $r \ll 1$, the initial 'sphere' will distort to an approximate ellipsoid, and for small distortions, E and F will be approximately quadratic in the normalized differences in the semimajor axes of the approximate ellipsoid and hence approximately quadratic in the metric perturbations in a synchronous gauge.

Given the $t = 0$ and $t = T$ points that can be taken to define the central timelike geodesics (at least in the absence of caustics), and given the initial radius r, the measures

[4] Recall that for a round sphere, $A = 4\pi r^2$, the Ricci scalar curvature is $2/r^2$, thus it can be easily obtained as $A^2\langle R^2\rangle = 64\pi^2$.

E and F of the distortion of the approximate sphere at time $t = T$ obtained by evolving the initial 'sphere' of proper radius r at time $t = 0$ are perfectly gauge-invariant quantities, but since the timelike geodesics of proper time length T that go into the definitions can be continually distorted by the fluctuating spacetime, these T-dependent gauge-invariant quantities E and F need not remain finite as T is taken to infinity. They are not localized quantities, or products of two localized quantities (like the product of two Weyl tensors at different points), so even in a de Sitter-invariant quantum state in which E and F do not depend on the initial $t = 0$ point or the tangent vector there of the central timelike geodesic, these nonlocal gauge-invariant quantities $E(T)$ and $F(T)$ can in principle increase without limit as T is taken to infinity. This does not mean that the de Sitter-invariant quantum state has local fluctuations that are in any sense growing from one location or time to another (which would violate de Sitter invariance) but just that gauge-invariant quantities like $E(T)$ and $F(T)$ that probe a whole region of spacetime, which gets larger and larger the larger one makes T, can grow indefinitely with T, analogously to the way that the product of differences of massless scalar field values can grow with the proper time separations between the points whose difference of field values are used in the product.

Chapter 3

Hořava-Lifshitz Gravity

3.1 Basics on Hořava-Lishitz Gravity

It is a well-known fact that GR is not renormalizable, which is also reflected by the fact that the gravitational coupling κ, or the Newton's constant G_N, has negative mass dimension[1] $[G_N] = M^{-2}$. As a non-renormalizable theory, it should be regarded as an effective field theory. That is, it is a working theory to predict low energy physics, but as the energy scale goes higher, it should be completed by some more fundamental theory. It has been known for long time that introducing higher derivatives can improve the renormalizability [50, 51]. However, naively adding higher-derivative interactions would inevitably lead to ghost degrees of freedom which causes violation of unitarity. This can be easily understood in the relativistic case, since adding higher-derivative terms, such as the squares of the curvature, would necessarily introduce higher order time derivatives, which give rise to a violation of unitarity.

The so-called Hořava-Lifshitz gravity [52, 18] is introduced as a possible UV completion to GR at the cost of breaking Lorentz invariance at high energy. Lorentz invariance is expected to be emergent as an accidental symmetry in the IR where GR is recovered. As we shall see, as a consequence of its non-relativistic nature, one can add higher-order spatial derivatives while maintaining time derivatives no higher than second order, therefore improving the UV behavior without violating unitarity.

Inspired by some condensed matter models and dynamics of critical systems [53, 54, 55],

[1] Note that in order to regard GR from the perspective of QFT and in particular to apply the techniques of dimensional analysis on renormalizability, one should use the units $c = \hbar = 1$ in this relativistic case. Therefore the dimension of G_N in ordinary units $[G_N] = M^{-1}L^3T^{-2}$ becomes $[G_N] = M^{-2} = L^2$. This should be distinguished from the "geometrized units" $G_N = c = 1$ usually used in GR or the Planck units $G_N = c = \hbar = 1$.

Hořava applied in the context of quantum gravity the idea of anisotropic scaling,

$$t \to \lambda^z t, \quad x^i \to \lambda x^i, \tag{3.1}$$

where space and time coordinates are scaled differently as characterized by the dynamical critical index z. Accordingly, the original diffeomorphism of GR is replaced by the so-called *foliation-preserving diffeomorphism* (FPD)

$$t \to \tilde{t}(t), \quad x^i \to \tilde{x}^i(t, x^j). \tag{3.2}$$

In other words, the spacetime as a differentiable manifold with a metric is assumed to have an additional structure, a fixed foliation, i.e. a particular way of slicing the spacetime into spatial leaves. The FPDs are just coordinate transformations which preserve this extra structure. In the following, the dimensions of objects are measured in units of spatial momentum. In particular, the dimensions of time and space coordinates are

$$[t] = -z, \quad [x^i] = -1. \tag{3.3}$$

The definitions of the dimensions here and below may seem 'artificial' and confusing. However, this is just a matter of convention in choosing the units. The physics underlying is the anisotropic scaling (3.1). We give a detailed explanation of this point in Appendix F, following [56].

As time and space are treated on a different footing, the natural and proper metric form is given by the ADM formalism

$$ds^2 = -N^2 c^2 dt^2 + \gamma_{ij}(dx^i + N^i dt)(dx^j + N^j dt), \tag{3.4}$$

where $[c] = z - 1$, and the dimensions of the spatial metric, lapse and shift are

$$[\gamma_{ij}] = [N] = 0, \quad [N_i] = [N^i] = z - 1. \tag{3.5}$$

Under infinitesimal FPDs

$$\delta t = \alpha(t), \quad \delta x^i = \beta^i(t, x^j), \tag{3.6}$$

these fields transform as

$$\delta N = \beta^k \partial_k N + \dot{\alpha} N + \alpha \dot{N}, \tag{3.7}$$

$$\delta N_i = \partial_i \beta^j N_j + \beta^j \partial_j N_i + \dot{\beta}^j \gamma_{ij} + \dot{\alpha} N_i + \alpha \dot{N}_i, \tag{3.8}$$

$$\delta \gamma_{ij} = \partial_i \beta^k \gamma_{kj} + \partial_j \beta^k \gamma_{ki} + \beta^k \partial_k \gamma_{ij} + \alpha \dot{\gamma}_{ij}, \tag{3.9}$$

where a dot indicates the time derivative. As we can see, γ_{ij}, N_i, N transform respectively as a tensor, a vector and a scalar under the time-independent pure spatial diffeomorphisms, while γ_{ij} transforms as a scalar and N and N_i as covectors under time reparametrizations.

According to the FPD, the time derivative of the spatial metric γ_{ij} should only appear in the extrinsic curvature

$$K_{ij} = \frac{1}{2N}(\dot{\gamma}_{ij} - D_i N_j - D_j N_i), \tag{3.10}$$

where D_i is the covariant derivative associated with γ_{ij}. Indeed, under FPD, $\dot{\gamma}_{ij}$ by itself is not covariant, whereas K_{ij} is covariant, i.e. it transform as a 3D tensor under the spatial diffeomorphisms, while transforming as a scalar under time reparametrizations.

In general, a function is called projectable if it depends only on the foliation hypersurface and thus is a function of t only. Otherwise it is non-projectable. Hořava's original model is a projectable one. But we will consider the non-projectable case in which N is allowed to be a function of both time and space.

The most general form of the action of the theory is

$$S = \frac{1}{2\kappa_H^2}\int dtd^3xN\sqrt{\gamma}(\mathcal{L}_K - \mathcal{L}_V), \tag{3.11}$$

where the constant κ_H can be related to the gravitational constant of HL theory G_H and the Planck scale M_p by

$$\kappa_H^2 = 8\pi G_H = M_p^{-2}. \tag{3.12}$$

The kinetic term is

$$\mathcal{L}_K = K_{ij}K^{ij} - \lambda K^2, \tag{3.13}$$

where $K = K_{ij}\gamma^{ij}$. The dimensionless coupling constant $\lambda \neq 1$ reflects the Lorentz-violation. $\lambda = 1$ corresponds to the case of GR, where the higher symmetry of the full diffeomorphism is recovered (assuming the potential part $\mathcal{L}_V$ has a suitable form). The dimension of the K squared term

$$[K_{ij}K^{ij}] = [K^2] = 2z, \tag{3.14}$$

together with that of the volume element

$$[dtd^3x] = -3 - z, \tag{3.15}$$

fix the dimension of the coupling constant κ_H as

$$[\kappa_H] = (z - 3)/2, \tag{3.16}$$

in order that the action be dimensionless. This implies that the theory in 3+1 dimension can be made power-counting renormalizable if $z = 3$ so that the coupling κ_H (or G_H) is dimensionless.

The kinetic term is generic in all versions within the framework of HL gravity. It is uniquely fixed by the FPD symmetry and the requirement that it contains only terms quadratic in the time derivative of the spatial metric $\dot{\gamma}_{ij}$. On the other hand, in the spirit of effective field theory, the potential term $\mathcal{L}_V$ should contain *all* terms constructed in a manner which (1) are allowed by the symmetries and (2) are of a dimension not higher than that of the kinetic term. In general, the number of possible potential terms is large, although it is finite. Hořava imposed the so-called detailed balance condition [18], which is inspired by condensed matter systems, to reduce the possible terms. There is, however, no fundamental reason to insist on such condition, which can be relaxed within the projectable version [57, 19]. Then the most general potential terms consist of the spatial metric and its spatial derivatives. Moreover, inspired by solving some problems associated with the projectable version, the non-projectable version with the most general potential terms are also studied [23, 58, 21]. The general form contains extra terms involving the spatial derivatives of the lapse function in addition to those in the projectable version.

In this way, the higher spatial derivative terms improve the UV behavior, and by choosing the appropriate critical exponent z the theory can be made at least power-counting renormalizable. Meanwhile, due to its non-relativistic nature, the action is still second order in time derivatives, therefore avoiding the non-unitary problem.

Since we are only concerned with the IR limit in the non-projectable version, the lowest order terms we consider are

$$-\mathcal{L}_V = \xi R + \eta A^i A_i, \tag{3.17}$$

where $A_i = \partial_i \ln N$ is the acceleration associated with the normal vector of the spatial slices, and we have dropped a constant term (cosmological constant) which is irrelevant for our purpose. If the parameter η is taken to ∞, it enforces $\partial_i N \to 0$, thus corresponding to the projectable version. The parameter ξ can be set to unity by a rescaling of the time when (and only when) matter is not present. We will consider pure gravity without matter, but still keep ξ as general parameter.

Due to the breaking of the full diffeomorphism group down to FPD, HL gravity is expected to have extra degrees of freedom in comparison with GR. Indeed, studies [19, 23] of linear perturbations on a Minkowski background reveal that, in addition to the usual two spin-2 modes as in GR, there is an extra spin-0 mode. This corresponds to the excitation of the foliation structure, which is essentially characterized by a scalar field in the gravitational sector. However, the extra scalar degree of freedom doesn't consistently decouple at low energy, and therefore the theory cannot flow to GR. There are some pathologies associated with this scalar degree of freedom, e.g. the strong coupling problem and the inconsistency of the phase space structure [59, 60, 61, 62]. It is still an open question whether the general framework of HL gravity should ultimately be regarded as a viable scheme to provide a UV completion of standard GR, or just as a phenomenological model for applying interesting ideas (c.f. [63, 64, 65] for some recent works in this direction).

Within the general framework of HL gravity, one can modify the action by extending the symmetry of FPD to include a local U(1) symmetry [66], whereby the extra degree of freedom can be eliminated. The original proposal [66] implies that the new U(1) symmetry forces λ to 1, but it was later argued [67] that it is possible for such a mechanism to hold for general λ. Also, this U(1) gauge symmetry can be helpful in solving some problems in the strong coupling problem in both the projectable version [68] and non-projectable version [69].

3.2 Einstein-aether Theory and Hořava-Lifshitz Gravity

In this section, we discuss the correspondence between the IR limit of HL gravity and Einstein-aether theory. This enables us to study HL gravity using some results already obtained in the latter context.

3.2.1 Einstein-aether Theory

The Einstein-aether theory [70, 71] (see [72] for a review) is a phenomenological model of gravity with Lorentz violation from the inclusion of a timelike unit vector field, i.e. the aether. It can be regarded as a 4-velocity field which fixes a preferred frame of reference thereby breaking the boost symmetry . Moreover, it must be dynamical as required by general covariance. In other words, it preserve the full diffeomorphism invariance, in contrast to the original idea of HL gravity. (As we shall see, however, the two theories are essentially closely related.) The Einstein-aether theory is a tensor-vector theory of gravity; in particular the Lorentz violation only effects the gravitational sector. Matter is usually assumed to be not coupled to the aether field directly so that there is no Lorentz violation in the matter sector, at least at the classical level. However, as mentioned in [72], Lorentz-violating effects in the gravitational sector may contaminate the matter sector via loop effects. But such effects in the matter sector are tightly constrained by observations [73]. In any case, the issue of matter coupling should not concern us, since we only consider pure gravity without matter.

The action of this theory is given by, following the convention of [74],

$$S = \frac{1}{16\pi G_{æ}} \int d^4x\sqrt{-g}[\ {}^{(4)}R + L_{æ} + Q(u^a u_a + 1)], \tag{3.18}$$

where Q is a Lagrange multiplier to impose the condition that u^a is a timelike unit vector, and[2]

$$L_{æ} = -K^{ab}{}_{cd}\nabla_a u^c \nabla_b u^d \equiv -[c_1 \nabla_a u_b \nabla^a u^b + c_2(\nabla_a u^a)^2 + c_3 \nabla_a u^b \nabla_b u^a - c_4 A^a A_a], \quad (3.19)$$

with $A^a \equiv u^b \nabla_b u^a$. For convenience, we also denote each term multiplied by c_i as K_i, e.g. $K_1 \equiv \nabla_a u_b \nabla^a u^b$, etc. In this thesis we consider Einstein-aether theory without the presence of matter. Varying the action with respect to the metric field and the aether field as well as the Lagrange multiplier gives the EOM as

$$G_{ab} = T^{æ}_{ab}, \quad (3.20)$$

$$\nabla_c J^c{}_a + Q u_a + c_4 A_c \nabla_a u^c = 0, \quad (3.21)$$

$$u^a u_a = -1, \quad (3.22)$$

where the energy-momentum tensor of the aether field is

$$\begin{aligned} T^{æ}_{ab} &= -\frac{1}{2} g_{ab} J^m{}_n \nabla_m u^n + c_4 A_a A_b + Q u_a u_b + c_1(\nabla_a u_c \nabla_b u^c - \nabla_c u_a \nabla^c u_b) \\ &+ \nabla_c(u^c J_{(ab)} + J^c{}_{(a} u_{b)} - u_{(a} J_{b)}{}^c), \end{aligned} \quad (3.23)$$

and where $J^a{}_b \equiv K^{ac}{}_{bd} \nabla_c u^d$.

3.2.2 Identification to the IR Limit of HL Gravity

The identification of the IR limit of HL gravity with Einstein-aether theory was first proposed by Jacobson in [76]. Similar ideas of regarding the HL theory as a generally covariant theory fixed in a particular gauge are also discussed in [20, 77]. The identification is established by imposing the following two conditions on the Einstein-aether theory:

1. Hypersurface orthogonality:

$$u_{[a} \nabla_b u_{c]} = 0. \quad (3.24)$$

   Locally this is equivalent to the existence of a scalar field ϕ such that u_a is the unit covector normal to the hypersurfaces $\phi = const$, i.e.

$$u_a = -\frac{\partial_a \phi}{\sqrt{-g^{ab}\partial_a \phi \partial_b \phi}}, \quad (3.25)$$

   where the minus sign is to make u^a point to the direction of increasing ϕ. It is

[2] The relative sign of c_4 w.r.t. the first three is just a matter of convention s.t. the parameters when identified to HL gravity are the same as those in the earlier papers in the $(+---)$ signature. In [75], there is a total minus sign in $L_{æ}$, and since $J^a{}_b$ is defined the same way in both papers, they can be converted to each other by simply changing $J^a{}_b$ to $-J^a{}_b$ in the EOM.

invariant under the reparameterization $\phi \to \tilde{\phi}(\phi)$. The scalar field ϕ fixes the foliation, in accordance with the HL gravity. Note that ϕ can also be identified as the Stückelberg field to restore general covariance, as will be discussed later. This condition eliminates two of the three degrees of freedom of the unit normal aether field u^a, leaving only one degree of freedom corresponding to the scalar field ϕ and therefore making the theory essentially a tensor-scalar theory of gravity.

2. Choose the coordinate system of the ADM 3+1 formalism, and in particular, to avoid higher (time) derivatives of ϕ, impose the so-called 'unitary gauge', i.e. $\phi = t$ as the time coordinate. Note that the invariance under reparameterization of the scalar field now becomes precisely one of the foliation-preserving diffeomorphism $t \to \tilde{t}(t)$ of the HL gravity.

Under the first condition, one has the following relations (more details of the relevant formulae used in this section are listed in Appendix G):

$$\nabla_a u_b = K_{ab} - u_a A_b, \tag{3.26}$$

where K_{ab} is the extrinsic curvature of the hypersurface $\phi = const$, and $K_{ab}u^a = A_a u^a = 0$. Note also that as a consequence of the hypersurface orthogonality condition

$$\omega_a \equiv \epsilon_{abcd} u^b \nabla^c u^d = 0, \tag{3.27}$$

the c_1, c_3 and c_4 terms in the action are no longer independent due to the relation[3]

$$0 = \omega_a \omega^a = \nabla_a u_b \nabla^a u^b - \nabla_a u_b \nabla^b u^a + A_a A^a. \tag{3.28}$$

So one can eliminate any one of the three terms. For example, in [25], the c_4 term in the action is eliminated.

One can rewrite each term in $L_{æ}$ as:

$$\begin{aligned} c_1 \nabla_a u_b \nabla^a u^b &= c_1 (K_{ab} K^{ab} - A_b A^b), && (3.29) \\ c_2 (\nabla_a u^a)^2 &= c_2 K^2, && (3.30) \\ c_3 \nabla_a u^b \nabla_b u^a &= c_3 K_{ab} K^{ab}. && (3.31) \end{aligned}$$

Putting these together one has

$$L_{æ} = -[c_{13} K_{ab} K^{ab} + c_2 K^2 - c_{14} A_a A^a], \tag{3.32}$$

[3] Using the relation $\epsilon_{\alpha\mu\nu\rho}\epsilon^{\alpha abc} = -3!\delta^{[abc]}_{\mu\nu\rho}$, $\omega_\alpha \omega^\alpha = -6 \cdot \frac{1}{6}(\delta^{abc}_{\mu\nu\rho} + \delta^{bca}_{\mu\nu\rho} + \delta^{cab}_{\mu\nu\rho} - \delta^{bac}_{\mu\nu\rho} - \delta^{acb}_{\mu\nu\rho} - \delta^{cba}_{\mu\nu\rho}) u^\mu \nabla^\nu u^\rho u_a \nabla_b u_c = -(-\nabla_a u_b \nabla^a u^b + 0 + 0 - A^a A_a - (-1)\nabla_a u_b \nabla^b u^a - 0)$ noticing $u_a \nabla_c u^a = 0$. One can also obtain the same result by directly evaluating $u_{[a} \nabla_b u_{c]} u^{[a} \nabla^b u^{c]} = 0$.

where the parameters $c_{ij} \equiv c_i + c_j$.

Note that the particular combinations of the four coefficients c_i are related to hypersurface orthogonality. Indeed, one can use Eq.(3.28) to change the form of $L_{æ}$

$$c_1K_1 + c_2K_2 + c_3K_3 - c_4K_4 \to (c_1 + c_4)K_1 + c_2K_2 + (c_3 - c_4)K_3.$$

This confirms again that c_4 can be set to zero under $c_1 \to c_1 + c_4$ and $c_3 \to c_3 - c_4$.

Inserting the 3+1 splitting of the 4D Ricci scalar

$$^{(4)}R = 2(G_{ab}u^au^b - \,^{(4)}R_{ab}u^au^b) = (R - K_{ab}K^{ab} + K^2) - 2\,^{(4)}R_{ab}u^au^b, \tag{3.33}$$

where $^{(4)}R_{ab}u^au^b$ is equal to a sum of the difference of the c_2 and c_3 terms and some total derivative terms,

$$^{(4)}R_{ab}u^au^b = -u^a(\nabla_a\nabla_b - \nabla_b\nabla_a)u^b = K^2 - K_{ab}K^{ab} - \nabla_a(u^a\nabla_bu^b) + \nabla_b(u^a\nabla_au^b). \tag{3.34}$$

Note these are purely geometrical relations independent of the dynamics. Neglecting the boundary terms, the action becomes (extremizing the Q term is implicit in the above calculation, so it doesn't have to be written in the action)

$$S = \frac{1}{16\pi G_{æ}}\int d^4x\sqrt{-g}[R + (1 - c_{13})K_{ab}K^{ab} - (1 + c_2)K^2 + c_{14}A_aA^a]. \tag{3.35}$$

Under the identifications

$$(1 - c_{13}) = \xi^{-1}, \quad (1 + c_2) = \lambda/\xi, \quad c_{14} = \eta/\xi, \quad G_{æ} = G_H/\xi, \tag{3.36}$$

the above action becomes

$$S = \frac{1}{16\pi G_H}\int d^4x\sqrt{-g}(\xi R + K_{ab}K^{ab} - \lambda K^2 + \eta A_aA^a), \tag{3.37}$$

which takes the similar form (4.5) of the IR limit of the non-projectable version of the HL gravity.

It should be noted, however, that this is *not* yet exactly the same as the latter. In the HL gravity, the action is written in terms of 3D spatial objects defined on the hypersurfaces of the fixed foliation, possessing only the foliation-preserving diffeomorphism. On the other hand, the above action consists of fully covariant objects. Thus the full diffeomorphism invariance is preserved. The action is independent of any particular foliation, although it can take the *same form* when expressed in terms of the *components* in a particular family of coordinates, where the theory becomes HL gravity.

Indeed, to establish the correspondence to HL gravity, one has to impose the second condition. Fixing this gauge reduces the full diffeomorphism to the foliation-preserving diffeomorphism. Now, in the particular foliation consisting of hypersurfaces orthogonal to u^a and with the coordinate system whose time coordinate is ϕ, the action takes the form

$$S = \frac{1}{16\pi G_H} \int dt d^3x N \sqrt{\gamma}(\xi R + K_{ij}K^{ij} - \lambda K^2 + \eta a_i a^i), \tag{3.38}$$

where K_{ij} and a_i can be regarded as either the coordinate components of the covariant 4D objects K_{ab} and A_a in the unitary gauge, or 3D non-relativistic objects by themselves, covariant under the foliation-preserving diffeomorphism. This is exactly the action (4.5).

In summary, the IR limit of HL gravity is equivalent to Einstein-aether theory in the unitary gauge with the aether field being hypersurface orthogonal. The advantage of this correspondence for the study of HL gravity is that one can directly use some of the previously obtained results in Einstein-aether theory to shed some light on HL gravity.

In particular, note that the constant $G_{æ}$ appearing in the action should not be directly identified with the Newton's constant G_N. In fact, with the presence of matter, by taking the slow-motion and weak field limit of Einstein-aether theory, the effective Newton's constant is [78, 72]

$$G_N = \frac{G_{æ}}{1 - c_{14}/2}. \tag{3.39}$$

This requires $c_{14} < 2$ in order to ensure a positive G_N. Moreover, the study of linear perturbations on a Minkowski background in Einstein-aether theory [79] implies that the squared speeds of spin-2 and spin-0 modes[4] in the aether's rest frame are

$$v^2_{spin-2} = \frac{1}{1 - c_{13}}, \quad v^2_{spin-0} = \frac{c_{123}(2 - c_{14})}{c_{14}(1 - c_{13})(2 + c_{13} + 3c_2)}. \tag{3.40}$$

Besides, there are also constraints coming from the positivity of the energy of these modes in linear theory [80, 81, 72]. It is found that the energy of spin-2 modes is positive definite, just as in pure GR, while the sign of the energy of the spin-0 modes is $c_{14}(2-c_{14})$. Combining these requirements of linear stability and energy positivity leads to

$$c_{13} < 1, \quad 0 < c_{14} < 2. \tag{3.41}$$

As we shall see below, one can use the correspondence between Einstein-aether theory and HL gravity to constrain the parameters of the latter. Note in particular that the condition $c_{13} < 1$ guarantees the positivity of the parameter ξ, η of the parameters of HL gravity, as can be seen immediately from (3.36).

[4] The spin-1 modes do not concern us since we are only interested in the case where the aether field obeys hypersurface orthogonality, a condition which effectively eliminates the spin-1 degree of freedom.

3.2.3 Restored General Covariance in Stückelberg Formalism

The method discussed above can be summarized as starting from a generally covariant theory with extra degrees of freedom, then imposing certain conditions to break the full diffeomorphism invariance resulting in a Lorentz-violating theory which can be identified as the low energy limit of the HL gravity. In fact, one can also reverse the logic and promote the HL gravity to a fully covariant theory. As argued in [20, 21, 77], the full HL graivty (including the higher-derivative terms) can be made into a fully spacetime covariant theory by using the Stückelberg formalism[5], where a scalar field ϕ is introduced to restore the full general covariance. It is essentially the same one as introduced in (3.25) to characterize the foliation structure. So the extra degree of freedom due to the breaking of the full diffeomorphism invariance is now transmuted explicitly to the scalar field and the spin-0 mode can be identified as the excitation of ϕ.

Using the normal vector u^a and the spacetime metric g_{ab}, the 3D metric h_{ij} can be promoted to be a covariant object

$$h_{ab} = g_{ab} + u_a u_b. \tag{3.42}$$

Similar manipulation can be applied to all other 3D quantities of HL gravity, in particular the extrinsic curvature K_{ij} and the 3D Riemann tensor becomes

$$K_{ab} = h^c_a \nabla_c u_b, \tag{3.43}$$

$$R_{abc}{}^d = h^e_a h^l_c h^f_b h^d_m \, {}^{(4)}R_{efl}{}^m - K_{ca}K^d_b + K_{cb}K^d_a. \tag{3.44}$$

Moreover, the 3D covariant derivative is

$$D_c T^{a\dots}_{b\dots} = h^a_m \dots h^n_b \dots h^q_c \nabla_q T^{m\dots}_{n\dots} \ . \tag{3.45}$$

Now the HL action

$$S \sim \int dt d^3x N\sqrt{h}(K_{ij}K^{ij} - \lambda K^2 + \xi R + \eta A^i A_i + \dots), \tag{3.46}$$

where the dots denote higher-derivative terms, can be written in a generally covariant form

$$S \sim \int d^4x \sqrt{-g}(\xi \, {}^{(4)}R + (1-\xi)K_{ab}K^{ab} - (\lambda - \xi)K^2 + \eta A^a A_a + \dots). \tag{3.47}$$

[5] In [77], the normal vector u^a is promoted to a dynamical field. This is equivalent to the Stückelberg formalism of [20].

Absorbing the factor ξ in the constant in front of the action, one has

$$S \sim \int d^4x\sqrt{-g}(\,^{(4)}R - c_1 K_{ab}K^{ab} - c_2 K^2 + c_{14}A^a A_a + \dots), \tag{3.48}$$

where

$$c_1 = 1 - \xi^{-1}, \quad c_2 = \lambda/\xi - 1, \quad c_{14} = \eta/\xi, \tag{3.49}$$

which are essentially consistent with (3.36)[6].

This promoted theory with full general covariance can be thought of as GR with a derivatively coupled scalar field, where the scalar field is contained in the covariant quantities such as K_{ab} and h_a^b in the form of u^a as given by (3.25). Due to the particular form of (3.25), in particular the 'kinetic term' $g^{ab}\partial_a\phi\partial_b\phi$ appearing under a square root in the denominator, it seems that the scalar field in the action contains higher order spacetime derivatives, in particular, time derivatives higher than second order, which would imply the presence of ghosts. However, this is not the case [20]. Precisely due to the particular form of (3.25), the higher derivatives in the action and the EOM can be eliminated by fixing the unitary gauge, i.e. choosing the coordinate systems with the time coordinate equal to the value of ϕ. This is preserved by the FPD and is just what is required by the key idea of HL theory. A detailed proof of the no-ghost theorem was given in [21].

[6] Note that using the condition (3.28) following from hypersurface orthogonality, one can always eliminate the c_3 term of the Einstein-aether theory. In particular, using $K_1 - K_3 + K_4 = 0$, the K_3 term in the action can be replaced by $K_1 + K_4$, and

$$c_1K_1 + c_2K_2 + c_3K_3 - c_4K_4 = c_{13}K_1 + c_2K_2 - (c_4 - c_3)K_4.$$

Relabelling the parameters as $c_{13} \to c_1$ and $c_4 - c_3 \to c_4$, and using the relation (3.29), one obtains the result in the main text.

Chapter 4

IR Limit of Hořava-Lifshitz Gravity

As we have discussed, the IR limit of HL gravity should be regarded as a Lorentz-violating theory, in contrast to standard GR. Before we proceed to study its solution, it is useful to note the fact [23] that, the effective gravitational constant appearing in the Newton's law obtained by taking the Newtonian limit is

$$G_N = \frac{G_H}{\xi(1-\frac{\eta}{2\xi})}, \tag{4.1}$$

and that appearing in the Friedmann equation by studying the homogeneous isotropic cosmology is

$$G_{cosmo} = \frac{2G_H}{3\lambda - 1}. \tag{4.2}$$

In general $G_N \neq G_{cosmo}$, unlike GR. Such a difference is a generic feature of more general gravitational theories with Lorentz violation, including the ghost condensate model [82] and the Einstein-aether theory. Indeed, we have mentioned the effective G_N in Einstein-aether theory in (3.39). In addition, the corresponding G_{cosmo} is given by [78, 72]

$$G_{cosmo} = \frac{G_{æ}}{1+\frac{1}{2}(c_{13}+3c_2)}. \tag{4.3}$$

Identifying the parameters via the correspondence (3.36), one can easily verify that G_N and G_{cosmo} are consistent with the correspondence between HL gravity and Einstein-aether theory. In any case, the discrepancy between the two values are tightly constrained by measurement of He^4 primordial abundance [78, 72] as

$$|G_{cosmo}/G_N - 1| \lesssim 1/8, \tag{4.4}$$

which is a constraint on the relations between the two parameters η and λ. Since we will consider the static case such that λ is not a parameter of the solution, this constraint will not concern us. However, to have a positive G_N, (4.1) does imply $\eta/\xi < 2$.

In addition, the parameterized post-Newtonian (PPN) formalism [83] is studied in [21], where it is shown that all except two PPN parameters are the same as those in GR, while these two PPN parameters, called α_1, α_2 which characterize preferred frame effects, are constrained by observations in order for the theory to be phenomenologically viable.

4.1 Static Spherically Symmetric Vacuum Solution

The general form of the IR limit of the non-projectable HL gravity is

$$S = \frac{1}{16\pi G_H}\int dt d^3x N\sqrt{\gamma}(K_{ij}K^{ij} - \lambda K^2 + \xi R + \eta a^i a_i), \tag{4.5}$$

where $a_i \equiv \partial_i \ln N$ and the indices are raised by γ^{ij}. For simplicity the cosmological constant which is irrelevant in the following is neglected. The projectable version corresponds to $\eta \to \infty$, and the GR limit is obtained by $\lambda, \xi \to 1$ and $\eta \to 0$.

In this notation, the Hamiltonian constraint is given by

$$-K_{ij}K^{ij} + \lambda K^2 + \xi R - \eta a_i a^i - 2\eta D_i a^i = 0, \tag{4.6}$$

the momentum constraint is

$$D_j(K^{ij} - \lambda K\gamma^{ij}) = 0, \tag{4.7}$$

and the evolution equation, obtained by varying the action with respect to γ_{ij}, is

$$\begin{aligned} &-\frac{1}{N}(\partial_t - N^m D_m)P^{ij} - KP^{ij} - 2K^{mi}P^j_m - \frac{1}{N}(P^{mi}D_m N^j + P^{mj}D_m N^i) \\ &+\frac{1}{2}\gamma^{ij}K^{mn}P_{mn} - \xi G^{ij} + \frac{\xi}{N}(D^iD^jN - \gamma^{ij}D_mD^mN) + \eta(\frac{1}{2}a_m a^m\gamma^{ij} - a^i a^j) = 0, \end{aligned} \tag{4.8}$$

where $P^{ij} \equiv K^{ij} - \lambda\gamma^{ij}K$. Note the following:

- In contrast to the non-projectable version that we consider here, in the projectable version, the variation with respect to $N(t)$ would not lead to a local constraint as above, but instead to a spatial integral. As a consequence, so far as local dynamics is concerned, the projectable version has one less constraint then the non-projectable one.

- Since the shift N_i only appears in the kinetic part, the momentum constraint is the same even if higher order derivative terms are included in the action.

Consider the ansatz for the solution which is static and spherically symmetric,

$$ds^2 = -h(r)dt^2 + g(r)dr^2 + r^2 d\Omega^2. \tag{4.9}$$

Note in this form, the geometrical meaning of the coordinate r is that $4\pi r^2$ is the area of the two dimensional sphere. Now the extrinsic curvature is vanishing. Then the Hamiltonian constraint (4.6) becomes

$$\xi R - \eta a_i a^i - 2\eta D_i a^i = 0, \tag{4.10}$$

the momentum constraint is trivial, and (4.8) becomes

$$\xi G^{ij} + \frac{\xi}{N}(D^i D^j N - \gamma^{ij} D_m D^m N) + \eta(\frac{1}{2} a_m a^m \gamma^{ij} - a^i a^j) = 0. \tag{4.11}$$

Inserting the ansatz (4.9), the evolution equation (4.11) gives an algebraic relation for $g(r)$ in terms of h and its derivatives

$$g = 1 + r\frac{h'}{h} + ar^2 \left(\frac{h'}{h}\right)^2, \tag{4.12}$$

where a prime denotes a derivative with respect to r, and $a \equiv \eta/(8\xi)$. Inserting this relation back into the Hamiltonian constraint (4.10) to eliminate g and its derivatives, one obtains a second order differential equation of $h(r)$

$$-r\frac{h''}{h} = a\left(\frac{h'}{h}\right)^3 r^2 + 2\frac{h'}{h}. \tag{4.13}$$

After introducing a function

$$z(r) \equiv rh'/h, \tag{4.14}$$

the above two equations (4.12) and (4.13) become

$$rz' = -(z + z^2 + az^3), \tag{4.15}$$
$$g = 1 + z + az^2. \tag{4.16}$$

Define new parameters

$$b \equiv (1 - \sqrt{1-4a})/2, \quad c \equiv \frac{1}{(1-2b)} = \frac{1}{\sqrt{1-4a}}, \tag{4.17}$$

such that

$$g = (1 + bz)[1 + (1-b)z], \tag{4.18}$$

and $a = b(1-b) = (c^2-1)/(4c^2)$. Note that the constants b, c are real numbers since we expect $a < 1/4$, and that all the constants a, b, c are essentially determined by the parameter of the theory η/ξ. Then regarding Eq. (4.15) as an differential equation for $r(z)$ instead of $z(r)$, i.e.

$$\frac{dr}{r} = -\frac{dz}{z(1+bz)[1+(1-b)z]}, \tag{4.19}$$

one can easily solve

$$r(z) = C_1 \frac{[1+(1-b)z]^{(1-b)c}}{z(1+bz)^{bc}}. \tag{4.20}$$

where C_1 is an integration constant. This implies that $z \to 0$ corresponds to $r \to \infty$.

Using (4.19), the definition (4.14) can be rewritten as

$$\frac{dh}{h} = -\frac{dz}{(1+bz)[1+(1-b)z]}, \tag{4.21}$$

so the metric component h can be solved as a function of z as, using (4.17),

$$h(z) = C_2 \left(\frac{1+bz}{1+(1-b)z} \right)^c. \tag{4.22}$$

Correspondingly,

$$g(r)dr^2 \to g_{zz}dz^2 = \frac{r^2(z)}{(1+z+az^2)z^2} dz^2. \tag{4.23}$$

Now, in the new coordinate system (t, z, Ω), different than (4.9), the solution is

$$ds^2 = -h(z)dt^2 + g_{zz}dz^2 + r^2(z)d\Omega^2, \tag{4.24}$$

where $r(z)$, $h(z)$ and g_{zz} are given by (4.20), (4.22) and (4.23), respectively, with the two integration constants C_1 and C_2 to be determined by further physical conditions.

This is essentially the same as the solution obtained in [24] in the context of HL gravity. Moreover, this is also equivalent to the spherically symmetric solution in static Einstein-aether theory obtained in [25], where the aether is chosen to be 'static', in the sense that it is parallel to the timelike Killing vector. Indeed, due to the spherical symmetry, the aether must be hypersurface orthogonal, which is just one essential condition required to establish the correspondence between Einstein-aether theory and HL gravity. Note that in the case of 'static' aether, there cannot exist a black hole solution, since the aether must be timelike. But if the aether field is allowed to be not aligned with the timelike Killing vector, black solutions can be obtained (numerically) [84, 85].

The solution in the above form is complicated, and the process to extract physical information of the solution in this form is cumbersome. In fact, the form of the metric can be much more simplified by defining a new variable ρ instead of z via

$$z = \frac{2Mc}{c\rho - (1+c)M}, \tag{4.25}$$

such that

$$h = (1 - \frac{2M}{c\rho})^c. \tag{4.26}$$

Correspondingly, $r(z)$ of (4.20) becomes

$$r(\rho) = \frac{C_1}{2M}\rho\left(1 - \frac{2M}{c\rho}\right)^{\frac{1-c}{2}}, \tag{4.27}$$

and

$$g_{zz}dz^2 \to g_{\rho\rho}d\rho^2 = (\frac{C_1}{2M})^2(1 - \frac{2M}{c\rho})^{-c}d\rho^2. \tag{4.28}$$

Now in this new coordinate system (t, ρ, Ω) the solution becomes

$$ds^2 = -C_2^2(1 - \frac{2M}{c\rho})^c dt^2 + (\frac{C_1}{2M})^2(1 - \frac{2M}{c\rho})^{-c}d\rho^2 + (\frac{C_1}{2M})^2\rho^2(1 - \frac{2M}{c\rho})^{1-c}d\Omega^2. \tag{4.29}$$

One can further fix the two integration constants as $C_1 = 2M$ and $C_2 = 1$ such that t and ρ become the standard 'time' and 'radius' coordinates in the asymptotically flat region as $\rho \to \infty$, and M is the mass parameter. This leads to the final form

$$ds^2 = -(1 - \frac{2M}{c\rho})^c dt^2 + (1 - \frac{2M}{c\rho})^{-c}d\rho^2 + \rho^2(1 - \frac{2M}{c\rho})^{1-c}d\Omega^2. \tag{4.30}$$

This solution takes the same form as that of the so-called the Fisher solution[1] or the Janis-Newman-Winicour solution [87, 88, 89, 90, 91, 92] (and a comprehensive analysis of the Fisher solution in arbitrary spacetime dimension was given in [86]). The Fisher solution is the static spherically symmetric solution to Einstein gravity plus a minimally-coupled massless scalar field system, i.e.

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}[\ ^{(4)}R - 2(\partial\phi)^2]. \tag{4.31}$$

The solution is the metric of the same form as our (4.30), together with the scalar field configuration

$$\phi(\rho) = \frac{c\Sigma}{2M}\ln(1 - \frac{2M}{c\rho}), \tag{4.32}$$

where M and Σ are two integration constants which can be respectively identified as the

[1] I thank S. Abdolrahimi for pointing out this and for helpful discussions regarding the paper [86].

Komar mass and the scalar charge. In this case, the constant c is not an independent parameter, but given by

$$c = \frac{M}{\sqrt{M^2 + \Sigma^2}}. \tag{4.33}$$

For real Σ, $c \in [0, 1]$. In particular, $c = 1$ corresponds to the Schwarzschild solution with no scalar hair, and $c = 0$ corresponds to scalar field in a background with vanishing Komar mass. Taking Σ to be imaginary is equivalent to changing the sign of the kinetic term of the scalar field, which turns the scalar field into a ghost. In this case, $c \in (1, \infty)$ if $-M^2 < \Sigma^2 < 0$; and c becomes imaginary if $\Sigma^2 < -M^2$ (this case is used to study traversable wormhole geometries in [93]).

However, for our solution in HL gravity, c is defined to be a real number, and it is constrained by physical requirements to be larger than unity. This makes our solution different from the Fisher solution. For example, following [86], we can consider the Misner-Sharp energy

$$M_{MS}(\rho) = \frac{r}{2}[1 - g^{\rho\rho}(r'(\rho))^2] = \frac{1}{8}\frac{\rho_0[4c\rho - (1+c)^2\rho_0]}{\rho^{(1-c)/2}(\rho - \rho_0)^{(1+c)/2}}. \tag{4.34}$$

The radius where M_{MS} becomes vanishing is

$$\rho_e = \rho_0 \frac{(1+c)^2}{4c}. \tag{4.35}$$

Although M_{MS} can become negative in a region $\rho_0 < \rho < \rho_e$ in the case of the Fisher solution [86] where $c \in [0, 1]$, it is always positive in our case since $\rho_e < \rho_0$. (Note that we only consider $\rho > \rho_0$.) But not all properties are different. Those which are insensitive to the parameter c are the same in both case, e.g. there is no closed trapped surface in both cases.

4.2 Using 3D Method to Obtain Static HL Gravity Solutions

In fact, we do not have to solve the EOM at all to obtain the static spherically symmetric solution. In this case, the 4D problem can be reduced to a problem of 3D Euclidean Einstein gravity minimally coupled to a massless scalar field. This contains the Schwarzschild solution of pure GR as a special case. So we can use the Schwarzschild solution to generate solutions of HL gravity in a very simple way.

In the static case, the shift vector N_i and the extrinsic curvature K_{ij} all vanish; therefore the action of HL gravity becomes[2]

$$S = \frac{1}{2\kappa_H^2}\int dtd^3xN\sqrt{\gamma}(\xi R + \eta a_i a^i). \tag{4.36}$$

Introduce a scalar field v on the spatial hypersurface via

$$N = e^{-2v}, \quad v \equiv -\frac{1}{2}\ln N. \tag{4.37}$$

Then the action becomes

$$S = \frac{1}{2\kappa_H^2}\int dtd^3x\sqrt{\gamma}e^{-2v}(\xi R + \eta 4\partial_i v\partial^i v) = \frac{1}{2\bar{\kappa}_H^2}\int d^3x\sqrt{\gamma}e^{-2v}(R + \eta' 4\partial_i v\partial^i v), \tag{4.38}$$

where $\eta' \equiv \eta/\xi$, and in the second step the time integral $V_t \equiv \int dt$ is absorbed in $\bar{\kappa}_H^{-2} \equiv \xi V_t \kappa_H^{-2}$. Now, the action takes a form analogous to the (Euclidean) low-energy effective string action in the 'string frame' with the 'dilaton' v and a 'wrong' sign for its kinetic term (see, e.g. [94]).

One can rewrite the action in the 'Einstein frame' by the following field redefinition

$$\tilde{\gamma}_{ij} = e^{-4v}\gamma_{ij} = N^2\gamma_{ij}. \tag{4.39}$$

Then

$$\tilde{g}_{ab} = e^{2\omega}g_{ab} \quad \Rightarrow \quad \tilde{R} = e^{-2\omega}(R - 2(D-1)\nabla^2\omega - (D-2)(D-1)\partial_a\omega\partial^a\omega. \tag{4.40}$$

With $D = 3$ and $\omega = -2v$, one has

$$\tilde{R} = e^{4v}(R + 8\nabla^2 v - 8\partial_i v\partial^i v). \tag{4.41}$$

In order to use this result to replace R in the action by an expression in terms of quantities all with tilde, one needs further to take care of the ∇^2 and γ^{ij} in the second two terms:

1. $(\partial v)^2 = e^{-4v}(\tilde{\partial} v)^2$.

2. Using the relation (c.f. Wald's book [95])

$$\tilde{g}_{ab} = \Omega^2 g_{ab} \quad \Rightarrow \quad \tilde{\nabla}^2\tilde{\Phi} = \Omega^{-2}\nabla^2\Phi + (D-2)\Omega^{-3}\partial_a\Omega\partial^a\Phi, \tag{4.42}$$

[2] It can be easily checked that the EOM obtained from this action is the same as the result of inserting $N_i = K_{ij} = 0$ in the EOM obtained from the action (4.5).

(here the weight of the scalar field Φ (i.e. v) is $s = 0$), one has

$$\tilde{\nabla}^2 v = e^{4v}(\nabla^2 v - 2\partial_a v \partial^a v) \;\Rightarrow\; \nabla^2 v = e^{-4v}\tilde{\nabla}^2 v + 2\partial_a v \partial^a v = e^{-4v}(\tilde{\nabla}^2 v + 2\tilde{\partial}_a v \tilde{\partial}^a v). \tag{4.43}$$

Finally one obtains

$$S = \frac{1}{2\bar{\kappa}_H^2}\int d^3x\sqrt{\tilde{\gamma}}[\tilde{R} - 8(1-\eta'/2)\tilde{\gamma}^{ij}\partial_i v \partial_j v]. \tag{4.44}$$

This is now in the 'Einstein frame' with the canonical kinetic term of the 'dilaton', as long as $1 - \eta'/2 > 0$, or in the above used notation, $a < 1/4$. Noting that $c = 1 - \eta'/2$, the above action can be written as

$$S = \frac{1}{2\bar{\kappa}_H^2}\int d^3x\sqrt{\tilde{\gamma}}[\tilde{R} - \frac{8}{c^2}\tilde{\gamma}^{ij}\partial_i v \partial_j v]. \tag{4.45}$$

In fact, the similar argument can be applied to pure GR in the static case. Given the 3+1 decomposition

$$ds^2 = -N^2(x)dt^2 + \gamma_{ij}(x)dx^i dx^j, \tag{4.46}$$

with again $2v = -\ln N$, the Ricci scalar becomes

$${}^{(4)}R = R - 8(\partial v)^2 + 4D^2 v. \tag{4.47}$$

So the Einstein-Hilbert action becomes

$$S = \frac{1}{2\kappa_{GR}^2}\int d^4x \; {}^{(4)}R = \frac{1}{2\bar{\kappa}_{GR}^2}\int d^3x\sqrt{\gamma}e^{-2v}[R - 8(\partial v)^2 + 4D^2 v], \tag{4.48}$$

where again a factor of time integral is absorbed in $\bar{\kappa}_{GR}^2$. Then following the above method to transform it into the 'Einstein' frame, dropping boundary terms, one has

$$S = \frac{1}{2\bar{\kappa}_{GR}^2}\int d^3x\sqrt{\tilde{\gamma}}[\tilde{R} - 8\tilde{\gamma}^{ij}\partial_i v \partial_j v]. \tag{4.49}$$

Comparing (4.49) with (4.45), and noting that the difference in the constant factor in front of the action integral is irrelevant to the EOM in our case, one can see that given a static solution in GR, a solution in HL gravity can be easily obtained by a constant rescaling $v \to vc$ while maintaining the same $\tilde{\gamma}_{ij}$. According to the Birkhoff's theorem, spherically symmetric vacuum solutions in GR are uniquely determined to be the Schwarzschild solution characterized by a parameter r_0

$$ds^2 = -Fdt^2 + F^{-1}dr^2 + r^2 d\Omega^2 = -Fdt^2 + F^{-1}(dr^2 + r^2 F d\Omega^2), \quad F(r) \equiv 1 - \frac{r_0}{r}, \tag{4.50}$$

where r_0 is related to its ADM or Komar mass M by $r_0 = 2M$.

Now the above argument implies that given parameter c, the corresponding solutions in HL gravity are given by, with r relabelled as ρ,

$$ds^2 = -F^c dt^2 + F^{-c}(d\rho^2 + \rho^2 F d\Omega^2), \quad F(\rho) = 1 - \frac{\rho_0}{\rho}. \tag{4.51}$$

Again by evaluating its Komar mass M, one finds

$$\rho_0 = 2M/c, \tag{4.52}$$

which is derived in Appendix H.

In fact, one can also easily solve the 3D Euclidean problem by directly deriving the 3D EOM and integrating them. This is illustrated in Appendix I.

4.3 Constraints on Parameters and Geometric Properties

As argued in [24], we require $\xi > 0$ for the positivity of the Planck scale. Direct analysis of the stability of fluctuations around Minkowski background in HL gravity requires [23]

$$0 < \eta/\xi < 2, \quad \Leftrightarrow 0 < a < 1/4, \text{ or } 1 < c < \infty. \tag{4.53}$$

As we discussed before, using the equivalence between the IR limit of HL gravtiy and Einstein-aether theory, we can also use the results in the latter as constraints in HL gravity. In particular, the constraints (3.41) now becomes $0 < c_{14} < 2$. Given the correspondence between parameters of the two theories (3.36), this implies

$$1 < c < \infty. \tag{4.54}$$

This is consistent with the constraints from direct analysis within HL gravity. In the following, we will consider (4.54) as the physical range of the parameter c, and analyze the geometric properties of the metric (4.51) .

First of all, the value of the Ricci scalar curvature (both the 4D and 3D ones are the same in this case) and its limit as $\rho \to \rho_0$ are

$$R = \frac{\rho_0^2(1-c^2)}{2\rho^{c+2}(\rho-\rho_0)^{2-c}} \to \begin{cases} 0 & c > 2 \text{ or } c = 1 \\ \infty & c < 2, c \neq 1 \end{cases}. \tag{4.55}$$

Of course, for $c = 1$, as a Ricci flat solution, R is vanishing everywhere. One can also calculate the squares of the curvature, such as $R_{ab}R^{ab}$ and $R_{abcd}R^{abcd}$, which all have the structure as a product of R^2 and a term regular at ρ_0. Thus, we can see that it is only in pure GR that $\rho = \rho_0$ corresponds to a horizon, where the geometry is regular. Otherwise, $\rho = \rho_0$ is a curvature singularity as long as $c \neq 1$. In addition, although the Ricci scalar vanishes near ρ_0 in the case of $c > 2$, the geometry there is still singular. Indeed, one can construct a parallel propagated orthonormal frame along a timelike geodesic. Then one can see that some components of the Riemann tensor are divergent (indicating divergent tidal forces) as ρ approaches ρ_0 in finite proper time. Details of calculation are given in Appendix J.

The factor in front of the $d\Omega^2$ part in the metric (4.51) indicates that the area of the 2D sphere at some fixed ρ is $4\pi r^2$, with the area radius r given as a function of ρ by

$$r(\rho) = \rho F^{-c/2+1/2} = \frac{\rho}{(1-\rho_0/\rho)^{(c-1)/2}}. \tag{4.56}$$

As $\rho \to \rho_0$,

$$r(\rho) \to \begin{cases} \infty & \text{if} \quad c > 1 \text{ HL} \\ \rho & \text{if} \quad c = 1 \text{ GR, Schwarzschild} \\ 0 & \text{if} \quad c < 1 \text{ Fisher.} \end{cases} \tag{4.57}$$

For $\rho \in (\rho_0, \infty)$, $r(\rho)$ may become non-monotonic. A minimal r_m can be obtained from $dr/d\rho = 0$ as

$$r_m = \rho_m \left(\frac{c-1}{1+c}\right)^{(1-c)/2}, \tag{4.58}$$

where $\rho_m \equiv \rho_0(1+c)/2$, so

$$\begin{aligned} \rho_m < \rho_0 \quad & \text{if} \quad c < 1 \text{ Fisher} \\ \rho_m = \rho_0 \quad & \text{if} \quad c = 1 \text{ Schw} \\ \rho_m > \rho_0 \quad & \text{if} \quad c > 1 \text{ HL.} \end{aligned} \tag{4.59}$$

In other words, when $c \leq 1$, the two sphere monotonically shrinks to a point (if $c < 1$) or to the minimal sphere (if $c = 1$) at $\rho = \rho_0$. For $c > 1$ in our case, the two sphere first shrinks to a minimal sphere at ρ_m, then expands to infinity at ρ_0. The typical behavior of the area radius in different cases of c is illustrated in Fig 4.1.

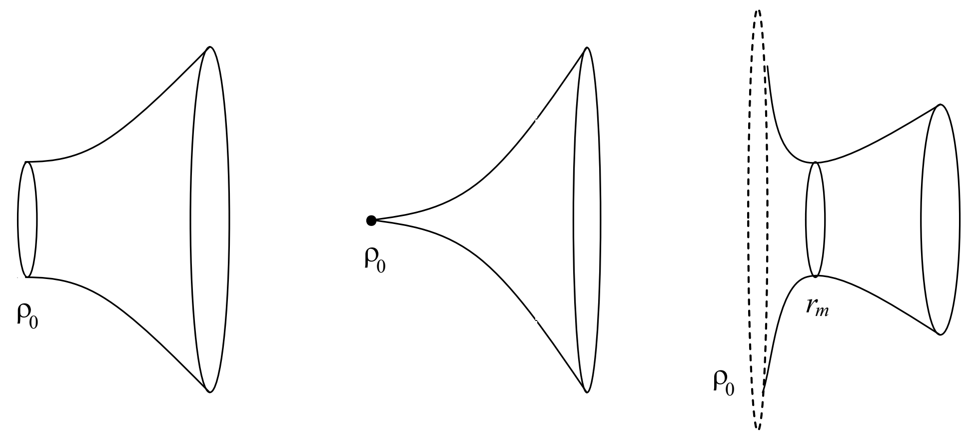
The proper radial distance is given by

$$dl = \frac{d\rho}{F^{c/2}}. \tag{4.60}$$

Let's consider the proper radial distance $l(\rho)$ measured from the minimal sphere at $\rho = \rho_m$, i.e.

$$l(\rho) \equiv \int_{\rho_m}^{\rho} \frac{d\rho}{F^{c/2}} = \int_{\rho_m}^{\rho} \frac{\rho^{c/2}}{(\rho-\rho_0)^{c/2}} d\rho. \tag{4.61}$$

As we decrease ρ from infinity to ρ_m, both $r(\rho)$ and $l(\rho)$ decrease monotonically to the



(a) Schwarzschild, $c = 1$. (b) Fisher, $0 \leq c < 1$ (c) HL, $c > 1$

Figure 4.1: Illustration for area radius r with different c. The dashed circle indicates the divergence of $r(\rho)$ at ρ_0.

point ρ_m where $r(\rho_m) = r_m$ and $l(\rho_m) = 0$. Then, as ρ continues decreasing, $r(\rho)$ begins to increase while $l(\rho)$ becomes negative. Within the range $1 < c < 2$, as ρ approaches ρ_0, $l(\rho)$ goes to a finite negative value and $r(\rho)$ diverges. Otherwise, if $c \geq 2$, the proper radial distance would diverge near the ρ_0 singularity. This divergence property can be easily seen from the asymptotic behavior of the integral as $\rho \to \rho_0$,

$$l \sim \int^{\rho_0} \frac{\rho_0^{c/2} d\rho}{(\rho - \rho_0)^{c/2}}, \tag{4.62}$$

where it clearly shows that as $c \geq 2$ the integral is divergent.

Because of the correspondence between HL gravity and Einstein-aether theory, the solution we obtained above is exactly the same solution obtained in [25], where the aether field is static, in the sense that it is aligned with the timelike Killing vector of the static spacetime. Here we have written the solution in a Fisher-like form (4.30), which makes it more convenient to analyze its geometrical property. As a comparison, we plot the relation between the area radius v.s. the proper radial distance in Fig. 4.2, where, following the parameter values $c_1 = 0,\ 0.1,\ 0.7,\ 1.9$ in [25], we take the corresponding values of parameters c as determined via $c = (1 - c_1/2)^{-1/2}$. Also we set the unit such that $2M = 1$, where M is the ADM mass. Then $\rho_0 = 1/c$. The plots in this figure clearly show the typical features of the solution in different cases of c as discussed above, and they are consistent with [25], as expected.

One can also consider the norm of the timelike Killing vector ∂_t

$$N(\rho) = \sqrt{-g_{tt}} = (1 - \frac{\rho_0}{\rho})^{c/2}, \tag{4.63}$$

which vanishes at $\rho = \rho_0$. In the Schwarzschild case, this simply says $\rho = \rho_0$ is a Killing horizon. For general $c \neq 1$, as we have seen there is a singularity at ρ_0. Yet we may still borrow the concept of the Killing horizon, in particular, the surface gravity

$$\kappa_{sg} = \frac{dN}{dl} = \frac{c\rho_0}{2\rho^2}(1 - \frac{\rho_0}{\rho})^{c-1}, \tag{4.64}$$

which vanishes as $\rho \to \rho_0$ for $c > 1$, and blows up for $c < 1$. When $c = 1$, one recovers the GR value $\kappa_{sg} = 1/(2\rho_0) = 1/(4M)$. In Fig. 4.3, we illustrate three cases with $c > 1$. Note that in this figure, the slope of the curve is κ_{sg}. Again for comparison, we take the same values of c according to those of c_1 in [25]. In all these cases, $N \to 0$ as $\rho \to \rho_0$, with the difference being that, for $c = 1$, the slope at ρ_0 is non-vanishing, for $1 < c < 2$ ($c_1 = 0.5$) the slope vanishes at finite l, and for $c > 2$ ($c_1 = 1.9$) the slope asymptotically approaches zero as $l \to \infty$. These features are also consistent with [25].

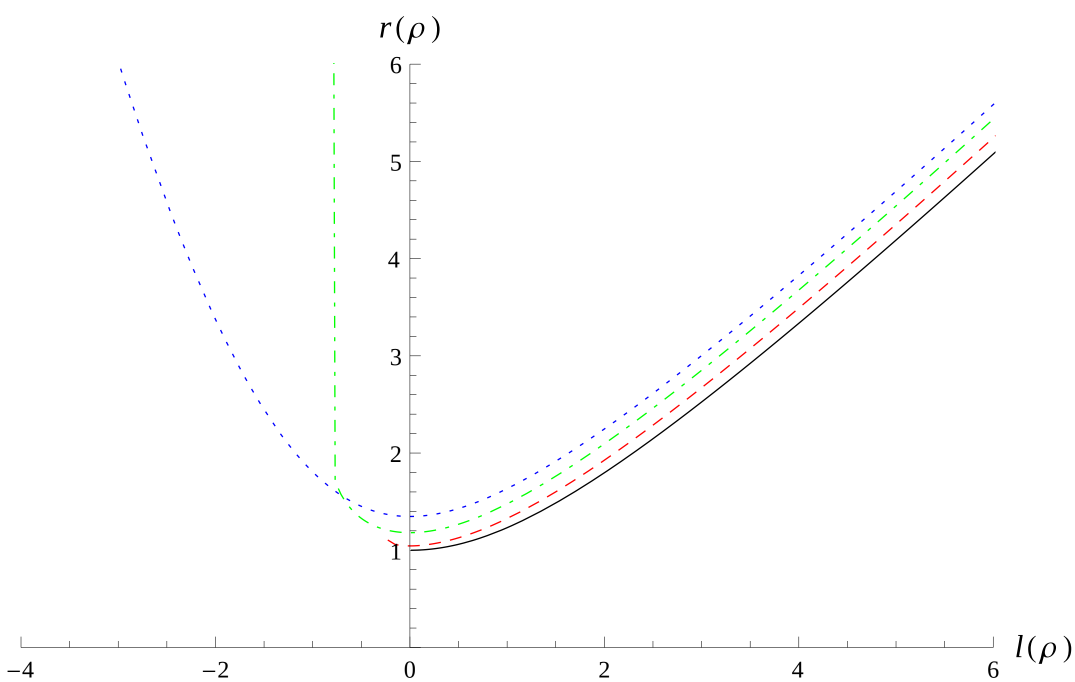


Figure 4.2: Area radius $r(\rho)$ v.s. proper radial distance $l(\rho)$ as measured with respect to ρ_m, with $\rho_0 = 1/c$. The solid (black), dashed (red), dot-dashed (green) and dotted (blue) lines correspond to $c_1 = 0$, 0.1, 0.7 and 1.9, respectively, for comparison with [25].

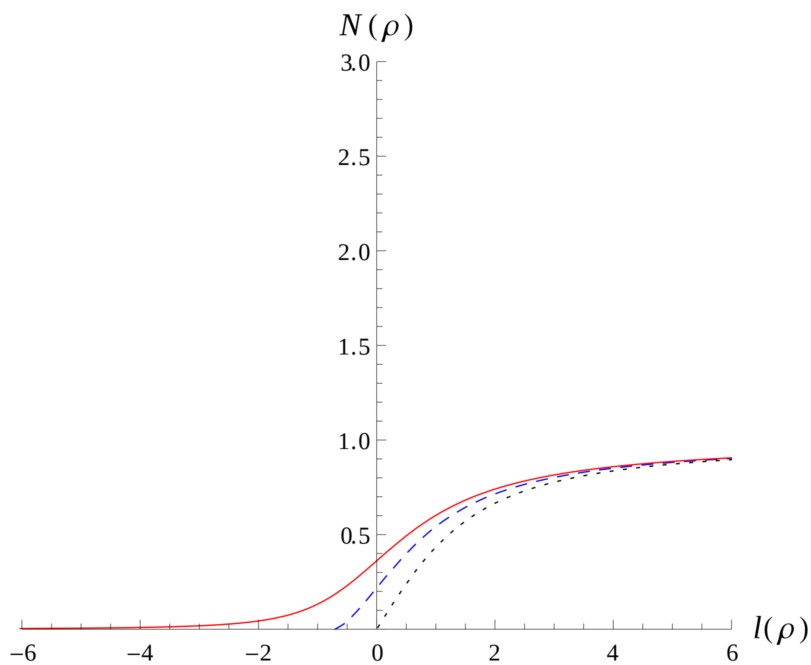


Figure 4.3: The norm of the timelike Killing vector $N(\rho)$ v.s. proper radial distance $l(\rho)$ as measured with respect to ρ_m, with $\rho_0 = 1/c$. The dotted (black), dashed (blue) and solid (red) lines correspond to $c_1 = 0$, 0.5 and 1.9, respectively, for comparison with [25].

Conclusion

Summary of Results

In conclusion, we have shown that for a minimally-coupled massless scalar field in de Sitter spacetime, there exists a de Sitter-invariant quantum state, so long as one only requires the state to give well-defined expectation values of shift-invariant operators. This is a poor-man's analogue for a de Sitter-invariant quantum state of the graviton that is only required to give well-defined expectation values for gauge-invariant operators. If these operators are products of localized gauge-invariant operators in two regions (like the Weyl tensor at two points), the cluster-decomposition property exhibited for the massless scalar field suggests that the expectation values of the corresponding gauge-invariant gravitational field operators will also tend to constants (presumably zero for quantities that vanish in classical de Sitter spacetime, such as products of Weyl tensors) in the limit that the two regions are pulled far apart, but if the operators involve a whole spacetime region of time period T, it would not be surprising for them to diverge in the limit that T is taken to infinity, just as the product of two scalar field differences does when the separations of the points within each cluster whose fields are differenced are taken to infinity.

In HL gravity, we introduced a coordinate transformation to put the static spherically symmetric vacuum solution in the IR limit into a simple form. Furthermore, we reduced the 4D problem to a 3D Euclidean Einstein gravity minimally coupled to a massless scalar field, whose solution is related to that of standard GR by a constant rescaling of the 3D scalar field. This simplifies the problem and can easily generate a family of static spherically symmetric solutions of HL gravity from the Schwarzschild solution of GR. We also considered the range of parameters of the model as constrained by various physical requirements. Within the allowed range, we analyzed the geometric properties of the solution.

Directions for Future Work

The vacuum state we obtained for the massless scalar field in the de Sitter spacetime is not a Fock state, and the quantum state space is not a Fock representation of the Hilbert space. It would be interesting to reformulate our result using the algebraic formalism of QFT (e.g. [96]) where the physical state space is given from a C*-algebra via the Gelfand-Naimark-Segal (GNS) construction, i.e. the state space is a GNS representation of the algebra, since in this formalism one can investigate states in Hilbert space without having to assume any particular representation.

It is possible for HL gravity as an interesting theory, at least at the phenomenological level, to be consistent with experiments and observations. However, it also suffers some other problems. It was suggested that there are no rotating black hole solutions [97]. Since these objects are most relevant to astrophysical observations, the inability to accommodate such solutions would be seen as a sign that the theory is not a good candidate as a UV completion of GR. Later, however, it is argued [107, 108, 109] that the result of [97] is not correct and that rotating black holes can be constructed (at least in the slow-rotation limit). It would be worth working along the same direction. Given the static solution, due to its singular behavior, it should be interpreted as describing the exterior of some spherical distribution of matter, such that the singularity should have been replaced by some interior solution of HL gravity coupled with matter. Then the exterior and interior solutions can be glued by the junction conditions (this was discussed in the HL context in [98, 99]). Note also that in HL gravity, due to the full diffeomorphism broken into FPD, there are new types of physical singularities [100], which in GR can be eliminated by coordinate transformations (in particular, those with space-dependent time transformations) and therefore are not considered as physical in GR . With the static solution obtained here, we can regard it as describing the exterior of some spherically distributed matter, such as a thin shell. It would be interesting to study whether such a solution is stable under perturbations, and whether it will develop any singularities, in particular those singularities which have no counterpart in Einstein gravity due to lack of the full diffeomorphism group.

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Appendices

A EOM of Linearized Gravity on a de Sitter Background

Consider the action

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}(R - 2\Lambda) \tag{A.1}$$

where the cosmological constant $\Lambda = 3H^2 = 3/L^2$. Let $g^{(0)}_{ab}$ be the metric for de Sitter, then inserting the perturbed metric

$$g_{ab} = g^{(0)}_{ab} + h_{ab} \tag{A.2}$$

into the action, the action quadratic in h_{ab} is given by (c.f. e.g. [101])

$$\begin{aligned} S_2 &= \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left[\frac{1}{2}\nabla^a h_{ac}\nabla_b h^{bc} + \frac{1}{4}\nabla_a h\nabla^a h - \frac{1}{2}\nabla^a h\nabla^b h_{ba}\right. \\ &- \left.\frac{1}{4}\nabla_a h_{bc}\nabla^a h^{bc} - \frac{1}{2}L^{-2}(h_{ab}h^{ab} + \frac{1}{2}h^2)\right], \end{aligned} \tag{A.3}$$

where h is the trace of h_{ab}, and we have dropped the label (0) of the background metric, with which indices are raised or lowered, and with respect to which ∇_a is defined. Then the EOM can be obtained as

$$\Box h_{ab} + \nabla_a\nabla_b h - (\nabla_a\nabla_c h^c_b + \nabla_b\nabla_c h^c_a) + (\nabla_c\nabla_d h^{cd} - \Box h)g_{ab} - 2L^{-2}(h_{ab} + \frac{h}{2}g_{ab}) = 0 \tag{A.4}$$

The physical gauge conditions are:

$$\text{transverse} \quad \nabla^a h_{ab} = 0 \tag{A.5}$$

$$\text{traceless} \quad h = 0 \tag{A.6}$$

$$\text{synchronous} \quad h_{ab}n^b = 0 \tag{A.7}$$

where n is the timelike unit vector which generates the time coordinate used in the background. The first two conditions reduce the EOM to

$$\Box h_{ab} - 2L^{-2}h_{ab} = 0 \tag{A.8}$$

which still has some gauge degree of freedom. Indeed, a pure gauge $\nabla_{(a}\xi_{b)}$ is also a solution, as long as ξ satisfies

$$\nabla^a \xi_a = 0, \quad (\Box + 3L^{-2})\xi_a = 0 \tag{A.9}$$

The synchronous condition eliminates this gauge degree of freedom, making h_{ab} contain only the two physical degrees of freedom.

In the spatially flat FRW coordinate system (choosing coordinates of the background with respect to which $\Box$ is defined), the non-vanishing components are h_{ij} which also satisfy the traceless and transverse conditions. Then the EOM for h^i_j becomes that of a minimally-coupled massless scalar field

$$\Box h^i_j = 0 \tag{A.10}$$

Note that:

1. h^i_j should be understood as the ij component of $h^\mu_\nu \equiv g^{\mu\rho}h_{\rho\nu}$. For $h_{\mu\nu}$ and $h^{\mu\nu}$, the EOM do not take this form.

2. The box operator $\Box \equiv \nabla^a\nabla_a$ applied on a arbitrary scalar, e.g. f, can be written as

$$\Box f = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu)f. \tag{A.11}$$

   In general, this equation do not hold when the scalar is replaced by an arbitrary tensor. In our case of (A.10), however, the box operator should be understood as the scalar d'Alembertian operator as if h^i_j were a scalar.

B Coordinate Systems of de Sitter

de Sitter space can be covered by different coordinate charts. These coordinates can be obtained from the Minkowski coordinates of the embedding space.

B.1 Global Coordinates

The global coordinates are introduced as

$$\begin{aligned} X^0 &= \sinh\tau, \\ X^1 &= \cosh\tau\cos\chi, \\ X^2 &= \cosh\tau\sin\chi\cos\theta, \\ X^3 &= \cosh\tau\sin\chi\sin\theta\cos\phi, \\ X^4 &= \cosh\tau\sin\chi\sin\theta\sin\phi, \end{aligned} \tag{B.1}$$

where $\tau \in (-\infty, +\infty)$, and $\phi \in [0, 2\pi]$ and $\theta, \chi \in [0, \pi]$ are three angular coordinates of the three-sphere. Illustrated in Fig. B.1. The metric is

$$ds^2 = -d\tau^2 + \cosh^2\tau(d\chi^2 + \sin^2\chi d\Omega_2^2), \tag{B.2}$$

where

$$d\Omega_2^2 = d\theta^2 + \sin^2\theta d\phi^2. \tag{B.3}$$

The spatial slices are compact S^3, shrinking from past infinity I^- to a minimal size at $\tau = 0$ and then expanding to future infinity I^+.

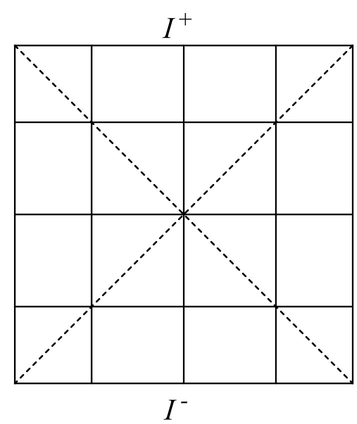

Figure B.1: Global chart of de Sitter. The dashed diagonal lines denote de Sitter horizon.

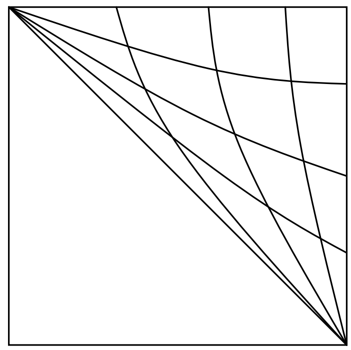

Figure B.2: Spatially flat chart of de Sitter.

B.2 Spatially Flat Coordinates

These coordinates cover half of the whole de Sitter space (e.g. the causal past of an observer on the south pole), given by

$$\begin{aligned} X^0 &= -\sinh t_0 - \frac{1}{2}\exp(t_0)\vec{x}\cdot\vec{x}, \\ X^4 &= \cosh t_0 - \frac{1}{2}\exp(t_0)\vec{x}\cdot\vec{x}, \\ X^i &= x^i \exp(t_0), \end{aligned} \tag{B.4}$$

where $t_0 \in (-\infty, +\infty)$, and x^i are the coordinates of the flat spatial slices. See Fig. B.2. The metric becomes

$$ds^2 = -dt_0^2 + e^{2t_0} d\vec{x}\cdot d\vec{x}. \tag{B.5}$$

In this case, one can obtain the conformally flat coordinates by further introducing $\eta = -e^{-t_0}$, such that $\eta \in (-\infty, 0)$ and

$$ds^2 = \frac{-d\eta^2 + d\vec{x}\cdot d\vec{x}}{\eta^2} \tag{B.6}$$

The other half can be covered by the same coordinates with $\eta \to -\eta$.

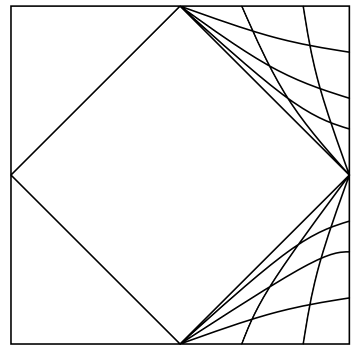

Figure B.3: Spatially open chart of de Sitter

B.3 Hyperbolic Coordinates

These patches cover parts of the whole de Sitter, see Fig. B.3. They are introduced by

$$\begin{aligned} X^0 &= \sinh\tau' \cosh\zeta, \\ X^1 &= \cosh\tau', \\ X^2 &= \sinh\tau' \sinh\zeta \cos\theta, \\ X^3 &= \sinh\tau' \sinh\zeta \sin\theta \cos\phi, \\ X^4 &= \sinh\tau' \sinh\zeta \sin\theta \sin\phi, \end{aligned} \tag{B.7}$$

where $\tau' \in (-\infty, +\infty)$ and $\zeta \in [0, \infty)$ together with ϕ, θ are the coordinates on the spatial slices H^3, which are non-compact. The metric becomes

$$ds^2 = -d\tau'^2 + \sinh^2\tau'(d\zeta^2 + \sinh^2\zeta d\Omega_2^2) \tag{B.8}$$

B.4 Static Coordinates

In this case, illustrated in Fig. B.4, there is locally a timelike Killing vector ∂_t. The coordinates covering the left or right wedge are given by

$$\begin{aligned} X^0 &= \sqrt{1-r^2}\sinh(t), \\ X^1 &= r\sin\theta\cos\phi, \\ X^2 &= r\sin\theta\sin\phi, \\ X^3 &= r\cos\theta, \\ X^4 &= \sqrt{1-r^2}\cosh(t), \end{aligned} \tag{B.9}$$

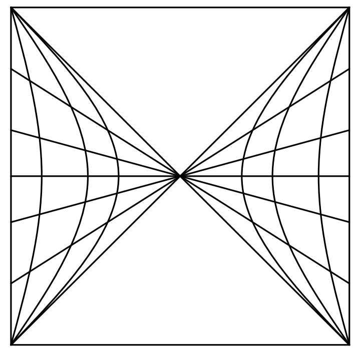

Figure B.4: Static chart of de Sitter

with $r \in [0,1]$. The other wedge has the same coordinates, with the timelike Killing vectoring having an opposite direction. The metric becomes

$$ds^2 - (1-r^2)dt^2 + (1-r^2)^{-1}dr^2 + r^2 d\Omega_2^2. \tag{B.10}$$

C Various Two-point Functions

We use $\langle \ldots \rangle$ for either vacuum expectation value $\langle 0| \ldots |0\rangle$, or $tr\rho \ldots$ at finite temperature. Following the convention of [28], various two-point functions can be defined:

1. The Wightman function

$$G(x,y) \equiv \langle \mathcal{O}(x)\mathcal{O}(y)\rangle \tag{C.1}$$

   One may also denote another $G^-(x,y) \equiv \langle \mathcal{O}(y)\mathcal{O}(x)\rangle$, but this is not necessary.

2. The commutator

$$iG_c(x,y) \equiv \langle [\mathcal{O}(x), \mathcal{O}(y)]\rangle = [\mathcal{O}(x), \mathcal{O}(y)] \tag{C.2}$$

   Note different conventions may differ in the i factor.

3. The symmetric, or Hadamard's, function

$$G^{(1)}(x,y) \equiv \langle \{\mathcal{O}(x), \mathcal{O}(y)\}\rangle = G(x,y) + G(y,x) \tag{C.3}$$

4. Retarded Green function

$$G_R(x,x') \equiv -\theta(t-t')G_c(x,x') = +i\theta(t-t')\langle [\mathcal{O}(x), \mathcal{O}(x')]\rangle. \tag{C.4}$$

5. Advanced Green function

$$G_A(x,x') \equiv \theta(t'-t)G_c(x,x') = -i\theta(t'-t)\langle[\mathcal{O}(x),\mathcal{O}(x')]\rangle. \tag{C.5}$$

6. The Feynman propagator

$$iG_F(x,x') = \langle T\mathcal{O}(x)\mathcal{O}(x')\rangle = \theta(t-t')G(x,x') + \theta(t'-t)G(x',x) \tag{C.6}$$

The first three two-point functions are solutions to the homogeneous EOM if $\mathcal{O}(x)$ is a field appearing in the action. The last three are solutions to the inhomogeneous EOM with a delta function source. These definitions differ by the i factor from the convention in, e.g. Peskin's book [102]. Due to this i (implicit in G_R and G_A) the inhomogeneous EOMs take the form

$$(\Box_x - m^2)G_F(x,x') = \delta(x-x') \tag{C.7}$$

$$(\Box_x - m^2)G_{R,A}(x,x') = -\delta(x-x') \tag{C.8}$$

Given the (in)homogeneous EOM, various two-point functions and the quantum states under which the expectation values are taken are further determined by different boundary conditions.

Comments on possible modifications:

- If the i factor is not present in the definitions, the RHS should be multiplied by a factor of i, as in [102] (2.56) for G_R
- Moreover, for curved spacetime, the delta function should be replaced by its covariant version with a $(-g)^{-1/2}$ factor times the products of the ordinary delta function for each coordinate, c.f. [38].
- For different metric signature, the box operator should be replaced by $-\Box$.
- In this convention, also shared by [103], ImG_R equal to π times the spectral density which is positive definite. An opposite convention is also used, e.g. in [104, 105], where the imaginary part of G_R is always negative.

D Solutions to the Hypergeometric Equation

In general, the hypergeometric equation is

$$z(1-z)F''(z) + [c-(1+a+b)z]F'(z) - abF = 0 \tag{D.1}$$

In the equation (2.29), $c = 2$, $ab = m^2$ and $1+a+b = 4$. In this case, one has $1+a+b-c = c$, and then the equation is invariant under the change $z \to 1 - z$. This means that both ${}_2F_1(a, b, c; z)$ and ${}_2F_1(a, b, c; 1 - z)$ are solutions, which are also linearly independent, and which are singular at $z = 0$ and $z = 1$, respectively. This property of the hypergeometric equation is physically related to the existence of the α-vacua. Thus the general solution can be constructed as $\alpha\, {}_2F_1(a, b, c; z)+\beta\, {}_2F_1(a, b, c; 1-z)$. In the context of the Wightman function in de Sitter, this general solution implies that the Wightman function would blow up if either one point is the on the light cone of the other, or if one point is on the light cone of the antipodal point to the other. Of course, one can always impose further conditions to fix the two constants α and β. If one requires only one singular point at $z = 0$, i.e. when the two points are null-separated, then $\alpha = 0$. Moreover, at short distances, the physics should be well approximated by that in flat space, and it is natural to require that the $z \to 1$ limit of the Wightman function should recover the flat space limit. The UV behavior of the Wightman function in flat space is

$$G_f \sim \frac{1}{4\pi^2}\frac{1}{\mu^2}, \tag{D.2}$$

where μ is the geodesic distance in flat space. The $z \to 1$ expansion of the hypergeometric function gives

$${}_2F_1(h_+, h_-, 2; 1 - z) \sim \frac{4}{\Gamma(h_+)\Gamma(h_-)}\frac{1}{\mu^2}. \tag{D.3}$$

where we have used the fact that $z \sim \mu^2/4$ as $z \to 0$, following from our definition (2.28). Comparing the above two results one immediately recovers $\beta = c_m$ as given in the main text.

In the Euclidean case, the equation is still the same, with the only difference being that the argument of z is restricted to $0 < z < 1$ $(-1 < Z < 1)$ since the S^4 is compact.

E Modes in Global and Flat Coordinates

In general, for a metric of the form

$$ds^2 = -dt^2 + a^2(t)g_{ij}dx^i dx^j \tag{E.1}$$

the box operator gives

$$\Box f(x) = (-\partial_t^2 - 3H\partial_t + a^{-2}\Delta)f \tag{E.2}$$

where $H \equiv \dot{a}/a$, and where Δ is the Laplacian in the 3-metric $g_{ij}dx^i dx^j$. For the metric in conformal form

$$ds^2 = a^2(\eta)(-d\eta^2 + g_{ij}dx^i dx^j) \tag{E.3}$$

one has

$$\Box f = a^{-2}(-2H\partial_\eta - \partial_\eta^2 + \Delta)f \tag{E.4}$$

E.1 Global Coordinates

In this chart, according to the symmetry on the constant time S^3, one can assume the form of the solution as, up to a normalization constant,

$$u_n \sim y(\tau)_L Y_L(\chi, \theta, \phi) \tag{E.5}$$

where Y_L are the harmonics on S^3 satisfying

$$\Delta Y_L = -L(L+2)Y_L \tag{E.6}$$

where under the same L, the degeneracy is $(L+1)^2$ and their indices are suppressed since y_L only depend on L. Then the EOM becomes a equation for $y_L(\tau)$,

$$y_L'' + 3\frac{\sinh\tau}{\cosh\tau}y_L' + \left[\frac{L(L+2)}{\cosh^2\tau} + m^2\right]y_L = 0. \tag{E.7}$$

Defining $w \equiv -\exp(2\tau)$, and introducing $f(z)$ via

$$y_L = (\cosh\tau)^L e^{(L+3/2-i\mu)\tau} f(w), \tag{E.8}$$

where $\mu \equiv \sqrt{m^2 - (3/2)^2}$, this equation can be put into the form of the standard hypergeometric equation

$$w(1-w)f'' + [c - (1+a+b)w]f' - abf = 0, \tag{E.9}$$

where $a \equiv L + 3/2$, $b \equiv L + 3/2 - i\mu$, and $c \equiv 1 - i\mu$.

Thus the we have the solution

$$y_L(\tau) = \mu^{-1/2}2^{L+1}(\cosh\tau)^L e^{(L+3/2-i\mu)\tau}\ {}_2F_1(L+3/2, L+3/2-i\mu; 1-i\mu; -e^{2\tau}) \tag{E.10}$$

where the normalization constant is determined from the normalization of the Klein-Gordon inner product which are orthonormal. Its complex conjugate $y_L^*(\tau)$ is the other linearly independent solution.

Near past infinity I^-, where the hypergeometric function goes to 1, the above solutions have the asymptotic behavior

$$y_L \to e^{(3/2-i\mu)\tau}, \quad y_L^* \to e^{(3/2+i\mu)\tau}, \quad \tau \to -\infty \tag{E.11}$$

Thus, at I^- and with respect to the global time τ, solutions $y_L Y_L$ can be regarded as the positive frequency modes, while $y_L^* Y_L$ are the negative frequency modes.

Since the EOM (E.7) is invariant under time reversal, one can obtain another pair of solutions by simply taking $\tau \to -\tau$ in the above solutions. Then the new modes have similar behavior near future infinity I^+, playing the role of positive/negative frequency modes there. Of course, the two pairs of modes are related by the Bogolyubov transformation and correspond to different vacua. The one is the in vacuum, i.e. a state with no incoming particles at I^-, while the other is the out vacuum, i.e. a state with no outgoing particles at I^+.

The modes for the Euclidean vacuum can be obtained by analytic continuation from the modes in Euclidean de Sitter, i.e. S^4. Then the Bogolyubov transformation (2.35) relating the Euclidean and in vacua can be obtained as [38]

$$u_L^{in} = \frac{1}{\sqrt{1-e^{-2\pi\mu}}}(u_L - e^{-\pi\mu} u_L^*), \tag{E.12}$$

i.e. Eq.(2.36) with $\alpha = -\pi\mu + i\pi/2$.

E.2 Conformally Flat Coordinates

In this chart, according to the $E(3)$ symmetry of the constant time slices, one can assume the solution

$$u_n \sim y_k(\eta) e^{i\vec{k}\cdot\vec{x}} \tag{E.13}$$

where $k \equiv \sqrt{\vec{k}^2} > 0$. Then the EOM becomes

$$y_k'' - \frac{2}{\eta} y_k' + (k^2 + \frac{m^2}{\eta^2}) y_k = 0, \tag{E.14}$$

whose general solutions can be expressed in terms of the Hankel functions

$$(-\eta)^{3/2} H_\nu^{(1)}(-k\eta), \quad (-\eta)^{3/2} H_\nu^{(2)}(-k\eta), \tag{E.15}$$

where $\nu \equiv \sqrt{(3/2)^2 - m^2}$.

For $m = 0$, the general solutions are

$$(-\eta)^{3/2} H_{3/2}^{(1)}(-k\eta), \quad (-\eta)^{3/2} H_{3/2}^{(2)}(-k\eta), \tag{E.16}$$

where, for $x = -k\eta$,

$$H_{3/2}^{(1)}(x) = -\sqrt{\frac{2}{\pi}} e^{ix}(x^{-1/2} + ix^{-3/2}), \quad H_{3/2}^{(2)}(x) = -\sqrt{\frac{2}{\pi}} e^{-ix}(x^{-1/2} - ix^{-3/2}) = H_{3/2}^{(1)*}(x). \tag{E.17}$$

Thus the solution

$$(-\eta)^{3/2}H^{(1)}_{3/2}(-k\eta)e^{i\vec{k}\cdot\vec{x}} \tag{E.18}$$

is the positive frequency solution with respect to the conformal time η, while its complex conjugate gives the negative frequency solution. In fact, one can rewrite the mode (with normalization constant C) as

$$C(-\eta)^{3/2}H^{(1)}_{3/2}(-k\eta)e^{i\vec{k}\cdot\vec{x}} = C\sqrt{\frac{2}{\pi}}\frac{-\eta}{\sqrt{k}}(1-\frac{i}{k\eta})e^{-ik\eta+i\vec{k}\cdot\vec{x}}. \tag{E.19}$$

In the UV as $k \to \infty$, one should recover the result in (conformal) Minkowski spacetime

$$\frac{1}{\sqrt{2\omega}}e^{-i\omega t+\vec{k}\cdot\vec{x}}. \tag{E.20}$$

which determines the constant $C = -\sqrt{\pi}/2$, such that the positive frequency mode of massless scalar field in a conformally flat chart is

$$-\eta\frac{1}{\sqrt{2k}}(1-\frac{i}{k\eta})e^{-ik\eta+i\vec{k}\cdot\vec{x}} \tag{E.21}$$

Note the factor $-\eta$ is due to the fact that this is really in a conformally flat chart.

F Dimensional Analysis of Field Theories at a Lifshitz Point

Following [56], we consider dimensional analysis of a theory with anisotropic scaling. We emphasize that changing the canonical dimensions of constants and operators is just a matter of convention for choosing certain units; it is the anisotropic scaling that is the underlying physics which makes all the difference.

First, let us consider the action in Minkowski spacetime,

$$S = \int dtd^dx(\dot{\phi}^2 - g\phi(-\Delta)^z\phi) \tag{F.1}$$

where $\Delta = \partial_i^2$, in contrast to a relativistic free theory

$$S = \int dtd^dx(\dot{\phi}^2 - c^2(\partial_i\phi)^2) = \int dtd^dx(\dot{\phi}^2 + c^2\phi\Delta\phi) \tag{F.2}$$

where in the relativistic case c is always set to unity, but here we keep it explicitly. Of course, the action (F.2) can be regarded as a particular case of (F.1) with $z = 1$ and c^2 playing the role of g, and in particular, if in some units c becomes dimensionless, so is g. But we will focus on general cases with $z > 1$. In general, one can have an action of the form

$$S = \int dt d^d x[\dot{\phi}^2 - \sum_{m=1}^{z} g_m \phi(-\Delta)^m \phi] \tag{F.3}$$

with coupling constants g_m where $g_1 \equiv c^2$ and $g_z \equiv g$.

Recall that the coupling constant c^2 of the second order spatial derivative term also appears in the spacetime line element

$$ds^2 = -c^2 dt^2 + (dx^i)^2. \tag{F.4}$$

It should be noted that in non-relativistic theories, this c should be regarded just as a quantity to make $[cdt] = [dx] = [ds]$, where we use [ ] to denote the *canonical* or *engineering dimension*. So in a theory given by (F.1) with $z > 1$, dimensions can be determined by combinations of the three dimensionful constants $\hbar, c, g$. We have chosen the units $\hbar \to 1$, as a consequence of which,

$$[\frac{1}{x}] = [\text{momentum}] = [p], \quad [\frac{1}{t}] = [\text{energy}] = [\mu], \tag{F.5}$$

where p and μ are introduced as two quantities with the dimensions of spatial momentum and energy, respectively. This can be easily seen from the basic relations $E = \hbar\omega$ and $\vec{p} = \hbar\vec{k}$. Note that the change of the dimension is just a matter of convention of units, not anything physical. For example, in units $\hbar = c = 1$, [length] = [mass^{-1}], whereas in $G = c = 1$, [length] = [mass] (c.f., the Schwarzschild radius $r_0 = 2M$).

To apply the power-counting analysis, one cannot set both $g, c \to 1$ simultaneously, in which case everything is dimensionless. In the following, we use the units with $g \to 1$. Then by comparing the two terms in the action (F.1), one can see immediately that (even without $\hbar \to 1$)

$$[\partial_t] = [\partial_i^z], \tag{F.6}$$

which is equivalent to

$$[dt] = [(dx^i)^z]. \tag{F.7}$$

Combining (F.5), we have

$$[\mu] = [p^z]. \tag{F.8}$$

From now on, we measure everything in the unit of p, as in Horava's original paper [18]. Then

$$[c] = [x/t] = [p^{z-1}] = [x^{1-z}] = z - 1. \tag{F.9}$$

As a result of $\hbar \to 1$ and $g \to 1$, one can determine the dimension of the field from either the 'kinetic' or the 'potential' term of the action (F.1) as

$$[\phi] = \frac{d-z}{2}. \tag{F.10}$$

If we use the same units for the general action (F.3), the dimensions of the other coupling constants are

$$[g_m] = 2(z-m) \geq 0. \tag{F.11}$$

In particular $[c] = z - 1$ and $[g] = 0$, which are consistent with (F.9) and the fact that we are in the units with $g \to 1$. This implies that all these g_m terms are renormalizable. In particular, the last g term is marginal, while the rest are relevant.

If instead we use the units with $\hbar \to 1$ and $c \to 1$, we have $[t] = [x]$ which further implies $[\mu] = [p]$ and $[g] = [p^{2(1-z)}]$. In this case, we can introduce an energy/mass/momentum scale ζ by $g = \zeta^{2(1-z)}$. In general, $g = \zeta^{2(1-z)}c^2$. This is related to the quantity Z in [19] as $g = Z^2$. Now the dimension of g is generally *negative*, which seems to indicate that the g term is non-renomalizable. However, it is not the case.

The key point lies in something physical, i.e. the idea of 'anisotropic scaling', which is somewhat 'hidden' in the units $\hbar = g = 1$ (therefore such units are natural and convenient for applying this physical idea), and which should be manifested in the units $\hbar = c = 1$.

To explicitly identify the anisotropic scaling, let's analyze the superficial degree of divergence, D, of a general Feynman diagram with L loops and I internal propagator. In either (F.1) or (F.3), as long as only the UV behavior is concerned, we can always write the propagator as, schematically,

$$G(\omega, k) = \frac{1}{\omega^2 - gk^{2z}} \tag{F.12}$$

This shows clearly that adding higher order spatial derivative terms physically changes the UV behavior of the propagator. The potential divergence of the diagram comes from the integral, again schematically,

$$\int (d\omega d^d k)^L G(\omega, k)^I. \tag{F.13}$$

In the units $\hbar = g = 1$, if we denote Λ as the cutoff momentum, then, due to (F.8), each loop integral volume element contributes

$$d\omega dk^d \sim \Lambda^{z+d}. \tag{F.14}$$

Each propagator contributes

$$G(\omega, k) = \frac{1}{\omega^2 - k^{2z}} \sim \Lambda^{-2z}. \tag{F.15}$$

Thus the superficial degree of divergence is

$$D = (z + d)L - 2Iz \tag{F.16}$$

and the UV behavior of the diagram is

$$\int^{\infty} \Lambda^{D-1} d\Lambda, \tag{F.17}$$

which is convergent for $D < 0$. Note that the physical idea of anisotropic scaling has been embodied in requiring that ω^2 and k^{2z} go to infinity in the same order (in contrast to $\omega^2 \sim k^2 \to \infty$ in the relativistic case), i.e. taking the cutoff as

$$\omega \sim \Lambda^z, \quad k \sim \Lambda \tag{F.18}$$

In the units $\hbar = c = 1$, on the other hand, the propagator becomes

$$G(\omega, k) = \frac{1}{\omega^2 - \zeta^{2-2z} k^{2z}}. \tag{F.19}$$

In this case, the physical idea of anisotropic scaling is embodied in that ω^2 and $\zeta^{2-2z}k^{2z}$ go to infinity of the same order, i.e

$$\omega \sim \zeta^{1-z}\Lambda^z, \quad k \sim \Lambda, \tag{F.20}$$

such that the propagator contributes

$$G(\omega, k) \sim \zeta^{2z-2}\Lambda^{-2z}. \tag{F.21}$$

Then the UV behavior of the diagram becomes

$$\int^{\infty} \zeta^{(1-z)(L-2I)} d(\Lambda^{(z+d)L-2Iz}) \sim \int^{\infty} \Lambda^{D-1} d\Lambda, \tag{F.22}$$

and we obtain the same degree of divergence (F.16).

It should be emphasized that (F.15), (F.19) and (F.17), (F.22) are physically equivalent results expressed in different conventions of units.

In sum, according to the critical exponent z in the anisotropic scaling, choosing the units such that g_z, the coupling constant of the term $\phi(-\Delta)^z\phi$, becomes dimensionless (or simply choose the units with $g_z = 1$), then interaction terms with coupling constants of non-negative dimensions in spatial momentum p measured *in the current units* are power-counting renormalizable. In other words, the relation between the dimensions of coupling constants and renormalizablity in a non-relativistic theory with anisotropic scaling is the same as that in a relativistic theory with isotropic scaling. In particular, those with positive dimensions are relevant, while the g_z term is marginal.

Now let's return to HL gravity. In perturbative quantum gravity, one always consider the graviton $h_{\mu\nu}$ as a perturbation around some background metric

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + \kappa h_{\mu\nu} \tag{F.23}$$

where $\kappa = \sqrt{8\pi G}$ introduced here is to make the graviton field dimensionful and to make κ disappear in the quadratic term. Then one can easily see that the powers of κ appear as the coupling of the interaction term for graviton. If, as usual, one considers isotropic scaling, and chooses the units $\hbar = c = 1$, then $\kappa = M^{-1}$ indicates that such theory is non-renormalizable. Now in HL gravity, we use the anisotropic scaling, with critical exponent z. Following the spirit of the above discussion, we should use the units such that the coupling constants of the terms containing $2z$ order spatial derivatives are dimensionless, (or simply choose the units such that one of them becomes unity). Then the dimension of the coupling constant of graviton interaction becomes $[\kappa] = (z-3)/2$. Thus, if we choose $z = 3$, the coupling becomes dimensionless, indicating the theory becomes power-counting renormalizable.

G Useful Formulae in the ADM Formalism

Here we list relevant formulae used in deriving the correspondence between Einstein-aether theory and HL gravity. We follow [106], taking care of a difference in the sign of the definition of the extrinsic curvature there with what we use in this thesis.

Let u be the hypersurface orthogonal aether field which should be identified with the unit (co)vector n normal to the hypersurface $t = const$. Consider the ADM form of the metric

$$ds^2 = (-N^2 + N^i N_i)dt^2 + 2N_i dx^i dt + \gamma_{ij} dx^i dx^j. \tag{G.1}$$

In the gauge $\phi = t$,

$$u_\mu = -N\delta^0_\mu, \quad u^\mu = g^{\mu\nu}u_\nu = -Ng^{\mu 0} = \frac{1}{N}(\delta^\mu_0 - N^i\delta^\mu_i). \tag{G.2}$$

Then some expressions relevant in deriving the results in the main text are

$$\Gamma^0_{00} = \frac{1}{N}\partial_0 N + \frac{N^i}{N} D_i N + \frac{N^i N^j}{N} K_{ij}, \quad \Gamma^0_{i0} = \frac{\partial_i N}{N} + K_{ij}\frac{N^j}{N}, \quad \Gamma^0_{ij} = \frac{1}{N} K_{ij}, \tag{G.3}$$

$$g^{0i} = \frac{N^i}{N^2}, \quad g^{00} = -\frac{1}{N^2}, \quad g^{ij} = \gamma^{ij} - \frac{N^i N^j}{N^2}. \tag{G.4}$$

Then the components of the acceleration are

$$A_\mu = u^\nu \nabla_\nu u_\mu = u^\nu(\partial_\nu u_\mu - \Gamma^\rho_{\nu\mu} u_\rho) = N g^{0\nu}\partial_\nu N \delta^0_\mu - N^2 g^{\nu 0}\Gamma^0_{\nu\mu}, \tag{G.5}$$

$$A_i = -N^2(g^{00}\Gamma^0_{0i} + g^{j0}\Gamma^0_{ji}) = \partial_i \ln N =: a_i, \tag{G.6}$$

$$A_0 = -\frac{1}{N}\partial_0 N + \Gamma^0_{00} + \frac{1}{N} N^i \partial_i N - N^i \Gamma^0_{i0} = N^i \partial_i \ln N = N^i a_i, \tag{G.7}$$

$$A^i = g^{i0} A_0 + g^{ij} A_j = \gamma^{ij}\partial_j \ln N = \partial^i \ln N =: a^i, \tag{G.8}$$

$$A^0 = g^{00} A_0 + g^{0i} A_i = 0. \tag{G.9}$$

In deriving the last equation, note the difference between g^{ij} and γ^{ij}.

It is straightforward to check $A_a u^a = 0$. Also it is easy to see that

$$A_a A^a = a_i a^i. \tag{G.10}$$

Note that the form of this equation holds *only* in the family of coordinate systems of the ADM formalism where u is the normal vector to the spacelike hypersurfaces. Under the foliation-preserving diffeomorphism, the RHS is invariant. However, in a different foliation where the hypersurfaces are not orthogonal to u, the form of the RHS will in general change. In other word, the RHS is not invariant under full diffeomorphism which changes the foliation.

H Calculation of the Komar Mass

In this Appendix, we calculate the Komar mass (or, equivalently the ADM mass in this case) corresponding to the solution (4.30) rewritten in the form

$$ds^2 = -F^c dt^2 + F^{-c} d\rho^2 + \rho^2 F^{1-c} d\Omega^2, \tag{H.1}$$

where $F = F(\rho) = (1 - \rho_0/\rho)$. The Komar mass is defined as

$$M = -\frac{1}{8\pi}\int_\infty \nabla^\alpha \xi^\beta dS_{\alpha\beta}, \tag{H.2}$$

where the integral is evaluated on the 2D closed spatial surface at infinity $\rho \to \infty$, $\xi^\alpha = \delta^\alpha_t$ is the timelike Killing vector, and the surface element 2-form is

$$dS_{\alpha\beta} = -2n_{[\alpha}r_{\beta]}\sqrt{\sigma}dy^2. \tag{H.3}$$

The timelike unit normal n^α and the unit spacelike normal to the 2D surface r^α are given as

$$n^\alpha = F^{-c/2}\delta^\alpha_t, \quad r^\alpha = F^{c/2}\delta^\alpha_\rho. \tag{H.4}$$

The measure is $\sqrt{\sigma}dy^2 = \rho^2 F^{1-c}\sin\theta d\theta d\phi$.

The integrand can be written as

$$-2\nabla^\alpha\xi^\beta n_{[\alpha}r_{\beta]} = 2\nabla^\beta\xi^\alpha n_\alpha r_\beta = 2\Gamma^\alpha_{\beta\gamma}n_\alpha r^\beta\xi^\gamma = -2\Gamma^t_{t\rho}F^c = -(1-\frac{\rho_0}{\rho})^{c-1}c\frac{\rho_0}{\rho^2}, \tag{H.5}$$

where in the last step we inserted the Christoffel symbol $\Gamma^t_{t\rho} = cF'/(2F)$. Inserting these back to the definition, we can easily evaluate the integral and obtain

$$M = \frac{c\rho_0}{2}, \tag{H.6}$$

i.e. $\rho_0 = 2M/c$. In fact, this can also be easily read from the asymptotic bebavior $g_{tt} \to 1 - c\rho_0/\rho$, as $\rho \to \infty$.

I Solving the 4D Static Problem in 3D

In this Appendix, we derive the static 4D solution by solving a 3D Euclidean problem. In particular, assume the 4D spherical static solution takes the form

$$ds^2 = -N^2dt^2 + \frac{1}{N^2}(d\rho^2 + \rho^2 F d\Omega^2), \tag{I.1}$$

where $N \equiv \exp(-2v)$. Then $v(\rho)$ and $F(\rho)$ are two functions to be solved.

As discussed in the main text, under this assumption, the effective 3D action of the form (4.45),

$$S_3 = \int d^3x\sqrt{\tilde{\gamma}}[\tilde{R} - \frac{8}{c^2}\tilde{\gamma}^{ij}\partial_i v\partial_j v], \tag{I.2}$$

can describe both GR and HL gravity. Note that here we have dropped a total constant which is irrelevant in the EOM. The reflection symmetry $v \to -v$ in S_3 corresponds to the freedom in the choice of $N = \exp(\pm 2v)$.

The EOM, i.e. the 3D Einstein equation and Klein-Gordon equation, are

$$\tilde{G}_{ij} = T_{ij}, \tag{I.3}$$

$$\tilde{D}_i\tilde{D}^i v = 0, \tag{I.4}$$

where

$$T_{ij} = \frac{8}{c^2}[\partial_i v\partial_j v - \frac{1}{2}\tilde{\gamma}_{ij}(\partial v)^2], \tag{I.5}$$

and $\tilde{D}_i$ is the covariant derivative associated with $\tilde{\gamma}_{ij}$, which is given by $d\tilde{s}^2 = d\rho^2+\rho F d\Omega^2$. Before solving the equations, note that these are second order equations of F and v; thus we expect 4 integration constants $C_{1,2,3,4}$, to be fixed by physical conditions.

Inserting the assumption into the EOM, the Klein-Gordon equation becomes

$$v'' + (\frac{2}{\rho} + \frac{F'}{F})v' = 0. \tag{I.6}$$

The independent Einstein equations are the $\rho\rho$ component

$$\frac{1}{\rho}\frac{F'}{F} + \frac{1}{4}(\frac{F'}{F})^2 + \frac{F-1}{\rho^2 F} - \frac{4}{c^2}(v')^2 = 0, \tag{I.7}$$

and the $\theta\theta$ component

$$\frac{1}{\rho}\frac{F'}{F} + \frac{F''}{2F} - \frac{1}{4}(\frac{F'}{F})^2 + \frac{4}{c^2}(v')^2 = 0. \tag{I.8}$$

Combining them to eliminate the v'-dependence leads to

$$\frac{\rho}{2}F'' + 2\rho F' + F - 1 = 0. \tag{I.9}$$

Note that the c^2 factor is absent. In fact this implies that the 3D geometry $\tilde{\gamma}_{ij}$ is independent of the constant scale of v. The solution of the above equation is

$$F(\rho) = 1 + \frac{C_1}{\rho} + \frac{C_2}{\rho^2} = \frac{(\rho-\rho_-)(\rho-\rho_+)}{\rho^2}, \tag{I.10}$$

where $\rho_\pm \equiv (-C_1 \pm \sqrt{C_1^2 - 4C_2})/2$. Inserting this F back into the Klein-Gordon equation gives the solution as

$$v(\rho) = C_3 + \frac{C_4}{4}\tanh^{-1}(\frac{2\rho + C_1}{\sqrt{C_1^2 - 4C_2}}). \tag{I.11}$$

A constant shift of v is a symmetry of S_3. Moreover, it corresponds to a constant rescaling of the time coordinate in 4D. Thus one can set $C_3 = 0$. Then use $\tanh^{-1}(z) = 1/2[\ln(1+$

$z) - \ln(1-z)]$ to rewrite it as

$$v(\rho) = \frac{C_4}{4} \ln \left| \frac{\sqrt{C_1^2 - 4C_2} + C_1 + 2\rho}{\sqrt{C_1^2 - 4C_2} - C_1 - 2\rho} \right| = \frac{C_4}{4} \ln \left| \frac{\rho - \rho_-}{\rho - \rho_+} \right| . \tag{I.12}$$

Recall that the equation (I.9) is obtained by combining the $\rho\rho$ and $\theta\theta$ components. In fact, each of them should be separately valid. This provides an additional condition to fix $C_4 = c$. In particular $C_4 = 1$ corresponds to Schwarzschild solution. The spacetime metric now becomes

$$ds^2 = - \left| \frac{\rho - \rho_-}{\rho - \rho_+} \right|^c dt^2 + \left| \frac{\rho - \rho_+}{\rho - \rho_-} \right|^{-c} \left[d\rho^2 + (\rho - \rho_-)(\rho - \rho_+) d\Omega^2 \right] . \tag{I.13}$$

Moreover, using the gauge degree of freedom of constant shift of the coordinate ρ, redefining $\rho \to \rho - \rho_+$ and $\rho_0 \equiv \rho_- - \rho_+$, the above form becomes (assuming $1 - \rho_0/\rho > 0$)

$$ds^2 = -(1 - \frac{\rho_0}{\rho})^c dt^2 + (1 - \frac{\rho_0}{\rho})^{-c} [d\rho^2 + \rho^2 (1 - \frac{\rho_0}{\rho}) d\Omega^2]. \tag{I.14}$$

Now the only constant ρ_0 is related to the Komar mass M by $\rho_0 = 2M/c$, as shown in Appendix H.

J Singularity at ρ_0 for $c > 2$

For $c > 2$, R, $R_{ab}R^{ab}$ and $R_{abcd}R^{abcd}$ are all vanishing. One can consider other scalar polynomials constructed from the Riemann curvature and its covariant derivatives. Here we will show that $\rho = \rho_0$ is a singularity by calculating the components of the Riemann tensor in a parallel propagated orthonormal frame along a timelike geodesic.

To this end, consider a timelike radial geodesic with a unit tangent vector

$$\partial_\tau = v = (\dot{t}, \dot{\rho}, 0, 0), \tag{J.1}$$

where a dot denotes a derivative with respect to the affine parameter τ, which is also the proper time in this case. Then its scalar product with the timelike Killing vector is a conserved quantity along the geodesic, denoted by E,

$$E = v_a (\partial_t)^a = -\dot{t} g_{tt} \quad \Rightarrow \quad \dot{t} = E F^{-c}. \tag{J.2}$$

Moreover, $\dot{\rho}$ can be determined from the unit norm of v,

$$-1 = v^a v_a = \dot{t}^2 g_{tt} + \dot{\rho}^2 g_{\rho\rho} \quad \Rightarrow \quad \dot{\rho}^2 = E^2 - F^c. \tag{J.3}$$

As $\rho \to \rho_0$, $F \to 0$. So $\rho(\tau)$ can be approximated by solving

$$\dot{\rho}^2 = E^2 \quad \Rightarrow \quad \int_{\rho_0}^{\rho} d\rho = \pm \int_{\tau_0}^{\tau} E d\tau. \tag{J.4}$$

Thus we obtain the near-ρ_0 behavior

$$\rho - \rho_0 \sim \pm E(\tau - \tau_0), \tag{J.5}$$

which states that a timelike geodesic can reach ρ_0 in finite proper time.

Now, construct a parallel propagated orthonormal frame along the timelike radial geodesic as

$$\begin{aligned} e_0 &= v = EF^{-c}\partial_t - E\sqrt{1 - F^c/E^2}\partial_\rho, \\ e_1 &= E\sqrt{1 - F^c/E^2}F^{-c}\partial_t - E\partial_\rho, \\ e_2 &= \frac{1}{\rho F^{(1-c)/2}}\partial_\theta, \\ e_3 &= \frac{1}{\rho \sin\theta F^{(1-c)/2}}\partial_\phi. \end{aligned} \tag{J.6}$$

The components of the Riemann tensor in this parallel propagated frame can be calculated. For example,

$$R_{0202} = -\frac{E^2\rho_0^2(c^2-1) + F^c(\rho_0^2(1+c) - 2c\rho\rho_0)}{4\rho^2(\rho-\rho_0)^2}. \tag{J.7}$$

This implies that for $c \neq 1$ (in particular for $c > 2$) the tidal forces diverge as $(\rho - \rho_0)^{-2}$.