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PLASTIC DESIGN OF REINFORCED CONCRETE SLABS

by

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ABSTRACT

This thesis presents a study of plastic design methods for reinforced concrete slabs. The ultimate limit state has been considered for both flexure and shear. The emphasis is on how the various plastic methods presented in the literature may be applied to the design of flat plates.

Both upper and lower-bound moment fields have been examined. Yield-line theory and the segment equilibrium method were the primary upper-bound solutions considered. A more efficient application of yield-line theory to the design of slabs is presented than given elsewhere in the literature. The segment equilibrium method is the logical result of such an approach and is thus, a special case of yield-line theory. Hillerborg's strip method and segment method were the primary lower-bound solutions which were discussed. These methods were examined systematically from the simple strip method to the load dispersion element and the advanced strip method, and finally, to the corner supported element and the segment method. Design examples have been included to illustrate the logical progression in the development of these lower-bound solutions.

The problem of shear and moment transfer in slab-column junctions of flat plates has been considered by several methods. The ACI approximate elastic analysis, yield-line analysis, and the beam analogy were compared. It was found that a plastic interaction equation which is a variation of beam analogy was, in the most instances, the best method for predicting the capacity of slab-column junctions transferring shear and moments. It was also found that there are cases where failure occurs by a yield-line mechanism around the column. For such cases, all methods except a yield-line analysis are unsafe.

The questions of pattern loading has been considered with regard to column moments and steel detailing requirements. Several plastic methods for dealing with pattern loading have been presented. It was found that while the cut-off points used in current practice cannot be justified by an elastic analysis, they can be justified by a plastic analysis.

ACKNOWLEDGEMENTS

This report is in essence a reprint to the thesis entitled "Plastic Design Methods for Reinforced Concrete Slabs" prepared by David M. Rogowsky in partial fulfilment of the requirements for an M.Sc. degree awarded in the fall of 1980. Supervisor for the thesis was Dr. S.H. Simmonds, Professor of Civil Engineering.

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LIST OF SYMBOLS

a	Distance from column centre-line to final cut-off point of top mat.
$A_{critical}$	Area of critical section for punching shear, based on the effective depth.
b	Distance from column centre-line to first cut-off point or edge of drop panel.
c	Side length of square load dispersion element.
c_x	Side length, parallel to x axis, of rectangular load dispersion element.
c_y	Side length, parallel to y axis, of rectangular load dispersion element.
c_1	Side length of rectangluar column in the direction of span being considered.
c_2	Side length of rectangular column perpendicular to the direction of span being considered.

d	Effective depth of slab, from slab surface to centroid of tensile reinforcement.
e	Eccentricity of the centoid of the critical section with respect to the centroid of the column.
f'_c	Specified strength of concrete.
f_y	Specified yield strength of reinforcement.
$J_{critical}$	Polar moment of inertia of the critical section with respect to the centroid of the critcal section.
K	A parameter used to determine wether or not there are significant positive moments within a preset moment field.
k_α	The value of K for moments in the y direction.
k_β	The value of K for moments in the x direction.

L Centre-to-centre distance between columns.

ℓ Clear span between columns.

m Positive moment (tension on underside of slab) per unit width of slab.

m' Negative moment (tension on top side of slab) per unit width of slab.

m_a Positive moment slab capacity per unit width of slab in zone between a and b from column centre-line.

m'_a Negative moment slab capacity per unit width of slab in zone between a and b from column centre-line.

m_r Radial moment per unit width of slab.

m_s Positive moment slab capacity per unit width of slab in zone within a distance b from the column centre-line.

m_s' Negative moment slab capacity per unit width of slab in zone within a distance b from the column centre-line.

m_t Tangential moment per unit width of slab.

m_x Moment in x direction, (reinforcement parallel to x axis) per unit width of slab.

m_y Moment in y direction, (reinforcement parallel to y axis) per unit width of slab.

m_{xs} The m_x moment along the edge of a load dispersion element which passes through the support.

m_{ys} The m_y moment along the edge of a load dispersion element which passes through the support.

$m_{xs,s}$ The m_{xs} moments due to the loads shown in Fig. 4.3.

$m_{ys,s}$ The m_{ys} moments due to the loads shown in Fig. 4.3.

$m_{x,1}$ That portion of $m_{xs,s}$ due to part 1 of the loading shown in Fig. 4.3.

$m_{x,2}$ That portion of $m_{xs,s}$ due to part 2 of the loading shown in Fig. 4.3.

$m_{x,3}$ That portion of $m_{xs,s}$ due to part 3 of the loading shown in Fig. 4.3.

m_{α} The value of m_{ys} for that portion of the edge within $\alpha\ c_x$ of the support.

$m_{1-\alpha}$ The value of m_{ys} for that portion of the edge beyond $\alpha\ c_x$ of the support.

m'_{α} The value of m_{α} after the superposition of a uniform positive moment field on the corner supported element.

m_β The value of m_{xs} for that portion of the edge within $\beta\ c_y$ from the support.

$m_{1-\beta}$ The value of m_{xs} for that portion of the edge beyond $\beta\ c_y$ from the support.

m'_β The values of m_β after the superposition of a uniform positive moment field on the corner supported element.

M Total moment transferred from slab to columns.

M_c Column strip moment.

M_c^+ Positive portion of M_c .

M_c^- Negative portion of M_c .

M_f Moment transferred to column by flexural reinforcement.

M_m Middle strip moment.

M_m^+ Positive portion of M_m .

M_m^- Negative portion of M_m .

M_o Total moment which can be transferred to column when no shear is transferred to column.

M_s Moment transferred to column by shear stressed in the concrete.

M_t Moment transferred to column by torsion on the side faces of the critical section.

M_{test} Moment transferred to column at failure of test specimen.

M_1 Column moment above slab.

M_2 Column moment below slab.

P Value of point support reaction acting on load dispersion element.

q	Uniformly distributed load applied to slab.
q_x	Portion of q carried by slab in the x direction
q_y	Portion of q carried by slab in the y direction
q_r	Polar-symetric distribution of load.
Q_ℓ	Lower-bound load.
Q_u	Upper-bound load.
r	Radius of polar-symetric distribution of loads and moments.
r	Radius of column or column capital.
r_q	Ratio of moveable load to permanent load.
R	Radius of primary conical mechanism.
R_0	An intermediate variable used in the solution of conical mechanisms, with no physical meaning. Defined in Equation 3.14.

R_1 An intermediate variable used in the solution of conical mechanisms, with no physical meaning. Defined in Equation 3.13.

R_2 An intermediate variable used in the solution of conical mechanisms, with no physical meaning. Defined in Equation 3.16.

T Torque transferred to column on a face of the critical section.

T_o The maximum torque which may be transferred to column if no shear is to be transferred.

v_1 Shear stress due to transfer of load into column.

v_2 Shear stress due to transfer of moment into column.

V Nodal force.

V Load transferred from slab to column.

V_{flex} Load transferred to column which produces yield-line failure.

V_o Shear which may be transferred to column if no moment is transferred.

V_{test} Load transferred to column at failure of the test specimen.

w Uniformily distributed load applied to slab.

α Parameter defining yield-line locations in example problem.

α Portion of c_x over which the maximum edge moments act on a rationalized load dispersion element.

α_{ec} Ratio of effective column stiffness to slab stiffness.

β Portion of c_y over which the maximum edge moments act on a rationalized load dispersion element.

γ Angle between co-ordinate axes of skew slab.

γ_f Portion of M transferred by flexure.

μ Ratio of moment capacities in co-oridinate directions of slab.

ρ Radius of secondary concial yield-line mechanism.

ψ Ratio of negative panel moment to positive panel moment.

X Distance from column centre-line to point of inflection.

1. INTRODUCTION

Reinforced concrete design procedures have recently evolved into limit state design procedures. This work concerns the design of reinforced concrete slabs, with particular reference to flat plates, for the ultimate or collapse limit state. The scope includes flexure as well as shear. At the ultimate limit state, slabs behave plastically thus, plastic methods of design are considered. Elastic procedures of analysis are used occasionally as benchmarks for comparison of the various methods.

The plasticity theorems and their implications in terms of reinforced concrete slab design are introduced. The upper-bound procedures of Johansen and Wiesinger are considered along with the lower-bound solutions of Hillerborg. Methods of checking the shear and moment transfer capacity of slab-column connections are compared. The question of accommodating pattern loading is also considered.

This thesis has been written to critically evaluate the above procedures from the point of view of a structural designer. Many of the procedures have been critically reviewed by others for their mathematical correctness, but this work will consider their strengths and weaknesses as design tools. The assumptions upon which the procedures are based will be discussed briefly, but the emphasis will be on: how the methods are related to each other, how the methods could or should be applied, the choices the structural designer is required to make, and how these choices will effect the overall design of the slab. Several portions of this discussion cover points not discussed elsewhere in the literature, but which are essential to a complete slab design.

The question of serviceability is not discussed specifically. Not that serviceability is not important, but because it is generally satisfied by some empirically determined minimum slab thickness. Similar empirical rules would still be required for plastic design methods. The present rules may require modification if the plastic designs used require significantly more plastic redistribution and cracking than present slab designs.

2. PLASTICITY THEOREMS

2.1 Introduction

A major task in the design of reinforced concrete slabs is to determine a distribution of moments which is in equilibrium with the applied loads. Elastic moment fields can be obtained with finite difference solutions or finite element solutions. However, since slabs do not behave elastically near ultimate load, and the bending capacity of the slab is based on a plastic analysis of a cross-section, it is reasonable to use methods of analysis and design which recognize plasticity. The plastic methods are somewhat approximate but, "Is it not just as valuable to know approximately how a real structure is going to behave as it is to know exactly how an approximate structure is going to behave?" (Lansdown 1967).

Yield-line theory, the segment equilibrium method and the strip method are methods of plastic design. The basis for each method will be examined along with the choices a designer is required to make for their use. The implications of these choices in design will be discussed.

2.2 Lower-Bound and Upper-Bound Theorems

All rational methods of plastic analysis fall into one of two categories, "upper-bound" solutions and "lower-bound" solutions. These categories are defined by the upper and lower-bound theorems of plasticity. These theorems as they apply to slabs have been stated by Hillerborg (Hillerborg 1975) as follows:

> *Lower-bound theorem:* If there is a load Q_ℓ for which it is possible to find a moment field which fulfills all equilibrium conditions and the moment at no point is higher than the yield moment, then Q_ℓ is a lower-bound value of the carrying capacity. The slab can *certainly* carry the load Q_ℓ .
>
> *Upper-bound theorem:* If, for a small virtual increment of deformation, the inner energy taken up by the slab on the assumption that the moment in every point where the curvature is changed equals the yield moment and this energy is found to equal the work performed by the load Q_u for the same increment of deformation, then Q_u is an upper-bound value of the carrying capacity. Loads greater than Q_u are *certainly* high enough to cause moment failure of the slab.

2.3 Characteristics of Lower-bound and Upper-bound Solutions

2.3.1 Safety and Economy

Lower-bound solutions underestimate the theoretical collapse load, therefore they are always on the safe side. The underestimate in the theoretical collapse load may be so large that the resulting design is uneconomic. On the other hand, upper-bound solutions overestimate the theoretical collapse load, therefore they are always theoretically on the unsafe side. Since upper-bound solutions always produce a higher estimate for the collapse load than lower-bound solutions, designs based on upper-bound solutions may be slightly more economical than designs based on lower-bound solutions.

2.3.2 Superposition

A lower-bound moment field in equilibrium with a given loading can be added to a second lower-bound moment field in equilibrium with a second set of loads. The resulting moment field will be in equilibrium with the combined loadings and it is thus still a lower-bound solution. Hence, the principle of superposition is valid for lower-bound solutions.

For upper-bound solutions, the sum of the ultimate moments for a series of loads is always greater than or equal to the ultimate moments for the sum of the loads (Johansen 1962). Thus, superposition is not generally correct, but the result is always more safe than each of the individual solutions being superposed.

2.3.3 Moments Calculated

With lower-bound solutions, the entire moment field is obtained. Thus, the cut-off points for the reinforcement may be readily determined. With upper-bound solutions, the moments are only known at discrete points (hopefully the critical sections). The determination of the cut-off points requires special investigation and is usually not covered in a "pure" upper-bound solution procedure since it is usually assumed that the reinforcement is continuous throughout the entire slab. Thus, perhaps the most important difference between practical lower-bound and upper-bound solutions is that one gives enough information to detail the reinforcement cut-off points while the other does not.

2.3.4 Analysis vs. Design

Lower-bound solutions remain on the safe side whether the solution is used for analysis or design. Upper-bound solutions remain on the "unsafe" side whether the solution is used for analysis or design. Thus, there is no difference between analysis and design as far as the plasticity theorems are concerned.

2.4 Implications of Approximate Lower-Bound and Upper-Bound Solutions Used for Design

For a given model of a structure, the "unique" solution (exact mathematical solution) is obtained when the lower and upper-bounds coincide. The unique solution can be obtained with either an exact lower-bound solution or an exact upper-bound solution. Unique solutions have been found for a few special cases (Wood 1961), but these cases are of little practical value in structural engineering. Therefore, one must make do with approximate solutions.

Fig. 2.1 gives a qualitative indication of the relationships between lower-bound, upper-bound, and unique solutions as well as the true load capacity that would be obtained in a load test. To the left of the figure, the predicted failure load decreases while to the right, the predicted failure load increases. The unique solution is obtained at the point where the lower and upper-bound meet. The most striking feature of this figure is that the unique (mathematically exact) solution underestimates the true load capacity. This implies that the usual mathematical model of the structure is not exact due to the neglect of inplane forces, strain hardening, etc. This also implies that upper-bound solutions are not always on the "unsafe" side of the true failure load. One can get acceptably safe designs with reasonable approximate upper-bound solutions, while poor approximate upper-bound solutions are unsafe. On the other hand, poor approximate lower-bound solutions are safe but uneconomical. Examples of good and poor approximations will be discussed when considering the details of each method of design.

2.5 Affinity Theorems

There exists a set of affinity theorems which are sometimes useful in slab design. These theorems were first proposed by Johansen (1943) and developed by Jones and Wood (1968). The theorems can be used to convert a skew or orthotropically reinforced slab into an affine slab such

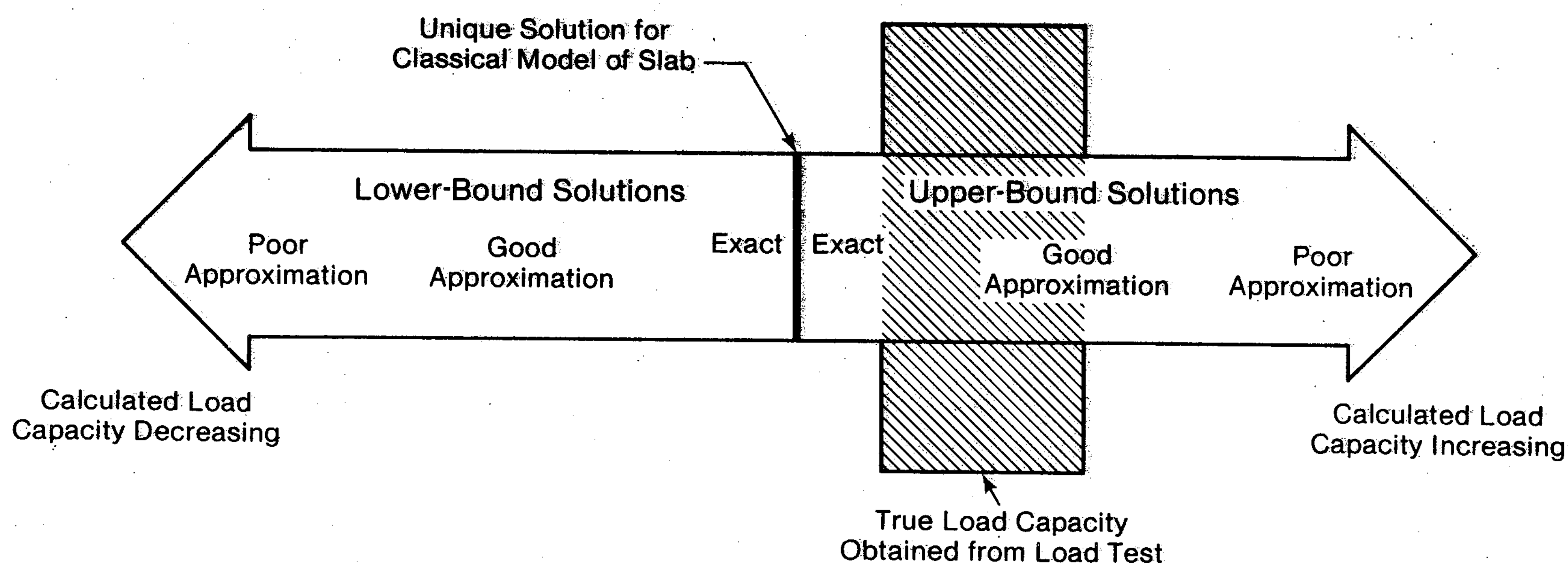


Figure 2.1 Implication of Upper and Lower-Bound Solutions For Reinforced Concrete Slabs

that the affine slab is isotropic and will have the same deflections at corresponding points as the original slab. In this way, solutions which have been derived for isotropic slabs can be extended to orthotropic and skew slabs if certain conditions listed later are satisfied.

The affinity theorems have been summarized by Mills (1970) as follows:

> If, at any point on the skew or orthotropic slab, the ultimate moments due to the separate bands of steel reinforcement are m and μm, then the strength of the isotropic slab at a corresponding point is m in all directions provided that:
>
> 1. The affine slab is drawn such that all distances measured in the direction of the m-reinforcement remain the same and this direction forms one co-ordinate axis for both slabs which will be called the first co-ordinate axis. See Fig. 2.2.
> 2. The other (second) co-ordinate axis follows the direction of the μm-reinforcement (and all measurements must be referred to these axes), but the affine slab is taken at right-angles to the first co-ordinate axis.
> 3. All distances in the affine slab shown in Fig. 2.2b are measured in the second co-ordinate direction and are obtained by dividing lengths in the original slab by $\sqrt{\mu}$.
> 4. All corresponding total loads in the affine slab

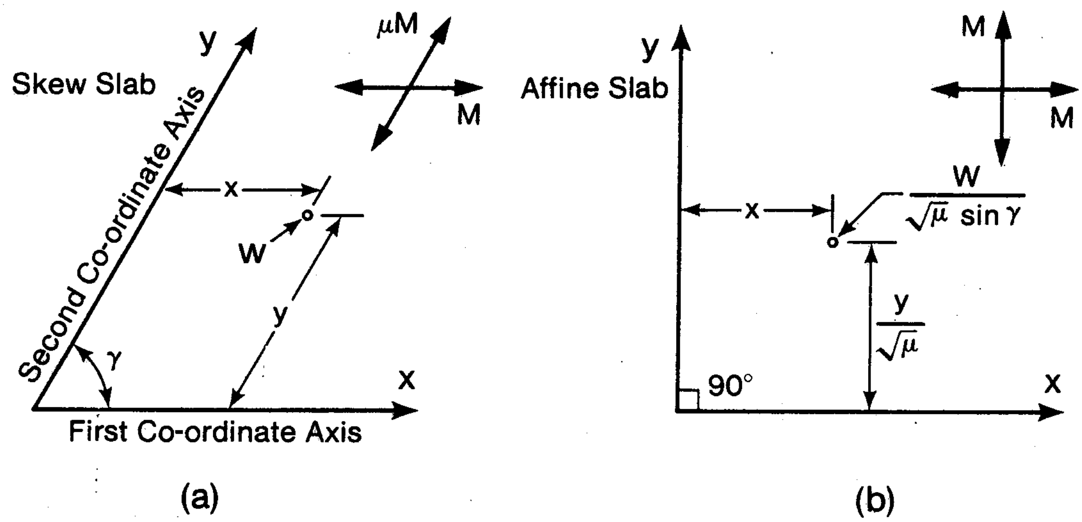


Figure 2.2 Original and Affine Slabs

are obtained by dividing the original total loads by $\sqrt{\mu}\ \sin\gamma$.

As a result of these points, the affine slab will not necessarily have the same geometry nor the same loading per unit area as the original slab. The transformation of a slab to an affine slab will thus leave the designer with little feeling for the behavior of the real slab.

It must be emphasized that in the original slab, the positive and negative moment capacities may be different, but μ for the positive moment capacities must be equal to μ for the negative moment capacities in order to make use of the affinity theorems. Also, it is assumed that the slab has homogeneous reinforcement, that is, the reinforcing mats must be uniform throughout the entire slab. As a result of these points, the application of the affinity theorems is somewhat restricted, but they can be used to advantage for some special cases. Such a case is pointed out in Section 4.3.

2.6 Elastic Solutions

Elastic solutions satisfy the requirements of the lower-bound theorem, but they also require geometric compatibility. There is only one elastic solution to each particular problem. Once the elastic properties are assumed or established, the designer is not required to use any judgement. However, fantastic judgement may be required in the choice of elastic properties as well as in interpreting and modifying the results of the analysis in order that a reasonable reinforcement layout can be used.

2.7 Basic Assumptions Common to All Procedures Considered

All of the procedures investigated assume that the slab is thin and that the displacements are small. It can be argued that each of these assumptions is on the conservative side. Plastic or elastic solutions can be obtained without these assumptions, but such solutions are very difficult to obtain.

By assuming small displacements, equilibrium can be formulated on the undeformed shape of the structure. The practical effect of this is that one ignores the possibility of load being carried by catenary or tensile membrane action, since the deflections required to develop significant membrane action are much larger than acceptable deflections even under full load. It is thus both safe and convenient to assume that the displacements are small.

Assuming that the slab is thin results in the neglect of inplane forces. Even with very small deflections, if the slab is relatively thick, inplane compressive forces can carry load by "dome action" and "slab jamming". For this type of behavior to occur, the slab or building surrounding the area under consideration must provide adequate boundary conditions for a compressive membrane with a rise equal to the slab thickness. Assuming that the slab is thin is the same as setting the rise of the compressive membrane equal to zero. Slab jamming can greatly enhance the load carrying capacity of a slab in some instances. Ockleston (1955) load tested a portion of a building to failure and found that at three times the predicted upper-bound failure load the slab had virtually no cracks and no deflection. At this point the slab snapped through and carried the load as a tensile membrane. This type of behavior has not been reported for flat pate structures, but it can occur under certain conditions. The possible gains in economy as a result of considering the effects of inplane compressive forces are enticing, but at present, practical analytical procedures are not adequate to consider this effect. Because of this, along with the uncertainty in actually obtaining the required boundary conditions, it is thus both safe and convenient to assume that the slab is thin.

All of the procedures require a "yield-criteria". That is, given the yield moment capacity in two directions, what is the effective yield moment in some intermediate direction, and do the moments in one direction affect the yield moment in some other direction. There has been a great deal of discussion on these points in the literature, but experimental and analytical studies (Lenschow and Sozen 1967) indicate that:

1. There is no appreciable increase in moment capacity of a reinforced concrete section as a result of biaxial bending. Methods used to determine the flexural capacity of beams can be used to evaluate the flexural strength

of a slab element in the direction of the reinforcing bars.

2. The yield moment along a line at some inclination to the reinforcing bars can be taken as the vector components of the yield moments crossing this line.
3. In practical ranges of bar and slab proportions, there is no appreciable effect on moment capacity of reorientation or "kinking" of the reinforcing bars at the yield-lines.

These points form what is know as the "square yield criteria". It should be noted that each assumption or observation is on the conservative side, so the square yield criteria is simple and safe to use. It will be used for the remainder of this work unless noted otherwise.

3. PLASTIC UPPER-BOUND SOLUTIONS

3.1 Historical Background

The yield-line theory is a theory which leads to upper-bound solutions for slabs in flexure, and will receive the bulk of the discussion in this chapter. It was the first ultimate strength procedure for slabs. It was proposed by Ingerslev (1923) and was highly developed by Johansen (1943). This method is extensively used in Scandinavian countries but is little used in North America. The literature contains considerable discussion of the details of yield-line theory to which one may refer. (Johansen 1962, 1972, Hognestad 1953, Wood 1961, Mills 1970, Simmonds and Ghali 1976).

3.2 Basic Assumption

The whole yield-line theory is built-up by assuming that the slab behaves as a rigid-plastic material in flexure. Fig. 3.1 compares the assumed moment-rotation curve to a typical slab moment-rotation curve. Once the slab has reached its yield moment, there is very little difference between the real and assumed behavior. Thus, slabs due to their inherently low steel ratio generally have sufficient ductility to permit the rotations required to develop failure mechanisms. This may not be true for slabs reinforced with welded wire fabric since some types of fabric do not possess significant ductility. For such slabs,

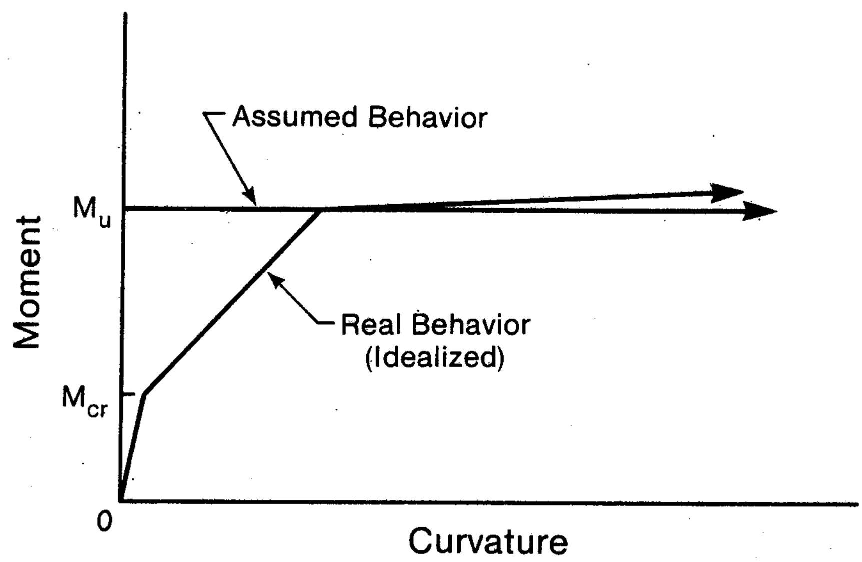


Figure 3.1 Typical Moment Curvature Diagram For Reinforced Concrete Slabs

one must ensure that very little plastic redistribution of moments will be required.

The rigid-plastic assumption leads to the following:

1. Since no deformations take place prior to yielding, and one would not expect yielding at service load, no information is provided with respect to service load behavior.
2. Curvatures only occur at points or lines of yielding, while the remainder of the slab remains plane.
3. Since planes intersect along straight lines, this implies a failure mechanism of plane segments of slab bounded by straight yield-lines of constant rotation and moment.
4. The slab is reduced from a continuum problem to a discrete failure mechanism with a few degrees of freedom.
5. For each failure mechanism, it is then possible to relate the loading and moment capacity from considerations of either equilibrium or energy.

3.3 Overview of Solution Procedure

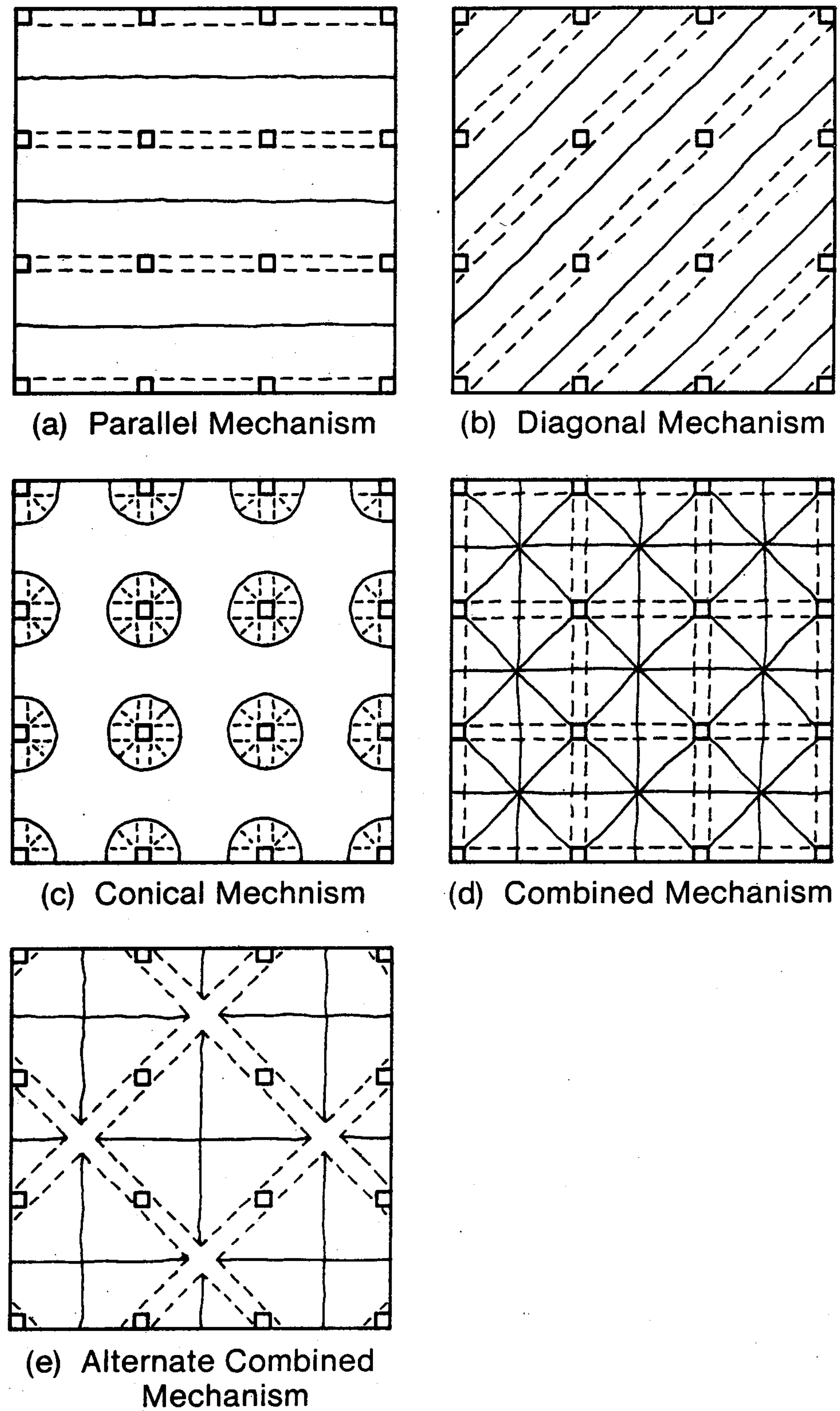
The formulation of the load-moment relationship may be done with an energy approach or an equilibrium approach. Both approaches produce the same results. The salient features of each formulation are discussed later in this thesis but first some terms and concepts

A yield-line "pattern" is a particular set of yield-lines which define a kinematically admissible mechanism. Each yield-line pattern has its own specific yield-line locations, while a yield-line "family" is a set of yield-line patterns which can be uniquely defined by the same set of parameters. Each yield-line family represents a different failure mechanism. Each of the patterns within a family will look more or less the same but will have the yield-lines in slightly different locations, hence, one must determine the pattern which produces the lowest load carrying capacity. All possible yield-line families should be investigated. While it is possible to prove mathematically that one has found the critical yield-line pattern within a particular family, there is no way of proving that one has found the most critical family. For unusual cases of geometry or loading, finding the critical yield-line family may be very difficult.

Fig. 3.2 shows several different yield-line families for a regular flat plate. The critical yield-line pattern within each family must be determined. In the parallel mechanism for example, the critical location of the positive yield-line in the exterior spans must be determined. In the conical mechanism, the diameter of the failure mechanism which produces failure at the lowest load must be determined.

Since yield-line patterns must be kinematically admissible, it is possible to give some rules for determining possible yield-line patterns (Johansen 1962).

1. The yield-line between two parts of a slab must pass



(a) Parallel Mechanism

(b) Diagonal Mechanism

(c) Conical Mechnism

(d) Combined Mechanism

(e) Alternate Combined Mechanism

Figure 3.2 Possible Failure Mechanisms for Flat Plates through the point of intersection of their axes of rotation.

2. For a given segment, an axis of rotation lies either along a line support, or passes over a column.
3. At a point where yield-lines of different signs meet, there can be yield-lines in not more than three directions.

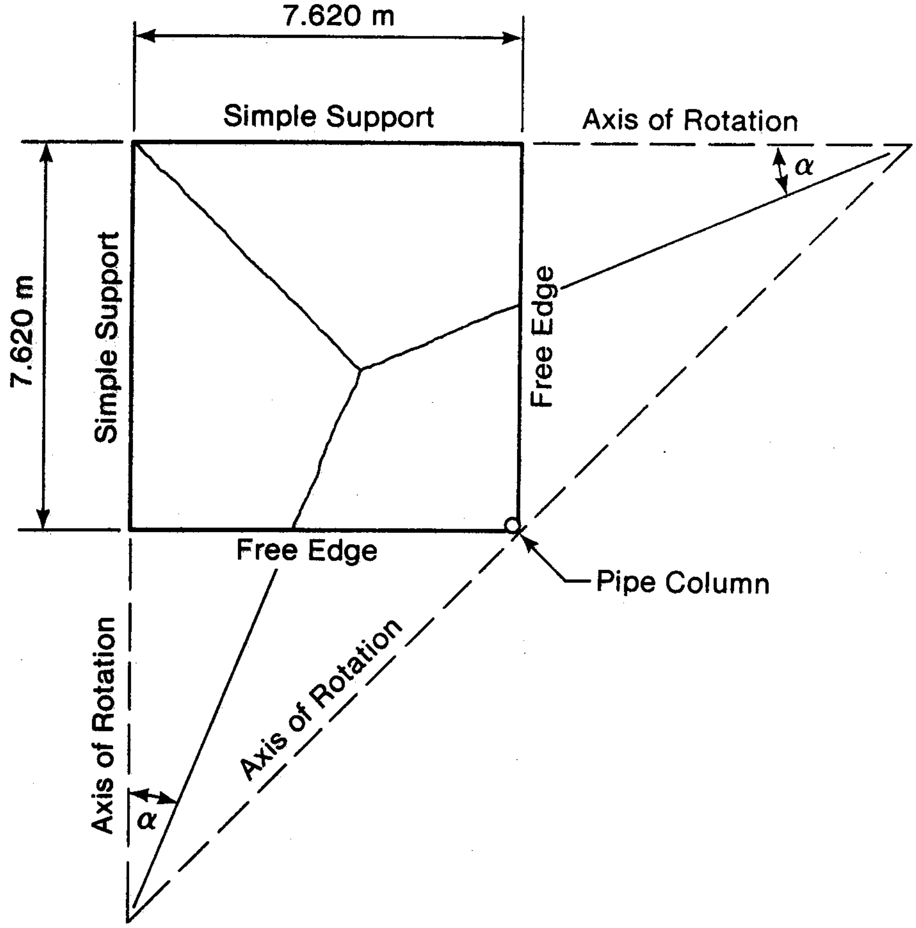
3.4 Energy Formulation

In an energy formulation, one takes a yield-line pattern and gives the mechanism a small virtual displacement. The external virtual work done by the external loads is equated to the internal virtual work which is produced along the yield-lines by the yield moments acting through the virtual rotations. From this, one obtains a relationship between the external load and the moment capacity of the slab for the particular yield-line pattern in question.

For each yield-line family, the geometry of the yield-lines is expressed by a set of parameters which must be optimized to obtain the criticial yield-line pattern for the particular family in question. In design, this would be the values of the parameters which produce the largest required yield moments in the slab. The optimum values can be found exactly with calculus or can be found approximately by trial.

With an energy formulation, the design moments required for the slab are generally not very sensitive to variations in the yield-line locations away from their critical locations. This can be demonstrated with the example slab shown in Fig. 3.3. The symmetry of the problem permits specifying the position of the axes of rotation for the different segments. Hence, the location of the yield-lines can be expressed as a function of the parameter α only. The relationship between moment and load as a function of α was obtained using an energy formulation. The required moments for various values of α are plotted in Fig. 3.4. The most important point illustrated in this figure is that the curve is quite flat around the optimum α, thus a relatively large change in α produces only a small change in the moment capacity required. This change in α corresponds to a change in yield-line location. Fig. 3.5 shows the range in yield-line locations which give moment capacities within 10% of the theoretically correct moment capacity for this family of yield-lines. This illustrates that, in general, the results obtained with an energy formulation are not significantly affected by moderate deviations of the yield-lines from their critical locations. Of course a completely different yield-line family such as a fan at the column may govern and should be investigated also.

Since any reasonable layout of yield-lines will produce acceptably accurate results, one could spend less time maximizing each yield-line family and spend more design time ensuring that all the possible families have been



Given : Uniformly Distributed Load = 24.0 kN/m^2
and Line Load on Free Edges = 2.2 kN/m

Determine : Uniform Positive Moment Capacity Required to Carry These Loads.

Figure 3.3 Slab with Yield-Line Pattern as a Function of x

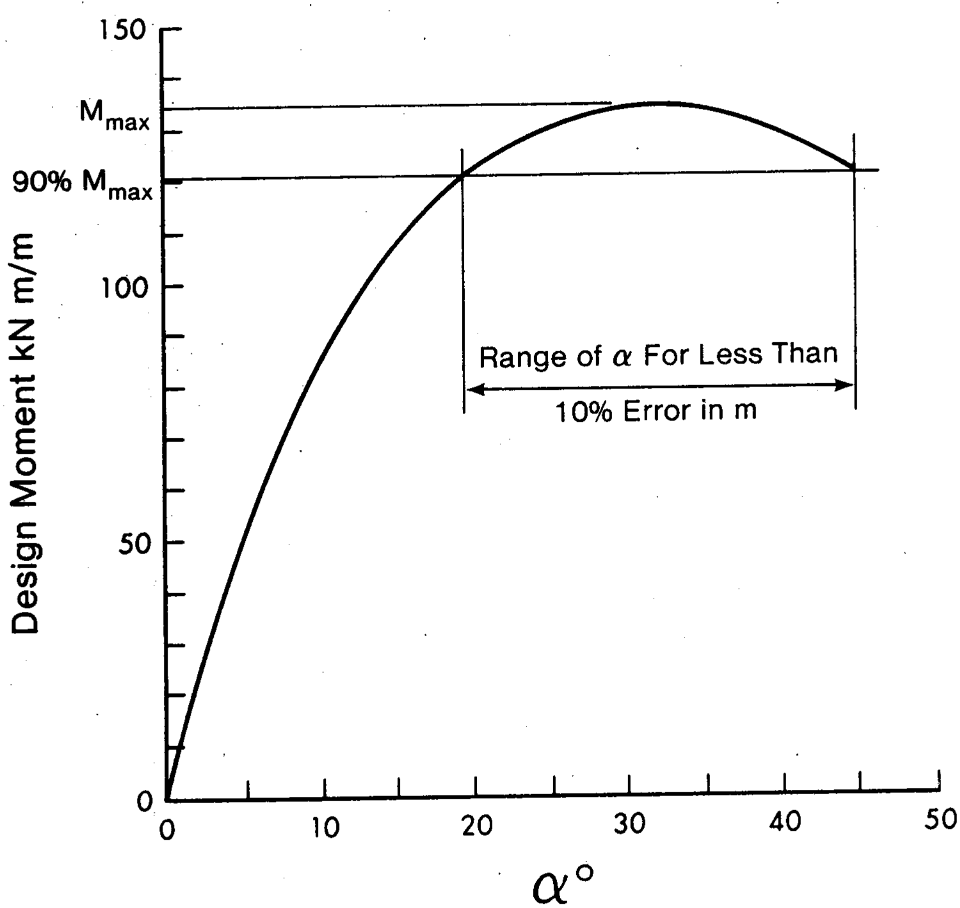
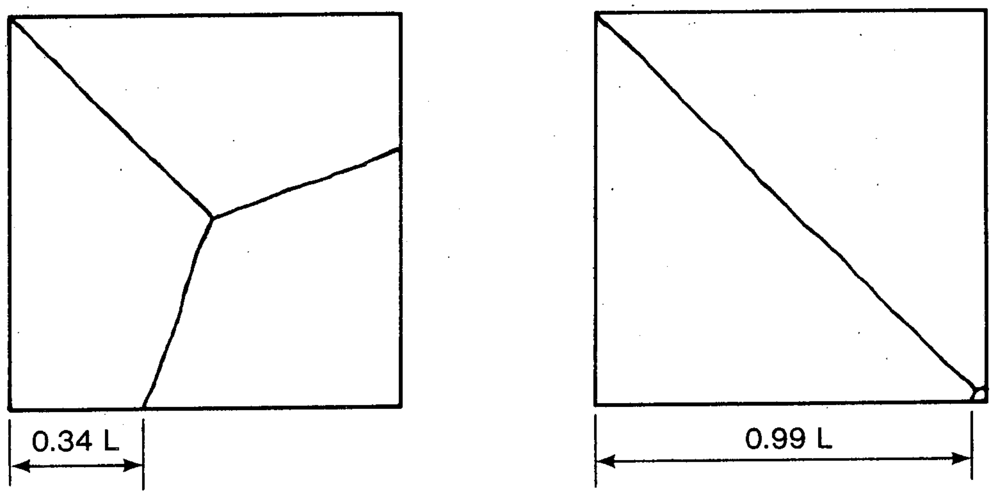
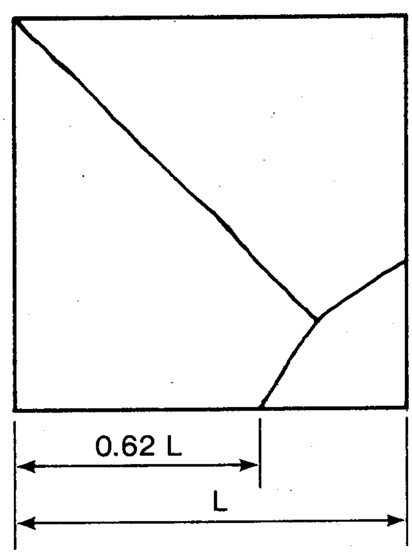


Figure 3.4 Required Slab Yield Moment Capacity as a Function of Yield-Line Location



(a) Range in Location of Yield Lines For 10% Error in Moments.



(b) Correct Yield Line Locations.

Figure 3.5 Insensitivity of Yield-Line Location in a Virtual Work Formulation investigated. It should be pointed out that the energy formulation is somewhat sensitive to the location of yield-lines in the vicinity of acute corners. Fig. 3.6 illustrates some areas which requires careful consideration.

3.5 Equilibrium Formulation

Equilibrium formulations have a number of features which are significantly different from the previously discussed energy formulations. On the basis of equilibrium between the external load and the moments along the yield-lines, one can develop the relationship between external load and moment capacity for each planar segment of the slab.

With an equilibrium formulation, if the yield-line pattern being considered is not the critical one, the results of the analysis provide an indication of which way the yield-lines should be moved in order to obtain the critical yield-line pattern for the particular family in question. In the design process, if the yield-line pattern is not correct, the yield-line moment required to equilibrate the loads applied to the slab segment on one side of the yield-line will be different from the yield-line moment required to equilibrate the loads applied to the slab segment on the other side of the yield-line. The yield-line should be moved towards the segment which required the larger yield-line moment so that this segment is made smaller. The analysis could then be repeated for the new

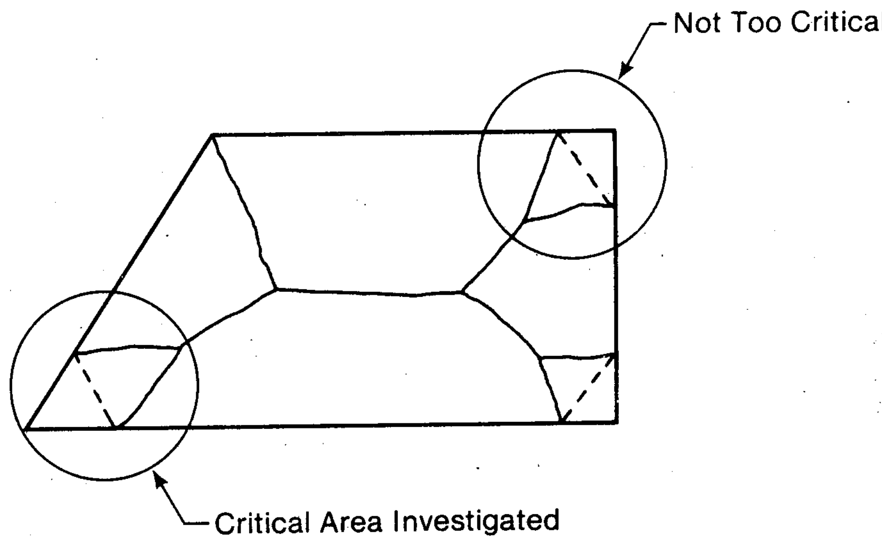


Figure 3.6 Critical Areas in a Virtual Work Formulation

yield-line locations. Unfortunately, the yield-line moments required by adjoining slab segments may differ significantly even for very small deviations from the correct yield-line locations. It has been suggested (Simmonds & Ghali 1976) that the equilibrium formulation should be used for one or two cycles of iteration to obtain a reasonably good estimate for the correct yield-line locations, then an energy formulation may be used to obtain the required design moments.

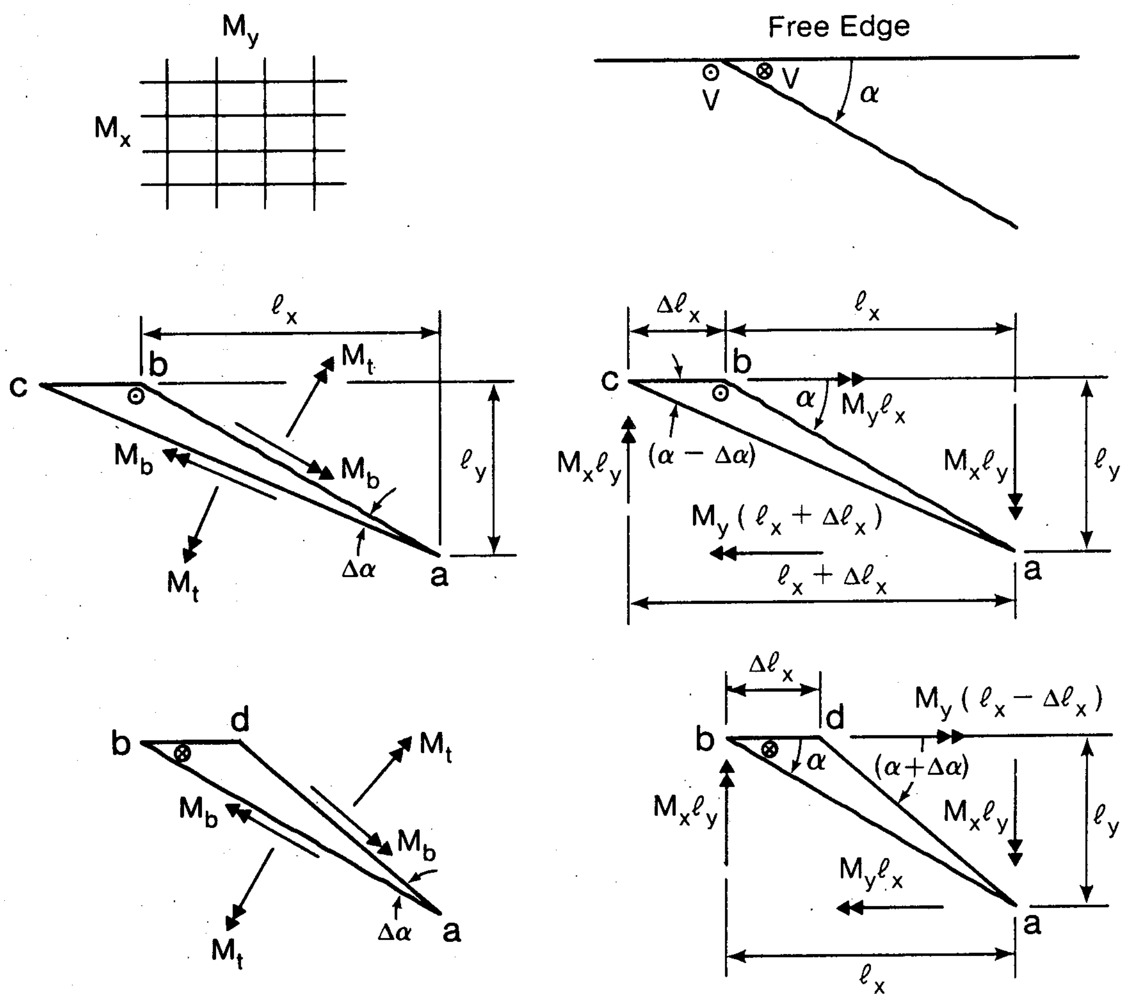
The above discussion assumes that the applied loads are given with the required moment capacity to be determined. A similar argument can be made for the case of analysis when the slab moment capacities are known and it is the collapse load which must be determined. In this case, while the moments will be equal on each side of a yield-line, the load required on each segment for equilibrium with these moments will be different. The yield-lines should be moved so that the segments with the largest distributed load are made larger, while the segments with the smallest distributed load are made smaller.

There can be shears as well as twisting moments along a yield-line. In an energy formulation, these shears and twisting moments produce zero net work and are ignored, but in an equilibrium formulation, these shears and twisting moments are required since they influence the equilibrium of each segment. The shears and twisting moments can be accounted for by the introduction of "nodal forces" which produce equilvalent equilibrium effects. A node is a point where yield-lines intersect or terminate. Nodal forces are applied to the slab segments very near nodes. The sum of the nodal forces at a node always equals zero.

Fig. 3.7 illustrates the case of a yield-line intersecting a free edge at an angle other than a right angle. The value of the nodal force V that produces the same effect as the twisting moment M_t is found by considering the equilibrium of the differential elements shown. A similar expression for the nodal forces can be obtained for the case of three yield-lines meeting.

Not all yield-lines have shears and twisting moments since symmetry will often indicate that these quantites are zero. For example, a yield-line which intersects a free edge at right angles has no nodal forces associated with it. Also, at a node where the yield-lines all have the same sign, all of the nodal forces are zero.

The nodal force method will not handle all possible cases. Fig. 3.8 shows two cases with incongruous nodal forces. This results when the corner of the opening is infinitely sharp, so that a different expression for equilibrium is required when the yield-line is moved away from the corner in either direction. The equilibrium equations have a discontinuity at the corner of the opening, and since the nodal force expressions have been derived for a continuous equilibrium equation, the nodal force



$\sum M$ Along ac

$$M_y\,(\Delta\ell_x)\cos(\alpha-\Delta\alpha) = V\,(\Delta\ell_x)\sin(\alpha-\Delta\alpha)$$

$\sum M$ Along ad

$$M_y\,(\Delta\ell_x)\cos(\alpha+\Delta\alpha) = V\,(\Delta\ell_x)\sin(\alpha+\Delta\alpha)$$

$$\therefore\ V = M_y\cot(\alpha-\Delta\alpha)\ \text{or}\ M_y\cot(\alpha+\Delta\alpha)$$

as $\Delta\alpha \rightarrow 0$ $\quad \boxed{V = M_y\cot\alpha}$

Figure 3.7 Development of Nodal Forces

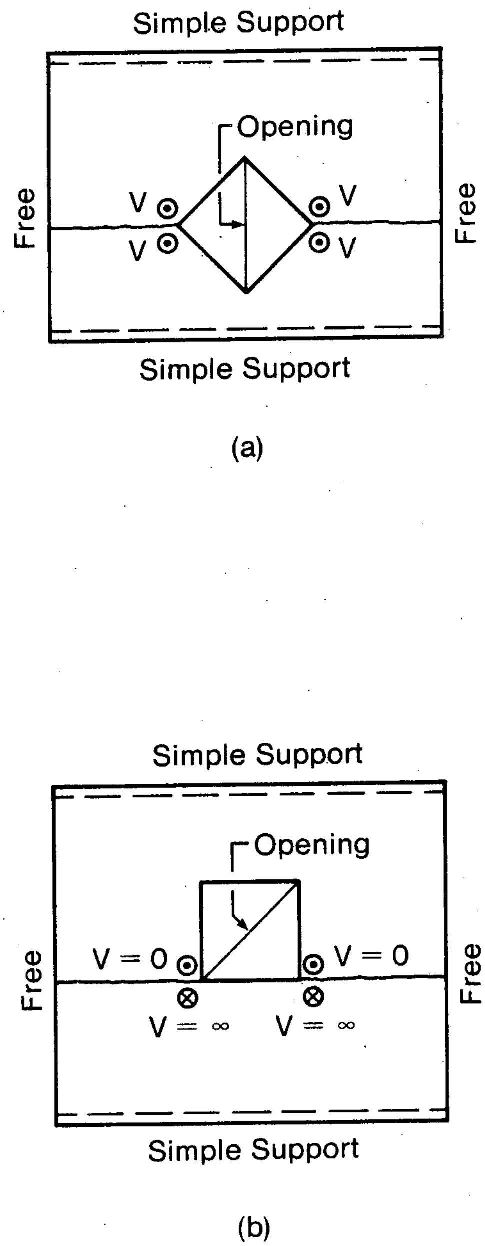


Figure 3.8 Cases of Incongruous Nodal Forces

expressions are no longer valid. A solution to this problem has not been found.

3.6 Probable Yield-line Families

The approach one uses to evaluate the capacity of a slab for each yield-line family is not nearly as important as ensuring that each possible yield-line family has been investigated. Evaluating a yield-line family approximately may produce moments which are 10% too low but neglecting to investigate the critical yield-line family can result in much greater errors. For some common cases it is possible to establish which yield-line family will govern the design. This will now be done for a flat plate on a regular column layout.

The yield-line families shown in Fig. 3.2 are typical of those which one would investigate for a regular flat plate. There are of course, other yield-line families which one should investigate, such as the parallel mechanisms acting perpendicular to those shown.

It is apparent that to investigate each family would require a great deal of work, but it can be shown that most of these patterns will not be critical. For square panels, the diagonal mechanism shown in Fig. 3.2b results in the same design moments as the parallel mechanism of Fig. 3.2a. So, the diagonal mechanism need not be investigated. Johansen has shown that the combined mechanism and the alternate combined mechanism of Figs. 3.2d and e are always less critical mechanisms than the parallel mechanism or conical mechanism shown in Fig. 3.2a and c. Thus, for interior panels of a regular flat plate, only the parallel and conical mechanisms need be investigated.

3.7 Conical Mechanisms

An understanding of conical mechanisms is essential in slab design. These mechanisms may form under point loads as well as over top of columns, and as indicated in Section 3.6, for flat plates, a conical mechanism may be the governing mechanism. The discussion will centre on the case of a typical interior column.

There appears to be a lack of agreement in the literature on the size of the conical mechanism which develops and the corresponding design moments. The work in the field can be divided into two distinct groups. The first uses an energy formulation and includes Wood, CEB, Gesund, Van den Beukle. The second group uses an equilibrium formulation and includes only Johansen. The two approaches differ on two points. The first relates to the assumption used regarding yield-lines in the vicinity of the column. The second point relates to the cut-off points of the reinforcement as they are affected by the conical mechanisms. The two approaches will be presented more or less in parallel in order to compare the differences in assumptions and results.

Consider a slab with isotropic top and bottom steel of flexural capacity per unit width of m′ and m respectively. It carries a uniformly distributed load w and is supported on circular columns of radius r spaced at a distance L centre-to-centre in each direction. If the reinforcement is continuous, the conical mechanism which will form is the "primary" conical mechanism. It will have a radius R, which is the parameter which must be optimized for the primary conical family of mechanisms.

The virtual work formulations (CEB) are based on the yield-line mechanism shown in Fig. 3.9a. The key assumption made is that a negative yield-line forms around the column, thus no work is done over top of the column. This assumption is probably true if there is a column above the slab, and would be a suitable assumption for the lower stories of a multi-story structure. The critical size of the conical mechanism is:

$$R = r\sqrt[3]{\frac{3L^2}{2\pi r^2} - \frac{1}{2}} \tag{3.1}$$

and the required design moments are:

$$m + m' = wL^2 \left\{ \frac{(1 - \frac{r}{R})}{2\pi} \left[1 - \frac{\pi R^2}{3L^2} - \frac{\pi r R}{3L^2} - \frac{\pi r^2}{3L^2} \right] \right\} \tag{3.2}$$

The equilibrium formulation (Johansen) is based on the yield-line mechanism shown in Fig. 3.9b. The major assumption is that the radial yield-lines go right to the

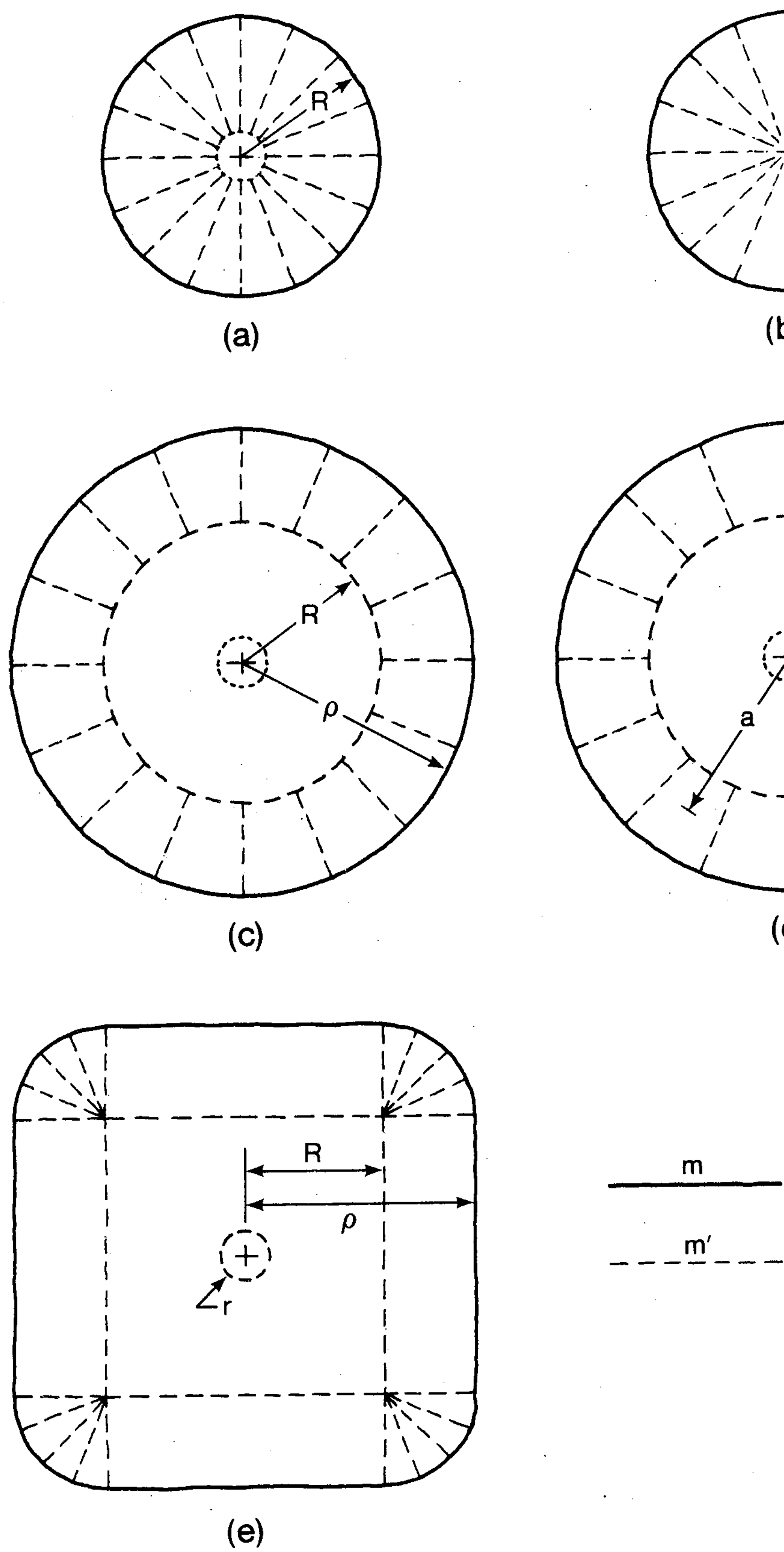


Figure 3.9 Possible Conical Mechanisms

centre of the column. This is generally true if there is no column above the slab and r/L is small, thus this would be a good assumption for top floors, roofs and single story slabs. Most flat slab tests have been conducted on such slabs and do in fact have numerous cracks directly over the column indicating yielding in this zone. The 3/4 scale slab tested by Guralnick and La Fraugh (1963), for example, clearly demonstrates this behavior. The results with this analysis indicate that:

$$R = r\sqrt[3]{\frac{L^2}{\pi r^2}} \tag{3.3}$$

and

$$m + m' = wL^2\left[\frac{1 - \sqrt[3]{\frac{\pi r^2}{L^2}}}{2\pi}\right] \tag{3.4}$$

A comparison of Equations 3.1 and 3.2 vs. 3.3 and 3.4 for typical values of r/L indicates that Johansen's primary concial mechanism has a radius about 12% smaller than the CEB primary conical mechanism. On the other hand, Johansen's design moments (m + m') are about 12% larger than the CEB design moments. There is clearly enough difference in the results to cause one to seriously consider which design equations are most appropriate for any particular slab.

There is however, a more important practical difference between the CEB recommendations which are simplificiations of Equations 3.1 and 3.2 and Johansen's work. CEB implies that as far as the conical mechanisms are concerned, some or all of the top steel may be cut-off at the edge of the primary conical mechanism since outside this zone it is not required to resist any moments. However, R is not a cut-off point. It is simply the size of the primary conical mechanism that would develop if the top mat was continuous. If the top bars are curtailed at a radius R, it is possible to form a "secondary" conical mechanism at a lower collapse load than required to form the primary conical mechanism. Such a secondary mechanism is illustrated in Fig. 3.9c.

Since neither CEB nor any other author using an energy approach has considered the possibility of a secondary conical mechanism, the following equations based on an energy formulation have been derived in order to provide a total set of design equations for the case of slabs with columns above.

An energy formulation for the yield-line mechanism shown in Fig. 3.9c with all of the top steel terminated at R such that $m'=0$ for all the negative yield-lines indicates that the size of the secondary conical mechanism is:

$$\rho = R\sqrt[3]{\frac{3L^2}{2\pi R^2} - \frac{1}{2}} \tag{3.5}$$

with the required bottom mat capacity of:

$$m = wL^2\left\{\frac{(1-\frac{R}{\rho})}{2\pi}\left[1 - \frac{\pi\rho^2}{3L^2} - \frac{\pi\rho R}{3L^2} - \frac{\pi R^2}{3L^2}\right]\right\} \tag{3.6}$$

Thus, given R, one can calculate ρ and m. The value of R is arbitrary but, should be greater than or equal to the value given in equation 3.1 in order to prevent a primary conical mechanism. For typical values of r/L, R will be such that ρ reaches midspan making the analysis somewhat invalid (but on the conservative side) since the shape of the mechanism is no longer completely circular.

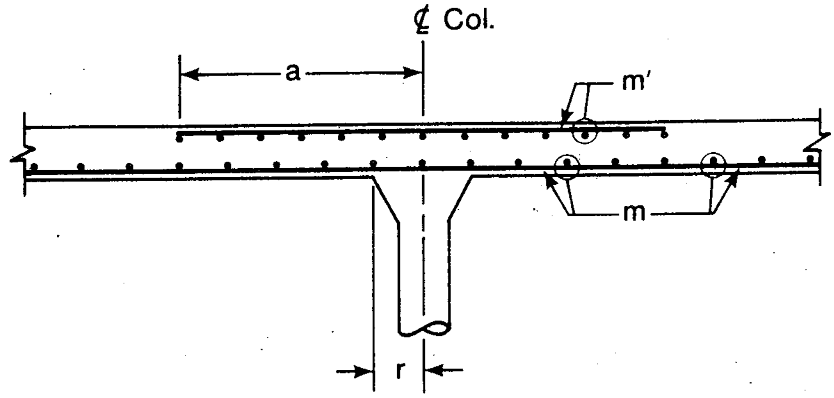
Johansen deals with the secondary conical mechanism in a somewhat different manner. He makes the top bars long enough to ensure that the secondary conical mechanism will form around the primary conical mechanism at a load equal to that of the primary conical mechanism collapse load. A cross-section of the type of reinforcement considered is shown in Fig. 3.10a. In this case, the yield-line mechanism is that shown in Fig. 3.9d. The important difference between this case and the previous one is that the top bars are cut-off at point a so that some of the negative yield-lines will have a moment capacity of m′. The results indicate that the size of the secondary conical mechanism will be:

$$\rho = \frac{2}{w}\left(\frac{wL^2}{2\pi} - m\right) \tag{3.7}$$

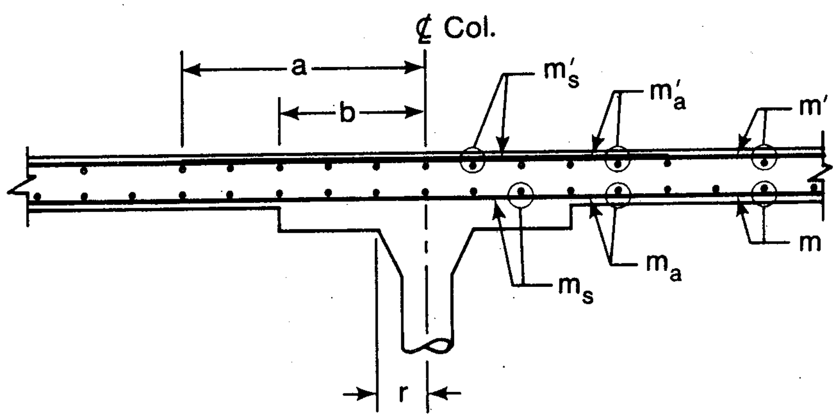
and the required cut-off point will be:

$$a = \frac{w}{3}\frac{(\rho^3 - R^3)}{m'} \tag{3.8}$$

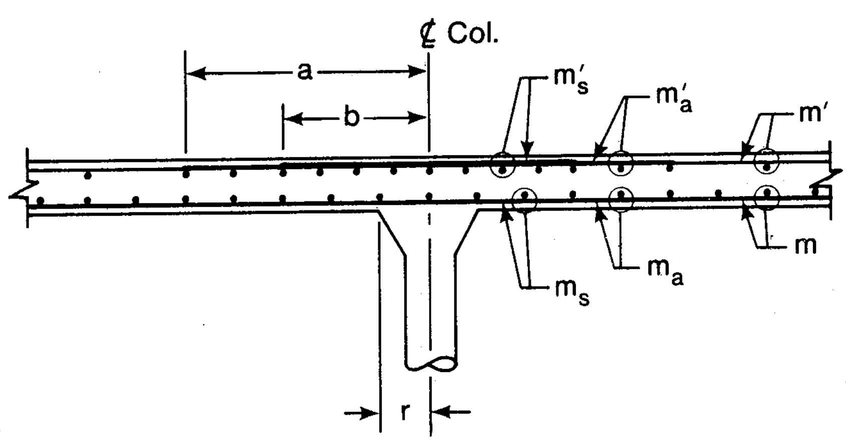
Thus, given r/L, one can calculate R by Equation 3.3. and (m



(a) Slab with Uniform Top Mat.



(b) Slab with Drop Panels.



(c) Slab with Non-Uniform Top Mats.

Figure 3.10 Moments, Drop Panels and Cut-off Points

+ m′) by Equation 3.4 and, having chosen the ratio between m and m′ , one can determine the required cut-off point a. For usual values of r/L and m/m′ , ρ usually reaches to approximately midspan and:

$$a \leq r + \frac{L}{3} \tag{3.9}$$

It is interesting to note that a is quite a bit longer than the ACI cut-off points.

The discussion thus far has related to circular top mats of orthogonal bars, however, current practice is to use square top mats. This will result in the yield-line mechanism shown in Fig. 3.9e. As previously pointed out, ρ usually reaches midspan or even slightly past so a value of L/2 for ρ was used. Given that all the top bars are cut-off in a square mat of side length 2R, and the mechanism forming outside the top mat, an energy formulation gives:

$$m = wL^2 \left\{ \frac{(\frac{1}{2} - \frac{R}{L}) \left[(1 - \frac{\pi}{12}) - (1 - \frac{\pi}{3}) \frac{R}{L} + (4 - \frac{\pi}{3})(\frac{R}{L})^2 \right]}{8(\frac{R}{L}) + \pi - 2\pi\,(\frac{R}{L})} \right\} \tag{3.10}$$

This is somewhat smaller than the moment capacity required by Equation 3.6 in the case of circular bar cut-off. The difference may be due to the fact that the mechanism assumed is not the optimum one, that is, it is possible for the yield-line to run across the corners of the top mat. This family was not investigated since even numerical solutions would be difficult to obtain. In addition, most top mats have two cut-off points and if all of the longer bars are placed in the middle of the mat with the shorter bars on each side of these, the corners of the mat will be "rounded" and the mat will be effectively circular and not square. It is important to note that *it is the extent of the lateral distribution of the top bars, and not length of the top bars which is important*. This is because the negative yield-lines which the top bars must resist at the edges of the mat are radial yield-lines. On this account, ACI detailing requirements fare somewhat better since they require top bars across the entire slab width due to the requirement for middle strip negative reinforcement.

Since the negative yield-lines are predominantly radial yield-lines in the case of Fig. 3.9a, and totally radial yield-lines in the case of Fig. 3.9b, one could provide circular top bars, which would cross the yield-lines at right angles. These reinforcement rings only require half as much steel as an orthogonal mat, since the orthogonal mat must resist m' in two directions while the rings only need to resist m' in one direction. If the rings were welded, there would also be a large saving in anchorage lengths. The cost of welding could be avoided if the top mat consisted of a flat spiral. Such "watch spring" top mats could be quickly rolled by machine and attached to a piece of welded wire fabric to facilitate handling. It should be pointed out that radial reinforcing bars are not required since, with

Johansen's formulation, there are no negative circular yield-lines which would mobilize radial bars. On the other hand, with the CEB formulation, such a negative circular yield-line occurs around the column, but the amount of work done by this yield-line is insignificant when compared to the total work done by the other portions of the concial mechanisms, and thus, the omission of the radial bars will have negligible effect on the capacity of the conical mechanism. Since top mats comprise about 50% of the total reinforcement in a slab, watch spring reinforcement which reduces the weight of top mats by half can potentially reduce the overall steel weight in a slab by about 25%. The economic advantages of watch spring top mats will no doubt spurn studies on the behavior of slabs with such top mats.

To this point, the discussion has dealt with top mats of uniform capacity curtailed at one point only. The case of two different moment capacities will now be considered. This case can result if drop panels or two different cut-off points are used as shown in Fig. 3.10b and c. Generally, the moment capacity will be maintained or reduced as one moves away from the column, that is $m'_s \geq m'_a \geq m'$ and $m_s \geq m_a \geq m$. From Fig. 3.10b, b is the point of the first change in moment capacity and a is the point of the second change in moment capacity such that; between the column centre-line and b, the slab has capacities m'_s and m_s; between b and a the slab has capacities m'_a and m_a; and past or outside a, the slab has capacities m′ and m. An equilibrium analysis gives:

$$m_s + m'_s = wL^2 \left[\frac{1 - \sqrt[3]{\frac{\pi r^2}{L^2}}}{2\pi} \right] \qquad (3.11)$$

$$b = \frac{w}{3} \frac{(R_1^3 - R_0^3)}{(m'_s - m'_a)} \qquad (3.12)$$

where:

$$R_1 = \sqrt{\frac{2}{w}\left(\frac{wL^2}{2\pi} - m_a - m'_a\right)} \qquad (3.13)$$

and:

$$R_0 = r\sqrt[3]{\frac{L^2}{\pi r^2}} \qquad (3.14)$$

Also:

$$a = \frac{w}{3} \frac{(R_2^3 - R_1^3)}{(m'_a - m')} \qquad (3.15)$$

Where:

$$R^2 = \sqrt{\frac{2}{w}\left(\frac{wL^2}{2\pi} - m - m'\right)} \qquad (3.16)$$

These equations are for a more general case than previously considered and can be simplified to obtain Equations 3.3 and 3.4 for that special case. An expanded deriviation of Equations 3.3, 3.4 and 3.11 through 3.16 can be found in Appendix A.

The equations are such that the maximum moments m'_s and m_s are determined based on the primary conical mechanism. A minimum slab thickness can be picked or checked on the basis of these moments. Then with chosen or trial values of m'_a and m_a, the first cut-off point is determined to prevent a secondary conical mechanism at a collapse load less than the primary mechanism. Alternatively, this cut-off point may be used to determined the size of the drop panel in the case shown in Fig. 3.10b since in this case there will be a change in moment capacity of the top steel at this point even though the steel remains the same. For known or trial values of m' and m, the next cut-off point can be determined in order to prevent a "tertiary" conical mechanism from forming around the first two conical mechanisms at a collapse load less than the two previous conical mechanisms. The moment capacities m' and m must extend throughout the remainder of the slab unless a check of additional conical mechanisms is made. The equations however, can be easily iterated to account for more cut-off points and more conical mechanisms.

Both approaches discussed can be extended to non-circular columns and non-square bays. Square and slightly rectangular columns can be treated as round columns of equal area. Highly rectangular columns must be investigated by rigorous analysis. Rectangular or other shape bays may be treated with the conical mechanism equations if one uses a square bay of equal area in order to determine an equivalent L.

The point of this discussion on conical mechanisms is that CEB and Johansen do not really solve the same problem. CEB considers the case of slabs with columns above slab while Johansen considers a slabs where there are no columns above. The CEB equations fall far short of a complete solution. They are based upon the assumption that the top mat is continuous. Since this is rarely the case, one must either resort to Johansen's equation or further develop the CEB equations. It should be pointed out that every slab during construction has no upper column until the next floor is added. Also every building has a top floor such that there are no upper columns. It is then reasonable to check all slabs at this stage of construction with Johansen's equations. Since the addition of the upper columns will tend to increase the strength of the slab, it would presumably be safe to use only Johansen's equations. Thus, the need to develop more general expressions for the CEB equations may not exist. Since Johansen's equations will be applicable to all slabs at some point in their life, and since the equations are very broad in their scope of application, they will be used through the remainder of the discussion on yield-line theory.

3.8 Conical vs. Parallel Mechanisms

It was pointed out in Section 3.6 that in flat plates both the conical and parallel mechanisms must be checked. Different cases will be examined in order to determine which mechanism will govern the design.

Consider the case of top mats consisting of isotropic reinforcement. The reinforcement may be curtailed to form either a square or circular top mat. The top mat should however be large enough to contain the primary conical mechanism. The bottom mat is assumed to be uniform and continuous. Thus, m' and m will be available to resist negative and positive panel moments in the parallel mechanism. The critical location for the negative parallel yield-line will be at the face of the column. Analysis indicates that for all cases with $r/L \geq 0.03$, the conical mechanism will require a larger moment capacity $(m + m')$ than the parallel mechanism. This conclusion is not very sensitive to the choice of the ratio of the negative panel moment to the positive panel moment, ψ . Fig. 3.11 gives K, an index of the required moment capacities as a function of r/L, for various ψ values for both the parallel and conical mechanisms. Using this figure, one proceeds vertically from the known r/L value up to the appropriate curve and then horizontally in order to obtain K. The required moments can then be computed from:

$$m + m' = KwL^2 \tag{3.17}$$

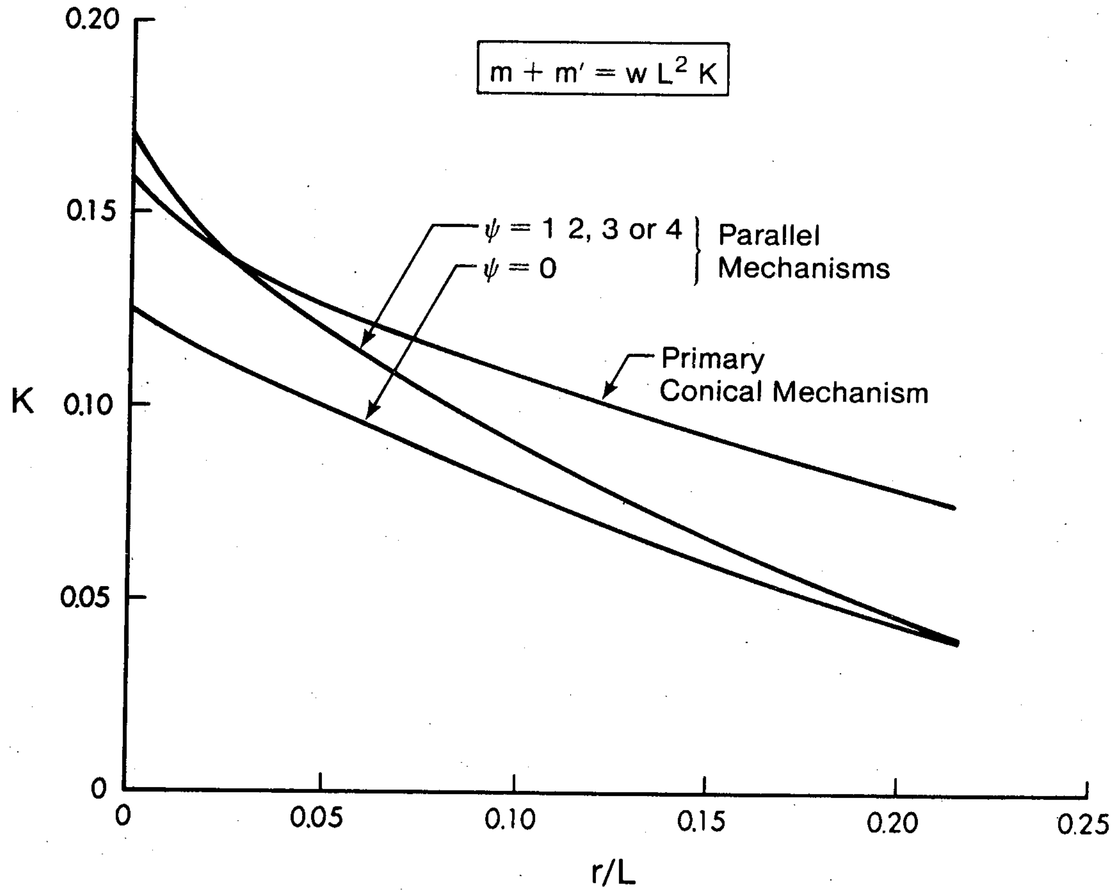


Figure 3.11 Comparison of Conical and Parallel Mechanisms with Orthogonally Reinforced Top Mats

A summary of some typical ratios of moments required by the parallel mechanism and conical mechanisms is given in Table 3.1.

The other case to be considered is that of ring reinforced or watch spring top mats. In this case only the appropriate component of the ring reinforcement which crosses the negative parallel yield-line is considered in resisting the negative panel moments. The critical location of the negative parallel yield-line is not obvious since, as one moves from the face of the column towards midspan, the negative yield-line capacity decreases. This occurs because the segment of the ring which is intersected becomes smaller and the reinforcement is intersected at a flatter angle. Thus the problem is that as one moves from the colmn toward midspan, the actual moment capacity reduces along with the required moment capacity, so that the critical location of the negative yield-line may not be at the face of the column. However, analysis indicates that the critical location of the negative parallel yield-line is still at the face of the columns. The following conclusions depend somewhat upon ψ , but for the most common values of ψ , the parallel mechanism will govern the design of the reinforcement. This is illustrated in Fig. 3.12 which corresponds to Fig. 3.11 and gives values for K. The curves for ψ = 1,2,3 and 4 are considerably more spread out on Fig. 3.12 than on Fig. 3.11 where they are so close that they are represented by a single curve. The value of

Table 3.1 Comparison of Conical and Parallel Mechanisms With Orthogonally Reinforced Top Mats

$\frac{r}{L}$	$\frac{(m+m')\ \text{Parallel}}{(m+m')\ \text{Conical}}$
0.05	0.95
0.10	0.85
0.15	0.74

Table 3.2 Comparison of Conical and Parallel Mechanisms With Ring or Watch Spring Top Mats

$\frac{r}{L}$	ψ	$\frac{(m+m')\ \text{Parallel}}{(m+m')\ \text{Conical}}$
0	1 to 4	1.06
0.05	1.80	1.09
0.10	1.87	1.11

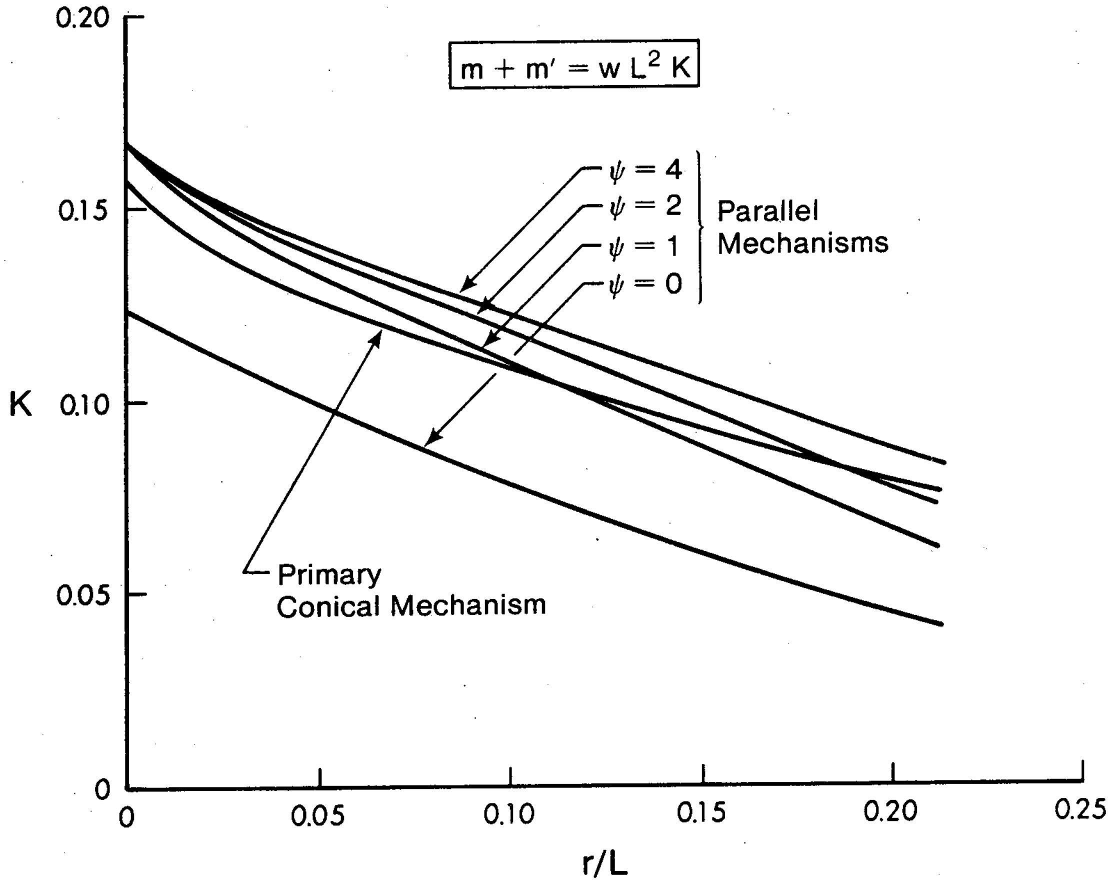


Figure 3.12 Comparison of Conical and Parallel Mechanisms with Ring or Watch Spring Top Mats

$(m+m')_{parallel}/(m+m')_{conical}$ decreases as r/L increases and as m′/m and ψ decrease. Table 3.2 indicates that reinforcement in addition to that required for the conical mechanism is required to resist the parallel mechanisms. Fortunately, the amount of extra reinforcement required is small. It may be added to the bottom mat or better still, it may be added to the top mat in the form of welded wire fabric which as previously stated, will aid in handling of the mat.

In order to get an appreciation for the magnitude of the differences in solutions obtained by the various methods, consider the following problem which was obtained from Simmonds and Ghali (1976). A flat plate which carries a total uniformly distributed load of 395 psf is supported on 24" square columns spaced at 20′ on centre each way. The results of designs based on the Direct Design Method, the CEB equations and Johansen's equations are summarized in Table 3.3.

It is apparent from Table 3.3 that the least conservative approach is the Direct Design Method which apparently does not adequately account for conical mechanisms in that the top mats are too narrow and there is insufficient moment capacity in the neighbourhood of the column. (This would be even more so when one calculates the (m′ + m) required for the actual size of the top mat used). This comparison suggests that either the assumptions used in the development of the conical mechanism equations are not valid or most slabs designed by the Direct Design Method are

Table 3.3 Comparison of Results for Various Design Methods for the Example Problem

	Johansen	CEB	Direct Design
Size of square top mat	15.4'	12.1'	10.0'
Average m'	13.33 'k/'	11.90 'k/'	10.78 'k/'
Average m	6.40 'k/'	5.95 'k/'	5.68 'k/'
Average (m+m')	19.73 'k/'	17.85 'k/'	16.46 'k/'

underdesigned and should be exhibiting evidence of conical failures. It is believed that the answer lies in the fact that the slab does not behave as a thin plate as far as the primary conical mechanism is concerned. The slab thickness relative to the size of the conical mechanism is much greater than the slab thickness relative to the total span. This enables the slab to jam against itself as illustrated in Fig. 3.13. This phenomenon no doubt accounts for the fact that conical yield-line failures of flat plates have never been reported in the literature. This is very similar to Ockleston's (1955) observations for slabs supported on beams. He built scale models of slab panels from plexiglass which were cut along the "yield-lines" so that the yield-line moment capacities were zero. The models carried considerable load in spite of the fact that a yield-line analysis, an upper-bound solution, indicated that the model would carry no load at all!

In light of the satisfactory performance of slabs designed by the Direct Design Method, it would appear that the conical yield-line mechanisms need not be considered. However, there may be cases such as exterior and corner columns where slab jamming may not be relied upon. In the case of interior columns, if the beneficial effects of slab jamming were not present, the collapse load would be about 10% less than predicted by Direct Design. Whether the difference in collapse load would be as small for exterior and corner panels is not known since no readily applicable

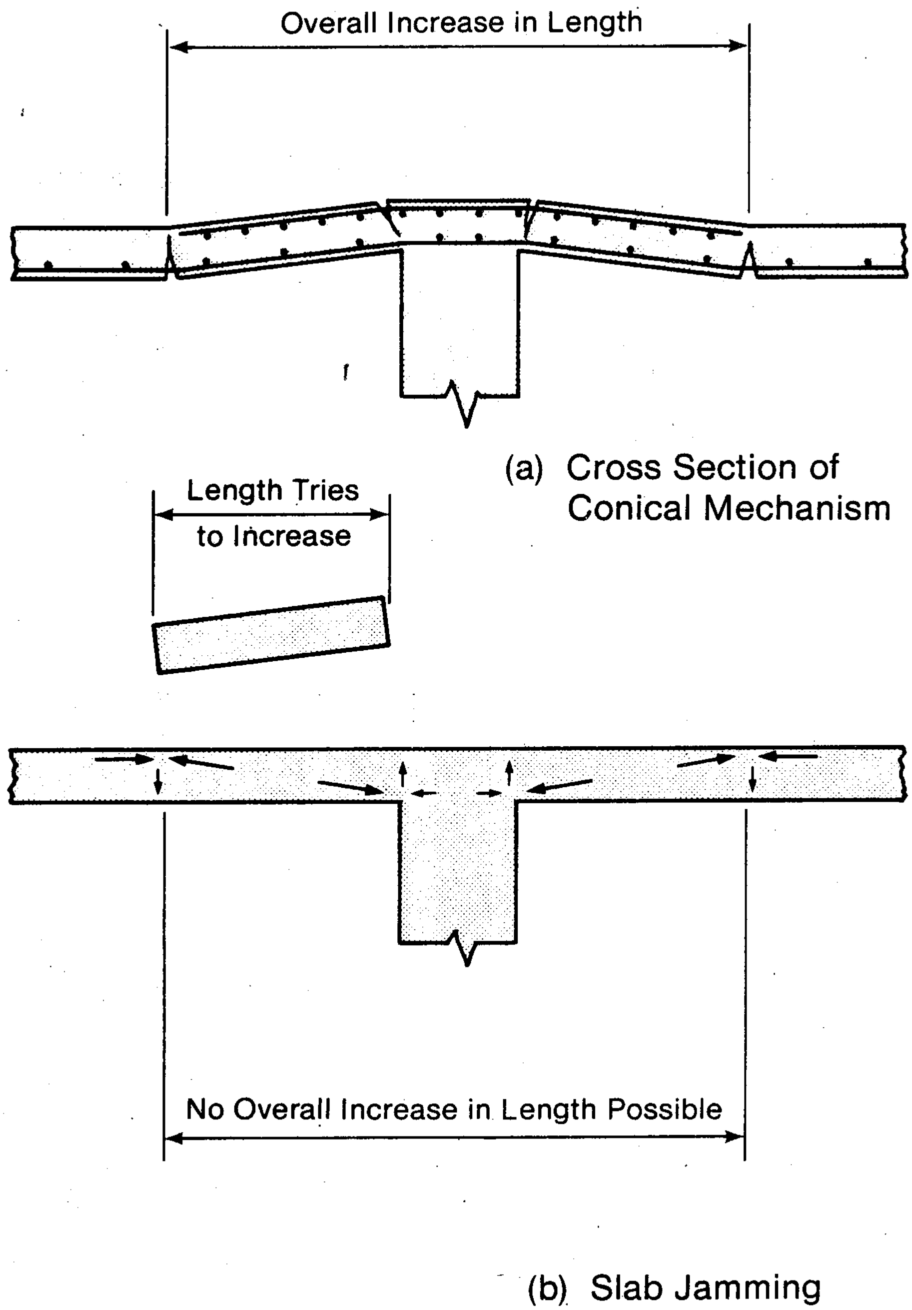


Figure 3.13 Slab Jamming in Conical Yield-Line Mechanisms

conical mechanisms have been developed for such cases. Further research in this area is required.

3.9 Segment Equilibrium

3.9.1 Historical Background

The segment equilibrium method of slab design leads to upper-bound solutions for slabs in flexure and is one of the newest ultimate strength procedures for slabs. It was proposed and developed almost entirely by Wiesinger (1973, 1975). This method was used for the design of millions of square feet of flat plates, most of which were 5 or 5.5 inches thick, and is well suited to flat plates with irregular as well as regular column layouts. As the name implies, the method is based on the equilibrium of slab segments.

3.9.2 Basic Assumptions

The segment equilibrium method assumes that the slab behaves as a rigid-plastic in flexure. See Fig. 3.1. This is the same assumption upon which yield-line theory is based, thus, with this method the slab is also divided into a series of plane segments which are bounded by straight lines of constant rotation and moment. However, Wiesinger referred to these lines as "zero shear lines" and not yield-lines.

Even though an equilibrium formulation is used, nodal forces are not used, that is, the effect of twisting moments is ignored. This makes the segment equilibrium method philosophically the same as the method used by Ingerslev (1923). It can be argued with the lower-bound theorem that it is safe to ignore the twisting moments since the moment field does not necessarily have to have twisting moments to meet the first requirement of the lower-bound theorem. The segment equilibrium method however, does not satisfy the second requirement of a lower-bound solution in that the moment field is not known everywhere throughout the slab. While it is possible that at some point within a segment the yield moment can be exceeded, it is assumed that the critical locations for moment are along the boundaries of the segments.

3.9.3 Overview of Solution Procedure

Wiesinger observed that yielding occurs at points of maximum moment and that in slabs as in beams, the points of maximum moment are also points of zero shear. By using some simple rules based on symmetry arguments, it is relatively easy to determine the location of the lines of zero shear in a slab.

The design begins with the establishment of the lines of zero shear. For a flat plate, the zero shear lines divide the entire plate into right triangular segments each bounded by lines of zero shear and supported at one of the acute corners. For each segment, one can determine the total moments required to maintain equilibrium in each of the directions parallel to the orthogonal sides of the segment.

As shown in Fig. 3.14, the moments in the direction of the span are referred to as column strip moments M_c^- and M_c^+, while the moments perpendicular to the span are referred to as middle strip moments M_m^- and M_m^+. In Fig. 3.14b, the moments are shown as vectors with half arrow heads. The reinforcing steel would lie parallel to these vectors with bottom steel for vectors which point towards the segment (positive moments) and top steel for vectors which point away from the segment (negative moments). For the moments in each direction, the designer may choose whatever distribution of positive and negative moments he desires as long as equilibrium of the segment is maintained. In addition, the moments chosen for adjoining segments should be the same on either side of a zero shear line in order to maintain equilibrium between segments. If you design for the larger moment you are safe since you will design for more than 100% of statics. If the mismatch in moments is too great, it indicates that the choice of zero shear lines is not very good.

The key to economy and safety with this method is in the distribution of moments. It is usually convenient to select the positive moments first, so that they correspond to minimum reinforcement, especially if the spans and loading are small.

Then, the negative moments required for the equilibrium of each segment are determined. For irregular or non-uniform column layouts, the negative moments between adjacent

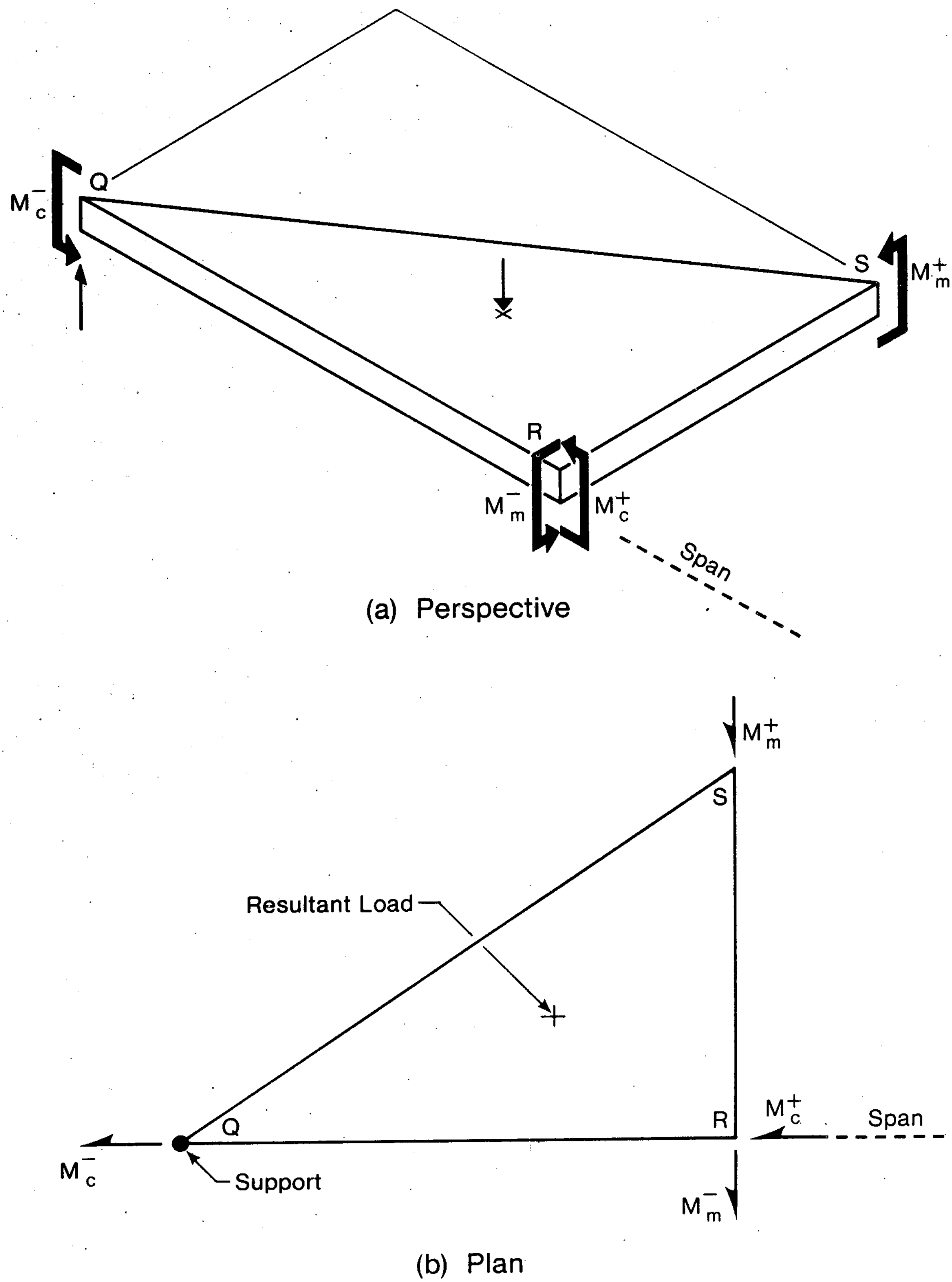


Figure 3.14 Moments Acting on a Triangular Segment

segments may not be equal, but since negative moments occur at supports, the unbalanced negative moment can be "dumped" into the support. The support must of course be designed for this moment.

Herein lies the principle difference between yield-line theory and segment equilibrium. In yield-line theory one chooses the support moments or the ratio of negative to positive slab moment capacities, then, after a suitable number of differentiations or trials, one determines the critical location of the yield-lines. On the other hand, with the segment equilibrium method, the location of the zero shear lines is set, suitable distributions of moments are chosen and the corresponding support moments are obtained from simple equilibrium. Even for the general case of irregular column layout there are simple procedures to pick lines of zero shear while it would be quite difficult to start with reinforcement ratios and yield-line theory. It should be apparent that the segment equilibrium method is better suited to design than the traditional yield-line approach. However, since the segment equilibrium method is a special case of yield-line theory, it must be concluded that the segment equilibrium method is yield-line theory applied to design.

3.9.4 Location of Zero Shear Lines

Any reasonable set of zero shear lines can be used to produce a satisfactory design. It is possible however, to produce some simple rules to help determine reasonable locations for lines of zero shear for the general case of a flat plate with an irregular column layout.

First, lines which join column centres are "lines of symmetry" and are therefore lines of zero shear. The entire slab can then be divided into triangular segments with columns at each corner as shown in Fig. 3.15a. The triangles should be such that the "circumcentre", that is, the centre of the circle which passes through each of the corners of the triangle falls within that triangle. This is illustrated in Fig. 3.15a.

The perpendicular bisectors of each of the previous lines are also "lines of symmetry" and therefore are also lines of zero shear. The perpendicular bisectors intersect at the "circumcentre" or panel centre as shown in Fig. 3.15b.

The lines joining the panel centres to the columns are also "lines of symmetry" and thus are lines of zero shear. The result is that the slab has been divided into right triangular segments bounded by lines of zero shear and supported by a column at one of the acute corners as shown in Fig. 3.15c.

The previous rules will not work when four adjacent columns define a quadralateral segment with two adjacent obtuse angles. Fig. 3.16a and b illustrate the problem with such a case while Fig. 3.16c illustrates a solution to the problem.

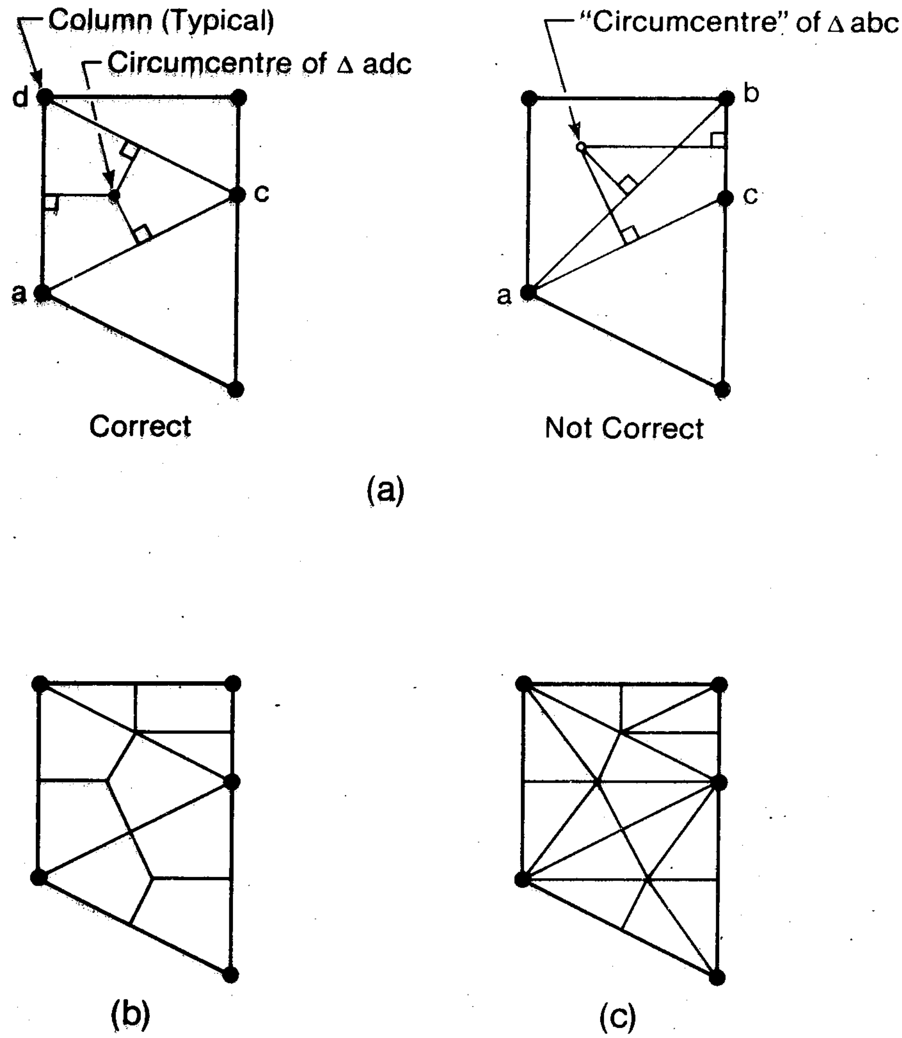


Figure 3.15 Construction of Zero Shear Lines

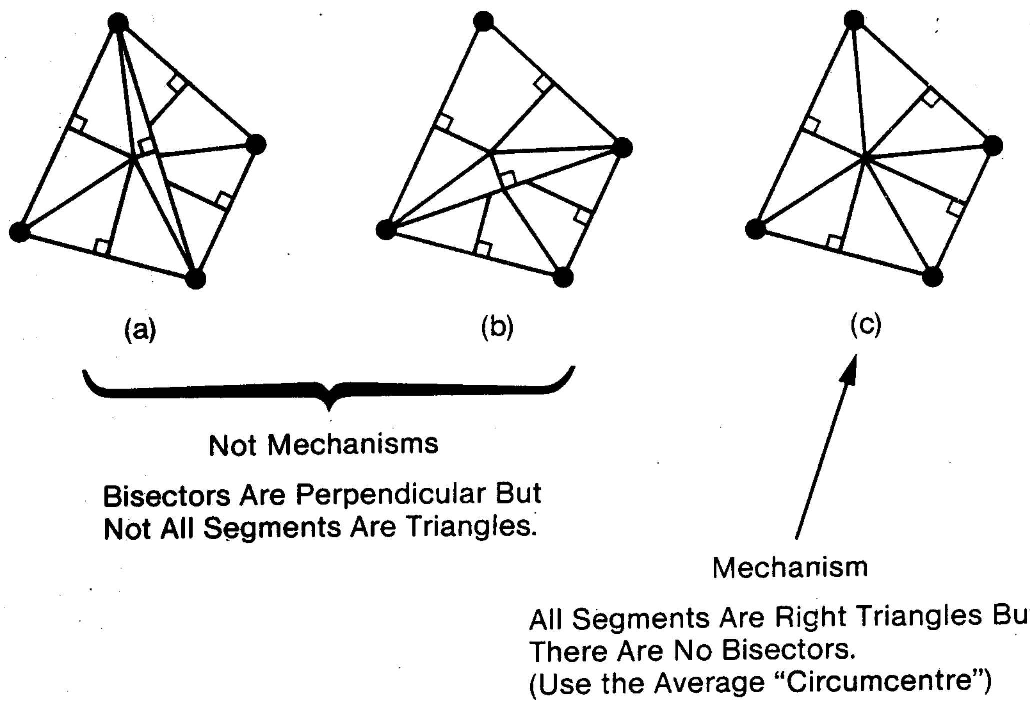


Figure 3.16 Special Case of Zero Shear Lines

For exterior spans, the perpendicular bisector may not be a line of zero shear. The line of zero shear may lie between 1/2 (for very stiff exterior columns), and 3/8 (for very flexible exterior columns) of the span from the exterior column, as shown in Fig. 3.17. Some judgement must be used in locating this zero shear line since the exterior column moment is somewhat sensitive to the choice of its location.

A further refinement can be made by considering the area of the columns, thus basing the calculations on the clear span rather than the centre-to-centre span. This can be done by converting the column into a column of "equivalent" area but with a shape geometrically similar to the panel around it bounded by the perpendicular bisectors. One can then consider the column strips of width equal to the "equivalent" column face as one way strips, while the remainder of the slab is still composed of right triangular segments which are slightly smaller than before.

A refined set of zero shear lines for a typical flat plate with a regular column layout is shown in Fig. 3.18. Note the comparison with Fig. 3.2d which is based on yield-line theory. The columns may be circular or even slightly rectangular, as long as they would have areas equal to those shown. In the lower right hand corner of Fig. 3.18 is another possible set of zero shear lines. These would correspond to a mechanism not considered by the rules for locating zero shear lines. This mechanism as well as

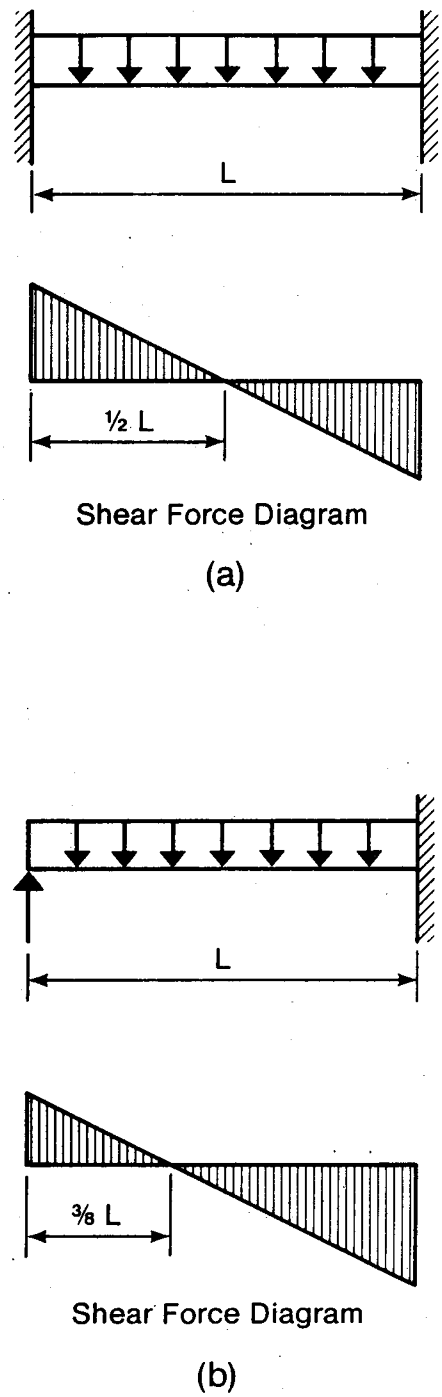


Figure 3.17 Influence of Exterior Support on Point of Zero Shear

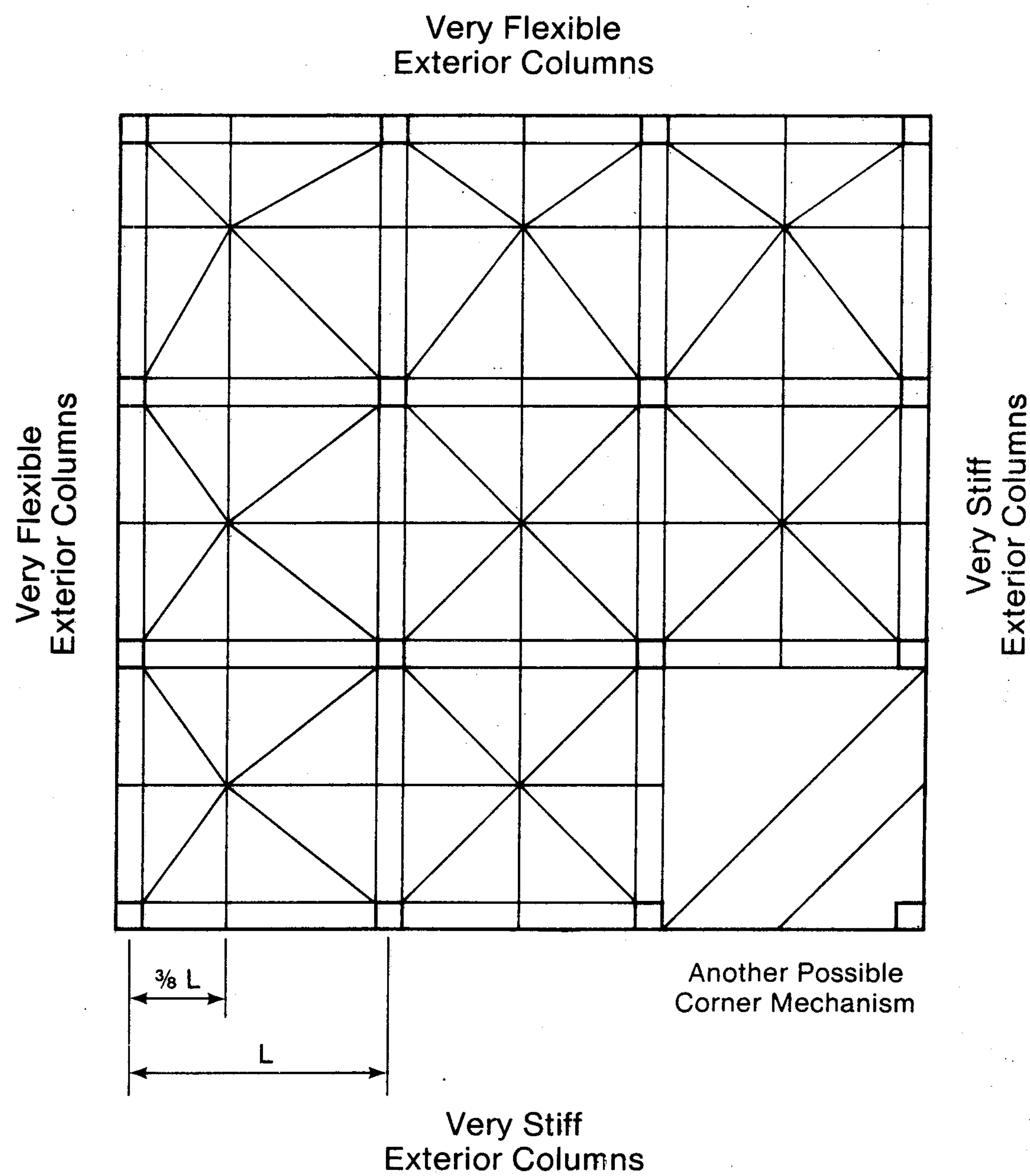


Figure 3.18 Lines of Zero Shear for Regular Column Layout

possible conical mechanisms should be checked after the initial design is done to ensure that they will not govern.

3.9.5 Distribution of Moments and Reinforcing

Once the lines of zero shear have been set and the moments required for equilibrium of each segment have been calculated, one must choose a suitable distribution of positive and negative moments. This choice is quite arbitrary, but one should try and approximate the elastic moment field in order to minimize the amount of redistribution required.

One can get exactly the same moments as obtained with the Direct Design Method if:

$$M_c^- = 0.70\ M_c$$

$$M_c^+ = 0.30\ M_c$$

$$M_m^- = 0.54\ M_c$$

$$M_m^+ = 0.46\ M_m$$

These correspond to ratios of $M_c^- / M_c^+ / M_m^- / M_m^+$ percentages of 49/21/16/14. By way of comparison, Weisinger used ratios ranging from 60/20/0/20 to 35/50/0/15. The behavior of these plates under construction loads and service load is, without exception, entirely satisfactory (Wiesinger 1973). A controlled test of a portion of a real structure also indicated satisfactory behavior. (Cardenas and Kaar 1971). There are no precise rules for distributing moments in a slab and as Wiesinger has pointed out, "It is a matter of individual engineering judgement, responsibility and background, to decide at present how far to go with the redistribution of moments".

One must also choose the width over which the positive and negative reinforcement are distributed. Again, one should try and approximate the elastic moment field. Wiesinger found it economical and practical to use a uniform isotropic bottom mat with occasional additional bottom bars as required in the column strips. Wiesinger used top reinforcement in the form of orthogonal mats over the columns and to simplify placing, there was no top steel in the middle strips. The width and length of the top mats extended to 1/3 of the span on either side of the column, which in accordance with Equation 3.9 ensures that the conical mechanisms will not govern. Fig. 3.19 compares the elastic moment field to the design moment fields for the Direct Design Method as well as Wiesinger's approach. Clearly, the Direct Design Method models the elastic moment field more closely than Wiesinger's approach, but experience indicates that his approach is good enough for the type of slabs which he designed. This indicates that the designer may have considerable latitude in the choice of moment fields so that he may at the expense of "exactness", choose to simplify the reinforcement layout in order to gain economy.

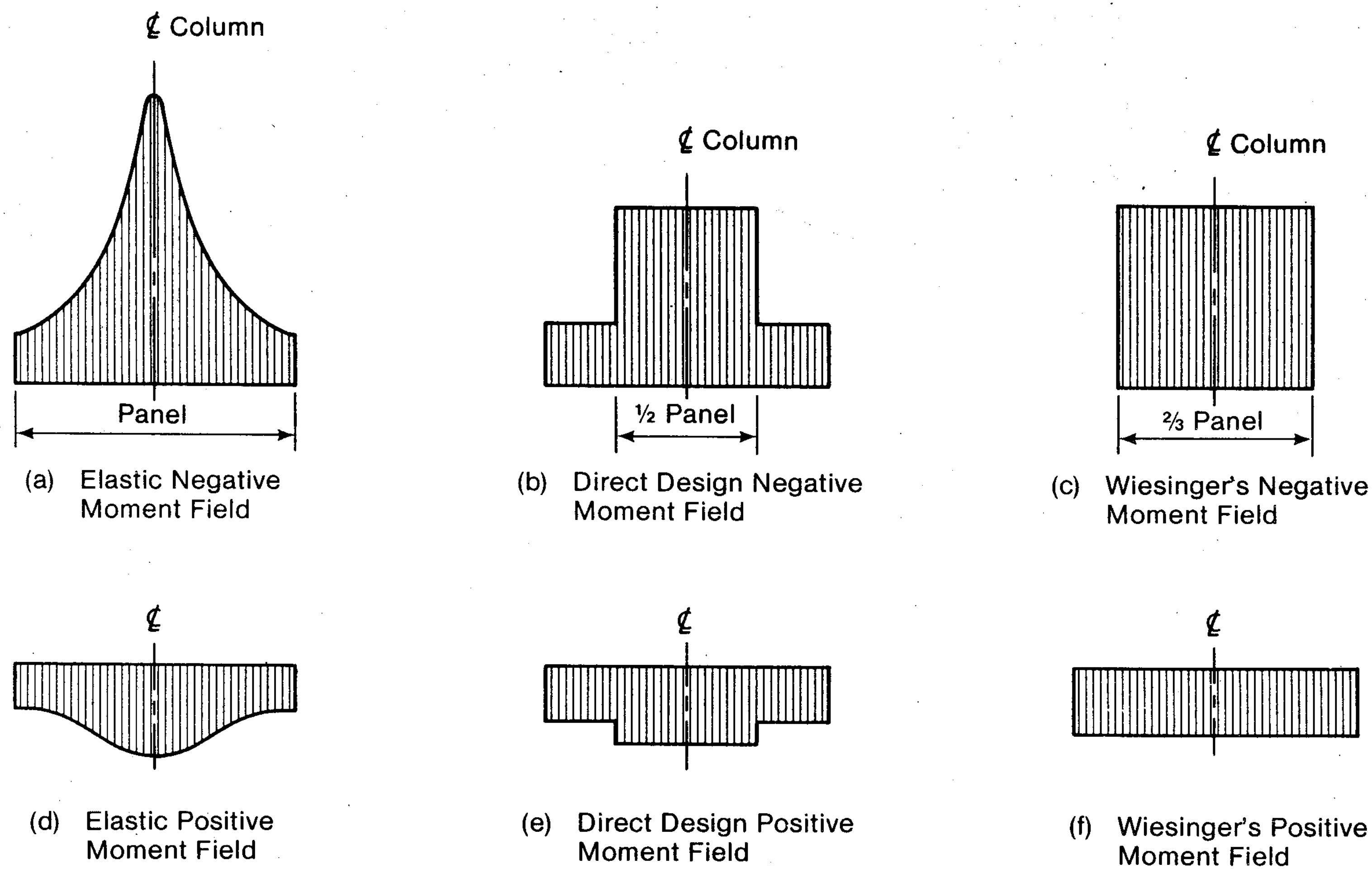


Figure 3.19 Design Moment Fields

3.9.6 Details of Wiesinger's Designs

Several aspects of Wiesinger's designs are unique and innovative, which in themselves have nothing to do with segment equilibrium, but which collectively lead to the successful application of the segment equilibrium method. These points will be discussed in order to better understand the advantages and perhaps the limitations of this method.

All of Wiesinger's slabs had a live load less than the dead load. This is significant because this reduces the influence of pattern loading. Since nothing in the segment equilibrium method accounts for pattern loading, it is important that the effects of pattern loading be small. One could of course investigate each case of pattern loading as a separate load case, but this would require a great deal of work and make the segment equilibrium totally impractical as a hand (noncomputer) solution.

The detailing of the top mats is particularly important. First, the top mats extend somewhat farther than the ACI requirements. This permits the points of inflection to move farther from the column under pattern loading so that this enhances the capacity of the design to resist pattern loading. The extent of the top bars is approximately in accordance with the cut-off point required by the conical mechanisms. The choice of moment field also increases the resistance to the conical mechanisms by providing a larger negative moment capacity over the whole top mat. Compare Figs. 3.19b and c. Since the Direct Design Method requirements are only slightly unconservative with respect to the conical mechanisns, one would expect Wiesinger's top mats to be satisfactory with respect to these mechanisms.

Frequently Wiesinger's columns were 3.5 inch diameter steel pipe columns. Single columns were used for four or five story structures while double columns were used in structures up to eight storys. The small diameter enabled them to be hidden in standard partition walls. This in turn permitted the structural designer to keep the spans small (but perhaps irregular) without interferring with the functional requirements of the building. Since the columns were relatively flexible in comparision to the slab, the column moments permitted were very small so moments were redistributed in order to keep the unbalanced negative moments at each column very small. Since the columns were steel, they were fabricated in single story sections with shear heads and an aligning device for the columns in the next story. The shear heads provided an efficient, ductile, and economical means of dealing with the shear and moment transfer problem which occurs at the slab column-junction. And finally, because of their flexibility, the columns were not used to resist lateral loads.

Wiesinger claims to have effected great economy with the use of welded wire fabric. The bottom mats consisted of welded wire fabric corresponding to minimum reinforcement throughout, with the addition of extra bottom bars as required in particular column strips. The same size welded wire fabric was also used to form the base of the top mats, to which the requisite number of additional bars were tied. This slightly modifies Fig. 3.19c so that it is more like the elastic moment field. The problems associated with the lack of ductility of welded wire fabric were reduced by using rather large diameter wires of about 0.263 inch diameter which are generally more ductile than smaller diameter wires. It should be pointed out that the slabs were generally built in nonseismic zones so that ductilities larger than those required for a moderate amount of redistribution were not required.

In applying the segment equilibrium method to irregular slabs, Wiesinger would resolve the negative column strip moments into two global directions. This makes the calculation of the column moments in each of these directions quite simple. It also made it easy to balance the moments in each of the directions so that an isotropic top mat could be used. If the top mat is isotropic, it can then be rotated in any direction, making placement of the steel somewhat easier.

4. PLASTIC LOWER-BOUND SOLUTIONS

4.1 Historical Background

Practical lower-bound solutions for reinforced concrete slabs originated with the development of the strip method of design. The method was developed almost entirely by Hillerborg (1956, 1959, 1960, 1975) in order to provide a design method which always produces designs on the safe side, (to the left in Fig. 2.1). Hillerborg's techniques are still the only practical lower-bound solutions for reinforced concrete slab design. His methods are extensively used in the Scandinavian countries, but are little used in North America. The literature contains a considerable discussion of the details and application of the simple strip method to which one may refer. (Crawford 1962, 1964, Armer 1968, Kemp 1970, 1971). Rozvany and others have made considerable use of this method in the study of optimum reinforcement layouts (Rozvany, 1968, 1971, 1972 etc.). However, Hillerborg remains essentially alone in the development of the central theme of the strip method, and his book "Strip Method of Design" (Hillerborg 1975) represents the state-of-the-art. In this work, Hillerborg presents not only the simple strip method but also the advanced strip method and what amounts to a segment method. The last two methods have received very little discussion in the literature inspite of the fact that they have been in existence for over twenty years, and represent a vastly superior method of solution to many types of slab problems.

4.2 Simple Strip Method

In addition to the assumptions given in section 2.3 which are common to all of the procedures considered, the strip method assumes that the slab behaves as an elastic-perfectly plastic material in flexure. In the simple strip method, the twisting moments are also neglected.

The result of the latter assumption is that the basic differential equation for the slab becomes:

$$\frac{\partial^2 m_x}{\partial x^2} + \frac{\partial^2 m_y}{\partial y^2} = q \tag{4.1}$$

This can be separated into two equations such that:

$$\frac{\partial^2 m_x}{\partial x^2} = q_x \tag{4.2 a}$$

$$\frac{\partial^2 m_y}{\partial y^2} = q_y \tag{4.2 b}$$

$$q_x + q_y = q \tag{4.2 c}$$

Equations 4.2a and 4.2b are the governing equations for one way beams in the x and y directions respectively. Each strip may be supported by reactions at the strip ends or by other strips which are crossed. The design of the slab is thus reduced to designing one way beams or strips. The load not carried by the strip spanning in the x directing must be carried by the strip spanning in the y direction. Hence, one safe load path is designed. The design may not be economical, but it will be safe.

The elastic-perfectly plastic assumption simplifies the solution of strips which are continuous or statically indeterminate. The plastic portion of this assumption also eliminates the need to ensure that the deflections of the x and y strips at a point are equal. Thus, the x and y strips need not carry load precisely in proportion to their relative stiffness, so the distribution of q into q_x and q_y is a relatively simple matter.

The following discussion presents a basic overview of the solution procedure. The actual details of the method will be considered in the discussion of some design examples. The basic procedure is to provide a safe load path so that all of the load is carried all of the way into the supports.

Firstly, the slab is divided into "beams" or strips in each direction. The portion of load assigned to each strip (q_x and q_y) must be set by the designer separately. Strips in one direction may be supported by strips acting in the other direction in the same manner as joists are supported by beams in typical simple span structural systems. The order in which the strips must be designed will be readily apparent. If the strip is statically determinate, the design is straight forward, however, if the strip is statically indeterminate, one may make use of plasticity as well as equilibrium to produce a design similar to that of a continuous beam by the mechanism method. It is also adequate and perhaps even preferable to use an approximate elastic moment field as long as equilibrium is maintained. The better the design moment field approximates the elastic moment field, the better will be the serviceability of the slab since less cracking and less redistribution of moment will result.

In general, it is assumed that the moments are uniformly distributed across the width of each strip. In some cases, such as non-parallel supports or supports which do not extend across the full width of the strip, there is a slight violation of the lower-bound theorem. However, it does result in uniformly spaced reinforcement across the width of each strip. The effects of this violation can be reduced by using more but narrower strips along with a more refined split of the load in the x and y directions as suggested by Kemp (1979).

Thus far the discussion has pertained to uniformly distributed loads and the simple strip method. Point loads and point supports may be accommodated in the simple strip method by the use of "strong bands". (Wood and Armer 1968, Kemp 1971). Hillerborg, however uses a different approach which he refers to as the advanced strip method.

4.3 The Advanced Strip Method

The advanced strip method is based on a load dispersion element which is used to convert a point load to a uniform patch load. This load dispersion element is rarely treated in print and is often misunderstood or misinterpreted by many authors. A complete derivation and discussion of the load dispersion element are in order.

Loads which are not uniformly distributed across the width of a strip produce a non-uniform distribution of moments across the width of the strip. The worst case occurs when the load is distributed over a very small area, that is, a "point" load. Left uninvestigated, this type of loading can produce a gross violation of the lower-bound theorem, but Hillerborg provides a lower-bound solution to this problem with the load dispersion element. The load dispersion element is equally applicable to point loads as well as point supports, however, the values of the support reaction must be known before the load dispersion element can be used. The discussion will be in terms of a point support but is equally applicable to point loads.

Conceptually, the load dispersion element can be thought of as an inverted footing which is between the slab and the column. The load dispersion element acts like a footing since it takes the column reaction and spreads it uniformly over the area of the load dispersion element. A moment field is generated in the imaginary footing, as it is in a regular footing. The distributed column reaction is then applied to the simple strips as an upward acting load, and an overall analysis is done to determine the simple strip moments. The simple strip moments are then superimposed on the moment field in the imaginary footing (the load dispersion element) in order to determine the total moment field in the vicinity of the column.

Now that the basic concept of the load dispersion element has been presented, it is necessary to further define or rather redefine the load dispersion element as one-quarter of the imaginary footing. This is quite natural since the imaginary footing may be made doubly symmetric. The load dispersion element is then supported at one corner only and has moments along only the edges which pass through the support. The load dispersion element has also been referred to as "type 3 element" and the "corner supported element".

The moment field in the load dispersion elements can be any lower-bound solution which satisfies the boundary conditions and is in equilibrium with the loading. Of the infinite number of lower-bound solutions, some are more suitable than others. One could use an elastic solution which results in the moment field as shown in Fig. 4.1. This figure gives the m_x moments in one load dispersion element with dimensions c_x in the x direction and c_y in the y direction, and the point support acting at the origin. As one would expect, the moments rise sharply over the column. To provide reinforcement to match this

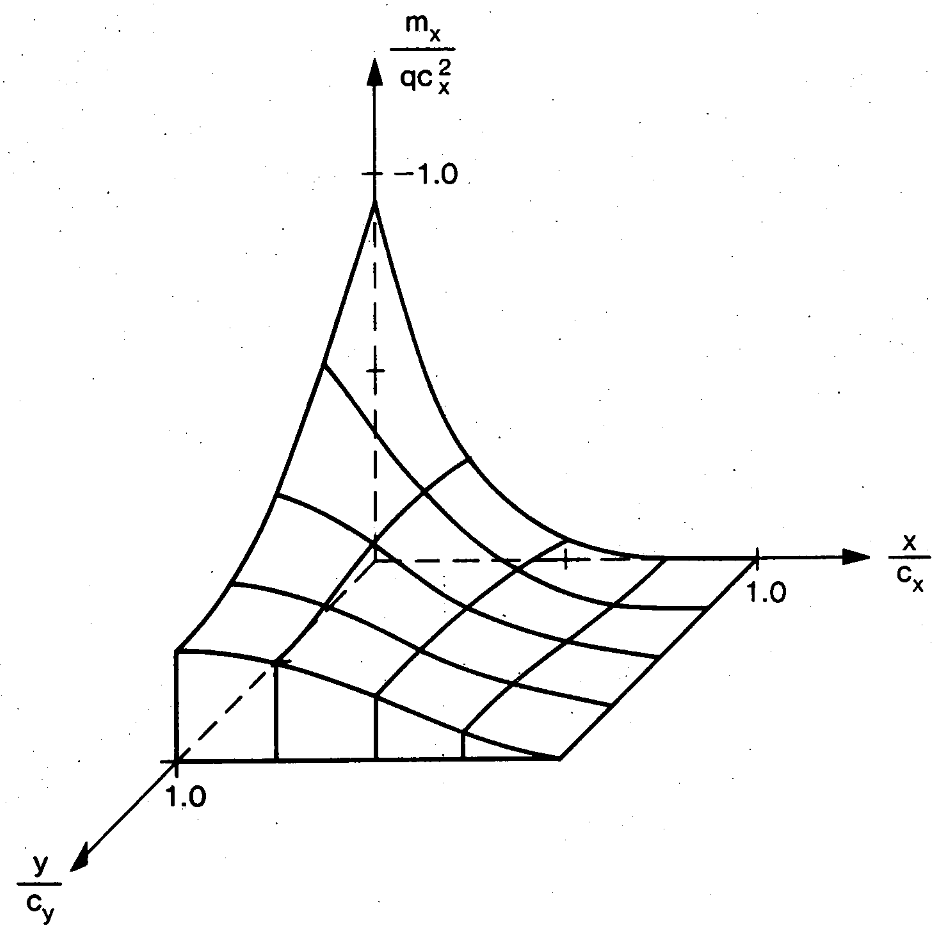


Figure 4.1 Elastic Moment Field Congruent to Hillerborg's Preset Moment Field

moment field would be awkward and generally would not be done. The elastic moment field is redistributed so that the maximum moment is constant or perhaps stepped across the width of the load dispersion element. Hillerborg has developed several lower-bound solutions with such maximum moment characteristics.

The simplest of Hillerborg's moment fields has a constant maximum moment. The derivation of this moment field is unique and deserves careful discussion. Consider for simplicity a square load dispersion element with side dimensions of c, carrying a uniformly distributed load q acting downward, and supported by edge moments m_{xs} and m_{ys} and point support reaction P. See Fig. 4.2. (The affinity theorems discussed in Section 2.5 can be used to extend the results to rectangular load dispersion elements). From equilibrium of the element as a whole:

$$P = qc^2 \tag{4.3}$$

$$m_{xs} = -\frac{1}{2}qc^2 \tag{4.4}$$

$$m_{ys} = -\frac{1}{2}qc^2 \tag{4.5}$$

The loading is divided into 3 parts as shown in Fig. 4.3:

1. One-half of the acting load q is carried by strips in

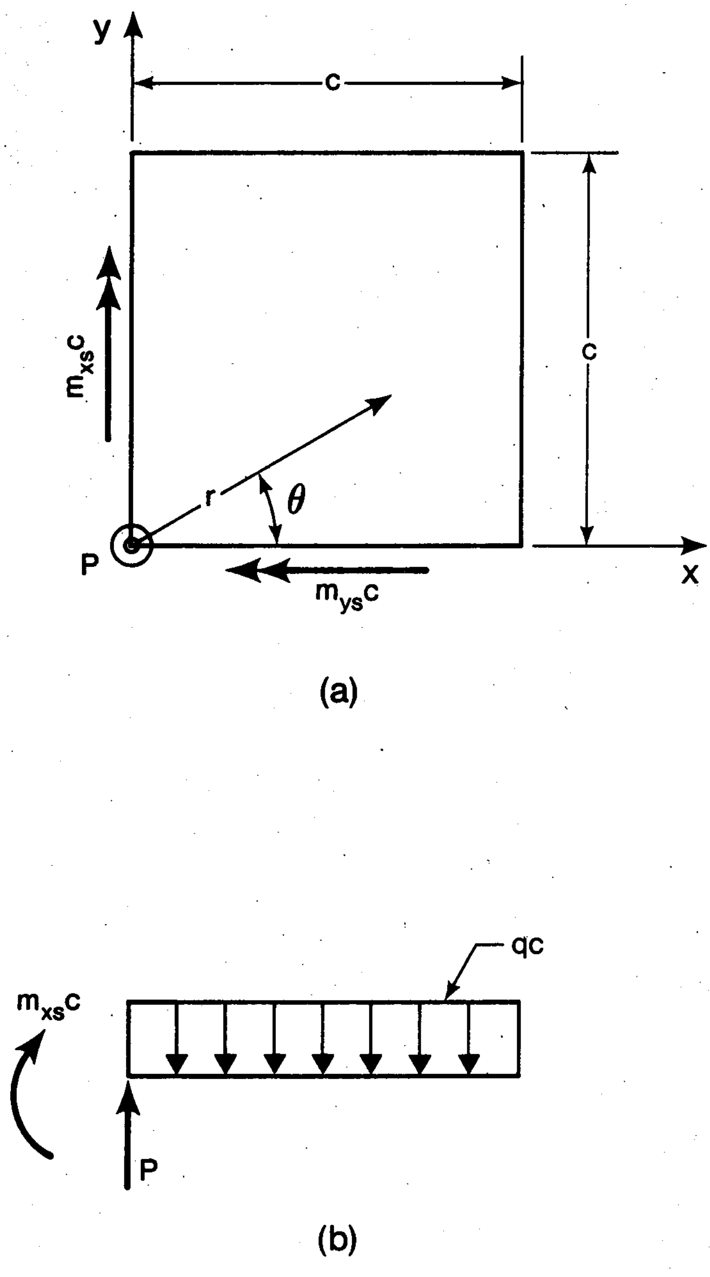


Figure 4.2 Load Dispersion Element

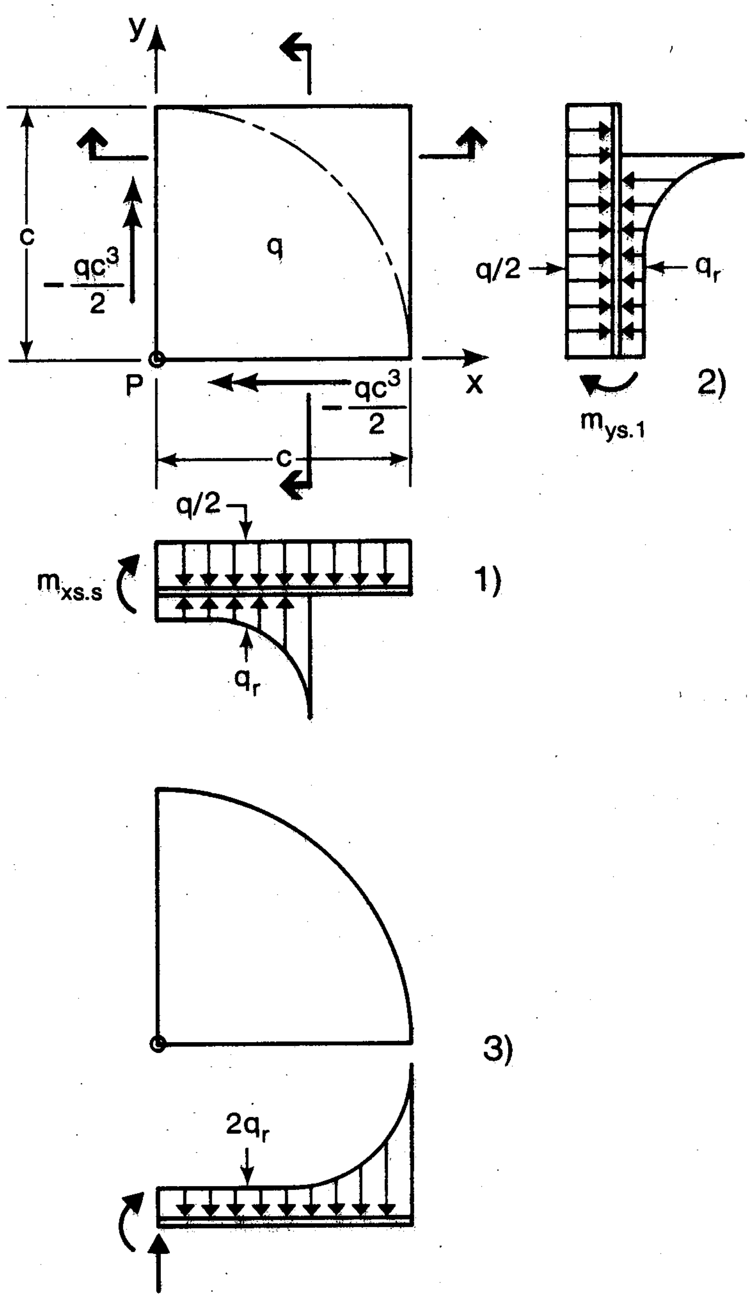


Figure 4.3 Division of Loading

the x direction, which are kept in equilibrium by a polar symmetrically distributed reaction q_r from the quadrantal area shown and by an edge moment $m_{xs,s}$, which varies along the edge as shown in Section 1 Fig. 4.3.

2. The other half of q is carried in the same manner by strips in the y direction (Section 2 Fig. 4.3).
3. The reaction q_r is then applied as a load on the quadrantal area is carried by the reaction P and the edge moment m_t. The subscript t indicates that it is a tangential moment in polar co-ordinates.

The reasons for introducing the polar symmetric distributed reaction are rather obscure but polar symmetric solutions are already known so this helps to "simplify" the calculations. Also, since the edge moments from parts (1),(2) and (3) result in a varying edge moment, a suitably varying tangential moment m_t in part (3) must be added in order to produce a uniform edge moment. A polar symmetric distribution of m_t has the advantage of being self equilibrating, but it does give rise to radial moments m_r which must be taken into account.

A polar symmetrical distribution of q_r must be found such that each strip is in vertical equilibrium. For x strips of unit width and part (1) of the loading:

$$\int_{x=0}^{x=\sqrt{c^2-y^2}} qr\,dx = \frac{qc}{2} \tag{4.6}$$

This results in:

$$q_r = \frac{qc}{\pi \sqrt{c^2 - x^2 - y^2}} \tag{4.7}$$

The process of determining the moment field in the load dispersion element may now begin. The general expression for the moment due to part (1) of the loading is:

$$m_{x,1} = -\frac{1}{4} q (c-x)^2 \tag{4.8}$$

Part (2) of the loading need not be considered since it produces moments in the y direction only, that is, $m_{x,2}$ =0. Part (3) of the loading, that is the polar symmetrically distributed reaction force q_r , produces a moment $m_{x,3}$ which can be determined by integrating the expression for q_r twice with respect to x.

$$\frac{d^2 m_{x,3}}{dx^2} = q_r \tag{4.9}$$

The constants of integration are determined from the conditions; $m_{x,3}$ =0 and $\frac{dm_{x,3}}{dx}$ =0 for .
This gives:

$$m_{x,3} = \frac{qc}{\pi} \left[x \sin^{-1}\left(\frac{x}{\sqrt{c^2-y^2}}\right) + \sqrt{c^2-x^2-y^2} - \frac{\pi x}{2} \right] \tag{4.10}$$

The edge moment of concern occurs at x=0, therefore:

$$m_{xs,s} = m_{x,1}\Big|_{x=0} + m_{x,2}\Big|_{x=0} + m_{x,3}\Big|_{x=0} \tag{4.11}$$

which yields:

$$m_{xs,s} = -\frac{1}{4}qc^2 + 0 + \frac{qc}{\pi}\sqrt{c^2 - y^2} \tag{4.12}$$

But we wish the total design edge moment m_{xs} to be constant, such that:

$$m_{xs} = -\frac{1}{2}qc^2 \tag{4.13}$$

One may add a moment m_t such that:

$$m_{xs} = m_{xs,s} + m_t \tag{4.14}$$

Equation 4.14 yields:

$$m_t = -\frac{1}{4}qc^2 - \frac{qc}{\pi}\sqrt{c^2 - y^2} \tag{4.15}$$

But from a knowledge of the polar symmetric case, this m_t and q_r give rise to a radial moment m_r, such that:

$$\frac{d\,(rm_r)}{dr} - m_t = \frac{2qc}{\pi}\sqrt{c^2 - r^2} \tag{4.16}$$

Integrating this equation yields:

$$m_r = -\frac{1}{4}qc^2 + \frac{qc}{2\pi}\sqrt{c^2 - r^2} + \frac{qc^3}{\pi r}\sin^{-1}\left(\frac{r}{c}\right) \tag{4.17}$$

The constant of intergration is found to be zero on the basis of boundary conditions. Since m_r is always positive and m_t is always negative, Hillerborg shows that the maximum positive and negative moments are given by:

$$m'_x = m_{x,1} + \cancelto{0}{m_{x,2}} + m_{x,3} + m_{ts} \tag{4.18a}$$

$$m_{xs} = m_{x,1} + \cancelto{0}{m_{x,2}} + m_{x,3} + m_r \tag{4.18b}$$

This simplification puts bounds on the moment field such that two bounds must be investigated. Equations 4.18 are represented by Fig. 4.4. This figure has perplexed many engineers since it implies that one can have a positive as well as a negative moment acting at the same point, in the same direction at the same time. This is not really the case since the surfaces represented are only bounds on the

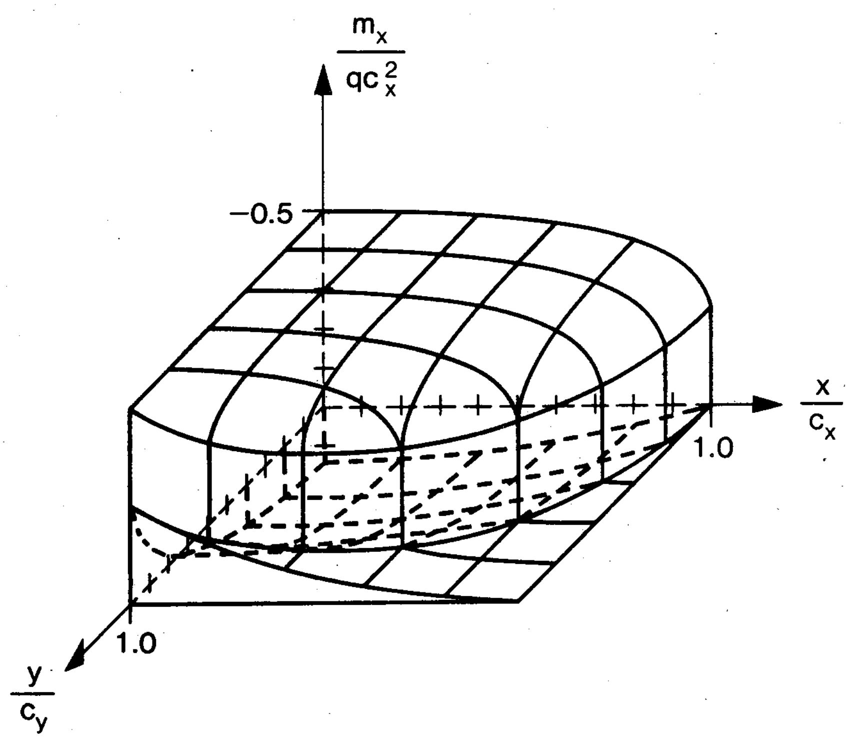


Figure 4.4 Hillerborg's Preset Moment Field

moments. In fact, Equation 4.18a is only correct when x=0, and Equation 4.18b is only correct when y=0.

The simplification introduced by Hillerborg is not necessary since one could simply superimpose $m_{x,1}$ and the appropriate components of $m_{x,2}$ and $m_{x,3}$. Thus:

$$m_x = m_{x,1} + \cancel{m_{x,2}}^{0} + m_{x,3} + m_t \sin^2\theta + m_r \cos^2\theta \qquad (4.19)$$

where:

$$\theta = \tan^{-1}(y/x)$$

Equation 4.19 is represented in Fig. 4.5 and Table 4.1. Inspite of its peculiar shape, this moment field requires a simpler reinforcement layout than the elastic moment field shown in Fig. 4.1. The only practical difference between Hillerborg's present moment field and the alternative preset moment field is in the cut-off points for the reinforcement. These are compared in Fig. 4.6 which illustrates the silhouettes of Figs. 4.4 and 4.5. The positive steel will have the same cut-off points for both solutions while Hillerborg requires considerably longer top reinforcement.

4.4 Strip Design Examples

The principles just discussed will now be applied to some examples. First, in order to solidify the concept of the simple strip method, consider the slab design given in

Table 4.1 Alternative Preset Moment Field

	Values of $\frac{M_x}{qc^2}$										
y/c \ x/c	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0	-0.500	0.135	0.131	0.124	0.114	0.101	0.086	0.068	0.047	0.024	0
0.1	-0.500	-0.183	-0.004	0.060	0.076	0.077	0.069	0.055	0.038	0.018	0
0.2	-0.500	-0.374	-0.187	-0.072	-0.013	0.014	0.024	0.022	0.014	0.000	0
0.3	-0.500	-0.438	-0.309	-0.194	-0.114	-0.065	-0.037	-0.025	-0.022	-0.025	0
0.4	-0.500	-0.464	-0.378	-0.282	-0.201	-0.142	-0.102	-0.076	-0.061	-0.049	0
0.5	-0.500	-0.477	-0.416	-0.341	-0.268	-0.207	-0.160	-0.125	-0.097	-0.003	0
0.6	-0.500	-0.484	-0.440	-0.380	-0.317	-0.258	-0.207	-0.162	-0.100	-0.003	0
0.7	-0.500	-0.488	-0.454	-0.405	-0.349	-0.292	-0.235	-0.168	-0.010	-0.003	0
0.8	-0.500	-0.490	-0.461	-0.418	-0.365	-0.303	-0.200	-0.023	-0.010	-0.003	0
0.9	-0.500	-0.490	-0.461	-0.414	-0.341	-0.063	-0.040	-0.023	-0.010	-0.010	0
1.0	-0.500	-0.203	-0.160	-0.123	-0.090	-0.063	-0.040	-0.023	-0.010	-0.003	0

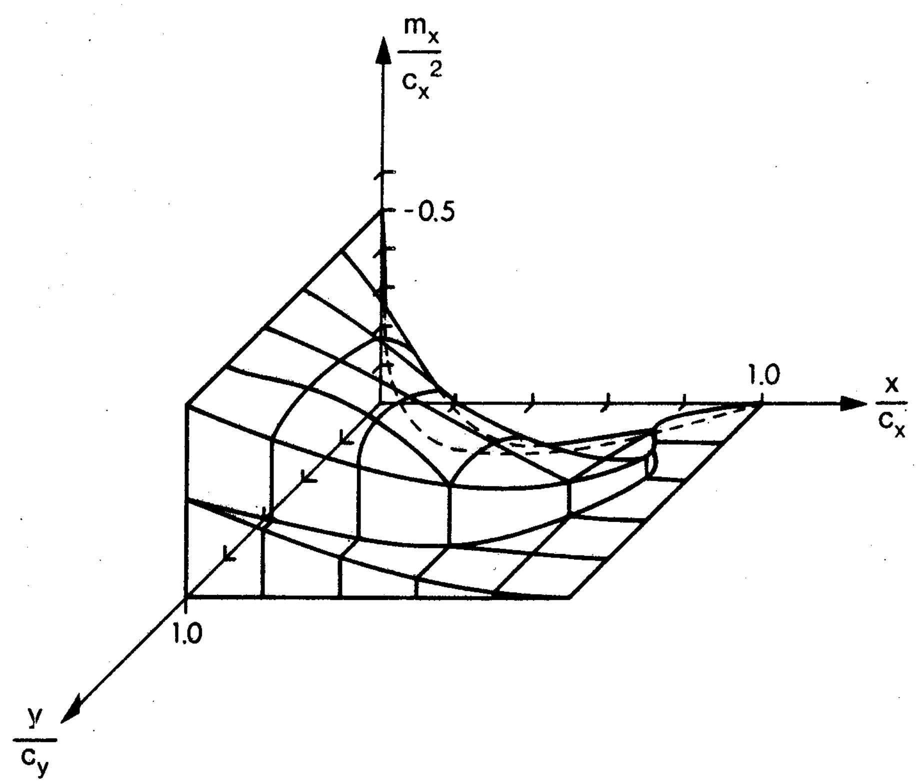


Figure 4.5 Alternative Preset Moment Field

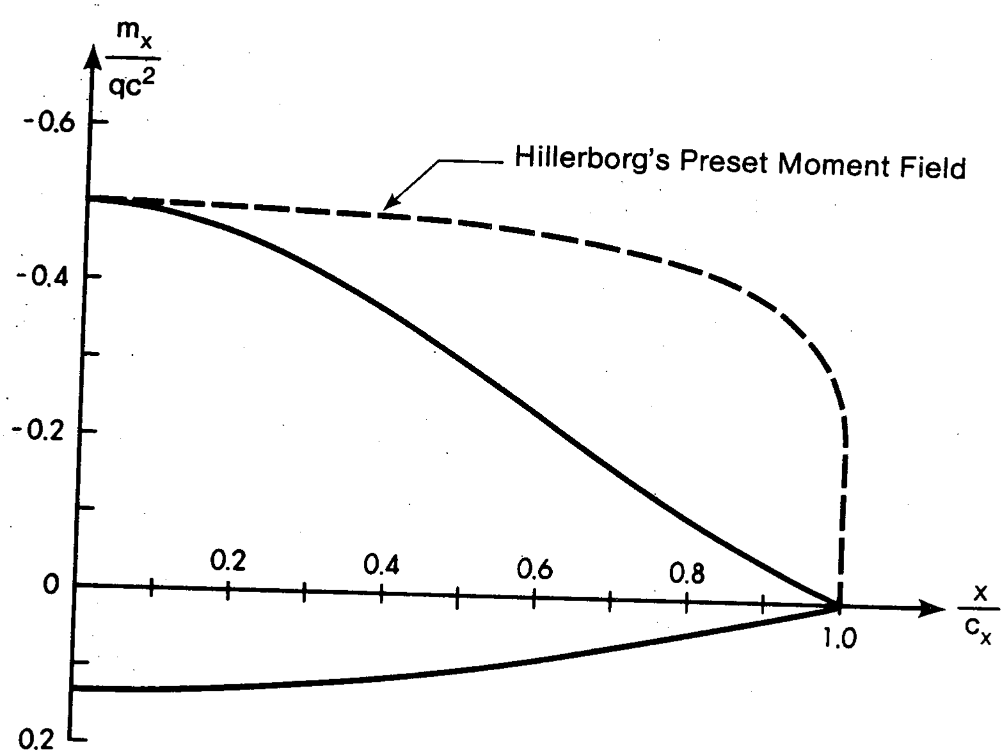


Figure 4.6 Silhouette of Preset Moment Fields

Example 1 Appendix B. In this simple case, the rectangular slab is to be designed for the uniform loading shown in Fig. B1.1.

The first step is to choose the load dispersion lines. This defines the strips in each direction. The choice of strips can influence the total amount of reinforcement required in of the slab, but it has a more pronounced effect on the layout of tne reinforcement since the reinforcement will be designed in bands corresponding to these strips. Simple reinforcement layouts can be achieved with a suitable choice of strips. The load dispersion lines which were chosen are shown in Fig. B1.2.

The second step requires that one choose the distribution of load carried in each direction, that is, how much of the load is carried by strips spanning in the x direction and how much of the load is carried by strips spanning in the y direction. The distribution of load can be done arbitrarily, but better serviceability (less plastic redistribution at service load), can be achieved by distributing the load in proportion to the stiffnesses of the strips. This can be done quantitatively by comparing deflections as suggested by Kemp (1979), or qualitatively as done in the example. The final distribution of loads used is shown in Fig. B1.3.

The third and final step is to design each of the strips for the chosen loadings on the basis of either unit width or total width of strip. Each strip can be treated as a separate beam. As pointed out by Armer (1960), if the reinforcement is chosen to exactly match the design moment field, the result would be the "unique" solution as previously defined in Section 2.4. However, if the reinforcement is proportioned so that the capacity is always greater than the design moment field, the solution is true lower-bound. Generally speaking, after calculating the required area of steel, one usually rounds up the steel requirements to the next convenient bar size and spacing. A convenient way of showing the design moment field is illustrated in Fig. B1.4. Only the maximum moments and their location are shown in this figure. One would have to look at the individual strip bending moment diagrams in order to determine the location of the cut-off points.

Example 2 in Appendix B considers the design of the same slab as in Example 1, but for two point loads rather than a uniformly distributed load. This problem demonstrates the use of load dispersion elements in conjunction with simple strips.

The first step is to choose load dispersion lines. In this design these lines define strips of slab which carry the loads to the supports. The area under the point loads where the strips in the x and y directions cross defines the size of the load dispersion elements. A poor choice for the width of the strips will be reflected in an unreasonable moment field. If the strip width is too narrow, the moment intensity per unit width will be too great for the slab thickness chosen. On the other hand, if the strips are too wide, the reinforcement in the edges of the strip will be too far away from where it is needed so that it will not become effective until large plastic deformations take place. Unfortunately there is no warning to the designer should this be the case. Also, the choice of wide strips results in somewhat larger total static design moments. So when in doubt, a designer should choose narrower strips. This will not only reduce the weight of steel required, but also some indication is given when the strip is too narrow.

The second step is to choose the amount of load carried by each of the strips. This choice is arbitrary, but the better it approximates an elastic solution, the better the serviceability will be. The distribution of load was chosen on the basis of the relative strip deflections at the point loads. The results indicate, as one would expect, that most of the load is carried by the slab in the short direction. The solution is based on 75% of the load carried in the short direction.

The third step is to determine the preset field which is required to convert the point loads into patch load. This has been done using Hillerborg's moment field the silhouette of which is shown in Fig. 4.6. This silhouette gives a conservative bending moment diagram in terms of moments as well as cut-off points.

The fourth step is to determine the moments in the simple strips due to the patch loads. It is at this stage that the load is distributed in each direction. In the calculations, the strips are treated as separate beams, however, the calculations may be done either on a unit width of strip or for the total strip width.

The last step is to superpose the preset moment field on the simple strip moment field in order to obtain the design moment field. From the design moment field shown in Fig. B2.4, it would appear that the moments are reasonable.

Example 3 in Appendix B considers the design of a slab with columns or point supports. The slab and loading are similar to those in Example 1, except that two interior columns have been added. Conceptually, perhaps the easiest way to design such a slab is to treat the columns as upward acting point loads. The slab is then designed for two load cases, first a downward acting uniformly distributed load, and second two upward acting point loads. These would correspond to Examples 1 and 2. In order to determine the design moment field, the results of both load cases must be superposed. The details of such an approach as well as the results shall now be examined.

The first step as always is to choose load dispersion lines. The North American concept of column strips and middle strips is helpful in selecting suitable strip widths. As pointed out previously, the concepts are not the same, but setting the simple strip widths approximately equal to the column and middle strips should result in reasonable banding of reinforcement.

The second step is to choose the distribution of load carried in each direction. This was done in more or less the same manner as in Examples 1 and 2. It should be pointed out that the uniform load can be distributed independently from that chosen for the upward acting patch load.

The third step is to choose column reactions. The column reactions are in fact "arbitrary" since they will be "statically correct" for the design moment fields. A poorly chosen reaction will result in an unreasonable moment field. It should be pointed out that in some structures such as footings and raft slabs, the column reactions are known and do not have to be assumed. The examples have been contrived so that the point loads of Example 2 are of the same magnitude as the column reactions in Example 3. The techniques and resulting moment fields can then be compared for point loads versus point supports.

The fourth step is to determine the preset moment field required to convert the point loads to two upward acting patch loads. Since the load dispersion elements are not square, the preset moment field must be determined in each direction.

The fifth step is to determine the simple strip moments due to the uniform load acting downwards.

The sixth step is to determine the simple strip moments due to the upward acting patch load. The patch loads are of course distributed as assumed in step 3.

The last step is to superimpose the preset moment field with the strip moments due to the uniform loads, and the strip moments, due to the upward acting patch loads, in order to obtain the design moment field. If the reinforcement is designed for this moment field, the results will be lower-bound.

In this particular example even though the results are lower-bound, they are not reasonable. For example, it is not reasonable that negative moments extend over the entire length of strips passing over the columns (2-2, 4-4 and 7-7). Elastic solutions indicate that there should be positive moments in the midspan regions. The lack of similarity between the design moment field and an elastic moment field results from the use of two load cases, one for the column reactions, and one for the loading. With such an approach, there is no real way of making choices which will result in a design moment field which will approximate the elastic moment field for the two span by three span structures except by trial and error.

Example 4 in Appendix B considers the design of the same slab as in Example 3. This time however, an approach will be used which enables one to control or influence the final moment field so that it may approach the elastic or any other desired distribution of moments. In this approach, the uniformly distributed load as well as the upward acting column reactions are considered in one load case. There will of course also be preset moment fields to convert the point support reactions to patch loads. Only those portions of the design which must be changed in order to illustrate the philosophical differences in this approach are presented.

The first step is to choose load dispersion lines. This is done in exactly the same manner as Example 3.

The second step is to choose the distribution of load carried in each direction. The uniformly distributed load has been distributed in exactly the same manner as in Example 3. No distribution is chosen for the patch loads at this time. When the strips in the x direction are designed, the portion of the patch load distributed in the x direction will be "adjusted" or chosen to produce reasonable design moments for these strips. The remaining portion of the patch loads must of course be included in the design of the strips in the y direction. Hopefully, if the final distribution of loads produce reasonable design moments in the x direction, the design moments in the y direction will also be reasonable.

The third and fourth steps in the design are the same as in Example 3. The column loads and preset moment fields can be the same since the only deviation from Example 3 thus far is in the distribution of the column reactions. This again emphasizes the fact that the load dispersion element acts as an imaginary footing which must be designed for the static moment in both directions and that the distribution of load is of no consequence to the preset moment field.

The fifth step is to determine the strip moments. First, the strips spanning in the x direction are manipulated until reasonable moment fields are obtained. As pointed out previously, for the strips which cross over the columns, one can use whatever portion of the patch load which is necessary to obtain reasonable moments. This can be solved for after one assumption or "criteria" for what consitutes a reasonable moment field has been chosen. One may set the point of zero shear, the point of inflection, the moment at any point in the span, or the ratio between moments within the span. Since the preset moments have already been calculated, one could fix the final negative design moments or even the ratio of the final negative design moment to the final positive design moments. The strips which do not pass over the columns can be assumed to be in contact with one or more strips which run in the y direction. One can "adjust" this contact pressure until the design moment field in the x direction is suitable. The strip running in the y direction must of course be designed for this contact pressure. In this way strips in the x direction may be supported to some extent by strips in the y direction and vice versa. After each of the x strips has been considered, the moment fields for the strips spanning in the y direction must be determined. For these strips, there are no choices to be made. One must simply consider all of the loads not carried in the x direction as well as any resulting contact pressures between strips.

The final step is to superimpose the preset moments on the moments from the previous step in order to obtain the design moment fields. In spite of the efforts to obtain a solution which approximates the elastic solution, it would appear that the design moment field does not. One would expect the strip spanning between the columns to have a positive moment at midspan while Fig. B4.2 indicates a small negative design moment. The assumption which should be changed as well as an outline of how and why it should be changed are given at the end of the examples. The reworking of the example is left to the reader.

The use of load dispersion elements in conjunction with simple strips is referred to as the advanced strip method. The examples have given a clear progression from the simple strip method through to the limits of the advanced strip method. The last example not only illustrates the techniques used to control the design moment field, but also illustrates the numerous calculations and recalculations and the detailed knowledge of slab behavior required to obtain a reasonable solution. The design calculations and choices become even more difficult if there are column moments or the columns are offset. Such cases can be handled with the advanced strip method but the calculations will be long and tedious with numerous recalculations required since the correct choices in various steps of the design will not always be readily apparent. The advanced strip may be useful for slabs carrying point loads and slabs with few or perhaps no columns at all, but the advanced strip method is not very suitable for slabs supported by more than a few columns.

4.5 Segment Method

One can overcome many of the difficulties with the strip method by eliminating the combined use of strips and load dispersion elements, and using only large load dispersion elements such that they contact each other. In this case, all the loads can be carried directly into the columns by the load dispersion elements with the distributed reaction pressure equal to the applied load. Used in this manner, the load dispersion element should more properly be called a "corner supported element" since it is no longer used to disperse a point load into a patch load. The use of corner supported elements enables lower-bound solutions to be obtained without superimposing simple strip solutions and without having the problems associated with choosing suitable distributions of load. This is no longer a strip method, it is a segment method.

The concept behind a lower-bound segment method of solution is to divide the slab into a number of segments, then by using standardized lower-bound moment fields (preset moment fields) one can obtain a lower-bound solution for the entire slab. Equilibrium must be maintained for each segment and also between segments. The preset moment fields are standardized in such a way that a uniform positive moment field may be superimposed on the preset moment field. This ensures that there is equilibrium between segments along edges of positive moment. There may not be equilibrium between segments along edges of negative moment, but since edges of negative moment occur at or over supports, the imbalance in moment may be assigned to the support. In this way, equilibrium is maintained throughout the entire slab. One only needs a set if standardized moment fields for various shaped segments.

The preset moment fields discussed to this point have had uniform edge moments. This is not the only or necessarily the most suitable moment field for design purposes. In many instances it is desirable to concentrate the reinforcement towards the point support. This can be done by having a stepped edge moment such that for some width near the support, the edge moment is larger and uniform, while for the remaining width away from the support the edge moment would also be a constant but of lower value as shown in Fig. 4.7. Hillerborg refers to such an element as a "rationalized load dispersion element" however, it is more meaningful to think of it as a corner supported element with a rational moment field.

The nomenclature used for the corner supported element is shown in Fig. 4.7. Any desired distribution of maximum (design) edge momments can be substituted into Equation 4.13 as long as overall equilibrium of the element is maintained. For example, the design edge moment m_x can be set equal to m_β for $y \leq \beta c_y$ and $m_{1-\beta}$ for $\beta c_y < y \leq c_y$. These values can then be used in Equation 4.14 to obtain the required distribution of m_t. The resulting distribution of m_r would have to be determined for the polar

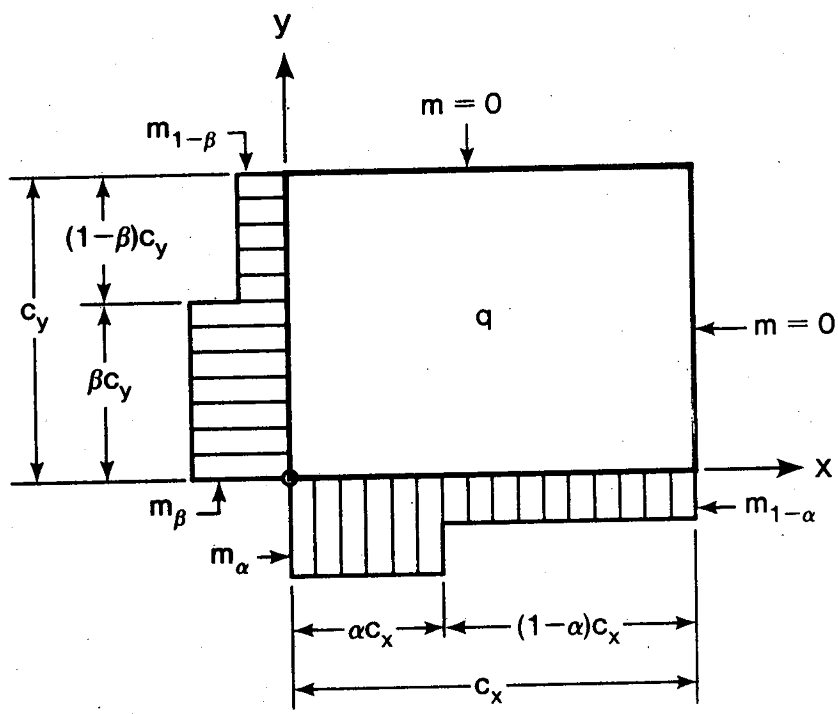


Figure 4.7 Corner Supported Element

symmetric case, and the new expressions for m_t and m_r would then be substituted into equation 4.19. This is not quite the manner in which Hillerborg approaches this problem but it does convey the desired notion, that is, that it is at least conceptually possible to obtain lower-bound solutions for any desired distribution of edge moments.

The following discussion will only deal with the rationalized preset moment fields as developed by Hillerborg but it should be pointed out that there are other means of obtaining rationalized preset moment fields. The approach pointed out in the previous paragraph as well as Hillerborg's solution are in a closed form, however any elastic plate analysis program could be used to obtain numerical solutions for any desired edge moments. Whatever technique is used to obtain the preset moment field, one should check to ensure that the edge moments are the design moments for the element, that is the interior moments should always be less than the edge moments. Numerically obtained, moment fields have the advantage of being continuous while the closed form solutions have discontinuous steps in the moment field. Fig. 4.8 illustrates a typical Hillerborg rationalized preset moment field. A numerically obtained moment field probably would not look as strange and would be conceptually easier to accept, but when Hillerborg did this work in the 1950's, computer programs for plate analysis were not readily available. To date no one has published or even discussed the possibility of substituting numerically

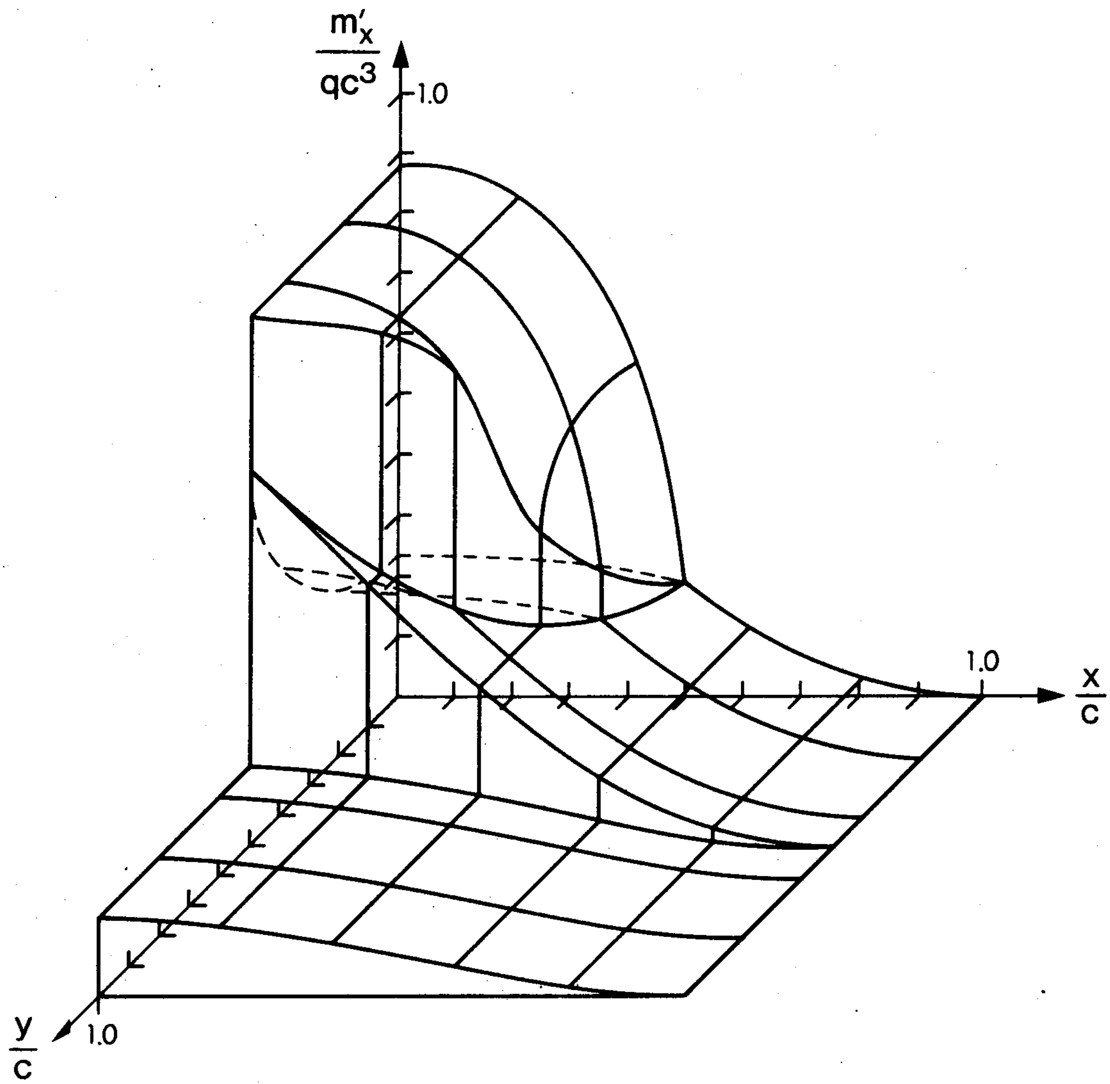


Figure 4.8 Typical Hillerborg Rationalized
Preset Moment Field

obtained preset moment fields for those obtained by Hillerborg. Further discussion of numerically obtained rationalized preset moment field must await further development.

At present, the only preset moment fields one can use are those developed by Hillerborg. While they appear to be rather complicated, they are relatively easy to apply. The first thing to note is that the mean value of the edge moment is still:

$$m = \frac{wc^2}{2} \tag{4.20}$$

that is, the total edge moment must still be able to keep the element in equilibrium, as in Equations 4.2 and 4.3. The designer can distribute this moment along the edges of the element in any manner desired as long as overall equilibrium is satisfied. For Hillerborg's preset moment field shown in Fig. 4.8, the design moments have two steps, m_β being the higher step, and $m_{1-\beta}$ the lower step. As in the case of Fig. 4.4, the m_x moments decrease to 0 at x/c=1.0, and the dashed lines indicate a possible positive moment field. For most practical distributions of edge moments there will be no significant positive moments anywhere within the element. This implies that only one mat of reinforcement need be used, and that only one design moment need be calculated for each direction on the element. However, for the type of edge distribution shown in Fig. 4.8, it is possible to select values of α , β , m'_α and m'_β such that sizeable positive moments are developed over some portions of the element. Hillerborg obtained two dimensionless parameters which indicate when the chosen moment field will have sizeable positive moment for which no reinforcement is provided. The dimensionless paramenters are:

$$K_x = \frac{\beta(-m'_\beta + m'_{1-\beta})}{\frac{1}{2} q c_x^2} \tag{4.21a}$$

$$K_y = \frac{\alpha(-m'_\alpha + m'_{1-\alpha})}{\frac{1}{2} q c_y^2} \tag{4.21b}$$

Hillerborg provided charts which indicate acceptable limits of these parameters. Fig. 4.9 is an example of one such chart. The limiting values for K_x and K_y are based on restricting the maximum positive moment anywhere within the element to 5% of the mean negative edge moment. Since only a small portion of element would have positive moment without positive reinforcement, this violation of the lower-bound theorem would probably reduce the capacity of the slab by less than 1%. Since there is nothing intrinsically special about the limit chosen by Hillerborg, the limiting values of K_x and K_y are also not particularly significant. Also, the regions of positive moment may not even exist since they are the result of peculiarities particular to Hillerborg's derivation. So the

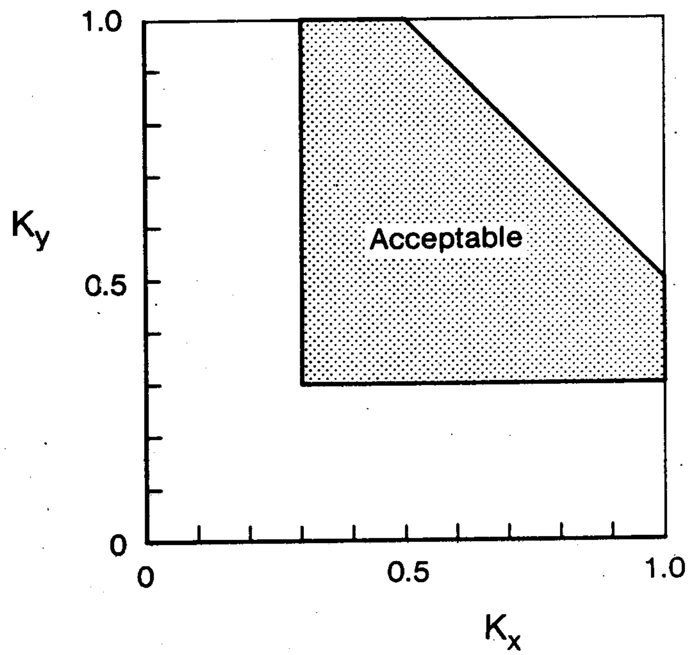


Figure 4.9 K-Limits for = =0.5

limits on K_x and K_y should not be viewed too strictly. In addition, it is usual to provide at least minimum reinforcing throughout the entire bottom mat. Because of the code restrictions on minimum and maximum reinforcement, it is likely that the bottom steel will resist a moment of about 10% of the mean negative design edge moment. Thus in practical design, the real range in acceptable K values will be much larger than Hillerborg suggests.

The guidelines which Hillerborg recommends will aid in producing designs with reinforcement rather similar to current practice, that is they will aid an inexperienced designer by indicating that the chosen moment field is unusual. With reference to Fig. 4.7, the first important recommendation is that values of α and β greater than 0.6 are seldom suitable. For the most common choice, $\alpha = \beta = 0.5$, the permissible range of K is quite large and varies from 0.3 to 0.75. Hillerborg claims that usually the values of K vary between 0.5 and 0.6 for most designs. However, an interior span of a slab designed by the Direct Design Method would have K values of 0.33. On the other hand, if all the middle strip negative reinforcement is moved into the column strip as is sometimes done, the K values would be 0.65. This implies that for the ratio of positive to negative panel moments chosen in the Direct Design Method, the lateral distribution of negative moments is about as uniform as one should go. The final noteworthy characteristic of the moment field is that for $\alpha = \beta = 0.5$, and approximately equal K values, the cut-off point for the m'_{β} and m'_{α} reinforcement is at about 0.5c to 0.6c.

The last point that need be discussed before presenting the final set of examples is the question of superimposed uniform positive moment fields. With the slab completely divided up into corner supported segments, and preset moment fields with negative moments only, one must superimpose uniform positive moment fields in order to obtain reasonable design moment fields. This is a simple matter since adjoining corner supported segments will meet in the midspan regions of the slab where it is easy to evaluate suitable moments. The existence of these positive moments along the previously "free" edges of the corner supported elements will make the limitations on K even more conseravative.

4.6 Segment Method of Design Examples

The following design examples will clarify the points discussed and will present a practical commentary on possible design techniques.

Example 5 in Appendix B deals with the redesign of the slab from Example 4. This time however, the "segment method" will be used rather than the strip method or advanced strip method. Basically, the slab will be divided into segments, most of which will be corner supported elements. Each segment must be in equilibrium on its own. The design then deals with the total static moment and hence the average static moments in each direction. A moment field is chosen such that at the boundaries between segments, the moments are equal in order to satisfy equlibrium between segments. The preset moment fields are used for each segment in order to ensure a lower-bound solution is obtained.

The first step is to qualitatively choose segements. The choice is qualitative because the actual location of the lines defining segments is not set. This step enables one to identify corner supported segments along the with other segments, and to consider how these segments will carry loads. The second step is to determine the mean segment moments. One usually starts by setting the average support moment for a corner supported segment. An indication of what might be a suitable value can be obtained by considering how the segment will act. Does the segment together with an adjacent segment act as a beam with the far end fixed or simply supported? What would be the required end moment in the adjacent span? The actual value of the average negative moment chosen will not influence the safety of the slab, since the positive moments will be determined from the requirements of static equilibrium. The choice does however influence serviceability. By choosing very large negative moments, the positive design moments will be made smaller, tending to cause more cracking on the underside of the slab at midspan. On the other hand, choosing very small negative moments, will result in more cracking on the top surface of the slab over the supports. The architectural finishes can have an influence on this choice since carpeting will make top surface cracks more acceptable while suspended ceilings will make bottom surface cracks more acceptable. One should usually try to approximate the elastic distribution of moments in order to minimize cracking, but when the designer is in doubt as to what the elastic moments might be, the above considerations will guide him towards an acceptable solution. Once the support moments have been set, it is a simple matter to find the points of zero shear which in turn define the size of the segments, since the lines between segments are generally lines of zero shear. The expression is:

$$c = \frac{L}{2} - \frac{(m'_{\text{near end}} - m'_{\text{far end}})}{wL} \qquad (4.22)$$

where c is the distance to the point of zero shear measured from the near end of a span with end moments of $m'_{\text{near end}}$ and $m'_{\text{far end}}$. The positive moments are then determined by equilibrium of the segment.

$$m = \frac{wc^2}{2} + m' \qquad (4.23)$$

In this manner, the moments in both directions can be determined in the corner supported segments. There are however, other types of segments which are required to fill in the boundaries of the slab. In this example, these other segments are simply supported on one or two adjacent edges, but many other types are possible. The main conditions for dealing with any segment are, the equilibrium is fulfilled for the elements as a whole and, that the distribution of moments chosen is reasonable. In this example, this condition is already fulfilled for the segments which are simply supported along one edge since they are linked to adjacent corner supported elements for which the positive moments have already been found. The moment field in the segments simply supported on two adjacent edges must be determined by some means. There are many possible solutions, but the final design is based on a preset moment field obtained by Hillerborg for such a segment. This is not the same preset moment field as for the corner supported element, but rather a moment field which satisfies the boundary condition of two adjacent simply supported edges. Even Hillerborg presents several lower-bound solutions to this case. The one chosen requires both top and botom reinforcement over the entire segment. While this solution may still lead to cracking due to twisting moments in the corner of the slab, it does agree with the current practice of providing top reinforcement in the corner of a simply supported slab. At the end of this step in the design, the size of each segment will be known along with the average positive and negative edge moments required to keep each segment in equilibrium.

The third step in the design is to rationalize the moment field, that is to laterally distribute the average edge moments previously obtained in a rational manner. This involves assigning more of the negative moments to the region of the segments near the corner supports as would normally occur in an elastic solution. One can also adjust the distribution of the positive moments so that they too are concentrated somewhat towards what are commonly know as the column strips. The suitablity of the chosen distribution of the moments is investigated by the parameters K_x and K_y . In this case, the parameters indicate that the lateral distributions are acceptable, and that the results may be used as the design moment field.

Example 6 in Appendix B considers the design of a considerably more complex slab than in the previous examples. The example demonstrates how the segment method can be applied rather easily to such complex cases and illustrates how offset columns and column moments may be dealt with.

The first step as before, is to qualitatively select segments. In this example, the column offset is larger than permitted by the Direct Design Method, but is moderate enough to present no problems in laying out segments. In the y direction the corner supported elements do not completely line up with each other, but this is no problem since the positive moments will be redistributed so that they will only exist where adjacent corner supported segments are in direct contact with each other. This will be done in one of the later steps.

The second step is to determine the mean segment moments required to maintain each segment in equilibrium. Again, this starts with setting reasonable negative edge moments for the segments. In this example however, because of uneven spans and support conditions the desirable negative moments on each side of the columns will be different. In an elastic analysis, this difference would be distributed between the column and slab in accordance with the relative member stiffnesses as in moment distribution, however, in plasticity any distribution of moments between the slab and the columns is acceptable as long as equilibrium is maintained and the members are capable of resisting the assigned loads.

In order to limit the amounts of plastic redistribution required in the structure, one should at least qualitatively consider the relative stiffness of the slab and columns. The CRSI handbook is very helpful in this regard since it gives values for α_{ec} . In this particular example it was decided that the columns would take about half of the unbalanced moment between the adjoining slab segments. As in Example 5, after the negative moments are set, the size of the segments is determined along with the positive moments. The corner segments have different boundary conditions than in Example 5, and thus will be treated in a different manner. By using a concept from the simple strip method, one can very easily deal with these segments. By assuming that half the load is carried in each direction, the moments in these segments will be half those in the adjoining segment parallel to it. Thus, without doing any further calculations, one would use half as much reinforcement per unit width parallel to the boundary of the slab in the exterior segments as in the adjoining interior segments.

The third and final step is to rationalize the moment field. This is done essentially as in Example 5, this time however, the positive moment between the columns is uniformly distributed only over the region of direct contact between the corner supported segments. The top steel in the column zones should of course be designed for the larger of the two design moments when they differ on either side of the column. The calculation of the column reactions has been included so that all of the forces and moments required to design the column and slab-column junction are known.

From Examples 5 and 6 it is clear that the segment method is distinctly different than the strip method and that for problems involving column supports, the segment method is the more appropriate method of solution. The segment method has great potential in the design of flat slabs.

5. SHEAR AND MOMENT TRANSFER

5.1 Introduction

In spite of all that has been said about the design of slabs for flexure, slabs rarely fail in flexure. In the past, numerous methods of design have been used resulting in a wide range in the amount of reinforcement required. Fig. 5.1 presents a comparison between steel requirements for various design methods (Sozen and Siess 1963). It indicates that the present day Direct Design Method requires approximately twice as much reinforcing steel as Turner used. In view of the fact that by 1910 Turner could claim to have successfully designed over 1000 flat slabs and the subsequent satisfactory performance of these slabs along with thousands of others, it would appear that there is significant room for variations from current practice without jeopardizing the safety of the slab. Sozen and Siess (1963) have said that "a slab will carry almost any uniform load that is put on it. The problem is to transmit the load out of the slab to the supporting members and to make the supporting members stiff and strong enough to provide the necessary reactions in the vertical and horizontal planes. The key to Turner's success was in his handling of these vitally important problems, unwisely considered by some as mere details."

The research on shear in slabs has been aimed at three different problems: point loads on slabs; shear in column

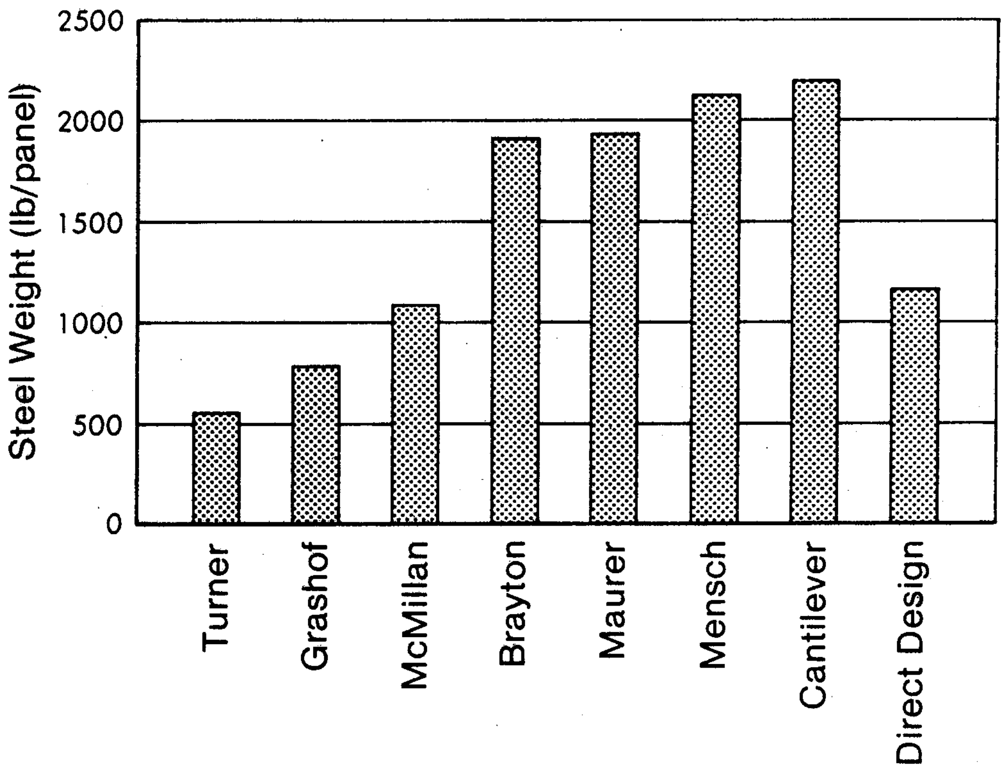


Figure 5.1 Weight of Steel Required for the Interior Panel of a Flat Slab by Various Design Methods

footings, and slab-column joints. In this discussion, only slab-column joints will be investigated. For this particular problem some moment can be transferred from the slab to the column directly by the flexural reinforcement while some moment may be transferred to the column by shear stresses in the slab so that there is an interaction between shear and moment transferred to the column.

There are many different approaches to this problem, but regardless of which method of analysis is used, there are certain concepts which are common. A free body diagram of a slab column junction is shown in Fig. 5.2(a). The slab transfers a vertical force V to the column as well as a moment. The moment transferred to the column is not necessarily the slab moment M but rather is the difference between the column moments above and below the slab $(M_1 - M_2)$. Some of this moment may be introduced into the column by the eccentricity of the shear force V. Some of this moment will come directly from the slab moments. And finally, some moments may be introduced into the column by torsional moments T on the side faces of the column. Regardless of where the forces come from, the free body diagram of the column in Fig. 5.2a must be in equilibrium along with a similar free body diagram for the slab. Various approaches differ on the choice of the total moment transferred to the column along with the magnitude of the moment transferred by each of the various mechanisms.

Three different approaches to the problem will be

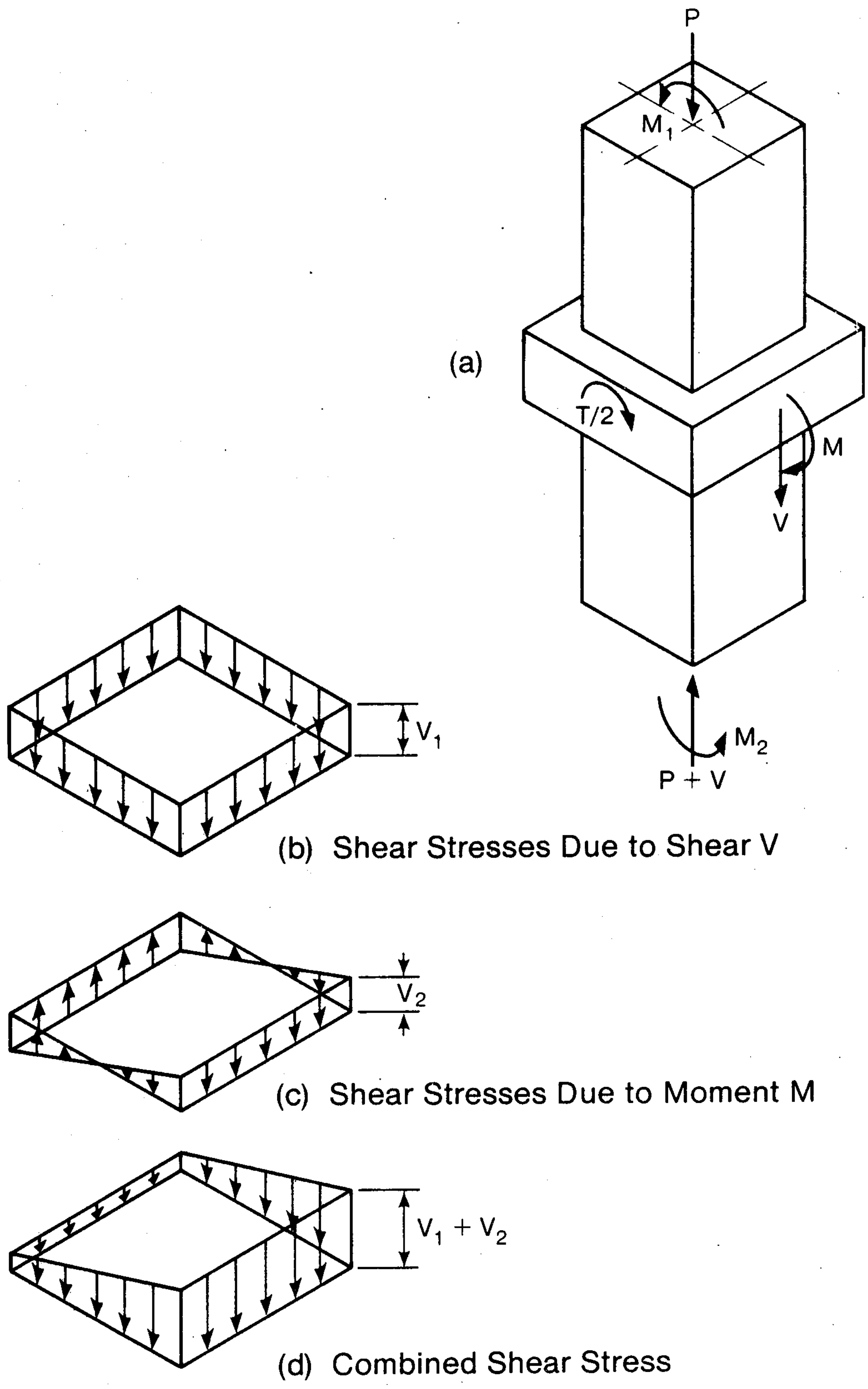


Figure 5.2 Elastic Analysis for Shear and Moment Transfer

discussed. The first is an elastic analysis with a brittle end point and is the approach currently used by ACI and CEB. The second approach to be considered is a plastic analysis which permits redistribution of stresses. This approach is typified by the beam analogies of Hawkins and Kanoh and Yoshizaki. The final approach is to recognize that the problem may not really be a shear problem in the concrete itself but rather is a conical yield-line mechanism. This last approach has been taken by Gesund in North America and by Dragosavic and Van den Beukel in Europe.

5.2 Elastic Analysis

This method of analysis is discussed because it represents current practice (ASCE-ACI Task Committee 426 1974). It was developed in the context of an overall elastic approach to design, including elastic moment fields. However, due to the complexity of exact elastic solutions only an approximate elastic solution is used.

Generally as a result of an equivalent frame analysis or some other elastic analysis, one starts with the known column moments and axial loads, above and below the slab. Then, as discussed in the previous section, one can determine how much shear and moment must be transferred from the slab to the column. The results obtained in this manner will not necessarily be the same as those obtained by taking the slab moments and shears at the column centreline and reducing them to the face of the support as in the case of a continuous beam. The difference is due to the fact that in the case of the beam, it is assumed the structure is cut in a straight line across the entire panel while in the former case, the structure is assumed to be cut around the column as shown on the free body diagram.

In accordance with the requirements of statics, this shear and moment must be transmitted across the boundary of the free body diagram in Fig. 5.2a. This boundary is referred to as the critical section since it is on this section that the stresses will be checked and compared against some specified maximum permissible stress. The choice of the location of the critical section is set by ACI as being located at a distance d/2 from the column face. It was found empirically that this location of the critical section produced a better fit of experimental results than other locations for the critical section.

The sum of the vertical shear stresses over the area of the critical section must equal V for vertical equilibrium. The average vertical shear stress is then:

$$v_1 = \frac{V}{A_{critical}} \tag{5.1}$$

as shown in Fig. 5.2b. If the centroid of the critical section does not correspond to the centroid of the column the eccentricity "e" of the resultant shear force V will introduce some moment into the column. This moment (Ve) may account for a considerable part of moment required to keep the column in equilibrium, thus reducing the amount of moment required to be transferred by other means. This possible reduction is generally not pointed out in the literature. While it is not rational in terms of statics, it is conservative to ignore this reduction.

Some of the moment to be transferred, either the correct moment or the unreduced moment, is transferred directly to the column by flexural reinforcement. The fraction of the moment transferred by direct flexure is given by γ_f where:

$$\gamma_f = 1 - \frac{1}{1 + \frac{2}{3}\sqrt{\frac{c_1+d}{c_2+d}}} \tag{5.2}$$

This equation is based on the work of DiStasio and Van Buren (1960) which indicated that 40% of the moment was transferred by flexure for a square column. Hawkins el al (1971) then modified the equation to account for column rectangularity. ACI requires that there be sufficient reinforcement within (3/2)h of the column to resist the moment transferred by direct flexure. The rationale for including reinforcement which does not pass through the critical section is that this additional moment transfer offsets the torsional moments on the side faces which are not fully utilized in this particular method of analysis (Hawkins 1979).

The remaining portion of the moment which must be transferred to the column is transferred by a linearly varying shear stress distribution such that the net vertical force is zero and the sum of the moments about any point equals the remaining moment to be transferred. Such a distribution is shown in Fig. 5.2c. Since the total moment to be transferred to the column for equilibrium of the column was M, and a portion of that was made up by the eccentricity of the critical section was (Ve), and some of the remaining moment was carried by flexure, the moment to be transferred by the linearly varying shear stress distribution M_s is:

$$M_s = (M - Ve)(1 - \gamma_f) \tag{5.3}$$

The linearly varying shear stress, v_2 is then:

$$v_2 = \frac{M_s c}{J_{critical}} \tag{5.4}$$

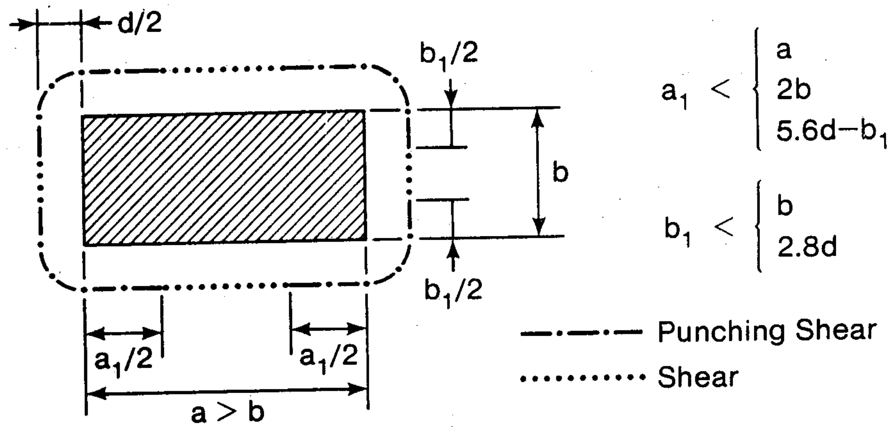
where, $J_{critical}$ is the polar moment of inertia of the critical section about its centroid, and c is the distance from the axis passing through the centroid of the critical section to the point in question.

The shear stresses v_1 and v_2 are then superimposed as shown in Fig. 5.2d to obtain the total shear stress distribution. The maximum shear stress must be less than the specified maximum permissible shear stress. From Fig. 5.2d it is apparent that much of the critical section is stressed well below the maximum permissible shear stress, thus much of the shear capacity is under utilized. This is analogous to the design of a short structural steel beam-column and limiting the maximum load to that causing yielding to occur in the most extreme fibre.

For simplicity, moments in one direction only have been considered. In most instances there can be moments acting in two directions so that another load case which transfers moment in the other direction must be added to Fig. 5.2. The stresses will then superimpose to give a maximum at one corner of the critical section. In such a case much of the critical section will be subject to low shear stresses. In fact, if the applied moments are large and the applied shear is small some portions of the critical section may be carrying upward acting shears when the connection "fails" downward. This would not appear to be very rational and is one of the difficulties with this approach.

For highly rectangular columns, the true shear stress distribution is not linear along the critical section. Most of the shear stress is concentrated towards the corners of the column, with the middle portions between the corners carrying little or no load. The result is that the apparent average limiting shear stress at failure for slab column connections loaded in pure shear decreases as the column rectangularily increases. The carrying capacity of the corners of the critical sections does not in fact reduce, but failure initiates at the corners before the concrete in the area of the middle of the column face begins to pick up any significant amount of load. The ACI accounts for this behavior by reducing the permissible shear stress while using the total length of the critical section. The CEB on the other hand uses the full limiting shear stress for that portion of the critical section near the column corners, and a permissible one way shear stress on the remainder of the critical section. This is illustrated in Fig. 5.3. The CEB would also reduce the limiting shear stress for portions of the critical section for very large square or circular columns, while ACI does not. There are other minor differences between ACI and CEB, but they both use the same philosophical approach.

The differences between ACI and CEB have been discussed in order to illustrate that the procedures used are by no means exact, and that there are several ways of idealizing the problem. With each method there will be a number of expressions and coefficients which must be empirically determined by experimental observations. The real test of a method of analysis is not how philosophically pure it is, but rather how well it predicts real behavior. Fig. 5.4 illustrates how well the ACI method predicts the failure load for slab-column joints transferring both shear and moment. The applied loads at actual failure are V_{test} and M_{test} while the connection capacity in pure shear and pure moment transfer are V_o and M_o . Fig. 5.4 represents interaction diagram with the theoretical failure line joining (V_{test}/V_o) =1.0 and (M_{test}/M_o) =1.0. It



Critical Section For An Elongated Area

For a Circular Column, Use A Punching Shear Strength For A Length of Critical Section $> 3.5\ d\pi$ And A Beam Shear Strength for the Remainder of the Critical Section.

Figure 5.3 CEB Critical Sections

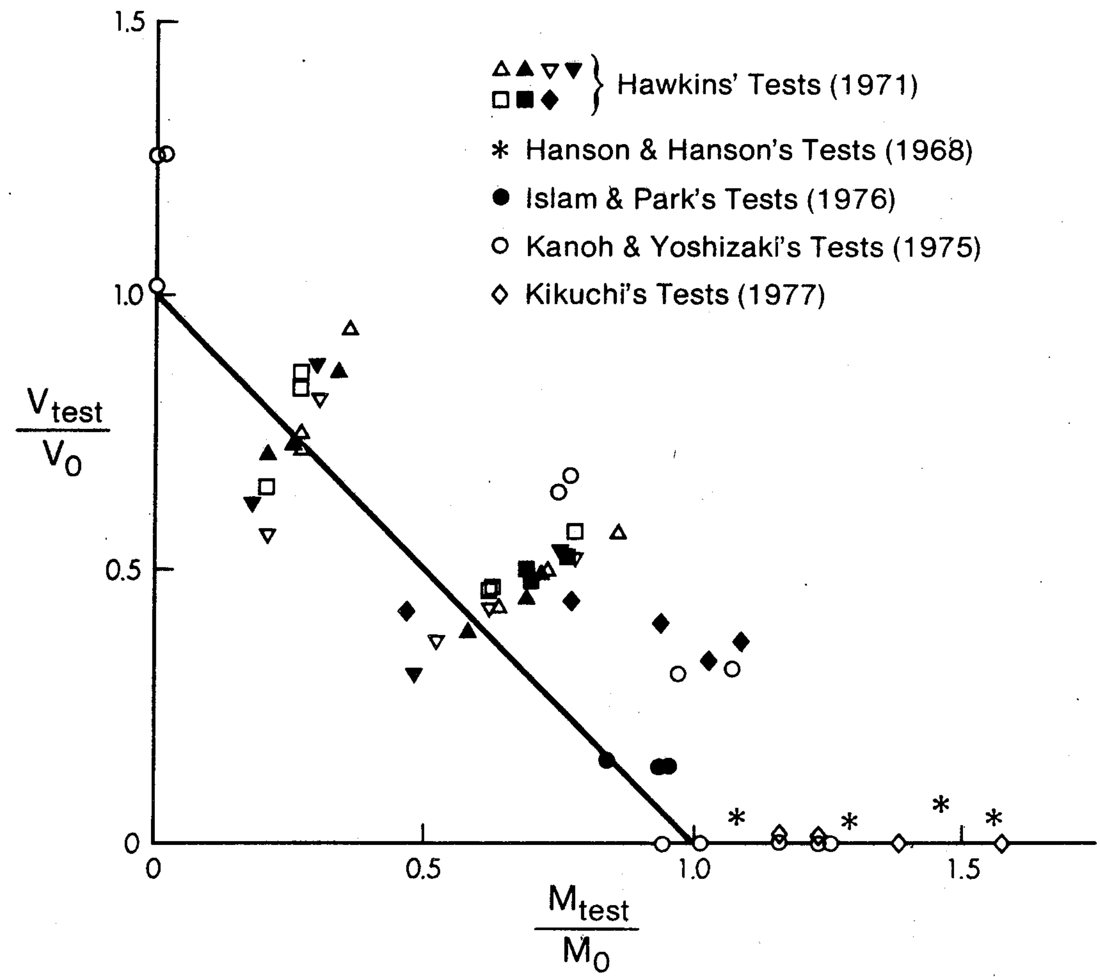


Figure 5.4 Comparison of ACI Approach and Test Results

is interesting to note that only the tests by Hawkins fall below the line predicting failure. An investigation into the reasons for such a large discrepancy between experimenters is beyond the scope of this work, but it should be done since it may identify particularly bad details or ranges of parameters where the ACI approach is not safe.

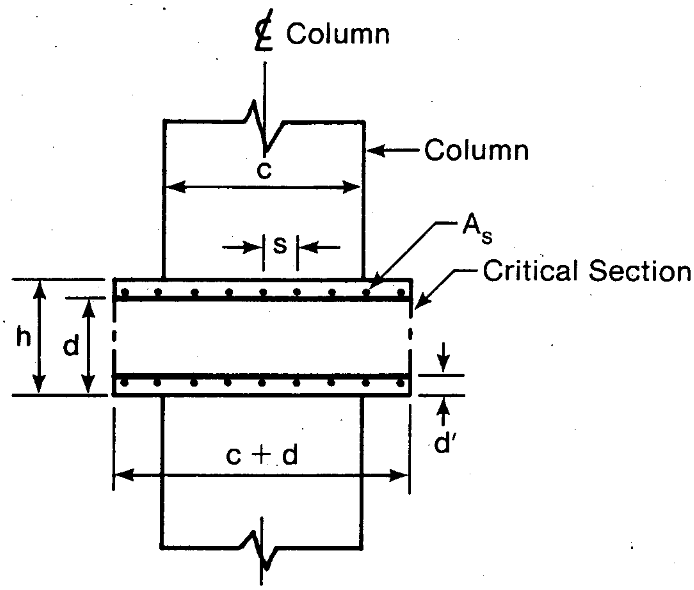
5.3 Plastic Analysis

This method of analysis is discussed because it is more representative of actual behavior than the previous elastic method. In some of its forms, the plastic approach is simpler to apply than the elastic approach.

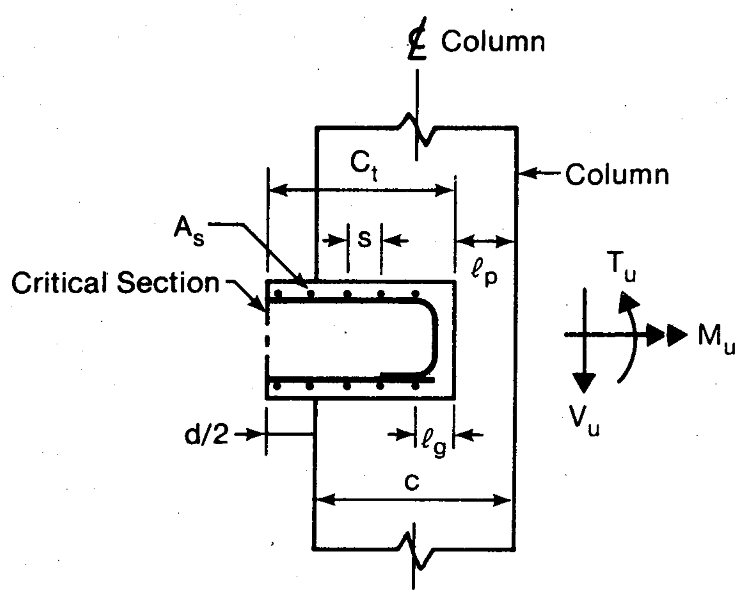
Plastic approaches are based on the assumption that the various mechanisms which transfer shear and moment to the column are capable of developing and maintaining their capacities in spite of plastic deformations. This is akin to the rigid-plastic assumption for slabs in flexure. The result is that rather than determining how much force is transferred by each of the mechanisms and then checking to ensure that the mechanism has not exceeded its permissible stress, one ensures that it is possible to transfer the required forces to the column by some combination of the various mechanisms without exceeding the plastic capacity of all the available mechanisms. One is no longer concerned with the "elastic" distribution of stresses. What is of concern is the maximum strength that can be developed on each face of the critical section with due regard to the interaction of the various mechanisms acting on each face.

The first significant method of plastic analysis proposed was Hawkins' beam analogy (1971). Each face of the critical section is assumed to behave as beam cross-section which can transmit moment, shear and torsion to the column. The forces which are to be transmitted to the column are distributed between the beam sections so that the connection develops the maximum strength possible for the particular failure mechanism or mode of deformation considered. This requires that one must consider all probable failure modes and maximize the joint strength for each mode. The mechanism with the lowest maximum capacity will govern the design. This is identical in philosophy to yield-line theory where all possible yield-line families must be investigated, each of which must be optimized, with the family of lowest capacity governing the design.

Hawkins intended that this method would be an alternative to, if not a replacement for the present ACI procedure, so he used many of the choices made by ACI in the application of his method. He uses the same critical section and essentially the same equations for the strength of each beam section as ACI presently does. However, only the steel actually crossing the critical section as shown in Fig. 5.5 is considered in the calculations. It is assumed that there is no interaction between the bending moment and the shear or torsion capacity of a beam section. There is however, an interaction between the shear and torsion capacity on a



(a) Interior Column



(b) Exterior Column

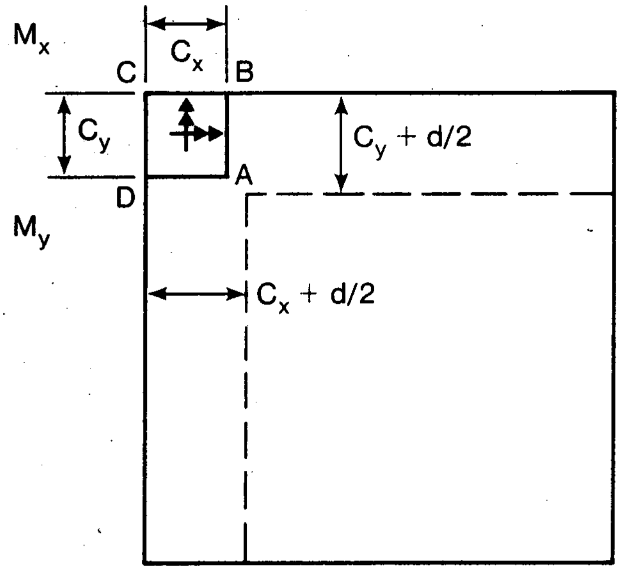
Figure 5.5 Critical Sections and Notation for Slab-Column Joints face. Hawkins uses the interaction equation:

$$\left(\frac{V}{Vo}\right)^2 + \left(\frac{T}{To}\right) \leq 1.0 \qquad (5.5)$$

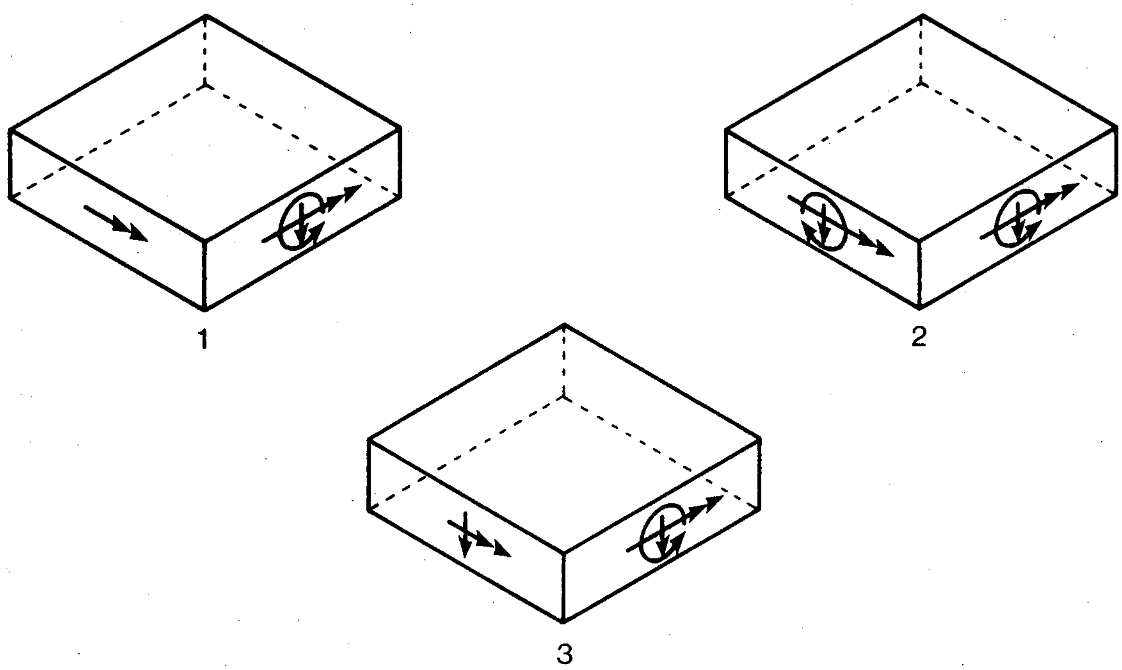
In which V and T are the shear force and torque assigned to the face in question, and V_o and T_o are the capacity of the face in question for pure shear and pure torsion, as computed by the ACI equations without performance factors.

The governing equations for various cases have been presented in the literature by Hawkins (1971) and thus will not be repeated here. However, the possible failure conditions for corner, exterior and interior columns are shown in Figs. 5.6, 5.7 and 5.8. From the large number of possible failure conditions which must be checked, especially for interior columns, it is clear that this is not a practical every day design method. This method can be used to advantage in certain situations since it does provide a better estimate of the failure capacity than the ACI method, and it also provides an indication of the mode of failure. This will provide an indication of the most direct way to increase the capacity of the slab-column connection.

Kanoh and Yoshizaki (1979) have recently proposed a simplified version of the beam analogy, which can best be described as a "plastic interaction equation" approach. This method is easier to apply than the Hawkins' beam analogy



(a) Geometry



(b) Failure Conditions

Figure 5.6 Corner Column Beam Analogy Failure Conditions

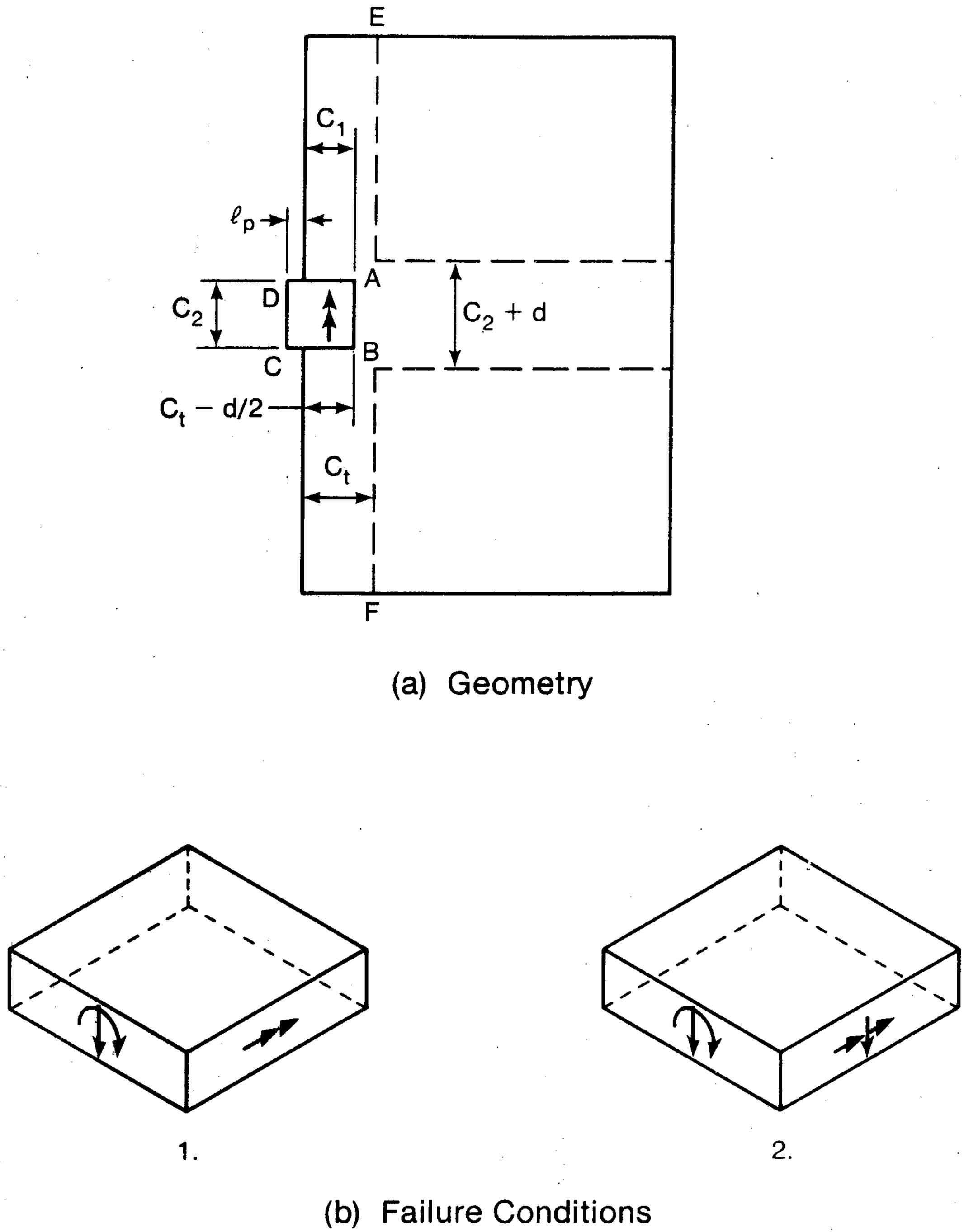


Figure 5.7 Exterior Column Beam Analogy Failure Conditions

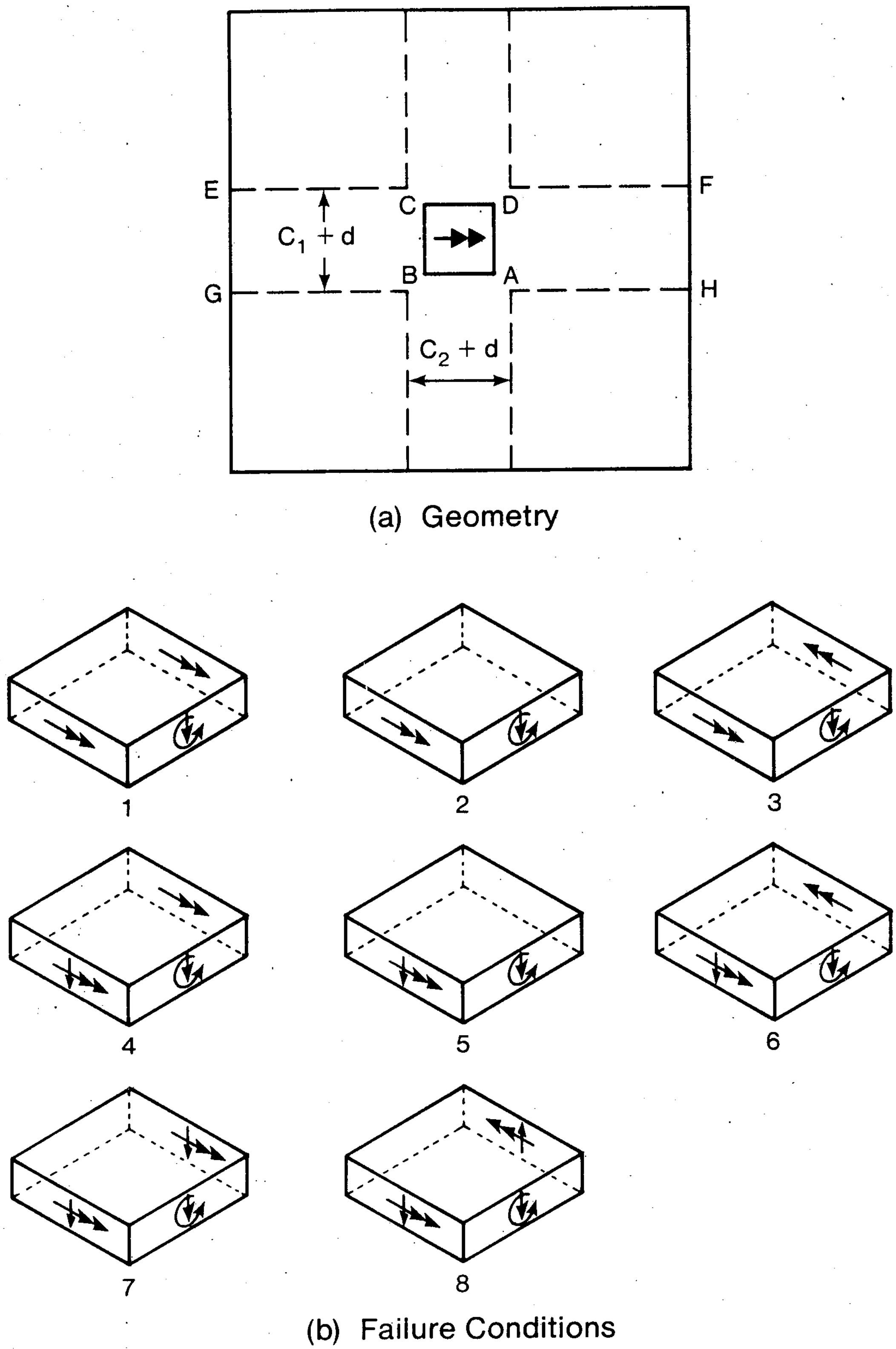


Figure 5.8 Interior Column Beam Analogy Failure Conditions

failure conditions. It also predicts the failure loads as well as or better than Hawkins' beam analogy. However, simplicity has it's price and in this case the plastic interaction equation gives no indication of the likely mode of failure. This approach uses a critical section located a distance d/2 from the columns face, and assumes that the effects of shear and moment interact as follows:

$$\frac{V}{Vo} + \frac{M}{Mo} \leq 1.0 \tag{5.6}$$

The values in the numerators of Equation 5.6 are V, the load transferred to the column, and M, the moment transferred to the column. The moment M can of course be reduced by V times e, the eccentricity of the critical section with respect to the column, as in the elastic approach.

V_o and M_o are the connection capacities for the pure shear case and the pure moment case respectively.

The pure shear capacity is simply the maximum permissible shear stress times the total area of the critical section, as shown in Fig. 5.9b. The plastic interaction equations have only been applied to relatively square columns thus far where the limiting concrete shear stress in Imperial units is 4 $\sqrt{f'_c}$. Since this method and the approximate elastic method use the same critical section and limiting shear stress, the resulting V_o will be identical, and both methods will have the same

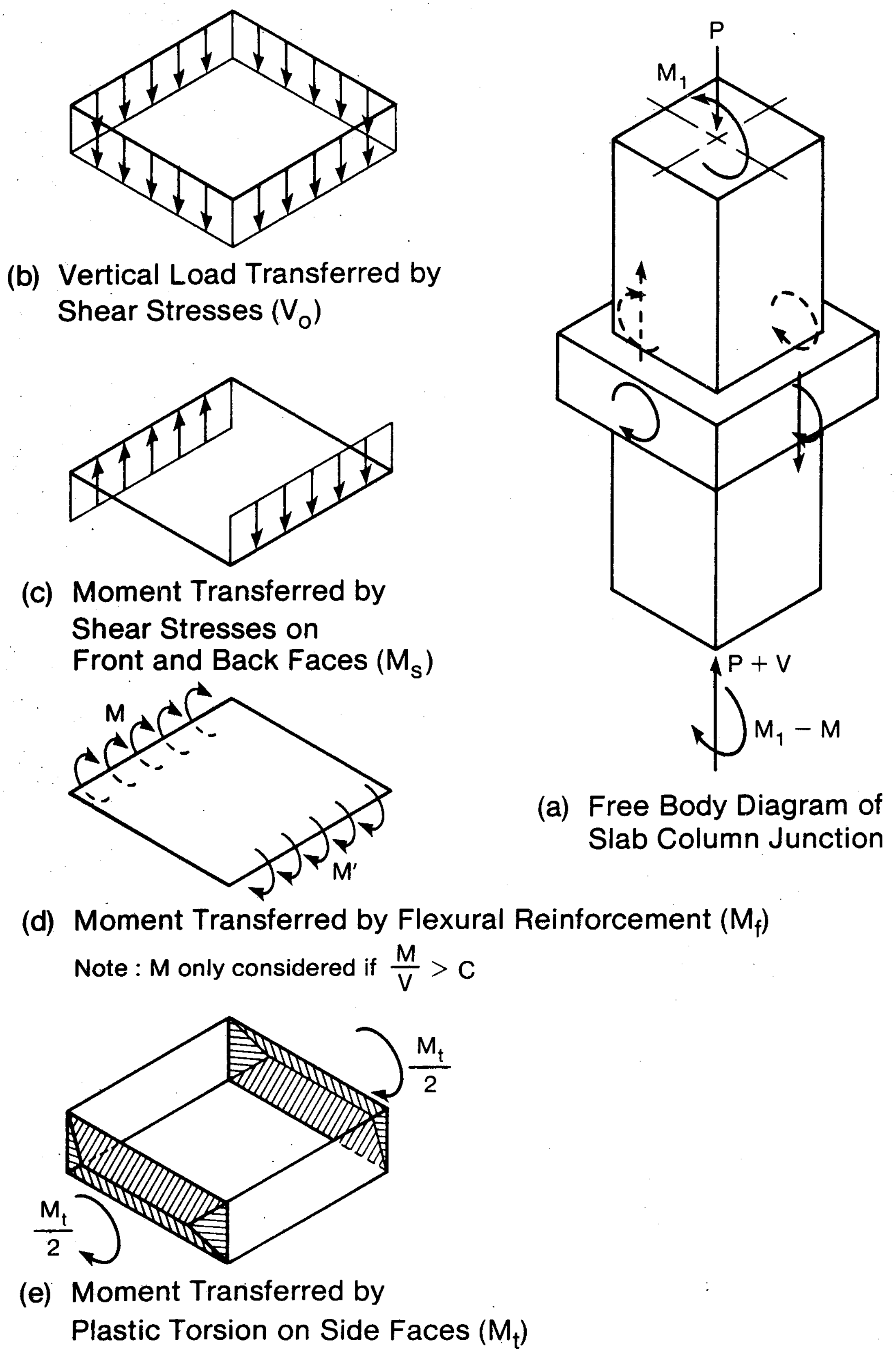


Figure 5.9 Plastic Interaction Equation Load Transfer Mechanisms

safety for the case of pure shear transfer and square columns. Further development of this method is required before it can be applied to highly rectangular columns.

The pure moment connection capacity M_o is composed of three portions:

$$M_o = M_{xs} + M_f + M_t \tag{5.7}$$

The first portion M_s is the moment transferred by shear stresses action on the front and back faces of the critical section as shown in Fig. 5.9c. In this case M_s is the sum of the moments due to the limiting shear stress acting on the front and back faces of the critical section taken about the centroid of the column. As in the case of pure shear, the plastic interaction equation has only been applied to square columns, and further work is required before the method can be applied to highly rectangular columns. It would appear that the CEB approach using different shear capacities for different portions of the critical section would be a more rational approach than ACI approach since the average limiting shear stress will be considerably different over the long and short faces of a highly rectangular column. It would however, be convenient to maintain the square corners on the critical section as used by ACI. Weisinger's brief discussion (1977) of some proposed revisions to ACI 318-71 suggests the use of such critical stresses and such a critical section and points out how these might be applied to cases with unusual column geometry and openings in slabs. The development of such a methodology in the context of the plastic interaction equation is beyond the scope of this work but it would appear to be worthy of further study. For the square columns under consideration, the shear stresses transferring vertical load to the column interact with the shear stresses transferring moment to the column as shown in Fig. 5.10a. This figure indicates that when the connection is transferring its full shear capacity, that is when the full limiting shear capacity is developed over the entire critical section, it is not possible to transfer any moment by shear stresses as shown in Fig. 5.9c, except for the moment transferred by Ve. On the other hand when the full M_s is developed, the side faces of the critical section can carry vertical shear, hence the vertical step in Fig. 5.10a.

The second component of M_o is M_f, the moment transferred to the column by reinforcement which crosses the critical section. As shown in Fig. 5.9d, it is possible for moment to be transferred to the column by top reinforcement on one face of the column and bottom reinforcement on the opposite face of the column. The capacity of the bottom reinforcement may be included when the ratio M/V is sufficient to cause tension on the bottom surface of the slab. This provides another illustration of the usefulness of bottom reinforcement in the column zone. It is assumed

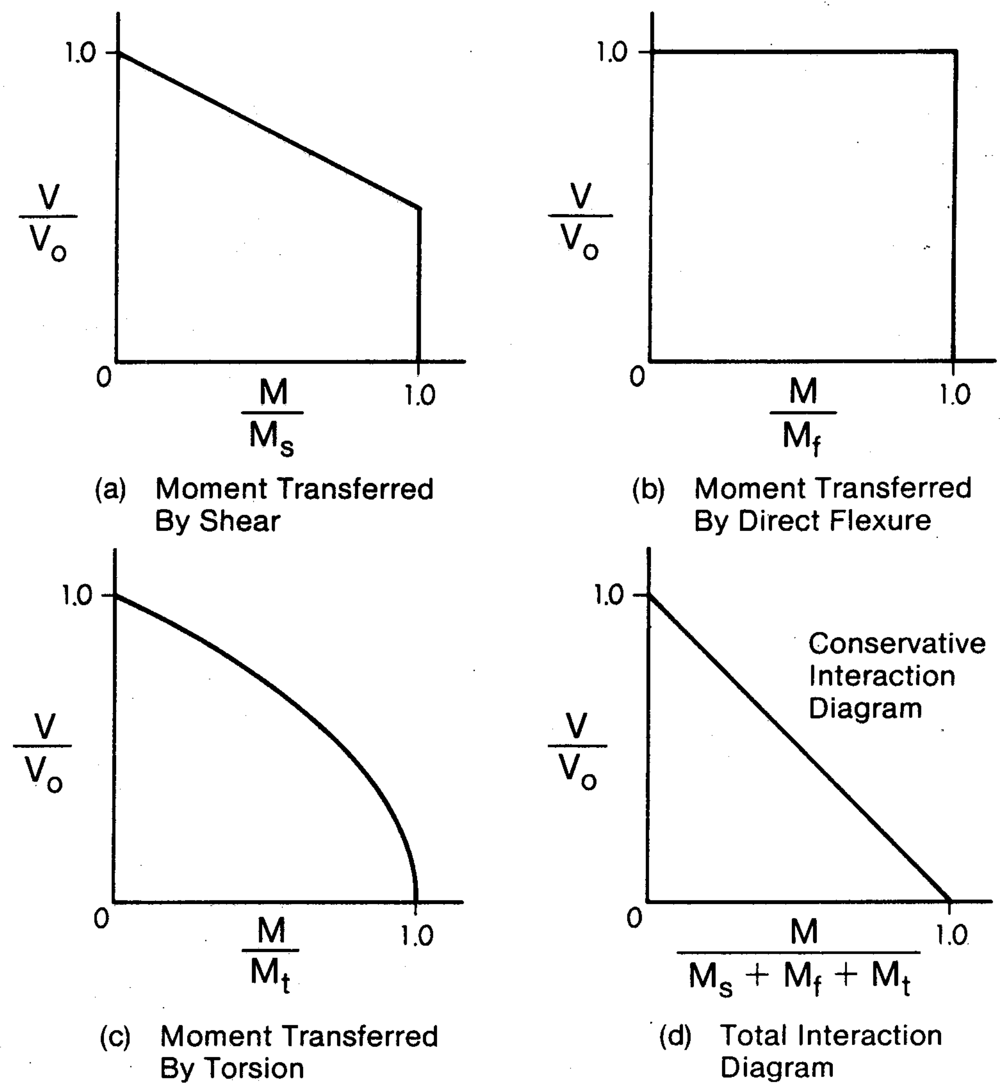


Figure 5.10 Plastic Interaction Equation Components

that there is no interaction between the vertical shear stresses and normal bending moment. This results in the interaction diagram shown in Fig. 5.10b.

The third portion of M_o is M_t, the moment trasferred by torsional stresses acting on the side faces of the critical section. Kanoh and Yoshizaki make use of a plastic torsional shear stress of 24 $\sqrt{f'_c}$ expressed in Imperial units. The sand heap analogy shown in Fig. 5.9e is used to represent the moment transferred by M_t which is twice the total volume of the sand heaps. At first sight, it would appear that a limiting torsional shear stress of 24 $\sqrt{f'_c}$ is far too large when the direct shear stress is limited to only 4 $\sqrt{f'_c}$. The limiting value for the torsional shear stress was obtained empirically for tests which were developed specifically to meausre this quantity. A reanalysis of the available data indicates that the limiting torsional shear stresses observed are very close to the shear friction values which could be mobilized on the critical sections for the particular amount of reinforcement present in the tests. Also, while the critical section is assumed to be plane, the cracking due to torsion will produce a warped failure plane, half of which is forced closed by gravity forces. So, until further data are available, 24 $\sqrt{f'_c}$ would appear to be a reasonable value for the plastic torsional shear stress. There is however, an interaction between the direct shear stresses and the torsional shear stresses acting on the critical section. This interaction is illustrated in Fig. 5.10c. It is assumed that there is no interaction between M_f and M_t since they do not even act on the same faces of the critical section.

The interaction diagrams in Figs. 5.10a, b and c are all "concave down", thus the total interaction diagram between V/V_o and M/M_o would also be "concave down", and can conservatively be represented by the straight line interaction diagram shown in Fig. 5.10d which represents Equation 5.6. A connection which has values which result in a point lying to the right of or above the interaction line would be unsafe.

As stated previously, this approach automatically redistributes the stresses on the critical section so that the maximum capacity is obtained. This method does not identify any one particular mechanism of failure, it simply ensures at least one load path to carry the required V and M. The validity of this approach can be demonstrated by plotting real (V_{test}/V_o) and (M_{test}/M_o) values obtained in experimental tests on the interaction diagram as shown in Fig. 5.11. This figure is compared to Fig. 5.4 in Section 5.5.

5.4 Yield-Line Analysis

This approach to explaining the phenomenon around slab-column joints is essentially pure yield-line theory. The slab-column punching shear tests have been analyzed by

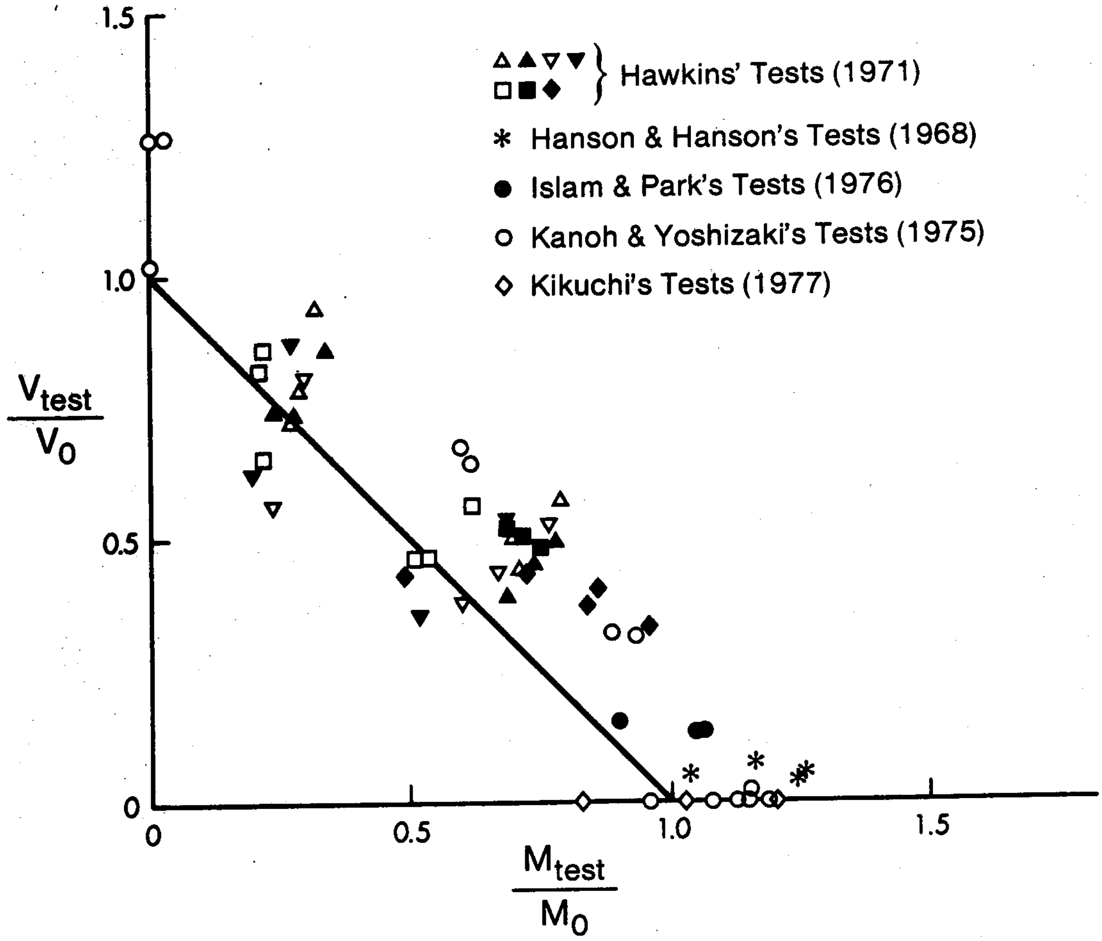


Figure 5.11 Comparison of Plastic Interaction Equation with Test Results

various researchers using yield-line theory. The governing yield-line mechanisms for each test set-up were determined. In most cases a conical yield-line mechanism was found to be the governing mechanism. The original analysis of many of the slab-column connection tests included a yield-line analysis, however, conical mechanisms were never investigated. The yield-line collapse loads which were based on the wrong mechanisms were too large, thus many failures which occurred at loads less than the improperly calculated yield-line collapse load were reported as shear failures when in fact they were conical yield-line failures.

Dragosavic and Van den Beukel (1974) analyzed the results of a number of punching shear tests. They determined the collapse load V_{flex} based on the governing yield-line mechanism, and V_o the theoretical punching shear load. These were compared to the real failure load V_{test}. The results are shown in Fig. 5.12. When V_{flex} is greater than V_o, failure should be punching shear, and (V_{test}/V_o) should equal one. Also, when V_{flex} is less than V_o, the failure should be in a flexural mode at a load less than V_o. These facts are clearly demonstrated in Fig. 5.12.

Gesund (1970) has performed a similar type of analysis of the punching shear test results. He has developed a parameter Q in order to differentiate between shear failures and flexural failures. This parameter is defined on Fig. 5.13 on which the yield-line collapse load divided by

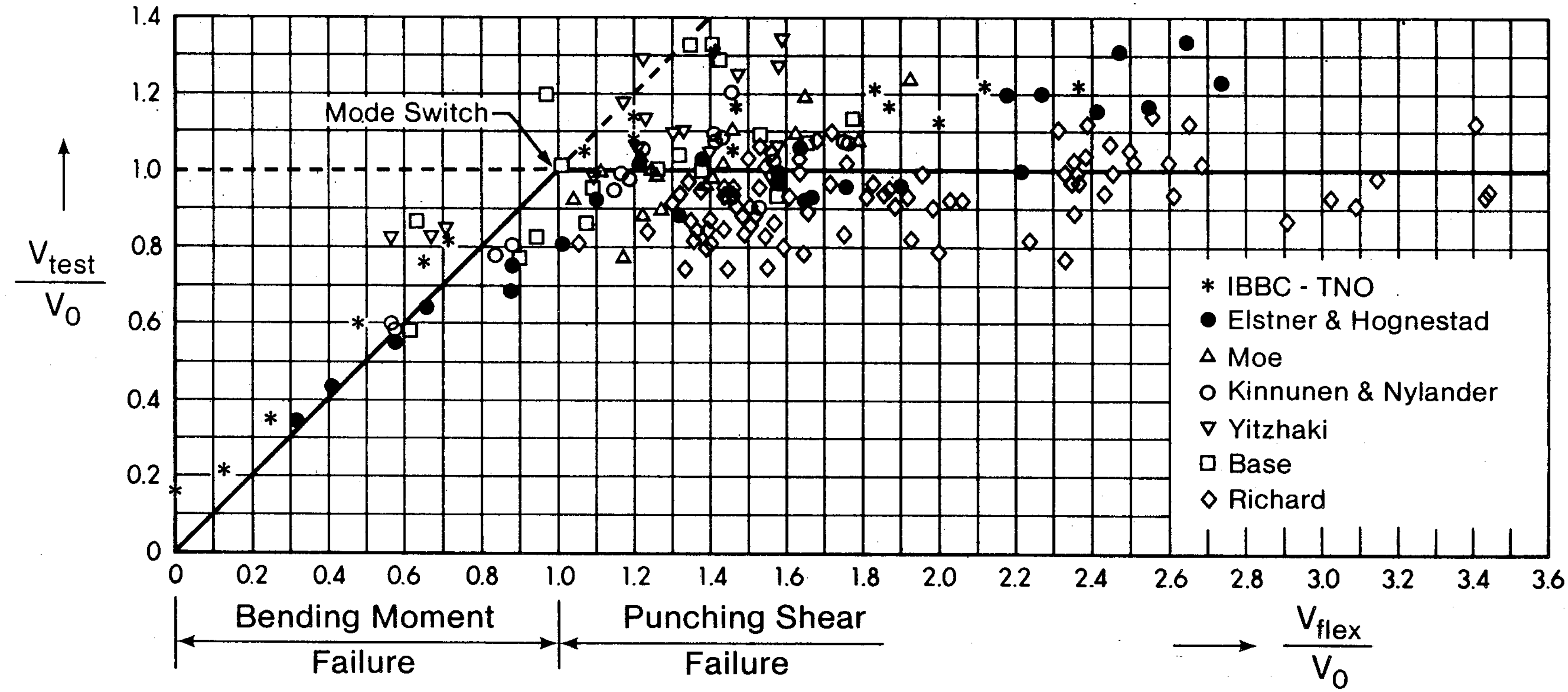


Figure 5.12 Comparison of Failure Modes for Punching Shear Tests

the test load is plotted against Q for a large number of punching shear tests. A test specimen will fail in a yield-line mechanism unless it fails by punching shear at a lower load. When a shear failure occurs, V_{flex}/V_{test} will be greater than 1.0. Gesund suggests on the basis of Fig. 5.13 that if Q is greater than about 4, the slab will fail in shear since most points with Q greater than 4 have V_{flex}/V_{test} greater than 1.0. He also suggests that when Q is less than about 2, the the slab will fail in flexure, and when Q is between 2 and 4, the failure will be combined shear and flexure. The rather large scatter of the data on Fig. 5.13 suggests that Q is not a very statistically strong parameter. It should be noted that Gesunds' yield-line analysis does not predict the yield-line collapse loads very well. This is clearly shown in Fig. 5.13 since many of the results had failure loads in excess of the calculated yield-line collapse load. These cases have V_{flex}/V_{test} significantly less than 1.0, while corrected predicted collapse loads would result in V_{flex}/V_{test} equal to exactly 1.0. This, together with the fact that Dragosavic and Van den Beukel had no similar difficulty predicting the flexural failure loads by yield-line theory, suggests that Gesund's analysis could be improved. In spite of this, Gesund should not be discounted since he still arrives at the correct conclusion, that is, that many of the punching shear tests which have been reported as shear failures when they were in fact flexural failures. Gesund (Goli and Gesund

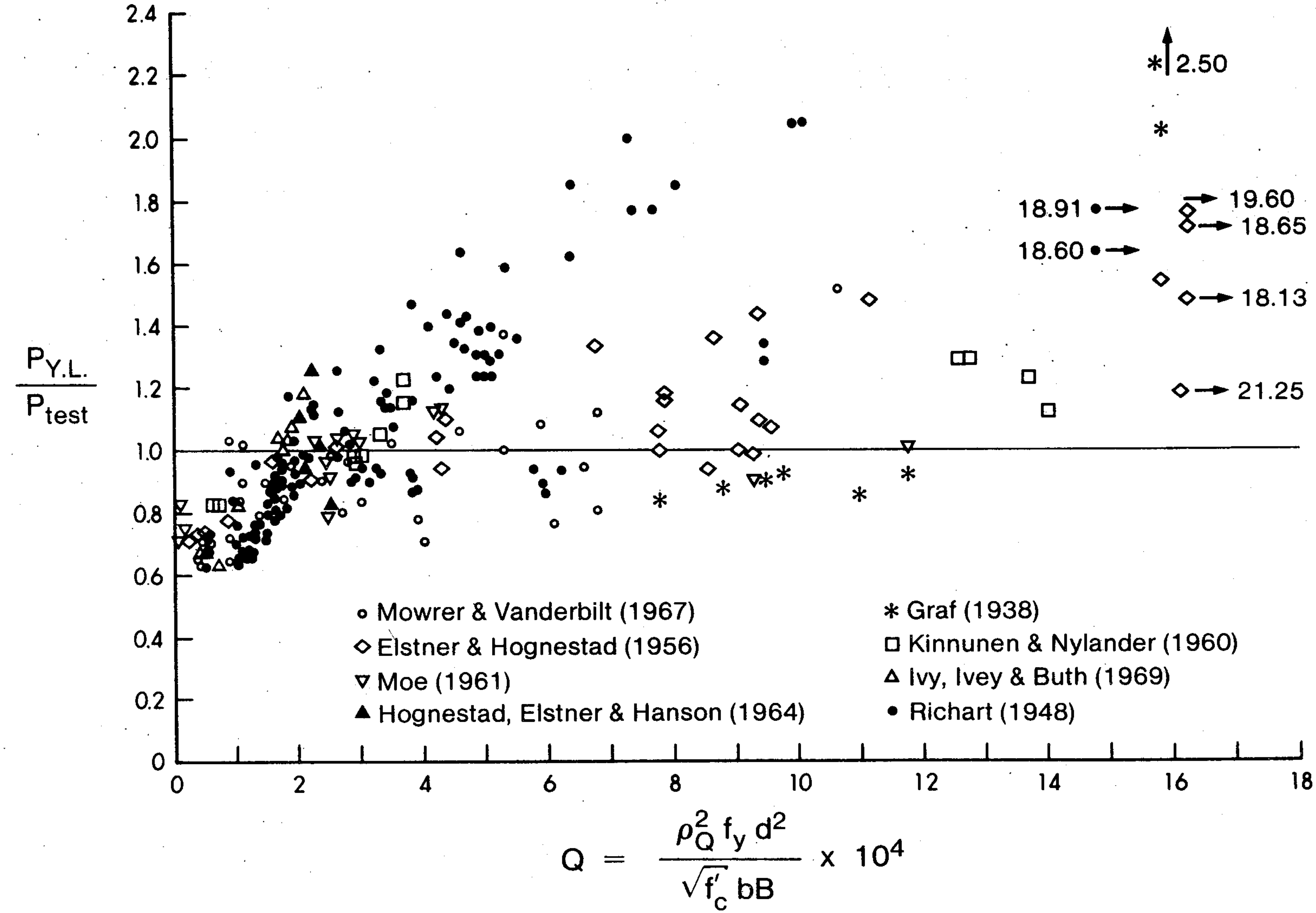


Figure 5.13 Differentiation Between Flexural Failures and Shear Failures with the Parameter Q

1979, Gesund 1979) has investigated interior, exterior and corner columns numerically, and has proposed a method of checking the slab-column connections for shear and moment transfer, recognizing that local flexural failures may occur. In view of the uncertainty in Gesund's analysis, and the complexity of his method for checking column connections, it will not be discussed further.

If one uses the yield-line approach for slab design, complete with a thorough investigation of conical mechanisms, the shear stresses acting on the critical section around a column will be small. Consider a flat plate with uniform square bays and circular columns, such that r/L is known. For a given uniform load, the total required (m+m') can be determined by Equation 3.4. For any particular concrete and steel strength, the minimum depth based on the maximum permitted steel ratio can be determined. This would require that m=m' in order to reduce this depth to an absolute minimum. In most practical cases m' is greater than m so the minimum thickness will be based on m' and will be greater than the absolute minimum thickness. In any event, the shear stress on the critical section at a distance d/2 from the column face can be determined. For a particular concrete and steel strength, the maximum shear stress is only a function of r/L and the value of the applied load w. Fig. 5.14 illustrates the maximum shear stresses which can be obtained of a slab designed "correctly" by yield-line theory. Curves for three different column sizes are shown.

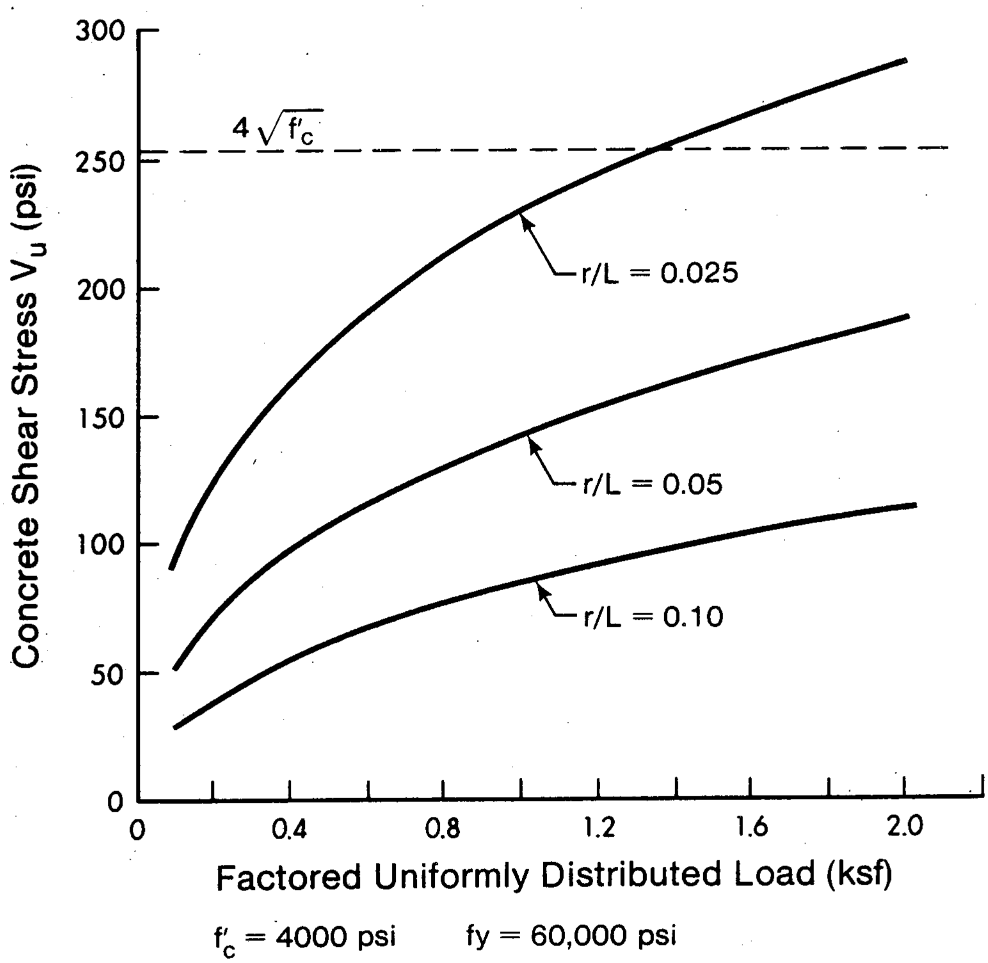


Figure 5.14 Theoretical Maximum Concrete Shear Stress

One starts with the factored uniformly distributed load, proceeds vertically to the applicable column curve and then horizontally to obtain the maximum possible shear stress which would result if the minimum depth was based on flexure. For typical loadings of 400 psf, the shear stresses around a very small column with r/L=0.025 is only about 150 psi. The limiting value of shear stress used by ACI, $4\ \sqrt{f'_c}$, is indicated by the horizontal dashed line. Clearly, for most practical cases of loading and column size, the shear stresses will never reach the limiting value. For very large distributed loads, as in the case of footings and raft foundations, the limiting shear stress may be reached if the columns are very small, however such small columns probably could not resist the large loading so that it is doubtful that one could ever reach the limiting shear stress in cases of uniform loading and spans. It should be noted again that Equation 3.4 requires larger design moments than most current designs provide for, thus minimum slab depths based on currently used moment fields will result in higher shear stresses. However, failures should be by concial yield-line mechanisms and not by punching shear.

5.5 Discussion and Comparison of Approaches

First, the elastic and plastic approaches will be discussed and compared since they both consider the phenomenon as involving shear in the concrete. The same data base has been used for Fig. 5.4 and Fig. 5.11 so that it is possible to compare the accuracy of the ACI approach and the plastic interaction equation. A comparison of these figures indicates that the scatter of the results is much less for the plastic interaction equation than it is for the ACI equations. The test results which fall to the right of the theoretical interaction line are predicted much better by the plastic interaction equation than the ACI equations while the same is true for the points which fall to the left of the interaction line, but to a lesser degree.

Most of the tests which plot on the unconservative side (below interaction line) of Figs. 5.4 and 5.11 are those reported by Hawkins. The results of 17 of these tests are summarized in Table 5.1. A test where the (test capacity/calculated capacity) is equal to 1.0 would fall on the interaction line. Results below 1.0 are unsafe and will plot to the left of the interaction line. The results shown in Table 5.1 speak for themselves. The beam analogy predicts the failure load more closely and with less variation than the other two approaches. However, the beam analogy has the greatest number of cases where the failure load is under estimated. It is true that two of the values are very close to 1.0 which would mean that ACI and the beam analogy would both have four unsafe cases. But by the same arguement, the plastic interaction equation would only give one unsafe result. The results of this particular test are underlined for all the methods considered. Clearly all of the approaches have difficulty predicting the failure load of

Table 5.1 Re-evaluation of Hawkins' Test

	Test Capacity / Calculated Capacity		Values of Test Capacity / Calculated Capacity < 1.0
	Mean	C.O.V.	
ACI 318 - 71	1.09	0.16	0.96 0.93 0.90 0.77
Beam Analogy	1.08	0.13	0.99 0.92 0.88 0.90 0.84 0.99
Plastic Interaction Equation	1.13	0.14	0.99 0.97 0.97 0.80

this test. This test was reported as a shear failure, but at the time the load dropped off, the slab deflection was 1.9" in a span of 7'. This test should have been excluded from the data since it probably was a flexurally induced failure. As an aside, Gesund would have classified all 17 of the tests as flexural failures since the maximum Q value in the test series was 1.17.

It is clear that the plastic interaction equation is the best method for predicting the lower-bound to the test data. It should be pointed out that there is a slight difference in the plastic interaction equations used for Fig. 5.11 and Table 5.1. The results in Table 5.1 neglect the moment transfer capacity of the bottom reinforcement as suggested in Fig. 5.9d for cases with large shear and low moment. On the other hand, Kanoh and Yoskizaki who are responsible for Fig. 5.11 did include the capacity of the bottom reinforcement in the calculation of M_f.

The yield-line treatment of the slab-column junction requires further development. Dragosavic and Van den Beukel have derived equations for some special cases of shear and moment transfer, and as stated previously, Gesund has developed a method for analysing connections with the use of some computer generated charts. The yield-line approach can not be directly compared to the other methods for shear and moment transfer. This method does, however, shed some doubt on the data base used for the establishment of limiting values of shear stress used in the other methods.

Specifically, it would indicate that the lower limit of the test data reported as shear failures were probably flexural failures. This conclusion is reinforced by Fig. 5.15 (Criswell 1970) which illustrates the load deflection curves for a series of tests which reported all failures as shear failures. Clearly, the tests which failed at the lower loads were flexural failures. This is not untypical of many of the test series reported in the literature. Since the limiting value of the shear stress used by ACI was based on the lower limit of the test results, it would appear to be based on flexural failures. One should re-evaluate the data base upon which the limiting values were determined, and filter out all of the test results which were flexural failures in order to determine what the limiting value of the concrete shear stress really is. A check against flexural failures should then be made with a flexural analysis, instead of hoping to account for this behavior by using a reduced permissible shear stress.

The type of test specimens used in the investigation of punching shear becomes an important consideration. Most tests have been conducted with a single column and a relatively small portion of slab. The intention is to model the slab-column joint to approximately the points of inflection. In these specimens, once yield starts, the moments cannot redistribute to other portions of the structure, so the plastic deformations increase until they are so gross that the concrete finally fails. Even in

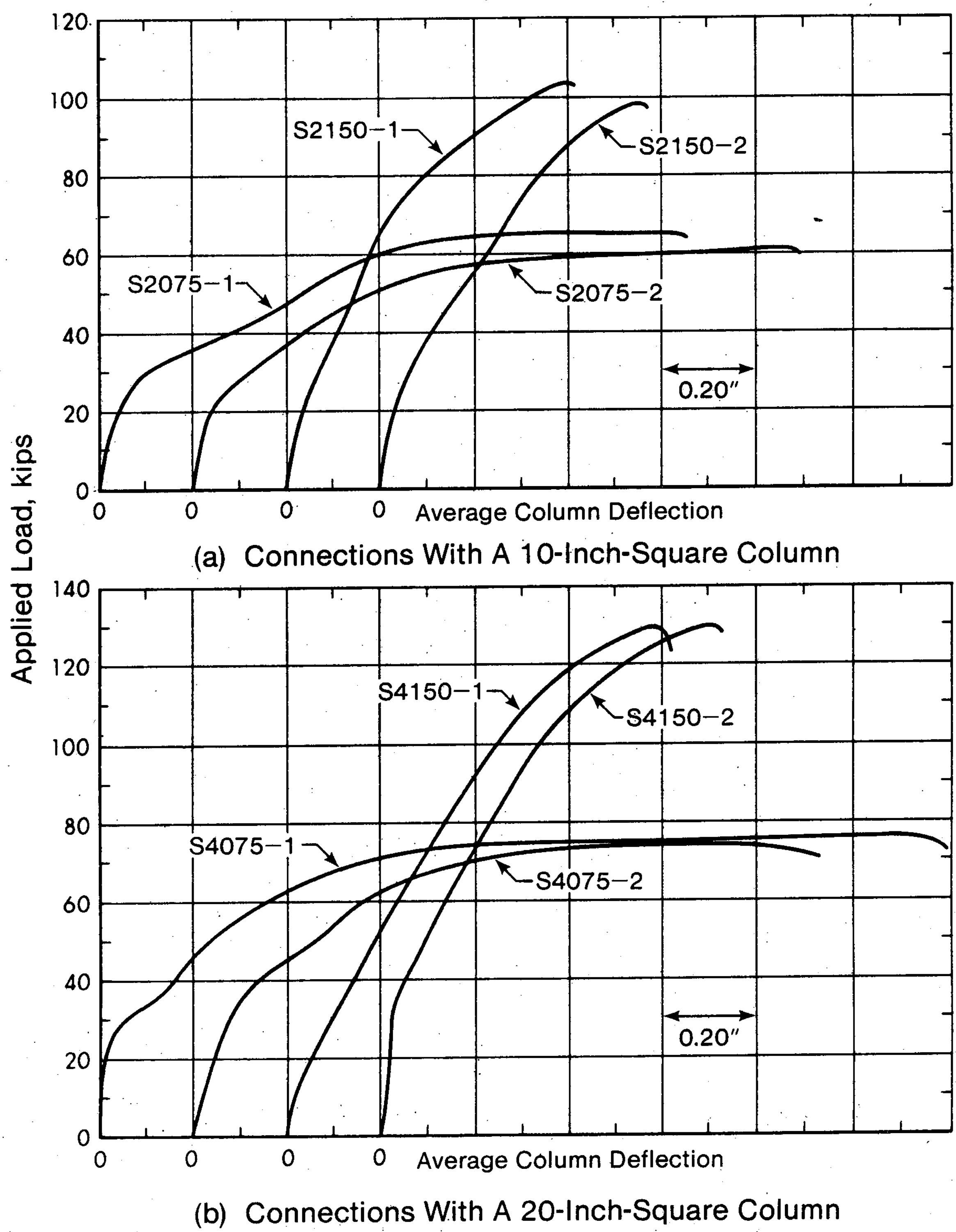


Figure 5.15 Load Deflection Curves for Punching Shear Tests

structures designed by yield-line theory, the required rotations would not be as large as in these tests, so one would expect that the shear and moment transfer capacity with realistic amounts of plastic rotation would be greater than the test specimens usually indicate. It should also be noted that the test specimens do not have the benefit of slab jamming against the surrounding slabs. More realistic data would be obtained by specimens with at least two columns so that the plastic rotations would be representative of those required in a real structure.

6. PATTERN LOADING

6.1 Introduction

The discussion to this point has dealt with one set of loads only, however most real structures are subject to some degree of pattern loading. Pattern loading is an important consideration since it will directly effect column moments, maximum positive moments, maximum negative moments and the location of the points of inflection. The points of inflection will in turn govern the length of the top bars (but not the width of the top mat). This is a simple but important detail which can effect significantly the economy as well as the safety of slabs. It is not intended that this discussion should conclude what loading patterns slabs should be designed for but rather, once the pattern loading is specified, how one would go about accounting for it with a plastic approach to slab design. The intent is to present various philosophical approaches to the problem and not necessarily detailed universal design equations.

The approaches discussed will make use of the principle of superposition which is valid for lower-bound moment fields, but not strictly valid for upper-bound solutions. However, as discussed in Section 2.3.2, superposition will result in a solution on the conservative side for upper-bound moment fields. As a result of this fact, the methods of handling pattern loadings which are discussed are generally applicable to both upper and lower-bound solutions.

Each different load pattern could be investigated as a separate load case with the design moment field being the moment field envelope of all the cases. This is often done in elastic analysis, but for a plastic analysis, this would be very complicated and time consuming. This analysis need not be done for slabs with simple geometry since plastic formulas have been derived to account for pattern loading. Unfortunately, for cases with complex or irregular geometry there are no closed form plastic solutions and a full plastic analysis of each load case is required.

Investigations involving elastic analysis of slabs subjected to various types of pattern loading have concluded that checkerboard loading is not as critical as strip loading (Jirsa et al 1969). Strip loading involves loading bays across the entire width of the structure. The analysis considers this type of pattern load and thus takes on a two dimensional form, very similar to that of a continuous beam. The case of two adjacent spans loaded will only govern the maximum negative moments, but these moments are limited to the plastic slab moment capacity so they are of little concern, and only alternate bay loading will be considered.

6.2 Degree of Pattern Loading

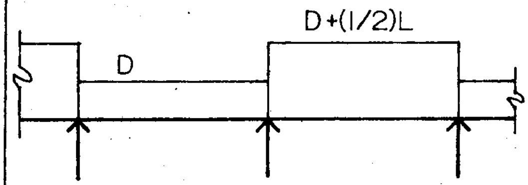
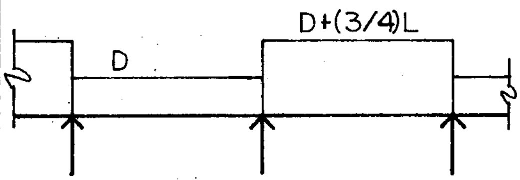
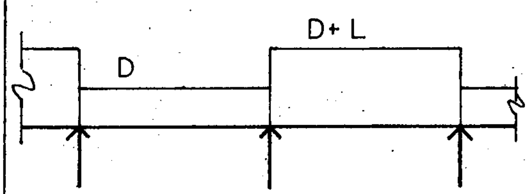
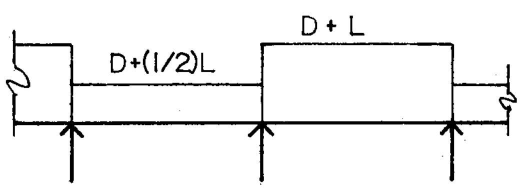
The degree of pattern loading can be described by:

$$r_q = \frac{\text{Movable Load}}{\text{Permanent Load}} \tag{6.1}$$

The "permanent load" is that portion of the load per unit length of span which remains on all of the spans. This would be the dead load plus perhaps some portion of the live load. The "movable load" per unit length of span is the fraction of the remaining portion of the live load which is applied to alternate spans. The loads under consideration are of course ultimate (factored) loads. The paramter r_q reflects not only the type of pattern loading, but also the effects of live to dead load ratio. Besides being a mathematically convenient parameter(Jofriet and McNeice 1971), it also gives some indication of how much the points of inflection differ from those obtained under uniform loading. Table 6.1 gives values of r_q for various load patterns and various live to dead load ratios.

The pattern loadings to be considered should be obtained from applicable building codes. The ACI 318-77 as well as CSA 23.1 are somewhat inconsistent in the pattern loading for which the slab should be designed. For example, ACI 318-77 Section 9.2.2 suggests that pattern (3) of Table 2.1 be used. ie. (D+L):(D), while Section 13.6.9.2 suggests that pattern (1) of Table 2.1 be used ie.

Table 6.1 Degree of Pattern Loading

No.	Load Pattern	L/D (service)	rq*
1		0.5 1.0 1.5 2.0 3.0	0.304 0.607 0.911 1.214 1.821
2		0.5 1.0 1.5 2.0 3.0	0.455 0.911 1.366 1.821 2.732
3		0.5 1.0 1.5 2.0 3.0	0.607 1.214 1.821 2.429 3.643
4		0.5 1.0 1.5 2.0 3.0	0.233 0.378 0.477 0.548 0.646

* rq has built into it a dead load factor of 1.4 and a live load factor of 1.7

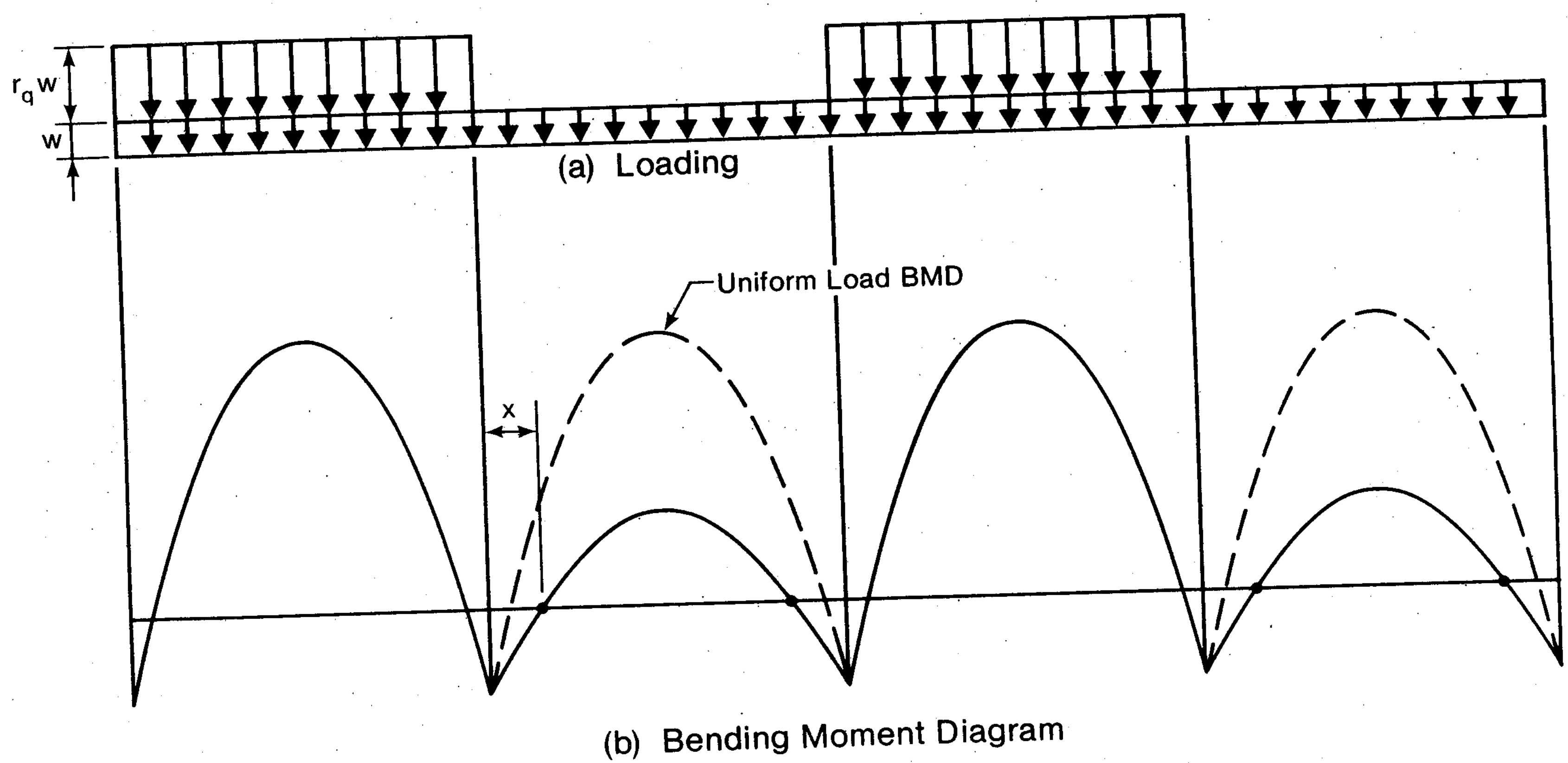
(D+0.5L):(D), and finally Section 13.7.6 suggests that pattern (2) of Table 2.1 be used ie. (D+0.75L):(D). Designers need better guidance than this on which to base rational slab designs. The specified pattern loads should take due regard of the likelihood or probability of various pattern loading cases.

6.3 Possible Plastic Solutions

The method used for the following solutions is based on the formation of plastic hinges at the face of the supports and at midspan, along with simple equilibrium. The only unknowns will be column moments and the location of the points of inflection. Once one is known or assumed the other may be obtained from simple statics. The column moment will of course always be the moment required to keep the beam end moments in equilibrium. However, the design moment for the column itself should also include the effect of the eccentricity of the end shear in the slab acting at the face of the column. A plastic analysis also requires that the value of ψ needs to be known in order to determine the positive and negative moments.

Four possible solutions have been investigated. They are:

1. Set the column moments equal to zero and determine the location of the points of inflection based on an end moment equal to that used had all the spans been loaded with permanent and movable load. See Fig. 6.1. This



$M_{col} = 0$

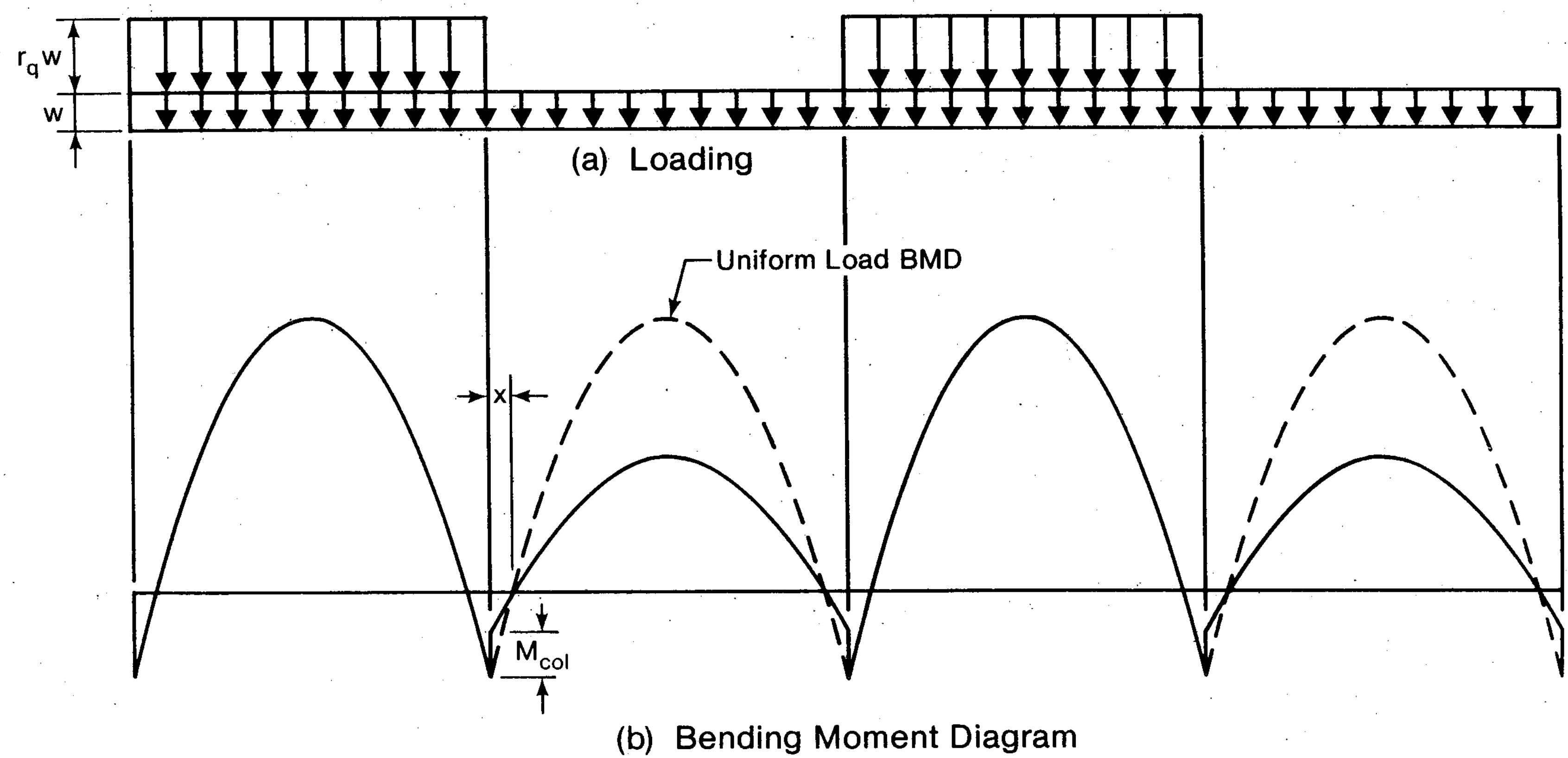
$$x = \frac{\ell}{2}\left[1 - \sqrt{1 - \left(\frac{\psi}{\psi + 1}\right)(1 + r_q)}\right]$$

Figure 6.1 Cut-Off Points for Pattern Loading with Zero Column Moments approach mimimizes the column moments, but increases the length of the top bars.

2. Set the points of inflection at the location they would assume under uniform loading and determine the column moments required to maintain equilibrium at the support joint. See Fig. 6.2. This approach minimizes the length of top bars required.

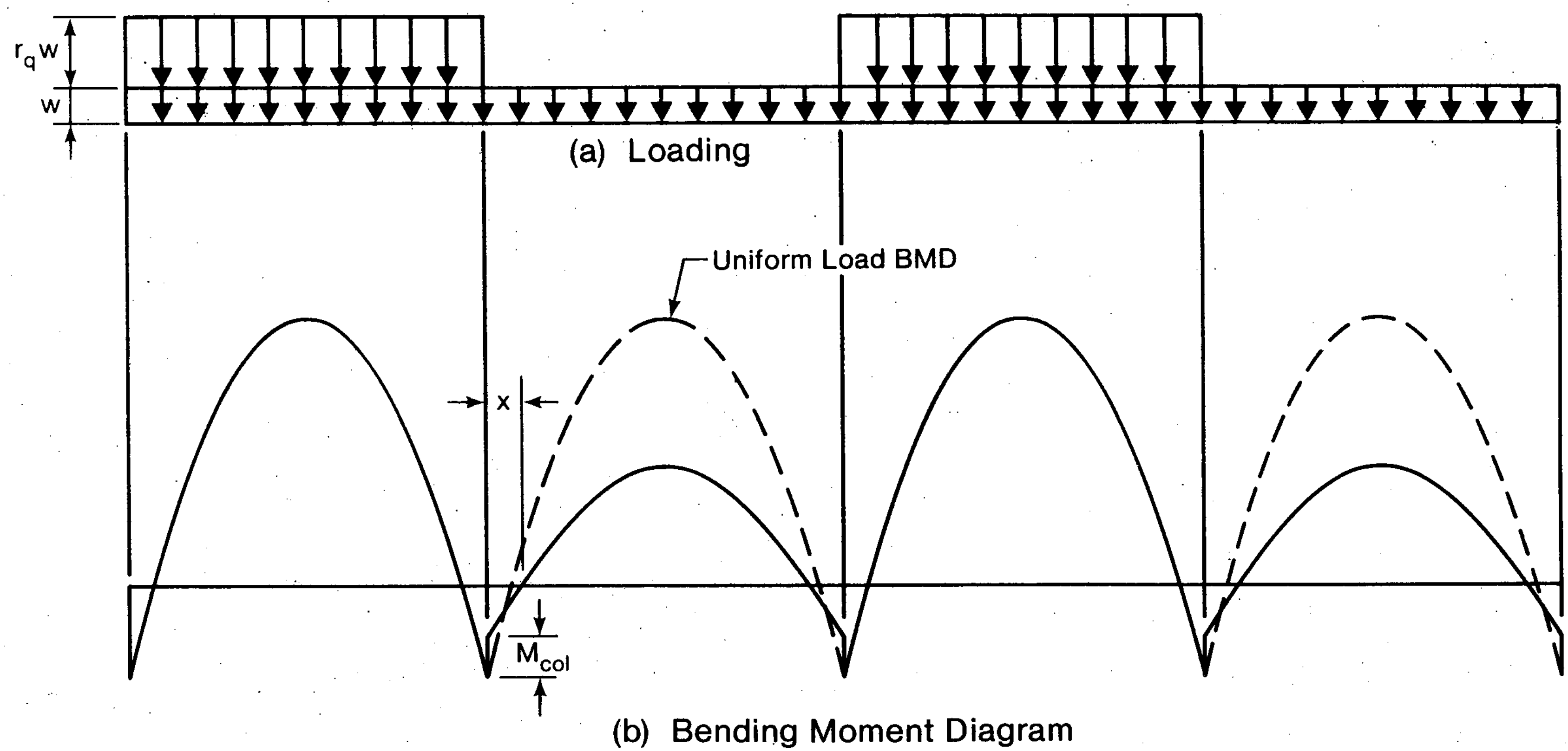
3. Set the points of inflection at some arbitrary point (but at least as far from the column as the points of inflection are under uniform load), and determine the column moments required for equilibrium. See Fig. 6.3. This is a more versatile approach than above since it permits one to reduce the column moments by increasing the length of the top bars.

4. Set the column moments equal to zero, use the same locations for the points of inflection as under uniform load, and set the end moments equal to the end moment in the unloaded span. The maximum midspan moment required for equilibrium in the loaded span can then be determined with simple statics. See Fig. 6.4. This permits one to account for the effects of pattern loading by the addition of some amount of bottom steel. The amount of extra reinforcement required will depend upon ψ, the type of pattern loading and the ratio of live to dead load. For example with $\psi = 1.0$, L/D=0.5 and pattern loading case (1), no extra bottom steel is required. However, for a more realistic ψ



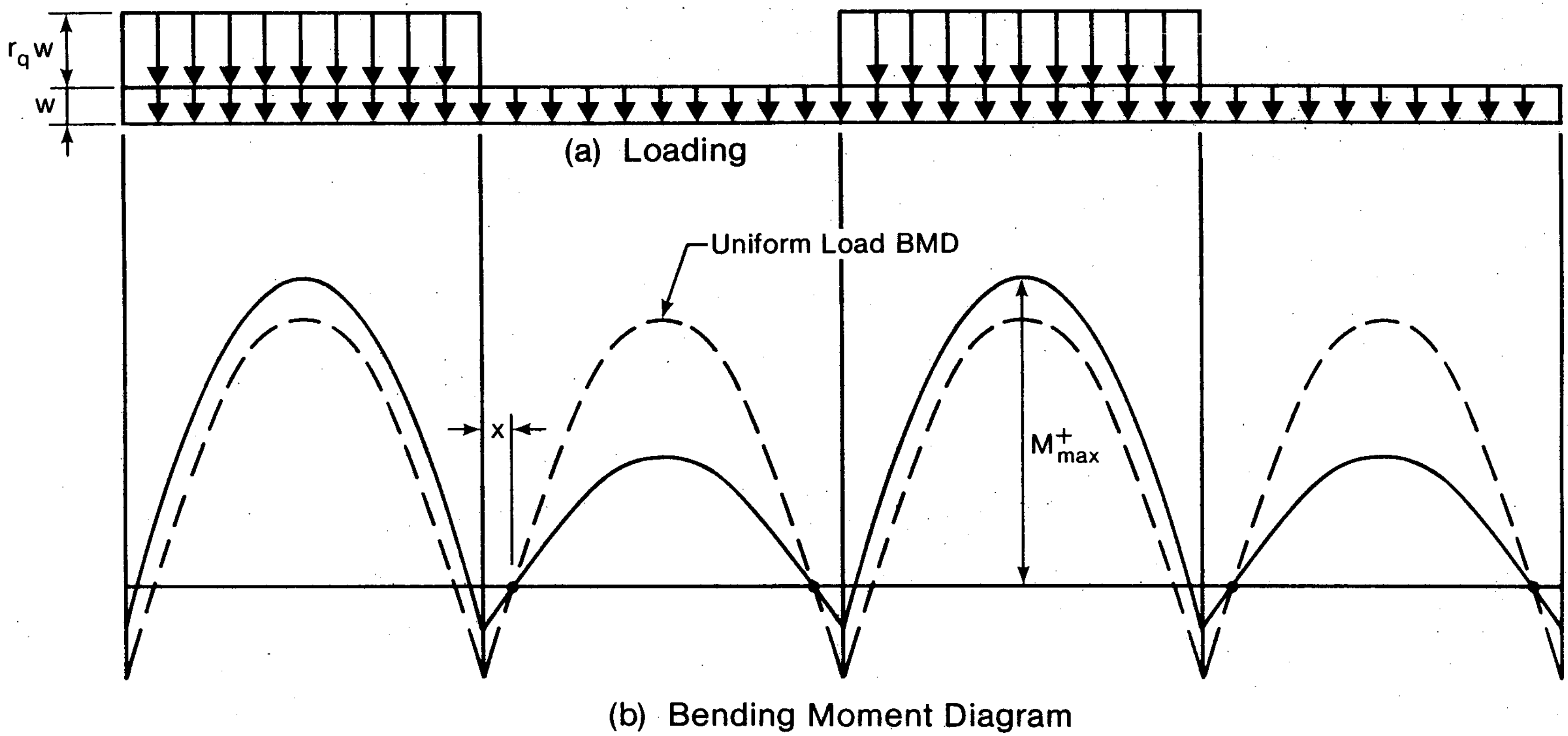
$$M_{col} = \left[\left(\frac{\psi}{\psi + 1}\right) r_q\right] \frac{w\ell^2}{8} \qquad x = \frac{\ell}{2}\left[1 - \sqrt{1 - \left(\frac{\psi}{\psi + 1}\right)}\right]$$

Figure 6.2 Column Moments for Pattern Loading with Non-Moving Points of Inflection



$$M_{col} = \frac{w x^2}{2} - \frac{w \ell x}{2} + \left(\frac{\psi}{\psi + 1}\right)(1 + r_q)\frac{w \ell^2}{8} \qquad x \text{ is arbitrary but } x \geq \frac{\ell}{2}\left[1 - \sqrt{1 - \left(\frac{\psi}{\psi + 1}\right)}\right]$$

Figure 6.3 Column Moments for Pattern Loading with Arbitrary Cut-Off Points



$$M_{col} = 0 \qquad M^+_{max} = \frac{w\ell^2(1 + r_q + r_q\psi)}{8(\psi + 1)} \qquad x = \frac{\ell}{2}\left[1 - \sqrt{1 - \left(\frac{\psi}{\psi + 1}\right)}\right]$$

Figure 6.4 Positive Panel Moments for Pattern Loading with Zero Column Moments of about 1.8, extra steel will always be required.

6.4 Elastic Solutions

Some of the possible solutions just discussed require a reasonable choice for the location of the points of inflection, it would therefore be of some benefit to examine the location of the points of inflection as obtained from elastic analysis. The references on pattern loading make no reference to the points of inflection so an analysis was conducted by means of indirect stiffness (slope deflection). Columns were assumed above and below the slab and were assumed to have various relative stiffnesses α_{ec}. The location of the points of inflection are then only a function of the relative column stiffness α_{ec} and the degree of pattern loading r_q. By modeling the system as a plane frame, one assumes that the points of inflection are in a straight line, whereas the location of the line of contraflexure varies across the width of the panel. The analysis used gives the "average" location of the point of inflection, since it will satisfy static equalibrium for the panel as a whole.

The results of the analysis for equal spans are shown in Fig. 6.5, while Fig. 6.6 shows the results of an analysis with alternating short and long spans with the short span = 2/3 of the long span. On these figures one starts with the known r_q, proceeds horizontally to the known α_{ec}, and then drops vertically to obtain x/L the location of

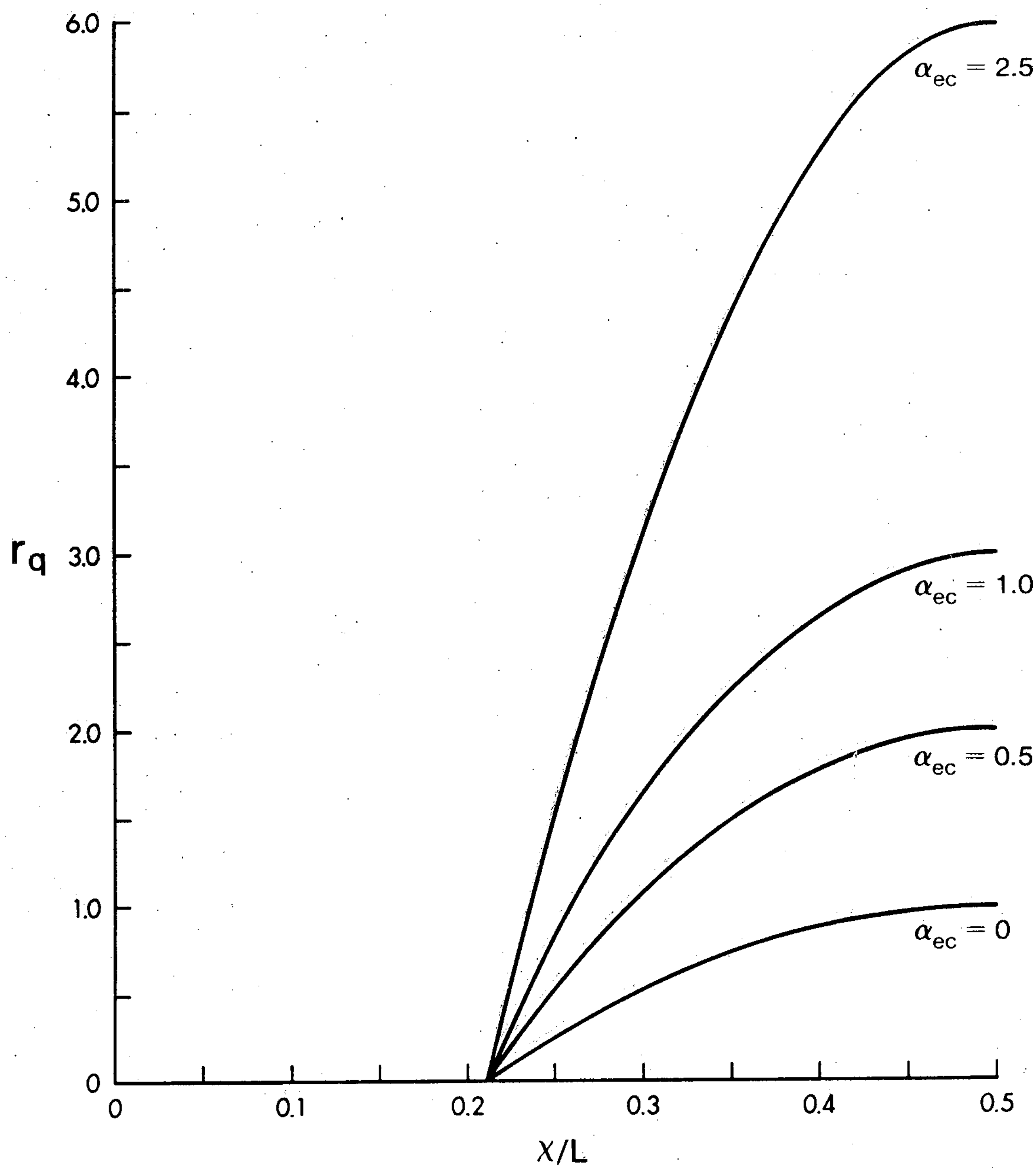


Figure 6.5 Effect of Pattern Loading and Column Stiffness on Points of Inflection

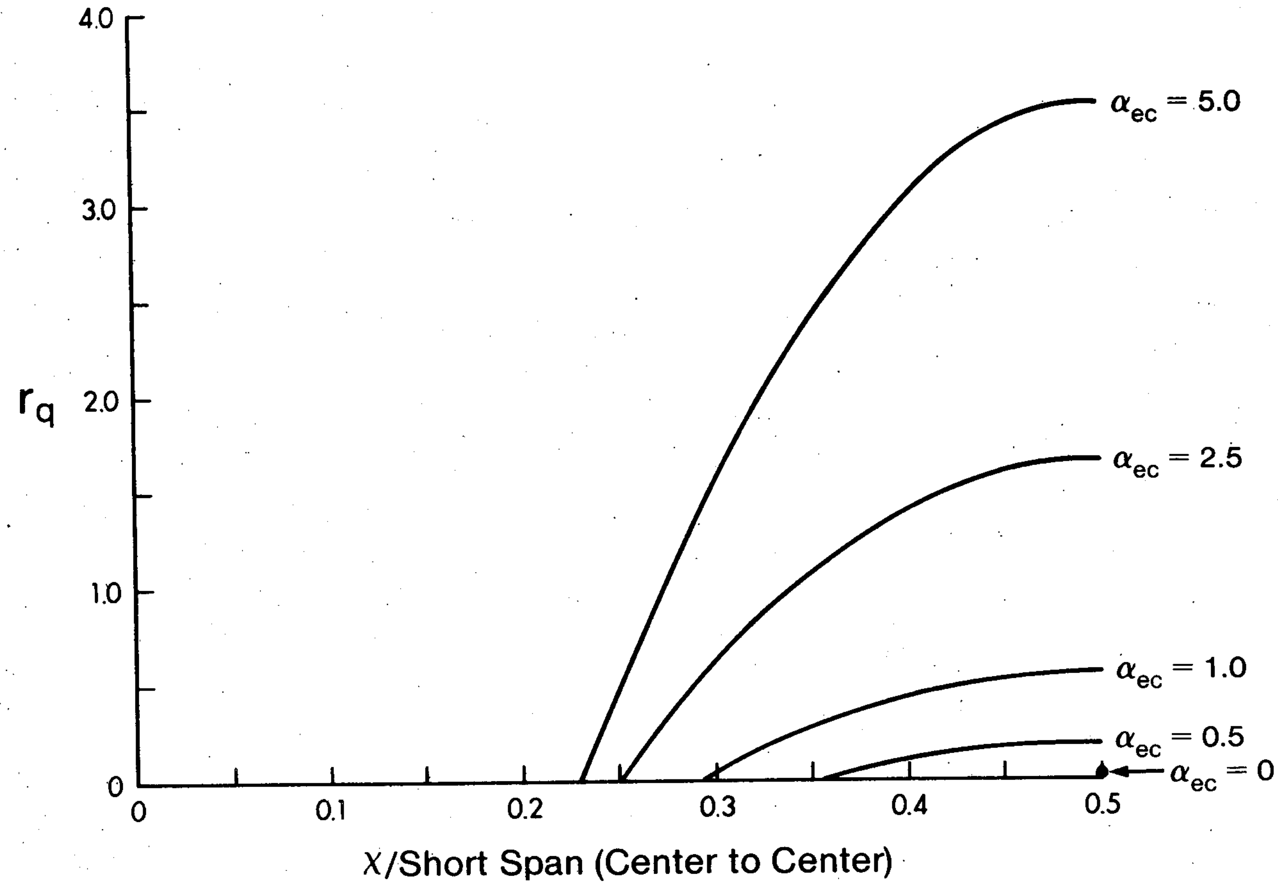


Figure 6.6 Effect of Pattern Loading and Column Stiffness on Points of Inflection with Short Span = 2/3 Long Span (Alternating)

the point of inflection which is measured from the column centre line and expressed as a faction of the centre-to-centre span.

Using Fig. 6.5, one can examine some typical cases. Typically α_{ec} varies between 0.5 and 1.25 and r_q is approximately equal to 1.0, thus X/L typically varies between 0.25 to 0.30. On the other hand, for the extreme limits of the Direct Design Method, (L=3D) with pattern load (D):(D+L), r_q = 3.64. With this value of r_q and equal spans, Fig. 6.5 indicates that negative moments will exist across the entire span for α_{ec} less than 1.25. For the case of unequal spans, Fig. 6.6 indicates that negative moments will exist across the entire short span for α_{ec} less than about 5. ACI detailing practice however is to stop the top bars at about X/L= 0.32. Thus, according to elastic analysis, ACI detailing requirements are probably safe for typical cases, but would be unsafe for cases approaching the limits for the use of the Direct Design Method.

7. DISCUSSION AND CONCLUSIONS

7.1 Flexure-General

All of the methods discussed in this work are based upon plasticity. At least some of the concepts presented are used daily in the design of slabs. Even if an elastic analysis were carried out, the elastic moment field would be redistributed across the width of the slab panels so that in various strips, the reinforcement will be of uniform spacing. The degree to which a designer makes use of plasticity is a matter of personal preferance but it will be influenced to some extent by serviceability requirements.

While all of the methods considered are general enough to be all-around design tools, some are better suited to some types of problems than others. Yield-line theory for example is the only "practical" method of analysis for slabs, while the other methods are only suitable for slab design. No one method is superior over all the others. One should use the method which is best for the given circumstances, or even a hybrid combination of methods for various portions of a slab design.

There is no real saving in reinforcement with any one method of design. Generally, the amount of reinforcement required will be at least as much as or slightly more than that required by the Direct Design Method. All the methods either require or benefit greatly from bottom reinforcement immediately around the column.

A few very general points must be considered. The first is that these design methods are only applicable to braced structures. The second point is that the choice of column moments is "arbitrary". It is possible to have many "correct" designs; one should however, try to approximate the elastic moment field in order to improve serviceability. Given a reasonable amount of engineering experience, all of the methods can be used by a designer to produce satisfactory designs. In view of Fig. 5.1, there would appear to be a rather wide range of suitable moment fields to choose from. This makes it relatively easy to produce a safe slab design without the use of a computer.

7.2 Upper-Bound Solutions

There is a great deal of discussion in the literature dealing with yield-line theory and slab analysis. There is however, very little discussion of yield-line theory and slab design. The computations required for design are greatly simplified if yield-line locations are set at the beginning of the design. Suitable positive slab moments are then chosen. Finally, the negative moments required to keep each of the segments in equilibrium are determined. Since the negative moments occur at supports, any difference in negative moments between adjoining segments can be assigned to the support. The location of the yield-lines originally chosen in the first step will then be the mathematically correct locations for that particular family at yield-line mechanisms. The significance of this result cannot be over emphasized since not only is the large amount of work required to optimize this yield-line family eliminated, but the critical yield-line pattern within the family is known with absolute certainty.

The design approach just suggested can be used to advantage if the yield-line family considered is the family which will govern the particular slab in question. For a flat plate on a regular column layout with square or rectangular top mats, the conical yield-line families should govern as shown in Table 3.1. However, this slab is typical of those designed by the Direct Design Method which only considers parallel yield-line mechanisms. It can be argued that on the basis of slab jamming, the strength of the conical yield-line mechanisms is enhanced to the point where these mechanisms are no longer the governing family of mechanisms. The satisfactory performance of slab designs which ignores the possibility of conical mechanisms suggests that where slab jamming can occur, the governing yield-line family will be that of the parallel mechanisms which extend across the entire width of the structure. It should be pointed out that while conical yield-line failures are generally not reported in test results, it does not mean that they do not exist. They are often either mistaken for, or cause punching shear failures so that the reported cause of failure is a punching shear failure.

Since the Direct Design Method is essentially a special case of yield-line theory with elastic embelishments, much of what has been said applies to the Direct Design Method. In particular, the Direct Design Method is satisfactory with respect to the design moments, cut-off points and column moments for the parallel yield-line mechanisms. However, the top mats are neither long enough nor do they have enough moment capacity to resist conical yield-line mechanisms.

Weisinger's segment equilibrium method is clearly a special case of yield-line theory and has been presented in order to illustrate the capabilities and usefulness of yield-line theory for the design of slabs. Much can be learned from a study of Weisinger's designs. Perhaps the most important point is the role played by banded reinforcement and the distribution of positive and negative moments in the serviceability of slabs. The use of yield-line theory as it has been discussed here only provides an indication of the total moment required along a yield-line. The moments and thus the reinforcement should be distributed along the yield-line approximately in accordance with elastic theory. Of course there is the possibility that yield-lines would occur somewhere within the slab segments between the assigned yield-lines, but then this possibility also exists with homogeneous isotropic reinforcement and is really just another family of yield-line mechanisms. Thus, banding of the reinforcement should not present any real safety problem if the slab was designed for the governing yield-line family in the first place.

7.3 Lower-Bound Solutions

The discussion of lower-bound design methods has, by necessity, been brief and somewhat incomplete. However, once the concepts used by Hillerborg are understood, they can easily be extended to other cases. Hillerborg's work includes the consideration of supports of finite area, and preset moment fields for many different types of segments and loadings such as: point loads within a corner supported segment; line loads and linearly varying distributed loads; segments supported on two or three adjacent sides; slabs with re-enterent corners; triangular segments; etc. It is clear that the development of lower-bound solutions has far outstripped their application in everday design, and that further discussion and clarification will be required before these techniques are assimilated by the design profession.

Chapter 4 has demonstrated that there are basically two different approaches: the strip method and the segment method which differ significantly in their application. The concept of using a previously developed lower-bound moment field, known as a "preset" moment field, is applicable to both the load dispersion elements which are combined with slab strips and slab segments or elements which carry the load directly to supports. While the strip method is more easily understood, it is not as useful or powerful a method as the segment method. For flat plates, the segment method is particularly versatile but it relies on the availability of suitable preset moment fields such that the maximum moments occur only along the boundaries of the slab segments. There is a great deal of potential for the development of more suitable and more easily understood preset moment fields with the use of the finite element method or other numerical methods. Such solutions would greatly aid the acceptance of the segment method by the design profession. For practical design however, the main condition is that equilibrium is fulfilled for the segments as a whole, and that the lateral distribution of moments chosen is reasonable.

7.4 Shear and Moment Transfer

It has been illustrated via Fig. 5.1 that slabs will generally carry almost any amount of load if the loads can be transferred to the columns. The slab column junction is often the most critical element in a slab design as far as the ultimate failure load is concerned. The interaction of shear and moment must be considered.

Of the various approaches to this problem which were considered, the plastic interaction equation would appear to be the best approach. For the simple cases considered, it is no more difficult and perhaps even easier to apply than the approximate elastic method used by ACI. There remains much work to be done on the plastic interaction equations before they can be applied as a general procedure. In particular, methods of handling round columns, highly rectangular columns, shear heads, shear reinforcement, prestressing effects, and openings near columns must be developed. The conceptual model used in the plastic interaction equation approach is so simple that practical solutions to these cases will no doubt soon be developed.

The punching shear test results indicate that even the plastic interaction equation can not predict conical yield-line failures around the columns. Further testing of specimens with at least two columns are required in order to model the behavior of complete structures and determine how the redistribution of moments effects the slab shear capacity. These tests would also indicate whether there is a need to specifically design for conical yield-line mechanisms.

7.5 Pattern Loading and Column Moments

Chapter 5 has discussed how one would go about ensuring that a particular slab design is safe for a specified loading pattern. The emphasis was on cut-off points for the top bars along with the design moments for the columns. This in turn could have a great effect on the shear and moment transfer problem in the slab column junction. This has not really been discussed since for cases where pattern loading produces significant column moments, the shear transferred to the column is reduced. That is, since some of the load is removed from the slab, in order to produce the pattern loading, the total load carried by the slab is reduced. Further research is required in order to determine the pattern loading which produces the worst probable combination of shear and moment to be transferred to the column. This must be based on statistical studies which consider the independence or lack of independence of loads on adjacent spans. This may in fact, depend upon the use or occupancy of the structure. In any event, until the proper pattern loads have been determined, it is safe to design all slabs for loading pattern 3 which is shown in Table 6.2. One should however, also check the plastic interaction equation for the case of full uniform load. It should be pointed out that many different cases can be checked easily with the plastic interaction equation since V_o and M_o need to be calculated only once. Each case of V and M can be quickly substituted into Equation 4.1.

It has been pointed out by Beeby (1980) that on the basis of probability studies, the simple detailing requirements found in British concrete design codes were satisfactory. Their codes, like ACI, have top bars which are too short according to the elastic moment envelope resulting from pattern loading. Beeby found that by assuming independence of live load on adjacent panels, the probable degree of pattern loading which results requires steel only slightly in excess of the current simplified detailing requirements. He also brings up a number of factors which increase the actual slab capacity but which are usually neglected in the design calculations. These include slab jamming as previously discussed, and tension stiffening due to the tensile strength of the concrete. He also considers several other factors, the most significant being the increase in the reliability of the tensile strength of the reinforcement when many bars are used. That is, the more bars which cross a yield-line, the greater will be the probable average yield-line capacity. Because of these neglected phenomenon which increase the slab strength, Beeby concluded that it would be safe to design a slab on the basis of only the uniform load case. He does not give any indication of what to do about column moments. It should be pointed out that Beeby arrived at his conclusions based on a slab supported by columns of zero flexural stiffness. For slabs with columns of finite stiffness, the effects of pattern loading on the detailing requirements of the slab steel will be even less critical. However, the question of probable column moments must be dealt with.

Contrary to Beeby, it is recommended that pattern loading be considered explicitly. If the columns are such that they will carry no moment, the pattern loading should be dealt with by method 1 or method 4 from Chapter 5. On the other hand, if the columns can resist moment, method 2 or method 3 should be used when considering pattern loading. This will ensure that the column and slab-column connection will have adequate strength. It can be argued that the analysis proposed in Chapter 5 is too simplistic since moments may leak around the column from one span to the next. However, in view of Beeby's paper, this can hardly make the proposed procedure unsafe.

The column moment used in the Direct Design Method is approximately in accordance with methods 2 and 3 given in Chapter 5. The column moment required is essentially that of method 2, except that the column moment is divided by $1 + \frac{1}{\alpha_{ec}}$. For infinitely stiff columns α_{ec} approaches infinity and the column moment required approaches that required by method 2. For relatively stiff columns, the column moment is reduced only slightly so that the length of the top bars must be increased slightly. This is done in the standard detailing practice by extending 50% of the top steel to 30% of the clear span from the column face so that the design satisfies method 3. However, for flexible columns, the column moment is substantially decreased so that the standard detailing requirements will not satisfy any of the methods in Chapter 5. Unfortunately, this is the case for many commonly used values of α_{ec} . Thus, the detailing requirements in the Direct Design Method cannot always be fully justified on the basis of plastic analysis. One must therefore look to the factors which Beeby considered in order to justify the current detailing requirements. An extension of Beeby's analysis to include columns with non-zero stiffness would be most interesting since it would shed some light on the probable combinations of column load and moment as well as on the location of the cut-off points.

The discussion thus far has been addressed mainly to interior columns. For exterior columns there is little doubt that there will be a column moment. The problem lies in the choice of a suitable column moment. This problem does not have a simple answer. Present design methods utilize column moments smaller than those obtained from an elastic analysis without any ill effect. The following discussion will not include firm recommendations but rather an indication of some of the factors which should be considered when choosing column moments.

In many instances, the column type or size will indicate the nature of the column moments one should use. The column size may be determined by architectural considerations or the axial design loads. Large columns will tend to be stiff and should be assigned relatively large moments. The converse would be true for small columns. For slabs supported on steel pipe columns as in many of Wiesinger's designs, the column moment would be set equal to zero.

In some instances it may be desirable to set the column moment equal to that resulting from the minimum eccentricity requirement for the column design. This could be considered a "free" moment since there is no extra cost in terms of the column design.

In some cases it may be desirable to use small column moments in order to minimize the problems associated with shear and moment transfer at the slab-column joint. This is particularly true at exterior columns.

When in doubt about what is a reasonable column moment, one is safer with a choice which is too large whan one which is too small. A choice of large column moments will produce a design in which yielding of the slab will occur before sudden failure of the column or slab-column junction. The inherent ductility of the slab will provide adequate warning in an overload situation, but there may be serviceability problems if large column rotations are required in order to develop the assigned moments.

The choice of column moment is somewhat self-fulfilling. If one chooses a large column moment, it will require the design of a large, stiff column. This stiff column will attract significant moment so that the elastic column moments will be in the neighbourhood of the moment chosen. By the same token, selecting a small column moment will result in the design of a slender flexible column which will not attract significant moments.

Special care must be taken at exterior columns when more steel is provided in the slab at this support than is required by the design moment field. In such a case, it is possible for the slab to transmit a moment to the column in excess of the design moments. One should thus *always* design exterior columns and the exterior slab column connection for the actual slab moment capacity which is provided. This approach will ensure that should a failure occur, it will be by yielding in the slab rather than failure in the column or slab-column connection.

7.6 Some Final Thoughts

In view of the contradiction between the conical yield-line analysis which is an upper-bound solution, and observed behavior which indicates that the upper-bound to the collapse load is usually exceeded, the slab probably should not be treated as a thin plate near point loads or columns. It appears to be too conservative to model this area of the slab as a thin plate. Further research is required in order to develop a convenient, more realistic method of dealing with this area of the slab.

Both Wiesinger's segment equilibrium method and Hillerborg's segment are very similar. While Wiesinger uses triangular segments with a moment field which is only known at discrete points, Hillerborg uses rectangular segments with a moment field which is known at every point. It should be possible to marry the two approaches by developing lower-bound moment fields for Wiesinger's triangular segments. The result would be a more general lower-bound solution method capable of handling most slab designs.

Since most slabs utilize reinforcement in bands of constant moment capacity, it would be desirable to constrain a finite element analysis so that the moments at critical sections are uniform across specified widths of the slab. Such an analysis would only require lateral redistribution of moments as is done presently with elastic analysis, but it would give an indication of what happens to the moment field between the critical sections as a result of the redistribution. This is not usually addressed, so that the results of an elastic analysis which have been smoothed off for easy reinforcement layout are no longer lower-bound.

Clearly more work must be done on the establishment of suitable criteria for determining whether a moment field is suitable. At present, this must be determined on the basis of good engineering judgement. Unfortunately, good judgement comes from failures which of course are the result of bad judgement. As more and more designs utilize plastic methods, the collective judgement of the design profession as a whole will increase and perhaps someday it will be possible to codify good judgement. Fortunately, even today the plastic methods of design considered in this paper are sufficiently developed so that the scope of the designs which can be carried out are limited by the designers' knowledge of the various methods and not by the state-of-the-art.

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APPENDIX A

EQUILIBRIUM DERIVATIONS OF CONICAL YIELD-LINE FORMULAS

Reference: "Yield-Line Theory", K.W. Johansen, 1962. (YLT)

Let: S = column reaction
c = column radius
p = uniformly distributed load on the slab

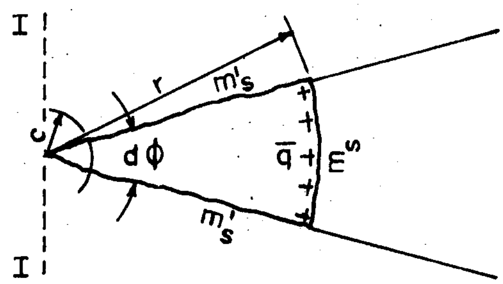


Figure AI (Equivalent to Fig. 69 YLT)

The reaction of the column is assumed to be uniformly distributed over the circle of radius c, and will have a stress of:

$$p_c = \frac{S}{\pi c^2}$$

For a circular yield-line of radius r, the shearing force along the yield-line is:

$$\bar{q} = \frac{S - p\pi r^2}{2\pi r} \qquad \text{(From vertical equilibrium)}$$

Also, $\bar{q} = \dfrac{m_s' + m_s}{r}$ (From the consideration of knot forces with equation 35 YLT)

Then: $$m_s' + m_s = \frac{S}{2\pi} - \frac{pr^2}{2} \qquad \text{(A I)}$$

For the small sector shown in Figure AI, with radius r, summing moments about I-I gives:

$$(m_s' + m_s) r\, d\Phi - r\bar{q} r\, d\Phi = \left(\frac{pr^3}{3} - \frac{p_c c^3}{3}\right) d\Phi$$

Since $\bar{q} = \dfrac{m_s' + m_s}{r}$ the left side of the above equation is zero.

And $$r = c\sqrt[3]{\frac{p_c}{p}} \qquad \text{(48 YLT)}$$

With this value of r substituted into equation A1,

$$m_s' + m_s = \frac{S}{2\pi} - \frac{p\,c^2}{2}\sqrt[3]{\frac{p_c}{p}}\sqrt[3]{\frac{p_c}{p}}$$

This simplifies to:

$$m_s' + m_s = \frac{S}{2\pi}\left(1 - \sqrt[3]{\frac{p}{p_c}}\right) \qquad (49\ \text{YLT})$$

Equation (48 YLT) gives us the required moment capacity over the column ($m_s' + m_s$). Since this generally larger than the usual values of m' and m away from the column, (i.e. m' and m for the parallel mechanism), we can obtain the required moment capacity by adding a drop panel. But, how large should the drop panel be, and how long should the top bars be?

With drop panels, the ultimate moment capacities do not have the same value along the radial yield-lines, and so the previous determination of $\bar{q}$ is no longer applicable. From the equation of moments for a small element at an arbitrary radius ρ, bounded by two consecutive circular yield-lines and by two consecutive radial yield-lines, it can easily be seen that we still have:

$$\bar{q} = \frac{m_\rho' + m_\rho}{\rho} \qquad \text{(This is still true because of equation 35 YLT)} \qquad (50\ \text{YLT})$$

And $$\bar{q} = \frac{S - \pi p \rho^2}{2\pi\rho} \qquad \text{(This is still true from vertical equilibrium)}$$

From these two equations one can obtain:

$$\rho^2 = \frac{2}{p}\left(\frac{S}{2\pi} - m_\rho' - m_\rho\right) \qquad (51\ \text{YLT})$$

This defines the yield-line in the zone with ultimate moment capacities m_ρ' and m_ρ. If, at a distance x, the ultimate moment capacities change from m_1' and m_1 to m_2' and m_2, x is determined from the moment about I-I for the element shown in Figure A2, bounded by two circular yield-lines corresponding to m_1 and m_2 and two consecutive radii. The cut off point x is determined so that the collapse load for this secondary conical mechanism is still p.

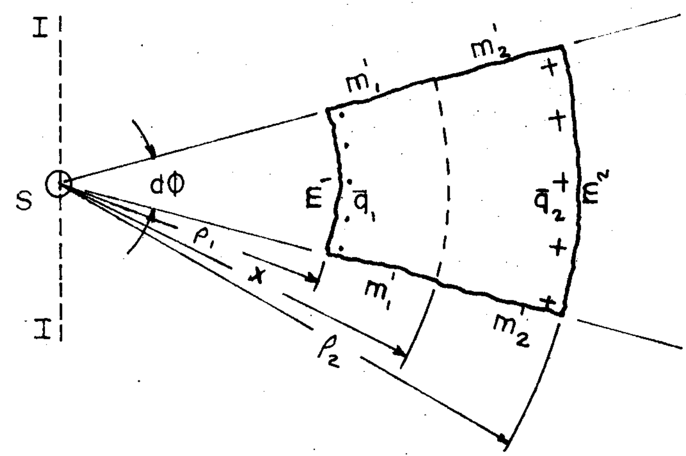


Figure A 2 (Equivalent to Fig 69 YLT)

It should be noted that the circular yield-line at ρ_1 is a positive yield line due to the primary conical mechanism. Consider equilibrium of the moments about I-I:

$$m_2\rho_2 d\phi - m_1\rho_1 d\phi + m_2'(\rho_2 - x)d\phi + m_1'(x - \rho_1)d\phi = \bar{q}_2\rho_2^2 d\phi - \bar{q}_1\rho_1^2 d\phi + \frac{p\rho_2^3 d\phi}{3} - \frac{p\rho_1^3 d\phi}{3}$$

Which gives:

$$x = \frac{p}{3}\frac{(\rho_2^3 - \rho_1^3)}{(m_1' - m_2')} \qquad \text{(52 YLT)}$$

In which ρ_1 and ρ_2 are determined from equation (51 YLT).

Referring to Fig 12b or c, the required slab capacities directly over the column or capital are, m_s' and m_s, and are determined from equation A1. The moment capacity is reduced from m_s' and m_s to m_a' and m_a at some point so that the collapse load will still be p. This cut off point is determined from equations 51 and 52 YLT. Thus:

$$b = \frac{p}{3}\frac{(R_1^3 - r^3)}{(m_s' - m_a')}$$

With $$R_1^2 = \frac{2}{p}\left(\frac{S}{2\pi} - m_a - m_a'\right)$$

And $$r = c\sqrt[3]{\frac{L^2}{\pi c^2}}$$

At the next reduction in moment capacity, that is, from m_a' and m_a to m' and m, the required cut off point will be:

$$a = \frac{p}{3}\frac{(R_2^3 - R_1^3)}{(m_a' - m')}$$

With $$R_2^2 = \frac{2}{p}\left(\frac{S}{2\pi} - m - m'\right)$$

These correspond to equations 53 YLT.

For the special case shown in Fig. 14c, equation A1 gives the required moment capacity:

$$m' + m = \frac{S}{2\pi} - \frac{pr^2}{2} = \frac{pL^2}{2\pi}\left(1 - \sqrt[3]{\frac{\pi c^2}{L^2}}\right)$$

And the cut off point is:

$$a = \frac{p}{3}\frac{(R_2^3 - r^3)}{m'}$$

With $$R_2^2 = \frac{2}{p}\left(\frac{S}{2\pi} - m\right)$$

And $$r = c\sqrt[3]{\frac{L^2}{\pi c^2}}$$

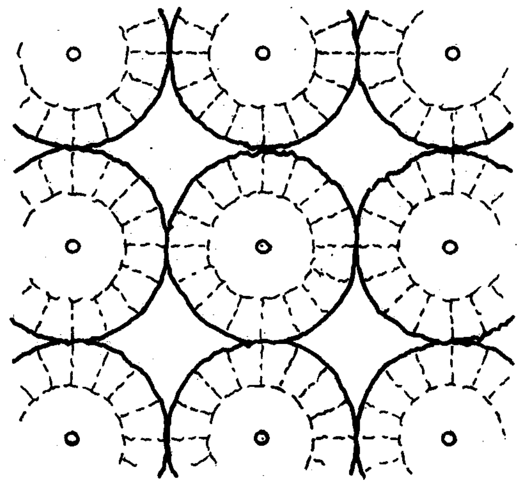
R_1 and R_2 are radii of circular yield-lines and are thus limited to $\leq (L/2)$. Johansen makes no mention of this limitation on the equations which have been developed. This can be dealt with in several ways.

1) Simply use $R_2 = (L/2)$ when equation 51 YLT yields a larger value.

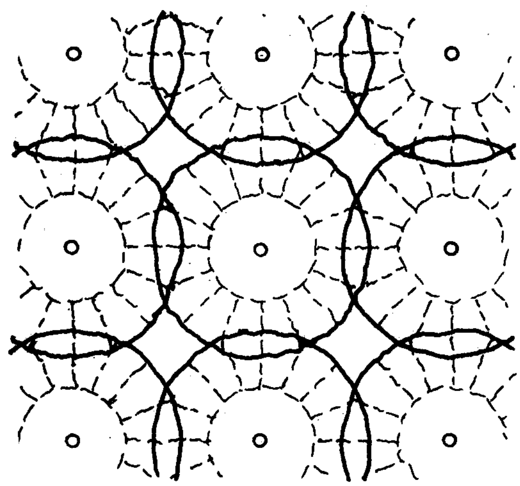
2) Simply use the values of R_1 and R_2 as they are.

3) Produce a new analysis which takes into account the fact that the exterior circular yield-line has flat spots.

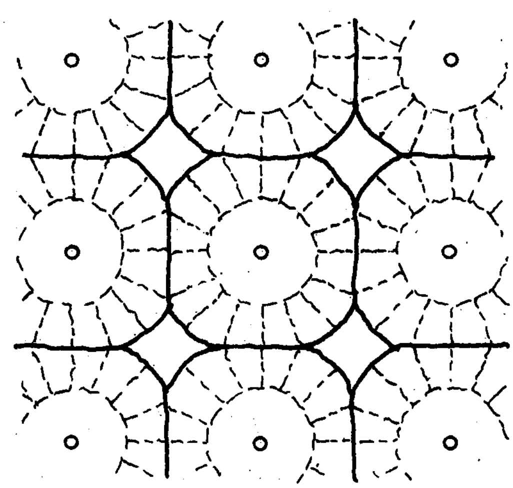
Mechanism corresponding to these cases are shown on the next page. Clearly, (1) is not the optimum mechanism. Thus it will under estimate the design moments. While (2) is not kinematically admissible, the design would be safe. On the other hand, when only one panel is loaded, (2) would be kinematically admissible and would govern the design. Finally, (3) is likely to be the collapse mechanism, but solutions for this mechanism have not been developed.



(1)



(2)



(3)

APPENDIX B

DESIGN EXAMPLES

EXAMPLE I.

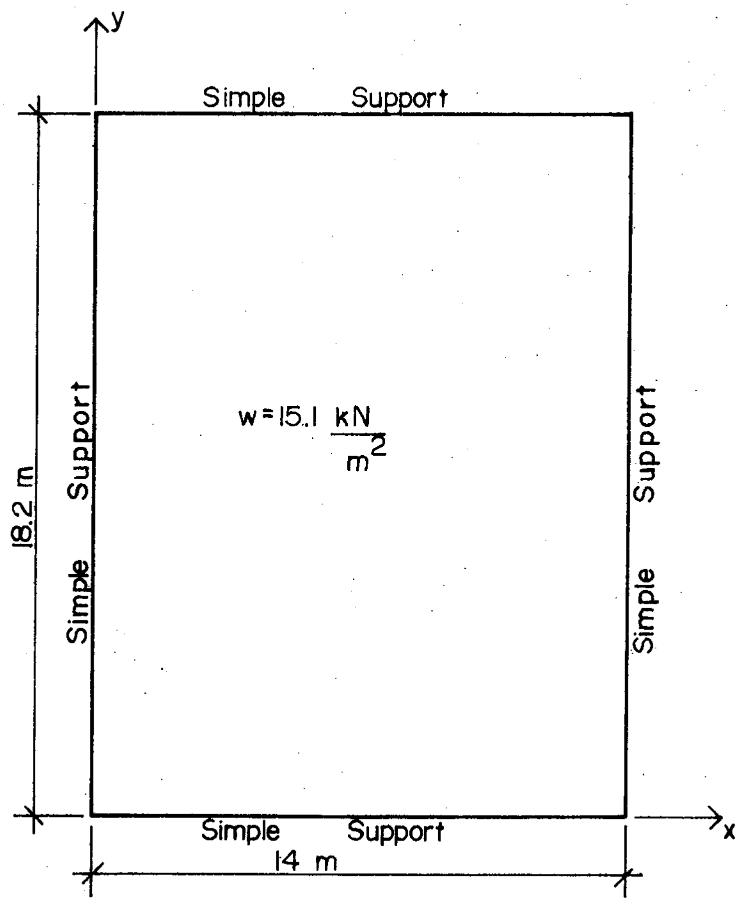


Figure BI.I Slab Layout

STEP I. CHOOSE LOAD DISPERSION LINES

Generally the interior half of the slab will have more or less constant moment across its width. It is also convenient to have the edge strips the same width in both directions. Basing the edge strip width on 1/4 of the narrow width will result in smaller moments in the long direction.

Thus: Edge Strip Width $= \frac{14}{4} = 3.50$ m

With: Interior Strip Widths $= 14 - 2(3.50) = 7.00$ m

And $= 18.2 - 2(3.50) = 11.20$ m

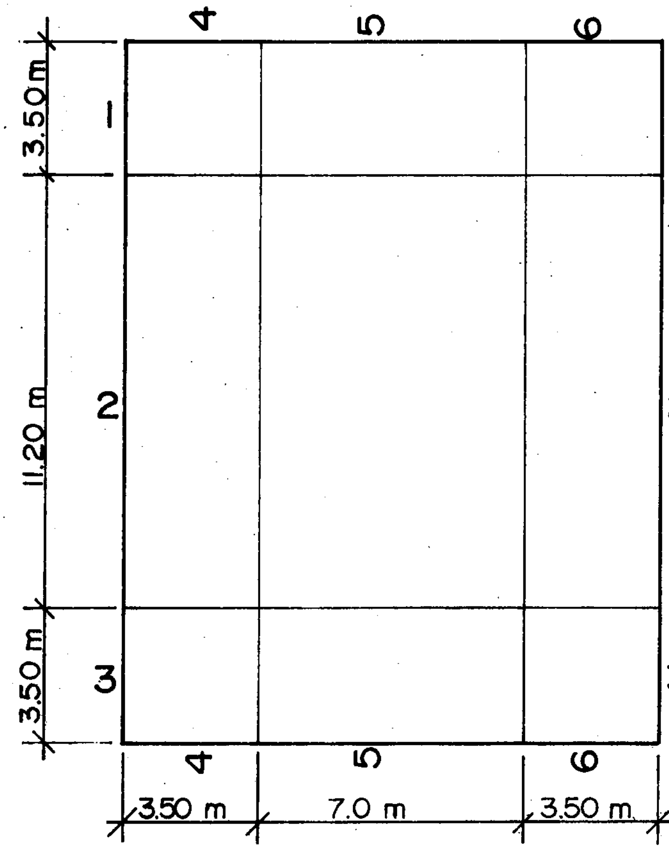


Figure B1.2 Load Dispersion Lines

STEP 2. CHOOSE DISTRIBUTION OF LOAD CARRIED IN EACH DIRECTION

Strip 2-2 is much stiffer than strip 5-5, therefore, let strip 2-2 carry 100 % of the load. In the corners, the load can reasonably be assumed to be carried equally in each direction. Where strip 5-5 crosses strip 1-1 and 3-3, it is convenient to split the load equally in each direction since this will produce a uniform design moment in the y direction across the entire slab. (Strips 4-4, 5-5 and 6-6 all have the same loading.)

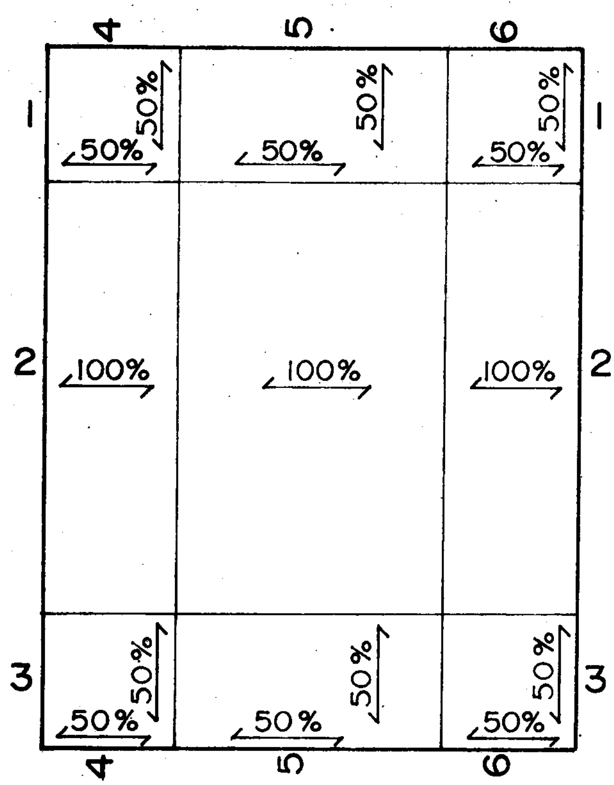
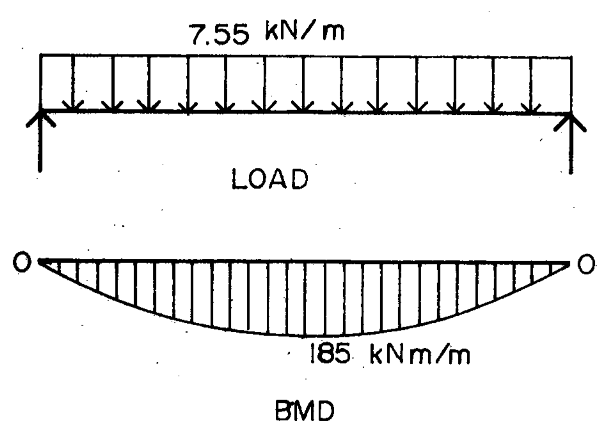


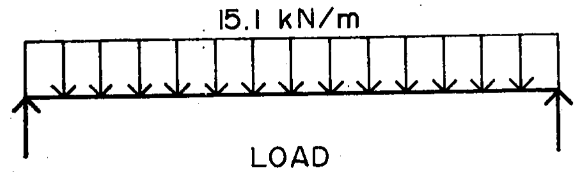
Figure B1.3 Distribution of Load

STEP 3. DESIGN STRIPS (Per m Width)

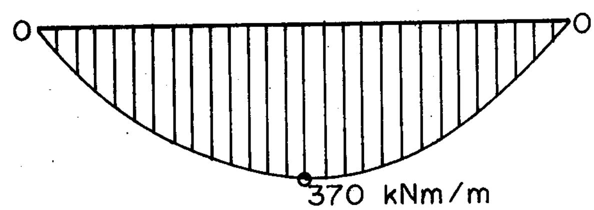
Strip 1-1 & 3-3



Strip 2-2

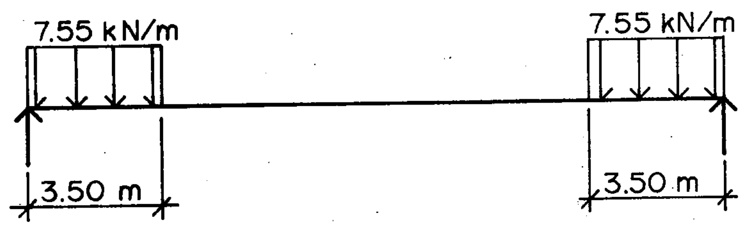


LOAD

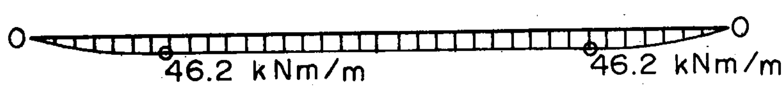


BMD

Strip 4-4, 5-5 & 6-6



LOAD



BMD

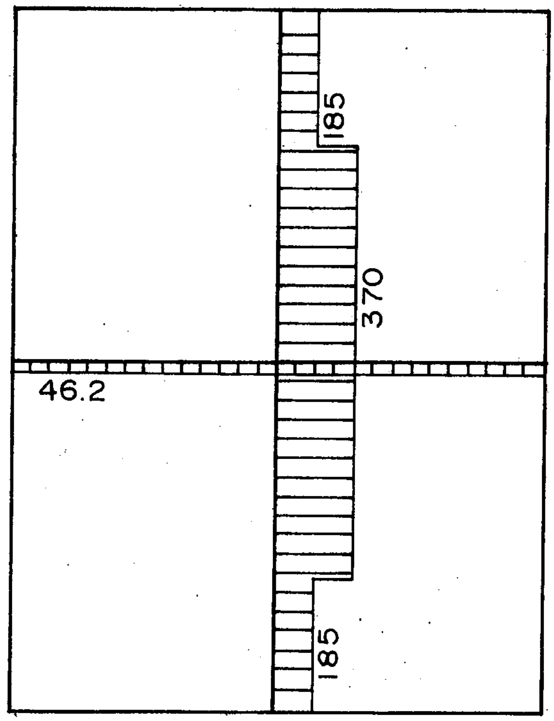


Figure B1.4 Design Moment Field (kNm/m)

EXAMPLE 2.

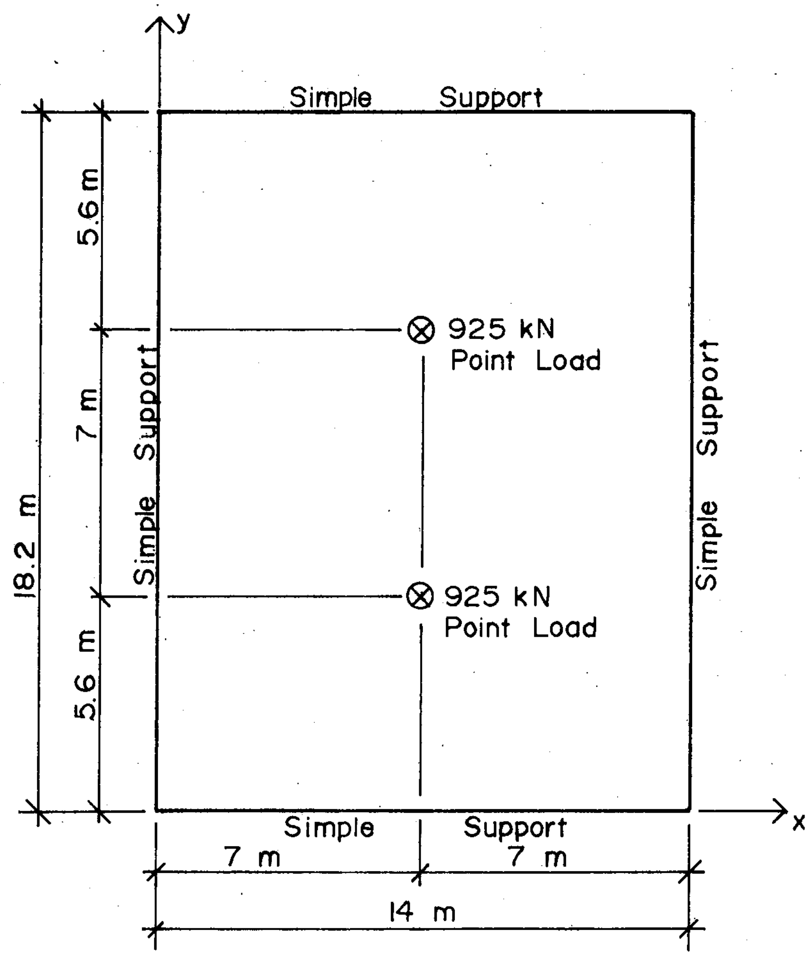


Figure B2.1 Slab Layout

STEP 1. CHOOSE LOAD DISPERSION LINES

The width of the strips carrying a concentrated load is rather arbitrary, but for the dimensions given, 3 m would seem reasonably conservative. This would be approximately 6 to 8 times the slab thickness.

The shaded areas correspond to the load dispersion elements which convert the point loads to patch loads which are applied to the simple strips.

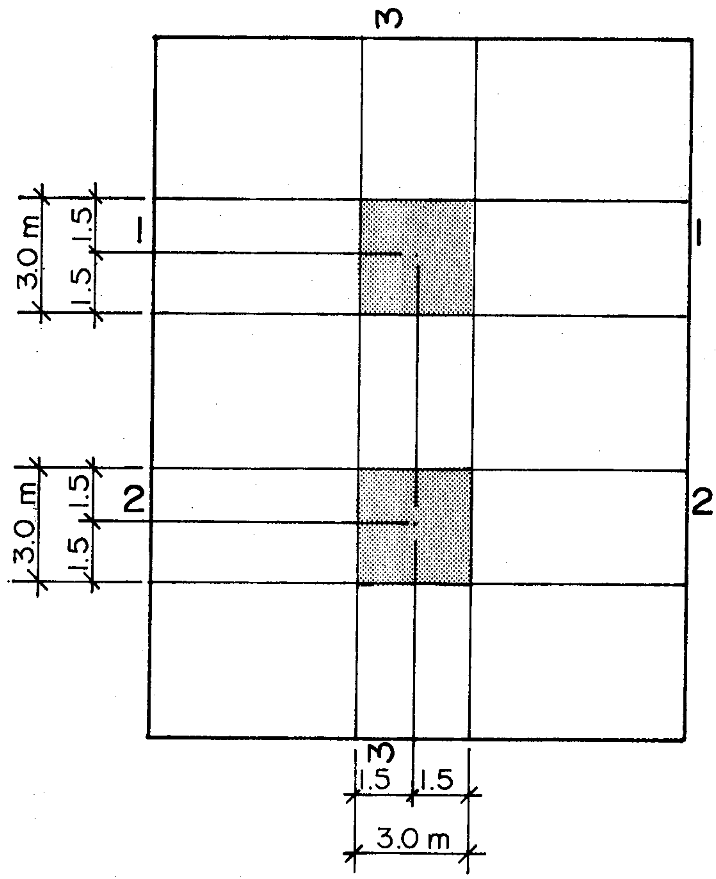


Figure B2.2 Load Dispersion Lines

STEP 2. CHOOSE DISTRIBUTION OF LOAD CARRIED BY STRIPS IN EACH DIRECTION

Deflection of strip 1-1 at point load P applied at midspan is approximately:

$$\frac{PL^3}{48EI} = \frac{P(14)^3}{48EI} = 57.2\,\frac{P}{EI}$$

Deflection of strip 3-3 at point loads P applied at load locations is approximately:

$$\frac{Pa(3La-3a^2-a^2)}{6EI} = \frac{5.6(3(18.2)(5.6)-3(5.6)^2-(5.6)^2)P}{6EI} = 168.3\,\frac{P}{EI}$$

Then, the load carried in the x direction is approximately:

$$\frac{1/57.2}{1/57.2 + 1/168.3} \times 100\% = 75\%$$

and load carried in y direction is 25%.

The preset moment field will however be distributed 50% - 50% since the load dispersion element has equal stiffness in each direction.

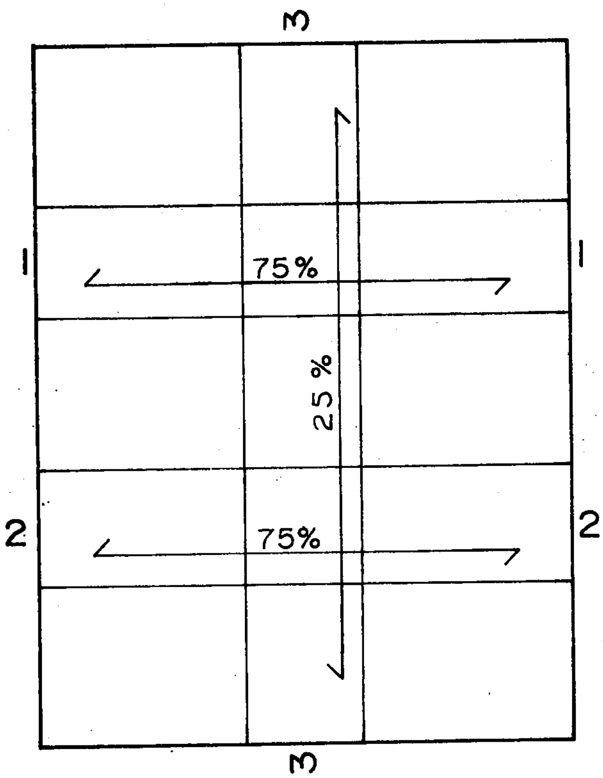


Figure B 2.3 Distribution of Load

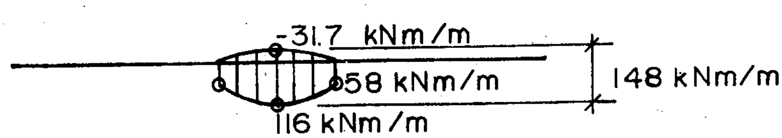
STEP 3. PRESET MOMENT FIELD

Patch load $w = \frac{925}{3 \times 3} = 102.8 \frac{kN}{m^2}$

Maximum preset moments

$m_x^+ = (0.5)\, wc^2 = 0.5\,(103)(1.5)^2 = 116 \text{ kN m/m}$

$m_x^- = (-0.137)\, wc^2 = -0.137(103)(1.5)^2 = -31.7 \text{ kNm/m}$

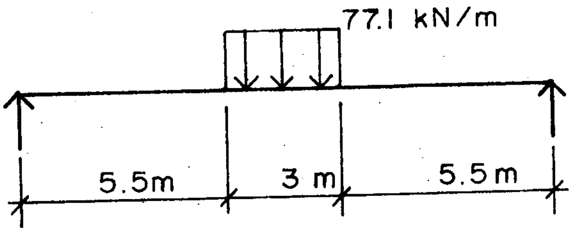


PRESET BMD

STEP 4. SIMPLE STRIP MOMENTS (per m width)

Strip 1-1 & 2-2

w = 0.75(102.8) = 77.1 kN/m



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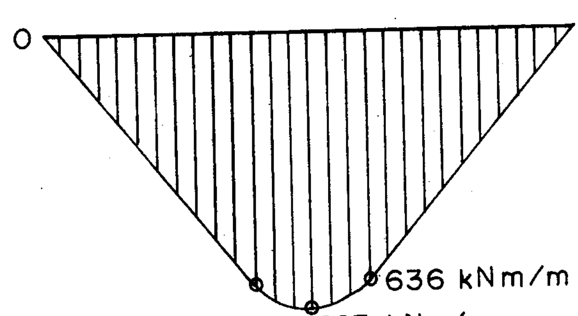
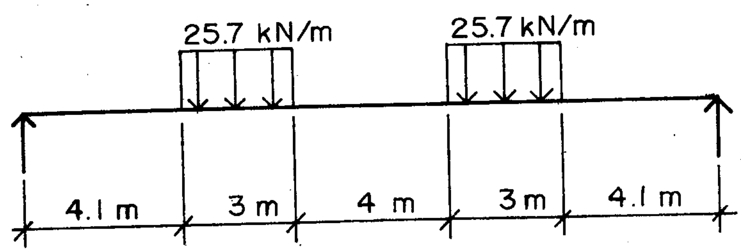
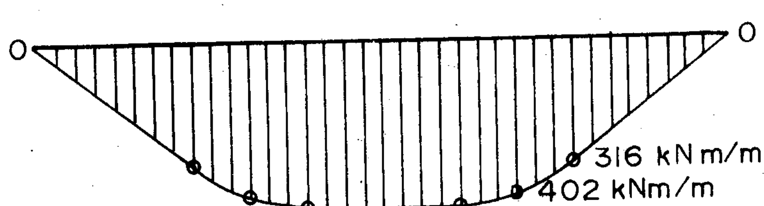


BMD

Strip 3-3



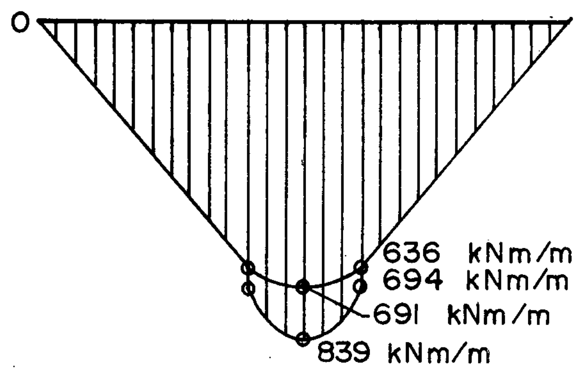
LOAD



BMD

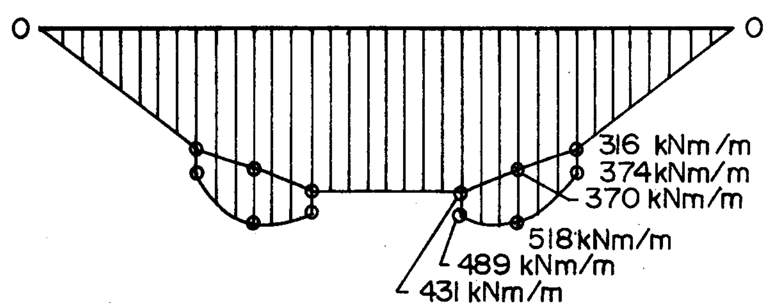
STEP 5. SUPERIMPOSE MOMENT FIELDS TO GET THE DESIGN MOMENT FIELD (per m width)

Strip 1-1 & 2-2



BMD

Strip 3-3



BMD

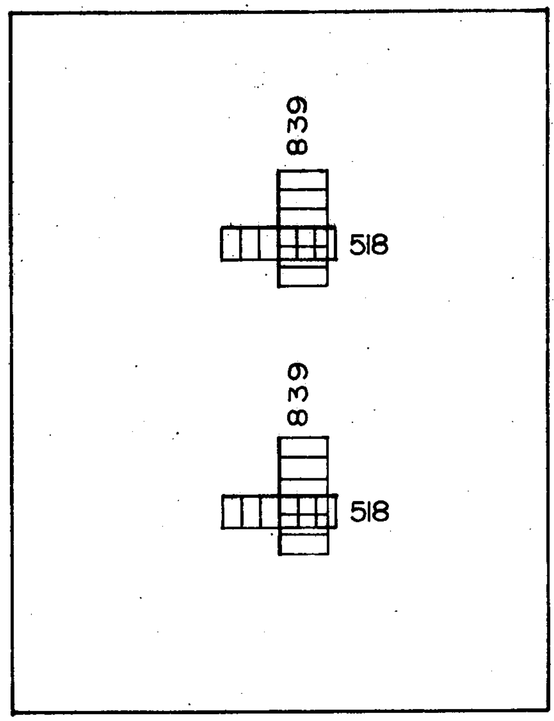


Figure B 2.4 Design Moment Field (kNm/m)

EXAMPLE 3.

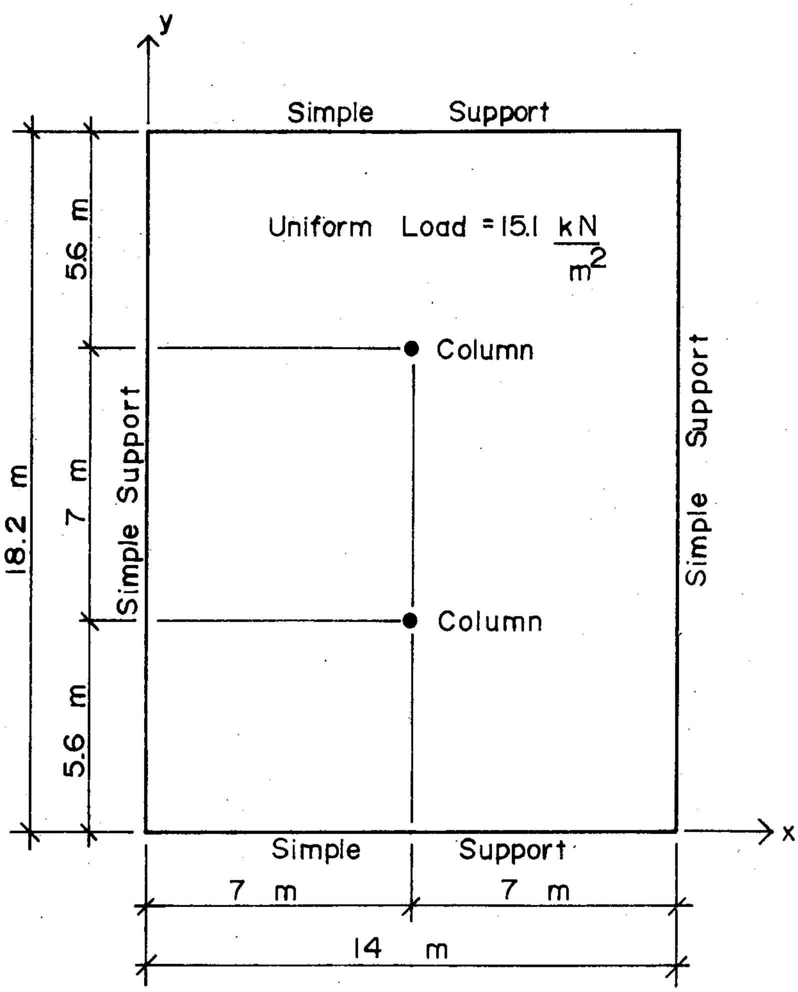


Figure B 3.1 Slab Layout

STEP 1. CHOOSE LOAD DISPERSION LINES

Since, the average span in the x direction is 7 m, let the width of the y strips equal:

$$\frac{7.0}{2} = 3.5 \text{ m}$$

Since the average span in the y direction is $\frac{7 + 5.6}{2} = 6.3$ m, let the x strip width equal:

$$\frac{6.3}{2} = 3.15 \text{ m} \quad \text{say } 3.2 \text{ m}$$

Centre the strips over the columns so that the load dispersion elements are balanced.

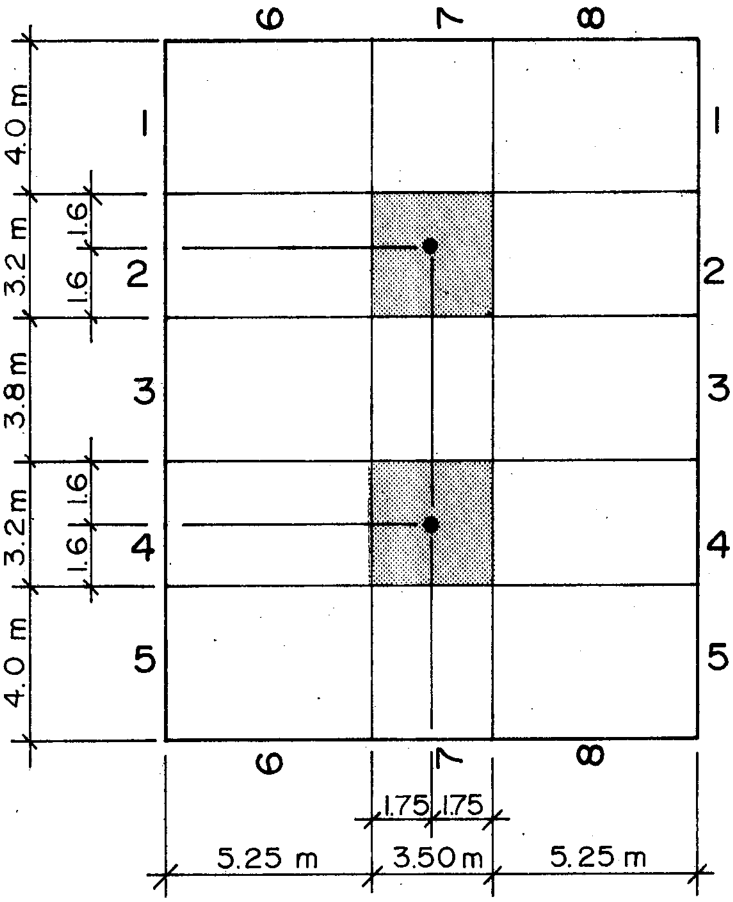


Figure B 3.2 Load Dispersion Lines

STEP 2. CHOOSE DISTRIBUTION OF LOAD CARRIED IN EACH DIRECTION

The uniformly distributed load is carried almost entirely in the x direction for strips 2-2, 3-3, and 4-4. In the corners, strip 1-1 is stiffer than strip 6-6 and will carry more load, say 60% of the uniformly distributed load. On the other hand, the opposite is probably true for strip 1-1 and 7-7.

The column load should be distributed similarly to the point loads of example 2, that is, 75% in the x direction and 25% in the y direction.

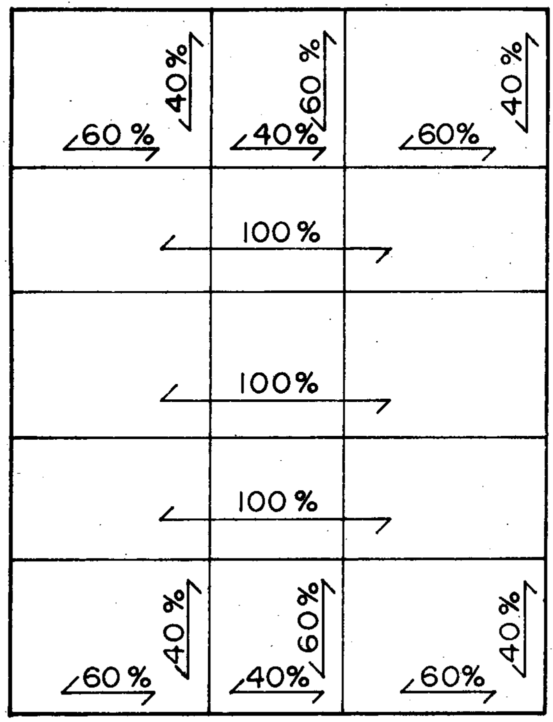


Figure B 3.3 Distribution of Uniform Load

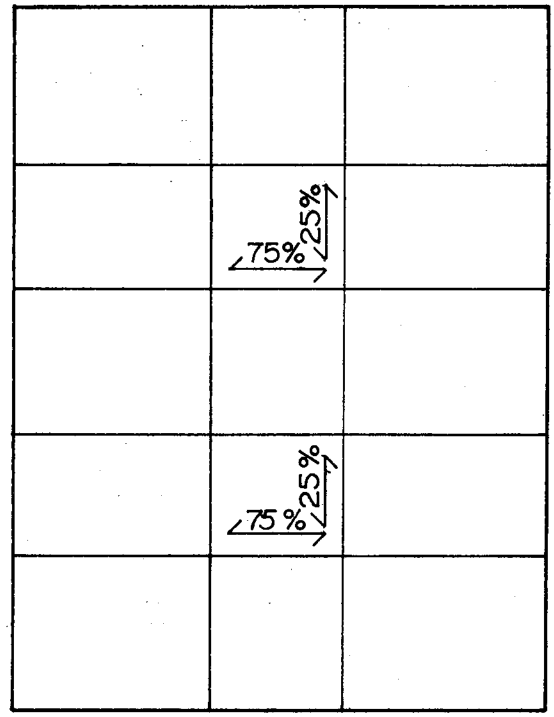


Figure B 3.4 Distribution of Upward Acting Reaction

STEP 3. CHOOSE COLUMN REACTIONS

Arrive at the column reactions on the basis of tributary area, taking into account the difference in end moments in exterior spans.

$$P \approx 15.1 \left[\left(1.25\frac{(7)}{2} + 1.25\frac{(7)}{2}\right) \times \left(1.25\frac{(5.6)}{2} + 1.0\frac{(7)}{2}\right) \right]$$

$$P \approx 925 \text{ kN}$$

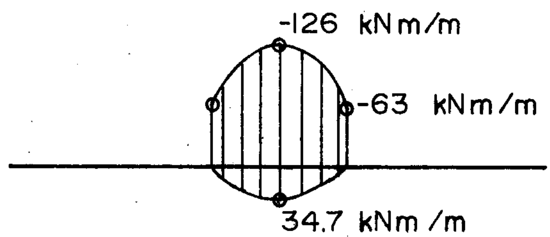
$$\text{Patch load} = \frac{925}{3.2 \times 3.5} = 82.6 \frac{\text{kN}}{\text{m}^2}$$

STEP 4. DETERMINE PRESET MOMENT FIELDS

Maximum preset x moments

$$m_x^- = (-0.500)wc_x^2 = (-0.500)(82.6)(1.75)^2 = -126 \text{ kN}\frac{m}{m}$$

$$m_x^+ = (0.137)\,wc_x^2 = (0.137)(82.6)(1.75)^2 = 34.7 \text{ kN}\frac{m}{m}$$

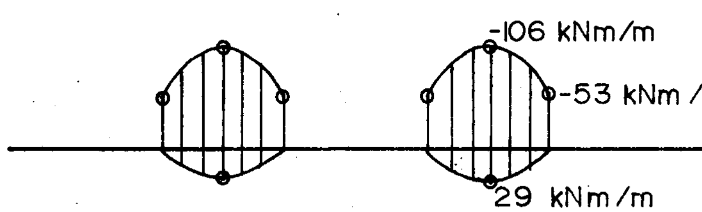


BM_xD

Maximum preset y moments

$$m_y^- = (-0.500)\,wc_y^2 = (-0.500)(82.6)(1.6)^2 = -106 \text{ kN}\frac{m}{m}$$

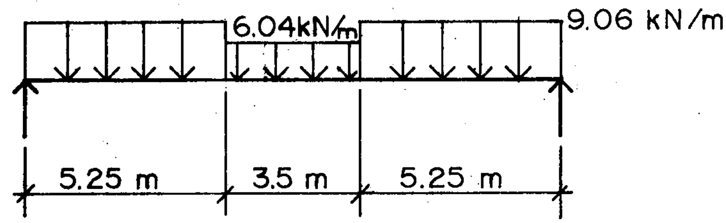
$$m_y^+ = (0.137)\,w\,c_y^2 = (0.137)(82.6)(1.6)^2 = 29.0 \text{ kN}\frac{m}{m}$$



BM_yD

STEP 5. DETERMINE STRIP MOMENTS DUE TO UNIFORM LOAD (per m width)

Strip 1-1 & 5-5



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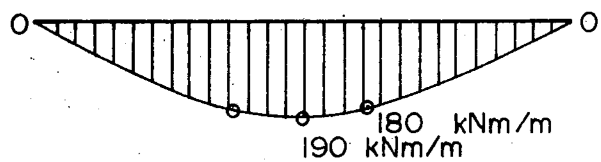
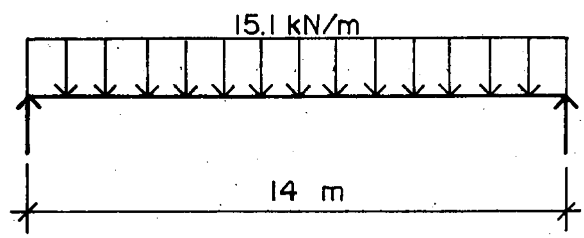


BMD

Strip 2-2, 3-3 & 4-4



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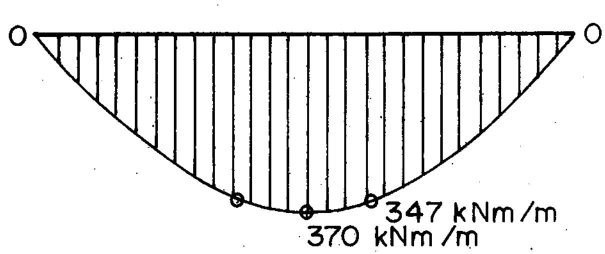
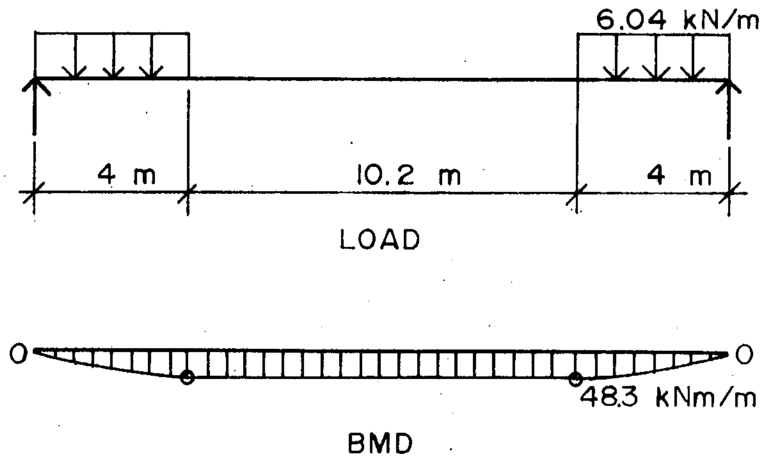
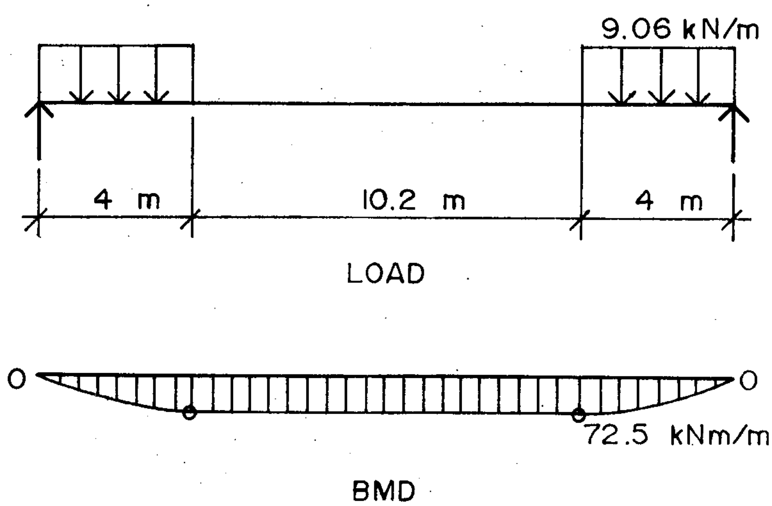


BMD

Strip 6-6 & 8-8

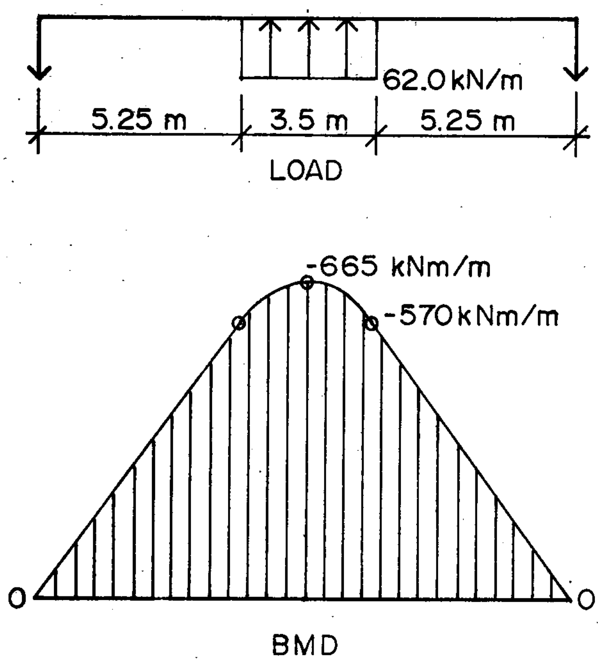


Strip 7-7

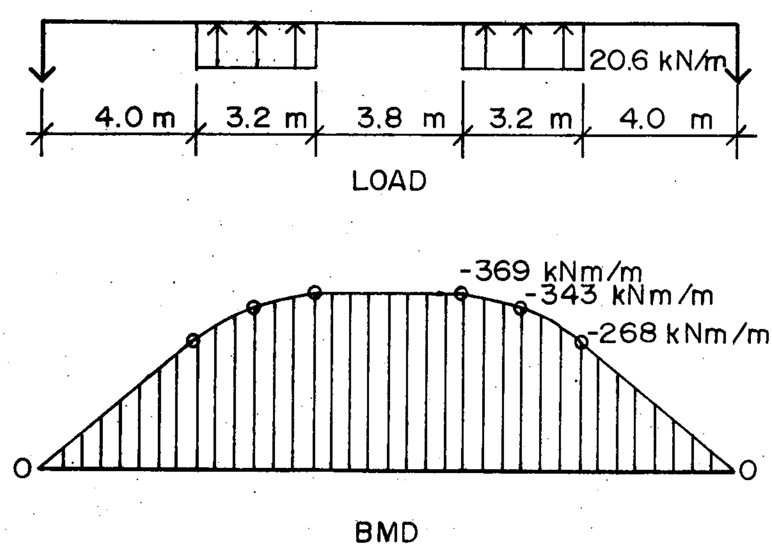


STEP 6. DETERMINE STRIP MOMENTS DUE TO UPWARD ACTING PATCH LOADS (per m width)

Strip 2-2 & 4-4

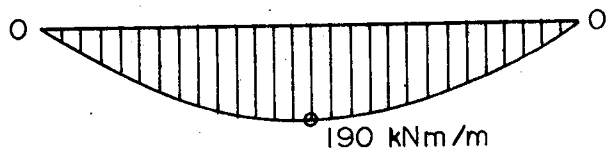


Strip 7-7



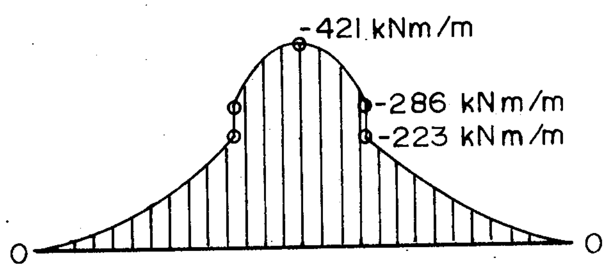
STEP 7. SUPERIMPOSE PRESET MOMENT FIELD, STRIP MOMENTS DUE TO UNIFORM LOAD, AND STRIP MOMENTS DUE TO UPWARD ACTING PATCH LOADS TO OBTAIN THE DESIGN MOMENT FIELD (per m width)

Strip 1-1 & 5-5



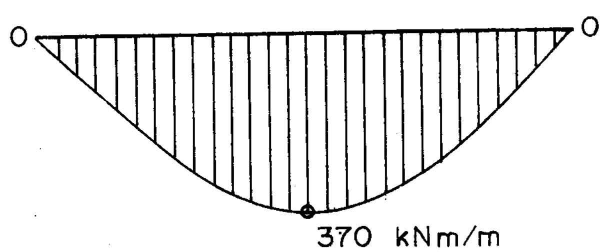
BMD

Strip 2-2 & 4-4



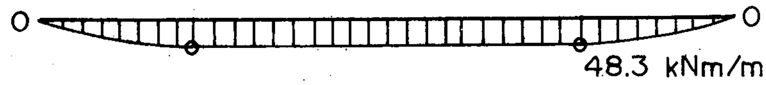
BMD

Strip 3-3



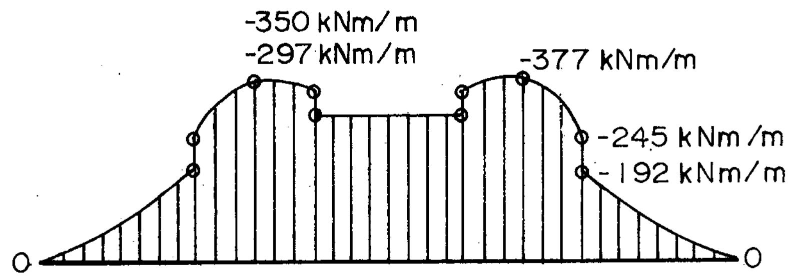
BMD

Strip 6-6 & 8-8



BMD

Strip 7-7



BMD

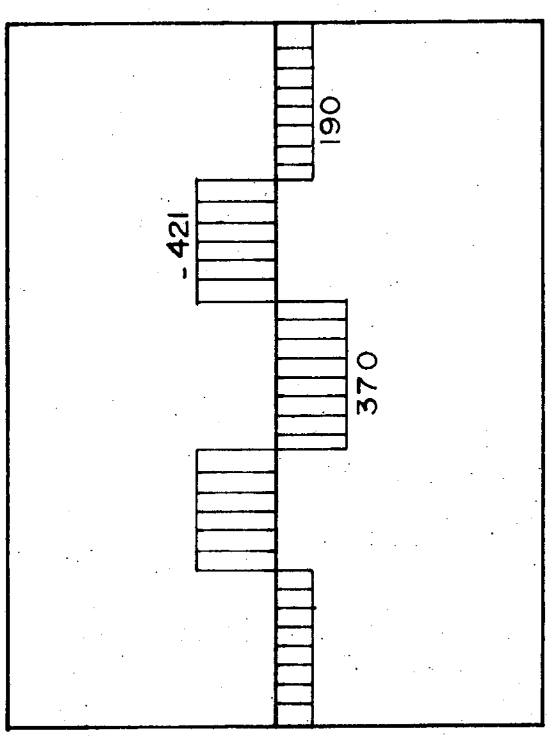


Figure B3.5 Design Moments in X Direction (kNm/m)

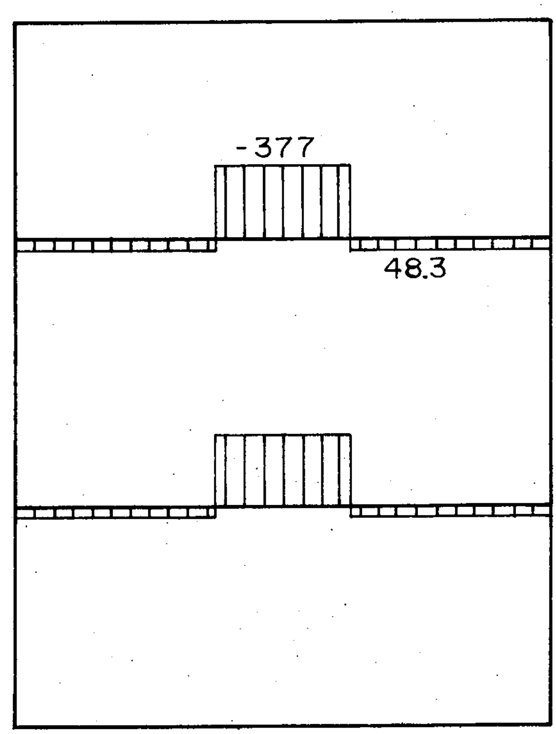


Figure B3.6 Design Moments in Y Direction (kNm/m)

EXAMPLE 4.

REDESIGN SLAB SHOWN IN FIGURE B3.I.

STEP I. CHOOSE LOAD DISPERSION LINES

Use the same load dispersion lines as in example 3. Therefore, Fig. B3.2 is applicable to this example also.

STEP 2. CHOOSE THE DISTRIBUTION OF LOAD CARRIED IN EACH DIRECTION

Use the same distribution of the uniformly distributed load as in example 3. The patch load carried in the x direction will be determined in order to obtain reasonable design moments in the x direction. The remainder of the patch load will of course be carried in the y direction. The design moments in the y strips must be checked for reasonableness with the distribution of the patch load adjusted, and the design repeated if necessary.

Thus, Fig. B3.3 is applicable to this example, while Fig. B3.4 is not.

STEP 3. CHOOSE COLUMN REACTIONS

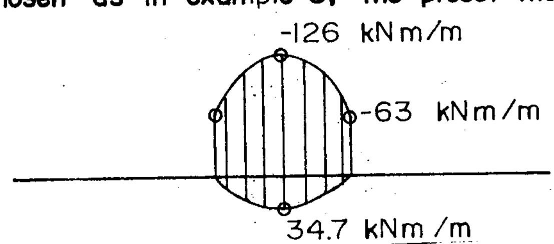
Use the same column reactions as in example 3.

$$\therefore P = 925 \text{ kN}$$

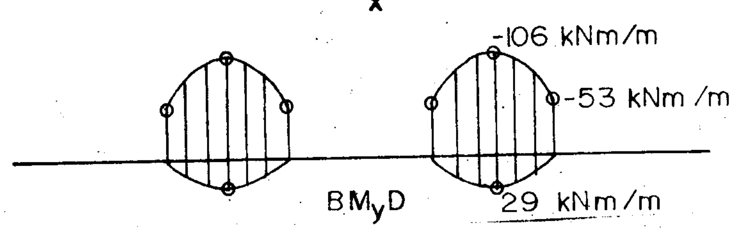
$$\text{Patch load} = 82.6 \ \frac{\text{kN}}{\text{m}^2}$$

STEP 4. DETERMINE PRESET MOMENT FIELDS

Since the same load dispersion lines and the same column reactions have been chosen as in example 3, the preset moment fields will also be the same.



BM_xD



BM_yD

STEP 5. DETERMINE STRIP MOMENTS (per m width)

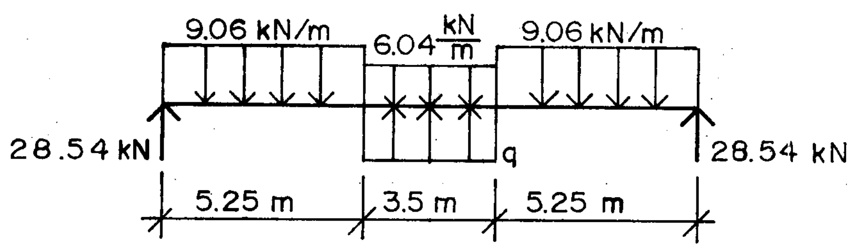
Strip 1-1 & 5-5

Let there be some contact pressure q between these strips and strip 7-7. It is not clear what the value of q should be since the extent to which strip 7-7 supports strip 1-1 and 5-5 is not known.

By assuming a reasonable location of maximum positive moment (point of zero shear), one can solve for q.

The point of zero shear will vary between (3/8)L and (1/2)L from the simple support.

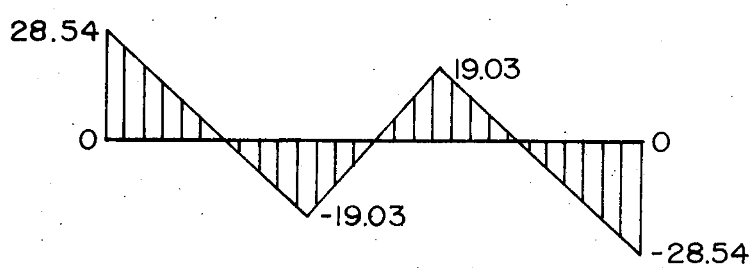
Set point of zero shear at (0.45)L . Therefore $x = (0.45)7 = 3.15$ m, and end reaction $= 3.15(9.06) = 28.54$ kN.



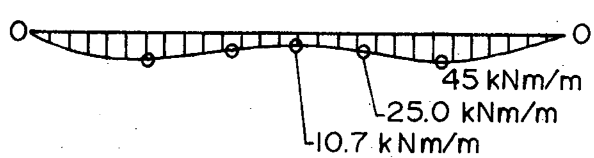
$\Sigma V = 0$ (↑+)

$$0 = 28.54 - (9.06)(5.25) - (6.04)(3.5) + (q)(3.5) - (9.06)(5.25) + 28.54$$

$$q = \frac{2(9.06)(5.25) + (6.04)(3.5) - 2(28.54)}{3.5} = 16.91 \ \frac{kN}{m}$$

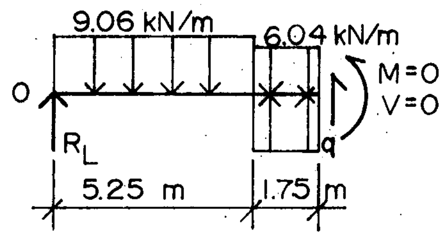


SFD (kN)



BMD

It would be advantageous to reduce the 10.7 kNm/m moment to zero since this will reduce the positive steel requirements without requiring negative steel. Consider the FBD of half the span.



FBD

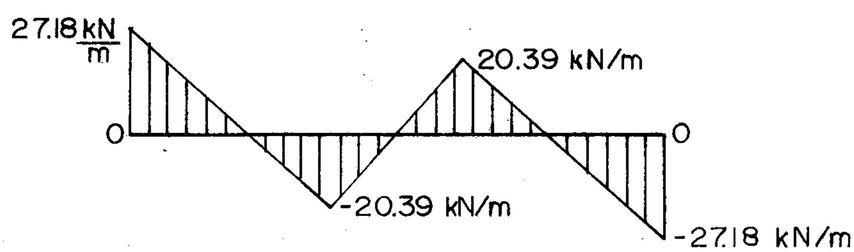
$\Sigma M_o = 0$ (↺)

$$0 = -9.06\frac{(5.25)^2}{2} - 6.04(1.75)(5.25 + \frac{1.75}{2}) + q(1.75)(5.25 + \frac{1.75}{2})$$

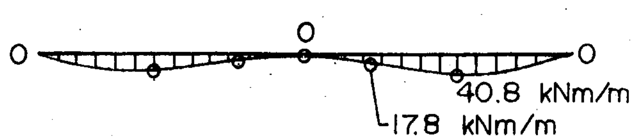
$$q = \frac{9.06\frac{(5.25)^2}{2} + 6.04(1.75)(5.25 + \frac{1.75}{2})}{(1.75)(5.25 + \frac{1.75}{2})} = 17.69 \quad \frac{kN}{m^2}$$

Left reaction = 9.06(5.25) + 6.04(1.75) - 17.69(1.75)

R_L = 27.18 kN

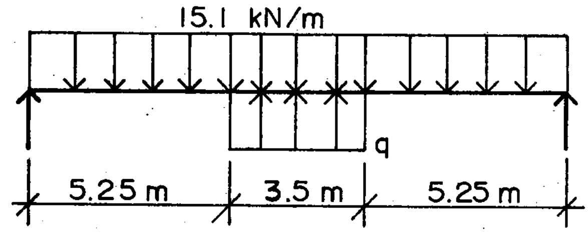


SFD



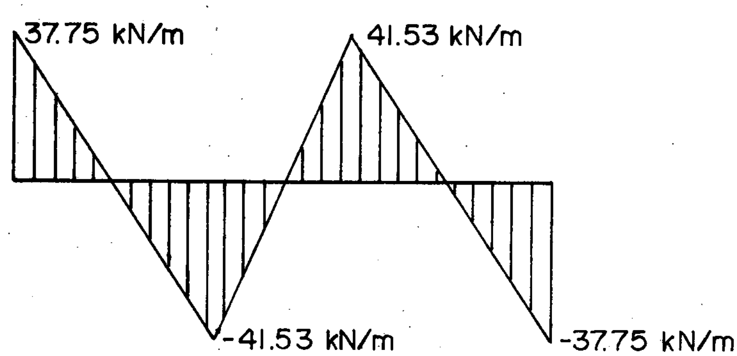
BMD

Strip 2-2 & 4-4

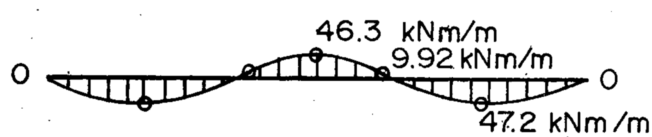


The point of zero shear should be further to the left than in the previous cases, say x = 2.5 m.

End reaction = 15.1(2.5) = 37.75 kN



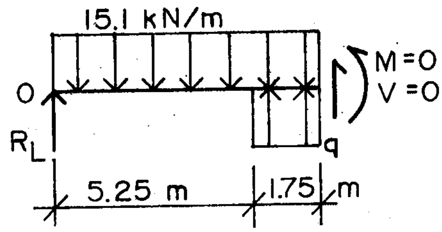
SFD



BMD

Strip 3-3

As in strips 1-1 and 5-5, set the interior moment on the column centre-line to zero and determine the upward reaction of strip 7-7.



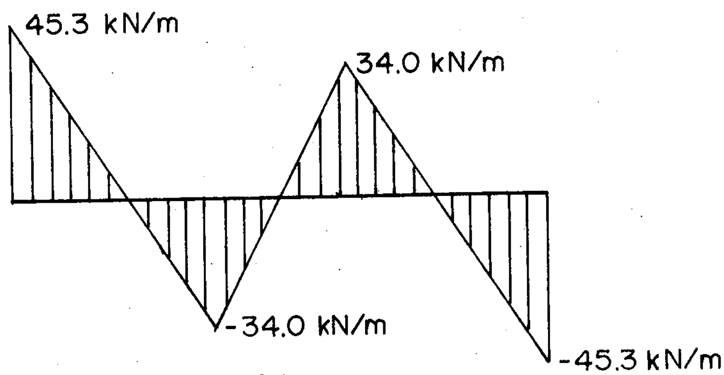
FBD

$\Sigma M_O = 0$ (+↺)

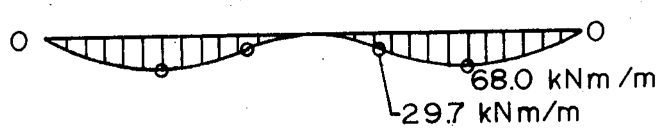
$$0 = (-15.1)\frac{(7)^2}{2} + q(1.75)\left(5.25 + \frac{1.75}{2}\right)$$

$$q = \frac{15.1(7)^2/2}{1.75\left(5.25 + \frac{1.75}{2}\right)} = 34.51 \ \frac{kN}{m^2}$$

$$R_L = 15.1(7) - 34.51(1.75) = 45.3 \ kN$$

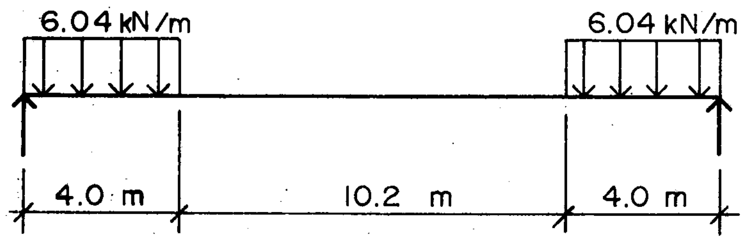


SFD



BMD

Strip 6-6 & 8-8



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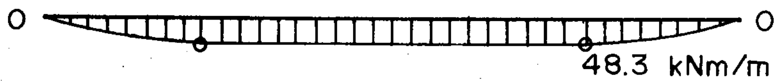
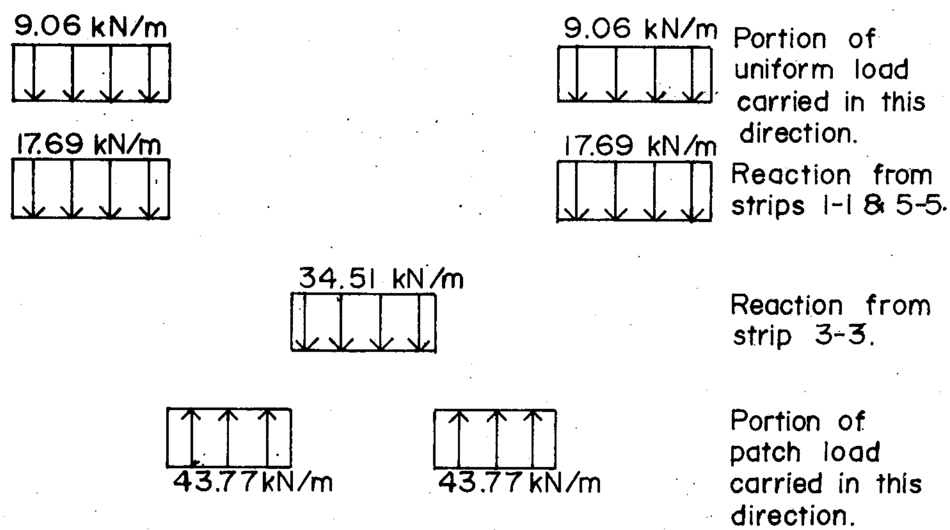
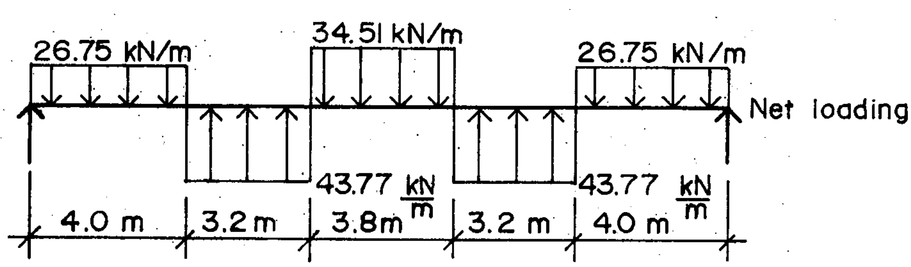


BMD

Strip 7-7





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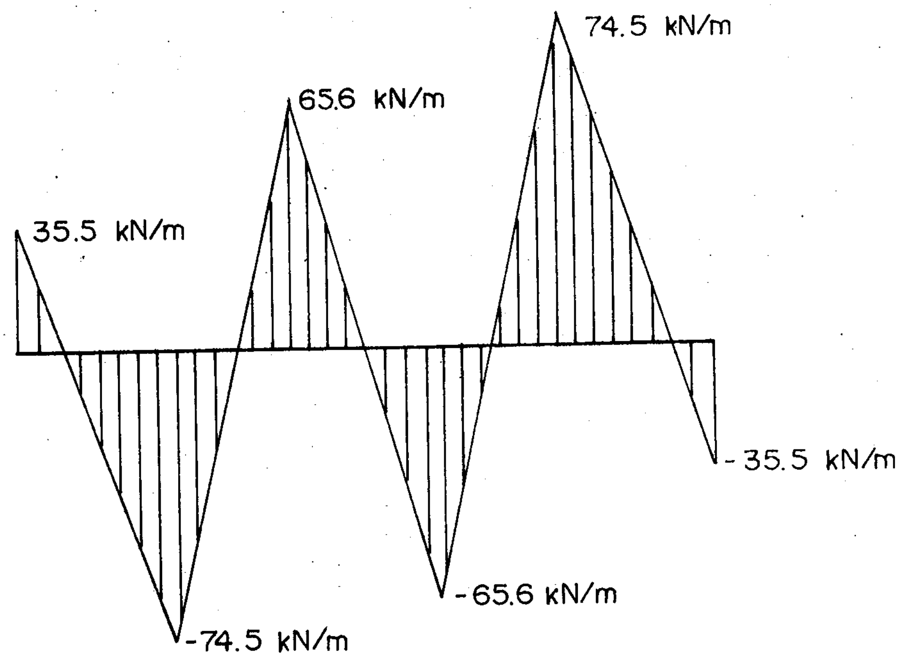
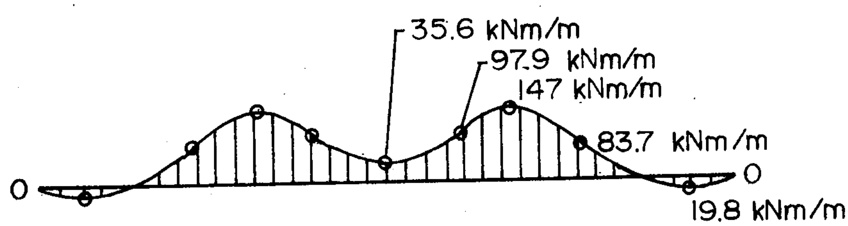

74.5 kN/m
65.6 kN/m
35.5 kN/m
-35.5 kN/m
-65.6 kN/m
-74.5 kN/m

SFD

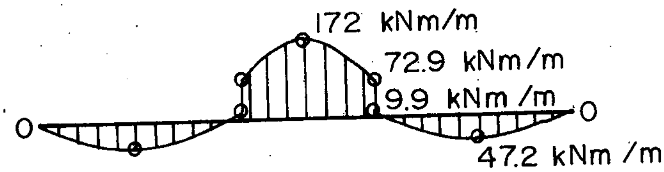

35.6 kNm/m
97.9 kNm/m
147 kNm/m
83.7 kNm/m
0
0
19.8 kNm/m

BMD

STEP 6. SUPERIMPOSE PRESET MOMENT FIELDS AND STRIP MOMENTS TO OBTAIN THE DESIGN MOMENTS

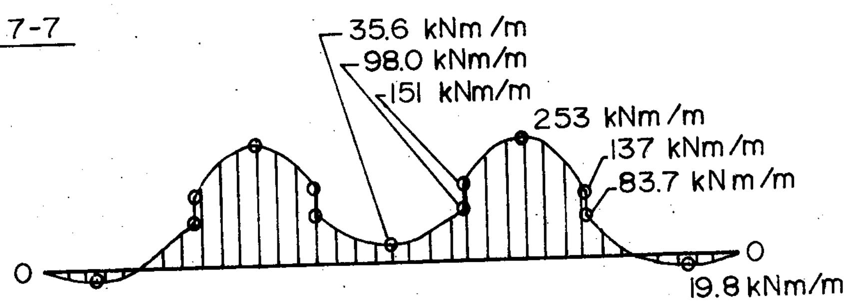
Strips 1-1, 3-3, 5-5, 6-6, and 8-8 do not have any preset moment field to superimpose with, thus they are unchanged.

Strip 2-2 & 4-4



BMD

Strip 7-7



BMD

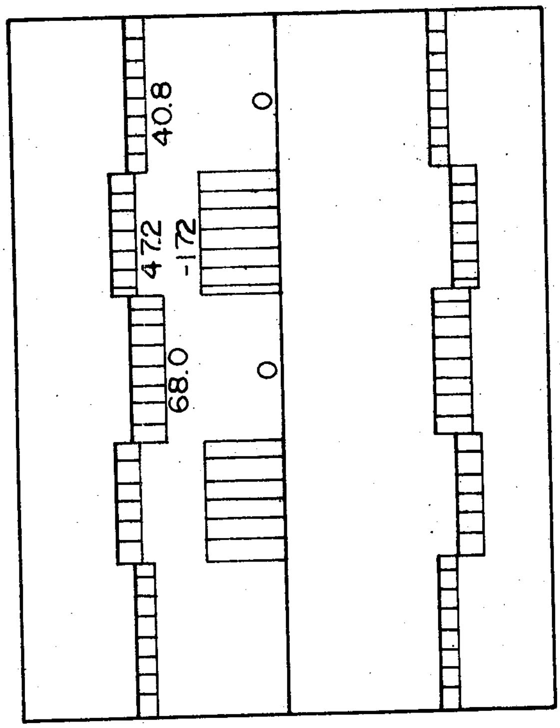


Figure B4.1 Design Moments in X Direction (kNm/m)

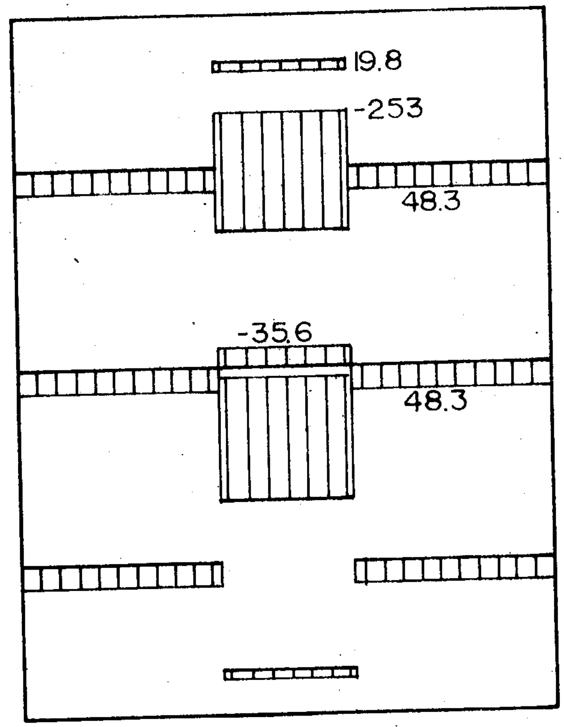


Figure B 4.2 Design Moments in Y Direction (kNm/m)

To get a positive design moment in strip 7-7 between the columns, one should change the moment field for strip 3-3. It would be more reasonable to set the moment in strip 3-3 over strip 7-7 at some small negative value, say 1/2 to 2/3 of the positive moment i.e. 34 to 45 kNm/m. This would result in a greater contact pressure on strip 7-7 which would inturn make the offending moment in strip 7-7 more positive.

EXAMPLE 5.

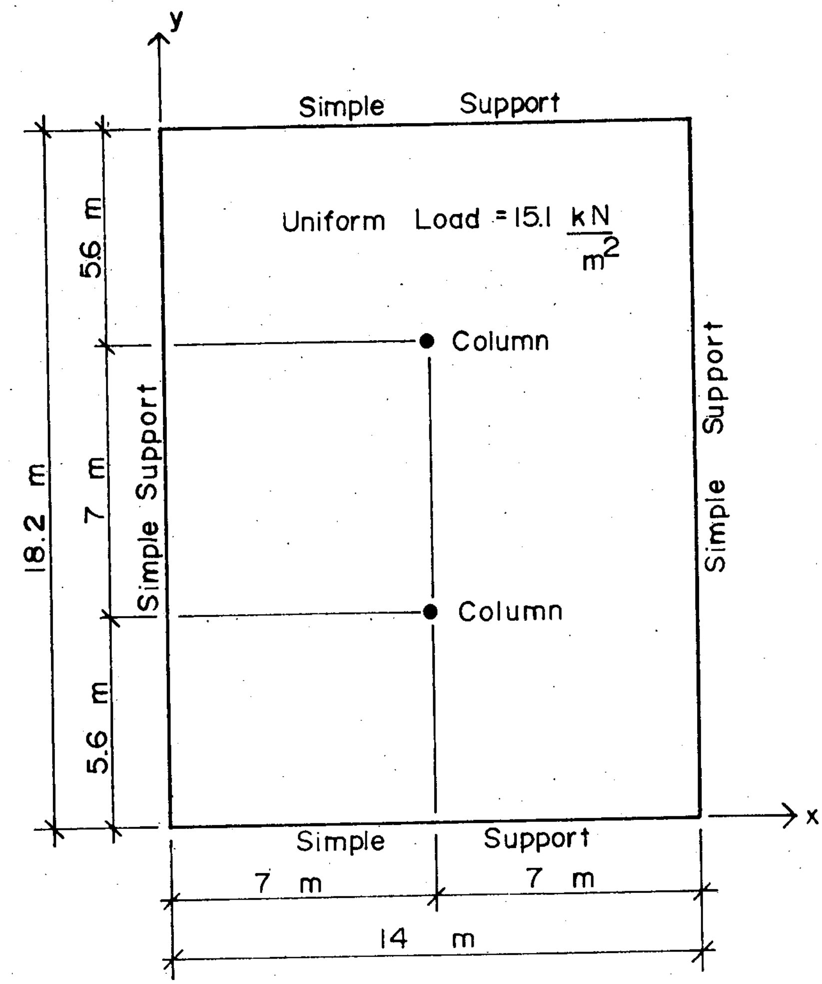


Figure B 5.1 Slab Layout

STEP 1. DEFINE SEGMENTS QUALITATIVELY

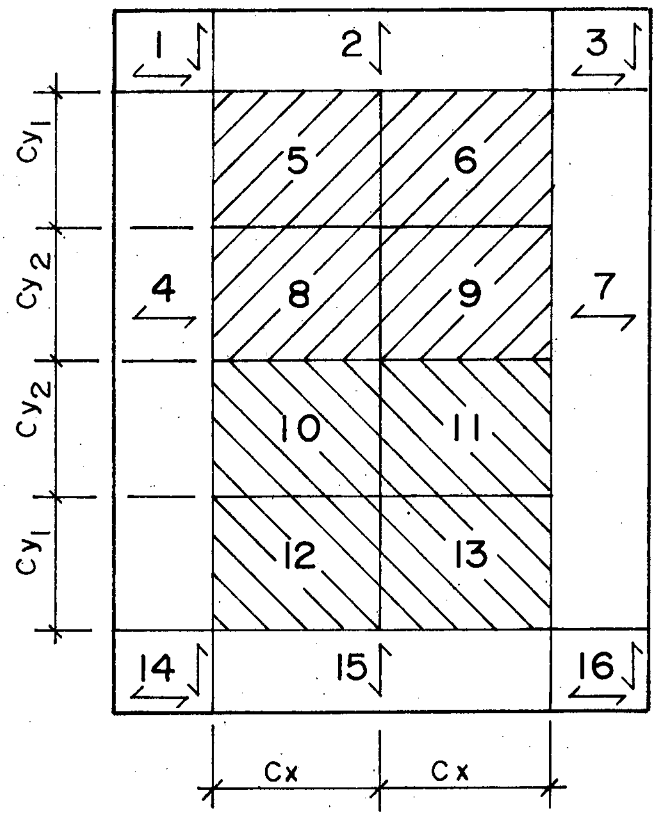


Figure B 5.2 Qualitative Segment Layout

STEP 2. DETERMINE MEAN SEGMENT MOMENTS

Corner Supported Segments x Direction

There is essentially a two span structure in the x direction, so suitable moments are:

$$\bar{m}_{x5} \approx \frac{wL^2}{8} = \frac{-(15.1)(7)^2}{8} = -92.5 \text{ kNm/m}$$

and $\bar{m}_{x4} = 0$

$$C_x = \frac{L}{2} - \frac{(m_{x5}^- - m_{x4}^-)}{w\ L} = \frac{7}{2} - \frac{(-92.5-0)}{15.1(7)} = 4.38\ m$$

$$m_{x5}^+ = \frac{wC^2}{2} + m_x^- = \frac{15.1(4.38)^2}{2} - 92.5 = 52.3\ kNm/m$$

From equilibrium between segments 5 and 4:

$$m_{x4}^+ = m_{x5}^+ = 52.3\ kNm/m$$

and from equilibrium between segments 5 and 6:

$$m_{x5}^+ = m_{x6}^- = -92.5 kNm/m$$

From symmetry, the m_x moments are known for segments 7 through 13.

<u>Corner Supported Segments y Direction</u>

There is essentially a three span structure in the y direction. The negative moment at the exterior face of the first interior support

$$m_y^- \approx \frac{wL^2}{8} = \frac{-(15.1)(5.6)^2}{8} = -59.2\ kNm/m$$

The negative moment at the interior face of the first interior support

$$m_y^- \approx \frac{wL^2}{12} = \frac{-(15.1)(7)^2}{12} = -61.7\ kNm/m$$

Set $m_{y12}^- = -60\ kNm/m = m_{y10}^-$, and $m_{y15}^- = 0$.

$$\therefore\ C_{y1} = \frac{5.6}{2} - \frac{(-60-0)}{15.1(5.6)} = 3.51\ m$$

$$C_{y2} = \frac{7}{2} - \frac{((-60)-(-60))}{15.1(7)} = 3.5\,m$$

$$m_{y12}^+ = \frac{15.1(3.51)^2}{2} - 60 = 33.0\ kNm/m$$

$$m_{y10}^+ = \frac{15.1(3.5)^2}{2} - 60 = 32.5\ kNm/m$$

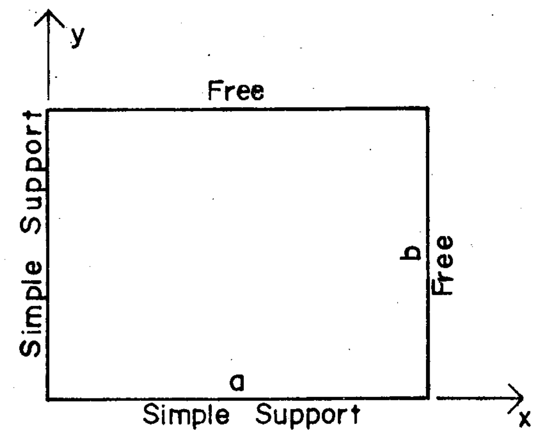
From equilibrium between segments

$$m_{y15}^+ = m_{y12}^+ = 33.0\ kNm/m \quad \text{and} \quad m_{y8}^+ = m_{y10}^+ = 32.5\,kNm/m$$

From symmetry the m moments are known for segments 2, 5, 6, 8, 9, 10, 11, 12, 13, and 15.

Corner Segments 1, 3, 14 and 16

These segments can be handled in exactly the same manner as in Example 1, that is, the load could be distributed in each direction so that it could be carried by positive steel running the entire length of the slab. However, we do know that twisting moments will produce negative moments in these segments so one should use some other moment field which would introduce top reinforcement into these segments. Hillerborg discusses several possible moment fields for such a case, but we will only consider one of them.



Segment Uniformly Loaded, No Edge Moments

Reinforcing the entire segment for:

$$m_x^+ = \frac{2wa^2}{9} \qquad m_x^- = \frac{-wa^2}{6}$$

$$m_y^+ = \frac{2wb^2}{9} \qquad m_y^- = \frac{-wb^2}{6}$$

will envelope a lower bound solution for the particular segment and boundary conditions.

$$m_{x1}^+ = \frac{2(15.1)(2.62)^2}{9} = 23.0 \text{ kNm/m}$$

$$m_{x1}^- = \frac{-(15.1)(2.62)^2}{6} = -17.3 \text{ kNm/m}$$

$$m_{y1}^+ = \frac{2(15.1)(2.09)^2}{9} = 14.7 \text{ kNm/m}$$

$$m_{y1}^- = \frac{-(15.1)(2.09)^2}{6} = -11.0 \text{ kNm/m}$$

A more detailed investigation into what the preset moment field actually looks like in order to determine cut off points is not warranted in this case since these design moments are very small, and the steel requirements will be based on minimum reinforcement.

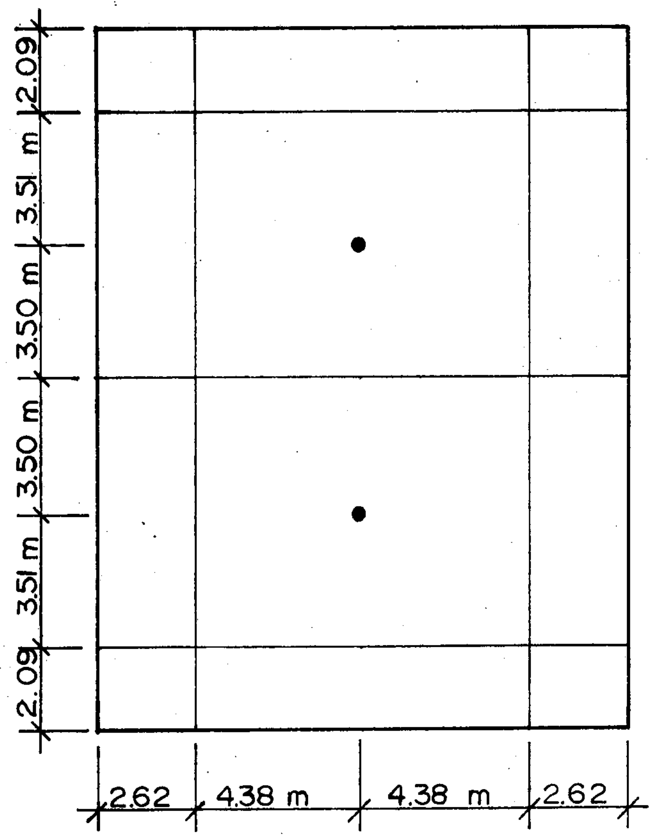


Figure B5.3 Quantitative Segment Layout

STEP 3. RATIONALIZE MOMENT FIELD

For the moment fields over the columns, use $\alpha = \beta = 1/2$. Therefore, the design negative moments will be twice as large per unit width as previously calculated but are only distributed over 1/2 the segment width.

Leave the positive moments in exterior spans unchanged.

Rationalize the positive moments between the columns by shifting 2/3 of the moment into the middle half of the segment.

Since $M^+_y = 2(C_x)(m^+_y) = 2(4.38)(32.5) = 284.7$ kN

$m^+_{y\alpha} = (2/3)(284.7/4.38) = 43.3$ kNm/m

$m^+_{y1-\alpha} = (1/3)(284.7/4.38) = 21.7$ kNm/m

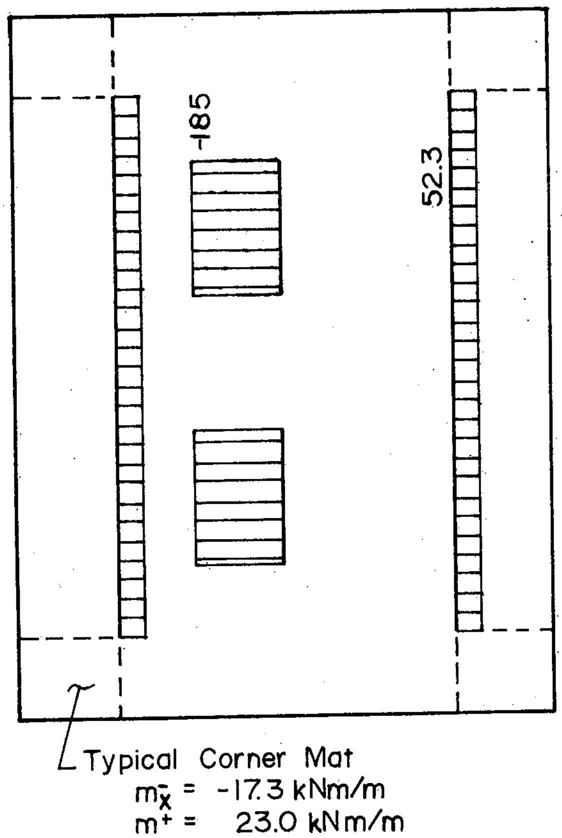


Figure B5.4 Design Moments in X Direction (kNm/m)

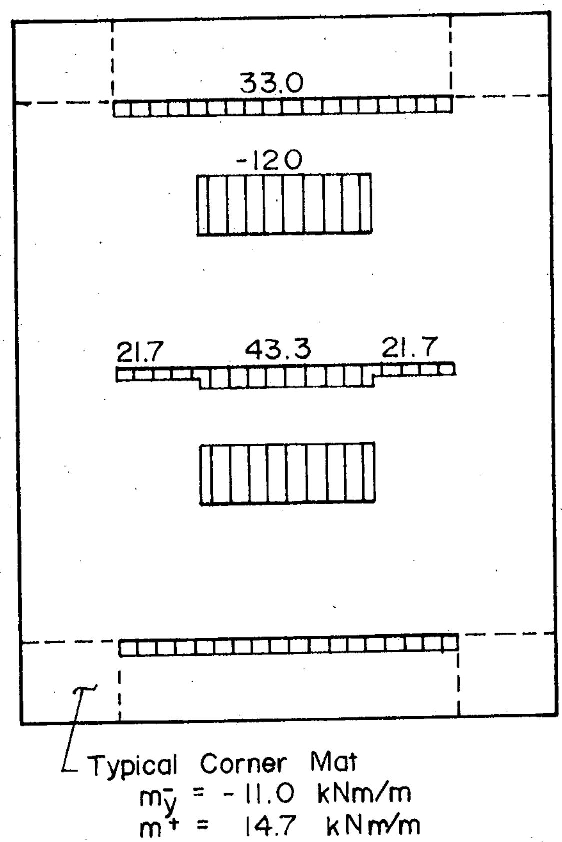


Figure B5.5 Design Moments in Y Direction (kNm/m)

Check the suitability of the rationalized moment fields with K_x and K_y.

$\alpha = \beta = 0.5$

$$K_x = \frac{\beta(-m^-_{x_\beta} + m^-_{x_{1-\beta}})}{(1/2)\,w\,C_x^2} = \frac{0.5(185+0)}{(1/2)(15.1)(4.38)^2} = 0.64 \qquad 0.30 < K < 0.75 \therefore OK$$

$$K_y = \frac{\alpha(-m^-_{y_\alpha} + m^-_{y_{1-\alpha}})}{(1/2)\,w\,C_y^2} = \frac{0.5(120+0)}{(1/2)(15.1)(3.51)^2} = 0.65 \qquad 0.30 < K < 0.75 \therefore OK$$

$$K_y = \frac{\alpha(-m^-_{y_\alpha} + m^-_{y_{1-\alpha}})}{(1/2)\,w\,C_y^2} = \frac{0.5(120+0)}{(1/2)(15.1)(3.51)^2} = 0.65 \qquad 0.30 < K < 0.75 \therefore OK$$

EXAMPLE 6.

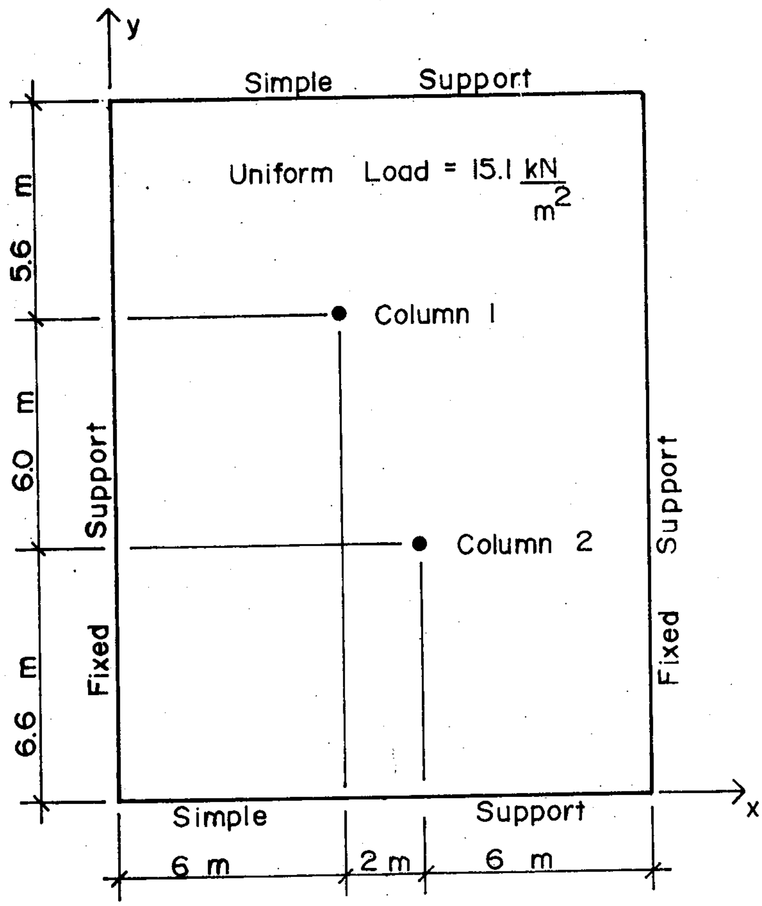


Figure B6.1 Slab Layout

STEP I. DEFINE SEGMENTS QUALITATIVELY

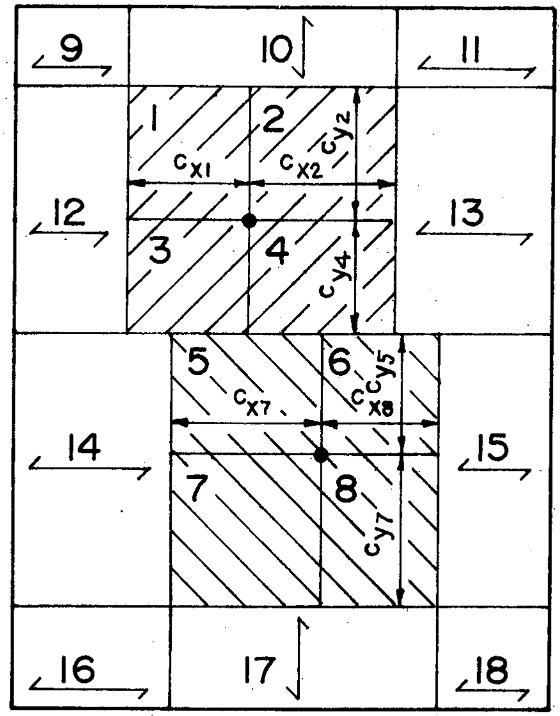


Figure B6.2 Qualitative Segment Layout

STEP 2. DETERMINE MEAN SEGMENT MOMENTS

Corner Supported Segments in x Direction

$$m_{x1}^- \approx \frac{-wL^2}{12} = \frac{-15.1(6)^2}{12} = -45.3 \text{ kNm/m}$$

$$m_{x2}^- \approx \frac{-wL^2}{12} = \frac{-15.1(8)^2}{12} = -80.5 \text{ kNm/m}$$

∴ Set $m_x^- = -55$ kNm/m (↑)

And $m_x^- = -70$ kNm/m (↓)

Then from equilibrium between segments 1 and 2, and column 1, the equivalent column x moment is 15 kNm/m (↑).

$$m_{x12}^- \approx \frac{-wL^2}{12} = \frac{-15.1(6)^2}{12} = -45.3 \text{ kNm/m}$$

Set $m_{x12}^- = -45$ kNm/m

Then $c_{x1} = \frac{6}{2} + \frac{(55-45)}{15.1(6)} = 3.11$ m

And $m_{x1}^+ = \frac{15.1(3.11)^2}{2} - 55 = 18.0$ kNm/m

$$m^-_{x13} \approx \frac{-wL^2}{12} = \frac{-15.1(8)^2}{12} = -80.5 \text{ kNm/m}$$

Set $m^-_{x13} = -80.0$ kNm/m

Then $c_{x2} = \dfrac{8}{2} + \dfrac{(70-80)}{15.1(8)} = 3.92$ m

And $m^+_{x2} = \dfrac{15.1(3.92)^2}{2} - 70 = 46.0$ kNm/m

Segments 3 to 8 are similar since the spans are also 6m and 8m.

<u>Corner Supported Segments in y Direction</u>

$m^-_{y17} = 0$ Since it is simply supported

$$m^-_{y7} \approx \frac{-wL^2}{8} = \frac{-(15.1)(6.6)^2}{8} = -82.2 \text{ kNm/m}$$

$$m^-_{y5} \approx \frac{-wL^2}{12} = \frac{-(15.1)(6)^2}{12} = -45.3 \text{ kNm/m}$$

Set $m^-_{y7} = -70$ kNm/m

And $m^-_{y5} = -55$ kNm/m

Then equivalent column y moment required for equilibrium is 15 kNm/m.

$$m^-_{y3} \approx \frac{-wL^2}{12} = \frac{-(15.1)(6)^2}{12} = -45.3 \text{ kNm/m}$$

$$m^-_{y1} \approx \frac{-wL^2}{8} = \frac{-(15.1)(5.6)^2}{8} = -59.2 \text{ kNm/m}$$

Set $m^-_{y3} = -45$ kNm/m

$m^-_{y1} = -60$ kNm/m

Then equivalent column y moment is 15 kNm/m —.

$m^-_{y10} = 0$

Then $c_{y7} = \dfrac{6.6}{2} + \dfrac{(70+0)}{15.1(6.6)} = 4.00$ m

$c_{y5} = \dfrac{6}{2} + \dfrac{(55-45)}{15.1(6)} = 3.11$ m

$c_{y3} = 6 - 3.11 = 2.89$ m

$$c_{y_1} = \frac{5.6}{2} + \frac{(60+0)}{15.1(5.6)} = 3.51 \text{ m}$$

$$m^{+}_{y_7} = \frac{15.1(4)^2}{2} - 70 = 50.8 \text{ kNm/m} = m^{+}_{y_{17}}$$

$$m^{+}_{y_5} = \frac{15.1(3.11)^2}{2} - 55 = 18.0 \text{ kNm/m} = m^{+}_{y_3}$$

$$m^{+}_{y_1} = \frac{15.1(3.51)^2}{2} - 45 = 48.0 \text{ kNm/m} = m^{+}_{y_{10}}$$

Corner Segments

A different approach shall be illustrated in this example. The load is to be distributed equally in each direction. For example, segment 9 would carry 7.55 kN/m^2 in each direction. Thus, the moments in the x direction will be exactly (1/2) of those in segment 12, and in the y direction the moments will be exactly (1/2) of those in segment 10. The positive steel from segment 9 in the x direction must be continued right across segment 10, and the positive steel in the y direction must be continued right across segment 12. Thus, there will be half as much reinforcement parallel to the boundaries of the slab in the exterior segments as in the adjoining interior segments.

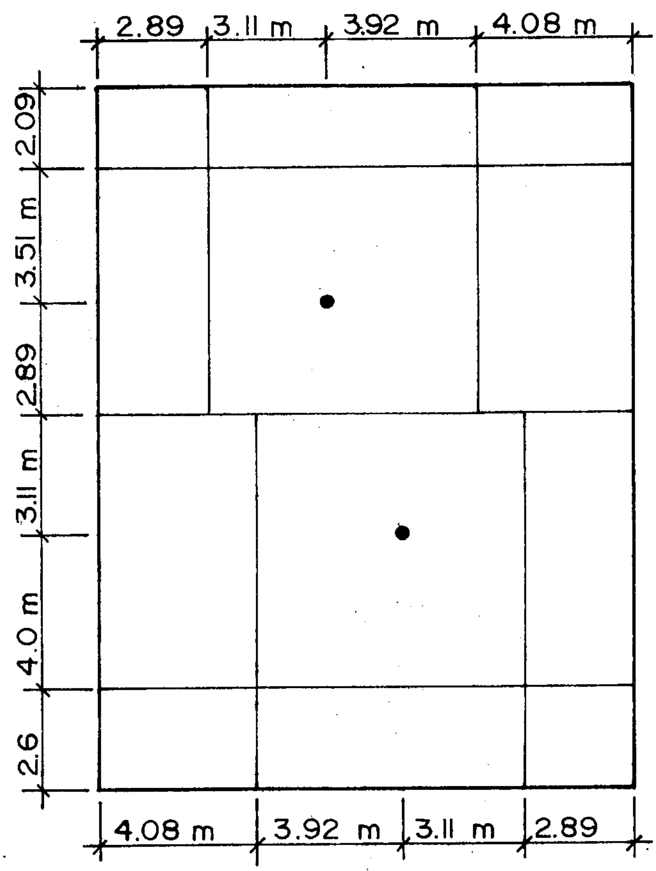


Figure B6.3 Quantitative Segment Layout

STEP 3. RATIONALIZE MOMENT FIELD

For the negative moments over columns, use $\alpha = \beta = 1/2$. Thus, the negative design moments are twice as large as the mean moments. (Base on the larger of the two moments.) The positive moments between the columns should be concentrated so that all of the positive moments occur where there is direct contact between corner supported segments. Therefore:

$$m^+_{y3} = m^+_{y4} = m^+_{y5} = m^+_{y6} = \frac{18.0(3.11 + 3.92)}{14 - (4.08 + 4.08)} = 21.7 \text{ kNm/m}$$

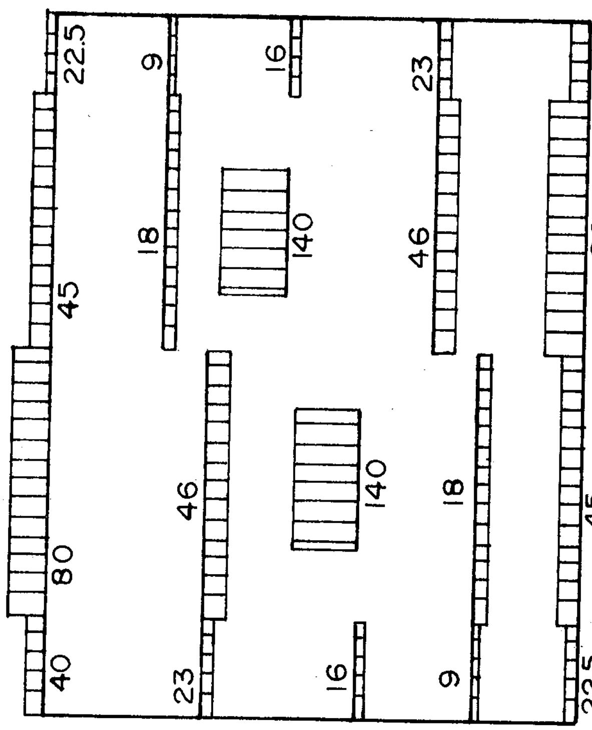


Figure B6.4 Design Moments in X Direction (kNm/m)

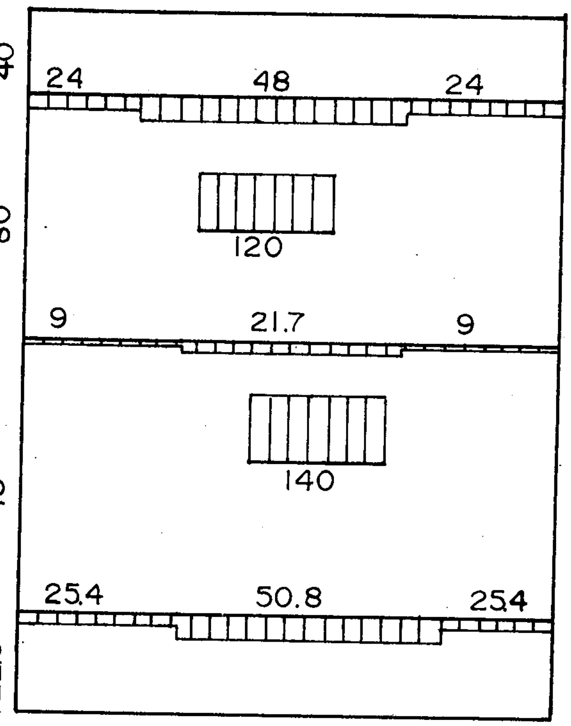


Figure B6.5 Design Moments in Y Direction (kNm/m)

STEP 4. DETERMINE COLUMN LOADS AND MOMENTS

Column 1. $P = 15.1(3.11+3.92)(2.89+3.51) = 679$ kN

$M_x = 15\ (2.89 + 3.51) = 96$ kNm ↑

$M_y = 1.5\ (3.11 + 3.92) = 105$ kNm ←

Column 2. $P = 15\ (3.92 + 3.11)(4.0 + 3.11) = 755$ kN

$M_x = 15\ (4.0 + 3.11) = 107$ kNm ↓

$M_y = 15\ (3.92 + 3.11) = 105$ kNm →