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Finiteness Conditions on Ordered Groups

BY

© Yangkok Kim

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TO MY SUPERVISOR, A. H. RHEMTULLA

ABSTRACT

In this thesis, we investigate some finiteness conditions on the subclass of torsion free groups.

In chapter 2, we point out that ordered groups satisfying bounded Engel condition are nilpotent. We then show that several other finiteness conditions also give a similar nice structure to orderable groups. The techniques used come from recent studies on residually finite p-groups and from results of Zelmanov for Engel groups. In following chapter, more relaxed finiteness conditions are introduced and we characterize orderable groups with these conditions.

In chapter 4, we call a group G restrained if there exists an integer n such that $\langle x^{\langle y \rangle} \rangle$ can be generated by n elements for all x, y in G. We show that a group G is polycyclic-by-finite if and only if G is a finitely generated restrained group in which every non-trivial finitely generated subgroup has a non-trivial finite quotient. This provides a general setting for various results in soluble and residually finite groups that have appeared recently.

In chapter 5, we introduce a new group concept, namely, finite base which is similar to the concept of finite rank for the class of orderable groups.

In last chapter, a group G is said to preserve the cardinality of 2-element subsets product under permutations, or G is a $PC(2,n)$-group if $G = 1$ or for each n-tuple $(S_1, \ldots, S_n)$ of 2-element subsets of G, there is a non-identity permutation σ in Σ_n such that the cardinalities of $S_1 S_2 \cdots S_n$ and $S_{\sigma(1)} S_{\sigma(2)} \cdots S_{\sigma(n)}$ are same. Some characterizations of $PC(2,n)$-groups are presented.

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CHAPTER 1

INTRODUCTION

In this thesis, we shall investigate some torsion-free groups with full orders under certain conditions. Each chapter contains the definitions and properties of the conditions needed. In this chapter we recall only the definitions and relations of groups with full order and describe the main theorems of each chapter.

In chapter 2, we study orderable groups satisfying an additional condition such as n-Engel, collapsing or PSP. A group G is said to be orderable (or an O-group) if there exists a full order relation $\leq$ on the set G such that $a \leq b$ implies $xay \leq xby$ for all a, b, x, y in G. The term "ordered group" is used to denote a group endowed with a fixed order. We shall denote the identity element of G by e. A subgroup C of an ordered group G is said to be convex under $\leq$ if $x \in C$ whenever $e \leq x \leq c$ for some $c \in C$.

In [R3], an O-group G was characterized in terms of the system of its subgroups. Convex subgroups of an ordered group G play important roles in studying the whole group G. For example, if the convex subgroups of G are all normal, then G' has a central system with torsion-free factors. So we start chapter 2 by showing this property. The main results of chapter 2 will be the following.

THEOREM 2.3. *A bounded Engel O-group is nilpotent.*

THEOREM 2.4. *A group is torsion-free nilpotent if and only if it is orderable and collapsing.*

THEOREM 2.5. *An orderable group is a PSP-group if and only if it is abelian.*

In chapter 3, we continue orderable groups under more relaxed conditions, the restricted Engel and $WPSP$ conditions. We have the following characterizations of orderable groups with these conditions.

THEOREM 3.5. *A restricted Engel O-group is locally nilpotent.*

THEOREM 3.12. *An orderable $WPSP$-group is abelian.*

Now we extend our study to the larger class of groups, for example, locally indicable groups or right-orderable groups. We give definitions and some relations of these groups and refer the reader to [B5] for a further reference.

A group G is called right-orderable (or an RO-group) if there exists a full order relation $\leq$ on the set G such that $a \leq b$ implies $ac \leq bc$ for all a, b, c in G. If $a \geq e$, a is called positive. A group G is called locally indicable if every finitely generated non-trivial subgroup of G has an infinite cyclic quotient.

Let G be a right-ordered group. Just as with orderable groups, we define a subgroup N of G to be convex if for every $g \in G$ and $c \in N$, $e \leq g \leq c$ implies $g \in N$. A group G is a C-group if it admits a right-order that satisfies the following property:

If $C \to D$ is a convex jump in G, then $C \lhd D$ and D/C is isomorphic to a subgroup of the additive group of the real numbers.

Let C^* denote the class of those right orderable groups in which every right-order has the above property. In [C3], P. Conrad showed that the above property is equivalent to the following one:

For each pair of positive elements a, b in G there exists a positive integer n such that $a^n b > a$.

Clearly the class of orderable groups is properly contained in the class of C-groups. Every C-group is locally indicable. And locally indicable groups are RO-groups. In [B1], G. Bergman gave examples of RO-groups which are not locally indicable.

Our theorems in chapter 4 include the following.

THEOREM 4.7. *A locally indicable bounded Engel group is nilpotent.*

THEOREM 4.11. *A finitely generated collapsing RO-group is nilpotent-by-finite.*

In chapter 5 we study a new group property, namely, finite base which is similar to the concept of finite rank for the class of orderable groups. One of the results is the following.

THEOREM 5.9. *If G is a C-group with finite base, then it is soluble.*

Recently there has been a big progress in the study of groups satisfying *"finiteness conditions"*. For example, groups with various permutability conditions were studied in [C2], [B2] and [R5]. Now we consider groups with another type of permutability condition. For an integer $n > 1$, a group G is said to preserve the cardinality of 2-element subsets product under permutations, or G is a $PC(2,n)$-group if $G = 1$ or for each n-tuple $(S_1, \ldots, S_n)$ of 2-element subsets of G, there is a permutation $\sigma(\neq 1)$ in Σ_n such that

$$|S_1 S_2 \cdots S_n| = |S_{\sigma(1)} S_{\sigma(2)} \cdots S_{\sigma(n)}|.$$

where $|S|$ means the cardinality of the set S. Let $PC(2)$ be the class $\bigcup_{n>1} PC(2,n)$. In chapter 6 we will show the following.

THEOREM 6.7. *A finitely generated $PC(2)$-group with finite local trace is center-by-finite.*

THEOREM 6.8. *A finitely generated non-periodic $PC(2)$-group G is center-by-finite.*

CHAPTER 2

BOUNDED ENGEL ORDERABLE GROUPS

We begin with the definitions and properties of classes of groups which we study in this chapter.

BOUNDED ENGEL CONDITION: Let n be a positive integer. A group G is called an *n-Engel* group if $[x,_n y] = 1$ for all x, y in G, where $[x,_n y]$ is defined inductively as follows: $[x,_1 y] = [x, y]$ and $[x,_i y] = [[x,_{i-1} y], y]$ for $i > 1$. G is a ***bounded Engel*** group if it is an n-Engel group for some n. For basic properties of this class of groups see [R6].

COLLAPSING CONDITION: A group G is said to be *n-collapsing* if for any n-element subset S of G, we have $|S^n| < n^n$. We say that G is *collapsing* if it is n-collapsing for some n. Note that G is n-collapsing if and only if for every $x_1, \ldots, x_n$ in G, we have $x_{f(1)} \cdots x_{f(n)} = x_{g(1)} \cdots x_{g(n)}$ for some distinct functions $f, g : \{1, \ldots, n\} \to \{1, \ldots, n\}$. In [S1], J. Semple and A. Shalev introduced this condition and proved the following theorem.

THEOREM. *Let G be a finitely generated residually finite group. Then G is collapsing if and only if G is nilpotent-by-finite.*

PSP CONDITION: Let G be a group. If there exists an integer $n > 1$ such that for each n-tuple $(H_1, ..., H_n)$ of subgroups of G, there is a permutation $\sigma(\neq 1)$ in Σ_n such that the two complexes $H_1 H_2 \cdots H_n$ and $H_{\sigma(1)} H_{\sigma(2)} \cdots H_{\sigma(n)}$ are equal, then G is said to have *the property of permutable subgroup products*, or G is a *PSP-group*. This condition was discussed by A. Rhemtulla and A. Weiss in [R2] where they proved the following result.

THEOREM. *A finitely generated soluble group* G *is a* PSP*-group if and only if it is finite-by-abelian.*

From now on we study orderable groups under each of the above conditions. As we promised in chapter 1, we start with following.

LEMMA 2.1. *Let* C *be a convex subgroup of an ordered group* $(G, \leq)$. *Then* C *is normal in each of the following cases.*

(i) G *is an Engel group.*

(ii) G *is* n*-collapsing for some positive integer* n.

(iii) G *is a* PSP*-group.*

Proof. (i) Suppose not. Then $C \lneq C^x$ or $C^x \lneq C$ for some x in G because the set of convex subgroups forms a chain. Let $C \lneq C^x$ and $a^x \in C^x \backslash C$ where a in C. Since G is an Engel group, there is an integer $n > 0$ such that $[a, {}_n x] = 1$. Hence $\langle a, a^x, a^{x^2}, \dots \rangle = \langle a, a^x, \dots, a^{x^{n-1}} \rangle$. Thus $a^{x^n} \in C^{x^{n-1}}$, and it follows that $a^x \in C$. This is a contradiction.

(ii) If C is not normal in G, then for some a in C and x in G, $a^x \in C^x \backslash C$. Let $S = \{ax^{-1}, ax^{-2}, \dots, ax^{-n}\}$. Then there exist two distinct functions f, g from $\{1, 2, ..., n\}$ to $\{-1, -2, ..., -n\}$ such that

$$\prod_{i=1,\dots,n} ax^{f(i)} = \prod_{i=1,\dots,n} ax^{g(i)}.$$

Let r be the largest integer such that $f(r) \neq g(r)$, let $s(i) = f(1) + \cdots + f(i)$ and $t(i) = g(1) + \cdots + g(i)$. Then we get the equality:

$$aa^{x^{-s(1)}} a^{x^{-s(2)}} \cdots a^{x^{-s(r-1)}} x^{s(r)} = aa^{x^{-t(1)}} a^{x^{-t(2)}} \cdots a^{x^{-t(r-1)}} x^{t(r)}.$$

Since $s(r) = t(r)$ and $f(r) \neq g(r)$, $s(r-1) \neq t(r-1)$ and hence one of the above is infinitely greater than the other. This is a contradiction. Thus C must be normal.

(iii) If C is not normal in G, then $a^x \in C^x \backslash C$ for some $a \in C$ and some $x \in G$. Let $p_1, \ldots, p_n$ be distinct primes and let $t_i = p_1 ... p_n / p_i$ for $i = 1, ..., n$. For each $i = 1, ..., n$, pick positive integers s_i to satisfy the condition $s_{i+1} > 2(s_1 + \cdots + s_i)$. Consider the n cyclic subgroups $H_i = \langle (a^{s_i} x^{-t_i}) \rangle$ of G. By hypothesis there is a permutation $\sigma (\neq 1)$ such that the two complexes $H_1 H_2 \cdots H_n$ and $H_{\sigma(1)} H_{\sigma(2)} \cdots H_{\sigma(n)}$ are equal. Let j be the smallest integer such that $\sigma(j) \neq j$. Say $\sigma(j) = k$, and pick an element $y = (a^{s_1} x^{-t_1}) \cdots (a^{s_k} x^{-t_k})$ in $H_1 H_2 \cdots H_n$. Since $y \in H_{\sigma(1)} H_{\sigma(2)} \cdots H_{\sigma(n)}$, we have $(a^{s_1} x^{-t_1}) \cdots (a^{s_k} x^{-t_k}) = (a^{s_{\sigma(1)}} x^{-t_{\sigma(1)}})^{r_{\sigma(1)}} \cdots (a^{s_{\sigma(n)}} x^{-t_{\sigma(n)}})^{r_{\sigma(n)}}$ for some integers r_i, $i = 1, ..., n$. From now on we write $a(s_i, t_i)$ for $(a^{s_i})^{x^{t_i}}$. Then, by collecting the x's to the right on both sides of the above equality, we obtain

$$a(s_1, 0) a(s_2, t_1) \cdots a(s_k, t_1 + \cdots + t_{k-1}) x^{-(t_1 + \cdots + t_k)} = a(\beta_1, \alpha_1) \cdots a(\beta_m, \alpha_m) x^{-t},$$

where $\beta_i \in \{ \pm s_1, \ldots, \pm s_n \}$ and α_i, t are integers. Since $t_1 + t_2 + \cdots + t_k = t$, we get $a(s_1, 0) a(s_2, t_1) \cdots a(s_k, t_1 + \cdots + t_{k-1}) = a(\beta_1, \alpha_1) \cdots a(\beta_m, \alpha_m)$. Let $\alpha = max\{\alpha_i; i = 1, \ldots, m\}$ and suppose there are q $a(\beta_i, \alpha_i)$'s with $\alpha_i = \alpha$, say, $a(\beta_{11}, \alpha), \ldots, a(\beta_{1q}, \alpha)$. Then by moving all these terms to the right end as above, we get

$$a(s_1, 0) \cdots a(s_k, t_1 + \cdots + t_{k-1}) = c_1 c_2 \cdots c_{m-q} a(\beta_{11} + \cdots + \beta_{1q}, \alpha),$$

where c_j are conjugates of $a(\beta_i, \alpha_i)$ with $\alpha_i < \alpha$. Here note that all β_{1i} are distinct and $\beta_{1i} + \beta_{1j} \neq 0$. Moreover we know that all $c_j \ll a(1, \alpha)$ and so $\alpha \geq t_1 + \cdots + t_{k-1}$. If $\alpha > t_1 + \cdots + t_{k-1}$, then $\beta_{11} + \cdots + \beta_{1q} = 0$. This is impossible by the choice of s_i. This means that $\alpha = \sum_{i \leq j} n_{\sigma(i)} t_{\sigma(i)} = m_1 t_1 + \cdots + m_{j-1} t_{j-1} + m_k t_k$ where $m_{\sigma(i)}$ are integers. So $t_j = \sum_{i \neq j} r_i t_i$ for some integers r_i contrary to the choice of t_j. This completes the proof. □

THEOREM(P. HALL). *Let G be a finitely generated group, let $N \lhd G$ and suppose that G/N is finitely presented. Then N is finitely generated as a G-group.*

LEMMA 2.2. *Let G be an ordered group in which every convex subgroup is normal. If D is a convex subgroup of G and D is finitely generated as a G-group, then there exists a convex subgroup C such that $C \to D$ is a jump.*

Proof. By hypothesis, $D = \langle X^G \rangle$ for some finite subset $X = \{x_1, ..., x_n\}$ of G. Suppose $|x_1| \leq \cdots \leq |x_n|$. Let C be the largest convex subgroup of G that does not contain x_n. Then $C \to D$ is a jump since D is the smallest convex subgroup that contains x_n. □

THEOREM 2.3. *A bounded Engel O-group G is nilpotent.*

Proof. Let G be finitely generated and let $G = G_0$. Then there exists a convex subgroup G_1 such that $G_1 \to G_0$ forms a jump since G_0 is finitely generated. Here G/G_1 is a finitely generated abelian group and so finitely presented. By Theorem(P. Hall) G_1 is finitely generated as a G-group. Moreover all convex subgroups are normal in G by Lemma 2.1. Hence the convex subgroup G_2, where $G_2 \to G_1$ forms a jump, exists by Lemma 2.2. Now G/G_2 is a finitely generated torsion-free soluble n-Engel group and hence nilpotent (see [R6], p.64). Note that finitely generated nilpotent groups are finitely presented (see [R5], p.33). So G_2 is finitely generated as a G-group and hence the convex subgroup G_3, where $G_3 \to G_2$ is a jump, also exists. This produces a descending central series $G = G_0 > G_1 > \cdots$, with torsion-free factors. Let N be the intersection of the G_i's so that G/N is a residually finite p-group for all primes p. Pick any odd prime p. Since G is an n-Engel group, it can not have any section isomorphic to the wreath product of a cyclic group of prime order p and a cyclic group of order greater than p^n. Thus it follows that G/N is linear (see [W], in the proof of Theorem 2). Now since nonabelian free groups are not Engel groups, it follows by Tits' Alternative [T1] that G/N is soluble-by-finite and so nilpotent. Thus $N = G_m$ for some m and hence $N = 1$ and G is nilpotent. Since G is torsion-free, the n-Engel condition implies that G is nilpotent of class depending only on n, independent of the number of generators of G, as shown by Zelmanov in [Z]. Thus an n-Engel orderable group is nilpotent of class $c = c(n)$. □

THEOREM 2.4. *A group G is torsion-free nilpotent if and only if it is orderable and collapsing.*

Proof. Let G be a finitely generated n-collapsing orderable group and let K be the isolator of G' in G. Then we can put an order $\leq$ on G so that $K \to G$ is a jump under this order. For any other jump $C \to D$, we get $[D, K] \leq C$. For the convex subgroups are normal in G by Lemma 2.1, and since the group of order preserving automorphisms of any subgroup of the additive group of reals is isomorphic to a subgroup of the multiplicative group of positive reals, the centralizer of D/C contains the isolator K of G'. Thus if J is any finitely generated subgroup of K, then under the restriction of the order $\leq$ to J, we get, as in the proof of Theorem 2.3, a descending central series $J = J_0 \geq J_1 \geq \cdots$, where the factors are all torsion-free. Let N be the intersection of the J_i's. Then J/N is a residually finite p-group. Now applying Corollary E in [S2] and, noting that if a group is orderable and a finite extension of a nilpotent class c group, then it is itself nilpotent of class c, it follows that J/N is nilpotent of class at most $c = c(n)$. Thus $N = 1$ and hence K is nilpotent of class at most c. Since K is finitely generated as a G-group, there is a subgroup K_1 such that $K_1 \to K$ is a jump under $\leq$. This follows by Lemma 2.1 and Lemma 2.2. Now G/K_1 is a finitely generated metabelian group. Thus by Theorem 4.2 of [S1], it is nilpotent-by-finite. But it is orderable, hence it is nilpotent. Thus G/K_1 is finitely presented and hence K_1 is finitely generated as a G-group, and there is a subgroup K_2 such that $K_2 \to K_1$ is a jump under $\leq$. Thus G/K_2 is a finitely generated abelian-by-nilpotent group. Repeat the above argument and deduce that G/K_2 is nilpotent. In this way we get G/K_m to be nilpotent for all positive integers m, where $G \geq K_1 \geq K_2 \geq \cdots$ is the descending chain of convex subgroups of G under $\leq$. Since G/K_m is nilpotent of class at most $c+1$ for all m, we get $K_m = 1$ for some m. We have so far shown that if G is a finitely generated ordered n-collapsing group then it is torsion-free nilpotent of class bounded by some function of n, independent of the number of generators of G. Thus we can remove the condition that G is finitely generated. To complete the proof we only need observe that a torsion-free nilpotent group is orderable, and n-collapsing for some n by Corollary 2.4 of [S1]. □

THEOREM 2.5. *An orderable group* G *is a* PSP*-group if and only if it is abelian.*

Proof. The proof of this theorem is similar to that of Theorem 2.4. The only changes are that we refer to the main result in [R2] where it is shown that a torsion-free soluble PSP-group is abelian, and to the main result in [L2] where it is shown that a torsion-free residually finite-p PSP-group is abelian. □

Recall that a group G is said to have *finite rank* if there is a positive integer d such that every finitely generated subgroup of G can be generated by d elements. The structure of soluble groups of finite rank is reasonably well known (see [R6]). More recently, A. Lubotzky and A. Mann have shown in [L5] that *a residually finite group of finite rank has a locally soluble subgroup of finite index.* The following result is one of the instances that the finite rank condition gives a nice structure to certain orderable groups.

THEOREM 2.6. *Let* G *be an ordered group of finite rank. If the convex subgroups of* G *are all normal then* G *is nilpotent-by-abelian.*

Proof. By hypothesis, there is a total order $\leq$ on G such that all the convex subgroups of G under this order are normal. Let H be a finitely generated subgroup of G and order H by restricting the order on G to H. Then the convex subgroups of H under $\leq$ are normal in H. Let K be the isolator of H' in H. Then the restriction of the order $\leq$ to K

gives an H-order on K. The convex subgroups of K under this order are again normal in H and as H/K is a finitely generated torsion-free abelian group, we can put an order on H so that $K \to H$ is a convex jump and all the other convex jumps in H arise from the convex jumps in K under $\leq$. Just as in the proof of Theorem 2.4, K centralizes all convex jumps and there is a descending central series $K = K_0 \geq K_1 \geq \cdots$, where the factors are all torsion-free. Let N be the intersection of the K_i's. Then K/N is a residually finite p-group of finite rank, and by the main result of [L5], it is soluble-by-finite of finite rank. Thus there are only finitely many convex subgroups of K and hence $N = K_i$ for some integer i, and hence $N = 1$. This implies that K is nilpotent and it follows that the derived subgroup G' is locally nilpotent. It is also torsion-free and of finite rank. Thus it is nilpotent by the result of Mal'cev in [M] and so G is an abelian extension of a nilpotent subgroup. This completes the proof. □

CHAPTER 3

$WPSP$-GROUPS

DEFINITIONS. A group G is called a *generalized Engel* group if for all x, y in G there exist positive integers r, s such that $[x, {}_r y^s] = 1$, where $[x, {}_r y^s]$ is defined inductively as follows: $[x, {}_1 y^s] = [x, y^s]$, and $[x, {}_i y^s] = [[x, {}_{i-1} y^s], y^s]$ for $i > 1$. In particular, if $s = 1$, we call G an *Engel* group. A group G is called a *restricted Engel* group if there exist positive integers r, s such that for all x, y in G, $[x, {}_r y^s] = 1$. In particular, if $s = 1$, we call G a *bounded Engel* group.

Clearly a restricted Engel group is a generalized Engel group.

LEMMA 3.1. *Let C be a convex subgroup of an ordered group G. Then C is normal if G is a generalized Engel group.*

Proof. Look at Lemma 2.1. □

Let A be a torsion-free abelian group of finite rank and $T \leq Aut(A)$. We extend the action of T to the rational vector space $V = A \otimes_{\mathbf{Z}} \mathbb{Q}$ in a natural way, where $\mathbb{Q}$ is the field of rational number. T acts *rationally irreducibly* on A if A/B is periodic whenever B is a nontrivial T-admissible subgroup of A. Then it is easy to see that T acts rationally irreducibly on A if and only if T is irreducible as a group of linear transformation of V.

LEMMA 3.2. *Let $G = \langle A, y \rangle$ where A is a torsion-free abelian group of finite rank on which $\langle y \rangle$ acts rationally irreducibly. If G is a generalized Engel O-group, then G is abelian.*

Proof. Assume, if possible, that $[a, y] \neq 1$ for some $a \in A$. Now $V = A \otimes_{\mathbb{Z}} \mathbb{Q}$ is an irreducible $\mathbb{Q}\langle y\rangle$-module, hence by Schur's Lemma, the centralizer ring $\Gamma = End_{\mathbb{Q}\langle y\rangle} V$ is a division ring, finite dimensional over $\mathbb{Q}$. The image of $\langle y\rangle$ in $End_{\mathbb{Q}} V$ clearly lies in and spans Γ so that Γ is an algebraic number field. Moreover, regarded as a Γ-space, V is one dimensional. Thus we may consider A to be an additive subgroup of $\mathbb{Q}(\eta)$ for some algebraic number η and the action of conjugation by y as multiplication by η. Since G is a generalized Engel group, $[a,_n y^k] = 1$. In additive notation, $0 = [a,_n y^k] = [-a + a^{y^k},_{n-1} y^k] = [(\eta^k - 1)a,_{n-1} y^k] = (\eta^k - 1)[a,_{n-1} y^k] = (\eta^k - 1)^2 [a,_{n-2} y^k] = a(\eta^k - 1)^n$. Thus $\eta^k = 1$. So $[a, y^k] = 1$ and $[a, y] = 1$ for G is an O-group. Thus y acts trivially on A. □

We recall that a soluble group G is said to be *minimax* if and only if it has a series $1 = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$ in which the factors are cyclic or quasicyclic. We shall call a group *constrained* if and only if there is no prime p for which it has a section isomorphic to $C_p \wr C_\infty$, the standard restricted wreath product of a cyclic group of order p by an infinite cyclic group. This terminology is due to P. H. Kropholler [K3], who proved that *every finitely generated constrained soluble group is minimax and hence has finite rank.*

LEMMA 3.3. *Let G be a finitely generated soluble O-group. If G is a generalized Engel group, then it is nilpotent.*

Proof. Note that G has no section isomorphic to $C_p \wr C_\infty$ because G is a generalized Engel group. Hence G has finite rank. Write $G = \langle A, x_1, \ldots, x_n\rangle$ where A is a non-trivial normal abelian subgroup of G. Let

$M_1(\neq 1)$ be a subgroup of A on which x_1 acts rationally irreducibly. By Lemma 3.2, $[M_1, x_1] = 1$. Let $M_2(\neq 1)$ be a subgroup of M_1 on which x_2 acts rationally irreducibly and then $[M_2, x_1] = [M_2, x_2] = 1$. After finitely many steps we get a non-trivial center Z of G. Recall that in an orderable group G, G/Z and G/H are orderable where Z is the center and H the hypercenter of G (see [F1]). So if necessary, we may replace G by G/Z. And repeat the above argument to deduce that G is hypercentral and so locally nilpotent. □

For a group G, $d(G)$ denotes the minimal cardinality of a set of generators and $G^n = \langle g^n : g \in G \rangle$. Recall that a finite p-group G is *powerful* if p is odd and G/G^p is abelian, or $p = 2$ and G/G^4 is abelian. In [L4] A. Lubotzky and A. Mann proved the following theorem: *if G is a powerful p-group and $H \leq G$ then $d(H) \leq d(G)$,* and in [L2], the authors showed that *if a finite p-group G has no section isomorphic to $C_p \wr C_{p^{k-1}}$, then $G^{p^{k+1}}$ is powerful.* In the following lemma, $\gamma_r(G) = [\gamma_{r-1}(G), G]$ and $\gamma_1(G) = G$.

LEMMA 3.4. *If G is a finitely generated residually finite p-group and restricted Engel group, then it is soluble-by-finite.*

Proof. Let $G_r = \gamma_r(G)G^{p^r}$ for all r and $d(G) = n$. Then G/G_r is a finitely generated nilpotent group with finite exponent and so it is a finite p-group and $\bigcap_r G_r = 1$. Since G/G^r is a restricted Engel group, it has no section isomorphic to $C_p \wr C_{p^{k-2}}$ for some k. By [L2], $G_r G^{p^k}/G_r$ is a powerful p-group for all r. Now we consider the subgroup $H = \bigcap_r G_r G^{p^k}$ of G. Then G/H is a finitely generated residually finite group with finite exponent p^k. By Zelmanov's theorem in [Z], G/H is finite and nilpotent. Let G/H be nilpotent of class c and $[G : H] = m$. Then for all $r \geq \ell \geq a = ck$, $d(G_r G^{p^k}/G_r) \leq d(G_r G^{p^k}) \leq 2nm$ for $[G : G_r G^{p^k}] \leq m$. Note that $\gamma_\ell(G) \subseteq G_r G^{p^k}$ and $G_\ell = \gamma_\ell(G) G^{p^\ell} \subseteq G_r G^{p^k}$ if ℓ is a multiple of k. Since $G_r G^{p^k}/G_r$ is powerful, $d(G_\ell/G_r) \leq d(G_r G^{p^k}/G_r) \leq 2nm$. Hence $G \supset G_a \supset G_{2a} \supset G_{3a} \supset \cdots$ forms a p-congruence structure with a bound $2nm$ introduced by A. Lubotzky [L3]. Thus G is a finitely generated linear group by the main theorem in [L3]. Since non-abelian free groups are not restricted Engel groups, G is soluble-by-finite by Tits' Alternative. □

Now the proof of the following theorem is quite similar to that of Theorem 2.3. However for the completeness, we include it.

THEOREM 3.5. *A restricted Engel O-group G is locally nilpotent.*

Proof. Let $G = G_0$ be finitely generated and let G_1 be a convex subgroup of G such that $G_1 \to G_0$ forms a jump. G_1 exists since G_0 is finitely generated. The convex subgroup G_2, where $G_2 \to G_1$ forms a jump, exists by Lemma 2.2, since all convex subgroups are normal in G by Lemma 3.1 and G_1 is finitely generated as a G-group. Now G/G_2 is a finitely generated restricted Engel soluble O-group. By Lemma 3.3, G/G_2 is nilpotent. Since finitely generated nilpotent groups are finitely presented, G_2 is finitely generated as a G-group and hence the convex subgroup G_3, where $G_3 \to G_2$ is a jump, also exists. This produces a descending central series $G = G_0 > G_1 > \cdots$, with torsion-free factors. Let N be the intersection of the G_i's so that G/N is a residually finite p-group for all primes p. By

Lemma 3.4, G/N is soluble-by-finite. Since G/N is orderable, it is nilpotent by Lemma 3.3. Thus $N = G_m$ for some m and hence $N = 1$. □

DEFINITION. Let G be a group. If there exists an integer $n > 1$ such that for each n-tuple $(H_1, ..., H_n)$ of subgroups of G, there are two distinct permutations σ, τ such that the two complexes $H_{\sigma(1)}H_{\sigma(2)} \cdots H_{\sigma(n)}$ and $H_{\tau(1)}H_{\tau(2)} \cdots H_{\tau(n)}$ are equal, then G is said to have *the property of weak permutable subgroup products*, or G is a *WPSP-group.*

LEMMA 3.6. *Let G be a $WPSP$ ordered group. Then the convex subgroups of G are all normal.*

Proof. If C is not normal in G, then $a^x \gg a$ for some $a \in C$ and $x \in G$. Let $p_1, p_2, \ldots, p_n$ be distinct primes and let $t_i = p_1 p_2 \cdots p_n / p_i$ for $i = 1, 2, ..., n$. For each $i = 1, 2, ..., n$, pick a positive integer s_i to satisfy the condition $s_{i+1} > 2(s_1 + s_2 + \cdots + s_i)$. Now we consider the n cyclic subgroups $H_i = \langle (a^{s_i} x^{-t_i}) \rangle$ of G. Since G is a $WPSP$-group, there are two distinct permutations σ, τ such that $H_{\sigma(1)}H_{\sigma(2)} \cdots H_{\sigma(n)}$ and $H_{\tau(1)}H_{\tau(2)} \cdots H_{\tau(n)}$ are same. Let j be the smallest integer such that $\sigma(j) \neq \tau(j)$ and let $\sigma(j) < \tau(j) = \sigma(q)$. Suppose there are k positive integers $y_1 < y_2 < \cdots < y_k$ such that $y_i < q$ and $\sigma(y_i) < \sigma(q)$. We pick an element

$$y = (a^{s_{\sigma(y_1)}} x^{-t_{\sigma(y_1)}}) \cdots (a^{s_{\sigma(y_k)}} x^{-t_{\sigma(y_k)}})(a^{s_{\sigma(q)}} x^{-t_{\sigma(q)}}) \text{ in } H_{\sigma(1)}H_{\sigma(2)} \cdots H_{\sigma(n)}.$$

Since $y \in H_{\tau(1)}H_{\tau(2)} \cdots H_{\tau(n)}$, $y = (a^{s_{\tau(1)}} x^{-t_{\tau(1)}})^{r_{\tau(1)}} \cdots (a^{s_{\tau(n)}} x^{-t_{\tau(n)}})^{r_{\tau(n)}}$ for some integers $r_i, i = 1, 2, \ldots, n$. We write $a(s_j, t_j)$ for $(a^{s_j})^{x^{t_j}}$. Then, by collecting the x's to the right on the above two expressions of y and cancelling x, we get

$$a(s_{\sigma(y_1)},0)a(s_{\sigma(y_2)},t_{\sigma(y_1)})\cdots a(s_{\sigma(q)},t_{\sigma(y_1)}+\cdots+t_{\sigma(y_k)}) = a(\beta_1,\alpha_1)\cdots a(\beta_m,\alpha_m),$$

where $\beta_i \in \{\ \pm s_1,\dots.\pm s_n\ \}$ and α_i are integers. Let $\alpha = max\{\alpha_i; i = 1,\dots,m\}$ and suppose there are d $a(\beta_i,\alpha_i)$'s with $\alpha_i=\alpha$, say, $a(\beta_{11},\alpha)$, $\dots, a(\beta_{1d},\alpha)$. Then by moving these d factors to the right end as above, we get

$$a(s_{\sigma(y_1)},0)\cdots a(s_{\sigma(q)},t_{\sigma(y_1)}+\cdots+t_{\sigma(y_k)}) = c_1c_2\cdots c_{m-d}a(\beta_{11}+\cdots+\beta_{1d},\alpha),$$

where c_j are conjugates of $a(\beta_i,\alpha_i)$ with $\alpha_i<\alpha$. Note that all β_{1i} are distinct and $\beta_{1i}+\beta_{1j}\neq 0$. Moreover $c_j \ll a(1,\alpha)$ and so $\alpha \geq t_{\sigma(y_1)}+\cdots+t_{\sigma(y_k)}$. If $\alpha > t_{\sigma(y_1)}+\cdots+t_{\sigma(y_k)}$, then $\beta_{11}+\cdots+\beta_{1d}=0$. This is impossible by the choice of s_i. So $\alpha = t_{\sigma(y_1)}+\cdots+t_{\sigma(y_k)}$. Hence $\beta_{11}+\cdots+\beta_{1d}=s_{\sigma(q)}$. By the choice of s_i, d must be 1, i.e., $\beta_{11}=s_{\sigma(q)}$. Furthermore

$$\begin{aligned}\alpha = \sum_{i\leq j} m_{\tau(i)}t_{\tau(i)} &= m_{\tau(1)}t_{\tau(1)}+\cdots+m_{\tau(j-1)}t_{\tau(j-1)}+m_{\tau(j)}t_{\tau(j)}\\ &= m_{\sigma(1)}t_{\sigma(1)}+\cdots+m_{\sigma(j-1)}t_{\sigma(j-1)}+m_{\sigma(q)}t_{\sigma(q)}.\end{aligned}$$

So $t_{\sigma(j)}=\sum_{i\neq\sigma(j)} r_it_i$ for some $r_i\in\mathbb{Z}$. This is a contradiction to the choice of t_i. □

LEMMA 3.7. *The wreath product of a cyclic group of order p and an infinite cyclic group is not a $WPSP$-group.*

Proof. Suppose that $G=\langle a\rangle \wr \langle x\rangle$ is a $WPSP$-group where $\langle a\rangle$ is a cyclic group of order p and $\langle x\rangle$ an infinite cyclic group. And we denote

$\mathrm{Dr}_{i\in\mathbf{Z}}\langle a_i\rangle$ as the base group of G. Let $p_1, p_2, \ldots, p_n$ be distinct primes greater than n, $t_i = p_1p_2\cdots p_n/p_i$ and $s_i = iu$ where a prime $u > p_1p_2\cdots p_n$ for $i = 1, 2, \ldots, n$. Now we consider the n cyclic subgroups $H_i = \langle (a_{s_i}^{-1}x^{-t_i}a_{s_i})\rangle$ of G. Then there are two distinct permutations σ, τ such that $H_{\sigma(1)}H_{\sigma(2)}\cdots H_{\sigma(n)}$ and $H_{\tau(1)}H_{\tau(2)}\cdots H_{\tau(n)}$ are same. Let j be the smallest integer such that $\sigma(j) \neq \tau(j)$ and let $\sigma(j) < \tau(j) = \sigma(q)$. Suppose there are k positive integers $y_1 < y_2 < \cdots < y_k$ such that $y_i < q$ and $\sigma(y_i) < \sigma(q)$. Note that $j = y_i$ for some i. We pick an element in $H_{\sigma(1)}H_{\sigma(2)}\cdots H_{\sigma(n)}$

$$y = (a_{s_{\sigma(y_1)}}^{-1}x^{-t_{\sigma(y_1)}}a_{s_{\sigma(y_1)}})\cdots(a_{s_{\sigma(y_k)}}^{-1}x^{-t_{\sigma(y_k)}}a_{s_{\sigma(y_k)}})(a_{s_{\sigma(q)}}^{-1}x^{-t_{\sigma(q)}}a_{s_{\sigma(q)}}).$$

Since y lies in $H_{\tau(1)}H_{\tau(2)}\cdots H_{\tau(n)}$, we have another expression for y, say

$$y = (a_{s_{\tau(x_1)}}^{-1}x^{-t_{\tau(x_1)}}a_{s_{\tau(x_1)}})^{r_{\tau(x_1)}}\cdots(a_{s_{\tau(x_m)}}^{-1}x^{-t_{\tau(x_m)}}a_{s_{\tau(x_m)}})^{r_{\tau(x_m)}}$$

for some integers $r_{\tau(x_i)} \neq 0$, $i = 1, 2, \ldots, m$. We write $a(s_j + t_j)$ for $(a_{s_j})^{x^{t_j}}$ and $a^{-1}(s_j + t_j)$ for $(a_{s_j}{}^{-1})^{x^{t_j}}$. Then, by collecting the x's to the right on the above two expressions of y and cancelling x, we get

$$\text{(3.8)}\quad a^{-1}(s_{\sigma(y_1)})a(s_{\sigma(y_1)} + t_{\sigma(y_1)})\cdots a(s_{\sigma(q)} + t_{\sigma(y_1)} + \cdots + t_{\sigma(y_k)} + t_{\sigma(q)})$$
$$= a^{-1}(s_{\tau(x_1)})a(s_{\tau(x_1)} + r_{\tau(x_1)}t_{\tau(x_1)})\cdots a(s_{\tau(x_m)} + r_{\tau(x_1)}t_{\tau(x_1)} + \cdots + r_{\tau(x_m)}t_{\tau(x_m)})$$

Note that $t_{\sigma(y_1)} + \cdots + t_{\sigma(y_k)} + t_{\sigma(q)} = r_{\tau(x_1)}t_{\tau(x_1)} + \cdots + r_{\tau(x_m)}t_{\tau(x_m)}$.

Case (i). $q = n$ and $\sigma(q) = n$.

Then $a(s_{\tau(j)} + r_{\tau(1)}t_{\tau(1)} + \cdots + r_{\tau(j-1)}t_{\tau(j-1)})$ is different from all other factors in the left hand side of (3.8).

Case (ii). In other case.

By the choice of t_i and s_i, all factors in each side of (3.8) are distinct. Moreover $a(s_{\sigma(y_1)})$ must be $a(s_{\tau(x_1)})$. So $\sigma(y_1) = \tau(x_1)$. Similarly we get

$$\sigma(y_1) = \tau(x_1),\ \sigma(y_2) = \tau(x_2),\ \ldots,\ \sigma(q) = \tau(x_m) = \tau(j).$$

Hence $\sigma(j) = \tau(x_\ell)$ for some $x_\ell < j$, a contradiction. □

Note that the above Lemma 3.7 implies that finitely generated soluble $WPSP$-groups have finite rank.

LEMMA 3.9. *Let $G = \langle A, t\rangle$ where A is a torsion-free abelian group of finite rank on which $\langle t\rangle$ acts rationally irreducibly. If G is a $WPSP$-group, then $\langle t\rangle$ acts trivially on A.*

Proof. Suppose that $[a,t] \neq 1$ for some $a \in A$. Then as in the proof Lemma 3.2, we may consider A to be an additive subgroup of $\mathbb{Q}(\eta)$ for some algebraic number η and the action of conjugation by t as multiplication by η. Now there are two cases according as η is a root of 1 or not.

If η is a root of 1, say it is a primitive k th root of 1, then pick any $1 \neq b$ in A. Let $a = [t,b] = b(1-\eta)$ and m a positive integer to be fixed. Now we consider n subgroups $H_i = \langle (a^{m^i} t^{-1})\rangle$. Since G is a $WPSP$-group, there are two distinct permutations σ, τ such that $H_{\sigma(1)}H_{\sigma(2)}\cdots H_{\sigma(n)}$ and $H_{\tau(1)}H_{\tau(2)}\cdots H_{\tau(n)}$ are same. Note that there are $i < j, i' < j'$ such that $q = \sigma(i) = \tau(j')$ and $\sigma(j) = \tau(i') = p$. Now we consider an element in $H_{\tau(1)}H_{\tau(2)}\cdots H_{\tau(n)}$, $y = (a^{m^{\tau(i')}}t^{-1})(a^{m^{\tau(j')}}t^{-1})^{-1} = b(m^{\tau(i')} - m^{\tau(j')})(1-\eta) =$

$b(m^p - m^q)(1-\eta)$. Since y lies in $H_{\sigma(1)}H_{\sigma(2)}\cdots H_{\sigma(n)}$, we have another expression of y, say,

$$\begin{aligned} y &= (a^{m^{\sigma(1)}}t^{-1})^{\lambda_{\sigma(1)}}(a^{m^{\sigma(2)}}t^{-1})^{\lambda_{\sigma(2)}}\cdots(a^{m^{\sigma(n)}}t^{-1})^{\lambda_{\sigma(n)}} \\ &= b\big(m^{\sigma(1)}(1-\eta^{[\lambda_{\sigma(1)}]}) + m^{\sigma(2)}(1-\eta^{[\lambda_{\sigma(2)}]})\eta^{[\lambda_{\sigma(1)}]} + \cdots \\ &\qquad \cdots + m^{\sigma(n)}(1-\eta^{[\lambda_{\sigma(n)}]})\eta^{[\lambda_{\sigma(1)}+\lambda_{\sigma(2)}+\cdots+\lambda_{\sigma(n-1)}]}\big). \end{aligned}$$

where [ ] denotes residue class modulo k. Note that each power of m has only finitely many possible coefficients which are independent on $\lambda_{\sigma(j)}$. Let S be the set of all possible coefficients of $m^{\sigma(i)}, i = 1, \ldots, n$. Now we can fix a required m such that if $s_1m^1 + \cdots + s_nm^n = s_1'm^1 + \cdots + s_n'm^n$, then $s_i = s_i'$ for all i where $s_i, s_i' \in S$. Then $[\lambda_\ell] = 0$ for $\ell \neq p, q$ and $[\lambda_p + \lambda_q] = 0$. And so we get

$$\begin{aligned} (m^p - m^q)(1-\eta) &= m^q(1-\eta^{[\lambda_q]}) + m^p(1-\eta^{[\lambda_p]})\eta^{[\lambda_q]} \\ &= m^q - m^q\eta^{[\lambda_q]} + m^p\eta^{[\lambda_q]} - m^p. \end{aligned}$$

So $2(m^p - m^q) = (m^p - m^q)(\eta + \eta^{[\lambda_q]})$. This is impossible.

Suppose that η is not a root of unity. Note that $\Gamma = End_{\mathbb{Q}\langle t\rangle}V$ can be embedded in $\mathbb{C}$ so that $|\eta| > 1$ (see [H1], p.122) where $V = A \otimes_{\mathbb{Z}} \mathbb{Q}$. So we assume $|\eta| > 1$. Let $m(i) = m^i$, $m > 2n$ and k be any integer not equal to zero. And consider n subgroups $H_i = \langle (t^k)^{a^{m(i)}} \rangle$ of G. Now we have two distinct permutations σ, τ such that $H_{\sigma(1)}H_{\sigma(2)}\cdots H_{\sigma(n)} = H_{\tau(1)}H_{\tau(2)}\cdots H_{\tau(n)}$. Note there are $i < j, i' < j'$ such that $\sigma(i) = \tau(j')$ and $\sigma(j) = \tau(i')$.

Consider an element $y = (t^{a^{m(\tau(i'))}})^{-k}(t^{a^{m(\tau(j'))}})^{k}$ in $H_{\tau(1)}H_{\tau(2)}\cdots H_{\tau(n)}$. In additive notation, this is

$$\begin{aligned} y &= -m(\tau(i'))a + m(\tau(j'))a + \big(m(\tau(i')) - m(\tau(j'))\big)a\eta^k \\ &= -m(\sigma(j))a + m(\sigma(i))a + \big(m(\sigma(j)) - m(\sigma(i))\big)a\eta^k \\ &= a\Big(-m(\sigma(j)) + m(\sigma(i)) + \big(m(\sigma(j)) - m(\sigma(i))\big)\eta^k\Big). \end{aligned}$$

Since y is in $H_{\sigma(1)}H_{\sigma(2)}\cdots H_{\sigma(n)}$, $y = (t^{a^{m(\sigma(i_1))}})^{k\gamma_{\sigma(i_1)}}\cdots(t^{a^{m(\sigma(i_r))}})^{k\gamma_{\sigma(i_r)}}$ where $\gamma_{\sigma(i_j)} \neq 0$, $j = 1, \ldots, r$. Then

$$\begin{aligned} y = a\Big(&-m(\sigma(i_1)) + \big(m(\sigma(i_1)) - m(\sigma(i_2))\big)\eta^{k\gamma_{\sigma(i_1)}} + \cdots \\ &\cdots + \big(m(\sigma(i_{r-1})) - m(\sigma(i_r))\big)\eta^{k(\gamma_{\sigma(i_1)}+\cdots+\gamma_{\sigma(i_{r-1})})} + m(\sigma(i_r))\Big). \end{aligned}$$

Hence for each k we have the following expression.

$$-m(\sigma(j)) + m(\sigma(i)) = c_1\eta^{ky_{k1}} + c_2\eta^{ky_{k2}} + \cdots + c_{\ell(k)}\eta^{ky_{k\ell(k)}}$$

where $y_{k1} < y_{k2} < \cdots < y_{k\ell(k)}$ and no subsum of the right hand side is zero. Note that there is an integer M such that $\sum_{i=1}^{\ell(k)}|c_i| \leq M$ for all k because each power of η has only finitely many possible coefficients which are independent on $\gamma_{\sigma(i)}$. Now pick k such that $|\eta^k| > M - 1 + |m(\sigma(i)) - m(\sigma(j))|$. Then we get a contradiction. □

Here we mention the following properties. The proofs are the same as the ones of corresponding ones in [R2].

LEMMA 3.10. *A torsion-free nilpotent* $WPSP$*-group is abelian.* □

THEOREM 3.11. *Let* G *be a finitely generated soluble group. Then* G *is a* $WPSP$*-group if and only if* G *is finite-by-abelian.* □

THEOREM 3.12. *An orderable* $WPSP$*-group is abelian.*

Proof. Follow the same procedure in the proof of Theorem 2.5 with above two results. □

CHAPTER 4

LOCALLY INDICABLE GROUPS

DEFINITIONS. A group G is said to be *weakly restrained* if $\langle x^{\langle y\rangle}\rangle$ is finitely generated for all x,y in G. And we call G *restrained* if there exists an integer n such that $\langle x^{\langle y\rangle}\rangle$ can be generated by n elements for all x,y in G. In this case, G would be called *n-restrained.*

LEMMA 4.1. *A group G is restrained if*

(i) it is an n-Engel group for some positive integer n , or

(ii) it is n-collapsing for some positive integer n.

Proof. (i) Let x,y be elements in G. Then the subgroup generated by $\{x,[x,y],\ldots,[x,{}_ry]\}$ is precisely the subgroup generated by $\{x,x^y,\ldots,x^{y^r}\}$. This can be easily seen by inducting on r. Thus if G is an n-Engel group, then $\langle x^{\langle y\rangle}\rangle=\langle x,x^y,\ldots,x^{y^{n-1}}\rangle$.

(ii) Let $S=\{\ xy^{-1},\ xy^{-2},\ldots,\ xy^{-n}\ \}$. Then there exist two distinct functions f,g from $\{1,2,...,n\}$ to $\{-1,-2,...,-n\}$ such that

$$\prod_{i=1,\ldots,n} xy^{f(i)}=\prod_{i=1,\ldots,n} xy^{g(i)}.$$

Let r be the largest integer such that $f(r)\neq g(r)$, let $s(i)=f(1)+\cdots+f(i)$ and $t(i)=g(1)+\cdots+g(i)$. Then we get the equality;

$$xx^{y^{-s(1)}}x^{y^{-s(2)}}\cdots x^{y^{-s(r-1)}}y^{s(r)}=xx^{y^{-t(1)}}x^{y^{-t(2)}}\cdots x^{y^{-t(r-1)}}y^{t(r)}.$$

If $s(r)\neq t(r)$, then $y^k\in\langle x^{\langle y\rangle}\rangle$ for some $k>0$. Letting m be the least positive integer such that $y^m\in\langle x^{\langle y\rangle}\rangle$, we get $\langle x^{\langle y\rangle}\rangle=\langle y^m,x^{y^i};0\leq i<m\rangle$.

Note that $m < n^2$. If $s(r) = t(r)$ and $f(r) \neq g(r)$, then $s(r-1) \neq t(r-1)$, say, $s(r-1) < t(r-1)$. Then $x^{y^{s(r-1)}} \in \langle x, x^y, \ldots, x^{y^{s(r-1)-1}} \rangle$ and $\langle x^{\langle y \rangle} \rangle = \langle x^{y^i}; s(r-1) < i < -s(r-1) \rangle$, requiring fewer than $2n^2$ generators. □

LEMMA 4.2. *A group G is weakly restrained if*

(i) it is a generalized Engel group, or

(ii) it is collapsing, or

(iii) it satisfies the maximal condition locally.

Proof. (i) and (iii) are clear. The proof of (ii) is identical to that of Lemma 4.1. □

LEMMA 4.3. *Let G be a finitely generated weakly restrained group. If H is a normal subgroup of G such that G/H is cyclic, then H is finitely generated.*

Proof. For some $g \in G$, we can write G in the form $H\langle g \rangle$. Since G is finitely generated, there exist $h_1, h_2, \ldots, h_r$ in H such that $G = \langle h_1, h_2, \ldots, h_r, g \rangle$ and $H = \langle h_1, h_2, \ldots, h_r \rangle^G$. For each $i = 1, \ldots, r$, $\langle h_i^{\langle g \rangle} \rangle$ is finitely generated, say, $\langle h_i^{\langle g \rangle} \rangle = \langle h_{i1}, h_{i2}, \ldots, h_{id(i)} \rangle$. Now let $H_1 = \langle h_{i\ell(i)}; 1 \leq i \leq r, 1 \leq \ell(i) \leq d(i) \rangle$. Then clearly g lies in $N_G(H_1)$, the normalizer of H_1 in G and $\langle h_1, \ldots, h_r \rangle \leq H_1$. Hence $N_G(H_1) = G$. This means that $H_1 = H$ and H is finitely generated. □

COROLLARY 4.4. *Let G be a finitely generated weakly restrained group. Then G' is finitely generated.*

Proof. This result follows readily from repeated use of Lemma 4.3. □

In particular if G is a finitely generated weakly restrained soluble group then G is polycyclic. Using Tits' Alternative it also follows that a finitely generated weakly restrained linear group is polycyclic-by-finite.

Now we need to recall the definition of twisted wreath product (se N3]) to introduce the theorem proved by J. Wilson in [W]. Let E, H be fi e groups and L a subgroup of H, and suppose that a faithful action σ of L on E is given. We may form the direct product B of $|H : L|$ copies of E, and define a faithful action of H on B in such a way that H permutes transitively the copies of E and L acts on the first of these copies according to the action σ. The split extension of B by H with this action is the *twisted wreath product* of E by H ; it will be denoted by $E\ twr_L H$ and reference to the action σ will be suppressed. In particular if $L = 1$, the group becomes the standard wreath product $E \wr H$.

THEOREM(J. WILSON). *Let G be a finitely generated residually finite group and let n be a positive integer. Suppose that G has no sections isomorphic to groups $E\ twr_L\ H$, with H finite and cyclic, E an elementary abelian p-group on which L acts faithfully and irreducibly, and $[H : L] > n$. Then G is a finite extension of a soluble minimax group.*

NOTE 4.5. *If $G = E\ twr_L H$ is a twisted wreath roduct with $[H : L] > n$, then G is not n-restrained.* □

DEFINITION. We say a group G has *finite local trace*, or G is a *locally graded group* if every finitely generated non-trivial subgroup of G has a non-trivial finite quotient.

Now we introduce an example in which the short Lemma 4.3 plays some role in the study of infinite groups which have appeared recently.

In [P], O. Puglisi and L. Spiezia have defined the class of groups, $\mathcal{E}_k^*$ which consists of groups satisfying the following property: For every pair X, Y of infinite subsets of G, there exist some x in X and y in Y, such that $[x, {}_k y] = 1$. And they proved that *every infinite hyperabelian $\mathcal{E}_k^*$-group is a k-Engel group.*

Now we prove that every infinite $\mathcal{E}_k^*$-group G with finite local trace is a k-Engel group. Note that every hyperabelian group has finite local trace. Here we can assume that G is not locally finite because of Theorem B in [P]. Let G be finitely generated and let N be the finite residual of G. Write $N = \bigcap_\lambda N_\lambda$ where $N_\lambda \lhd G$ and G/N_λ is finite. Since G is infinite, N_λ is infinite. Hence G/N_λ is a k-Engel group for all λ and so is G/N. Thus G/N has no section isomorphic to $E\ twr_L H$ with $[H : L] > k$. By Theorem(J. Wilson) G/N is soluble-by-finite and nilpotent. Note that G is weakly restrained. So N is finitely generated by Lemma 4.3. Let N be infinite. Since N has finite local trace, there is a normal subgroup K of N such that N/K is finite. Since K is infinite, N/K is a k-Engel group and so nilpotent. Thus $N \neq N'$ and G/N' is abelian-by-nilpotent. Hence G/N' is residually finite. This contradicts to the choice of N. So N is finite. Thus $C_G(N)$ has finite index in G. So it contains an infinite abelian subgroup A. For every x, y in N, $[xa, {}_k(yb)] = [x, {}_k y] = 1$ for some a, b in A. Thus N is a finite k-Engel group and nilpotent. Again G/N' is residually finite. Hence G is a k-Engel group. □

Definition. Let G be an infinite group. For every pair X, Y of infinite subsets of G, there exist some x in X and y in Y, such that $\langle x^{\langle y \rangle} \rangle$ is generated by n elements. Then G is called an $R_n^\sharp$-group.

Clearly every infinite restrained group is an $R_n^\sharp$-group for some n. Let G be an $R_n^\sharp$-group and x, y in G. If y has finite order or some power of y centralizes x, then $\langle x^{\langle y \rangle} \rangle$ is finitely generated. In the other case we consider two infinite subsets of G, $X = \{x, x^y, x^{y^2}, \dots\}$ and $Y = \{y, y^2, \dots\}$. Then $\langle (x^{y^m})^{\langle y^\ell \rangle} \rangle$ is finitely generated for some m, ℓ and so is $\langle x^{\langle y \rangle} \rangle$. Thus we get the following.

Note 4.6. *If G is an $R_n^\sharp$-group, then it is weakly restrained.* □

Theorem 4.7. *An infinite group G is polycyclic-by-finite if and only if it is a finitely generated $R_n^\sharp$-group with finite local trace.*

Proof. If H is polycyclic and r is the length of a series from 1 to H with cyclic factors, then every subgroup H can be generated by r elements as can be seen via induction on r. Now if $H \lhd G$ and G/H is of order s then every subgroup of G can be generated by $s + r$ elements and G is restrained. That a polycyclic-by-finite group G is residually finite is well known. Thus we have shown one way implication.

First we assume that G is a finitely generated residually finite group. We claim that G has no section isomorphic to $E \ twr_L H$ with $[H : L] > n$. Suppose that G has a section $K \lhd M$ such that $M/K \cong E \ twr_L H$. Let M be infinite. For every $x, y \in M \backslash K$, consider two infinite sets Kx and Ky. Since G is an $R_n^\sharp$-group, there exist k_1, k_2 in K such that $\langle (k_1 x)^{\langle (k_2 y) \rangle} \rangle$ is generated by n elements. Modulo K, M/K is n-restrained and so is $E\ twr_L H$. This is a contradiction. Suppose that M is finite. Write $M = \{x_1, x_2, \ldots, x_r\}$. Since G is residually finite, there is $N_i \lhd G$ such that $x_i \notin N_i$ and G/N_i is finite. Let $N = \bigcap_{i=1}^{r} N_i$. Then $M \cap N = 1$ and G/N is finite. And $MN/N \cong M/M \cap N = M$. Now $MN/KN \cong M/K \cong E\ twr_L H$. If we replace M by MN and K by KN, then we are in the above case. So by Theorem(J. Wilson) G has a soluble subgroup of finite index. By the remark following Corollary 4.4, we conclude that G is polycyclic-by-finite.

Now let G be a given group. Let R be the finite residual of G. Then G/R is polycyclic-by-finite by the above argument. Since G is weakly restrained, R is finitely generated by Lemma 4.3. Here G has finite local trace. So $R = 1$. □

THEOREM 4.8. *A locally indicable n-Engel group G is nilpotent.*

Proof. Let G be finitely generated. Then G has a subgroup G_1 such that G/G_1 is infinite cyclic. Hence G_1 is finitely generated by Lemma 4.3. Now let $G_2 = I_{G_1}(G_1')$, the isolator of G_1' in G_1. Since G is locally indicable, $G_2 \neq G_1$ and $G_2 \lhd G$. Furthermore there is a series, $G_2 \lhd G_{21} \lhd \cdots \lhd G_{2s(1)} \lhd G_1$ with infinite cyclic factors. By repeated applications of Lemma 4.3 we get G_2 is finitely generated. Let $G_3 = I_{G_2}(G_2')$, the isolator of G_2' in G_2. This procedure gives us a normal series $G = G_0 \rhd G_1 \rhd G_2 \rhd \cdots$ with torsion-free abelian factors. Note that G/G_i is finitely generated soluble, hence nilpotent. Let N be the intersection of the G_i's so that G/N is residually torsion-free nilpotent for all i. Now since G/N is an n-Engel group, by Zelmanov's theorem, G/N is nilpotent. Thus $N = G_m$ for some

m. This means that $N = 1$ and G is nilpotent. Since G is torsion-free, the n-Engel condition implies that G is nilpotent of class depending only on n, independent of the number of generators of G, as shown by Zelmanov in [Z]. Thus a locally indicable n-Engel group is nilpotent. □

Note that in [C1], the authors proved the following result: *if G is a finitely generated generalized Engel soluble group, then it is nilpotent-by-finite.* Using this result, we can get the following analogous result of Theorem 4.8 under a restricted Engel condition. Of course, Lemma 3.3 could also be obtained as an easy consequence of that.

COROLLARY 4.9. *Let G be a finitely generated group with finite local trace. If G is a restricted Engel group, it is nilpotent-by-finite.*

Proof. Let R be the finite residual of G. Then G/R is residually finite. Since G is a restricted Engel group, G/R is n-restrained for some n. By Note 4.5 G/R has no section isomorphic to $E \ twr_L \ H$ where $[H : L] > n$. Hence G/R is soluble-by-finite and polycyclic-by-finite by the remark following Lemma 4.4. Let $R \neq 1$. Since R is finitely generated, it has a proper finite quotient R/M. Then $core_G(M) \lhd G$ and $R/core_G(M)$ is finite. Thus $G/core_G(M)$ is polycyclic-by-finite and so residually finite, contrary to the choice of R. Thus G is polycyclic-by-finite and nilpotent-by-finite. □

COROLLARY 4.10. *Let G be a torsion-free n-Engel group. If G has finite local trace, then G is nilpotent.*

Proof. We claim that G is a locally indicable group. Then it is nilpotent by Theorem 4.8. Let H be a finitely generated non-trivial subgroup and let N be the finite residual of H. Then H/N is a finitely generated residually finite group and so nilpotent. If H/N is infinite, the quotient group by the torsion subgroup is torsion-free. If H/N is finite, then N is finitely generated. Let K be the finite residual of N. Since N is the finite residual of H, $N = K$, a contradiction. □

LEMMA 4.11. *If G is a collapsing RO-group, then G is a C^*-group.*

Proof. Let P be the positive cone of a given right-order on G and a, b in P. Suppose, if possible, that $a^m b < a$ for all positive integers, m. Consider the set $S = \{ba, ba^2, \ldots, ba^n\}$ where n is an integer such that $|S^n| < n^n$. Since G is n-collapsing, there exist two distinct functions f, g on the set $\{1, 2, ..., n\}$ such that

$$\prod_{i=1,\ldots,n} ba^{f(i)} = \prod_{i=1,\ldots,n} ba^{g(i)}.$$

Hence for some $0 < r \leq n$ we have $ba^{f(1)}ba^{f(2)}\ldots ba^{f(r)} = ba^{g(1)}ba^{g(2)}\ldots ba^{g(r)}$ and $f(r) \neq g(r)$. Say $f(r) < g(r)$ and let $s = g(r) - f(r)$. Then we have $ba^{f(1)} \cdots b = ba^{g(1)} \cdots ba^s$ and $a^{f(1)} \cdots b = a^{g(1)} \cdots ba^s$. Now $a^m b < a$ for all $m > 0$ implies $a^{f(1)}ba^{f(2)} \cdots b < aa^{f(2)}ba^{f(3)} \cdots b < aa^{f(3)}b \cdots b < \cdots < a$. On the other hand, $a^{g(1)} \cdots b \geq e$ so that $a^{g(1)} \cdots ba^s \geq a^s \geq a$, giving the required contradiction. □

THEOREM 4.12. *A finitely generated RO-group G is collapsing if and only if it is nilpotent-by-finite.*

Proof. Suppose G is collapsing. Then by Lemma 4.11 it is locally indicable. Hence by the same argument as in Theorem 4.8, we get a normal series $G = G_0 \rhd G_1 \rhd G_2 \rhd \cdots$ with torsion-free abelian factors. Note that G/G_i is finitely generated soluble and collapsing, hence nilpotent-by-finite for all i by [S1]. Hence G/G_i is residually finite. Let N be the intersection of the G_i's so that G/N is residually finite. Now since G/N is collapsing, by [S1] G/N is nilpotent-by-finite. Thus $N = G_m$ for some m. This means that $N = 1$ and G is nilpotent-by-finite. The converse is clear. □

CHAPTER 5

FINITE BASE

DEFINITION. Let N, K be subgroups of a group G and $N \leq K$. We denote by $R^1_K(N)$ the subgroup generated by the set $\{ g \in K; g^n \in N \text{ for some } n \geq 1 \}$. Let $R^{i+1}_K(N) = R^1_K(R^i_K(N))$ and $R_K(N) = \bigcup_{i=1}^{\infty} R^i_K(N)$. We call $R_K(N)$ the *root* of N in K.

Note $R_K(N)$ is isolated for if $x^r \in R_K(N)$, then $x^r \in R^i_K(N)$ for some i and $x \in R^{i+1}_K(N)$. Hence $x \in R_K(N)$. So $R_K(N)$ is exactly the smallest isolated subgroup of K containing N. The following example is one which shows how the root grows.

EXAMPLE. Let $N = \langle\ a, b\ ; [a,b] = c, [c,a] = [c,b] = e\ \rangle$ be a free nilpotent group of class 2 and let t be the automorphism of N defined by $a^t = b, b^t = (ba)^{-1}$. Then $G = \langle\ a, b, t\ ; a^t = b, b^t = (ba)^{-1}, t^3 = c\ \rangle$ is a polycyclic group. Now we consider the root $R_G(\langle e \rangle)$ of $\langle e \rangle$ in G. We claim that $R^1_G(\langle e \rangle) \neq R^2_G(\langle e \rangle)$ and $R_G(\langle e \rangle) = R^2_G(\langle e \rangle) = G$. Let P be $R^1_G(\langle e \rangle)$ which is a subgroup generated by all periodic elements of G. We show that G/P is isomorphic to a cyclic group of order 3.

NOTE 5.1. *The group G as above has the following properties.*

(i) $a^n b^m = b^m a^n c^{nm}$ *for all integers,* n, m.

(ii) $(a^{-1}b^{-1})^n = b^{-n}a^{-n}c^{n(n+1)/2}$ *for all integers,* n.

(iii) a^3, b^3 *and* t^3 *lie in* P.

Proof. (ii) Let $n > 0$. Induction on n.

For $n = 1$, $a^{-1}b^{-1} = b^{-1}a^{-1}[a^{-1}, b^{-1}] = b^{-1}a^{-1}c$.

Suppose that $(a^{-1}b^{-1})^{n-1} = b^{-n+1}a^{-n+1}c^{n(n-1)/2}$. Then

$$\begin{aligned}(a^{-1}b^{-1})^n &= (a^{-1}b^{-1})\cdot(a^{-1}b^{-1})^{n-1} = b^{-1}a^{-1}c\cdot b^{-n+1}a^{-n+1}c^{n(n-1)/2}\\ &= b^{-1}a^{-1}b^{-n+1}a^{-n+1}c^{n(n-1)/2+1} = b^{-1}b^{-n+1}a^{-1}a^{-n+1}c^{n-1+n(n-1)/2+1}\\ &= b^{-n}a^{-n}c^{n(n+1)/2}.\end{aligned}$$

Similarly for $n < 0$.

(iii) $(at^{-1})^3 = aa^ta^{t^2}t^{-3} = a\cdot b\cdot(ba)^{-1}c^{-1} = e$ by (i). So $at^{-1} \in P$. Hence $(at^{-1})^t = bt^{-1}$, $(bt^{-1})^t = a^{-1}b^{-1}t^{-1}$, and $(at^{-1})^{a^{-1}} = a^2b^{-1}t^{-1}$ lie in P. So $a^2b^{-1}t^{-1}\cdot tba = a^3 \in P$ and so does $b^3 = (a^3)^t$. Moreover $at^{-1}\cdot(bt^{-1})^{-1} = ab^{-1}$, $(bt^{-1})^t\cdot(bt^{-1})^{-1} = a^{-1}b^{-2} \in P$ and $a^{-3}\cdot ab^{-1}\cdot(b^{-3})^t\cdot a^{-1}b^{-2} = a^{-1}bab^{-1} = c^{-1} \in P$. So $t^3 \in P$. □

Let $K = \langle a^3, b^3, t^3\rangle^G \leq N$ and $Q = \langle K, at^{-1}\rangle^G \leq P$. Now we claim $P = Q$ by showing that every periodic element of G is in Q. Let $x = c^\ell a^m b^n t^\epsilon$, where $\epsilon = 1, -1$, be a periodic element of G. Let $\epsilon = -1$.

Then $x^2 = c^\ell a^m b^n t^{-1} c^\ell a^m b^n t^{-1} = c^{2\ell+mn+n(n+1)/2}a^{m-n}b^m t^{-2}$ and

$x^3 = c^{3\ell+mn+n(n+1)/2+m(m+1)/2-1}$ by using Note 5.1.

Since x is periodic, $3\ell + mn + n(n+1)/2 + m(m+1)/2 - 1 = 0$. We consider this relation modulo 3. There are nine possible cases of m, n that must be checked. For example, suppose $m \equiv 0$ and $n \equiv 1 \ mod\ 3$. Then $x = c^\ell a^{3m'}b^{3n'+1}t^{-1}$ for some integers m', n'. Since $bt^{-1} = (at^{-1})^t \in Q$ and $K \subset Q$, $x \in Q$. We can apply the same argument for $\epsilon = 1$ to get $P = Q$. Recall that if a group G is a 2-Engel group then $\langle x\rangle^G$ is abelian for all $x \in G$. Since G/K has an exponent 3 and $|G/K| = 27$, it is a 2-Engel group. Note $P/K = \langle (at^{-1}) \rangle^{G/K}$ and so P/K is abelian. If $G = P$, then G/K is abelian. This is a contradiction. From the proof of (iii), we have $a^{-1}b^{-2} \cdot b^3 = [a,t] \in P$. Moreover $[b,a] \cdot a^{-1}b^{-1}t^{-1} \cdot (at^{-1})^{-1} = b^{-1}a^{-2} \in P$. So $(b^{-1}a^{-2})^t \cdot b^3 = bab = [t,b] \in P$. Hence $P \geq G'$ and G/P is elementary abelian. In fact $G/K \cong (C_3 \times C_3) \rtimes C_3$ and $G/P = \langle (aP) \rangle \cong C_3$. $\square$

DEFINITION. We will say that a group G has *finite base* n if there is an integer n such that every subgroup H of G has an n-generated subgroup K of H with $R_H(K) = H$ and if n is the least integer with the property. In particular, if $H = R_H(\langle e \rangle)$ for each subgroup H of G then we shall say that G has base 0.

Clearly the additive group of rational numbers has finite base 1. Since all periodic groups have finite base 0, this concept is not useful in studying periodic groups. So it makes sense to restrict our groups to torsion-free groups, for example, O-groups. Here we have the closure properties of groups with finite base. We follow the notations in [R5].

LEMMA 5.2. *The class of groups with finite base is* S, H *and* P*-closed*

Proof. S and H-closure are immediate. For P-closure, let $N \lhd G$ and suppose that N has finite base r and G/N has finite base s. Let H be a subgroup of G and $H \cap N = L$. Then modulo L, $\overline{H} = R_{\overline{H}}(\langle \overline{x}_1, \ldots, \overline{x}_s \rangle)$ for some $x_i \in H$ and $L = R_L(\langle y_1, \ldots, y_r \rangle)$ for some $y_i \in L$. Clearly $L \subset R_H(\langle y_1, \ldots, y_r, x_1, \ldots, x_s \rangle)$. For $x \in H \backslash L$, $\overline{x} \in R_{\overline{H}}(\langle \overline{x}_1, ..., \overline{x}_s \rangle)$. Let $\overline{x} \in R^1_{\overline{H}}(\langle \overline{x}_1, ..., \overline{x}_s \rangle)$. Then $\overline{x} = \overline{g}_1 \cdots \overline{g}_r$ and $x = g_1 \cdots g_r \ell_1$. where $\overline{g}_i^m \in \langle \overline{x}_1, ..., \overline{x}_s \rangle$. So $g_i^m = x_{i_1} \cdots x_{i_n} \ell_2 \in \langle x_1, ..., x_s, L \rangle$. Thus $x \in$

$R_H^1(\langle x_1, ..., x_s, L\rangle) \subset R_H(\langle x_1, ..., x_s, L\rangle)$. Suppose that if $\overline{x} \in R_H^{i-1}(\langle \overline{x}_1, ..., \overline{x}_s\rangle)$, then $x \in R_H(\langle x_1, ..., x_s, L\rangle)$. Let $\overline{x} \in R_H^i(\langle \overline{x}_1, ..., \overline{x}_s\rangle)$. Then $\overline{x} = \overline{g}_1 \cdots \overline{g}_s$ and $x = g_1 \cdots g_s \ell_3$ where $\overline{g}_i^m \in R_H^{i-1}(\langle \overline{x}_1, ..., \overline{x}_s\rangle)$. By induction hypothesis, $g_i^m \in R_H(\langle x_1, ..., x_s, L\rangle)$. Thus $x \in R_H^1(R_H(\langle x_1, ..., x_s, L\rangle)) = R_H(\langle x_1, ..., x_s, L\rangle)$. So if $\overline{x} \in R_H(\langle \overline{x}_1, ..., \overline{x}_s\rangle)$, then $x \in R_H(\langle x_1, ..., x_s, L\rangle)$.

Note that $R_H(\langle x_1, ..., x_s, y_1, ..., y_r\rangle)$ contains $x_1, ..., x_s, L$ and isolated. Thus $R_H(\langle x_1, ..., x_s, L\rangle) = R_H(\langle x_1, ..., x_s, y_1, ..., y_r\rangle)$. □

LEMMA 5.3. *Let G be a torsion-free abelian group with finite base 1. Then G is isomorphic to a subgroup of the additive group of rational numbers.*

Proof. Since $b(G) = 1$, $G = R_G(\langle x\rangle)$ for some x in G. For $g \in G$, $g^\ell = x^n$ for some ℓ, n. Hence G is indecomposable. Let H be a finitely generated subgroup of G. Then H is also indecomposable, i.e., $H \cong \mathbb{Z}$. This means that G is locally cyclic and torsion-free. Hence G is isomorphic to a subgroup of $\mathbb{Q}$. □

LEMMA 5.4. *Let G be a torsion-free nilpotent group with base 1. Then it is isomorphic to a subgroup of the additive group of rational numbers.*

Proof. Induction on the nilpotent class n. By Lemma 5.3, it is clear if $n = 1$. Suppose that it is true if $n < r$. For a group G of the nilpotent class r, $\langle G', g\rangle$ has the nilpotent class $< r$ for all g in G. In particular for $g \in Z(G)$, the center of G, $\langle G', g\rangle$ is isomorphic to a subgroup of $\mathbb{Q}$. Hence $x^\ell = g^n$ for some $x \in G'$. For all $y \in G$, $\langle G', y\rangle$ is isomorphic to a subgroup of $\mathbb{Q}$. Hence $y^\alpha = x^\beta$ and $y^{\alpha\ell} = x^{\beta\ell} = g^{n\beta}$. So $y^{\alpha\ell}$ lies in $Z(G)$. Since the center of a torsion-free nilpotent group is isolated, y lies in $Z(G)$. Hence G is abelian. □

Clearly a torsion-free locally nilpotent group with base 1 is isomorphic to a subgroup of $\mathbb{Q}$. We can not extend the above argument to the larger class of groups even for polycyclic groups for $G = \langle x, y : x = y^2, x^y = x^{-1} \rangle$ is an easy example.

LEMMA 5.5. *If a group G has finite base, then it satisfies maximal condition on isolated subgroups.*

Proof. Suppose that there is a proper ascending series of isolated subgroups, $H_1 < H_2 < \cdots$. Let $H = \bigcup_{i=1}^{\infty} H_i$. Then there is an n-generated subgroup $K = \langle x_1, \ldots, x_n \rangle$ of H with $R_H(K) = H$. Take a subgroup H_ℓ containing $x_1, \ldots, x_n$ and then $H_\ell = R_H(H_\ell) = R_H(K) = H$. □

Since every relatively convex subgroup of an O-group is isolated, we get

COROLLARY 5.6. *If G is an orderable group with finite base, then every relatively convex subgroup is normal in G.* □

Let G be an abelian group with finite rank r. Then the factor group of G with respect to its torsion subgroup is isomorphic with a subgroup of a direct product of $r_0 (\leq r)$ copies of the additive group of rational numbers. Hence G has finite base. From this and the P-closure it follows that soluble groups with finite rank have finite base. For a locally soluble group G with finite rank r, there is an integer n depending only on r such that $G^{(n)}$ is periodic (see [R6]). Hence a locally soluble group G with finite rank has finite base. Moreover it follows from [L5] that a residually finite group with finite rank has finite base. Conversely let G be a torsion-free abelian group with finite base and A a subgroup such that $G/A = \langle \bar{t} \rangle$ is infinite cyclic. If s is the smallest integer such that $A = R_A(\langle x_1, \ldots, x_s \rangle)$, $x_i \in A$, then G has finite base $\geq s$. Now suppose G has finite base s. Then $\langle A, t \rangle = R_{\langle A,t \rangle}(\langle t^{\alpha_1} a_1, \ldots, t^{\alpha_p} a_p, a_{p+1}, \ldots, a_s \rangle)$ where $a_i \in A$. Here we have an expression for t in the right hand side. Since t is torsion-free, we have $1 = a_1^{b_1} \cdots a_p^{b_p} a_{p+1}^{b_{p+1}} \cdots a_s^{b_s}$ where $b_i \in \mathbb{Z}$ and $b_i \neq 0$ for some i. Hence $A = R_A(\langle a_1, \ldots, a_{i-1}, a_{i+1}, a_s \rangle)$, a contradiction. Thus G cannot have infinitely many independent elements. This means that G has finite rank. So it follows that a torsion-free nilpotent group with finite base has finite rank.

Recall that for an abelian group A, the 0-*rank* of A is the cardinal of a maximal independent set consisting of elements of A with infinite order.

COROLLARY 5.7. *If G is a locally nilpotent group, the following properties of G are equivalent.*

(i) G has finite base.

(ii) Each abelian subgroup of G has finite 0*-rank.*

(iii) The factor group G by its torsion subgroup is a torsion-free nilpotent group of finite rank. □

COROLLARY 5.8. *A torsion-free locally nilpotent group with finite base has finite rank.* □

THEOREM 5.9. *If G is a C-group with finite base, then it is soluble.*

Proof. Since G has finite base, there is a finitely generated subgroup $H = \langle x_1, \ldots, x_r \rangle$ with $e < x_1 < \cdots < x_r$ such that $G = R_G(H)$. Note that x_r determines a jump $G_1 \to G_0$ and $G_0 = G$. Now let $G_2 = I_{G_1}(G'_1)$, the isolator of G'_1 in G_1. Since G_1 is a C-group with finite base, $G_2 \lhd G$ and

$G_2 \neq G_1$ by the above argument. Repeat this setting $G_3 = I_{G_2}(G_2')$. Then we get a normal series $G = G_0 \rhd G_1 \rhd G_2 \rhd \cdots$ with torsion-free abelian factors. For each positive integer m, modulo G_m let A be a maximal normal abelian subgroup of $\overline{G}$. Since $\overline{G}$ has finite base, A has finite rank and is torsion-free. Hence G/A can be considered a subgroup of $\mathrm{GL}(n, \mathbb{Q})$. By the Mal'cev's theorem, G/A has a bounded soluble length dependent on n only and so does G. This means $G/\cap G_m$ is soluble. Hence G is soluble. □

LEMMA 5.10. *If G is an O-group with finite base, then it has finite rank.*

Proof. Note that G is soluble by Theorem 5.9. Let s be the solubility length of G. By Lemma 5.5 and Corollary 5.6, the isolator J of G' in G has a descending central series $C_1 > C_2 > \cdots$. We claim that for every m, J/C_m has finite rank r, depending only on base n and solubility length s. Write $H = J/C_m$. Note that H is torsion-free nilpotent and so an O-group. Moreover H/L has a solubility length $< s$ where $L = I(H^{(s-1)})$ is the isolator of $H^{(s-1)}$ in H. By induction H/L has finite rank. Since H is an O-group, L is torsion-free abelian. Thus H has finite rank. □

In [B6], B. Mura and A. Rhemtulla introduced the class O_2-groups of all ordered soluble groups satisfying the maximal condition on isolated subgroups. By Lemma 5.5 and Theorem 5.9, an O-group with finite base is an O_2-group. And the converse is also true by Theorem 2 in [B6]. The following are immediate corollaries of the results in [B6].

THEOREM 5.11. *If G is an O-group with finite base, then it is nilpotent-by-abelian.* □

THEOREM 5.12. *If G is an O-group with finite base, then every torsion-free quotient of G is orderable.* □

We recall that a group G is called an *R^*-group* if $g^{x_1} \cdots g^{x_n} = e$ implies $g = e$ for all $g, x_1, \ldots, x_n$ in G. Clearly an O-group is an R^*-group. Here we mention another property of finite base which corresponds to analogous one of finite rank for orderable groups. The proof is quite similar to Theorem 4.1.1 in [B5].

THEOREM 5.13. *Let G be a group with finite base. Then G is orderable if and only if it is a soluble R^*-group.* □

There exist polycyclic O-groups with trivial center. As an example, let $H = \langle 1, \frac{1+\sqrt{5}}{2} \rangle$ be a subgroup of $(R, +)$ and τ an automorphism of H which acts on H as multiplication by $\frac{1+\sqrt{5}}{2}$. Then the split extension of H by $\langle \tau \rangle$ is a required one. So we get a certain limitation to the above Theorem 5.9 and Corollary 5.11. An O-group with finite base need not be soluble minimax for $(\mathbb{Q}, +)$ is an easy example. But using Theorem 5.9 and Lemma 5.10, we have that a finitely generated O-group with finite base is a soluble minimax group.

CHAPTER 6

PC-GROUPS

In the previous chapters, we discussed groups with the property of permutable subgroup products. Now we consider a similar notion of permutable products, for 2-element subsets of G instead of subgroups of G.

NOTATIONS. For subsets $S, S_1, \dots, S_n$ of G and g in G, $S \cdot g = \{sg; s \in S\}$, $g \cdot S = \{gs; s \in S\}$ and $S_1 S_2 \cdots S_n = \{s_1 \cdots s_n; s_i \in S_i\}$. And $|S|$ means the cardinality of the set S.

DEFINITION. For an integer $n > 1$, a group G is said to *preserve the cardinality of 2-element subsets product under permutations*, or G is a *$PC(2,n)$-group* if $G = 1$ or for each n-tuple $(S_1, \dots, S_n)$ of 2-element subsets of G, there is a permutation $\sigma (\neq 1)$ in Σ_n such that

$$(6.0) \qquad |S_1 S_2 \cdots S_n| = |S_{\sigma(1)} S_{\sigma(2)} \cdots S_{\sigma(n)}|.$$

Let $PC(2)$ be the class $\bigcup_{n>1} PC(2,n)$. The following note is one which explains why it makes sense to fix one side of 6.0.

NOTE 6.1. *For $n \geq 3$, a non-trivial group G has the following property ; For each n-tuple $(S_1, \dots, S_n)$ of 2-element subsets of G, there exist distinct permutations $\sigma, \tau \in \Sigma_n$ such that the cardinalities of $S_{\sigma(1)} \cdots S_{\sigma(n)}$ and $S_{\tau(1)} \cdots S_{\tau(n)}$ are same.*

Proof. Note $|S_1 S_2 \cdots S_n| \leq 2^n$. If $n \geq 4$, then $n! > 2^n$. So the number of permutations is strictly greater than the number of possible cardinalities of all permutable products. Hence there are two distinct permutations with above property. Suppose $n = 3$. Let S_1, S_2 and S_3 be three given 2-element subsets of G. If $|S_{\sigma(1)}S_{\sigma(2)}S_{\sigma(3)}| \neq 2, 3$ for all $\sigma \in \Sigma_3$, we are already done. So we can assume $|S_{\sigma(1)}S_{\sigma(2)}S_{\sigma(3)}| = 2$ or 3 for some $\sigma \in \Sigma_3$. Write $S_1 = \{x_1, x_1x\}, S_2 = \{y, y_1\}$ and $S_3 = \{z_1, zz_1\}$. Suppose $|S_1S_2S_3| = 2$. Then $|S_1S_2| = |S_2S_3| = 2$. Now by the simple calculation, we get $|S_3S_1S_2|$ and $|S_2S_3S_1|$ are 2 or 4. Let $|S_1S_2S_3| = 3$. Write $S_1' = \{1, x\}$ and $S_3' = \{1, z\}$. If $|S_1S_2| = |S_1'S_2| = 2$, then we have $y = xy_1$ and $y_1 = xy$. Moreover $S_1'S_2S_3' = \{y,\ y_1,\ yz,\ y_1z\}$. Since $|S_1'S_2S_3'| = 3$, we have $y = y_1z$ or $y_1 = yz$. Note that $y = y_1z \Leftrightarrow xy_1 = y_1z = xyz \Leftrightarrow y_1 = yz$. Hence $|S_1S_2S_3| = 2$, a contradiction. So $|S_1'S_2| = |\{y, y_1, xy, xy_1\}| = 3$. Without loss of generality, we can assume $y = xy_1$. Since $S_1'S_2S_3' = S_1'S_2 \cup S_1'S_2 \cdot z$, there are two cases to examine.

Case (i). $y = xyz$, $y_1 = yz$ and $xy = y_1z$.

Then $y = xy \cdot z = y_1z \cdot z = yz^3$ and $y = xyz = xxy_1z = x^3y$. Thus $x^3 = z^3 = 1$. Note that $S_2S_3S_1 = S_2S_3 \cdot x_1 \cup S_2S_3 \cdot x_1x$ and $S_2S_3 = \{yz_1,\ yzz_1,\ yzzz_1\}$. Now suppose $|S_2S_3S_1| < 6$. Then at least one element in $S_2S_3 \cdot x_1$ lies in $S_2S_3 \cdot x_1x$. Note that $yzz_1x_1 = yz_1x_1x \Leftrightarrow zz_1x_1 = z_1x_1x \Leftrightarrow yzzz_1x_1 = yzz_1x_1x \Leftrightarrow yz_1x_1 = yzzzz_1x_1 = yzzz_1x_1x$ and $yzzz_1x_1 = yz_1x_1x \Leftrightarrow yz_1x_1 = yzz_1x_1x \Leftrightarrow yzz_1x_1 = yzzz_1x_1x$. So that one element in $S_2S_3 \cdot x_1$ lies in $S_2S_3 \cdot x_1x$ implies that other two elements in $S_2S_3 \cdot x_1$ belong to $S_2S_3 \cdot x_1x$. Hence $|S_2S_3S_1| = 6$ or 3. Similarly we can show $|S_3S_1S_2| = 6$ or 3.

Case (ii). $y = y_1z$, $y_1 = xyz$ and $xy = yz$.

This case can be checked by the same argument as in case (i). □

Clearly $PC(2)$ contains all finite groups. So for a given n, it seems hard to characterize $PC(2,n)$-group. However in a very particular case, we have an affirmative answer.

LEMMA 6.2. *Let* G *be a* $PC(2,2)$ *or* $PC(2,3)$*-group. Then*

(i) if $x^2 = 1$, *then* $x \in Z(G)$, *the center of* G,

(ii) if $[x,y] \neq 1$, *then* $x^y = x^{-1}$,

(iii) if $[x,y] \neq 1$, *then* $\langle x,y \rangle$ *is a quaternion group of order 8.*

Proof. (i) If x has an order 2 and $[x,y] \neq 1$, take $S_1 = \{1,\ x\}, S_2 = \{xy,\ y\}$ and $S_3 = \{1,\ y^{-1}xy\}$. Then $|S_1S_2S_3| \neq |S_{\sigma(1)}S_{\sigma(2)}S_{\sigma(3)}|$ for all $\sigma(\neq 1) \in \Sigma_3$ and $|S_1S_2| \neq |S_2S_1|$.

(ii) Let G be a $PC(2,3)$-group. For $S_1 = \{1,x\}, S_2 = \{y, x^{-1}y\}$ and $S_3 = \{1, y^{-1}xy\}$, there is $\sigma(\neq 1) \in \Sigma_3$ such that $|S_1S_2S_3| = |S_{\sigma(1)}S_{\sigma(2)}S_{\sigma(3)}|$.

There are five cases to check. We consider one of them (the others are similar). Suppose $|S_1S_2S_3| = |S_3S_1S_2| \leq 4$. If $|S_1S_2| = 2$, $x^2 = 1$ and so $x \in Z(G)$, a contradiction. Hence $|S_1S_2| = |\{y, xy, x^{-1}y\}| = 3$. Note that $S_3S_1S_2 = S_1S_2 \cup y^{-1}xy \cdot S_1S_2$. So at least two elements in $y^{-1}xy \cdot S_1S_2$ are in S_1S_2. The non-trivial possible cases are (i) $y = y^{-1}xyxy$, (ii) $xy = y^{-1}xyx^{-1}y$, (iii) $x^{-1}y = y^{-1}xyy$ and (iv) $x^{-1}y = y^{-1}xyxy$. Note that (i) or (iii) is equivalent to the relation we want. If (ii) and (iv) are true, then $y^{-1}xy = x^{-2} = x^2$. Since x^2 lies in the center of G, $y^{-1}xy = x^2$ gives a contradiction. If G is a $PC(2,2)$-group, take $S_1 = \{1,\ x\}$ and $S_2 = \{xy,\ y\}$. We then get the same result by the simple calculation.

(iii) Note that in (ii) we only used the non-commutativity of x and y. If we consider $[xy,y]\neq 1$ and $[x^{-1}y,y]\neq 1$, then we get $x^2=y^2$ and $x^4=1$. Since $\langle x,y\rangle$ is nonabelian, we are done. □

THEOREM 6.3. *G is a $PC(2,2)$ or $PC(2,3)$-group if and only if G is abelian or the direct product of a quaternion group of order 8 and an elementary abelian 2-group.*

Proof. Let $x,y\in G$ and $[x,y]\neq 1$. Then Q$=\langle x,y : x^4=1, x^2=y^2, xy=yx^{-1}\rangle$ is a quaternion group by (6.2)Lemma. Now we claim $G=CQ$ where C is the centralizer of Q in G. Suppose $g\in G\backslash CQ$. Then $[g,x]\neq 1$ or $[g,y]\neq 1$. Let $[g,y]\neq 1$ and $y^g=y^{-1}$. Then $y^{gx}=(y^{-1})^x=y$, i.e., $[gx,y]=1$. Since $g\in G\backslash CQ$, $[gx,x]\neq 1$. Hence $[x,gxy]=[y,gxy]=1$ and so $g\in CQ$, a contradiction. So $G=CQ$. Suppose C has an element g of order 4. Then $[x,gy]=[x,y]\neq 1$ and $(gy)^4=1$. So $(gy)^x=(gy)^{-1}$ and $[gy,x]=(gy)^{-2}=g^{-2}y^{-2}$. Since $[gy,x]=[y,x]=y^{-2}$, we get $g^2=1$, a contradiction. Note that C lies in $Z(G)$. For, if $g\in C\backslash Z(G)$, then $[g,w]\neq 1$ for some w and so $g^4=1$, contrary to the above argument. Suppose that C has an element g of order prime $p\neq 2$. We consider 2-element subsets $S_1=\{x,gxy\}$ and $S_2=\{g^{-1}y,g^{-2}\}$ if G is a $PC(2,2)$-group and $S_1=\{1,gx\}, S_2=\{gy,y^{-1}x^{-1}\}$ and $S_3=\{g^{-1},x^{-1}\}$ if G is a $PC(2,3)$-group. Then $|S_1S_2|\neq|S_2S_1|$ and $|S_1S_2S_3|\neq|S_{\sigma(1)}S_{\sigma(2)}S_{\sigma(3)}|$ for all $\sigma(\neq 1)\in\Sigma_3$. Hence C is an elementary abelian 2-group. Let $z=[x,y]$ and D be the maximal subgroup of C not containing z. Then $z\in\langle D,w\rangle$ for all $w\in C\backslash D$. Since $w^2=z^2=1$, $\langle D,z\rangle=\langle D,w\rangle$. Hence $C=\langle D,z\rangle$

and $D \cap \langle z \rangle = 1$. So $C = D \times \langle z \rangle$. Moreover $Q \cap D = 1$ and $QD = QC$. Hence $G = Q \times D$, where D is a subgroup of C.

For the converse, let $G = Q \times D$ where D is an elementary abelian 2-group and Q a quaternion group of order 8. First we show that G is in $PC(2,3)$. Let A, B and C be three given 2-element subsets of G. Write $A = \{g_1, g_1ax\}, B = \{by, cz\}$ and $C = \{g_2, dwg_2\}$, where $a, b, c, d \in Q$, $x, y, z, w \in D$ and $g_1, g_2 \in G$. Then $|ABC| = |A'BC'|$ and $|CAB| = |C''A'B|$, where $A' = \{1, ax\}, C' = \{1, dw\}$ and $C'' = \{1, d^{\epsilon}w\}$. Note that in C'', $\epsilon = 1$ if g_2g_1 lies in the centeralizer of d, and $\epsilon = -1$ if not.

Case (i). $|AB| = 4$.

Since $C' = \{1, dw\}$ and $C'' = \{1, d^{\epsilon}w\}$, $A'BC' = A'B \cup A'B \cdot dw$ and $C''A'B = A'B \cup d^{\epsilon}w \cdot A'B$. Note that if there is one element in $A'B \cdot dw$ which is in $A'B$, then there is one element in $d^{\epsilon}w \cdot A'B$ which is in $A'B$. The converse is also true. For example, suppose that $by = abdxyw$. Then $by = abdxyw \Leftrightarrow d^{\eta}abxyw \Leftrightarrow d^{\epsilon}abxyw$ if $\epsilon = \eta$, and $d^{\epsilon}by = abxyw \Leftrightarrow d^{\epsilon}byw = abxy$ if not. This means $|A'BC'| = |C''A'B|$ and so $|ABC| = |CAB|$.

Case (ii). $|AB| = 3$.

This case can be checked by the same argument as in case (i).

Case (iii). $|AB| = 2$.

Since $|A'B| = |\{1, ax\}\{by, cz\}| = 2$, we have $b = ac$ and $c = ab$. So $c = ab = aac$ and $a^2 = 1$. Hence A' lies in the center of G. Thus $|A'BC'| = |BC'A'|$. Clearly $|BC'A'| = |BCA|$.

It is shorter to show that G is in $PC(2,2)$. Let $A = \{g_1, g_1ax\}, B = \{g_2, byg_2\}$ be two given 2-element subsets of G, where $a, b \in Q$, $x, y \in D$

and $g_1, g_2 \in G$. Write $A' = \{1, ax\}$, $A'' = \{1, a^\epsilon x\}$ and $B' = \{1, by\}$ where $\epsilon = 1$ if $g_2 g_1$ lies in the centeralizer of a, and $\epsilon = -1$ if not. Then $|AB| = |A'B'|$ and $A'B' = \{1, by, ax, abxy\}$. And $|BA| = |B'A''|$ and $B'A'' = B' \cup B' \cdot a^\epsilon x = \{1, by, a^\epsilon x, ba^\epsilon xy\}$. If $\epsilon = 1$, then $abxy = 1 \Leftrightarrow baxy = 1$. If not, $by = ax \Leftrightarrow 1 = ba^{-1}xy$ and $1 = abxy \Leftrightarrow by = a^{-1}x$. □

LEMMA 6.4. *A $PC(2,n)$-group is center-by-(finite exponent $f(n)$).*

Proof. We claim that there exists an integer k such that $[y^k, x] = 1$ for all $x, y \in G$. Let $x, y \in G$. We consider a n-tuple $(S_1, \ldots, S_n)$ of 2-element subsets of G where $S_i = \{y,\ y^{1-i}xy^i\}$. Since G is a $PC(2,n)$-group, there is a permutation $\sigma(\neq 1) \in \Sigma_n$ such that $|S_1 S_2 \cdots S_n| = |S_{\sigma(1)} S_{\sigma(2)} \cdots S_{\sigma(n)}|$ and $|S_1 S_2 \cdots S_n| = min(|x|, n+1)$. Write $g(i,j) = S_{\sigma(i)} S_{\sigma(i+1)} \cdots S_{\sigma(j)}$ for $i \leq j$.

If $|g(i,\ell)|$ and $|g(\ell,j)|$ are strictly increasing functions of i, j for all ℓ, then for an integer j such that $\sigma(j) + 1 \neq \sigma(j+1)$, $|S_{\sigma(j)} S_{\sigma(j+1)}| < 4$. Here $S_{\sigma(j)} = \{y,\ y^{1-\sigma(j)} x y^{\sigma(j)}\}$ and $S_{\sigma(j+1)} = \{y,\ y^{1-\sigma(j+1)} x y^{\sigma(j+1)}\}$. So we have a relation $x = x^{y^s}$ where $s(\neq 0)$ depends on σ and so x, y. However note that there are only finitely many choices of s independent of x, y, say, $s_1, \ldots, s_m$. Let $k = l.c.m.\{s_i : i = 1, \ldots, m\}$. Then $[x, y^k] = 1$ for all x, y with $|x| > n+1$.

Suppose that $|g(i,\ell)|$ or $|g(\ell,j)|$ is not strictly increasing.

Case (i). $|x| > n+1$.

Let $|g(\ell,j)| = |g(\ell,j+1)|$. Then $g(\ell,j+1) = g(\ell,j) \cdot y \cup g(\ell,j) x^{y^{\sigma(j+1)-1}} \cdot y$ and so $g(\ell,j) = g(\ell,j) x^{y^{\sigma(j+1)-1}}$. Since $|g(\ell,j)| \leq n+1$, $|x| \leq n+1$. This is a contradiction. The other case is similar.

Case (ii). $|x| \le n+1$.

For $S_{\sigma(1)}S_{\sigma(2)}\cdots S_{\sigma(n)}$, let j be an integer such that $\sigma(j)+1 \neq \sigma(j+1)$. Now we can assume that $|S_{\sigma(j)}S_{\sigma(j+1)}| = 4$. Then since $|S_1S_2\cdots S_n| = |x|$, we can find p, q with $p \le j < j+1 \le q$ such that $|g(p,q)| = |g(p,q+1)|$ or $|g(p-1,q)| = |g(p,q)|$. Let $|g(p,q)| = |g(p,q+1)|$. The other case is similar. Without loss of generality, we can assume $S_{\sigma(p)}\cdots S_{\sigma(q)} = \{y^m, x^{y^{\alpha-m}}y^m, x^{y^{\beta-m}}y^m, c_1y^m, \ldots, c_iy^m\}$ where $m = q-p+1$, and c_i is a product of conjugates of x. Then $|g(p,q)| = |g(p,q+1)|$ gives relation $x^{y^\beta} = (x^{y^r})^a$ or $x^{y^\alpha} = (x^{y^r})^b$ where $2 \le a, b < |x|$ and $r = \sigma(q+1)-1$. In any case we have $x^{y^s} = x^d$ for some $2 \le d < |x|$. Since $|x| \le n+1$, $[y^k, x] = 1$ for some k. In every case our s and k depend on x, y. However there are still only finitely many choices of s and k that are independent of x, y. This completes the proof. □

Now we mention the closure properties of $PC(2)$ as immediate consequences of Lemma 6.4. As before, we follow the notations in [R5]. Consider a restricted direct product $G = DrA_n$, where A_n is an alternating group of degree $n > 4$. Then G is locally finite but has no center. Clearly the standard wreath product of two infinite cyclic groups is not center-by-(finite exponent). Neither is a free product of two infinite cyclic groups. Moreover there exist finitely generated non-abelian torsion-free nilpotent groups. So

COROLLARY 6.5. *(i) A $PC(2)$-group is collapsing.*

(ii) A $PC(2)$-group is n-restrained for some positive integer n.

(ii) The class of $PC(2)$-groups is not closed under any of the closure operations P, D, C, W, F, R, L. □

COROLLARY 6.6. *A finitely generated soluble $PC(2)$-group G is center-by-finite.*

Proof. By Lemma 6.4, G is center-by-(finite exponent.) And a finitely generated soluble group with finite exponent is finite. □

Note that the converse is not true. An infinite dihedral group has a trivial center. So it cannot be a $PC(2)$-group by Lemma 6.4.

THEOREM 6.7. *A finitely generated $PC(2)$-group G with finite local trace is center-by-finite.*

Proof. Let N be the finite residual of G. By Lemma 6.4 G is center-by-(finite exponent). Thus G/N is a finitely generated residually finite center-by-(finite exponent). Since a finitely generated residually finite group of finite exponent is finite by Zelmanov's theorem, G/N is center-by-finite. G is restrained and so N is finitely generated by Lemma 4.3. Let $N \neq 1$. Since G has finite local trace, N has a non-trivial finite factor group N/K. But then $N/core_G(K)$ is finite and $G/core_G(K)$ is finite-by-(center-by-finite). This group is polycyclic-by-finite and so it is residually finite, contrary to the choice of N. □

An element g of a group G is called an *FC-element* if it has only a finite number of conjugates in G. In particular if there is a positive integer m such that no element of G has more than m conjugates, then G is called a *BFC-group*. In [N1], B. H. Neumann gave a precise description of *BFC*-groups; *a group G is a BFC-group if and only if the commutator subgroup G' is finite.*

THEOREM 6.8. *A finitely generated non-periodic $PC(2)$-group G is center-by-finite.*

Proof. Let $G = \langle x_1, x_2, \ldots, x_r \rangle$ be a $PC(2,n)$-group and let z be an element of infinite order in $Z(G)$, the center of G. For $w \in G$, let Ny be a right coset of N, the normalizer of $\langle x \rangle = \langle wz \rangle$ if w has finite order, and $\langle x \rangle = \langle w \rangle$ if not. Suppose that y is reduced and $\ell(y) = m \geq n$ where $\ell(y)$ denotes the length of the shortest word for y. Write $S = \{x_i^{\pm 1} : i = 1, ..., r\}$ and $y = y_1 y_2 \cdots y_m$, where $y_i \in S$. If $m \geq n$, then we consider a n-tuple $(S_1, \ldots, S_n)$ of 2-element subsets of G where $S_i = \{y_i,\ x^{\pi_{i-1}} y_i\}$, $\pi_0 = 1$, $\pi_j = y_1 y_2 \cdots y_j$. Since G is a $PC(2,n)$-group, there is $\sigma(\neq 1) \in \Sigma_n$ such that $|S_1 S_2 \cdots S_n| = |S_{\sigma(1)} S_{\sigma(2)} \cdots S_{\sigma(n)}|$. Write $g(i,j) = S_{\sigma(i)} S_{\sigma(i+1)} \cdots S_{\sigma(j)}$ for $i \leq j$. Since x is of infinite order, $|g(i,\ell)|$ and $|g(\ell,j)|$ are strictly increasing functions of i, j for all ℓ. Let j be an integer $\sigma(j)+1 \neq \sigma(j+1)$. Note that $|S_1 S_2 \cdots S_n| = n+1$. Hence $|S_{\sigma(j)} S_{\sigma(j+1)}| < 4$. Since $S_{\sigma(j)} = \{y_{\sigma(j)},\ x^{\pi_{\sigma(j)-1}} y_{\sigma(j)}\}$ and $S_{\sigma(j+1)} = \{y_{\sigma(j+1)},\ x^{\pi_{\sigma(j+1)-1}} y_{\sigma(j+1)}\}$, we get $x^{\pi_{\sigma(j)}} = x^{\pi_{\sigma(j+1)-1}}$, or $(x^{-1})^{\pi_{\sigma(j)}} = x^{\pi_{\sigma(j+1)-1}}$. Hence $\pi_{\sigma(j)} \pi_{\sigma(j+1)-1}^{-1}$ lies in N. So $N\pi_{\sigma(j)} = N\pi_{\sigma(j+1)-1}$. By the repeated applications of the above argument, we can assume that $Ny = Ny'$, where $\ell(y') < n$. Hence N has finite index in G and so does $C(wz) = C(w)$. In fact $|G : N| < (2r)^n$ and $N/C(wz)$ is isomorphic with a subgroup of the automorphism group of $\langle (wz) \rangle$.

Thus $|G : C(w)| < 2(2r)^n$ for all $w \in G$. Hence G is a BFC-group. Since G is finitely generated, it is center-by-finite. □

COROLLARY 6.9. *A torsion-free $PC(2)$-group is abelian.* □

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