Spectral Factorization of Matrices of Laurent Polynomials and Construction of Quasi-tight Framelets

by

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Abstract

As a generalization of orthonormal wavelets, tight framelets (also called tight wavelet frames) are of importance in both wavelet analysis and applied sciences due to their many desirable properties in applications. However, tight framelets are often derived from particular refinable functions satisfying certain stringent conditions. Hence, we generalize the notion of tight framelets to quasi-tight framelets, which is essentially a dual framelet system, but behaves quite similar to tight framelets. This thesis makes a comprehensive study of the construction of Oblique Extension Principle (OEP) based compactly supported quasi-tight framelets.

For univariate cases, we show that the construction of quasi-tight framelets is much more flexible than that of tight framelets. As a matter of fact, we can always derive a quasi-tight framelet system with high order of vanishing moments from refinable functions associated with any arbitrary compactly supported refinement masks. Also, it is much easier to design moment correcting filters for the quasi-tight framelet filter banks. We provide detailed algorithms to construct quasi-tight framelets in Chapter 2 and Chapter 3, where the highest order of vanishing moments and the smallest number of framelet generators can easily be achieved.

Symmetry is also a desirable property in the construction of framelet systems. So we construct univariate (anti-)symmetric quasi-tight framelets in Chapter 4. We completely characterize the OEP-based (anti-)symmetric compactly supported quasi-tight framelet systems with two generators.

For the multivariate framelets, it is known in the literature that the problems of constructing tight framelets / dual framelets with vanishing moments from general (nonseparable) refinable functions are quite hard. We propose solutions to the problems using quasi-tight framelets. We constructively prove that it is very easy to derive multivariate quasi-tight framelets with directionality/high order of vanishing moments, from any arbitrary M-refinable function, with any dilation matrix M.

The constructions of quasi-tight framelets are directly linked to the mathematical problem of (indefinite) spectral factorizations of matrices of Laurent polynomials. We study/solve the spectral factorization problem in different settings in each Chapter 2 to 5.

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Table of Contents

List of Figures

Chapter 1

Introduction to Quasi-tight Framelets

1.1 Introduction to Wavelets and Framelets

Wavelets and framelets have been widely studied in the mathematical and engineering communities, due to their applications in many areas such as image processing and numerical algorithms. To introduce the work of this thesis, we provide some necessary background of wavelets/framelets, and some notations used here.

Let $L_2(\mathbb{R}^d)$ denote the set of all square integrable functions on $\mathbb{R}^d$. A set of functions $F \subset L_2(\mathbb{R}^d)$ is called ***a frame in*** $L_2(\mathbb{R}^d)$ if there exist some positive constants $C_1, C_2 > 0$, such that

$$C_1 \|f\|^2_{L_2(\mathbb{R}^d)} \leqslant \sum_{h \in F} |\langle f, h \rangle|^2 \leqslant C_2 \|f\|^2_{L_2(\mathbb{R}^d)}, \qquad \forall\, f \in L_2(\mathbb{R}^d).$$

C_1 and C_2 are called the lower and upper frame bound respectively. In the case that $C_1 = C_2 = 1$, we call it a ***(normalized) tight frame in*** $L_2(\mathbb{R}^d)$. The idea of frames is a generalization of an orthonormal basis in a Hilbert space, while a frame system will generally be overcomplete (has redundancy). To give a trivial example, if E is an orthonormal basis in $L_2(\mathbb{R}^d)$ and G is a finite subset of $L_2(\mathbb{R}^d)$, then $F = E \cup G$ must be a frame in $L_2(\mathbb{R}^d)$. See [8] for a comprehensive review of the theory of frames in Hilbert spaces.

Framelet (wavelet frame) is a special type of frames in $L_2(\mathbb{R}^d)$, where F consists of an affine system. Suppose U is a $d \times d$ real valued matrix, we use the following notation for the dilations and shifts of a function f:

$$f_{U;k}(x) := |\det(U)|^{1/2} f(Ux - k), \qquad x, k \in \mathbb{R}^d.$$

In one-dimensional case, given a set of functions $\Psi \subset L_2(\mathbb{R})$, we call

$$\begin{aligned}\mathsf{AS}(\Psi) :=& \{\psi^\ell_{2^j;k} \ : \ \psi^\ell \in \Psi, k \in \mathbb{Z}, j \in \mathbb{Z}\} \\ :=& \{2^{j/2}\psi^\ell(2^j \cdot -k) \ : \ \psi^\ell \in \Psi, k \in \mathbb{Z}, j \in \mathbb{Z}\}\end{aligned}$$

the ***(dyadic) affine system*** generated by Ψ. If $\mathsf{AS}(\Psi)$ is a (tight) frame in $L_2(\mathbb{R})$, then Ψ is called a ***homogeneous (tight) framelet in*** $L_2(\mathbb{R})$.

For the multidimensional case, we call a $d \times d$ matrix M a ***dilation matrix*** if it is an integer matrix such that all its eigenvalues are greater than one in modulus. Given a dilation matrix M, an M***-affine system*** generated by $\Psi \subset L_2(\mathbb{R}^d)$ is defined as

$$\begin{aligned}\mathsf{AS}^{\mathsf{M}}(\Psi) :=& \{\psi^\ell_{\mathsf{M}^j;k} \ : \ \psi^\ell \in \Psi, k \in \mathbb{Z}^d, j \in \mathbb{Z}\} \\ :=& \{|\det(\mathsf{M})|^{j/2}\psi^\ell(\mathsf{M}^j \cdot -k) \ : \ \psi^\ell \in \Psi, k \in \mathbb{Z}^d, j \in \mathbb{Z}\}.\end{aligned}$$

Ψ is called a ***homogeneous (tight)*** M***-framelet in*** $L_2(\mathbb{R}^d)$ if $\mathsf{AS}^{\mathsf{M}}(\Psi)$ is a (tight) frame in $L_2(\mathbb{R}^d)$. For generality, we use the notation in the multidimensional case $L_2(\mathbb{R}^d)$ throughout this chapter. For the one-dimensional problem ($d = 1$), although the case for general integer dilation $\mathsf{M} \geqslant 2$ (also called M-band wavelets/framelets) is also considered in the literature (e.g. see [82, 12, 3, 84, 83, 38]), for simplicity, our study focuses on the case that $\mathsf{M} = 2$ in this thesis.

Framelets are usually constructed by multiresolution analysis (MRA). We only consider MRA with one generator here. A sequence $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces in $L_2(\mathbb{R}^d)$ forms a ***multiresolution analysis (MRA) of*** $L_2(\mathbb{R}^d)$ if for some $\phi \in L_2(\mathbb{R}^d)$

(1) $V_j = \overline{span\{\phi(\mathsf{M}^j \cdot -k) \ : \ k \in \mathbb{Z}^d\}}$ and $V_j \subset V_{j+1}$ for all integers $j \in \mathbb{Z}$;

(2) $\overline{\cup_{j\in\mathbb{Z}} V_j} = L_2(\mathbb{R}^d)$.

Note that $V_0 \subset V_1$ implies that there exists some sequence $a = \{a(k)\}_{k\in\mathbb{Z}^d}$, such that

$$\phi = |\det(\mathsf{M})| \sum_{k\in\mathbb{Z}^d} a(k)\phi(\mathsf{M}\cdot -k). \tag{1.1.1}$$

Hence, ϕ is called an M***-refinable function*** (or ***scaling function*** in wavelet analysis), and a is called a ***refinement mask (filter)***. The first condition in MRA shows that V_j is shift invariant in its scale level, i.e., $f \in V_j$ implies $f(\cdot - \mathsf{M}^{-j}k) \in V_j$ for all $k \in \mathbb{Z}^d$. MRA is used to approximate functions in $L_2(\mathbb{R}^d)$ with different scale/resolution levels. For a function $f \in L_2(\mathbb{R}^d)$, let $P_j f$ be the orthogonal projection of f into the space V_j, then $P_j f \to f$ in $L_2(\mathbb{R}^d)$ as $j \to \infty$. For a framelet Ψ in $L_2(\mathbb{R}^d)$, if there exists an MRA generated by some $\phi \in L_2(\mathbb{R}^d)$, such that $\Psi \subset V_1$, then Ψ is called ***an MRA-based framelet***. That is, for each $\psi^\ell \in \Psi$, there exists some sequence $b_\ell = \{b_\ell(k)\}_{k\in\mathbb{Z}}$, such that

$$\psi^\ell = |\det(\mathsf{M})| \sum_{k\in\mathbb{Z}^d} b_\ell(k)\phi(\mathsf{M}\cdot -k), \qquad \forall \psi_\ell \in \Psi. \tag{1.1.2}$$

For detailed studies about shift invariant spaces/MRA-based framelets, see [15, 9, 45, 17] and many references therein.

For an integrable function $f \in L_1(\mathbb{R}^d)$, its Fourier transform $\widehat{f}$ is defined to be $\widehat{f}(\xi) := \int_{\mathbb{R}^d} f(x)e^{-ix\cdot\xi}dx$ for $\xi \in \mathbb{R}^d$. The Fourier transform can be naturally extended to square integrable functions in $L_2(\mathbb{R}^d)$ and tempered distributions. By $l_0(\mathbb{Z}^d)$ we denote the set of all finitely supported sequences/filters $a = \{a(k)\}_{k\in\mathbb{Z}^d} : \mathbb{Z}^d \to \mathbb{C}$ on $\mathbb{Z}^d$. For a filter $a \in l_0(\mathbb{Z}^d)$, its support is defined as $\operatorname{supp}(a) := \{k \in \mathbb{Z}^d \ : \ a(k) \neq 0\}$ and its Fourier series (or symbol) is defined to be $\widehat{a}(\xi) := \sum_{k\in\mathbb{Z}^d} a(k)e^{-ik\cdot\xi}$ for $\xi \in \mathbb{R}^d$, which is a $2\pi\mathbb{Z}^d$-periodic trigonometric polynomial in d variables. In particular, by $\boldsymbol{\delta}$ we denote ***the Dirac sequence*** such that $\boldsymbol{\delta}(0) = 1$ and $\boldsymbol{\delta}(k) = 0$ for all $\mathbb{Z}^d\backslash\{0\}$. For $\gamma \in \mathbb{Z}^d$, we also use the notation $\boldsymbol{\delta}_\gamma$ to stand for the sequence $\boldsymbol{\delta}(\cdot - \gamma)$, i.e., $\boldsymbol{\delta}_\gamma(\gamma) = 1$ and $\boldsymbol{\delta}_\gamma(k) = 0$ for all $k \in \mathbb{Z}^d\backslash\{\gamma\}$. Note that $\widehat{\boldsymbol{\delta}_\gamma}(\xi) = e^{-i\gamma\cdot\xi}$. In frequency domain, equations

(1.1.1) and (1.1.2) can be written as

$$\widehat{\phi}(\mathsf{M}^{\mathsf{T}}\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi), \quad \widehat{\psi^\ell}(\mathsf{M}^{\mathsf{T}}\xi) = \widehat{b_\ell}(\xi)\widehat{\phi}(\xi).$$

Let $\psi^1, \ldots, \psi^s, \tilde{\psi}^1, \ldots, \tilde{\psi}^s \in L_2(\mathbb{R}^d)$. We say that $(\{\tilde{\psi}^1, \ldots, \tilde{\psi}^s\}, \{\psi^1, \ldots, \psi^s\})$ is ***a homogeneous dual*** M***-framelet*** in $L_2(\mathbb{R}^d)$ if both $\{\tilde{\psi}^1, \ldots, \tilde{\psi}^s\}$ and $\{\psi^1, \ldots, \psi^s\}$ are homogeneous M-framelets in $L_2(\mathbb{R}^d)$ such that

$$\langle f, g \rangle = \sum_{j\in\mathbb{Z}} \sum_{\ell=1}^{s} \sum_{k\in\mathbb{Z}^d} \langle f, \tilde{\psi}^\ell_{\mathsf{M}^j;k} \rangle \langle \psi^\ell_{\mathsf{M}^j;k}, g \rangle, \quad \forall\, f, g \in L_2(\mathbb{R}^d) \tag{1.1.3}$$

with the above series converging absolutely. It follows directly from (1.1.3) that every function $f \in L_2(\mathbb{R}^d)$ has the following multiscale framelet representation:

$$f = \sum_{j\in\mathbb{Z}} \sum_{\ell=1}^{s} \sum_{k\in\mathbb{Z}^d} \langle f, \tilde{\psi}^\ell_{\mathsf{M}^j;k} \rangle \psi^\ell_{\mathsf{M}^j;k} \tag{1.1.4}$$

with the series converging unconditionally in $L_2(\mathbb{R}^d)$. Also, if $\{\psi^1, \ldots, \psi^s\}$ is a homogeneous tight M-framelet in $L_2(\mathbb{R}^d)$, then it is self-dual, i.e., $(\{\psi^1, \ldots, \psi^s\}, \{\psi^1, \ldots, \psi^s\})$ is a homogeneous dual M-framelet in $L_2(\mathbb{R}^d)$.

Compactly supported MRA-based dual M-framelets can be obtained by a general procedure called ***Oblique Extension Principle (OEP)*** introduced in [11, 17]. Given $\Theta, a, b_1, \ldots, b_s, \tilde{a}, \tilde{b_1}, \ldots, \tilde{b_s} \in l_0(\mathbb{Z}^d)$, where $\widehat{a}(0) = \widehat{\tilde{a}}(0) = 1$, we can define

$$\widehat{\phi}(\xi) := \prod_{j=1}^{\infty} \widehat{a}((\mathsf{M}^{\mathsf{T}})^{-j}\xi), \quad \widehat{\tilde{\phi}}(\xi) := \prod_{j=1}^{\infty} \widehat{\tilde{a}}((\mathsf{M}^{\mathsf{T}})^{-j}\xi), \qquad \xi \in \mathbb{R}^d. \tag{1.1.5}$$

Then it is trivial to observe that ϕ and $\tilde{\phi}$ are both M-refinable functions/distributions satisfying

$$\widehat{\phi}(\mathsf{M}^{\mathsf{T}}\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi), \quad \widehat{\tilde{\phi}}(\mathsf{M}^{\mathsf{T}}\xi) = \widehat{\tilde{a}}(\xi)\widehat{\tilde{\phi}}(\xi).$$

Define

$$\widehat{\psi^\ell}(\mathsf{M}^\mathsf{T}\xi) = \widehat{b_\ell}(\xi)\widehat{\phi}(\xi), \quad \widehat{\tilde{\psi}^\ell}(\mathsf{M}^\mathsf{T}\xi) = \widehat{\tilde{b}_\ell}(\xi)\widehat{\tilde{\phi}}(\xi), \qquad a.e.\, \xi \in \mathbb{R}^d, \ell = 1, \ldots, s. \tag{1.1.6}$$

Then $(\{\tilde{\psi}^1, \ldots, \tilde{\psi}^s\}, \{\psi^1, \ldots, \psi^s\})$ is a homogeneous dual M-framelet in $L_2(\mathbb{R}^d)$ if and only if $\phi, \tilde{\phi} \in L_2(\mathbb{R}^d)$, $\widehat{\Theta}(0) = 1$,

$$\widehat{b_1}(0) = \cdots = \widehat{b_s}(0) = \widehat{\tilde{b}_1}(0) = \cdots = \widehat{\tilde{b}_s}(0) = 0, \tag{1.1.7}$$

and $(\{\tilde{a}; \tilde{b}_1, \ldots, \tilde{b}_s\}, \{a; b_1, \ldots, b_s\})_\Theta$ is ***a dual*** M***-framelet filter bank***, i.e.,

$$\widehat{\Theta}(\mathsf{M}^\mathsf{T}\xi)\widehat{\tilde{a}}(\xi)\overline{\widehat{a}(\xi + 2\pi\omega)} + \widehat{\tilde{b}_1}(\xi)\overline{\widehat{b_1}(\xi + 2\pi\omega)} + \cdots + \widehat{\tilde{b}_s}(\xi)\overline{\widehat{b_s}(\xi + 2\pi\omega)} = \boldsymbol{\delta}(\omega)\widehat{\Theta}(\xi),$$
$$\forall\, \omega \in \Omega_\mathsf{M}, \xi \in \mathbb{R}^d, \tag{1.1.8}$$

where $\Omega_\mathsf{M} := [(\mathsf{M}^\mathsf{T})^{-1}\mathbb{Z}^d] \cap [0, 1)^d$. A filter bank $\{a; b_1, \ldots, b_s\}_\Theta$ is called ***a tight*** M***-framelet filter bank*** if $(\{a; b_1, \ldots, b_s\}, \{a; b_1, \ldots, b_s\})_\Theta$ is a dual M-framelet filter bank.

It is known in [45, Theorem 7.1.8] that $\{\psi^1, \ldots, \psi^s\}$ is a tight M-framelet in $L_2(\mathbb{R}^d)$ if and only if $\{a; b_1, \ldots, b_s\}_\Theta$ is a tight M-framelet filter bank and $\widehat{\Theta}(0) = 1$. Hence one does not need to check the necessary conditions $\phi \in L_2(\mathbb{R}^d)$ and $\widehat{b_1}(0) = \cdots = \widehat{b_s}(0) = 0$ in advance for tight framelets. Consequently, the construction of dual/tight framelets boils down to the construction of dual/tight framelet filter banks. One-dimensional tight framelets have been well investigated and constructed in the literature, to only mention a few, see [11, 16, 17, 35, 39, 43, 44, 45, 48, 58, 73, 78] and many references therein. In particular, one-dimensional tight 2-framelets with symmetry property have been extensively studied in [40, 43, 48] and references therein.

Define $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For $\mu = (\mu_1, \ldots, \mu_d)^\mathsf{T} \in \mathbb{N}_0^d$ and $x = (x_1, \ldots, x_d)^\mathsf{T} \in \mathbb{R}^d$, we define $|\mu| := \mu_1 + \cdots + \mu_d$, $x^\mu := x_1^{\mu_1} \cdots x_d^{\mu_d}$ and $\partial^\mu := \partial_1^{\mu_1} \cdots \partial_d^{\mu_d}$. For $m \in \mathbb{N}_0$ and smooth functions f and g on $\mathbb{R}^d$, we shall simply use the big $\mathcal{O}$ notation $f(\xi) = g(\xi) + \mathcal{O}(\|\xi\|^m)$ as $\xi \to 0$ to stand for $\partial^\mu f(0) = \partial^\mu g(0)$ for all $\mu \in \mathbb{N}_0^d$ with $|\mu| < m$. We say that the sequence a satisfies ***order*** m ***sum***

rules with respect to M, if

$$\widehat{a}(\xi + 2\pi\omega) = \mathcal{O}(\|\xi\|^m), \quad \xi \to 0, \forall\, \omega \in \Omega_{\mathsf{M}} \backslash \{0\}. \tag{1.1.9}$$

We define $\operatorname{sr}(a, \mathsf{M})$ to be the largest nonnegative integer m satisfying (1.1.9). Since $\widehat{a}(0) = \widehat{\tilde{a}}(0) = 1$, we now trivially see that (1.1.7) and (1.1.8) imply $\operatorname{sr}(a, \mathsf{M}) \geqslant 1$. In our study of one-dimensional ($d = 1$) dyadic ($\mathsf{M} = 2$) framelets, we simply use the notation $\operatorname{sr}(a) := \operatorname{sr}(a, 2)$. Given a sequence $a \in l_0(\mathbb{Z}^d)$ and $\widehat{a}(0) = 1$, suppose $\phi \in L_2(\mathbb{R}^d)$ is generated from a through (1.1.5), then $\operatorname{sr}(a, \mathsf{M})$ is related to the approximation property of the shift invariant space generated by ϕ (e.g., see [57]). The sequence a satisfying $\widehat{a}(0) = 1$ and (1.1.9) with $m \geqslant 1$ is also called a ***low-pass filter*** in the engineering community.

In order to check the technical condition $\phi \in L_2(\mathbb{R}^d)$, we can calculate the smoothness exponent. Let $a \in l_0(\mathbb{Z}^d)$ with $\widehat{a}(0) = 1$ and $m := \operatorname{sr}(a, \mathsf{M})$. For $1 \leqslant p \leqslant \infty$, we now introduce a technical quantity (see [37]):

$$\operatorname{sm}_p(a, \mathsf{M}) := \frac{d}{p} - \log_{\rho(\mathsf{M})} \rho_m(a, M)_p \quad \text{and} \quad \operatorname{sm}(a, \mathsf{M}) := \operatorname{sm}_2(a, \mathsf{M}), \tag{1.1.10}$$

where $\rho(\mathsf{M})$ is the spectral radius of M and

$$\rho_m(a, \mathsf{M})_p := |\det(\mathsf{M})| \sup\{\lim_{n\to\infty} \|a_n * (\nabla^\mu \boldsymbol{\delta})\|_{l_p(\mathbb{Z}^d)}^{1/n} \;:\; \mu \in \mathbb{N}_0^d, |\mu| = m\}$$

with $\widehat{a_n}(\xi) := \widehat{a}(\xi)\widehat{a}(\mathsf{M}^\mathsf{T}\xi) \cdots \widehat{a}((\mathsf{M}^\mathsf{T})^{n-1}\xi)$. Let ϕ be defined in (1.1.5). If $\operatorname{sm}(a, \mathsf{M}) > 0$, then $\phi \in L_2(\mathbb{R}^d)$ and moreover, $\int_{\mathbb{R}^d} |\widehat{\phi}(\xi)|^2 (1 + \|\xi\|^2)^\tau d\xi < \infty$ for all $0 \leqslant \tau < \operatorname{sm}(a, \mathsf{M})$.

For a compactly supported function $\psi \in L_2(\mathbb{R}^d)$, we say that ψ has ***order*** m ***vanishing moments*** if

$$\int_{\mathbb{R}^d} \psi(x) x^\mu dx = 0, \quad \forall\, \mu \in \mathbb{N}_0^d,\ |\mu| < m,$$
$$\text{or equivalently,} \quad \widehat{\psi}(\xi) = \mathcal{O}(\|\xi\|^m), \quad \xi \to 0. \tag{1.1.11}$$

In particular, we define $\operatorname{vm}(\psi) := m$ for the largest possible integer m in

(1.1.11). For a sequence $b \in l_0(\mathbb{Z}^d)$, we similarly define $\operatorname{vm}(b)$ to be the largest possible integer $m \in \mathbb{N}_0$ such that $\widehat{b}(\xi) = \mathcal{O}(\|\xi\|^m)$ as $\xi \to 0$. That is, the sequence b has the order $\operatorname{vm}(b)$ vanishing moments. The sequence b with $\operatorname{vm}(b) \geqslant 1$ is called a ***high-pass filter*** in the engineering community. If ψ is derived from a function ϕ with $\widehat{\phi}(0) \neq 0$ through $\widehat{\psi}(\mathsf{M}^\mathsf{T}\xi) = \widehat{b}(\xi)\widehat{\phi}(\xi)$ for some $b \in l_0(\mathbb{Z}^d)$, then it is trivial to see that $\operatorname{vm}(\psi) = \operatorname{vm}(b)$. Note that the condition in (1.1.7) is equivalent to saying that all the filters $b_1, \dots, b_s, \tilde{b}_1, \dots, \tilde{b}_s$ have order one vanishing moment (i.e., the basic vanishing moment). The notion of vanishing moments plays the key role for the sparsity of a framelet representation in (1.1.4). Therefore, vanishing moments are one of the most desirable properties of wavelets and framelets.

1.2 From Tight Framelets to Quasi-tight Framelets

By definition, homogeneous tight M-framelet $\{\psi^1, \dots, \psi^s\}$ satisfies

$$C_1\|f\|^2_{L_2(\mathbb{R}^d)} \leqslant \sum_{j\in\mathbb{Z}} \sum_{\ell=1}^{s} \sum_{k\in\mathbb{Z}^d} |\langle f, \psi^\ell_{\mathsf{M}^j;k}\rangle|^2 \leqslant C_2\|f\|^2_{L_2(\mathbb{R}^d)}, \qquad \forall\, f \in L_2(\mathbb{R}^d). \tag{1.2.1}$$

with the two frame bounds $C_1 = C_2 = 1$. So the inequalities in (1.2.1) become equalities. Besides this energy preservation property, another desirable property is that tight framelets are self-dual, i.e., (1.1.4) holds with $\{\tilde{\psi}^1, \dots, \tilde{\psi}^s\}$ the same as $\{\psi^1, \dots, \psi^s\}$. For a given dual M-framelet $(\{\tilde{\psi}^1, \dots, \tilde{\psi}^s\}, \{\psi^1, \dots, \psi^s\})$ in $L_2(\mathbb{R}^d)$, every function $f \in L_2(\mathbb{R}^d)$ has the representation in (1.1.4). In many application problems of framelets, each $\tilde{\psi}^\ell$ models some desired feature capturing certain key singularities. For example, in images, $\tilde{\psi}^\ell$ may behave like an edge or texture. If f contains such (scaled and shifted) feature $\tilde{\psi}^\ell_{\mathsf{M}^j;k}$ for some $1 \leqslant \ell \leqslant s$, $j \in \mathbb{N}_0$ and $k \in \mathbb{Z}^d$, then the coefficient $\langle f, \tilde{\psi}^\ell_{\mathsf{M}^j;k}\rangle$ has a large significant magnitude. Therefore, we can capture such desired feature in f by observing a significant coefficient $\langle f, \tilde{\psi}^\ell_{\mathsf{M}^j;k}\rangle$. However, we reconstruct f through (1.1.4) by using $\langle f, \tilde{\psi}^\ell_{\mathsf{M}^j;k}\rangle \psi^\ell_{\mathsf{M}^j;k}$. If ψ^ℓ is very similar/close to $\tilde{\psi}^\ell$, then we indeed are able to capture the desired feature $\tilde{\psi}^\ell_{\mathsf{M}^j;k}$ in f. However, if this is not the case, then the representation in (1.1.4), which can exactly reconstruct f, do not make sense to extract and represent features in f. This is probably the main reason that only homogeneous dual framelets $(\{\tilde{\psi}^1,\dots,\tilde{\psi}^s\},\{\psi^1,\dots,\psi^s\})$ in $L_2(\mathbb{R}^d)$, with $\tilde{\psi}^\ell$ being similar to ψ^ℓ for all $\ell=1,\dots,s$, are useful in applications. The ideal case that $\tilde{\psi}^\ell=\psi^\ell$ for all $\ell=1,\dots,s$ leads to tight framelets.

However, constructing tight framelets is not always an easy task. Let us introduce the difficulties and challenges here.

For a matrix or matrix function $A(\xi)$, we define $A^\star(\xi):=\overline{A(\xi)}^{\mathsf{T}}$, the transpose of the complex conjugate of $A(\xi)$. In the construction of univariate ($d=1$) dyadic ($\mathsf{M}=2$) framelets, we can rewrite (1.1.8) as

$$\begin{bmatrix}\widehat{\tilde{b}}_1(\xi) & \cdots & \widehat{\tilde{b}}_s(\xi)\\ \widehat{\tilde{b}}_1(\xi+\pi) & \cdots & \widehat{\tilde{b}}_s(\xi+\pi)\end{bmatrix}\begin{bmatrix}\widehat{b}_1(\xi) & \cdots & \widehat{b}_s(\xi)\\ \widehat{b}_1(\xi+\pi) & \cdots & \widehat{b}_s(\xi+\pi)\end{bmatrix}^\star=\mathcal{M}_{a,\tilde{a},\Theta}(\xi), \tag{1.2.2}$$

where $\mathcal{M}_{a,\tilde{a},\Theta}(\xi)$ is defined as:

$$\mathcal{M}_{a,\tilde{a},\Theta}(\xi):=\begin{bmatrix}\widehat{\Theta}(\xi)-\widehat{\Theta}(2\xi)\widehat{\tilde{a}}(\xi)\overline{\widehat{a}(\xi)} & -\widehat{\Theta}(2\xi)\widehat{\tilde{a}}(\xi)\overline{\widehat{a}(\xi+\pi)}\\ -\widehat{\Theta}(2\xi)\widehat{\tilde{a}}(\xi+\pi)\overline{\widehat{a}(\xi)} & \widehat{\Theta}(\xi+\pi)-\widehat{\Theta}(2\xi)\widehat{\tilde{a}}(\xi+\pi)\overline{\widehat{a}(\xi+\pi)}\end{bmatrix}. \tag{1.2.3}$$

For the case that the primal functions ψ^ℓ and dual functions $\tilde{\psi}^\ell$ are derived from the same refinable function ϕ, i.e., $a=\tilde{a}$, we simply write $\mathcal{M}_{a,\Theta}(\xi):=\mathcal{M}_{a,\tilde{a},\Theta}(\xi)$. It is easy to conclude from (1.2.2) that for tight framelet filter banks, $\mathcal{M}_{a,\Theta}(\xi)$ has to be a positive semi-definite matrix for all $\xi\in\mathbb{R}$. In the special case that $\widehat{\Theta}(\xi)=1$, that is, $\Theta=\boldsymbol{\delta}$, the positive semi-definiteness of $\mathcal{M}_{a,\boldsymbol{\delta}}(\xi)$ reduces to the restriction on a:

$$\det(\mathcal{M}_{a,\boldsymbol{\delta}}(\xi))=1-|\widehat{a}(\xi)|^2-|\widehat{a}(\xi+\pi)|^2\geqslant 0. \tag{1.2.4}$$

Such framelet filter banks with $\Theta=\boldsymbol{\delta}$ are called ***Unitary Extension Principle (UEP)-based framelet filter banks*** (e.g., see [72]). Otherwise, we call them ***Oblique Extension Principle (OEP)-based framelet filter banks***. The restriction in (1.2.4) can be avoided by considering the OEP-based tight framelets. OEP-based tight framelets have been studied in [17, 11], where the authors constructed tight framelets with highest possible order of vanishing moments. Hence, Θ is also called a ***moment correcting filter*** (or vanishing moment recovery function in [11]). It has been proved in [11] that for any given finitely supported refinable mask $a \in l_0(\mathbb{Z})$ such that $\widehat{a}(\xi) = 1$, there always exists a compactly supported $\Theta \in l_0(\mathbb{Z})$, such that $\mathcal{M}_{a,\Theta}(\xi)$ is positive semi-definite for all $\xi \in \mathbb{R}$, as long as the integer shifts of ϕ (refinable function generated by a through (1.1.5)) are stable. It is known that without the stability condition of the shifts of ϕ, it might not be possible to construct compactly supported tight framelets from such a mask a by OEP in general (see [49, Example 4.5]). So we have to consider other types of dual framelets in the general cases.

In the multivariate case, we have much more challenges in constructing tight framelets and dual framelets with high order of vanishing moments. For simplicity, we only consider the case that $\Theta = \boldsymbol{\delta}$ be the Dirac sequence, and omit the subscript Θ here. That is, we denote the dual M-framelet filter bank $(\{\tilde{a}; \tilde{b}_1, \dots, \tilde{b}_s\}, \{a; b_1, \dots, b_s\}) := (\{\tilde{a}; \tilde{b}_1, \dots, \tilde{b}_s\}, \{a; b_1, \dots, b_s\})_{\boldsymbol{\delta}}$, and the tight M-framelet filter bank $\{a; b_1, \dots, b_s\} := \{a; b_1, \dots, b_s\}_{\boldsymbol{\delta}}$. Let us first rewrite the equations in (1.1.8) for a dual M-framelet filter bank $(\{\tilde{a}; \tilde{b}_1, \dots, \tilde{b}_s\}, \{a; b_1, \dots, b_s\})$. Define

$$\mathsf{d}_{\mathsf{M}} := |\det(\mathsf{M})| \quad \text{and} \quad \{\omega_1, \dots, \omega_{\mathsf{d}_{\mathsf{M}}}\} := \Omega_{\mathsf{M}} := [(\mathsf{M}^{\mathsf{T}})^{-1}\mathbb{Z}^d] \cap [0,1)^d. \quad (1.2.5)$$

Denote $b_0 := a$ and $\tilde{b}_0 := \tilde{a}$. Then it is not difficult to observe that (1.1.8) is equivalent to

$$\sum_{\ell=0}^{s} \left[\widehat{\tilde{b}_\ell}(\xi + 2\pi\omega_1), \dots, \widehat{\tilde{b}_\ell}(\xi + 2\pi\omega_{\mathsf{d}_{\mathsf{M}}})\right]^{\star} \left[\widehat{b_\ell}(\xi + 2\pi\omega_1), \dots, \widehat{b_\ell}(\xi + 2\pi\omega_{\mathsf{d}_{\mathsf{M}}})\right] = I_{\mathsf{d}_{\mathsf{M}}}, \quad (1.2.6)$$

where $I_{\mathsf{d}_{\mathsf{M}}}$ is the $\mathsf{d}_{\mathsf{M}} \times \mathsf{d}_{\mathsf{M}}$ identity matrix. For $u \in l_0(\mathbb{Z}^d)$ and $\gamma \in \mathbb{Z}^d$, ***its*** γ***-coset sequence*** $u^{[\gamma]}$ is defined to be $u^{[\gamma]} := \{u(\gamma + \mathsf{M}k)\}_{k \in \mathbb{Z}^d}$. Define

$$\{\gamma_1, \dots, \gamma_{\mathsf{d}_{\mathsf{M}}}\} := \Gamma_{\mathsf{M}} := [\mathsf{M}[0,1)^d] \cap \mathbb{Z}^d. \quad (1.2.7)$$

Then $\widehat{u}(\xi) = \sum_{\gamma\in\Gamma_{\mathsf{M}}} e^{-i\gamma\cdot\xi}\widehat{u^{[\gamma]}}(\mathsf{M}^{\mathsf{T}}\xi)$ and we have

$$[\widehat{u}(\xi+2\pi\omega_1),\ldots,\widehat{u}(\xi+2\pi\omega_{\mathsf{d_M}})] = \left[\widehat{u^{[\gamma_1]}}(\mathsf{M}^{\mathsf{T}}\xi),\ldots,\widehat{u^{[\gamma_{\mathsf{d_M}}]}}(\mathsf{M}^{\mathsf{T}}\xi)\right]E(\xi)U, \quad (1.2.8)$$

where

$$E(\xi) := \operatorname{diag}(e^{-i\gamma_1\cdot\xi},\ldots,e^{-i\gamma_{\mathsf{d_M}}\cdot\xi}) \quad \text{and} \quad U := (e^{-i\gamma_j\cdot 2\pi\omega_k})_{1\leqslant j,k\leqslant \mathsf{d_M}}. \quad (1.2.9)$$

Because $E(\xi)E^\star(\xi) = I_{\mathsf{d_M}}$ and $UU^\star = \mathsf{d_M} I_{\mathsf{d_M}}$ for all $\xi\in\mathbb{R}^d$, it is now straightforward to deduce that (1.2.6) is equivalent to

$$\sum_{\ell=1}^{s}\left[\widehat{\tilde{b}_\ell^{[\gamma_1]}}(\xi),\ldots,\widehat{\tilde{b}_\ell^{[\gamma_{\mathsf{d_M}}]}}(\xi)\right]^\star\left[\widehat{b_\ell^{[\gamma_1]}}(\xi),\ldots,\widehat{b_\ell^{[\gamma_{\mathsf{d_M}}]}}(\xi)\right] = \mathsf{d_M}^{-1}\mathcal{N}_{\tilde{a},a}(\xi) \quad (1.2.10)$$

with

$$\mathcal{N}_{\tilde{a},a}(\xi) := I_{\mathsf{d_M}} - \mathsf{d_M}\left[\widehat{\tilde{a}^{[\gamma_1]}}(\xi),\ldots,\widehat{\tilde{a}^{[\gamma_{\mathsf{d_M}}]}}(\xi)\right]^\star\left[\widehat{a^{[\gamma_1]}}(\xi),\ldots,\widehat{a^{[\gamma_{\mathsf{d_M}}]}}(\xi)\right]. \quad (1.2.11)$$

In particular, a filter bank $\{a; b_1,\ldots,b_s\}$ is a tight M-framelet filter bank if and only if

$$\sum_{\ell=1}^{s}\left[\widehat{b_\ell^{[\gamma_1]}}(\xi),\ldots,\widehat{b_\ell^{[\gamma_{\mathsf{d_M}}]}}(\xi)\right]^\star\left[\widehat{b_\ell^{[\gamma_1]}}(\xi),\ldots,\widehat{b_\ell^{[\gamma_{\mathsf{d_M}}]}}(\xi)\right] = \mathsf{d_M}^{-1}\mathcal{N}_a(\xi)$$
$$\text{with} \quad \mathcal{N}_a(\xi) := \mathcal{N}_{a,a}(\xi). \quad (1.2.12)$$

Since $\mathcal{N}_a$ has the eigenvalue $1-\mathsf{d_M}\sum_{j=1}^{\mathsf{d_M}}|\widehat{a^{[\gamma_j]}}(\xi)|^2$ and all its other eigenvalues are 1, we have $\det(\mathcal{N}_a(\xi)) = 1-\mathsf{d_M}\sum_{j=1}^{\mathsf{d_M}}|\widehat{a^{[\gamma_j]}}(\xi)|^2$. Take the determinants on both sides of (1.2.12) and apply the Cauchy-Binet formula to the left-hand side of (1.2.12). It is observed in [6] that

$$\mathcal{A}(\xi) := \det(\mathcal{N}_a(\xi)) = 1-\mathsf{d_M}\sum_{j=1}^{\mathsf{d_M}}|\widehat{a^{[\gamma_j]}}(\xi)|^2 = \sum_{\ell=1}^{s_a}|\widehat{u_\ell}(\xi)|^2, \qquad \forall\,\xi\in\mathbb{R}^d, \quad (1.2.13)$$

for some finitely supported sequences $u_1,\ldots,u_{s_a}\in l_0(\mathbb{Z}^d)$ with $s_a = \binom{s}{\mathsf{d_M}}$. That is, the nonnegative $2\pi\mathbb{Z}^d$-periodic trigonometric polynomial $\mathcal{A}$ in (1.2.13) can be written as a sum of Hermitian squares of $2\pi\mathbb{Z}^d$-periodic trigonometric polynomials. By (1.2.8), one can easily observe that $\mathcal{A}(\mathsf{M}^{\mathsf{T}}\xi) = 1 - \sum_{\omega\in\Omega_{\mathsf{M}}} |\widehat{a}(\xi + 2\pi\omega)|^2$. Conversely, suppose that there exist $u_1, \ldots, u_{s_a} \in l_0(\mathbb{Z}^d)$ for some integer s_a such that $\mathcal{A}$ can be written as a sum of Hermitian squares as in (1.2.13). Then it is known in [63] that one can always construct a tight M-framelet filter bank $\{a; b_1, \ldots, b_s\}$. In dimension one, due to the Fejér-Riesz lemma, for a nonnegative 2π-periodic trigonometric polynomial $\mathcal{A}$ in (1.2.13), there always exists $u_1 \in l_0(\mathbb{Z})$ such that (1.2.13) is satisfied with $s_a = 1$. However, as discussed in [6], the problem on sums of Hermitian squares in (1.2.13) is much more complicated in dimension higher than one and is known to be a challenging problem in real algebraic geometry. See [6, 7, 63] for a detailed discussion on (1.2.13) and its applications to the construction of multivariate tight framelet filter banks. However, even if the sum of Hermitian squares in (1.2.13) exists, this general method in [6, 63] has several drawbacks. First, all the constructed high-pass filters in [63] have (much) longer supports than that of the low-pass filter a, while short support of filters is a highly desired property in their applications. Secondly, the relation between s_a in (1.2.13) and a low-pass filter a is unknown and the numbers s_a and s could be very large even for low-pass filters with short support. Thirdly, to the best of our knowledge, there is currently no known method/algorithm to efficiently compute $u_1, \ldots, u_{s_a} \in l_0(\mathbb{Z}^d)$ in (1.2.13). So far, all currently known constructions of multivariate nonseparable tight framelets are developed for special low-pass filters (in particular, from low-pass filters for $2I_d$-refinable box splines), for example, see [6, 7, 10, 27, 36, 42, 46, 62, 63, 74, 76, 77, 80] and references therein. Despite recent progresses and enormous effort, construction of multivariate tight framelets still remains as a challenging problem and most constructed multivariate tight framelets in the literature lack some desirable properties such as directionality, vanishing moments or symmetry.

Though dual framelet filter banks offer flexibility over tight framelet filter banks, it is not easy to construct them either. To construct a dual M-framelet filter bank, one has to factorize the matrix $\mathcal{N}_{\tilde{a},a}$ in (1.2.10) so that all the high-pass filters satisfy the basic vanishing moment in (1.1.7). For a one-dimensional

2π-periodic trigonometric polynomial $\mathcal{A}(\xi)$, due to the fundamental theorem of algebra, if $\mathcal{A}(\xi)$ has a zero with multiplicity $m \in \mathbb{N}$ at $\xi = 0$, then one can always write $\mathcal{A}(\xi) = (1 - e^{-i\xi})^m \mathcal{B}(\xi)$ for some 2π-periodic trigonometric polynomial $\mathcal{B}$ with $\mathcal{B}(0) \neq 0$. This factorization technique for separating out the special factor $(1 - e^{-i\xi})^m$ is the key for constructing one-dimensional dual framelet filter banks with high vanishing moments (e.g., see [14, 44] for details). However, such a factorization technique is not available for dimensions higher than one; there are also no special multivariate trigonometric polynomials playing the role of $(1 - e^{-i\xi})^m$ for us to generalize the construction of one-dimensional dual framelet filter banks to multiple dimensions. In fact, most (or generic) multivariate trigonometric polynomials cannot be factorized into products of two nontrivial trigonometric polynomials for dimensions higher than one. Consequently, it is often difficult to construct a dual M-framelet filter bank satisfying the basic vanishing moment condition in (1.1.7). Indeed, to the best of our knowledge, so far [22, 23] are the only known papers to have a general construction of dual M-framelet filter banks with the basic vanishing moment in (1.1.7). However, the construction in [22, 23] is linked to syzygy modules of multivariate Laurent polynomials in algebra and the constructed high-pass filters also have much larger supports than that of their associated low-pass filter.

Now let us introduce the notion of ***quasi-tight framelets*** and explain our motivations. As we explained before, in many real applications, we would like the primal functions ψ^ℓ and dual functions $\tilde{\psi}^\ell$ to be similar to each other. Instead of requiring $\tilde{\psi}^\ell = \psi^\ell$ as in a tight framelet, we can easily achieve our objective by naturally considering $\tilde{\psi}^\ell = \epsilon_\ell \psi^\ell$ with $\epsilon_\ell \in \{-1, 1\}$. This motivates us to introduce the notion of quasi-tight framelets. For $\phi, \psi^1, \dots, \psi^s \in L_2(\mathbb{R}^d)$ and $\epsilon_1, \dots, \epsilon_s \in \{-1, 1\}$, where $\psi_\ell \in \overline{span\{\phi(\mathsf{M} \cdot -k) \ : \ k \in \mathbb{Z}^d\}} \subset L_2(\mathbb{R}^d)$, $\ell = 1, \dots, s$, we say that $\{\psi^1, \dots, \psi^s\}_{(\epsilon_1, \dots, \epsilon_s)}$ is ***a homogeneous quasi-tight*** M***-framelet*** in $L_2(\mathbb{R}^d)$ if $(\{\epsilon_1\psi^1, \dots \epsilon_s\psi^s\}, \{\psi^1, \dots, \psi^s\})$ is a homogeneous dual M-framelet in $L_2(\mathbb{R}^d)$. Consequently, every function $f \in L_2(\mathbb{R}^d)$

has the following representation:

$$f=\sum_{j\in\mathbb{Z}}\sum_{\ell=1}^{s}\sum_{k\in\mathbb{Z}^d}\epsilon_\ell\langle f,\psi^\ell_{\mathsf{M}^j;k}\rangle\psi^\ell_{\mathsf{M}^j;k}. \tag{1.2.14}$$

When $\epsilon_1=\cdots=\epsilon_s=1$, a quasi-tight M-framelet becomes a tight M-framelet. A quasi-tight framelet is often obtained from a quasi-tight framelet filter bank. For $\Theta, a, b_1,\ldots,b_s\in l_0(\mathbb{Z}^d)$ and $\epsilon_1,\ldots,\epsilon_s\in\{-1,1\}$, we say that $\{a;b_1,\ldots,b_s\}_{\Theta,(\epsilon_1,\ldots,\epsilon_s)}$ is ***a quasi-tight*** M***-framelet filter bank*** if $(\{a;\epsilon_1 b_1,\ldots,\epsilon_s b_s\},\{a;b_1,\ldots,b_s\})_\Theta$ is a dual M-framelet filter bank, i.e.,

$$\widehat{\Theta}(\mathsf{M}^{\mathsf{T}}\xi)\overline{\widehat{a}(\xi)}\widehat{a}(\xi+2\pi\omega)+\varepsilon_1\overline{\widehat{b_1}(\xi)}\widehat{b_1}(\xi+2\pi\omega)+\cdots+\varepsilon_s\overline{\widehat{b_s}(\xi)}\widehat{b_s}(\xi+2\pi\omega)=\boldsymbol{\delta}(\omega)\widehat{\Theta}(\xi),$$
$$\forall\,\omega\in\Omega_{\mathsf{M}},\xi\in\mathbb{R}^d. \tag{1.2.15}$$

In the case that $\Theta=\boldsymbol{\delta}$, we simply omit the subscript Θ, $\{a;b_1,\ldots,b_s\}_{(\epsilon_1,\ldots,\epsilon_s)}:=\{a;b_1,\ldots,b_s\}_{\boldsymbol{\delta},(\epsilon_1,\ldots,\epsilon_s)}$.

[45, Example 3.2.2] probably is the first to provide an example of quasi-tight 2-framelets $\{\psi^1,\psi^2\}_{(-1,1)}$ and quasi-tight 2-framelet filter bank $\{a;b_1,b_2\}_{(-1,1)}$ in one dimension, where $a=\{-\frac{1}{16},\frac14,\frac58,\frac14,-\frac{1}{16}\}_{[-2,2]}$ and

$$b_1=\{-\tfrac{1}{16},\tfrac14,-\tfrac38,\tfrac14,-\tfrac{1}{16}\}_{[-2,2]},\quad b_2=\{-\tfrac{\sqrt2}{4},\tfrac{\sqrt2}{2},-\tfrac{\sqrt2}{4}\}_{[0,2]}.$$

Define $\widehat{\phi}(\xi):=\prod_{j=1}^\infty\widehat{a}(2^{-j}\xi)$, $\widehat{\psi^1}(\xi):=\widehat{b_1}(\xi/2)\widehat{\phi}(\xi/2)$, and $\widehat{\psi^2}(\xi):=\widehat{b_2}(\xi/2)\widehat{\phi}(\xi/2)$ with $\phi,\psi^1,\psi^2\in L_2(\mathbb{R})$. The above example in [45, Example 3.2.2] was obtained by applying the general algorithm developed in [44] for constructing dual framelet filter banks to the above low-pass filter a.

Let $\{a;b_1,\ldots,b_s\}_{\Theta,(\epsilon_1,\ldots,\epsilon_s)}$ be a quasi-tight M-framelet filter bank with $\widehat{a}(0)=1$. Let m denote the smallest order of the vanishing moments among the high-pass filters, i.e., $m:=\min(\mathrm{vm}(b_1),\ldots,\mathrm{vm}(b_s))$. For $\omega\in\Omega_{\mathsf{M}}\backslash\{0\}$, we easily deduce from (1.2.15) that $\widehat{\Theta}(\mathsf{M}^{\mathsf{T}}\xi)\overline{\widehat{a}(\xi)}\widehat{a}(\xi+2\pi\omega)+\mathcal{O}(\|\xi\|^m)=0$ as $\xi\to0$. Consequently, by $\widehat{a}(0)=1$ and $\widehat{\Theta}(0)=1$, we get $\widehat{a}(\xi+2\pi\omega)=\mathcal{O}(\|\xi\|^m)$ as $\xi\to0,\forall\,\omega\in\Omega_{\mathsf{M}}\backslash\{0\}$. That is, the filter a must satisfy order m sum rules with respect to M:

$$\mathrm{sr}(a)\geqslant m. \tag{1.2.16}$$

We can also trivially deduce from (1.2.15) with $\omega = 0$ that

$$\widehat{\Theta}(\xi) - \widehat{\Theta}(\mathsf{M}^\mathsf{T}\xi)|\widehat{a}(\xi)|^2 = \mathcal{O}(\|\xi\|^{2m}), \qquad \xi \to 0. \tag{1.2.17}$$

Consequently, for a quasi-tight M-framelet filter bank $\{a; b_1, \ldots, b_s\}_{\Theta,(\epsilon_1,\ldots,\epsilon_s)}$, we always have

$$\min(\mathrm{vm}(b_1), \ldots, \mathrm{vm}(b_s)) \leqslant \min(\mathrm{sr}(a, \mathsf{M}), \tfrac{1}{2}\,\mathrm{vm}(u_{\Theta,a})),$$
$$\text{with } \widehat{u_{\Theta,a}}(\xi) := \widehat{\Theta}(\xi) - \widehat{\Theta}(\mathsf{M}^\mathsf{T}\xi)|\widehat{a}(\xi)|^2. \tag{1.2.18}$$

Given a low-pass filter $a \in l_0(\mathbb{Z}^d)$ and moment correcting filter $\Theta \in l_0(\mathbb{Z}^d)$ (hence we get the underlying MRA), the above inequality gives the maximum order of vanishing moments that OEP-based quasi-tight framelets can achieve. For the simple case that $\Theta = \boldsymbol{\delta}$, we just denote $u_a := u_{\boldsymbol{\delta},a}$. For $a \in l_0(\mathbb{Z}^d)$ and $c \in \mathbb{R}^d$, we say that a has ***order*** m ***linear-phase moments*** with phase c if $\widehat{a}(\xi) = e^{-ic\cdot\xi} + \mathcal{O}(\|\xi\|^m)$ as $\xi \to 0$. We define $\mathrm{lpm}(a) := m$ for the largest possible integer m. Then we can easily verify that $\mathrm{vm}(u_a) \geqslant \mathrm{lpm}(a)$. If in addition a has symmetry satisfying $a(c_a - k) = \overline{a(k)}$ for all $k \in \mathbb{Z}^d$ for some $c_a \in \mathbb{Z}^d$, then it is known in [41, Proposition 5.3] that $\mathrm{lpm}(a) = \mathrm{vm}(u_a)$ and (1.2.18) becomes

$$\min(\mathrm{vm}(b_1), \ldots, \mathrm{vm}(b_s)) \leqslant \min(\mathrm{sr}(a, \mathsf{M}), \tfrac{1}{2}\,\mathrm{lpm}(a)).$$

1.3 Introduction to Spectral Factorizations of Matrices of Laurent Polynomials

The construction of quasi-tight framelet filter banks relies on the problem of generalized ***spectral factorizations of matrices of Laurent polynomials*** (or trigonometric polynomials in frequency domain). Let us introduce the problem here.

For $z = (z_1, \ldots, z_d)^\mathsf{T} \in \mathbb{C}^d \setminus \{0\}$ and $k = (k_1, \ldots, k_d)^\mathsf{T} \in \mathbb{Z}^d$, denote $z^k := z_1^{k_1} \cdots z_d^{k_d}$. Suppose $\mathsf{A}(z) = \sum_{k\in\mathbb{Z}^d} A(k) z^k$ is an $n \times n$ matrix of Laurent polynomials, which is Hermitian on $\mathbb{T}^d$, that is,

$$\overline{\mathsf{A}(z)}^\mathsf{T} = \mathsf{A}(z), \qquad \forall z \in \mathbb{T}^d := \{z \in \mathbb{C}^d \ : \ |z_1| = \ldots = |z_d| = 1\}.$$

We want to find another matrix $\mathsf{U}(z) = \sum_{k\in\mathbb{Z}^d} U(k)z^k$ of Laurent polynomials, such that

$$\mathsf{A}(z) = \mathsf{U}(z) \begin{bmatrix} \mathbf{I_{m_1}} & \\ & -\mathbf{I_{m_2}} \end{bmatrix} \overline{\mathsf{U}(z)}^\mathsf{T}, \qquad \forall z \in \mathbb{T}^d, \tag{1.3.1}$$

where $\mathbf{I_{m_1}}, \mathbf{I_{m_2}}$ are identity matrices of size m_1 and m_2 respectively. We call (1.3.1) the spectral factorization of the matrix $\mathsf{A}(z)$.

In one-dimensional case ($d = 1$), related problems have been studied in different settings. The most famous result is the matrix-valued Fejér-Riesz lemma (see Theorem 2.3.2), which considers the special case that $\mathsf{A}(z)$ is positive semi-definite on $z \in \mathbb{T}$. In this case, the theorem says that the factorization in (1.3.1) always exists for $m_1 = n$ and $m_2 = 0$. This positive semi-definite situation has been studied in many different settings and different literatures (e.g. harmonic analysis, operator theory, control theory, algebra, engineering, etc.). For example, see [86, 87, 75, 54, 28, 25, 32, 21] and many references therein. Similar results also hold if we require all the matrices to be of rational functions [88, 70] or polynomials [28, 24, 85] rather than Laurent polynomials.

The general indefinite setting in (1.3.1) brings much more difficulties than the positive semi-definite case. The problem has been studied in terms of polynomial matrices, that is, $\mathsf{A}(z)$ and $\mathsf{U}(z)$ are matrices of polynomials and the result holds for $z \in \mathbb{R}$ or $z \in i\mathbb{R}$. In late 20th century, I. Gohberg, P. Lancaster, L. Rodman and a lot of researchers made comprehensive studies on polynomial matrices and indefinite systems. The tools they built such as standard pairs/triples and sign characteristic inspired enormous results in control theory, numerical computation and other different areas of applied math today. See their books [31, 30, 1] for a detailed introduction/review. The indefinite spectral factorization (1.3.1) in the polynomial setting (also called J-spectral factorization) has been solved in [29, 69, 71], using the tool of "sign characteristic" introduced in the phenomenal paper [28], which describes the sign change of the eigenvalues of the matrix $\mathsf{A}(z)$.

In the multivariate ($d > 1$) setting, the problem is known to be much harder even for the positive semi-definite case. If we require $\mathsf{A}(z)$ to be positive semi-definite and $m_2 = 0$, it is related to the polynomial SOS problem (Hilbert's 17th problem), and it is known to be hard, and a lower bound for m_1 is usually huge.

1.4 Contributions and Outline of This Thesis

In this thesis, we study the matrix spectral factorization problem in (1.3.1) in both univariate and multivariate case, and use it to make a systematic study of the construction of quasi-tight framelets.

Chapters 2 to 4 deal with the one dimensional case ($d = 1$). In chapters 2 and 3, we firstly solved the matrix spectral factorization problem in the indefinite setting as (1.3.1). Since the problem has some essential difficulties compared to the positive semi-definite case, we used the idea of sign characteristic and generalized the methods of [29] and [71] (spectral factorization for polynomial matrices) into analytic settings, and completely solved (1.3.1) for Laurent polynomials. More precisely, the necessary and sufficient conditions for the factorization and the lower bounds for m_1 and m_2 have been found in Theorem 2.2.2, Theorem 2.3.1 and Theorem 3.1.2. Also, detailed algorithms to solve the spectral factorization in (1.3.1) are provided. Using the spectral factorization result, we completely characterized the quasi-tight framelet filter banks with two high-pass filters $\{a; b_1, b_2\}_{\Theta,(\epsilon_1,\epsilon_2)}$ in chapter 2. In chapter 3, we proved that from any arbitrary compactly supported refinable function in $L_2(\mathbb{R})$, we can always construct a compactly supported quasi-tight framelet having the minimum number of generators and the highest possible order of vanishing moments as in (1.2.18). As we can see, compared to the OEP-based tight framelets, the construction of quasi-tight framelets is much more flexible that we can choose a much wider class of low-pass filters, and we don't need the requirement that the shifts of the refinable function ϕ need to be stable for the choice of Θ.

Chapter 4 studies the construction of quasi-tight framelets with symmetry. Tight framelets with symmetry have been studied by a lot of papers in the literature, such as [44, 40, 48, 68, 49, 43, 79]. A complete characterization of OEP-based compactly supported tight framelets with two generators and symmetry has been derived in [48, 40]. In chapter 4, we give the necessary and sufficient condition for the construction of the quasi-tight framelet filter bank $\{a; b_1, b_2\}_{\Theta,(1,-1)}$ with symmetry. The result is based on the indefinite spectral factorization of Laurent polynomial matrices with symmetry. To the best of our knowledge, similar factorization problems have never been investigated in the literature.

Chapter 5 studies the construction of multivariate quasi-tight framelets. As we reviewed in the previous two sections, constructions of tight framelets and dual framelets with vanishing moments from a general (nonseparable) refinable function with a general dilation matrix M are very hard problems. Actually, there are not many examples available in the current literature. In chapter 5, we firstly prove that for an arbitrary compactly supported M-refinable function $\phi \in L_2(\mathbb{R}^d)$, we can easily derive a quasi-tight framelet, where all the high-pass filters are only supported on two points (so the framelet naturally has directionality). If in addition all the coefficients of the associated low-pass filter are nonnegative, such a quasi-tight M-framelet becomes a directional tight M-framelet in $L_2(\mathbb{R}^d)$. Secondly, we show in two constructive methods that we can always derive from an arbitrary compactly supported M-refinable function ϕ a compactly supported quasi-tight M-framelet in $L_2(\mathbb{R}^d)$ with the highest possible order of vanishing moments. Our constructions are based on a special indefinite spectral factorization result, where each of the column vectors of $\mathsf{U}(z)$ in (1.3.1) contains at most 2 nonzero terms.

Several examples are also provided in each chapter to illustrate the theoretical results and constructive algorithms.

Chapter 2

Quasi-tight Framelets with Two Generators in One Dimension

In this chapter, we study the construction of quasi-tight framelets in one-dimensional case. As discussed in Chapter 1, given a refinable function $\phi \in L_2(\mathbb{R})$ generated by a low-pass filter $a \in l_0(\mathbb{Z})$ and a moment correcting filter $\Theta \in l_0(\mathbb{Z})$, the construction of OEP-based compactly supported quasi-tight framelets reduces to finding high-pass filters $b_1, \ldots, b_s \in l_0(\mathbb{Z})$, such that $\{a; b_1, \ldots, b_s\}_{\Theta,(\epsilon_1,\ldots,\epsilon_s)}$ is a quasi-tight framelet filter bank. It turns out that the problem is linked to the spectral factorization of matrices of Laurent polynomials. We start this chapter by reviewing some properties of Laurent polynomials in Section 2.1. In Section 2.2, we find a lower bound s of the minimum number of framelet generators in a quasi-tight framelet filter bank. In Section 2.3, we prove a general theorem on the spectral factorization of matrices of Laurent polynomials with constant signature. Based on this theorem, an algorithm is provided in Section 2.4 for the construction of quasi-tight framelet filter banks with two high-pass filters and maximum order of vanishing moments. Some illustrative examples are provided in Section 2.5. The results of this chapter and the next chapter are summarized in [18].

2.1 Quasi-tight Framelet Filter Banks with Laurent Polynomials

Let $l(\mathbb{Z})$ be the linear space of all sequences $u = \{u(k)\}_{k\in\mathbb{Z}} : \mathbb{Z} \to \mathbb{C}$ of complex numbers on $\mathbb{Z}$. By $l_0(\mathbb{Z})$ we denote the space of all finitely supported sequences, $u = \{u(k)\}_{k\in\mathbb{Z}} : \mathbb{Z} \to \mathbb{C}$ such that $\{k \in \mathbb{Z} : \ u(k) \neq 0\}$ is a finite set. For a sequence $u = \{u(k)\}_{k\in\mathbb{Z}} \in l_0(\mathbb{Z})$, we can define its associated Laurent polynomial (or z-transform in engineering literature) as $\mathsf{u}(z) := \sum_{k\in\mathbb{Z}} u(k)z^k$ for $z \in \mathbb{C} \setminus \{0\}$. Similarly, for a finitely supported sequence of r by s matrices $u = \{u(k)\}_{k\in\mathbb{Z}} \in (l_0(\mathbb{Z}))^{r\times s}$, where $u(k)$ is an r by s matrix for all $k \in \mathbb{Z}$, its associated matrix of Laurent polynomials is defined to be $\mathsf{u}(z) = \sum_{k\in\mathbb{Z}} u(k)z^k$. We also define the associated adjoint sequence of u for $u \in (l_0(\mathbb{Z}))^{r\times s}$ (or $l_0(\mathbb{Z})$) to be $u^\star(k) := \overline{u(-k)}^T$, $k \in \mathbb{Z}$. In terms of Laurent polynomials, $\mathsf{u}^\star(z) := \sum_{k\in\mathbb{Z}} \overline{u(k)}^T z^{-k}$.

For a finitely supported Laurent polynomial $\mathsf{u}(z)$, if $\mathsf{u}(z)$ is not identically zero, we define its length as $\mathrm{len}(\mathsf{u}(z)) := l_2 - l_1$, where l_1, l_2 are integers uniquely defined by: $u(k) = 0$ for all $k > l_2$ and $k < l_1$, $u(l_1) \neq 0$, $u(l_2) \neq 0$. If $\mathsf{u}(z) = 0$, then we define $\mathrm{len}(\mathsf{u}(z)) = -\infty$. Under this notation, it is easy to check $\mathrm{len}(\mathsf{u}(z)\mathsf{v}(z)) = \mathrm{len}(\mathsf{u}(z)) + \mathrm{len}(\mathsf{v}(z))$ holds for all finitely supported Laurent polynomials $\mathsf{u}(z)$ and $\mathsf{v}(z)$.

For reader's convenience, we state some simple properties that will be useful later and can be verified by direct calculations.

Proposition 2.1.1. *Let* $\mathsf{A}(z) = \sum_{k=m_1}^{n_1} A(k)z^k$ *and* $\mathsf{B}(z) = \sum_{k=m_2}^{n_2} B(k)z^k$ *to be two matrices of Laurent polynomials. Then the followings hold.*

(1) $(\mathsf{AB})^\star(z) = \mathsf{B}^\star(z)\mathsf{A}^\star(z)$.

(2) Suppose $\mathsf{A}(z)$ *is an* $n \times n$ *square matrix, then* $\det(\mathsf{A}^\star(z)) = (\det(\mathsf{A}(z)))^\star$.

(3) Suppose $\mathsf{A}(z)$ *is an* $n \times n$ *square matrix. Then* $(\mathsf{A}(z))^{-1}$ *is also an* $n \times n$ *matrix of Laurent polynomials if and only if* $\det(\mathsf{A}(z))$ *is a nonzero monomial (Laurent polynomial with only one term). In this case, we say that* $\mathsf{A}(z)$ *is* **invertible** *or* **unimodular** *and write its inverse as* $\mathsf{A}^{-1}(z)$.

(4) Suppose $\mathsf{A}(z)$ *is invertible, then* $\mathsf{A}^\star(z)$ *is also invertible, and* $(\mathsf{A}^\star)^{-1}(z) = (\mathsf{A}^{-1})^\star(z)$. *So we can write it as* $\mathsf{A}^{-\star}(z)$.

(5) $\mathsf{A}^\star(z) = \mathsf{A}(z)$ *if and only if* $\mathsf{A}(z)$ *is a Hermitian matrix for all* $z \in \mathbb{T} := \{z \in \mathbb{C} \ : \ |z| = 1\}$. *We say that* $\mathsf{A}(z)$ *is* **(para-)Hermitian** *(or* **selfadjoint***) in this case.*

For $z \in \mathbb{T}$, we can take $z = e^{-i\xi}$, where i is the imaginary unit and $\xi \in \mathbb{R}$. Then a Laurent polynomial can be written as a 2π-periodic function of $\xi \in \mathbb{R}$:

$$\widehat{u}(\xi) := \mathsf{u}(e^{-i\xi}) = \sum_{k \in \mathbb{Z}} u(k) e^{-ik\xi},$$

which is the Fourier series (symbol) of the filter u. Note that $\mathsf{u}^\star(e^{-i\xi}) = \overline{\widehat{u}(\xi)}$. Also, if the sequence $u \in (l_0(\mathbb{Z}))^{s \times t}$ is finitely supported, then $\widehat{u}(\xi)$ is an analytic function matrix of ξ for all $\xi \in \mathbb{R}$. (By saying that a function matrix is analytic, we mean that the function in each element of the matrix is analytic.)

For a Laurent polynomial $\mathsf{p}(z)$ and $z_0 \in \mathbb{C} \setminus \{0\}$, we use $\mathsf{Z}(\mathsf{p}(z), z_0)$ to denote the multiplicity of zeros of $\mathsf{p}(z)$ at z_0. Similar notation is also adopted for analytic functions. If $f(\xi)$ is a function (or function matrix) analytic at ξ_0, we define $\mathsf{Z}(f(\xi), \xi_0) = \infty$ if $f(\xi)$ is identically zero. Otherwise, $\mathsf{Z}(f(\xi), \xi_0)$ is defined to be the integer such that $f(\xi) = (\xi - \xi_0)^{\mathsf{Z}(f(\xi), \xi_0)} g(\xi)$, where $g(\xi)$ is also analytic at ξ_0 and $g(\xi_0) \neq 0$. Since analytic functions can always be locally expressed as power series, $\mathsf{Z}(f(\xi), \xi_0)$ is well defined. Notice that for $u \in l_0(\mathbb{Z})$, $z_0 = e^{-i\xi_0} \in \mathbb{C} \setminus \{0\}$, our notations for multiplicity of zeros of Laurent polynomials in z-domain and that of analytic functions in the frequency domain coincide:

$$\mathsf{Z}(\mathsf{u}(z), z_0) = \mathsf{Z}(\mathsf{u}(e^{-i\xi}), \xi_0).$$

Hence, the order of sum rules of a low-pass filter a can be calculated as

$$\mathrm{sr}(\mathsf{a}(z)) := \mathrm{sr}(\widehat{a}(\xi)) := \mathrm{sr}(a) = \mathsf{Z}(\mathsf{a}(z), -1) = \mathsf{Z}(\widehat{a}(\xi), \pi). \tag{2.1.1}$$

The order of vanishing moments of a high-pass filter b can be calculated as

$$\mathrm{vm}(\mathsf{b}(z)) := \mathrm{vm}(\widehat{b}(\xi)) := \mathrm{vm}(b) = \mathsf{Z}(\mathsf{b}(z), 1) = \mathsf{Z}(\widehat{b}(\xi), 0). \tag{2.1.2}$$

The conditions 1.2.15 for quasi-tight framelet filter bank $\{a; b_1, \dots, b_s\}_{\Theta,(\epsilon_1,\dots,\epsilon_s)}$ can be written with Laurent polynomials as

$$\begin{aligned}\Theta(z^2)\mathsf{a}(z)\mathsf{a}^\star(z) + \epsilon_1 \mathsf{b}_1(z)\mathsf{b}_1^\star(z) + \ldots + \epsilon_s \mathsf{b}_s(z)\mathsf{b}_s^\star(z) &= \Theta(z),\\ \Theta(z^2)\mathsf{a}(z)\mathsf{a}^\star(-z) + \epsilon_1 \mathsf{b}_1(z)\mathsf{b}_1^\star(-z) + \ldots + \epsilon_s \mathsf{b}_s(z)\mathsf{b}_s^\star(-z) &= 0.\end{aligned}$$

Or in the matrix form

$$\begin{bmatrix} \mathsf{b}_1(z) & \cdots & \mathsf{b}_s(z) \\ \mathsf{b}_1(-z) & \cdots & \mathsf{b}_s(-z) \end{bmatrix} \begin{bmatrix} \epsilon_1 & & \\ & \ddots & \\ & & \epsilon_s \end{bmatrix} \begin{bmatrix} \mathsf{b}_1(z) & \cdots & \mathsf{b}_s(z) \\ \mathsf{b}_1(-z) & \cdots & \mathsf{b}_s(-z) \end{bmatrix}^\star = \mathcal{M}_{\mathsf{a},\Theta}(z), \tag{2.1.3}$$

where

$$\mathcal{M}_{\mathsf{a},\Theta}(z) := \begin{bmatrix} \Theta(z) - \Theta(z^2)\mathsf{a}(z)\mathsf{a}^\star(z) & -\Theta(z^2)\mathsf{a}(z)\mathsf{a}^\star(-z) \\ -\Theta(z^2)\mathsf{a}(-z)\mathsf{a}^\star(z) & \Theta(-z) - \Theta(z^2)\mathsf{a}(-z)\mathsf{a}^\star(-z) \end{bmatrix}, \tag{2.1.4}$$

and $\epsilon_j = \pm 1$ for $j = 1, \dots, s$. If $\epsilon_j = 1$ for all $j = 1, \dots, s$, we get a tight framelet filter bank.

For a quasi-tight framelet filter bank, without loss of generality, we can arrange the order of $\mathsf{b}_1(z), \dots, \mathsf{b}_s(z)$, such that the first n_+ of ϵ_j are $+1$ and the last n_- of ϵ_j are -1. Call the following matrix the ***signature matrix of the quasi-tight framelet filter bank*** $\{a; b_1, \dots, b_s\}_{\Theta,(\epsilon_1,\dots,\epsilon_s)}$:

$$J := \begin{bmatrix} \epsilon_1 & & \\ & \ddots & \\ & & \epsilon_s \end{bmatrix} = \begin{bmatrix} \mathbf{I}_{n_+} & \\ & -\mathbf{I}_{n_-} \end{bmatrix},$$

where $\mathbf{I}_{n_+}$ and $\mathbf{I}_{n_-}$ denote identity matrices of size n_+ and n_- respectively.

For tight framelet filter banks, i.e., $\epsilon_1 = \ldots = \epsilon_s = 1$, (2.1.3) implies that the Hermitian matrix $\mathcal{M}_{\mathsf{a},\Theta}(z)$ is necessarily positive semidefinite for all $z \in \mathbb{T}$. As we mentioned in Chapter 1, given a low-pass filter $a \in l_0(\mathbb{Z})$, in order to show that there exists a moment correcting filter $\Theta \in l_0(\mathbb{Z})$ such that $\mathcal{M}_{\mathsf{a},\Theta}(z)$ is positive semidefinite for all $z \in \mathbb{T}$, we have to assume that the associated refinable function ϕ has stable shifts. (See [11] and [49, Example 4.5]) The generalized construction of quasi-tight framelet filter bank in (2.1.3) allows us to use a much wider class of filters a and Θ. As a matter of fact, we will prove in the next chapter that as long as $\mathcal{M}_{\mathsf{a},\Theta}(z)$ is Hermitian on $z \in \mathbb{T}$ (which only requires $\Theta^\star = \Theta$ by Theorem 2.2.4), we can construct a quasi-tight framelet filter bank with maximum possible order of vanishing moments as in (1.2.18).

Given a moment correcting filter $\Theta \in l_0(\mathbb{Z})$ and a low-pass filter $a \in l_0(\mathbb{Z})$ such that $\mathsf{a}(1) = 1$ and $\Theta(1) = 1$, we want to solve the high-pass filters $b_1, \ldots, b_s$ with some minimum n_b order of vanishing moments. That is, we need to find Laurent polynomials $\mathring{\mathsf{b}}_1(z), \ldots, \mathring{\mathsf{b}}_s(z)$, such that

$$\mathsf{b}_l(z) = (1 - z^{-1})^{n_b} \mathring{\mathsf{b}}_l(z), \qquad l = 1, \ldots, s. \tag{2.1.5}$$

The condition (2.1.3) becomes

$$\begin{bmatrix} \mathring{\mathsf{b}}_1(z) & \cdots & \mathring{\mathsf{b}}_s(z) \\ \mathring{\mathsf{b}}_1(-z) & \cdots & \mathring{\mathsf{b}}_s(-z) \end{bmatrix} \begin{bmatrix} \epsilon_1 & & \\ & \ddots & \\ & & \epsilon_s \end{bmatrix} \begin{bmatrix} \mathring{\mathsf{b}}_1(z) & \cdots & \mathring{\mathsf{b}}_s(z) \\ \mathring{\mathsf{b}}_1(-z) & \cdots & \mathring{\mathsf{b}}_s(-z) \end{bmatrix}^* = \mathcal{M}_{\mathsf{a},\Theta|n_b}(z),$$

where

$$\mathcal{M}_{\mathsf{a},\Theta|n_b}(z) := \begin{bmatrix} (1 - z^{-1})^{-n_b} & \\ & (1 + z^{-1})^{-n_b} \end{bmatrix} \mathcal{M}_{\mathsf{a},\Theta}(z) \begin{bmatrix} (1 - z)^{-n_b} & \\ & (1 + z)^{-n_b} \end{bmatrix} \tag{2.1.6}$$

$$= \begin{bmatrix} \mathsf{A}(z) & \mathsf{B}(z) \\ \mathsf{B}(-z) & \mathsf{A}(-z) \end{bmatrix}$$

with

$$\mathsf{A}(z) := \frac{\Theta(z) - \Theta(z^2)\mathsf{a}(z)\mathsf{a}^\star(z)}{(1 - z)^{n_b}(1 - z^{-1})^{n_b}}, \qquad \mathsf{B}(z) := \frac{-\Theta(z^2)\mathsf{a}(z)\mathsf{a}^\star(-z)}{(1 + z)^{n_b}(1 - z^{-1})^{n_b}}. \tag{2.1.7}$$

From the definition of vanishing moments and sum rules in (2.1.2) and (2.1.1), we can see that $\mathsf{A}(z)$ and $\mathsf{B}(z)$ are well-defined Laurent polynomials as long as the nonnegative integer n_b satisfies:

$$0 \leqslant n_b \leqslant \min\left\{\mathrm{sr}(a), \frac{1}{2}\,\mathrm{vm}\left(\Theta(z) - \Theta(z^2)\mathsf{a}(z)\mathsf{a}^*(z)\right)\right\}.$$

Notice that this is also the upper bound for the maximum possible order of vanishing moments calculated in (1.2.18).

The relationship between $\mathring{\mathsf{b}}_l(z)$ and $\mathring{\mathsf{b}}_l(-z)$ in the two rows makes the matrix equation hard to solve. Given a sequence $u \in l(\mathbb{Z})$, we define its ***coset sequence*** $u^{[\gamma]} := \{u(\gamma + 2k)\}_{k\in\mathbb{Z}}$. Then the z-transform of u can be written as: $\mathsf{u}(z) = \mathsf{u}^{[0]}(z^2) + z\mathsf{u}^{[1]}(z^2)$. Also, $\mathsf{u}(-z) = \mathsf{u}^{[0]}(z^2) - z\mathsf{u}^{[1]}(z^2)$. Using the coset sequences, the filter bank can be decoupled as:

$$\begin{bmatrix} \mathring{\mathsf{b}}_1(z) & \cdots & \mathring{\mathsf{b}}_s(z) \\ \mathring{\mathsf{b}}_1(-z) & \cdots & \mathring{\mathsf{b}}_s(-z) \end{bmatrix} = \begin{bmatrix} 1 & z \\ 1 & -z \end{bmatrix} \begin{bmatrix} \mathring{\mathsf{b}}_1^{[0]}(z^2) & \cdots & \mathring{\mathsf{b}}_s^{[0]}(z^2) \\ \mathring{\mathsf{b}}_1^{[1]}(z^2) & \cdots & \mathring{\mathsf{b}}_s^{[1]}(z^2) \end{bmatrix}.$$

The matrix $\begin{bmatrix} \mathring{\mathsf{b}}_1^{[0]}(z) & \cdots & \mathring{\mathsf{b}}_s^{[0]}(z) \\ \mathring{\mathsf{b}}_1^{[1]}(z) & \cdots & \mathring{\mathsf{b}}_s^{[1]}(z) \end{bmatrix}$ is called the ***polyphase matrix*** of the filter bank $\{\mathring{b}_1, \ldots, \mathring{b}_s\}$. Let us equivalently write the condition (2.1.3) in terms of polyphase matrices:

$$\begin{bmatrix} \mathring{\mathsf{b}}_1^{[0]}(z) & \cdots & \mathring{\mathsf{b}}_s^{[0]}(z) \\ \mathring{\mathsf{b}}_1^{[1]}(z) & \cdots & \mathring{\mathsf{b}}_s^{[1]}(z) \end{bmatrix} \begin{bmatrix} \epsilon_1 & & \\ & \ddots & \\ & & \epsilon_s \end{bmatrix} \begin{bmatrix} \mathring{\mathsf{b}}_1^{[0]}(z) & \cdots & \mathring{\mathsf{b}}_s^{[0]}(z) \\ \mathring{\mathsf{b}}_1^{[1]}(z) & \cdots & \mathring{\mathsf{b}}_s^{[1]}(z) \end{bmatrix}^{\star} = \mathcal{N}_{\mathsf{a},\Theta|n_b}(z), \tag{2.1.8}$$

where $\mathcal{N}_{\mathsf{a},\Theta|n_b}(z)$ is calculated from:

$$\mathcal{M}_{\mathsf{a},\Theta|n_b}(z) = \begin{bmatrix} 1 & z \\ 1 & -z \end{bmatrix} \mathcal{N}_{\mathsf{a},\Theta|n_b}(z^2) \begin{bmatrix} 1 & z \\ 1 & -z \end{bmatrix}^{\star}, \tag{2.1.9}$$

that is,

$$\mathcal{N}_{\mathsf{a},\Theta|n_b}(z) := \frac{1}{2} \begin{bmatrix} \mathsf{A}^{[0]}(z) + \mathsf{B}^{[0]}(z) & z(\mathsf{A}^{[1]}(z) - \mathsf{B}^{[1]}(z)) \\ \mathsf{A}^{[1]}(z) + \mathsf{B}^{[1]}(z) & \mathsf{A}^{[0]}(z) - \mathsf{B}^{[0]}(z) \end{bmatrix}, \tag{2.1.10}$$

for $\mathsf{A}(z)$ and $\mathsf{B}(z)$ defined in (2.1.7).

The main problem in constructing quasi-tight framelet filter banks with some required order of vanishing moments becomes how to factorize the matrix $\mathcal{N}_{\mathsf{a},\Theta|n_b}(z)$ into the form of (2.1.8).

Theorem 2.1.2. *Suppose* $\{a; b_1, \ldots, b_s\}_{\Theta,(\epsilon_1,\ldots,\epsilon_s)}$ *is a quasi-tight framelet filter bank, and all the high-pass filters have at least* n_b *order of vanishing moments, then* (2.1.8) *holds, where* $\mathcal{N}_{\mathsf{a},\Theta|n_b}(z)$ *is defined in* (2.1.10) (2.1.7)*, and* $\mathring{\mathsf{b}}_1, \ldots, \mathring{\mathsf{b}}_s$ *are defined in* (2.1.5)*.*

On the other hand, suppose $\mathcal{N}_{\mathsf{a},\Theta|n_b}(z)$ *can be factorized as*

$$\mathcal{N}_{\mathsf{a},\Theta|n_b}(z) = \mathsf{U}(z) \begin{bmatrix} \epsilon_1 & & \\ & \ddots & \\ & & \epsilon_s \end{bmatrix} \mathsf{U}^\star(z), \tag{2.1.11}$$

for some $2 \times s$ *matrix* $\mathsf{U}(z)$ *of Laurent polynomials, and some* $\epsilon_j = \pm 1$, $j = 1, \ldots, s$*. Define high-pass filters* $\mathsf{b}_l(z)$, $l = 1, \ldots, s$, *as:*

$$\begin{bmatrix} \mathsf{b}_1(z) & \cdots & \mathsf{b}_s(z) \\ \mathsf{b}_1(-z) & \cdots & \mathsf{b}_s(-z) \end{bmatrix} = \begin{bmatrix} (1-z^{-1})^{n_b} & \\ & (1+z^{-1})^{n_b} \end{bmatrix} \begin{bmatrix} 1 & z \\ 1 & -z \end{bmatrix} \mathsf{U}(z^2). \tag{2.1.12}$$

Then $\{a; b_1, \ldots, b_s\}_{\Theta,(\epsilon_1,\ldots,\epsilon_s)}$ *is a quasi-tight framelet filter bank, with at least* n_b *order of vanishing moments.*

Proof. The first part of the theorem can be verified directly by the condition (2.1.3). For the proof of the second part, direct calculations using (2.1.12) shows the equations in (2.1.3) hold. So $\{a; b_1, \ldots, b_s\}_{\Theta,(\epsilon_1,\ldots,\epsilon_s)}$ is a quasi-tight framelet filter bank, with all high-pass filters having vanishing moments of order $\mathrm{vm}(b_\ell) \geqslant n_b$, $\ell = 1, \ldots, s$. ■

Based on Theorem 2.1.2, the construction of quasi-tight framelet filter banks is equivalent to solving the generalized spectral factorization problem of $\mathcal{N}_{\mathsf{a},\Theta|n_b}(z)$ in (2.1.11) .

2.2 Minimum Number of High-pass Filters

In this section, we study the minimum number s of high-pass filters needed in the construction of a quasi-tight framelet filter bank. We will prove in Section 2.4 that there always exists a quasi-tight framelet filter bank with highest possible order of vanishing moments and this minimum number of high-pass filters.

In order to construct a quasi-tight framelet filter bank $\{a; b_1, \ldots, b_s\}_{\Theta,(\epsilon_1,\ldots,\epsilon_s)}$ from given $\Theta, a \in l_0(\mathbb{Z})$, we just need to solve the following decomposition of $\mathcal{M}_{\mathsf{a},\Theta}(z)$:

$$\begin{bmatrix} \mathsf{b}_1(z) & \cdots & \mathsf{b}_s(z) \\ \mathsf{b}_1(-z) & \cdots & \mathsf{b}_s(-z) \end{bmatrix} \begin{bmatrix} \epsilon_1 & & \\ & \ddots & \\ & & \epsilon_s \end{bmatrix} \begin{bmatrix} \mathsf{b}_1(z) & \cdots & \mathsf{b}_s(z) \\ \mathsf{b}_1(-z) & \cdots & \mathsf{b}_s(-z) \end{bmatrix}^\star = \mathcal{M}_{\mathsf{a},\Theta}(z). \tag{2.2.1}$$

The freedom of using different ϵ_j here allows us to remove the requirement that the matrix $\mathcal{M}_{\mathsf{a},\Theta}(z)$ on the right-hand-side has to be positive semi-definite for all $z \in \mathbb{T}$. As a matter of fact, the number of positive/negative eigenvalues of $\mathcal{M}_{\mathsf{a},\Theta}(z)$ for all $z \in \mathbb{T}$ has a direct link to the number of $+1$ and -1 appears in the signature matrix $\operatorname{diag}(\epsilon_1, \ldots, \epsilon_s)$. This gives us a necessary condition on the minimum number s of high-pass filters needed. Let us introduce some notations first.

If a constant matrix A is Hermitian, basic results from linear algebra tell us that A can be diagonalized by a unitary matrix, i.e., there exists a unitary matrix U, such that $A = U\Lambda U^\star$, where $\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is a diagonal matrix of all the eigenvalues of A. Also, all the eigenvalues of a Hermitian matrix are real. We use $\nu_+(A)$ to denote the number of its positive eigenvalues, and $\nu_-(A)$ to denote the number of its negative eigenvalues. The ***signature*** $\operatorname{sig}(A)$ of A is defined as their difference:

$$\operatorname{sig}(A) := \nu_+(A) - \nu_-(A).$$

We also call $J = \operatorname{diag}(\mathbf{I}_{\nu_+}, -\mathbf{I}_{\nu_-}, \mathbf{0}_{n_0})$ the ***signature matrix*** of A, where $\mathbf{I}_{\nu_+}$ and $\mathbf{I}_{\nu_-}$ are identity matrices of sizes $\nu_+(A)$ and $\nu_-(A)$ respectively, and $\mathbf{0}_{n_0}$

is the square zero matrix of size equal to the dimension of the null space of A.

Theorem 2.2.1. *Suppose an $n \times n$ Hermitian matrix A can be decomposed in the following way*

$$A = U \begin{bmatrix} \mathbf{I}_{m_+} & \\ & -\mathbf{I}_{m_-} \end{bmatrix} U^\star, \tag{2.2.2}$$

where U is an $n \times m$ matrix, and $\mathbf{I}_{m_+}$, $\mathbf{I}_{m_-}$ are identity matrices of size m_+ and m_- respectively, such that $m_+ + m_- = m$. Then

$$m_+ \geqslant \nu_+(A), \qquad m_- \geqslant \nu_-(A).$$

Proof. Firstly, we consider the case that A is nonsingular. In this case, the decomposition (2.2.2) forces that all the three matrices on the right hand side of (2.2.2) have rank at least n. So $m \geqslant n$, and U must have full row rank.

If $m = n$, then U is a nonsingular square matrix. By Sylvester's law of inertia,

$$m_+ = \nu_+(A), \qquad m_- = \nu_-(A).$$

If $m > n$, since U has full row rank, we can add $m - n$ more rows to U to get $\widetilde{U}$, such that $\widetilde{U} := \begin{bmatrix} U \\ V \end{bmatrix}$ is an $m \times m$ nonsingular square matrix. Then the $m \times m$ matrix $\widetilde{A} := \widetilde{U} \begin{bmatrix} \mathbf{I}_{m_+} & \\ & -\mathbf{I}_{m_-} \end{bmatrix} \widetilde{U}^\star$ has A on the top left corner:

$$\widetilde{A} := \widetilde{U} \begin{bmatrix} \mathbf{I}_{m_+} & \\ & -\mathbf{I}_{m_-} \end{bmatrix} \widetilde{U}^\star = \begin{bmatrix} U \\ V \end{bmatrix} \begin{bmatrix} \mathbf{I}_{m_+} & \\ & -\mathbf{I}_{m_-} \end{bmatrix} \begin{bmatrix} U^\star & V^\star \end{bmatrix} = \begin{bmatrix} A & B^\star \\ B & C \end{bmatrix} \tag{2.2.3}$$

for some $(m - n) \times n$ matrix B, and some $(m - n) \times (m - n)$ matrix C.

Define the nonsingular $m \times m$ matrix $W := \begin{bmatrix} \mathbf{I}_n & \mathbf{0} \\ -BA^{-1} & \mathbf{I}_{m-n} \end{bmatrix}$, and let $\mathring{A} := W\widetilde{A}W^\star$. Plugging in (2.2.3), we can directly calculate that

$$\mathring{A} := W\widetilde{A}W^\star = \begin{bmatrix} \mathbf{I}_n & \mathbf{0} \\ -BA^{-1} & \mathbf{I}_{m-n} \end{bmatrix} \begin{bmatrix} A & B^\star \\ B & C \end{bmatrix} \begin{bmatrix} \mathbf{I}_n & -A^{-\star}B^\star \\ \mathbf{0} & \mathbf{I}_{m-n} \end{bmatrix} = \begin{bmatrix} A & \mathbf{0} \\ \mathbf{0} & D \end{bmatrix}, \tag{2.2.4}$$

where the $(m-n) \times (m-n)$ matrix $D := C - BA^{-1}B^\star$.

From (2.2.4), we can see that the eigenvalues of $\mathring{A}$ are just the eigenvalues of A combined with the eigenvalues of D. So

$$\nu_+(\mathring{A}) \geqslant \nu_+(A), \qquad \nu_-(\mathring{A}) \geqslant \nu_-(A). \tag{2.2.5}$$

Also, from the definition of $\widetilde{A}$ and $\mathring{A}$ in (2.2.3) and (2.2.4), we can see that

$$\mathring{A} = W\widetilde{A}W^\star = W\widetilde{U}\begin{bmatrix} \mathbf{I}_{m_+} & \\ & -\mathbf{I}_{m_-} \end{bmatrix}\widetilde{U}^\star W^\star = W\widetilde{U}\begin{bmatrix} \mathbf{I}_{m_+} & \\ & -\mathbf{I}_{m_-} \end{bmatrix}(W\widetilde{U})^\star. \tag{2.2.6}$$

Since $W\widetilde{U}$ is an $m \times m$ nonsingular matrix, by Sylvester's law of inertia again, (2.2.6) implies that

$$\nu_+(\mathring{A}) = m_+, \qquad \nu_-(\mathring{A}) = m_-. \tag{2.2.7}$$

Combining (2.2.5) and (2.2.7), we get

$$m_+ \geqslant \nu_+(A), \qquad m_- \geqslant \nu_-(A).$$

This proves the theorem for the case that A is nonsingular.

For the case that A is singular, we can find its eigenvalue decomposition first:

$$PAP^\star = \begin{bmatrix} \Lambda & \\ & \mathbf{0} \end{bmatrix},$$

where Λ is a $k \times k$ nonsingular diagonal matrix containing all the nonzero eigenvalues of A, while P is an $n \times n$ unitary matrix. Rewrite the above decomposition using (2.2.2):

$$\begin{bmatrix} \Lambda & \\ & \mathbf{0} \end{bmatrix} = PAP^\star = PU\begin{bmatrix} \mathbf{I}_{m_+} & \\ & -\mathbf{I}_{m_-} \end{bmatrix}U^\star P^\star = Q\begin{bmatrix} \mathbf{I}_{m_+} & \\ & -\mathbf{I}_{m_-} \end{bmatrix}Q^\star,$$

where $Q := PU$. We define $\widetilde{Q}$ by deleting the last $n-k$ rows of Q, then the above equation implies:

$$\Lambda = \widetilde{Q} \begin{bmatrix} \mathbf{I}_{m_+} & \\ & -\mathbf{I}_{m_-} \end{bmatrix} \widetilde{Q}^\star.$$

Since Λ is nonsingular, we know from the previously proved case that

$$m_+ \geqslant \nu_+(\Lambda) = \nu_+(A), \qquad m_- \geqslant \nu_-(\Lambda) = \nu_-(A).$$

This proves the theorem for the case that A is a singular matrix. ■

In terms of Hermitian matrices of Laurent polynomials, we have the following theorem.

Theorem 2.2.2. *Suppose $\mathsf{A}(z)$ is an $n \times n$ matrix of Laurent polynomials satisfying $\mathsf{A}^\star(z) = \mathsf{A}(z)$. If $\mathsf{A}(z)$ has decomposition*

$$\mathsf{A}(z) = \mathsf{U}(z) \begin{bmatrix} \mathbf{I}_{s_+} & \\ & -\mathbf{I}_{s_-} \end{bmatrix} \mathsf{U}^\star(z) \tag{2.2.8}$$

for some $n \times s$ matrix of Laurent polynomials $\mathsf{U}(z)$, where $\mathbf{I}_{s_+}$, $\mathbf{I}_{s_-}$ are identity matrices of size s_+ and s_- respectively, such that $s_+ + s_- = s$, then

$$s_+ \geqslant \max_{z \in \mathbb{T}} \nu_+(\mathsf{A}(z)), \qquad s_- \geqslant \max_{z \in \mathbb{T}} \nu_-(\mathsf{A}(z)),$$

which implies

$$s \geqslant \max_{z \in \mathbb{T}} \nu_+(\mathsf{A}(z)) + \max_{z \in \mathbb{T}} \nu_-(\mathsf{A}(z)). \tag{2.2.9}$$

Proof. Since $\mathsf{A}^\star(z) = \mathsf{A}(z)$ implies that for all $z_0 \in \mathbb{T}$, $\mathsf{A}(z_0)$ is a Hermitian matrix, $(\mathsf{A}(z_0))^\star = \mathsf{A}(z_0)$. And $\mathsf{U}^\star(z_0) = (\mathsf{U}(z_0))^\star$ holds. We know from Theorem 2.2.1 that $s_+ \geqslant \nu_+(\mathsf{A}(z_0))$ and $s_- \geqslant \nu_-(\mathsf{A}(z_0))$. Considering all $z_0 \in \mathbb{T}$, we get

$$s_+ \geqslant \max_{z \in \mathbb{T}} \nu_+(\mathsf{A}(z)), \qquad s_- \geqslant \max_{z \in \mathbb{T}} \nu_-(\mathsf{A}(z)).$$

Since $s = s_+ + s_-$, the above two inequalities add up to

$$s \geqslant \max_{z \in \mathbb{T}} \nu_+(\mathsf{A}(z)) + \max_{z \in \mathbb{T}} \nu_-(\mathsf{A}(z)).$$

This completes the proof. ■

Theorem 2.2.2 shows that (2.2.9) is a necessary condition for the decomposition (2.2.8) to be possible. In Chapter 3, we will prove that it is also a sufficient condition. That is, for all $s \in \mathbb{N}$ satisfying (2.2.9), we can find an $n \times s$ matrix of Laurent polynomials $\mathsf{U}(z)$, such that (2.2.8) holds. See Theorem 3.1.2 for details.

Given $a, \Theta \in l_0(\mathbb{Z})$, we can define

$$s^+_{a,\Theta} := \max_{z \in \mathbb{T}} \nu_+(\mathcal{M}_{\mathsf{a},\Theta}(z)), \qquad s^-_{a,\Theta} := \max_{z \in \mathbb{T}} \nu_-(\mathcal{M}_{\mathsf{a},\Theta}(z)). \tag{2.2.10}$$

Building on Theorem 2.2.2, we can calculate the minimum number of high-pass filters needed in the construction of a quasi-tight framelet filter bank.

Corollary 2.2.3. *Given a quasi-tight framelet filter bank* $\{a; b_1, \ldots, b_s\}_{\Theta,(\epsilon_1,\ldots,\epsilon_s)}$*, such that neither* a *nor* Θ *is identically zero. Suppose* $\mathcal{M}_{\mathsf{a},\Theta}(z)$ *defined in* (2.1.4) *is a Hermitian matrix. Then the number* s *of high-pass filters must satisfy*

$$s \geqslant s^+_{a,\Theta} + s^-_{a,\Theta} = \max_{z \in \mathbb{T}} \nu_+(\mathcal{M}_{\mathsf{a},\Theta}(z)) + \max_{z \in \mathbb{T}} \nu_-(\mathcal{M}_{\mathsf{a},\Theta}(z)). \tag{2.2.11}$$

Proof. The condition for a quasi-tight framelet filter bank is

$$\begin{bmatrix} \mathsf{b}_1(z) & \cdots & \mathsf{b}_s(z) \\ \mathsf{b}_1(-z) & \cdots & \mathsf{b}_s(-z) \end{bmatrix} \begin{bmatrix} \epsilon_1 & & \\ & \ddots & \\ & & \epsilon_s \end{bmatrix} \begin{bmatrix} \mathsf{b}_1(z) & \cdots & \mathsf{b}_s(z) \\ \mathsf{b}_1(-z) & \cdots & \mathsf{b}_s(-z) \end{bmatrix}^\star = \mathcal{M}_{\mathsf{a},\Theta}(z).$$

Without loss of generality, we can rearrange the order of high-pass filters so that we can assume that

$$\begin{bmatrix} \epsilon_1 & & \\ & \ddots & \\ & & \epsilon_s \end{bmatrix} = \begin{bmatrix} \mathbf{I}_{s_+} & \\ & -\mathbf{I}_{s_-} \end{bmatrix},$$

for some nonnegative integers s_+ and s_-, where $s_+ + s_- = s$. Since $\mathcal{M}_{\mathsf{a},\Theta}(z)$ is Hermitian, by Theorem 2.2.2 we know that (2.2.11) holds. ■

Theorem 2.2.2 and Corollary 2.2.3 only provide a theoretical lower bound for s as in (2.2.9) and (2.2.11). The sufficiency for the lower bound to be achieved will be established later in this chapter (for the case $s^+_{a,\Theta} + s^-_{a,\Theta} = 2$) and in the next chapter (general case).

The following Lemma would be useful in our later discussions.

Lemma 2.2.1. *Let* $\mathsf{A}(z)$ *be an* $n \times n$ *Hermitian matrix of Laurent polynomials, and* B *be a finite subset of* $\mathbb{T}$. *Then*

$$\max_{z\in\mathbb{T}} \nu_+(\mathsf{A}(z)) = \max_{z\in\mathbb{T}\backslash B} \nu_+(\mathsf{A}(z)), \qquad \max_{z\in\mathbb{T}} \nu_-(\mathsf{A}(z)) = \max_{z\in\mathbb{T}\backslash B} \nu_-(\mathsf{A}(z)).$$

Proof. Since $\mathsf{A}(z)$ is an $n \times n$ Hermitian matrix of Laurent polynomials, its n eigenvalues, $\lambda_1(z), \ldots, \lambda_n(z)$, which are the n roots of the polynomial $\det(\lambda\mathbf{I_n} - \mathsf{A}(z))$, can be chosen as real continuous functions of $z \in \mathbb{T}$. (They are actually algebraic functions which are global analytic.) Hence, for all $j = 1, \ldots, n$, $\{z \in \mathbb{T} \,:\, \lambda_j(z) > 0\}$ must be an open set. Denote $n_+ := \max_{z\in\mathbb{T}} \nu_+(\mathsf{A}(z))$, we can see that

$$\{z \in \mathbb{T} \,:\, \nu_+(\mathsf{A}(z)) = n_+\} = \bigcup_{\substack{J\subseteq\{1,\cdots,n\} \\ |J|=n_+}} \bigcap_{j\in J} \{z \in \mathbb{T} \,:\, \lambda_j(z) > 0\}.$$

Since the unions and intersections are taken over a finite number of open sets, the above set is still open. By the definition of n_+, we know that $\nu_+(\mathsf{A}(z)) = n_+$ is achieved for some $z \in \mathbb{T}$, so the above set is also nonempty. Therefore, the nonempty open set $\{z \in \mathbb{T} \,:\, \nu_+(\mathsf{A}(z)) = n_+\}$ contains infinite number of points. So $\{z \in \mathbb{T} \,:\, \nu_+(\mathsf{A}(z)) = n_+\}\backslash B$ must be nonempty. This implies that

$$\max_{z\in\mathbb{T}\backslash B} \nu_+(\mathsf{A}(z)) \geqslant n_+ = \max_{z\in\mathbb{T}} \nu_+(\mathsf{A}(z)).$$

Since $\mathbb{T}\backslash B$ is a subset of $\mathbb{T}$, the inequality of the other directon is obvious. Therefore, we proved

$$\max_{z\in\mathbb{T}} \nu_+(\mathsf{A}(z)) = \max_{z\in\mathbb{T}\backslash B} \nu_+(\mathsf{A}(z)).$$

The identity $\max_{z\in\mathbb{T}} \nu_-(\mathsf{A}(z)) = \max_{z\in\mathbb{T}\backslash B} \nu_-(\mathsf{A}(z))$ can be proved similarly. ■

For an $n \times n$ square matrix $\mathsf{A}(z)$ of Laurent polynomials, we define its ***spectrum*** $\sigma(\mathsf{A}(z))$ as

$$\sigma(\mathsf{A}(z)) := \{z \in \mathbb{C} \setminus \{0\} \,:\, \det(\mathsf{A}(z)) = 0\}.$$

If $\det(\mathsf{A}(z))$ is not identically zero, we know that $\sigma(\mathsf{A}(z))$ is a finite set. In this chapter, we only solve the spectral factorization of a Hermitian matrix $\mathsf{A}(z)$ of Laurent polynomials with constant signature for $z \in \mathbb{T} \setminus \sigma(\mathsf{A}(z))$. That is, $\nu_+(\mathsf{A}(z))$ and $\nu_-(\mathsf{A}(z))$ are constant for $z \in \mathbb{T}\setminus\sigma(\mathsf{A}(z))$. In this case, according to Lemma 2.2.1, we know that

$$\max_{z\in\mathbb{T}} \nu_+(\mathsf{A}(z)) + \max_{z\in\mathbb{T}} \nu_-(\mathsf{A}(z)) = \max_{z\in\mathbb{T}\setminus\sigma(\mathsf{A}(z))} \nu_+(\mathsf{A}(z)) + \max_{z\in\mathbb{T}\setminus\sigma(\mathsf{A}(z))} \nu_-(\mathsf{A}(z)) = n.$$

The following theorem allows us to compute the signature of $\mathcal{M}_{\mathsf{a},\Theta}(z)$ on $z \in \mathbb{T} \setminus \sigma(\mathcal{M}_{\mathsf{a},\Theta}(z))$ directly from $\mathsf{a}(z)$ and $\Theta(z)$. The proof of the positive semi-definite case (case (1)) is given in [45, Lemma 1.4.5.].

Theorem 2.2.4. *Given filters $a, \Theta \in l_0(\mathbb{Z})$, neither of them is identically zero. Then the matrix $\mathcal{M}_{\mathsf{a},\Theta}(z)$ defined in* (2.1.4) *is Hermitian on $z \in \mathbb{T}$ if and only if $\Theta^\star(z) = \Theta(z)$. Moreover, if $\Theta^\star(z) = \Theta(z)$ and*

$$\det(\mathcal{M}_{\mathsf{a},\Theta}(z)) = \Theta(z)\Theta(-z) - \Theta(z^2)\left(\Theta(-z)\mathsf{a}(z)\mathsf{a}^\star(z) - \Theta(z)\mathsf{a}(-z)\mathsf{a}^\star(-z)\right)$$

is not identically zero, then:

(1) $\mathcal{M}_{\mathsf{a},\Theta}(z)$ *is positive semi-definite on* $z \in \mathbb{T}$ *(which implies* $\operatorname{sig}(\mathcal{M}_{\mathsf{a},\Theta}(z)) = 2$ *for all* $z \in \mathbb{T} \setminus \sigma(\mathcal{M}_{\mathsf{a},\Theta}(z))$*) if and only if*

$$\Theta(z) \geqslant 0, \qquad \det(\mathcal{M}_{\mathsf{a},\Theta}(z)) \geqslant 0, \qquad \forall z \in \mathbb{T}.$$

(2) $\mathcal{M}_{\mathsf{a},\Theta}(z)$ *is negative semi-definite on* $z \in \mathbb{T}$ *(which implies* $\operatorname{sig}(\mathcal{M}_{\mathsf{a},\Theta}(z)) = -2$ *for all* $z \in \mathbb{T} \setminus \sigma(\mathcal{M}_{\mathsf{a},\Theta}(z))$*) if and only if*

$$\Theta(z) \leqslant 0, \qquad \det(\mathcal{M}_{\mathsf{a},\Theta}(z)) \geqslant 0, \qquad \forall z \in \mathbb{T}.$$

(3) $\mathcal{M}_{\mathsf{a},\Theta}(z)$ *has one positive and one negative eigenvalue for all* $z \in \mathbb{T} \setminus \sigma(\mathcal{M}_{\mathsf{a},\Theta}(z))$ *(i.e.* $\operatorname{sig}(\mathcal{M}_{\mathsf{a},\Theta}(z)) = 0$ *for all* $z \in \mathbb{T} \setminus \sigma(\mathcal{M}_{\mathsf{a},\Theta}(z))$*) if and only if*

$$\det(\mathcal{M}_{\mathsf{a},\Theta}(z)) \leqslant 0, \qquad \forall z \in \mathbb{T}.$$

(4) *Otherwise (beyond the three cases above),* $\operatorname{sig}(\mathcal{M}_{\mathsf{a},\Theta}(z))$ *varies on* $z \in \mathbb{T} \setminus \sigma(\mathcal{M}_{\mathsf{a},\Theta}(z))$*, and* $s^{+}_{a,\Theta} + s^{-}_{a,\Theta} > 2$*.*

Proof. If $\mathcal{M}_{\mathsf{a},\Theta}(z)$ is Hermitian for all $z \in \mathbb{T}$, then from

$$[\mathcal{M}_{\mathsf{a},\Theta}]^{\star}_{1,2}(z) = \left(-\Theta(z^2)\mathsf{a}(z)\mathsf{a}^{\star}(-z)\right)^{\star} = -\Theta^{\star}(z^2)\mathsf{a}^{\star}(z)\mathsf{a}(-z),$$

$$[\mathcal{M}_{\mathsf{a},\Theta}]_{2,1}(z) = -\Theta(z^2)\mathsf{a}^{\star}(z)\mathsf{a}(-z),$$

we can see that $[\mathcal{M}_{\mathsf{a},\Theta}]^{\star}_{1,2}(z) = [\mathcal{M}_{\mathsf{a},\Theta}]_{2,1}(z)$ implies $\Theta^{\star} = \Theta$. Conversely, if $\Theta^{\star} = \Theta$, by direct calculation, we can check $\mathcal{M}^{\star}_{\mathsf{a},\Theta}(z) = \mathcal{M}_{\mathsf{a},\Theta}(z)$. So $\mathcal{M}^{\star}_{\mathsf{a},\Theta}(z) = \mathcal{M}_{\mathsf{a},\Theta}(z)$ if and only if $\Theta^{\star} = \Theta$ holds.

To prove item (3), we know that the determinant of $\mathcal{M}_{\mathsf{a},\Theta}(z)$ is equal to the product of its two eigenvalues. So $\operatorname{sig}(\mathcal{M}_{\mathsf{a},\Theta}(z)) = 0$ for all $z \in \mathbb{T} \setminus \sigma(\mathcal{M}_{\mathsf{a},\Theta}(z))$ if and only if $\det(\mathcal{M}_{\mathsf{a},\Theta}(z)) < 0$ for all $z \in \mathbb{T} \setminus \sigma(\mathcal{M}_{\mathsf{a},\Theta}(z))$, or equivalently, $\det(\mathcal{M}_{\mathsf{a},\Theta}(z)) \leqslant 0$ for all $z \in \mathbb{T}$.

To prove item (2), let us write the matrix $\mathcal{M}_{\mathsf{a},\Theta}(z)$ in the frequency domain.

Take $z = e^{-i\xi}$, for $\xi \in \mathbb{R}$, then

$$\mathcal{M}_{\mathsf{a},\Theta}(e^{-i\xi}) = \begin{bmatrix} \widehat{\Theta}(\xi) - \widehat{\Theta}(2\xi)|\widehat{a}(\xi)|^2 & -\widehat{\Theta}(2\xi)\widehat{a}(\xi)\overline{\widehat{a}(\xi+\pi)} \\ -\widehat{\Theta}(2\xi)\widehat{a}(\xi+\pi)\overline{\widehat{a}(\xi)} & \widehat{\Theta}(\xi+\pi) - \widehat{\Theta}(2\xi)|\widehat{a}(\xi+\pi)|^2 \end{bmatrix}.$$

For necessity: Since $\det(\mathcal{M}_{\mathsf{a},\Theta}(e^{-i\xi})) \geqslant 0$ is a necessary condition for $\mathcal{M}_{\mathsf{a},\Theta}(e^{-i\xi})$ to be negative semi-definite on $\xi \in \mathbb{R}$, we only need to prove $\widehat{\Theta}(\xi) \leqslant 0$ for all $\xi \in \mathbb{R}$. Suppose there exists some $\xi_0 \in \mathbb{R}$, such that $\widehat{\Theta}(\xi_0) > 0$. Since $\widehat{\Theta}(\xi)$ is a continuous function on $\mathbb{R}$, we can find some nonempty open interval (c, d) such that $\widehat{\Theta}(\xi) > 0$ for any $\xi \in (c, d)$. The negative semi-definite matrix $\mathcal{M}_{\mathsf{a},\Theta}(e^{-i\xi})$ must follow:

$$\left[\mathcal{M}_{\mathsf{a},\Theta}(e^{-i\xi})\right]_{1,1} = \widehat{\Theta}(\xi) - \widehat{\Theta}(2\xi)|\widehat{a}(\xi)|^2 \leqslant 0, \qquad \forall \xi \in \mathbb{R}. \tag{2.2.12}$$

Hence, $0 < \widehat{\Theta}(\xi) \leqslant \widehat{\Theta}(2\xi)|\widehat{a}(\xi)|^2$ for $\xi \in (c, d)$. This implies $\widehat{\Theta}(2\xi) > 0$ for $\xi \in (c, d)$, i.e., $\widehat{\Theta}(\xi) > 0$ for any $\xi \in (2c, 2d)$. Inductively, we can prove $\widehat{\Theta}(\xi) > 0$ for any $\xi \in (2^n c, 2^n d)$, $n \in \mathbb{N}$. Since $\widehat{\Theta}(\xi)$ is 2π-periodic, and the length of the interval $(2^n c, 2^n d)$ becomes arbitrarily large as n increases, we know that $\widehat{\Theta}(\xi) > 0$ for all $\xi \in \mathbb{R}$. Thus, $\widehat{\Theta}(\xi)\widehat{\Theta}(2\xi) > 0$ for any $\xi \in \mathbb{R}$. The determinant

$$\det(\mathcal{M}_{\mathsf{a},\Theta}(e^{-i\xi})) = \widehat{\Theta}(\xi+\pi)\left[\widehat{\Theta}(\xi) - \widehat{\Theta}(2\xi)|\widehat{a}(\xi)|^2\right] - \widehat{\Theta}(\xi)\widehat{\Theta}(2\xi)|\widehat{a}(\xi+\pi)|^2 \geqslant 0$$

implies that

$$\widehat{\Theta}(\xi+\pi)\left[\widehat{\Theta}(\xi) - \widehat{\Theta}(2\xi)|\widehat{a}(\xi)|^2\right] \geqslant \widehat{\Theta}(\xi)\widehat{\Theta}(2\xi)|\widehat{a}(\xi+\pi)|^2 \geqslant 0. \tag{2.2.13}$$

By $\widehat{\Theta}(\xi+\pi) > 0$, we know $\left[\mathcal{M}_{\mathsf{a},\Theta}(e^{-i\xi})\right]_{1,1} = \widehat{\Theta}(\xi) - \widehat{\Theta}(2\xi)|\widehat{a}(\xi)|^2 \geqslant 0$ for any $\xi \in \mathbb{R}$. Compare with the reverse inequality in (2.2.12), we get $\widehat{\Theta}(\xi) - \widehat{\Theta}(2\xi)|\widehat{a}(\xi)|^2 = 0$ for all $\xi \in \mathbb{R}$. So $\det(\mathcal{M}_{\mathsf{a},\Theta}(e^{-i\xi})) = -\widehat{\Theta}(\xi)\widehat{\Theta}(2\xi)|\widehat{a}(\xi+\pi)|^2 \leqslant 0$, $\forall\, \xi \in \mathbb{R}$. Since the reverse inequality also holds, we must have $\det(\mathcal{M}_{\mathsf{a},\Theta}(e^{i\xi})) = 0$ for all $\xi \in \mathbb{R}$, which is a contradiction to the setting of the problem. So the necessity part of (2) is proved.

For sufficiency: From $\det(\mathcal{M}_{\mathsf{a},\Theta}(e^{-i\xi})) \geqslant 0$, we know that (2.2.13) holds.

Since $\widehat{\Theta}(\xi+\pi) \leqslant 0$, and it is not identically zero, we can conclude that $\left[\mathcal{M}_{\mathsf{a},\Theta}(e^{-i\xi})\right]_{1,1} = \widehat{\Theta}(\xi) - \widehat{\Theta}(2\xi)|\widehat{a}(\xi)|^2 \leqslant 0$. Also, $\left[\mathcal{M}_{\mathsf{a},\Theta}(e^{-i\xi})\right]_{2,2} = \widehat{\Theta}(\xi+\pi) - \widehat{\Theta}(2\xi)|\widehat{a}(\xi+\pi)|^2 \leqslant 0$ holds as well. This shows $\mathcal{M}_{\mathsf{a},\Theta}(e^{-i\xi})$ is negative semi-definite for all $\xi \in \mathbb{R}$. We proved the sufficiency part of (2).

Item (1) can be proved similarly as item (2).

Since items (1)(2) and (3) give necessary and sufficient conditions to all cases that $\operatorname{sig}(\mathcal{M}_{\mathsf{a},\Theta}(z))$ is constant on $z \in \mathbb{T} \setminus \sigma(\mathcal{M}_{\mathsf{a},\Theta}(z))$, we know that the signature varies if and only if none of the conditions in items (1), (2) or (3) is satisfied. Thus item (4) is proved. ■

The special simple case that $\Theta = \boldsymbol{\delta}$ is widely discussed in the literature. In terms of Laurent polynomials, $\Theta = \boldsymbol{\delta}$ if and only if $\Theta(z) = 1$. We present the following proposition for this special case.

Proposition 2.2.5. *Let $\Theta = \boldsymbol{\delta}$, and a be a finitely supported filter such that $\mathsf{a}(z)$ is not identically zero. Then the matrix*

$$\mathcal{M}_{\mathsf{a}}(z) := \mathcal{M}_{\mathsf{a},1}(z) = \begin{bmatrix} 1-\mathsf{a}(z)\mathsf{a}^\star(z) & -\mathsf{a}(z)\mathsf{a}^\star(-z) \\ -\mathsf{a}(-z)\mathsf{a}^\star(z) & 1-\mathsf{a}(-z)\mathsf{a}^\star(-z) \end{bmatrix}$$

is Hermitian for all $z \in \mathbb{T}$. Moreover,

(1) $s^+_{a,\Theta} = 1$ and $s^-_{a,\Theta} = 0$ if and only if

$$\mathsf{a}(z)\mathsf{a}^\star(z) + \mathsf{a}(-z)\mathsf{a}^\star(-z) = 1, \qquad \forall\, z \in \mathbb{T}.$$

(2) $s^+_{a,\Theta} = 2$ and $s^-_{a,\Theta} = 0$ if and only if

$$\mathsf{a}(z)\mathsf{a}^\star(z) + \mathsf{a}(-z)\mathsf{a}^\star(-z) \leqslant 1, \qquad \forall z \in \mathbb{T},$$

and the above inequality is not identity.

(3) $s^+_{a,\Theta} = 1$ and $s^-_{a,\Theta} = 1$ if and only if

$$\mathsf{a}(z)\mathsf{a}^\star(z) + \mathsf{a}(-z)\mathsf{a}^\star(-z) \geqslant 1, \qquad \forall z \in \mathbb{T},$$

and the above inequality is not identity.

(4) Otherwise (beyond the three cases above), $s^+_{a,\Theta} = 2$ and $s^-_{a,\Theta} = 1$.

Proof. Notice that "1" is always an eigenvalue of $\mathcal{M}_{\mathsf{a}}(z)$. Hence, the other eigenvalue of $\mathcal{M}_{\mathsf{a}}(z)$ is equal to $\det(\mathsf{A}(z))$. Also, we can easily calculate

$$\det(\mathcal{M}_{\mathsf{a}}(z)) = 1 - \mathsf{a}(z)\mathsf{a}^\star(z) - \mathsf{a}(-z)\mathsf{a}^\star(-z).$$

The calculation of $s^+_{a,\Theta}$ and $s^-_{a,\Theta}$ in the four cases above follows directly from the sign of $\det(\mathcal{M}_{\mathsf{a}}(z))$. ■

2.3 The Spectral Decomposition of a Matrix of Laurent Polynomials with Constant Signature

As discussed in Theorem 2.1.2, the problem of constructing quasi-tight framelet filter banks is the same as to find a spectral decomposition of a matrix $\mathcal{N}_{\mathsf{a},\Theta|n_b}(z)$ of Laurent polynomials. The main theorem we want to establish in this section is the following:

Theorem 2.3.1. *Let $\mathsf{A}(z) = \sum_{k=-L}^{L} A(k)z^k$ be an $n \times n$ matrix of Laurent polynomials, such that $\mathsf{A}(z)$ is Hermitian, and $\det(\mathsf{A}(z))$ is not identically zero. If $\operatorname{sig}(\mathsf{A}(z))$ is constant for all $z \in \mathbb{T} \setminus \sigma(\mathsf{A}(z))$, that is, $\nu_+(\mathsf{A}(z))$ and $\nu_-(\mathsf{A}(z))$ are both constants for all $z \in \mathbb{T} \setminus \sigma(\mathsf{A}(z))$ (we denote them by ν_+ and ν_- respectively), then there exists an $n \times n$ matrix $\mathsf{U}(z)$ of Laurent polynomials, such that*

$$\mathsf{A}(z) = \mathsf{U}(z)\mathsf{D}\mathsf{U}^\star(z), \tag{2.3.1}$$

where $\mathsf{D} := \operatorname{diag}(\mathbf{I}_{\nu_+}, -\mathbf{I}_{\nu_-})$ is an $n \times n$ constant diagonal matrix. We call D the **signature matrix** *of $\mathsf{A}(z)$.*

For the special case that $\mathsf{A}(z)$ is positive semi-definite for all $z \in \mathbb{T}$ (which means $\operatorname{sig}(\mathsf{A}(z)) = n$, $\forall z \in \mathbb{T} \setminus \sigma(\mathsf{A}(z))$), the above result is known as the following famous matrix-valued Fejér-Riesz lemma:

Theorem 2.3.2 (Matrix-Valued Fejér-Riesz Lemma)**.** *Let* $\mathsf{A}(z) = \sum_{k=-L}^{L} A(k)z^k$ *be an* $n \times n$ *Hermitian matrix of Laurent polynomials. If* $\mathsf{A}(z)$ *is positive semi-definite for all* $z \in \mathbb{T}$*, then there exists an* $n \times n$ *matrix* $\mathsf{U}(z) = \sum_{k=0}^{L} U(k)z^k$ *of Laurent polynomials, such that* $\mathsf{A}(z) = \mathsf{U}(z)\mathsf{U}^\star(z)$*.*

This matrix-valued Fejér-Riesz lemma is well known in both mathematics and engineering literature. For example, see [75, 54, 25] and many references therein. Similar results also hold if we require all the matrices to be of rational functions rather than Laurent polynomials [88, 70].

Another interesting relevant problem is to consider the spectral factorization of a polynomial matrix $\mathsf{A}(z)$, which is Hermitian for all z belonging to either the real axis or the imaginary axis. This problem has important applications in both engineering and control theory. Spectral factorizations in this setting have been solved beautifully using the tool of sign characteristic, see [69, 71, 29, 28]. The concept of sign characteristic for matrix polynomials was introduced and analyzed in [28] from an algebraic approach, which requires a lot of knowledge in matrix polynomials to be illustrated thoroughly. To avoid the complicated algebraic discussions, our proof below for the Laurent polynomials problem develops the concepts of the partial multiplicity and the sign characteristic for matrices of Laurent polynomials in a much simpler way. (However, our definitions are consistent with the original definitions for polynomials problem in [28], according to Theorem 3.7 in [28].)

The structure of the proof of Theorem 2.3.1 is similar to that of the polynomial problem in [29], which is inductive on the length of determinant of $\mathsf{A}(z)$. Firstly, we prove in Section 2.3.1 that Theorem 2.3.1 holds if $\det(A(z))$ is a nonzero monomial, i.e., $\operatorname{len}(\det(A(z))) = 0$. Then we show that if $\operatorname{len}(\det(\mathsf{A}(z))) > 0$, it can be reduced inductively to the special case that $\operatorname{len}(\det(\mathsf{A}(z))) = 0$. A complete proof of Theorem 2.3.1 is provided in Section 2.3.2.

2.3.1 Empty Spectrum Case

If the Hermitian matrix $\mathsf{A}(z)$ has empty spectrum (determinant is a nonzero monomial), the factorization in Theorem 2.3.1 is studied in the general ring with involution. For example, see [29, 65, 66, 21]. Our constructive proof below follows similar procedure in [13, 29], which are originally designed for polynomial problems.

Theorem 2.3.3. *Let* $\mathsf{A}(z) = \sum_{k=-L}^{L} A(k)z^k$ *be an* $n \times n$ *matrix of Laurent polynomials, such that* $\mathsf{A}(z)$ *is Hermitian. If* $\det(A(z))$ *is a nonzero monomial, then there exists an* $n \times n$ *matrix* $\mathsf{U}(z)$ *of Laurent polynomials, and an* $n \times n$ *constant diagonal matrix* $\mathsf{D} = \operatorname{diag}(\mathbf{I}_{\nu_+}, -\mathbf{I}_{\nu_-})$*, for some nonnegative integers* ν_+ *and* ν_- *satisfying* $\nu_+ + \nu_- = n$*, such that* (2.3.1) *holds.*

We use Algorithm 2.1 to realize the decomposition in (2.3.1). The Theorem 2.3.3 is proved by showing that the Algorithm 2.1 is feasible and will terminate in finite steps.

In order to make the Algorithm 2.1 easier to read, we provide some lemmas first, which serve as useful sub-steps in Algorithm 2.1. We prove these lemmas in an algorithmic way, so that they could be used directly in Algorithm 2.1.

For a Laurent polynomial $\mathsf{u}(z)$, we use $\deg(\mathsf{u}(z))$ to denote its highest degree, and use $\operatorname{ldeg}(\mathsf{u}(z))$ to denote its lowest degree. Its fsupp is defined as the interval: $\operatorname{fsupp}(\mathsf{u}(z)) := [\operatorname{ldeg}(\mathsf{u}(z)),\ \deg(\mathsf{u}(z))]$.

For a $k \times k$ matrix $\mathsf{Q}(z)$ of Laurent polynomials, we call it ***diagonally dominant at diagonal*** s if

(1) for all $i \neq s$:

$$\operatorname{fsupp}(\mathsf{Q}_{i,s}(z)) \subsetneq \operatorname{fsupp}(\mathsf{Q}_{s,s}(z)), \quad \text{and } \operatorname{fsupp}(\mathsf{Q}_{s,i}(z)) \subsetneq \operatorname{fsupp}(\mathsf{Q}_{s,s}(z)); \tag{2.3.2}$$

(2) for all $i > s$:

$$\deg(\mathsf{Q}_{s,i}(z)) < \deg(\mathsf{Q}_{s,s}(z)). \tag{2.3.3}$$

$\mathsf{Q}(z)$ is called ***diagonally dominant*** if it is diagonally dominant at all its diagonals $s = 1, \ldots, k$.

Lemma 2.3.1. *Suppose* $\mathsf{Q}(z)$ *is a* $k \times k$ *Hermitian matrix of Laurent polynomials, which is also diagonally dominant, and the lengths of the diagonal*

Laurent polynomials are nondecreasing:

$$\mathrm{len}(\mathsf{Q}_{1,1}(z)) \leqslant \mathrm{len}(\mathsf{Q}_{2,2}(z)) \leqslant \ldots \leqslant \mathrm{len}(\mathsf{Q}_{k,k}(z)). \tag{2.3.4}$$

Let $\mathsf{b}(z)$ *be a column vector of Laurent polynomials with size* k*, satisfying*

$$\mathrm{ldeg}(\mathsf{b}_l(z)) \geqslant \mathrm{ldeg}(\mathsf{Q}_{l,l}(z)), \qquad l = 1, \ldots, k. \tag{2.3.5}$$

Then there exists a column vector $X(z)$ *of Laurent polynomials with size* k*, such that* $Y(z) := \mathsf{b}(z) - \mathsf{Q}(z)X(z)$ *satisfies*

$$\mathrm{fsupp}(Y_l(z)) \subsetneq \mathrm{fsupp}(\mathsf{Q}_{l,l}(z)), \;\; \deg(Y_l(z)) < \deg(\mathsf{Q}_{l,l}(z)), \qquad l = 1, \ldots, k. \tag{2.3.6}$$

Proof. If $\mathsf{b}(z)$ already satisfies $\deg(\mathsf{b}_l(z)) < \deg(\mathsf{Q}_{l,l}(z))$ for all $l = 1, \ldots, k$, then we can just take $X(z) = 0$, and the result is true. So we just need to consider the case that there exists some $s \in \{1, \ldots, k\}$, such that

$$\deg(\mathsf{b}_s(z)) \geqslant \deg(\mathsf{Q}_{s,s}(z)). \tag{2.3.7}$$

Since $\mathsf{Q}(z)$ is Hermitian, we can denote the fsupp of its diagonal elements as

$$[-n_l, n_l] := \mathrm{fsupp}(\mathsf{Q}_{l,l}), \qquad l = 1, \ldots, k.$$

From (2.3.4), we know that $n_1 \leqslant \ldots \leqslant n_k$. Define

$$\mathsf{D}(z) := \begin{bmatrix} \left[(z+1)(z^{-1}+1)\right]^{n_k-n_1} & & & \\ & \left[(z+1)(z^{-1}+1)\right]^{n_k-n_2} & & \\ & & \ddots & \\ & & & 1 \end{bmatrix},$$

$\widetilde{\mathsf{Q}}(z) := \mathsf{D}(z)\mathsf{Q}(z)$, and $\widetilde{\mathsf{b}}(z) := \mathsf{D}(z)\mathsf{b}(z)$. We can see from (2.3.5) that for all $l = 1, \ldots, k$:

$$\mathrm{fsupp}(\widetilde{\mathsf{Q}}_{l,l}(z)) = [-n_k, n_k], \qquad \mathrm{ldeg}(\widetilde{\mathsf{b}}_l(z)) \geqslant \mathrm{ldeg}(\widetilde{\mathsf{Q}}_{l,l}(z)) = -n_k. \tag{2.3.8}$$

Since $\mathsf{Q}(z)$ is diagonally dominant, from (2.3.2) we can see that $\text{fsupp}(\widetilde{\mathsf{Q}}_{l,i}(z)) \subsetneq \text{fsupp}(\widetilde{\mathsf{Q}}_{l,l}(z)) = [-n_k, n_k]$ for all $l = 1, \ldots, k,\ i \neq l$. Thus we can write $\widetilde{\mathsf{Q}}(z)$ as

$$\widetilde{\mathsf{Q}}(z) = \sum_{l=-n_k}^{n_k} \widetilde{Q}_l z^l. \tag{2.3.9}$$

Also, from (2.3.3) we know that $\deg(\widetilde{\mathsf{Q}}_{l,i}(z)) < \deg(\widetilde{\mathsf{Q}}_{l,l}(z)) = n_k$ for all $l = 1, \ldots, k,\ i > l$. So the coefficient matrix $\widetilde{Q}_{n_k}$ in (2.3.9) is lower triangular, which by (2.3.8) also has nonzero diagonal elements. Therefore $\widetilde{Q}_{n_k}$ is nonsingular.

From (2.3.7), we also know that there exists some $s \in \{1, \ldots, k\}$, such that $\deg(\widetilde{\mathsf{b}}_s(z)) \geqslant \deg(\widetilde{\mathsf{Q}}_{s,s}(z)) = n_k$. So by (2.3.8), we can write $\widetilde{\mathsf{b}}(z)$ as

$$\widetilde{\mathsf{b}}(z) = \sum_{l=-n_k}^{M} \widetilde{b}_l z^l,$$

with $M \geqslant n_k$.

Let us parameterize the unknown $X(z) = \sum_{l=0}^{M-n_k} X_l z^l$, and take $\widetilde{Y}(z) := \widetilde{\mathsf{b}}(z) - \widetilde{\mathsf{Q}}(z)X(z)$. By this definition, we know that $\text{fsupp}(\widetilde{Y}(z)) \subset [-n_k, M]$. Write $\widetilde{Y}(z) = \sum_{l=-n_k}^{M} \widetilde{Y}_l z^l$, we want to solve for $X(z)$, such that the coefficients $\widetilde{Y}_l = 0$ for all $l = n_k, n_k + 1, \ldots, M$. Notice that we have $M - n_k + 1$ matrix equations to solve for $M - n_k + 1$ unknowns. The equations could be formulated as the following Toeplitz form

$$\begin{bmatrix} \widetilde{Q}_{n_k} & \widetilde{Q}_{n_k-1} & \cdots & \widetilde{Q}_{2n_k-M} \\ & \ddots & \ddots & \vdots \\ & & \widetilde{Q}_{n_k} & \widetilde{Q}_{n_k-1} \\ & & & \widetilde{Q}_{n_k} \end{bmatrix} \begin{bmatrix} X_0 \\ \vdots \\ X_{M-n_k-1} \\ X_{M-n_k} \end{bmatrix} = \begin{bmatrix} \widetilde{b}_{n_k} \\ \vdots \\ \widetilde{b}_{M-1} \\ \widetilde{b}_M \end{bmatrix},$$

where we use $\widetilde{Q}_j = 0$ if $j < -n_k$. Since $\widetilde{Q}_{n_k}$ is nonsingular, we can solve the above system from the last equation, and use backward substitution to find all $X_0, \ldots, X_{M-n_k}$.

Now, we found a vector $X(z)$ of Laurent polynomials such that

$$\widetilde{Y}(z) = \widetilde{\mathsf{b}}(z) - \widetilde{\mathsf{Q}}(z)X(z) = \mathsf{D}(z)\Big(\mathsf{b}(z) - \mathsf{Q}(z)X(z)\Big)$$

satisfies $\operatorname{fsupp}(\widetilde{Y}_l(z)) \subset [-n_k,\ n_k - 1]$, for all $l = 1, \dots, k$. Take $Y(z) := \mathsf{b}(z) - \mathsf{Q}(z)X(z) = \mathsf{D}^{-1}(z)\widetilde{Y}(z)$, we will prove that it satisfies (2.3.6). Notice that $\mathsf{D}(z)$ is a diagonal matrix with diagonals $\mathsf{D}_l(z) = \big[(z+1)(z^{-1}+1)\big]^{n_k - n_l}$, for all $l = 1, \dots, k$. So the fsupp of $Y_l(z) = \widetilde{Y}_l(z)/\mathsf{D}_l(z)$ is contained in $\operatorname{fsupp}(Y_l(z)) \subset [-n_l, n_l - 1]$, for all $l = 1, \dots, k$. So (2.3.6) holds. This completes the proof of the lemma. ■

Lemma 2.3.2. *Let $\mathsf{Q}(z)$ be a $k \times k$ Hermite matrix of Laurent polynomials, and* (2.3.4) *is satisfied. Suppose $\mathsf{Q}(z)$ is diagonally dominant at first s diagonals, for some $s < k$. (If $\mathsf{Q}(z)$ is not diagonally dominant at the first diagonal, just take $s = 0$.) Then there exists a $k \times k$ invertible matrix $\mathsf{U}(z)$ of Laurent polynomials, such that $\widetilde{\mathsf{Q}}(z) := \mathsf{U}(z)\mathsf{Q}(z)\mathsf{U}^\star(z)$ is diagonally dominant at first $(s+1)$ diagonals.*

Proof. Write $\mathsf{Q}(z) = \begin{bmatrix} \mathsf{A}(z) & \mathsf{B}(z) \\ \mathsf{B}^\star(z) & \mathsf{C}(z) \end{bmatrix}$, where $\mathsf{A}(z)$ is an $(s+1) \times (s+1)$ Hermitian matrix of Laurent polynomials. Since $\mathsf{Q}(z)$ is diagonally dominant at diagonals $1, \dots, s$, from (2.3.2) and (2.3.4) we know that for all $i < (s+1)$, $\operatorname{fsupp}(\mathsf{Q}_{i,(s+1)}(z)) \subsetneq \operatorname{fsupp}(\mathsf{Q}_{i,i}(z)) \subset \operatorname{fsupp}(\mathsf{Q}_{(s+1),(s+1)})$ and $\operatorname{fsupp}(\mathsf{Q}_{(s+1),i}(z)) \subsetneq \operatorname{fsupp}(\mathsf{Q}_{i,i}(z)) \subset \operatorname{fsupp}(\mathsf{Q}_{(s+1),(s+1)}(z))$. So $\mathsf{A}(z)$ is a diagonally dominant matrix. Hence, for $s = k-1$, the lemma is true with $\mathsf{U}(z) = \mathbf{I_k}$.

For $s < k-1$, we can find integers $\lambda_{s+2}, \dots, \lambda_k$, such that

$$\widetilde{\mathsf{B}}(z) := \mathsf{B}(z) \begin{bmatrix} z^{-\lambda_{s+2}} & & \\ & \ddots & \\ & & z^{-\lambda_k} \end{bmatrix}$$

satisfies $\operatorname{ldeg}(\widetilde{\mathsf{B}}_{l,i}(z)) \geqslant \operatorname{ldeg}(\mathsf{A}_{l,l}(z))$, for all $l = 1, \dots, (s+1)$, and $i = 1, \dots, k-(s+1)$. Write $\widetilde{\mathsf{B}}(z)$ as column vectors $\widetilde{\mathsf{B}}(z) = \Big[\mathsf{b}^{(s+2)}(z) \quad \dots \quad \mathsf{b}^{(k)}(z)\Big]$, we can see that

$$\mathrm{ldeg}(\mathsf{b}_l^{(i)}(z)) \geqslant \mathrm{ldeg}(\mathsf{A}_{l,l}(z)), \qquad \text{for all } i = (s+2), \dots, k, \quad l = 1, \dots, (s+1).$$

Using Lemma 2.3.1, for each $i = (s+2), \dots, k$, we can solve vectors $x^{(i)}(z)$, such that $y^{(i)}(z) := \mathsf{b}^{(i)}(z) - \mathsf{A}(z)x^{(i)}(z)$ satisfies

$$\mathrm{fsupp}(y_l^{(i)}(z)) \subsetneq \mathrm{fsupp}(\mathsf{A}_{l,l}(z)), \ \deg(y_l^{(i)}(z)) < \deg(\mathsf{A}_{l,l}(z)), \qquad l = 1, \dots, k. \tag{2.3.10}$$

Denote $\widetilde{X}(z) := \begin{bmatrix} x^{(s+2)}(z) & \dots & x^{(k)}(z) \end{bmatrix}$, $Y(z) := \begin{bmatrix} y^{(s+2)}(z) & \dots & y^{(k)}(z) \end{bmatrix} = \widetilde{\mathsf{B}}(z) - \mathsf{A}(z)\widetilde{X}(z)$, and $\Lambda(z) := \mathrm{diag}(z^{\lambda_{s+2}}, \dots, z^{\lambda_k})$, we know that

$$\begin{aligned}
&\begin{bmatrix} \mathbf{I_{s+1}} & \\ -\widetilde{X}^\star(z) & \mathbf{I_{k-(s+1)}} \end{bmatrix} \begin{bmatrix} \mathbf{I_{s+1}} & \\ & \Lambda(z) \end{bmatrix} \mathsf{Q}(z) \begin{bmatrix} \mathbf{I_{s+1}} & \\ & \Lambda^\star(z) \end{bmatrix} \begin{bmatrix} \mathbf{I_{s+1}} & -\widetilde{X}(z) \\ & \mathbf{I_{k-(s+1)}} \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{I_{s+1}} & \\ -\widetilde{X}^\star(z) & \mathbf{I_{k-(s+1)}} \end{bmatrix} \begin{bmatrix} \mathsf{A}(z) & \widetilde{\mathsf{B}}(z) \\ \widetilde{\mathsf{B}}^\star(z) & \Lambda(z)\mathsf{C}(z)\Lambda^\star(z) \end{bmatrix} \begin{bmatrix} \mathbf{I_{s+1}} & -\widetilde{X}(z) \\ & \mathbf{I_{k-(s+1)}} \end{bmatrix} \\
&= \begin{bmatrix} \mathsf{A}(z) & \widetilde{\mathsf{B}}(z) - \mathsf{A}(z)\widetilde{X}(z) \\ \widetilde{\mathsf{B}}^\star(z) - \widetilde{X}^\star(z)\mathsf{A}^\star(z) & \mathsf{E}(z) \end{bmatrix} = \begin{bmatrix} \mathsf{A}(z) & Y(z) \\ Y^\star(z) & \mathsf{E}(z) \end{bmatrix},
\end{aligned} \tag{2.3.11}$$

where $\mathsf{E}(z) := \Lambda(z)\mathsf{C}(z)\Lambda^\star(z) - \widetilde{\mathsf{B}}^\star(z)\widetilde{X}(z) - \widetilde{X}^\star(z)\widetilde{\mathsf{B}}(z)$. From (2.3.10), we can see that the above matrix $\widetilde{\mathsf{Q}}(z) := \begin{bmatrix} \mathsf{A}(z) & Y(z) \\ Y^\star(z) & \mathsf{E}(z) \end{bmatrix}$ is diagonally dominant at the first $(s+1)$ diagonals. Taking

$$\mathsf{U}(z) := \begin{bmatrix} \mathbf{I_{s+1}} & \\ -\widetilde{X}^\star(z) & \mathbf{I_{k-(s+1)}} \end{bmatrix} \begin{bmatrix} \mathbf{I_{s+1}} & \\ & \Lambda(z) \end{bmatrix} = \begin{bmatrix} \mathbf{I_{s+1}} & \\ -\widetilde{X}^\star(z) & \Lambda(z) \end{bmatrix},$$

the equality (2.3.11) implies that $\widetilde{\mathsf{Q}}(z) = \mathsf{U}(z)\mathsf{Q}(z)\mathsf{U}^\star(z)$. This completes the proof of the lemma. ■

Lemma 2.3.3. *Let $\mathsf{Q}(z)$ be a $k \times k$ matrix of Laurent polynomials, such that $\mathsf{Q}(z)$ is Hermitian and $\det(\mathsf{Q}(z))$ is a nonzero monomial. If its first diagonal element $\mathsf{Q}_{1,1}(z) = 0$, then there exists a $k \times k$ invertible matrix $\mathsf{U}(z)$ of Laurent polynomials, and a $(k-1) \times (k-1)$ matrix $\widetilde{\mathsf{Q}}(z)$ of Laurent polynomials, such* *that*

$$\mathsf{Q}(z) = \mathsf{U}(z) \begin{bmatrix} 1 & \\ & \widetilde{\mathsf{Q}}(z) \end{bmatrix} \mathsf{U}^\star(z).$$

Proof. Since $\mathsf{Q}(z)$ is invertible, we can calculate $\mathsf{Q}^{-1}(z) = \operatorname{adj}(\mathsf{Q}(z))/\det(\mathsf{Q}(z))$. Write $\mathsf{Q}(z)$ and $\mathsf{Q}^{-1}(z)$ as

$$\mathsf{Q}(z) = \begin{bmatrix} 0 & \mathsf{a}^\star(z) \\ \mathsf{a}(z) & \mathsf{E}(z) \end{bmatrix}, \qquad \mathsf{Q}^{-1}(z) = \begin{bmatrix} \mathsf{b}(z) & \mathsf{c}^\star(z) \\ \mathsf{c}(z) & \mathsf{F}(z) \end{bmatrix},$$

where $\mathsf{a}(z)$ and $\mathsf{c}(z)$ are both vectors of Laurent polynomials of size $(k-1)$, $\mathsf{E}(z)$ and $\mathsf{F}(z)$ are matrices of Laurent polynomials of size $(k-1)\times(k-1)$, and $\mathsf{b}(z)$ is a scalar Laurent polynomial satisfying $\mathsf{b}^\star(z) = \mathsf{b}(z)$. From $\mathsf{Q}(z)\mathsf{Q}^{-1}(z) = \begin{bmatrix} \mathsf{a}^\star\mathsf{c} & \mathsf{a}^\star\mathsf{F} \\ \mathsf{ab} + \mathsf{Ec} & \mathsf{ac}^\star + \mathsf{EF} \end{bmatrix} = \mathbf{I_k}$, the first column gives us

$$\mathsf{a}^\star\mathsf{c} = 1, \qquad \mathsf{ab} + \mathsf{Ec} = 0.$$

The first equation above also implies that $\mathsf{c}^\star\mathsf{a} = 1^\star = 1$. Multiply $\mathsf{c}^\star$ on the left of the second equation, we get $\mathsf{c}^\star\mathsf{ab} + \mathsf{c}^\star\mathsf{Ec} = 0$, that is, $\mathsf{c}^\star\mathsf{Ec} = -\mathsf{b}$.

Set

$$y(z) := \frac{1}{2}\left(\mathsf{b}(z) + 1\right), \quad x(z) := \frac{1}{2}\left(\mathsf{b}(z) - 1\right)\mathsf{a}(z), \quad \mathsf{V}(z) := \begin{bmatrix} y & \mathsf{c}^\star \\ x & \mathbf{I_{k-1}} \end{bmatrix}.$$

We can calculate

$$\begin{aligned} \mathsf{V}(z)\mathsf{Q}(z)\mathsf{V}^\star(z) &= \begin{bmatrix} y & \mathsf{c}^\star \\ x & \mathbf{I_{k-1}} \end{bmatrix} \begin{bmatrix} 0 & \mathsf{a}^\star(z) \\ \mathsf{a}(z) & \mathsf{E}(z) \end{bmatrix} \begin{bmatrix} y & x^\star \\ \mathsf{c} & \mathbf{I_{k-1}} \end{bmatrix} \\ &= \begin{bmatrix} \mathsf{c}^\star\mathsf{a}y + y\mathsf{a}^\star\mathsf{c} + \mathsf{c}^\star\mathsf{Ec} & \mathsf{c}^\star\mathsf{a}x^\star + y\mathsf{a}^\star + \mathsf{c}^\star\mathsf{E} \\ x\mathsf{a}^\star\mathsf{c} + y\mathsf{a} + \mathsf{Ec} & \mathsf{a}x^\star + x\mathsf{a}^\star + \mathsf{E} \end{bmatrix} \end{aligned}$$

If we check each term, we get

$$\begin{aligned} \mathsf{c}^\star\mathsf{a}y + y\mathsf{a}^\star\mathsf{c} + \mathsf{c}^\star\mathsf{Ec} =& 2y + \mathsf{c}^\star\mathsf{Ec} = \mathsf{b} + 1 - \mathsf{b} = 1. \\ \mathsf{c}^\star\mathsf{a}x^\star + y\mathsf{a}^\star + \mathsf{c}^\star\mathsf{E} =& x^\star + y\mathsf{a}^\star + \mathsf{c}^\star\mathsf{E} = \frac{1}{2}(\mathsf{b}-1)\mathsf{a}^\star + \frac{1}{2}(\mathsf{b}+1)\mathsf{a}^\star + \mathsf{c}^\star\mathsf{E} \end{aligned}$$

$$=\mathsf{b}\mathsf{a}^\star + \mathsf{c}^\star\mathsf{E} = (\mathsf{a}\mathsf{b}+\mathsf{E}\mathsf{c})^\star = 0.$$

$$\mathsf{a}x^\star + x\mathsf{a}^\star + \mathsf{E} = \frac{1}{2}(\mathsf{b}-1)\,\mathsf{a}\mathsf{a}^\star + \frac{1}{2}(\mathsf{b}-1)\,\mathsf{a}\mathsf{a}^\star + \mathsf{E} = (\mathsf{b}-1)\mathsf{a}\mathsf{a}^\star + \mathsf{E}.$$

Thus, $\mathsf{V}(z)\mathsf{Q}(z)\mathsf{V}^\star(z) = \begin{bmatrix}1 & 0\\ 0 & (\mathsf{b}(z)-1)\mathsf{a}(z)\mathsf{a}(z)^\star + \mathsf{E}(z)\end{bmatrix}$. Also, notice that $\det(\mathsf{V}(z)) = y(z) - \mathsf{c}^\star(z)x(z) = \frac{1}{2}(\mathsf{b}(z)+1) - \frac{1}{2}(\mathsf{b}(z)-1) = 1$, so $\mathsf{V}(z)$ is invertible. We can define $\mathsf{U}(z) := \mathsf{V}^{-1}(z)$, and $\widetilde{\mathsf{Q}}(z) := (\mathsf{b}(z)-1)\mathsf{a}(z)\mathsf{a}(z)^\star + \mathsf{E}(z)$. It is straightforward to check that $\mathsf{Q}(z) = \mathsf{U}(z)\begin{bmatrix}1 & \\ & \widetilde{\mathsf{Q}}(z)\end{bmatrix}\mathsf{U}^\star(z)$ satisfies all the requirements in the lemma. ■

Now, we are ready to present the algorithm for the decomposition of matrices of Laurent polynomials with empty spectrum.

Algorithm 2.1. *Let $\mathsf{A}(z)$ be an $n \times n$ matrix of Laurent polynomials, such that $\mathsf{A}^\star(z) = \mathsf{A}(z)$, and $\det(\mathsf{A}(z))$ is a nonzero monomial.*

(S0) *Initialization. Set $\mathsf{U}(z) := \mathbf{I}_n$ to be the $n \times n$ identity matrix. Let $\mathsf{Q}(z) := \mathsf{A}(z)$ and $k := n$.*

(S1) *Find a permutation matrix $\widetilde{\mathsf{U}}$, such that $\widetilde{\mathsf{Q}}(z) := \widetilde{\mathsf{U}}\mathsf{Q}(z)\widetilde{\mathsf{U}}^\star$ satisfies*

$$\mathrm{len}(\widetilde{\mathsf{Q}}_{1,1}(z)) \leqslant \mathrm{len}(\widetilde{\mathsf{Q}}_{2,2}(z)) \leqslant \ldots \leqslant \mathrm{len}(\widetilde{\mathsf{Q}}_{k,k}(z)).$$

Update $\mathsf{U}(z) := \mathsf{U}(z)\begin{bmatrix}\mathbf{I}_{n-k} & \\ & \widetilde{\mathsf{U}}^{-1}\end{bmatrix}$, and set $\mathsf{Q}(z) := \widetilde{\mathsf{Q}}(z)$.

(S2) *If the first diagonal element $\mathsf{Q}_{1,1}(z) \neq 0$, go to step **(S3)**. Otherwise, use Lemma 2.3.3 to find a $k \times k$ matrix $\widetilde{\mathsf{U}}(z)$ of Laurent polynomials, such that $\widetilde{\mathsf{U}}(z)\mathsf{Q}(z)\widetilde{\mathsf{U}}^\star(z) = \begin{bmatrix}1 & 0\\ 0 & \widetilde{\mathsf{Q}}(z)\end{bmatrix}$, for some $(k-1)\times(k-1)$ matrix $\widetilde{\mathsf{Q}}(z)$ of Laurent polynomials.*

*Update $\mathsf{U}(z) := \mathsf{U}(z)\begin{bmatrix}\mathbf{I}_{n-k} & \\ & \widetilde{\mathsf{U}}^{-1}(z)\end{bmatrix}$, and $\mathsf{Q}(z) := \widetilde{\mathsf{Q}}(z)$. Set $k := k-1$. Restart from **(S1)**.*

(S3) *For $\mathsf{Q}_{1,1}(z) \neq 0$, if $\mathsf{Q}(z)$ is a diagonally dominant matrix, go to step **(S6)**. Otherwise, find the number s, such that $\mathsf{Q}(z)$ is diagonally dominant at first s diagonals, but not at diagonal $s+1$. If it is not diagonally dominant at the first diagonal, just take $s = 0$. Go to step **(S4)**.*

(S4) *Use Lemma 2.3.2 to find a $k \times k$ matrix $\widetilde{\mathsf{U}}(z)$ of Laurent polynomials, such that $\widetilde{\mathsf{Q}}(z) := \widetilde{\mathsf{U}}(z)\mathsf{Q}(z)\widetilde{\mathsf{U}}^{\star}(z)$ is diagonally dominant at first $(s+1)$ diagonals.*

Update $\mathsf{U}(z) := \mathsf{U}(z)\begin{bmatrix} \mathbf{I}_{n-k} & \\ & \widetilde{\mathsf{U}}^{-1}(z) \end{bmatrix}$, *and* $\mathsf{Q}(z) := \widetilde{\mathsf{Q}}(z)$.

(S5) *After **(S4)**, if the length of diagonal elements in $\mathsf{Q}(z)$ is not non-decreasing any more, that is,*

$$\operatorname{len}(\mathsf{Q}_{1,1}(z)) \leqslant \operatorname{len}(\mathsf{Q}_{2,2}(z)) \leqslant \ldots \leqslant \operatorname{len}(\mathsf{Q}_{k,k}(z))$$

*is not satisfied, restart from **(S1)** to sort them again. Otherwise, repeat from **(S3)**.*

(S6) *If $\mathsf{Q}(z)$ is diagonally dominant, then $\mathsf{Q}(z)$ must be a constant matrix. Compute its eigenvalue decomposition $\mathsf{Q} = \widetilde{\mathsf{U}}\Lambda\widetilde{\mathsf{U}}^{\star}$, where $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_k)$. Without loss of generality, we assume the first $(k - \nu_-)$ of the eigenvalues are positive, and the last ν_- of them are negative. Redefine $\widetilde{\mathsf{U}} := \widetilde{\mathsf{U}}(z)\operatorname{diag}(\sqrt{|\lambda_1|}, \ldots, \sqrt{|\lambda_k|})$, we have $\mathsf{Q} = \widetilde{\mathsf{U}}\operatorname{diag}(\mathbf{I}_{k-\nu_-}, -\mathbf{I}_{\nu_-})\widetilde{\mathsf{U}}^{\star}$.*

Update $\mathsf{U}(z) := \mathsf{U}(z)\begin{bmatrix} \mathbf{I}_{n-k} & \\ & \widetilde{\mathsf{U}} \end{bmatrix}$, *and define* $\mathsf{D} := \begin{bmatrix} \mathbf{I}_{n-k} & & \\ & \mathbf{I}_{k-\nu_-} & \\ & & -\mathbf{I}_{\nu_-} \end{bmatrix}$.

Such output $\mathsf{U}(z)$ and D satisfy (2.3.1), and all the requirements in Theorem 2.3.3.

Proof. Every time we update $\mathsf{U}(z)$ and $\mathsf{Q}(z)$ in steps **(S1)(S2)(S4)(S6)**, we are actually factoring out some matrices from the original $\mathsf{Q}(z)$. The update of $\mathsf{U}(z)$ is just absorbing the factored out matrices into the left factor $\mathsf{U}(z)$. The update of $\mathsf{Q}(z)$ is just setting the new $\mathsf{Q}(z)$ to be the matrix left after the factorization. So we can see if the algorithm can finalize in **(S6)**, the decomposition $\mathsf{A}(z) = \mathsf{U}(z)\mathsf{D}\mathsf{U}^\star(z)$ must hold.

Let us prove that all the steps in the algorithm is feasible and it will terminate in finite steps.

Step **(S2)** is proved by Lemma 2.3.3. Notice that Lemma 2.3.3 guarantees that $\widetilde{\mathsf{U}}(z)$ is invertible. Therefore, the update of $\mathsf{U}(z)$ is feasible.

Step **(S4)** is proved by Lemma 2.3.2, notice that the matrix $\widetilde{\mathsf{U}}(z)$ in **(S4)** is also invertible. Hence, the update of $\mathsf{U}(z)$ here is also feasible.

In **(S6)**, we will show that if $\mathsf{Q}(z)$ is diagonally dominant, then $\mathsf{Q}(z)$ has to be a constant matrix. In this case, $\mathrm{len}(\det(\mathsf{Q}(z))) = \sum_{l=1}^{k} \mathrm{len}(\mathsf{Q}_{l,l}(z))$. By construction, we can see $\mathsf{A}(z) = \mathsf{U}(z) \begin{bmatrix} I_{n-k} & \\ & \mathsf{Q}(z) \end{bmatrix} \mathsf{U}^\star(z)$. Thus $\det(\mathsf{A}(z)) = \det(\mathsf{U}(z))\det(\mathsf{Q}(z))\det(\mathsf{U}^\star(z))$, which implies $\det(\mathsf{Q}(z)) | \det(\mathsf{A}(z))$. Since $\det(\mathsf{A}(z))$ is a nonzero monomial, we know $\mathrm{len}(\det(\mathsf{Q}(z))) = 0$. Hence $\sum_{l=1}^{k} \mathrm{len}(\mathsf{Q}_{l,l}(z)) = 0$, which implies that all the diagonal elements of $\mathsf{Q}(z)$ has to be monomials. By our construction, we know $\mathsf{Q}(z)$ has to be a Hermitian matrix, so all the diagonal elements must be nonzero constants. We also know that $\mathsf{Q}(z)$ is diagonally dominant, so $\mathsf{Q}(z)$ must be a diagonal constant matrix.

Finally, we prove that the algorithm will stop after finite iterations. The algorithm might restart from **(S1)** in **(S2)(S5)** and also restart from **(S3)** in **(S5)**.

When the restart from **(S1)** in **(S2)** occurs, the size of $\mathsf{Q}(z)$, k, will decrease by 1. So it could happen only finite times.

In order to show that the algorithm could only restarts from **(S1)** in **(S5)** for finite times, let us use the lexicographic order of sequences of length k. For any 2 sequences of nonnegative integers with length k: $\{\alpha_j\}_{j=1}^k, \{\beta_j\}_{j=1}^k \in \mathbb{N}_0^k$, we say $\{\alpha_j\}_{j=1}^k$ is smaller than $\{\beta_j\}_{j=1}^k$ if there exists some index $j_0 \in \{1, 2, \cdots, k\}$, such that $\alpha_j = \beta_j$ for all $j < j_0$, and $\alpha_{j_0} < \beta_{j_0}$. $\{\alpha_j\}_{j=1}^k$ is equal to $\{\beta_j\}_{j=1}^k$ if $\alpha_j = \beta_j$, $\forall\, j = 1, 2, \cdots, k$. It's easy to see that $\mathbb{N}_0^k$ is a well-ordered set under this lexicographic order. The sequence $\{\mathrm{len}(\mathsf{Q}_{i,i}(z))\}_{i=1}^k \in \mathbb{N}_0^k$. Every time the algorithm restarts from **(S1)** in **(S5)**, the lexicographic order of $\{\mathrm{len}(\mathsf{Q}_{i,i}(z))\}_{i=1}^k \in \mathbb{N}_0^k$ will decrease. Since the sequence is lower bounded by the sequence $\{0, \ldots, 0\}$, the restarts can occur only finite times.

Every time the algorithm restarts from **(S3)** in **(S5)**, s will increase by at least 1, until the matrix $\mathsf{Q}(z)$ becomes diagonally dominant. So these iterations can only happen for finite number of times.

This completes the proof of the algorithm and Theorem 2.3.3. ■

2.3.2 General Case

Now we study the general case of the spectral decomposition problem as Theorem 2.3.1. That is, the case that $\det(\mathsf{A}(z))$ is not a monomial.

Recall that the theory of the Smith normal form says that any $n \times n$ matrix $\mathsf{A}(z)$ of Laurent polynomials can be factorized into

$$\mathsf{A}(z) = \mathsf{E}(z)\mathsf{D}(z)\mathsf{F}(z),$$

where $\mathsf{E}(z)$ and $\mathsf{F}(z)$ are invertible matrices of Laurent polynomials (determinants are nonzero monomials), and $\mathsf{D}(z) = \mathrm{diag}(d_1(z), d_2(z), \ldots, d_n(z))$ is a diagonal matrix with $d_i(z)|d_{i+1}(z)$ for all $i = 1, 2, \ldots, n-1$. $\mathsf{D}(z)$ is called the ***Smith Normal Form*** of $\mathsf{A}(z)$. If we require all $d_i(z)$, $i = 1, \ldots, n$, to be monic polynomials (polynomials with leading coefficient equal to 1), and to have nonzero constant term:

$$d_i(z) = z^{k_i} + c_{i,k_i-1}z^{k_i-1} + \cdots + c_{i,0}, \quad \text{with } c_{i,0} \neq 0, \qquad \forall\ i = 1, 2, \ldots, n,$$

then $\mathsf{D}(z)$ is unique. (For simplicity, we will always require this condition in our later discussions.) The polynomials $d_i(z)$ are called the ***invariant polynomials*** of $\mathsf{A}(z)$. The product $d_1(z)d_2(z)\cdots d_k(z)$ is essentially the greatest common divisor of the determinants of all $k \times k$ submatrices in $\mathsf{A}(z)$. Let us factorize the invariant polynomials in $\mathbb{C}$:

$$d_i(z) = \prod_{k=1}^{n_i}(z - z_{i,k})^{\alpha_{i,k}}, \qquad i = 1, 2, \ldots, n.$$

The collection of all the factors $(z - z_{i,k})^{\alpha_{i,k}}$, $k = 1, 2, \ldots, n_i$, $i = 1, 2, \ldots, n$, where each factor could repeat as many times as it occurs, is called the ***elementary divisors*** of $\mathsf{A}(z)$. For each $i = 1, 2, \ldots, n$, since we require $d_i(z)$

to have nonzero constant terms, $d_i(z)$ has no root at 0. Thus there won't be any $(z-0)^{\alpha_{i,k}}$ terms in the elementary divisors. Also, by $d_i(z)|d_{i+1}(z)$ for all $i=1,\ldots,n-1$, we can see that the Smith Normal Form $\mathsf{D}(z)$ of $\mathsf{A}(z)$ is uniquely determined by its elementary divisors.

As to the determinant, observe that $\det(\mathsf{A}(z)) = \det(\mathsf{E}(z))\det(\mathsf{D}(z))\det(\mathsf{F}(z))$. Since $\det(\mathsf{E}(z))$ and $\det(\mathsf{F}(z))$ are both nonzero monomials, we can see that the determinant of $\mathsf{A}(z)$ is essentially the product of all its invariant polynomials, or the product of all its elementary divisors, up to a multiplication by some nonzero monomials:

$$\det(\mathsf{A}(z)) = c_A z^{k_A} \prod_{i=1}^{n} d_i(z) = c_A z^{k_A} \prod_{i=1}^{n}\prod_{k=1}^{n_i} (z - z_{i,k})^{\alpha_{i,k}}, \tag{2.3.12}$$

for some nonzero constant $c_A \in \mathbb{C}$, and some integer $k_A \in \mathbb{Z}$. Hence, all the roots information of $\det(\mathsf{A}(z))$ is contained in the roots of the elementary divisors.

Suppose $\mathsf{A}(z)$ is an $n\times n$ Hermitian matrix of Laurent polynomials. Denote its invariant polynomials as $d_1(z), d_2(z), \ldots, d_n(z)$, then there exist invertible matrices of Laurent polynomials $\mathsf{E}(z)$ and $\mathsf{F}(z)$, such that

$$\mathsf{A}(z) = \mathsf{E}(z)\,\mathrm{diag}(d_1(z), d_2(z), \cdots, d_n(z))\mathsf{F}(z).$$

Multiply $\mathsf{E}^{-1}(z)$ and $\mathsf{E}^{-\star}(z)$ on the left and the right side of $\mathsf{A}(z)$ respectively, we get:

$$\mathring{\mathsf{A}}(z) := \mathsf{E}^{-1}(z)\mathsf{A}(z)\mathsf{E}^{-\star}(z) = \begin{bmatrix} d_1(z) & & & \\ & d_2(z) & & \\ & & \ddots & \\ & & & d_n(z) \end{bmatrix} \mathsf{F}(z)\mathsf{E}^{-\star}(z). \tag{2.3.13}$$

Since $\mathsf{A}(z)$ is Hermitian, the matrix $\mathring{\mathsf{A}}(z)$ is also a Hermitian matrix of Laurent polynomials. From the above equation we can see that for each invariant polynomial $d_i(z)$ of $\mathsf{A}(z)$, $d_i(z)$ divides the i-th row of $\mathring{\mathsf{A}}(z)$, $i=1,2,\ldots,n$.

Now we are ready to prove the following theorem.

Theorem 2.3.4. *Let* $\mathsf{A}(z) = \sum_{k=-L}^{L} A(k)z^k$ *be an* $n \times n$ *matrix of Laurent polynomials, such that* $\mathsf{A}(z)$ *is Hermitian, and* $\operatorname{len}(\det(A(z))) > 0$*. If* $\mathsf{A}(z)$ *has some elementary divisor* $(z - z_0)^\alpha$ *satisfying either one of the two situations:*

(1) $z_0 \in \mathbb{C} \setminus \mathbb{T} \setminus \{0\}$*, and* $\alpha \geqslant 1$*;*

(2) $z_0 \in \mathbb{T}$*, and* $\alpha \geqslant 2$*;*

then there exist two $n \times n$ *matrices* $\mathsf{U}(z)$ *and* $\widetilde{\mathsf{A}}(z)$ *of Laurent polynomials, such that*

$$\mathsf{A}(z) = \mathsf{U}(z)\widetilde{\mathsf{A}}(z)\mathsf{U}^\star(z),$$

where $\widetilde{\mathsf{A}}^\star(z) = \widetilde{\mathsf{A}}(z)$*, and* $\operatorname{len}(\det(\widetilde{\mathsf{A}}(z))) \leqslant \operatorname{len}(\det(\mathsf{A}(z))) - 2$*.*

Proof. Denote the invariant polynomials of $\mathsf{A}(z)$ as $d_1(z), d_2(z), \ldots, d_n(z)$, then there exist invertible matrices $\mathsf{E}(z)$ and $\mathsf{F}(z)$ of Laurent polynomials, such that

$$\mathsf{A}(z) = \mathsf{E}(z)\operatorname{diag}(d_1(z), d_2(z), \cdots, d_n(z))\mathsf{F}(z).$$

Since $(z - z_0)^\alpha$ is an elementary divisor of $\mathsf{A}(z)$, we know that there exists some $d_k(z)$, such that $(z - z_0)^\alpha | d_k(z)$. Define $\mathring{\mathsf{A}}(z)$ as (2.3.13), we can see that $d_k(z)$ divides the k-th row of $\mathring{\mathsf{A}}(z)$. So $(z - z_0)^\alpha$ also divides the k-th row of $\mathring{\mathsf{A}}(z)$. Moreover, by the definition (2.3.13), we can see that $\mathring{\mathsf{A}}(z)$ is also Hermitian, which implies that $((z - z_0)^\alpha)^\star = (z^{-1} - \overline{z_0})^\alpha = (-\overline{z_0})^\alpha z^{-\alpha}(z - \overline{z_0}^{-1})^\alpha$ divides the k-th column of $\mathring{\mathsf{A}}(z)$. Since we only care about the monic divisors, it is equivalent to say $(z - \overline{z_0}^{-1})^\alpha$ divides the k-th column of $\mathring{\mathsf{A}}(z)$.

For the first situation in the theorem, we have $z_0 \in \mathbb{C} \setminus \mathbb{T} \setminus \{0\}$. $z_0 \notin \mathbb{T}$ gives us $|z_0|^2 \neq 1$, which implies that $\overline{z_0}^{-1} \neq z_0$. So $(z - z_0)^\alpha$ and $(z - \overline{z_0}^{-1})^\alpha$ are different polynomials. Since they divide the k-th row and the k-th column of $\mathring{\mathsf{A}}(z)$ respectively, we can see that $(z - z_0)^\alpha(z - \overline{z_0}^{-1})^\alpha$ (or equivalently $((z - z_0)^\alpha)^\star (z - z_0)^\alpha$) divides the (k, k) element of the matrix $\mathring{\mathsf{A}}(z)$. So we can factor out $(z - z_0)^\alpha$ from the k-th row and $((z - z_0)^\alpha)^\star$ from the k-th column of $\mathring{\mathsf{A}}(z)$ simultaneously. It yields the following factorization:

$$\mathring{\mathsf{A}}(z) = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & (z-z_0)^\alpha & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix} \widetilde{\mathsf{A}}(z) \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & (z-z_0)^\alpha & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}^\star,$$

where $\widetilde{\mathsf{A}}(z)$ is an $n \times n$ Hermitian matrix of Laurent polynomials. Use $\mathsf{D}_{k,\alpha}(z)$ to denote the diagonal matrix with the k-th diagonal element equal to $(z-z_0)^\alpha$, and all other diagonal elements equal to 1, i.e.,

$$\mathsf{D}_{k,\alpha}(z) := \operatorname{diag}(1, \dots, 1, (z-z_0)^\alpha, 1, \dots, 1). \tag{2.3.14}$$

The above factorization implies $\mathring{\mathsf{A}}(z) = \mathsf{E}^{-1}(z)\mathsf{A}(z)\mathsf{E}^{-\star}(z) = \mathsf{D}_{k,\alpha}(z)\widetilde{\mathsf{A}}(z)\mathsf{D}^\star_{k,\alpha}(z)$. So $\mathsf{A}(z)$ can be written as

$$\mathsf{A}(z) = \mathsf{E}(z)\mathring{\mathsf{A}}(z)\mathsf{E}^\star(z) = \mathsf{E}(z)\mathsf{D}_{k,\alpha}(z)\widetilde{\mathsf{A}}(z)\mathsf{D}^\star_{k,\alpha}(z)\mathsf{E}^\star(z).$$

Let $\mathsf{U}(z) := \mathsf{E}(z)\mathsf{D}_{k,\alpha}(z)$, we get $\mathsf{A}(z) = \mathsf{U}(z)\widetilde{\mathsf{A}}(z)\mathsf{U}^\star(z)$.

Since $\det(\mathsf{E}(z))$ is a nonzero monomial, $\det(\mathsf{U}(z)) = \det(\mathsf{E}(z))\det(\mathsf{D}_{k,\alpha}(z)) = c_U z^{k_U}(z-z_0)^\alpha$ for some $c_U \neq 0$ and $k_U \in \mathbb{Z}$. Thus $\operatorname{len}(\det(\mathsf{U}(z))) = \alpha$. Also, $\operatorname{len}(\det(\mathsf{U}^\star(z))) = \operatorname{len}((\det(\mathsf{U}(z)))^\star) = \operatorname{len}(\det(\mathsf{U}(z))) = \alpha$. So

$$\begin{aligned} \operatorname{len}(\det(\widetilde{\mathsf{A}}(z))) &= \operatorname{len}(\det(\mathsf{A}(z))) - \operatorname{len}(\det(\mathsf{U}(z))) - \operatorname{len}(\det(\mathsf{U}^\star(z))) \\ &= \operatorname{len}(\det(\mathsf{A}(z))) - \alpha - \alpha \leqslant \operatorname{len}(\det(\mathsf{A}(z))) - 2. \end{aligned}$$

For the second situation in the theorem, let $\beta := \lfloor \alpha/2 \rfloor$, which implies $\beta \geqslant 1$ and $2\beta \leqslant \alpha$. As we discussed before, $(z-z_0)^\alpha$ divides the k-th row of $\mathring{\mathsf{A}}(z)$, and $(z-\overline{z_0}^{-1})^\alpha$ divides the k-th column of $\mathring{\mathsf{A}}(z)$. Notice that for $z_0 \in \mathbb{T}$, $|z_0|^2 = 1$ implies $\overline{z_0}^{-1} = z_0$. So $(z-\overline{z_0}^{-1})^\alpha$ and $(z-z_0)^\alpha$ are the same polynomial. We have $(z-z_0)^\alpha$ divides both the k-th row and the k-th column of $\mathring{\mathsf{A}}(z)$. From $\beta \leqslant \alpha$, we can see that $(z-z_0)^\beta$ divides the k-th row and $((z-z_0)^\beta)^\star = (-\overline{z_0})^\beta z^{-\beta}(z-\overline{z_0}^{-1})^\beta = (-\overline{z_0})^\beta z^{-\beta}(z-z_0)^\beta$ divides the

k-th column of $\mathring{\mathsf{A}}(z)$. Moreover, since $2\beta \leqslant \alpha$, we get $(z-z_0)^{2\beta}$ divides the (k,k) element of $\mathring{\mathsf{A}}(z)$. So we can factor out $(z-z_0)^{\beta}$ from the k-th row and $((z-z_0)^{\beta})^{\star}$ from the k-th column at the same time, to get

$$\mathring{\mathsf{A}}(z) = \mathsf{D}_{k,\beta}(z)\widetilde{\mathsf{A}}(z)\mathsf{D}^{\star}_{k,\beta}(z),$$

where $\widetilde{\mathsf{A}}(z)$ is an $n \times n$ Hermitian matrix of Laurent polynomials, and $\mathsf{D}_{k,\beta}(z)$ is defined as (2.3.14). Similar to the previous situation, we get the factorization of $\mathsf{A}(z)$: $\mathsf{A}(z) = \mathsf{E}(z)\mathring{\mathsf{A}}(z)\mathsf{E}^{\star}(z) = \mathsf{E}(z)\mathsf{D}_{k,\beta}(z)\widetilde{\mathsf{A}}(z)\mathsf{D}^{\star}_{k,\beta}(z)\mathsf{E}^{\star}(z)$.

Define $\mathsf{U}(z) := \mathsf{E}(z)\mathsf{D}_{k,\beta}(z)$, we have $\mathsf{A}(z) = \mathsf{U}(z)\widetilde{\mathsf{A}}(z)\mathsf{U}^{\star}(z)$. By similar discussions as the previous case, we can see $\mathrm{len}(\det(\mathsf{U}(z))) = \mathrm{len}(\det(\mathsf{D}_{k,\beta}(z))) + \mathrm{len}(\det(\mathsf{F}(z))) = \beta$. Also, $\mathrm{len}(\det(\mathsf{U}^{\star}(z))) = \mathrm{len}((\det(\mathsf{U}(z)))^{\star}) = \mathrm{len}(\det(\mathsf{U}(z))) = \beta$. So

$$\begin{aligned}\mathrm{len}(\det(\widetilde{\mathsf{A}}(z))) &= \mathrm{len}(\det(\mathsf{A}(z))) - \mathrm{len}(\det(\mathsf{U}(z))) - \mathrm{len}(\det(\mathsf{U}^{\star}(z))) \\ &= \mathrm{len}(\det(\mathsf{A}(z))) - \beta - \beta \leqslant \mathrm{len}(\det(\mathsf{A}(z))) - 2.\end{aligned}$$

This completes the proof of the theorem. ■

The idea of extraction of elementary divisors is also used to factorize the positive semi-definite matrices of rational functions (for example, see [88]). Notice that the above theorem cannot extract the elementary divisor $(z-z_0)^{\alpha}$ if $z_0 \in \mathbb{T}$ and $\alpha = 1$. Fortunately, if $\mathsf{A}(z)$ is positive semi-definite for all $z \in \mathbb{T}$, this will never happen, since all its elementary divisors $(z-z_0)^{\alpha}$ with $z_0 \in \mathbb{T}$ will have even degree α (see Corollary 2.3.7). However, $\alpha = 1$ could happen if the matrix $\mathsf{A}(z)$ is not positive semi-definite. See the following example.

Example 2.1. *Consider the matrix*

$$\mathsf{A}(z) = \begin{bmatrix} z^{-1}(z-1)^2 & (z-1)(z+1) \\ (z^{-1}-1)(z^{-1}+1) & -z^{-1}(z-1)^2 \end{bmatrix}.$$

By direct calculations we can see $\mathsf{A}^{\star}(z) = \mathsf{A}(z)$, *and* $\det(\mathsf{A}(z)) = \frac{4(z-1)^2}{z} = -\mathsf{d}(z)\mathsf{d}^{\star}(z)$, *where* $\mathsf{d}(z) = 2(z-1)$. *So* $\det(\mathsf{A}(z)) \leqslant 0$ *for all* $z \in \mathbb{T}$. *Since the determinant is equal to the product of all the eigenvalues of* $\mathsf{A}(z)$, *we know that*

$\mathsf{A}(z)$ *always has 1 positive and 1 negative eigenvalue for all* $z \in \mathbb{T} \setminus \sigma(\mathsf{A}(z))$. *Hence,* $\mathrm{sig}(\mathsf{A}(z)) = 0$ *is constant for all* $z \in \mathbb{T} \setminus \sigma(\mathsf{A}(z))$.

As to the Smith Normal Form of $\mathsf{A}(z)$*, let*

$$\mathsf{E}(z) := \begin{bmatrix} -\frac{2z^3-4z^2-z+1}{z} & -2z(z-2) \\ \frac{2z^3-z-1}{z^2} & 2z \end{bmatrix}, \quad \mathsf{F}(z) := \begin{bmatrix} 1 & 2z^2 - z \\ -1 & -2z^2+z+1 \end{bmatrix},$$

$$\mathsf{D}(z) := \begin{bmatrix} z-1 & 0 \\ 0 & z-1 \end{bmatrix}.$$

We can directly check that $\mathsf{A}(z) = \mathsf{E}(z)\mathsf{D}(z)\mathsf{F}(z)$. $\mathsf{E}(z)$ *and* $\mathsf{F}(z)$ *are both invertible matrices since* $\det(\mathsf{E}(z)) = 4z^{-1}$*, and* $\det(\mathsf{F}(z)) = 1$. *So* $\mathsf{D}(z)$ *is the Smith Normal Form of* $\mathsf{A}(z)$. *We can see that* $\mathsf{A}(z)$ *contains 2 elementary divisors with degree equal to 1:* $(z-1)$.

The following theorem handles the $z_0 \in \mathbb{T}$ and $\alpha = 1$ case. It shows that if the signature of $\mathsf{A}(z)$ is constant for $z \in \mathbb{T} \setminus \sigma(\mathsf{A}(z))$, the elementary divisors with degree equal to 1 can still be extracted out.

Theorem 2.3.5. *Let* $\mathsf{A}(z) = \sum_{k=-L}^{L} A(k)z^k$ *be an* $n \times n$ *matrix of Laurent polynomials, such that* $\mathsf{A}(z)$ *is Hermitian, and* $\mathrm{len}(\det(A(z))) > 0$. *If* $\mathsf{A}(z)$ *also satisfies:*

(1) $\mathrm{sig}(A(z))$ *is constant for all* $z \in \mathbb{T} \setminus \sigma(\mathsf{A}(z))$*;*

(2) there exists some $z_0 \in \sigma(\mathsf{A}(z)) \bigcap \mathbb{T}$*, and all the elementary divisors of* $\mathsf{A}(z)$ *with root* z_0 *have degree equal to* 1*;*

then there exist two $n \times n$ *matrices* $\mathsf{U}(z)$ *and* $\widetilde{\mathsf{A}}(z)$ *of Laurent polynomials, such that*

$$\mathsf{A}(z) = \mathsf{U}(z)\widetilde{\mathsf{A}}(z)\mathsf{U}^\star(z),$$

where $\widetilde{\mathsf{A}}^\star(z) = \widetilde{\mathsf{A}}(z)$*, and* $\mathrm{len}(\det(\widetilde{\mathsf{A}}(z))) \leqslant \mathrm{len}(\det(\mathsf{A}(z))) - 2$.

We need the following two lemmas to prove the Theorem 2.3.5.

Lemma 2.3.4. *Let $A(\xi)$ be an $n \times n$ matrix of analytic functions depending on $\xi \in \mathbb{R}$ (or $\mathbb{C}$). For some $\xi_0 \in \mathbb{R}$ (or $\mathbb{C}$), suppose $A(\xi)$ can be factorized as following in some neighborhood of ξ_0:*

$$A(\xi) = E_{\xi_0}(\xi) \begin{bmatrix} (\xi-\xi_0)^{\alpha_1} & & & \\ & (\xi-\xi_0)^{\alpha_2} & & \\ & & \ddots & \\ & & & (\xi-\xi_0)^{\alpha_n} \end{bmatrix} F_{\xi_0}(\xi),$$

where $E_{\xi_0}(\xi)$ and $F_{\xi_0}(\xi)$ satisfy:

(1) *$E_{\xi_0}(\xi)$ and $F_{\xi_0}(\xi)$ are both $n \times n$ analytic matrices in some neighborhood of ξ_0;*

(2) *$E_{\xi_0}(\xi_0)$ and $F_{\xi_0}(\xi_0)$ are both nonsingular;*

(3) *the integer sequence $\{\alpha_j\}_{j=1}^n$ is nondecreasing, i.e., $0 \leqslant \alpha_1 \leqslant \ldots \leqslant \alpha_n$.*

Then the sequence $\{\alpha_j\}_{j=1}^n$ is unique (independent of the factorization we use). We call it the **partial multiplicities** *of $A(\xi)$ at ξ_0.*

Before the proof, we provide a simple fact about analytic matrices: If $C(\xi)$ is an $n \times n$ matrix, which is analytic in some neighborhood of ξ_0, and $\det(C(\xi_0)) \neq 0$, then $C^{-1}(\xi) = \frac{1}{\det(C(\xi))} \operatorname{adj}(C(\xi))$ is also an analytic matrix in some neighborhood of ξ_0.

Proof. Suppose we have the following two different factorizations of $A(\xi)$, both satisfy the 3 conditions in the lemma:

$$\begin{aligned} A(\xi) &= E_{\xi_0}(\xi) \begin{bmatrix} (\xi-\xi_0)^{\alpha_1} & & \\ & \ddots & \\ & & (\xi-\xi_0)^{\alpha_n} \end{bmatrix} F_{\xi_0}(\xi) \\ &= \widetilde{E}_{\xi_0}(\xi) \begin{bmatrix} (\xi-\xi_0)^{\widetilde{\alpha}_1} & & \\ & \ddots & \\ & & (\xi-\xi_0)^{\widetilde{\alpha}_n} \end{bmatrix} \widetilde{F}_{\xi_0}(\xi). \end{aligned}$$

Then we have

$$\begin{aligned}
&\begin{bmatrix} (\xi-\xi_0)^{\widetilde{\alpha}_1} & & \\ & \ddots & \\ & & (\xi-\xi_0)^{\widetilde{\alpha}_n} \end{bmatrix} \\
=&\widetilde{E}_{\xi_0}^{-1}(\xi)E_{\xi_0}(\xi)\begin{bmatrix} (\xi-\xi_0)^{\alpha_1} & & \\ & \ddots & \\ & & (\xi-\xi_0)^{\alpha_n} \end{bmatrix} F_{\xi_0}(\xi)\widetilde{F}_{\xi_0}^{-1}(\xi) \\
=&P(\xi)\begin{bmatrix} (\xi-\xi_0)^{\alpha_1} & & \\ & \ddots & \\ & & (\xi-\xi_0)^{\alpha_n} \end{bmatrix} Q(\xi),
\end{aligned} \tag{2.3.15}$$

where $P(\xi) := \widetilde{E}_{\xi_0}^{-1}(\xi)E_{\xi_0}(\xi)$ and $Q(\xi) := F_{\xi_0}(\xi)\widetilde{F}_{\xi_0}^{-1}(\xi)$. By the fact we mentioned before the proof, $P(\xi)$ and $Q(\xi)$ are both analytic matrices in some neighborhood of ξ_0.

For all $k \leqslant n$, check the top left $k \times k$ submatrix of (2.3.15):

$$\begin{aligned}
&\begin{bmatrix} (\xi-\xi_0)^{\widetilde{\alpha}_1} & & \\ & \ddots & \\ & & (\xi-\xi_0)^{\widetilde{\alpha}_k} \end{bmatrix} \\
=&P_{r,1:k}(\xi)\begin{bmatrix} (\xi-\xi_0)^{\alpha_1} & & \\ & \ddots & \\ & & (\xi-\xi_0)^{\alpha_n} \end{bmatrix} Q_{c,1:k}(\xi) = R_k(\xi)Q_{c,1:k}(\xi),
\end{aligned} \tag{2.3.16}$$

where $P_{r,1:k}(\xi)$ is the $k \times n$ submatrix of $P(\xi)$, constructed by taking the first k rows of $P(\xi)$, and $Q_{c,1:k}(\xi)$ is the $n \times k$ submatrix of $Q(\xi)$, constructed by taking the first k columns of $Q(\xi)$. $R_k(\xi) := P_{r,1:k}(\xi)\,\mathrm{diag}((\xi-\xi_0)^{\alpha_1}, \dots, (\xi-\xi_0)^{\alpha_n})$ is a $k \times n$ matrix. From the definition, we can see that the s-th column of $R_k(\xi)$ is in $\mathcal{O}((\xi-\xi_0)^{\alpha_s})$ as $\xi \to \xi_0$, for all $s = 1, \dots, n$.

Taking the determinant of (2.3.16), by the Cauchy-Binet Formula,

$$(\xi-\xi_0)^{\widetilde{\alpha}_1+\widetilde{\alpha}_2\ldots+\widetilde{\alpha}_k} = \det\begin{bmatrix}(\xi-\xi_0)^{\widetilde{\alpha}_1} & & \\ & \ddots & \\ & & (\xi-\xi_0)^{\widetilde{\alpha}_k}\end{bmatrix}$$
$$= \sum_{J\subseteq\{1,2,\ldots,n\},}^{|J|=k} \det([R_k]_{c,J}(\xi))\det([Q_{c,1:k}]_{r,J}(\xi)), \qquad (2.3.17)$$

where $[R_k]_{c,J}(\xi)$ is the $k\times k$ submatrix of $R_k(\xi)$, constructed by taking the columns with indices belonging to J; $[Q_{c,1:k}]_{r,J}(\xi)$ is the $k\times k$ submatrix of $Q_{c,1:k}(\xi)$, constructed by taking the rows with indices belonging to J. The summation is taken over all indices sets J, whose size is equal to k. Since all the elements in the s-th column of $[R_k]_{c,J}(\xi)$ are in $\mathcal{O}((\xi-\xi_0)^{\alpha_s})$ as $\xi\to\xi_0$, for all $s=1,\ldots,n$, and the sequence $\{\alpha_j\}_{j=1}^n$ is nondecreasing, we can see

$$\mathsf{Z}(\det([R_k]_{c,J}(\xi)),\xi_0)\geqslant \alpha_1+\ldots+\alpha_k, \qquad \text{for all } \; J\subseteq\{1,\ldots,n\},\; |J|=k.$$

Hence, each term in the summation on the right-hand-side of (2.3.17) is in $\mathcal{O}((\xi-\xi_0)^{\alpha_1+\ldots+\alpha_k})$, as $\xi\to\xi_0$. So $\mathsf{Z}((\xi-\xi_0)^{\widetilde{\alpha}_1+\ldots+\widetilde{\alpha}_k},\xi_0)\geqslant \alpha_1+\ldots+\alpha_k$. This implies

$$\alpha_1+\ldots+\alpha_k\leqslant \widetilde{\alpha}_1+\ldots+\widetilde{\alpha}_k, \qquad \text{for all } \; k=1,2,\ldots,n.$$

Similarly, we can prove $\widetilde{\alpha}_1+\ldots+\widetilde{\alpha}_k\leqslant \alpha_1+\ldots+\alpha_k$ also holds for all $k=1,\ldots,n$. The two inequalities give that

$$\widetilde{\alpha}_1+\ldots+\widetilde{\alpha}_k = \alpha_1+\ldots+\alpha_k, \qquad \text{for all } \; k=1,2,\ldots,n.$$

So $\{\alpha_j\}_{j=1}^n$ and $\{\widetilde{\alpha}_j\}_{j=1}^n$ are the same sequence. ■

Lemma 2.3.5. *Let $A(\xi)$ be an $n\times n$ matrix of analytic functions depending on $\xi\in\mathbb{R}$. If $A(\xi)$ is Hermitian, i.e., $(A(\xi))^\star = A(\xi)$, for all $\xi\in\mathbb{R}$, then there exists an $n\times n$ matrix $U(\xi)$ and scalar functions $\lambda_1(\xi),\lambda_2(\xi),\ldots,\lambda_n(\xi)$, satisfying*

(1) $U(\xi)$ *and* $\lambda_1(\xi), \lambda_2(\xi), \ldots, \lambda_n(\xi)$ *are all analytic for* $\xi \in \mathbb{R}$*;*

(2) $U(\xi)$ *is unitary, i.e.,* $U(\xi)(U(\xi))^\star = \mathbf{I_n}$ *for all* $\xi \in \mathbb{R}$*;*

and the factorization

$$A(\xi) = U(\xi) \begin{bmatrix} \lambda_1(\xi) & & & \\ & \lambda_2(\xi) & & \\ & & \ddots & \\ & & & \lambda_n(\xi) \end{bmatrix} U^\star(\xi)$$

holds for all $\xi \in \mathbb{R}$*.*

Lemma 2.3.5 is due to Rellich in his study of perturbation theory. The proof of it can be found in [30, 59]. This lemma tells us that the eigenvalues and eigenvectors of an analytic Hermitian matrix can also be expressed as analytic functions. Also, since all the eigenvalues of a Hermitian matrix are real, we know that $\lambda_1(\xi), \ldots, \lambda_n(\xi)$ in the above lemma are all real functions.

Furthermore, since all the eigenvalues $\lambda_1(\xi), \ldots, \lambda_n(\xi)$ of $A(\xi)$ in Lemma 2.3.5 are continuous functions of $\xi \in \mathbb{R}$, we know that if $\det(A(\xi)) \neq 0$ for ξ belongs to some interval (c_1, c_2), then all the eigenvalues $\lambda_j(\xi)$ will not change signs, i.e., $\mathrm{sig}(A(\xi))$ will be a constant for $\xi \in (c_1, c_2)$.

By Lemma 2.3.4 and Lemma 2.3.5, we can build the following theorem.

Theorem 2.3.6. *Suppose* $\mathsf{A}(z)$ *is an* $n \times n$ *matrix of Laurent polynomials, such that* $\mathsf{A}(z)$ *is Hermitian, i.e.,* $\mathsf{A}^\star(z) = \mathsf{A}(z)$*. Take* $z_0 = e^{-i\xi_0} \in \mathbb{T}$*, where* $\xi_0 \in \mathbb{R}$*. Let* $\mathsf{d}_1(z), \ldots, \mathsf{d}_n(z)$ *be the invariant polynomials of* $\mathsf{A}(z)$*, and define the sequence* $\{\alpha_j\}_{j=1}^n$ *by*

$$\alpha_j = \mathsf{Z}(\mathsf{d}_j(z), z_0), \qquad j = 1, \ldots, n.$$

Also, let $\lambda_1(\xi), \ldots, \lambda_n(\xi)$ *be the eigenvalues of the matrix* $\mathsf{A}(e^{-i\xi})$*, chosen according to Lemma 2.3.5. That is,* $\lambda_1(\xi), \ldots, \lambda_n(\xi)$ *are analytic functions for all* $\xi \in \mathbb{R}$*. Define the sequence* $\{\beta_j\}_{j=1}^n$ *by*

$$\beta_j = \mathsf{Z}(\lambda_j(\xi), \xi_0), \qquad j = 1, \ldots, n.$$

Without loss of generality, we can assume $\beta_1 \leqslant \ldots \leqslant \beta_n$.

Then the sequence $\{\alpha_j\}_{j=1}^n$ and the sequence $\{\beta_j\}_{j=1}^n$ are the same.

Proof. The invariant polynomials $\mathsf{d}_j(z)|\mathsf{d}_{j+1}(z)$ hold for all $j = 1, 2, \ldots, n-1$. Hence, we know that $\alpha_1 \leqslant \ldots \leqslant \alpha_n$.

Also, we can write $\mathsf{A}(z)$ into its Smith Normal Form. We know that there exist $n \times n$ invertible matrices of Laurent polynomials $\mathsf{E}(z)$ and $\mathsf{F}(z)$, such that

$$\mathsf{A}(z) = \mathsf{E}(z)\,\mathrm{diag}(\mathsf{d}_1(z), \mathsf{d}_2(z), \ldots, \mathsf{d}_n(z))\mathsf{F}(z). \tag{2.3.18}$$

Take $z = e^{-i\xi}$, $\xi \in \mathbb{R}$, we can see that all the invariant polynomials $\mathsf{d}_j(e^{-i\xi})$ are analytic functions of ξ, and $\mathsf{Z}(\mathsf{d}_j(e^{-i\xi}), \xi_0) = \mathsf{Z}(\mathsf{d}_j(z), z_0) = \alpha_j$, for all $j = 1, 2, \ldots, n$.

Denote $\mathsf{d}_j(e^{-i\xi}) = (\xi - \xi_0)^{\alpha_j}\widetilde{d}_j(\xi)$, then $\widetilde{d}_j(\xi_0) \neq 0$. We can rewrite equation (2.3.18) as

$$\begin{aligned}
\mathsf{A}(e^{-i\xi}) =& \mathsf{E}(e^{-i\xi})\,\mathrm{diag}(\mathsf{d}_1(e^{-i\xi}), \mathsf{d}_2(e^{-i\xi}), \ldots, \mathsf{d}_n(e^{-i\xi}))\mathsf{F}(e^{-i\xi}) \\
=& \mathsf{E}(e^{-i\xi}) \begin{bmatrix} (\xi-\xi_0)^{\alpha_1} & & \\ & \ddots & \\ & & (\xi-\xi_0)^{\alpha_n} \end{bmatrix} \begin{bmatrix} \widetilde{d}_1(\xi) & & \\ & \ddots & \\ & & \widetilde{d}_n(\xi) \end{bmatrix} \mathsf{F}(e^{-i\xi}) \\
=& E_{\xi_0}(\xi) \begin{bmatrix} (\xi-\xi_0)^{\alpha_1} & & \\ & \ddots & \\ & & (\xi-\xi_0)^{\alpha_n} \end{bmatrix} F_{\xi_0}(\xi),
\end{aligned}$$

where $E_{\xi_0}(\xi) := \mathsf{E}(e^{-i\xi})$, and $F_{\xi_0}(\xi) := \mathrm{diag}(\widetilde{d}_1(\xi), \cdots, \widetilde{d}_n(\xi))\mathsf{F}(e^{-i\xi})$. From the definition, $E_{\xi_0}(\xi)$ and $F_{\xi_0}(\xi)$ are both analytic matrices, and $\det(E_{\xi_0}(\xi_0)) \neq 0$, $\det(F_{\xi_0}(\xi_0)) \neq 0$. Hence, the matrices $E_{\xi_0}(\xi), F_{\xi_0}(\xi)$ and the sequence $\{\alpha_j\}_{j=1}^n$ satisfy all the conditions in Lemma 2.3.4. So the partial multiplicities of $\mathsf{A}(e^{-i\xi})$ at ξ_0 are $\{\alpha_j\}_{j=1}^n$.

By Lemma 2.3.5, the analytic Hermitian matrix $\mathsf{A}(e^{-i\xi})$ can also be factorized as

$$\mathsf{A}(e^{-i\xi}) = W(\xi)\begin{bmatrix}\lambda_1(\xi) & & \\ & \ddots & \\ & & \lambda_n(\xi)\end{bmatrix}(W(\xi))^\star, \tag{2.3.19}$$

where $W(\xi)$ is a unitary analytic matrix and the eigenvalues $\lambda_1(\xi),\ldots,\lambda_n(\xi)$ are analytic functions of $\xi\in\mathbb{R}$. Since $\lambda_k(\xi)$ can be reordered by permutations of the columns of $W(\xi)$, we can just assume that $\beta_1\leqslant\beta_2\leqslant\ldots\leqslant\beta_n$.

Since $\beta_j = \mathsf{Z}(\lambda_j(\xi),\xi_0)$, we can denote $\lambda_j(\xi) = (\xi-\xi_0)^{\beta_j}f_j(\xi)$, where $f_j(\xi_0)\neq 0$, for all $j=1,\ldots,n$. The factorization (2.3.19) becomes

$$\begin{aligned}\mathsf{A}(e^{-i\xi}) =& W(\xi)\begin{bmatrix}(\xi-\xi_0)^{\beta_1} & & \\ & \ddots & \\ & & (\xi-\xi_0)^{\beta_n}\end{bmatrix}\begin{bmatrix}f_1(\xi) & & \\ & \ddots & \\ & & f_n(\xi)\end{bmatrix}(W(\xi))^\star\\ =&\widetilde{E}_{\xi_0}(\xi)\begin{bmatrix}(\xi-\xi_0)^{\beta_1} & & \\ & \ddots & \\ & & (\xi-\xi_0)^{\beta_n}\end{bmatrix}\widetilde{F}_{\xi_0}(\xi)\end{aligned}$$

where $\widetilde{E}_{\xi_0}(\xi) := W(\xi)$, and $\widetilde{F}_{\xi_0}(\xi) := \operatorname{diag}(f_1(\xi),\cdots,f_n(\xi))(W(\xi))^\star$. From the definition, $\widetilde{E}_{\xi_0}(\xi)$ and $\widetilde{F}_{\xi_0}(\xi)$ are both analytic matrices, and $\det(\widetilde{E}_{\xi_0}(\xi_0))\neq 0$, $\det(\widetilde{F}_{\xi_0}(\xi_0))\neq 0$. Hence, the matrices $\widetilde{E}_{\xi_0}(\xi),\widetilde{F}_{\xi_0}(\xi)$ and the sequence $\{\beta_j\}_{j=1}^n$ satisfy all the conditions in Lemma 2.3.4. So $\{\beta_j\}_{j=1}^n$ are also the partial multiplicities of $\mathsf{A}(e^{-i\xi})$ at ξ_0. By Lemma 2.3.4, we know that $\{\beta_j\}_{j=1}^n = \{\alpha_j\}_{j=1}^n$. This completes the proof of the lemma. ■

For a Hermitian matrix $\mathsf{A}(z)$ of Laurent polynomials, although we know that the eigenvalues $\lambda_1(\xi),\ldots,\lambda_n(\xi)$ of $\mathsf{A}(e^{-i\xi})$ are analytic functions of $\xi\in\mathbb{R}$, we cannot expect them to be Laurent polynomials in z-domain. Actually, the following example shows that the analytic functions $\lambda_1(\xi),\ldots,\lambda_n(\xi)$ might not be 2π-periodic functions of $\xi\in\mathbb{R}$.

Example 2.2. *Consider the same matrix $\mathsf{A}(z)$ as in Example 2.1. Solving $\det(\mathsf{A}(e^{-i\xi})-\lambda\mathbf{I_2})=0$, we can find two analytic functions that are eigenvalues of $\mathsf{A}(e^{-i\xi})$: $\lambda_{1,2}(\xi) = \pm 4\sin(\xi/2)$. They are both 4π-periodic functions of $\xi\in\mathbb{R}$, and we cannot find 2 eigenvalues of $\mathsf{A}(e^{-i\xi})$ that are both analytic and*

2π-periodic functions of $\xi \in \mathbb{R}$. Also, as calculated in Example 2.1, the 2 invariant polynomials of $\mathsf{A}(z)$ are $\mathsf{d}_1(z) = \mathsf{d}_2(z) = z - 1$. Take $\xi_0 = 0$ and $z_0 = 1$, we can calculate $\alpha_j := \mathsf{Z}(\mathsf{d}_j(z), 1) = 1$ and $\beta_j := \mathsf{Z}(\lambda_j(\xi), 0) = 1$, $j = 1, 2$.

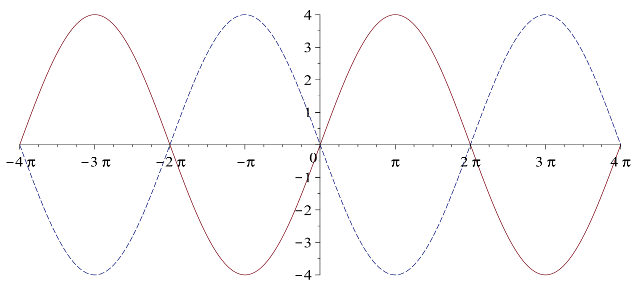


Figure 2.1: Plot of the eigenvalue functions in Example 2.2. The solid line is $\lambda_1(\xi) = 4\sin(\xi/2)$, the dashed line is $\lambda_2(\xi) = -4\sin(\xi/2)$.

Now, we are ready to prove the Theorem 2.3.5.

Proof of Theorem 2.3.5. Denote the invariant polynomials of the matrix $\mathsf{A}(z)$ by $\mathsf{d}_1(z), \dots, \mathsf{d}_n(z)$. For $z_0 \in \sigma(\mathsf{A}(z)) \bigcap \mathbb{T}$, from (2.3.12) we can see that there exists some $k \in \{1, 2, \dots, n\}$, such that $(z - z_0) | \mathsf{d}_k(z)$. Define the sequence $\{\alpha_j\}_{j=1}^n$ by

$$\alpha_j := \mathsf{Z}(\mathsf{d}_j(z), z_0), \qquad j = 1, 2, \dots, n.$$

From the condition (2), we know all $\alpha_j \leqslant 1$. Also, by $\mathsf{d}_j(z) | \mathsf{d}_{j+1}(z)$ for all $j = 1, 2, \dots, n-1$, we know that $\alpha_1 \leqslant \dots \leqslant \alpha_n$. Thus the sequence must be

$$\{\alpha_j\}_{j=1}^n = \{0, \dots, 0, 1, \dots, 1\}.$$

Taking $z = e^{-i\xi}$, we get a matrix $\mathsf{A}(e^{-i\xi})$ that is analytic of $\xi \in \mathbb{R}$. By Lemma 2.3.5, the analytic Hermitian matrix $\mathsf{A}(e^{-i\xi})$ can also be factorized as

$$\mathsf{A}(e^{-i\xi}) = W(\xi) \begin{bmatrix} \lambda_1(\xi) & & \\ & \ddots & \\ & & \lambda_n(\xi) \end{bmatrix} (W(\xi))^\star, \tag{2.3.20}$$

where $W(\xi)$ is a unitary analytic matrix and $\lambda_1(\xi), \ldots, \lambda_n(\xi)$ are analytic functions of $\xi \in \mathbb{R}$.

Since $z_0 \in \mathbb{T}$, we can find some $\xi_0 \in [-\pi, \pi)$, such that $z_0 = e^{-i\xi_0}$, where i is the imaginary unit. Define the sequence $\{\beta_j\}_{j=1}^n$ by

$$\beta_j := \mathsf{Z}(\lambda_j(\xi), \xi_0), \qquad \text{for all} \ \ j = 1, \ldots, n.$$

Without loss of generality, we can choose the factorization such that $\beta_1 \leqslant \ldots \leqslant \beta_n$. According to Theorem 2.3.6, we can see that

$$\{\beta_j\}_{j=1}^n = \{\alpha_j\}_{j=1}^n = \{0, \ldots, 0, 1, \ldots, 1\}.$$

Let K be the number of times "1" appears in $\{\beta_j\}_{j=1}^n$ or $\{\alpha_j\}_{j=1}^n$. Recall from the definition of $\{\alpha_j\}_{j=1}^n$ that each "1" corresponds to an elementary divisor $(z - z_0)$. So $K > 0$ is the number of times the elementary divisor $(z - z_0)$ appears. Let us see how the signs of the eigenvalues $\lambda_j(\xi)$ change from the left to the right side of ξ_0.

For $j = 1, \ldots, n - K$, we have $\beta_j = 0$. So $\lambda_j(\xi_0) \neq 0$. Since the eigenvalue $\lambda_j(\xi)$ is a continuous function of $\xi \in \mathbb{R}$, it will not change its sign between two sides of ξ_0, i.e., $\text{sign}(\lambda_j(\xi_0^-)) = \text{sign}(\lambda_j(\xi_0^+))$.

For $j = n - K + 1, \ldots, n$, we have $\beta_j = 1$. In this case, $\lambda_j(\xi_0) = 0$ and $\lambda_j'(\xi_0) \neq 0$. We know that the eigenvalues of a Hermitian matrix are all real, so $\lambda_j(\xi)$ is a real function of $\xi \in \mathbb{R}$. This implies that $\lambda_j'(\xi)$ is also a real function. Hence, $\lambda_j'(\xi_0)$ is a nonzero real number. We have the following two possible situations.

(1) If $\lambda_j'(\xi_0) > 0$, we know that $\lambda_j(\xi)$ is increasing near ξ_0. So

$$\lambda_j(\xi_0^-) < 0, \qquad \text{and} \quad \lambda_j(\xi_0^+) > 0.$$

(2) If $\lambda_j'(\xi_0) < 0$, we know that $\lambda_j(\xi)$ is decreasing near ξ_0. So

$$\lambda_j(\xi_0^-) > 0, \qquad \text{and} \quad \lambda_j(\xi_0^+) < 0.$$

Since the signature of $\mathsf{A}(z)$ is constant for all $z \in \mathbb{T} \backslash \sigma(\mathsf{A}(z))$, we know that the number of positive eigenvalues and the number of negative eigenvalues of $\mathsf{A}(e^{-i\xi})$ will remain unchanged between the two sides of ξ_0. So the above two cases must happen exactly the same number of times. That is, K has to be an even integer. And there are exactly $K/2$ number of $\lambda_j(\xi)$, such that $\lambda_j(\xi_0) = 0$ and $\lambda_j'(\xi_0) > 0$; meanwhile, there are exactly $K/2$ number of $\lambda_j(\xi)$, such that $\lambda_j(\xi_0) = 0$ and $\lambda_j'(\xi_0) < 0$. (The sign of $\lambda_j'(\xi_0)$ here are called the ***sign characteristic***. In [28], the authors introduced this concept in an algebraic way to study the self-adjoint matrix polynomials. In the light of Theorem 3.7 of [28], our definition here is a natural generalization of the concept into the Laurent polynomial matrices problem.)

Since $K > 0$, there exist some $j_1, j_2 \geqslant n - K + 1$, such that

$$\lambda_{j_1}(\xi_0) = 0, \qquad \lambda_{j_1}'(\xi_0) = \gamma_1^2 > 0, \qquad \lambda_{j_2}(\xi_0) = 0, \qquad \lambda_{j_2}'(\xi_0) = -\gamma_2^2 < 0,$$

for some real $\gamma_1, \gamma_2 \neq 0$. Thus, we can write $\lambda_{j_1}(\xi)$ and $\lambda_{j_2}(\xi)$ as

$$\lambda_{j_1}(\xi) = \gamma_1^2(\xi - \xi_0) + \mathcal{O}((\xi - \xi_0)^2), \qquad \lambda_{j_2}(\xi) = -\gamma_2^2(\xi - \xi_0) + \mathcal{O}((\xi - \xi_0)^2),$$

as $\xi \to \xi_0$.

In the eigenvalue decomposition (2.3.20), since $W(\xi)$ is unitary on $\xi \in \mathbb{R}$, it would be invertible at ξ_0. $W(\xi)$ is also an analytic matrix implies that $W^{-1}(\xi_0)W(\xi) = \mathbf{I_n} + \mathcal{O}((\xi - \xi_0))$, as $\xi \to \xi_0$. So, there exists an $n \times n$ analytic matrix $G(\xi)$, such that

$$W^{-1}(\xi_0)W(\xi) = \mathbf{I_n} + (\xi - \xi_0)G(\xi), \qquad \big(W^{-1}(\xi_0)W(\xi)\big)^\star = \mathbf{I_n} + (\xi - \xi_0)(G(\xi))^\star.$$

Multiply constant matrices $(W(\xi_0))^{-1}$ and $(W(\xi_0))^{-\star}$ on the left and the right side of (2.3.20) respectively, we define $\mathring{\mathsf{A}}(e^{-i\xi})$ as

$$\begin{aligned}\mathring{\mathsf{A}}(e^{-i\xi}) :=& W^{-1}(\xi_0)\mathsf{A}(e^{-i\xi})(W(\xi_0))^{-\star} \\ =& W^{-1}(\xi_0)W(\xi)\begin{bmatrix}\lambda_1(\xi) & & \\ & \ddots & \\ & & \lambda_n(\xi)\end{bmatrix}(W^{-1}(\xi_0)W(\xi))^\star\end{aligned}$$

$$
\begin{aligned}
&= (\mathbf{I_n} + (\xi-\xi_0)G(\xi)) \begin{bmatrix} \lambda_1(\xi) & & \\ & \ddots & \\ & & \lambda_n(\xi) \end{bmatrix} (\mathbf{I_n} + (\xi-\xi_0)(G(\xi))^\star) \\
&= \Lambda(\xi) + (\xi-\xi_0)G(\xi)\Lambda(\xi) + (\xi-\xi_0)\Lambda(\xi)(G(\xi))^\star \\
&\quad + (\xi-\xi_0)^2 G(\xi)\Lambda(\xi)(G(\xi))^\star, \qquad (2.3.21)
\end{aligned}
$$

where $\Lambda(\xi) := \mathrm{diag}(\lambda_1(\xi), \ldots, \lambda_n(\xi))$. Plugging in $\xi = \xi_0$, we can directly get

$$
\mathring{\mathsf{A}}(e^{-i\xi_0}) = \Lambda(\xi_0) = \begin{bmatrix} \lambda_1(\xi_0) & & & & & \\ & \ddots & & & & \\ & & \lambda_{n-K}(\xi_0) & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix}, \qquad (2.3.22)
$$

As we picked $j_1, j_2 \geqslant n-K+1$, the j_1-th and the j_2-th rows, as well as the j_1-th and the j_2-th columns of $\mathring{\mathsf{A}}(e^{-i\xi})$ are all in $\mathcal{O}((\xi-\xi_0))$, as $\xi \to \xi_0$.

Now, we will check the lower right $K \times K$ submatrix of $\mathring{\mathsf{A}}(e^{-i\xi})$ from (2.3.21). Since $\lambda_{n-K+1}(\xi), \ldots, \lambda_n(\xi)$ are in $\mathcal{O}((\xi-\xi_0))$, as $\xi \to \xi_0$, we can see that the lower right $K \times K$ submatrix of the second and the third term on the right-hand-side of (2.3.21) are both in $\mathcal{O}((\xi-\xi_0)^2)$, as $\xi \to \xi_0$. Hence, the summation of the 4 terms on the right-hand-side of (2.3.21) yields:

$$
\begin{aligned}
&\mathring{\mathsf{A}}_{(n-K+1):n,(n-K+1):n}(e^{-i\xi}) \\
&= \begin{bmatrix} \lambda_{n-K+1}(\xi) & & \\ & \ddots & \\ & & \lambda_n(\xi) \end{bmatrix} + \mathcal{O}((\xi-\xi_0)^2) + \mathcal{O}((\xi-\xi_0)^2) + \mathcal{O}((\xi-\xi_0)^2)
\end{aligned}
$$

$$= \begin{bmatrix} \lambda_{n-K+1}(\xi) & & & & & & \\ & \ddots & & & & & \\ & & \gamma_1^2(\xi-\xi_0) & & & & \\ & & & \ddots & & & \\ & & & & -\gamma_2^2(\xi-\xi_0) & & \\ & & & & & \ddots & \\ & & & & & & \lambda_n(\xi) \end{bmatrix} + \mathcal{O}((\xi-\xi_0)^2), \tag{2.3.23}$$

as $\xi \to \xi_0$. The $\gamma_1^2(\xi-\xi_0)$ and $-\gamma_2^2(\xi-\xi_0)$ terms appear at the j_1 and the j_2 diagonal positions respectively.

Now, we can use the following matrix V to cancel the first order term at the (j_1, j_1) position of $\mathring{\mathsf{A}}(e^{-i\xi})$. Define the $n \times n$ matrix V as

$$V := \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & \gamma_1^{-1} & & \gamma_2^{-1} & & \\ & & & \ddots & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}, \tag{2.3.24}$$

which corresponds to dividing the j_1-th row by γ_1, then adding γ_2^{-1} times the j_2-th row to the j_1-th row. Taking symmetric operations on both rows and columns of $\mathring{\mathsf{A}}(e^{-i\xi})$, we define $\breve{\mathsf{A}}(e^{-i\xi}) := V\mathring{\mathsf{A}}(e^{-i\xi})V^\star$.

From the definition, we can see that similar to $\mathring{\mathsf{A}}(e^{-i\xi})$, the j_1-th and the j_2-th rows, as well as the j_1-th and the j_2-th columns of $\breve{\mathsf{A}}(e^{-i\xi})$ are still in $\mathcal{O}((\xi-\xi_0))$, as $\xi \to \xi_0$. Also, the lower-right $K \times K$ submatrix of $\breve{\mathsf{A}}(e^{-i\xi})$ gives:

$\breve{\mathsf{A}}_{(n-K+1):n,(n-K+1):n}(e^{-i\xi})$

$$= \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & \gamma_1^{-1} & & \gamma_2^{-1} & & \\ & & & \ddots & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix} \mathring{\mathsf{A}}_{(n-K+1):n,(n-K+1):n}(e^{-i\xi}) \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & \gamma_1^{-1} & & & & \\ & & & \ddots & & & \\ & & \gamma_2^{-1} & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_{n-K+1}(\xi) & & & & & & \\ & \ddots & & & & & \\ & & 0 & & -\gamma_2(\xi-\xi_0) & & \\ & & & \ddots & & & \\ & & -\gamma_2(\xi-\xi_0) & & -\gamma_2^2(\xi-\xi_0) & & \\ & & & & & \ddots & \\ & & & & & & \lambda_n(\xi) \end{bmatrix} + \mathcal{O}((\xi-\xi_0)^2).$$

Thus, the (j_1, j_1) diagonal element of $\breve{\mathsf{A}}(e^{-i\xi})$ is in $\mathcal{O}((\xi-\xi_0)^2)$, as $\xi \to \xi_0$.

Finally, we can change back to z-domain. The matrix $\breve{\mathsf{A}}(z)$ of Laurent polynomials is written as

$$\breve{\mathsf{A}}(z) = V\mathring{\mathsf{A}}(z)V^\star = VW^{-1}(\xi_0)\mathsf{A}(z)W^{-\star}(\xi_0)V^\star.$$

Since the j_1-th row and the j_1-th column of $\breve{\mathsf{A}}(e^{-i\xi})$ are in $\mathcal{O}((\xi-\xi_0))$, we know that $(z-z_0)$ divides both the j_1-th row and the j_1-th column of $\breve{\mathsf{A}}(z)$. Also, the fact that the (j_1, j_1) element of $\breve{\mathsf{A}}(e^{-i\xi})$ is in $\mathcal{O}((\xi-\xi_0)^2)$ implies that $(z-z_0)^2$ divides the (j_1, j_1) element of $\breve{\mathsf{A}}(z)$. So we can factor out $(z-z_0)$ from the j_1-th row and $(z-z_0)^\star$ from the j_1-th column simultaneously, to get

$$\breve{\mathsf{A}}(z) = \mathsf{D}_{j_1,1}(z)\widetilde{\mathsf{A}}(z)\mathsf{D}_{j_1,1}^\star(z),$$

for some $n \times n$ Hermitian matrix $\widetilde{\mathsf{A}}(z)$ of Laurent polynomials, and $\mathsf{D}_{j_1,1}(z)$ is defined as (2.3.14). Thus, we have

$$\mathsf{A}(z) = W(\xi_0)V^{-1}\breve{\mathsf{A}}(z)V^{-\star}W^\star(\xi_0)$$

$$=W(\xi_0)V^{-1}\mathsf{D}_{j_1,1}(z)\widetilde{\mathsf{A}}(z)\mathsf{D}^{\star}_{j_1,1}(z)V^{-\star}W^{\star}(\xi_0) = \mathsf{U}(z)\widetilde{\mathsf{A}}(z)\mathsf{U}^{\star}(z),$$

where $\mathsf{U}(z) := W(\xi_0)V^{-1}\mathsf{D}_{j_1,1}(z)$.

$$\begin{aligned}\operatorname{len}(\det(\widetilde{\mathsf{A}}(z))) &= \operatorname{len}(\det(\mathsf{A}(z))) - \operatorname{len}(\det(\mathsf{U}(z))) - \operatorname{len}(\det(\mathsf{U}^{*}(z)))\\ &= \operatorname{len}(\det(\mathsf{A}(z))) - 2.\end{aligned}$$

So $\mathsf{U}(z)$ and $\widetilde{\mathsf{A}}(z)$ satisfy all the requirements of the theorem. This completes the proof of the Theorem 2.3.5. ■

From the above proof, we can see that under the assumptions of the Theorem 2.3.5, there will always be a pair of eigenvalues $\lambda_{j_1}(\xi)$ and $\lambda_{j_2}(\xi)$, that will both change signs across ξ_0. So the situation in the Theorem 2.3.5 can never happen if $\mathsf{A}(z)$ is positive semi-definite. We summarize this observation as the following corollary.

Corollary 2.3.7. *Suppose $\mathsf{A}(z)$ is a Hermitian matrix of Laurent polynomials, that is also positive semi-definite for all $z \in \mathbb{T}$. Then all its elementary divisors $(z - z_0)^{\alpha}$ with $z_0 \in \mathbb{T}$ will have even degree, i.e., $\alpha \in 2\mathbb{Z}$.*

Proof. Since $z_0 \in \mathbb{T}$, we can find some $\xi_0 \in \mathbb{R}$, such that $z_0 = e^{-i\xi_0}$. Suppose $\lambda_1(\xi), \ldots, \lambda_n(\xi)$ are the eigenvalues of $\mathsf{A}(e^{-i\xi})$, that are also analytic functions of $\xi \in \mathbb{R}$. Define the sequences $\{\alpha_j\}_{j=1}^{n}$ and $\{\beta_j\}_{j=1}^{n}$ as in Theorem 2.3.6. By Theorem 2.3.6, we can see that $\{\beta_j\}_{j=1}^{n} = \{\alpha_j\}_{j=1}^{n}$.

Since $\mathsf{A}(e^{-i\xi})$ is positive semi-definite for all $\xi \in \mathbb{R}$, that is, $\lambda_j(\xi)$ will not change sign across ξ_0, for all $j = 1, \ldots, n$, we know that

$$\beta_j = \mathsf{Z}(\lambda_j(\xi), \xi_0) \in 2\mathbb{Z}, \qquad \forall\ j = 1, \ldots, n.$$

So $\alpha_j \in 2\mathbb{Z}$ for all $j = 1, \ldots, n$. From the definition of α_j, we know that $\{\alpha_j\}_{j=1}^{n}$ are just the degrees of elementary divisors $(z - z_0)^{\alpha}$ in each invariant polynomial. So all such α satisfy $\alpha \in 2\mathbb{Z}$. ■

Now we are ready to prove Theorem 2.3.1.

Proof of Theorem 2.3.1. If $\operatorname{len}(\det(\mathsf{A}(z))) > 0$, from equation (2.3.12), we can see that $\mathsf{A}(z)$ has some elementary divisor $(z-z_0)^{\alpha}$, $z_0 \neq 0$. Let $\mathsf{A}_0(z) := \mathsf{A}(z)$.

For $j \geqslant 0$, if $\mathsf{A}_j(z)$ has some elementary divisor $(z - z_0)^\alpha$, with $z_0 \in \mathbb{C} \setminus \mathbb{T} \setminus \{0\}$, or $\alpha > 1$, apply the Theorem 2.3.4 to get a factorization of $\mathsf{A}_j(z)$ as $\mathsf{A}_j(z) = \mathsf{U}_{j+1}(z)\mathsf{A}_{j+1}(z)\mathsf{U}^\star_{j+1}(z)$, for some $n \times n$ matrices $\mathsf{U}_{j+1}(z)$ and $\mathsf{A}_{j+1}(z)$ of Laurent polynomials, satisfying $\mathsf{A}^\star_{j+1}(z) = \mathsf{A}_{j+1}(z)$, and $\operatorname{len}(\mathsf{A}_{j+1}(z)) < \operatorname{len}(\mathsf{A}_j(z))$. If all the elementary divisors $(z - z_0)^\alpha$ of $\mathsf{A}_j(z)$ has degree $\alpha = 1$ and $z_0 \in \mathbb{T}$, we can apply Theorem 2.3.5 to still get the factorization $\mathsf{A}_j(z) = \mathsf{U}_{j+1}(z)\mathsf{A}_{j+1}(z)\mathsf{U}^\star_{j+1}(z)$, with $\mathsf{A}^\star_{j+1}(z) = \mathsf{A}_{j+1}(z)$ and $\operatorname{len}(\mathsf{A}_{j+1}(z)) < \operatorname{len}(\mathsf{A}_j(z))$. Reset $j := j + 1$ and repeat the steps, until $\operatorname{len}(\det(\mathsf{A}_j(z))) = 0$.

This iteration will stop after finite number of steps, since $\operatorname{len}(\det(\mathsf{A}(z)))$ is finite and $\operatorname{len}(\det(\mathsf{A}_j(z)))$ is strictly decreasing after each step. Hence, we can get a factorization as

$$\mathsf{A}(z) = \mathsf{U}_1(z) \cdots \mathsf{U}_k(z)\mathsf{A}_k(z)\mathsf{U}^\star_k(z) \cdots \mathsf{U}^\star_1(z),$$

where $\mathsf{A}_k(z)$ has no elementary divisors, i.e., $\operatorname{len}(\det(\mathsf{A}_k(z))) = 0$.

In this case, it is proved by Theorem 2.3.3 that $\mathsf{A}_k(z)$ can be factorized as $\mathsf{A}_k(z) = \mathsf{U}_{k+1}(z)\mathsf{D}\mathsf{U}^\star_{k+1}(z)$, for some $n \times n$ matrix $\mathsf{U}_{k+1}(z)$ of Laurent polynomials, and $\mathsf{D} = \operatorname{diag}(\mathbf{I}_{\nu_+}, -\mathbf{I}_{\nu_-})$ is an $n \times n$ constant diagonal matrix, for some nonnegative integers ν_+ and ν_-, such that $\nu_+ + \nu_- = n$. Define $\mathsf{U}(z) := \mathsf{U}_1(z) \cdots \mathsf{U}_{k+1}(z)$, then

$$\mathsf{A}(z) = \mathsf{U}(z)\mathsf{D}\mathsf{U}^\star(z)$$

holds. In order to complete the proof, we notice that for all $z_0 \in \mathbb{T} \setminus \sigma(\mathsf{A}(z))$, Theorem 2.2.1 shows that

$$\nu_+ \geqslant \nu_+(\mathsf{A}(z_0)), \qquad \nu_- \geqslant \nu_-(\mathsf{A}(z_0)).$$

Since $\nu_+ + \nu_- = n = \nu_+(\mathsf{A}(z_0)) + \nu_-(\mathsf{A}(z_0))$, the above two inequalities have to be equalities. That is, $\nu_+ = \nu_+(\mathsf{A}(z_0))$ and $\nu_- = \nu_-(\mathsf{A}(z_0))$ hold for all $z_0 \in \mathbb{T} \setminus \sigma(\mathsf{A}(z))$.

This completes the proof of the theorem. ■

2.3.3 The Algorithm for Spectral Decomposition of a Matrix of Laurent Polynomials with Constant Signature

Since the proof of Theorem 2.3.1 follows a constructive approach by applying Theorem 2.3.4, 2.3.5, 2.3.3 repeatedly, we can use the following algorithm to solve it.

Most of the steps in the algorithm will be simply following the proofs of the above theorems. The only non-constructive step in the proof is to find $W(\xi)$ in (2.3.20), which cannot be solved explicitly. However, it is still possible to find a matrix with similar properties as $\mathring{\mathsf{A}}(z)$ in (2.3.21), by diagonalizing the coefficient matrix of the first order expansion term: $\frac{d}{d\xi}\mathsf{A}(e^{-i\xi})|_{\xi_0} = -iz_0\mathsf{A}'(z_0)$, where $z_0 = e^{-i\xi_0}$.

Algorithm 2.2. *Input an $n \times n$ Hermitian matrix $\mathsf{A}(z)$ of Laurent polynomials with constant signature on $z \in \mathbb{T} \setminus \sigma(\mathsf{A}(z))$, such that $\det(\mathsf{A}(z))$ is not identically 0.*

(S0) *Initialization. Set $\widetilde{\mathsf{A}}(z) := \mathsf{A}(z)$, $\mathsf{U}(z) := \mathbf{I_n}$.*

(S1) *Compute the Smith Normal Form $\mathsf{D}(z)$ of $\widetilde{\mathsf{A}}(z)$, to get a decomposition $\widetilde{\mathsf{A}}(z) = \mathsf{E}(z)\mathsf{D}(z)\mathsf{F}(z)$, where $\mathsf{E}(z)$ and $\mathsf{F}(z)$ are invertible matrices of Laurent polynomials, and $\mathsf{D}(z) = \operatorname{diag}(\mathsf{d}_1(z), \ldots, \mathsf{d}_n(z))$.*

(S2) *If $\mathsf{D}(z)$ is a constant matrix, go to **(S5)**. Otherwise, redefine*

$$\widetilde{\mathsf{A}}(z) := \mathsf{E}^{-1}(z)\widetilde{\mathsf{A}}(z)\mathsf{E}^{-\star}(z) = \operatorname{diag}(\mathsf{d}_1(z), \ldots, \mathsf{d}_n(z))\mathsf{F}(z)\mathsf{E}^{-\star}(z),$$

and $\mathsf{U}(z) := \mathsf{U}(z)\mathsf{E}(z)$.

(S3) ***For** j **from** 1 **to** n:*

Factorize $\mathsf{d}_j(z) = \prod_{k=1}^{n_j}(z - z_{j,k})^{\alpha_{j,k}}$.

***If** there exists some factor $(z - z_{j,k})^{\alpha_{j,k}}$, $z_{j,k} \in \mathbb{C} \setminus \mathbb{T} \setminus \{0\}$:*

(1) redefine $\mathsf{U}(z)$ by multiplying its j-th column by $(z - z_{j,k})^{\alpha_{j,k}}$;

(2) redefine $\widetilde{\mathsf{A}}(z)$ *by dividing its* j*-th row by* $(z - z_{j,k})^{\alpha_{j,k}}$*, and dividing its* j*-th column by* $(z^{-1} - \overline{z_{j,k}})^{\alpha_{j,k}}$*;*

(3) ***break*** *the* ***for*** *loop, and go back to* ***(S1)****;*

else if *there exists some factor* $(z - z_{j,k})^{\alpha_{j,k}}$*,* $z_{j,k} \in \mathbb{T}, \alpha_{j,k} \geqslant 2$*:*

(1) redefine $\mathsf{U}(z)$ *by multiplying its* j*-th column by* $(z - z_{j,k})^{\lfloor \alpha_{j,k}/2 \rfloor}$*;*

(2) redefine $\widetilde{\mathsf{A}}(z)$ *by dividing its* j*-th row by* $(z - z_{j,k})^{\lfloor \alpha_{j,k}/2 \rfloor}$*, and dividing its* j*-th column by* $(z^{-1} - \overline{z_{j,k}})^{\lfloor \alpha_{j,k}/2 \rfloor}$*;*

(3) ***break*** *the* ***for*** *loop, and go back to* ***(S1)****;*

end if;

end for;

(S4) *If the* ***for*** *loop doesn't break from any conditions in* ***(S3)****, then all the elementary divisors will have roots on* $\mathbb{T}$ *with degree equal to* 1*. Pick one of the elementary divisors* $(z - z_0)$*. Suppose it is contained in the last* K *invariant polynomials* $d_{n-K+1}(z), \ldots, d_n(z)$*:*

(1) From the construction of $\widetilde{\mathsf{A}}(z)$*, we can see the last* K *columns and the last* K *rows of* $\widetilde{\mathsf{A}}(z_0)$ *have to be* 0*. Consider the constant matrix* $-iz_0\widetilde{\mathsf{A}}'(z_0)$*, which is also Hermitian. Take its lower right* $K \times K$ *submatrix, denoted as* A_K*, and find its eigenvalue decomposition as* $\mathsf{A}_K := \mathsf{U}_1 \Gamma \mathsf{U}_1^\star$*, for some unitary matrix* U_1 *and* $\Gamma = \operatorname{diag}(\gamma_1^2, -\gamma_2^2, \cdots, \gamma_K)$*. Then the eigenvalues in* Γ *must be all nonzero, while* $K/2$ *of them are positive and* $K/2$ *of them are negative. Arrange them such that the first one is positive and the second one is negative. Redefine* $\widetilde{\mathsf{A}}(z) := \operatorname{diag}(\mathbf{I_{n-K}}, \mathsf{U}_1^{-1})\widetilde{\mathsf{A}}(z) \operatorname{diag}(\mathbf{I_{n-K}}, \mathsf{U}_1^{-\star})$*, and* $\mathsf{U}(z) := \mathsf{U}(z) \operatorname{diag}(\mathbf{I_{n-K}}, \mathsf{U}_1)$*.*

(2) Take $\mathsf{U}_2 := \operatorname{diag}\left(\mathbf{I_{n-K}}, \begin{bmatrix} \gamma_1^{-1} & \gamma_2^{-1} \\ 0 & 1 \end{bmatrix}, \mathbf{I_{K-2}}\right)$*. Redefine* $\widetilde{\mathsf{A}}(z) := \mathsf{U}_2 \widetilde{\mathsf{A}}(z) \mathsf{U}_2^\star$*, and* $\mathsf{U}(z) := \mathsf{U}(z)\mathsf{U}_2^{-1}$*.*

(3) Redefine $\widetilde{\mathsf{A}}(z)$ *by dividing its* $(n-K+1)$*-th row by* $(z - z_0)$*, and dividing its* $(n-K+1)$*-th column by* $(z^{-1} - \overline{z_0})$*; redefine* $\mathsf{U}(z)$ *by multiplying its* $(n-K+1)$*-th column by* $(z - z_0)$*.*

*Go back to **(S1)**.*

(S5) *Finalize: Since* $\widetilde{\mathsf{A}}(z)$ *has empty spectrum, apply Algorithm 2.1 to get the factorization* $\widetilde{\mathsf{A}}(z) = \widetilde{\mathsf{U}}(z)\mathsf{D}\widetilde{\mathsf{U}}^\star(z)$. *Redefine* $\mathsf{U}(z) := \mathsf{U}(z)\widetilde{\mathsf{U}}(z)$.
Output $\mathsf{U}(z)$ *and* D, *the equation* $\mathsf{A}(z) = \mathsf{U}(z)\mathsf{D}\mathsf{U}^\star(z)$ *will hold.*

Proof. Steps **(S1)(S2)(S3)** simply follow the proof of the Theorem 2.3.4, while step **(S5)** follows Algorithm 2.1. We only need to prove that the step **(S4)** is feasible.

Suppose $z_0 = e^{-i\xi_0}$ for some $\xi_0 \in \mathbb{R}$. By (2.3.21), (2.3.22) and (2.3.23) from the proof of Theorem 2.3.5, we can see that there exists some constant unitary matrix $W_0 := W^{-1}(\xi_0)$, such that $\mathring{\mathsf{A}}(z) := W_0\widetilde{\mathsf{A}}(z)W_0^\star$ satisfies:

$$W_0\widetilde{\mathsf{A}}(z_0)W_0^\star = \mathring{\mathsf{A}}(z_0) = \begin{bmatrix} \lambda_1 & & & \\ & \ddots & & \\ & & \lambda_{n-K} & \\ & & & \mathbf{0}_{K\times K} \end{bmatrix}, \tag{2.3.25}$$

where $\lambda_1, \cdots, \lambda_{n-K} \neq 0$ are the nonzero eigenvalues of $\widetilde{\mathsf{A}}(z_0)$. And

$$\begin{aligned} W_0\left(-iz_0\widetilde{\mathsf{A}}'(z_0)\right)W_0^\star =& W_0\frac{d}{d\xi}\widetilde{\mathsf{A}}(e^{-i\xi})\Big|_{\xi_0}W_0^\star = \frac{d}{d\xi}\mathring{\mathsf{A}}(e^{-i\xi})\Big|_{\xi_0} \\ =& \left[\begin{array}{c|ccccc} * & & & * & & \\ \hline & \gamma_1^2 & & & & \\ & & -\gamma_2^2 & & & \\ * & & & \ddots & & \\ & & & & \gamma_{K-1} & \\ & & & & & \gamma_K \end{array}\right], \end{aligned} \tag{2.3.26}$$

where the lower right $K \times K$ submatrix is diagonal, with $K/2$ positive and $K/2$ negative diagonal elements.

From (2.3.25) we can see that the eigenspace of $\widetilde{\mathsf{A}}(z_0)$ corresponding to eigenvalue 0 is of dimension K. It must also be the span of the last K column vectors of $W_0^\star$:

$$E_0 = span\{w_{n-K+1}, \ldots, w_n\}.$$

Also, by the construction of $\widetilde{\mathsf{A}}(z)$ in **(S2)**, we can see that the last K columns and last K rows of $\widetilde{\mathsf{A}}(z_0)$ must be zero:

$$\widetilde{\mathsf{A}}(z_0) = \begin{bmatrix} * & \mathbf{0}_{(n-K)\times K} \\ \mathbf{0}_{K\times(n-K)} & \mathbf{0}_{K\times K} \end{bmatrix}.$$

So E_0 is also the span of K natural basis vectors $E_0 = span\{e_{n-K+1}, \dots, e_n\}$. This implies that $span\{w_{n-K+1}, \dots, w_n\} = span\{e_{n-K+1}, \dots, e_n\}$, so the matrix $W_0^\star$ of the eigenvectors of $\widetilde{\mathsf{A}}(z_0)$ has the form $W_0^\star = \begin{bmatrix} W_{01} & \mathbf{0}_{(n-K)\times K} \\ W_{02} & \widetilde{W_0} \end{bmatrix}$, for some matrix W_{01}, W_{02} and $\widetilde{W_0}$, while $\widetilde{W_0}$ is a $K \times K$ unitary matrix.

From (2.3.26):

$$W_0 \left(-iz_0\widetilde{\mathsf{A}}'(z_0)\right) W_0^\star = \begin{bmatrix} W_{01}^* & W_{02}^* \\ \mathbf{0}_{(n-K)\times K} & \widetilde{W_0}^* \end{bmatrix} \begin{bmatrix} * & * \\ * & \mathsf{A}_K \end{bmatrix} \begin{bmatrix} W_{01} & \mathbf{0}_{(n-K)\times K} \\ W_{02} & \widetilde{W_0} \end{bmatrix}$$

$$= \left[\begin{array}{c|ccccc} * & & & * & & \\ \hline & \gamma_1^2 & & & & \\ & & -\gamma_2^2 & & & \\ * & & & \ddots & & \\ & & & & \gamma_{K-1} & \\ & & & & & \gamma_K \end{array}\right],$$

we get $\widetilde{W_0}^\star \mathsf{A}_K \widetilde{W_0} = \operatorname{diag}(\gamma_1^2, -\gamma_2^2, \dots, \gamma_{K-1}, \gamma_K)$. Hence, the lower right $K \times K$ submatrix of $\left(-iz_0\widetilde{\mathsf{A}}'(z_0)\right)$, A_K, has $K/2$ positive and $K/2$ negative eigenvalues. This proves that item (1) in step **(S4)** is feasible.

From the construction, we can see that the redefined $\widetilde{\mathsf{A}}(z)$ after item (1) in step **(S4)** satisfies:

$$\widetilde{\mathsf{A}}(z_0) = \begin{bmatrix} * & \mathbf{0}_{(n-K)\times K} \\ \mathbf{0}_{K\times(n-K)} & \mathbf{0}_{K\times K} \end{bmatrix}, \qquad -iz_0\widetilde{\mathsf{A}}'(z_0) = \left[\begin{array}{c|ccccc} * & & & * & & \\ \hline & \gamma_1^2 & & & & \\ & & -\gamma_2^2 & & & \\ * & & & \ddots & & \\ & & & & \gamma_{K-1} & \\ & & & & & \gamma_K \end{array}\right].$$

The design of U_2 in item (2) of step **(S4)** is similar to the matrix V in (2.3.24). We can verify that the redefined $\widetilde{\mathsf{A}}(z)$ after item (2) satisfies:

$$\widetilde{\mathsf{A}}(z_0) = \begin{bmatrix} * & \mathbf{0}_{(n-K)\times K} \\ \mathbf{0}_{K\times(n-K)} & \mathbf{0}_{K\times K} \end{bmatrix}, \qquad -iz_0\widetilde{\mathsf{A}}'(z_0) = \left[\begin{array}{c|ccccc} * & & & * & & \\ \hline & 0 & -\gamma_2 & & & \\ & -\gamma_2 & -\gamma_2^2 & & & \\ * & & & \ddots & & \\ & & & & \gamma_{K-1} & \\ & & & & & \gamma_K \end{array}\right].$$

The above equality shows that $(z-z_0)$ divides both the $(n-K+1)$-th row and the $(n-K+1)$-th column of $\widetilde{\mathsf{A}}(z)$, meanwhile, $(z-z_0)^2$ divides the $(n-K+1)$-th diagonal element of $\widetilde{\mathsf{A}}(z)$. Thus the item (3) in **(S4)** is feasible. ■

Example 2.3 (Revisit Example 2.1)**.** *Let us compute the spectral factorization of the following matrix*

$$\mathsf{A}(z) = \begin{bmatrix} z^{-1}(z-1)^2 & (z-1)(z+1) \\ (z^{-1}-1)(z^{-1}+1) & -z^{-1}(z-1)^2 \end{bmatrix}.$$

As we showed before, the matrix yields a Smith Normal Form $\mathsf{D}(z)$ *with decomposition* $\mathsf{A}(z) = \mathsf{E}(z)\mathsf{D}(z)\mathsf{F}(z)$*, where*

$$\mathsf{E}(z) := \begin{bmatrix} -\frac{2z^3-4z^2-z+1}{z} & -2z(z-2) \\ \frac{2z^3-z-1}{z^2} & 2z \end{bmatrix}, \quad \mathsf{F}(z) := \begin{bmatrix} 1 & 2z^2-z \\ -1 & -2z^2+z+1 \end{bmatrix},$$

$$\mathsf{D}(z) := \begin{bmatrix} z-1 & 0 \\ 0 & z-1 \end{bmatrix}.$$

So $\mathsf{A}(z)$ *has two elementary divisors* $(z-1)$*. Moreover, we can check that*

$$-iz_0\mathsf{A}'(z_0)\Big|_{z_0=1} = \begin{bmatrix} 0 & -2i \\ 2i & 0 \end{bmatrix}.$$

We can see the above matrix has two eigenvalues with different signs, $\lambda_1, \lambda_2 = \pm 2$*. Applying the Algorithm 2.2, we can get*

$$\mathsf{U}(z) = \begin{bmatrix} 0 & z-1 \\ 2 & -\frac{z+1}{z} \end{bmatrix}, \qquad \mathsf{D} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

and $\mathsf{A}(z) = \mathsf{U}(z)\mathsf{D}\mathsf{U}^\star(z)$ *holds.*

2.4 Algorithm for Constructing Quasi-tight Framelet Filter Banks with Two High-pass Filters

In this section, we provide an algorithm to construct quasi-tight framelet filter banks with two high-pass filters and high order of vanishing moments.

Theorem 2.4.1. *Given a nonzero moment correcting filter* Θ *and a low-pass filter* a*, such that* $\Theta^\star = \Theta$*, and*

$$\det(\mathcal{M}_{\mathsf{a},\Theta})(z) = \Theta(z)\Theta(-z) - \Theta(z^2)\left(\Theta(-z)\mathsf{a}(z)\mathsf{a}^*(z) - \Theta(z)\mathsf{a}(-z)\mathsf{a}^*(-z)\right)$$

is not identically zero. For any integer n_b *satisfying*

$$1 \leqslant n_b \leqslant \min\left\{\operatorname{sr}(a), \frac{1}{2}\operatorname{vm}\left(\Theta(z) - \Theta(z^2)\mathsf{a}(z)\mathsf{a}^\star(z)\right)\right\}, \qquad (2.4.1)$$

(1) if

$$\Theta(z) \geqslant 0, \qquad \det(\mathcal{M}_{\mathsf{a},\Theta}(z)) \geqslant 0, \qquad \forall z \in \mathbb{T},$$

then there exist two high-pass filters b_1, b_2, both with at least n_b order of vanishing moments, such that $\{a; b_1, b_2\}_\Theta$ is a tight framelet filter bank;

(2) if

$$\Theta(z) \leqslant 0, \qquad \det(\mathcal{M}_{\mathsf{a},\Theta}(z)) \geqslant 0, \qquad \forall z \in \mathbb{T},$$

then there exist two high-pass filters b_1, b_2, both with at least n_b order of vanishing moments, such that $\{a; b_1, b_2\}_{\Theta,(-1,-1)}$ is a quasi-tight framelet filter bank;

(3) if

$$\det(\mathcal{M}_{\mathsf{a},\Theta}(z)) \leqslant 0, \qquad \forall z \in \mathbb{T},$$

then there exist two high-pass filters b_1, b_2, both with at least n_b order of vanishing moments, such that $\{a; b_1, b_2\}_{\Theta,(1,-1)}$ is a quasi-tight framelet filter bank;

(4) otherwise, it is not possible to construct a quasi-tight framelet filter bank from such Θ and a with two high-pass filters.

Also, for the case (1)(2)(3), it is not possible to find a quasi-tight framelet filter bank from such Θ and a with only one high-pass filter.

Proof. To prove item (4), according to item (4) of Theorem 2.2.4, we know that $\operatorname{sig}(\mathcal{M}_{\mathsf{a},\Theta}(z))$ varies on $z \in \mathbb{T} \setminus \sigma(\mathcal{M}_{\mathsf{a},\Theta}(z))$ and $s^+_{a,\Theta} + s^-_{a,\Theta} > 2$. By Corollary 2.2.3, we need $s > 2$ high-pass filters to construct a quasi-tight framelet filter bank.

In the cases of (1)(2)(3), according to (1)(2)(3) of Theorem 2.2.4, we know that $\operatorname{sig}(\mathcal{M}_{\mathsf{a},\Theta}(z))$ is constant on $z \in \mathbb{T}\setminus\sigma(\mathcal{M}_{\mathsf{a},\Theta}(z))$. Thus, by Corollary 2.2.3, we need at least $s \geqslant 2$ high-pass filters to construct a quasi-tight framelet filter bank.

To prove (1)(2)(3), since $\operatorname{sig}(\mathcal{M}_{\mathsf{a},\Theta}(z))$ is constant on $z \in \mathbb{T} \setminus \sigma(\mathcal{M}_{\mathsf{a},\Theta}(z))$, from (2.1.6) and (2.1.9), we can see that $\operatorname{sig}(\mathcal{N}_{\mathsf{a},\Theta|n_b}(z))$ is also constant on $\mathbb{T} \setminus \sigma(\mathcal{N}_{\mathsf{a},\Theta|n_b}(z))$. Then Theorem 2.3.1 tells us that there exists a 2×2 matrix

$\mathsf{U}(z)$ of Laurent polynomials, such that

$$\mathcal{N}_{\mathsf{a},\Theta|n_b}(z) = \mathsf{U}(z) \begin{bmatrix} \epsilon_1 & \\ & \epsilon_2 \end{bmatrix} \mathsf{U}^\star(z), \qquad \epsilon_1, \epsilon_2 = \pm 1.$$

By Theorem 2.1.2, we can use such $\mathsf{U}(z)$ to construct a quasi-tight framelet filter bank $\{a; b_1, b_2\}_{\Theta,(\epsilon_1,\epsilon_2)}$, while both b_1 and b_2 have at least n_b order of vanishing moments.

The signs of ϵ_1, ϵ_2 here depend on the signs of the eigenvalues of $\mathcal{M}_{\mathsf{a},\Theta}(z)$. According to Theorem 2.2.2,

$$\nu_+\left(\begin{bmatrix} \epsilon_1 & \\ & \epsilon_2 \end{bmatrix}\right) \geqslant \max_{z\in\mathbb{T}} \nu_+(\mathcal{M}_{\mathsf{a},\Theta}(z)), \qquad \nu_-\left(\begin{bmatrix} \epsilon_1 & \\ & \epsilon_2 \end{bmatrix}\right) \geqslant \max_{z\in\mathbb{T}} \nu_-(\mathcal{M}_{\mathsf{a},\Theta}(z)).$$

From Theorem 2.2.4, we know that for item (1), $\epsilon_1 = \epsilon_2 = 1$; for item (2), $\epsilon_1 = \epsilon_2 = -1$; for item (3), $\epsilon_1 = 1$ and $\epsilon_2 = -1$. This completes the proof of the theorem. ■

We can use the following algorithm to construct quasi-tight framelet filter banks in the case (1)(2)(3) of the above theorem.

Algorithm 2.3. *Input a nonzero moment correcting filter* Θ *and a low-pass filter* a*, such that* $\Theta^\star = \Theta$*, and*

$$\det(\mathcal{M}_{\mathsf{a},\Theta})(z) = \Theta(z)\Theta(-z) - \Theta(z^2)\left(\Theta(-z)\mathsf{a}(z)\mathsf{a}^*(z) - \Theta(z)\mathsf{a}(-z)\mathsf{a}^*(-z)\right)$$

is not identically zero. Input an integer n_b*, satisfying*

$$0 \leqslant n_b \leqslant \min\left\{\operatorname{sr}(a), \frac{1}{2}\operatorname{vm}\left(\Theta(z) - \Theta(z^2)\mathsf{a}(z)\mathsf{a}^*(z)\right)\right\}.$$

Also, suppose either one of items (1)(2)(3) in Theorem 2.4.1 holds.

(S1) *Calculate* $\mathcal{N}_{\mathsf{a},\Theta|n_b}(z)$ *as defined in* (2.1.10) (2.1.7).

(S2) *Use Algorithm 2.2 to find a* 2×2 *matrix* $\mathsf{U}(z)$ *of Laurent polynomials,* *such that the spectral decomposition of* $\mathcal{N}_{\mathsf{a},\Theta|n_b}(z)$ *holds as*

$$\mathcal{N}_{\mathsf{a},\Theta|n_b}(z) = \mathsf{U}(z) \begin{bmatrix} \epsilon_1 & \\ & \epsilon_2 \end{bmatrix} \mathsf{U}^\star(z),$$

where $\epsilon_1, \epsilon_2 = \pm 1$.

(S3) *Define high-pass filters* b_1 *and* b_2 *as:*

$$\begin{bmatrix} \mathsf{b}_1(z) & \mathsf{b}_2(z) \end{bmatrix} = (1 - z^{-1})^{n_b} \begin{bmatrix} 1 & z \end{bmatrix} \mathsf{U}(z^2).$$

Then, $\{a; b_1, b_2\}_{\Theta,(\epsilon_1,\epsilon_2)}$ *is a quasi-tight framelet filter bank.*

2.5 Illustrative Examples

Since the problem on construction of tight-framelet filter banks has been widely discussed in the literature, we only provide examples of quasi-tight framelets with $\epsilon_1 = 1$ and $\epsilon_2 = -1$.

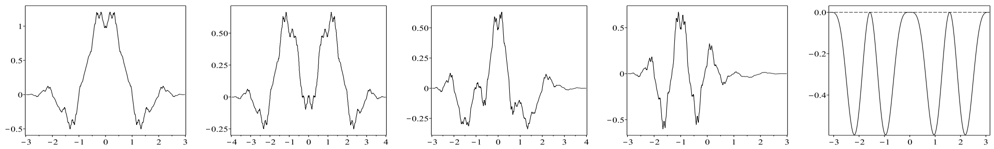
Example 2.4. Consider $\Theta(z) = \frac{1}{2}(z + \frac{1}{z})$ and the interpolatory low-pass filter

$$\mathsf{a}(z) = \tfrac{1}{2} + \tfrac{3}{8}(z + z^{-1}) - \tfrac{1}{8}(z^3 + z^{-3}).$$

We see from Figure 2.2 that $\det(\mathcal{M}_{\mathsf{a},\Theta}(z)) \leqslant 0$ for all $z \in \mathbb{T}$. Therefore, $s^+_{a,\Theta} = s^-_{a,\Theta} = 1$. Note that $\operatorname{sr}(\mathsf{a}) = 2$ and $\operatorname{vm}(\Theta(z) - \Theta(z^2)\mathsf{a}(z)\mathsf{a}^\star(z)) = 4$. Hence, the maximum order of vanishing moments is two. Taking $n_b = 2$, we obtain a quasi-tight framelet filter bank $\{a; b_1, b_2\}_{\Theta,(1,-1)}$ as follows:

$$\mathsf{b}_1(z) = \tfrac{1}{32} z^{-3}(z-1)^2(z^6 + 2z^5 - 4z^4 - 14z^3 - 23z^2 - 16z - 8),$$
$$\mathsf{b}_2(z) = -\tfrac{1}{32} z^{-3}(z-1)^2(z^4 + 2z^3 + 4z^2 + 2z + 9),$$

with $\operatorname{vm}(b_1) = \operatorname{vm}(b_2) = 2$. Since $\operatorname{sm}(a) = 1$, the refinable function ϕ belongs to $L_2(\mathbb{R})$. Define $\tilde{\eta} := (\phi(\cdot + 1) + \phi(\cdot - 1))/2$. Therefore, $\{\phi, \tilde{\eta}; \psi^1, \psi^2\}_{(1,-1)}$ a quasi-tight framelet in $L_2(\mathbb{R})$ and $\{\psi^1, \psi^2\}_{(1,-1)}$ is a homogeneous quasi-tight framelet in $L_2(\mathbb{R})$, where ψ^1, ψ^2 have at least two vanishing moments.



(a) ϕ **(b)** $\tilde{\eta}$ **(c)** ψ^1 **(d)** ψ^2 **(e)** $\det(\mathcal{M}_{\mathsf{a},\Theta})$

Figure 2.2: The quasi-tight framelet $\{\phi, \tilde{\eta}; \psi^1, \psi^2\}_{(1,-1)}$ and the homogeneous quasi-tight framelet $\{\psi^1, \psi^2\}_{(1,-1)}$ in $L_2(\mathbb{R})$ obtained in Example 2.4. (A) is the refinable function $\phi \in L_2(\mathbb{R})$. (B) is the function $\tilde{\eta} := (\phi(\cdot + 1) + \phi(\cdot - 1))/2$. (C) and (D) are the framelet functions ψ^1 and ψ^2. (E) is $\det(\mathcal{M}_{\mathsf{a}}(e^{-i\xi}))$ for $\xi \in [-\pi, \pi]$.

Example 2.5. *Choose* $\Theta = \boldsymbol{\delta}$ *and the low-pass filter*

$$\mathsf{a}(z) = \frac{1}{64}(z^4 - 8z^3 + 30z^2 - 8z + 1)(1+z)^2 z^{-3}.$$

We have $\mathrm{sm}(a) = 0.5573$*. Notice that* $\mathrm{sr}(\mathsf{a}) = 2$ *and* $\mathrm{vm}(1 - \mathsf{a}\mathsf{a}^\star) = 6$*. Take* $n_b = 2$*, then the constructed quasi-tight framelet filter bank* $\{a; b_1, b_2\}_{\Theta,\{1,-1\}}$ *is given by:*

$$\begin{aligned}\mathsf{b}_1(z) =& \tfrac{(6\sqrt{17}+17\sqrt{2})}{544z^3}(z-1)^2(z^2 - 4z + 1)(z^2 + 35 - 6\sqrt{34}),\\ \mathsf{b}_2(z) =& \tfrac{\sqrt{17}}{1088}(z-1)^2(5z^4 - 20z^3 + 78z^2 - 20z + 5).\end{aligned}$$

We have $\mathrm{vm}(b_1) = \mathrm{vm}(b_2) = 2$*.*

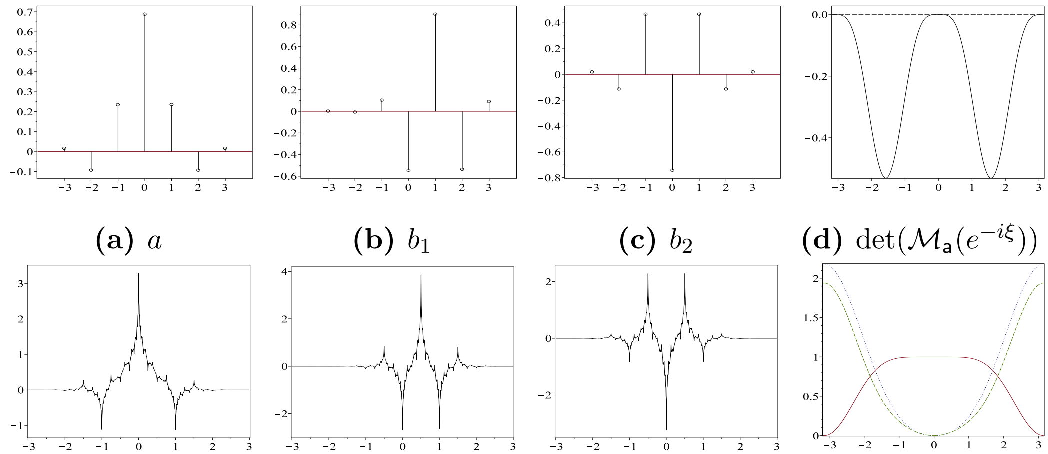
Example 2.6. *Choose* $\Theta = \boldsymbol{\delta}$ *and the low-pass filter*

$$\widehat{a}(\xi) = \frac{1 + e^{-i\xi}}{2}\cos^2(\xi/2)\left(1 + 2\sin^2(\xi/2)\right).$$

We have $\mathrm{sm}(a) = 1.4408$*. Notice that* $\mathrm{sr}(\mathsf{a}) = 3$ *and* $\mathrm{vm}(1 - \mathsf{a}\mathsf{a}^\star) = 2$*. Take* $n_b = 1$*, then the constructed quasi-tight framelet filter bank* $\{a; b_1, b_2\}_{\Theta,\{1,-1\}}$ *is given by:*

$$\begin{aligned}\mathsf{b}_1(z) =& \tfrac{-\sqrt{2}}{512z^2}(z-1)(16z^5 - 271z^4 + 16z^3 + 16z^2 - 1),\\ \mathsf{b}_2(z) =& \tfrac{\sqrt{2}}{512z^2}(z-1)(16z^5 + 241z^4 + 16z^3 + 16z^2 - 1).\end{aligned}$$

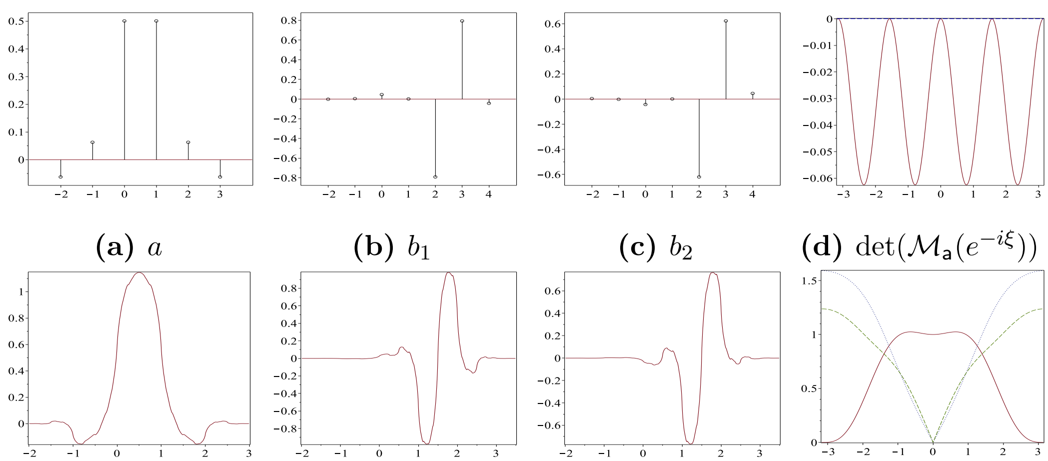
We have $\mathrm{vm}(b_1) = \mathrm{vm}(b_2) = 1$*.*



(a) a **(b)** b_1 **(c)** b_2 **(d)** $\det(\mathcal{M}_{\mathsf{a}}(e^{-i\xi}))$

(e) ϕ **(f)** ψ^1 **(g)** ψ^2 **(h)** $|\widehat{a}|, |\widehat{b_1}|, |\widehat{b_2}|$

Figure 2.3: In Example 2.5: (a),(b) and (c) are the graphs of the filters a, b_1, b_2. (d) $\det(\mathcal{M}_{\mathsf{a}}(e^{-i\xi}))$ for $\xi \in [-\pi, \pi]$. (e) scaling function ϕ. (f) wavelet function ψ^1. (g) wavelet function ψ^2. (h) magnitude of $|\widehat{a}(\xi)|$ (in solid line), $|\widehat{b_1}(\xi)|$ (in dotted line) and $|\widehat{b_2}(\xi)|$ (in dashed line) in for $\xi \in [-\pi, \pi]$.



(a) a **(b)** b_1 **(c)** b_2 **(d)** $\det(\mathcal{M}_{\mathsf{a}}(e^{-i\xi}))$

(e) ϕ **(f)** ψ^1 **(g)** ψ^2 **(h)** $|\widehat{a}|, |\widehat{b_1}|, |\widehat{b_2}|$

Figure 2.4: In Example 2.6: (a),(b) and (c) are the graphs of the filters a, b_1, b_2. (d) $\det(\mathcal{M}_{\mathsf{a}}(e^{-i\xi}))$ for $\xi \in [-\pi, \pi]$. (e) scaling function ϕ. (f) wavelet function ψ^1. (g) wavelet function ψ^2. (h) magnitude of $|\widehat{a}(\xi)|$ (in solid line), $|\widehat{b_1}(\xi)|$ (in dotted line) and $|\widehat{b_2}(\xi)|$ (in dashed line) in for $\xi \in [-\pi, \pi]$.

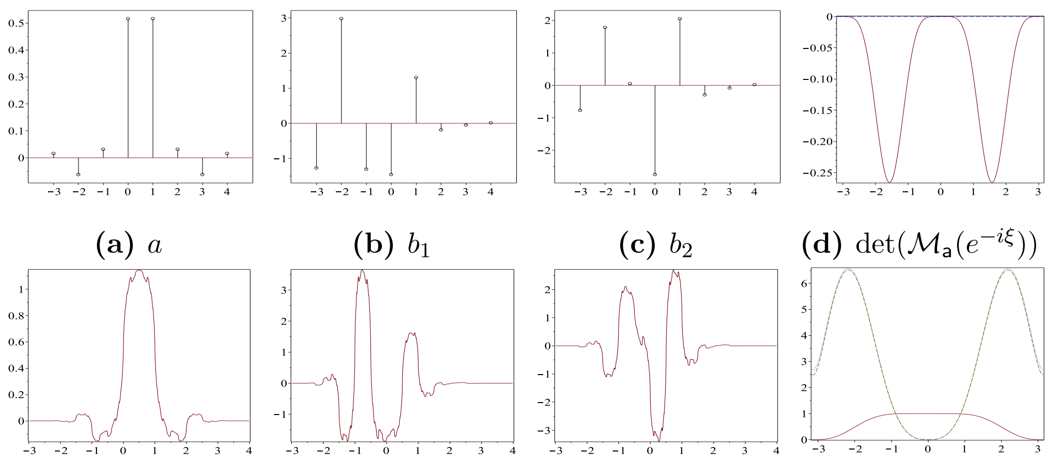
Example 2.7. *Choose* $\Theta = \boldsymbol{\delta}$ *and the low-pass filter*

$$\widehat{a}(\xi) = \frac{1 + e^{-i\xi}}{2} \cos^2(\xi/2) \left(1 + \frac{3}{2}\sin^2(\xi/2) + 2\sin^4(\xi/2)\right).$$

We have $\mathrm{sm}(a) = 1.1268$. *Notice that* $\mathrm{sr}(\mathsf{a}) = 3$ *and* $\mathrm{vm}(1 - \mathsf{a}\mathsf{a}^\star) = 4$. *Take* $n_b = 2$, *then the constructed quasi-tight framelet filter bank* $\{a; b_1, b_2\}_{\Theta,\{1,-1\}}$ *is given by:*

$$\mathsf{b}_1(z) = \tfrac{\sqrt{119327}}{7636928z^3}(z-1)^2(289z^5 - 578z^4 - 5667z^3 + 17981z^2 + 9342z - 28223),$$
$$\mathsf{b}_2(z) = \tfrac{-\sqrt{119327}}{1909232z^3}(z-1)^2(23z^3 - 46z^2 - 418z + 1365)\left((\sqrt{17}+1)z^2 - \sqrt{17} + 1\right).$$

We have $\mathrm{vm}(b_1) = \mathrm{vm}(b_2) = 2$.



(a) a (b) b_1 (c) b_2 (d) $\det(\mathcal{M}_{\mathsf{a}}(e^{-i\xi}))$

(e) ϕ (f) ψ^1 (g) ψ^2 (h) $|\widehat{a}|, |\widehat{b_1}|, |\widehat{b_2}|$

Figure 2.5: In Example 2.7: (a),(b) and (c) are the graphs of the filters a, b_1, b_2. (d) $\det(\mathcal{M}_{\mathsf{a}}(e^{-i\xi}))$ for $\xi \in [-\pi, \pi]$. (e) scaling function ϕ. (f) wavelet function ψ^1. (g) wavelet function ψ^2. (h) magnitude of $|\widehat{a}(\xi)|$ (in solid line), $|\widehat{b_1}(\xi)|$ (in dotted line) and $|\widehat{b_2}(\xi)|$ (in dashed line) in for $\xi \in [-\pi, \pi]$.

Example 2.8. *Choose* $\Theta = \boldsymbol{\delta}$ *and the low-pass filter*

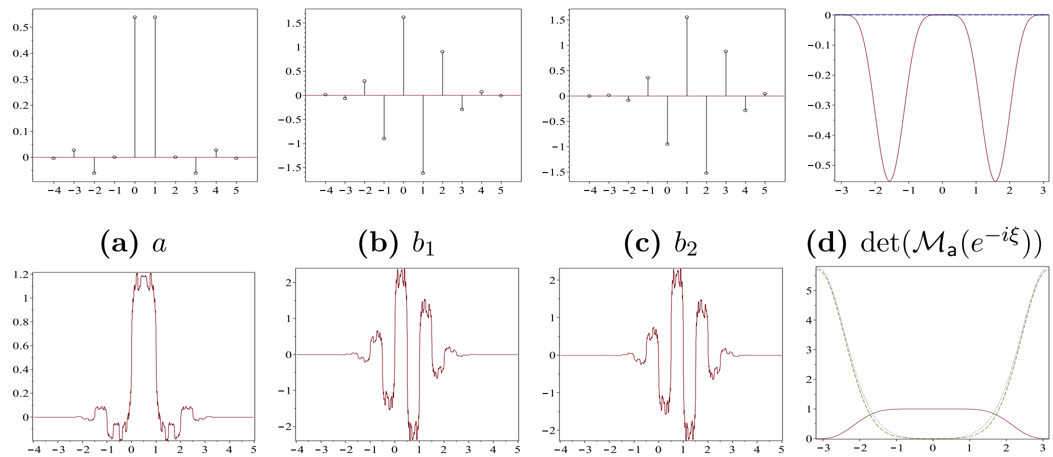
$$\widehat{a}(\xi) = \frac{1 + e^{-i\xi}}{2}\cos^2(\xi/2)\left(1 + \frac{3}{2}\sin^2(\xi/2) + \frac{15}{8}\sin^4(\xi/2) + \frac{35}{16}\sin^6(\xi/2)\right).$$

We have $\mathrm{sm}(a) = 0.8297$. *Notice that* $\mathrm{sr}(\mathsf{a}) = 3$ *and* $\mathrm{vm}(1 - \mathsf{a}\mathsf{a}^\star) = 8$. *Take* $n_b = 3$, *then the constructed quasi-tight framelet filter bank* $\{a; b_1, b_2\}_{\Theta,\{1,-1\}}$ *is given by:*

$$\mathsf{b}_1(z) = \tfrac{-(z-1)^3}{94208z^4}(1015z^6 - 3480z^5 + 14361z^4 - 30512z^3 + 14361z^2 - 3480z + 1015),$$

$$\mathsf{b}_2(z) = \frac{15\sqrt{7}(z-1)^3}{188416z^4}(21z^4 - 72z^3 + 134z^2 - 72z + 21)\left((2\sqrt{7}+\sqrt{23})z^2 + 2\sqrt{7} - \sqrt{23}\right).$$

We have $\operatorname{vm}(b_1) = \operatorname{vm}(b_2) = 3$.



(a) a **(b)** b_1 **(c)** b_2 **(d)** $\det(\mathcal{M}_{\mathsf{a}}(e^{-i\xi}))$

(e) ϕ **(f)** ψ^1 **(g)** ψ^2 **(h)** $|\widehat{a}|, |\widehat{b_1}|, |\widehat{b_2}|$

Figure 2.6: In Example 2.8: (a),(b) and (c) are the graphs of the filters a, b_1, b_2. (d) $\det(\mathcal{M}_{\mathsf{a}}(e^{-i\xi}))$ for $\xi \in [-\pi, \pi]$. (e) scaling function ϕ. (f) wavelet function ψ^1. (g) wavelet function ψ^2. (h) magnitude of $|\widehat{a}(\xi)|$ (in solid line), $|\widehat{b_1}(\xi)|$ (in dotted line) and $|\widehat{b_2}(\xi)|$ (in dashed line) in for $\xi \in [-\pi, \pi]$.

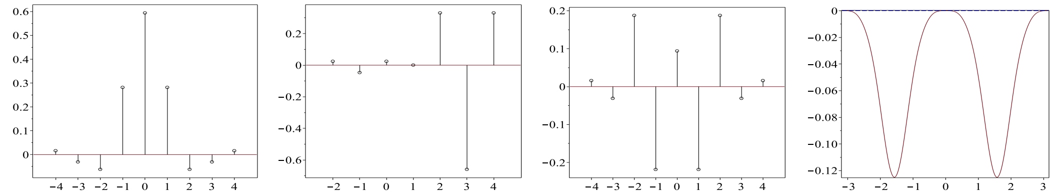
Example 2.9. *Choose* $\Theta = \boldsymbol{\delta}$ *and the low-pass filter*

$$\widehat{a}(\xi) = \cos^4(\xi/2)\left(1 + 2\sin^2(\xi/2) + 4\sin^4(\xi/2)\right).$$

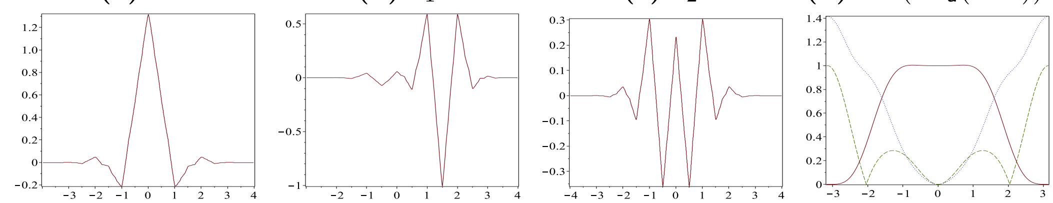
We have $\operatorname{sm}(a) = 1.6297$. *Notice that* $\operatorname{sr}(\mathsf{a}) = 4$ *and* $\operatorname{vm}(1 - \mathsf{a}\mathsf{a}^\star) = 4$. *Take* $n_b = 2$, *then the constructed quasi-tight framelet filter bank* $\{a; b_1, b_2\}_{\Theta,\{1,-1\}}$ *is given by:*

$$\mathsf{b}_1(z) = \frac{\sqrt{2}}{16z^2}(z-1)^2\left((2+\sqrt{3})z^4 + 2 - \sqrt{3}\right),$$
$$\mathsf{b}_2(z) = \frac{1}{64z^4}(z-1)^2(z^6 + 11z^4 + 8z^3 + 11z^2 + 1).$$

We have $\operatorname{vm}(b_1) = \operatorname{vm}(b_2) = 2$.



(a) a **(b)** b_1 **(c)** b_2 **(d)** $\det(\mathcal{M}_{\mathsf{a}}(e^{-i\xi}))$



(e) ϕ **(f)** ψ^1 **(g)** ψ^2 **(h)** $|\widehat{a}|, |\widehat{b_1}|, |\widehat{b_2}|$

Figure 2.7: In Example 2.9: (a),(b) and (c) are the graphs of the filters a, b_1, b_2. (d) $\det(\mathcal{M}_{\mathsf{a}}(e^{-i\xi}))$ for $\xi \in [-\pi, \pi]$. (e) scaling function ϕ. (f) wavelet function ψ^1. (g) wavelet function ψ^2. (h) magnitude of $|\widehat{a}(\xi)|$ (in solid line), $|\widehat{b_1}(\xi)|$ (in dotted line) and $|\widehat{b_2}(\xi)|$ (in dashed line) in for $\xi \in [-\pi, \pi]$.

Example 2.10. *Choose* $\Theta = \boldsymbol{\delta}$ *and the low-pass filter*

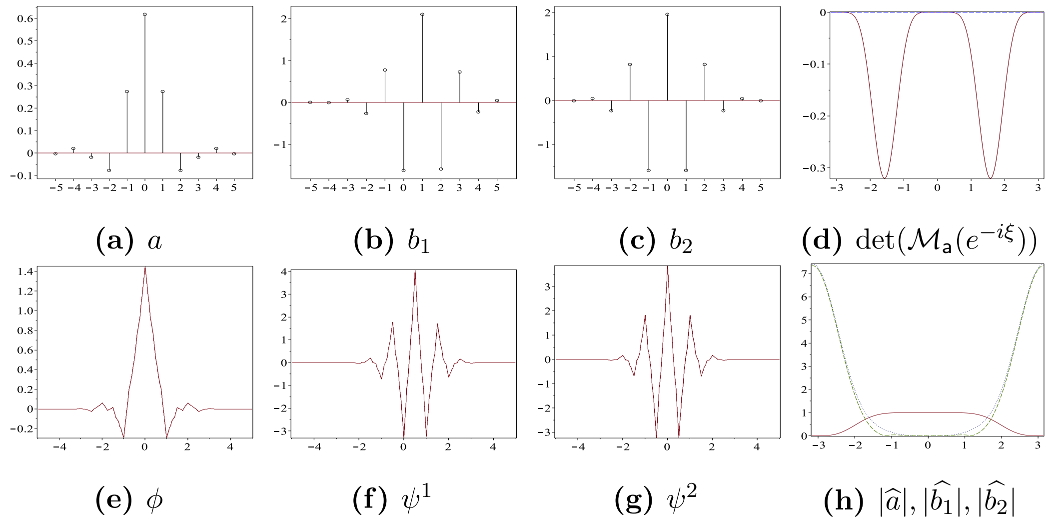
$$\widehat{a}(\xi) = \cos^4(\xi/2)\left(1 + 2\sin^2(\xi/2) + 3\sin^4(\xi/2) + 4\sin^6(\xi/2)\right).$$

We have $\operatorname{sm}(a) = 1.3516$. *Notice that* $\operatorname{sr}(\mathsf{a}) = 4$ *and* $\operatorname{vm}(1 - \mathsf{a}\mathsf{a}^\star) = 8$. *Take* $n_b = 4$, *then the constructed quasi-tight framelet filter bank* $\{a; b_1, b_2\}_{\Theta,\{1,-1\}}$ *is given by:*

$$\begin{aligned}
\mathsf{b}_1(z) =& \tfrac{\sqrt{82}(z-1)^4}{2624 z^5}(z^4 - z^3 + 6z^2 - z + 1)\left((3\sqrt{5} + \sqrt{41})z^2 + 3\sqrt{5} - \sqrt{41}\right),\\
\mathsf{b}_2(z) =& \tfrac{-\sqrt{41}(z-1)^4}{10496 z^5}(13z^6 - 13z^5 + 255z^4 - 190z^3 + 255z^2 - 13z + 13).
\end{aligned}$$

We have $\operatorname{vm}(b_1) = \operatorname{vm}(b_2) = 4$.

Since the construction of quasi-tight framelet filter banks only depends on the spectral decomposition result of Hermitian matrices of Laurent polynomials, we can easily generalize our results to the construction of quasi-tight framelets with a general integer dilation $\mathsf{M} \geqslant 2$. See [18] for the theoretical details. We only provide an example here to illustrate it.



(a) a **(b)** b_1 **(c)** b_2 **(d)** $\det(\mathcal{M}_{\mathsf{a}}(e^{-i\xi}))$

(e) ϕ **(f)** ψ^1 **(g)** ψ^2 **(h)** $|\widehat{a}|, |\widehat{b_1}|, |\widehat{b_2}|$

Figure 2.8: In Example 2.10: (a),(b) and (c) are the graphs of the filters a, b_1, b_2. (d) $\det(\mathcal{M}_{\mathsf{a}}(e^{-i\xi}))$ for $\xi \in [-\pi, \pi]$. (e) scaling function ϕ. (f) wavelet function ψ^1. (g) wavelet function ψ^2. (h) magnitude of $|\widehat{a}(\xi)|$ (in solid line), $|\widehat{b_1}(\xi)|$ (in dotted line) and $|\widehat{b_2}(\xi)|$ (in dashed line) in for $\xi \in [-\pi, \pi]$.

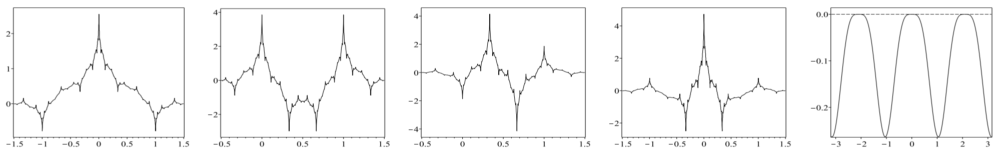
Example 2.11. Let $\mathsf{M} = 3$ be a dilation factor. Consider $\Theta(z) = 1$ and the low-pass filter

$$\mathsf{a}(z) = -\tfrac{1}{27} z^{-3}(1 + z + z^2)^2(2z^2 - 7z + 2).$$

The three eigenvalues of $\mathcal{M}_{\mathsf{a}}(z)$ are $1, 1$ and $\det(\mathcal{M}_{\mathsf{a}}(z))$. We see from Figure 2.9 that $\det(\mathcal{M}_{\mathsf{a}}(z)) \leqslant 0$ on $\mathbb{T}$. Hence $s^+_{a,\Theta} = 2$ and $s^-_{a,\Theta} = 1$. Note that $\operatorname{sr}(\mathsf{a}, 3) = 2$ and $\operatorname{vm}(1 - \mathsf{a}\mathsf{a}^\star) = 4$. Therefore, the maximum order of vanishing moments is two. Taking $n_b = 2$, we obtain a quasi-tight 3-framelet filter bank $\{a; b_1, b_2, b_3\}_{\Theta,(1,1,-1)}$ as follows:

$$\mathsf{b}_1(z) = \tfrac{\sqrt{6}}{6}(z-1)^2(z+1), \qquad \mathsf{b}_2(z) = \tfrac{\sqrt{6}}{18}(z-1)^3,$$
$$\mathsf{b}_3(z) = \tfrac{1}{27} z^{-3}(z-1)^4(2z^2 + 5z + 2),$$

with $\operatorname{vm}(b_1) = 2$, $\operatorname{vm}(b_2) = 3$ and $\operatorname{vm}(b_3) = 4$. Since $\operatorname{sm}(a, 3) \approx 0.6599$, the refinable function ϕ belongs to $L_2(\mathbb{R})$. Therefore, $\{\phi, \phi; \psi^1, \psi^2, \psi^3\}_{(1,1,-1)}$ is a quasi-tight 3-framelet in $L_2(\mathbb{R})$ and $\{\psi^1, \psi^2, \psi^3\}_{(1,1,-1)}$ is a homogeneous quasi-tight 3-framelet in $L_2(\mathbb{R})$, where ψ^1, ψ^2 have at least two vanishing moments.



(a) ϕ **(b)** ψ^1 **(c)** ψ^2 **(d)** ψ^3 **(e)** $\det(\mathcal{M}_{\mathsf{a}})$

Figure 2.9: The quasi-tight 3-framelet $\{\phi, \phi; \psi^1, \psi^2, \psi^3\}_{(1,1,-1)}$ in $L_2(\mathbb{R})$ and the homogeneous quasi-tight 3-framelet $\{\psi^1, \psi^2, \psi^3\}_{(1,1,-1)}$ in $L_2(\mathbb{R})$ obtained in Example 2.11. (A) is the refinable function $\phi \in L_2(\mathbb{R})$. (B), (C) and (D) are the framelet functions ψ^1, ψ^2 and ψ^3. (E) is $\det(\mathcal{M}_{\mathsf{a}}(e^{-i\xi}))$ for $\xi \in [-\pi, \pi]$.

Chapter 3

Quasi-tight Framelets with Minimum Number of Generators in One Dimension

3.1 Spectral Decomposition of a Matrix of Laurent Polynomials with Non-constant Signature

In Section 2.3.2, we proved that an $n \times n$ Hermitian matrix of Laurent polynomials $\mathsf{A}(z)$ can have a spectral decomposition $\mathsf{A}(z) = \mathsf{U}(z)\mathsf{D}\mathsf{U}^\star(z)$, where $\mathsf{U}(z)$ is an $n \times n$ matrix of Laurent polynomials, and D is an $n \times n$ constant matrix, as long as $\operatorname{sig}(\mathsf{A}(z))$ is constant for all $z \in \mathbb{T} \setminus \sigma(\mathsf{A}(z))$. It is easy to see that this condition is also necessary: For $z \in \mathbb{T} \setminus \sigma(\mathsf{A}(z))$, since $\mathsf{U}(z)$ is nonsingular, by the Sylvester's law of inertia, $\operatorname{sig}(\mathsf{A}(z)) = \operatorname{sig}(\mathsf{D})$ must be constant.

For the case that $\operatorname{sig}(\mathsf{A}(z))$ is not constant, we can still obtain a similar spectral decomposition of $\mathsf{A}(z)$, $\mathsf{A}(z) = \mathsf{U}(z)\mathsf{D}\mathsf{U}^\star(z)$. However, according to Theorem 2.2.2, we will need the matrix D to have a bigger size. We provide Theorem 3.1.1 and 3.1.2 to illustrate that the necessary lower bound of size D in Theorem 2.2.2 can always be achieved. Theorem 3.1.1 gives the result with the additional assumption that $\det(\mathsf{A}(z))$ is not identically zero. The general situation without this assumption is proved in Theorem 3.1.2. The structure of the proof is inspired by [71], which gives similar results for spectral decompositions of a Hermitian matrix of polynomials defined on $\mathbb{R}$ or $i\mathbb{R}$.

Theorem 3.1.1. *Let* $\mathsf{A}(z)$ *be an* $n \times n$ *Hermitian matrix of Laurent polynomials, i.e.,* $\mathsf{A}^\star(z) = \mathsf{A}(z)$*, and* $\det(\mathsf{A}(z))$ *is not identically zero. Then there exists some* $n \times m$ *matrix* $\mathsf{U}(z)$ *of Laurent polynomials and some* $m \times m$ *constant diagonal matrix* $\mathsf{D} = \operatorname{diag}(\mathbf{I_{m_1}}, -\mathbf{I_{m_2}})$ *with* $m_1 + m_2 = m$ *and* $m_1, m_2 \in \mathbb{N} \cup \{0\}$ *such that* $\mathsf{A}(z) = \mathsf{U}(z)\mathsf{D}\mathsf{U}^\star(z)$ *holds if and only if*

$$m_1 \geqslant \max_{z \in \mathbb{T}} \nu_+(\mathsf{A}(z)), \qquad m_2 \geqslant \max_{z \in \mathbb{T}} \nu_-(\mathsf{A}(z)). \tag{3.1.1}$$

Proof. The necessity part is proved in Theorem 2.2.2, we only need to prove the sufficiency part.

Suppose that the claim holds for

$$m_1 = \max_{z \in \mathbb{T}} \nu_+(\mathsf{A}(z)), \qquad m_2 = \max_{z \in \mathbb{T}} \nu_-(\mathsf{A}(z)),$$

then $\mathsf{A}(z) = \widetilde{\mathsf{U}}(z)\widetilde{\mathsf{D}}\widetilde{\mathsf{U}}^\star(z)$ is obviously still true with $\widetilde{\mathsf{U}}(z) := [\mathbf{0_{n\times s_1}}, \mathsf{U}(z), \mathbf{0_{n\times s_2}}]$ and $\widetilde{\mathsf{D}} := \begin{bmatrix} \mathbf{I_{s_1+m_1}} & \\ & -\mathbf{I_{s_2+m_2}} \end{bmatrix}$, for any integer $s_1, s_2 \geqslant 0$. Therefore, we only need to prove the theorem in the case that m_1, m_2 equal to the lower bound in (3.1.1).

Denote

$$n_+ := \max_{z \in \mathbb{T}} \nu_+(\mathsf{A}(z)), \qquad n_- := \max_{z \in \mathbb{T}} \nu_-(\mathsf{A}(z)). \tag{3.1.2}$$

If $\operatorname{sig}(\mathsf{A}(z))$ is constant on $\mathbb{T} \setminus \sigma(\mathsf{A}(z))$, since $\sigma(\mathsf{A}(z))$ is a finite set, according to Lemma 2.2.1,

$$n_+ = \max_{z \in \mathbb{T}\setminus\sigma(\mathsf{A}(z))} \nu_+(\mathsf{A}(z)) = \nu_+(\mathsf{A}(z_0)), \quad n_- = \max_{z \in \mathbb{T}\setminus\sigma(\mathsf{A}(z))} \nu_-(\mathsf{A}(z)) = \nu_-(\mathsf{A}(z_0)),$$

for all $z_0 \in \mathbb{T}\setminus\sigma(\mathsf{A}(z))$. Thus, $n_+ + n_- = n$. Therefore, the result is proved by Theorem 2.3.1. If $\operatorname{sig}(\mathsf{A}(z))$ is not constant on $\mathbb{T} \setminus \sigma(\mathsf{A}(z))$, we can see that $m_0 := n_+ + n_- > n$. In the following, we will construct $(m_0 - n)$ functions

$\mu_1(z), \ldots, \mu_{m_0-n}(z)$, such that the Hermitian matrix

$$\widetilde{\mathsf{A}}(z) := \begin{bmatrix} \mathsf{A}(z) & & & \\ & \mu_1(z) & & \\ & & \ddots & \\ & & & \mu_{m_0-n}(z) \end{bmatrix} \tag{3.1.3}$$

has constant signature on $\mathbb{T} \setminus \sigma(\widetilde{\mathsf{A}}(z))$.

Since $\det(\mathsf{A}(z))$ is a Laurent polynomial that is not identically zero, $\{z_1, \ldots, z_K\} := \sigma(\mathsf{A}(z)) \bigcap \mathbb{T}$ contains only finite number of points on $\mathbb{T}$. So $\{z_1, \ldots, z_K\}$ cuts $\mathbb{T}$, which is the unit circle in complex plane, into K connected open segments: $\Gamma_1, \Gamma_2 \cdots, \Gamma_K$, such that they satisfy

(1) $\bigcup_{j=1}^{K} \Gamma_j \bigcup \{z_l\}_{l=1}^{K} = \mathbb{T}$;

(2) Pairwise disjoint: $\Gamma_j \bigcap \{z_l\}_{l=1}^{K}$ is empty, $\Gamma_j \bigcap \Gamma_k$ is empty, for all $j, k = 1, \ldots, K$, $j \neq k$;

(3) Both endpoints of Γ_j are contained inside $\{z_l\}_{l=1}^{K}$, denote them by $z_{j,1}$ and $z_{j,2}$, for all $j = 1, 2, \ldots, K$.

By Lemma 2.3.5, we can choose all the eigenvalues $\lambda_1(\xi), \ldots, \lambda_n(\xi)$ of $\mathsf{A}(e^{-i\xi})$ to be continuous functions of $\xi \in \mathbb{R}$. In each Γ_j, since $\det(\mathsf{A}(e^{-i\xi})) = \prod_{k=1}^{n} \lambda_k(\xi) \neq 0$, none of the $\lambda_k(\xi)$ will attain zero. As nonzero continuous functions on an open interval, all $\lambda_k(\xi)$ will not change signs within each Γ_j. Thus $\nu_+(\mathsf{A}(z))$ and $\nu_-(\mathsf{A}(z))$ remain constant on each Γ_j.

For each Γ_j, define a function

$$\eta_j(z) := (z_{j,1} z_{j,2})^{-\frac{1}{2}} z^{-1} (z - z_{j,1})(z - z_{j,2}), \qquad j = 1, \ldots, K.$$

The square root of $z_{j,1} z_{j,2}$ is chosen in the complex plane, where the two possible solutions only differ by a " $-$ " sign. For both solutions, we can directly verify that $\eta_j^\star(z) = \eta_j(z)$. So $\eta_j(z)$ is a real function for all $z \in \mathbb{T}$.

Since the signature of $\mathsf{A}(z)$ is not constant for all $z \in \mathbb{T} \setminus \sigma(\mathsf{A}(z))$, we know that $\mathbb{T}$ contains more than one open segments Γ_j. So $z_{j,1} \neq z_{j,2}$, and both $z_{j,1}$ and $z_{j,2}$ are single roots of $\eta_j(z)$. Hence $\eta_j(z)$ will have different signs

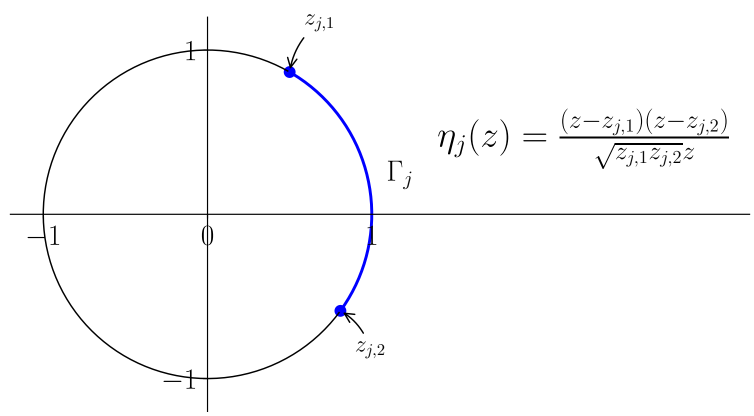


Figure 3.1: $\eta_j(z) > 0$ for all $z \in \Gamma_j$, and $\eta_j(z) < 0$ for all $z \in \Gamma_l, l \neq j$.

between two sides of $z_{j,1}$ and $z_{j,2}$. Therefore, in calculation of the square root of $z_{j,1}z_{j,2}$, we can just choose the solution such that $\eta_j(z) > 0$ for all $z \in \Gamma_j$, and $\eta_j(z) < 0$ for all $z \in \mathbb{T} \setminus \Gamma_j \setminus \{z_{j,1}, z_{j,2}\}$. In summary, $\eta_j(z)$ satisfies

(1) $\eta_j(z)$ is real for all $z \in \mathbb{T}$;

(2) $\eta_j(z) > 0$ for all $z \in \Gamma_j$, and $\eta_j(z) < 0$ for all $z \in \Gamma_k$, $k \neq j$.

Let us construct functions $\mu_k(z)$ recursively for $k = 1, \dots, m_0 - n$, such that (3.1.3) has constant signature on $z \in \mathbb{T} \setminus \sigma(\widetilde{\mathsf{A}}(z))$. Start with $\mathsf{A}_0(z) := \mathsf{A}(z)$, $k = 1$. In order to have the algorithm work, we only need to verify the following two conditions before the start of each new recursion:

(1) $\mathsf{A}_{k-1}(z)$ is a Hermitian matrix of Laurent polynomials, satisfying $\max_{z\in\mathbb{T}} \nu_+(\mathsf{A}_{k-1}(z)) = n_+$ and $\max_{z\in\mathbb{T}} \nu_-(\mathsf{A}_{k-1}(z)) = n_-$, where n_+ and n_- are defined in (3.1.2).

(2) $k \leqslant m_0 - n$.

They are obviously true for $k = 1$.

Define an index set $J := \{j \ : \ \nu_-(\mathsf{A}_{k-1}(z)) = n_- \text{ for all } z \in \Gamma_j\}$. Now, take

$$\mu_k(z) := (-1)^{|J|+1} \prod_{j\in J} \eta_j(z), \qquad \mathsf{A}_k(z) := \begin{bmatrix} \mathsf{A}_{k-1}(z) & \\ & \mu_k(z) \end{bmatrix}.$$

We can see that since $\eta_j(z)$ are real functions on $z \in \mathbb{T}$, $\mu_k^\star(z) = \mu_k(z)$ is also real on $\mathbb{T}$. From $\mathsf{A}_{k-1}^\star(z) = \mathsf{A}_{k-1}(z)$ in condition (1), we know that the matrix $\mathsf{A}_k(z)$ is also a Hermitian matrix of Laurent polynomials. By the construction of $\mu_k(z)$, we can directly verify from the sign of $\eta_j(z)$ that $\mu_k(z) > 0$ for all $z \in \bigcup_{j\in J} \Gamma_j$, and $\mu_k(z) < 0$ for all $z \in \bigcup_{j\notin J} \Gamma_j$. For $z \in \mathbb{T}$, the eigenvalues of $\mathsf{A}_k(z)$ are just all the eigenvalues of $\mathsf{A}_{k-1}(z)$, combined with $\mu_k(z)$. Now, let us calculate $\nu_+(\mathsf{A}_k(z))$ and $\nu_-(\mathsf{A}_k(z))$ on each Γ_j.

- For $z \in \bigcup_{j\in J} \Gamma_j$, since $\mu_k(z) > 0$, we have

$$\nu_-(\mathsf{A}_k(z)) = \nu_-(\mathsf{A}_{k-1}(z)) = n_-.$$

  By condition (2), we know that $k \leqslant m_0 - n = n_+ + n_- - n$, so

$$\begin{aligned}\nu_+(\mathsf{A}_k(z)) &= (n+k) - \nu_-(\mathsf{A}_k(z)) = (n+k) - n_- \\ &\leqslant n + (n_+ + n_- - n) - n_- = n_+.\end{aligned}$$

- For $z \in \bigcup_{j\notin J} \Gamma_j$, since $\mu_k(z) < 0$, and $\nu_-(\mathsf{A}_{k-1}(z)) < n_-$, we have

$$\nu_-(\mathsf{A}_k(z)) = \nu_-(\mathsf{A}_{k-1}(z)) + 1 \leqslant n_-.$$

  Meanwhile, $\nu_+(\mathsf{A}_k(z)) = \nu_+(\mathsf{A}_{k-1}(z)) \leqslant n_+$.

Combining the two cases, we showed that $\max_{z\in\mathbb{T}\setminus\sigma(\mathsf{A}(z))} \nu_+(\mathsf{A}_k(z)) \leqslant n_+$, and $\max_{z\in\mathbb{T}\setminus\sigma(\mathsf{A}(z))} \nu_-(\mathsf{A}_k(z)) \leqslant n_-$. The inequalities of the other direction is obvious, since $\max_{z\in\mathbb{T}\setminus\sigma(\mathsf{A}(z))} \nu_+(\mathsf{A}_k(z)) \geqslant \max_{z\in\mathbb{T}\setminus\sigma(\mathsf{A}(z))} \nu_+(\mathsf{A}_{k-1}(z)) = n_+$, and $\max_{z\in\mathbb{T}\setminus\sigma(\mathsf{A}(z))} \nu_-(\mathsf{A}_k(z)) \geqslant \max_{z\in\mathbb{T}\setminus\sigma(\mathsf{A}(z))} \nu_-(\mathsf{A}_{k-1}(z)) = n_-$. So,

$$\max_{z\in\mathbb{T}\setminus\sigma(\mathsf{A}(z))} \nu_+(\mathsf{A}_k(z)) = n_+, \qquad \max_{z\in\mathbb{T}\setminus\sigma(\mathsf{A}(z))} \nu_-(\mathsf{A}_k(z)) = n_-.$$

By Lemma 2.2.1, we get

$$\max_{z\in\mathbb{T}} \nu_+(\mathsf{A}_k(z)) = n_+, \qquad \max_{z\in\mathbb{T}} \nu_-(\mathsf{A}_k(z)) = n_-. \tag{3.1.4}$$

Now we can take $k := k+1$, and repeat the above procedure recursively to

Example:

$\mathsf{A}_{k-1}(z)$ is a 2×2 matrix.

$n_+ := \max_{z\in\mathbb{T}} \nu_+(\mathsf{A}_{k-1}(z)) = 2.$

$n_- := \max_{z\in\mathbb{T}} \nu_-(\mathsf{A}_{k-1}(z)) = 2.$

$J :=\{j \ : \ \nu_-(\mathsf{A}_{k-1}(z)) = 2,$
$\quad$ for all $z \in \Gamma_j\}$
$=\{2, 4\}.$

Define

$$\mu_k(z) := -\eta_2(z)\eta_4(z).$$

We get

$\mu_k(z) > 0, \quad \forall z \in \Gamma_2 \cup \Gamma_4,$

$\mu_k(z) < 0, \quad \forall z \in \Gamma_1 \cup \Gamma_3.$

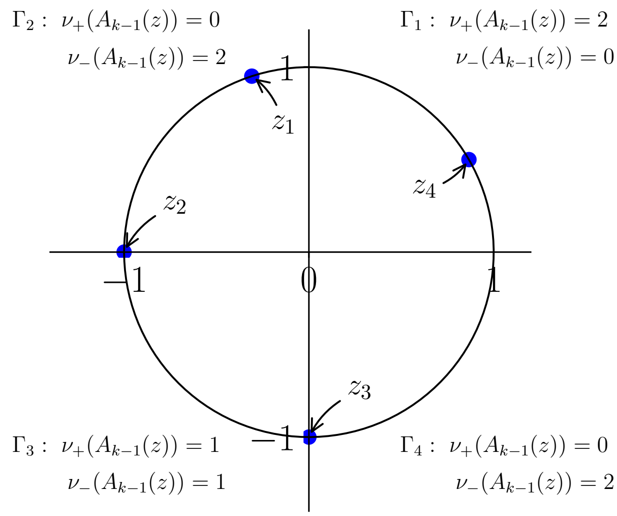


Figure 3.2: A simple example of the construction of $\mu_k(z)$ from $\mathsf{A}_{k-1}(z)$.

construct all the Laurent polynomials $\mu_1(z), \dots, \mu_{m_0-n}(z)$. Equalities (3.1.4) guarantee that the condition (1) will always hold in the new iteration.

Therefore, we can repeat our constructions until the condition (2) is violated. Take $\widetilde{\mathsf{A}}(z) := \mathsf{A}_{m_0-n}(z)$ to be the last matrix constructed. According to (3.1.4), it is an $m_0 \times m_0$ Hermitian matrix of Laurent polynomials still satisfying

$$\max_{z\in\mathbb{T}\setminus\sigma(\widetilde{\mathsf{A}}(z))} \nu_+(\widetilde{\mathsf{A}}(z)) = n_+, \qquad \max_{z\in\mathbb{T}\setminus\sigma(\widetilde{\mathsf{A}}(z))} \nu_-(\widetilde{\mathsf{A}}(z)) = n_-.$$

Since $n_+ + n_- = m_0$, both $\nu_+(\widetilde{\mathsf{A}}(z))$ and $\nu_-(\widetilde{\mathsf{A}}(z))$ must be constant for all $z \in \mathbb{T} \setminus \sigma(\widetilde{\mathsf{A}}(z))$. Hence, $\mathrm{sig}(\widetilde{\mathsf{A}}(z))$ is constant on $\mathbb{T} \setminus \sigma(\widetilde{\mathsf{A}}(z))$. By Theorem 2.3.1, there exists an $m_0 \times m_0$ matrix $\widetilde{\mathsf{U}}(z)$ of Laurent polynomials such that

$$\widetilde{\mathsf{A}}(z) = \widetilde{\mathsf{U}}(z)\mathsf{D}\widetilde{\mathsf{U}}^\star(z) \tag{3.1.5}$$

holds with $\mathsf{D} = \mathrm{diag}(\mathbf{I}_{\mathbf{n}_+}, -\mathbf{I}_{\mathbf{n}_-})$ being the $m_0 \times m_0$ constant diagonal matrix.

From the structure of $\widetilde{\mathsf{A}}(z)$ in (3.1.3), we conclude that $\mathsf{A}(z)$ can be reconstructed by deleting the last m_0-n rows and columns of $\widetilde{\mathsf{A}}(z)$. This corresponds to deleting the last $m_0 - n$ rows of $\widetilde{\mathsf{U}}(z)$ and the last $m_0 - n$ columns of $\widetilde{\mathsf{U}}^\star(z)$ in the above factorization (3.1.5). So, define $\mathsf{U}(z)$ to be the $n \times m_0$ matrix of Laurent polynomials constructed by deleting the last $m_0 - n$ rows of $\widetilde{\mathsf{U}}(z)$, we get the desired factorization

$$\mathsf{A}(z) = \mathsf{U}(z)\mathsf{D}\mathsf{U}^\star(z).$$

This completes the proof for the sufficiency part of the theorem. ■

Theorem 3.1.1 shows that the necessary lower bound $s \geqslant \max_{z\in\mathbb{T}} \nu_+(\mathsf{A}(z)) + \max_{z\in\mathbb{T}} \nu_-(\mathsf{A}(z))$ of the spectral factorization in Theorem 2.2.2 is always achievable, as long as $\det(\mathsf{A}(z))$ is not identically zero. To complete the theory, we provide the following theorem to show that this lower bound can still be reached even for the degenerate case that $\det(\mathsf{A}(z)) = 0$.

For a matrix $\mathsf{A}(z)$ of Laurent polynomials, if its invariant polynomials are $\mathsf{d}_1(z), \cdots, \mathsf{d}_n(z)$, then we call the number of $\mathsf{d}_j(z)$ that are not identically zero the ***general rank*** of $\mathsf{A}(z)$.

Theorem 3.1.2. *Let $\mathsf{A}(z)$ be an $n \times n$ Hermitian matrix of Laurent polynomials. Then there exists some $n \times m$ matrix $\mathsf{U}(z)$ of Laurent polynomials and some $m \times m$ constant diagonal matrix $\mathsf{D} = \operatorname{diag}(\mathbf{I_{m_1}}, -\mathbf{I_{m_2}})$ with $m_1 + m_2 = m$ and $m_1, m_2 \in \mathbb{N} \cup \{0\}$ such that $\mathsf{A}(z) = \mathsf{U}(z)\mathsf{D}\mathsf{U}^\star(z)$ holds if and only if*

$$m_1 \geqslant \max_{z\in\mathbb{T}} \nu_+(\mathsf{A}(z)), \qquad m_2 \geqslant \max_{z\in\mathbb{T}} \nu_-(\mathsf{A}(z)).$$

Proof. The necessity part and the sufficiency part in the case that $\det(\mathsf{A}(z)) \not\equiv 0$ is proved by Theorem 3.1.1. We only need to prove the sufficiency part for the degenerate case. Let us write $\mathsf{A}(z)$ into its Smith Normal Form:

$$\mathsf{A}(z) = \mathsf{E}(z) \begin{bmatrix} \mathsf{d}_1(z) & & & & & \\ & \ddots & & & & \\ & & \mathsf{d}_r(z) & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix} \mathsf{F}(z),$$

where r is the general rank of $\mathsf{A}(z)$, $\mathsf{d}_1(z), \ldots, \mathsf{d}_r(z)$ are the first r invariant polynomials of $\mathsf{A}(z)$ that are not zero, and $\mathsf{E}(z)$ and $\mathsf{F}(z)$ are invertible matrices of Laurent polynomials. Define

$$\mathring{\mathsf{A}}(z) := \mathsf{E}^{-1}(z)\mathsf{A}(z)\mathsf{E}^{-\star}(z) = \begin{bmatrix} \mathsf{d}_1(z) & & & & & \\ & \ddots & & & & \\ & & \mathsf{d}_r(z) & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix} \mathsf{F}(z)\mathsf{E}^{-\star}(z).$$

Then $\mathring{\mathsf{A}}(z)$ is Hermitian and its last $(n-r)$ rows are zero. This implies that its last $(n-r)$ columns must also be zero. Hence,

$$\mathring{\mathsf{A}}(z) = \begin{bmatrix} \widetilde{\mathsf{A}}(z) & \\ & \mathbf{0} \end{bmatrix},$$

where $\widetilde{\mathsf{A}}(z)$ is an $r \times r$ Hermitian matrix of Laurent polynomials. Since the invariant polynomials of $\mathring{\mathsf{A}}(z)$ are the same as that of $\mathsf{A}(z)$, which are $\mathsf{d}_1(z), \ldots, \mathsf{d}_r(z), 0, \ldots, 0$, we know that the invariant polynomials of $\widetilde{\mathsf{A}}(z)$ must be $\mathsf{d}_1(z), \ldots, \mathsf{d}_r(z)$. So $\det(\widetilde{\mathsf{A}}(z))$ is not identically zero. Also, for all $z \in \mathbb{T}$, since $\mathsf{E}^{-1}(z)$ is nonsingular, we get

$$\nu_+(\widetilde{\mathsf{A}}(z)) = \nu_+(\mathring{\mathsf{A}}(z)) = \nu_+(\mathsf{A}(z)), \qquad \nu_-(\widetilde{\mathsf{A}}(z)) = \nu_-(\mathring{\mathsf{A}}(z)) = \nu_-(\mathsf{A}(z)).$$

Applying Theorem 3.1.1, we know that for every

$$m \geqslant \max_{z\in\mathbb{T}} \nu_+(\widetilde{\mathsf{A}}(z)) + \max_{z\in\mathbb{T}} \nu_-(\widetilde{\mathsf{A}}(z)) = \max_{z\in\mathbb{T}} \nu_+(\mathsf{A}(z)) + \max_{z\in\mathbb{T}} \nu_-(\mathsf{A}(z)),$$

there exists an $r \times m$ matrix of Laurent polynomials $\widetilde{\mathsf{U}}(z)$, and an $m \times m$ constant diagonal matrix $\mathsf{D} = \operatorname{diag}(\mathbf{I_{m_1}}, -\mathbf{I_{m_2}})$, for some integers m_1 and m_2 satisfying $m_1 + m_2 = m$, such that

$$\widetilde{\mathsf{A}}(z) = \widetilde{\mathsf{U}}(z)\mathsf{D}\widetilde{\mathsf{U}}^\star(z).$$

Adding $(n-r)$ more rows of zeros to $\widetilde{\mathsf{U}}(z)$ yields an $n \times m$ matrix $\mathsf{V}(z) := \begin{bmatrix} \widetilde{\mathsf{U}}(z) \\ \mathbf{0_{(n-r)\times m}} \end{bmatrix}$. We can directly verify that $\mathring{\mathsf{A}}(z) = \mathsf{V}(z)\mathsf{D}\mathsf{V}^\star(z)$. Define $\mathsf{U}(z) := \mathsf{E}(z)\mathsf{V}(z)$, we know $\mathsf{A}(z) = \mathsf{U}(z)\mathsf{D}\mathsf{U}^\star(z)$ holds. This completes the proof of the theorem. ■

3.2 Algorithm for Constructing Quasi-tight Framelet Filter Banks with Minimum Number of High-pass Filters and High Order of Vanishing Moments

Since the proof of Theorem 3.1.2 is constructive, adopting the procedures in Theorem 2.1.2, we can use it to construct quasi-tight framelet filter banks directly. Hence, combining Theorem 2.1.2 and Theorem 3.1.2, we get the following theorem.

Theorem 3.2.1. *Let $a, \Theta \in l_0(\mathbb{Z})\backslash\{0\}$ be two finitely supported not-identically-zero filters such that $\Theta^\star = \Theta$. Suppose n_b is a positive integer satisfying*

$$1 \leqslant n_b \leqslant \min(\operatorname{sr}(a), \tfrac{1}{2}\operatorname{vm}(\Theta(z) - \Theta(z^2)\mathsf{a}(z)\mathsf{a}^\star(z))). \qquad (3.2.1)$$

Let the quantities $s^+_{a,\Theta}, s^-_{a,\Theta}, s_{a,\Theta}$ *be defined as*

$$s^+_{a,\Theta} := \max_{z\in\mathbb{T}} \nu_+(\mathcal{M}_{a,\Theta}(z)), \quad s^-_{a,\Theta} := \max_{z\in\mathbb{T}} \nu_-(\mathcal{M}_{a,\Theta}(z)), \quad s_{a,\Theta} := s^+_{a,\Theta} + s^-_{a,\Theta}, \tag{3.2.2}$$

and $\mathcal{M}_{a,\Theta}(z)$ *be defined in* (2.1.4). *Then for* $s = s_{a,\Theta}$ *there exist* $b_1, \ldots, b_s \in l_0(\mathbb{Z})$, *and* $\epsilon_1 = \ldots = \epsilon_{s^+_{a,\Theta}} = 1$, $\epsilon_{s^+_{a,\Theta}+1} = \ldots = \epsilon_s = -1$ *such that* $\{a; b_1, \ldots, b_s\}_{\Theta,(\epsilon_1,\ldots,\epsilon_s)}$ *is a quasi-tight framelet filter bank with* $\min\{\mathrm{vm}(b_1), \ldots, \mathrm{vm}(b_s)\} \geqslant n_b$. *Moreover, for* $1 \leqslant s < s_{a,\Theta}$, *there does not exist a quasi-tight framelet filter bank* $\{a; b_1, \ldots, b_s\}_{\Theta,(\epsilon_1,\ldots,\epsilon_s)}$ *with* $b_1, \ldots, b_s \in l_0(\mathbb{Z})$ *and* $\epsilon_1, \ldots, \epsilon_s \in \{-1, 1\}$.

Furthermore, if $\mathsf{a}(1) = \Theta(1) = 1$, $\mathsf{b}_1(1) = \cdots = \mathsf{b}_s(1) = 0$, *and* $\phi \in L_2(\mathbb{R})$ *with* ϕ *being defined as* $\widehat{\phi}(\xi) := \prod_{j=1}^{\infty} \widehat{a}(2^{-j}\xi)$, *then* $\{\psi^1, \ldots, \psi^s\}_{(\epsilon_1,\ldots,\epsilon_s)}$ *is a homogeneous quasi-tight framelet in* $L_2(\mathbb{R})$, *where* $\psi^1, \ldots, \psi^s \in L_2(\mathbb{R})$ *are defined in* $\widehat{\psi^\ell}(2\xi) = \widehat{b_\ell}(\xi)\widehat{\phi}(\xi)$.

The following algorithm helps us to find a quasi-tight framelet filter bank with high order of vanishing moments and minimum number of high-pass filters.

Algorithm 3.1. *Input nonzero filters* Θ, a, *such that* $\mathsf{a}(1) = \Theta(1) = 1$, $\Theta^\star = \Theta$. *Choose integers* n_b *satisfying* (3.2.1) *and* $s \geqslant s_{a,\Theta}$, *where* $s_{a,\Theta}$ *is defined in* (3.2.2).

(S1) *Calculate* $\mathcal{N}_{\mathsf{a},\Theta|n_b}(z)$ *as defined in* (2.1.10) *and* (2.1.7).

(S2) *Use Theorem 3.1.2 to find a spectral decomposition of* $\mathcal{N}_{\mathsf{a},\Theta|n_b}(z)$:

$$\mathcal{N}_{\mathsf{a},\Theta|n_b}(z) = \mathsf{U}(z) \begin{bmatrix} \epsilon_1 & & \\ & \ddots & \\ & & \epsilon_s \end{bmatrix} \mathsf{U}^\star(z)$$

where $\mathsf{U}(z)$ *is a* $2 \times s$ *matrix of Laurent polynomials, and* $\epsilon_1, \ldots, \epsilon_s = \pm 1$.

(S3) *Define high-pass filters* $b_1, \cdots, b_s$ *as:*

$$\begin{bmatrix} \mathsf{b}_1(z) & \cdots & \mathsf{b}_s(z) \end{bmatrix} = (1 - z^{-1})^{n_b} \begin{bmatrix} 1 & z \end{bmatrix} \mathsf{U}(z^2).$$

Then $\{a; b_1, \dots, b_s\}_{\Theta,(\epsilon_1,\dots,\epsilon_s)}$ *is a quasi-tight framelet filter bank with at least* n_b *order of vanishing moments.*

3.3 Illustrative Examples

Example 3.1. *Choose the low-pass filter*

$$\widehat{a}(\xi) = \tfrac{1}{26}(e^{-i\xi}+1)(2\cos(\xi)-1)(18\cos(\xi)-5).$$

Notice that the refinable function ϕ *associated with the low-pass filter* a *does not have stable shifts. Also, we can calculate that* $|\widehat{a}(2\pi/3)| = \frac{14}{13} > 1$*, and* $|\widehat{a}(2\pi/3)| \notin \{2^j \;:\; j \in \mathbb{N}\}$*. Hence, by [49, Proposition 4.4], there does not exist a Laurent polynomial* Θ *with real coefficients, such that* $\mathcal{M}_{a,\Theta}(z)$ *is positive semidefinite. Hence, it is not possible use OEP to construct a real-valued tight framelet from such low-pass filter* a*. Notice that* $\mathrm{sr}(a) = 1$ *and* $\mathrm{vm}(1-\mathsf{a}\mathsf{a}^\star) = 2$*, and we have* $\mathrm{sm}(a) = 0.7693$*. Take* $\Theta(z) = 1$ *and* $n_b = 1$*, we can get quasi-tight framelet filter bank* $\{a; b_1, b_2, b_3\}_{\Theta,(-1,1,1)}$ *as*

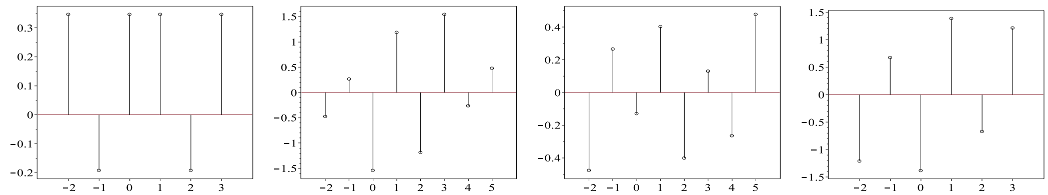
$$\begin{aligned}
\mathsf{b}_1(z) &= \tfrac{\sqrt{2}}{97344} z^{-2}(z-1)(9z^2+4z+9)(3645z^4+9782z^2+3645),\\
\mathsf{b}_2(z) &= \tfrac{\sqrt{2}}{97344} z^{-2}(z-1)(9z^2+4z+9)(3645z^4-1034z^2+3645),\\
\mathsf{b}_3(z) &= \tfrac{1}{52} z^{-2}(z-1)(63z^4+28z^3+100z^2+28z+63).
\end{aligned}$$

We have $\mathrm{vm}(b_1) = \mathrm{vm}(b_2) = \mathrm{vm}(b_3) = 1$*.*

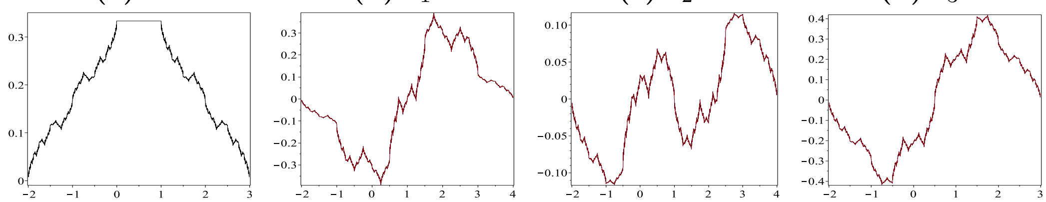
Example 3.2. *Choose* $\widehat{\Theta}(\xi) = 1$ *and the low-pass filter*

$$\widehat{a}(\xi) = \cos^2(\xi/2)\left(1 + \left(2\sqrt{6}-4\right)\sin^2(\xi/2)\right).$$

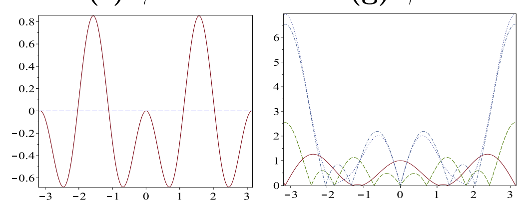
Notice that $\mathrm{sr}(\mathsf{a}) = 2$*,* $\mathrm{vm}(1-\mathsf{a}\mathsf{a}^\star) = 2$*, and we have* $\mathrm{sm}(a) = 0.9382$*. From the frequency plot of* $\det(\mathcal{M}_{\mathsf{a}}(\xi)) = 1 - |\widehat{a}(\xi)|^2 - |\widehat{a}(\xi+\pi)|^2$ *in Figure 3.4(i), we can see that its sign is not fixed. Hence we need at least three high-pass filters to construct a quasi-tight framelet filter bank. Take* $n_b = 1$*, then the*



(a) a **(b)** b_1 **(c)** b_2 **(d)** b_3



(e) ϕ **(f)** ψ^1 **(g)** ψ^2 **(h)** ψ^3



(i) $\det(\mathcal{M}_{\mathsf{a}}(\xi))$ **(j)** $|\widehat{a}|, |\widehat{b_1}|, |\widehat{b_2}|, |\widehat{b_3}|$

Figure 3.3: In Example 3.1: (a),(b),(c) and (d) are the graphs of the filters a, b_1, b_2 and b_3. (e) Scaling function ϕ. (f) - (h) Wavelet functions ψ^1, ψ^2 and ψ^3. (i) $\det(\mathcal{M}_{\mathsf{a}}(\xi)) = 1 - |\widehat{a}(\xi)|^2 - |\widehat{a}(\xi+\pi)|^2$, for $\xi \in [-\pi, \pi]$, where the dashed line is $y = 0$. (j) $|\widehat{a}(\xi)|$ (in solid line), $|\widehat{b_1}(\xi)|$ (in dotted line), $|\widehat{b_2}(\xi)|$ (in dashed line) and $|\widehat{b_3}(\xi)|$ (in dash-dotted line) for $\xi \in [-\pi, \pi]$.

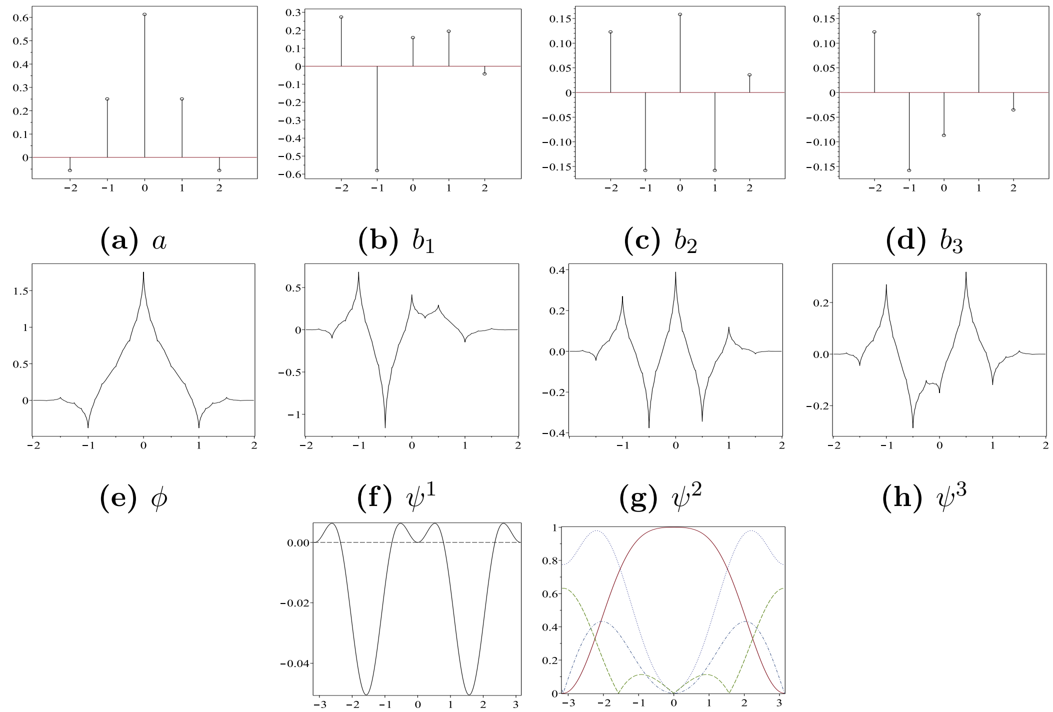
constructed quasi-tight framelet filter bank $\{a; b_1, b_2, b_3\}_{\Theta,\{1,1,-1\}}$ *is given by:*

$$\begin{aligned} b_1 =& \tfrac{\sqrt{10}}{40}\{\sqrt{6}+1,\ \ -3\sqrt{6},\ 2,\ \sqrt{6},\ \sqrt{6}-3\}_{[-2,2]},\\ b_2 =& \tfrac{\sqrt{10}}{40}\{4-\sqrt{6},\ \ -2,\ 2,\ \ -2,\ \sqrt{6}-2\}_{[-2,2]},\\ b_3 =& \tfrac{\sqrt{10}}{40}\{4-\sqrt{6},\ \ -2,\ 2\sqrt{6}-6,\ 2,\ 2-\sqrt{6}\}_{[-2,2]}. \end{aligned}$$

We have $\mathrm{vm}(b_1) = \mathrm{vm}(b_2) = 1,$ *while* $\mathrm{vm}(b_3) = 2.$

Example 3.3. *Choose* $\widehat{\Theta}(\xi) = 1$ *and the low-pass filter*

$$\widehat{a}(\xi) = \frac{1+e^{-i\xi}}{2}\cos^2(\xi/2)\left(1 + \left(2\sqrt{15}-6\right)\sin^2(\xi/2)\right).$$



(a) a **(b)** b_1 **(c)** b_2 **(d)** b_3

(e) ϕ **(f)** ψ^1 **(g)** ψ^2 **(h)** ψ^3

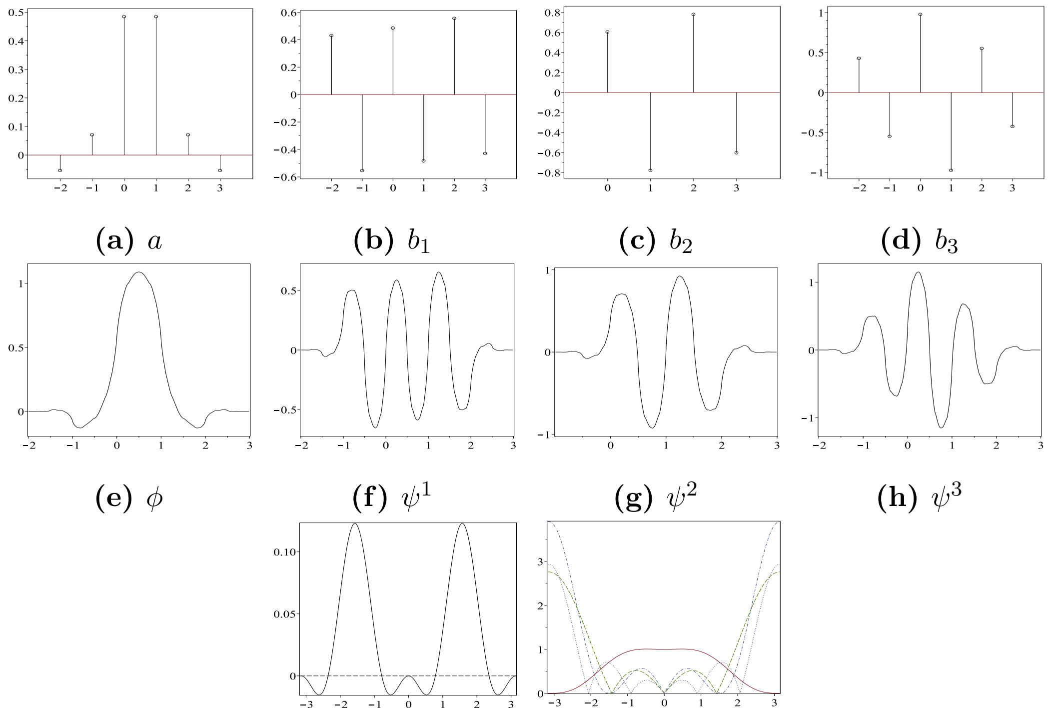
(i) $\det(\mathcal{M}_{\mathsf{a}}(\xi))$ **(j)** $|\widehat{a}|, |\widehat{b_1}|, |\widehat{b_2}|, |\widehat{b_3}|$

Figure 3.4: In Example 3.2: (a),(b),(c) and (d) are the graphs of the filters a, b_1, b_2 and b_3. (e) Scaling function ϕ. (f) - (h) Wavelet functions ψ^1, ψ^2 and ψ^3. (i) $\det(\mathcal{M}_{\mathsf{a}}(\xi)) = 1 - |\widehat{a}(\xi)|^2 - |\widehat{a}(\xi+\pi)|^2$, for $\xi \in [-\pi, \pi]$, where the dashed line is $y = 0$. (j) $|\widehat{a}(\xi)|$ (in solid line), $|\widehat{b_1}(\xi)|$ (in dotted line), $|\widehat{b_2}(\xi)|$ (in dashed line) and $|\widehat{b_3}(\xi)|$ (in dash-dotted line) for $\xi \in [-\pi, \pi]$.

Notice that $\operatorname{sr}(\mathsf{a}) = 3$*,* $\operatorname{vm}(1 - \mathsf{a}\mathsf{a}^\star) = 2$*, and we have* $\operatorname{sm}(a) = 1.5420$*. From the frequency plot of* $\det(\mathcal{M}_{\mathsf{a}}(\xi)) = 1 - |\widehat{a}(\xi)|^2 - |\widehat{a}(\xi+\pi)|^2$ *in Figure 3.5(i), we can see that its sign is not fixed. Hence we need at least three high-pass filters to construct a quasi-tight framelet filter bank. Take* $n_b = 1$*, then the constructed quasi-tight framelet filter bank* $\{a; b_1, b_2, b_3\}_{\Theta,\{1,1,-1\}}$ *is given by:*

$$
\begin{aligned}
b_1 =& \tfrac{1}{16}\{\sqrt{15}+3,\ \ -\sqrt{15}-5,\ 2\sqrt{15},\ \ -2\sqrt{15},\ \sqrt{15}+5,\ \ -3-\sqrt{15}\}_{[-2,3]},\\
b_2 =& \tfrac{\sqrt[4]{15}\sqrt{2}}{8}\{\sqrt{3},\ \ -\sqrt{5},\ \sqrt{5},\ \ -\sqrt{3}\}_{[0,3]},\\
b_3 =& \tfrac{\sqrt[4]{15}}{8}\{\sqrt{3},\ \ -\sqrt{5},\ \sqrt{5}+\sqrt{3},\ \ -\sqrt{5}-\sqrt{3},\ \sqrt{5},\ \ -\sqrt{3}\}_{[-2,3]}.
\end{aligned}
$$

We have $\operatorname{vm}(b_1) = \operatorname{vm}(b_2) = \operatorname{vm}(b_3) = 1$.



(a) a (b) b_1 (c) b_2 (d) b_3

(e) ϕ (f) ψ^1 (g) ψ^2 (h) ψ^3

(i) $\det(\mathcal{M}_{\mathsf{a}}(\xi))$ (j) $|\widehat{a}|, |\widehat{b_1}|, |\widehat{b_2}|, |\widehat{b_3}|$

Figure 3.5: In Example 3.3: (a),(b),(c) and (d) are the graphs of the filters a, b_1, b_2 and b_3. (e) Scaling function ϕ. (f) - (h) Wavelet functions ψ^1, ψ^2 and ψ^3. (i) $\det(\mathcal{M}_{\mathsf{a}}(\xi)) = 1 - |\widehat{a}(\xi)|^2 - |\widehat{a}(\xi+\pi)|^2$, for $\xi \in [-\pi, \pi]$, where the dashed line is $y = 0$. (j) $|\widehat{a}(\xi)|$ (in solid line), $|\widehat{b_1}(\xi)|$ (in dotted line), $|\widehat{b_2}(\xi)|$ (in dashed line) and $|\widehat{b_3}(\xi)|$ (in dash-dotted line) for $\xi \in [-\pi, \pi]$.

Example 3.4. *Choose* $\widehat{\Theta}(\xi) = 1$ *and the low-pass filter*

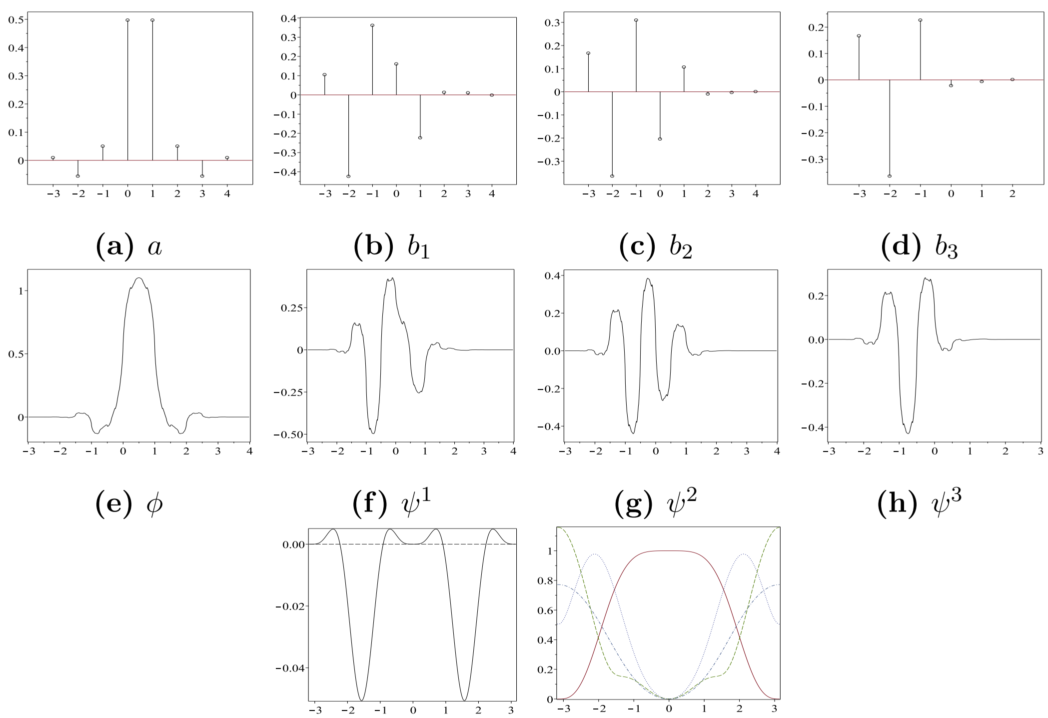
$$\widehat{a}(\xi) = \frac{1 + e^{-i\xi}}{2} \cos^2(\xi/2) \left(1 + \tfrac{3}{2}\sin^2(\xi/2) + \tfrac{6}{5}\sin^4(\xi/2)\right).$$

Notice that $\mathrm{sr}(\mathsf{a}) = 3$, $\mathrm{vm}(1 - \mathsf{a}\mathsf{a}^\star) = 4$, *and we have* $\mathrm{sm}(a) = 1.3125$. *From the frequency plot of* $\det(\mathcal{M}_{\mathsf{a}}(\xi)) = 1 - |\widehat{a}(\xi)|^2 - |\widehat{a}(\xi+\pi)|^2$ *in Figure 3.6(i), we can see that its sign is not fixed. Hence we need at least three high-pass filters to construct a quasi-tight framelet filter bank. Take* $n_b = 2$, *then the constructed quasi-tight framelet filter bank* $\{a; b_1, b_2, b_3\}_{\Theta,\{1,1,-1\}}$ *is given by:*

$$b_1 = \tfrac{\sqrt{2741311}}{877219520}\{55379, -224754, 191376, 85039, -118449, 6864, 5454, -909\}_{[-3,4]},$$

$b_2 = \frac{3\sqrt{8223933}}{109652440}\{2122, -4652, 3943, -2608, 1357, -127, -42, 7\}_{[-3,4]}$,
$b_3 = \frac{3\sqrt{8223933}}{54826220}\{1061, -2326, 1441, -141, -42, 7\}_{[-3,2]}$.

We have $\mathrm{vm}(b_1) = \mathrm{vm}(b_2) = \mathrm{vm}(b_3) = 2$.



(a) a **(b)** b_1 **(c)** b_2 **(d)** b_3

(e) ϕ **(f)** ψ^1 **(g)** ψ^2 **(h)** ψ^3

(i) $\det(\mathcal{M}_{\mathsf{a}}(\xi))$ **(j)** $|\widehat{a}|, |\widehat{b_1}|, |\widehat{b_2}|, |\widehat{b_3}|$

Figure 3.6: In Example 3.4: (a),(b),(c) and (d) are the graphs of the filters a, b_1, b_2 and b_3. (e) Scaling function ϕ. (f) - (h) Wavelet functions ψ^1, ψ^2 and ψ^3. (i) $\det(\mathcal{M}_{\mathsf{a}}(\xi)) = 1 - |\widehat{a}(\xi)|^2 - |\widehat{a}(\xi+\pi)|^2$, for $\xi \in [-\pi, \pi]$, where the dashed line is $y = 0$. (j) $|\widehat{a}(\xi)|$ (in solid line), $|\widehat{b_1}(\xi)|$ (in dotted line), $|\widehat{b_2}(\xi)|$ (in dashed line) and $|\widehat{b_3}(\xi)|$ (in dash-dotted line) for $\xi \in [-\pi, \pi]$.

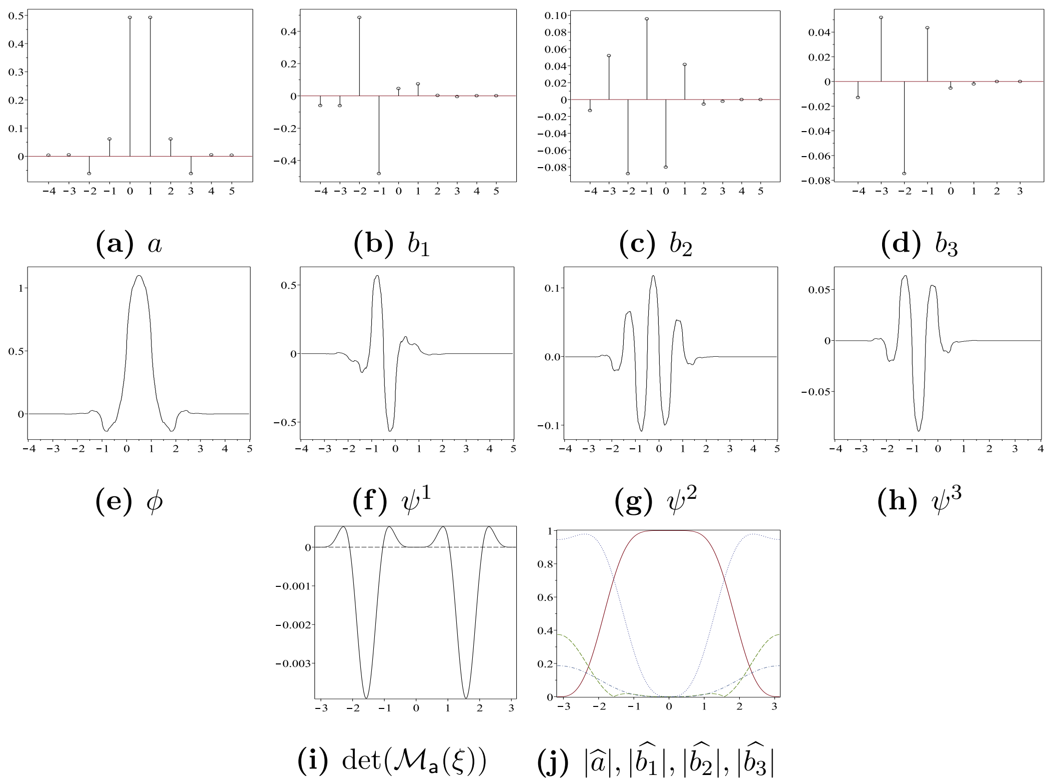
Example 3.5. *Choose* $\widehat{\Theta}(\xi) = 1$ *and the low-pass filter*

$$\widehat{a}(\xi) = \frac{1+e^{-i\xi}}{2}\cos^2(\xi/2)\left(1 + \tfrac{3}{2}\sin^2(\xi/2) + \tfrac{15}{8}\sin^4(\xi/2) - \tfrac{55}{32}\sin^6(\xi/2)\right).$$

Notice that $\mathrm{sr}(\mathsf{a}) = 3$, $\mathrm{vm}(1 - \mathsf{a}\mathsf{a}^\star) = 6$, *and we have* $\mathrm{sm}(a) = 1.4862$. *From the frequency plot of* $\det(\mathcal{M}_{\mathsf{a}}(\xi)) = 1 - |\widehat{a}(\xi)|^2 - |\widehat{a}(\xi+\pi)|^2$ *in Figure 3.7(i), we can see that its sign is not fixed. Hence we need at least three high-pass* *filters to construct a quasi-tight framelet filter bank. Take* $n_b = 3$*, then the constructed quasi-tight framelet filter bank* $\{a; b_1, b_2, b_3\}_{\Theta,\{1,1,-1\}}$ *is given by:*

$$\begin{aligned}
\mathsf{b}_1(z) &= \tfrac{5\sqrt{4517491323641}(z-1)^3}{37007288923267072z^4}(698885z^6 + 3049680z^5 - 9360861z^4 - 28741088z^3 \\
&\qquad + 201849675z^2 + 839872848z + 209192269), \\
\mathsf{b}_2(z) &= -\tfrac{\sqrt{745386068400765}(z^2+1)(z-1)^3}{74014577846534144z^4}(163295z^4 + 712560z^3 + 7290978z^2 \\
&\qquad + 34643824z - 35434017), \\
\mathsf{b}_3(z) &= \tfrac{\sqrt{185217144269281}(z-1)^3}{37007288923267072z^4}(163295z^4 + 712560z^3 + 7290978z^2 \\
&\qquad + 34643824z - 35434017).
\end{aligned}$$

We have $\mathrm{vm}(b_1) = \mathrm{vm}(b_2) = \mathrm{vm}(b_3) = 3$.



(a) a (b) b_1 (c) b_2 (d) b_3

(e) ϕ (f) ψ^1 (g) ψ^2 (h) ψ^3

(i) $\det(\mathcal{M}_{\mathsf{a}}(\xi))$ (j) $|\widehat{a}|, |\widehat{b_1}|, |\widehat{b_2}|, |\widehat{b_3}|$

Figure 3.7: In Example 3.5: (a),(b),(c) and (d) are the graphs of the filters a, b_1, b_2 and b_3. (e) Scaling function ϕ. (f) - (h) Wavelet functions ψ^1, ψ^2 and ψ^3. (i) $\det(\mathcal{M}_{\mathsf{a}}(\xi)) = 1 - |\widehat{a}(\xi)|^2 - |\widehat{a}(\xi + \pi)|^2$, for $\xi \in [-\pi, \pi]$, where the dashed line is $y = 0$. (j) $|\widehat{a}(\xi)|$ (in solid line), $|\widehat{b_1}(\xi)|$ (in dotted line), $|\widehat{b_2}(\xi)|$ (in dashed line) and $|\widehat{b_3}(\xi)|$ (in dash-dotted line) for $\xi \in [-\pi, \pi]$.

Chapter 4

Quasi-tight Framelets with Symmetry

In many applications of wavelets/framelets, since the signals we want to analyze have certain structures, we hope that the wavelets/framelets we constructed have symmetry. This motivates the study of quasi-tight framelet filter banks with symmetry. This chapter characterizes the existence of (anti-)symmetric quasi-tight framelet filter banks with two generators $\{a; b_1, b_2\}_{\Theta,(\epsilon_1,\epsilon_2)}$. Since the tight framelet case ($\epsilon_1 = \epsilon_2 = 1$) has been studied in [40, 48], we only investigate the case with signature $(\epsilon_1, \epsilon_2) = (1, -1)$. We start from introducing some properties of (anti-)symmetric Laurent polynomials in Section 4.1. In Section 4.2 and Section 4.3, we study the problem of matrix spectral factorization with symmetry. Necessary and sufficient conditions on spectral factorizations of 2×2 matrices of Laurent polynomials with symmetry are derived. The (constructive) proof starts with the simple case that the matrix has empty spectrum in Section 4.2, and the result for the general case is given in Section 4.3. In Section 4.4, we apply the spectral factorization theorem to characterize the existence of quasi-tight framelet filter banks with symmetry. Since the proofs in this chapter are all constructive, we can use them directly as algorithms to calculate some illustrative examples, which are given in the last section of this chapter. The results in this chapter are summarized in [19].

4.1 Introduction to Laurent Polynomials with Symmetry

We say a filter (or sequence) $u = \{u(k)\}_{k\in\mathbb{Z}} : \mathbb{Z} \to \mathbb{C}$ has ***symmetry*** if

$$u(c-k) = \epsilon u(k), \qquad \forall\ k\ \in \mathbb{Z}, \tag{4.1.1}$$

holds with some $c \in \mathbb{Z}$ and $\epsilon \in \{-1, 1\}$. $\frac{c}{2}$ is called the ***symmetry center***. For compactly supported filters, it is also the center of the filter support $\mathrm{fsupp}(u)$. If $\epsilon = 1$, we call u to be ***symmetric***; if $\epsilon = -1$, we call it ***antisymmetric***. For filter $u \neq 0$ having symmetry as (4.1.1), we define the symmetry operator S in both frequency domain and z-domain to address the symmetry type:

$$\mathsf{S}\widehat{u}(\xi) := \frac{\widehat{u}(\xi)}{\widehat{u}(-\xi)} = \epsilon e^{-ic\xi}, \qquad \xi \in \mathbb{R}, \tag{4.1.2}$$

$$\mathsf{Su}(z) := \frac{\mathsf{u}(z)}{\mathsf{u}(z^{-1})} = \epsilon z^c, \qquad z \in \mathbb{C} \setminus \{0\}. \tag{4.1.3}$$

For filter $u \in l_0(\mathbb{Z})$ and $u \neq 0$, it is straightforward to verify that (4.1.1)(4.1.2) and (4.1.3) are equivalent representations of symmetry in time/frequency/z-domain respectively. So, in this case, we would generalize the definition and say ***the Laurent polynomial*** $\mathsf{u}(z)$ ***has symmetry with type*** ϵz^c.

Sometimes we will need to deal with the sequence which is identically $\mathbf{0}$. We just say $\mathbf{0}$ has symmetry of any type. We still use $\mathsf{S}\mathbf{0} = \epsilon z^c$ to denote it, although it cannot be calculated from (4.1.2) or (4.1.3). This would be compatible with all the calculations we perform later.

For an integer c, define $\mathrm{odd}(c) := \begin{cases} 0, & \text{if } c \text{ is even,} \\ 1, & \text{if } c \text{ is odd} \end{cases}$. Then we have the following proposition.

Proposition 4.1.1. *Suppose $u, v \in l_0(\mathbb{Z})$, and they both have symmetry with types $\epsilon_u z^{c_u}$ and $\epsilon_v z^{c_v}$, respectively. Then $\mathsf{u}(z)\mathsf{v}(z)$, $\mathsf{u}^\star(z)$ and $\mathsf{u}(z)/\mathsf{v}(z)$ (if $\mathsf{v}(z)|\mathsf{u}(z)$) all have symmetry, with types*

(1) $\mathsf{S}\left(\mathsf{u}(z)\mathsf{v}(z)\right) = \epsilon_u \epsilon_v z^{c_u + c_v}$;

(2) $\mathsf{S}\left(\mathsf{u}(z)/\mathsf{v}(z)\right) = \epsilon_u \epsilon_v z^{c_u - c_v}$;

(3) $\mathsf{S}\mathsf{u}^\star(z) = [\mathsf{S}\mathsf{u}(z)]^\star = \frac{1}{\mathsf{S}\mathsf{u}(z)} = \epsilon_u z^{-c_u}$;

(4) *if* $\mathsf{u}(z) \neq 0$, *we have* $c_u = \operatorname{ldeg}(\mathsf{u}(z)) + \deg(\mathsf{u}(z))$;

(5) *if* $\mathsf{u}(z) \neq 0$, *we have* $\operatorname{odd}(\operatorname{len}(\mathsf{u}(z))) = \operatorname{odd}(c_u)$.

Proof. If either $u = 0$ or $v = 0$, the results can be verified directly. Otherwise, items (1)(2)(3) can be proved by straightforward calculations using (4.1.3). To prove item (4), we can see from the definition of symmetry in time domain (4.1.1) that $c_u/2$ is the symmetry center of $\mathsf{u}(z)$. That is, $\frac{c_u}{2} = \frac{\operatorname{ldeg}\mathsf{u}(z) + \deg(\mathsf{u}(z))}{2}$, which implies item (4). For item (5), we can see from item (4) that

$$\begin{aligned}\operatorname{len}(\mathsf{u}(z)) =& \deg(\mathsf{u}(z)) - \operatorname{ldeg}(\mathsf{u}(z)) = \deg(\mathsf{u}(z)) + \operatorname{ldeg}(\mathsf{u}(z)) - 2\operatorname{ldeg}(\mathsf{u}(z)) \\ =& c_u - 2\operatorname{ldeg}(\mathsf{u}(z)).\end{aligned}$$

The above equation shows that the item (5) is true. ■

The symmetry property of a Laurent polynomial is also related to its root information. We have the following lemma which would be useful later. The proof can be found in Proposition 2.2 and Lemma 2.4 of [40].

Lemma 4.1.1. *Let* $\mathsf{p}_1(z), \dots, \mathsf{p}_n(z)$ *be* n *Laurent polynomials.*

(1) $\mathsf{p}_1(z)$ *has symmetry if and only if* $\mathsf{Z}(\mathsf{p}_1, z) = \mathsf{Z}(\mathsf{p}_1, z^{-1})$, $\forall z \in \mathbb{C} \setminus \{0\}$.

(2) *If* $\mathsf{p}_1(z), \dots, \mathsf{p}_n(z)$ *all have symmetry, then* $\gcd(\mathsf{p}_1(z), \dots, \mathsf{p}_n(z))$ *also has symmetry.*

(3) *There exists some* $c \in \mathbb{Z}$ *and* $\epsilon \in \mathbb{T}$, *such that* $\mathsf{p}(z) = \epsilon z^c \mathsf{p}^\star(z)$ *if and only if* $\mathsf{Z}(\mathsf{p}, z) = \mathsf{Z}(\mathsf{p}, \overline{z}^{-1})$, $\forall z \in \mathbb{C} \setminus \{0\}$.

Also, the symmetry type of a Laurent polynomial is determined by the multiplicities of its roots at ± 1. See the following lemma.

Lemma 4.1.2. *Given a Laurent polynomial* $\mathsf{p}(z) \neq 0$ *with symmetry:* $\mathsf{S}\mathsf{p}(z) = \epsilon z^c$, $\epsilon \in \{1, -1\}$, $c \in \mathbb{Z}$. *Then the following results hold.*

(1) $\epsilon = (-1)^{\mathsf{Z}(\mathsf{p}(z),1)}$.

(2) $\mathrm{odd}(c) = \mathrm{odd}(\mathsf{Z}(\mathsf{p}(z),1) + \mathsf{Z}(\mathsf{p}(z),-1))$.

Proof. Since $\mathsf{p}(z) \neq 0$, we can write $\mathsf{p}(z) = (z-1)^{\mathsf{Z}(\mathsf{p}(z),1)}\mathsf{q}(z)$ for some Laurent polynomial $\mathsf{q}(z)$ satisfying $\mathsf{q}(1) \neq 0$. So

$$\epsilon z^c = \mathsf{Sp}(z) = \frac{\mathsf{p}(z)}{\mathsf{p}(z^{-1})} = (-z)^{\mathsf{Z}(\mathsf{p}(z),1)}\frac{\mathsf{q}(z)}{\mathsf{q}(z^{-1})}, \qquad z \in \mathbb{C} \setminus \{0\}.$$

Plugging in $z = 1$, we get $\epsilon = (-1)^{\mathsf{Z}(\mathsf{p}(z),1)}$. This proves the item (1).

Since $\mathsf{p}(z)$ has symmetry, by the item (1) of Lemma 4.1.1, we know that

$$\mathsf{Z}(\mathsf{p}(z), z_0) = \mathsf{Z}(\mathsf{p}(z), z_0^{-1}), \qquad \forall\ z_0 \in \mathbb{C} \setminus \{0\}. \tag{4.1.4}$$

Denote $\sigma(\mathsf{p}) := \{z_0 \in \mathbb{C} \setminus \{0\}\ :\ \mathsf{p}(z_0) = 0\}$. Partition $\sigma(\mathsf{p})$ into a disjoint union as: $\sigma(\mathsf{p}) = \sigma_{in} \cup \sigma_{out} \cup \sigma_{up} \cup \sigma_{down} \cup \sigma_{\pm 1}$, where

$$\sigma_{in} := \{z_0 \in \sigma(\mathsf{p})\ :\ |z_0| < 1\}, \qquad \sigma_{out} := \{z_0 \in \sigma(\mathsf{p})\ :\ |z_0| > 1\},$$
$$\sigma_{up} := \{z_0 \in \sigma(\mathsf{p}) \cap \mathbb{T}\ :\ \mathrm{Im}(z_0) > 0\}, \quad \sigma_{down} := \{z_0 \in \sigma(\mathsf{p}) \cap \mathbb{T}\ :\ \mathrm{Im}(z_0) < 0\},$$

$$\sigma_{\pm 1} := \sigma(\mathsf{p}) \cap \{1, -1\}.$$

The map $\eta : z \to z^{-1}$ is a bijection between $\{z_0 \in \mathbb{C} \setminus \{0\}\ :\ |z_0| < 1\}$ and $\{z_0 \in \mathbb{C} \setminus \{0\}\ :\ |z_0| > 1\}$. So (4.1.4) implies that η is also a bijection between σ_{in} and σ_{out}. Moreover, we have $\sum_{z_0 \in \sigma_{out}} \mathsf{Z}(\mathsf{p}, z_0) = \sum_{z_0^{-1} \in \sigma_{in}} \mathsf{Z}(\mathsf{p}, z_0) = \sum_{z_0^{-1} \in \sigma_{in}} \mathsf{Z}(\mathsf{p}, z_0^{-1}) = \sum_{z_0 \in \sigma_{in}} \mathsf{Z}(\mathsf{p}, z_0)$.

Similarly, the map $\eta : z \to z^{-1}$ is also a bijection between $\{z_0 \in \mathbb{T}\ :\ \mathrm{Im}(z_0) > 0\}$ and $\{z_0 \in \mathbb{T}\ :\ \mathrm{Im}(z_0) < 0\}$. So (4.1.4) implies that η is also a bijection between σ_{up} and σ_{down}. Moreover, we have $\sum_{z_0 \in \sigma_{down}} \mathsf{Z}(\mathsf{p}, z_0) = \sum_{z_0^{-1} \in \sigma_{up}} \mathsf{Z}(\mathsf{p}, z_0) = \sum_{z_0^{-1} \in \sigma_{up}} \mathsf{Z}(\mathsf{p}, z_0^{-1}) = \sum_{z_0 \in \sigma_{up}} \mathsf{Z}(\mathsf{p}, z_0)$.

Now, from

$$\sum_{z_0 \in \sigma(\mathsf{p})} \mathsf{Z}(\mathsf{p}, z_0) = \sum_{z_0 \in \sigma_{in}} \mathsf{Z}(\mathsf{p}, z_0) + \sum_{z_0 \in \sigma_{out}} \mathsf{Z}(\mathsf{p}, z_0) + \sum_{z_0 \in \sigma_{up}} \mathsf{Z}(\mathsf{p}, z_0)$$
$$+ \sum_{z_0 \in \sigma_{down}} \mathsf{Z}(\mathsf{p}, z_0) + \mathsf{Z}(\mathsf{p}, 1) + \mathsf{Z}(\mathsf{p}, -1)$$

$$=2\sum_{z_0\in\sigma_{in}} \mathsf{Z}(\mathsf{p}, z_0) + 2\sum_{z_0\in\sigma_{up}} \mathsf{Z}(\mathsf{p}, z_0) + \mathsf{Z}(\mathsf{p}, 1) + \mathsf{Z}(\mathsf{p}, -1),$$

we can see that $\operatorname{odd}(\operatorname{len}(\mathsf{p})) = \operatorname{odd}\left(\sum_{z_0\in\sigma(\mathsf{p})} \mathsf{Z}(\mathsf{p}, z_0)\right) = \operatorname{odd}(\mathsf{Z}(\mathsf{p}, 1) + \mathsf{Z}(\mathsf{p}, -1))$. According to item (5) of Proposition 4.1.1, we know that $\operatorname{odd}(c) = \operatorname{odd}(\operatorname{len}(\mathsf{p})) = \operatorname{odd}(\mathsf{Z}(\mathsf{p}, 1) + \mathsf{Z}(\mathsf{p}, -1))$. This proves the item (2). ■

To deal with matrices generated by filters with symmetry, we generalize the operator S for an $r \times s$ matrix $\mathsf{A}(z)$ of Laurent polynomials: If all the elements of $\mathsf{A}(z)$ have symmetry, then $\mathsf{SA}(z)$ is defined to be an $r \times s$ matrix of monomials where $[\mathsf{SA}]_{i,j}(z) := \mathsf{S}[\mathsf{A}]_{i,j}(z)$, $i = 1, \ldots, r, j = 1, \ldots, s$.

In order to make the matrix operations such as $\mathsf{A}(z) \pm \mathsf{B}(z)$ and $\mathsf{A}(z)\mathsf{B}(z)$ be closed under symmetry, we need to define the compatibility. If $r \times s$ matrices $\mathsf{A}(z)$ and $\mathsf{B}(z)$ both have symmetry, and the symmetry types are the same, then $\mathsf{A}(z) + \mathsf{B}(z)$ and $\mathsf{A}(z) - \mathsf{B}(z)$ also have symmetry, and the symmetry types are unchanged. In this case, we say ***the operations*** $\mathsf{A}(z) + \mathsf{B}(z)$ ***and*** $\mathsf{A}(z) - \mathsf{B}(z)$ ***are compatible***.

It is a little bit more complicated to define the compatibility for matrix multiplications. For an $r \times s$ matrix $\mathsf{P}(z)$ with symmetry, we say the symmetry type of $\mathsf{P}(z)$ is ***compatible*** or $\mathsf{P}(z)$ has ***compatible symmetry*** if

$$\mathsf{SP}(z) = (\mathsf{S}\theta_1)^\star(z)\mathsf{S}\theta_2(z) \tag{4.1.5}$$

holds for some $1 \times r$ and $1 \times s$ row vectors of Laurent polynomials $\theta_1(z)$ and $\theta_2(z)$ with symmetry. It is easy to see from the definition (4.1.5) that $\mathsf{S}\theta_1(z)$ gives the symmetry relationship between the rows of $\mathsf{P}(z)$, while $\mathsf{S}\theta_2(z)$ gives the symmetry relationship between the columns of $\mathsf{P}(z)$:

$$\frac{\mathsf{SP}_{j,i}(z)}{\mathsf{SP}_{k,i}(z)} = \frac{\mathsf{S}\theta^\star_{1,j}(z)}{\mathsf{S}\theta^\star_{1,k}(z)}, \quad \forall\, i \in \{1, \ldots, s\}, \quad \forall\, j, k \in \{1, \ldots, r\}, \tag{4.1.6}$$

$$\frac{\mathsf{SP}_{i,j}(z)}{\mathsf{SP}_{i,k}(z)} = \frac{\mathsf{S}\theta_{2,j}(z)}{\mathsf{S}\theta_{2,k}(z)}, \quad \forall\, i \in \{1, \ldots, r\}, \quad \forall\, j, k \in \{1, \ldots, s\}, \tag{4.1.7}$$

where $\theta_{i,j}(z)$ is the j-th component of the vector $\theta_i(z)$, $i = 1, 2$.

If an $n \times n$ square matrix $\mathsf{P}(z)$ of Laurent polynomials has compatible symmetry as (4.1.5), it can be verified with straightforward calculations that $\det(\mathsf{P}(z))$ also has symmetry, with type

$$\mathsf{S}(\det(\mathsf{P}(z))) = \prod_{j=1}^{n} \mathsf{S}\theta_{1,j}^{\star}(z)\mathsf{S}\theta_{2,j}(z). \tag{4.1.8}$$

For an $r \times s$ matrix $\mathsf{P}(z)$ and an $s \times t$ matrix $\mathsf{Q}(z)$ of Laurent polynomials, we say ***the multiplication*** $\mathsf{P}(z)\mathsf{Q}(z)$ ***is compatible*** if

$$\mathsf{SP}(z) = \mathsf{S}\theta_1^{\star}(z)\mathsf{S}\theta_2(z), \qquad \mathsf{SQ}(z) = \mathsf{S}\theta_2^{\star}(z)\mathsf{S}\theta_3(z),$$

for some $1 \times r$, $1 \times s$ and $1 \times t$ row vectors $\theta_1(z)$, $\theta_2(z)$ and $\theta_3(z)$ of Laurent polynomials with symmetry. In this case, $\mathsf{P}(z)\mathsf{Q}(z)$ also has compatible symmetry:

$$\mathsf{S}(\mathsf{PQ})(z) = (\mathsf{SP})(z)(\mathsf{SQ})(z) = \mathsf{S}\theta_1^{\star}(z)\mathsf{S}\theta_3(z).$$

If an $n \times n$ matrix $\mathsf{P}(z)$ of Laurent polynomials is invertible (determinant equal to a nonzero monomial), and it has compatible symmetry as (4.1.5), we can compute its inverse as $\mathsf{P}^{-1}(z) = \frac{1}{\det(\mathsf{P}(z))}\operatorname{adj}(\mathsf{P}(z))$. From (4.1.8), we know that the (i, j) cofactor of $\mathsf{P}(z)$ has symmetry with type:

$$\begin{aligned}\mathsf{S}\left[\operatorname{adj}(\mathsf{P}(z))\right]_{j,i} =& \prod_{k\neq i} \mathsf{S}\theta_{1,k}^{\star}(z) \prod_{l\neq j} \mathsf{S}\theta_{2,l}(z) \\ =& \frac{\mathsf{S}\left(\det(\mathsf{P}(z))\right)}{\mathsf{S}\theta_{1,i}^{\star}(z)\mathsf{S}\theta_{2,j}(z)} = \mathsf{S}\left(\det(\mathsf{P}(z))\right)\mathsf{S}\theta_{1,i}(z)\mathsf{S}\theta_{2,j}^{\star}(z).\end{aligned}$$

So $\mathsf{P}^{-1}(z)$ also has compatible symmetry and

$$\mathsf{SP}^{-1}(z) = \mathsf{S}\theta_2^{\star}(z)\mathsf{S}\theta_1(z). \tag{4.1.9}$$

Hence, if $\mathsf{P}(z), \mathsf{Q}(z)$ and $\mathsf{R}(z)$ are matrices of Laurent polynomials of sizes $n \times n$, $n \times m$ and $m \times n$, respectively, where $\mathsf{P}(z)$ is invertible, then the multiplication $\mathsf{P}(z)\mathsf{Q}(z) = \mathsf{A}(z)$ is compatible implies that the multiplication $\mathsf{P}^{-1}(z)\mathsf{A}(z)$ is also compatible; and the multiplication $\mathsf{R}(z)\mathsf{P}(z) = \mathsf{A}(z)$ is compatible implies that the multiplication $\mathsf{A}(z)\mathsf{P}^{-1}(z)$ is also compatible.

Special cases that would be useful later are similar to elementary matrices. We summarize them as the following proposition.

Proposition 4.1.2. *Suppose* $\mathsf{A}(z)$ *is an* $r \times s$ *matrix of Laurent polynomials with compatible symmetry:* $\mathsf{SA}(z) = \mathsf{S}\theta_1^\star(z)\mathsf{S}\theta_2(z)$.

(1) For an $s \times t$ *permutation matrix* $\mathsf{P}_{i,j}$ *and a* $t \times r$ *permutation matrix* $\widetilde{\mathsf{P}}_{i,j}$*, where* $\widetilde{\mathsf{P}}_{i,j}\mathsf{A}(z)$ *and* $\mathsf{A}(z)\mathsf{P}_{i,j}$ *correspond to switching the* i*-th and the* j*-th row and column of* $\mathsf{A}(z)$ *respectively, we know that* $\widetilde{\mathsf{P}}_{i,j}\mathsf{A}(z)$ *and* $\mathsf{A}(z)\mathsf{P}_{i,j}$ *both have compatible symmetry, with types*

$$\mathsf{S}(\widetilde{\mathsf{P}}_{i,j}\mathsf{A})(z) = (\mathsf{S}(\theta_1\widetilde{\mathsf{P}}_{i,j}^{\mathsf{T}}))^\star(z)\mathsf{S}\theta_2(z), \qquad \mathsf{S}(\mathsf{AP}_{i,j})(z) = (\mathsf{S}\theta_1)^\star(z)\mathsf{S}(\theta_2\mathsf{P}_{i,j})(z).$$

(2) If all $\mathsf{d}_j(z)$ *have symmetry,* $j = 1, \ldots, n$*, then* $\mathsf{D}_n(z) := \operatorname{diag}(\mathsf{d}_1(z), \cdots, \mathsf{d}_n(z))$ *is compatible with any symmetry type on either side. That is,*

$$\mathsf{S}(\mathsf{D}_r\mathsf{A})(z) = (\mathsf{S}(\theta_1\mathsf{D}_r^\star))^\star(z)\mathsf{S}\theta_2(z), \qquad \mathsf{S}(\mathsf{AD}_s)(z) = (\mathsf{S}\theta_1)^\star(z)\mathsf{S}(\theta_2\mathsf{D}_s)(z).$$

(3) Suppose $\mathsf{U}(z)$ *is a* $t \times r$ *matrix of Laurent polynomials, where each of the entries of* $\mathsf{U}(z)$ *has symmetry. If for some* $k \in \{1, \ldots, s\}$*, it satisfies*

$$\mathsf{SU}_{i,1}(z)\mathsf{SA}_{1,k}(z) = \mathsf{SU}_{i,2}(z)\mathsf{SA}_{2,k}(z) = \ldots = \mathsf{SU}_{i,r}(z)\mathsf{SA}_{r,k}(z), \quad \text{for all } i = 1, \ldots, t, \quad (4.1.10)$$

then we can find some $1 \times t$ *row vector* $\theta_3(z)$ *of Laurent polynomials with symmetry, such that* $\mathsf{S}\theta_{3,i}^\star(z) = \frac{\mathsf{SU}_{i,1}(z)}{\mathsf{S}\theta_{1,1}(z)}$*, and* $\mathsf{SU}(z) = \mathsf{S}\theta_3^\star(z)\mathsf{S}\theta_1(z)$*. That is,* $\mathsf{U}(z)$ *has compatible symmetry and the multiplication* $\mathsf{U}(z)\mathsf{A}(z)$ *is compatible. On the other hand, if the multiplication* $\mathsf{U}(z)\mathsf{A}(z)$ *is compatible, then* (4.1.10) *holds for all* $k \in \{1, \ldots, s\}$.

(4) Suppose $\mathsf{U}(z)$ *is an* $s \times t$ *matrix of Laurent polynomials, where each of the entries of* $\mathsf{U}(z)$ *has symmetry. If for some* $k \in \{1, \ldots, r\}$*, it satisfies*

$$\mathsf{SA}_{k,1}(z)\mathsf{SU}_{1,i}(z) = \mathsf{SA}_{k,2}(z)\mathsf{SU}_{2,i}(z) = \ldots = \mathsf{SA}_{k,s}(z)\mathsf{SU}_{s,i}(z), \quad \text{for all } i = 1, \ldots, t, \quad (4.1.11)$$

then we can find some $1 \times t$ row vector $\theta_3(z)$ of Laurent polynomials with symmetry, such that $\mathsf{S}\theta_{3,i}(z) = \frac{\mathsf{SU}_{1,i}(z)}{\mathsf{S}\theta^\star_{2,1}(z)}$, and $\mathsf{SU}(z) = \mathsf{S}\theta^\star_2(z)\mathsf{S}\theta_3(z)$. That is, $\mathsf{U}(z)$ has compatible symmetry and the multiplication $\mathsf{A}(z)\mathsf{U}(z)$ is compatible. On the other hand, if the multiplication $\mathsf{A}(z)\mathsf{U}(z)$ is compatible, then (4.1.11) holds for all $k \in \{1, \dots, r\}$.

Proof. Items (1) and (2) can be seen by straightforward calculations using the definition (4.1.5). The proof of the item (4) is quite similar to that of the item (3). We only prove the item (3) here.

From (4.1.10), we know that

$$\mathsf{SU}_{i,l}(z)\mathsf{SA}_{l,k}(z) = \mathsf{SU}_{i,1}(z)\mathsf{SA}_{1,k}(z), \qquad \forall\, l \in \{1, \dots, r\},\ \forall\, i \in \{1, \dots, t\}.$$

Since $\mathsf{A}(z)$ has compatible symmetry, by the symmetry type of $\mathsf{A}(z)$, we can see that the above equality implies

$$\frac{\mathsf{SU}_{i,l}(z)}{\mathsf{SU}_{i,1}(z)} = \frac{\mathsf{SA}_{1,k}(z)}{\mathsf{SA}_{l,k}(z)} = \frac{\mathsf{S}\theta^\star_{1,1}(z)}{\mathsf{S}\theta^\star_{1,l}(z)} = \frac{\mathsf{S}\theta_{1,l}(z)}{\mathsf{S}\theta_{1,1}(z)}, \qquad \forall\, l \in \{1, \dots, r\},\ \forall\, i \in \{1, \dots, t\}. \tag{4.1.12}$$

We can find some $1 \times t$ row vector $\theta_3(z)$ of Laurent polynomials with symmetry, such that $\mathsf{S}\theta^\star_{3,i}(z) = \frac{\mathsf{SU}_{i,1}(z)}{\mathsf{S}\theta_{1,1}(z)}$, for all $i = 1, \dots, t$. From (4.1.12), we can see that

$$\mathsf{SU}_{i,l}(z) = \frac{\mathsf{S}\theta_{1,l}(z)}{\mathsf{S}\theta_{1,1}(z)}\mathsf{SU}_{i,1}(z) = \frac{\mathsf{S}\theta_{1,l}(z)}{\mathsf{S}\theta_{1,1}(z)}\mathsf{S}\theta^\star_{3,i}(z)\mathsf{S}\theta_{1,1}(z) = \mathsf{S}\theta^\star_{3,i}(z)\mathsf{S}\theta_{1,l}(z), \qquad \forall\, l \in \{1, \dots, r\},\ \forall\, i \in \{1, \dots, t\}.$$

So $\mathsf{SU}(z) = \mathsf{S}\theta^\star_3(z)\mathsf{S}\theta_1(z)$. That is, $\mathsf{U}(z)$ has compatible symmetry and the multiplication $\mathsf{U}(z)\mathsf{A}(z)$ is compatible.

On the other hand, if $\mathsf{SU}(z) = \mathsf{S}\theta^\star_3(z)\mathsf{S}\theta_1(z)$ for some $1 \times t$ row vector $\theta_3(z)$ of Laurent polynomials with symmetry, then

$$\mathsf{S}\theta^\star_{3,i}(z)\mathsf{S}\theta_{2,k}(z) = \mathsf{SU}_{i,1}(z)\mathsf{SA}_{1,k}(z) = \mathsf{SU}_{i,2}(z)\mathsf{SA}_{2,k}(z) = \dots = \mathsf{SU}_{i,r}(z)\mathsf{SA}_{r,k}(z)$$

holds for all $k \in \{1, \dots, s\}$, $i \in \{1, \dots, t\}$. This proves the equation (4.1.10) for all $k \in \{1, \dots, s\}$. ■

The construction of (anti-)symmetric tight framelet filter banks with two high-pass filters has been discussed in [40][48]. We would like to discuss the construction of the quasi-tight case with signature matrix $\operatorname{diag}(1, -1)$.

4.2 Spectral Decomposition of Matrices of Laurent Polynomials with Symmetry and Empty Spectrum

The main theorem we want to prove in this section is the following:

Theorem 4.2.1. *Suppose* $\mathsf{A}(z)$ *is a* 2×2 *matrix of Laurent polynomials with compatible symmetry, satisfying* $\mathsf{A}^\star(z) = \mathsf{A}(z)$. *Also* $\det(\mathsf{A}(z)) = C$ *is a constant satisfying* $C < 0$. *Denote the symmetry type of* $\mathsf{A}(z)$ *by* $\mathsf{SA}(z) = \begin{bmatrix} 1 & \alpha(z) \\ \alpha^\star(z) & 1 \end{bmatrix}$. *Then we can find a matrix* $\mathsf{U}(z) = \begin{bmatrix} \mathsf{U}_{1,1}(z) & \mathsf{U}_{1,2}(z) \\ \mathsf{U}_{2,1}(z) & \mathsf{U}_{2,2}(z) \end{bmatrix}$ *of Laurent polynomials with compatible symmetry, where the symmetry type satisfies*

$$\frac{\mathsf{SU}_{1,1}(z)}{\mathsf{SU}_{2,1}(z)} = \frac{\mathsf{SU}_{1,2}(z)}{\mathsf{SU}_{2,2}(z)} = \alpha(z), \tag{4.2.1}$$

and $\mathsf{A}(z) = \mathsf{U}(z) \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \mathsf{U}^\star(z)$ *holds.*

Denote the symmetry type $\alpha(z) = \epsilon z^c$, where $\epsilon \in \{1, -1\}$ and $c \in \mathbb{Z}$. Then we have four possible cases:

(1) $\epsilon = 1$, $c \in 2\mathbb{Z} + 1$;

(2) $\epsilon = -1$, $c \in 2\mathbb{Z} + 1$;

(3) $\epsilon = 1$, $c \in 2\mathbb{Z}$;

(4) $\epsilon = -1$, $c \in 2\mathbb{Z}$.

For cases (1)(2)(3), the result in Theorem 4.2.1 is proved by Theorem 4.2.2. For the case (4), we prove it in Theorem 4.2.3.

We introduce the following lemmas as sub-steps to prove the Theorem 4.2.2. The proofs of the lemmas are all constructive, which can be used as algorithms directly.

Lemma 4.2.1. *Suppose* $\mathsf{a}(z)$ *and* $\mathsf{b}(z)$ *are both Laurent polynomials with symmetry:* $\mathsf{Sa}(z) = \epsilon_a z^{c_a}$, $\mathsf{Sb}(z) = \epsilon_b z^{c_b}$, *for some* $c_a, c_b \in \mathbb{Z}$, *and* $\epsilon_a, \epsilon_b \in \{1, -1\}$. *Also, assume* $\mathsf{b}(z) \neq 0$, *and* $\operatorname{len}(\mathsf{a}(z)) > \operatorname{len}(\mathsf{b}(z))$. *Then there exists a Laurent polynomial* $\mathsf{q}_1(z)$ *with symmetry, such that*

$$\mathsf{a}_1(z) := \mathsf{a}(z) - \mathsf{b}(z)\mathsf{q}_1(z) \tag{4.2.2}$$

satisfies $\operatorname{len}(\mathsf{a}_1(z)) < \operatorname{len}(\mathsf{a}(z))$, *and* $\mathsf{Sa}_1(z) = \mathsf{Sa}(z) = \mathsf{Sb}(z)\mathsf{Sq}_1(z)$.

Proof. Let us write the Laurent polynomials as

$$\mathsf{a}(z) = \sum_{k=\operatorname{ldeg}(\mathsf{a})}^{\deg(\mathsf{a})} a(k)z^k, \qquad \mathsf{b}(z) = \sum_{k=\operatorname{ldeg}(\mathsf{b})}^{\deg(\mathsf{b})} b(k)z^k.$$

Define

$$\mathsf{q}_1(z) := \frac{a(\deg(\mathsf{a}))}{b(\deg(\mathsf{b}))} z^{\deg(\mathsf{a})-\deg(\mathsf{b})} + \frac{a(\operatorname{ldeg}(\mathsf{a}))}{b(\operatorname{ldeg}(\mathsf{b}))} z^{\operatorname{ldeg}(\mathsf{a})-\operatorname{ldeg}(\mathsf{b})}. \tag{4.2.3}$$

From the symmetry types of $\mathsf{a}(z)$ and $\mathsf{b}(z)$, we know that

$$a(\operatorname{ldeg}(\mathsf{a})) = \epsilon_a a(\deg(\mathsf{a})), \qquad b(\operatorname{ldeg}(\mathsf{b})) = \epsilon_b b(\deg(\mathsf{b})),$$

$$\deg(\mathsf{a}) + \operatorname{ldeg}(\mathsf{a}) = c_a, \qquad \deg(\mathsf{b}) + \operatorname{ldeg}(\mathsf{b}) = c_b.$$

Using (4.2.3), we can calculate

$$\begin{aligned}
\mathsf{Sq}_1(z) =& \mathsf{q}_1(z)/\mathsf{q}_1(z^{-1}) = \frac{\frac{a(\deg(\mathsf{a}))}{b(\deg(\mathsf{b}))} z^{\deg(\mathsf{a})-\deg(\mathsf{b})} + \frac{a(\operatorname{ldeg}(\mathsf{a}))}{b(\operatorname{ldeg}(\mathsf{b}))} z^{\operatorname{ldeg}(\mathsf{a})-\operatorname{ldeg}(\mathsf{b})}}{\frac{a(\deg(\mathsf{a}))}{b(\deg(\mathsf{b}))} z^{-\deg(\mathsf{a})+\deg(\mathsf{b})} + \frac{a(\operatorname{ldeg}(\mathsf{a}))}{b(\operatorname{ldeg}(\mathsf{b}))} z^{-\operatorname{ldeg}(\mathsf{a})+\operatorname{ldeg}(\mathsf{b})}} \\
=& \frac{\frac{a(\deg(\mathsf{a}))}{b(\deg(\mathsf{b}))} z^{\deg(\mathsf{a})-\deg(\mathsf{b})} + \epsilon_a\epsilon_b \frac{a(\deg(\mathsf{a}))}{b(\deg(\mathsf{b}))} z^{(c_a-c_b)-(\deg(\mathsf{a})-\deg(\mathsf{b}))}}{\frac{a(\deg(\mathsf{a}))}{b(\deg(\mathsf{b}))} z^{-(\deg(\mathsf{a})-\deg(\mathsf{b}))} + \epsilon_a\epsilon_b \frac{a(\deg(\mathsf{a}))}{b(\deg(\mathsf{b}))} z^{(\deg(\mathsf{a})-\deg(\mathsf{b}))-(c_a-c_b)}}
\end{aligned}$$

$$=\epsilon_a \epsilon_b z^{c_a - c_b} = \mathsf{Sa}(z)/\mathsf{Sb}(z).$$

So $\mathsf{Sb}(z)\mathsf{Sq}_1(z) = \mathsf{Sa}(z) = \mathsf{Sa}_1(z)$.

Since $\text{len}(\mathsf{a}(z)) = \deg(\mathsf{a}(z)) - \text{ldeg}(\mathsf{a}(z)) > \text{len}(\mathsf{b}(z)) = \deg(\mathsf{b}(z)) - \text{ldeg}(\mathsf{b}(z))$, we can see $\deg(\mathsf{a}) - \deg(\mathsf{b}) > \text{ldeg}(\mathsf{a}) - \text{ldeg}(\mathsf{b})$. Therefore, from the definition of $\mathsf{q}_1(z)$ in (4.2.3), we can get $\text{ldeg}(\mathsf{q}_1) = \text{ldeg}(\mathsf{a}) - \text{ldeg}(\mathsf{b})$, and $\deg(\mathsf{q}_1) = \deg(\mathsf{a}) - \deg(\mathsf{b})$. So

$$\text{ldeg}(\mathsf{b}\mathsf{q}_1) = \text{ldeg}(\mathsf{b}) + \text{ldeg}(\mathsf{q}_1) = \text{ldeg}(\mathsf{a}),$$

$$\deg(\mathsf{b}\mathsf{q}_1) = \deg(\mathsf{b}) + \deg(\mathsf{q}_1) = \deg(\mathsf{a}).$$

As to the coefficients corresponding to the lowest and the highest degree terms of $\mathsf{b}\mathsf{q}_1$, we can calculate directly from (4.2.3):

$$\text{coefficient of } \mathsf{b}(z)\mathsf{q}_1(z) \text{ at lowest degree:} \quad b(\text{ldeg}(\mathsf{b}))q_1(\text{ldeg}(\mathsf{q}_1)) = a(\text{ldeg}(\mathsf{a})),$$

$$\text{coefficient of } \mathsf{b}(z)\mathsf{q}_1(z) \text{ at highest degree:} \quad b(\deg(\mathsf{b}))q_1(\deg(\mathsf{q}_1)) = a(\deg(\mathsf{a})).$$

So both the lowest and highest degree terms cancel in the subtraction $\mathsf{a}_1(z) = \mathsf{a}(z) - \mathsf{b}(z)\mathsf{q}_1(z)$. Hence, $\text{ldeg}(\mathsf{a}_1(z)) > \text{ldeg}(\mathsf{a}(z))$, and $\deg(\mathsf{a}_1(z)) < \deg(\mathsf{a}(z))$. That is, $\text{len}(\mathsf{a}_1(z)) < \text{len}(\mathsf{a}(z))$. This finished the proof of the lemma. ■

Lemma 4.2.2 (Long Division of Laurent Polynomials with Symmetry)**.** *Suppose* $\mathsf{a}(z)$ *and* $\mathsf{b}(z)$ *are both Laurent polynomials with symmetry:* $\mathsf{Sa}(z) = \epsilon_a z^{c_a}$, $\mathsf{Sb}(z) = \epsilon_b z^{c_b}$*, for some* $c_a, c_b \in \mathbb{Z}$*, and* $\epsilon_a, \epsilon_b \in \{1, -1\}$*. Also, assume* $\mathsf{b}(z) \neq 0$*. We have the following 4 possible cases for the symmetry type of* $\mathsf{a}(z)$ *and* $\mathsf{b}(z)$*:*

(i) $\epsilon_a \epsilon_b = -1$*, and* $c_a - c_b \in 2\mathbb{Z}$*;*

(ii) $\epsilon_a \epsilon_b = -1$*, and* $c_a - c_b \in 2\mathbb{Z} + 1$*;*

(iii) $\epsilon_a \epsilon_b = 1$*, and* $c_a - c_b \in 2\mathbb{Z}$*;*

(iv) $\epsilon_a \epsilon_b = 1$*, and* $c_a - c_b \in 2\mathbb{Z} + 1$*.*

We can prove the following two results.

(1) For case (i), we can find a Laurent polynomial $\mathsf{q}(z)$ *with symmetry, such that*

$$\mathsf{r}(z) := \mathsf{a}(z) - \mathsf{b}(z)\mathsf{q}(z) \tag{4.2.4}$$

satisfies $\operatorname{len}(\mathsf{r}(z)) \leqslant \operatorname{len}(\mathsf{b}(z))$, *and* $\mathsf{Sr}(z) = \mathsf{Sa}(z) = \mathsf{Sb}(z)\mathsf{Sq}(z)$.

(2) For cases (ii)(iii)(iv), we can find a Laurent polynomial $\mathsf{q}(z)$ *with symmetry, such that the* $\mathsf{r}(z)$ *defined in* (4.2.4) *satisfies* $\operatorname{len}(\mathsf{r}(z)) < \operatorname{len}(\mathsf{b}(z))$, *and* $\mathsf{Sr}(z) = \mathsf{Sa}(z) = \mathsf{Sb}(z)\mathsf{Sq}(z)$.

Proof. In all the 4 possible cases of symmetry types, if $\operatorname{len}(\mathsf{a}(z)) < \operatorname{len}(\mathsf{b}(z))$, we can take $\mathsf{q}(z) := 0$, and $\mathsf{r}(z) := \mathsf{a}(z)$. Then $\mathsf{q}(z)$ and $\mathsf{r}(z)$ satisfy all the requirements in the items (1) and (2). The proof is completed. So we only need to consider that $\operatorname{len}(\mathsf{a}(z)) \geqslant \operatorname{len}(\mathsf{b}(z))$.

If $\operatorname{len}(\mathsf{a}(z)) > \operatorname{len}(\mathsf{b}(z))$, we can define $\mathsf{a}_0(z) := \mathsf{a}(z)$, and repeatedly apply the Lemma 4.2.1. That is, if $\operatorname{len}(\mathsf{a}_j(z)) > \operatorname{len}(\mathsf{b}(z))$, $j = 0, 1, \ldots$, we can find a Laurent polynomial $\mathsf{q}_j(z)$ with symmetry, such that

$$\mathsf{a}_{j+1}(z) := \mathsf{a}_j(z) - \mathsf{b}(z)\mathsf{q}_j(z)$$

satisfies $\operatorname{len}(\mathsf{a}_{j+1}(z)) < \operatorname{len}(\mathsf{a}_j(z))$, and $\mathsf{Sa}_{j+1}(z) = \mathsf{Sa}_j(z) = \mathsf{Sb}(z)\mathsf{Sq}_j(z)$. This process cannot iterate forever since $\operatorname{len}(\mathsf{a}_j(z))$ is strictly decreasing and we require $\operatorname{len}(\mathsf{a}_j(z)) > \operatorname{len}(\mathsf{b}(z))$. Suppose it finalizes at some $j = K - 1$, then

$$\begin{aligned}
\mathsf{a}_K(z) =& \mathsf{a}_{K-1}(z) - \mathsf{b}(z)\mathsf{q}_{K-1}(z)\\
=& \mathsf{a}_{K-2}(z) - \mathsf{b}(z)\mathsf{q}_{K-2}(z) - \mathsf{b}(z)\mathsf{q}_{K-1}(z)\\
=& \ldots\\
=& \mathsf{a}_0(z) - \mathsf{b}(z)\left(\mathsf{q}_{K-1}(z) + \ldots + \mathsf{q}_0(z)\right)
\end{aligned}$$

satisfies $\operatorname{len}(\mathsf{a}_K(z)) \leqslant \operatorname{len}(\mathsf{b}(z))$. Also, the symmetry types satisfy

$$\mathsf{Sa}_K(z) = \mathsf{Sa}_{K-1}(z) = \ldots = \mathsf{Sa}_0(z) = \mathsf{Sa}(z),$$

$$\mathsf{Sq}_j(z) = \mathsf{Sa}_{j+1}(z)/\mathsf{Sb}(z) = \mathsf{Sa}(z)/\mathsf{Sb}(z), \qquad \forall\ j = 0, \ldots, K-1.$$

Define

$$\widetilde{\mathsf{q}}(z) := \mathsf{q}_{K-1}(z) + \ldots + \mathsf{q}_0(z), \tag{4.2.5}$$

we can see

$$\mathsf{S}\widetilde{\mathsf{q}}(z) = \mathsf{Sa}(z)/\mathsf{Sb}(z), \tag{4.2.6}$$

$$\mathsf{a}_K(z) = \mathsf{a}(z) - \mathsf{b}(z)\widetilde{\mathsf{q}}(z). \tag{4.2.7}$$

For the symmetry type in the case (i), if $\mathrm{len}(\mathsf{a}(z)) = \mathrm{len}(\mathsf{b}(z))$, we can take $\mathsf{q}(z) := 0$, and $\mathsf{r}(z) := \mathsf{a}(z)$. They satisfy all the requirements in the item (1). Otherwise if $\mathrm{len}(\mathsf{a}(z)) > \mathrm{len}(\mathsf{b}(z))$, just apply the above algorithm and define $\mathsf{q}(z) := \widetilde{\mathsf{q}}(z)$ as in (4.2.5). Then, from (4.2.7) we can see $\mathsf{r}(z) := \mathsf{a}(z) - \mathsf{b}(z)\mathsf{q}(z) = \mathsf{a}_K(z)$ satisfies $\mathrm{len}(\mathsf{r}(z)) \leqslant \mathrm{len}(\mathsf{b}(z))$. And the equation (4.2.6) implies that $\mathsf{Sr}(z) = \mathsf{Sa}(z) = \mathsf{Sb}(z)\mathsf{Sq}(z)$. This finishes the proof of the item (1).

For the symmetry types in cases (ii) and (iv), we have $c_a - c_b \in 2\mathbb{Z} + 1$. That is, $\mathrm{odd}(c_a) \neq \mathrm{odd}(c_b)$. By the item (5) of Proposition 4.1.1, we know that $\mathrm{odd}(\mathrm{len}(\mathsf{a}(z))) \neq \mathrm{odd}(\mathrm{len}(\mathsf{b}(z)))$, which implies that $\mathrm{len}(\mathsf{a}(z)) \neq \mathrm{len}(\mathsf{b}(z))$. So, we only need to consider the situation that $\mathrm{len}(\mathsf{a}(z)) > \mathrm{len}(\mathsf{b}(z))$. Let us apply the above algorithm, and define $\mathsf{q}(z) := \widetilde{\mathsf{q}}(z)$ as in (4.2.5). Then, from (4.2.7) and (4.2.6), we can see $\mathsf{r}(z) := \mathsf{a}(z) - \mathsf{b}(z)\mathsf{q}(z) = \mathsf{a}_K(z)$ satisfies the symmetry type result $\mathsf{Sr}(z) = \mathsf{Sa}_K(z) = \mathsf{Sa}(z) = \mathsf{Sb}(z)\mathsf{Sq}(z)$. Furthermore, from $\epsilon_r z^{c_r} := \mathsf{Sr}(z) = \mathsf{Sa}(z) = \epsilon_a z^{c_a}$, we know that $c_r = c_a$. So $\mathrm{odd}(c_r) = \mathrm{odd}(c_a) \neq \mathrm{odd}(c_b)$. By the item (5) of Proposition 4.1.1, we get $\mathrm{odd}(\mathrm{len}(\mathsf{r}(z))) \neq \mathrm{odd}(\mathrm{len}(\mathsf{b}(z)))$, so $\mathrm{len}(\mathsf{r}) \neq \mathrm{len}(\mathsf{b})$. Since $\mathrm{len}(\mathsf{r}) = \mathrm{len}(\mathsf{a}_K) \leqslant \mathrm{len}(\mathsf{b})$, we know $\mathrm{len}(\mathsf{r}) < \mathrm{len}(\mathsf{b})$. This finishes the proof of the cases (ii) and (iv) in the item (2).

For the symmetry type in the case (iii), if $\mathrm{len}(\mathsf{a}(z)) > \mathrm{len}(\mathsf{b}(z))$, we can apply the algorithm above to find the Laurent polynomial $\widetilde{\mathsf{q}}(z)$ as in (4.2.5). If $\mathrm{len}(\mathsf{a}(z)) = \mathrm{len}(\mathsf{b}(z))$, we just use $\widetilde{\mathsf{q}}(z) = 0$ here. For both situations, we define $\mathsf{a}_K(z)$ as

$$\mathsf{a}_K(z) = \mathsf{a}(z) - \mathsf{b}(z)\widetilde{\mathsf{q}}(z),$$

then $\mathrm{len}(\mathsf{a}_K(z)) \leqslant \mathrm{len}(\mathsf{b}(z))$, and $\mathsf{Sa}_K(z) = \mathsf{Sa}(z) = \mathsf{Sb}(z)\mathsf{S}\widetilde{\mathsf{q}}(z)$. (In the situation of $\mathrm{len}(\mathsf{a}(z)) = \mathrm{len}(\mathsf{b}(z))$, just take $K = 1$.) If $\mathrm{len}(\mathsf{a}_K(z)) < \mathrm{len}(\mathsf{b}(z))$, we can just take $\mathsf{q}(z) := \widetilde{\mathsf{q}}(z)$, and $\mathsf{r}(z) := \mathsf{a}_K(z)$. They satisfy all the requirements in the item (2). And the proof of the theorem is completed. If $\mathrm{len}(\mathsf{a}_K(z)) = \mathrm{len}(\mathsf{b}(z))$, define

$$\mathring{\mathsf{q}}(z) := \frac{a_K(\deg(\mathsf{a}_K))}{b(\deg(\mathsf{b}))} z^{\deg(\mathsf{a}_K)-\deg(\mathsf{b})}.$$

We will prove that

$$\mathsf{r}(z) := \mathsf{a}_K(z) - \mathsf{b}(z)\mathring{\mathsf{q}}(z) \tag{4.2.8}$$

satisfies

$$\mathrm{len}(\mathsf{r}) < \mathrm{len}(\mathsf{a}_K) = \mathrm{len}(\mathsf{b}), \qquad \mathsf{S}\mathsf{b}\mathsf{S}\mathring{\mathsf{q}} = \mathsf{S}\mathsf{a}_K = \mathsf{S}\mathsf{r}. \tag{4.2.9}$$

From $\mathrm{len}(\mathsf{a}_K(z)) = \mathrm{len}(\mathsf{b}(z))$, we know that

$$\deg(\mathsf{a}_K) - \mathrm{ldeg}(\mathsf{a}_K) = \mathrm{len}(\mathsf{a}_K(z)) = \mathrm{len}(\mathsf{b}(z)) = \deg(\mathsf{b}) - \mathrm{ldeg}(\mathsf{b}).$$

Hence, $\deg(\mathsf{a}_K) - \deg(\mathsf{b}) = \mathrm{ldeg}(\mathsf{a}_K) - \mathrm{ldeg}(\mathsf{b})$. So

$$\begin{aligned}
\deg(\mathsf{b}\mathring{\mathsf{q}}) &= \deg(\mathsf{b}) + \deg(\mathring{\mathsf{q}}) = \deg(\mathsf{a}_K),\\
\mathrm{ldeg}(\mathsf{b}\mathring{\mathsf{q}}) &= \mathrm{ldeg}(\mathsf{b}) + \mathrm{ldeg}(\mathring{\mathsf{q}}) = \mathrm{ldeg}(\mathsf{b}) + (\deg(\mathsf{a}_K) - \deg(\mathsf{b}))\\
&= \mathrm{ldeg}(\mathsf{b}) + (\mathrm{ldeg}(\mathsf{a}_K) - \mathrm{ldeg}(\mathsf{b})) = \mathrm{ldeg}(\mathsf{a}_K).
\end{aligned}$$

This tells us that $\mathrm{fsupp}(\mathsf{b}\mathring{\mathsf{q}}) = \mathrm{fsupp}(\mathsf{a}_K)$. The coefficient of the highest degree term of $\mathsf{b}(z)\mathring{\mathsf{q}}(z)$ is $b(\deg(\mathsf{b}))\mathring{q}(\deg(\mathring{\mathsf{q}})) = a_K(\deg(\mathsf{a}_K))$. Therefore, the terms corresponding to the highest degree cancel in the subtraction $\mathsf{r}(z) = \mathsf{a}_K(z) - \mathsf{b}(z)\mathring{\mathsf{q}}(z)$. So $\mathrm{len}(\mathsf{r}) < \mathrm{len}(\mathsf{a}_K)$.

As to the symmetry type, we can calculate that $\mathsf{S}\mathring{\mathsf{q}}(z) = z^{2\deg(\mathsf{a}_K)-2\deg(\mathsf{b})}$. Also, from $\epsilon_{a_K}\epsilon_b = \epsilon_a\epsilon_b = 1$, we know that $\epsilon_{a_K} = \epsilon_b$. So

$$\begin{aligned}
\mathsf{S}\mathsf{b}\mathsf{S}\mathring{\mathsf{q}}(z) =& \epsilon_b z^{c_b+2\deg(\mathsf{a}_K)-2\deg(\mathsf{b})} = \epsilon_b z^{\deg(\mathsf{b})+\mathrm{ldeg}(\mathsf{b})+2\deg(\mathsf{a}_K)-2\deg(\mathsf{b})}\\
=& \epsilon_b z^{2\deg(\mathsf{a}_K)-(\deg(\mathsf{b})-\mathrm{ldeg}(\mathsf{b}))} = \epsilon_b z^{2\deg(\mathsf{a}_K)-(\deg(\mathsf{a}_K)-\mathrm{ldeg}(\mathsf{a}_K))}\\
=& \epsilon_{a_K} z^{\deg(\mathsf{a}_K)+\mathrm{ldeg}(\mathsf{a}_K)} = \epsilon_{a_K} z^{c_{a_K}} = \mathsf{S}\mathsf{a}_K(z).
\end{aligned}$$

This finishes the proof of (4.2.9). Define $\mathsf{q}(z) := \widetilde{\mathsf{q}}(z) + \mathring{\mathsf{q}}(z)$, (4.2.7) and (4.2.8) imply that (4.2.4) holds. From (4.2.6) and (4.2.9), the symmetry types satisfy

$\mathsf{Sr}(z) = \mathsf{Sa}(z) = \mathsf{Sb}(z)\mathsf{Sq}(z)$. This completes the proof of the item (2) in the case of symmetry type (iii). ■

Using Lemma 4.2.2, we can prove the following theorem.

Theorem 4.2.2. *Suppose* $\mathsf{A}(z)$ *is a* 2×2 *Hermitian matrix of Laurent polynomials with compatible symmetry, where* $\det(\mathsf{A}(z)) = C$ *is a constant satisfying* $C < 0$. *Assume the symmetry type of* $\mathsf{A}(z)$ *is* $\mathsf{SA}(z) = \begin{bmatrix} 1 & \alpha(z) \\ \alpha^\star(z) & 1 \end{bmatrix}$, *where* $\alpha(z)$ *is one of the followings:*

(1) $\alpha(z) = \epsilon z^{2c+1}, \quad \epsilon \in \{1, -1\},\ c \in \mathbb{Z}$;

(2) $\alpha(z) = z^{2c}, \quad c \in \mathbb{Z}$.

Then we can find a matrix $\mathsf{U}(z) = \begin{bmatrix} \mathsf{U}_{1,1}(z) & \mathsf{U}_{1,2}(z) \\ \mathsf{U}_{2,1}(z) & \mathsf{U}_{2,2}(z) \end{bmatrix}$ *of Laurent polynomials with compatible symmetry, where the symmetry type satisfies*

$$\frac{\mathsf{SU}_{1,1}(z)}{\mathsf{SU}_{2,1}(z)} = \frac{\mathsf{SU}_{1,2}(z)}{\mathsf{SU}_{2,2}(z)} = \alpha(z), \tag{4.2.10}$$

and $\mathsf{A}(z) = \mathsf{U}(z) \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \mathsf{U}^\star(z)$ *holds.*

Proof. Firstly, we iteratively construct a sequence of invertible matrices $\mathsf{V}^{(0)}(z), \ldots, \mathsf{V}^{(K-1)}(z)$ with compatible symmetry, such that

$$\mathsf{A}^{(K)}(z) := \mathsf{V}^{(K-1)}(z) \cdots \mathsf{V}^{(0)}(z)\mathsf{A}(z)\mathsf{V}^{(0)\star}(z) \cdots \mathsf{V}^{(K-1)\star}(z)$$

satisfies

(1) all the multiplications are compatible;

(2) there exists some element in $\mathsf{A}^{(K)}(z)$ that is zero.

If there already exists some element in $\mathsf{A}(z)$ that is zero, just take $K = 1$, $\mathsf{V}^{(0)}(z) = \mathbf{I_2}$, and $\mathsf{A}^{(1)}(z) := \mathsf{A}(z)$. The construction is completed. Otherwise, we define $\mathsf{A}^{(0)}(z) := \mathsf{A}(z)$. For $j = 0, \ldots, K-1$, we will iteratively solve for $\mathsf{V}^{(j)}(z)$, and define $\mathsf{A}^{(j+1)}(z) := \mathsf{V}^{(j)}(z)\mathsf{A}^{(j)}(z)\mathsf{V}^{(j)\star}(z)$.

Our construction of $\mathsf{V}^{(j)}(z)$ below requires the following two conditions for $\mathsf{A}^{(j)}(z)$, which are obviously true for $\mathsf{A}^{(0)}(z) := \mathsf{A}(z)$.

(i) $\mathsf{A}^{(j)\star}(z) = \mathsf{A}^{(j)}(z)$, and $\det(\mathsf{A}^{(j)}(z)) = \det(\mathsf{A}(z)) = C$ is a constant;

(ii) $\mathsf{S}\mathsf{A}^{(j)}(z) = \mathsf{S}\mathsf{A}(z)$.

Assume the above two items are true for some $j = 0, \ldots, K-1$. If none of the four elements in $\mathsf{A}^{(j)}(z)$ is zero, we will prove

$$\operatorname{len}(\mathsf{A}^{(j)}_{1,1}) + \operatorname{len}(\mathsf{A}^{(j)}_{2,2}) = \operatorname{len}(\mathsf{A}^{(j)}_{1,2}) + \operatorname{len}(\mathsf{A}^{(j)}_{2,1}) = 2\operatorname{len}(\mathsf{A}^{(j)}_{2,1}). \tag{4.2.11}$$

Since $\mathsf{A}^{(j)}(z)$ is Hermitian, we know that $\left(\mathsf{A}^{(j)}_{1,1}\mathsf{A}^{(j)}_{2,2}\right)^\star = \mathsf{A}^{(j)}_{1,1}\mathsf{A}^{(j)}_{2,2}$, and $\left(\mathsf{A}^{(j)}_{1,2}\mathsf{A}^{(j)}_{2,1}\right)^\star = \mathsf{A}^{(j)}_{1,2}\mathsf{A}^{(j)}_{2,1}$. So $\operatorname{fsupp}\left(\mathsf{A}^{(j)}_{1,1}\mathsf{A}^{(j)}_{2,2}\right)$ and $\operatorname{fsupp}\left(\mathsf{A}^{(j)}_{1,2}\mathsf{A}^{(j)}_{2,1}\right)$ are both symmetric intervals about the center 0. Define

$$[-m, m] := \operatorname{fsupp}(\mathsf{A}^{(j)}_{1,1}\mathsf{A}^{(j)}_{2,2}), \qquad [-n, n] := \operatorname{fsupp}(\mathsf{A}^{(j)}_{1,2}\mathsf{A}^{(j)}_{2,1}).$$

Since $\det(\mathsf{A}^{(j)}(z)) = \mathsf{A}^{(j)}_{1,1}(z)\mathsf{A}^{(j)}_{2,2}(z) - \mathsf{A}^{(j)}_{1,2}(z)\mathsf{A}^{(j)}_{2,1}(z) = C$ is a constant, and none of the 4 elements in $\mathsf{A}(z)$ is 0, we know that all the terms in $\mathsf{A}^{(j)}_{1,1}\mathsf{A}^{(j)}_{2,2}$ and $\mathsf{A}^{(j)}_{1,2}\mathsf{A}^{(j)}_{2,1}$ with degree not equal to 0 must cancel in the subtraction. Therefore, we have $m = n$. From

$$\begin{aligned}\operatorname{len}(\mathsf{A}^{(j)}_{1,1}) + \operatorname{len}(\mathsf{A}^{(j)}_{2,2}) &= \operatorname{len}(\mathsf{A}^{(j)}_{1,1}\mathsf{A}^{(j)}_{2,2}) = 2m = 2n \\ &= \operatorname{len}(\mathsf{A}^{(j)}_{1,2}\mathsf{A}^{(j)}_{2,1}) = \operatorname{len}(\mathsf{A}^{(j)}_{1,2}) + \operatorname{len}(\mathsf{A}^{(j)}_{2,1}) = 2\operatorname{len}(\mathsf{A}^{(j)}_{2,1}),\end{aligned}$$

we proved (4.2.11). From (4.2.11), we get

$$\min\left\{\operatorname{len}(\mathsf{A}^{(j)}_{1,1}(z)),\ \operatorname{len}(\mathsf{A}^{(j)}_{2,2}(z))\right\} \leqslant \operatorname{len}(\mathsf{A}^{(j)}_{2,1}(z)). \tag{4.2.12}$$

Based on (4.2.12) and Lemma 4.2.2, we can use the following 2 steps to find an invertible matrix $\mathsf{V}^{(j)}(z)$ to reduce the length of the $(2,1)$ element of $\mathsf{A}^{(j)}(z)$. That is, we can find an invertible matrix $\mathsf{V}^{(j)}(z)$ with compatible symmetry, such that $\mathsf{A}^{(j+1)}(z) := \mathsf{V}^{(j)}(z)\mathsf{A}^{(j)}(z)\mathsf{V}^{(j)\star}(z)$ satisfies $\operatorname{len}(\mathsf{A}^{(j+1)}_{2,1}(z)) <$

$\mathrm{len}(\mathsf{A}^{(j)}_{2,1}(z))$. Notice that we have $\mathsf{SA}^{(j)}(z) = \mathsf{SA}(z)$, that is, $\mathsf{SA}^{(j)}_{1,1} = 1$ and $\mathsf{SA}^{(j)}_{2,1} = \alpha^\star(z)$.

Step 1 If $\mathrm{len}(\mathsf{A}^{(j)}_{1,1}(z)) \leqslant \mathrm{len}(\mathsf{A}^{(j)}_{2,1}(z))$, we can use Lemma 4.2.2 (item (2)) to find a Laurent polynomial $\mathsf{q}(z)$ with symmetry, such that

$$\mathsf{r}(z) := \mathsf{A}^{(j)}_{2,1}(z) - \mathsf{A}^{(j)}_{1,1}(z)\mathsf{q}(z)$$

satisfies $\mathrm{len}(\mathsf{r}(z)) < \mathrm{len}(\mathsf{A}^{(j)}_{1,1}(z))$, and $\mathsf{Sr}(z) = \mathsf{Sq}(z)\mathsf{SA}^{(j)}_{1,1}(z) = \mathsf{SA}^{(j)}_{2,1}(z)$. Define $\mathsf{V}^{(j)}(z) := \begin{bmatrix} 1 & \\ -\mathsf{q}(z) & 1 \end{bmatrix}$, then

$$\mathsf{A}^{(j+1)}(z) := \mathsf{V}^{(j)}(z)\mathsf{A}^{(j)}(z)\mathsf{V}^{(j)\star}(z) = \begin{bmatrix} \mathsf{A}^{(j)}_{1,1}(z) & \mathsf{r}^\star(z) \\ \mathsf{r}(z) & \mathsf{A}^{(j+1)}_{2,2}(z) \end{bmatrix}$$

satisfies $\mathsf{SA}^{(j+1)}(z) = \mathsf{SA}^{(j)}(z)$, $\det(\mathsf{A}^{(j+1)}(z)) = \det(\mathsf{A}^{(j)}(z))$, and

$$\mathrm{len}(\mathsf{A}^{(j+1)}_{2,1}(z)) = \mathrm{len}(\mathsf{r}(z)) < \mathrm{len}(\mathsf{A}^{(j)}_{1,1}(z)) \leqslant \mathrm{len}(\mathsf{A}^{(j)}_{2,1}(z)). \tag{4.2.13}$$

Also, we have $\mathsf{SV}^{(j)}(z) = \mathsf{SA}^{(j)}(z) = \mathsf{S}\theta_1^\star(z)\mathsf{S}\theta_1(z)$, where $\mathsf{S}\theta_1(z) := [1, \alpha(z)]$. So the multiplications are compatible.

Step 2 If $\mathrm{len}(\mathsf{A}^{(j)}_{2,2}(z)) \leqslant \mathrm{len}(\mathsf{A}^{(j)}_{2,1}(z))$, we can use Lemma 4.2.2(item (2)) to find a Laurent polynomial $\mathsf{q}(z)$ with symmetry, such that

$$\mathsf{r}(z) := \mathsf{A}^{(j)}_{1,2}(z) - \mathsf{A}^{(j)}_{2,2}(z)\mathsf{q}(z)$$

satisfies $\mathrm{len}(\mathsf{r}(z)) < \mathrm{len}(\mathsf{A}^{(j)}_{2,2}(z))$, and $\mathsf{Sr}(z) = \mathsf{Sq}(z)\mathsf{SA}^{(j)}_{2,2}(z) = \mathsf{SA}^{(j)}_{1,2}(z)$. Define $\mathsf{V}^{(j)}(z) := \begin{bmatrix} 1 & -\mathsf{q}(z) \\ & 1 \end{bmatrix}$, then

$$\mathsf{A}^{(j+1)}(z) := \mathsf{V}^{(j)}(z)\mathsf{A}(z)\mathsf{V}^{(j)\star}(z) = \begin{bmatrix} \mathsf{A}^{(j+1)}_{1,1}(z) & \mathsf{r}(z) \\ \mathsf{r}^\star(z) & \mathsf{A}^{(j)}_{2,2}(z) \end{bmatrix}$$

satisfies $\mathsf{SA}^{(j+1)}(z) = \mathsf{SA}^{(j)}(z)$, $\det(\mathsf{A}^{(j+1)}(z)) = \det(\mathsf{A}^{(j)}(z))$, and

$$\operatorname{len}(\mathsf{A}^{(j+1)}_{2,1}(z)) = \operatorname{len}(\mathsf{r}(z)) < \operatorname{len}(\mathsf{A}^{(j)}_{2,2}(z)) \leqslant \operatorname{len}(\mathsf{A}^{(j)}_{2,1}(z)). \tag{4.2.14}$$

Also, we have $\mathsf{SV}^{(j)}(z) = \mathsf{SA}^{(j)}(z) = \mathsf{S}\theta_1^{\star}(z)\mathsf{S}\theta_1(z)$, where $\mathsf{S}\theta_1(z) := [1, \alpha(z)]$. So the multiplications are compatible.

When we use Lemma 4.2.2 in the above two steps, notice that the symmetry type $\alpha(z)$ in item (1) corresponds to the case (ii) or (iv) in Lemma 4.2.2, and the symmetry type $\alpha(z)$ in item (2) corresponds to the case (iii) in Lemma 4.2.2. So we can always use the item (2) of Lemma 4.2.2 in the two steps above.

After we apply either step 1 or step 2 above, we can see that $\mathsf{A}^{(j+1)}(z)$ is still Hermitian, and both the determinant and the symmetry type are unchanged:

$$\begin{aligned}
\det(\mathsf{A}^{(j+1)}(z)) &= \det(\mathsf{V}(z))\det(\mathsf{A}^{(j)}(z))\det(\mathsf{V}^{\star}(z)) \\
&= \det(\mathsf{A}^{(j)}(z)) = \det(\mathsf{A}(z)) = C, \\
\mathsf{SA}^{(j+1)}(z) &= \mathsf{SA}^{(j)}(z).
\end{aligned}$$

Therefore, $\mathsf{A}^{(j+1)}(z)$ also satisfies the conditions (i) and (ii) we assumed for $\mathsf{A}^{(j)}(z)$. So we can redefine $j := j + 1$, and repeat the above procedure to construct the next $\mathsf{V}^{(j)}(z)$.

By (4.2.13) and (4.2.14), we know that the length of $\mathsf{A}^{(j)}_{2,1}(z)$ is strictly decreasing:

$$\operatorname{len}(\mathsf{A}^{(0)}_{2,1}(z)) > \operatorname{len}(\mathsf{A}^{(1)}_{2,1}(z)) > \operatorname{len}(\mathsf{A}^{(2)}_{2,1}(z)) > \dots.$$

So the procedure cannot last forever. That is, there exists some integer K, such that at least one of the four elements in $\mathsf{A}^{(K)}(z)$ is zero. By our construction, we have $\mathsf{A}^{(K)}(z) = \mathsf{V}^{(K-1)}(z)\cdots\mathsf{V}^{(0)}(z)\mathsf{A}(z)\mathsf{V}^{(0)\star}(z)\cdots\mathsf{V}^{(K-1)\star}(z)$. Define

$$\mathsf{S}\theta_1(z) := [1, \alpha(z)], \tag{4.2.15}$$

we can see that

$$\mathsf{SA}^{(K)}(z) = \mathsf{SA}(z) = \mathsf{SV}^{(j)}(z) = \mathsf{S}\theta_1^\star(z)\mathsf{S}\theta_1(z), \quad j = 1, \dots, K-1.$$

Define the invertible matrix $\mathsf{W}(z) := \big(\mathsf{V}^{(K-1)}(z)\cdots\mathsf{V}^{(0)}(z)\big)^{-1}$, we get

$$\mathsf{A}(z) = \mathsf{W}(z)\mathsf{A}^{(K)}(z)\mathsf{W}^\star(z), \qquad \mathsf{SW}(z) = \mathsf{SA}(z) = \mathsf{SA}^{(K)}(z) = \mathsf{S}\theta_1^\star(z)\mathsf{S}\theta_1(z). \tag{4.2.16}$$

We now discuss the three possible cases:

(1) If $\mathsf{A}_{1,2}^{(K)}(z) = \mathsf{A}_{2,1}^{(K)}(z) = 0$, from $\mathsf{A}_{1,1}^{(K)}(z)\mathsf{A}_{2,2}^{(K)}(z) = \det(\mathsf{A}^{(K)}(z)) = C$, we know that $\mathsf{A}_{1,1}^{(K)}(z)$ and $\mathsf{A}_{2,2}^{(K)}(z)$ must both be nonzero monomials. From their symmetry types $\mathsf{SA}_{1,1}^{(K)}(z) = \mathsf{SA}_{2,2}^{(K)}(z) = 1$, we know that $\mathsf{A}_{1,1}^{(K)}(z) = c_1$ and $\mathsf{A}_{2,2}^{(K)}(z) = c_2$ must be both constant, and $c_1c_2 = C < 0$. For the case that $c_1 > 0$ and $c_2 < 0$, we define $\widetilde{\mathsf{U}}(z) := \begin{bmatrix} \sqrt{c_1} & \\ & \sqrt{-c_2} \end{bmatrix}$, then $\mathsf{A}^{(K)}(z) = \widetilde{\mathsf{U}}(z)\begin{bmatrix} 1 & \\ & -1 \end{bmatrix}\widetilde{\mathsf{U}}^\star(z)$ holds. Since 0 can be defined to be any symmetry type, we can see that $\mathsf{S}\widetilde{\mathsf{U}}(z) = \mathsf{S}\theta_1^\star(z)\mathsf{S}\theta_1(z)$ holds, where $\mathsf{S}\theta_1(z)$ is defined in (4.2.15).

For the case that $c_1 < 0$ and $c_2 > 0$, we define $\widetilde{\mathsf{U}}(z) := \begin{bmatrix} & \sqrt{-c_1} \\ \sqrt{c_2} & \end{bmatrix}$, then $\mathsf{A}^{(K)}(z) = \widetilde{\mathsf{U}}(z)\begin{bmatrix} 1 & \\ & -1 \end{bmatrix}\widetilde{\mathsf{U}}^\star(z)$ still holds. Since 0 can be defined to be any symmetry type, we can see that $\mathsf{S}\widetilde{\mathsf{U}}(z) = \mathsf{S}\theta_1^\star(z)\mathsf{S}\theta_2(z)$ holds, where $\mathsf{S}\theta_1(z)$ is defined in (4.2.15), and $\mathsf{S}\theta_2(z) := [\alpha(z), 1]$.

(2) If $\mathsf{A}_{1,1}^{(K)}(z) = 0$, take $\widetilde{\mathsf{U}}(z) := \frac{1}{\sqrt{2}}\begin{bmatrix} \mathsf{A}_{1,2}^{(K)}(z) & \mathsf{A}_{1,2}^{(K)}(z) \\ 1 + \frac{1}{2}\mathsf{A}_{2,2}^{(K)}(z) & -1 + \frac{1}{2}\mathsf{A}_{2,2}^{(K)}(z) \end{bmatrix}$, then $\mathsf{A}^{(K)}(z) = \widetilde{\mathsf{U}}(z)\begin{bmatrix} 1 & \\ & -1 \end{bmatrix}\widetilde{\mathsf{U}}^\star(z)$ holds. Also, we have $\mathsf{S}\widetilde{\mathsf{U}}(z) = \mathsf{S}\theta_1^\star(z)\mathsf{S}\theta_3(z)$, where $\mathsf{S}\theta_1(z)$ is defined in (4.2.15) and $\mathsf{S}\theta_3(z) := [\alpha(z), \alpha(z)]$.

(3) If $\mathsf{A}_{2,2}^{(K)}(z) = 0$, take $\widetilde{\mathsf{U}}(z) := \frac{1}{\sqrt{2}}\begin{bmatrix} 1 + \frac{1}{2}\mathsf{A}_{1,1}^{(K)}(z) & -1 + \frac{1}{2}\mathsf{A}_{1,1}^{(K)}(z) \\ \mathsf{A}_{2,1}^{(K)}(z) & \mathsf{A}_{2,1}^{(K)}(z) \end{bmatrix}$. We

can directly verify that $\mathsf{A}^{(K)}(z) = \widetilde{\mathsf{U}}(z) \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \widetilde{\mathsf{U}}^\star(z)$. Also, we have $\mathsf{S}\widetilde{\mathsf{U}}(z) = \mathsf{S}\theta_1^\star(z)\mathsf{S}\theta_4(z)$, where $\mathsf{S}\theta_1(z)$ is defined in (4.2.15) and $\mathsf{S}\theta_4(z) := [1, 1]$.

For all the three cases, from (4.2.16), we get

$$\mathsf{A}(z) = \mathsf{W}(z)\mathsf{A}^{(K)}(z)\mathsf{W}^\star(z) = \mathsf{W}(z)\widetilde{\mathsf{U}}(z) \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \widetilde{\mathsf{U}}^\star(z)\mathsf{W}^\star(z).$$

That is, $\mathsf{A}(z) = \mathsf{U}(z) \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \mathsf{U}^\star(z)$, where $\mathsf{U}(z) := \mathsf{W}(z)\widetilde{\mathsf{U}}(z)$. Also, from the symmetry type of $\mathsf{W}(z)$ and $\widetilde{\mathsf{U}}(z)$, we can see that the symmetry type of $\mathsf{U}(z)$ is compatible: $\frac{\mathsf{SU}_{1,1}(z)}{\mathsf{SU}_{2,1}(z)} = \frac{\mathsf{SU}_{1,2}(z)}{\mathsf{SU}_{2,2}(z)} = \alpha(z)$. This completes the proof of the theorem. ■

The case that $\alpha(z) = -z^{2c}$ is considered in the following theorem.

Theorem 4.2.3. *Suppose* $\mathsf{A}(z)$ *is a* 2×2 *Hermitian matrix of Laurent polynomials with compatible symmetry. Also* $\det(\mathsf{A}(z)) = -d^2 < 0$, *where* $d \neq 0$ *is a real constant. Assume the symmetry type of* $\mathsf{A}(z)$ *is* $\mathsf{SA}(z) = \begin{bmatrix} 1 & -z^{2c} \\ -z^{-2c} & 1 \end{bmatrix}$, *for some* $c \in \mathbb{Z}$. *Then we can find a matrix* $\mathsf{U}(z) = \begin{bmatrix} \mathsf{U}_{1,1}(z) & \mathsf{U}_{1,2}(z) \\ \mathsf{U}_{2,1}(z) & \mathsf{U}_{2,2}(z) \end{bmatrix}$ *of Laurent polynomials with compatible symmetry, where the symmetry type satisfies*

$$\frac{\mathsf{SU}_{1,1}(z)}{\mathsf{SU}_{2,1}(z)} = \frac{\mathsf{SU}_{1,2}(z)}{\mathsf{SU}_{2,2}(z)} = -z^{2c}, \tag{4.2.17}$$

and $\mathsf{A}(z) = \mathsf{U}(z) \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \mathsf{U}^\star(z)$.

To prove the Theorem 4.2.3, we introduce the following algorithm to find $\mathsf{U}(z)$. Theorem 4.2.3 is proved by showing the Algorithm 4.1 is feasible.

Algorithm 4.1. *Given a matrix* $\mathsf{A}(z)$ *of Laurent polynomials satisfying the assumptions in Theorem 4.2.3.*

(S1) *Define* $[-n,n] := \mathrm{fsupp}(\mathsf{A}_{1,1}(z))$. *Write* $\mathsf{U}_{1,1}(z) = \sum_{j=0}^{n} t_j z^j$ *and* $\mathsf{U}_{1,2}(z) = \sum_{j=0}^{n} \widetilde{t}_j z^j$, *where* $\{t_0,\ldots,t_n,\widetilde{t_0},\ldots,\widetilde{t_n}\}$ *is a nontrivial solution to the homogeneous system* X *of* $2n$ *equations induced by* $\mathcal{R}(z)\equiv 0$, *where* $\mathcal{R}(z)$ *and* $\mathsf{U}_{2,1}(z)$ *are uniquely determined through long division using* $\mathsf{A}_{1,1}(z)$ *by*

$$\mathsf{A}_{2,1}(z)\mathsf{U}_{1,1}(z) + z^{n-c} d\mathsf{U}^{\star}_{1,2}(z) = \mathsf{A}_{1,1}(z)\mathsf{U}_{2,1}(z) + \mathcal{R}(z), \quad \textit{with } \mathrm{fsupp}(\mathcal{R}(z)) \subseteq [-n, n-1]. \tag{4.2.18}$$

The space of all solutions to X *has dimension at least two. So we can always find nontrivial solutions.*

(S2) *If* $\mathsf{SU}_{1,1}(z) = z^n$ *and* $\mathsf{SU}_{1,2}(z) = -z^n$, *go to* **(S3)**. *Otherwise, redefine*

$$\mathsf{U}_{1,1}(z) := \Big[\mathsf{U}_{1,1}(z) - z^n \mathsf{U}^{\star}_{1,1}(z)\Big]/2, \qquad \mathsf{U}_{1,2}(z) := \Big[\mathsf{U}_{1,2}(z) + z^n \mathsf{U}^{\star}_{1,2}(z)\Big]/2. \tag{4.2.19}$$

We have $\mathsf{U}_{1,1}(z)$ *and* $\mathsf{U}_{1,2}(z)$ *not simultaneously identical to zero. Also, the symmetry types satisfy* $\mathsf{SU}_{1,1}(z) = -z^n$ *and* $\mathsf{SU}_{1,2}(z) = z^n$.

(S3) *There must exist a nonzero real number* λ, *such that*

$$\mathsf{A}_{1,1}(z) = \lambda\Big[\mathsf{U}_{1,1}(z)\mathsf{U}^{\star}_{1,1} - \mathsf{U}_{1,2}(z)^{\star}\mathsf{U}_{1,2}(z)\Big]. \tag{4.2.20}$$

If $\lambda > 0$, *redefine*

$$\mathsf{U}_{1,1}(z) := \sqrt{\lambda}\mathsf{U}_{1,1}(z), \qquad \mathsf{U}_{1,2}(z) := \sqrt{\lambda}\mathsf{U}_{1,2}(z). \tag{4.2.21}$$

Otherwise, redefine

$$\mathsf{U}_{1,1}(z) := \sqrt{-\lambda}\mathsf{U}_{1,2}(z), \qquad \mathsf{U}_{1,2}(z) := \sqrt{-\lambda}\mathsf{U}_{1,1}(z). \tag{4.2.22}$$

$\mathsf{U}_{1,1}(z)$ *and* $\mathsf{U}_{1,2}(z)$ *now have symmetry*

$$\mathsf{SU}_{1,1}(z) = \epsilon z^n, \qquad \mathsf{SU}_{1,2}(z) = -\epsilon z^n, \tag{4.2.23}$$

for some $\epsilon \in \{1, -1\}$.

(S4) *Define*

$$\mathsf{U}_{2,1}(z) := \frac{\mathsf{A}_{2,1}(z)\mathsf{U}_{1,1}(z) + z^{n-c}d\mathsf{U}^{\star}_{1,2}(z)}{\mathsf{A}_{1,1}(z)}, \tag{4.2.24}$$

$$\mathsf{U}_{2,2}(z) := \frac{\mathsf{A}_{2,1}(z)\mathsf{U}_{1,2}(z) + z^{n-c}d\mathsf{U}^{\star}_{1,1}(z)}{\mathsf{A}_{1,1}(z)}. \tag{4.2.25}$$

Then $\mathsf{U}_{2,1}(z)$ *and* $\mathsf{U}_{2,2}(z)$ *here are well-defined Laurent polynomials with symmetry:*

$$\mathsf{SU}_{2,1}(z) = -\epsilon z^{n-2c}, \qquad \mathsf{SU}_{2,2}(z) = \epsilon z^{n-2c}, \tag{4.2.26}$$

where the ϵ *here is the same as that in* (4.2.23).

(S5) *Define*

$$\mathsf{U}(z) := \begin{bmatrix} \mathsf{U}_{1,1}(z) & \mathsf{U}_{1,2}(z) \\ \mathsf{U}_{2,1}(z) & \mathsf{U}_{2,2}(z) \end{bmatrix}.$$

Then $\mathsf{A}(z) = \mathsf{U}(z) \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \mathsf{U}^{\star}(z)$, *and* (4.2.17) *holds.*

Proof of Algorithm 4.1. Firstly, we will prove that $\mathsf{A}_{1,1}(z)$ is not identically zero. Suppose $\mathsf{A}_{1,1}(z) = 0$, then $\det(\mathsf{A}(z)) = -\mathsf{A}_{1,2}(z)\mathsf{A}_{2,1}(z) = -d^2$. So $\mathsf{A}_{1,2}(z)$ must be a nonzero monomial, which cannot have symmetry type $-z^{2c}$. Hence, we proved that $\mathsf{A}_{1,1}(z)$ is not identically zero.

In ***(S1)***, it can be verified directly that X is a homogeneous linear system. Since $\operatorname{fsupp}(\mathcal{R}(z)) \subseteq [-n, n-1]$, X has $2n$ equations with $2n+2$ unknowns $\{t_0, \dots, t_n, \widetilde{t_0}, \dots, \widetilde{t_n}\}$. So the space of all solutions to X has dimension at least two. Hence, we can find some nontrivial solution pair $\{\mathsf{U}_{1,1}(z), \mathsf{U}_{1,2}(z)\}$ to X. Since $\mathcal{R} \equiv 0$, after ***(S1)***, we have

$$\mathsf{A}_{2,1}(z)\mathsf{U}_{1,1}(z) + z^{n-c}d\mathsf{U}^{\star}_{1,2}(z) = \mathsf{A}_{1,1}(z)\mathsf{U}_{2,1}(z), \tag{4.2.27}$$

for some Laurent polynomial $\mathsf{U}_{2,1}(z)$.

Take Hermitian conjugate on both sides of (4.2.27), notice that $\mathsf{A}^{\star}_{1,1}(z) =$

$\mathsf{A}_{1,1}(z)$ and $\mathsf{A}^\star_{2,1}(z) = -z^{2c}\mathsf{A}_{2,1}(z)$, we get

$$-z^{2c}\mathsf{A}_{2,1}(z)\mathsf{U}^\star_{1,1}(z) + z^{c-n}d\mathsf{U}_{1,2}(z) = \mathsf{A}_{1,1}(z)\mathsf{U}^\star_{2,1}(z).$$

Multiplying z^{n-2c} to the above equation yields

$$-z^n\mathsf{A}_{2,1}(z)\mathsf{U}^\star_{1,1}(z) + z^{-c}d\mathsf{U}_{1,2}(z) = z^{n-2c}\mathsf{A}_{1,1}(z)\mathsf{U}^\star_{2,1}(z). \tag{4.2.28}$$

Averaging (4.2.27) and (4.2.28), we get

$$\begin{aligned}\mathsf{A}_{2,1}(z)\frac{1}{2}\Big[\mathsf{U}_{1,1}(z) - z^n\mathsf{U}^\star_{1,1}(z)\Big] + z^{n-c}d\frac{1}{2}\Big[\mathsf{U}^\star_{1,2}(z) + z^{-n}\mathsf{U}_{1,2}\Big]& \\ = \mathsf{A}_{1,1}(z)\frac{1}{2}\Big[\mathsf{U}_{2,1}(z) + z^{n-2c}\mathsf{U}^\star_{2,1}(z)\Big].& \end{aligned} \tag{4.2.29}$$

After step ***(S1)***, if $\mathsf{SU}_{1,1}(z) \neq z^n$ or $\mathsf{SU}_{1,2}(z) \neq -z^n$, the redefined $\mathsf{U}_{1,1}(z)$ and $\mathsf{U}_{1,2}(z)$ in (4.2.19) are not simultaneously equal to zero. Moreover, since the old $\mathsf{U}_{1,1}(z)$ and $\mathsf{U}_{1,2}(z)$ satisfy $\text{fsupp}(\mathsf{U}_{1,1}(z)) \subseteq [0,n]$ and $\text{fsupp}(\mathsf{U}_{1,2}(z)) \subseteq [0,n]$, by the definition in (4.2.19), the redefined $\mathsf{U}_{1,1}(z)$ and $\mathsf{U}_{1,2}(z)$ also satisfy $\text{fsupp}(\mathsf{U}_{1,1}(z)) \subseteq [0,n]$ and $\text{fsupp}(\mathsf{U}_{1,2}(z)) \subseteq [0,n]$. Also, we can verify with direct calculation that the redefined $\mathsf{U}_{1,1}(z)$ and $\mathsf{U}_{1,2}(z)$ have symmetry: $\mathsf{SU}_{1,1}(z) = -z^n$, and $\mathsf{SU}_{1,2}(z) = z^n$. Furthermore, by (4.2.29), we can see that the new pair $\{\mathsf{U}_{1,1}(z), \mathsf{U}_{1,2}(z)\}$ satisfy

$$\mathsf{A}_{1,1}(z) \mid \mathsf{A}_{2,1}(z)\mathsf{U}_{1,1}(z) + z^{n-c}d\mathsf{U}^\star_{1,2}(z).$$

Redefine $\mathsf{U}_{2,1}(z) := \frac{\mathsf{A}_{2,1}(z)\mathsf{U}_{1,1}(z)+z^{n-c}d\mathsf{U}^\star_{1,2}(z)}{\mathsf{A}_{1,1}(z)}$. (4.2.29) implies that (4.2.27) still holds after step ***(S2)***.

To prove step ***(S3)***, we can see from (4.2.27) that

$$\begin{aligned}&\mathsf{A}_{1,2}(z)\Big(\mathsf{A}_{2,1}(z)\mathsf{U}_{1,2}(z) + z^{n-c}d\mathsf{U}^\star_{1,1}(z)\Big)\\ =&\mathsf{A}_{1,2}(z)\mathsf{A}_{2,1}(z)\mathsf{U}_{1,2}(z) + z^{n-c}d\mathsf{U}^\star_{1,1}(z)\mathsf{A}_{1,2}(z)\\ =&\Big(\mathsf{A}_{1,1}(z)\mathsf{A}_{2,2}(z) - \det(\mathsf{A}(z))\Big)\mathsf{U}_{1,2}(z) + z^{n-c}d\mathsf{U}^\star_{1,1}(z)\mathsf{A}_{1,2}(z)\\ =&\Big(\mathsf{A}_{1,1}(z)\mathsf{A}_{2,2}(z) + d^2\Big)\mathsf{U}_{1,2}(z) + z^{n-c}d\mathsf{U}^\star_{1,1}(z)\mathsf{A}_{1,2}(z)\end{aligned}$$

$$=\mathsf{A}_{1,1}(z)\mathsf{A}_{2,2}(z)\mathsf{U}_{1,2}(z)+z^{n-c}d\Big(z^{c-n}d\mathsf{U}_{1,2}(z)+\mathsf{U}^{\star}_{1,1}(z)\mathsf{A}_{1,2}(z)\Big)$$
$$=\mathsf{A}_{1,1}(z)\mathsf{A}_{2,2}(z)\mathsf{U}_{1,2}(z)+z^{n-c}d\Big(z^{n-c}d\mathsf{U}^{\star}_{1,2}(z)+\mathsf{U}_{1,1}(z)\mathsf{A}^{\star}_{1,2}(z)\Big)^{\star}$$
$$=\mathsf{A}_{1,1}(z)\mathsf{A}_{2,2}(z)\mathsf{U}_{1,2}(z)+z^{n-c}d\mathsf{A}_{1,1}(z)\mathsf{U}^{\star}_{2,1}(z)$$
$$=\mathsf{A}_{1,1}(z)\Big(\mathsf{A}_{2,2}(z)\mathsf{U}_{1,2}(z)+z^{n-c}d\mathsf{U}^{\star}_{2,1}(z)\Big).$$

Since $\det(\mathsf{A}(z))$ is a constant, we know that $\gcd(\mathsf{A}_{1,1}(z),\mathsf{A}_{1,2}(z))=1$. So, the above equation implies that $\mathsf{A}_{1,1}(z)\mid\mathsf{A}_{2,1}(z)\mathsf{U}_{1,2}(z)+z^{n-c}d\mathsf{U}^{\star}_{1,1}(z)$. Define $\mathsf{U}_{2,2}(z):=\frac{\mathsf{A}_{2,1}(z)\mathsf{U}_{1,2}(z)+z^{n-c}d\mathsf{U}^{\star}_{1,1}(z)}{\mathsf{A}_{1,1}(z)}$, we can see that

$$\mathsf{A}_{2,1}(z)\mathsf{U}_{1,2}(z)+z^{n-c}d\mathsf{U}^{\star}_{1,1}(z)=\mathsf{A}_{1,1}(z)\mathsf{U}_{2,2}(z). \tag{4.2.30}$$

Combine (4.2.27) and (4.2.30), we proved

$$\begin{bmatrix}\mathsf{U}_{2,2}(z) & -\mathsf{U}_{1,2}(z)\\ -\mathsf{U}_{2,1}(z) & \mathsf{U}_{1,1}(z)\end{bmatrix}\begin{bmatrix}\mathsf{A}_{1,1}(z)\\ \mathsf{A}_{2,1}(z)\end{bmatrix}=z^{n-c}d\begin{bmatrix}\mathsf{U}^{\star}_{1,1}(z)\\ -\mathsf{U}^{\star}_{1,2}(z)\end{bmatrix}. \tag{4.2.31}$$

Multiply $\begin{bmatrix}\mathsf{U}^{\star}_{1,2}(z) & \mathsf{U}^{\star}_{1,1}(z)\end{bmatrix}$ to the left on both sides of (4.2.31), we get

$$\Big(\mathsf{U}_{2,2}(z)\mathsf{U}^{\star}_{1,2}(z)-\mathsf{U}_{2,1}(z)\mathsf{U}^{\star}_{1,1}(z)\Big)\mathsf{A}_{1,1}(z)+\Big(\mathsf{U}_{1,1}(z)\mathsf{U}^{\star}_{1,1}(z)-\mathsf{U}_{1,2}(z)\mathsf{U}^{\star}_{1,2}(z)\Big)\mathsf{A}_{2,1}(z)=0.$$

That is,

$$\Big(\mathsf{U}_{1,1}(z)\mathsf{U}^{\star}_{1,1}(z)-\mathsf{U}_{1,2}(z)\mathsf{U}^{\star}_{1,2}(z)\Big)\mathsf{A}_{2,1}(z)=\Big(\mathsf{U}_{2,1}(z)\mathsf{U}^{\star}_{1,1}(z)-\mathsf{U}_{2,2}(z)\mathsf{U}^{\star}_{1,2}(z)\Big)\mathsf{A}_{1,1}(z). \tag{4.2.32}$$

Since $\gcd(\mathsf{A}_{1,1}(z),\mathsf{A}_{2,1}(z))=1$, the above equation implies

$$\mathsf{A}_{1,1}(z)\mid\mathsf{U}_{1,1}(z)\mathsf{U}^{\star}_{1,1}(z)-\mathsf{U}_{1,2}(z)\mathsf{U}^{\star}_{1,2}(z). \tag{4.2.33}$$

Furthermore, from $\operatorname{fsupp}(\mathsf{U}_{1,1}(z))\subseteq[0,n]$ and $\operatorname{fsupp}(\mathsf{U}_{1,2}(z))\subseteq[0,n]$, we can see $\operatorname{fsupp}(\mathsf{U}_{1,1}\mathsf{U}^{\star}_{1,1})\subseteq[-n,n]$, and $\operatorname{fsupp}(\mathsf{U}_{1,2}\mathsf{U}^{\star}_{1,2})\subseteq[-n,n]$. This implies that

$$\operatorname{fsupp}(\mathsf{U}_{1,1}\mathsf{U}^{\star}_{1,1}-\mathsf{U}_{1,2}\mathsf{U}^{\star}_{1,2})\subseteq\operatorname{fsupp}(\mathsf{A}_{1,1}). \tag{4.2.34}$$

To prove (4.2.20), we can see that after step ***(S2)***, the symmetry types of $\mathsf{U}_{1,1}(z)$ and $\mathsf{U}_{1,2}(z)$ have two possible cases:

(1) $\mathsf{SU}_{1,1}(z) = z^n$, $\mathsf{SU}_{1,2}(z) = -z^n$;

(2) $\mathsf{SU}_{1,1}(z) = -z^n$, $\mathsf{SU}_{1,2}(z) = z^n$.

For the case (1), by item (1) of Lemma 4.1.2, we know $\mathsf{Z}(\mathsf{U}^\star_{1,1}(z), 1) = \mathsf{Z}(\mathsf{U}_{1,1}(z), 1) \in 2\mathbb{Z}$, and $\mathsf{Z}(\mathsf{U}^\star_{1,2}(z), 1) = \mathsf{Z}(\mathsf{U}_{1,2}(z), 1) \in 2\mathbb{Z}+1$. So $\mathsf{Z}(\mathsf{U}^\star_{1,1}(z)\mathsf{U}_{1,1}(z), 1) \in 4\mathbb{Z}$, while $\mathsf{Z}(\mathsf{U}^\star_{1,2}(z)\mathsf{U}_{1,2}(z), 1) \in 4\mathbb{Z}+2$. Hence, $\mathsf{U}_{1,1}(z)\mathsf{U}^\star_{1,1}(z) \neq \mathsf{U}_{1,2}(z)\mathsf{U}^\star_{1,2}(z)$. Similarly, for the case (2), we have $\mathsf{Z}(\mathsf{U}^\star_{1,1}(z)\mathsf{U}_{1,1}(z), 1) \in 4\mathbb{Z}+2$, while $\mathsf{Z}(\mathsf{U}^\star_{1,2}(z)\mathsf{U}_{1,2}(z), 1) \in 4\mathbb{Z}$. We also get $\mathsf{U}_{1,1}(z)\mathsf{U}^\star_{1,1}(z) \neq \mathsf{U}_{1,2}(z)\mathsf{U}^\star_{1,2}(z)$. So for both cases, we get $\mathsf{U}_{1,1}(z)\mathsf{U}^\star_{1,1}(z) - \mathsf{U}_{1,2}(z)\mathsf{U}^\star_{1,2}(z) \neq 0$. Therefore, by (4.2.33) and (4.2.34), we can conclude that there exists a nonzero constant number λ, such that (4.2.20) holds. Since $\mathsf{A}_{1,1}(z)$ and $\mathsf{U}_{1,1}(z)\mathsf{U}^\star_{1,1}(z) - \mathsf{U}_{1,2}(z)\mathsf{U}^\star_{1,2}(z)$ are both real functions on $\mathbb{T}$, we can see that λ must be a real number.

If $\lambda > 0$, normalizing the solution pair $\{\mathsf{U}_{1,1}(z), \mathsf{U}_{1,2}(z)\}$ with (4.2.21) will give

$$\mathsf{A}_{1,1}(z) = \mathsf{U}_{1,1}(z)\mathsf{U}^\star_{1,1}(z) - \mathsf{U}_{1,2}(z)\mathsf{U}^\star_{1,2}(z). \tag{4.2.35}$$

Notice that by (4.2.27) and (4.2.30),

$$\mathsf{A}_{1,1}(z) \mid \mathsf{A}_{2,1}(z)\mathsf{U}_{1,1}(z) + z^{n-c}d\mathsf{U}^\star_{1,2}(z), \quad \mathsf{A}_{1,1}(z) \mid \mathsf{A}_{2,1}(z)\mathsf{U}_{1,2}(z) + z^{n-c}d\mathsf{U}^\star_{1,1}(z) \tag{4.2.36}$$

still hold after the normalization.

If $\lambda < 0$, redefining the solution pair $\{\mathsf{U}_{1,1}(z), \mathsf{U}_{1,2}(z)\}$ with (4.2.22) will also give (4.2.35). Furthermore, notice that switching $\mathsf{U}_{1,1}(z)$ and $\mathsf{U}_{1,2}(z)$ corresponds to switching the two relations in (4.2.36). So (4.2.36) still holds for the new solution pair $\{\mathsf{U}_{1,1}(z), \mathsf{U}_{1,2}(z)\}$ in this case.

Since the normalizations in step ***(S3)*** do not change the symmetry types of $\mathsf{U}_{1,1}(z)$ and $\mathsf{U}_{1,2}(z)$, they still have symmetry inherited from step ***(S2)***. This proved (4.2.23).

By (4.2.36), $\mathsf{U}_{2,1}(z)$ and $\mathsf{U}_{2,2}(z)$ we set in (4.2.24) and (4.2.25) are well-defined Laurent polynomials. The symmetry types of $\mathsf{U}_{2,1}(z)$ and $\mathsf{U}_{2,2}(z)$ in (4.2.26) can be proved by direct calculations using their definitions in (4.2.24) and (4.2.25), and the symmetry types of $\mathsf{U}_{1,1}(z), \mathsf{U}_{1,2}(z)$ from (4.2.23).

Furthermore, the definitions of $\mathsf{U}_{2,1}(z)$ and $\mathsf{U}_{2,2}(z)$ in (4.2.24) and (4.2.25) imply that (4.2.31) holds. Notice that (4.2.31) implies (4.2.32). By (4.2.35), we know that (4.2.32) yields

$$\mathsf{U}_{2,1}(z)\mathsf{U}^{\star}_{1,1}(z) - \mathsf{U}_{2,2}(z)\mathsf{U}^{\star}_{1,2}(z) = \mathsf{A}_{2,1}(z). \tag{4.2.37}$$

Multiplying $\begin{bmatrix}\mathsf{U}_{1,1}(z) & \mathsf{U}_{1,2}(z)\end{bmatrix}$ to the left on both sides of (4.2.31), we have

$$\Big(\mathsf{U}_{1,1}(z)\mathsf{U}_{2,2}(z) - \mathsf{U}_{1,2}(z)\mathsf{U}_{2,1}(z)\Big)\mathsf{A}_{1,1}(z) = z^{n-c}d\Big(\mathsf{U}_{1,1}(z)\mathsf{U}^{\star}_{1,1}(z) - \mathsf{U}_{1,2}(z)\mathsf{U}^{\star}_{1,2}(z)\Big).$$

Combining the above equation with (4.2.35), we can conclude

$$\det(\mathsf{U}(z)) = \mathsf{U}_{1,1}(z)\mathsf{U}_{2,2}(z) - \mathsf{U}_{1,2}(z)\mathsf{U}_{2,1}(z) = z^{n-c}d. \tag{4.2.38}$$

Multiplying $\begin{bmatrix}\mathsf{U}^{\star}_{2,2}(z) & \mathsf{U}^{\star}_{2,1}(z)\end{bmatrix}$ to the left on both sides of (4.2.31), we get

$$\begin{aligned}\Big(\mathsf{U}_{2,2}(z)\mathsf{U}^{\star}_{2,2}(z) - \mathsf{U}_{2,1}(z)\mathsf{U}^{\star}_{2,1}(z)\Big)\mathsf{A}_{1,1}(z) + \Big(\mathsf{U}_{1,1}(z)\mathsf{U}^{\star}_{2,1}(z) - \mathsf{U}_{1,2}(z)\mathsf{U}^{\star}_{2,2}(z)\Big)\mathsf{A}_{2,1}(z)\\ = z^{n-c}d\Big(\mathsf{U}^{\star}_{1,1}(z)\mathsf{U}^{\star}_{2,2}(z) - \mathsf{U}^{\star}_{1,2}(z)\mathsf{U}^{\star}_{2,1}(z)\Big) = z^{n-c}d(\det(\mathsf{U}(z)))^{\star}.\end{aligned}$$

Using (4.2.37) and (4.2.38), we can simplify the above equation as

$$\begin{aligned}\Big(\mathsf{U}_{2,2}(z)\mathsf{U}^{\star}_{2,2}(z) - \mathsf{U}_{2,1}(z)\mathsf{U}^{\star}_{2,1}(z)\Big)\mathsf{A}_{1,1}(z) + \mathsf{A}^{\star}_{2,1}(z)\mathsf{A}_{2,1}(z) =& z^{n-c}dz^{c-n}d,\\ =& d^2 = -\det(\mathsf{A}(z))\\ =& \mathsf{A}_{1,2}(z)\mathsf{A}_{2,1}(z) - \mathsf{A}_{1,1}(z)\mathsf{A}_{2,2}(z).\end{aligned}$$

Since $\mathsf{A}^{\star}(z) = \mathsf{A}(z)$, we have $\mathsf{A}^{\star}_{2,1}(z) = \mathsf{A}_{1,2}(z)$. So we can cancel the term $\mathsf{A}_{1,2}(z)\mathsf{A}_{2,1}(z)$ on both sides of the above equation, and divide by $\mathsf{A}_{1,1}(z)$ to get

$$\mathsf{U}_{2,2}(z)\mathsf{U}^{\star}_{2,2}(z) - \mathsf{U}_{2,1}(z)\mathsf{U}^{\star}_{2,1}(z) = -\mathsf{A}_{2,2}(z).$$

That is,

$$\mathsf{U}_{2,1}(z)\mathsf{U}^{\star}_{2,1}(z) - \mathsf{U}_{2,2}(z)\mathsf{U}^{\star}_{2,2}(z) = \mathsf{A}_{2,2}(z). \tag{4.2.39}$$

Combining (4.2.35) (4.2.37) and (4.2.39), we proved $\mathsf{A}(z) = \mathsf{U}(z) \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \mathsf{U}^\star(z)$. The symmetry type of $\mathsf{U}(z)$ in (4.2.17) follows from (4.2.23) and (4.2.26). This finished the proof of the algorithm. ■

Theorem 4.2.2 and Theorem 4.2.3 together proved the result of Theorem 4.2.1 in all the four possible symmetry cases of $\alpha(z)$. So we proved Theorem 4.2.1.

4.3 General Case

In this section, we study the factorization in Theorem 4.2.1 without requiring $\det(\mathsf{A}(z))$ to be a constant. As a matter of fact, necessary and sufficient conditions for spectral factorizations of general 2×2 matrices of Laurent polynomials with symmetry are related to the *gcd* of all the 4 elements in the matrix $\mathsf{A}(z)$. In Section 4.3.1, we prove that the spectral factors can be extracted out for the case that gcd $= 1$. In Section 4.3.2, we study the DOS(Difference of Squares) property of Laurent polynomials with symmetry. The necessary and sufficient conditions of spectral factorizations of general 2×2 matrices of Laurent polynomials with symmetry are proved as Theorem 4.3.6 in Section 4.3.3.

4.3.1 $GCD = 1$ Case

It is well-known that the Smith Normal Form of matrices of Laurent polynomials can be calculated by performing long divisions repeatedly. However, directly using long divisions iteratively might destroy the symmetry structures. Fortunately, we can still build the following Theorem 4.3.1 for Extended Euclidean Algorithm with symmetry.

Theorem 4.3.1 (Extended Euclidean Algorithm for Laurent polynomials with symmetry)**.** *Given two Laurent polynomials* $\mathsf{a}(z)$ *and* $\mathsf{b}(z)$ *with symmetry, define* $\mathsf{r}(z) := \gcd(\mathsf{a}(z), \mathsf{b}(z))$. *Then there exist Laurent polynomials* $\mathsf{u}(z)$ *and* $\mathsf{v}(z)$ *with symmetry, such that*

$$\mathsf{a}(z)\mathsf{u}(z) + \mathsf{b}(z)\mathsf{v}(z) = \mathsf{r}(z), \tag{4.3.1}$$

and $\mathsf{Sa}(z)\mathsf{Su}(z) = \mathsf{Sb}(z)\mathsf{Sv}(z) = \mathsf{Sr}(z)$. *Also we have* $\gcd(\mathsf{u}(z), \mathsf{v}(z)) = 1$.

Proof. The theorem can be proved constructively, so that we can follow the steps and use it as an algorithm.

If $\mathsf{a}(z) = 0$, we can define $\mathsf{u}(z) = 0$ and $\mathsf{v}(z) = 1$; otherwise if $\mathsf{b}(z) = 0$, we can define $\mathsf{v}(z) = 0$ and $\mathsf{u}(z) = 1$, then the result of the theorem can be verified directly. We only consider the case that $\mathsf{a}(z) \neq 0$ and $\mathsf{b}(z) \neq 0$.

Let $\mathsf{r}(z) := \gcd(\mathsf{a}(z), \mathsf{b}(z))$, we can use the original Extended Euclidean Algorithm (without symmetry) to find Laurent polynomials $\mathsf{u}_1(z)$ and $\mathsf{v}_1(z)$, such that

$$\mathsf{a}(z)\mathsf{u}_1(z) + \mathsf{b}(z)\mathsf{v}_1(z) = \mathsf{r}(z). \tag{4.3.2}$$

$\mathsf{u}_1(z), \mathsf{v}_1(z)$ may not have symmetry here. We can see that (4.3.2) also implies

$$\mathsf{a}(z^{-1})\mathsf{u}_1(z^{-1}) + \mathsf{b}(z^{-1})\mathsf{v}_1(z^{-1}) = \mathsf{r}(z^{-1}).$$

Plugging in $\mathsf{a}(z^{-1}) = \mathsf{a}(z)/\mathsf{Sa}(z)$, $\mathsf{b}(z^{-1}) = \mathsf{b}(z)/\mathsf{Sb}(z)$, and $\mathsf{r}(z^{-1}) = \mathsf{r}(z)/\mathsf{Sr}(z)$ to the above equation, we get

$$\mathsf{a}(z)\mathsf{u}_1(z^{-1})/\mathsf{Sa}(z) + \mathsf{b}(z)\mathsf{v}_1(z^{-1})/\mathsf{Sb}(z) = \mathsf{r}(z)/\mathsf{Sr}(z). \tag{4.3.3}$$

Multiplying $\mathsf{Sr}(z)$ on (4.3.3), and averaging it with (4.3.2) gives us

$$\mathsf{a}(z)\frac{\mathsf{u}_1(z^{-1})\mathsf{Sr}(z)/\mathsf{Sa}(z) + \mathsf{u}_1(z)}{2} + \mathsf{b}(z)\frac{\mathsf{v}_1(z^{-1})\mathsf{Sr}(z)/\mathsf{Sb}(z) + \mathsf{v}_1(z)}{2} = \mathsf{r}(z). \tag{4.3.4}$$

Define

$$\mathsf{u}(z) := \frac{\mathsf{u}_1(z^{-1})\mathsf{Sr}(z)/\mathsf{Sa}(z) + \mathsf{u}_1(z)}{2},$$
$$\mathsf{v}(z) := \frac{\mathsf{v}_1(z^{-1})\mathsf{Sr}(z)/\mathsf{Sb}(z) + \mathsf{v}_1(z)}{2}.$$

Equation (4.3.4) implies that (4.3.1) holds. Direct calculations give us $\mathsf{Su}(z) = \mathsf{u}(z)/\mathsf{u}(z^{-1}) = \mathsf{Sr}(z)/\mathsf{Sa}(z)$ and $\mathsf{Sv}(z) = \mathsf{v}(z)/\mathsf{v}(z^{-1}) = \mathsf{Sr}(z)/\mathsf{Sb}(z)$. Hence we have $\mathsf{Sa}(z)\mathsf{Su}(z) = \mathsf{Sb}(z)\mathsf{Sv}(z) = \mathsf{Sr}(z)$.

Let $\mathsf{d}(z) := \gcd(\mathsf{u}(z), \mathsf{v}(z))$. Since $\mathsf{r}(z) = \gcd(\mathsf{a}(z), \mathsf{b}(z))$, we know that $\mathsf{d}(z)\mathsf{r}(z) | \mathsf{a}(z)\mathsf{u}(z)$ and $\mathsf{d}(z)\mathsf{r}(z) | \mathsf{b}(z)\mathsf{v}(z)$. Hence $\mathsf{d}(z)\mathsf{r}(z)$ divides the left hand side of (4.3.1), which implies that $\mathsf{d}(z)\mathsf{r}(z)|\mathsf{r}(z)$. Therefore, $\gcd(\mathsf{u}(z), \mathsf{v}(z)) = \mathsf{d}(z) = 1$. This completes the proof of the theorem. ■

Using the Theorem 4.3.1, we can build the following lemma.

Lemma 4.3.1. *Suppose $\mathsf{N}_1(z), \mathsf{N}_2(z) \neq 0$ are two Laurent polynomials with symmetry. Then we can find an invertible 2×2 matrix $\mathsf{P}(z)$ of Laurent polynomials with compatible symmetry, such that*

$$\mathsf{P}(z) \begin{bmatrix} \mathsf{N}_1(z) \\ \mathsf{N}_2(z) \end{bmatrix} = \begin{bmatrix} \mathsf{r}(z) \\ 0 \end{bmatrix}, \tag{4.3.5}$$

where $\mathsf{r}(z) = \gcd(\mathsf{N}_1(z), \mathsf{N}_2(z))$. Also the above matrix multiplication is compatible.

Proof. From Theorem 4.3.1, we can find Laurent polynomials $\mathsf{u}(z)$ and $\mathsf{v}(z)$ with symmetry, such that

$$\mathsf{u}(z)\mathsf{N}_1(z) + \mathsf{v}(z)\mathsf{N}_2(z) = \mathsf{r}(z), \tag{4.3.6}$$

with symmetry type

$$\mathsf{Su}(z)\mathsf{SN}_1(z) = \mathsf{Sv}(z)\mathsf{SN}_2(z) = \mathsf{Sr}(z). \tag{4.3.7}$$

Moreover, by $\gcd(\mathsf{u}(z), \mathsf{v}(z)) = 1$, we can use Theorem 4.3.1 again, to find Laurent polynomials $\mathsf{s}(z)$ and $\mathsf{t}(z)$ with symmetry, such that

$$\mathsf{u}(z)\mathsf{t}(z) + \mathsf{v}(z)\left(-\mathsf{s}(z)\right) = 1, \tag{4.3.8}$$

with symmetry type

$$\mathsf{Su}(z)\mathsf{St}(z) = \mathsf{Sv}(z)\mathsf{Ss}(z) = 1. \tag{4.3.9}$$

Define $\mathsf{P}_1(z) := \begin{bmatrix} \mathsf{u}(z) & \mathsf{v}(z) \\ \mathsf{s}(z) & \mathsf{t}(z) \end{bmatrix}$. We can see from (4.3.7) and (4.3.9) that $\frac{\mathsf{Ss}(z)}{\mathsf{St}(z)} = \frac{\mathsf{Su}(z)}{\mathsf{Sv}(z)} = \frac{\mathsf{SN}_2(z)}{\mathsf{SN}_1(z)}$, that is, $\mathsf{Ss}(z)\mathsf{SN}_1(z) = \mathsf{St}(z)\mathsf{SN}_2(z)$. By the item (3) of Proposition 4.1.2, this equality and (4.3.7) imply that $\mathsf{P}_1(z)$ has compatible symmetry, and the multiplication in (4.3.10) is compatible. From (4.3.8), we get $\det(\mathsf{P}_1(z)) = \mathsf{u}(z)\mathsf{t}(z) - \mathsf{v}(z)\mathsf{s}(z) = 1$, which implies that $\mathsf{P}_1(z)$ is invertible. Also, according to (4.3.6)

$$\mathsf{P}_1(z)\begin{bmatrix}\mathsf{N}_1(z)\\ \mathsf{N}_2(z)\end{bmatrix} = \begin{bmatrix}\mathsf{u}(z) & \mathsf{v}(z)\\ \mathsf{s}(z) & \mathsf{t}(z)\end{bmatrix}\begin{bmatrix}\mathsf{N}_1(z)\\ \mathsf{N}_2(z)\end{bmatrix} = \begin{bmatrix}\mathsf{r}(z)\\ \mathsf{s}(z)\mathsf{N}_1(z)+\mathsf{t}(z)\mathsf{N}_2(z)\end{bmatrix}. \quad (4.3.10)$$

Since $\mathsf{r}(z) = \gcd(\mathsf{N}_1(z), \mathsf{N}_2(z))$, we get $\mathsf{r}(z) \mid \mathsf{s}(z)\mathsf{N}_1(z) + \mathsf{t}(z)\mathsf{N}_2(z)$. Define $\mathsf{p}(z) := \frac{\mathsf{s}(z)\mathsf{N}_1(z)+\mathsf{t}(z)\mathsf{N}_2(z)}{\mathsf{r}(z)}$, and $\mathsf{P}_2(z) := \begin{bmatrix}1 & 0\\ -\mathsf{p}(z) & 1\end{bmatrix}$. We can see $\det(\mathsf{P}_2(z)) = 1$, and

$$\mathsf{P}_2(z)\begin{bmatrix}\mathsf{r}(z)\\ \mathsf{s}(z)\mathsf{N}_1(z)+\mathsf{t}(z)\mathsf{N}_2(z)\end{bmatrix} = \begin{bmatrix}1 & 0\\ -\mathsf{p}(z) & 1\end{bmatrix}\begin{bmatrix}\mathsf{r}(z)\\ \mathsf{s}(z)\mathsf{N}_1(z)+\mathsf{t}(z)\mathsf{N}_2(z)\end{bmatrix} = \begin{bmatrix}\mathsf{r}(z)\\ 0\end{bmatrix}. \quad (4.3.11)$$

According to item (3) of Proposition 4.1.2, we know that $\mathsf{P}_2(z)$ has compatible symmetry, and the above matrix multiplication is also compatible. By defining $\mathsf{P}(z) := \mathsf{P}_2(z)\mathsf{P}_1(z)$, (4.3.10) and (4.3.11) imply that (4.3.5) holds, and the matrix multiplication in (4.3.5) is compatible. $\det(\mathsf{P}(z)) = \det(\mathsf{P}_1(z))\det(\mathsf{P}_2(z)) = 1$ shows that $\mathsf{P}(z)$ is invertible. ■

Although it is not easy to reduce $\mathsf{A}(z)$ to its Smith Normal Form with invertible matrices $\mathsf{E}(z)$ and $\mathsf{F}(z)$ having compatible symmetry, we can still diagonalize it using the following theorem.

Theorem 4.3.2. *Suppose* $\mathsf{A}(z) = \begin{bmatrix}\mathsf{A}_{1,1}(z) & \mathsf{A}_{1,2}(z)\\ \mathsf{A}_{2,1}(z) & \mathsf{A}_{2,2}(z)\end{bmatrix}$ *is a matrix of Laurent polynomials with compatible symmetry. Then there exist invertible* 2×2 *matrices* $\mathsf{P}(z)$ *and* $\mathsf{Q}(z)$ *of Laurent polynomials with compatible symmetry, such that* $\mathsf{P}(z)\mathsf{A}(z)\mathsf{Q}(z)$ *is a diagonal matrix:*

$$\mathsf{D}(z) = \begin{bmatrix}\mathsf{e}_1(z) & \\ & \mathsf{e}_2(z)\end{bmatrix} := \mathsf{P}(z)\mathsf{A}(z)\mathsf{Q}(z), \quad (4.3.12)$$

where the multiplications are both compatible, and $\mathsf{e}_1(z)$, $\mathsf{e}_2(z)$ *both have symmetry. Furthermore, suppose* $\mathsf{d}_1(z)$ *and* $\mathsf{d}_2(z)$ *are the invariant polynomi-*

als of $\mathsf{A}(z)$*, then for all* $z_0 \in \mathbb{C} \setminus \{0\}$*, the sequences* $\{\mathsf{Z}(\mathsf{e}_i(z), z_0)\}_{i=1}^2$ *and* $\{\mathsf{Z}(\mathsf{d}_i(z), z_0)\}_{i=1}^2$*, up to a possible permutation, are the same.*

The proof of Theorem 4.3.2 will be constructive, so we can use it directly as an algorithm to find $\mathsf{D}(z)$, $\mathsf{E}(z)$ and $\mathsf{F}(z)$. We build the following lemma as a useful sub-step in our construction.

Lemma 4.3.2. *Suppose* $\mathsf{A}(z)$ *is a* 2×2 *matrix of Laurent polynomials with compatible symmetry.*

(1) If $\mathsf{A}_{2,1}(z) \neq 0$*, then we can find an invertible matrix* $\widetilde{\mathsf{P}}(z)$ *of Laurent polynomials with compatible symmetry, such that*

$$\widetilde{\mathsf{A}}(z) := \widetilde{\mathsf{P}}(z)\mathsf{A}(z) = \begin{bmatrix} \mathsf{r}(z) & \widetilde{\mathsf{A}}_{1,2}(z) \\ 0 & \widetilde{\mathsf{A}}_{2,2}(z) \end{bmatrix}, \tag{4.3.13}$$

where $\mathsf{r}(z) := \gcd(\mathsf{A}_{1,1}(z), \mathsf{A}_{2,1}(z)) \neq 0$*, and the above multiplication is compatible.*

(2) If $\mathsf{A}_{1,2}(z) \neq 0$*, then we can find an invertible matrix* $\widetilde{\mathsf{Q}}(z)$ *of Laurent polynomials with compatible symmetry, such that*

$$\widetilde{\mathsf{A}}(z) := \mathsf{A}(z)\widetilde{\mathsf{Q}}(z) = \begin{bmatrix} \mathsf{r}(z) & 0 \\ \widetilde{\mathsf{A}}_{2,1}(z) & \widetilde{\mathsf{A}}_{2,2}(z) \end{bmatrix}, \tag{4.3.14}$$

where $\mathsf{r}(z) := \gcd(\mathsf{A}_{1,1}(z), \mathsf{A}_{1,2}(z)) \neq 0$*, and the above multiplication is compatible.*

Proof. To prove item (1), for the case that $\mathsf{A}_{1,1}(z) = 0$, we just take $\widetilde{\mathsf{P}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ to be the permutation matrix. Then (4.3.13) will be satisfied. Notice that according to the item (1) of Proposition 4.1.2, the permutation matrix is compatible with any symmetry type.

For the case that $\mathsf{A}_{1,1}(z) \neq 0$, we can use Lemma 4.3.1 to find an invertible matrix $\widetilde{\mathsf{P}}(z)$ with compatible symmetry, such that

$$\widetilde{\mathsf{P}}(z) \begin{bmatrix} \mathsf{A}_{1,1}(z) \\ \mathsf{A}_{2,1}(z) \end{bmatrix} = \begin{bmatrix} \mathsf{r}(z) \\ 0 \end{bmatrix}.$$

So (4.3.13) will hold with such $\widetilde{\mathsf{P}}(z)$. According to the item (3) of Proposition 4.1.2, we know that the multiplication in (4.3.13) is compatible.

Similarly, to prove item (2), for the case that $\mathsf{A}_{1,1}(z) = 0$, we just take $\widetilde{\mathsf{Q}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ to be the permutation matrix. Then (4.3.14) will be satisfied. Notice that according to the item (1) of Proposition 4.1.2, the permutation matrix is compatible with any symmetry type.

For the case that $\mathsf{A}_{1,1}(z) \neq 0$, we can use Lemma 4.3.1 to find an invertible matrix $\widetilde{\mathsf{Q}}(z)$ with compatible symmetry, such that

$$\widetilde{\mathsf{Q}}^{\star}(z) \begin{bmatrix} \mathsf{A}^{\star}_{1,1}(z) \\ \mathsf{A}^{\star}_{1,2}(z) \end{bmatrix} = \begin{bmatrix} \mathsf{r}^{\star}(z) \\ 0 \end{bmatrix}.$$

So (4.3.14) will hold with this $\widetilde{\mathsf{Q}}(z)$. According to the item (4) of Proposition 4.1.2, we know that the multiplication in (4.3.14) is compatible. ■

Proof of Theorem 4.3.2. For the trivial case that $\mathsf{A}(z) = \mathbf{0}$, we can just take $\mathsf{E} = \mathsf{F} = \mathbf{I_2}$, and $\mathsf{D} = \mathbf{0}$. So we only consider the nontrivial case.

If $\mathsf{A}(z)$ is already diagonal, just take $\mathsf{P}(z) = \mathsf{Q}(z) = \mathbf{I_2}$, and $\mathsf{D}(z) = \mathsf{A}(z)$. Otherwise, since some of the off-diagonal elements in $\mathsf{A}(z)$ is non-zero, we can apply the two items in Lemma 4.3.2 repeatedly, to get a sequence of matrices $\widetilde{\mathsf{A}}^{(1)}(z), \widetilde{\mathsf{A}}^{(2)}(z), \ldots$, where

$$\widetilde{\mathsf{A}}^{(k_1+k_2)}(z) := \widetilde{\mathsf{P}}_{k_1} \ldots \widetilde{\mathsf{P}}_1(z)\mathsf{A}(z)\widetilde{\mathsf{Q}}_1(z) \ldots \widetilde{\mathsf{Q}}_{k_2}(z).$$

The process will terminate if $\widetilde{\mathsf{A}}^{(k_1+k_2)}(z)$ becomes diagonal, i.e., $\widetilde{\mathsf{A}}^{(k_1+k_2)}_{1,2}(z) = \widetilde{\mathsf{A}}^{(k_1+k_2)}_{2,1}(z) = 0$. Now we prove that the process will terminate after a finite number of steps.

After applying item (1) of Lemma 4.3.2, for some integer k, we will get a matrix $\widetilde{\mathsf{A}}^{(k)}(z)$ with $\widetilde{\mathsf{A}}^{(k)}_{1,1}(z) \neq 0$ and $\widetilde{\mathsf{A}}^{(k)}_{2,1}(z) = 0$. If $\widetilde{\mathsf{A}}^{(k)}_{1,1}(z) \mid \widetilde{\mathsf{A}}^{(k)}_{1,2}(z)$, we can use $\widetilde{\mathsf{Q}}(z) := \begin{bmatrix} 1 & -\widetilde{\mathsf{A}}^{(k)}_{1,2}(z)/\widetilde{\mathsf{A}}^{(k)}_{1,1}(z) \\ 0 & 1 \end{bmatrix}$ in item (2) of Lemma 4.3.2, to generate $\widetilde{\mathsf{A}}^{(k+1)}(z) := \widetilde{\mathsf{A}}^{(k)}(z)\widetilde{\mathsf{Q}}(z)$. Then we can verify $\widetilde{\mathsf{A}}^{(k+1)}_{1,2}(z) = \widetilde{\mathsf{A}}^{(k+1)}_{2,1}(z) = 0$. That is, $\widetilde{\mathsf{A}}^{(k+1)}(z)$ is diagonal and the algorithm will terminate. Otherwise, if

$\widetilde{\mathsf{A}}^{(k)}_{1,1}(z) \nmid \widetilde{\mathsf{A}}^{(k)}_{1,2}(z)$, then $\mathrm{len}(\gcd(\widetilde{\mathsf{A}}^{(k)}_{1,1}(z), \widetilde{\mathsf{A}}^{(k)}_{1,2}(z))) < \mathrm{len}(\widetilde{\mathsf{A}}^{(k)}_{1,1}(z))$. Hence, after applying the item (2) of Lemma 4.3.2 to get $\widetilde{\mathsf{A}}^{(k+1)}(z) := \widetilde{\mathsf{A}}^{(k)}(z)\widetilde{\mathsf{Q}}(z)$, we have $\mathrm{len}(\widetilde{\mathsf{A}}^{(k+1)}_{1,1}(z)) = \mathrm{len}(\gcd(\widetilde{\mathsf{A}}^{(k)}_{1,1}(z), \widetilde{\mathsf{A}}^{(k)}_{1,2}(z))) < \mathrm{len}(\widetilde{\mathsf{A}}^{(k)}_{1,1}(z))$. That is, the length of the $(1,1)$ element will strictly decrease.

Similarly, after applying item (2) of Lemma 4.3.2, for some integer k, we will get a matrix $\widetilde{\mathsf{A}}^{(k)}(z)$ with $\widetilde{\mathsf{A}}^{(k)}_{1,1}(z) \neq 0$ and $\widetilde{\mathsf{A}}^{(k)}_{1,2}(z) = 0$. If $\widetilde{\mathsf{A}}^{(k)}_{1,1}(z) \mid \widetilde{\mathsf{A}}^{(k)}_{2,1}(z)$, we can use $\widetilde{\mathsf{P}}(z) := \begin{bmatrix} 1 & 0 \\ -\widetilde{\mathsf{A}}^{(k)}_{2,1}(z)/\widetilde{\mathsf{A}}^{(k)}_{1,1}(z) & 1 \end{bmatrix}$ in item (1) of Lemma 4.3.2, to generate $\widetilde{\mathsf{A}}^{(k+1)}(z) := \widetilde{\mathsf{P}}(z)\widetilde{\mathsf{A}}^{(k)}(z)$. Then we can verify $\widetilde{\mathsf{A}}^{(k+1)}_{1,2}(z) = \widetilde{\mathsf{A}}^{(k+1)}_{2,1}(z) = 0$. That is, $\widetilde{\mathsf{A}}^{(k+1)}(z)$ is diagonal and the algorithm will terminate. Otherwise, if $\widetilde{\mathsf{A}}^{(k)}_{1,1}(z) \nmid \widetilde{\mathsf{A}}^{(k)}_{2,1}(z)$, then $\mathrm{len}(\gcd(\widetilde{\mathsf{A}}^{(k)}_{1,1}(z), \widetilde{\mathsf{A}}^{(k)}_{2,1}(z))) < \mathrm{len}(\widetilde{\mathsf{A}}^{(k)}_{1,1}(z))$. Hence, after applying the item (1) of Lemma 4.3.2 to get $\widetilde{\mathsf{A}}^{(k+1)}(z) = \widetilde{\mathsf{P}}(z)\widetilde{\mathsf{A}}^{(k)}(z)$, we have $\mathrm{len}(\widetilde{\mathsf{A}}^{(k+1)}_{1,1}(z)) = \mathrm{len}(\gcd(\widetilde{\mathsf{A}}^{(k)}_{1,1}(z), \widetilde{\mathsf{A}}^{(k)}_{2,1}(z))) < \mathrm{len}(\widetilde{\mathsf{A}}^{(k)}_{1,1}(z))$. That is, the length of the $(1,1)$ element will also strictly decrease.

Therefore, by repeatedly applying the 2 items in Lemma 4.3.2, before it stops at some diagonal matrix $\widetilde{\mathsf{A}}^{(k_1+k_2)}(z) := \widetilde{\mathsf{P}}_{k_1} \dots \widetilde{\mathsf{P}}_1(z)\mathsf{A}(z)\widetilde{\mathsf{Q}}_1(z) \dots \widetilde{\mathsf{Q}}_{k_2}(z)$, the length of the $(1,1)$ element will strictly decrease. So the process cannot last forever and must stop after a finite number of iterations. Hence, we proved that there exist invertible matrices $\mathsf{P}(z) := \widetilde{\mathsf{P}}_{k_1} \dots \widetilde{\mathsf{P}}_1(z)$, and $\mathsf{Q}(z) := \widetilde{\mathsf{Q}}_1(z) \dots \widetilde{\mathsf{Q}}_{k_2}(z)$ with compatible symmetry, such that $\mathsf{D}(z)$ defined in (4.3.12) is a diagonal matrix, and the multiplications in (4.3.12) are compatible.

Now, let $\mathsf{d}_1(z)$ and $\mathsf{d}_2(z)$ be the two invariant polynomials of $\mathsf{A}(z)$. That is, there exist 2×2 invertible matrices $\mathsf{E}(z)$ and $\mathsf{F}(z)$ of Laurent polynomials, such that

$$\mathsf{A}(z) = \mathsf{E}(z) \begin{bmatrix} \mathsf{d}_1(z) & \\ & \mathsf{d}_2(z) \end{bmatrix} \mathsf{F}(z). \tag{4.3.15}$$

For some arbitrary $z_0 \in \mathbb{C} \setminus \{0\}$, write $\mathsf{d}_i(z) = (z - z_0)^{\mathsf{Z}(\mathsf{d}_i(z), z_0)}\mathsf{p}_i(z)$, and $\mathsf{e}_i(z) = (z - z_0)^{\mathsf{Z}(\mathsf{e}_i(z), z_0)}\mathsf{q}_i(z)$, for $i = 1, 2$. From (4.3.15),

$$\begin{aligned} \mathsf{A}(z) =& \mathsf{E}(z) \begin{bmatrix} (z-z_0)^{\mathsf{Z}(\mathsf{d}_1(z),z_0)} & \\ & (z-z_0)^{\mathsf{Z}(\mathsf{d}_2(z),z_0)} \end{bmatrix} \begin{bmatrix} \mathsf{p}_1(z) & \\ & \mathsf{p}_2(z) \end{bmatrix} \mathsf{F}(z) \\ =& \mathsf{U}_1(z) \begin{bmatrix} (z-z_0)^{\mathsf{Z}(\mathsf{d}_1(z),z_0)} & \\ & (z-z_0)^{\mathsf{Z}(\mathsf{d}_2(z),z_0)} \end{bmatrix} \mathsf{V}_1(z) \end{aligned} \tag{4.3.16}$$

holds, where $\mathsf{U}_1(z) := \mathsf{E}(z)$ and $\mathsf{V}_1(z) := \begin{bmatrix} \mathsf{p}_1(z) & \\ & \mathsf{p}_2(z) \end{bmatrix} \mathsf{F}(z)$ are both analytic matrices in variable z in a neighborhood of z_0, and are both nonsingular at z_0. Since $\mathsf{d}_1(z)$ and $\mathsf{d}_2(z)$ are the invariant polynomials of $\mathsf{A}(z)$, we know that $\mathsf{Z}(\mathsf{d}_1(z), z_0) \leqslant \mathsf{Z}(\mathsf{d}_2(z), z_0)$.

Similarly, if $\mathsf{Z}(\mathsf{e}_1(z), z_0) \leqslant \mathsf{Z}(\mathsf{e}_2(z), z_0)$, we denote $\mathsf{U}_2(z) := \mathsf{P}^{-1}(z)$, and $\mathsf{V}_2(z) := \begin{bmatrix} \mathsf{q}_1(z) & \\ & \mathsf{q}_2(z) \end{bmatrix} \mathsf{Q}^{-1}(z)$. Otherwise, we denote $\mathsf{U}_2(z) := \mathsf{P}^{-1}(z) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and $\mathsf{V}_2(z) := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \mathsf{q}_1(z) & \\ & \mathsf{q}_2(z) \end{bmatrix} \mathsf{Q}^{-1}(z)$. From (4.3.12), we know that

$$\mathsf{A}(z) = \mathsf{U}_2(z) \begin{bmatrix} (z - z_0)^{\min\{\mathsf{Z}(\mathsf{e}_1(z), z_0), \mathsf{Z}(\mathsf{e}_2(z), z_0)\}} & \\ & (z - z_0)^{\max\{\mathsf{Z}(\mathsf{e}_1(z), z_0), \mathsf{Z}(\mathsf{e}_2(z), z_0)\}} \end{bmatrix} \mathsf{V}_2(z) \tag{4.3.17}$$

holds, where $\mathsf{U}_2(z)$ and $\mathsf{V}_2(z)$ are both analytic matrices in variable z in a neighborhood of z_0, and are both nonsingular at z_0. We can see both of the factorizations in (4.3.16) and (4.3.17) satisfy all the requirements in the Lemma 2.3.4, so the sequences $\{\mathsf{Z}(\mathsf{e}_i(z), z_0)\}_{i=1}^2$ and $\{\mathsf{Z}(\mathsf{d}_i(z), z_0)\}_{i=1}^2$, up to a possible permutation, are the same.

This completes the proof of the theorem. ∎

Theorem 4.3.3. *Suppose* $\mathsf{A}(z)$ *is a* 2×2 *Hermitian matrix of Laurent polynomials with compatible symmetry. If*

(1) $\gcd(\mathsf{A}_{1,1}(z), \mathsf{A}_{1,2}(z), \mathsf{A}_{2,1}(z), \mathsf{A}_{2,2}(z)) = 1$*;*

(2) $\det(\mathsf{A}(z)) = -\mathsf{d}(z)\mathsf{d}^\star(z)$ *for some Laurent polynomial* $\mathsf{d}(z) \neq 0$ *with symmetry;*

then we can find 2×2 *matrices* $\mathsf{U}(z)$ *and* $\mathsf{B}(z)$ *of Laurent polynomials with compatible symmetry, such that* $\det(\mathsf{B}(z)) = -C$ *is a negative constant,* $\mathsf{B}^\star(z) = \mathsf{B}(z)$*, and* $\mathsf{A}(z) = \mathsf{U}(z)\mathsf{B}(z)\mathsf{U}^\star(z)$ *holds, where the multiplications are compatible.*

Proof. If $\det(\mathsf{A}(z)) = -C$ is a negative constant, we can just define $\mathsf{U} := \mathbf{I_2}$ and $\mathsf{B}(z) := \mathsf{A}(z)$. Then the proof is completed. So we only need to consider the case that $\sigma(\mathsf{A}(z))$ is not empty.

Define $\mathsf{A}^{(0)}(z) := \mathsf{A}(z)$. We will provide a procedure to iteratively construct a sequence of matrices $\{\mathsf{A}^{(j)}(z)\}_{j=0}^{K}$, whose lengths of the determinants are decreasing. For some 2×2 matrix $\mathsf{A}^{(j)}(z)$ of Laurent polynomials with $\sigma(\mathsf{A}^{(j)}(z))$ nonempty, suppose $\mathsf{A}^{(j)}(z)$ satisfies the following 3 conditions, which are obviously true for $\mathsf{A}^{(0)}(z) := \mathsf{A}(z)$:

(i) $\mathsf{A}^{(j)\star}(z) = \mathsf{A}^{(j)}(z)$, and $\mathsf{A}^{(j)}(z)$ has compatible symmetry;

(ii) $\gcd(\mathsf{A}^{(j)}_{1,1}(z), \mathsf{A}^{(j)}_{1,2}(z), \mathsf{A}^{(j)}_{2,1}(z), \mathsf{A}^{(j)}_{2,2}(z)) = 1$;

(iii) $\det(\mathsf{A}^{(j)}(z)) = -\mathsf{d}^{(j)}(z)\mathsf{d}^{(j)\star}(z)$ for some Laurent polynomial $\mathsf{d}^{(j)}(z) \neq 0$ with symmetry.

From such $\mathsf{A}^{(j)}(z)$, the following procedure will help us to find 2×2 matrices $\mathsf{U}^{(j)}(z)$ and $\mathsf{A}^{(j+1)}(z)$ of Laurent polynomials with compatible symmetry, such that

(I) $\mathsf{A}^{(j)}(z) = \mathsf{U}^{(j)}(z)\mathsf{A}^{(j+1)}(z)\mathsf{U}^{(j)\star}(z)$, and the multiplications are compatible;

(II) the conditions (i)(ii) and (iii) above are also satisfied for $\mathsf{A}^{(j+1)}(z)$;

(III) $\sigma(\mathsf{A}^{(j+1)}(z)) \subsetneq \sigma(\mathsf{A}^{(j)}(z))$.

For $\mathsf{A}^{(j)}(z)$ satisfying (i)(ii) and (iii), by Theorem 4.3.2, we can find invertible matrices $\mathsf{E}(z)$ and $\mathsf{F}(z)$ of Laurent polynomials with compatible symmetry, such that $\mathsf{A}^{(j)}(z) = \mathsf{E}(z)\mathsf{D}(z)\mathsf{F}(z)$, where $\mathsf{D}(z) = \begin{bmatrix} \mathsf{e}_1(z) & \\ & \mathsf{e}_2(z) \end{bmatrix}$ is a diagonal matrix of Laurent polynomials with symmetry. Also, by the fact that $\mathsf{E}(z)$ and $\mathsf{F}(z)$ are both invertible, without loss of generality, we can assume $\det(\mathsf{E}(z)) = \det(\mathsf{F}(z)) = 1$, hence

$$\mathsf{e}_1(z)\mathsf{e}_2(z) = \det(\mathsf{D}(z)) = \det(\mathsf{A}^{(j)}(z)) = -\mathsf{d}^{(j)}(z)\mathsf{d}^{(j)\star}(z). \tag{4.3.18}$$

Moreover, since the Smith Normal Form of $\mathsf{A}^{(j)}(z)$ is $\begin{bmatrix} 1 & \\ & \det(\mathsf{A}^{(j)}(z)) \end{bmatrix}$, from Theorem 4.3.2, we know that

$$\mathsf{Z}(\mathsf{e}_i(z), z_0) = 0 \text{ or } \mathsf{Z}(\mathsf{e}_i(z), z_0) = \mathsf{Z}(\det(\mathsf{A}^{(j)}(z)), z_0),$$
$$\text{for all } z_0 \in \mathbb{C} \setminus \{0\}, i = 1, 2. \quad (4.3.19)$$

Now, we can define the Hermitian matrix $\mathring{\mathsf{A}}(z)$ of Laurent polynomials as

$$\mathring{\mathsf{A}}(z) := \mathsf{E}^{-1}(z)\mathsf{A}^{(j)}(z)\mathsf{E}^{-\star}(z) = \mathsf{D}(z)\mathsf{F}(z)\mathsf{E}^{-\star}(z) = \begin{bmatrix} \mathsf{e}_1(z) & \\ & \mathsf{e}_2(z) \end{bmatrix} \mathsf{F}(z)\mathsf{E}^{-\star}(z). \tag{4.3.20}$$

Notice that all the matrix multiplications in (4.3.20) are compatible.

Since $\sigma(\mathsf{A}^{(j)}(z))$ is nonempty, we can pick some $z_0 \in \sigma(\mathsf{A}^{(j)}(z))$. By (4.3.18), there exists some $k \in \{1, 2\}$, such that $(z - z_0) \mid \mathsf{e}_k(z)$. Define

$$\alpha := \mathsf{Z}(\mathsf{e}_k(z), z_0) > 0,$$

by (4.3.19), we know that $\alpha = \mathsf{Z}(\det(\mathsf{A}^{(j)}(z)), z_0)$. According to (4.3.18), we can see that $\mathsf{Z}(\mathsf{e}_l(z), z_0) = \mathsf{Z}(\det(\mathsf{A}^{(j)}(z)), z_0) - \mathsf{Z}(\mathsf{e}_k(z), z_0) = \alpha - \alpha = 0$, for $l \neq k$. We have the following 3 possible cases for the locations of z_0.

(1) If $z_0 \in \mathbb{C} \setminus \{0\} \setminus \mathbb{T} \setminus \mathbb{R}$, we know that z_0, z_0^{-1}, $\overline{z_0}$ and $\overline{z_0}^{-1}$ are four different points on the complex plane. Define $\mathsf{p}(z) := (z - z_0)^\alpha (z - z_0^{-1})^\alpha$, then according to Lemma 4.1.1, $\mathsf{p}(z)$ has symmetry. Since $\mathsf{e}_k(z)$ also has symmetry, by Lemma 4.1.1 again, we can see that $\mathsf{Z}(\mathsf{e}_k(z), z_0^{-1}) = \mathsf{Z}(\mathsf{e}_k(z), z_0) = \alpha$. So $\mathsf{p}(z) \mid \mathsf{e}_k(z)$. From (4.3.20), we get $\mathsf{p}(z)$ divides the k-th row of $\mathring{\mathsf{A}}(z)$. Moreover, as $\mathring{\mathsf{A}}(z)$ is a Hermitian matrix, we can see that $\mathsf{p}^\star(z)$ divides the k-th column of $\mathring{\mathsf{A}}(z)$. According to the definition of $\mathsf{p}(z)$, we know that $\mathsf{p}^\star(z) = z^{-2\alpha}(z - \overline{z_0})^\alpha (z - \overline{z_0}^{-1})^\alpha$. Since $z_0, z_0^{-1}, \overline{z_0}$ and $\overline{z_0}^{-1}$ are different points, we can see that $\gcd(\mathsf{p}(z), \mathsf{p}^\star(z)) = 1$. Hence, $\mathsf{p}(z) \mid \mathring{\mathsf{A}}_{k,k}(z)$ and $\mathsf{p}^\star(z) \mid \mathring{\mathsf{A}}_{k,k}(z)$ imply that $\mathsf{p}(z)\mathsf{p}^\star(z) \mid \mathring{\mathsf{A}}_{k,k}(z)$. Define

$$\mathsf{V}_{1,\mathsf{p}}(z) := \operatorname{diag}(\mathsf{p}(z), 1), \qquad \mathsf{V}_{2,\mathsf{p}}(z) := \operatorname{diag}(1, \mathsf{p}(z)). \tag{4.3.21}$$

We can factor out $\mathsf{p}(z)$ from the k-th row of $\mathring{\mathsf{A}}(z)$, and factor out $\mathsf{p}^\star(z)$ from the k-th column of $\mathring{\mathsf{A}}(z)$, to get

$$\mathring{\mathsf{A}}(z) = \mathsf{V}_{k,\mathsf{p}}(z)\mathsf{A}^{(j+1)}(z)\mathsf{V}^\star_{k,\mathsf{p}}(z) \tag{4.3.22}$$

for some 2×2 matrix $\mathsf{A}^{(j+1)}(z)$ of Laurent polynomials. Define $\mathsf{U}^{(j)}(z) := \mathsf{E}(z)\mathsf{V}_{k,\mathsf{p}}(z)$, then (4.3.20) and (4.3.22) imply that

$$\begin{aligned}\mathsf{A}^{(j)}(z) =&\mathsf{E}(z)\mathring{\mathsf{A}}(z)\mathsf{E}^\star(z) = \mathsf{E}(z)\mathsf{V}_{k,\mathsf{p}}(z)\mathsf{A}^{(j+1)}(z)\mathsf{V}^\star_{k,\mathsf{p}}(z)\mathsf{E}^\star(z)\\ =&\mathsf{U}^{(j)}(z)\mathsf{A}^{(j+1)}(z)\mathsf{U}^{(j)\star}(z).\end{aligned} \tag{4.3.23}$$

Since $\mathsf{p}(z)$ has symmetry and $\mathring{\mathsf{A}}(z)$ has compatible symmetry, we know from (4.3.22) that $\mathsf{A}^{(j+1)}(z)$ has compatible symmetry as well. Also, the multiplications are compatible. This proves that the condition (i) is satisfied for $\mathsf{A}^{(j+1)}(z)$. Also, because $\mathsf{E}(z)$ is invertible, by (4.3.20), we can see that the Smith Normal Form of $\mathring{\mathsf{A}}(z)$ is the same as that of $\mathsf{A}^{(j)}(z)$, which is $\begin{bmatrix}1 & \\ & \det(\mathsf{A}^{(j)}(z))\end{bmatrix}$. So $\gcd(\mathring{\mathsf{A}}_{1,1}(z), \mathring{\mathsf{A}}_{1,2}(z), \mathring{\mathsf{A}}_{2,1}(z), \mathring{\mathsf{A}}_{2,2}(z)) = 1$. This implies that

$$\begin{aligned}&\gcd\left(\mathsf{A}^{(j+1)}_{1,1}(z), \mathsf{A}^{(j+1)}_{1,2}(z), \mathsf{A}^{(j+1)}_{2,1}(z), \mathsf{A}^{(j+1)}_{2,2}(z)\right)\\ =&\gcd\left(\frac{\mathring{\mathsf{A}}_{k,k}(z)}{\mathsf{p}(z)\mathsf{p}^\star(z)}, \frac{\mathring{\mathsf{A}}_{k,l}(z)}{\mathsf{p}(z)}, \frac{\mathring{\mathsf{A}}_{l,k}(z)}{\mathsf{p}^\star(z)}, \mathring{\mathsf{A}}_{l,l}(z)\right) = 1.\end{aligned}$$

Hence, the condition (ii) is satisfied for $\mathsf{A}^{(j+1)}(z)$. From

$$\begin{aligned}\det(\mathsf{A}^{(j)}(z)) =&\det(\mathring{\mathsf{A}}(z)) = \det(\mathsf{V}_{k,\mathsf{p}}(z))\det(\mathsf{A}^{(j+1)}(z))\det(\mathsf{V}^\star_{k,\mathsf{p}}(z))\\ =&\mathsf{p}(z)\mathsf{p}^\star(z)\det(\mathsf{A}^{(j+1)}(z)),\end{aligned}$$

we get $\det(\mathsf{A}^{(j+1)}(z)) = \frac{\det(\mathsf{A}^{(j)}(z))}{\mathsf{p}(z)\mathsf{p}^\star(z)} = \frac{-\mathsf{d}^{(j)}(z)\mathsf{d}^{(j)\star}(z)}{\mathsf{p}(z)\mathsf{p}^\star(z)}$. Since $\mathsf{d}^{(j)}(z)$ has symmetry, by Lemma 4.1.1, we can define

$$\begin{aligned}\beta_1 :=&\mathsf{Z}(\mathsf{d}^{(j)}(z), z_0) = \mathsf{Z}(\mathsf{d}^{(j)}(z), z_0^{-1}),\\ \beta_2 :=&\mathsf{Z}(\mathsf{d}^{(j)}(z), \overline{z_0}) = \mathsf{Z}(\mathsf{d}^{(j)}(z), \overline{z_0}^{-1}),\\ \mathsf{q}(z) :=&(z-z_0)^{\beta_1}(z-z_0^{-1})^{\beta_1}(z^{-1}-\overline{z_0})^{\beta_2}(z^{-1}-\overline{z_0}^{-1})^{\beta_2}.\end{aligned}$$

Then, $\mathsf{q}(z) \mid \mathsf{d}^{(j)}(z)$, and by Lemma 4.1.1, we know that $\mathsf{q}(z)$ has symmetry. Define $\mathsf{d}^{(j+1)}(z) := \frac{\mathsf{d}^{(j)}(z)}{\mathsf{q}(z)}$. Because $\mathsf{d}^{(j)}(z)$ and $\mathsf{q}(z)$ both have symmetry, we know that $\mathsf{d}^{(j+1)}(z)$ has symmetry as well. Also, we have $\mathsf{Z}(\mathsf{d}^{(j)\star}, z_0) = \mathsf{Z}(\mathsf{d}^{(j)}, \overline{z_0}^{-1}) = \beta_2$, which implies that

$$\begin{aligned}\alpha &= \mathsf{Z}(\det(\mathsf{A}^{(j)}(z), z_0)) = \mathsf{Z}(-\mathsf{d}^{(j)}\mathsf{d}^{(j)\star}, z_0) \\ &= \mathsf{Z}(\mathsf{d}^{(j)}, z_0) + \mathsf{Z}(\mathsf{d}^{(j)\star}, z_0) = \beta_1 + \beta_2.\end{aligned}$$

Hence,

$$\begin{aligned}\mathsf{q}(z)\mathsf{q}^\star(z) =&(z - z_0)^{\beta_1+\beta_2}(z - z_0^{-1})^{\beta_1+\beta_2}(z^{-1} - \overline{z_0})^{\beta_1+\beta_2}(z^{-1} - \overline{z_0}^{-1})^{\beta_1+\beta_2} \\ =&(z - z_0)^{\alpha}(z - z_0^{-1})^{\alpha}(z^{-1} - \overline{z_0})^{\alpha}(z^{-1} - \overline{z_0}^{-1})^{\alpha} = \mathsf{p}(z)\mathsf{p}^\star(z).\end{aligned}$$

Now, we can see that

$$\det(\mathsf{A}^{(j+1)}(z)) = \frac{-\mathsf{d}^{(j)}(z)\mathsf{d}^{(j)\star}(z)}{\mathsf{p}(z)\mathsf{p}^\star(z)} = \frac{-\mathsf{d}^{(j)}(z)\mathsf{d}^{(j)\star}(z)}{\mathsf{q}(z)\mathsf{q}^\star(z)} = -\mathsf{d}^{(j+1)}(z)\mathsf{d}^{(j+1)\star}(z).$$

Therefore, we showed that the condition (iii) is also satisfied for $\mathsf{A}^{(j+1)}(z)$.

Notice that (4.3.23) implies that $\sigma(\mathsf{A}^{(j+1)}(z)) \subseteq \sigma(\mathsf{A}^{(j)}(z))$. Also, $z_0 \in \sigma(\mathsf{A}^{(j)}(z))$, but

$$\begin{aligned}\mathsf{Z}(\det(\mathsf{A}^{(j+1)}), z_0) =&\mathsf{Z}\left(\frac{\det(\mathsf{A}^{(j)})}{\mathsf{p}(z)\mathsf{p}^\star(z)}, z_0\right) \\ =&\mathsf{Z}(\det(\mathsf{A}^{(j)}), z_0) - \mathsf{Z}(\mathsf{p}(z), z_0) - \mathsf{Z}(\mathsf{p}^\star(z), z_0) \\ =&\alpha - \alpha - 0 = 0.\end{aligned}$$

Hence, $z_0 \notin \sigma(\mathsf{A}^{(j+1)}(z))$. That is, $\sigma(\mathsf{A}^{(j+1)}(z)) \subsetneq \sigma(\mathsf{A}^{(j)}(z))$. This finishes the proof of the case $z_0 \in \mathbb{C} \setminus \{0\} \setminus \mathbb{T} \setminus \mathbb{R}$.

(2) If $z_0 \in \mathbb{T} \bigcup \mathbb{R} \setminus \{0, \pm 1\}$, we know that $z_0^{-1} \neq z_0$.

For the situation that $z_0 \in \mathbb{T} \setminus \{\pm 1\}$, we can see $\overline{z_0}^{-1} = z_0$. So $\mathsf{Z}(\mathsf{d}^{(j)\star}(z), z_0) = \mathsf{Z}(\mathsf{d}^{(j)\star}, \overline{z_0}^{-1}) = \mathsf{Z}(\mathsf{d}^{(j)}(z), z_0)$. For the situation that $z_0 \in \mathbb{R} \setminus \{0, \pm 1\}$, we can see $\overline{z_0} = z_0$. So $\mathsf{Z}(\mathsf{d}^{(j)\star}(z), z_0) = \mathsf{Z}(\mathsf{d}^{(j)}, \overline{z_0}^{-1}) = \mathsf{Z}(\mathsf{d}^{(j)}(z), z_0^{-1}) = \mathsf{Z}(\mathsf{d}^{(j)}(z), z_0)$, where the last equality comes from the fact that $\mathsf{d}^{(j)}(z)$ has symmetry. Thus, in both situations, we get

$$\mathsf{Z}(\mathsf{d}^{(j)\star}(z), z_0) = \mathsf{Z}(\mathsf{d}^{(j)}(z), z_0),$$

which implies that

$$\begin{aligned} \alpha =& \mathsf{Z}(\det(\mathsf{A}^{(j)}(z)), z_0) = \mathsf{Z}(-\mathsf{d}^{(j)}(z)\mathsf{d}^{(j)\star}(z), z_0) \\ =& \mathsf{Z}(\mathsf{d}^{(j)}(z), z_0) + \mathsf{Z}(\mathsf{d}^{(j)\star}(z), z_0) = 2\mathsf{Z}(\mathsf{d}^{(j)}(z), z_0) \in 2\mathbb{Z}. \end{aligned} \tag{4.3.24}$$

Define

$$\begin{aligned} \mathsf{p}(z) :=& (z - z_0)^{\alpha/2}(z - z_0^{-1})^{\alpha/2} z^{-\alpha/2} \\ =& \begin{cases} (z + z^{-1} - 2\operatorname{Re}(z_0))^{\alpha/2}, & \text{if } z_0 \in \mathbb{T} \setminus \{\pm 1\}, \\ (z + z^{-1} - (z_0 + z_0^{-1}))^{\alpha/2}, & \text{if } z_0 \in \mathbb{R} \setminus \{0, \pm 1\}. \end{cases} \end{aligned}$$

For both situations, we can see that $\mathsf{Sp}(z) = 1$, and $\mathsf{p}^\star(z) = \mathsf{p}(z)$. Since $\alpha = \mathsf{Z}(\mathsf{e}_k(z), z_0)$, we know that $(z - z_0)^\alpha \mid \mathsf{e}_k(z)$. Also, by the fact that $\mathsf{e}_k(z)$ has symmetry, we get $\mathsf{Z}(\mathsf{e}_k(z), z_0^{-1}) = \mathsf{Z}(\mathsf{e}_k(z), z_0) = \alpha$. So $(z - z_0^{-1})^\alpha \mid \mathsf{e}_k(z)$. Given that $z_0 \neq \pm 1$, that is, $z_0^{-1} \neq z_0$, we can conclude that $(z - z_0)^\alpha (z - z_0^{-1})^\alpha \mid \mathsf{e}_k(z)$. That is, $\mathsf{p}(z)\mathsf{p}^\star(z) = \mathsf{p}^2(z) \mid \mathsf{e}_k(z)$. According to (4.3.20), we know that $\mathsf{p}(z)$ divides the k-th row of $\mathring{\mathsf{A}}(z)$, $\mathsf{p}^\star(z)$ divides the k-th column of the Hermitian matrix $\mathring{\mathsf{A}}(z)$, and $\mathsf{p}(z)\mathsf{p}^\star(z)$ divides $\mathring{\mathsf{A}}_{k,k}(z)$. So we can factor out $\mathsf{p}(z)$ from the k-th row of $\mathring{\mathsf{A}}(z)$, and factor out $\mathsf{p}^\star(z)$ from the k-th column of $\mathring{\mathsf{A}}(z)$ simultaneously. Define $\mathsf{V}_{k,\mathsf{p}}(z)$ as in (4.3.21), we can see that (4.3.22) holds for some 2×2 matrix $\mathsf{A}^{(j+1)}(z)$ of Laurent polynomials with compatible symmetry. Define $\mathsf{U}^{(j)}(z) := \mathsf{E}(z)\mathsf{V}_{k,\mathsf{p}}(z)$, then (4.3.20) and (4.3.22) imply that (4.3.23) holds, and the multiplications in (4.3.23) are compatible. By the same argument as in case (1), we can see the conditions (i) and (ii) are true for the matrix $\mathsf{A}^{(j+1)}(z)$.

To prove condition (iii), we can see from (4.3.24) that $\mathsf{Z}(\mathsf{d}^{(j)}(z), z_0) = \alpha/2$. As $\mathsf{d}^{(j)}(z)$ has symmetry, by Lemma 4.1.1, we also know that $\mathsf{Z}(\mathsf{d}^{(j)}(z), z_0^{-1}) = \mathsf{Z}(\mathsf{d}^{(j)}(z), z_0) = \alpha/2$. So $\mathsf{p}(z) \mid \mathsf{d}^{(j)}(z)$. Define $\mathsf{d}^{(j+1)}(z) :=$

$\frac{\mathsf{d}^{(j)}(z)}{\mathsf{p}(z)}$. Because $\mathsf{d}^{(j)}(z)$ and $\mathsf{p}(z)$ both have symmetry, we know that $\mathsf{d}^{(j+1)}(z)$ also has symmetry. Moreover, we get from (4.3.23) that

$$\begin{aligned}\det(\mathsf{A}^{(j+1)}(z)) =& \frac{\det(\mathsf{A}^{(j)}(z))}{\det(\mathsf{U}^{(j)}(z))\det(\mathsf{U}^{(j)\star}(z))} = \frac{\det(\mathsf{A}^{(j)}(z))}{\det(\mathsf{V}_{k,\mathsf{p}}(z))\det(\mathsf{V}^{\star}_{k,\mathsf{p}}(z))}\\ =& \frac{\det(\mathsf{A}^{(j)}(z))}{\mathsf{p}(z)\mathsf{p}^{\star}(z)} = \frac{-\mathsf{d}^{(j)}(z)\mathsf{d}^{(j)\star}(z)}{\mathsf{p}(z)\mathsf{p}^{\star}(z)} = -\mathsf{d}^{(j+1)}(z)\mathsf{d}^{(j+1)\star}(z).\end{aligned} \tag{4.3.25}$$

Hence, we proved that the condition (iii) also holds for $\mathsf{A}^{(j+1)}(z)$.

Furthermore, the above equation gives $\det(\mathsf{A}^{(j+1)}(z)) = \frac{\det(\mathsf{A}^{(j)}(z))}{\mathsf{p}(z)\mathsf{p}^{\star}(z)}$, which implies that

$$\begin{aligned}\mathsf{Z}(\det(\mathsf{A}^{(j+1)}(z)), z_0) =& \mathsf{Z}(\det(\mathsf{A}^{(j)}(z)), z_0) - \mathsf{Z}(\mathsf{p}(z), z_0) - \mathsf{Z}(\mathsf{p}^{\star}(z), z_0)\\ =& \alpha - \alpha/2 - \alpha/2 = 0,\end{aligned}$$

where we used $\mathsf{p}^{\star}(z) = \mathsf{p}(z)$ and the definition of $\mathsf{p}(z)$ in the above calculation. So $z_0 \in \sigma(\mathsf{A}^{(j)}(z))$, but $z_0 \notin \sigma(\mathsf{A}^{(j+1)}(z))$. That is, $\sigma(\mathsf{A}^{(j+1)}(z)) \subsetneq \sigma(\mathsf{A}^{(j)}(z))$. This finished the proof of the case that $z_0 \in \mathbb{T}\bigcup\mathbb{R}\setminus\{0, \pm 1\}$.

(3) If $z_0 \in \{\pm 1\}$, we know that

$$z_0 = z_0^{-1} = \overline{z_0}^{-1} = \overline{z_0}. \tag{4.3.26}$$

In this case, $\mathsf{Z}(\mathsf{d}^{(j)\star}(z), z_0) = \mathsf{Z}(\mathsf{d}^{(j)}(z), \overline{z_0}^{-1}) = \mathsf{Z}(\mathsf{d}^{(j)}(z), z_0)$, which implies that (4.3.24) holds. That is, $\alpha \in 2\mathbb{Z}$. Define $\mathsf{p}(z) := (z - z_0)^{\alpha/2}$. Using (4.3.26), we can directly calculate that $\mathsf{Sp}(z) = (-z_0)^{\alpha/2} z^{\alpha/2}$, and $\mathsf{p}^{\star}(z) = (-z_0 z)^{-\alpha/2}(z - z_0)^{\alpha/2}$. So $\mathsf{p}(z)$ has symmetry. Since $\alpha = \mathsf{Z}(\mathsf{e}_k(z), z_0)$, we know that $(z - z_0)^{\alpha} \mid \mathsf{e}_k(z)$. That is, $\mathsf{p}(z)\mathsf{p}^{\star}(z) = (-z_0 z)^{-\alpha/2}(z - z_0)^{\alpha} \mid \mathsf{e}_k(z)$. According to (4.3.20), $\mathsf{p}(z) \mid \mathsf{e}_k(z)$ implies that $\mathsf{p}(z)$ divides the k-th row of $\mathring{\mathsf{A}}(z)$, and $\mathsf{p}^{\star}(z)$ divides the k-th column of the Hermitian matrix $\mathring{\mathsf{A}}(z)$. Also, $\mathsf{p}(z)\mathsf{p}^{\star}(z) \mid \mathsf{e}_k(z)$ implies that $\mathsf{p}(z)\mathsf{p}^{\star}(z)$ divides $\mathring{\mathsf{A}}_{k,k}(z)$. So we can factor out $\mathsf{p}(z)$ from the k-th row of $\mathring{\mathsf{A}}(z)$, and factor out $\mathsf{p}^{\star}(z)$ from the k-th column of $\mathring{\mathsf{A}}(z)$ simultaneously.

Define $\mathsf{V}_{k,\mathsf{p}}(z)$ as in (4.3.21), we can see that (4.3.22) holds for some 2×2 matrix $\mathsf{A}^{(j+1)}(z)$ of Laurent polynomials with compatible symmetry. Define $\mathsf{U}^{(j)}(z) := \mathsf{E}(z)\mathsf{V}_{k,\mathsf{p}}(z)$, then (4.3.20) and (4.3.22) imply that (4.3.23) holds, and the multiplications in (4.3.23) are compatible. By the same argument as in case (1), we can see the conditions (i) and (ii) are true for the matrix $\mathsf{A}^{(j+1)}(z)$.

To prove the condition (iii), we can see from (4.3.24) that $\mathsf{Z}(\mathsf{d}^{(j)}(z), z_0) = \alpha/2$. So $\mathsf{p}(z) \mid \mathsf{d}^{(j)}(z)$. Define $\mathsf{d}^{(j+1)}(z) := \frac{\mathsf{d}^{(j)}(z)}{\mathsf{p}(z)}$. Because $\mathsf{d}^{(j)}(z)$ and $\mathsf{p}(z)$ both have symmetry, we know that $\mathsf{d}^{(j+1)}(z)$ also has symmetry. Similar to the case (2), we get from (4.3.23) that (4.3.25) holds. Hence, we proved that the condition (iii) also holds for $\mathsf{A}^{(j+1)}(z)$.

Furthermore, as (4.3.25) gives $\det(\mathsf{A}^{(j+1)}(z)) = \frac{\det(\mathsf{A}^{(j)}(z))}{\mathsf{p}(z)\mathsf{p}^\star(z)}$, we can calculate that

$$\begin{aligned}\mathsf{Z}(\det(\mathsf{A}^{(j+1)}(z)), z_0) =& \mathsf{Z}(\det(\mathsf{A}^{(j)}(z)), z_0) - \mathsf{Z}(\mathsf{p}(z), z_0) - \mathsf{Z}(\mathsf{p}^\star(z), z_0)\\ =& \alpha - \alpha/2 - \alpha/2 = 0.\end{aligned}$$

So $z_0 \in \sigma(\mathsf{A}^{(j)}(z))$, but $z_0 \notin \sigma(\mathsf{A}^{(j+1)}(z))$. That is, $\sigma(\mathsf{A}^{(j+1)}(z)) \subsetneq \sigma(\mathsf{A}^{(j)}(z))$. This finished the proof of the case that $z_0 \in \{\pm 1\}$.

Hence, we showed that for all possible locations of $z_0 \in \mathbb{C}\setminus\{0\}$, we can find matrices $\mathsf{A}^{(j+1)}(z)$ and $\mathsf{U}^{(j)}(z)$, such that (I)(II) and (III) are satisfied. Since the conditions (i)(ii) and (iii) are still satisfied for $\mathsf{A}^{(j+1)}(z)$, as long as $\sigma(\mathsf{A}^{(j+1)}(z))$ is not empty, we can iteratively apply this procedure to generate a sequence of matrices of Laurent polynomials $\{\mathsf{A}^{(j)}(z)\}_{j=0}^{K}$, until $\sigma(\mathsf{A}^{(K)}(z))$ becomes empty. We get

$$\mathsf{A}(z) = \mathsf{A}^{(0)}(z) = \mathsf{U}^{(0)}(z)\cdots\mathsf{U}^{(K-1)}(z)\mathsf{A}^{(K)}(z)\mathsf{U}^{(K-1)}(z)\cdots\mathsf{U}^{(0)}(z).$$

Define $\mathsf{B}(z) := \mathsf{A}^{(K)}(z)$, and $\mathsf{U}(z) := \mathsf{U}^{(0)}(z)\cdots\mathsf{U}^{(K-1)}(z)$. We can see $\mathsf{U}(z)$ and $\mathsf{B}(z)$ both have compatible symmetry, and the multiplications in $\mathsf{A}(z) = \mathsf{U}(z)\mathsf{B}(z)\mathsf{U}^\star(z)$ are compatible. Moreover, $\mathsf{B}(z) = \mathsf{A}^{(K)}(z)$ is a Hermitian matrix of Laurent polynomials, and

$$\det(\mathsf{B}(z)) = \det(\mathsf{A}^{(K)}(z)) = -\mathsf{d}^{(K)}(z)\mathsf{d}^{(K)\star}(z), \qquad (4.3.27)$$

is also Hermitian. Since $\sigma(\mathsf{B}(z))$ is empty, we know that $\det(\mathsf{B}(z))$ is a nonzero monomial. So (4.3.27) implies that the Hermitian Laurent polynomial $\det(\mathsf{B}(z))$ must be a negative constant. This completes the proof of the theorem. ■

4.3.2 Difference of Squares of Laurent Polynomials with Symmetry

In this section we study the DOS property of a Laurent polynomial. Given a Laurent polynomial $\mathsf{p}(z)$, we say $\mathsf{p}(z)$ has the ***DOS (difference of squares) property with respect to symmetry type*** ϵz^c, where $\epsilon \in \{-1, 1\}$, and $c \in \mathbb{Z}$, if there exist two Laurent polynomials $\mathsf{p}_1(z)$ and $\mathsf{p}_2(z)$ having symmetry, such that

$$\mathsf{p}_1(z)\mathsf{p}_1^\star(z) - \mathsf{p}_2(z)\mathsf{p}_2^\star(z) = \mathsf{p}(z), \qquad \text{and} \quad \frac{\mathsf{Sp}_1(z)}{\mathsf{Sp}_2(z)} = \epsilon z^c. \qquad (4.3.28)$$

Theorem 4.3.4. *Suppose* $\mathsf{p}_1(z), \mathsf{p}_2(z), \mathsf{p}_3(z)$ *and* $\mathsf{p}_4(z)$ *are Laurent polynomials with symmetry, denote the symmetry types by*

$$\mathsf{Sp}_j(z) = \epsilon_j z^{c_j}, \qquad \epsilon_j \in \{1, -1\},\ c_j \in \mathbb{Z},\ j = 1, 2, 3, 4.$$

It satisfies $\mathsf{Sp}_1(z)/\mathsf{Sp}_2(z) = \mathsf{Sp}_3(z)/\mathsf{Sp}_4(z)$. *Define*

$$\begin{aligned}\mathsf{p}_5(z) :=& \mathsf{p}_1(z)\mathsf{p}_3(z) + z^{c_1}\mathsf{p}_2^\star(z)\mathsf{p}_4(z),\\ \mathsf{p}_6(z) :=& \mathsf{p}_2(z)\mathsf{p}_3(z) + z^{c_1}\mathsf{p}_1^\star(z)\mathsf{p}_4(z).\end{aligned}$$

Then $\mathsf{Sp}_5(z)$ *and* $\mathsf{Sp}_6(z)$ *satisfy*

$$\mathsf{Sp}_5(z) = \epsilon_1\epsilon_3 z^{c_1+c_3}, \qquad \mathsf{Sp}_6(z) = \epsilon_2\epsilon_3 z^{c_2+c_3}, \qquad \frac{\mathsf{Sp}_5(z)}{\mathsf{Sp}_6(z)} = \frac{\mathsf{Sp}_1(z)}{\mathsf{Sp}_2(z)}, \quad (4.3.29)$$

and

$$\mathsf{p}_5\mathsf{p}_5^\star - \mathsf{p}_6\mathsf{p}_6^\star = \left(\mathsf{p}_1\mathsf{p}_1^\star - \mathsf{p}_2\mathsf{p}_2^\star\right)\left(\mathsf{p}_3\mathsf{p}_3^\star - \mathsf{p}_4\mathsf{p}_4^\star\right). \tag{4.3.30}$$

Proof. From $\mathsf{Sp}_1(z)/\mathsf{Sp}_2(z) = \mathsf{Sp}_3(z)/\mathsf{Sp}_4(z)$, we can see that $\epsilon_1\epsilon_2 = \epsilon_3\epsilon_4$, and $c_1 - c_2 = c_3 - c_4$. Then we can calculate the symmetry types:

$$\begin{aligned}
\mathsf{S}(\mathsf{p}_1\mathsf{p}_3)(z) =& \mathsf{Sp}_1(z)\mathsf{Sp}_3(z) = \epsilon_1\epsilon_3 z^{c_1+c_3}, \\
\mathsf{S}\left(z^{c_1}\mathsf{p}_2^\star\mathsf{p}_4\right)(z) =& z^{2c_1}\epsilon_2 z^{-c_2}\epsilon_4 z^{c_4} = \epsilon_2\epsilon_4 z^{c_1+c_3} = \epsilon_1\epsilon_3 z^{c_1+c_3}, \\
\mathsf{S}\left(\mathsf{p}_2\mathsf{p}_3\right)(z) =& \mathsf{Sp}_2(z)\mathsf{Sp}_3(z) = \epsilon_2\epsilon_3 z^{c_2+c_3}, \\
\mathsf{S}\left(z^{c_1}\mathsf{p}_1^\star\mathsf{p}_4\right)(z) =& z^{2c_1}\epsilon_1 z^{-c_1}\epsilon_4 z^{c_4} = \epsilon_1\epsilon_4 z^{c_2+c_3} = \epsilon_2\epsilon_3 z^{c_2+c_3}.
\end{aligned}$$

So we can see that $\mathsf{Sp}_5(z) = \epsilon_1\epsilon_3 z^{c_1+c_3}$, $\mathsf{Sp}_6(z) = \epsilon_2\epsilon_3 z^{c_2+c_3}$, and $\mathsf{Sp}_5(z)/\mathsf{Sp}_6(z) = \epsilon_1\epsilon_2 z^{c_1-c_2} = \mathsf{Sp}_1(z)/\mathsf{Sp}_2(z)$. Hence, we proved (4.3.29).

In order to prove (4.3.30), we can rewrite the definition of $\mathsf{p}_5(z)$ and $\mathsf{p}_6(z)$ as

$$\begin{bmatrix}\mathsf{p}_5 \\ \mathsf{p}_6\end{bmatrix} = \begin{bmatrix}\mathsf{p}_1 & \mathsf{p}_2^\star \\ \mathsf{p}_2 & \mathsf{p}_1^\star\end{bmatrix}\begin{bmatrix}\mathsf{p}_3 \\ z^{c_1}\mathsf{p}_4\end{bmatrix}.$$

Therefore, we can directly calculate that

$$\begin{aligned}
\mathsf{p}_5\mathsf{p}_5^\star - \mathsf{p}_6\mathsf{p}_6^\star &= \begin{bmatrix}\mathsf{p}_5^\star & \mathsf{p}_6^\star\end{bmatrix}\begin{bmatrix}1 & \\ & -1\end{bmatrix}\begin{bmatrix}\mathsf{p}_5 \\ \mathsf{p}_6\end{bmatrix} \\
&= \begin{bmatrix}\mathsf{p}_3^\star & z^{-c_1}\mathsf{p}_4^\star\end{bmatrix}\begin{bmatrix}\mathsf{p}_1^\star & \mathsf{p}_2^\star \\ \mathsf{p}_2 & \mathsf{p}_1\end{bmatrix}\begin{bmatrix}1 & \\ & -1\end{bmatrix}\begin{bmatrix}\mathsf{p}_1 & \mathsf{p}_2^\star \\ \mathsf{p}_2 & \mathsf{p}_1^\star\end{bmatrix}\begin{bmatrix}\mathsf{p}_3 \\ z^{c_1}\mathsf{p}_4\end{bmatrix} \\
&= \begin{bmatrix}\mathsf{p}_3^\star & z^{-c_1}\mathsf{p}_4^\star\end{bmatrix}\left(\mathsf{p}_1\mathsf{p}_1^\star - \mathsf{p}_2\mathsf{p}_2^\star\right)\begin{bmatrix}1 & \\ & -1\end{bmatrix}\begin{bmatrix}\mathsf{p}_3 \\ z^{c_1}\mathsf{p}_4\end{bmatrix} \\
&= \left(\mathsf{p}_1\mathsf{p}_1^\star - \mathsf{p}_2\mathsf{p}_2^\star\right)\begin{bmatrix}\mathsf{p}_3^\star & z^{-c_1}\mathsf{p}_4^\star\end{bmatrix}\begin{bmatrix}1 & \\ & -1\end{bmatrix}\begin{bmatrix}\mathsf{p}_3 \\ z^{c_1}\mathsf{p}_4\end{bmatrix} \\
&= \left(\mathsf{p}_1\mathsf{p}_1^\star - \mathsf{p}_2\mathsf{p}_2^\star\right)\left(\mathsf{p}_3\mathsf{p}_3^\star - \mathsf{p}_4\mathsf{p}_4^\star\right).
\end{aligned}$$

This completes the proof of the theorem. ■

Lemma 4.3.3. *Suppose* $\mathsf{p}(z)$ *is a Laurent polynomial satisfying* $\mathsf{p}^\star(z) = \mathsf{p}(z)$. *Define* $\sigma(\mathsf{p}) := \{z_0 \in \mathbb{C} \setminus \{0\} \ : \ \mathsf{p}(z) = 0\}$. *Then* $\sum_{z_0 \in \mathbb{T} \cap \sigma(\mathsf{p})} \mathsf{Z}(\mathsf{p}, z_0) \in 2\mathbb{Z}$.

Proof. Since $\mathsf{p}^\star(z) = \mathsf{p}(z)$, we know that in time domain $p(-k) = \overline{p(k)}$ for all $k \in \mathbb{Z}$. So fsupp($\mathsf{p}(z)$) is a symmetric interval with center 0, and len(p) $\in 2\mathbb{Z}$. By the fundamental theorem of algebra, we can see that $\sum_{z_0 \in \sigma(\mathsf{p})} \mathsf{Z}(\mathsf{p}, z_0) \in 2\mathbb{Z}$.

Partition $\sigma(\mathsf{p})$ into a disjoint union as: $\sigma(\mathsf{p}) = \sigma_{in} \cup \sigma_{out} \cup \sigma_{\mathbb{T}}$, where

$$\begin{aligned} \sigma_{in} :=& \{z_0 \in \sigma(\mathsf{p}) \ : \ |z_0| < 1\}, \\ \sigma_{out} :=& \{z_0 \in \sigma(\mathsf{p}) \ : \ |z_0| > 1\}, \\ \sigma_{\mathbb{T}} :=& \{z_0 \in \sigma(\mathsf{p}) \ : \ |z_0| = 1\}. \end{aligned}$$

According to $\mathsf{p}^\star(z) = \mathsf{p}(z)$, we know that

$$\mathsf{Z}(\mathsf{p}, z_0) = \mathsf{Z}(\mathsf{p}^\star, \overline{z_0}^{-1}) = \mathsf{Z}(\mathsf{p}, \overline{z_0}^{-1}), \qquad \forall\, z_0 \in \mathbb{C} \setminus \{0\}. \tag{4.3.31}$$

The map $\eta : z \to \overline{z}^{-1}$ is a bijection between $\{z_0 \in \mathbb{C} \setminus \{0\} \ : \ |z_0| < 1\}$ and $\{z_0 \in \mathbb{C} \setminus \{0\} \ : \ |z_0| > 1\}$. So (4.3.31) implies that η is also a bijection between σ_{in} and σ_{out}. Moreover, we have $\sum_{z_0 \in \sigma_{out}} \mathsf{Z}(\mathsf{p}, z_0) = \sum_{\overline{z_0}^{-1} \in \sigma_{in}} \mathsf{Z}(\mathsf{p}, z_0) = \sum_{z_0 \in \sigma_{in}} \mathsf{Z}(\mathsf{p}, \overline{z_0}^{-1}) = \sum_{z_0 \in \sigma_{in}} \mathsf{Z}(\mathsf{p}, z_0)$. Now, from

$$\begin{aligned} \sum_{z_0 \in \sigma(\mathsf{p})} \mathsf{Z}(\mathsf{p}, z_0) =& \sum_{z_0 \in \sigma_{in}} \mathsf{Z}(\mathsf{p}, z_0) + \sum_{z_0 \in \sigma_{out}} \mathsf{Z}(\mathsf{p}, z_0) + \sum_{z_0 \in \sigma_{\mathbb{T}}} \mathsf{Z}(\mathsf{p}, z_0) \\ =& 2 \sum_{z_0 \in \sigma_{out}} \mathsf{Z}(\mathsf{p}, z_0) + \sum_{z_0 \in \sigma_{\mathbb{T}}} \mathsf{Z}(\mathsf{p}, z_0) \in 2\mathbb{Z}, \end{aligned}$$

we can see that $\sum_{z_0 \in \mathbb{T} \cap \sigma(\mathsf{p})} \mathsf{Z}(\mathsf{p}, z_0) = \sum_{z_0 \in \sigma_{\mathbb{T}}} \mathsf{Z}(\mathsf{p}, z_0) \in 2\mathbb{Z}$.

This finishes the proof of the lemma. ■

The following theorem characterizes the DOS property of a Laurent polynomial.

Theorem 4.3.5. *Suppose $\epsilon \in \{1, -1\}$ and $c \in \mathbb{Z}$. A Laurent polynomial $\mathsf{p}(z)$ has the DOS property with respect to the symmetry type ϵz^c if and only if*

(1) $\mathsf{p}(z)$ has real coefficients and $\mathsf{p}^\star(z) = \mathsf{p}(z)$;

(2) $\mathsf{p}(z)$ satisfies the following technical conditions of the DOS property:

[i] if $\epsilon = 1$ and $c \in 2\mathbb{Z}$, then there is no condition;

[ii] if $\epsilon = 1$ *and* $c \in 2\mathbb{Z}+1$, *then* $\mathsf{Z}(\mathsf{p}, x) \in 2\mathbb{Z}$ *for all* $x \in (-1, 0)$;

[iii] if $\epsilon = -1$ *and* $c \in 2\mathbb{Z}$, *then* $\mathsf{Z}(\mathsf{p}, x) \in 2\mathbb{Z}$ *for all* $x \in (-1, 0) \bigcup (0, 1)$;

[iv] if $\epsilon = -1$ *and* $c \in 2\mathbb{Z}+1$, *then* $\mathsf{Z}(\mathsf{p}, x) \in 2\mathbb{Z}$ *for all* $x \in (0, 1)$.

Proof. Firstly, we prove the necessity. Suppose we can find Laurent polynomials $\mathsf{p}_1(z)$ and $\mathsf{p}_2(z)$ with symmetry, such that (4.3.28) holds. Then taking the Hermitian conjugate on both sides of the first equation in (4.3.28), we can see that $\mathsf{p}^\star(z) = \mathsf{p}(z)$.

As $\mathsf{S}(\mathsf{p}_1\mathsf{p}_1^\star)(z) = \mathsf{S}\mathsf{p}_1(z)\mathsf{S}\mathsf{p}_1^\star(z) = 1$, similarly, $\mathsf{S}(\mathsf{p}_2\mathsf{p}_2^\star)(z) = 1$. So the first equation in (4.3.28) implies that $\mathsf{S}\mathsf{p}(z) = 1$. In time domain, it can be written as

$$p(k) = p(-k), \qquad \forall\ k \in \mathbb{Z}. \tag{4.3.32}$$

Also, $\mathsf{p}^\star(z) = \mathsf{p}(z)$ implies that in time domain, we have

$$p(k) = \overline{p(-k)}, \qquad \forall\ k \in \mathbb{Z}. \tag{4.3.33}$$

Equations (4.3.32) and (4.3.33) imply $p(k) = \overline{p(k)}$ for all $k \in \mathbb{Z}$. Hence we know that all the coefficients of $\mathsf{p}(z)$ are real. This proves the item (1).

To prove the item (2), we can see from the symmetry types of $\mathsf{p}_1(z)$ and $\mathsf{p}_2(z)$ that

$$\mathsf{p}_1^\star(z) = \overline{\mathsf{p}_1(\bar{z}^{-1})} = \frac{\overline{\mathsf{p}_1(\bar{z})}}{\overline{\mathsf{S}\mathsf{p}_1(\bar{z})}} = \frac{\overline{\mathsf{p}_1(\bar{z})}}{\mathsf{S}\mathsf{p}_1(z)}, \qquad \mathsf{p}_2^\star(z) = \frac{\overline{\mathsf{p}_2(\bar{z})}}{\mathsf{S}\mathsf{p}_2(z)} = \overline{\mathsf{p}_2(\bar{z})}\frac{\epsilon z^c}{\mathsf{S}\mathsf{p}_1(z)}.$$

For $z = x \in \mathbb{R} \setminus \{0\}$, we can see

$$\begin{aligned}\mathsf{p}(x) =& \mathsf{p}_1(x)\mathsf{p}_1^\star(x) - \mathsf{p}_2(x)\mathsf{p}_2^\star(x) = |\mathsf{p}_1(x)|^2 / \mathsf{S}\mathsf{p}_1(x) - |\mathsf{p}_2(x)|^2 \epsilon x^c / \mathsf{S}\mathsf{p}_1(x) \\ =& \left(|\mathsf{p}_1(x)|^2 - \epsilon x^c |\mathsf{p}_2(x)|^2\right) / \mathsf{S}\mathsf{p}_1(x).\end{aligned}$$

When $\epsilon = 1$ and $c \in 2\mathbb{Z}+1$, since $\epsilon x^c < 0$ for all $x \in (-1, 0)$, we can see that $\mathsf{Z}(\mathsf{p}, x) = 2\min(\mathsf{Z}(\mathsf{p}_1, x), \mathsf{Z}(\mathsf{p}_2, x)) \in 2\mathbb{Z}$ for all $x \in (-1, 0)$. Thus, we proved the item [ii]. The items [iii] and [iv] can be proved similarly. This proves the necessity part of the theorem.

Now, we prove the sufficiency. Notice that if (4.3.28) holds, then for any integer k, we have

$$\left(z^k \mathsf{p}_1(z)\right)\left(z^k \mathsf{p}_1(z)\right)^\star - \mathsf{p}_2(z)\mathsf{p}_2^\star(z) = \mathsf{p}(z), \qquad \text{and} \quad \frac{\mathsf{S}\left(z^k \mathsf{p}_1(z)\right)}{\mathsf{S}\mathsf{p}_2(z)} = \epsilon z^{c+2k}.$$

Hence, we only need to prove the DOS property for the symmetry type ϵz^c, where $\epsilon \in \{1, -1\}$ and $c \in \{0, 1\}$.

Since $\mathsf{p}^\star(z) = \mathsf{p}(z)$, we know that in time domain, $p(-k) = \overline{p(k)}$, $\forall k \in \mathbb{Z}$. Also, as $\mathsf{p}(z)$ has real coefficients, we can conclude that $p(-k) = p(k)$ for all $k \in \mathbb{Z}$. That is, $\mathsf{S}\mathsf{p}(z) = 1$. By Lemma 4.1.1, we know that $\mathsf{Z}(\mathsf{p}(z), z_0^{-1}) = \mathsf{Z}(\mathsf{p}(z), z_0)$, for all $z_0 \in \mathbb{C} \setminus \{0\}$. Also, from $\mathsf{p}^\star(z) = \mathsf{p}(z)$, we can see that $\mathsf{Z}(\mathsf{p}(z), \overline{z_0}^{-1}) = \mathsf{Z}(\mathsf{p}^\star(z), z_0) = \mathsf{Z}(\mathsf{p}(z), z_0)$ for all $z_0 \in \mathbb{C} \setminus \{0\}$. Hence, we get

$$\mathsf{Z}(\mathsf{p}(z), z_0) = \mathsf{Z}(\mathsf{p}(z), z_0^{-1}) = \mathsf{Z}(\mathsf{p}(z), \overline{z_0}^{-1}) = \mathsf{Z}(\mathsf{p}(z), \overline{z_0}), \qquad \forall\, z_0 \in \mathbb{C} \setminus \{0\}. \tag{4.3.34}$$

Furthermore, according to Lemma 4.1.2, we can conclude from $\mathsf{S}\mathsf{p}(z) = 1$ that

$$\mathsf{Z}(\mathsf{p}, 1) \in 2\mathbb{Z} \qquad \text{and} \quad \mathsf{Z}(\mathsf{p}, -1) \in 2\mathbb{Z}. \tag{4.3.35}$$

By (4.3.34) and (4.3.35), we know that there exists a real number A, such that $A\mathsf{p}(z)$ can be written into the product of the following 4 types of factors:

(a) $\mathsf{q}(z) = z^{-1}(z - z_0)^2 = z - 2z_0 + z^{-1}$, where $z_0 \in \sigma_{\pm 1}$;

(b) $\mathsf{q}(z) = z^{-2}(z - z_0)(z - z_0^{-1})(z - \overline{z_0})(z - \overline{z_0}^{-1})$, where $z_0 \in \sigma_{in,up}$;

(c) $\mathsf{q}(z) = z^{-1}(z - z_0)(z - \overline{z_0})$, where $z_0 \in \sigma_{\mathbb{T},up}$;

(d) $\mathsf{q}(z) = z^{-1}(z - z_0)(z - z_0^{-1})$, where $z_0 \in \sigma_{\mathbb{R},in}$;

where we denote $\sigma(\mathsf{p}) := \{z_0 \in \mathbb{C} \setminus \{0\} \;:\; \mathsf{p}(z_0) = 0\}$, and

$$\begin{aligned}
\sigma_{\pm 1} :=& \sigma(\mathsf{p}) \cap \{-1, 1\},\\
\sigma_{in,up} :=& \sigma(\mathsf{p}) \cap \{z_0 \in \mathbb{C} \setminus \{0\} \;:\; |z_0| < 1, \operatorname{Im}(z_0) > 0\},\\
\sigma_{\mathbb{T},up} :=& \sigma(\mathsf{p}) \cap \{z_0 \in \mathbb{C} \setminus \{0\} \;:\; |z_0| = 1, \operatorname{Im}(z_0) > 0\},\\
\sigma_{\mathbb{R},in} :=& \sigma(\mathsf{p}) \cap \{z_0 \in \mathbb{R} \;:\; 0 < |z_0| < 1\}.
\end{aligned}$$

By Theorem 4.3.4, in order to prove the DOS property of $\mathsf{p}(z)$, we only need to prove that all of the factors above have the DOS property with respect to the specific symmetry type ϵz^c. We will discuss them one by one.

Type (a) For $z_0 \in \sigma_{\pm 1}$, we can see that

$$z^{-1}(z-1)^2 = 0^2 - (z-1)(z-1)^\star, \quad z^{-1}(z+1)^2 = (z+1)(z+1)^\star - 0^2.$$

Since 0 has symmetry of any type, we can see this factor has DOS property with respect to all the symmetry types $\epsilon \in \{-1, 1\}$ and $c \in \{0, 1\}$.

Type (b) For $z_0 \in \sigma_{in,up}$, we can see that

$$\begin{aligned}\mathsf{q}(z) =& z^{-2}(z-z_0)(z-z_0^{-1})(z-\overline{z_0})(z-\overline{z_0}^{-1}) \\ =& \left[|z_0|^{-1}z^{-1}(z-z_0)(z-\overline{z_0})\right]\left[|z_0|^{-1}z^{-1}(z-z_0)(z-\overline{z_0})\right]^\star - 0^2.\end{aligned}$$

Since 0 has symmetry of any type, we can see this factor has DOS property with respect to all the symmetry types $\epsilon \in \{-1, 1\}$ and $c \in \{0, 1\}$.

Type (c) For $z_0 \in \sigma_{\mathbb{T},up}$, we know that

$$\mathsf{q}(z) = z^{-1}(z-z_0)(z-\overline{z_0}) = z - 2\operatorname{Re}(z_0) + z^{-1},$$

where $|\operatorname{Re}(z_0)| < 1$.

For the symmetry type $c = 1$ and $\epsilon = 1$, we can take $\mathsf{q}_1(z) := z + 1$, and $\mathsf{q}_2(z) := \sqrt{2 + 2\operatorname{Re}(z_0)}$. Then $\mathsf{q} = \mathsf{q}_1\mathsf{q}_1^\star - \mathsf{q}_2\mathsf{q}_2^\star$, and $\mathsf{Sq}_1(z)/\mathsf{Sq}_2(z) = z$.

For the symmetry type $c = 1$ and $\epsilon = -1$, we can take $\mathsf{q}_1(z) := \sqrt{2 - 2\operatorname{Re}(z_0)}$, and $\mathsf{q}_2(z) := 1 - z^{-1}$. Then $\mathsf{q} = \mathsf{q}_1\mathsf{q}_1^\star - \mathsf{q}_2\mathsf{q}_2^\star$, and $\mathsf{Sq}_1(z)/\mathsf{Sq}_2(z) = -z$.

For the symmetry type $c = 0$ and $\epsilon = -1$, we can take $\mathsf{q}_1(z) := \sqrt{\frac{1-\operatorname{Re}(z_0)}{2}}(z+1)$, and $\mathsf{q}_2(z) := \sqrt{\frac{1+\operatorname{Re}(z_0)}{2}}(z-1)$. Then $\mathsf{q} = \mathsf{q}_1\mathsf{q}_1^\star - \mathsf{q}_2\mathsf{q}_2^\star$, and $\mathsf{Sq}_1(z)/\mathsf{Sq}_2(z) = -1$.

For the symmetry type $c = 0$ and $\epsilon = 1$, we can take $\mathsf{q}_1(z) := \frac{z - 4\operatorname{Re}(z_0) + 2 + z^{-1}}{2\sqrt{2 - 2\operatorname{Re}(z_0)}}$, and $\mathsf{q}_2(z) := \frac{z - 2 + z^{-1}}{2\sqrt{2 - 2\operatorname{Re}(z_0)}}$. Then $\mathsf{q} = \mathsf{q}_1\mathsf{q}_1^\star - \mathsf{q}_2\mathsf{q}_2^\star$, and $\mathsf{Sq}_1(z)/\mathsf{Sq}_2(z) = 1$.

Type (d) For $z_0 \in \sigma_{\mathbb{R},in}$, that is, $z_0 \in (-1,0) \cup (0,1)$, we know that

$$\mathsf{q}(z) = z^{-1}(z - z_0)(z - z_0^{-1}) = z - (z_0 + z_0^{-1}) + z^{-1}.$$

If $\mathsf{Z}(\mathsf{p}, z_0) \in 2\mathbb{Z}$, we only need to consider the factor $\mathsf{q}^2(z)$:

$$\mathsf{q}^2(z) = \mathsf{q}(z)\mathsf{q}^\star(z) - 0^2.$$

Hence, we get $\mathsf{q}^2 = \mathsf{q}_1\mathsf{q}_1^\star - \mathsf{q}_2\mathsf{q}_2^\star$, where $\mathsf{q}_1(z) := \mathsf{q}(z)$ and $\mathsf{q}_2(z) := 0$. Since $\mathsf{Sq}(z) = 1$ and 0 can be taken as any symmetry type, we have $\mathsf{Sp}_1(z)/\mathsf{Sp}_2(z) = \epsilon z^c$ for all $\epsilon \in \{1, -1\}$ and $c \in \{0, 1\}$.

Now we consider the case that $\mathsf{Z}(\mathsf{p}, z_0) \in 2\mathbb{Z} + 1$.

For the case $\epsilon = 1$ and $c = 0$, define $\mathsf{q}_1(z) := z + z^{-1} - (z_0 + z_0^{-1}) + \frac{1}{4}$, and $\mathsf{q}_2(z) := z + z^{-1} - (z_0 + z_0^{-1}) - \frac{1}{4}$. Then $\mathsf{q} = \mathsf{q}_1\mathsf{q}_1^\star - \mathsf{q}_2\mathsf{q}_2^\star$, and $\mathsf{Sq}_1(z)/\mathsf{Sq}_2(z) = 1$.

For the case $\epsilon = 1$ and $c = 1$, by the item [ii], we only need to consider $z_0 \in (0, 1)$. So we can define $\mathsf{q}_1(z) := z + 1$, and $\mathsf{q}_2(z) := \sqrt{2 + z_0 + z_0^{-1}}$. Then $\mathsf{q} = \mathsf{q}_1\mathsf{q}_1^\star - \mathsf{q}_2\mathsf{q}_2^\star$, and $\mathsf{Sq}_1(z)/\mathsf{Sq}_2(z) = z$.

For the case $\epsilon = -1$ and $c = 0$, by the item [iii], we have $\mathsf{Z}(\mathsf{p}, z_0) \in 2\mathbb{Z}$. So this situation will not happen.

For the case $\epsilon = -1$ and $c = 1$, by the item [iv], we only need to consider $z_0 \in (-1, 0)$. So we can define $\mathsf{q}_1(z) := \sqrt{2 - z_0 - z_0^{-1}}$, and $\mathsf{q}_2(z) := z^{-1} - 1$. Then $\mathsf{q} = \mathsf{q}_1\mathsf{q}_1^\star - \mathsf{q}_2\mathsf{q}_2^\star$, and $\mathsf{Sq}_1(z)/\mathsf{Sq}_2(z) = -z$.

This completes the proof of the sufficiency part of the theorem. ∎

4.3.3 Spectral Factorization of Matrices of Laurent Polynomials with Symmetry

In this section, we prove the necessary and sufficient conditions of the spectral factorizations of 2×2 matrices of Laurent polynomials with symmetry. We need the following lemma first.

Lemma 4.3.4. *If* $\mathsf{p}(z)$ *and* $\mathsf{q}(z)$ *are two Laurent polynomials with symmetry, satisfying* $\mathsf{q}(z)\mathsf{q}^\star(z)|\mathsf{p}(z)\mathsf{p}^\star(z)$, *then there exists a Laurent polynomial* $\mathsf{d}(z)$ *with symmetry, such that*

$$\mathsf{d}(z)\mathsf{d}^\star(z) = \frac{\mathsf{p}(z)\mathsf{p}^\star(z)}{\mathsf{q}(z)\mathsf{q}^\star(z)}. \tag{4.3.36}$$

Furthermore, for each Laurent polynomial $\mathsf{d}(z)$ *with symmetry satisfying* (4.3.36), *there exists an integer* k, *such that* $\mathsf{Sd}(z) = z^{2k}\frac{\mathsf{Sp}(z)}{\mathsf{Sq}(z)}$.

Proof. Denote the Laurent polynomial $\mathsf{a}(z) := \frac{\mathsf{p}(z)\mathsf{p}^\star(z)}{\mathsf{q}(z)\mathsf{q}^\star(z)}$. According to [45, Theorem 3.1.8], there exists a Laurent polynomial $\mathsf{d}(z)$ with symmetry, such that $\mathsf{a}(z) = \mathsf{d}(z)\mathsf{d}^\star(z)$ if and only if the following conditions hold:

(1) $\mathsf{Sa}(z) = 1$ and all the coefficients of $\mathsf{a}(z)$ are real numbers;

(2) $\mathsf{a}(z) \geqslant 0$ for all $z \in \mathbb{T}$;

(3) for all real numbers $x \in (-1, 0) \cup (0, 1)$, $\mathsf{Z}(\mathsf{a}, x) \in 2\mathbb{Z}$.

From the definition of $\mathsf{a}(z)$, we can directly calculate that $\mathsf{Sa}(z) = 1$. That is,

$$a(-k) = a(k), \qquad \forall\, k \in \mathbb{Z}. \tag{4.3.37}$$

Also, $\mathsf{a}^\star(z) = \mathsf{a}(z)$ implies that

$$\overline{a(-k)} = a(k), \qquad \forall\, k \in \mathbb{Z}. \tag{4.3.38}$$

Equations (4.3.37) and (4.3.38) imply all the coefficients of $\mathsf{a}(z)$ are real numbers. Hence, we proved the item (1).

As $\mathsf{p}(z)\mathsf{p}^\star(z) \geqslant 0$ and $\mathsf{q}(z)\mathsf{q}^\star(z) \geqslant 0$ for all $z \in \mathbb{T}$, we can see that $\mathsf{a}(z) = \mathsf{p}(z)\mathsf{p}^\star(z)/\mathsf{q}(z)\mathsf{q}^\star(z) \geqslant 0$ for all $z \in \mathbb{T}$. This proves the item (2).

For an arbitrary real number $x \in (-1, 0) \cup (0, 1)$, since p and q both have symmetry, we know

$$\mathsf{Z}(\mathsf{p}^\star, x) = \mathsf{Z}(\mathsf{p}, x^{-1}) = \mathsf{Z}(\mathsf{p}, x), \qquad \mathsf{Z}(\mathsf{q}^\star, x) = \mathsf{Z}(\mathsf{q}, x^{-1}) = \mathsf{Z}(\mathsf{q}, x).$$

Therefore,

$$\begin{aligned}\mathsf{Z}(a,x) =& \mathsf{Z}\left(\frac{\mathsf{pp}^\star}{\mathsf{qq}^\star}, x\right) = \mathsf{Z}(\mathsf{p},x) + \mathsf{Z}(\mathsf{p}^\star,x) - \mathsf{Z}(\mathsf{q},x) - \mathsf{Z}(\mathsf{q}^\star,x) \\ =& 2\mathsf{Z}(\mathsf{p},x) - 2\mathsf{Z}(\mathsf{q},x) \in 2\mathbb{Z}.\end{aligned}$$

This proves the item (3). Hence, there exists a Laurent polynomial $\mathsf{d}(z)$ with symmetry, such that $\mathsf{a}(z) = \mathsf{d}(z)\mathsf{d}^\star(z)$.

To calculate the symmetry type of d, denote $\mathsf{Sd}(z) = \epsilon_d z^{c_d}$, $\mathsf{Sp}(z) = \epsilon_p z^{c_p}$ and $\mathsf{Sq}(z) = \epsilon_q z^{c_q}$, for $\epsilon_d, \epsilon_p, \epsilon_q \in \{1, -1\}$ and $c_d, c_p, c_q \in \mathbb{Z}$. According to Lemma 4.1.2 (1), we know that $\epsilon_d = (-1)^{\mathsf{Z}(\mathsf{d},1)} = (-1)^{\mathsf{Z}(\mathsf{p},1)-\mathsf{Z}(\mathsf{q},1)} = \epsilon_p/\epsilon_q$. From Proposition 4.1.1 (5), we get

$$\begin{aligned}\mathrm{odd}(c_d) =& \mathrm{odd}(\mathrm{len}(\mathsf{d})) = \mathrm{odd}(\mathrm{len}(\mathsf{p}) - \mathrm{len}(\mathsf{q})) = \mathrm{odd}(\mathrm{odd}(\mathrm{len}(\mathsf{p})) - \mathrm{odd}(\mathrm{len}(\mathsf{q}))) \\ =& \mathrm{odd}(\mathrm{odd}(c_p) - \mathrm{odd}(c_q)) = \mathrm{odd}(c_p - c_q).\end{aligned}$$

Therefore, we can find an integer k, such that $\mathsf{Sd}(z) = z^{2k}\frac{\mathsf{Sp}(z)}{\mathsf{Sq}(z)}$. This completes the proof of the lemma. ■

Theorem 4.3.6. *Suppose* $\mathsf{A}(z)$ *is a* 2×2 *Hermitian matrix of Laurent polynomials with compatible symmetry, denote its symmetry type by* $\mathsf{SA}(z) = \begin{bmatrix} 1 & \alpha(z) \\ \alpha^\star(z) & 1 \end{bmatrix}$. *Then we can find a matrix* $\mathsf{U}(z) = \begin{bmatrix} \mathsf{U}_{1,1}(z) & \mathsf{U}_{1,2}(z) \\ \mathsf{U}_{2,1}(z) & \mathsf{U}_{2,2}(z) \end{bmatrix}$ *of Laurent polynomials with compatible symmetry, such that* $\mathsf{A}(z) = \mathsf{U}(z)\begin{bmatrix} 1 & \\ & -1 \end{bmatrix}\mathsf{U}^\star(z)$ *holds, and the symmetry type satisfies*

$$\frac{\mathsf{SU}_{1,1}(z)}{\mathsf{SU}_{2,1}(z)} = \frac{\mathsf{SU}_{1,2}(z)}{\mathsf{SU}_{2,2}(z)} = \alpha(z), \tag{4.3.39}$$

if and only if the following two conditions hold:

(1) $\det(\mathsf{A}(z)) = -\mathsf{d}(z)\mathsf{d}^\star(z)$ *for some Laurent polynomial* $\mathsf{d}(z) \neq 0$ *with symmetry.*

(2) Define $\mathsf{p}_0(z) := \gcd(\mathsf{A}_{1,1}(z), \mathsf{A}_{1,2}(z), \mathsf{A}_{2,1}(z), \mathsf{A}_{2,2}(z))$*, and*

$$\mathsf{p}(z) := \frac{\mathsf{p}_0(z)}{(z-1)^{\mathsf{Z}(\mathsf{p}_0,1)}(z+1)^{\mathsf{Z}(\mathsf{p}_0,-1)}}.$$

Then $\mathsf{p}(z)$ *satisfies the DOS (Difference of Squares) condition with respect to type* $\mathsf{Sd}(z)\alpha(z)$*.*

Proof. Firstly, we prove the sufficiency part by constructing $\mathsf{U}(z)$ in the following 4 steps.

Step 1 Since $\mathsf{A}(z)$ is Hermitian, we know that $\mathsf{A}^\star_{1,1}(z) = \mathsf{A}_{1,1}(z)$. Together with $\mathsf{SA}_{1,1}(z) = 1$, we can conclude from Lemma 4.3.3 that $\mathsf{Z}(\mathsf{A}_{1,1}(z), 1) \in 2\mathbb{Z}$ and $\mathsf{Z}(\mathsf{A}_{1,1}(z), -1) \in 2\mathbb{Z}$. Let

$$\beta_1 := \min\left\{\frac{1}{2}\mathsf{Z}(\mathsf{A}_{1,1}(z), 1),\ \mathsf{Z}(\mathsf{A}_{1,2}(z), 1)\right\},$$
$$\beta_2 := \min\left\{\frac{1}{2}\mathsf{Z}(\mathsf{A}_{1,1}(z), -1),\ \mathsf{Z}(\mathsf{A}_{1,2}(z), -1)\right\},$$

and define $f(z) := (z-1)^{\beta_1}(z+1)^{\beta_2}$. Then we know that f divides the first row of $\mathsf{A}(z)$. Since $\mathsf{A}(z)$ is Hermitian, $f^\star(z)$ also divides the first column of $\mathsf{A}(z)$. Moreover, $f(z)f^\star(z) = (-1)^{\beta_1}(z-1)^{2\beta_1}(z+1)^{2\beta_2}z^{-\beta_1-\beta_2}$ divides $\mathsf{A}_{1,1}(z)$. Define

$$\widetilde{\mathsf{A}}(z) := \begin{bmatrix} \frac{\mathsf{A}_{1,1}(z)}{f(z)f^\star(z)} & \frac{\mathsf{A}_{1,2}(z)}{f(z)} \\ \frac{\mathsf{A}_{2,2}(z)}{f^\star(z)} & \mathsf{A}_{2,2}(z) \end{bmatrix}, \qquad \mathsf{F}(z) := \begin{bmatrix} f(z) & 0 \\ 0 & 1 \end{bmatrix}. \tag{4.3.40}$$

We can see that $\mathsf{A}(z) = \mathsf{F}(z)\widetilde{\mathsf{A}}(z)\mathsf{F}^\star(z)$, and $\widetilde{\mathsf{A}}(z)$ is a Hermitian matrix of Laurent polynomials with compatible symmetry.

Step 2 Define

$$\widetilde{\mathsf{p}}(z) := \gcd\left(\widetilde{\mathsf{A}}_{1,1}, \widetilde{\mathsf{A}}_{1,2}, \widetilde{\mathsf{A}}_{2,1}, \widetilde{\mathsf{A}}_{2,2}\right). \tag{4.3.41}$$

By the definition of β_1, we can see

$$\min\left\{\mathsf{Z}(\widetilde{\mathsf{A}}_{1,1}, 1),\ \mathsf{Z}(\widetilde{\mathsf{A}}_{1,2}, 1)\right\} = \min\left\{\mathsf{Z}\left(\frac{\mathsf{A}_{1,1}}{ff^\star}, 1\right),\ \mathsf{Z}\left(\frac{\mathsf{A}_{1,2}}{f}, 1\right)\right\}$$

$$= \min\left\{\mathsf{Z}\left(\mathsf{A}_{1,1}, 1\right) - \mathsf{Z}(ff^\star, 1),\ \mathsf{Z}\left(\mathsf{A}_{1,2}, 1\right) - \mathsf{Z}(f, 1)\right\}$$
$$= \min\left\{\mathsf{Z}\left(\mathsf{A}_{1,1}, 1\right) - 2\beta_1,\ \mathsf{Z}\left(\mathsf{A}_{1,2}, 1\right) - \beta_1\right\} = 0.$$

Hence,

$$\mathsf{Z}(\widetilde{\mathsf{p}}(z), 1) = 0. \tag{4.3.42}$$

Similarly, from the definition of β_2, we can get

$$\mathsf{Z}(\widetilde{\mathsf{p}}(z), -1) = 0. \tag{4.3.43}$$

Moreover, for all $z_0 \in \mathbb{C} \setminus \{0, 1, -1\}$,

$$\begin{aligned}
\mathsf{Z}(\widetilde{\mathsf{p}}(z), z_0) &= \min\left\{\mathsf{Z}\left(\widetilde{\mathsf{A}}_{1,1}, z_0\right), \mathsf{Z}\left(\widetilde{\mathsf{A}}_{1,2}, z_0\right), \mathsf{Z}\left(\widetilde{\mathsf{A}}_{2,1}, z_0\right), \mathsf{Z}\left(\widetilde{\mathsf{A}}_{2,2}, z_0\right)\right\} \\
&= \min\left\{\mathsf{Z}\left(\mathsf{A}_{1,1}, z_0\right) - 0, \mathsf{Z}\left(\mathsf{A}_{1,2}, z_0\right) - 0, \mathsf{Z}\left(\mathsf{A}_{2,1}, z_0\right) - 0, \mathsf{Z}\left(\mathsf{A}_{2,2}, z_0\right) - 0\right\} \\
&= \min\left\{\mathsf{Z}\left(\mathsf{A}_{1,1}, z_0\right), \mathsf{Z}\left(\mathsf{A}_{1,2}, z_0\right), \mathsf{Z}\left(\mathsf{A}_{2,1}, z_0\right), \mathsf{Z}\left(\mathsf{A}_{2,2}, z_0\right)\right\} \\
&= \mathsf{Z}(\mathsf{p}_0(z), z_0).
\end{aligned} \tag{4.3.44}$$

From (4.3.42)(4.3.43)(4.3.44), we can see that $\widetilde{\mathsf{p}}(z) = \frac{\mathsf{p}_0(z)}{(z-1)^{\mathsf{Z}(\mathsf{p}_0,1)}(z+1)^{\mathsf{Z}(\mathsf{p}_0,-1)}} = \mathsf{p}(z)$. According to the item (2), $\widetilde{\mathsf{p}}(z)$ satisfies the DOS condition with respect to type $\mathsf{Sd}(z)\alpha(z)$. By the item (1) of Theorem 4.3.5, we can see that $\widetilde{\mathsf{p}}^\star(z) = \widetilde{\mathsf{p}}(z)$, and $\mathsf{S}\widetilde{\mathsf{p}}(z) = 1$. Define $\mathring{\mathsf{A}}(z) := \frac{1}{\widetilde{\mathsf{p}}(z)}\widetilde{\mathsf{A}}(z)$, then $\mathring{\mathsf{A}}(z)$ is a Hermitian matrix of Laurent polynomials and $\mathsf{S}\mathring{\mathsf{A}}(z) = \mathsf{S}\widetilde{\mathsf{A}}(z)$. Also,

$$\det(\mathring{\mathsf{A}}(z)) = \frac{\widetilde{\mathsf{A}}(z)}{\widetilde{\mathsf{p}}(z)\widetilde{\mathsf{p}}(z)} = \frac{\mathsf{A}(z)}{\det(\mathsf{F}(z))\det(\mathsf{F}^\star(z))\widetilde{\mathsf{p}}(z)\widetilde{\mathsf{p}}^\star(z)} = \frac{-\mathsf{d}(z)\mathsf{d}^\star(z)}{(f(z)\widetilde{\mathsf{p}}(z))\,(f(z)\widetilde{\mathsf{p}}(z))^\star}. \tag{4.3.45}$$

From Lemma 4.3.4, we know that there exists a Laurent polynomial $\mathring{\mathsf{d}}(z)$ with symmetry, such that $\det(\mathring{\mathsf{A}}(z)) = -\mathring{\mathsf{d}}(z)\mathring{\mathsf{d}}^\star(z)$.

Step 3 Since $\gcd(\mathring{\mathsf{A}}_{1,1}(z), \mathring{\mathsf{A}}_{1,2}(z), \mathring{\mathsf{A}}_{2,1}(z), \mathring{\mathsf{A}}_{2,2}(z)) = 1$, we can see that $\mathring{\mathsf{A}}(z)$ satisfies all the conditions in Theorem 4.3.3. Thus, we can use Theorem 4.3.3 to find matrices $\breve{\mathsf{A}}(z)$ and $\breve{\mathsf{U}}(z)$ of Laurent polynomials with compatible symmetry, such that $\mathring{\mathsf{A}}(z) = \breve{\mathsf{U}}(z)\breve{\mathsf{A}}(z)\breve{\mathsf{U}}^\star(z)$, where the multiplications are both compatible and $\det(\breve{\mathsf{A}}(z)) = -C$ is a negative constant.

Step 4 Now, we can verify that $\breve{\mathsf{A}}(z)$ satisfies all the requirements in Theorem 4.2.1. Applying the Theorem4.2.1, we can find a matrix $\mathsf{V}(z)$ of Laurent polynomials with compatible symmetry, such that $\breve{\mathsf{A}}(z) = \mathsf{V}(z)\begin{bmatrix}1 & \\ & -1\end{bmatrix}\mathsf{V}^\star(z)$. Combining all the above constructions, we get

$$\begin{aligned}\mathsf{A}(z) =& \mathsf{F}(z)\widetilde{\mathsf{A}}(z)\mathsf{F}^\star(z) = \widetilde{\mathsf{p}}(z)\mathsf{F}(z)\mathring{\mathsf{A}}(z)\mathsf{F}^\star(z) \\ =& \widetilde{\mathsf{p}}(z)\mathsf{F}(z)\breve{\mathsf{U}}(z)\breve{\mathsf{A}}(z)\breve{\mathsf{U}}^\star(z)\mathsf{F}^\star(z) \\ =& \widetilde{\mathsf{p}}(z)\mathsf{F}(z)\breve{\mathsf{U}}(z)\mathsf{V}(z)\begin{bmatrix}1 & \\ & -1\end{bmatrix}\mathsf{V}^\star(z)\breve{\mathsf{U}}^\star(z)\mathsf{F}^\star(z). \end{aligned} \tag{4.3.46}$$

Define $\widetilde{\mathsf{U}}(z) := \mathsf{F}(z)\breve{\mathsf{U}}(z)\mathsf{V}(z)$. Then from the construction of $\mathsf{F}(z), \breve{\mathsf{U}}(z)$ and $\mathsf{V}(z)$, we can see that the multiplications are compatible, hence $\widetilde{\mathsf{U}}(z)$ has compatible symmetry. Furthermore, (4.3.46) implies that

$$\mathsf{A}(z) = \widetilde{\mathsf{p}}(z)\widetilde{\mathsf{U}}(z)\begin{bmatrix}1 & \\ & -1\end{bmatrix}\widetilde{\mathsf{U}}^\star(z), \tag{4.3.47}$$

thus $\frac{\mathsf{S}\widetilde{\mathsf{U}}_{1,1}(z)}{\mathsf{S}\widetilde{\mathsf{U}}_{2,1}(z)} = \frac{\mathsf{S}\mathsf{A}_{1,1}(z)}{\mathsf{S}\mathsf{A}_{2,1}(z)} = \alpha(z)$, and $\det(\mathsf{A}(z)) = -\widetilde{\mathsf{p}}(z)\widetilde{\mathsf{p}}^\star(z)\det(\widetilde{\mathsf{U}}(z))\det(\widetilde{\mathsf{U}}(z))^\star$. By the item (1), we know that

$$\det(\widetilde{\mathsf{U}}(z))\det(\widetilde{\mathsf{U}}(z))^\star = \frac{-\det(\mathsf{A}(z))}{\widetilde{\mathsf{p}}(z)\widetilde{\mathsf{p}}^\star(z)} = \frac{\mathsf{d}(z)\mathsf{d}^\star(z)}{\widetilde{\mathsf{p}}(z)\widetilde{\mathsf{p}}^\star(z)}.$$

Since $\det(\widetilde{\mathsf{U}}(z))$ also has symmetry, according to Lemma 4.3.4, we can see that there exists some integer k, such that $\mathsf{S}\det(\widetilde{\mathsf{U}}(z)) = z^{2k}\frac{\mathsf{Sd}(z)}{\mathsf{S}\widetilde{\mathsf{p}}(z)} = z^{2k}\mathsf{Sd}(z)$. Denote $\mathsf{S}\widetilde{\mathsf{U}}_{1,1}(z) = \epsilon_1 z^{k_1}$, where $\epsilon_1 \in \{1, -1\}$ and $k_1 \in \mathbb{Z}$, we can calculate that

$$\begin{aligned}\frac{\mathsf{S}\widetilde{\mathsf{U}}_{1,2}(z)}{\mathsf{S}\widetilde{\mathsf{U}}_{1,1}(z)} =& \frac{\mathsf{S}\widetilde{\mathsf{U}}_{1,2}(z)\mathsf{S}\widetilde{\mathsf{U}}_{2,1}(z)}{\mathsf{S}\widetilde{\mathsf{U}}_{1,1}(z)\mathsf{S}\widetilde{\mathsf{U}}_{2,1}(z)} = \frac{\mathsf{S}\det(\widetilde{\mathsf{U}}(z))}{\mathsf{S}\widetilde{\mathsf{U}}_{1,1}(z)\mathsf{S}\widetilde{\mathsf{U}}_{2,1}(z)} \\ =& \frac{z^{2k}\mathsf{Sd}(z)}{\left(\mathsf{S}\widetilde{\mathsf{U}}_{1,1}(z)\right)^2}\frac{\mathsf{S}\widetilde{\mathsf{U}}_{1,1}(z)}{\mathsf{S}\widetilde{\mathsf{U}}_{2,1}(z)} = z^{2k-2k_1}\mathsf{Sd}(z)\alpha(z). \end{aligned} \tag{4.3.48}$$

Since $\widetilde{\mathsf{p}}(z)$ has the DOS property with respect to type $\mathsf{Sd}(z)\alpha(z)$, by Theorem 4.3.5, we know that $\widetilde{\mathsf{p}}(z)$ also has the DOS property with respect to type $z^{2k-2k_1}\mathsf{Sd}(z)\alpha(z)$. Hence, there exist Laurent polynomials $\mathsf{p}_1(z)$ and $\mathsf{p}_2(z)$ with symmetry, such that

$$\widetilde{\mathsf{p}}(z) = \mathsf{p}_1(z)\mathsf{p}_1^\star(z) - \mathsf{p}_2(z)\mathsf{p}_2^\star(z), \qquad \frac{\mathsf{Sp}_1(z)}{\mathsf{Sp}_2(z)} = z^{2k-2k_1}\mathsf{Sd}(z)\alpha(z). \tag{4.3.49}$$

Equations in (4.3.49) implies that

$$\begin{bmatrix} \widetilde{\mathsf{p}}(z) & \\ & -\widetilde{\mathsf{p}}(z) \end{bmatrix} = \begin{bmatrix} \mathsf{p}_1(z) & \mathsf{p}_2^\star(z) \\ \mathsf{p}_2(z) & \mathsf{p}_1^\star(z) \end{bmatrix} \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \begin{bmatrix} \mathsf{p}_1(z) & \mathsf{p}_2^\star(z) \\ \mathsf{p}_2(z) & \mathsf{p}_1^\star(z) \end{bmatrix}^\star, \tag{4.3.50}$$

and $\begin{bmatrix} \mathsf{p}_1(z) & \mathsf{p}_2^\star(z) \\ \mathsf{p}_2(z) & \mathsf{p}_1^\star(z) \end{bmatrix}$ has compatible symmetry. Define

$$\mathsf{U}(z) := \widetilde{\mathsf{U}}(z) \begin{bmatrix} \mathsf{p}_1(z) & \mathsf{p}_2^\star(z) \\ \mathsf{p}_2(z) & \mathsf{p}_1^\star(z) \end{bmatrix}, \tag{4.3.51}$$

we can see from (4.3.47) and (4.3.50) that

$$\mathsf{A}(z) = \widetilde{\mathsf{U}}(z) \begin{bmatrix} \widetilde{\mathsf{p}}(z) & \\ & -\widetilde{\mathsf{p}}(z) \end{bmatrix} \widetilde{\mathsf{U}}^\star(z) = \mathsf{U}(z) \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \mathsf{U}^\star(z).$$

From the symmetry types relationship in (4.3.48) and (4.3.49), we know that $\mathsf{S}\widetilde{\mathsf{U}}_{1,1}(z)\mathsf{Sp}_1(z) = \mathsf{S}\widetilde{\mathsf{U}}_{1,2}(z)\mathsf{Sp}_2(z)$. By the item (4) of Proposition 4.1.2, we can directly verify that the multiplication in (4.3.51) is compatible. Hence, $\mathsf{U}(z)$ has compatible symmetry and (4.3.39) holds. This finishes the proof of the sufficiency part.

Now, we will prove the necessity part of the theorem. Suppose $\mathsf{A}(z) = \mathsf{U}(z) \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \mathsf{U}^\star(z)$, where $\mathsf{U}(z)$ has compatible symmetry and (4.3.39) holds. Take $\mathsf{d}(z) := \det(\mathsf{U}(z))$, then $\mathsf{d}(z)$ has symmetry and $\det(\mathsf{A}(z)) = -\mathsf{d}(z)\mathsf{d}^\star(z)$. This proves the item (1).

To prove the item (2), we can repeat the Step 1 above in the sufficiency proof. (Notice that the construction in Step 1 does not use any assumptions in the item (2).) Using the Step 1, we can define $\widetilde{\mathsf{A}}(z)$ and $\mathsf{F}(z)$ as in (4.3.40), where $\widetilde{\mathsf{A}}(z)$ is a matrix of Laurent polynomials with compatible symmetry, and $\mathsf{A}(z) = \mathsf{F}(z)\widetilde{\mathsf{A}}(z)\mathsf{F}^\star(z)$ holds. Then, we can define $\widetilde{\mathsf{p}}(z)$ as in (4.3.41). By the same argument as in Step 2 above, we can see that (4.3.42), (4.3.43) and (4.3.44) hold. That is, $\widetilde{\mathsf{p}}(z) = \frac{\mathsf{p}_0(z)}{(z-1)^{\mathsf{Z}(\mathsf{p}_0,1)}(z+1)^{\mathsf{Z}(\mathsf{p}_0,-1)}} = \mathsf{p}(z)$. Thus, to prove item (2), we only need to show that $\widetilde{\mathsf{p}}(z)$ has the DOS property with respect to type $\mathsf{Sd}(z)\alpha(z)$.

Since all the four elements in the matrix $\widetilde{\mathsf{A}}(z)$ have symmetry, according to the definition of $\widetilde{\mathsf{p}}(z)$ in (4.3.41) and item (2) in Lemma 4.1.1, we know that $\widetilde{\mathsf{p}}(z)$ also has symmetry. Denote $\mathsf{S}\widetilde{\mathsf{p}}(z) = \epsilon_0 z^{c_0}$, where $\epsilon_0 \in \{1,-1\}$ and $c_0 \in \mathbb{Z}$. From (4.3.42) and (4.3.43), we know that $\mathsf{Z}(\widetilde{\mathsf{p}},1) = 0 \in 2\mathbb{Z}$ and $\mathsf{Z}(\widetilde{\mathsf{p}},-1) = 0 \in 2\mathbb{Z}$. By Lemma 4.1.2, we get $\epsilon_0 = 1$ and $c_0 \in 2\mathbb{Z}$. Without loss of generality, we can redefine $\widetilde{\mathsf{p}}(z)$ by $\widetilde{\mathsf{p}}(z) := z^{-c_0/2}\widetilde{\mathsf{p}}(z)$, and get $\mathsf{S}\widetilde{\mathsf{p}}(z) = 1$.

For all $z_0 \in \mathbb{C}\setminus\{0\}$, we can calculate from the definition of $\widetilde{\mathsf{p}}(z)$ in (4.3.41) that

$$\begin{aligned}
\mathsf{Z}(\widetilde{\mathsf{p}}(z), \overline{z_0}^{-1}) &= \min\left\{\mathsf{Z}\left(\widetilde{\mathsf{A}}_{1,1}, \overline{z_0}^{-1}\right), \mathsf{Z}\left(\widetilde{\mathsf{A}}_{1,2}, \overline{z_0}^{-1}\right), \mathsf{Z}\left(\widetilde{\mathsf{A}}_{2,1}, \overline{z_0}^{-1}\right), \mathsf{Z}\left(\widetilde{\mathsf{A}}_{2,2}, \overline{z_0}^{-1}\right)\right\}\\
&= \min\left\{\mathsf{Z}\left(\widetilde{\mathsf{A}}^\star_{1,1}, z_0\right), \mathsf{Z}\left(\widetilde{\mathsf{A}}^\star_{1,2}, z_0\right), \mathsf{Z}\left(\widetilde{\mathsf{A}}^\star_{2,1}, z_0\right), \mathsf{Z}\left(\widetilde{\mathsf{A}}^\star_{2,2}, z_0\right)\right\}\\
&= \min\left\{\mathsf{Z}\left(\widetilde{\mathsf{A}}_{1,1}, z_0\right), \mathsf{Z}\left(\widetilde{\mathsf{A}}_{2,1}, z_0\right), \mathsf{Z}\left(\widetilde{\mathsf{A}}_{1,2}, z_0\right), \mathsf{Z}\left(\widetilde{\mathsf{A}}_{2,2}, z_0\right)\right\}\\
&= \mathsf{Z}(\widetilde{\mathsf{p}}(z), z_0).
\end{aligned}$$

In the third equality, we used the fact that $\widetilde{\mathsf{A}}(z)$ is Hermitian, that is, $\widetilde{\mathsf{A}}^\star_{1,1} = \widetilde{\mathsf{A}}_{1,1}$, $\widetilde{\mathsf{A}}^\star_{1,2} = \widetilde{\mathsf{A}}_{2,1}$, $\widetilde{\mathsf{A}}^\star_{2,1} = \widetilde{\mathsf{A}}_{1,2}$, and $\widetilde{\mathsf{A}}^\star_{2,2} = \widetilde{\mathsf{A}}_{2,2}$. According to item (3) of Lemma 4.1.1, we can see that the above equality implies that there exist some $\widetilde{c} \in \mathbb{Z}$ and $\widetilde{\epsilon} \in \mathbb{T}$, such that $\widetilde{\mathsf{p}}(z) = \widetilde{\epsilon} z^{\widetilde{c}} \widetilde{\mathsf{p}}^\star(z)$. Since $\mathsf{S}\widetilde{\mathsf{p}}(z) = 1$, we know that fsupp$(\widetilde{\mathsf{p}}(z))$ is a symmetric interval with center 0. Hence, fsupp$(\widetilde{\mathsf{p}}^\star(z)) =$ fsupp$(\widetilde{\mathsf{p}}(z))$. This implies that $\widetilde{c} = 0$. If $\widetilde{\epsilon} \neq 1$, without loss of generality, we can redefine $\widetilde{\mathsf{p}}(z) := \widetilde{\epsilon}^{-1/2}\widetilde{\mathsf{p}}(z)$. Then $\widetilde{\mathsf{p}}^\star(z) = \widetilde{\mathsf{p}}(z)$. Notice that multiplying the constant $\widetilde{\epsilon}^{-1/2}$ does not change the symmetry type of $\widetilde{\mathsf{p}}(z)$, that is, we still have $\mathsf{S}\widetilde{\mathsf{p}}(z) = 1$. Therefore, we proved that $\widetilde{\mathsf{p}}(z)$ satisfies the item (1) of the Theorem 4.3.5. To prove that $\widetilde{\mathsf{p}}(z)$ has the DOS property, we only need to prove that it also satisfies the item (2) of the Theorem 4.3.5.

Now, we can define $\mathring{\mathsf{A}}(z) := \frac{1}{\widetilde{\mathsf{p}}(z)}\widetilde{\mathsf{A}}(z)$. Since $\widetilde{\mathsf{p}}^\star(z) = \widetilde{\mathsf{p}}(z)$, we know that $\mathring{\mathsf{A}}(z)$ is a Hermitian matrix of Laurent polynomials. Also, by $\mathsf{S}\widetilde{\mathsf{p}}(z) = 1$, we can see that $\mathring{\mathsf{A}}(z)$ has compatible symmetry with $\mathsf{S}\mathring{\mathsf{A}}(z) = \mathsf{S}\widetilde{\mathsf{A}}(z)$. Also, (4.3.45) holds. From Lemma 4.3.4, we know that there exists a Laurent polynomial $\mathring{\mathsf{d}}(z)$ with symmetry, such that $\det(\mathring{\mathsf{A}}(z)) = -\mathring{\mathsf{d}}(z)\mathring{\mathsf{d}}^\star(z)$. Together with the fact that $\gcd\left(\mathring{\mathsf{A}}_{1,1}(z), \mathring{\mathsf{A}}_{1,2}(z), \mathring{\mathsf{A}}_{2,1}(z), \mathring{\mathsf{A}}_{2,2}(z)\right) = 1$, which satisfies the DOS condition with respect to any type, we know that the matrix $\mathring{\mathsf{A}}(z)$ satisfies both the conditions in items (1) and (2). So, by to the sufficiency part of the theorem, which has already been established above, we know that there exists a 2×2 matrix $\mathring{\mathsf{U}}(z)$ of Laurent polynomials with compatible symmetry, such that $\mathring{\mathsf{A}}(z) = \mathring{\mathsf{U}} \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \mathring{\mathsf{U}}^\star(z)$.

Using the factorization $\mathsf{A}(z) = \mathsf{U}(z) \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \mathsf{U}^\star(z)$, we have

$$\begin{aligned} \mathsf{U}(z) \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \mathsf{U}^\star(z) =& \mathsf{A}(z) = \mathsf{F}(z)\widetilde{\mathsf{A}}(z)\mathsf{F}^\star(z) = \widetilde{\mathsf{p}}(z)\mathsf{F}(z)\mathring{\mathsf{A}}(z)\mathsf{F}^\star(z) \\ =& \widetilde{\mathsf{p}}(z)\mathsf{F}(z)\mathring{\mathsf{U}}(z) \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \mathring{\mathsf{U}}^\star(z) \\ =& \widetilde{\mathsf{p}}(z)\widetilde{\mathsf{U}}(z) \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \widetilde{\mathsf{U}}^\star(z), \end{aligned}$$

where $\widetilde{\mathsf{U}}(z) := \mathsf{F}(z)\mathring{\mathsf{U}}(z)$ has compatible symmetry. Define $\mathsf{Q}(z) := \mathrm{adj}(\widetilde{\mathsf{U}}(z))\mathsf{U}(z)$, and $\widetilde{\mathsf{d}}(z) := \det(\widetilde{\mathsf{U}}(z))$. Then $\mathsf{Q}(z)$ has compatible symmetry and $\widetilde{\mathsf{d}}(z)$ has symmetry. We can see from the above equation that

$$\begin{aligned} \mathsf{Q}(z) \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \mathsf{Q}^\star(z) =& \mathrm{adj}(\widetilde{\mathsf{U}}(z))\mathsf{U}(z) \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \mathsf{U}^\star(z)\,\mathrm{adj}(\widetilde{\mathsf{U}}(z))^\star \\ =& \widetilde{\mathsf{p}}(z)\,\mathrm{adj}(\widetilde{\mathsf{U}}(z))\widetilde{\mathsf{U}}(z) \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \widetilde{\mathsf{U}}^\star(z)\,\mathrm{adj}(\widetilde{\mathsf{U}}(z))^\star \end{aligned}$$

$$=\widetilde{\mathsf{p}}(z)\widetilde{\mathsf{d}}(z)\widetilde{\mathsf{d}}^\star(z)\begin{bmatrix}1 & \\ & -1\end{bmatrix}.$$

Checking the top-left entry of the above matrix equation, we get $\widetilde{\mathsf{p}}(z)\widetilde{\mathsf{d}}(z)\widetilde{\mathsf{d}}^\star(z) = \mathsf{Q}_{1,1}(z)\mathsf{Q}^\star_{1,1}(z) - \mathsf{Q}_{1,2}(z)\mathsf{Q}^\star_{1,2}(z)$. Denote $\mathsf{SU}_{1,2}(z) = \epsilon_{1,2}z^{c_{1,2}}$, where $\epsilon_{1,2} \in \{1,-1\}$ and $c_{1,2} \in \mathbb{Z}$. The symmetry type

$$\begin{aligned}\frac{\mathsf{SQ}_{1,1}(z)}{\mathsf{SQ}_{1,2}(z)} =& \frac{\mathsf{SU}_{1,1}(z)}{\mathsf{SU}_{1,2}(z)} = \frac{\mathsf{SU}_{1,1}(z)\mathsf{SU}_{2,2}(z)}{\mathsf{SU}_{1,2}(z)\mathsf{SU}_{2,2}(z)} = \frac{\mathsf{Sd}(z)}{\mathsf{SU}_{1,2}(z)\mathsf{SU}_{2,2}(z)}\\ =& \frac{\mathsf{Sd}(z)}{(\mathsf{SU}_{1,2}(z))^2}\frac{\mathsf{SU}_{1,2}(z)}{\mathsf{SU}_{2,2}(z)} = z^{-2c_{1,2}}\mathsf{Sd}(z)\alpha(z),\end{aligned}$$

where we used $\mathsf{Sd}(z) = \mathsf{SU}_{1,1}(z)\mathsf{SU}_{2,2}(z)$, and $\frac{\mathsf{SU}_{1,2}(z)}{\mathsf{SU}_{2,2}(z)} = \alpha(z)$. Therefore, $\widetilde{\mathsf{p}}(z)\widetilde{\mathsf{d}}(z)\widetilde{\mathsf{d}}^\star(z)$ has the DOS property with respect to type $z^{-2c_{1,2}}\mathsf{Sd}(z)\alpha(z)$. By Theorem 4.3.5, we know that $\widetilde{\mathsf{p}}(z)\widetilde{\mathsf{d}}(z)\widetilde{\mathsf{d}}^\star(z)$ also has the DOS property with respect to type $\mathsf{Sd}(z)\alpha(z)$. More precisely, it satisfies the condition in item (2) of Theorem 4.3.5. For all $z_0 \in \mathbb{R}\setminus\{0\}$, we can check that $\mathsf{Z}(\widetilde{\mathsf{d}}^\star, z_0) = \mathsf{Z}(\widetilde{\mathsf{d}}, \overline{z_0}^{-1}) = \mathsf{Z}(\widetilde{\mathsf{d}}, z_0^{-1}) = \mathsf{Z}(\widetilde{\mathsf{d}}, z_0)$. So, $\mathsf{Z}(\widetilde{\mathsf{d}}(z)\widetilde{\mathsf{d}}^\star(z), z_0) \in 2\mathbb{Z}$, and

$$\operatorname{odd}\left(\mathsf{Z}\left(\widetilde{\mathsf{p}}(z), z_0\right)\right) = \operatorname{odd}\left(\mathsf{Z}\left(\widetilde{\mathsf{p}}(z)\widetilde{\mathsf{d}}(z)\widetilde{\mathsf{d}}^\star(z), z_0\right)\right), \qquad \forall\, z_0 \in \mathbb{R}\setminus\{0\}.$$

Hence, we proved that $\widetilde{\mathsf{p}}(z)$ also satisfies the item (2) of Theorem 4.3.5. Therefore, $\widetilde{\mathsf{p}}(z)$ has the DOS property with respect to type $\mathsf{Sd}(z)\alpha(z)$. This completes the proof of the necessity part of the theorem. ∎

4.4 Construction of Quasi-tight Framelet Filter Banks with Two High-pass Filters and Symmetry

In this section, we will try to characterize quasi-tight framelet filter banks $\{a; b_1, b_2\}_{\Theta,(\epsilon_1,\epsilon_2)}$ with symmetry property. As the case of tight framelets ($\epsilon_1 = \epsilon_2 = 1$) has already been studied in [48, 40], we only investigate the case that $(\epsilon_1, \epsilon_2) = (1, -1)$ here.

Given a low-pass filter a and a moment correcting filter Θ with symmetry, firstly we discuss the possible symmetry types of the high-pass filters.

Theorem 4.4.1. *Let $\{a; b_1, b_2\}_{\Theta,(\epsilon_1,\epsilon_2)}$ be a quasi-tight framelet filter bank, such that all the filters Θ, a, b_1, b_2 are not identically zero and have symmetry:*

$$\mathsf{S}\Theta(z) = 1, \quad \mathsf{Sa}(z) = \epsilon z^c, \quad \mathsf{Sb}_1(z) = \epsilon_1 z^{c_1}, \quad \mathsf{Sb}_2(z) = \epsilon_2 z^{c_2} \tag{4.4.1}$$

for some $\epsilon, \epsilon_1, \epsilon_2 \in \{1, -1\}$, and $c, c_1, c_2 \in \mathbb{Z}$. Then the symmetry centers satisfy

$$c_1 + c \in 2\mathbb{Z}, \qquad c_2 + c \in 2\mathbb{Z}. \tag{4.4.2}$$

Proof. Recall the condition of the quasi-tight framelet filter bank $\{a; b_1, b_2\}_{\Theta,(\epsilon_1,\epsilon_2)}$ requires:

$$\Theta(z^2)\mathsf{a}(z)\mathsf{a}^\star(-z) + \mathsf{b}_1(z)\mathsf{b}_1^\star(-z) - \mathsf{b}_2(z)\mathsf{b}_2^\star(-z) = 0. \tag{4.4.3}$$

For the symmetry types of the three terms in (4.4.3), we have

$$\begin{aligned}
\mathsf{S}\big(\Theta(z^2)\mathsf{a}(z)\mathsf{a}^\star(-z)\big) =& \mathsf{S}\Theta(z^2)\mathsf{Sa}(z)\mathsf{Sa}^\star(-z) = \epsilon z^c \epsilon(-z)^{-c} = (-1)^c, \\
\mathsf{S}\big(\mathsf{b}_1(z)\mathsf{b}_1^\star(-z)\big) =& \mathsf{Sb}_1(z)\mathsf{Sb}_1^\star(-z) = \epsilon_1 z^{c_1} \epsilon_1(-z)^{-c_1} = (-1)^{c_1}, \\
\mathsf{S}\big(\mathsf{b}_2(z)\mathsf{b}_2^\star(-z)\big) =& \mathsf{Sb}_2(z)\mathsf{Sb}_2^\star(-z) = \epsilon_2 z^{c_2} \epsilon_2(-z)^{-c_2} = (-1)^{c_2}.
\end{aligned}$$

Because c, c_1 and c_2 are all integers, at least two of the three numbers $(-1)^c$, $(-1)^{c_1}$, $(-1)^{c_2}$ must be the same. We now prove that all the three numbers must be equal. Suppose $(-1)^c = (-1)^{c_1} \neq (-1)^{c_2}$. Moving the $\mathsf{b}_2(z)\mathsf{b}_2^\star(-z)$ term from the left to the right-hand-side of the equation (4.4.3), we can see that the left-hand-side of the equation has symmetry type $(-1)^c$, and the right-hand-side has symmetry type $(-1)^{c_2}$. Since $\mathsf{b}_2(z)\mathsf{b}_2^\star(-z) \neq 0$, it cannot have different symmetry types. So $(-1)^{c_2} = (-1)^c = (-1)^{c_1}$. The other cases can be proved similarly. ■

Since our construction of quasi-tight framelet filter banks is based on the factorization of the matrix $\mathcal{N}_{\mathsf{a},\Theta|n_b}(z)$ of Laurent polynomials into the polyphase matrix of high-pass filters, we need the following lemma for the symmetry type of coset sequences.

Lemma 4.4.1. *Suppose $u \in l_0(\mathbb{Z})$ has symmetry: $\mathsf{S}\mathsf{u}(z) = \epsilon z^c$, where $\epsilon \in \{-1, 1\}$ and $c \in \mathbb{Z}$. Then the following results are true.*

(1) If c is an even integer, then $u^{[0]}(z)$ and $u^{[1]}(z)$ both have symmetry:

$$\mathsf{S}\mathsf{u}^{[0]}(z) = \epsilon z^{c/2}, \qquad \mathsf{S}\mathsf{u}^{[1]}(z) = \epsilon z^{c/2-1}, \qquad \frac{\mathsf{S}\mathsf{u}^{[0]}(z)}{\mathsf{S}\mathsf{u}^{[1]}(z)} = z.$$

(2) If c is an odd integer, for all $k \in \mathbb{Z}$, we can define

$$\mathsf{P}_k(z) := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & z^k \\ 1 & -z^k \end{bmatrix}, \qquad k \in \mathbb{Z}, \tag{4.4.4}$$

and
$$\begin{bmatrix} \mathsf{v}_1(z) \\ \mathsf{v}_2(z) \end{bmatrix} := \mathsf{P}_k(z) \begin{bmatrix} \mathsf{u}^{[0]}(z) \\ \mathsf{u}^{[1]}(z) \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} \mathsf{u}^{[0]}(z) + z^k \mathsf{u}^{[1]}(z) \\ \mathsf{u}^{[0]}(z) - z^k \mathsf{u}^{[1]}(z) \end{bmatrix}.$$

Then v_1, v_2 have symmetry:

$$\mathsf{S}\mathsf{v}_1(z) = \epsilon z^{\frac{c-1}{2}+k}, \qquad \mathsf{S}\mathsf{v}_2(z) = -\epsilon z^{\frac{c-1}{2}+k}, \qquad \frac{\mathsf{S}\mathsf{v}_1(z)}{\mathsf{S}\mathsf{v}_2(z)} = -1.$$

On the other hand, suppose $v_1, v_2 \in l_0(\mathbb{Z})$ both have symmetry. The following results are true.

(3) If for some $c \in \mathbb{Z}$,

$$\mathsf{S}\mathsf{v}_1(z) = \epsilon z^{c+1}, \qquad \mathsf{S}\mathsf{v}_2(z) = \epsilon z^c, \qquad \frac{\mathsf{S}\mathsf{v}_1(z)}{\mathsf{S}\mathsf{v}_2(z)} = z$$

hold, then $\mathsf{u}(z) := \mathsf{v}_1(z^2) + z\mathsf{v}_2(z^2)$ also has symmetry: $\mathsf{S}\mathsf{u}(z) = \epsilon z^{2c+2}$.

(4) If for some $c \in \mathbb{Z}$,

$$\mathsf{S}\mathsf{v}_1(z) = \epsilon z^c, \qquad \mathsf{S}\mathsf{v}_2(z) = -\epsilon z^c, \qquad \frac{\mathsf{S}\mathsf{v}_1(z)}{\mathsf{S}\mathsf{v}_2(z)} = -1$$

hold, using $\mathsf{P}_k(z)$ *as* (4.4.4)*, we can define*

$$\begin{bmatrix}\mathsf{w}_1(z)\\ \mathsf{w}_2(z)\end{bmatrix} := \mathsf{P}_k^{-1}(z)\begin{bmatrix}\mathsf{v}_1(z)\\ \mathsf{v}_2(z)\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ z^{-k} & -z^{-k}\end{bmatrix}\begin{bmatrix}\mathsf{v}_1(z)\\ \mathsf{v}_2(z)\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}\mathsf{v}_1(z)+\mathsf{v}_2(z)\\ z^{-k}\mathsf{v}_1(z)-z^{-k}\mathsf{v}_2(z)\end{bmatrix}.$$

Then $\mathsf{u}(z) := \mathsf{w}_1(z^2) + z\mathsf{w}_2(z^2)$ *also has symmetry:* $\mathsf{Su}(z) = \epsilon z^{2(c-k)+1}$.

Proof. Notice that for any Laurent polynomial $\mathsf{u}(z)$, we have

$$\mathsf{u}^{[0]}(z^2) = \frac{1}{2}\left[\mathsf{u}(z) + \mathsf{u}(-z)\right], \qquad \mathsf{u}^{[1]}(z^2) = \frac{1}{2}z^{-1}\left[\mathsf{u}(z) - \mathsf{u}(-z)\right].$$

Using these two identities, the 4 items in the lemma can be proved by direct calculations. ■

Based on Theorem 4.3.6, we can give the necessary and sufficient condition for the existence of quasi-tight framelet filter banks with symmetry.

Theorem 4.4.2. *Let* $a, \Theta \in l_0(\mathbb{Z})$ *be given filters having symmetry:* $\mathsf{Sa}(z) = \epsilon z^c$ *and* $\mathsf{S}\Theta(z) = 1$. $\Theta^\star = \Theta$. *Let* n_b *be an integer number satisfying* (2.4.1). *Define* $\mathcal{N}_{\mathsf{a},\Theta|n_b}(z)$ *as in* (2.1.10)*. Then there exists a quasi-tight framelet filter bank* $\{a; b_1, b_2\}_{\Theta,(1,-1)}$ *with symmetry and* n_b *order of vanishing moments if and only if the following two items hold:*

(1) $\det(\mathcal{N}_{\mathsf{a},\Theta|n_b}(z)) = -\mathsf{d}_{n_b}(z)\mathsf{d}^\star_{n_b}(z)$ *for a Laurent polynomial* $\mathsf{d}_{n_b}(z)$ *having symmetry.*

(2) $\mathsf{p}(z)$ *has the DOS (Difference Of Squares) property with respect to symmetry type* $(-1)^{c+n_b} z^{\operatorname{odd}(c+n_b)-1}\mathsf{Sd}_{n_b}(z)$*, where* $\mathsf{p}(z)$ *is defined as*

$$\mathsf{p}_0(z) := \gcd\left(\left[\mathcal{N}_{\mathsf{a},\Theta|n_b}(z)\right]_{1,1}, \left[\mathcal{N}_{\mathsf{a},\Theta|n_b}(z)\right]_{1,2}, \left[\mathcal{N}_{\mathsf{a},\Theta|n_b}(z)\right]_{2,1}, \left[\mathcal{N}_{\mathsf{a},\Theta|n_b}(z)\right]_{2,2}\right),$$
$$\mathsf{p}(z) := \frac{\mathsf{p}_0(z)}{(z-1)^{\mathsf{Z}(\mathsf{p}_0,1)}(z+1)^{\mathsf{Z}(\mathsf{p}_0,-1)}}.$$

In case of $\Theta = \boldsymbol{\delta}$*, we have* $\mathsf{p}_0(z) = \mathsf{p}(z) = 1$*. So this condition is automatically satisfied.*

Proof. Recall the definition of $\mathcal{N}_{\mathsf{a},\Theta|n_b}(z)$

$$\mathcal{N}_{\mathsf{a},\Theta|n_b}(z) := \frac{1}{2}\begin{bmatrix} \mathsf{A}^{[0]}(z)+\mathsf{B}^{[0]}(z) & z(\mathsf{A}^{[1]}(z)-\mathsf{B}^{[1]}(z)) \\ \mathsf{A}^{[1]}(z)+\mathsf{B}^{[1]}(z) & \mathsf{A}^{[0]}(z)-\mathsf{B}^{[0]}(z) \end{bmatrix}, \tag{4.4.5}$$

where

$$\mathsf{A}(z) := \frac{\Theta(z)-\Theta(z^2)\mathsf{a}(z)\mathsf{a}^\star(z)}{(1-z)^{n_b}(1-z^{-1})^{n_b}}, \qquad \mathsf{B}(z) := \frac{-\Theta(z^2)\mathsf{a}(z)\mathsf{a}^\star(-z)}{(1+z)^{n_b}(1-z^{-1})^{n_b}}. \tag{4.4.6}$$

From the symmetry types of $\mathsf{a}(z)$ and $\Theta(z)$, it is easy to calculate $\mathsf{SA}(z) = 1$ and $\mathsf{SB}(z) = (-1)^{c+n_b}$. By Lemma 4.4.1(1), we know that

$$\mathsf{SA}^{[0]}(z) = 1, \quad \mathsf{SA}^{[1]}(z) = z^{-1}, \quad \mathsf{SB}^{[0]}(z) = (-1)^{c+n_b}, \quad \mathsf{SB}^{[1]}(z) = (-1)^{c+n_b}z^{-1}. \tag{4.4.7}$$

Firstly, we prove the necessity part of the theorem. Suppose there exists a quasi-tight framelet filter bank $\{a; b_1, b_2\}_{\Theta,(1,-1)}$ with symmetry, where $\mathsf{Sb}_1(z) = \epsilon_1 z^{c_1}$ and $\mathsf{Sb}_2(z) = \epsilon_2 z^{c_2}$. Since both high-pass filters have at least n_b order of vanishing moments, we can write

$$\mathsf{b}_\ell(z) = (1-z^{-1})^{n_b}\mathring{\mathsf{b}}_\ell(z), \qquad \ell = 1, 2, \tag{4.4.8}$$

for some Laurent polynomials $\mathring{\mathsf{b}}_1(z)$ and $\mathring{\mathsf{b}}_2(z)$. From the symmetry types of b_1 and b_2, we can see $\mathsf{S}\mathring{\mathsf{b}}_1(z) = \epsilon_1 z^{c_1+n_b}$ and $\mathsf{S}\mathring{\mathsf{b}}_2(z) = \epsilon_2 z^{c_2+n_b}$.

If $c + n_b \in 2\mathbb{Z}$, we know from (4.4.7) that $\mathsf{S}\mathcal{N}_{\mathsf{a},\Theta|n_b}(z) = \begin{bmatrix} 1 & z \\ z^{-1} & 1 \end{bmatrix}$. By Theorem 4.4.1, we can see that $c+n_b \in 2\mathbb{Z}$ implies $c_1+n_b \in 2\mathbb{Z}$ and $c_2+n_b \in 2\mathbb{Z}$. Hence, according to the symmetry types of $\mathring{\mathsf{b}}_1, \mathring{\mathsf{b}}_2$ and Lemma 4.4.1(1), we know that

$$\mathsf{U}(z) := \begin{bmatrix} \mathring{\mathsf{b}}_1^{[0]}(z) & \mathring{\mathsf{b}}_2^{[0]}(z) \\ \mathring{\mathsf{b}}_1^{[1]}(z) & \mathring{\mathsf{b}}_2^{[1]}(z) \end{bmatrix} \tag{4.4.9}$$

has compatible symmetry $\mathsf{SU}(z) = \begin{bmatrix} \epsilon_1 z^{(c_1+n_b)/2} & \epsilon_2 z^{(c_2+n_b)/2} \\ \epsilon_1 z^{(c_1+n_b)/2-1} & \epsilon_2 z^{(c_2+n_b)/2-1} \end{bmatrix}$. According to Theorem 2.1.2, $\{a; b_1, b_2\}_{\Theta,(1,-1)}$ is a quasi-tight framelet filter bank with n_b

order of vanishing moments implies that

$$\mathcal{N}_{\mathsf{a},\Theta|n_b}(z) = \mathsf{U}(z)\begin{bmatrix}1 & \\ & -1\end{bmatrix}\mathsf{U}^\star(z) \tag{4.4.10}$$

holds. That is, $\mathcal{N}_{\mathsf{a},\Theta|n_b}(z)$ has a spectral factorization with symmetry as in Theorem 4.3.6. From the necessity part of Theorem 4.3.6, we know that the items (1) and (2) hold.

If $c+n_b \in 2\mathbb{Z}+1$, for any integer k, we can take $\mathsf{P}_k(z)$ as (4.4.4), and define $\mathsf{N}(z) := \mathsf{P}_k(z)\mathcal{N}_{\mathsf{a},\Theta|n_b}\mathsf{P}_k^\star(z)$. From (4.4.5) and (4.4.7), it is straightforward to calculate that $\mathsf{N}(z)$ has compatible symmetry: $\mathsf{SN}(z) = \begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$. Denote $\mathsf{U}(z)$ as in (4.4.9). By Theorem 4.4.1, we can see that $c+n_b \in 2\mathbb{Z}+1$ implies $c_1+n_b \in 2\mathbb{Z}+1$ and $c_2+n_b \in 2\mathbb{Z}+1$. Hence, according to the symmetry types of $\mathring{\mathsf{b}}_1, \mathring{\mathsf{b}}_2$ and Lemma 4.4.1(2), we know that $\widetilde{\mathsf{U}}(z) := \mathsf{P}_k(z)\mathsf{U}(z)$ has compatible symmetry: $\mathsf{S}\widetilde{\mathsf{U}}(z) = \begin{bmatrix}\epsilon_1 z^{\frac{c_1+n_b-1}{2}+k} & \epsilon_2 z^{\frac{c_2+n_b-1}{2}+k}\\ -\epsilon_1 z^{\frac{c_1+n_b-1}{2}+k} & -\epsilon_2 z^{\frac{c_2+n_b-1}{2}+k}\end{bmatrix}$. For the quasi-tight framelet filter bank $\{a; b_1, b_2\}_{\Theta,(1,-1)}$, (4.4.10) must hold. (4.4.10) implies that

$$\begin{aligned}\mathsf{N}(z) =& \mathsf{P}_k(z)\mathcal{N}_{\mathsf{a},\Theta|n_b}\mathsf{P}_k^\star(z) = \mathsf{P}_k(z)\mathsf{U}(z)\begin{bmatrix}1 & \\ & -1\end{bmatrix}\mathsf{U}^\star(z)\mathsf{P}_k^\star(z)\\ =& \widetilde{\mathsf{U}}(z)\begin{bmatrix}1 & \\ & -1\end{bmatrix}\widetilde{\mathsf{U}}^\star(z).\end{aligned}$$

That is, $\mathsf{N}(z)$ has a spectral factorization with symmetry as in Theorem 4.3.6. Notice that $\mathsf{P}_k(z)$ is a unimodular matrix, i.e., the Smith Normal Form of $\mathsf{N}(z)$ and $\mathcal{N}_{\mathsf{a},\Theta|n_b}(z)$ are the same. So the gcd of the 4 elements in $\mathsf{N}(z)$ and $\det(\mathsf{N}(z))$, up to multiplications by some monomial, must be the same as those of $\mathcal{N}_{\mathsf{a},\Theta|n_b}(z)$. Therefore, from the necessity part of Theorem 4.3.6, we know that the items (1) and (2) in the theorem hold. This proves the necessity part of the theorem.

Now we will prove the sufficiency part.

For the case that $c+n_b \in 2\mathbb{Z}$, from (4.4.5) and (4.4.7), we know that

$\mathsf{S}\mathcal{N}_{\mathsf{a},\Theta|n_b}(z) = \begin{bmatrix} 1 & z \\ z^{-1} & 1 \end{bmatrix}$. From the sufficiency part of Theorem 4.3.6, the two items in the theorem imply that there exists a 2×2 matrix $\mathsf{U}(z)$ of Laurent polynomials with compatible symmetry, such that (4.4.10) holds, and $\frac{\mathsf{SU}_{1,1}(z)}{\mathsf{SU}_{2,1}(z)} = \frac{\mathsf{SU}_{1,2}(z)}{\mathsf{SU}_{2,2}(z)} = z$. Define $\mathring{\mathsf{b}}_1(z)$ and $\mathring{\mathsf{b}}_2(z)$ as $\begin{bmatrix} \mathring{\mathsf{b}}_1^{[0]}(z) & \mathring{\mathsf{b}}_2^{[0]}(z) \\ \mathring{\mathsf{b}}_1^{[1]}(z) & \mathring{\mathsf{b}}_2^{[1]}(z) \end{bmatrix} := \mathsf{U}(z) = \begin{bmatrix} \mathsf{U}_{1,1}(z) & \mathsf{U}_{1,2}(z) \\ \mathsf{U}_{2,1}(z) & \mathsf{U}_{2,2}(z) \end{bmatrix}$. According to Lemma 4.4.1(3), $\mathring{\mathsf{b}}_1$ and $\mathring{\mathsf{b}}_2$ both have symmetry. Define filers b_1 and b_2 as (4.4.8), we know that b_1 and b_2 are high-pass filters with symmetry as well. According to Theorem 2.1.2, $\{a; b_1, b_2\}_{\Theta,(1,-1)}$ is a quasi-tight framelet filter bank with symmetry and n_b order of vanishing moments.

For the case that $c + n_b \in 2\mathbb{Z} + 1$, for any integer k, we can take $\mathsf{P}_k(z)$ as (4.4.4), and define $\mathsf{N}(z) := \mathsf{P}_k(z)\mathcal{N}_{\mathsf{a},\Theta|n_b}\mathsf{P}_k^\star(z)$. From (4.4.5) and (4.4.7), it is straightforward to calculate that $\mathsf{N}(z)$ has compatible symmetry: $\mathsf{SN}(z) = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. Since $\mathsf{N}(z)$ and $\mathcal{N}_{\mathsf{a},\Theta|n_b}(z)$ has the same Smith Normal Form, we know that $\mathsf{N}(z)$ satisfies the 2 conditions in Theorem 4.3.6 with $\alpha(z) = -1$. By the sufficiency part of Theorem 4.3.6, we can find a 2×2 matrix $\widetilde{\mathsf{U}}(z)$ of Laurent polynomials with compatible symmetry, such that $\mathsf{N}(z) = \widetilde{\mathsf{U}}(z) \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \widetilde{\mathsf{U}}^\star(z)$ holds, and $\frac{\mathsf{SU}_{1,1}(z)}{\mathsf{SU}_{2,1}(z)} = \frac{\mathsf{SU}_{1,2}(z)}{\mathsf{SU}_{2,2}(z)} = -1$. Define $\mathring{\mathsf{b}}_1$ and $\mathring{\mathsf{b}}_2$ as $\begin{bmatrix} \mathring{\mathsf{b}}_1^{[0]}(z) & \mathring{\mathsf{b}}_2^{[0]}(z) \\ \mathring{\mathsf{b}}_1^{[1]}(z) & \mathring{\mathsf{b}}_2^{[1]}(z) \end{bmatrix} := \mathsf{P}_k^{-1}(z)\widetilde{\mathsf{U}}(z)$. According to Lemma 4.4.1(4), $\mathring{\mathsf{b}}_1$ and $\mathring{\mathsf{b}}_2$ both have symmetry. Define filers b_1 and b_2 as (4.4.8), we know that b_1 and b_2 are high-pass filters with symmetry as well. Since $\mathsf{P}_k(z)$ is invertible and

$$\begin{aligned} \mathsf{P}_k(z)\mathcal{N}_{\mathsf{a},\Theta|n_b}\mathsf{P}_k^\star(z) =& \mathsf{N}(z) = \widetilde{\mathsf{U}}(z) \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \widetilde{\mathsf{U}}^\star(z) \\ =& \mathsf{P}_k(z) \begin{bmatrix} \mathring{\mathsf{b}}_1^{[0]}(z) & \mathring{\mathsf{b}}_2^{[0]}(z) \\ \mathring{\mathsf{b}}_1^{[1]}(z) & \mathring{\mathsf{b}}_2^{[1]}(z) \end{bmatrix} \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \begin{bmatrix} \mathring{\mathsf{b}}_1^{[0]}(z) & \mathring{\mathsf{b}}_2^{[0]}(z) \\ \mathring{\mathsf{b}}_1^{[1]}(z) & \mathring{\mathsf{b}}_2^{[1]}(z) \end{bmatrix}^\star \mathsf{P}_k^\star(z), \end{aligned}$$

we know that $\mathcal{N}_{\mathsf{a},\Theta|n_b}(z) = \begin{bmatrix} \mathring{\mathsf{b}}_1^{[0]}(z) & \mathring{\mathsf{b}}_2^{[0]}(z) \\ \mathring{\mathsf{b}}_1^{[1]}(z) & \mathring{\mathsf{b}}_2^{[1]}(z) \end{bmatrix} \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \begin{bmatrix} \mathring{\mathsf{b}}_1^{[0]}(z) & \mathring{\mathsf{b}}_2^{[0]}(z) \\ \mathring{\mathsf{b}}_1^{[1]}(z) & \mathring{\mathsf{b}}_2^{[1]}(z) \end{bmatrix}^\star$ holds. According to Theorem 2.1.2, $\{a; b_1, b_2\}_{\Theta,(1,-1)}$ is a quasi-tight framelet filter bank with symmetry and n_b order of vanishing moments. This completes the proof of the sufficient part of the theorem. ■

4.5 Illustrative Examples

Example 4.1. *Choose* $\Theta = \boldsymbol{\delta}$ *and low-pass filter*

$$\mathsf{a}(z) = -\tfrac{1}{16}\left(z^2 - 6\,z + 1\right)(1+z)^2 z^{-2} + \lambda z^{-3}(1+z)^2(1-z)^4,$$

where $\lambda = -\frac{3}{32} + \frac{\sqrt{2}}{16}$. *We have* $\mathrm{sm}_2 = 1.0193$, *and* $\mathsf{Sa}(z) = 1$. *Notice that* $\mathrm{sr}(\mathsf{a}) = 2$ *and* $\mathrm{vm}(1 - \mathsf{a}\mathsf{a}^\star) = 4$. *Take* $n_b = 2$, *then the constructed quasi-tight framelet filter bank* $\{a; b_1, b_2\}_{\Theta,\{1,-1\}}$ *is given by:*

$$\begin{aligned}\mathsf{b}_1(z) = \tfrac{(z-1)^2}{2048}\Big[&(4-3\sqrt{2})(z^3+z^{-3}) - 2\sqrt{2}(z^2+z^{-2}) \\ &-(2068-1559\sqrt{2})(z+z^{-1}) + 1084\sqrt{2}\Big],\\ \mathsf{b}_2(z) = \tfrac{(z-1)^2}{2048}\Big[&(3\sqrt{2}-4)(z^3+z^{-3}) + 2\sqrt{2}(z^2+z^{-2}) \\ &-(2028-1513\sqrt{2})(z+z^{-1}) + 964\sqrt{2}\Big].\end{aligned}$$

We have $\mathrm{vm}(b_1) = \mathrm{vm}(b_2) = 2$. *The symmetry types are* $\mathsf{Sb}_1(z) = \mathsf{Sb}_2(z) = z^2$.

Example 4.2. *Choose* $\Theta = \boldsymbol{\delta}$ *and* $\lambda = 0$ *in the above example. We get the low-pass filter* $a = \left\{-\frac{1}{16}, \frac{1}{4}, \frac{5}{8}, \frac{1}{4}, -\frac{1}{16}\right\}_{[-2,2]}$. *We have* $\mathrm{sm}_2 = 0.8853$, *and* $\mathsf{Sa}(z) = 1$. *Notice that* $\mathrm{sr}(\mathsf{a}) = 2$ *and* $\mathrm{vm}(1 - \mathsf{a}\mathsf{a}^\star) = 4$. *Take* $n_b = 2$, *then the constructed quasi-tight framelet filter bank* $\{a; b_1, b_2\}_{\Theta,\{1,-1\}}$ *is given by:*

$$b_1 = \tfrac{\sqrt{2}}{4}\{-1, 2, -1\}_{[0,2]}, \qquad b_2 = \tfrac{1}{16}\{1, -4, 6, -4, 1\}_{[-2,2]}.$$

We have $\mathrm{vm}(b_1) = 2$ *and* $\mathrm{vm}(b_2) = 4$. *The symmetry types are* $\mathsf{Sb}_1(z) = z^2$ *and* $\mathsf{Sb}_2(z) = 1$.

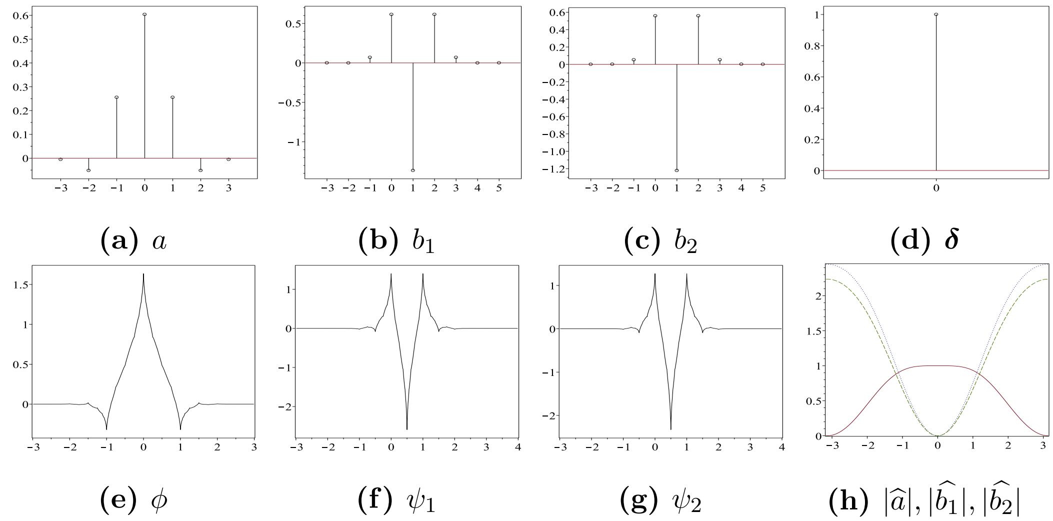


Figure 4.1: In Example 4.1: (a),(b),(c) and (d) are the graphs of the filters a, b_1, b_2 and Θ. (e) scaling function ϕ. (f) wavelet function ψ_1. (g) wavelet function ψ_2. (h) magnitude of $|\widehat{a}(\xi)|$ (in solid line), $|\widehat{b}_1(\xi)|$ (in dotted line) and $|\widehat{b}_2(\xi)|$ (in dashed line) in for $\xi \in [-\pi, \pi]$.

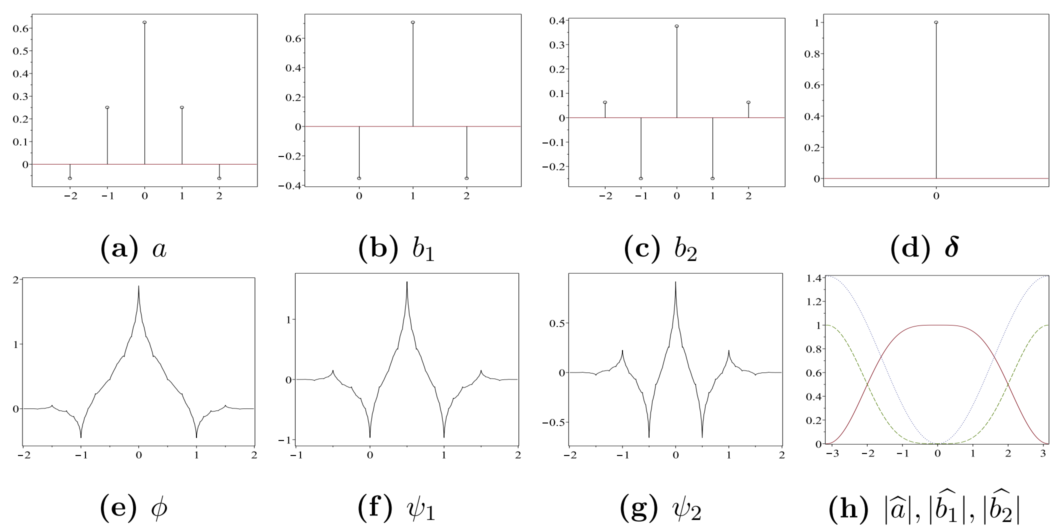


Figure 4.2: In Example 4.2: (a),(b),(c) and (d) are the graphs of the filters a, b_1, b_2 and Θ. (e) scaling function ϕ. (f) wavelet function ψ_1. (g) wavelet function ψ_2. (h) magnitude of $|\widehat{a}(\xi)|$ (in solid line), $|\widehat{b}_1(\xi)|$ (in dotted line) and $|\widehat{b}_2(\xi)|$ (in dashed line) in for $\xi \in [-\pi, \pi]$.

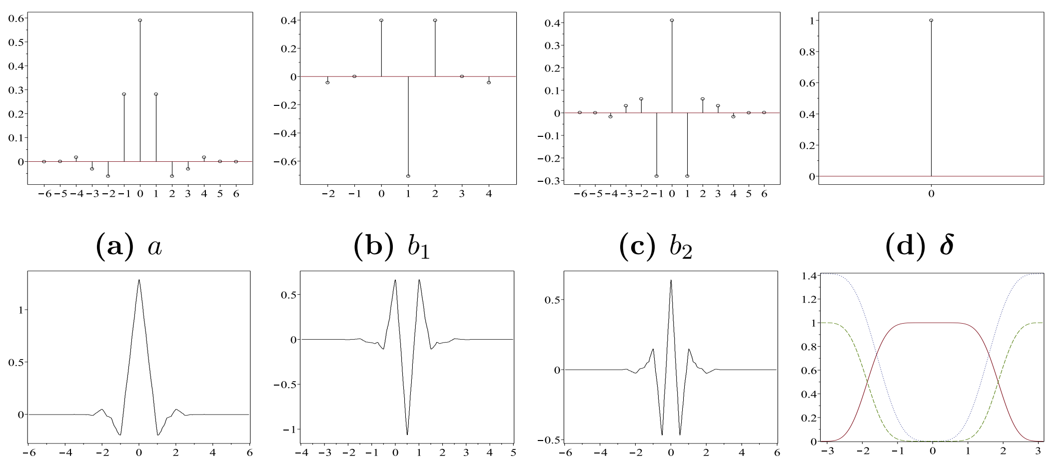
Example 4.3. *Choose* $\Theta = \boldsymbol{\delta}$ *and the low-pass filter*

$$a = \frac{1}{1024}\{-1, 0, 18, -32, -63, 288, 604, 288, -63, -32, 18, 0, -1\}_{[-6,6]}.$$

We have $\mathrm{sm}_2 = 1.6821$*, and* $\mathsf{Sa}(z) = 1$*. Notice that* $\mathrm{sr}(\mathsf{a}) = 4$ *and* $\mathrm{vm}(1 - \mathsf{a}\mathsf{a}^\star) = 8$*. Take* $n_b = 4$*, then the constructed quasi-tight framelet filter bank* $\{a; b_1, b_2\}_{\Theta,\{1,-1\}}$ *is given by:*

$$b_1 = \tfrac{\sqrt{2}}{32}\{-1, 0, 9, -16, 9, 0, -1\}_{[-2,4]},$$
$$b_2 = \tfrac{1}{1024}\{1, 0, -18, 32, 63, -288, 420, -288, 63, 32, -18, 0, 1\}_{[-6,6]}.$$

We have $\mathrm{vm}(b_1) = 4$ *and* $\mathrm{vm}(b_2) = 8$*. The symmetry types are* $\mathsf{Sb}_1(z) = z^2$ *and* $\mathsf{Sb}_2(z) = 1$*.*



(a) a **(b)** b_1 **(c)** b_2 **(d)** $\boldsymbol{\delta}$

(e) ϕ **(f)** ψ_1 **(g)** ψ_2 **(h)** $|\widehat{a}|, |\widehat{b_1}|, |\widehat{b_2}|$

Figure 4.3: In Example 4.3: (a),(b),(c) and (d) are the graphs of the filters a, b_1, b_2 and Θ. (e) scaling function ϕ. (f) wavelet function ψ_1. (g) wavelet function ψ_2. (h) magnitude of $|\widehat{a}(\xi)|$ (in solid line), $|\widehat{b_1}(\xi)|$ (in dotted line) and $|\widehat{b_2}(\xi)|$ (in dashed line) in for $\xi \in [-\pi, \pi]$.

Example 4.4. *Choose* $\Theta = \boldsymbol{\delta}$ *and the low-pass filter*

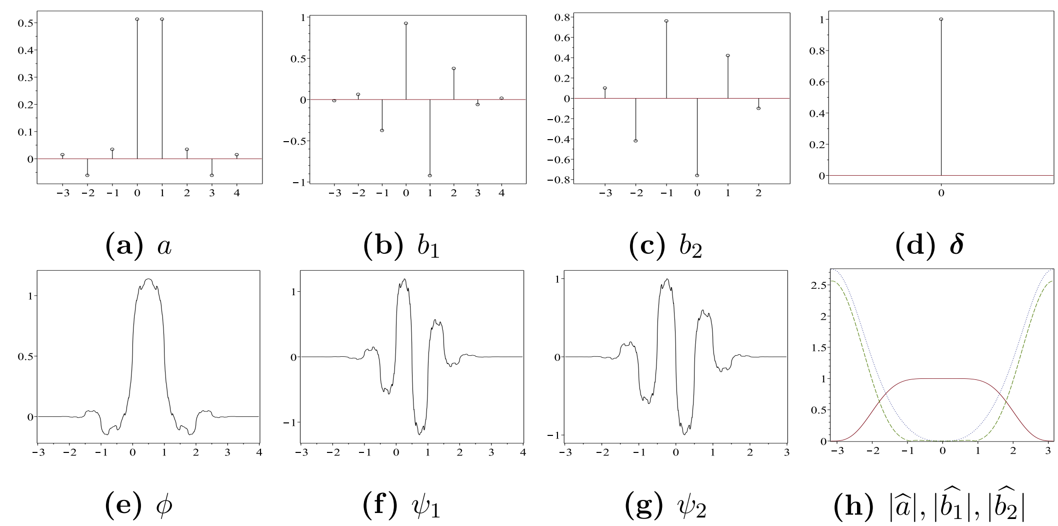
$$a = \tfrac{1}{1024}\{15, -63, 35, 525, 525, 35, -63, 15\}_{[-3,4]}.$$

We have $\mathrm{sm}_2 = 1.1543$*, and* $\mathsf{Sa}(z) = z$*. Notice that* $\mathrm{sr}(\mathsf{a}) = 3$ *and* $\mathrm{vm}(1 - \mathsf{a}\mathsf{a}^\star) = 6$*. Take* $n_b = 3$*, then the constructed quasi-tight framelet filter bank* $\{a; b_1, b_2\}_{\Theta,\{1,-1\}}$ *is given by:*

$$b_1 = \tfrac{1}{1024}\{-15, 63, -385, 945, -945, 385, -63, 15\}_{[-3,4]},$$

$$b_2 = \frac{\sqrt{105}}{512}\{5, -21, 38, -38, 21, -5\}_{[-1,4]}.$$

We have $\mathrm{vm}(b_1) = \mathrm{vm}(b_2) = 3$*. The symmetry types are* $\mathsf{S}\mathsf{b}_1(z) = -z$ *and* $\mathsf{S}\mathsf{b}_2(z) = -z^3$*.*



(a) a (b) b_1 (c) b_2 (d) $\boldsymbol{\delta}$

(e) ϕ (f) ψ_1 (g) ψ_2 (h) $|\widehat{a}|, |\widehat{b_1}|, |\widehat{b_2}|$

Figure 4.4: In Example 4.4: (a),(b),(c) and (d) are the graphs of the filters a, b_1, b_2 and Θ. (e) scaling function ϕ. (f) wavelet function ψ_1. (g) wavelet function ψ_2. (h) magnitude of $|\widehat{a}(\xi)|$ (in solid line), $|\widehat{b}_1(\xi)|$ (in dotted line) and $|\widehat{b}_2(\xi)|$ (in dashed line) in for $\xi \in [-\pi, \pi]$.

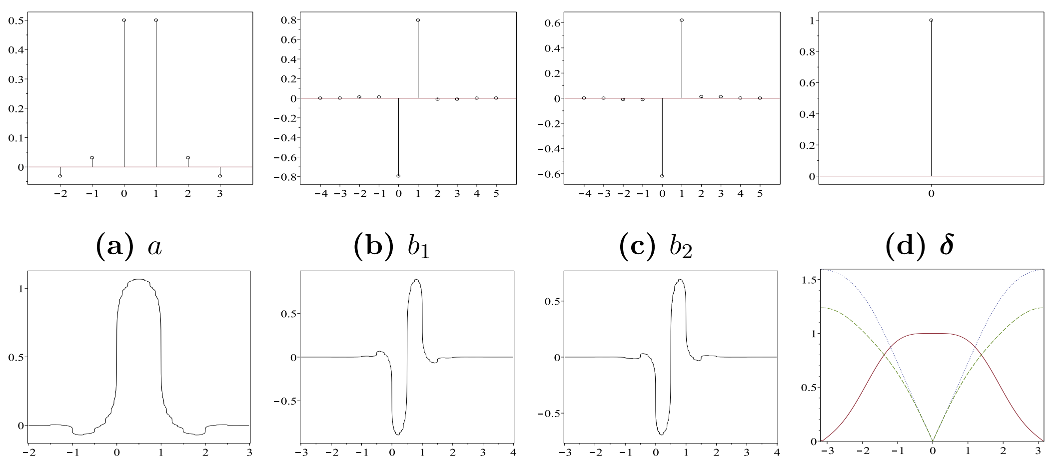
Example 4.5. *Choose* $\Theta = \boldsymbol{\delta}$ *and the low-pass filter*

$$a = \frac{1}{32}\left\{-1, 1, 16, 16, 1, -1\right\}_{[-2,3]}.$$

We have $\mathrm{sm}_2 = 0.7184$*, and* $\mathsf{Sa}(z) = z$*. Notice that* $\mathrm{sr}(\mathsf{a}) = 1$ *and* $\mathrm{vm}(1 - \mathsf{a}\mathsf{a}^\star) = 4$*. Take* $n_b = 1$*, then the constructed quasi-tight framelet filter bank* $\{a; b_1, b_2\}_{\Theta,\{1,-1\}}$ *is given by:*

$$b_1 = \frac{\sqrt{2}}{4096}\{-1, 1, 32, 32, -2302, 2302, -32, -32, -1, 1\}_{[-4,5]},$$
$$b_2 = \frac{\sqrt{2}}{4096}\{1, -1, -32, -32, -1794, 1794, 32, 32, 1, -1\}_{[-4,5]}.$$

We have $\mathrm{vm}(b_1) = \mathrm{vm}(b_2) = 1$*. The symmetry types are* $\mathsf{S}\mathsf{b}_1(z) = \mathsf{S}\mathsf{b}_2(z) = -z$*.*



(a) a **(b)** b_1 **(c)** b_2 **(d)** $\boldsymbol{\delta}$

(e) ϕ **(f)** ψ_1 **(g)** ψ_2 **(h)** $|\widehat{a}|, |\widehat{b_1}|, |\widehat{b_2}|$

Figure 4.5: In Example 4.5: (a),(b),(c) and (d) are the graphs of the filters a, b_1, b_2 and Θ. (e) scaling function ϕ. (f) wavelet function ψ_1. (g) wavelet function ψ_2. (h) magnitude of $|\widehat{a}(\xi)|$ (in solid line), $|\widehat{b_1}(\xi)|$ (in dotted line) and $|\widehat{b_2}(\xi)|$ (in dashed line) in for $\xi \in [-\pi, \pi]$.

Chapter 5

Quasi-tight Framelets in High Dimension

In this chapter, we study multivariate quasi-tight M-framelets derived from an arbitrary refinable function $\phi \in L_2(\mathbb{R}^d)$. Separable multivariate wavelets and framelets with a diagonal dilation matrix M can be trivially constructed through tensor product from one-dimensional wavelets and framelets. However, such separable wavelets and framelets are known to give preferences to the axis coordinate directions and they are only a very special family of multivariate wavelets and framelets. It is important but often challenging to study nonseparable/general multivariate wavelets and framelets in both theory and applications. Currently, there is a growing interest in wavelet analysis on studying and constructing (nonseparable) multivariate wavelets and framelets. As we reviewed in Chapter 1, construction of multivariate wavelets and framelets are widely known as a challenging problem in the literature. In this chapter, we show that given an arbitrary M-refinable function $\phi \in L_2(\mathbb{R}^d)$ derived from some real-valued low-pass filter $a \in l_0(\mathbb{Z}^d)$, we can easily construct quasi-tight M-framelets with either of the two properties:

(1) directionality;

(2) highest possible order of vanishing moments.

Although the constructions are still valid with arbitrary real-valued moment correcting filter $\Theta \in l_0(\mathbb{Z}^d)$ such that $\widehat{\Theta}(0) = 1$ and $\Theta^\star = \Theta$, for simplicity, we state the results in this chapter with $\Theta = \boldsymbol{\delta}$, i.e., $\widehat{\Theta}(\xi) = 1$. Such framelets are called Unitary Extension Principle based framelets (See [72, 73]).

This chapter is organized as follows. In Section 5.1, we shall prove Theorem 5.1.1, which shows that given an arbitrary real-valued low-pass filter $a \in l_0(\mathbb{Z}^d)$ such that $\widehat{a}(0) = 1$, we can always construct a quasi-tight M-framelet filter bank with directionality, where all the high-pass filters are supported on only two points. We will also obtain a general result on factorizing a Hermite matrix of $2\pi\mathbb{Z}^d$-periodic trigonometric polynomials. In Section 5.2, we study the quasi-tight framelets with high order of vanishing moments. By proving some auxiliary results first, we shall prove Theorem 5.2.1, which shows that given an arbitrary real-valued low-pass filter $a \in l_0(\mathbb{Z}^d)$ such that $\widehat{a}(0) = 1$, we can always construct a quasi-tight M-framelet filter bank with highest possible order of vanishing moments

$$\min(\mathrm{vm}(b_1), \ldots, \mathrm{vm}(b_s)) \leqslant \min(\mathrm{sr}(a, \mathsf{M}), \tfrac{1}{2}\,\mathrm{vm}(u_a)), \quad \text{with } \widehat{u_a}(\xi) := 1 - |\widehat{a}(\xi)|^2. \tag{5.0.1}$$

We prove Theorem 5.2.1 with two constructive methods (algorithms). In Section 5.3, we provide several examples of quasi-tight framelets and quasi-tight framelet filter banks with directionality/high order of vanishing moments.

The results in this chapter are summarized in [20].

5.1 Multivariate Quasi-tight Framelets with Directionality

Multivariate framelets with directionality have been studied extensively in the literature, due to their wide applications in signal (such as image) processing. For example, natural images (2D signals) can have quite complicated structures such as edges and textures. In order to make the framelets capture the directional structure in the signal, a lot of framelets have been designed in the literature, such as Curvelets, Shearlets, Contourlets, DT-$\mathbb{C}$WF, TP-$\mathbb{C}$TF, etc. To only mention a few, see [4, 5, 34, 53, 33, 67, 81, 55, 60, 50, 61, 52, 2, 51] and many references therein.

With quasi-tight framelets, we can easily achieve directionality by the following theorem.

Theorem 5.1.1. *Let* M *be a* $d \times d$ *dilation matrix and let* $a \in l_0(\mathbb{Z}^d)$ *be a finitely supported real-valued sequence on* $\mathbb{Z}^d$ *satisfying the basic sum rule condition (*$\operatorname{sr}(a, \mathsf{M}) \geqslant 1$*). Then there always exist finitely supported real-valued high-pass filters* $b_1, \ldots, b_s \in l_0(\mathbb{Z}^d)$ *and* $\epsilon_1, \ldots, \epsilon_s \in \{-1, 1\}$ *such that* $\{a; b_1, \ldots, b_s\}_{(\epsilon_1,\ldots,\epsilon_s)}$ *is a quasi-tight* M*-framelet filter bank and every high-pass filter* b_ℓ *takes the form* $c_\ell(\boldsymbol{\delta}_{\alpha_\ell} - \boldsymbol{\delta}_{\beta_\ell})$ *for some* $c_\ell \in \mathbb{R}$ *and* $\alpha_\ell, \beta_\ell \in \operatorname{supp}(a)$ *(hence* b_ℓ *naturally has directionality, basic vanishing moments and symmetry property) for all* $\ell = 1, \ldots, s$*. Moreover,*

(1) if in addition $\phi \in L_2(\mathbb{R}^d)$*, then* $\{\phi; \psi^1, \ldots, \psi^s\}_{(\epsilon_1,\ldots,\epsilon_s)}$ *is a (directional) quasi-tight* M*-framelet in* $L_2(\mathbb{R}^d)$*, where*

$$\widehat{\phi}(\xi) := \prod_{j=1}^{\infty} \widehat{a}((\mathsf{M}^{\mathsf{T}})^{-j}\xi) \quad \textit{and} \quad \widehat{\psi^\ell}(\mathsf{M}^{\mathsf{T}}\xi) := \widehat{b_\ell}(\xi)\widehat{\phi}(\xi),$$
$$\xi \in \mathbb{R}^d, \ell = 1, \ldots, s; \quad (5.1.1)$$

(2) if in addition the following condition holds:

$$\textit{all the coefficients in } \overline{\widehat{a^{[\gamma_j]}}(\xi)}\widehat{a^{[\gamma_k]}}(\xi) \textit{ are nonnegative for all } j, k = 1, \ldots, \mathsf{d}_{\mathsf{M}}, \quad (5.1.2)$$

(for example, the above condition in (5.1.2) *is satisfied if the filter* a *has nonnegative coefficients.) then we can take* $\epsilon_1 = \cdots = \epsilon_s = 1$ *and therefore,* $\{a; b_1, \ldots, b_s\}$ *is a (directional) tight* M*-framelet filter bank and* $\{\phi; \psi^1, \ldots, \psi^s\}$ *is a (directional) tight* M*-framelet in* $L_2(\mathbb{R}^d)$*.*

Before the proof of the theorem, recall that for a real number c, the sign function is defined to be

$$\operatorname{sgn}(c) := \begin{cases} 1 & \text{if } c > 0, \\ 0 & \text{if } c = 0, \\ -1 & \text{if } c < 0. \end{cases} \quad (5.1.3)$$

Proof of Theorem 5.1.1. By definition, we notice that $\{a; b_1, \ldots, b_s\}_{(\epsilon_1,\ldots,\epsilon_s)}$ is a quasi-tight M-framelet filter bank if and only if

$$\sum_{\ell=1}^{s} \epsilon_\ell \mathsf{d_M} \left[\widehat{b_\ell^{[\gamma_1]}}(\xi), \ldots, \widehat{b_\ell^{[\gamma_{\mathsf{d_M}}]}}(\xi)\right]^\star \left[\widehat{b_\ell^{[\gamma_1]}}(\xi), \ldots, \widehat{b_\ell^{[\gamma_{\mathsf{d_M}}]}}(\xi)\right] = \mathcal{N}_a(\xi) \tag{5.1.4}$$

with

$$\mathcal{N}_a(\xi) := I_{\mathsf{d_M}} - \mathsf{d_M} \left[\widehat{a^{[\gamma_1]}}(\xi), \ldots, \widehat{a^{[\gamma_{\mathsf{d_M}}]}}(\xi)\right]^\star \left[\widehat{a^{[\gamma_1]}}(\xi), \ldots, \widehat{a^{[\gamma_{\mathsf{d_M}}]}}(\xi)\right]. \tag{5.1.5}$$

We now construct the desired high-pass filters $b_1, \ldots, b_s$ by a recursive algorithm. Let $\mathcal{N} = \mathcal{N}_a$. The main idea of the following proof has three steps: (1) Eliminate the nonzero terms in the off-diagonal entries of $\mathcal{N}$ one by one so that all the off-diagonal entries in the updated $\mathcal{N}$ are identically zero. (2) Eliminate the nonzero nonconstant terms in the diagonal entries of the updated $\mathcal{N}$ one by one so that the final updated $\mathcal{N}$ is a constant diagonal matrix. (3) Prove that the constant diagonal matrix $\mathcal{N}$ is the zero matrix.

Let $\ell := 1$. Suppose that some of the off-diagonal entries $[\mathcal{N}]_{j,k}$ are not identically zero for some $1 \leqslant j < k \leqslant \mathsf{d_M}$. Then $[\mathcal{N}]_{j,k}$ has a nonzero term $ce^{-i\gamma\cdot\xi}$ with $c \neq 0$ and $\gamma \in \mathbb{Z}^d$. By the definition of $\mathcal{N}$, we observe that the term $ce^{-i\gamma\cdot\xi}$ must appear as one of the terms in $-\overline{\widehat{a^{[\gamma_j]}}(\xi)}\widehat{a^{[\gamma_k]}}(\xi)$. Therefore, there must exist $\alpha, \beta \in \mathbb{Z}^d$ such that $a^{[\gamma_j]}(\alpha)a^{[\gamma_k]}(\beta) \neq 0$ (i.e., $\{\gamma_j + \mathsf{M}\alpha, \gamma_k + \mathsf{M}\beta\} \subset \operatorname{supp}(a)$) and $\beta - \alpha = \gamma$. Define

$$b_\ell := \sqrt{|c|/\mathsf{d_M}}(\boldsymbol{\delta}_{\gamma_j+\mathsf{M}\alpha} - \boldsymbol{\delta}_{\gamma_k+\mathsf{M}\beta}), \qquad \epsilon_\ell := -\operatorname{sgn}(c). \tag{5.1.6}$$

If the additional condition in (5.1.2) holds, then c must be a negative number and hence, $\epsilon_\ell = 1$. It follows directly from the definition of b_ℓ in (5.1.6) that

$$\widehat{b_\ell^{[\gamma_j]}}(\xi) = \sqrt{|c|/\mathsf{d_M}}e^{-i\alpha\cdot\xi}, \quad \widehat{b_\ell^{[\gamma_k]}}(\xi) = -\sqrt{|c|/\mathsf{d_M}}e^{-i\beta\cdot\xi} \quad \text{and} \quad \widehat{b_\ell^{[\gamma_p]}}(\xi) = 0, \quad \forall\, p \in \{1, \ldots, \mathsf{d_M}\}\backslash\{j, k\}.$$

Consequently, the $\mathsf{d_M} \times \mathsf{d_M}$ matrix

$$B_\ell(\xi) := \epsilon_\ell \mathsf{d_M} \left[\widehat{b_\ell^{[\gamma_1]}}(\xi), \ldots, \widehat{b_\ell^{[\gamma_{\mathsf{d_M}}]}}(\xi)\right]^\star \left[\widehat{b_\ell^{[\gamma_1]}}(\xi), \ldots, \widehat{b_\ell^{[\gamma_{\mathsf{d_M}}]}}(\xi)\right] \qquad (5.1.7)$$

has only four nonzero entries with

$$\begin{aligned}
&[B_\ell(\xi)]_{j,j} = \epsilon_\ell \mathsf{d_M} |\widehat{b_\ell^{[\gamma_j]}}(\xi)|^2 = \epsilon_\ell |c| = -c, \\
&[B_\ell(\xi)]_{k,k} = \epsilon_\ell \mathsf{d_M} |\widehat{b_\ell^{[\gamma_k]}}(\xi)|^2 = \epsilon_\ell |c| = -c, \\
&[B_\ell(\xi)]_{j,k} = \epsilon_\ell \mathsf{d_M} \overline{\widehat{b_\ell^{[\gamma_j]}}(\xi)} \widehat{b_\ell^{[\gamma_k]}}(\xi) = -\epsilon_\ell |c| e^{-i(\beta-\alpha)\cdot\xi} = c e^{-i\gamma\cdot\xi}, \\
&[B_\ell(\xi)]_{k,j} = \epsilon_\ell \mathsf{d_M} \overline{\widehat{b_\ell^{[\gamma_k]}}(\xi)} \widehat{b_\ell^{[\gamma_j]}}(\xi) = -\epsilon_\ell |c| e^{-i(\alpha-\beta)\cdot\xi} = c e^{i\gamma\cdot\xi}.
\end{aligned} \qquad (5.1.8)$$

Now replace/update $\mathcal{N}$ by $\mathcal{N} - B_\ell$ and replace ℓ by $\ell + 1$ (i.e., increase ℓ by one). Because $\mathcal{N}^\star(\xi) = \mathcal{N}(\xi)$, by the above four identities, we conclude that the term $ce^{-i\gamma\cdot\xi}$ does not appear in the (j,k)-entry of the updated $\mathcal{N}$ and only the constant terms in the diagonal entries of the previous $\mathcal{N}$ are modified. Hence, we can repeat this procedure until all the off-diagonal entries in $\mathcal{N}$ are identically zero.

Now we deal with the diagonal matrix $\mathcal{N}$. Suppose that some of the diagonal entries $[\mathcal{N}]_{j,j}$ are not constant for some $1 \leqslant j \leqslant \mathsf{d_M}$. Then $[\mathcal{N}]_{j,j}$ has a nonzero nonconstant term $ce^{-i\gamma\cdot\xi}$ with $c \neq 0$ and $\gamma \in \mathbb{Z}^d \backslash \{0\}$. By the definition and construction of $\mathcal{N}$, we observe that the term $ce^{-i\gamma\cdot\xi}$ must appear as one of the terms in $-\overline{\widehat{a^{[\gamma_j]}}(\xi)}\widehat{a^{[\gamma_j]}}(\xi)$. Therefore, there must exist $\alpha, \beta \in \mathbb{Z}^d$ such that $a^{[\gamma_j]}(\alpha) a^{[\gamma_j]}(\beta) \neq 0$ (i.e., $\{\gamma_j + \mathsf{M}\alpha, \gamma_j + \mathsf{M}\beta\} \subset \operatorname{supp}(a)$) and $\beta - \alpha = \gamma \neq 0$. Define

$$b_\ell := \sqrt{|c|/\mathsf{d_M}}(\boldsymbol{\delta}_{\gamma_j + \mathsf{M}\alpha} - \boldsymbol{\delta}_{\gamma_j + \mathsf{M}\beta}), \qquad \epsilon_\ell := -\operatorname{sgn}(c). \qquad (5.1.9)$$

If the additional condition in (5.1.2) holds, then c must be a negative number and hence, $\epsilon_\ell = 1$. It follows directly from the definition of b_ℓ in (5.1.9) that

$$\widehat{b_\ell^{[\gamma_j]}}(\xi) = \sqrt{|c|/\mathsf{d_M}}(e^{-i\alpha\cdot\xi} - e^{-i\beta\cdot\xi}) \quad \text{and} \quad \widehat{b_\ell^{[\gamma_p]}}(\xi) = 0 \qquad \forall\, p \in \{1, \ldots, \mathsf{d_M}\} \backslash \{j\}.$$

Consequently, the $\mathsf{d_M} \times \mathsf{d_M}$ matrix $B_\ell(\xi)$ defined in (5.1.7) has only one nonzero entry at

$$[B_\ell(\xi)]_{j,j} = \epsilon_\ell \mathsf{d_M} |\widehat{b_\ell^{[\gamma_j]}}(\xi)|^2 = \epsilon_\ell |c|(2 - e^{-i(\alpha-\beta)\cdot\xi} - e^{i(\alpha-\beta)\cdot\xi}) = c(e^{-i\gamma\cdot\xi} + e^{i\gamma\cdot\xi} - 2). \tag{5.1.10}$$

Now replace/update $\mathcal{N}$ by $\mathcal{N} - B_\ell$ and replace ℓ by $\ell+1$ (i.e., increase ℓ by one). Because $\mathcal{N}^\star(\xi) = \mathcal{N}(\xi)$, by the above identity, we conclude that both the term $ce^{-i\gamma\cdot\xi}$ and $ce^{i\gamma\cdot\xi}$ do not appear in the (j,j)-entry of the updated $\mathcal{N}$ and only the constant term in the (j,j)-entry of the previous $\mathcal{N}$ is modified. We can repeat this procedure until all the nonzero nonconstant terms in the updated $\mathcal{N}$ are identically zero. We set $s := \ell - 1$.

Therefore, we end up with

$$\mathcal{N}_a(\xi) = \mathcal{N} + \sum_{\ell=1}^{s} B_\ell(\xi) \tag{5.1.11}$$

such that $\mathcal{N}$ is a diagonal matrix of constants. We now prove that $\mathcal{N} = 0$. Note that both (5.1.8) and (5.1.10) trivially imply that the sum of every row of $B_\ell(0)$ must be zero. Since the filter a satisfies the basic sum rule condition $\operatorname{sr}(a, \mathsf{M}) \geqslant 1$, we can see that

$$\sum_{k\in\mathbb{Z}^d} a(\gamma + \mathsf{M}k) = |\det(\mathsf{M})|^{-1}, \qquad \forall\, \gamma \in \Gamma_{\mathsf{M}} := [\mathsf{M}[0,1)^d] \cap \mathbb{Z}^d. \tag{5.1.12}$$

From (5.1.12), we have $\widehat{a^{[\gamma_1]}}(0) = \cdots = \widehat{a^{[\gamma_{\mathsf{d_M}}]}}(0) = \mathsf{d_M}^{-1}$. Now we trivially deduce from the definition of $\mathcal{N}_a$ in (5.1.5) that all the diagonal entries of $\mathcal{N}_a(0)$ are $1 - \mathsf{d_M}^{-1}$ and all the off-diagonal entries of $\mathcal{N}_a(0)$ are $-\mathsf{d_M}^{-1}$. Consequently the sum of every row of $\mathcal{N}_a(0)$ is $(1-\mathsf{d_M}^{-1}) + (\mathsf{d_M}-1)(-\mathsf{d_M}^{-1}) = 0$. Therefore, we conclude from (5.1.11) that the sum of every row of $\mathcal{N}$ must be zero. However, $\mathcal{N}$ is a diagonal matrix of constants and thus, we must have $\mathcal{N} = 0$. Since $\mathcal{N} = 0$, by our definition of B_ℓ in (5.1.7) and using (5.1.11), we conclude that (5.1.4) is satisfied and $\{a; b_1, \ldots, b_s\}_{(\epsilon_1,\ldots,\epsilon_s)}$ is a quasi-tight M-framelet filter bank.

If the additional condition in (5.1.2) is satisfied, by our above construction we have $\epsilon_1 = \cdots = \epsilon_s = 1$ and hence $\{a; b_1, \ldots, b_s\}$ is a tight M-framelet filter bank. ■

For $u \in l_0(\mathbb{Z}^d)$, by $N(\widehat{u})$ we denote the total number of nonzero terms in the $2\pi\mathbb{Z}^d$-periodic trigonometric polynomial $\widehat{u}$. That is, $N(\widehat{u}) = \#\mathrm{supp}(u)$, the cardinality of the support of the filter u. From the above proof of Theorem 5.1.1, it is not difficult to conclude that the number s of high-pass filters in Theorem 5.1.1 is given by

$$s = \sum_{1 \leqslant j < k \leqslant \mathsf{d_M}} N\left(\overline{\widehat{a^{[\gamma_j]}}(\xi)}\widehat{a^{[\gamma_k]}}(\xi)\right) + \frac{1}{2}\sum_{j=1}^{\mathsf{d_M}} \left(N\left(|\widehat{a^{[\gamma_j]}}(\xi)|^2\right) - 1\right).$$

Moreover, by the special structure of the high-pass filters in Theorem 5.1.1, we also have $s \leqslant \binom{\#\,\mathrm{supp}(a)}{2}$.

Suppose that $\{a; b_1, \ldots, b_s\}_{(\epsilon_1,\ldots,\epsilon_s)}$ is a quasi-tight M-framelet filter bank. Then it is trivial to observe that $\{a; b_1(\cdot + \mathsf{M}k_1), \ldots, b_s(\cdot + \mathsf{M}k_s)\}_{(\epsilon_1,\ldots,\epsilon_s)}$ are quasi-tight M-framelet filter banks for all $k_1, \ldots, k_s \in \mathbb{Z}^d$. If $b_1 = cb_2$ for some $c \in \mathbb{R}$, then $\{a; \sqrt{|\epsilon_1 c^2 + \epsilon_2|}b_2, \ldots, b_s\}_{(\mathrm{sgn}(\epsilon_1 c^2+\epsilon_2),\epsilon_3,\ldots,\epsilon_s)}$ is also a quasi-tight M-framelet filter bank with at most $s-1$ number of high-pass filters. Up to the above trivial variants, from our proof of Theorem 5.1.1, one can conclude that the constructed quasi-tight M-framelet filter bank in Theorem 5.1.1 is essentially unique.

To get a tight M-framelet filter bank, the above proof of Theorem 5.1.1 shows that the additional condition in (5.1.2) can be weakened by requiring that all the coefficients in $\overline{\widehat{a^{[\gamma_j]}}(\xi)}\widehat{a^{[\gamma_k]}}(\xi)$ are nonnegative for all $1 \leqslant j < k \leqslant \mathsf{d_M}$ and all the nonconstant coefficients in $|\widehat{a^{[\gamma_j]}}(\xi)|^2$ are nonnegative for all $1 \leqslant j \leqslant \mathsf{d_M}$. If all the coefficients of a low-pass filter a are nonnegative, then it is trivial that the additional condition in (5.1.2) is satisfied. Consequently, Theorem 5.1.1 recovers or improves the special construction of directional tight $2I_d$-framelets and tight $2I_d$-framelet filter banks in [47, 64] for box spline refinable functions, because all the low-pass filters for box spline refinable functions have nonnegative coefficients.

For $1 \leqslant j \leqslant r$, by e_j we denote the jth unit coordinate column vector in $\mathbb{R}^r$, i.e., e_j is the $r \times 1$ vector with its only nonzero element being 1 at the

jth entry. Using the idea in the proof of Theorem 5.1.1, we now establish a general result similar to the matrix spectral factorization.

Theorem 5.1.2. *Let $\mathcal{A}$ be an $r \times r$ matrix of $2\pi\mathbb{Z}^d$-periodic trigonometric polynomials in d variables with real coefficients such that $\mathcal{A}^\star(\xi) = \mathcal{A}(\xi)$ for all $\xi \in \mathbb{R}^d$. For $1 \leqslant j \leqslant r$, let κ_r be the sum of the jth column of $\mathcal{A}(0)$, i.e., $(\kappa_1, \dots, \kappa_r) := (1, \dots, 1)\mathcal{A}(0)$. Then there exist $\epsilon_0, \dots, \epsilon_s \in \{-1, 1\}$ and $r \times 1$ vectors $\mathbf{u}_1, \dots, \mathbf{u}_s$ of $2\pi\mathbb{Z}^d$-periodic trigonometric polynomials with real coefficients such that*

$$\mathcal{A}(\xi) = sgn(\kappa_1)|\kappa_1|e_1e_1^\star + \dots + sgn(\kappa_r)|\kappa_r|e_re_r^\star + \varepsilon_1\mathbf{u}_1(\xi)\mathbf{u}_1^\star(\xi) + \dots + \varepsilon_s\mathbf{u}_s(\xi)\mathbf{u}_s^\star(\xi), \tag{5.1.13}$$

and each vector function of $\mathbf{u}_1, \dots, \mathbf{u}_s$ has only two nonzero entries with one being $c_\ell e^{-i\alpha_\ell\cdot\xi}$ and the other being $-c_\ell e^{-i\beta_\ell\cdot\xi}$ for some $c_\ell \in \mathbb{R}$ and $\alpha_\ell, \beta_\ell \in \mathbb{Z}^d$. In terms of the matrix form, (5.1.13) *can be rewritten as*

$$\mathcal{A}(\xi) = \mathcal{B}(\xi) \begin{bmatrix} sgn(\kappa_1) & & & & & \\ & \ddots & & & & \\ & & sgn(\kappa_r) & & & \\ & & & \varepsilon_1 & & \\ & & & & \ddots & \\ & & & & & \varepsilon_s \end{bmatrix} \mathcal{B}^\star(\xi), \tag{5.1.14}$$

where $\mathcal{B}(\xi) := [\sqrt{|\kappa_1|}e_1, \dots, \sqrt{|\kappa_r|}e_r, \mathsf{u}_1, \dots, \mathsf{u}_s]$ is an $r\times(r+s)$ matrix obtained by putting all the column vectors $e_1, \dots, e_r, \mathsf{u}_1, \dots, \mathsf{u}_s$ together.

Proof. To prove the claim, we construct the desired vectors $\mathbf{u}_1, \dots, \mathbf{u}_s$ of $2\pi\mathbb{Z}^d$-periodic trigonometric polynomials by following a similar recursive algorithm as in the proof of Theorem 5.1.1.

Let $\ell := 1$ and $\mathcal{N} := \mathcal{A}$. Suppose that some of the off-diagonal entries $[\mathcal{N}]_{j,k}$ are not identically zero for some $1 \leqslant j < k \leqslant r$. Then $[\mathcal{N}]_{j,k}$ has a nonzero term $ce^{-i\gamma\cdot\xi}$ with $c \neq 0$ and $\gamma \in \mathbb{Z}^d$. Define

$$\mathbf{u}_\ell(\xi) := \sqrt{|c|}e_j - \sqrt{|c|}e^{i\gamma\cdot\xi}e_k \quad \text{and} \quad \varepsilon_\ell := -\text{sgn}(c).$$

Consequently, the $r \times r$ matrix $\varepsilon_\ell \mathbf{u}_\ell(\xi)\mathbf{u}_\ell^\star(\xi)$ has only four nonzero entries $-c, -c, ce^{-i\gamma\cdot\xi}$ and $ce^{i\gamma\cdot\xi}$ at the positions $(j,j), (k,k), (j,k)$ and (k,j), respectively. Now replace/update $\mathcal{N}$ by $\mathcal{N} - \varepsilon_\ell \mathbf{u}_\ell(\xi)\mathbf{u}_\ell^\star(\xi)$ and replace ℓ by $\ell+1$ (i.e., increase ℓ by one). Because $\mathcal{N}^\star(\xi) = \mathcal{N}(\xi)$, we conclude that the term $ce^{-i\gamma\cdot\xi}$ does not appear in the (j,k)-entry of the updated $\mathcal{N}$ and only the constant terms in the diagonal entries of the previous $\mathcal{N}$ are modified. Hence, we can repeat this procedure until all the off-diagonal entries in $\mathcal{N}$ are identically zero.

Now we deal with the diagonal matrix $\mathcal{N}$. Suppose that some of the diagonal entries $[\mathcal{N}]_{j,j}$ are not constant for some $1 \leqslant j \leqslant \mathsf{d_M}$. Then $[\mathcal{N}]_{j,j}$ has a nonzero nonconstant term $ce^{-i\gamma\cdot\xi}$ with $c \neq 0$ and $\gamma \in \mathbb{Z}^d \backslash \{0\}$. Define

$$\mathbf{u}_\ell(\xi) := \sqrt{|c|}(1 - e^{-i\gamma\cdot\xi})e_j \quad \text{and} \quad \varepsilon_\ell := -\operatorname{sgn}(c).$$

Then $\varepsilon_\ell \mathbf{u}_\ell(\xi)\mathbf{u}_\ell^\star(\xi)$ has only one nonzero entry $c(e^{-i\gamma\cdot\xi} + e^{i\gamma\cdot\xi} - 2)$ at the (j,j)-entry. Now replace/update $\mathcal{N}$ by $\mathcal{N} - \varepsilon_\ell \mathbf{u}_\ell(\xi)\mathbf{u}_\ell^\star(\xi)$ and replace ℓ by $\ell + 1$. Because $\mathcal{N}^\star(\xi) = \mathcal{N}(\xi)$, we conclude that both the term $ce^{-i\gamma\cdot\xi}$ and $ce^{i\gamma\cdot\xi}$ do not appear in the (j,j)-entry of the updated $\mathcal{N}$ and only the constant term in the (j,j)-entry of the previous $\mathcal{N}$ is modified. We can repeat this procedure until all the nonzero nonconstant terms in the updated $\mathcal{N}$ are identically zero. We set $s := \ell - 1$.

Therefore, we end up with $\mathcal{A}(\xi) = \mathcal{N} + \sum_{\ell=1}^{s} \varepsilon_\ell \mathbf{u}_\ell(\xi)\mathbf{u}_\ell^\star(\xi)$ such that $\mathcal{N}$ is a diagonal matrix of constants. Note that the sum of every column in $\varepsilon_\ell \mathbf{u}_\ell(0)\mathbf{u}_\ell^\star(0)$ is zero. Consequently, we must have $\mathcal{N} = \operatorname{diag}(\kappa_1, \ldots, \kappa_r)$. This completes the proof. ∎

Though the sum of Hermitian squares of $2\pi\mathbb{Z}^d$-periodic trigonometric polynomials is a challenging problem in real algebraic geometric, as a direct consequence of Theorem 5.1.2, we have the following result on quasi-sum of Hermitian squares of $2\pi\mathbb{Z}^d$-periodic trigonometric polynomials.

Corollary 5.1.3. *Let $\mathcal{A}$ be a $2\pi\mathbb{Z}^d$-periodic trigonometric polynomial in d variables with real coefficients such that $\overline{\mathcal{A}(\xi)} = \mathcal{A}(\xi)$. Then there exist $\varepsilon_1, \ldots, \varepsilon_s \in \{-1, 1\}$ and $2\pi\mathbb{Z}^d$-periodic trigonometric polynomials $\mathbf{u}_1, \ldots, \mathbf{u}_s$ with real co-*

efficients such that

$$\mathcal{A}(\xi) = sgn(\mathcal{A}(0))(\sqrt{|\mathcal{A}(0)|})^2 + \varepsilon_1|\mathbf{u}_1(\xi)|^2 + \cdots + \varepsilon_s|\mathbf{u}_s(\xi)|^2, \qquad \forall\, \xi \in \mathbb{R}^d \tag{5.1.15}$$

and each function $\mathbf{u}_1, \ldots, \mathbf{u}_s$ *has only two nonzero entries with one being* $c_\ell e^{-i\alpha_\ell\cdot\xi}$ *and the other being* $-c_\ell e^{-i\beta_\ell\cdot\xi}$ *for some* $c_\ell \in \mathbb{R}$ *and* $\alpha_\ell, \beta_\ell \in \mathbb{Z}^d$.

We will provide a few examples to illustrate Theorem 5.1.1 in Section 5.3. Before finishing this section, we show an example demonstrating that the condition in (5.1.2) can be also satisfied by some low-pass filters having negative coefficients as well.

Example 5.1. For $d = 1$ and $\mathsf{M} = 2$, we consider the following low-pass filter

$$a = \{\tfrac{5}{29}, \tfrac{5}{29}, -\tfrac{1}{58}, \underline{-\tfrac{\mathbf{1}}{\mathbf{58}}}, \tfrac{5}{29}, \tfrac{5}{29}, \tfrac{5}{29}, \tfrac{5}{29}\}_{[-3,4]}.$$

Then clearly a satisfies the basic sum rules and the additional condition in (5.1.2) is satisfied: By the definition of the filter a, we have

$$\overline{\widehat{a^{[0]}}(\xi)}\widehat{a^{[1]}}(\xi) = \tfrac{1}{3364}(100e^{-2i\xi} + 90e^{-i\xi} + 80 + 301e^{i\xi} + 80e^{2i\xi} + 90e^{3i\xi} + 100e^{4i\xi}),$$
$$|\widehat{a^{[0]}}(\xi)|^2 = |\widehat{a^{[1]}}(\xi)|^2 = \tfrac{25}{841}e^{-3i\xi} + \tfrac{45}{1682}e^{-2i\xi} + \tfrac{20}{841}e^{-i\xi} + \tfrac{301}{3364} + \tfrac{20}{841}e^{i\xi} + \tfrac{45}{1682}e^{2i\xi} + \tfrac{25}{841}e^{3i\xi}.$$

Note that not all coefficients of the filter a are nonnegative but (5.1.2) is satisfied. By Theorem 5.1.1, we obtain a tight 2-framelet filter bank $\{a, b_1, \ldots, b_{13}\}$ given by

$$b_1 = \{\underline{\tfrac{\mathbf{2\sqrt{5}}}{\mathbf{29}}}, -\tfrac{2\sqrt{5}}{29}\}_{[0,1]}, \quad b_2 = \{\underline{\tfrac{\mathbf{2\sqrt{5}}}{\mathbf{29}}}, 0, -\tfrac{2\sqrt{5}}{29}\}_{[0,2]}, \quad b_3 = \{\tfrac{2\sqrt{5}}{29}, \underline{\mathbf{0}}, -\tfrac{2\sqrt{5}}{29}\}_{[-1,1]},$$
$$b_4 = \{\tfrac{\sqrt{301}}{58}, \underline{-\tfrac{\mathbf{\sqrt{301}}}{\mathbf{58}}}\}_{[-1,0]}, \quad b_5 = \{\tfrac{2\sqrt{5}}{29}, \underline{\mathbf{0}}, 0, -\tfrac{2\sqrt{5}}{29}\}_{[-1,2]},$$
$$b_6 = \{\tfrac{3\sqrt{10}}{58}, 0, \underline{\mathbf{0}}, -\tfrac{3\sqrt{10}}{58}\}_{[-2,1]}, \quad b_7 = \{\tfrac{3\sqrt{10}}{58}, 0, \underline{\mathbf{0}}, 0, -\tfrac{3\sqrt{10}}{58}\}_{[-2,2]},$$
$$b_8 = \{\tfrac{3\sqrt{10}}{58}, \underline{\mathbf{0}}, 0, 0, -\tfrac{3\sqrt{10}}{58}\}_{[-1,3]}, \quad b_9 = \{\tfrac{3\sqrt{10}}{58}, \underline{\mathbf{0}}, 0, 0, 0, -\tfrac{3\sqrt{10}}{58}\}_{[-1,4]},$$
$$b_{10} = \{\tfrac{5}{29}, 0, \underline{\mathbf{0}}, 0, 0, -\tfrac{5}{29}\}_{[-2,3]}, \quad b_{11} = \{\tfrac{5}{29}, 0, \underline{\mathbf{0}}, 0, 0, 0, -\tfrac{5}{29}\}_{[-2,4]},$$
$$b_{12} = \{\tfrac{5}{29}, 0, 0, \underline{\mathbf{0}}, 0, 0, -\tfrac{5}{29}\}_{[-3,3]}, \quad b_{13} = \{\tfrac{5}{29}, 0, 0, \underline{\mathbf{0}}, 0, 0, 0, -\tfrac{5}{29}\}_{[-3,4]}.$$

By calculation, we have $\mathrm{sm}(a, 2) \approx 0.992335$. Then $\{\phi; \psi^1, \ldots, \psi^{13}\}$ is a tight

2-framelet in $L_2(\mathbb{R})$, where $\phi, \psi^1, \ldots, \psi^{13}$ are defined in (5.1.1) with $\mathsf{M} = 2$ and $s = 13$.

5.2 Multivariate Quasi-tight Framelets with High Order of Vanishing Moments

Due to the special structure/construction, all the directional high-pass filters in Theorem 5.1.1 have only order one vanishing moment. Due to the importance of vanishing moments, it is natural and important to ask whether one can construct a quasi-tight M-framelet filter bank achieving the highest possible order $\min(\operatorname{sr}(a, \mathsf{M}), \frac{1}{2}\operatorname{vm}(u_a))$ vanishing moments in (5.0.1) with $\widehat{u_a}(\xi) := 1 - |\widehat{a}(\xi)|^2$. This question is satisfactorily answered by the following result, for which we shall provide two constructive different proofs.

Theorem 5.2.1. *Let M be a $d \times d$ dilation matrix and let $a \in l_0(\mathbb{Z}^d)$ be a finitely supported real-valued sequence on $\mathbb{Z}^d$ satisfying the basic sum rule condition ($\operatorname{sr}(a, \mathsf{M}) \geqslant 1$). Then there always exist finitely supported real-valued high-pass filters $b_1, \ldots, b_s \in l_0(\mathbb{Z}^d)$ and $\epsilon_1, \ldots, \epsilon_s \in \{-1, 1\}$ such that $\{a; b_1, \ldots, b_s\}_{(\epsilon_1, \ldots, \epsilon_s)}$ is a quasi-tight M-framelet filter bank and all the high-pass filters $b_1, \ldots, b_s$ have order m vanishing moments with $m := \min(\operatorname{sr}(a, \mathsf{M}), \frac{1}{2}\operatorname{vm}(u_a)) \geqslant 1$ and $\widehat{u_a}(\xi) = 1 - |\widehat{a}(\xi)|^2$. Define $\phi, \psi^1, \ldots, \psi^s$ as in (5.1.1). If $\phi \in L_2(\mathbb{R}^d)$, then $\{\phi; \psi^1, \ldots, \psi^s\}_{(\epsilon_1, \ldots, \epsilon_s)}$ is a quasi-tight M-framelet in $L_2(\mathbb{R}^d)$ such that all the generators $\psi^1, \ldots, \psi^s$ have at least order m vanishing moments.*

To prove Theorem 5.2.1, we need a few auxiliary results and recall some necessary notations. For $k \in \mathbb{Z}^d$ and $u \in l_0(\mathbb{Z}^d)$, the difference operator $\nabla_k u$ is defined to be $\nabla_k u := u - u(\cdot - k)$. For $\nu = (\nu_1, \ldots, \nu_d)^\mathsf{T} \in \mathbb{N}_0^d$, we define $\nabla^\nu := \nabla_{e_1}^{\nu_1} \cdots \nabla_{e_d}^{\nu_d}$. Recall that $\boldsymbol{\delta}$ is the Dirac sequence such that $\boldsymbol{\delta}(0) = 1$ and $\boldsymbol{\delta}(k) = 0$ for all $k \in \mathbb{Z}^d \backslash \{0\}$. Therefore, for $u \in l_0(\mathbb{Z}^d)$,

$$\widehat{\nabla^\nu u}(\xi) = \widehat{\nabla^\nu \boldsymbol{\delta}}(\xi)\widehat{u}(\xi) = (1 - e^{-i\xi_1})^{\nu_1} \cdots (1 - e^{-i\xi_d})^{\nu_d}\widehat{u}(\xi), \qquad \xi = (\xi_1, \ldots, \xi_d)^\mathsf{T} \in \mathbb{R}^d. \quad (5.2.1)$$

The following result is known in [37, Theorem 3.6] and [45, Lemma 7.2.2]. For the convenience of the reader, we provide a slightly modified proof from [45, Lemma 7.2.2] here.

Lemma 5.2.1. *Let $m \in \mathbb{N}$ and $u = \{u(k)\}_{k\in\mathbb{Z}^d} \in l_0(\mathbb{Z}^d)$. Then u has order m vanishing moments (i.e., $\widehat{u}(\xi) = \mathcal{O}(\|\xi\|^m)$ as $\xi \to 0$) if and only if there exist a $u_\nu \in l_0(\mathbb{Z}^d)$ for each $\nu \in \mathbb{N}_0^d$ with $|\nu| = m$ such that $u = \sum_{\nu\in\mathbb{N}_0^d,|\nu|=m} \nabla^\nu u_\nu$, that is, $\widehat{u}(\xi) = \sum_{\nu\in\mathbb{N}_0^d,|\nu|=m} \widehat{\nabla^\nu \boldsymbol{\delta}}(\xi)\widehat{u_\nu}(\xi)$. Moreover, if the filter u has real coefficients, then all filters u_ν have real coefficients.*

Proof. The sufficiency part is trivial, since $\widehat{\nabla^\nu \boldsymbol{\delta}}(\xi) = \mathcal{O}(\|\xi\|^{|\nu|})$ as $\xi \to 0$ (in fact $\operatorname{vm}(\nabla^\nu \boldsymbol{\delta}) = |\nu|$) for all $\nu \in \mathbb{N}_0^d$. Consequently, it is trivial that $\widehat{u}(\xi) = \sum_{\nu\in\mathbb{N}_0^d,|\nu|=m} \widehat{\nabla^\nu \boldsymbol{\delta}}(\xi)\widehat{u_\nu}(\xi) = \mathcal{O}(\|\xi\|^m)$ as $\xi \to 0$.

For $r \in \mathbb{N}_0$, we define $\Lambda_r := \{\mu \in \mathbb{N}_0^d \ : \ |\mu| \leqslant r\}$. Let $v := u$. To prove the necessity part, without loss of generality, by shifting the filter v, we can assume that $\operatorname{supp}(v) \subseteq \Lambda_r$ but $\operatorname{supp}(v)$ is not contained inside Λ_{r-1} for some $r \in \mathbb{N}_0$. Suppose that $r \geqslant m$ and $v(k) \neq 0$ for some $k \in \mathbb{N}_0^d$ with $|k| = r$. Then we can easily write $k = \nu + j$ with $\nu, j \in \mathbb{N}_0^d$ and $|\nu| = m$. We replace/update v by $v - v(k)(-1)^m[\nabla^\nu \boldsymbol{\delta}](\cdot - j)$. Since $\operatorname{supp}([\nabla^\nu \boldsymbol{\delta}](\cdot - j)) \subseteq \Lambda_{r-1} \cup \{k\}$, we conclude that the updated filter v is still supported inside Λ_r, $v(k) = 0$, and the updated filter v preserves the values as the previous filter at the set $\{n \in \mathbb{N}_0^d \ : \ |n| = r, n \neq k\}$. Therefore, we can continue this procedure for other $n \in \mathbb{N}_0^d$ with $|n| = r$ so that finally the updated filter v has support inside Λ_{r-1}. We can continue this procedure until $r < m$. Note that the Fourier series of $v(k)[\nabla^\nu \boldsymbol{\delta}](\cdot - j)$ is simply $v(k)e^{-ij\cdot\xi}\widehat{\nabla^\nu \boldsymbol{\delta}}(\xi)$.

Consequently, we can write $\widehat{u} = \widehat{v} + \sum_{\nu\in\mathbb{N}_0^d,|v|=m} \widehat{\nabla^\nu \boldsymbol{\delta}}(\xi)\widehat{u_\nu}(\xi)$ for some sequences $u_\nu \in l_0(\mathbb{Z}^d), \nu \in \mathbb{N}_0^d$ with $|\nu| = m$ and $\operatorname{supp}(v) \subseteq \Lambda_{m-1}$. Since $\widehat{u}(\xi) = \mathcal{O}(\|\xi\|^m)$ as $\xi \to 0$, we trivially have $\widehat{v}(\xi) = \mathcal{O}(\|\xi\|^m)$ as $\xi \to 0$, that is, the filter v has order m vanishing moments. Hence,

$$\sum_{k\in\Lambda_{m-1}} v(k)k^\mu = 0, \qquad \forall\, \mu \in \Lambda_{m-1}.$$

It is well known that the above system of linear equations can only have the trivial solution: $v(k) = 0$ for all $k \in \Lambda_{m-1}$. Therefore, $v = 0$ and we proved the claim. ■

Using Lemma 5.2.1, we have the following result, which improves Corollary 5.1.3.

Lemma 5.2.2. *Let $m \in \mathbb{N}_0$ and $u \in l_0(\mathbb{Z}^d)$ with real coefficients. Then*

$$\overline{\widehat{u}(\xi)} = \widehat{u}(\xi) \quad \text{and} \quad \widehat{u}(\xi) = \mathcal{O}(\|\xi\|^{2m}), \quad \xi \to 0 \tag{5.2.2}$$

if and only if there exist $\varepsilon_1, \ldots, \varepsilon_s \in \{-1, 1\}$ and $u_1, \ldots, u_s \in l_0(\mathbb{Z}^d)$ with real coefficients satisfying

$$\widehat{u}(\xi) = \sum_{\ell=1}^{s} \varepsilon_\ell |\widehat{u_\ell}(\xi)|^2 \quad \text{with} \quad \widehat{u_\ell}(\xi) = \mathcal{O}(\|\xi\|^m), \quad \xi \to 0, \forall\, \ell = 1, \ldots, s. \tag{5.2.3}$$

Proof. The sufficiency part is trivial and we only need to prove the necessity part. Suppose that (5.2.2) holds. By Lemma 5.2.1, there exist $u_\nu \in l_0(\mathbb{Z}^d), |\nu| = 2m$ with real coefficients such that $\widehat{u}(\xi) = \sum_{\nu \in \mathbb{N}_0^d, |\nu|=2m} 2\widehat{\nabla^\nu \boldsymbol{\delta}}(\xi)\widehat{u_\nu}(\xi)$. Since $\overline{\widehat{u}(\xi)} = \widehat{u}(\xi)$, we conclude that

$$\widehat{u}(\xi) = \sum_{\nu \in \mathbb{N}_0^d, |\nu|=2m} \theta_\nu(\xi) \quad \text{with} \quad \theta_\nu(\xi) := \widehat{\nabla^\nu \boldsymbol{\delta}}(\xi)\widehat{u_\nu}(\xi) + \overline{\widehat{\nabla^\nu \boldsymbol{\delta}}(\xi)}\, \overline{\widehat{u_\nu}(\xi)}. \tag{5.2.4}$$

For $\nu \in \mathbb{N}_0^d$ with $|\nu| = 2m$, we consider two cases.

Case 1. $\nu \in \mathbb{N}_0^d$ with $|\nu| = 2m$ but $\nu \notin 2\mathbb{N}_0^d$. Then there exist $\alpha, \beta \in \mathbb{N}_0^d$ such that $\alpha + \beta = \nu$ and $|\alpha| = |\beta| = m$. Define

$$\widehat{u_{\nu,\alpha,\beta}}(\xi) := \overline{\widehat{\nabla^\alpha \boldsymbol{\delta}}(\xi)} + \widehat{\nabla^\beta \boldsymbol{\delta}}(\xi)\widehat{u_\nu}(\xi).$$

By $|\alpha| = |\beta| = m$, we see that $\overline{\widehat{\nabla^\alpha \boldsymbol{\delta}}(\xi)} = \mathcal{O}(\|\xi\|^m)$ and $\widehat{\nabla^\beta \boldsymbol{\delta}}(\xi)\widehat{u_\nu}(\xi) = \mathcal{O}(\|\xi\|^m)$ as $\xi \to 0$. Consequently, we have $\widehat{u_{\nu,\alpha,\beta}}(\xi) = \mathcal{O}(\|\xi\|^m)$ as $\xi \to 0$. By calculation, we have

$$\begin{aligned} |\widehat{u_{\nu,\alpha,\beta}}(\xi)|^2 = |\widehat{\nabla^\alpha \boldsymbol{\delta}}(\xi)|^2 + |\widehat{\nabla^\beta \boldsymbol{\delta}}(\xi)\widehat{u_\nu}(\xi)|^2 \\ + \left(\widehat{\nabla^\alpha \boldsymbol{\delta}}(\xi)\widehat{\nabla^\beta \boldsymbol{\delta}}(\xi)\widehat{u_\nu}(\xi) + \overline{\widehat{\nabla^\alpha \boldsymbol{\delta}}(\xi)\widehat{\nabla^\beta \boldsymbol{\delta}}(\xi)\widehat{u_\nu}(\xi)}\right). \end{aligned}$$

Since $\alpha+\beta=\nu$, the last term in the above identity is simply $\theta_\nu(\xi)$. Consequently, we have

$$\theta_\nu(\xi) = |\widehat{u_{\nu,\alpha,\beta}}(\xi)|^2 - |\widehat{\nabla^\alpha \boldsymbol{\delta}}(\xi)|^2 - |\widehat{\nabla^\beta \boldsymbol{\delta}}(\xi)\widehat{u_\nu}(\xi)|^2.$$

To reduce the total number of filters, instead we can also combine the last two terms in the above identity with the terms in Case 2 discussed below.

Case 2: $\nu \in 2\mathbb{N}_0^d$, that is, $\nu = 2\mu$ for some $\mu \in \mathbb{N}_0^d$ with $|\mu| = m$. By $\widehat{\nabla^{2\mu}\boldsymbol{\delta}}(\xi) = |\widehat{\nabla^\mu \boldsymbol{\delta}}(\xi)|^2(-1)^{|\mu|}e^{-i\mu\cdot\xi}$, we deduce that

$$\theta_\nu(\xi) = \theta_{2\mu}(\xi) = |\widehat{\nabla^\mu \boldsymbol{\delta}}(\xi)|^2 \eta_\mu(\xi) \qquad \text{with} \quad \eta_\mu(\xi) := (-1)^{|\mu|}\Big(e^{-i\mu\cdot\xi}\widehat{u_{2\mu}}(\xi) + e^{i\mu\cdot\xi}\overline{\widehat{u_{2\mu}}(\xi)}\Big).$$

Note that $\overline{\eta_\mu(\xi)} = \eta_\mu(\xi)$ and η_μ has real coefficients. By Corollary 5.1.3, there exist $\varepsilon_0, \ldots, \varepsilon_s \in \{-1, 1\}$ and $2\pi\mathbb{Z}^d$-periodic trigonometric polynomials $\mathbf{u}_0, \ldots, \mathbf{u}_s$ with real coefficients such that $\eta_\mu(\xi) = \varepsilon_0|\mathbf{u}_0(\xi)|^2 + \cdots + \varepsilon_s|\mathbf{u}_s(\xi)|^2$. That is, we proved

$$\theta_\nu(\xi) = \varepsilon_0|\widehat{u_{\mu,0}}(\xi)|^2 + \cdots + \varepsilon_s|\widehat{u_{\mu,s}}(\xi)|^2 \qquad \text{with} \quad \widehat{u_{\mu,\ell}}(\xi) := \widehat{\nabla^\mu \boldsymbol{\delta}}(\xi)\mathbf{u}_\ell(\xi), \quad \ell = 0, \ldots, s.$$

Note that $\widehat{u_{\mu,\ell}}(\xi) = \mathcal{O}(\|\xi\|^m)$ as $\xi \to 0$ for all $\ell = 0, \ldots, s$ by $|\mu| = m$. Now the conclusion follows trivially from (5.2.4). ■

We now prove Theorem 5.2.1 by the first constructive method as follows.

Proof of Theorem 5.2.1. Define $\mathcal{A}(\xi) := 1 - \mathsf{d_M}\sum_{j=1}^{\mathsf{d_M}} |\widehat{a^{[\gamma_j]}}(\xi)|^2$. Obviously, $\overline{\mathcal{A}(\xi)} = \mathcal{A}(\xi)$. By the definition of the coset sequences, we observe that $\mathcal{A}(\mathsf{M}^\mathsf{T}\xi) = 1 - \sum_{\omega\in\Omega_\mathsf{M}} |\widehat{a}(\xi+2\pi\omega)|^2$. By the definition $m = \min(\operatorname{sr}(a, \mathsf{M}), \frac{1}{2}\operatorname{vm}(u_a))$ with $\widehat{u_a}(\xi) := 1 - |\widehat{a}(\xi)|^2$, we have $\operatorname{sr}(a, \mathsf{M}) \geqslant m$ and $\operatorname{vm}(u_a) \geqslant 2m$. Consequently, we have $|\widehat{a}(\xi+2\pi\omega)|^2 = \mathcal{O}(\|\xi\|^{2m})$ as $\xi \to 0$ for all $\omega \in \Omega_\mathsf{M}\backslash\{0\}$, and $1 - |\widehat{a}(\xi)|^2 = \widehat{u_a}(\xi) = \mathcal{O}(\|\xi\|^{2m})$ as $\xi \to 0$. That is, we must have

$$\widehat{\mathcal{A}}(\mathsf{M}^\mathsf{T}\xi) = 1 - |\widehat{a}(\xi)|^2 - \sum_{\omega\in\Omega_\mathsf{M}\backslash\{0\}} |\widehat{a}(\xi+2\pi\omega)|^2 = \mathcal{O}(\|\xi\|^{2m}), \qquad \xi \to 0.$$

Since M is an invertible matrix, consequently we must have $\mathcal{A}(\xi) = \mathcal{O}(\|\xi\|^{2m})$ as $\xi \to 0$. By Lemma 5.2.2, since $\mathcal{A}$ has real coefficients, there exist $\varepsilon_1, \ldots, \varepsilon_t \in \{-1, 1\}$ and $u_1, \ldots, u_t \in l_0(\mathbb{Z}^d)$ with real coefficients such that

$$\mathcal{A}(\xi) = \sum_{\ell=1}^{t} \varepsilon_\ell |\widehat{u_\ell}(\xi)|^2 \quad \text{with} \quad \widehat{u_\ell}(\xi) = \mathcal{O}(\|\xi\|^m), \quad \xi \to 0, \forall\, \ell = 1, \ldots, t. \tag{5.2.5}$$

Employing a similar idea as in [63], we now define the high-pass filters $b_1, \ldots, b_s \in l_0(\mathbb{Z}^d)$ with $s := t + \mathsf{d_M}$ as follows:

$$\widehat{b_\ell}(\xi) := \widehat{a}(\xi)\widehat{u_\ell}(\mathsf{M}^\mathsf{T}\xi), \qquad \ell = 1, \ldots, t \tag{5.2.6}$$

and

$$\widehat{b_{t+j}}(\xi) := \mathsf{d_M}^{-1/2} e^{-i\gamma_j \cdot \xi} - \mathsf{d_M}^{1/2} \widehat{a}(\xi) \overline{\widehat{a^{[\gamma_j]}}(\mathsf{M}^\mathsf{T}\xi)}, \quad j = 1, \ldots, \mathsf{d_M}. \tag{5.2.7}$$

Define $\varepsilon_{t+1} = \cdots = \varepsilon_{t+\mathsf{d_M}} := 1$. We now prove that $\{a; b_1, \ldots, b_s\}_{(\varepsilon_1, \ldots, \varepsilon_s)}$ is a quasi-tight M-framelet filter bank and $\operatorname{vm}(b_\ell) \geqslant m$ for all $\ell = 1, \ldots, s$.

Let B_ℓ be defined as in (5.1.7). We now calculate B_ℓ for the high-pass filters b_ℓ defined in (5.2.6) and (5.2.7). Let b_ℓ be defined in (5.2.6). Then $\widehat{b_\ell^{[\gamma]}}(\xi) = \widehat{a^{[\gamma]}}(\xi)\widehat{u_\ell}(\xi)$ for $\gamma \in \Gamma_\mathsf{M}$. Therefore, by (5.2.5), we have

$$\begin{aligned}\sum_{\ell=1}^{t} B_\ell(\xi) &= \sum_{\ell=1}^{t} \varepsilon_\ell |\widehat{u_\ell}(\xi)|^2 \mathsf{d_M} \Big[\widehat{a^{[\gamma_1]}}(\xi), \ldots, \widehat{a^{[\gamma_{\mathsf{d_M}}]}}(\xi)\Big]^\star \Big[\widehat{a^{[\gamma_1]}}(\xi), \ldots, \widehat{a^{[\gamma_{\mathsf{d_M}}]}}(\xi)\Big] \\ &= \mathsf{d_M} \mathcal{A}(\xi) \Big[\widehat{a^{[\gamma_1]}}(\xi), \ldots, \widehat{a^{[\gamma_{\mathsf{d_M}}]}}(\xi)\Big]^\star \Big[\widehat{a^{[\gamma_1]}}(\xi), \ldots, \widehat{a^{[\gamma_{\mathsf{d_M}}]}}(\xi)\Big].\end{aligned} \tag{5.2.8}$$

Let b_ℓ be defined in (5.2.7) with $\ell = t + j$. Then $\widehat{b_\ell^{[\gamma]}}(\xi) = \mathsf{d_M}^{-1/2} \boldsymbol{\delta}(\gamma - \gamma_j) - \mathsf{d_M}^{1/2} \widehat{a^{[\gamma]}}(\xi) \overline{\widehat{a^{[\gamma_j]}}(\xi)}$ for $\gamma \in \Gamma_\mathsf{M}$. Hence,

$$\Big[\widehat{b_\ell^{[\gamma_1]}}(\xi), \ldots, \widehat{b_\ell^{[\gamma_{\mathsf{d_M}}]}}(\xi)\Big] = \mathsf{d_M}^{-1/2} e_j^\mathsf{T} - \mathsf{d_M}^{1/2} \overline{\widehat{a^{[\gamma_j]}}(\xi)} \Big[\widehat{a^{[\gamma_1]}}(\xi), \ldots, \widehat{a^{[\gamma_{\mathsf{d_M}}]}}(\xi)\Big].$$

Therefore, by $\varepsilon_{t+1} = \cdots = \varepsilon_{t+\mathsf{d_M}} = 1$,

$$\sum_{\ell=t+1}^{t+\mathsf{d_M}} B_\ell(\xi) = \sum_{j=1}^{\mathsf{d_M}} \Big(e_j e_j^{\mathsf{T}} - \mathsf{d_M} e_j \overline{\widehat{a^{[\gamma_j]}}(\xi)} \Big[\widehat{a^{[\gamma_1]}}(\xi), \ldots, \widehat{a^{[\gamma_{\mathsf{d_M}}]}}(\xi)\Big]$$

$$-\mathsf{d_M} \Big[\widehat{a^{[\gamma_1]}}(\xi), \ldots, \widehat{a^{[\gamma_{\mathsf{d_M}}]}}(\xi)\Big]^\star \widehat{a^{[\gamma_j]}}(\xi) e_j^{\mathsf{T}}$$

$$+\mathsf{d_M^2} |\widehat{a^{[\gamma_j]}}(\xi)|^2 \Big[\widehat{a^{[\gamma_1]}}(\xi), \ldots, \widehat{a^{[\gamma_{\mathsf{d_M}}]}}(\xi)\Big]^\star \Big[\widehat{a^{[\gamma_1]}}(\xi), \ldots, \widehat{a^{[\gamma_{\mathsf{d_M}}]}}(\xi)\Big] \Big)$$

$$= I_{\mathsf{d_M}} + \Big(-2\mathsf{d_M} + \mathsf{d_M^2} \sum_{j=1}^{\mathsf{d_M}} |\widehat{a^{[\gamma_j]}}(\xi)|^2 \Big) \times$$

$$\Big[\widehat{a^{[\gamma_1]}}(\xi), \ldots, \widehat{a^{[\gamma_{\mathsf{d_M}}]}}(\xi)\Big]^\star \Big[\widehat{a^{[\gamma_1]}}(\xi), \ldots, \widehat{a^{[\gamma_{\mathsf{d_M}}]}}(\xi)\Big],$$

where $I_{\mathsf{d_M}}$ stands for the $\mathsf{d_M} \times \mathsf{d_M}$ identity matrix. Since $\mathcal{A}(\xi) = 1 - \mathsf{d_M} \sum_{j=0}^{\mathsf{d_M}-1} |\widehat{a^{[\gamma_j]}}(\xi)|^2$, we have

$$-2\mathsf{d_M} + \mathsf{d_M^2} \sum_{j=1}^{\mathsf{d_M}} |\widehat{a^{[\gamma_j]}}(\xi)|^2 = -2\mathsf{d_M} + \mathsf{d_M}(1 - \mathcal{A}(\xi)) = -\mathsf{d_M}(1 + \mathcal{A}(\xi)).$$

In other words, we obtain

$$\sum_{\ell=t+1}^{t+\mathsf{d_M}} B_\ell(\xi) = I_{\mathsf{d_M}} - \mathsf{d_M}(1 + \mathcal{A}(\xi)) \Big[\widehat{a^{[\gamma_1]}}(\xi), \ldots, \widehat{a^{[\gamma_{\mathsf{d_M}}]}}(\xi)\Big]^\star \Big[\widehat{a^{[\gamma_1]}}(\xi), \ldots, \widehat{a^{[\gamma_{\mathsf{d_M}}]}}(\xi)\Big].$$

Combining the above identity with (5.2.8) and noting that $s = t + \mathsf{d_M}$, we have

$$\sum_{\ell=1}^{s} B_\ell(\xi) = I_{\mathsf{d_M}} - \mathsf{d_M} \Big[\widehat{a^{[\gamma_1]}}(\xi), \ldots, \widehat{a^{[\gamma_{\mathsf{d_M}}]}}(\xi)\Big]^\star \Big[\widehat{a^{[\gamma_1]}}(\xi), \ldots, \widehat{a^{[\gamma_{\mathsf{d_M}}]}}(\xi)\Big] = \mathcal{N}_a(\xi).$$

That is, we verified the identity in (5.1.4). This proves that $\{a; b_1, \ldots, b_s\}_{(\varepsilon_1, \ldots, \varepsilon_s)}$ is a quasi-tight M-framelet filter bank.

We now prove that $\widehat{b_\ell}(\xi) = \mathcal{O}(\|\xi\|^m)$ as $\xi \to 0$ for all $\ell = 1, \ldots, s$. By the second identity in (5.2.5) and the definition of b_ℓ in (5.2.6), we trivially have $\widehat{b_\ell}(\xi) = \mathcal{O}(\|\xi\|^m)$ as $\xi \to 0$ for all $\ell = 1, \ldots, t$.

Let $\{\omega_1, \ldots, \omega_{\mathsf{d_M}}\} = \Omega_{\mathsf{M}}$ with $\omega_1 := 0$. By the identity in (1.2.8) with $E(\xi)$

and U being defined in (1.2.9), we have

$$\Big[\widehat{a}(\xi+2\pi\omega_1),\ldots,\widehat{a}(\xi+2\pi\omega_{\mathsf{d_M}})\Big]=\Big[e^{-i\gamma_1\cdot\xi}\widehat{a^{[\gamma_1]}}(\mathsf{M}^{\mathsf{T}}\xi),\ldots,e^{-i\gamma_{\mathsf{d_M}}\cdot\xi}\widehat{a^{[\gamma_{\mathsf{d_M}}]}}(\mathsf{M}^{\mathsf{T}}\xi)\Big]U \tag{5.2.9}$$

and $UU^\star=\mathsf{d_M}I_{\mathsf{d_M}}$. Note that $\widehat{a}(0)=1$ and $\omega_1=0$. Since $\operatorname{sr}(a,\mathsf{M})\geqslant m$, by the definition of sum rules in (1.1.9), we deduce from the above identity in (5.2.9) that

$$\Big[e^{-i\gamma_1\cdot\xi}\widehat{a^{[\gamma_1]}}(\mathsf{M}^{\mathsf{T}}\xi),\ldots,e^{-i\gamma_{\mathsf{d_M}}\cdot\xi}\widehat{a^{[\gamma_{\mathsf{d_M}}]}}(\mathsf{M}^{\mathsf{T}}\xi)\Big]= \\ \Big[\widehat{a}(\xi),\mathcal{O}(\|\xi\|^m),\ldots,\mathcal{O}(\|\xi\|^m)\Big]U^{-1},\qquad \xi\to 0.$$

Since $U^{-1}=\mathsf{d_M^{-1}}U^\star=(\mathsf{d_M^{-1}}e^{-i2\pi\gamma_j\cdot\omega_k})_{1\leqslant k,j\leqslant \mathsf{d_M}}$ and $\omega_1=0$, all the entries in the first row of U^{-1} are $\mathsf{d_M^{-1}}$. Therefore, we conclude from the above identity that

$$\Big[e^{-i\gamma_1\cdot\xi}\widehat{a^{[\gamma_1]}}(\mathsf{M}^{\mathsf{T}}\xi),\ldots,e^{-i\gamma_{\mathsf{d_M}}\cdot\xi}\widehat{a^{[\gamma_{\mathsf{d_M}}]}}(\mathsf{M}^{\mathsf{T}}\xi)\Big]=\mathsf{d_M^{-1}}\widehat{a}(\xi)[1,\ldots,1]+\mathcal{O}(\|\xi\|^m),\quad \xi\to 0.$$

That is, we proved

$$e^{-i\gamma_j\cdot\xi}\widehat{a^{[\gamma_j]}}(\mathsf{M}^{\mathsf{T}}\xi)=\mathsf{d_M^{-1}}\widehat{a}(\xi)+\mathcal{O}(\|\xi\|^m),\qquad \xi\to 0, j=1,\ldots,\mathsf{d_M}. \tag{5.2.10}$$

For b_ℓ defined in (5.2.7) with $\ell=t+j$, we deduce from the above identity that

$$\begin{aligned}\widehat{b_\ell}(\xi)&=\mathsf{d_M^{-1/2}}e^{-i\gamma_j\cdot\xi}\overline{(1-\mathsf{d_M}\overline{\widehat{a}(\xi)}e^{-i\gamma_j\cdot\xi}\widehat{a^{[\gamma_j]}}(\mathsf{M}^{\mathsf{T}}\xi))}\\ &=\mathsf{d_M^{-1/2}}e^{-i\gamma_j\cdot\xi}(1-|\widehat{a}(\xi)|^2)+\mathcal{O}(\|\xi\|^m)\\ &=\mathsf{d_M^{-1/2}}e^{-i\gamma_j\cdot\xi}\mathcal{O}(\|\xi\|^{2m})+\mathcal{O}(\|\xi\|^m)=\mathcal{O}(\|\xi\|^m)\end{aligned}$$

as $\xi\to 0$, where we used our assumption $1-|\widehat{a}(\xi)|^2=\widehat{u_a}(\xi)=\mathcal{O}(\|\xi\|^{2m})$ as $\xi\to 0$. This proves that all the high-pass filters $b_1,\ldots,b_s$ have at least order m vanishing moments. ∎

One shortcoming of the quasi-tight M-framelet filter bank $\{a;b_1,\ldots,b_s\}_{(\varepsilon_1,\ldots,\varepsilon_s)}$ with high vanishing moments in Theorem 5.2.1 is that the supports of all the high-pass filters $b_1,\ldots,b_s$, constructed through (5.2.6) and (5.2.7), are much larger than that of the low-pass filter a. This problem and the above proof of Theorem 5.2.1 motivate us to propose an algorithm solving only linear equations for constructing quasi-tight framelet filter banks with vanishing moments. To do so, let us introduce some notations. For $\mu=(\mu_1,\dots,\mu_d)^{\mathsf{T}}, \nu=(\nu_1,\dots,\nu_d)^{\mathsf{T}}\in\mathbb{N}_0^d$, we say that $\mu<\nu$ if either $|\mu|<|\nu|$ or $|\mu|=|\nu|$ and $\mu_j=\nu_j$ for $j=1,\dots,s-1$ but $\mu_s<\nu_s$ for some $1\leqslant s\leqslant d$. Note that Ω_{M} is a complete set of representatives of distinct cosets of the quotient group $[(\mathsf{M}^{\mathsf{T}})^{-1}\mathbb{Z}^d]/\mathbb{Z}^d$. Therefore, Ω_{M} can be regarded as an additive group under modulo $\mathbb{Z}^d$.

Lemma 5.2.3. *Let $b\in l_0(\mathbb{Z}^d)$ and $\beta\in\Omega_{\mathsf{M}}$. Define $F(\xi):=(e^{-i\gamma_j\cdot(\xi+2\pi\omega)})_{1\leqslant j\leqslant \mathsf{d_M},\omega\in\Omega_{\mathsf{M}}}$ and the $\mathsf{d_M}\times\mathsf{d_M}$ matrix $D_{b,\beta}$ by*

$$[D_{b,\beta}(\xi)]_{\omega,\eta}=\begin{cases}\widehat{b}(\xi+2\pi\omega), & \text{if } \omega+\beta-\eta\in\mathbb{Z}^d,\\ 0, & \text{if } \omega+\beta-\eta\notin\mathbb{Z}^d,\end{cases}\qquad \omega,\eta\in\Omega_{\mathsf{M}}. \tag{5.2.11}$$

Then

$$F(\xi)D_{b,\beta}(\xi)F^\star(\xi)=\mathsf{d_M}E_{b,\beta}(\mathsf{M}^{\mathsf{T}}\xi)\quad \text{with}\quad E_{b,\beta}(\xi):=\left(\widehat{b^{[\gamma_k-\gamma_j]}}(\xi)e^{i\gamma_k\cdot 2\pi\beta}\right)_{1\leqslant j,k\leqslant \mathsf{d_M}}. \tag{5.2.12}$$

Proof. Let $j,k=1,\dots,\mathsf{d_M}$. We now compute the (j,k)-entry of the matrix on the left-hand side of (5.2.12). Note that $\widehat{b}(\xi)=\sum_{p=1}^{\mathsf{d_M}}\widehat{b^{[\gamma_p]}}(\mathsf{M}^{\mathsf{T}}\xi)e^{-i\gamma_p\cdot\xi}$.

$$\begin{aligned}[F(\xi)D_{b,\beta}(\xi)F^\star(\xi)]_{j,k}&=\sum_{\omega\in\Omega_{\mathsf{M}}}[F(\xi)]_{j,\omega}[D_{b,\beta}(\xi)]_{\omega,\omega+\beta}[F^\star(\xi)]_{\omega+\beta,k}\\&=\sum_{\omega\in\Omega_{\mathsf{M}}}e^{-i\gamma_j\cdot(\xi+2\pi\omega)}\widehat{b}(\xi+2\pi\omega)e^{i\gamma_k\cdot(\xi+2\pi\omega+2\pi\beta)}\\&=\sum_{\omega\in\Omega_{\mathsf{M}}}\sum_{p=1}^{\mathsf{d_M}}e^{-i\gamma_j\cdot(\xi+2\pi\omega)}\widehat{b^{[\gamma_p]}}(\mathsf{M}^{\mathsf{T}}\xi)e^{-i\gamma_p\cdot(\xi+2\pi\omega)}e^{i\gamma_k\cdot(\xi+2\pi\omega+2\pi\beta)}\\&=\sum_{p=1}^{\mathsf{d_M}}\widehat{b^{[\gamma_p]}}(\mathsf{M}^{\mathsf{T}}\xi)e^{-i(\gamma_p+\gamma_j-\gamma_k)\cdot\xi}e^{i\gamma_k\cdot 2\pi\beta}\sum_{\omega\in\Omega_{\mathsf{M}}}e^{-i(\gamma_p+\gamma_j-\gamma_k)\cdot 2\pi\omega}.\end{aligned}$$

Note that the last sum in the above identity is equal to $\mathsf{d_M}$ if $\gamma_p+\gamma_j-\gamma_k\in\mathsf{M}\mathbb{Z}^d$ and 0 otherwise. Hence, we deduce from the above identity that

$$[F(\xi)D_{b,\beta}(\xi)F^\star(\xi)]_{j,k}=\mathsf{d_M}\widehat{b^{[\gamma_p]}}(\mathsf{M}^{\mathsf{T}}\xi)e^{-i\mathsf{M}\alpha_{j,k}\cdot\xi}e^{i\gamma_k\cdot 2\pi\beta}=\mathsf{d_M}\widehat{b^{[\gamma_k-\gamma_j]}}(\mathsf{M}^{\mathsf{T}}\xi)e^{i\gamma_k\cdot 2\pi\beta},$$

where we used the identity $\widehat{u^{[\gamma+\mathsf{M}\alpha]}}(\xi) = \widehat{u^{[\gamma]}}(\xi)e^{i\alpha\cdot\xi}$ and $\alpha_{j,k} \in \mathbb{Z}^d$ is the unique integer such that $\gamma_p = \gamma_k - \gamma_j + \mathsf{M}\alpha_{j,k}$ for the unique $\gamma_p \in \Gamma_{\mathsf{M}}$ satisfying $\gamma_p + \gamma_j - \gamma_k \in \mathsf{M}\mathbb{Z}^d$. This proves (5.2.12). ■

Now we are ready to state another method for constructing quasi-tight framelet filter banks with vanishing moments by solving only linear equations. For the convenience of the reader, we state the following result in an algorithmic way. We now prove Theorem 5.2.1 by the second constructive method as follows.

Theorem 5.2.2. *Let M be a $d \times d$ dilation matrix and let $a \in l_0(\mathbb{Z}^d)$ be a finitely supported real-valued sequence on $\mathbb{Z}^d$. Let $m \in \mathbb{N}_0$ such that $m \leqslant \min(\operatorname{sr}(a,\mathsf{M}), \frac{1}{2}\operatorname{vm}(u_a))$, where $\widehat{u_a}(\xi) := 1 - |\widehat{a}(\xi)|^2$. Define*

$$F(\xi) := (e^{-i\gamma_j\cdot(\xi+2\pi\omega)})_{1\leqslant j\leqslant \mathsf{d_M}, \omega\in\Omega_{\mathsf{M}}}. \tag{5.2.13}$$

Define $\mathcal{N}_a$ as in (5.1.5) and $E_\mu(\xi) := E_{\nabla^\mu \boldsymbol{\delta},0}(\xi)$ as in (5.2.12).

(S1) Solve the system X of linear equations induced by

$$\begin{aligned}\mathsf{d}_{\mathsf{M}}^{-1}\mathcal{N}_a(\xi) &= \sum_{|\mu|=m} E_\mu^\star(\xi)A_{\mu,\mu}(\xi)E_\mu(\xi)\\ &+ \sum_{\mu<\nu, |\mu|=|\nu|=m} \Big(E_\mu^\star(\xi)A_{\mu,\nu}(\xi)E_\nu(\xi) + E_\nu^\star(\xi)A_{\mu,\nu}^\star(\xi)E_\mu(\xi)\Big)\end{aligned} \tag{5.2.14}$$

and

$$A_{\mu,\mu}^\star(\xi) = A_{\mu,\mu}(\xi), \qquad \forall\, \mu \in \mathbb{N}_0^d, |\mu| = m, \tag{5.2.15}$$

for the coefficients in all the entries of the matrices $A_{\mu,\nu}$ of $2\pi\mathbb{Z}^d$-periodic trigonometric polynomials with $|\mu| = |\nu| = m$ and $\mu \leqslant \nu$. The linear system X always has a solution of $A_{\mu,\nu}$ with real coefficients for $|\mu| = |\nu| = m$ and $\mu \leqslant \nu$, as long as the supports of their coefficients are large enough.

(S2) For every (μ,ν) with $|\mu| = |\nu| = m$ and $\mu < \nu$, factorize $A_{\mu,\nu}(\xi) = A_{\mu,\nu,1}^\star(\xi)A_{\mu,\nu,2}(\xi)$ (e.g., $A_{\mu,\nu,1}(\xi) = I_{\mathsf{d_M}}$ and $A_{\mu,\nu,2} = A_{\mu,\nu}(\xi)$) for some

$\mathsf{d_M} \times \mathsf{d_M}$ *matrices* $A_{\mu,\nu,1}$ *and* $A_{\mu,\nu,2}$ *of* $2\pi\mathbb{Z}^d$*-periodic trigonometric polynomials with real coefficients. Define*

$$[\widehat{b_{\mu,\nu,1}}(\xi),\ldots,\widehat{b_{\mu,\nu,\mathsf{d_M}}}(\xi)]^{\mathsf{T}} := A_{\mu,\nu,1}(\mathsf{M}^{\mathsf{T}}\xi)F(\xi)e_1\widehat{\nabla^{\mu}\boldsymbol{\delta}}(\xi) + A_{\mu,\nu,2}(\mathsf{M}^{\mathsf{T}}\xi)F(\xi)e_1\widehat{\nabla^{\nu}\boldsymbol{\delta}}(\xi) \quad (5.2.16)$$

and $s_{\mu,\nu} := \mathsf{d_M}$ *and* $\varepsilon_{\mu,\nu,\ell} := 1$ *for all* $\ell = 1,\ldots,\mathsf{d_M}$*. Replace/update* $A_{\mu,\mu}$ *and* $A_{\nu,\nu}$ *by* $A_{\mu,\mu} - A^{\star}_{\mu,\nu,1}A_{\mu,\nu,1}$ *and* $A_{\nu,\nu} - A^{\star}_{\mu,\nu,2}A_{\mu,\nu,2}$*, respectively.*

(S3) For every $\mu \in \mathbb{N}_0^d$ *with* $|\mu| = m$*, apply Theorem 5.1.2 to the updated* $A_{\mu,\mu}$ *so that* $A_{\mu,\mu}(\xi) = \sum_{\ell=1}^{s_{\mu,\mu}} \varepsilon_{\mu,\mu,\ell}\mathbf{u}_{\ell}^{\star}(\xi)\mathbf{u}_{\ell}(\xi)$*, where* $\varepsilon_{\mu,\mu,\ell} \in \{-1,1\}$ *and* $\mathbf{u}_{\ell}$ *is a* $1\times\mathsf{d_M}$ *row vector of* $2\pi\mathbb{Z}^d$*-periodic trigonometric polynomials with real coefficients for* $\ell = 1,\ldots,s_{\mu,\mu}$*. Define*

$$\widehat{b_{\mu,\mu,\ell}}(\xi) := \widehat{\nabla^{\mu}\boldsymbol{\delta}}(\xi)\mathbf{u}_{\ell}(\mathsf{M}^{\mathsf{T}}\xi)F(\xi)e_1, \qquad \ell = 1,\ldots,s_{\mu,\mu}. \qquad (5.2.17)$$

Define

$$\{(b_1,\varepsilon_1),\ldots,(b_s,\varepsilon_s)\} := \{(b_{\mu,\nu,\ell},\varepsilon_{\mu,\nu,\ell}) \ : \ \mu,\nu \in \mathbb{N}_0^d, |\mu| = |\nu| = m, \mu \leqslant \nu, \ell = 1,\ldots,s_{\mu,\nu}\}.$$

Then $\{a;b_1,\ldots,b_s\}_{(\varepsilon_1,\ldots,\varepsilon_s)}$ *is a quasi-tight* M*-framelet filter bank such that all the high-pass filters have at least order* m *vanishing moments, i.e.,* $\operatorname{vm}(b_\ell) \geqslant m$ *for all* $\ell = 1,\ldots,s$*.*

Proof. Let $\{\omega_1,\ldots,\omega_{\mathsf{d_M}}\} := \Omega_{\mathsf{M}}$ with $\omega_1 := 0$. For $b \in (l_0(\mathbb{Z}^d))^r$, for simplicity of presentation, we define $r \times \mathsf{d_M}$ matrices

$$G_b(\xi) := [\widehat{b^{[\gamma_1]}}(\xi),\ldots,\widehat{b^{[\gamma_{\mathsf{d_M}}]}}(\xi)] \quad \text{and} \quad H_b(\xi) := [\widehat{b}(\xi+2\pi\omega_1),\ldots,\widehat{b}(\xi+2\pi\omega_{\mathsf{d_M}})].$$

By (1.2.8), we have $H_b(\xi) = G_b(\mathsf{M}^{\mathsf{T}}\xi)F(\xi)$.

Define $D_\mu(\xi) := D_{\nabla^\mu\boldsymbol{\delta},0}(\xi)$ as in (5.2.11). For $\mu < \nu$ in (S2), by the definition of $b_{\mu,\nu,\ell}$ in (5.2.16) and the identity in (1.2.8), we have

$$H_{[b_{\mu,\nu,1},\ldots,b_{\mu,\nu,\mathsf{d_M}}]^{\mathsf{T}}}(\xi) = A_{\mu,\nu,1}(\mathsf{M}^{\mathsf{T}}\xi)F(\xi)D_\mu(\xi) + A_{\mu,\nu,2}(\mathsf{M}^{\mathsf{T}}\xi)F(\xi)D_\nu(\xi).$$

Since $F(\xi)F^\star(\xi) = \mathsf{d_M} I_{\mathsf{d_M}}$, we deduce from the identity $H_b(\xi) = G_b(\mathsf{M}^\mathsf{T}\xi)F(\xi)$ and (5.2.12) that

$$\begin{aligned} G_{[b_{\mu,\nu,1},\ldots,b_{\mu,\nu,\mathsf{d_M}}]^\mathsf{T}}(\mathsf{M}^\mathsf{T}\xi) =& \mathsf{d_M}^{-1} H_{[b_{\mu,\nu,1},\ldots,b_{\mu,\nu,\mathsf{d_M}}]^\mathsf{T}}(\xi)F^\star(\xi) \\ =& A_{\mu,\nu,1}(\mathsf{M}^\mathsf{T}\xi)E_\mu(\mathsf{M}^\mathsf{T}\xi) + A_{\mu,\nu,2}(\mathsf{M}^\mathsf{T}\xi)E_\nu(\mathsf{M}^\mathsf{T}\xi). \end{aligned}$$

That is, we proved

$$G_{[b_{\mu,\nu,1},\ldots,b_{\mu,\nu,\mathsf{d_M}}]^\mathsf{T}}(\xi) = A_{\mu,\nu,1}(\xi)E_\mu(\xi) + A_{\mu,\nu,2}(\xi)E_\nu(\xi).$$

Therefore, by $\varepsilon_{\mu,\nu,\ell} = 1$ for all $\ell = 1, \ldots, \mathsf{d_M}$, we deduce from the above identity that

$$\begin{aligned} \sum_{\ell=1}^{\mathsf{d_M}} & \varepsilon_{\mu,\nu,\ell} G^\star_{b_{\mu,\nu,\ell}}(\xi) G_{b_{\mu,\nu,\ell}}(\xi) = G^\star_{[b_{\mu,\nu,1},\ldots,b_{\mu,\nu,\mathsf{d_M}}]^\mathsf{T}}(\xi) G_{[b_{\mu,\nu,1},\ldots,b_{\mu,\nu,\mathsf{d_M}}]^\mathsf{T}}(\xi) \\ &= \Big(E^\star_\mu A^\star_{\mu,\nu,1}(\xi) + E^\star_\nu(\xi) A^\star_{\mu,\nu,2}(\xi)\Big)\Big(A_{\mu,\nu,1}(\xi)E_\mu(\xi) + A_{\mu,\nu,2}(\xi)E_\nu(\xi)\Big) \\ &= E^\star_\mu(\xi) A^\star_{\mu,\nu,1}(\xi) A_{\mu,\nu,1}(\xi) E_\mu(\xi) + E^\star_\nu(\xi) A^\star_{\mu,\nu,2}(\xi) A_{\mu,\nu,2}(\xi) E_\nu(\xi) \\ &\quad + \Big(E^\star_\mu(\xi) A^\star_{\mu,\nu,1}(\xi) A_{\mu,\nu,2}(\xi) E_\nu(\xi) + E^\star_\nu(\xi) A^\star_{\mu,\nu,2}(\xi) A_{\mu,\nu,1}(\xi) E_\mu(\xi)\Big). \end{aligned}$$

As we shall see below, the first two terms in the last expression of the last identity have been handled by the updated $A_{\mu,\mu}$ and $A_{\nu,\nu}$ in (S2) (see proof below).

For $\mu \in \mathbb{N}_0^d$ with $|\mu| = m$, we have $H_{b_{\mu,\mu,\ell}}(\xi) = \mathbf{u}_\ell(\mathsf{M}^\mathsf{T}\xi)F(\xi)D_\mu(\xi)$. Therefore, $G_{b_{\mu,\mu,\ell}}(\xi) = \mathbf{u}_\ell(\xi)E_\mu(\xi)$ for all $\ell = 1, \ldots, s_{\mu,\mu}$. Hence,

$$\sum_{\ell=1}^{s_{\mu,\mu}} \varepsilon_{\mu,\mu,\ell} G^\star_{b_{\mu,\mu,\ell}}(\xi) G_{b_{\mu,\mu,\ell}}(\xi) = E^\star_\mu(\xi) \sum_{\ell=1}^{s_{\mu,\mu}} \varepsilon_{\mu,\mu,\ell} \mathbf{u}^\star_\ell(\xi)\mathbf{u}_\ell(\xi) E_\mu(\xi) = E^\star_\mu(\xi) A_{\mu,\mu}(\xi) E_\mu(\xi),$$

where $A_{\mu,\mu}$ is the updated version in (S2). Therefore, we proved

$$\begin{aligned} \sum_{\ell=1}^{s} \varepsilon_s G^\star_{b_\ell}(\xi) G_{b_\ell}(\xi) = & \sum_{|\mu|=m} E^\star_\mu(\xi) A_{\mu,\mu}(\xi) E_\mu(\xi) \\ & + \sum_{\mu<\nu, |\mu|=|\nu|=m} (E^\star_\mu(\xi) A_{\mu,\nu}(\xi) E_\nu(\xi) + E^\star_\nu(\xi) A^\star_{\mu,\nu}(\xi) E_\mu(\xi)) \end{aligned}$$

$$= \mathsf{d}_{\mathsf{M}}^{-1}\mathcal{N}_a(\xi).$$

Hence, we verified the condition in (5.1.4) and consequently, $\{a; b_1, \dots, b_s\}_{(\varepsilon_1,\dots,\varepsilon_s)}$ is a quasi-tight M-framelet filter bank.

Since $\widehat{\nabla^\mu \boldsymbol{\delta}}(\xi) = \mathcal{O}(\|\xi\|^{|\mu|})$ as $\xi \to 0$ for all $\mu \in \mathbb{N}_0^d$, it follows directly from (5.2.16) and (5.2.17) that $\widehat{b_\ell}(\xi) = \mathcal{O}(\|\xi\|^m)$ as $\xi \to 0$ for all $\ell = 1, \dots, s$. Hence, all the high-pass filters have at least order m vanishing moments.

To complete the proof, we now prove the existence of a desired solution to the linear system X induced by (5.2.14) and (5.2.15). We first prove that X must have a solution (probably with complex coefficients) and then we prove that X must have a solution with real coefficients. Define

$$\widehat{a_1}(\xi) := 1 - |\widehat{a}(\xi)|^2 \quad \text{and} \quad \widehat{a_j}(\xi) := -\overline{\widehat{a}(\xi)}\widehat{a}(\xi + 2\pi\omega_j), \qquad j = 2, \dots, \mathsf{d}_{\mathsf{M}}. \tag{5.2.18}$$

By $\omega_1 = 0$ and the definition of the matrices $D_{b,\beta}$ in (5.2.11), it is straightforward to observe that

$$\begin{aligned}\mathcal{N}(\xi) :=& I_{\mathsf{d}_{\mathsf{M}}} - \left[\widehat{a}(\xi + 2\pi\omega_1), \dots, \widehat{a}(\xi + 2\pi\omega_{\mathsf{d}_{\mathsf{M}}})\right]^\star \left[\widehat{a}(\xi + 2\pi\omega_1), \dots, \widehat{a}(\xi + 2\pi\omega_{\mathsf{d}_{\mathsf{M}}})\right] \\ =& \sum_{j=1}^{\mathsf{d}_{\mathsf{M}}} D_{a_j,\omega_j}(\xi).\end{aligned}$$

Since $F(\xi)F^\star(\xi) = \mathsf{d}_{\mathsf{M}} I_{\mathsf{d}_{\mathsf{M}}}$, we deduce from (1.2.8), (5.2.12) and the above identity that

$$\mathcal{N}_a(\mathsf{M}^{\mathsf{T}}\xi) = \mathsf{d}_{\mathsf{M}}^{-1} F(\xi)\mathcal{N}(\xi)F^\star(\xi) = \mathsf{d}_{\mathsf{M}}^{-1} \sum_{j=1}^{\mathsf{d}_{\mathsf{M}}} F(\xi) D_{a_j,\omega_j}(\xi) F^\star(\xi). \tag{5.2.19}$$

Suppose that we can prove

$$\widehat{a_j}(\xi) = \sum_{\mu,\nu \in \mathbb{N}_0^d, |\mu|=|\nu|=m} \overline{\widehat{\nabla^\mu \boldsymbol{\delta}}(\xi)}\, \widehat{\nabla^\nu \boldsymbol{\delta}}(\xi + 2\pi\omega_j) \widehat{u_{j,\mu,\nu}}(\xi), \qquad j = 1, \dots, \mathsf{d}_{\mathsf{M}}, \tag{5.2.20}$$

for some $u_{j,\mu,\nu} \in l_0(\mathbb{Z}^d)$. Then by the definition in (5.2.11), we must have

$$D_{a_j,\omega_j}(\xi) = \sum_{\mu,\nu\in\mathbb{N}_0^d,|\mu|=|\nu|=m} D^\star_{\nabla^\mu \boldsymbol{\delta},0}(\xi) D_{u_{j,\mu,\nu},\omega_j}(\xi) D_{\nabla^\nu \boldsymbol{\delta},0}(\xi).$$

Consequently, by (5.2.12) and $E_\mu := E_{\nabla^\mu \boldsymbol{\delta},0}$, we deduce that

$$\begin{aligned}
&F(\xi)D_{a_j,\omega_j}(\xi)F^\star(\xi)\\
=&\mathsf{d}_\mathsf{M}^{-2} \sum_{\mu,\nu\in\mathbb{N}_0^d,|\mu|=|\nu|=m} F(\xi)D^\star_{\nabla^\mu \boldsymbol{\delta},0}(\xi)F^\star(\xi)F(\xi)D_{u_{j,\mu,\nu},\omega_j}(\xi)F^\star(\xi)F(\xi)D_{\nabla^\nu \boldsymbol{\delta},0}(\xi)F^\star(\xi)\\
=&\mathsf{d}_\mathsf{M} \sum_{\mu,\nu\in\mathbb{N}_0^d,|\mu|=|\nu|=m} E^\star_\mu(\mathsf{M}^\mathsf{T}\xi)E_{u_{j,\mu,\nu},\omega_j}(\mathsf{M}^\mathsf{T}\xi)E_\nu(\mathsf{M}^\mathsf{T}\xi).
\end{aligned}$$

Now we deduce from (5.2.19) that

$$\mathsf{d}_\mathsf{M}^{-1}\mathcal{N}_a(\mathsf{M}^\mathsf{T}\xi) = \mathsf{d}_\mathsf{M}^{-1}\sum_{j=1}^{\mathsf{d}_\mathsf{M}} \sum_{\mu,\nu\in\mathbb{N}_0^d,|\mu|=|\nu|=m} E^\star_\mu(\mathsf{M}^\mathsf{T}\xi)E_{u_{j,\mu,\nu},\omega_j}(\mathsf{M}^\mathsf{T}\xi)E_\nu(\mathsf{M}^\mathsf{T}\xi).$$

Therefore, we proved

$$\mathsf{d}_\mathsf{M}^{-1}\mathcal{N}_a(\xi) = \mathsf{d}_\mathsf{M}^{-1}\sum_{j=1}^{\mathsf{d}_\mathsf{M}} \sum_{\mu,\nu\in\mathbb{N}_0^d,|\mu|=|\nu|=m} E^\star_\mu(\xi)E_{u_{j,\mu,\nu},\omega_j}(\xi)E_\nu(\xi). \tag{5.2.21}$$

Note that $\mathcal{N}_a^\star(\xi) = \mathcal{N}_a(\xi)$. Define

$$A_{\mu,\nu}(\xi) := \frac{1}{2\mathsf{d}_\mathsf{M}}\sum_{j=1}^{\mathsf{d}_\mathsf{M}} \Big(E_{u_{j,\mu,\nu}}(\xi) + E^\star_{u_{j,\nu,\mu}}(\xi)\Big), \qquad \mu,\nu\in\mathbb{N}_0^d, |\mu|=|\nu|=m, \mu \leqslant \nu.$$

From (5.2.21), it is trivial to verify that these $A_{\mu,\nu}$ satisfy both (5.2.14) and (5.2.15). That is, we proved that the linear system X induced by (5.2.14) and (5.2.15) must have a solution (but probably with complex coefficients).

For a $2\pi\mathbb{Z}^d$-periodic trigonometric polynomial $\widehat{u}$, it is straightforward to see that $\widehat{u}$ has real coefficients if and only if $\overline{\widehat{u}(-\xi)} = \widehat{u}(\xi)$. Since the low-pass filter a and all the filters $\nabla^\mu \boldsymbol{\delta}$ have real coefficients, we observe that $\mathcal{N}_a$ and E_μ have real coefficients. Changing ξ into $-\xi$ and applying complex conjugate to (5.2.14) , it is trivial to see that (5.2.14) still holds if we replace all $A_{\mu,\nu}(\xi)$ by $\overline{A_{\mu,\nu}(-\xi)}$, respectively. Consequently, if we replace $A_{\mu,\nu}(\xi)$ by $\frac{1}{2}(A_{\mu,\nu}(\xi) + \overline{A_{\mu,\nu}(-\xi)})$, then (5.2.14) and (5.2.15) still hold. Since all $\frac{1}{2}(A_{\mu,\nu}(\xi)+\overline{A_{\mu,\nu}(-\xi)})$ have real coefficients, we proved that the linear system X induced by (5.2.14) and (5.2.15) must have a solution with real coefficients.

To complete the proof, we now prove (5.2.20). From the definition in (5.2.18), we have $\widehat{a_1}(\xi) = \mathcal{O}(\|\xi\|^{2m})$ as $\xi \to 0$. Note that $\overline{\widehat{\nabla^\mu \boldsymbol{\delta}}(\xi)}\widehat{\nabla^\nu \boldsymbol{\delta}}(\xi) = \widehat{\nabla^{\mu+\nu}\boldsymbol{\delta}}(\xi)(-1)^{|\mu|}e^{i\mu\cdot\xi}$. Hence, we conclude from Lemma 5.2.1 that (5.2.20) holds for $j = 1$. For $j = 2, \ldots, \mathsf{d_M}$, the sum rule condition of a implies that $\widehat{a}(\xi \pm 2\pi\omega_j) = \mathcal{O}(\|\xi\|^m)$. According to Lemma 5.2.1, there exists some $u_{j,\mu}, v_{j,\mu} \in l_0(\mathbb{Z}^d)$ for each $\mu \in \mathbb{N}_0^d$, $|\mu| = m$, such that

$$\widehat{a}(\xi+2\pi\omega_j) = \sum_{\mu\in\mathbb{N}_0^d,|\mu|=m} \widehat{\nabla^\mu\boldsymbol{\delta}}(\xi)\widehat{u_{j,\mu}}(\xi), \qquad \widehat{a}(\xi-2\pi\omega_j) = \sum_{\nu\in\mathbb{N}_0^d,|\nu|=m} \widehat{\nabla^\nu\boldsymbol{\delta}}(\xi)\widehat{v_{j,\nu}}(\xi)$$

hold. The above second identity implies that $\widehat{a}(\xi) = \sum_{\nu\in\mathbb{N}_0^d,|\nu|=m} \widehat{\nabla^\nu\boldsymbol{\delta}}(\xi + 2\pi\omega_j)\widehat{v_{j,\nu}}(\xi + 2\pi\omega_j)$. Therefore,

$$\begin{aligned}
&\overline{\widehat{a}(\xi)}\widehat{a}(\xi + 2\pi\omega_j)\\
&= \sum_{\mu,\nu\in\mathbb{N}_0^d,\,|\mu|=|\nu|=m} \overline{\widehat{\nabla^\nu\boldsymbol{\delta}}(\xi + 2\pi\omega_j)\widehat{v_{j,\nu}}(\xi + 2\pi\omega_j)}\widehat{\nabla^\mu\boldsymbol{\delta}}(\xi)\widehat{u_{j,\mu}}(\xi)\\
&= \sum_{\mu,\nu\in\mathbb{N}_0^d,\,|\mu|=|\nu|=m} \overline{\widehat{\nabla^\mu\boldsymbol{\delta}}(\xi)}\widehat{\nabla^\nu\boldsymbol{\delta}}(\xi + 2\pi\omega_j)e^{-i\mu\cdot\xi}e^{i\nu\cdot(\xi+2\pi\omega_j)}\widehat{u_{j,\mu}}(\xi)\overline{\widehat{v_{j,\nu}}(\xi + 2\pi\omega_j)}.
\end{aligned}$$

Define $\widehat{u_{j,\mu,\nu}}(\xi) := -e^{-i\mu\cdot\xi}e^{i\nu\cdot(\xi+2\pi\omega_j)}\widehat{u_{j,\mu}}(\xi)\overline{\widehat{v_{j,\nu}}(\xi + 2\pi\omega_j)}$, we proved (5.2.20) for $j = 2, \ldots, \mathsf{d_M}$. ■

As a special case of Theorem 5.2.2, we have the following result.

Corollary 5.2.3. *Let* M *be a* $d \times d$ *dilation matrix and let* $a \in l_0(\mathbb{Z}^d)$ *be a finitely supported real-valued sequence on* $\mathbb{Z}^d$. *Let* $m \in \mathbb{N}_0$ *such that* $m \leqslant \min(\operatorname{sr}(a, \mathsf{M}), \frac{1}{2}\operatorname{vm}(u_a))$, *where* $\widehat{u_a}(\xi) := 1 - |\widehat{a}(\xi)|^2$. *Define* $\mathcal{N}_a$ *as in* (5.1.5), $F(\xi)$ *as in* (5.2.13) *and* $E_\mu(\xi) := E_{\nabla^\mu\boldsymbol{\delta},0}(\xi)$ *as in* (5.2.12). *If there exist* $A_{\mu,\mu}$ *with real coefficients for* $|\mu| = m$ *satisfying*

$$\mathsf{d}_{\mathsf{M}}^{-1}\mathcal{N}_a(\xi) = \sum_{|\mu|=m} E_\mu^\star(\xi) A_{\mu,\mu}(\xi) E_\mu(\xi) \quad \textit{and} \quad A_{\mu,\mu}^\star(\xi) = A_{\mu,\mu}(\xi),$$
$$|\mu| = m, \mu \in \mathbb{N}_0^d, \quad (5.2.22)$$

(Such a solution to (5.2.22) *always exists in dimension one, i.e.,* $d = 1$*), then there exist* $b_1, \dots, b_s \in l_0(\mathbb{Z}^d)$ *with real coefficients and* $\varepsilon_1, \dots, \varepsilon_s \in \{-1, 1\}$ *such that* $\{a; b_1, \dots, b_s\}_{(\varepsilon_1,\dots,\varepsilon_s)}$ *is a quasi-tight* M*-framelet filter bank such that all high-pass filters* $b_\ell, \ell = 1, \dots, s$ *have at least order* m *vanishing moments and all the high-pass filters take the form either* $\widehat{b_\ell}(\xi) = c_\ell e^{-i\alpha_\ell\cdot\xi}\widehat{\nabla^\mu\boldsymbol{\delta}}(\xi)$ *(i.e.,* $b_\ell = c_\ell(\nabla^\mu\boldsymbol{\delta}(\cdot - \alpha_\ell))$*) or* $\widehat{b_\ell}(\xi) = c_\ell e^{-i\alpha_\ell\cdot\xi}\widehat{\nabla^\mu\boldsymbol{\delta}}(\xi)\widehat{\nabla_{\beta_\ell}\boldsymbol{\delta}}(\xi)$ *(i.e.,* $b_\ell = c_\ell[(\nabla^\mu\boldsymbol{\delta})(\cdot - \alpha_\ell) - (\nabla^\mu\boldsymbol{\delta})(\cdot - \alpha_\ell - \beta_\ell)]$*) for some* $c_\ell \in \mathbb{R}$, $\alpha_\ell, \beta_\ell \in \mathbb{Z}^d$ *and some* $\mu \in \mathbb{N}_0^d$ *with* $|\mu| = m$.

Proof. For the one dimensional case $d = 1$, there are no terms satisfying $\mu < \nu$ and $|\mu| = |\nu|$. Consequently, (5.2.14) becomes (5.2.22). Therefore, the existence of a solution to (5.2.22) with $d = 1$ is guaranteed by Theorem 5.2.2. The claim follows directly by applying Theorem 5.1.2 to each $A_{\mu,\mu}$. ■

We call the high-pass filters constructed in Corollary 5.2.3 as differencing filters since all of them takes the form $\nabla^\mu\boldsymbol{\delta}$ or their differences.

5.3 Illustrative Examples

5.3.1 Examples of Multivariate Quasi-tight Framelets with Directionality

In this section, we provide several examples of directional quasi-tight or tight framelets. Using the algorithm appeared in the proof of Theorem 5.1.1, we can recover all the examples of directional tight $2I_d$-framelets obtained in [47] derived from box-spline refinable functions. We now provide some other examples using different dilation matrix M.

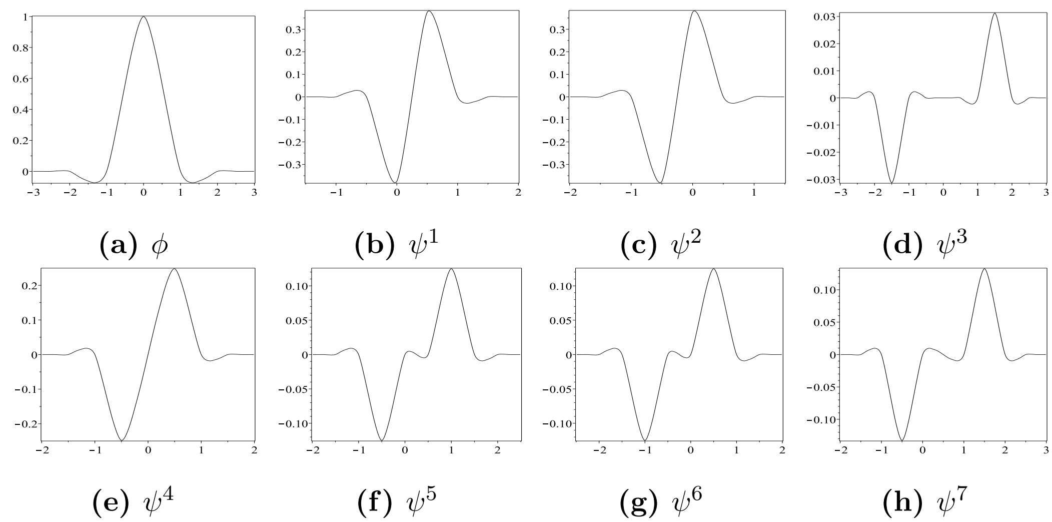
Example 5.2. Let $d = 1$ and $\mathsf{M} = 2$. We consider the following interpolatory filter

$$a = \{-\tfrac{1}{32}, 0, \tfrac{9}{32}, \underline{\mathbf{\tfrac{1}{2}}}, \tfrac{9}{32}, 0, -\tfrac{1}{32}\}_{[-3,3]}. \quad (5.3.1)$$

Using Theorem 5.1.1, we have a quasi-tight 2-framelet filter bank $\{a; b_1, \ldots, b_7\}_{(\epsilon_1,\ldots,\epsilon_7)}$, where

$$b_1 = \{\underline{-\boldsymbol{\tfrac{3}{8}}}, \tfrac{3}{8}\}_{[0,1]}, \quad b_2 = \{-\tfrac{3}{8}, \underline{\boldsymbol{\tfrac{3}{8}}}\}_{[-1,0]}, \quad b_3 = \{-\tfrac{1}{32}, 0, 0, \underline{\mathbf{0}}, 0, 0, \tfrac{1}{32}\}_{[-3,3]},$$
$$b_4 = \{-\tfrac{3\sqrt{7}}{32}, \underline{\mathbf{0}}, \tfrac{3\sqrt{7}}{32}\}_{[-1,1]}, \quad b_5 = \{-\tfrac{1}{8}, \underline{\mathbf{0}}, 0, \tfrac{1}{8}\}_{[-1,2]}, \quad b_6 = \{-\tfrac{1}{8}, 0, \underline{\mathbf{0}}, \tfrac{1}{8}\}_{[-2,1]},$$
$$b_7 = \{-\tfrac{3\sqrt{2}}{32}, \underline{\mathbf{0}}, 0, 0, \tfrac{3\sqrt{2}}{32}\}_{[-1,3]}$$

with $\epsilon_1 = \cdots = \epsilon_4 = 1$ and $\epsilon_5 = \cdots = \epsilon_7 = -1$. Since $\operatorname{sm}(a, 2) \approx 2.440765$, $\{\phi; \psi^1, \ldots, \psi^7\}$ is a quasi-tight 2-framelet in $L_2(\mathbb{R})$, where $\phi, \psi^1, \ldots, \psi^7$ are defined in (5.1.1) with $\mathsf{M} = 2$ and $s = 7$.



(a) ϕ **(b)** ψ^1 **(c)** ψ^2 **(d)** ψ^3

(e) ψ^4 **(f)** ψ^5 **(g)** ψ^6 **(h)** ψ^7

Figure 5.1: In Example 5.2: (a) Refinable function ϕ. (b) - (h) Framelet functions $\psi^1, \ldots, \psi^7$ corresponding to the high-pass filters.

Example 5.3. For $d = 2$, we consider the quincunx dilation matrix $\mathsf{M}_{\sqrt{2}}$ and a low-pass filter a:

$$\mathsf{M}_{\sqrt{2}} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \qquad a = \begin{bmatrix} 0 & \frac{1}{8} & 0 \\ \frac{1}{8} & \underline{\boldsymbol{\frac{1}{2}}} & \frac{1}{8} \\ 0 & \frac{1}{8} & 0 \end{bmatrix}_{[-1,1]\times[-1,1]}. \tag{5.3.2}$$

Using Theorem 5.1.1, we have a directional tight $\mathsf{M}_{\sqrt{2}}$-framelet filter bank

$\{a; b_1, \ldots, b_8\}$, where

$$b_1 = \begin{bmatrix} \underline{-\mathbf{\frac{1}{4}}} & \frac{1}{4} \end{bmatrix}_{[0,1]\times[0,0]}, \quad b_2 = \begin{bmatrix} -\frac{1}{4} & \frac{1}{4} \end{bmatrix}_{[0,1]\times[-1,-1]}, \quad b_3 = \begin{bmatrix} -\frac{1}{4} \\ \frac{1}{4} \end{bmatrix}_{[1,1]\times[-1,0]},$$

$$b_4 = \begin{bmatrix} -\frac{1}{4} \\ \frac{1}{4} \end{bmatrix}_{[1,1]\times[0,1]}, \quad b_5 = \begin{bmatrix} -\frac{1}{8} & \underline{\mathbf{0}} & \frac{1}{8} \end{bmatrix}_{[-1,1]\times[0,0]}, \quad b_6 = \begin{bmatrix} -\frac{1}{8} \\ \underline{\mathbf{0}} \\ \frac{1}{8} \end{bmatrix}_{[0,0]\times[-1,1]},$$

$$b_7 = \begin{bmatrix} \underline{\mathbf{0}} & -\frac{\sqrt{2}}{8} \\ \frac{\sqrt{2}}{8} & 0 \end{bmatrix}_{[0,1]\times[-1,0]}, \quad b_8 = \begin{bmatrix} -\frac{\sqrt{2}}{8} & 0 \\ \underline{\mathbf{0}} & \frac{\sqrt{2}}{8} \end{bmatrix}_{[0,1]\times[0,1]}.$$

Note that $\mathrm{sr}(a, \mathsf{M}_{\sqrt{2}}) = 2$. Since $\mathrm{sm}(a, \mathsf{M}_{\sqrt{2}}) \approx 1.577645$, $\{\phi; \psi^1, \ldots, \psi^8\}$ is a (directional) tight $\mathsf{M}_{\sqrt{2}}$-framelet in $L_2(\mathbb{R}^2)$, where $\phi, \psi^1, \ldots, \psi^8$ are defined in (5.1.1) with $\mathsf{M} = \mathsf{M}_{\sqrt{2}}$ and $s = 7$.

Example 5.4. For $d = 2$, we consider the dilation matrix $\mathsf{M}_{\sqrt{3}}$ and a low-pass filter a as follows:

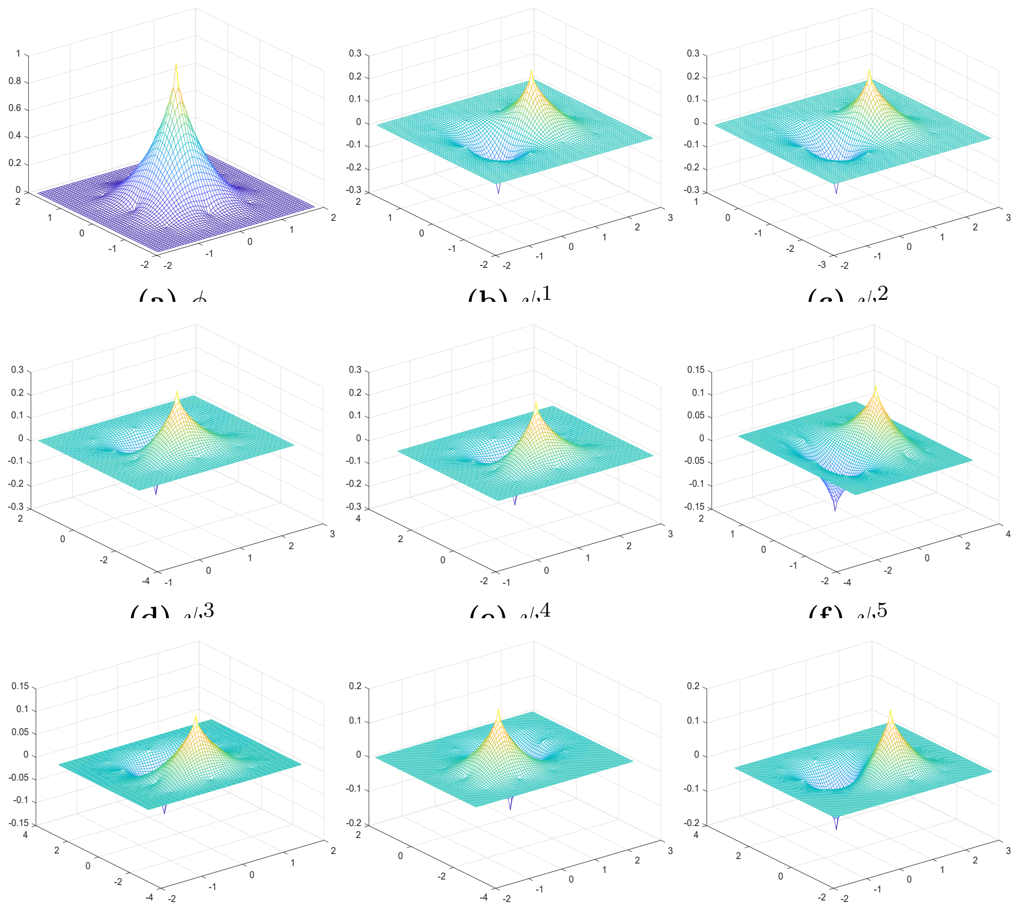
$$\mathsf{M}_{\sqrt{3}} = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}, \qquad a = \begin{bmatrix} 0 & \frac{1}{9} & \frac{1}{9} \\ \frac{1}{9} & \underline{\mathbf{\frac{1}{3}}} & \frac{1}{9} \\ \frac{1}{9} & \frac{1}{9} & 0 \end{bmatrix}_{[-1,1]\times[-1,1]}.$$

Using Theorem 5.1.1, we have a directional tight $\mathsf{M}_{\sqrt{3}}$-framelet filter bank $\{a; b_1, \ldots, b_{18}\}$, where

$$b_1 = \begin{bmatrix} \underline{-\mathbf{\frac{\sqrt{3}}{9}}} & \frac{\sqrt{3}}{9} \end{bmatrix}_{[0,1]\times[0,0]}, \quad b_2 = \begin{bmatrix} -\frac{\sqrt{3}}{9} & \frac{\sqrt{3}}{9} \end{bmatrix}_{[1,2]\times[1,1]}, \quad b_3 = \begin{bmatrix} -\frac{\sqrt{3}}{9} \\ \frac{\sqrt{3}}{9} \end{bmatrix}_{[1,1]\times[-1,0]},$$

$$b_4 = \begin{bmatrix} -\frac{\sqrt{3}}{9} \\ \frac{\sqrt{3}}{9} \end{bmatrix}_{[1,1]\times[1,2]}, b_5 = \begin{bmatrix} -\frac{\sqrt{2}}{9} \\ \frac{\sqrt{2}}{9} \end{bmatrix}_{[1,1]\times[0,1]}, \quad b_6 = \begin{bmatrix} -\frac{\sqrt{2}}{9} & \frac{\sqrt{2}}{9} \end{bmatrix}_{[0,1]\times[1,1]},$$

$$b_7 = \begin{bmatrix} 0 & -\frac{\sqrt{3}}{9} \\ \frac{\sqrt{3}}{9} & 0 \end{bmatrix}_{[1,2]\times[0,1]}, \quad b_8 = \begin{bmatrix} 0 & -\frac{\sqrt{3}}{9} \\ \underline{\mathbf{\frac{\sqrt{3}}{9}}} & 0 \end{bmatrix}_{[0,1]\times[0,1]}, \quad b_9 = \begin{bmatrix} 0 & -\frac{\sqrt{2}}{9} \\ \frac{\sqrt{2}}{9} & 0 \end{bmatrix}_{[1,2]\times[1,2]},$$



(a) ϕ (b) ψ^1 (c) ψ^2

(d) ψ^3 (e) ψ^4 (f) ψ^5

(g) ψ^6 (h) ψ^7 (i) ψ^8

Figure 5.2: In Example 5.3: (a) Refinable function ϕ (b) - (i) Framelet functions $\psi^1, \ldots, \psi^8$ corresponding to the high-pass filters.

$$b_{10} = \begin{bmatrix} -\frac{1}{9} & 0 & \frac{1}{9} \end{bmatrix}_{[1,3]\times[1,1]}, \quad b_{11} = \begin{bmatrix} -\frac{1}{9} \\ 0 \\ \frac{1}{9} \end{bmatrix}_{[1,1]\times[1,3]}, \quad b_{12} = \begin{bmatrix} -\frac{1}{9} & 0 \\ 0 & \frac{1}{9} \end{bmatrix}_{[1,2]\times[-1,0]},$$

$$b_{13} = \begin{bmatrix} -\frac{1}{9} & 0 \\ 0 & \frac{1}{9} \end{bmatrix}_{[1,2]\times[0,1]}, \quad b_{14} = \begin{bmatrix} \underline{\mathbf{0}} & -\frac{1}{9} \\ 0 & 0 \\ \frac{1}{9} & 0 \end{bmatrix}_{[0,1]\times[-2,0]}, \quad b_{15} = \begin{bmatrix} 0 & -\frac{1}{9} \\ \underline{\mathbf{0}} & 0 \\ \frac{1}{9} & 0 \end{bmatrix}_{[0,1]\times[-1,1]},$$

$$b_{16} = \begin{bmatrix} 0 & 0 & -\frac{1}{9} \\ \frac{1}{9} & 0 & 0 \end{bmatrix}_{[1,3]\times[0,1]}, \quad b_{17} = \begin{bmatrix} 0 & 0 & -\frac{1}{9} \\ \frac{1}{9} & 0 & 0 \end{bmatrix}_{[1,3]\times[1,2]}, \quad b_{18} = \begin{bmatrix} 0 & 0 & -\frac{1}{9} \\ 0 & \underline{\mathbf{0}} & 0 \\ \frac{1}{9} & 0 & 0 \end{bmatrix}_{[-1,1]\times[-1,1]}.$$

Since $\operatorname{sm}(a, \mathsf{M}_{\sqrt{3}}) \approx 1.657138$, $\{\phi; \psi^1, \ldots, \psi^{18}\}$ is a (directional) tight $\mathsf{M}_{\sqrt{3}}$-framelet in $L_2(\mathbb{R}^2)$, where $\phi, \psi^1, \ldots, \psi^{18}$ are defined in (5.1.1) with $\mathsf{M} = \mathsf{M}_{\sqrt{3}}$ and $s = 18$.

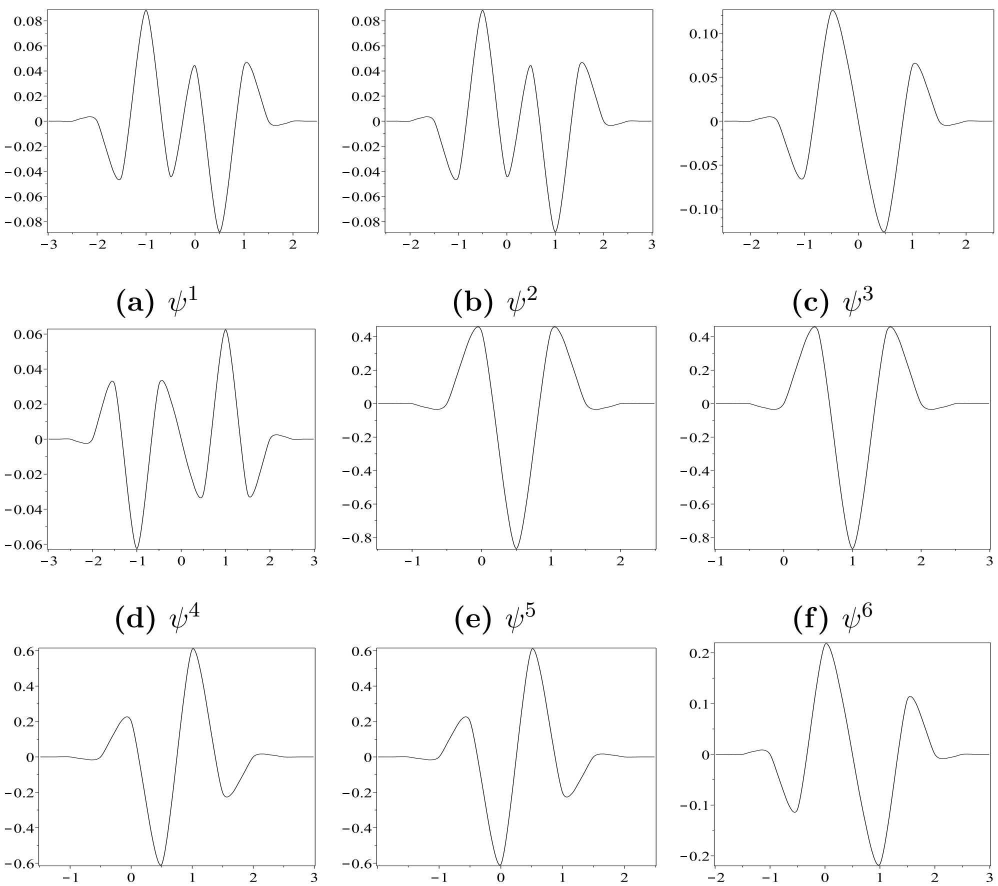
5.3.2 Examples of Multivariate Quasi-tight Framelets with High Order of Vanishing Moments

In this section we shall illustrate Theorem 5.2.2 and Corollary 5.2.3 for constructing quasi-tight framelets with high vanishing moments from arbitrary refinable functions. Let us first present a one-dimensional example to illustrate Corollary 5.2.3 for constructing quasi-tight framelets with all high-pass filters being special differencing filters. Recall that $\widehat{u_a}(\xi) := 1 - |\widehat{a}(\xi)|^2$.

Example 5.5. Consider the interpolatory low-pass filter a in (5.3.1) of Example 5.2. Since $\operatorname{sr}(a, 2) = 4$ and $\operatorname{vm}(u_a) = 4$, according to the inequality in (5.0.1), the highest order of vanishing moments that we can achieve is 2. Using Corollary 5.2.3 with $m = 2$, we have a quasi-tight 2-framelet filter bank $\{a; b_1, \ldots, b_9\}_{(\epsilon_1,\ldots,\epsilon_9)}$, where all the high-pass filters are differencing filters given by

$$
\begin{aligned}
&\widehat{b_1}(\xi) = \tfrac{\sqrt{2}}{32}(1 - e^{-i\xi})^2(1 - e^{3i\xi}), \quad \widehat{b_2}(\xi) = \tfrac{\sqrt{2}}{32}(1 - e^{-i\xi})^2(e^{-i\xi} - e^{2i\xi}),\\
&\widehat{b_3}(\xi) = \tfrac{1}{16}(1 - e^{-i\xi})^2(1 - e^{2i\xi}), \quad \widehat{b_4}(\xi) = \tfrac{1}{32}(1 - e^{-i\xi})^2(e^{3i\xi} - e^{-i\xi}),\\
&\widehat{b_5}(\xi) = \tfrac{\sqrt{3}}{4}(1 - e^{-i\xi})^2, \quad \widehat{b_6}(\xi) = \tfrac{\sqrt{3}}{4}(1 - e^{-i\xi})^2 e^{-i\xi}, \quad \widehat{b_7}(\xi) = \tfrac{\sqrt{42}}{32}(1 - e^{-i\xi})^3,\\
&\widehat{b_8}(\xi) = \tfrac{\sqrt{42}}{32}(1 - e^{-i\xi})^3 e^{i\xi}, \quad \widehat{b_9}(\xi) = \tfrac{\sqrt{3}}{16}(1 - e^{-i\xi})^2(e^{-i\xi} - e^{i\xi}),
\end{aligned}
$$

and $\epsilon_1 = \cdots \epsilon_6 = 1$ and $\epsilon_7 = \cdots = \epsilon_9 = -1$. Note that the high-pass filters b_5 and b_6 have 2 vanishing moments, while all other high-pass filters have 3 vanishing moments. Since $\operatorname{sm}(a, 2) \approx 2.440765$, $\{\phi; \psi^1, \ldots, \psi^9\}_{(\epsilon_1,\ldots,\epsilon_9)}$ is a quasi-tight 2-framelet in $L_2(\mathbb{R})$, where $\phi, \psi^1, \ldots, \psi^9$ are defined in (5.1.1) with $\mathsf{M} = 2$ and $s = 9$. Note that all the functions $\psi^1, \ldots, \psi^9$ have at least 2 vanishing moments.



(a) ψ^1 (b) ψ^2 (c) ψ^3

(d) ψ^4 (e) ψ^5 (f) ψ^6

(g) ψ^7 (h) ψ^8 (i) ψ^9

Figure 5.3: In Example 5.5: (a) - (i) Framelet functions $\psi^1, \ldots, \psi^9$ corresponding to the high-pass filters.

Example 5.6. Consider the bivariate low-pass filter

$$a = \begin{bmatrix} -\frac{1}{16} & \frac{1}{8} & -\frac{1}{16} \\ \frac{1}{8} & \underline{\mathbf{\frac{3}{4}}} & \frac{1}{8} \\ -\frac{1}{16} & \frac{1}{8} & -\frac{1}{16} \end{bmatrix}_{[-1,1]\times[-1,1]}.$$

Since $\operatorname{sr}(a, \mathsf{M}_{\sqrt{2}}) = 2$ and $\operatorname{vm}(u_a) = \operatorname{lpm}(a) = 4$, according to the inequality in (5.0.1), the highest order of vanishing moments that we can achieve is 2. Using Theorem 5.2.2 with $m = 2$, we obtain a quasi-tight $\mathsf{M}_{\sqrt{2}}$-framelet filter bank $\{a; b_1, \ldots, b_7\}_{(\epsilon_1,\ldots,\epsilon_7)}$, where

$$
\begin{aligned}
\widehat{b_1}(\xi) &= \tfrac{\sqrt{2}}{2}(1-e^{-i\xi_2})^2 - \tfrac{\sqrt{2}}{256}(1-e^{-i\xi_1})(1-e^{-i\xi_2})(1-e^{-i(\xi_2-\xi_1)}),\\
\widehat{b_2}(\xi) &= \tfrac{\sqrt{2}}{2}(1-e^{-i\xi_2})^2 - \tfrac{\sqrt{2}}{512}(1-e^{-i\xi_1})^2(e^{-2i\xi_2}-1-16e^{-i(\xi_2-\xi_1)}),\\
\widehat{b_3}(\xi) &= \tfrac{129\sqrt{2}}{256}(1-e^{-i\xi_1})^2 + \tfrac{\sqrt{2}}{256}(1-e^{-i\xi_1})(1-e^{-i\xi_2})(e^{-i\xi_1}-2+e^{-i(\xi_1-2\xi_2)}),\\
\widehat{b_4}(\xi) &= \tfrac{\sqrt{2}}{2}(1-e^{-i\xi_2})^2 + \tfrac{\sqrt{2}}{256}(1-e^{-i\xi_1})(1-e^{-i\xi_2})(1-e^{-i(\xi_2-\xi_1)}),\\
\widehat{b_5}(\xi) &= \tfrac{\sqrt{2}}{2}(1-e^{-i\xi_2})^2 + \tfrac{\sqrt{2}}{512}(1-e^{-i\xi_1})^2(e^{-2i\xi_2}-1-16e^{-i(\xi_2-\xi_1)}),\\
\widehat{b_6}(\xi) &= \tfrac{127\sqrt{2}}{256}(1-e^{-i\xi_1})^2 - \tfrac{\sqrt{2}}{256}(1-e^{-i\xi_1})(1-e^{-i\xi_2})(e^{-i\xi_1}-2+e^{-i(\xi_1-2\xi_2)}),\\
\widehat{b_7}(\xi) &= \tfrac{1}{8}(1-e^{-i\xi_1})^2e^{-i\xi_1}
\end{aligned}
$$

with $\epsilon_1 = \epsilon_2 = \epsilon_3 = 1$, and $\epsilon_4 = \epsilon_5 = \epsilon_6 = \epsilon_7 = -1$. All the high-pass filters have at least 2 vanishing moments. Since $\mathrm{sm}(a, \mathsf{M}_{\sqrt{2}}) \approx 0.235724$, $\{\phi; \psi^1, \ldots, \psi^7\}_{(\epsilon_1,\ldots,\epsilon_7)}$ is a quasi-tight $\mathsf{M}_{\sqrt{2}}$-framelet in $L_2(\mathbb{R}^2)$, where $\phi, \psi^1, \ldots, \psi^7$ are defined in (5.1.1) with $\mathsf{M} = \mathsf{M}_{\sqrt{2}}$ and $s = 7$. Note that all the functions $\psi^1, \ldots, \psi^7$ have at least 2 vanishing moments.

Example 5.7. For $d = 2$, we consider the low-pass filter

$$
a = \begin{bmatrix}
0 & 0 & -\frac{1}{16} & 0 & 0\\
0 & \frac{1}{16} & \frac{1}{8} & \frac{1}{16} & 0\\
-\frac{1}{16} & \frac{1}{8} & \underline{\mathbf{\frac{1}{2}}} & \frac{1}{8} & -\frac{1}{16}\\
0 & \frac{1}{16} & \frac{1}{8} & \frac{1}{16} & 0\\
0 & 0 & -\frac{1}{16} & 0 & 0
\end{bmatrix}_{[-2,2]\times[-2,2]}.
$$

Since $\mathrm{sr}(a, \mathsf{M}_{\sqrt{2}}) = 2$ and $\mathrm{vm}(u_a) = \mathrm{lpm}(a) = 4$, according to the inequality in (5.0.1), the highest order of vanishing moments that we can achieve is 2. Using Corollary 5.2.3 with $m = 2$, we obtain a quasi-tight $\mathsf{M}_{\sqrt{2}}$-framelet filter bank $\{a; b_1, \ldots, b_{19}\}_{(\epsilon_1,\ldots,\epsilon_{19})}$, where all the high-pass filters are differencing filters given by

$$
\begin{aligned}
&\widehat{b_1}(\xi) = \tfrac{1}{16}(1-e^{2i\xi_2})(1-e^{-i\xi_2})^2, \qquad \widehat{b_2}(\xi) = \tfrac{\sqrt{6}}{8}(1-e^{-i\xi_2})^2,\\
&\widehat{b_3}(\xi) = \tfrac{\sqrt{2}}{8}e^{-i\xi_1}(1-e^{-i\xi_2})^2, \quad \widehat{b_4}(\xi) = \tfrac{3}{16}(1-e^{-i(\xi_2-\xi_1)})(1-e^{-i\xi_1})(1-e^{-i\xi_2}),
\end{aligned}
$$

$\widehat{b_5}(\xi) = \frac{3}{16}(e^{-i\xi_1} - e^{i\xi_2})(1 - e^{-i\xi_1})(1 - e^{-i\xi_2}), \quad \widehat{b_6}(\xi) = \frac{\sqrt{3}}{16}(1 - e^{-2i\xi_2})(1 - e^{-i\xi_1})^2,$
$\widehat{b_7}(\xi) = \frac{1}{16}(1 - e^{2i\xi_1})(1 - e^{-i\xi_1})^2, \qquad \widehat{b_8}(\xi) = \frac{\sqrt{6}}{8}(1 - e^{-i\xi_1})^2,$
$\widehat{b_9}(\xi) = \frac{\sqrt{2}}{8}e^{-i\xi_1}(1 - e^{-i\xi_1})^2, \qquad \widehat{b_{10}}(\xi) = \frac{\sqrt{2}}{16}(1 - e^{-i\xi_1})(1 - e^{-i\xi_2})^2,$
$\widehat{b_{11}}(\xi) = \frac{\sqrt{2}}{16}(1 - e^{i(\xi_1+\xi_2)})(1 - e^{-i\xi_2})^2, \qquad \widehat{b_{12}}(\xi) = \frac{\sqrt{2}}{16}(1 - e^{-i(\xi_2-\xi_1)})(1 - e^{-i\xi_2})^2,$
$\widehat{b_{13}}(\xi) = \frac{\sqrt{10}}{32}(e^{-i\xi_1} - e^{i\xi_1})(1 - e^{-i\xi_2})^2, \qquad \widehat{b_{14}}(\xi) = \frac{\sqrt{2}}{16}(e^{-2i\xi_1} - e^{-i\xi_1})(1 - e^{-i\xi_2})^2,$
$\widehat{b_{15}}(\xi) = \frac{\sqrt{2}}{16}(1 - e^{i(\xi_1+\xi_2)})(1 - e^{-i\xi_1})^2, \qquad \widehat{b_{16}}(\xi) = \frac{1}{4}(1 - e^{-i\xi_1})(1 - e^{-i\xi_2}),$
$\widehat{b_{17}}(\xi) = \frac{1}{4}e^{-i\xi_1}(1 - e^{-i\xi_1})(1 - e^{-i\xi_2}), \quad \widehat{b_{18}}(\xi) = \frac{\sqrt{2}}{16}(1 - e^{-i(\xi_2-\xi_1)})(1 - e^{-i\xi_1})^2,$
$\widehat{b_{19}}(\xi) = \frac{\sqrt{26}}{32}(e^{-i(\xi_1+2\xi_2)} - e^{-i\xi_1})(1 - e^{-i\xi_1})^2,$

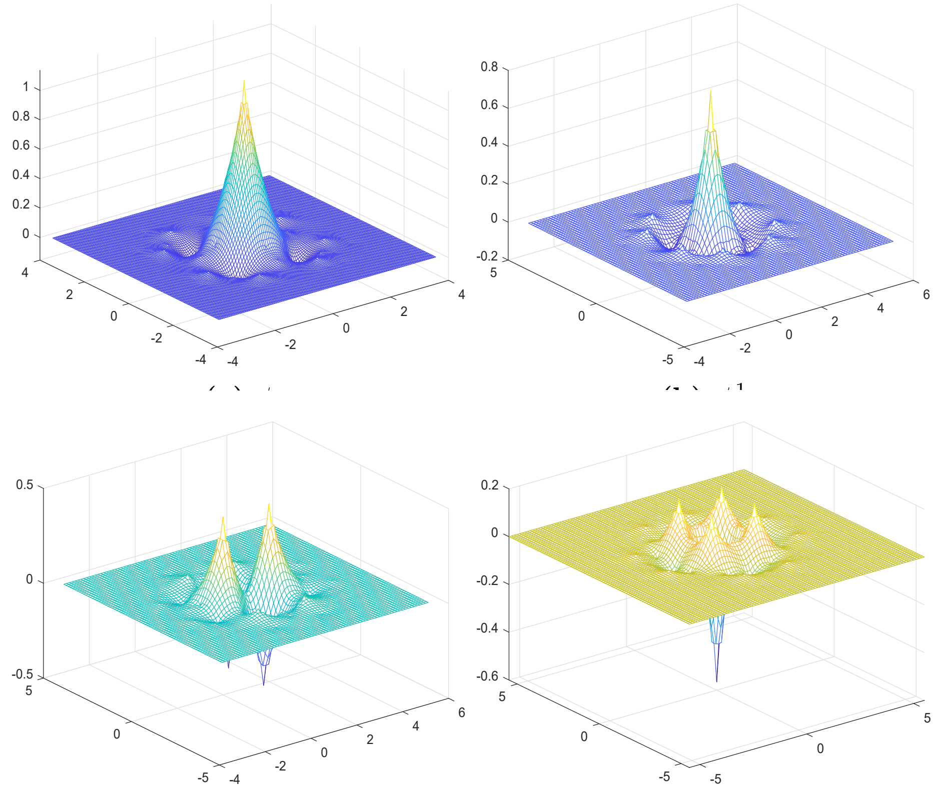
and $\epsilon_1 = \cdots = \epsilon_9 = 1$ and $\epsilon_{10} = \cdots = \epsilon_{19} = -1$. Since $\mathrm{sm}(a, \mathsf{M}_{\sqrt{2}}) \approx 1.801593$, $\{\phi; \psi^1, \ldots, \psi^{19}\}_{(\epsilon_1,\ldots,\epsilon_{19})}$ is a quasi-tight $\mathsf{M}_{\sqrt{2}}$-framelet in $L_2(\mathbb{R}^2)$, where $\phi, \psi^1, \ldots, \psi^{19}$ are defined in (5.1.1) with $\mathsf{M} = \mathsf{M}_{\sqrt{2}}$ and $s = 19$. Note that all the functions $\psi^1, \ldots, \psi^{19}$ have at least 2 vanishing moments.

Without requiring differencing high-pass filters, we can obtain a quasi-tight $\mathsf{M}_{\sqrt{2}}$-framelet filter bank $\{a; b_1, b_2, b_3\}_{(1,1,-1)}$, where

$$b_1 = \begin{bmatrix} 0 & -\frac{\sqrt{2}}{8} & 0 \\ \underline{-\frac{\sqrt{\mathbf{2}}}{\mathbf{8}}} & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{8} \\ 0 & -\frac{\sqrt{2}}{8} & 0 \end{bmatrix}_{[0,2]\times[-1,1]}, \qquad b_2 = \begin{bmatrix} 0 & -\frac{\sqrt{6}}{8} & 0 \\ \underline{\frac{\sqrt{\mathbf{6}}}{\mathbf{8}}} & 0 & \frac{\sqrt{6}}{8} \\ 0 & -\frac{\sqrt{6}}{8} & 0 \end{bmatrix}_{[0,2]\times[-1,1]},$$

$$b_3 = \begin{bmatrix} 0 & 0 & -\frac{1}{16} & 0 & 0 \\ 0 & \frac{1}{16} & \frac{1}{8} & \frac{1}{16} & 0 \\ -\frac{1}{16} & \frac{1}{8} & \underline{-\frac{\mathbf{1}}{\mathbf{2}}} & \frac{1}{8} & -\frac{1}{16} \\ 0 & \frac{1}{16} & \frac{1}{8} & \frac{1}{16} & 0 \\ 0 & 0 & -\frac{1}{16} & 0 & 0 \end{bmatrix}_{[-2,2]\times[-2,2]}$$

with $\mathrm{vm}(b_1) = \mathrm{vm}(b_2) = 2$, and $\mathrm{vm}(b_3) = 4$. Since $\mathrm{sm}(a, \mathsf{M}_{\sqrt{2}}) \approx 1.801593$, $\{\phi; \psi^1, \psi^2, \psi^3\}_{(1,1,-1)}$ is a quasi-tight $\mathsf{M}_{\sqrt{2}}$-framelet in $L_2(\mathbb{R}^2)$, where $\phi, \psi^1, \psi^2, \psi^3$ are defined in (5.1.1) with $\mathsf{M} = \mathsf{M}_{\sqrt{2}}$ and $s = 3$. Note that all the functions ψ^1, ψ^2, ψ^3 have at least 2 vanishing moments.



(c) ψ^2 **(d)** ψ^3

Figure 5.4: In Example 5.7: (a) Refinable function ϕ. (b) - (d) Framelet functions ψ^1, ψ^2, ψ^3 corresponding to the high-pass filters.

Example 5.8. For $d = 2$, we consider the following bivariate low-pass filter

$$a = \begin{bmatrix} 0 & 0 & -\frac{1}{64} & -\frac{1}{32} & -\frac{1}{64} \\ 0 & -\frac{1}{32} & \frac{5}{32} & \frac{5}{32} & -\frac{1}{32} \\ -\frac{1}{64} & \frac{5}{32} & \underline{\mathbf{\frac{11}{32}}} & \frac{5}{32} & -\frac{1}{64} \\ -\frac{1}{32} & \frac{5}{32} & \frac{5}{32} & -\frac{1}{32} & 0 \\ -\frac{1}{64} & -\frac{1}{32} & -\frac{1}{64} & 0 & 0 \end{bmatrix}_{[-2,2]\times[-2,2]}.$$

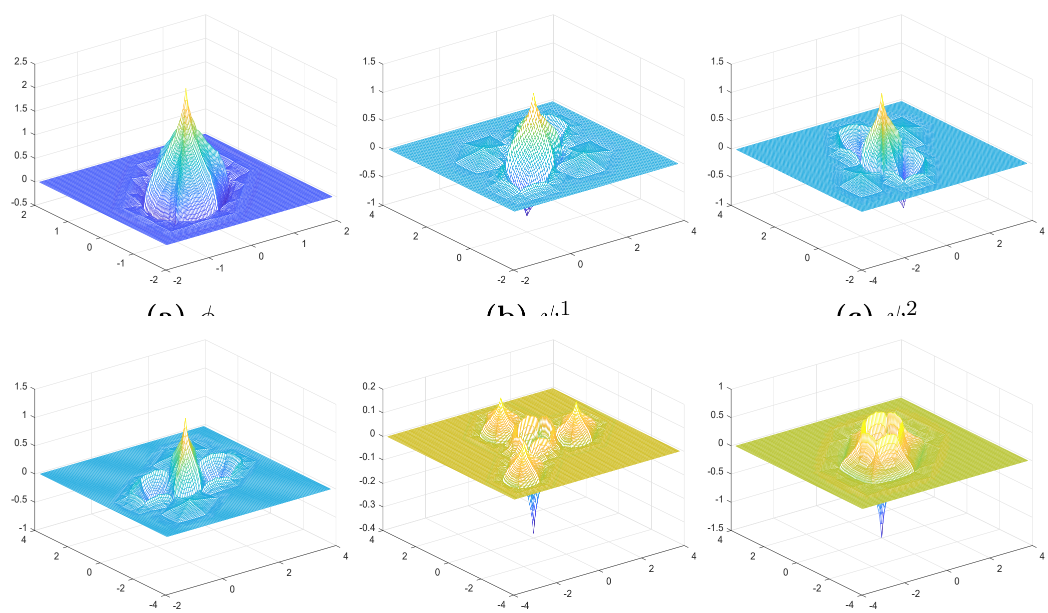
Since $\operatorname{sr}(a, 2I_2) = 2$ and $\operatorname{vm}(u_a) = \operatorname{lpm}(a) = 4$, according to the inequality in (5.0.1), the highest order of vanishing moments that we can achieve is 2. Using Theorem 5.2.1 with $m = 2$, we obtain a quasi-tight $2I_2$-framelet filter bank $\{a; b_1, \ldots, b_5\}_{(\epsilon_1,\ldots,\epsilon_5)}$, where

$$b_1 = \begin{bmatrix} \frac{1}{16} & 0 & -\frac{5}{16} \\ 0 & \frac{1}{2} & 0 \\ \underline{-\boldsymbol{\frac{5}{16}}} & 0 & \frac{1}{16} \end{bmatrix}_{[0,2]\times[0,2]}, \quad b_2 = \begin{bmatrix} 0 & 0 & -\frac{5}{16} & 0 & \frac{1}{16} \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{16} & 0 & \underline{-\boldsymbol{\frac{5}{16}}} & 0 & 0 \end{bmatrix}_{[-2,2]\times[0,2]},$$

$$b_3 = \begin{bmatrix} 0 & 0 & \frac{1}{16} \\ 0 & 0 & 0 \\ \underline{-\boldsymbol{\frac{5}{16}}} & \frac{1}{2} & -\frac{5}{16} \\ 0 & 0 & 0 \\ \frac{1}{16} & 0 & 0 \end{bmatrix}_{[0,2]\times[-2,2]}, \quad b_4 = \begin{bmatrix} 0 & 0 & \frac{1}{16} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \underline{-\boldsymbol{\frac{3}{16}}} & 0 & \frac{1}{16} \\ 0 & 0 & 0 & 0 & 0 \\ \frac{1}{16} & 0 & 0 & 0 & 0 \end{bmatrix}_{[-2,2]\times[-2,2]},$$

$$b_5 = \begin{bmatrix} 0 & 0 & -\frac{1}{64} & -\frac{1}{32} & -\frac{1}{64} \\ 0 & -\frac{1}{32} & \frac{5}{32} & \frac{5}{32} & -\frac{1}{32} \\ -\frac{1}{64} & \frac{5}{32} & \underline{-\boldsymbol{\frac{21}{32}}} & \frac{5}{32} & -\frac{1}{64} \\ -\frac{1}{32} & \frac{5}{32} & \frac{5}{32} & -\frac{1}{32} & 0 \\ -\frac{1}{64} & -\frac{1}{32} & -\frac{1}{64} & 0 & 0 \end{bmatrix}_{[-2,2]\times[-2,2]}$$

and $\epsilon_1 = \epsilon_2 = \epsilon_3 = 1$ and $\epsilon_4 = \epsilon_5 = -1$. Note that $\mathrm{vm}(b_1) = \ldots = \mathrm{vm}(b_4) = 2$ and $\mathrm{vm}(b_5) = 4$. Since $\mathrm{sm}(a, 2I_2) \approx 0.885296$, $\{\phi; \psi^1, \ldots, \psi^5\}_{(\epsilon_1,\ldots,\epsilon_5)}$ is a quasi-tight $2I_2$-framelet in $L_2(\mathbb{R}^2)$, where $\phi, \psi^1, \ldots, \psi^5$ are defined in (5.1.1) with $\mathsf{M} = 2I_2$ and $s = 5$. Note that all the functions $\psi^1, \ldots, \psi^5$ have at least 2 vanishing moments.



(a) ϕ (b) ψ^1 (c) ψ^2

(d) ψ^3 (e) ψ^4 (f) ψ^5

Figure 5.5: In Example 5.8: (a) Refinable function ϕ. (b) - (f) Framelet functions $\psi^1, \ldots, \psi^5$ corresponding to the high-pass filters.

Chapter 6

Conclusions and Future Work

In this thesis, we studied the construction of OEP-based quasi-tight framelets in both univariate and multivariate cases. For the one-dimensional case studied in Chapter 2 and Chapter 3, we proved that given a refinable function $\phi \in L_2(\mathbb{R})$ derived from an arbitrary low-pass filter $a \in l_0(\mathbb{Z})$, $\widehat{a}(0) = 1$ and an arbitrary moment correcting filter $\Theta \in l_0(\mathbb{Z})$, $\Theta^\star = \Theta$, $\widehat{\Theta}(0) = 1$, we can always derive a homogeneous quasi-tight framelet $\{\psi_1, \ldots, \psi_s\}_{(\epsilon_1,\ldots,\epsilon_s)} \subset L_2(\mathbb{R}^d)$ with highest possible order of vanishing moments, for all $s \geqslant \max_{z\in\mathbb{T}} \nu_+(\mathcal{M}_{\mathsf{a},\Theta}(z)) + \max_{z\in\mathbb{T}} \nu_-(\mathcal{M}_{\mathsf{a},\Theta}(z))$. Our construction is based on the generalized spectral factorization of Hermitian matrices of Laurent polynomials in Theorem 2.3.1 and Theorem 3.1.2. In Chapter 4, we studied the quasi-tight framelet system $\{\psi_1, \psi_2\}_{(1,-1)}$ with symmetry. Given a low-pass filter $a \in l_0(\mathbb{Z})$ and a moment correcting filter $\Theta \in l_0(\mathbb{Z})$ with symmetry, such that $\mathsf{a}(1) = \Theta(1) = 1$ and $\Theta^\star = \Theta$, we find the necessary and sufficient conditions for the existence of quasi-tight framelet filter bank $\{a; b_1, b_2\}_{\Theta,(1,-1)}$ with symmetry in Theorem 4.4.2. The construction is also based on the result of spectral factorizations of 2×2 matrices of Laurent polynomials with symmetry in Theorem 4.3.6. In the multivariate case studied in Chapter 5, given an arbitrary dilation matrix M and an M-refinable function $\phi \in L_2(\mathbb{R}^d)$ derived from some real-valued low-pass filter $a \in l_0(\mathbb{Z}^d)$, $\widehat{a}(0) = 1$, we proved that we can always derive a quasi-tight framelet $\{\phi; \psi_1, \ldots, \psi_s\} \subset L_2(\mathbb{R}^d)$, with directionality or highest possible order of vanishing moments.

There are a few related questions that are still unresolved, which could be future research problems.

In the univariate case, we might still be able to get some stronger versions of the Theorem 2.3.1 and Theorem 3.1.2 on spectral factorizations of matrices of Laurent polynomials. For the positive semi-definite version of Theorem 2.3.1 (known as Matrix-Valued Fejér-Riesz Lemma), it can be shown (see [54]) that if the given Hermitian matrix $\mathsf{A}(z)$ of Laurent polynomials has real coefficients, then the factorized matrix $\mathsf{U}(z)$ of Laurent polynomials can also have real coefficients. However, the construction procedure in our proof of Theorem 2.3.1 relies on the Theorem 2.3.5, which cannot guarantee real factorizations in general. We might need some new construction ideas to get real solutions.

Also, the construction algorithms we built in Chapters 2, 3 and 4 generally require steps of finding roots of Laurent polynomials in $\mathbb{C} \setminus \{0\}$. This type of algorithms is good enough in the theoretical proof of the existence of the solution. However, they are not numerically stable in the computation. Numerical aspects of spectral factorizations of matrices of polynomials/Laurent polynomials have been studied using different approaches in the literature (for example, see [56, 26, 89, 1] and many references therein). But to the best of our knowledge, there is no result built in the setting of Theorem 2.3.1 or Theorem 3.1.2. This could also be a future research topic.

Although there are quite a few results in constructing framelets with symmetry, characterizing all possible framelets with symmetry is still not an easy task. Using two framelet generators, [40, 48] characterized all the possible tight framelet filter banks $\{a; b_1, b_2\}_\Theta$ with symmetry, where the conditions are similar to those of our Theorem 4.3.6 and Theorem 4.4.2 for quasi-tight framelet filter bank $\{a; b_1, b_2\}_{\Theta,(1,-1)}$. For tight framelets with three generators, [43] classified the possible symmetry types of the high-pass filters into two cases, and fully characterized one of them. Therefore, we can see that the complete characterizations of general tight/quasi-tight framelets with symmetry (or spectral factorizations of matrices of Laurent polynomials with symmetry) are still unknown in the literature.

In the multivariate case, since our construction is so flexible that we can use any arbitrary low-pass filter a and dilation matrix M to construct high-pass filters with directionality, we might try to apply such transforms on signal processing tasks. Besides the directionality property, another benefit of using the framelets we constructed in Section 5.1 is that the high-pass filters are only supported on two points. In the signal processing applications, we need to calculate the discrete framelet/wavelet transform of the 2D/3D signal using the filter bank. The bottleneck of the computational speed is usually the lack of efficiency in the convolution step of the signal x and the filter b_ℓ. For nonseparable filters, if the size of the support of the filter b_ℓ is large, we normally have to use FFT(Fast Fourier Transform) to convert the calculation into frequency domain. As a matter of fact, in many applications, to get translation invariance, people often use undecimated transforms, where the filters need to be upsampled after each level/scale. So the size of the support of the filters grows exponentially for multi-scale transforms. However, the high-pass filters we constructed in Section 5.1 are only supported on two points. So to compute the convolution $x * b_\ell$, we just need to shift the signal x twice and compute their linear combination. This would be very efficient for high dimensional signal processing problems. As the mechanism to use quasi-tight framelets with general dilation matrices is still unknown, this could also lead to future research problems.

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