University of Alberta

FØLNER CONDITIONS AND SEMIDIRECT PRODUCTS RELATED TO AMENABILITY OF SEMIGROUPS AND GROUPS

by

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Abstract

This thesis examines relationships between various Følner-type conditions and amenability, with a focus on semidirect products.

Chapter two deals with semidirect products of locally compact groups. Two methods are developed for finding Følner nets for semidirect products based on Følner nets for the original groups.

Chapter three looks at Sorenson's conjecture. Klawe provided a counterexample to the conjecture that all left amenable right cancellative semigroups are left cancellative. Her example is shown to be weakly left cancellative. An example of a left amenable, right cancellative semigroup which is not weakly left cancellative is given. The final result is a necessary condition for a semidirect product of semigroups to be amenable.

The fourth chapter generalizes various concepts to left semigroup actions. A proof of Dixmier's condition is given. Using this, a semigroup is shown to be amenable exactly when all left semigroup actions of the semigroup are also amenable.

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Chapter 1

Introduction and Background

1.1 Introduction

A semigroup is a set equipped with an associative multiplication. We call a semigroup, S, left amenable if there exists a left invariant mean on the space of bounded functions, $\ell^\infty(S)$. A left invariant mean is a positive linear functional of norm one on $\ell^\infty(S)$ which provides 'average values' for bounded functions and is invariant under left translations. This definition was introduced by M. Day in [3], but the underlying concepts can be traced back to invariant measure theory studied by Lebesgue in the early 1900s. Results such as the famous Banach-Tarski paradox are related in large part to the study of amenability.

The body of this thesis contains three chapters. Each of these chapters deals with a different type of abstraction or example of left amenable semigroups. Firstly, in chapter two, the existence of inverses and an identity are imposed on the algebraic structure and a topology is introduced for discussion of amenability on a locally compact group, G. In chapter three, the results apply to all semigroups, but some remarks are made with special regard to cancellative semigroups. Finally, in chapter four, the concept of a left semigroup action on a set, X, is introduced and the definition of amenability is naturally extended to this generalization.

The definition of amenability involves properties of $\ell^\infty(S)$ (or $L^\infty(G)$ or $\ell^\infty(X)$) and finding a left invariant mean often involves nonconstructive methods such as the application of the Hahn-Banach theorem. In light of this, it is useful to consider properties of the internal structure of the semigroup itself which imply amenability, rather than rely on the structure of the more complicated $\ell^\infty(S)$. Work in this area was done by Følner [6] when he showed that a discrete group is left amenable if and only if it satisfies the, so called, Følner condition. Namioka extended Følner's work in [13] and came up with several related conditions. The key to showing that S satisfies one of these Følner-type conditions is to find a subset of S which overlaps largely with translated copies of itself.

Determining the structure of S and determining whether it satisfies any of the Følner-type conditions is generally not an easy task. This task is simplified if S is the semidirect product of simpler semigroups. This method of combining two semigroups to create another semigroup is useful for creating a semigroup with more complicated structure than either of the original two. In this thesis, several examples of of semidirect products are considered. We also show how amenability and Følner-type conditions can be extended to a semidirect product.

In chapter two, we develop two methods for finding Følner nets for a semidirect product of amenable locally compact groups. Both methods provide a Følner net for a semidirect product based upon Følner nets for the original two groups. The first method is a generalization of some recent results of Janzen [11]. This method provides a Følner net which is 'rectangular' in shape, making it easy to work with, but requires some additional conditions on the groups for the given construction to yield a Følner net. The second method is a more detailed examination of a method described briefly by Greenleaf in [8]. This method works for every semidirect product of locally compact amenable groups, but results in a Følner net with a 'trapezoidal', rather than rectangular, shape.

The third chapter deals with Følner-type conditions and related properties on semigroups, and provides some new insights on a conjecture of Sorenson. Sorenson conjectured that all right cancellative left amenable semigroups were left cancellative. Klawe disproved this conjecture by providing a counterexample in [12]. We show that Klawe's counterexample is weakly left cancellative. We then consider a weaker version of Sorenson's conjecture - that all right cancellative left amenable semigroups are weakly left cancellative - and disprove it. The final section of the chapter is dedicated to the proof of a necessary condition for a semidirect product of semigroups to be left amenable. This condition was inspired by a sufficient condition given by Klawe, and the two conditions are equivalent if one of the semigroups is right cancellative.

The fourth chapter considers replacing the role of the semigroup multiplication by the action of a semigroup on a set X. Two of the results of Namioka presented in [13] are generalized. The chapter concludes by proving that a semigroup is left amenable if and only if all left semigroup actions involving that semigroup are amenable.

In the fifth, and final, chapter we give some suggestions for future work in this area. We also provide unanswered problems relating to the previous chapters.

1.2 Background

Definition 1.2.1. Let X be a set. We use $\ell^\infty(X)$ to denote the Banach space of real-valued bounded functions on X with the standard supremum norm. Let $\mathcal{Y}$ be a closed subspace of $\ell^\infty(X)$ which contains all the constant functions. A functional in the dual space of $\mathcal{Y}$, $m \in \mathcal{Y}^*$, is called a *mean* on $\mathcal{Y}$ if

$$\|m\| = 1 = m(\chi_X)$$

where χ_X is the characteristic function for the whole set; in other words, χ_X is the constant function which is defined to be 1 at every point in X.

Definition 1.2.2. A semigroup S is *left amenable* if there exists a mean, m, on $\ell^\infty(S)$ which is left invariant. A mean, m, is left invariant if for any $f \in \ell^\infty(S)$ we have $m(f) = m(l_s f)$ for any $s \in S$, where $l_s f(t) = f(st)$.

Remark 1.2.3. *Let S be a semigroup, $A \subset S$ and $s \in S$. We write $s^{-1}A$ to denote the set $\{x \in S : \ sx \in A\}$.*

Remark 1.2.4. *A mean $m \in \ell^\infty(S)^*$ is left invariant if and only if $m(\chi_{s^{-1}A}) = m(\chi_A)$ for any $A \subset S$ and any $s \in S$.*

Example 1.2.5. *Let $\mathbb{N}$ be the set of natural numbers equipped with the standard addition. $\mathbb{N}$ is a semigroup since addition is associative. However, $\mathbb{N}$ is not a group since it does not contain the additive inverse of any given element. It is well known that every abelian semigroup is left amenable (this is shown in 3.4.1), so $\mathbb{N}$ is left amenable.*

In fact, it is not necessary to use this result to see that $\mathbb{N}$ is left amenable. To show left amenability consider a Banach limit on $\ell^\infty(\mathbb{N})$. A Banach limit, LIM, is a bounded linear functional satisfying:

$$\liminf_{n\to\infty}(x_n) \leq LIM((x_n)_{n=1}^\infty) \leq \limsup_{n\to\infty}(x_n)$$

and

$$LIM((x_n)_{n=1}^\infty) = LIM((x_{n+1})_{n=1}^\infty).$$

The existence of such a functional is a straightforward consequence of the Hahn-Banach Theorem. It is easy to see that a Banach limit is a left invariant mean.

Definition 1.2.6. A semigroup S is said to satisfy the *Følner condition* if for any finite $F \subset S$, and any $\epsilon > 0$ there exists a finite and non-empty $A \subset S$ such that:

$$|sA \backslash A| < \epsilon |A| \quad \forall s \in F.$$

Definition 1.2.7. A semigroup S is said to satisfy the *strong Følner condition* if for any finite $F \subset S$, and any $\epsilon > 0$ there exists a finite and non-empty $A \subset S$ such that:

$$|A \backslash sA| < \epsilon |A| \quad \forall s \in F.$$

The following two results connect the Følner conditions and left amenability.

Theorem 1.2.8. *Let S be a left amenable semigroup. Then S satisfies the Følner condition.*

Proof. See Theorem 3.5 in [13]. □

Proposition 1.2.9. *Let S be a semigroup satisfying the strong Følner condition. Then S is left amenable.*

Proof. See Section 2 in [1]. □

Remark 1.2.10. *If S is left cancellative (eg. if S is a group), then the strong Følner and Følner conditions are equivalent because $|A| = |sA|$, hence:*

$$|A \backslash sA| = |A| - |A \cap sA| = |sA| - |A \cap sA| = |sA \backslash A|.$$

Chapter 2

Følner Nets and Semidirect Products of Locally Compact Groups

2.1 Introduction

In this chapter we examine properties of semidirect products of amenable groups. In particular, we look at how the modular function of the semidirect product group depends on the modular functions of the groups which compose the semidirect product. We also examine the concept of Følner nets and look at the structure of products of Følner nets in a semidirect product group.

This chapter extends work done by Janzen, presented in [11]. Janzen examined the case where the groups are all unimodular. We look at non-unimodular groups.

2.2 Definitions and Background

In this section we provide the background and definitions necessary to examine the topics of this chapter. We review four main ideas relating to locally compact groups. The first of these is the left Haar measure for a locally compact group. The second is the concept of a Følner net, which is fundamentally the same as the Følner condition. The third is the definition of the modular function. This is a way of expressing how close the left Haar measure is to being right invariant. The final concept is that of semidirect products of groups. Once we have reviewed these concepts we will, in later sections, see how the modular function of a semidirect product relies on the modular functions of the subgroups, and how one might create a Følner net for a semidirect product using Følner nets for the subgroups.

Definition 2.2.1. A *locally compact group*, G, is a group equipped with a Hausdorff topology such that the maps:

$$G \times G \to G : (x, y) \mapsto xy$$

and

$$G \to G : x \mapsto x^{-1}$$

are continuous.

2.2.1 Left Haar Measure

Definition 2.2.2. A *left Haar measure* on G is a non-zero positive measure, λ_G, on the $\sigma-$algebra of Borel subsets of G, $\mathcal{B}(G)$, satisfying all of the following:

- outer regular on all Borel sets, ie, the measure of a Borel set can be approximated from the outside by open sets;
- inner regular on all open sets, ie, the measure of an open set can be approximated from the inside by compact sets;
- finite on all compact sets;
- invariant under left translation, ie, $\lambda_G(E) = \lambda_G(xE)\ \ \forall E \in \mathcal{B}(G), \forall x \in G$.

Example 2.2.3. *For any discrete group, G, the counting measure is a left Haar measure.*

Example 2.2.4. *For the real numbers, $\mathbb{R}$, the Lebesgue measure is a left Haar measure.*

There are a few well known results regarding the existence and uniqueness of left Haar measures on locally compact groups. For a locally compact group G, we shall denote the left Haar measure of G by λ_G, or if there is no confusion as to which group we are considering, λ.

Theorem 2.2.5. *If G is a locally compact group, then there exists a left Haar measure λ on G. Furthermore, if μ is also a left Haar measure on G, then there exists a constant $c \in \mathbb{R}^+$ such that $\lambda = c\mu$.*

Proof. See Theorem 15.5 [10]. □

Remark 2.2.6. *We occasionally refer to* the *left Haar measure of a locally compact group although uniqueness only holds up to a constant multiple.*

Remark 2.2.7. *The analog of the above theorem is true for the existence and uniqueness of a right Haar measure. However, we warn the reader that except in the case where G is abelian, discrete, or compact, it is unlikely that the left and right Haar measures are the same. For example, the left Haar measure of the 'ax+b' group(as defined in Example 2.3.3) is $\frac{1}{a}$ times the right Haar measure.*

2.2.2 Følner Nets

Definition 2.2.8. A locally compact group, G, is (left) amenable if there exists a left invariant mean on $L^\infty(G)$. Here, $L^\infty(G)$ is the space of equivalence classes of λ_G-measurable, essentially bounded functions.

Remark 2.2.9. *Definition 2.2.8 is the topological version of left amenability we saw earlier for semigroups. If G is discrete, then $L^\infty(G)$ can be identified with $\ell^\infty(G)$ and the definitions of amenability agree.*

Remark 2.2.10. *We often omit the* left *in front of the amenable when discussing groups. This is because a locally compact group is left amenable if and only if it is right amenable.(see Section 2.2 in [8])*

Definition 2.2.11. A locally compact group, G, satisfies the *topological version of the Følner condition* if for any compact $F \subset G$ and any $\varepsilon > 0$, there exists an $A \subset G, 0 < \lambda_G(A) < \infty$ such that:

$$\frac{\lambda_G(xA \backslash A)}{\lambda_G(A)} < \varepsilon, \ \ \forall x \in F.$$

Remark 2.2.12. *Definition 2.2.11 of the Følner condition for locally compact groups is very similar to Definition 1.2.6 of the Følner condition for discrete semigroups. As we move from the realm of discrete semigroups to the realm of locally compact groups we replace finite subsets by, in one case, subsets of finite measure, and in another case, by compact sets. In the case where the finite subset, A, is replaced by a set of finite measure, Greenleaf([8], Section 3.6) points out that the regularity of the Haar measure ensures that we can choose A to be compact. When we replace the finite subset, F, we need to restrict ourselves to compact sets since, as in Example 2.2.13, if we allow F to be any subset of finite measure, we no longer have that left amenability implies that the Følner condition is satisfied.*

Example 2.2.13. *Let $G = \mathbb{R}^2$ with the usual topology. Then the Haar measure is the standard Lebesgue measure and G is amenable. Consider the set of zero measure $F := \{(x, 0) \mid x \in \mathbb{R}\}$ and $\varepsilon = \frac{1}{2}$. Suppose that A is a subset of G with non-zero finite measure. Since λ_G is inner regular (the Lebesgue measure is inner regular on all Borel sets), we can find a compact $K \subset A$ such that $\lambda_G(A \backslash K) < \frac{1}{4}\lambda_G(A)$. Since K is bounded, we can take $x \in F$ such that $\|x\| > 2\sup\{\|y\| : y \in K\}$. Then we have:*

$$\begin{aligned}
\lambda_G(xA \backslash A) &\geq \lambda_G(xK \backslash A) \\
&\geq \lambda_G(xK \backslash K) - \lambda_G(A \backslash K) \\
&> \lambda_G(xK) - \frac{1}{4}\lambda_G(A) \\
&= \lambda_G(K) - \frac{1}{4}\lambda_G(A) \\
&= \lambda_G(A) - \lambda_G(A \backslash K) - \frac{1}{4}\lambda_G(A) \\
&> \lambda_G(A) - 2\frac{1}{4}\lambda_G(A)
\end{aligned}$$

So we get that:

$$\frac{\lambda_G(xA \backslash A)}{\lambda_G(A)} > \frac{1}{2} = \varepsilon\,.$$

We now extend the concept of the Følner condition to the notion of a Følner net. Consider the directed set Λ, where each element of Λ is a pair consisting of a compact subset $F \subset G$, and an $\varepsilon > 0$. The order we put on Λ is $\preceq$ where $(F_1, \varepsilon_1) \preceq (F_2, \varepsilon_2)$ if $F_1 \subset F_2$ and $\varepsilon_1 \geq \varepsilon_2$. With this order, $\preceq$, we can find a net of measurable subsets of an amenable locally compact group, G, satisfying the Følner condition for the index values of F and ε. We do not need to consider Λ as our only index set, so we create the following definition.

Definition 2.2.14. Let G be a locally compact group. A net, (A_α), of measurable subsets of G such that $0 < \lambda_G(A_\alpha) < \infty$ is called a *Følner net* if for any $\varepsilon > 0$, and any compact $F \subset G$, there exists β such that, for $\alpha > \beta$

$$\frac{\lambda_G(xA_\alpha \backslash A_\alpha)}{\lambda_G(A_\alpha)} < \varepsilon \ \forall x \in F$$

This definition is equivalent to the definition given in [11], but is easier to work with. Janzen's definition involved the symmetric difference of A and xA rather than $xA \backslash A$, but since λ_G is a left Haar measure, the only difference between the definitions is a factor of 2.

Example 2.2.15. *Let* $\mathbb{R}$ *be the real numbers equipped with addition. Let* $A_n = [-n, n]$ *for* $n \in \mathbb{N}$. *Then* $(A_n)_{n=1}^{\infty}$ *is a Følner net for* $\mathbb{R}$.

2.2.3 Modular Function

We will now define the modular function, Δ_G, of a locally compact group G. The modular function is a function which expresses how close λ_G is to being right-invariant. If Δ_G is the constant function equal to 1, then the left Haar measure of G is also invariant under right translation, hence is the right Haar measure of G. In this case we say that G is *unimodular.*

Definition 2.2.16. For $x \in G$ consider the measure $\lambda_{G,x}$ defined via: $\lambda_{G,x}(E) = \lambda_G(Ex)$. $\lambda_{G,x}$ is a left Haar measure and hence is a scalar multiple of λ_G. The *modular function* of G, Δ_G is defined via:

$$\lambda_{G,x} = \Delta_G(x)\lambda_G.$$

Remark 2.2.17. *The modular function Δ_G satisfies the following properties(see Section 2.4 in [5] or Theorem 15.11 in [10]):*

- $\Delta_G(xy) = \Delta_G(x)\Delta_G(y) \quad \forall x, y \in G$*;*
- Δ_G *is continuous;*
- Δ_G *takes on only positive values;*
- $\Delta_G(x) \int f(y)dy = \int f(yx^{-1})dy, \quad \forall f \in C_{00}(G)$*;*
- *if* μ_G *is the right Haar measure of G defined by* $\mu_G(E) = \lambda_G(E^{-1})$*, then*

$$d\mu(x) = \Delta_G(x^{-1})d\lambda(x).$$

Example 2.2.18. *Let G be a locally compact abelian group. In this case, left multiplication is the same as right multiplication so any left Haar measure of G is also a right Haar measure. Hence G is unimodular.*

Example 2.2.19. *Let G be a compact group. Since* Δ_G *is a continuous group homomorphism,* $\Delta_G(G)$ *is a compact subgroup of* $(0, \infty)$*. Since the only compact subgroup of* $(0, \infty)$ *is* $\{1\}$*,* $\Delta_G \equiv 1$*. Thus G is unimodular.*

2.2.4 Semidirect Products

The final definition of this section is that of a semidirect product. There are two methods for defining the semidirect product of groups. Definition 2.2.20 uses the *external* method for defining semidirect products.

Definition 2.2.20. Let N and H be locally compact groups. Let τ be a group homomorphism from H to $\mathrm{Aut}(N)$ such that $(n, h) \to \tau_h(n)$ is continuous in the product topology $N \times H$ where $\mathrm{Aut}(N)$ is the group of continuous automorphisms of N. We say that $G := N \rtimes_\tau H$ is the *semidirect product* of N and H with respect to τ if G is the group consisting of elements of the form (n, h) where $n \in N$ and $h \in H$ equipped with multiplication given by:

$$(n_1, h_1)(n_2, h_2) = (n_1\tau_{h_1}(n_2), h_1h_2)$$

Note that if G is equipped with the product topology then G is locally compact.

The second method used to define a semidirect product is called the internal method. The internal method considers a group with two subgroups satisfying certain conditions and uses conjugation by one of the subgroups on the other as the τ given in 2.2.20. For this method, we begin with a group action which determines τ rather than constructing the group action from an arbitrary τ.

2.3 Følner Net Construction

In this section we present two methods for finding Følner nets for semidirect products. The first method extends the work of Janzen, presented in [11]. The second method extends a result of Greenleaf presented in [8]. We begin by reviewing some basic results regarding the semidirect product and the modular function. We then build upon these results and examine a few interesting examples.

The following lemma is useful for working with the Haar measure of a semidirect product. The original proof of this lemma which uses Haar integrals can be found in [10] 15.29(a). We present an alternative proof using Haar measures rather than Haar integrals. Because of the way that multiplication is defined for a semidirect product of two locally compact groups, the right Haar measure of the semidirect product is the product of the right Haar measures of the two groups.

Lemma 2.3.1. *Let μ_N and μ_H be right Haar measures for N and H respectively. Then $\mu_N \times \mu_H$ is a right Haar measure for $N \rtimes_\tau H$.*

Proof. By the definition of the product measure, as seen in Theorem B, Section 35 in [9] we have that for $E \subset N \rtimes_\tau H$,

$$\mu_N \times \mu_H(E) = \int_{h \in H} \mu_N(E^h) d\mu_H(h)$$

where $E^h = \{x | (x, h) \in E\}$.

To see that $\mu_N \times \mu_H$ is a right invariant measure, consider $(a, b) \in N \rtimes_\tau H$. Then:

$$\begin{aligned}
\mu_N \times \mu_H(E(a,b)) &= \int_{h\in H} \mu_N((E(a,b))^h) d\mu_H(h) && (2.1)\ (2.1) \\
&= \int_{h\in H} \mu_N(\{(x,y)(a,b) | (x,y) \in E\}^h) d\mu_H(h) && (2.2)\ (2.2) \\
&= \int_{h\in H} \mu_N(\{(x\tau_y(a), yb) | (x,y) \in E\}) d\mu_H(h) && (2.3)\ (2.3) \\
&= \int_{h\in H} \mu_N(\{x\tau_y(a) | (x,y) \in E, h = yb\}) d\mu_H(h) && (2.4)\ (2.4) \\
&= \int h \in H \mu_N(\{x\tau_h(a) | (x,h) \in E\}) d\mu_H(h) && (2.5) \\
&= \int_{h\in H} \mu_N(E^h \tau_h(a)) d\mu_H(h) && (2.6) \\
&= \int_{h\in H} \mu_N(E^h) d\mu_H(h) && (2.7) \\
&= \mu_N \times \mu_H(E) && (2.8)
\end{aligned}$$

Therefore $\mu_N \times \mu_H$ is a right invariant measure for $N \rtimes_\tau H$.

The regularity requirements for $\mu_N \times \mu_H$ to be a Haar measure follow from the fact that the topology on $N \rtimes_\tau H$ is the product topology. Also, compact subsets of $N \rtimes_\tau H$ are contained in 'rectangles'. These rectangles have sides which are compact in N or H. Hence compact subsets of $N \rtimes_\tau H$ have finite measure.

Therefore $\mu_N \times \mu_H$ is a right Haar measure for $N \rtimes_\tau H$. □

The next step is to compare the modular functions of N and H to the modular function of $N \rtimes_\tau H$. To do this we first define a function, δ, which expresses how close λ_N is to being invariant under actions of elements of H.

Consider a semidirect product of locally compact groups $N \rtimes_\tau H$. Let $h \in H$. Consider the measure, $\lambda_{N,h}$, on N given by:

$$\lambda_{N,h}(E) = \lambda_N(\tau_h(E)),$$

for each measurable $E \subset N$. Notice that $\lambda_{N,h}$ is left invariant, since

$$\lambda_{N,h}(nE) = \lambda_N(\tau_h(n)\tau_h(E)) = \lambda_N(\tau_h(E)).$$

Since $\lambda_{N,h}$ is a left Haar measure, and hence a constant multiple of λ_N, the function $\delta : H \to \mathbb{R}^+$ given by:

$$\delta(h) = \frac{\lambda_N}{\lambda_{N,h}}$$

is well defined.

The definition of the δ function is similar to that of the modular function and the functions have similar properties. We will see shortly how δ relates the modular function of a semidirect product of two locally compact groups to the modular functions of those two groups.

Proposition 2.3.2. *The function δ as defined above has the following properties:*

1. *For any positive, nonzero $f \in C_c(N)$ we have that:*
$$\delta(h) = \frac{\int_N f(\tau_h(n))d\mu(n)}{\int_N f(n)d\mu(n)};$$
2. $\delta(h_1h_2) = \delta(h_1)\delta(h_2)$ *;*
3. $\delta(h) > 0$*;*
4. *δ is continuous; and*
5. $\Delta_{N\rtimes_\tau H}(n,h) = \delta(h)\Delta_N(n)\Delta_H(h)$ *where $\Delta_{N\rtimes_\tau H}$ denotes the modular function for $N \rtimes_\tau H$.*

Proof. The first 4 properties are straightforward and similar to the corresponding proofs for the modular function. For details, see 15.29(b) [10].

To prove the fifth property, observe that for $E \subset N \rtimes_\tau H,\ n \in N,\ h \in H$ we have:

$$\Delta_{N\rtimes_\tau H}(n,h) = \frac{\lambda_{N\rtimes_\tau H}(E(n,h))}{\lambda_{N\rtimes_\tau H}(E)} \tag{2.9}$$

$$= \frac{\mu_{N\rtimes_\tau H}((n,h)^{-1}E^{-1})}{\mu_{N\rtimes_\tau H}(E^{-1})} \tag{2.10}$$

$$= \frac{\mu_N \times \mu_H((n,h)^{-1}E^{-1})}{\mu_N \times \mu_H(E^{-1})} \tag{2.11}$$

$$= \frac{\int_{y\in H} \mu_N(((n,h)^{-1}E^{-1})^y)d\mu_H(y)}{\int_{y\in H} \mu_N((E^{-1})^y)d\mu_H(y)}. \tag{2.12}$$

However,

$$\begin{aligned}((n,h)^{-1}E^{-1})^y &= \{x|(x,y)\in(n,h)^{-1}E^{-1}\} && (2.13)\\ &= \{x|(n,h)(x,y)\in E^{-1}\} && (2.14)\\ &= \{x|(n\tau_h(x),hy)\in E^{-1}\} && (2.15)\\ &= \{\tau_{h^{-1}}(n^{-1}x)|(x,hy)\in E^{-1}\} && (2.16)\\ &= \tau_{h^{-1}}(n^{-1}(E^{-1})^{hy}), && (2.17)\end{aligned}$$

and

$$\begin{aligned}((E^{-1})^{hy})^{-1} &= \{x^{-1}|(x,hy)\in E^{-1}\} && (2.18)\\ &= \{x|(x^{-1},hy)\in E^{-1}\} && (2.19)\\ &= \{x|(\tau_{y^{-1}h^{-1}}(x),y^{-1}h^{-1})\in E\} && (2.20)\\ &= \tau_{hy}(E^{y^{-1}h^{-1}}). && (2.21)\end{aligned}$$

Therefore

$$\begin{aligned}\Delta_{N\rtimes_\tau H}(n,h) &= \frac{\int_{y\in H}\mu_N(\tau_{h^{-1}}(n^{-1}(E^{-1})^{hy}))d\mu_H(y)}{\int_{y\in H}\mu_N((E^{-1})^y)d\mu_H(y)} && (2.22)\\ &= \frac{\int_{y\in H}\lambda_N((\tau_{h^{-1}}(n^{-1}(E^{-1})^{hy}))^{-1})d\mu_H(y)}{\int_{y\in H}\mu_N((E^{-1})^y)d\mu_H(y)} && (2.23)\\ &= \frac{\int_{y\in H}\lambda_N(\tau_{h^{-1}}(\tau_{h^{-1}}(\tau_{hy}(E)^{y^{-1}h^{-1}})n))d\mu_H(y)}{\int_{y\in H}\mu_N((E^{-1})^y)d\mu_H(y)} && (2.24)\\ &= \frac{\delta(h)\Delta_N(n)\int_{y\in H}\lambda_N(\tau_{hy}((E)^{y^{-1}h^{-1}}))d\mu_H(y)}{\int_{y\in H}\mu_N((E^{-1})^y)d\mu_H(y)} && (2.25)\\ &= \frac{\delta(h)\Delta_N(n)\int_{y\in H}\mu_N((E^{-1})^{hy})d\mu_H(y)}{\int_{y\in H}\mu_N((E^{-1})^y)d\mu_H(y)} && (2.26)\\ &= \delta(h)\Delta_N(n)\Delta_H(h). && (2.27)\end{aligned}$$

□

Example 2.3.3 ('ax+b' Group). *Let $\mathbb{R}$ be the real numbers equipped with addition, and $\mathbb{R}^+$ the positive reals with multiplication. Both groups are equipped with their respective standard topologies. Define τ via: $\tau_a(b) = ab$ for $a \in \mathbb{R}^+$, and $b \in \mathbb{R}$. Then $G = \mathbb{R} \rtimes_\tau \mathbb{R}^+$ is the '$ax + b$' group.*

The right Haar measures for $\mathbb{R}$ and $\mathbb{R}^+$ are given by

$$\begin{aligned} d\mu_{\mathbb{R}}(b) &= dm(b), \text{ and} && (2.28) \\ d\mu_{\mathbb{R}^+}(a) &= \frac{1}{a}dm(a), && (2.29) \end{aligned}$$

where m is the standard Lebesgue measure.

So, for the δ function, we get:

$$\begin{aligned} \delta(a) &= \frac{\mu_{\mathbb{R}}([0,1])}{\mu_{\mathbb{R}}(\tau_a([0,1])} && (2.30) \\ &= \frac{\mu_{\mathbb{R}}([0,1])}{\mu_{\mathbb{R}}([0,a])} && (2.31) \\ &= \frac{1}{a}. && (2.32) \end{aligned}$$

The modular functions for both $\mathbb{R}$, and $\mathbb{R}^+$ are 1 since $\mathbb{R}$, and $\mathbb{R}^+$ are both abelian groups. Therefore

$$\begin{aligned} \Delta_G(b,a) &= \delta(a)\Delta_{\mathbb{R}}(b)\Delta_{\mathbb{R}^+}(a) && (2.33) \\ &= \frac{1}{a}. && (2.34) \end{aligned}$$

For the left Haar measure of G, we have

$$\begin{aligned} d\lambda_G(b,a) &= \Delta_G(b,a)d\mu_G(b,a) && (2.35) \\ &= \frac{1}{a}d\mu_G(b,a) && (2.36) \\ &= \frac{1}{a}d\mu_{\mathbb{R}}(b)d\mu_{\mathbb{R}^+}(a) && (2.37) \\ &= \frac{1}{a^2}dm(b)dm(a). && (2.38) \end{aligned}$$

Remark 2.3.4. *The reason that this group is called the 'ax+b' group is that we can express the element, (a, b), of it as the 2×2 matrix of the form*

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}.$$

Written in this way, the multiplication on the group is given by standard matrix multiplication. The name 'ax+b' group comes from considering the matrices of this form as affine transformations of the real line. By representing elements of the real line as 2×1 *vectors with a* 1 *in the second entry, the matrix above sends* x *to* $ax + b$.

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} ax+b \\ 1 \end{pmatrix}$$

Example 2.3.5. *Now we will look at an example of a unimodular semidirect product with* $\delta \neq 1$.

Let G *and* $\mathbb{R}$ *be as in Example 2.3.3. Define* ρ *via:* $\rho_{(b,a)}(c) = \frac{c}{a}$. *Let* $F = \mathbb{R} \rtimes_\rho G$. *Then* $\delta(b, a) = a$, *and* $\Delta_F(c, (b, a)) = \delta(b, a)\Delta_{\mathbb{R}}(c)\Delta_G(b, a) = a\frac{1}{a} = 1$.

So F *is unimodular, but the left Haar measure of* F *is not the direct product of the left Haar measures of* G *and* $\mathbb{R}$.

The following results are modifications to results from Janzen's paper [11]. He proved them in the case where G is the unimodular semidirect product of two unimodular groups. These results are interesting because in general it is not easy to find Følner nets for a particular group, but by using these results we find that we are sometimes able to construct product nets. In other words if the groups, nets, and semidirect product satisfy the conditions of Theorem 2.3.7 then the 'rectangular' net is a Følner net for the semidirect product. Janzen provides several examples in the unimodular case, and also points out that [8] has shown that a 'rectangular' net will never be a Følner net for the 'ax+b' group.

Proposition 2.3.6. *If* $G = N \rtimes_\tau H$ *is the semidirect product of locally compact groups, then:*

1. *For every measurable* $A \subset N$, $y \in H$, *we get that:*

$$\lambda_N(\tau_y(A)) = \delta(y^{-1})\lambda_N(A)$$

2. *If (A_α) is a Følner net for N, then for any compact $F \subset G$, and $\varepsilon > 0$, there exists α_0 such that for $\alpha > \alpha_0$*

$$\frac{\lambda_N(x(\tau_y(A_\alpha))\backslash\tau_y(A_\alpha))}{\lambda_N(A_\alpha)} < \varepsilon \ \forall (x,y) \in F$$

Proof. Part 1 of this proposition follows immediately from the definition of the δ function. It is included here because in the case where G is the unimodular semi-direct product of unimodular groups, we get the result of [11] that $\lambda_N(\tau_y(A)) = \lambda_N(A)$.

For 2, let $F \subset G$ be compact, and let $\varepsilon > 0$. Let $F^H = \{y \in H \mid \exists x \in N \text{ s.t. } (x,y) \in F\}$. Then F^H is compact in H. Since δ is continuous, it achieves its maximum and minimum values.

$$\begin{aligned} \lambda_N(x(\tau_y(A_\alpha))\backslash\tau_y(A_\alpha)) &= \lambda_N(\tau_y((\tau_{y^{-1}}(x))A_\alpha\backslash A_\alpha)) && (2.39)\\ &= \delta(y^{-1})\lambda_N((\tau_{y^{-1}}(x))A_\alpha\backslash A_\alpha) && (2.40)\end{aligned}$$

Since $\{\tau_{y^{-1}}(x) | (x,y) \in F\}$ is compact in N, and A_α is a Følner net for N we get a α_0 such that, for $\alpha > \alpha_0$ and $(x,y) \in F$:

$$\frac{\lambda_N((\tau_{y^{-1}}(x))A_\alpha\backslash A_\alpha)}{\lambda_N(A_\alpha)} < \frac{\varepsilon}{\max\{\delta(y^{-1}\colon y \in F^H\}} \qquad (2.41)$$

So now, for $(x,y) \in F$ and $\alpha > \alpha_0$ we see that:

$$\begin{aligned} \frac{\lambda_N(x(\tau_y(A_\alpha))\backslash\tau_y(A_\alpha))}{\lambda_N(A_\alpha)} &\leq \frac{\delta(y^{-1})\lambda_N((\tau_{y^{-1}}(x))A_\alpha\backslash A_\alpha)}{\lambda_N(A_\alpha)} && (2.42)\\ &\leq \max\{\delta(y^{-1})\colon y \in F^H\}\frac{\lambda_N((\tau_{y^{-1}}(x))A_\alpha\backslash A_\alpha)}{\lambda_N(A_\alpha)} && (2.43)\\ &< \max\{\delta(y^{-1})\colon y \in F^H\}\frac{\varepsilon}{\max\{\delta(y^{-1}\colon y \in F^H\}} && (2.44)\\ &= \varepsilon && (2.45)\end{aligned}$$

□

We will use juxtaposition to denote the multiplication in N and H and the symbol $*$ to denote multiplication in G.

Theorem 2.3.7. *Let $G = N \rtimes_\tau H$, for N and H locally compact amenable groups. Let $(A_\alpha)_\alpha$ be a Følner net for N and $(B_\beta)_\beta$ be a Følner net for H. If the following two conditions are met, then $(A_\alpha \times B_\beta)$ is a Følner net for G.*

1.

$$\frac{\lambda_N(\tau_y(A_\alpha)\backslash A_\alpha)}{\lambda_N(A_\alpha)} \to 0$$

uniformly in y on compact sets of H.

2.

$$\frac{\int_{yB_\beta\backslash B_\beta} \delta(t)d\lambda_H(t)}{\int_{B_\beta} \delta(t)d\lambda_H(t)} \to 0$$

uniformly in y on compact sets of H.

Proof. First, notice that for $(x, y) \in N \rtimes_\tau H$, $A \subset N$, and $B \subset H$ we have that:

$$(x, y) * (A \times B) = (x\tau_y(A) \times yB)$$

and so

$$\begin{aligned} ((x,y) * (A \times B))\backslash(A \times B) &= ((x,y) * (A \times B))\backslash(N \times B \cap A \times H) &(2.46)\\ &= (((x,y) * (A \times B))\backslash N \times B) &(2.47)\\ &\quad \cup (((x,y) * (A \times B))\backslash A \times H) &(2.48)\\ &= (x\tau_y(A) \times (yB\backslash B)) \cup ((x\tau_y(A)\backslash A) \times yB) &(2.49)\end{aligned}$$

Now let $\varepsilon > 0$ and $F \subset G$ be compact. Then $F_N := \{x \mid (x, y) \in F\}$ is compact in N, and $F^H := \{y \mid (x, y \in F\}$ is compact in H. So, since $(A_\alpha)_\alpha$ is a Følner net for N, by 2.3.6, there exists α_0 such that for $\alpha \geq \alpha_0$ we have:

$$\frac{\lambda_N(x\tau_y A_\alpha \backslash \tau_y A_\alpha)}{\lambda_N(A_\alpha)} < \frac{\varepsilon}{3\max\{\delta(h) \mid h \in F^H\}} \quad \forall (x, y) \in F \qquad (2.50)$$

similarly, by condition (1) of the theorem there exists α_1 such that for $\alpha \geq \alpha_1$ we have:

$$\frac{\lambda_N(\tau_y(A_\alpha)\backslash A_\alpha)}{\lambda_N(A_\alpha)} < \frac{\varepsilon}{3\max\{\delta(h) \mid h \in F^H\}} \quad \forall y \in F^H \qquad (2.51)$$

and by condition (2) there exists β_0 such that for $\beta \geq \beta_0$ we have:

$$\frac{\int_{yB_\beta \backslash B_\beta} \delta(t) d\lambda_H(t)}{\int_{B_\beta} \delta(t) d\lambda_H(t)} < \frac{\varepsilon}{3 \max\{\delta(h^{-1}) \mid h \in F^H\}} \quad \forall y \in F^H \tag{2.52}$$

Now let

$$K_{\alpha,\beta} = \frac{\lambda_G((x,y) * (A_\alpha \times B_\beta) \backslash (A_\alpha \times B_\beta))}{\lambda_G(A_\alpha \times B_\beta)}$$

So for $\alpha \geq \max\{\alpha_0, \alpha_1\}$ and $\beta \geq \beta_0$ we have:

$$K_{\alpha,\beta} \leq \frac{\lambda_G([x\tau_y A_\alpha \times (yB_\beta \backslash B_\beta)]) + \lambda_G([(x\tau_y A_\alpha \backslash A_\alpha) \times yB_\beta])}{\lambda_G(A_\alpha \times B_\beta)} \tag{2.53}$$

$$= \frac{\int_{[x\tau_y A_\alpha \times (yB_\beta \backslash B_\beta)]} \Delta_G(n,h) d\mu_G(n,h)}{\int_{A_\alpha \times B_\beta} \Delta_G(n,h) d\mu_G(n,h)} \tag{2.54}$$

$$+ \frac{\int_{[(x\tau_y A_\alpha \backslash A_\alpha) \times yB_\beta]} \Delta_G(n,h) d\mu_G(n,h)}{\int_{A_\alpha \times B_\beta} \Delta_G(n,h) d\mu_G(n,h)} \tag{2.55}$$

$$= \frac{\int_{x\tau_y(A_\alpha)} \int_{yB_\beta \backslash B_\beta} \delta(h) \Delta_N(n) \Delta_H(h) d\mu_H(h) d\mu_N(n)}{\int_{A_\alpha} \int_{B_\beta} \delta(h) \Delta_N(n) \Delta_H(h) d\mu_H(h) d\mu_N(n)} \tag{2.56}$$

$$+ \frac{\int_{x\tau_y(A_\alpha) \backslash A_\alpha} \int_{yB_\beta} \delta(h) \Delta_N(n) \Delta_H(h) d\mu_H(h) d\mu_N(n)}{\int_{A_\alpha} \int_{B_\beta} \delta(h) \Delta_N(n) \Delta_H(h) d\mu_H(h) d\mu_N(n)} \tag{2.57}$$

$$= \frac{\lambda_N(x\tau_y(A_\alpha)) \int_{yB_\beta \backslash B_\beta} \delta(h) d\lambda_H(h)}{\lambda_N(A_\alpha) \int_{B_\beta} \delta(h) d\lambda_H(h)} \tag{2.58}$$

$$+ \frac{\lambda_N(x\tau_y(A_\alpha) \backslash A_\alpha) \int_{yB_\beta} \delta(h) d\lambda_H(h)}{\lambda_N(A_\alpha) \int_{B_\beta} \delta(h) d\lambda_H(h)} \tag{2.59}$$

$$= \frac{\delta(y^{-1}) \lambda_N(A_\alpha) \int_{yB_\beta \backslash B_\beta} \delta(h) d\lambda_H(h)}{\lambda_N(A_\alpha) \int_{B_\beta} \delta(h) d\lambda_H(h)} \tag{2.60}$$

$$+ \frac{\lambda_N(x\tau_y(A_\alpha) \backslash A_\alpha)}{\lambda_N(A_\alpha)} \frac{\int_{yB_\beta} \delta(h) d\lambda_H(h)}{\int_{B_\beta} \delta(h) d\lambda_H(h)} \tag{2.61}$$

$$\leq \frac{\delta(y^{-1}) \int_{yB_\beta \backslash B_\beta} \delta(h) d\lambda_H(h)}{\int_{B_\beta} \delta(h) d\lambda_H(h)} \tag{2.62}$$

$$+ \frac{\lambda_N(x\tau_y(A_\alpha) \backslash \tau_y A_\alpha) + \lambda_N(\tau_y(A_\alpha) \backslash A_\alpha)}{\lambda_N(A_\alpha)} \delta(y) \tag{2.63}$$

$$< \frac{\delta(y^{-1})\varepsilon}{3 \max\{\delta(h^{-1}) \mid h \in F^H\}} + \frac{2\varepsilon\delta(y)}{3 \max\{\delta(h) \mid h \in F^H\}} \tag{2.64}$$

$$\leq \varepsilon \tag{2.65}$$

So $(A_\alpha \times B_\beta)_{\alpha,\beta}$ is a Følner net for G. $\square$

The final result of this chapter is a method for constructing a Følner net for any semidirect product $N \rtimes_\rho H$ given Følner nets for N and H. Greenleaf gives the main idea of this proof in [8] by discussing the 'ax+b' group as an example. One drawback of this result is that the resulting Følner net is not rectangular as it is in the previous theorem, but rather trapezoidal. For example, a Følner net for the 'ax+b' group is $(\{(ab, a) \mid a \in [\frac{1}{n}, n], b \in [-n, n]\})_{n=1}^{\infty}$. Another drawback to this construction is that we can only show that a subnet of the constructed net is a Følner net for the semidirect product rather than the entire net.

Theorem 2.3.8. *Let $G = N \rtimes_\rho H$ for N, H locally compact amenable groups. Let $(A_\alpha)_\alpha$ be a Følner net for N, and $(B_\beta)_\beta$ be a Følner net for H such that each B_β is compact (See Remark 2.2.12). Then there is a subnet of*

$$(\{(\rho_h(n), h) \mid n \in A_\alpha, h \in B_\beta\})_{\alpha,\beta}$$

which is a Følner net for G.

Proof. Consider $A \subset N$, $B \subset H$, $(n, h) \in G$. Then

$$(n, h) * B * A = \{(x\rho_{hb}(a), hb) \mid a \in A, b \in B\} \tag{2.66}$$
$$= \{(\tilde{a}, \tilde{b}) \mid \tilde{a} \in x\rho_{\tilde{b}}(A), \tilde{b} \in hB\} \tag{2.67}$$

and

$$((n, h) * B * A) \setminus (B * A) = ((n, h) * B * A) \setminus (B{*}N) \cup ((n, h) * B * A) \setminus (H{*}A).$$

This yields:

$$((n, h) * B * A) \setminus (B * A) = \{(\tilde{a}, \tilde{b}) \mid \tilde{a} \in x\rho_{\tilde{b}}(A), \tilde{b} \in hB \setminus B\} \tag{2.68}$$
$$\cup \{(\tilde{a}, \tilde{b}) \mid \tilde{a} \in x\rho_{\tilde{b}}(A) \setminus \rho_{\tilde{b}}(A), \tilde{b} \in hB\} \tag{2.69}$$

Now let $\varepsilon > 0$ and $F \subset G$ be compact. Then $F_N := \{n \mid (n, h) \in F\}$ is compact in N, and $F^H := \{h \mid (n, h) \in F\}$ is compact in H. So, since $(B_\beta)_\beta$ is a Følner net for H there exists $\beta_{F,\varepsilon}$ such that for every $h \in F^H$:

$$\frac{\lambda_H(hB_{\beta_{F,\varepsilon}}\backslash B_{\beta_{F,\varepsilon}})}{\lambda_H(B_{\beta_{F,\varepsilon}})} \leq \frac{\varepsilon}{2}$$

Similarly, since $(A_\alpha)_\alpha$ is a Følner net for N, and $B_{\beta_{F,\varepsilon}}$ is compact, we can find an $\alpha_{F,\varepsilon}$ such that for every $n \in F_N$, $h \in F^H$, and $b \in B_{\beta_{F,\varepsilon}}$:

$$\frac{\lambda_N(\rho_{(hb)^{-1}}(n)(A_{\alpha_{F,\varepsilon}})\backslash A_{\alpha_{F,\varepsilon}})}{\lambda_N(A_{\alpha_{F,\varepsilon}})} \leq \frac{\varepsilon}{2}$$

Choose $(n, h) \in F$. From now on, to decrease the amount of subscripts, we will replace $A_{\alpha_{F,\varepsilon}}$ by A, and $B_{\beta_{F,\varepsilon}}$ by B. Now, letting $k = \frac{\lambda_G(((n,h)*B*A)\backslash(B*A))}{\lambda_G(B*A)}$:

$$k \leq \frac{\int_{\{(\tilde{a},\tilde{b})|\tilde{a}\in x\rho_{\tilde{b}}(A),\tilde{b}\in hB\backslash B\}} \Delta_G d\mu_G + \int_{\{(\tilde{a},\tilde{b})|\tilde{a}\in n\rho_{\tilde{b}}(A)\backslash\rho_{\tilde{b}}(A),\tilde{b}\in hB\}} \Delta_G d\mu_G}{\lambda_G(B * A)} \tag{2.70}$$

$$= \frac{\int_{\tilde{b}\in hB\backslash B}\int_{\tilde{a}\in n\rho_{\tilde{b}}(A)} \delta(\tilde{b})\Delta_N(\tilde{a})\Delta_H(\tilde{b})d\mu_N(\tilde{a})d\mu_H(\tilde{b})}{\int_{\tilde{b}\in B}\int_{\tilde{a}\in\rho_{\tilde{b}}(A)} \delta(\tilde{b})\Delta_H(\tilde{b})\Delta_N(\tilde{a})d\mu_N(\tilde{a})d\mu_H(\tilde{b})} \tag{2.71}$$

$$+ \frac{\int_{\tilde{b}\in hB}\int_{\tilde{a}\in n\rho_{\tilde{b}}(A)\backslash\rho_{\tilde{b}}(A)} \delta(\tilde{b})\Delta_N(\tilde{a})\Delta_H(\tilde{b})d\mu_N(\tilde{a})d\mu_H(\tilde{b})}{\int_{\tilde{b}\in B}\int_{\tilde{a}\in\rho_{\tilde{b}}(A)} \delta(\tilde{b})\Delta_H(\tilde{b})\Delta_N(\tilde{a})d\mu_N(\tilde{a})d\mu_H(\tilde{b})} \tag{2.72}$$

$$= \frac{\int_{\tilde{b}\in hB\backslash B} \delta(\tilde{b})\Delta_H(\tilde{b})\lambda_N(n\rho_{\tilde{b}}(A))d\mu_H(\tilde{b})}{\int_{\tilde{b}\in B} \lambda_N(\rho_{\tilde{b}}(A))\delta(\tilde{b})\Delta_H(\tilde{b})d\mu_H(\tilde{b})} \tag{2.73}$$

$$+ \frac{\int_{\tilde{b}\in hB} \delta(\tilde{b})\Delta_H(\tilde{b})\lambda_N(\rho_{\tilde{b}}((\rho_{\tilde{b}^{-1}}(x))A\backslash A))d\mu_H(\tilde{b})}{\int_{\tilde{b}\in B} \lambda_N(\rho_{\tilde{b}}(A))\delta(\tilde{b})\Delta_H(\tilde{b})d\mu_H(\tilde{b})} \tag{2.74}$$

$$= \frac{\int_{\tilde{b}\in hB\backslash B} \delta(\tilde{b})\Delta_H(\tilde{b})\delta(\tilde{b}^{-1})\lambda_N(A)d\mu_H(\tilde{b})}{\int_{\tilde{b}\in B} \lambda_N(A)\delta(\tilde{b}^{-1})\delta(\tilde{b})\Delta_H(\tilde{b})d\mu_H(\tilde{b})} \tag{2.75}$$

$$+ \frac{\int_{\tilde{b}\in hB} \delta(\tilde{b})\Delta_H(\tilde{b})\delta(\tilde{b}^{-1})\lambda_N((\rho_{\tilde{b}^{-1}}(n))A\backslash A)d\mu_H(\tilde{b})}{\int_{\tilde{b}\in B} \lambda_N(A)\delta(\tilde{b}^{-1})\delta(\tilde{b})\Delta_H(\tilde{b})d\mu_H(\tilde{b})} \tag{2.76}$$

$$= \frac{\int_{\tilde{b}\in hB\backslash B} \Delta_H(\tilde{b})d\mu_H(\tilde{b})}{\int_{\tilde{b}\in B} \Delta_H(\tilde{b})d\mu_H(\tilde{b})} \tag{2.77}$$

$$+ \frac{\int_{\tilde{b}\in hB} \Delta_H(\tilde{b})\lambda_N((\rho_{\tilde{b}^{-1}}(x))A\backslash A)d\mu_H(\tilde{b})}{\int_{\tilde{b}\in B} \lambda_N(A)\Delta_H(\tilde{b})d\mu_H(\tilde{b})} \tag{2.78}$$

$$\leq \frac{\lambda_H(hB\backslash B)}{\lambda_H(B)} + \frac{\int_{b\in B} \lambda_N(A)\frac{\varepsilon}{2}d\lambda_H(b)}{\int_{b\in B} \lambda_N(A)d\lambda_H(b)} \tag{2.79}$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \tag{2.80}$$

So $(B_{\beta_{F,\varepsilon}} * A_{\alpha_{F,\varepsilon}})_{F,\varepsilon}$ is a Følner net for G.

□

Chapter 3

Discrete Semigroups

3.1 Introduction

In this chapter we will investigate various Følner type conditions as well as results on semidirect products of semigroups. We will summarize known relationships among the various conditions. We will examine Sorenson's conjecture, including a weaker version than the original. We will develop further results on semidirect products of semigroups relating to the Følner conditions.

3.2 Følner Type Conditions and Related Definitions

This section consists of the definitions of additional Følner-type conditions beyond the Følner condition and strong Følner condition (See section 1.2). Other related definitions are also presented. In the following section we will see how they relate to each other.

Definition 3.2.1. A semigroup S is said to satisfy the *weak Følner condition* if there exists a real number k, $0 < k < 1$, such that for any choice of $s_1, s_2, \ldots, s_n \in S$, not necessarily distinct, there exists an $A \subset S$, finite and non-empty such that:

$$\frac{1}{n}\sum_{i=1}^{n} |A \backslash s_i A| \leq k|A|.$$

Definition 3.2.2. A semigroup S is said to satisfy the *strong Namioka-Følner condition* if there exists a real number k, $0 < k < \frac{1}{2}$, such that for any choice of

$s_1, s_2, \ldots, s_n \in S$ not necessarily distinct, there exists an $A \subset S$ which is finite and non-empty such that:

$$\frac{1}{n}\sum_{i=1}^{n} |A \backslash s_i A| \leq k|A|.$$

Definition 3.2.3. A semigroup S is said to satisfy the *weak Namioka-Følner condition* if there exists a real number r, $0 < r < 1$ such that for any choice of $s_1, \ldots, s_n, t_1, \ldots, t_n \in S$ there exists an $A \subset S$ which is finite and non-empty such that:

$$\frac{1}{n}\sum_{i=1}^{n} |s_i A \cap t_i A| \geq r|A|.$$

Definition 3.2.4. A semigroup S is *left measurable* if it admits a left reversible invariant mean. That is, a $m \in \ell^\infty(S)^*$ with $\|m\| = 1 = m(\chi_S)$ such that $m(\chi_{sA}) = m(\chi_A)$ for any $A \subset S$ and any $s \in S$.

Remark 3.2.5. *Compare the above to Remark 1.2.4 to see why this definition is made.*

Definition 3.2.6. A semigroup S is said to satisfy *Dixmier's condition* if there are finitely many bounded functions $u_1, u_2, \ldots, u_n \in \ell^\infty(S)$ and $t_1, t_2, \ldots, t_n \in S$ such that

$$\inf\left\{\sum_{i=1}^{n} u_i(t_i s) - u_i(s) \colon s \in S\right\} > 0.$$

Definition 3.2.7. Let S be a semigroup and $k > 0$. We say S has property (F_k) if for any $s_1, \ldots, s_n \in S$ (not necessarily distinct), there is a finite, nonempty $A \subset S$ such that:

$$\frac{1}{n}\sum_{i=1} n|A \backslash s_i A| \leq k|A|.$$

The *Følner number* of S is defined by

$$\varphi(S) = \inf\{k \mid 0 < k \leq 1 \text{ and } S \text{ has property } (F_k)\}$$

This definition of Følner number was originally used by J.C.S. Wong and published by Yang in [15].

Definition 3.2.8. Let S be a semigroup. We say that S is *left* [right] *cancellative* if, for each $s, t, x \in S$ we have that:

$$xs = xt \Rightarrow s = t \ \ [sx = tx \Rightarrow s = t].$$

Example 3.2.9. *Let G be a group, and S a subsemigroup of G. Then S is both left and right cancellative.*

Definition 3.2.10. Let S be a semigroup. We say that S is *weakly left* [right] *cancellative* if, for each $s, t \in S$ we have that

$$|s^{-1}t| < \infty \ \ \left[|ts^{-1}| < \infty\right]$$

Remark 3.2.11. *Here it is important to recall that the notation $s^{-1}t$ does not necessarily refer to the product of the inverse of s with t. If such an inverse exists, then the two notations agree, but if s does not have an inverse then $s^{-1}t$ refers to the set $\{x \in S : sx = t\}$. In a left cancellative semigroup, this set has either zero or one element. So a left [right] cancellative semigroup is automatically weakly left [right] cancellative.*

Example 3.2.12. *Let S be a finite semigroup. Then for each $s, t \in S$ $s^{-1}t$ is contained in S, hence is finite. So S is weakly cancellative.*

Definition 3.2.13. Let S be a semigroup. We say that S has the *finite intersection property for right ideals* if for each choice of finitely many right ideals $I_1, I_2, \ldots, I_n$ of S, their intersection is nonempty. That is,

$$\bigcap_{i=1}^{n} I_i \neq \emptyset.$$

Remark 3.2.14. *To show that S has the finite intersection property for right ideals, it is equivalent to show that for any $a, b \in S$ $aS \cap bS \neq \emptyset$.*

Definition 3.2.15. Let S be a semigroup having the finite intersection property for right ideals. We define an equivalence relation, $\sim$, on S by

$$s \sim t \iff \exists x \in S, sx = tx.$$

Then $S' := S/\sim$ is a right cancellative semigroup, called the *right cancellative quotient semigroup* of S.

This definition does require some work to show that S' is indeed a semigroup. Full details can be found in [7].

3.3 Known Relations Between the Above Conditions

This table indicates what various Følner number values mean. The first result can be found in [15] (Proposition 2.1), the second and third are immediately clear from the definitions. Indeed the definitions of the strong Namioka-Følner condition and the weak Følner condition were likely the inspiration for the definition of the Følner number.

$$\begin{array}{lll} \varphi(S) = 0 & \Leftrightarrow & \text{SFC} \\ \varphi(S) < \frac{1}{2} & \Leftrightarrow & \text{SNFC} \\ \varphi(S) < 1 & \Leftrightarrow & \text{WFC} \end{array}$$

The following diagram indicates known implications for general semigroups. These results are all shown in [13] or are clear from the definitions:

$$\begin{array}{ccccccc} \text{SFC} & \Rightarrow & \text{SNFC} & \Rightarrow & \text{WNFC} & \Rightarrow & \text{WFC} \\ & & & & \Downarrow & & \\ & & & & \text{LA} & \Rightarrow & \text{FC} \end{array}.$$

For left cancellative semigroups, we know that the Følner condition is equivalent to the strong Følner condition(Remark 1.2.10), so we have:

$$\text{SFC} \Leftrightarrow \text{SNFC} \Leftrightarrow \text{WNFC} \Leftrightarrow \text{LA} \Leftrightarrow \text{FC} \Rightarrow \text{WFC}.$$

For (2-sided) cancellative semigroups, Yang showed that the weak Følner condition implies the strong Følner condition ([15], Theorem 2.7); so we have:

$$\text{SFC} \Leftrightarrow \text{SNFC} \Leftrightarrow \text{WNFC} \Leftrightarrow \text{LA} \Leftrightarrow \text{FC} \Leftrightarrow \text{WFC}.$$

For finite semigroups, the Følner condition is always satisfied (take A to be the whole semigroup), and Yang showed that left amenability implies the strong Følner condition([15], Theorem 2.3). So we have:

$$\text{SFC} \Leftrightarrow \text{SNFC} \Leftrightarrow \text{WNFC} \Leftrightarrow \text{LA} \Rightarrow \text{WFC} .$$

Proposition 3.3.1. *A semigroup, S, satisfies Dixmier's condition if and only if S is not left amenable. ([13], 2.3)*

Proposition 3.3.2. *If a semigroup, S, is left measurable then S satisfies the strong Følner condition. ([12], 5.3)*

Proposition 3.3.3. *If S is a semigroup, and there is a homomorphism $h : S \longrightarrow h(S)$ where $h(S)$ is a finite semigroup, then $\varphi(S) \geq \varphi(h(S))$. ([15], 2.6)*

Proposition 3.3.4. *Let S be a semigroup with the finite intersection property for right ideals. Then $\varphi(S) \leq \varphi(S')$. ([15], 2.9)*

3.4 Some Preliminaries

The following results are interesting in their own right, along with being particularly useful in identifying and creating examples of left amenable semigroups.

Proposition 3.4.1. *Let S be an abelian semigroup. Then S is left amenable.*

Proof. It is shown in [1] that every abelian semigroup satisfies the strong Følner condition. Hence every abelian semigroup is left amenable. □

Proposition 3.4.2. *Let S be a left amenable semigroup. Then S has the finite intersection property for right ideals.*

Proof. By noting that the intersection of two right ideals of a semigroup, is again a right ideal, it suffices to show that the intersection of two right ideals of a left amenable semigroup is nonempty.

Assume for contradiction that I and J are two nonempty right ideals of S and that $I \cap J = \emptyset$. Now consider the characteristic functions on these two ideals χ_I, χ_J

and let $s \in I$, $t \in J$. Observe that these functions and elements show that S satisfies Dixmier's condition since for all $x \in S$:

$$\chi_I(sx) - \chi_I(x) + \chi_J(tx) - \chi_J(x) = 2 - \chi_I(x) - \chi_J(x). \qquad (3.1)$$

Since $I \cap J = \emptyset$ the infimum of 3.1 must be 1. Since $1 > 0$, we have that S satisfies Dixmier's condition contradicting 3.3.1. □

Proposition 3.4.3. *Let S be a semigroup with a zero element. Let $z \in S$ such that for any $x \in S$, we have $zx = z = xz$. Then S satisfies the strong Følner condition, and hence is left amenable.*

Proof. Let $A = \{z\}$. Then A is a finite, non-empty subset of S and for any $s \in S$, $\varepsilon > 0$ we have:

$$|A \backslash sA| = |\{z\} \backslash \{z\}| = 0 < \varepsilon = \varepsilon|A|$$

Hence S satisfies the strong Følner condition. □

Remark 3.4.4. *In the case where S has a zero element, z, the left invariant mean on $\ell^\infty(S)$ is just the point evaluation functional at z.*

Proposition 3.4.5. *Let S be a semigroup which is both left amenable and left cancellative. Then S is left measurable.*

Proof. S is left cancellative, so $s^{-1}(sA) = A$ for all $s \in S, A \subset S$. Since S is left amenable, there exists a left invariant mean $m \in \ell_\infty(S)^*$. This tells us that

$$m(\chi_A) = m(\chi_{s^{-1}(sA)}) = m(\chi_{sA}) \quad \forall s \in S, A \subset S.$$

□

3.5 Weakly Cancellative Semigroups

We have seen that all groups are cancellative semigroups and all [left, right] cancellative semigroups are weakly [left, right] cancellative. It is natural to wonder what properties for groups or left cancellative semigroups can be extended to weakly left cancellative semigroups. We give some examples of weakly left cancellative semigroups. We will see that even for a simple example of a weakly left cancellative semigroup, the Følner condition is not equivalent to the strong Følner condition. Analysis on the class of weakly cancellative semigroups has recently been studied extensively by Dales, Lau, and Strauss [2].

Consider the following examples.

Example 3.5.1. *Let $S := (\mathbb{N}, \min)$ be the semigroup of natural numbers with the semigroup action given by taking the minimum of the two elements. (Here we use the standard order on $\mathbb{N}$)*

Then it is clear that $n^{-1}n = \{m \in \mathbb{N} : m \geq n\}$ so $|n^{-1}n| = \infty$.

On the other hand, S is clearly abelian, so S is left amenable.

Example 3.5.2. *Now let S be the semigroup of only two elements with multiplication given by:*

$$aa = a, \quad ab = a, \quad ba = b, \quad bb = b.$$

Since S is finite it is weakly left cancellative and satisfies the Følner condition, but $aS = \{a\}$ and $bS = \{b\}$ so S does not satisfy the finite intersection property for right ideals and hence is not left amenable. Thus S does not satisfy the strong Følner condition.

Remark 3.5.3. *To see how the Følner condition and strong Følner condition differ for weakly left cancellative semigroups, notice that we have the following equalities:*

$$A \backslash sA = A \backslash (A \cap sA) \tag{3.2}$$

$$|A \backslash sA| = |A| - |A \cap sA| \tag{3.3}$$

and similarly

$$|sA \backslash A| = |sA| - |sA \cap A| \tag{3.4}$$

So this tells us that

$$|A \backslash sA| = |A| + |sA \backslash A| - |sA| \tag{3.5}$$

When the semigroup is left cancellative, then $|A| = |sA|$, *so that* $|A \backslash sA| = |sA \backslash A|$ *and the Følner condition is equivalent to the strong Følner condition. However, if the semigroup is merely weakly left cancellative, then for the Følner condition and strong Følner condition to be equivalent, we need to be able to make* $\frac{|A|-|sA|}{|A|}$ *arbitrarily small.*

3.6 Sorenson's Conjecture

Sorenson stated that every right cancellative left amenable semigroup is left cancellative. In [12], Klawe gave an example of a left amenable, right cancellative semi-group which is not left cancellative, disproving Sorenson's conjecture. We show that her example is weakly left cancellative, demonstrating that there exists a left amenable, right cancellative, weakly left cancellative semigroup which is not left cancellative. It is then natural to consider the weakened version of Sorenson's conjecture: 'every right cancellative left amenable semigroup is weakly left cancellative'. We then construct a left amenable, right cancellative semi-group which is not weakly left cancellative, disproving this weaker version of Sorenson's conjecture.

The examples that will be examined in this section are generated by taking semidirect products of semigroups. The semidirect product of two semigroups is very similar to the semidirect product of two locally compact groups, with two notable differences. The first is that in this case we are dealing with discrete semigroups and so we do not require any continuity conditions. The second is that when we were dealing with groups, we were restricted to automorphisms. Since we are now dealing with semigroups, we can consider all endomorphisms (homomorphisms from a semigroup to itself). In light of this, we use the notations $\mathrm{End}(U)$ to denote the set of endomorphisms of a semigroup U, $\mathrm{Sur}(U)$ to denote the surjective endomorphisms, and $\mathrm{Inj}(U)$ to denote the injective endomorphisms.

Definition 3.6.1. Let U and T be semigroups. Let ρ be a semigroup homomorphism from T to $\mathrm{End}(U)$. We say that $S = U \rtimes_\rho T$ is the *semidirect product* of U and

T with respect to ρ if S is the semigroup consisting of elements of the form (u, t) where $u \in U$ and $t \in T$ equipped with multiplication given by:

$$(u_1, t_1)(u_2, t_2) = (u_1 \rho_{t_1}(u_2), t_1 t_2)$$

The following two results are proved in Klawe's paper [12] as Lemma 3.2, and Proposition 3.4.

Lemma 3.6.2. *If U and T are right cancellative semigroups with a homomorphism $\rho : T \to \mathrm{End}(U)$, then $S = U \rtimes_\rho T$ is right cancellative.*

Proposition 3.6.3. *If U and T are left amenable semigroups with a homomorphism $\rho : T \to \mathrm{Sur}(U)$, then $S = U \rtimes_\rho T$ is left amenable.*

Example 3.6.4. *Klawe's counterexample to Sorenson's conjecture is the semidirect product $S = U \rtimes_\rho T$, where U is the free abelian semigroup generated by $\{u_i \mid i = 0, 1, 2, 3, \dots\}$, T is the cyclic semigroup generated by $\{a\}$ and ρ is given by:*

$$\rho_a(u_0) = u_0 \tag{3.6}$$

$$\rho_a(u_i) = u_{i-1} \quad \text{for } i \geq 1 \tag{3.7}$$

By the above results, this S is left amenable and right cancellative.

Notice that

$$(u_0, a)(u_0, a) = (u_0^2, a^2)$$

and

$$(u_0, a)(u_1, a) = (u_0^2, a^2)$$

So S is not left cancellative.

We will now show that this S is weakly left cancellative.

Consider two arbitrary elements of the semigroup $x, y \in S$. We can write $x = (u_0^{e_0} u_1^{e_1} u_2^{e_2} \dots, a^n), y = (u_0^{f_0} u_1^{f_1} u_2^{f_2} \dots, a^m)$. It is sufficient to show that the set $x^{-1}y$ is finite.

By definition we have that:

$$x^{-1}y = \{z \in S \mid xz = y\} \tag{3.8}$$
$$= \{(u_0^{g_0}u_1^{g_1}u_2^{g_2}\dots, a^l) \mid (u_0^{e_0}u_1^{e_1}u_2^{e_2}\dots, a^n)(u_0^{g_0}u_1^{g_1}u_2^{g_2}\dots, a^l) \tag{3.9}$$
$$= (u_0^{f_0}u_1^{f_1}u_2^{f_2}\dots, a^m)\} \tag{3.10}$$
$$= \{(u_0^{g_0}u_1^{g_1}u_2^{g_2}\dots, a^l) \mid (u_0^{e_0+\sum_{i=0}^n g_i}u_1^{e_1+g_{1+n}}u_2^{e_2+g_{2+n}}\dots, a^{n+l}) \tag{3.11}$$
$$= (u_0^{f_0}u_1^{f_1}u_2^{f_2}\dots, a^m)\} \tag{3.12}$$
$$= \{(u_0^{g_0}u_1^{g_1}u_2^{g_2}\dots, a^l) \mid f_0 = e_0 + \sum_{i=0}^{n} g_i, f_i = e_i + g_{i+n},\ i \geq 1, m = n + l\} \tag{3.13}$$
$$= \{(u_0^{g_0}u_1^{g_1}u_2^{g_2}\dots, a^l) \mid l = m - n, \sum_{i=0}^{n} g_i = f_0 - e_0, g_{i+n} = f_i - e_i,\ i \geq 1\} \tag{3.14}$$

Note that if $f_i < e_i$ for any $i \geq 0$ or $m \leq n$ then there is no way to satisfy the above conditions, so $x^{-1}y = \emptyset$.

But, provided that $f_i \geq e_i$ for $i \geq 0$ and $m - n \geq 1$ we can determine that the size of $x^{-1}y$ is the number of unique ways that the condition $\sum_{i=0}^{n} g_i = f_0 - e_0$ can be satisfied. In other words, we are looking for the number of compositions of $f_0 - e_0$ into $n+1$ parts allowing parts of size zero. This is precisely $\left[x^{f_0-e_0}\right](1-x)^{-(n+1)}$, the coefficient of $x^{f_0-e_0}$ in the formal power series $(1-x)^{-(n+1)}$. We can calculate this value using the negative binomial theorem. This gives:

$$|x^{-1}y| = \binom{n + f_0 - e_0}{f_0 - e_0}$$

In short, this value is finite; hence S is weakly left cancellative.

We will now construct a semigroup which is left amenable and right cancellative, but not weakly left cancellative.

Example 3.6.5. *Let U be the free abelian semigroup generated by $\{u_{i,j} \mid i = 1, 2, 3\dots, j = 1, 2, 3, \dots\}$ with a two sided identity e. Let T be the cyclic semigroup generated by $\{a\}$. Define ρ via:*

$$\rho_a(u_{i,j}) = u_{i-1,j} \text{ if } i > 1, \tag{3.15}$$

$$\rho_a(u_{1,j}) = e, \tag{3.16}$$

$$\rho_a(e) = e. \tag{3.17}$$

U and T are abelian, hence left amenable, and $\rho(T) \subset \mathrm{Sur}(U)$. This implies that $S = U \rtimes_\rho T$ is left amenable. Also, since U and T are right cancellative, so is S. To see that S is not weakly left cancellative, notice that:

$$(e, a^2) = (e, a)(u_{1,j}, a) \text{ for any choice of } j \geq 0$$

Hence $|(e,a)^{-1}(e,a^2)| = \infty$.

3.7 A Necessary Condition for Amenability of a Semidirect Product

In this section, we examine the condition of proposition (3.6.3). Klawe showed that for the semidirect product of two left amenable semigroups to be left amenable, it is sufficient that $\rho : T \to \mathrm{Sur}(U)$.

It would be nice if the condition in the above proposition were necessary as well as sufficient. However, as Klawe points out, if U, T both have zeroes, and $\rho_t(u) = 0$ for any $t \in T, u \in U$ then S has a zero as well and hence is left amenable. Instead we will look at the semidirect product of the right cancellative quotient semigroup U' and T. This gives us the desired result in the case where U is right cancellative, namely that $\rho : T \to \mathrm{Sur}(U)$ is necessary and sufficient.

Definition 3.7.1. Let U, T be semigroups. Let ρ be a semigroup homomorphism from T to $\mathrm{End}(U)$. We define $\overline{\rho} : T \to \mathrm{End}(U')$ by setting $\overline{\rho}_t(\overline{u}) := \overline{\rho_t(u)}$.

Lemma 3.7.2. *$\overline{\rho}$ is well defined.*

Proof. Let $u, v \in U$ be in the same equivalence class. Then there exists an $x \in U$ such that $ux = vx$. Since ρ_t is a semigroup homomorphism,

$$\rho_t(u)\rho_t(x) = \rho_t(ux) = \rho_t(vx) = \rho_t(v)\rho_t(x)$$

But this means that $\rho_t(u)$ is equivalent to $\rho_t(v)$. Hence $\overline{\rho}_t(\overline{u})$ is well defined. $\square$

Proposition 3.7.3. *Let U, T be semigroups and let $\rho : T \to \mathrm{End}(U)$ be a semigroup homomorphism. If $U \rtimes_\rho T$ is left amenable, then $\overline{\rho} : T \to \mathrm{Sur}(U')$.*

Proof. We begin by noting that if $U \rtimes_\rho T$ is left amenable, then so are U and T. Now since U is left amenable, it has the finite intersection property for right ideals, so U' exists and the notation in our proposition makes sense.

Suppose for contradiction that $\overline{\rho}$ does not send T to $\mathrm{Sur}(U')$. Then there exist $a \in T$, and $u \in U$ such that

$$\overline{u} \notin \overline{\rho}_a(U')$$

or, written slightly differently,

$$\rho_a(U) \cap \overline{u} = \emptyset.$$

We will show that $U \rtimes_\rho T$ satisfies Dixmier's condition which will contradict it being left amenable. To do this we will define the following functions and elements of the semigroup.

Now let $f_1, f_2 \in \ell^\infty(U \rtimes_\rho T)$ be given by:

$$f_1 = \chi_{\overline{u} \times T} \tag{3.18}$$

$$f_2 = \chi_{\rho_a(U) \times T} \tag{3.19}$$

and let

$$s_1 = (u, a) \tag{3.20}$$

$$s_2 = (\rho_a(u), a). \tag{3.21}$$

Observe that for $(v, b) \in U \rtimes_\rho T$

$$f_1((u,a)(v,b)) = f_1(u\rho_a(v), ab) \quad (3.22)$$

$$= 1, \quad (3.23)$$

$$(3.24)$$

$$f_1(v,b) = \begin{cases} 1 & v \in \overline{u} \\ 0 & v \notin \overline{u} \end{cases}, \quad (3.25)$$

$$(3.26)$$

$$f_2((\rho_a(u),a)(v,b)) = f_2(\rho_a(u)\rho_a(v), ab) \quad (3.27)$$

$$= f_2(\rho_a(uv), ab) \quad (3.28)$$

$$= 1, \quad (3.29)$$

$$(3.30)$$

$$f_2(v,b) = \begin{cases} 1 & v \in \rho_a(U) \\ 0 & v \notin \rho_a(U) \end{cases}. \quad (3.31)$$

$$(3.32)$$

So that

$$\sup_{(v,b)\in U\rtimes_\rho T} f_1(v,b) + f_2(v,b) = 1 \quad (3.33)$$

and

$$\inf_{(v,b)\in U\rtimes_\rho T} f_1(s_1(v,b)) - f_1(v,b) + f_2(s_2(v,b)) - f_2(v,b) = 1 > 0. \quad (3.34)$$

Thus $U \rtimes_\rho T$ satisfies Dixmier's condition, which is a contradiction. Hence if $U \rtimes_\rho T$ then $\overline{\rho}$ must send T to the surjective endomorphisms of (U'). □

Chapter 4

Semigroup Actions

4.1 Introduction

The multiplication of a semigroup, S, is a mapping from $S \times S$ to S which satisfies $r(st) = (rs)t,\ \forall r, s, t \in S$. We can also consider mappings from $S \times X$ to X, $(s, x) \mapsto s \cdot x$ for an arbitrary set X. Then if $s \cdot (t \cdot x) = (st) \cdot x,\ \forall s, t \in S,\ x \in X$, we call this a left semigroup action. If we think of the multiplication of a semigroup as an action of the semigroup on itself, it is natural to consider the generalization of various concepts to semigroup actions on sets. This chapter examines how concepts such as amenability and Dixmier's condition can be extended to left semigroup actions.

Definition 4.1.1. Let S be a semigroup, X a set. If we have an action:

$$\cdot : S \times X \to X, (s, x) \mapsto s \cdot x$$

such that we have $s \cdot (t \cdot x) = (st) \cdot x$ for every choice of $s, t \in S, x \in X$, then we call the triple $(S, X, \cdot)$ a *left semigroup action* of S on X.

Definition 4.1.2. Let $(S, X, \cdot)$ be a left semigroup action. We say that $(S, X, \cdot)$ is *amenable* if there exists a mean, m, on $\ell^\infty(X)$ such that for $f \in \ell^\infty(X), s \in S$, we have $m(f) = m(l_s f)$. Here we use the notation l_s in the following fashion:

$$l_s f(x) = f(s \cdot x).$$

4.2 Dixmier's Condition

Dixmier's condition can be extended to a left semigroup action. All that needs to be changed is that the functions in $\ell^\infty(S)$ are replaced by functions in $\ell^\infty(X)$ and multiplication is replaced by the left action. The proof that a left semigroup action satisfies Dixmier's condition if and only if it is not amenable is quite similar to that given by Namioka in [13](Remark 2.3). But since this a new extension going beyond semigroups, we will reproduce the proof here.

Definition 4.2.1. Let $f \in \ell^1(X)$. f is called a *finite mean* if $f(x) \geq 0$, $\forall x \in X$, the set $\{x \in X \mid f(x) > 0\}$ is finite, and $\|f\|_1 = \sum_{x \in X} f(x) = 1$. We shall denote the set of all finite means by Φ.

Remark 4.2.2. *Note that the final condition above ensures that finite means are indeed means (when embedded into the second dual of $\ell^1(X)$).*

Given a left semigroup action, $(S, X, \cdot)$, we can define a left semigroup action of S on $\ell^1(X)$. This action is similar to convolution when we are dealing with a semigroup acting on itself by multiplication. This action is given by:

$$(s \cdot f)(x) := \sum_{y \in X, s \cdot y = x} f(y)$$

If $s \cdot y = x$ has no solution, then $(s \cdot f)(x) = 0$. One can check that $s \cdot f$ is indeed in $\ell^1(X)$ and that $t \cdot (s \cdot f) = (ts) \cdot f$. Additionally, it is useful to notice that the dual of this action is the left shift $l_s : \ell^\infty(X) \to \ell^\infty(X)$.

Lemma 4.2.3. *Let $(S, X, \cdot)$ be a left semigroup action. $(S, X, \cdot)$ is amenable if and only if there exists a net of finite means, $(f_\alpha)_\alpha \subset \Phi$ such that:*

$$w\text{--}\lim_\alpha s \cdot f_\alpha - f_\alpha = 0 \quad \forall s \in S$$

Proof. First we notice that the weak topology on $\ell^1(X)$ is the same as the relative topology induced by the weak-* topology on $\ell^\infty(X)^*$. Throughout this proof we will refer to this topology as the τ-topology

$\Rightarrow$ Assume that $(S, X, .)$ is amenable. Then there exists a S-invariant mean, m on $\ell^\infty(X)$.

We will show that Φ is τ-dense in the set of all means of X. Once we have this, we will be able to find a net of finite means, $(f_\alpha)_\alpha$ tending to m with respect to τ. Since $l_s^* m - m = 0$, we have that:

$$\tau - \lim_\alpha s.f_\alpha - f_\alpha = 0$$

To show that Φ is τ-dense, assume for contradiction that there exists a mean $m \notin \overline{\Phi}^\tau$. Then by a corollary to the Hahn-Banach theorem ([4], V.2.10) there exist $\delta > 0$ and $g \in \ell^\infty(X)^{**}$ which is τ-continuous such that:

$$\langle m, g\rangle \geq \delta + \langle f, g\rangle \tag{4.1}$$

for all finite means f. But since g is τ-continuous, $g \in \ell^\infty(X)$ and 4.1 yields:

$$m(g) \geq \delta + g(x) \tag{4.2}$$

for all $x \in X$. But 4.2 clearly contradicts m being a mean since it would imply $\|m\| > 1$.

$\Leftarrow$ Assume that $(f_\alpha)_\alpha$ is a net of finite means which converge to left invariance with respect to τ.

The set of all means of X is a τ-closed subset of the unit ball of $\ell^\infty(X)^*$. Hence, by Alaoglu's Theorem ([10], B 25) it is τ-compact. This implies that $(f_\alpha)_\alpha$ has a subnet $(f_\beta)_\beta$ which converges to a mean m with respect to τ. But since this subnet converges to left invariance, m must be a left invariant mean. □

Theorem 4.2.4. *Let $(S, X, \cdot)$ be a left semigroup action. Then $(S, X, \cdot)$ is not amenable if and only if there exist $n \in \mathbb{N}$, $f_1, \ldots, f_n \in \ell^\infty(X)$ and $s_1, \ldots, s_n \in S$ such that:*

$$\inf_{x \in X} \sum_{i=1}^{n} f_i(s_i \cdot x) - f_i(x) > 0 \tag{4.3}$$

Proof. By the lemma, $(S, X, \cdot)$ is not amenable, if and only if there does not exist a net of finite means which tend to left invariance weakly.

Let $E := (\ell^1(X))^S$ be equipped with the product of the norm topologies. Now define $T : \ell^1(X) \to E$ by $Tf(s) = s \cdot f - f$.

So now we have that $(S, X, \cdot)$ is not amenable if and only if

$$0 \notin \overline{T(\Phi)}^{\tau}$$

$$\Leftrightarrow \quad \exists \psi \in ((\ell^1(X))^S)^* \text{ such that } \inf\{|\psi(Tf)| : f \in \Phi\} > 0$$

$$\Leftrightarrow \quad \exists \psi \in ((\ell^1(X))^S)^* \text{ such that } \inf\{|\psi(Tf)| : f \in \Phi\} > 0$$

$$\Leftrightarrow \quad \exists u_1, \ldots, u_n \in \ell^1(X)^*, t_1, \ldots, t_n \in S \text{ such that } \inf\{\textstyle\sum_{i=1}^n u_i(T\delta_x(t_i)) : x \in X\} > 0$$

$$\Leftrightarrow \quad \exists u_1, \ldots, u_n \in \ell^1(X)^*, t_1, \ldots, t_n \in S \text{ such that } \inf\{\textstyle\sum_{i=1}^n u_i(t_i\delta_x - \delta_x) : x \in X\} > 0$$

$$\Leftrightarrow \quad \exists u_1, \ldots, u_n \in \ell^1(X)^*, t_1, \ldots, t_n \in S \text{ such that } \inf\{\textstyle\sum_{i=1}^n u_i(t_i.x) - u_i(x) : x \in X\} > 0.$$

□

4.3 Relation between Amenability of Left Semigroup Actions and Semigroup Left Amenability

Here we will present a result that shows if a semigroup is left amenable, then all left actions of that semigroup are amenable. The converse is also true, however there are semigroups which are not left amenable, but which have amenable left actions. Indeed, every semigroup has an amenable left action, namely the trivial action where $s.x = x$ for each choice of $s \in S$, $x \in X$. We also give another, slightly more interesting example of a left semigroup action which is amenable for a semigroup which is not left amenable.

Theorem 4.3.1. *Let S be a semigroup. S is left amenable if and only if every left action, $(S, X, \cdot)$, of S on a set X is amenable.*

Proof. $\Leftarrow$ Simply take $X = S$, and let $\cdot$ be the semigroup multiplication.

$\Rightarrow$ Assume for contradiction that there exists a left action $(S, X, \cdot)$ which is not amenable. Then there are $u_1, \ldots, u_n \in l_\infty(X)$ and $s_1, \ldots, s_n \in S$ such that:

$$m := \inf_{x \in X} \sum_{i=1}^{n} u_i(s_i.x) - u_i(x) > 0.$$

Fix $x_0 \in X$, for $i = 1, \ldots, n$ let $\hat{u}_i \in l_\infty(S)$ be given by:

$$\hat{u}_i(s) := u_i(s.x_0)$$

Then for $s \in S$

$$\sum_{i=1}^{n} \hat{u}_i(s_i s) - \hat{u}_i(s) = \sum_{i=1}^{n} u_i(s_i(s.x_0)) - u_i(s.x_0) \tag{4.4}$$

$$\geq m \tag{4.5}$$

So $\inf_{s \in S} \sum_{i=1}^{n} \hat{u}_i(s_i s) - \hat{u}_i(s) \geq m > 0$.

So S is not left amenable. This is a contradiction proving the theorem. □

Example 4.3.2. *Let F_2 be the free semigroup with two generators, a and b. It is well known that F_2 is not amenable. Let A_2 be the free abelian semigroup with two generators $\tilde{a}$ and $\tilde{b}$. Let $(F_2, A_2, \cdot)$ be the left action given by $a \cdot x = \tilde{a}x$, $b \cdot x = \tilde{b}x$ for all $x \in A_2$. Then $(F_2, A_2, \cdot)$ is amenable.*

Proof. Assume for contradiction that $(F_2, A_2, \cdot)$ is not amenable. By Dixmier's condition for left semigroup actions, theorem 4.2.4, there exist $n \in \mathbb{N}$, $f_1, \ldots, f_n \in \ell^\infty(A_2)$ and $s_1, \ldots, s_n \in F_2$ such that

$$\inf_{x \in A_2} \sum_{i=1}^{n} f_i(s_i \cdot x) - f_i(x) > 0.$$

Now consider $\tilde{s_1}, \ldots, \tilde{s_n} \in A_2$ realized under the homomorphism from F_2 to A_2 which sends a to $\tilde{a}$ and b to $\tilde{b}$. Since A_2 is abelian, it is also left amenable (Theorem 3.4.1) but the existence of $f_1, \ldots, f_n$ and $\tilde{s_1}, \ldots, \tilde{s_n}$ contradicts Dixmier's condition for semigroups (Proposition 3.3.1) implying that $(F_2, A_2, \cdot)$ is amenable. □

Chapter 5

Further Work

5.1 Introduction

This chapter is a collection of questions which arose during the creation of this thesis. Some are questions posed by other authors that are related to, but beyond the scope of this work. Others are natural continuations of some of the results presented herein.

5.2 Questions

Question 1. *Can the δ function which arose from semidirect products be defined in a more general sense? Are there applications for this δ function beyond calculating the modular function of a semidirect product?*

All that is apparently needed to define the δ function is a semigroup action on a measure space with a measure such that if the measure is composed with the action of an element of the semigroup, the resulting set function is a constant multiple of the original measure. For example, consider a semigroup (using matrix multiplication) of 2×2 invertible matrices acting on the measure space $\mathbb{R}^2$. The δ function in this case corresponds to the inverse of the absolute value of the determinant of the matrix.

Question 2. *In the case of a unimodular semidirect product of unimodular groups, Janzen showed that the listed conditions of Theorem 2.3.7 are both necessary and* *sufficient for the product of two Følner nets to be a Følner net for the semidirect product (Condition 2 of the theorem corresponds precisely to (B_β) being a Følner net for H). Can necessity be extended to the general case?*

We know that the 'ax+b' group can not have a rectangular Følner net, but neither does it satisfy the conditions of the theorem. So, while it is far from providing conclusive evidence that this question can be answered positively, at least it does not provide a counterexample.

Question 3. *Does there exist a left amenable semigroup with Følner number strictly between* 0 *and* 1*?*

This question has been posed previously by Yang [15]. He showed that if such a semigroup exists, it cannot be finite, abelian, left cancellative, or a semidirect product of these. If it can be shown that no such semigroup exists, then the strong Følner condition is equivalent to the weak Følner condition plus left amenability. One step in this direction might be showing that this is true for weakly left cancellative semigroups. This particular subcollection of semigroups is interesting because it contains all finite and all left cancellative semigroups.

Question 4. *Does there exist a semigroup, S, with the finite intersection property for right ideals satisfying:*

$$\phi(S) < \phi(S')?$$

By a theorem of Yang ([15], Theorem 2.9), we know that $\phi(S) \leq \phi(S')$. It is also known that S is left amenable if and only if S' is ([14], Proposition 1.25). Given the above problem, any example of this will either answer the previous question, or $\phi(S) = 0$ and $\phi(S') = 1$.

Question 5. *Can the condition of proposition 3.7.3 be shown to be sufficient as well as necessary?*

Proposition 3.7.3 tells us that $\overline{\rho}$ must map to the surjective endomorphisms of U' for a semidirect product of semigroups to be left amenable. Klawe showed that if ρ maps to the surjective endomorphisms of U this is sufficient for the semidirect product of left amenable semigroups to be left amenable. Certainly these conditions are equivalent if U is right cancellative, but this is not true in the general case.

Question 6. *We have generalized some results previously known for semigroups to semigroup actions. What other results can be generalized in this way?*

In ([13] 4.1), Namioka proved that the weak Namioka-Følner condtion implies left amenability. His proof was adapted from a result of Følner([6] Section 3). If we try to extend this result to the semigroup action case, we find that a multiplication is required on X.

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