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A SINGULAR BOUNDARY VALUE PROBLEM

by

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A THESIS

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled A SINGULAR BOUNDARY VALUE PROBLEM submitted by F. BRENT SATO in partial fulfilment of the requirements for the degree of Master of Science.

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ABSTRACT

The differential equation

$$w''' + ww'' + \lambda\left(1 - (w')^2\right) = 0,$$

where $\lambda \geq 0$, is an important equation in boundary layer theory. Various techniques have been used to prove the existence of solutions on $[0,\infty)$ satisfying various boundary conditions. In particular, in the context of an intermediate value property for operators on partially ordered sets developed by Muldowney and Willett [8], Gamlen and Muldowney [4] demonstrated the existence of a solution satisfying $w(0) = a$, $w'(0) = b$, $w'(\infty) = 1$, where $0 < b < 1$. In the same context, this result is extended in this thesis.

The introduction, Chapter I, contains a brief outline of the derivation of the boundary value problem under consideration. In Chapter II, the intermediate value property is defined and several examples of operators with this property are given. The basic theorem used to prove the final existence results is stated and proved as II.1. Other preliminary results required are also developed in Chapter II. The final results are presented in Chapter III.

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CHAPTER I

INTRODUCTION

Consider the flow of a fluid past a smooth solid surface. If the fluid is assumed to be perfect, that is, frictionless and incompressible, then on the boundary between the fluid and the surface, the fluid has, in general, a nonzero velocity. However, in real fluids, the existence of intermolecular attractions causes the fluid to adhere to the surface so that frictional forces retard its motion in a thin layer near the wall. The velocity of the fluid increases from zero at the boundary to the full value corresponding to external frictionless flow. This layer under consideration is called the boundary layer.

The frictional forces in a real fluid which give rise to the boundary layer are due to a physical phenomenon of fluids called viscosity. Viscosity in fluid flow is the analogue of friction in the motion of solids. Certain fluids, called Newtonian fluids, have a physical, temperature-dependent quantity associated with them called the coefficient of viscosity, μ.

In order to write the two dimensional equations of motion of a real, incompressible, Newtonian fluid, let (x,y) be a Cartesian co-ordinate system and τ = time. Let $\left(u(x,y,\tau), v(x,y,\tau)\right)$ be the velocity vector in the flow field, $p(x,y,\tau)$ the pressure in the fluid, and ρ = constant, the density. Then the fundamental differential equations which form the basis for fluid mechanics, the Navier-Stokes equations, give,

$$\rho\left(\frac{\partial u}{\partial \tau} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\rho\left(\frac{\partial v}{\partial \tau} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 ,$$

with the condition that the velocity is identically zero on solid surfaces.

If the surface is flat and coincides with the x-axis, and the frictionless flow outside the boundary layer is a steady flow $U(x)$, then several simplifying assumptions can be made to deduce Prandtl's boundary layer equations for steady flow,

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{dU}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \tag{1}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 , \tag{2}$$

with $\nu = \frac{\mu}{\rho}$ and boundary conditions $u = 0 = v$, when $y = 0$, and $u = U(x)$, when $y = \infty$.

Now, consider the flow past a wedge of a real, incompressible, Newtonian fluid flowing linearly toward it at a constant velocity, and in the direction which bisects the wedge angle. Let (x,y) be a Cartesian co-ordinate system whose x-axis coincides with one edge of the wedge and whose origin coincides with the tip. It is known that for small x, the external frictionless flow is $U(x) = u_1 x^m$, for some constant u_1, where the wedge angle is $\pi\lambda$ and $\lambda = \frac{2m}{m+1}$. In determining the boundary layer flow, Falkner and Skan [3] were able to reduce the equations (1) and (2) to a single ordinary differential equation by a suitable change of variables as follows.

Introduce a "stream" function $\psi(x,y)$ defined by $u = \frac{\partial\psi}{\partial y}$, $v = -\frac{\partial\psi}{\partial x}$. Then ψ determines a solution of (1) and (2) if

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\,\partial y} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2} = u_1^2 m x^{2m-1} + \nu\frac{\partial^3\psi}{\partial y^3}, \tag{3}$$

with $\frac{\partial\psi}{\partial y} = 0 = \frac{\partial\psi}{\partial x}$, when $y = 0$, and $\frac{\partial\psi}{\partial y} = u_1 x^m$, when $y = \infty$.

In particular, let

$$t = y\sqrt{\frac{(m+1)u_1}{2\nu}}\, x^{\frac{m-1}{2}}$$

and solve for $w = w(t)$ such that

$$\psi(x,y) = \sqrt{\frac{2\nu u_1}{m+1}}\, x^{\frac{m+1}{2}}\, w(t)$$

satisfies (3). The velocity components become

$$\begin{cases} u = u_1 x^m w'(t) \\ v = -\sqrt{\frac{m+1}{2}\nu u_1 x^{m-1}}\left(w(t) + \frac{m-1}{m+1} t w'(t)\right), \end{cases}$$

and (3) becomes

$$w'''(t) + w(t)w'(t) + \lambda\left(1 - (w'(t))^2\right) = 0 \tag{4}$$

with boundary conditions $w = 0 = w'$, when $t = 0$, and $w' = 1$, when $t = \infty$.

The solution of the differential equation (4) with various boundary conditions has been the subject of extensive literature since it was presented in 1931. Hartman's book [5, pp. 519-537] contains a summary of most of the results obtained. Recently, Gamlen and Muldowney [4] proved an intermediate value theorem which could be used to quickly and easily demonstrate the existence of a solution, for $\lambda \geq 0$, satisfying $w(0) = a$, $w'(0) = b$, $w'(\infty) = 1$, where $0 < b < 1$ and $w'(t) \geq 0$ on $[0,\infty)$. In what follows, the same technique is used to answer questions of existence for similar boundary conditions, but with $|b| < 1$.

CHAPTER II

THE INTERMEDIATE VALUE PROPERTY AND PRELIMINARY RESULTS

The problem of existence will be considered in terms of an intermediate value property introduced by Muldowney and Willett [8].

Definition: Let X and S be partially ordered sets. A map $T:X \to S$ has the intermediate value property with respect to $s \in S$ if $\alpha, \beta \in X$, $\alpha \leq \beta$, and $T\alpha \leq s \leq T\beta$ implies that there exists $z \in X$ such that $\alpha \leq z \leq \beta$ and $Tz = s$.

In the examples that follow assume that the partial ordering of real function spaces is the usual order holding pointwise a.e. on the domain of the functions, and assume componentwise partial ordering for product spaces.

Regarding notation, let $L^1(S)$, $C^k(S)$, and $AC^k(S)$ denote the real valued functions having finite Lebesgue integrals, their first k derivatives continuous, and their first k derivatives absolutely continuous on S, respectively. The prefix "loc" means the identifying property needs to hold only locally on S.

Example 1: A well-known result on differential inequalities (cf. [2, p. 30]) implies the following. Suppose $f:[0,h) \times \mathbb{R} \to \mathbb{R}$, $0 < h \leq \infty$, is continuous, and $\alpha, \beta \in \text{loc}\,AC[0,h)$ satisfy

$$\alpha'(t) \leq f(t,\alpha(t)), \quad \beta'(t) \geq f(t,\beta(t)), \quad \text{a.e. } t \in [0,h),$$

$$\alpha(0) \leq z_0 \leq \beta(0), \quad \alpha(t) \leq \beta(t), \quad t \in [0,h).$$

Then there exists $z \in C^1(0,h)$ such that

$$z'(t) = f(t,z(t)), \quad z(0) = z_0 ,$$

$$\alpha(t) \le z(t) \le \beta(t), \quad t \in [0,h).$$

It follows that the operator $T: \text{loc } AC[0,h) \to \text{loc } L^1[0,h) \times \mathbb{R}$ defined by

$$(Tx)(t) = \big(x'(t) - f(t,x(t)),\ x(0)\big) ,$$

has the intermediate value property with respect to $(0,z_0)$.

The intuitive case $f \equiv 0$ is illustrated below in Figure 1.

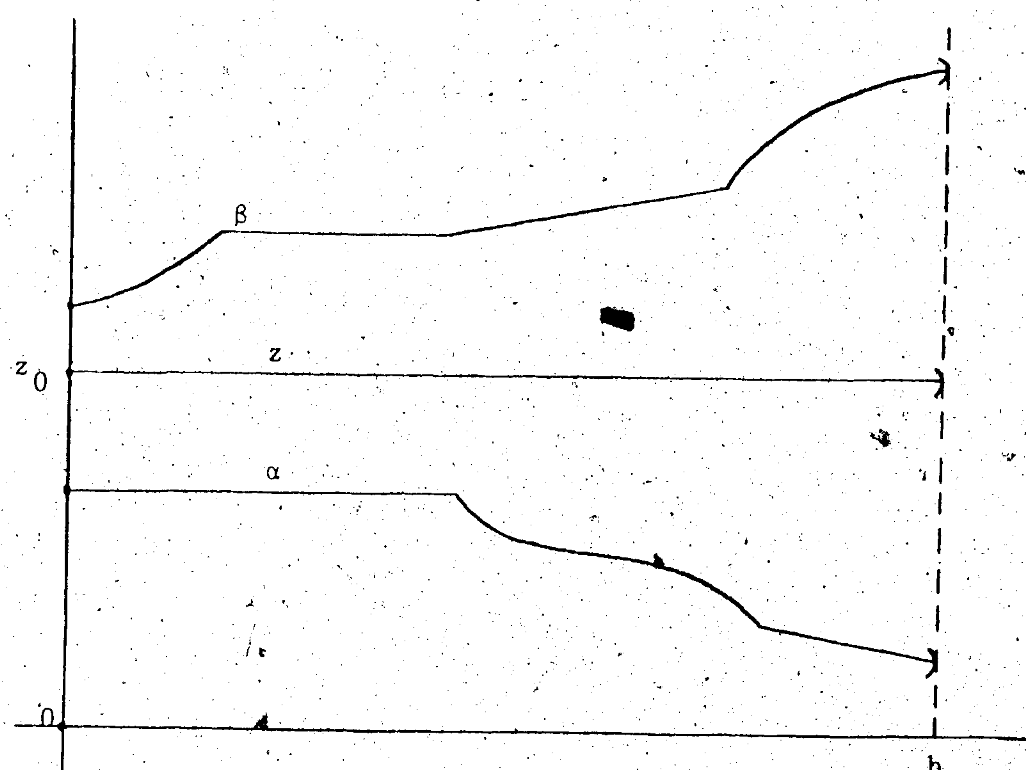

Figure 1

Example 2: A result of Jackson [6, p. 354] implies the following. Suppose $f:[0,h] \times \mathbb{R} \to \mathbb{R}$ is continuous and $\alpha, \beta \in AC^1[0,h]$ satisfy

$$\alpha''(t) \geq f(t,\alpha(t)), \quad \beta''(t) \leq f(t,\beta(t)), \quad \text{a.e. } t \in [0,h],$$

$$\alpha(0) \leq z_0 \leq \beta(0), \quad \alpha(h) \leq z_1 \leq \beta(h), \quad \alpha(t) \leq \beta(t), \quad t \in [0,h].$$

Then there exists $z \in C^2[0,h]$ such that

$$z''(t) = f(t,z(t)), \quad z(0) = z_0, \quad z(h) = z_1,$$

$$\alpha(t) \leq z(t) \leq \beta(t), \quad t \in [0,h].$$

It follows that the operator $T:AC^1[0,h] \to L^1[0,h] \times \mathbb{R}^2$ defined by

$$(Tx)(t) = \big(f(t,x(t)) - x''(t), \; x(0), \; x(h)\big),$$

has the intermediate value property with respect to $(0,z_0,z_1)$.

Figure 2 below illustrates the again intuitive case $f \equiv 0$.

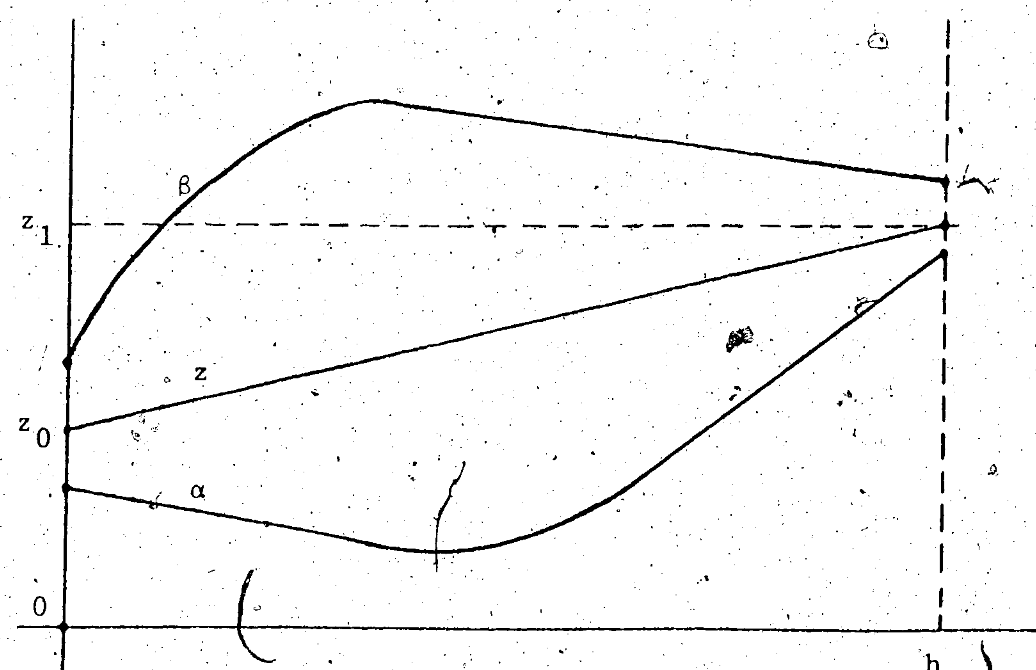

Figure 2

It is also apparent that a continuous map $f:I \to \mathbb{R}$ on an interval I has the intermediate value property with respect to $s \in f(I)$. Furthermore, if $g:I \to \mathbb{R}$ is decreasing, then $f+g$ has the intermediate value property with respect to $s \in (f+g)(I)$. In fact, suppose $a,b \in I$, $a \le b$, and,

$$f(a) + g(a) \le s \le f(b) + g(b) .$$

Let $c = \sup\{x \in [a,b]:f(x) + g(x) \le s\}$. Then $f(c) + g(c) = s$. For otherwise,

$$f(c) < s - g(c) \le s - g(b) \le f(b).$$

So, by the continuity of f, there exists $d \in (c,b]$ such that

$$f(d) = s - g(c) \le s - g(d) ,$$

contradicting the choice of c.

The following theorem gives a principle initially developed by Muldowney and Willett [8] which generalizes the preceding observation to show when the intermediate value property is preserved for monotone perturbations of maps between partially ordered sets. This general formulation of the principle is due to Gamlen and Muldowney [4].

II.1. Theorem: For partially ordered sets X and S, let $K:X \times X \to S$ be such that:

(i) $K(\cdot,x):X \to S$ has the intermediate value property with respect to $s \in S$ for each $x \in X$;

(ii) $K(x,\cdot):X \to S$ is decreasing for each $x \in X$.

Suppose further that for $s \in S$, $u,v \in X$, $u \le v$, $[u,v] = \{x \in X:u \le x \le v$

and $X_s[u,v] = \{x \in [u,v] : \exists y \in [u,v] \ni K(x,y) = s\}$:

(iii) well-ordered subsets of $X_s[u,v]$ have suprema in X ;

(iv) if W is a well-ordered subset of $X_s[u,v]$ such that $w = \sup W$ and $K(x,w) \leq s$ for each $x \in W$, then $K(w,w) \leq s$.

Then the map $x \mapsto K(x,x)$ has the intermediate value property with respect to $s \in S$.

Proof. Suppose that for some $\alpha, \beta \in X$ and $s \in S$,

$$\alpha \leq \beta, \quad K(\alpha,\alpha) \leq s \leq K(\beta,\beta).$$

By (ii),

$$K(\alpha,\alpha) \leq s \leq K(\beta,\alpha),$$

so it follows from (i) that there exists $\alpha_1 \in [\alpha,\beta]$ such that $K(\alpha_1,\alpha) = s$. Then (ii) implies that $K(\alpha_1,\alpha_1) \leq s$ so that

$$X_s^*[\alpha,\beta] = \{x \in X_s[\alpha,\beta] : K(x,x) \leq s\} \neq \phi .$$

Now, let W be a well-ordered subset of $X_s^*[\alpha,\beta]$ and $w = \sup W$. From (ii), $K(x,w) \leq s$ for each $x \in W$, so (iv) implies that $K(w,w) \leq s$. Thus,

$$w \leq \beta, \quad K(w,w) \leq s \leq K(\beta,\beta) ,$$

so (i) and (ii) imply, as above, that there exists $w_1 \in [w,\beta]$ such that $K(w_1,w_1) \leq s$, that is, $w = \sup W \leq w_1 \in X_s^*[\alpha,\beta]$. A lemma of Bourbaki [1] states that a partially ordered set has a maximal element if it contains an upper bound for each well-ordered subset. Therefore, $X_s^*[a,b]$ has a maximal element z. $K(z,z) \leq s$ and in fact $K(z,z) = s$. For otherwise,

$$z \leq \beta, \quad K(z,z) < s \leq K(\beta,\beta),$$

from which it follows by (i) and (ii) again that there exists $z_1 \in (z,\beta]$ such that $K(z_1,z_1) \leq s$. Then $z \geq \alpha$ implies that $z_1 \in X_s^*[\alpha,\beta]$, which contradicts the maximality of z.

Further examples of operators with the intermediate value property can be determined by referring to Examples 1 and 2 together with the above theorem. The following well-known result will be required (cf.[2,p.7]).

Ascoli's Theorem: Let F be a family of functions which is bounded and equicontinuous at every point of an interval I. Then every sequence of functions in F contains a subsequence which is uniformly convergent on every compact subinterval of I.

Example 3: Suppose $f:[0,h) \times \mathbb{R} \times \text{loc } AC[0,h) \to \mathbb{R}$, $0 < h \leq \infty$, is such that $f(\cdot,\cdot,y(\cdot))$ is continuous for each $y \in \text{loc } AC[0,h)$, and $f(t,x,\cdot)$ is increasing for each $(t,x) \in [0,h) \times \mathbb{R}$. Then, for $X = \text{loc } AC[0,h)$ and $S = \text{loc } L^1[0,h) \times \mathbb{R}$, the map $x \to K(x,x)$ has the intermediate value property with respect to $(0,z_0) \in S$, where $K:X \times X \to S$ is defined by

$$K(x,y)(t) = \big(x'(t)-f(t,x(t),y(\cdot)),\ x(0)\big).$$

In order to see this, it suffices to show that conditions (iii) and (iv) of II.1 are satisfied, since, by referring to Example 1, it is clear that (i) and (ii) of II.1 are satisfied.

For any $u,v \in X$, $u \leq v$, suppose $x,y \in X$, $u \leq x,y \leq v$, and

$$x(0) = z_0,\ x'(t) = f\big(t,x(t),y(\cdot)\big), \quad \text{a.e. } t \in [0,h).$$

By the continuity of u, v and f, on any compact interval $I \subset [0,h)$, there exists $M, N \in \mathbb{R}$ such that

$$|x(t)| \le M, \quad |x'(t)| \le N, \text{ a.e. } t \in I.$$

Thus $X_{(0,z_0)}[u,v]$ is uniformly bounded and equicontinuous locally on $[0,h)$. Now, if W is a well-ordered subset of $X_{(0,z_0)}[u,v]$, then there is a sequence $\{x_n\} \subset W$ such that $x_n \uparrow w = \sup W$. By Ascoli's Theorem there is a subsequence which converges uniformly to w on compact subintervals of $[0,h)$, and so, since $\{x_n\}$ is in fact equi-absolutely continuous locally on $[0,h)$, it follows that $w \in X$, that is, (iii) holds.

Suppose $K(x,w) \le (0,z_0)$ for each $x \in W$. Then for each $[t_1,t_2] \subset [0,h)$,

$$x(0) = z_0, \quad x(t_2) - x(t_1) \le \int_{t_1}^{t_2} f(s,x(s),w(\cdot))ds.$$

So, the Lebesgue Dominated Convergence Theorem applied to $\{x_n\}$ implies,

$$w(0) = z_0, \quad w(t_2) - w(t_1) \le \int_{t_1}^{t_2} f(s,w(s),w(\cdot))ds.$$

Then $w \in \text{loc } AC[0,h)$ implies

$$w(0) = z_0, \quad w'(t) \le f(t,w(t),w(\cdot)), \text{ a.e. } t \in [0,h)\ ,$$

equivalently, $K(w,w) \le (0,z_0)$, that is, (iv) holds.

Example 4: Suppose $f:[0,h] \times \mathbb{R} \times AC^1[0,h] \to \mathbb{R}$ is such that $f(\cdot,\cdot,y(\cdot))$ is continuous for each $y \in AC^1[0,h]$, and $f(t,x,\cdot)$ is decreasing for each $(t,x) \in [0,h] \times \mathbb{R}$. Then, for $X = AC^1[0,h]$ and $S = L^1[0,h] \times \mathbb{R}^2$, the map $x \to K(x,x)$ has the intermediate value property with respect to

$(0,z_0,z_1) \in S$, where $K: X \times X \to S$ is defined by

$$K(x,y)(t) = \big(f(t,x(t),y(\cdot)) - x''(t),\ x(0),\ x(h)\big).$$

To see this, it suffices to show that $\bar{K}$, the restriction of K to $Y \times Y$, has the intermediate value property with respect to $(0,z_0,z_1) \in S$, where $Y = \{x \in X : x(0) = z_0,\ x(h) = z_1\}$. For, suppose that this is the case, and $\alpha,\beta \in X$, $\alpha \leq \beta$, satisfy

$$K(\alpha,\alpha) \leq (0,z_0,z_1) \leq K(\beta,\beta).$$

In view of Example 2, it is clear that K satisfies (i) and (ii) of II.1. By (ii) it follows that

$$K(\alpha,\beta) \leq (0,z_0,z_1) \leq K(\beta,\beta).$$

So (i) implies that there exists $\beta_1 \in Y$, $\alpha \leq \beta_1 \leq \beta$, such that $K(\beta_1,\beta) = (0,z_0,z_1)$, and then (ii) implies

$$K(\alpha,\alpha) \leq (0,z_0,z_1) \leq \bar{K}(\beta_1,\beta_1).$$

By a similar argument, it follows that there exists $\alpha_1 \in Y$, $\alpha \leq \alpha_1 \leq \beta_1$, such that

$$\bar{K}(\alpha_1,\alpha_1) \leq (0,z_0,z_1) \leq \bar{K}(\beta_1,\beta_1).$$

Hence, there exists $z \in Y$, $\alpha_1 \leq z \leq \beta_1$, such that $\bar{K}(z,z) = (0,z_0,z_1)$, and the desired conclusion is obtained.

Now, to show that $\bar{K}$ has the intermediate value property with respect to $(0,z_0,z_1)$, it is only necessary to show that (iii) and (iv) of II.1 are satisfied.

For any $u,v \in Y$, $u \leq v$, suppose $x,y \in Y$, $u \leq x,y \leq v$, and,

$$x(0) = z_0,\ x(h) = z_1,\ x''(t) = f(t,x(t),y(\cdot)), \quad \text{a.e. } t \in [0,h].$$

By the continuity of u,v and f, there exists $M,N \in \mathbb{R}$ such that,

$$|x(t)| \leq M, \quad t \in [0,h],$$
$$|x''(t)| \leq N, \quad \text{a.e. } t \in [0,h].$$

Since $Y \subset C^1[0,h]$ and $u(0) = x(0) = v(0)$, there exists $Q \in \mathbb{R}$ such that

$$|x'(0)| \leq Q = \max\{|u'(0)|, |v'(0)|\}.$$

It then follows that

$$|x'(t)| \leq Nh + Q, \quad t \in [0,h].$$

Thus, $Y_{(0,z_0,z_1)}[u,v]$ and $\{x' \in AC[0,h] : x \in Y_{(0,z_0,z_1)}[u,v]\}$ are uniformly bounded and equi-absolutely continuous.

If W is a well-ordered subset of $Y_{(0,z_0,z_1)}[u,v]$, then there is a sequence $\{x_n\} \subset W$ such that $x_n \uparrow w = \sup W$. By Ascoli's Theorem there is a subsequence which converges uniformly to w on $[0,h]$ and thus $w \in AC[0,h]$. A further application of Ascoli's Theorem gives a subsequence $\{x_{n_k}\}$ and $w^* \in AC[0,h]$ such that $x_{n_k} \to w$ and $x'_{n_k} \to w^*$ uniformly on $[0,h]$. The Lebesgue Dominated Convergence Theorem implies

$$x_{n_k}(t) \to w(0) + \int_0^t w^*(s)ds, \quad t \in [0,h],$$

and so

$$w(t) = w(0) + \int_0^t w^*(s)ds, \quad t \in [0,h].$$

Since $w^* \in C[0,h]$, it follows that $w'(t) = w^*(t)$, $t \in [0,h]$. Therefore, $w \in AC^1[0,h]$, that is, $w \in Y$ and (iii) holds.

Suppose $\bar{K}(x,w) \leq (0,z_0,z_1)$ for each $x \in W$. Then for each $[t_1,t_2] \subset [0,h]$,

$$x(t_2) - x(t_1) \geq x'(0)(t_2-t_1) + \int_{t_1}^{t_2}\left[\int_0^s f\big(r,x(r),w(\cdot)\big)dr\right]ds ,$$

$$x(0) = z_0, \quad x(h) = z_1 .$$

So the Lebesgue Dominated Convergence Theorem applied to $\{x_{n_k}\}$ implies,

$$w(t_2) - w(t_1) \geq w'(0)(t_2-t_1) + \int_{t_1}^{t_2}\left[\int_0^s f\big(r,w(r),w(\cdot)\big)dr\right]ds ,$$

$$w(0) = z_0, \quad w(h) = z_1 .$$

Since $w \in AC^1[0,h]$, then

$$w''(t) \geq f\big(t,w(t),w'(t)\big), \quad \text{a.e. } t \in [0,h] ,$$

that is, $\bar{K}(w,w) \leq 0$, and (iv) holds.

If $f(t,x,y(\cdot))$ is replaced by $f(t,x,x',y(\cdot))$ in Example 4, then another valid example can be obtained provided that the rate of growth of solutions to $x'' = f(t,x,x',y(\cdot))$ is appropriately restricted for each y. In fact, the existence results in the next chapter require an explicit example as such for their proof. In the remainder of this chapter, preliminary results necessary to verify the example are developed. In doing so, two further examples of operators with the intermediate value property are found.

The following theorem is a new result whose importance here is that a previous result of Muldowney [7], which will be used in the proof of II.6, follows as an easy corollary.

A function $f:I \times \mathbb{R} \to \mathbb{R}$, where I is an interval in $\mathbb{R}$, satisfies Carathéodory's conditions if:

(i) $f(t,\cdot):\mathbb{R} \to \mathbb{R}$ is continuous for each $t \in I$;

(ii) $f(\cdot,x):I \to \mathbb{R}$ is Lebesgue measurable for each $x \in \mathbb{R}$;

(iii) for each compact $S \subset \mathbb{R}$, there exists $m \in \text{loc } L^1(I)$ such that $|f(t,x)| \leq m(t)$, for each $x \in S$.

A solution $x(t)$ of $x' = f(t,x)$, is called a solution in the sense of Carathéodory if it is loc $AC(I)$, on its interval of existence I, and $x'(t) = f(t,x(t))$, a.e. $t \in I$.

II.2. <u>Theorem</u>: Suppose $f:[0,h) \times \mathbb{R} \to \mathbb{R}$, $0 < h \leq \infty$, satisfies Carathéodory's conditions and $\alpha, \beta:[0,h) \to \mathbb{R}$ satisfy:

(i) $\alpha(0) \leq z_0 \leq \beta(0)$; $\alpha(t) \leq \beta(t)$, $t \in [0,h)$;

(ii) $f(t,\alpha(t))$, $f(t,\beta(t)) \in \text{loc } L^1[0,h)$;

(iii) $\alpha(t_2) - \alpha(t_1) \leq \int_{t_1}^{t_2} f(s,\alpha(s))ds$,

$$\beta(t_2) - \beta(t_1) \geq \int_{t_1}^{t_2} f(s,\beta(s))ds, \quad [t_1,t_2] \subset [0,h].$$

Then there is a solution, $z:[0,h) \to \mathbb{R}$, in the sense of Carathéodory, of

$$z' = f(t,z), \quad z(0) = z_0$$

such that $\alpha(t) \leq z(t) \leq \beta(t)$, $t \in [0,h]$.

Proof. Define $\bar{f}:[0,h)\times \mathbb{R} \to \mathbb{R}$ by

$$\bar{f}(t,x) = \begin{cases} f(t,\alpha(t)); & x \le \alpha(t) \\ f(t,x) \quad ; & \alpha(t) \le x \le \beta(t) \\ f(t,\beta(t)); & \beta(t) \le x. \end{cases}$$

Choose any $\delta > 0$. By the Carathéodory conditions, there exists $m \in \text{loc } L^1[0,h)$ such that $|\bar{f}(t,x)| \le m(t)$ on $[0,h)$, whenever $|x-z_0| \le \delta$. If $M(t) = \int_0^t m(s)ds$, $t \in [0,h)$, then M is non-decreasing, continuous and $M(0) = 0$, so there exists $0 < \tau < h$ such that $M(t) \le \delta$, $0 \le t \le \tau$. For any positive integer, n, define $x_n:[-\frac{1}{n},\tau] \to \mathbb{R}$ by

$$x_n(t) = \begin{cases} z_0 & ; -\frac{1}{n} \le t \le 0 \\ z_0 + \int_0^t \bar{f}\left(s,x_n\left(s-\frac{1}{n}\right)\right)ds & ; 0 \le t \le \tau. \end{cases}$$

It is clear that x_n is well defined. On $\left[-\frac{1}{n}, 0\right]$, $|x_n(t)-z_0| \le \delta$. Suppose that for some positive integer k, $\frac{k}{n} < \tau$, and $|x_n(t)-z_0| \le \delta$ on $\left[\frac{k-1}{n}, \frac{k}{n}\right]$. Then,

$$|x_n(t)-z_0| \le M(t) \le \delta, \quad t \in \left[\frac{k}{n}, N_k\right],$$

where $N_k = \min\{\tau, \frac{k+1}{n}\}$. It follows by induction that

$$|x_n(t)| \le |z_0| + \int_0^t m(s)\, ds$$

$$\le |z_0| + \delta, \quad t \in [0,\tau].$$

Therefore, the sequence $\{x_n\}$ is uniformly bounded and equicontinuous on $[0,\tau]$. By Ascoli's Theorem, there is a subsequence of $\{x_n\}$ which converges uniformly on $[0,\tau]$ to a continuous function $z:[0,\tau] \to \mathbb{R}$.

By the Lebesgue Dominated Convergence Theorem it follows that

$$z(t) = z_0 + \int_0^t \bar{f}\big(s,z(s)\big)ds, \quad t \in [0,\tau].$$

Choose $\tau_1 > \tau$ such that z is not right continuable on $[0,\tau_1)$. Note that if $\tau_1 < h$, then $\lim_{t \to \tau_1^-} |z(t)| = \infty$ since z is of bounded variation on $[0,\tau_1)$ (cf.[10,p.95]).

It is now only necessary to show that $\alpha(t) \leq z(t) \leq \beta(t)$ on $[0,\tau_1)$, for in this case, $\bar{f}\big(t,z(t)\big) = f\big(t,z(t)\big)$, and $\tau_1 = h$. The latter follows by the semicontinuity conditions required of α and β by hypothesis, which imply that α and β are bounded on $[0,\tau_1]$ if $\tau_1 < h$.

Suppose the contrary and without loss of generality assume that $z(t_1) < \alpha(t_1)$ for some $0 < t_1 < \tau_1$. By (iii), α is left lower semicontinuous on $(0,h)$ and right upper semicontinuous on $[0,h)$, so that

$$\liminf_{t \to t_1^-} \alpha(t) \geq \alpha(t_1),$$

$$\limsup_{t \to t_1^+} \alpha(t) \leq \alpha(t_1).$$

Thus,

$$t_0 = \sup\{t < t_1 : z(t) \geq \alpha(t)\} < t_1,$$

and $\alpha(t_0) = z(t_0)$, since z is continuous. Then,

$$\begin{aligned} z(t_1) - z(t_0) &= \int_{t_0}^{t_1} \bar{f}\big(s,z(s)\big)ds \\ &= \int_{t_0}^{t_1} f\big(s,\alpha(s)\big)ds \\ &\geq \alpha(t_1) - \alpha(t_0), \end{aligned}$$

which implies the contradiction $z(t_1) \geq \alpha(t_1)$.

II.3. <u>Corollary</u>: Suppose $f:[0,h) \times \mathbb{R} \to \mathbb{R}$, $0 < h \leq \infty$, satisfies Carathéodory's conditions and $\alpha:[0,h) \to \mathbb{R}$ satisfies:

(i) $\alpha(0) \leq z_0$, $|z_0| < \infty$;

(ii) $f(t,\alpha(t)) \in \text{loc } L^1[0,h)$;

(iii) $\alpha(t_2) - \alpha(t_1) \leq \int_{t_1}^{t_2} f(s,\alpha(s))ds$, $[t_1,t_2] \subset [0,h)$.

Then for some $0 < \tau < h$, there is a solution $z:[0,\tau) \to \mathbb{R}$, in the sense of Carathéodory, of

$$z' = f(t,z), \quad z(0) = z_0 ,$$

such that, $\alpha(t) \leq z(t)$ on $[0,\tau)$.

<u>Proof</u>. Choose $\delta > 0$, $m \in \text{loc } L^1[0,h)$, and $0 < \tau < h$ as in the proof of II.2. Define $\beta:[0,\tau) \to \mathbb{R}$ by

$$\beta(t) = z_0 + \int_0^t m(s)ds, \quad t \in [0,\tau).$$

Then, $|\beta(t) - z_0| \leq \delta$ on $[0,\tau)$ so that

$$\beta(t_2) - \beta(t_1) = \int_{t_1}^{t_2} m(s)ds$$

$$\geq \int_{t_1}^{t_2} f(s,\beta(s))ds, \quad [t_1,t_2] \subset [0,\tau).$$

The result now follows by arguments similar to those in the previous proof.

Remark: It is clear that the inequalities may be reversed in II.3. In this case, define $\beta:[0,\tau) \to \mathbb{R}$ by

$$\beta(t) = z_0 - \int_0^t m(s)ds, \quad t \in [0,\tau).$$

The following example is an immediate consequence of II.2.

Example 5: Suppose $f:[0,h) \times \mathbb{R} \to \mathbb{R}$, $0 < h \leq \infty$, satisfies Carathéodory's conditions and let $X = \{x:[0,h) \to \mathbb{R}: f(t,x(t)) \in \text{loc } L^1[0,h)\}$, $S = \{(t_1,t_2) \in \mathbb{R}^2: 0 \leq t_1 \leq t_2 < h\}$, $Y = \{x:S \to \mathbb{R}\}$. Then the operator $T:X \to Y \times \mathbb{R}$ defined by

$$(Tx)(t_1,t_2) = \left(x(t_2) - x(t_1) - \int_{t_1}^{t_2} f(s,x(s))ds,\ x(0)\right),$$

has the intermediate value property with respect to $(0,z_0)$.

The case $f \equiv 0$ is illustrated below in Figure 3. A comparison of Figures 1 and 3 indicates how this example extends Example 1, that is, continuity requirements are removed.

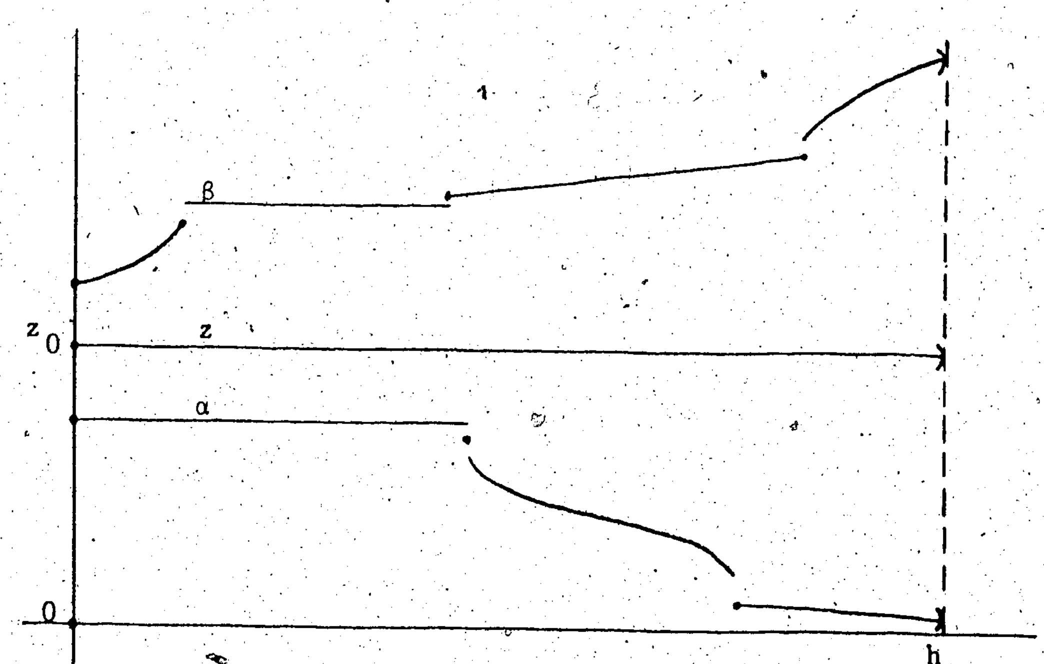

Figure 3

The remaining results of this chapter involve the second order differential equation, $x'' = f(t,x,x')$, where $f:[0,\infty) \times \mathbb{R}^2 \to \mathbb{R}$ is assumed continuous. The first of these can be found in Hartman's book [5, p. 424].

II.4. Theorem: Suppose that $|f(t,x,y)| \leq m$, for $t \in [0,h]$, $(x,y) \in \mathbb{R}^2$. Then for any $z_0, z_1 \in \mathbb{R}$, there is a solution $z \in C^2[0,h]$ of the boundary value problem

$$z'' = f(t,z,z'), \quad z(0) = z_0, \quad z(h) = z_1 .$$

Proof: For any positive integer n, let $x(t) = x(t,\mu)$ be the unique solution of the initial value problem

$$x''(t) = f\left(t, x\left(t - \frac{1}{n}\right), x'\left(t - \frac{1}{n}\right)\right), \quad t \in [0,h],$$

$$x'(t) = \mu, \quad x(t) = z_0, \quad t \in [-\frac{1}{n}, 0].$$

Then,

$$x(t) = z_0 + \mu t + \int_0^t (t-s) f\left(s, x\left(s - \frac{1}{n}\right), x'\left(s - \frac{1}{n}\right)\right) ds, \quad t \in [0,h],$$

implies

$$\lim_{\mu \to +\infty} x(h,\mu) = +\infty, \quad \lim_{\mu \to -\infty} x(h,\mu) = -\infty,$$

since f is bounded. Also, the map $\mu \to x(h,\mu)$ is continuous (cf. [2, p. 18]. Thus, there exists μ_n such that $x(h,\mu_n) = z_1$. Let $x_n(t) = x(t,\mu_n)$. For each n,

$$x_n(t) = z_0 + \mu_n t + \int_0^t (t-s) f\left(s, x_n\left(s - \frac{1}{n}\right), x_n'\left(s - \frac{1}{n}\right)\right) ds, \quad t \in [0,h],$$

$$x_n(0) = z_0, \quad x_n(h) = z_1,$$

implies

$$|\mu_n| \le \frac{|z_1-z_0|}{h} + mh$$

so that

$$|x_n''(t)| \le m, \quad |x_n'(t)| \le \frac{|z_1-z_0|}{h} + 2mh, \quad t \in [0,h].$$

Therefore, the sequences $\{x_n\}$ and $\{x_n'\}$ are uniformly bounded and equicontinuous on $[0,h]$. By Ascoli's Theorem, there is a subsequence $\{x_{n_k}\}$ and $z, z^* \in C[0,h]$ such that $x_{n_k} \to z$ and $x_{n_k}' \to z^*$ uniformly on $[0,h]$. The Lebesgue Dominated Convergence Theorem applied to $\{x_{n_k}'\}$ implies

$$x_{n_k}(t) \to z_0 + \int_0^t z^*(s)ds,$$

and so

$$z(t) = z_0 + \int_0^t z^*(s)ds, \quad t \in [0,h].$$

By the continuity of z^* it follows that $z'(t) = z^*(t)$ on $[0,h]$. Since it is clear that there exists μ^* such that $\mu_{n_k} \to \mu^*$, then

$$z(t) = z_0 + \mu^* t + \int_0^t (t-s)f\big(s,z(s),z'(s)\big)ds, \quad t \in [0,h],$$

or equivalently,

$$z''(t) = f\big(t,z(t),z'(t)\big), \quad t \in [0,h].$$

The definition and lemma which follow can be found in Jackson [6, p. 353]. The lemma shows that the Nagumo condition defined on f is sufficient to bound the first derivatives of certain solutions of $x'' = f(t,x,x')$. Such growth restrictions are used in II.6–II.8.

Definition: $f:[0,h] \times \mathbb{R}^2 \to \mathbb{R}$ satisfies a Nagumo condition on $[0,h]$ with respect to $\alpha,\beta \in C[0,h]$ if $\alpha(t) \le \beta(t)$ on $[0,h]$, and there exists a positive continuous function γ on $[0,\infty)$ such that $|f(t,x,y)| \le \gamma(|y|)$, for all $t \in [0,h]$, $\alpha(t) \le x \le \beta(t)$, $|y| < \infty$, and

$$\int_L^\infty \frac{s\,ds}{\gamma(s)} > \sup_{0 \le t \le h} \beta(t) - \inf_{0 \le t \le h} \alpha(t) ,$$

where $Lh = \max\{|\alpha(0) - \beta(h)|, |\alpha(h) - \beta(0)|\}$.

II.5. Lemma: Suppose f satisfies a Nagumo condition on $[0,h]$ with respect to $\alpha,\beta \in C[0,h]$. Then for any $z \in C^2[0,h]$ satisfying

$$z'' = f(t,z,z'), \quad \alpha(t) \le z(t) \le \beta(t), \quad t \in [0,h],$$

there exists $N > 0$, depending on α, β and γ, such that $|z'(t)| \le N$ on $[0,h]$.

Proof: Choose $N > L$ such that

$$\int_L^N \frac{s\,ds}{\gamma(s)} > \sup_{0 \le t \le h} \beta(t) - \inf_{0 \le t \le h} \alpha(t).$$

Suppose that there exists $x_0 \in [0,h]$ such that $|z'(x_0)| > N$, and without loss of generality, assume $z'(x_0) > N$. Since $z \in C[0,h]$, there exists $x_1 \in [0,h]$ such that

$$h\, z'(x_1) = z(h) - z(0) ,$$

and, by definition, $|z'(x_1)| \leq L$. It then follows that for some $(c,d) \subset [0,h]$, either

$$z'(c) = L, \quad z'(d) = N, \quad L < z'(t) < N, \quad t \in (c,d),$$

or

$$z'(c) = N, \quad z'(d) = L, \quad L < z'(t) < N, \quad t \in (c,d).$$

Again, without loss of generality, assume the former. Then,

$$|z''(t)| = |f(t,z(t),z'(t))| \leq \gamma(z'(t)), \quad t \in [c,d],$$

implies

$$\int_L^N \frac{s\, ds}{\gamma(s)} = \int_c^d \frac{z''(t)z'(t)dt}{\gamma(z'(t))}$$

$$\leq \int_c^d \frac{|z''(t)|z'(t)dt}{\gamma(z'(t))}$$

$$\leq \int_c^d z'(t)dt$$

$$\leq \sup_{0\leq t\leq h} \beta(t) - \inf_{0\leq t\leq h} \alpha(t) ,$$

contradicting the choice of N.

II.6-II.8 are new results which extend some results of Jackson [6, pp. 354-355].

II.6. Theorem: Given $\alpha_1, \beta_1 \in AC[0,h]$ and $\alpha_2, \beta_2 \in L^1[0,h]$, suppose that

(i) $\alpha_1'(t) = \alpha_2(t)$ and $\beta_1'(t) = \beta_2(t)$, a.e. $t \in [0,h]$;

(ii) $\alpha_1(t) \leq \beta_1(t)$, $t \in [0,h]$;

(iii) $\alpha_2(t_2) - \alpha_2(t_1) \geq \int_{t_1}^{t_2} f\big(s, \alpha_1(s), \alpha_2(s)\big) ds$, and

$$\beta_2(t_2) - \beta_2(t_1) \leq \int_{t_1}^{t_2} f\big(s, \beta_1(s), \beta_2(s)\big) ds, \quad [t_1, t_2] \subset [0,h];$$

(iv) $f(t,x,y)$ is locally uniformly Lipschitz in y on $[0,h] \times \mathbb{R}^2$;

(v) $f(t,x,y)$ satisfies a Nagumo condition on $[0,h]$ with respect to α_1, β_1;

(vi) $f_1(t,y) = f\big(t, \alpha_1(t), y\big)$ and $f_2(t,y) = f\big(t, \beta_1(t), y\big)$ satisfy Caratheodory's conditions on $[0,h] \times \mathbb{R}$.

Then for any $z_0, z_1 \in \mathbb{R}$ such that

$$\alpha_1(0) \leq z_0 \leq \beta_1(0), \quad \alpha_1(h) \leq z_1 \leq \beta_1(h) ,$$

there is a solution $z \in C^2[0,h]$ of the boundary value problem,

$$z'' = f(t,z,z'), \quad z(0) = z_0, \quad z(h) = z_1 ,$$

such that

$$\alpha_1(t) \leq z(t) \leq \beta_1(t), \quad t \in [0,h] .$$

Proof: Choose $N > 0$ as in II.5 and $C \geq N$ such that

$$|\alpha_2(t)| \le C, \quad |\beta_2(t)| \le C, \quad \text{a.e. } t \in [0,h].$$

Define $\bar{f}, f^*: [0,h] \times \mathbb{R}^2 \to \mathbb{R}$ by

$$\bar{f}(t,x,y) = \begin{cases} f(t,x,-C); & y \le -C \\ f(t,x,y); & |y| \le C \\ f(t,x,C); & y \ge C \end{cases}$$

$$\bar{f}^*(t,x,y) = \begin{cases} \bar{f}(t,\alpha_1(t),y) & ; x \le \alpha_1(t) \\ \bar{f}(t,x,y) & ; \alpha_1(t) \le x \le \beta_1(t) \\ \bar{f}(t,\beta_1(t),y) & ; x \ge \beta_1(t). \end{cases}$$

On $[0,h] \times \mathbb{R}^2$,

$$|\bar{f}^*(t,x,y)| \le \sup\{|f(t,x,y)|: \ t \in [0,h], \ \alpha_1(t) \le x \le \beta_1(t), \ |y| \le C\} < \infty ,$$

so by II.4, there is a solution $z \in C^2[0,h]$ of

$$z'' = \bar{f}^*(t,z,z'), \quad z(0) = z_0, \quad z(h) = z_1 .$$

It remains only to show that

$$\alpha_1(t) \le z(t) \le \beta_1(t), \quad t \in [0,h],$$

since, in this case, $|z'(t)| \le N$, so that

$$\bar{f}^*(t,z(t),z'(t)) = f(t,z(t),z'(t)).$$

Suppose the contrary, and without loss of generality, assume that there exists $[c,d] \subset [0,h]$ such that

$$z(c) = \beta_1(c), \quad z(d) = \beta_1(d), \quad z(t) > \beta_1(t), \quad t \in (c,d).$$

$z - \beta_1$ has a maximum in (c,d) at q, say. Since $z - \beta_1 \in AC[c,d]$, there is a subset of (c,q) of positive Lebesgue measure on which $z'(t) > \beta_1'(t)$, and a similar subset of (q,d) on which $z'(t) < \beta_1'(t)$ (cf. [9, p. 246]). Then $\beta_1'(t) = \beta_2(t)$ a.e. $t \in [0,h]$ implies that there exists $p \in (c,q)$ and $r \in (q,d)$ such that

$$z'(p) > \beta_2(p), \quad z'(r) < \beta_2(r).$$

But,

$$z''(t) = \bar{f}\big(t,\beta_1(t),z'(t)\big), \quad t \in [p,d),$$

$$\beta_2(t_2)-\beta_2(t_1) \le \int_{t_1}^{t_2} \bar{f}\big(s,\beta_1(s),\beta_2(s)\big)ds, \quad [t_1,t_2] \subset [p,d),$$

and $z'(p) > \beta_2(p)$, together with (iv) (which implies uniqueness) and (vi), imply that $z'(t) \ge \beta_2(t)$ on (p,d), by II.3. This contradicts $z'(r) < \beta_2(r)$.

The above theorem immediately extends Example 2 by relaxing the continuity and differentiability requirements on the functions α and β. In practice, it allows certain jump discontinuities in α' and β'. The type of discontinuity allowed is illustrated by comparing Figure 4 below and Figure 2.

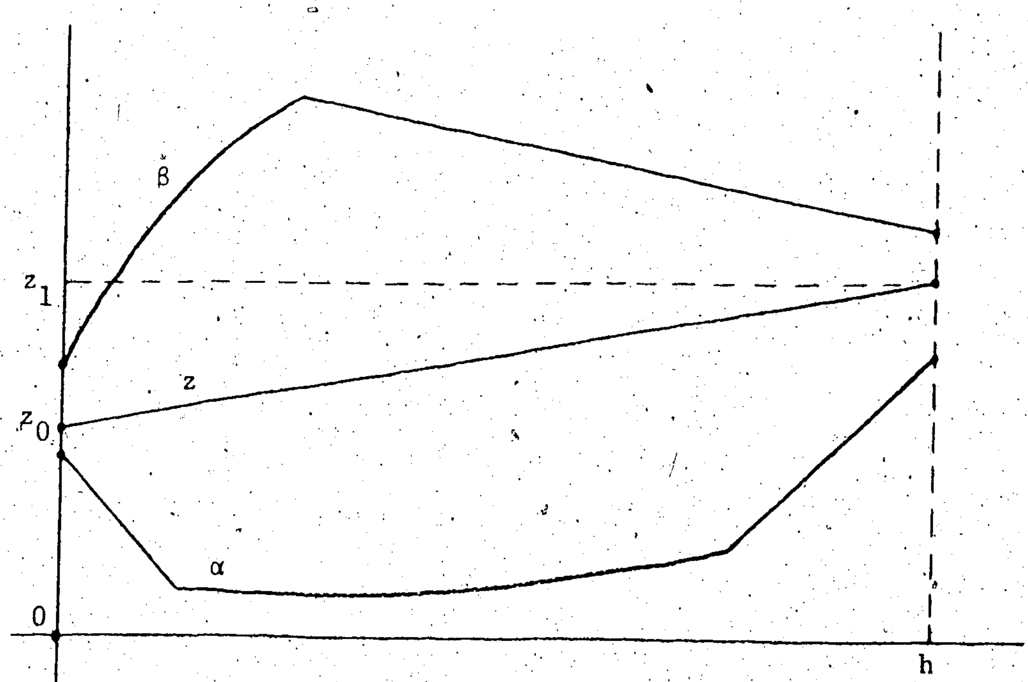

Figure 4

In II.7 the results of II.6 are extended to $[0,\infty)$. II.8 makes explicit the types of discontinuity allowed in α' and β' in practice.

II.7 Theorem: Given $\alpha_1,\beta_1 \in \text{loc AC}[0,\infty)$ and $\alpha_2,\beta_2 \in \text{loc } L^1[0,\infty)$, suppose that:

(i) $\alpha_1'(t) = \alpha_2(t)$, $\beta_1'(t) = \beta_2(t)$, a.e. $t \in [0,\infty)$;

(ii) $\alpha_1(t) \le \beta_1(t)$, $t \in [0,\infty)$;

(iii) $\alpha_2(t_2) - \alpha_2(t_1) \ge \int_{t_1}^{t_2} f(s,\alpha_1(s),\alpha_2(s))ds$;

$$\beta_2(t_2) - \beta_2(t_1) \le \int_{t_1}^{t_2} f(s,\beta_1(s),\beta_2(s))ds, \quad [t_1,t_2] \subset [0,\infty);$$

(iv) $f(t,x,y)$ is locally uniformly Lipschitz in y on $[0,h] \times \mathbb{R}^2$, $h > 0$;

(v) $f(t,x,y)$ satisfies a Nagumo condition on $[0,h]$, $h > 0$, with respect to α_1, β_1;

(vi) $f_1(t,y) = f\big(t,\alpha_1(t),y\big)$ and $f_2(t,y) = f\big(t,\beta_1(t),y\big)$ satisfy Carathéodory's conditions on $[0,\infty) \times \mathbb{R}$.

Then for any $z_0 \in \mathbb{R}$ such that $\alpha_1(0) \le z_0 \le \beta_1(0)$, there is a solution $z \in C^2[0,\infty)$ of

$$z'' = f(t,z,z'), \quad z(0) = z_0 ,$$

such that

$$\alpha_1(t) \le z(t) \le \beta_1(t), \quad t \in [0,\infty).$$

Proof: By II.5 and II.6, for each positive integer n there exists $z_n \in C^2[0,n]$ such that

$$z_n(0) = z_0 , \quad z_n(n) = \beta_1(n) ,$$

$$z_n''(t) = f\big(t,z_n(t),z_n'(t)\big) ,$$

$$\alpha_1(t) \le z_n(t) \le \beta_1(t), \quad t \in [0,n] ,$$

and there exists $N_n > 0$ such that $|y'(t)| \le N_n$ on $[0,n]$ for any $y \in C^2[0,n]$ satisfying

$$y''(t) = f\big(t,y(t),y'(t)\big), \quad \alpha_1(t) \le y(t) \le \beta_1(t), \quad t \in [0,n].$$

So for fixed n, the sequences $\{z_m\}$ and $\{z_m'\}$, $m \ge n$, are uniformly bounded and equicontinuous on $[0,n]$. By Ascoli's Theorem, there is a subsequence $\{z_{m_k}\}$ and $z, z^* \in C[0,\infty)$ such that $z_{m_k} \to z$ and $z'_{m_k} \to z^*$ uniformly on compact subintervals of $[0,\infty)$, and

$$\alpha_1(t) \le z(t) \le \beta_1(t), \quad t \in [0,\infty).$$

From the convergence $z'_{m_k} \to z^*$, the Lebesgue Dominated Convergence Theorem implies that

$$z_{m_k}(t) \to z(0) + \int_0^t z^*(s)ds ,$$

and so,

$$z(t) = z(0) + \int_0^t z^*(s)ds, \quad t \in [0,\infty).$$

By the continuity of z^* it follows that $z'(t) = z^*(t)$ on $[0,\infty)$.

Then,

$$z'_{m_k}(t) = z_{m_k}(0) + \int_0^t f\big(s, z_{m_k}(s), z'_{m_k}(s)\big)ds$$

implies

$$z'(t) = z(0) + \int_0^t f\big(s, z(s), z'(s)\big)ds, \quad t \in [0,\infty).$$

Finally, by the continuity of f,

$$z''(t) = f\big(t, z(t), z'(t)\big), \quad t \in [0,\infty).$$

II.8 <u>Corollary</u>: Given $\alpha_1, \beta_1 \in \text{loc } AC[0,\infty)$ and α_1, β_1 piecewise loc $AC^1[0,\infty)$, suppose that:

(i) $\alpha_1(t) \le \beta_1(t), \quad t \in [0,\infty)$;

(ii) $\alpha''_1(t) \ge f\big(t, \alpha_1(t), \alpha'_1(t)\big)$,

$$\beta_1''(t) \le f(t,\beta_1(t),\beta_1'(t)), \quad \text{a.e. } t \in [0,\infty);$$

(iii) at each discontinuity τ_1 of α_1', $\alpha_1'(\tau_1^-) \le \alpha_1'(\tau_1^+)$,

at each discontinuity τ_2 of β_1', $\beta_1'(\tau_2^-) \ge \beta_1'(\tau_2^+)$;

(iv) $f(t,x,y)$ is locally uniformly Lipschitz in y on $[0,h] \times \mathbb{R}^2$, $h > 0$;

(v) $f(t,x,y)$ satisfies a Nagumo condition on $[0,h]$, $h > 0$, with respect to α_1, β_1;

(vi) $f_1(t,y) = f(t,\alpha_1(t),y)$ and $f_2(t,y) = f(t,\beta_1(t),y)$ satisfy Carathéodory's conditions on $[0,\infty) \times \mathbb{R}$.

Then for any $z_0 \in \mathbb{R}$ such that $\alpha_1(0) \le z_0 \le \beta_1(0)$, there is a solution $z \in C^2[0,\infty)$ of

$$z'' = f(t,z,z'), \quad z(0) = z_0,$$

such that

$$\alpha_1(t) \le z(t) \le \beta_1(t), \quad t \in [0,\infty).$$

<u>Proof</u>: Choose $\alpha_2, \beta_2 \in \text{loc } L^1[0,\infty)$ such that

$$\alpha_2(t) = \alpha_1'(t), \quad \beta_2(t) = \beta_1'(t),$$

whenever the derivatives exist. The result follows from the previous theorem if condition (iii) of that theorem is satisfied. In particular, since α_2 and β_2 are piecewise continuous, it suffices to show that (iii) in II.7 is satisfied on intervals $[t_1,t_2] \subset [0,\infty)$ containing precisely one discontinuity τ_1 of α_2 or τ_2 of β_2. But in this case it is clear that

$$\int_{t_1}^{t_2} f\big(s,\alpha_1(s),\alpha_2(s)\big) \le \alpha_2(\tau_1^-)-\alpha_2(t_1) + \alpha_2(t_2)-\alpha_2(\tau_1^+)$$

$$\le \alpha_2(t_2)-\alpha_2(t_1),$$

or

$$\int_{t_1}^{t_2} f\big(s,\beta_1(s),\beta_2(s)\big)ds \ge \beta_2(\tau_2^-)-\beta_2(t_1) + \beta_2(t_2)-\beta_2(\tau_2^+)$$

$$\ge \beta_2(t_2)-\beta_2(t_1).$$

The next example is an immediate consequence of the above corollary.

Example 6: Suppose $f:[0,h] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and let $S = \{(t_1,t_2) \in \mathbb{R}^2 : 0 \le t_1 \le t_2 < \infty\}$, $Y = \{x:S \to \mathbb{R}\}$. Then the operator T:piecewise loc $AC^1[0,\infty) \to Y \times \mathbb{R}$ defined by

$$(Tx)(t_1,t_2) = \left[\int_{t_1}^{t_2} f\big(s,x(s),x'(s)\big)ds - x'(t_2^-) + x'(t_1^+),\ x(0)\right],$$

has the intermediate value property with respect to $(0,z_0)$.

CHAPTER III

FINAL RESULTS

The intermediate value theorem presented in the previous chapter as II.1 is used in this chapter to prove the existence of solutions to the boundary value problem.

$$w''' + ww'' + \lambda\left(1-(w')^2\right) = 0, \tag{5}$$

$$w(0) = a, \quad w'(0) = b, \quad w'(\infty) = 1, \tag{6}$$

where $\lambda \geq 0$ and $|b| < 1$. The particular results are contained in a subsequent theorem whose proof also uses the preliminary results II.7 and II.8. The technique used is analagous to that used by Gamlen and Muldowney [4] in an application to the above problem.

By the change of variable $z = w'$, the boundary value problem (5)-(6) becomes

$$z'' = F\big(t,z,z',z(\cdot)\big), \quad z(0) = b, \quad z(\infty) = 1,$$

where $F:[0,\infty) \times \mathbb{R}^2 \times \text{loc } AC^1[0,\infty) \to \mathbb{R}$ is defined by

$$F\big(t,x,x',y(\cdot)\big) = -\left(a + \int_0^t y(s)ds\right)x' - \lambda(1-x^2).$$

Assume that $\lambda \geq 0$ and $|b| < 1$, and let

$$\mathcal{X} = \{y \in \text{loc } AC^1[0,\infty) : |y(t)| \leq 1, \ t \in [0,\infty)\},$$

$$Y = \{x \in X : x(0) = b, \ x(\infty) = 1, \ x'(t) \geq 0, \ t \in [0,\infty)\},$$

$$S = \text{loc } L^1[0,\infty) \times \mathbb{R}.$$

Define $K: X \times X \to S$ by

$$K(x,y) = \big(F(t,x(t),x'(t),y(\cdot)) - x''(t),\ x(0)\big),$$

and let $\bar{K}$ be the restriction of K to $Y \times Y$.

(a) For fixed $y \in X$ and $\alpha_1, \beta_1 \in \text{loc } AC^1[0,\infty)$, it is easy to see that $f(t,x,x') = F\big(t,x,x',y(\cdot)\big)$ satisfies conditions (iv)-(vi) of II.7. Then II.7 implies that $K(\cdot,y)$ has the intermediate value property with respect to $(0,b) \in S$, for each $y \in X$. In fact, $\bar{K}(\cdot,y)$ has the intermediate value property with respect to $(0,b) \in S$, for each $y \in Y$. For, suppose $\alpha,\beta \in Y$, $\gamma \in X$, $K(\gamma,y) = (0,b)$, $\alpha \leq \gamma \leq \beta$, but for some $t_0 \geq 0$, $\gamma'(t_0) < 0$. Then,

$$\gamma''(t) + \left(a + \int^t y(s)ds\right)\gamma'(t) = -\lambda\big(1-\gamma^2(t)\big) \leq 0, \quad \text{a.e. } t \in [0,\infty),$$

implies that $\gamma'(t$ [illegible] (t_0,∞), and hence the contradiction $\gamma(\infty) < 1$.

(b) For each fi[illegible] $x \in Y$, $K(x,\cdot): X \to S$ is decreasing.

(c) For any u,v [illegible] Y, $u \leq v$, suppose $x,y \in Y$, $u \leq x,y \leq v$, and

$$x(0) = b, \quad x''(t) = F\big(t,x(t),x'(t),y(\cdot)\big), \quad \text{a.e. } t \in [0,\infty).$$

For any $h > 0$, by the continuity of F, there exists $M \in \mathbb{R}$ such that

$$|x''(t)| \leq M, \quad \text{a.e. } t \in [0,h].$$

Since $Y \subset C^1[0,\infty)$ and $u(0) = x(0) = v(0)$, there exists $Q \in \mathbb{R}$ such that $|x'(0)| \leq Q$. Then

$$|x'(t)| \leq Mh + Q, \quad t \in [0,h],$$

and it follows that $Y_{(0,b)}[u,v]$ and $\{x' \in \text{loc } AC[0,\infty) : x \in Y_{(0,b)}[u,v]\}$ are uniformly bounded and equi-absolutely continuous on $[0,h]$.

If W is a well-ordered subset of $Y_{(0,b)}[u,v]$, then there is a sequence $\{x_n\} \subset W$ such that $x_n \uparrow w = \sup W$. By Ascoli's Theorem there is a subsequence which converges uniformly to w on compact subintervals of $[0,\infty)$. It is then clear that $w \in \text{loc } AC[0,\infty)$ and

$$w(t) \leq 1, \quad w'(t) \geq 0, \quad \text{a.e. } t \in [0,\infty).$$

A further application of Ascoli's Theorem gives a subsequence $\{x_{n_k}\}$ and $w^* \in \text{loc } AC[0,\infty)$ such that $x_{n_k} \to w$ and $x'_{n_k} \to w^*$ uniformly on compact subintervals of $[0,\infty)$. By the Lebesgue Dominated Convergence Theorem and the continuity of w^*, it follows that $w'(t) = w^*(t)$ on $[0,\infty)$. Therefore, $w \in \text{loc } AC^1[0,\infty)$, and so $w \in Y$.

(d) For W, w, and $\{x_{n_k}\}$ as in (c), suppose that $\bar{K}(x,w) \leq (0,b)$ for each $x \in W$. Then for $[t_1,t_2] \subset [0,\infty)$,

$$x(t_2)-x(t_1) \geq x'(0)(t_2-t_1) + \int_{t_1}^{t_2}\left[\int_0^s F\big(r,x(r),x'(r),w(\cdot)\big)dr\right]ds.$$

Since $w \in \text{loc } AC^1[0,\infty)$, it follows from an application of the Lebesgue Dominated Convergence Theorem to $\{x_{n_k}\}$ and $\{x'_{n_k}\}$, that

$$w''(t) \geq F\big(t,w(t),w'(t),w(\cdot)\big), \quad \text{a.e. } t \in [0,\infty),$$

that is, $\bar{K}(w,w) \leq (0,b)$.

From (a)-(d), it follows by II.1 that the map $x \to \bar{K}(x,x)$ has the intermediate value property with respect to $(0,b) \in S$. In fact, if $\alpha \in Y$ and $\beta \in X$ are such that $\alpha \le \beta$, $\beta(\infty) = 1$, and

$$\bar{K}(\alpha,\alpha) \le (0,b) \le K(\beta,\beta),$$

then there exists $z \in Y$, $\alpha \le z \le \beta$, such that $\bar{K}(z,z) = (0,b)$. For, in this case, (b) implies

$$K(\alpha,\beta) \le (0,b) \le K(\beta,\beta),$$

so by (a), there exists $\beta_1 \in X$ satisfying

$$\alpha \le \beta_1 \le \beta, \quad K(\beta_1,\beta) = (0,b).$$

As in (a),

$$\beta_1''(t) + \left(a + \int_0^t \beta(s)ds\right)\beta_1'(t) \le 0, \quad \text{a.e. } t \in [0,\infty)$$

implies that $\beta_1'(t) \ge 0$ on $[0,\infty)$, and it follows that $\beta_1 \in Y$. Finally, by (b),

$$\bar{K}(\alpha,\alpha) \le (0,b) \le \bar{K}(\beta_1,\beta_1),$$

so that z can be determined by the intermediate value property.

The main results of this thesis are contained in the following theorem, whose proof makes use of the preceding observation.

<u>Theorem</u>: The boundary value problem

$$w''' + ww'' + \lambda\left(1-(w')^2\right) = 0, \tag{5}$$

$$w(0) = a, \quad w'(0) = b, \quad w'(\infty) = 1, \tag{6}$$

has a solution $w \in \text{loc } AC^2[0,\infty)$ which satisfies (5) almost everywhere and $w''(t) \geq 0$ on $[0,\infty)$, for each of the following cases:

(i) $\lambda \geq 0$, $0 < b < 1$, $-\infty < a < +\infty$;

(ii) $\lambda > 0$, $-1 < b \leq 0$, $-\infty < a < +\infty$;

(iii) $\lambda = 0$, $-1 < b \leq 0$, $-k_0 + \frac{1}{k_0}\left(b - \ln(b+1)\right) \leq a < +\infty$, for some $k_0 > 0$.

Proof: Make the change of variable $z = w'$ in (5) and define F, X, Y, S, K, and $\bar{K}$ as above. If $\beta \equiv 1$ and $|b| < 1$, then it is clear that $\beta \in X$, $\beta(\infty) = 1$, and $K(\beta,\beta) \geq (0,b)$. Therefore, it follows from the discussion above, that determining $\alpha \in Y$ such that $\bar{K}(\alpha,\alpha) \leq (0,b)$ implies the existence of $z \in Y$, $\alpha \leq z \leq \beta$, satisfying

$$z'' = -\left(a + \int_0^t z(s)ds\right)z'(t) - \lambda\left(1-z^2(t)\right), \quad \text{a.e. } t \in [0,\infty),$$

where $\lambda \geq 0$, $|b| < 1$. But, clearly,

$$w(t) = a + \int_0^t z(s)ds, \quad t \in [0,\infty)$$

satisfies (6) and (5) a.e. $t \in [0,\infty)$. So the theorem follows if for each case, an $\alpha \in \text{loc } AC^1[0,\infty)$ is determined, which satisfies:

(A) $\alpha'(t) \geq 0$, $t \in [0,\infty)$;

(B) $\alpha(0) = b$, $\alpha(\infty) = 1$;

(C) $\alpha''(t) \geq -\left(a + \int_0^t \alpha(s)ds\right)\alpha'(t) - \lambda\left(1-\alpha^2(t)\right)$, a.e. $t \in [0,\infty)$.

Case (i): Define

$$\alpha(t) = (1-b)\left[\frac{\int_0^t \exp\left(-\int_0^s a+br\,dr\right)ds}{\int_0^\infty \exp\left(-\int_0^s a+br\,dr\right)ds}\right] + b, \quad t \in [0,\infty).$$

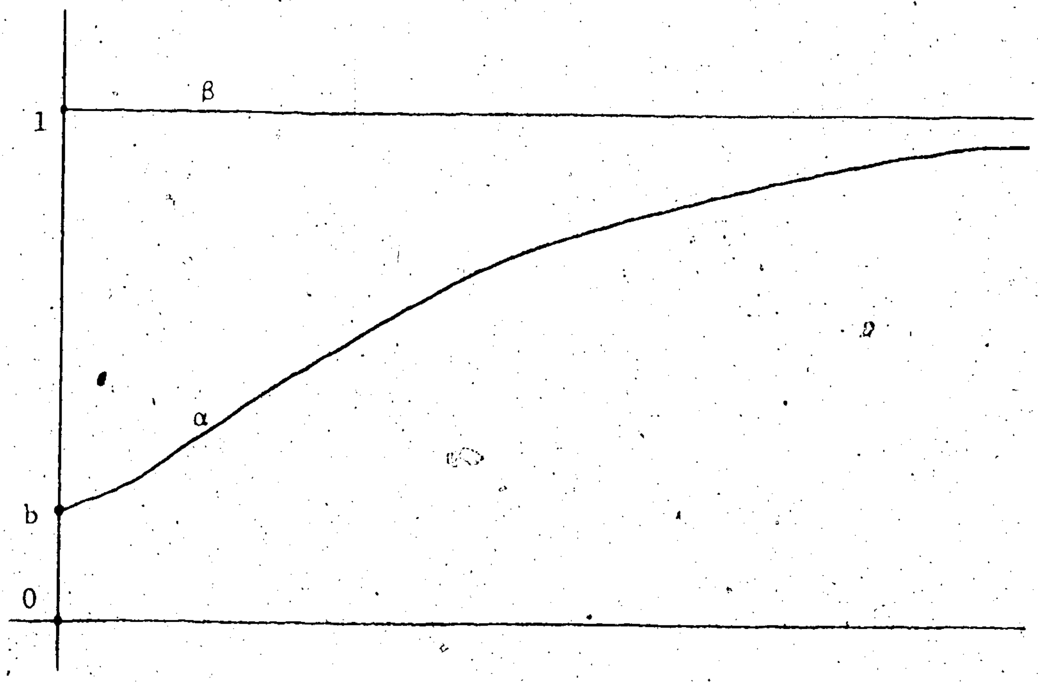

Figure 5

Then clearly, (A) and (B) are satisfied, and

$$\alpha''(t) = -(a + bt)\alpha'(t)$$

$$\geq -\left(a + \int_0^t \alpha(s)ds\right)\alpha'(t) - \lambda\left(1 - \alpha^2(t)\right).$$

Case (ii): Choose $c > 0$ such that $-1 < -c \leq b$ and consider the problem

$$z'' + (a-ct)z' + \lambda(1-c^2) = 0, \quad z(0) = b, \quad t \in [0,\infty).$$

A solution α_2 is of the form

$$\alpha_2(t) = -\lambda(1-c^2)\int_0^t e^{-r(s)}\left(\int_0^s e^{r(u)}du\right)ds + \alpha_2'(0)\int_0^t e^{-r(s)}ds + b,$$

where $r(t) = at - \frac{ct^2}{2}$. Since

$$\alpha_2'(t) = -e^{-r(t)}\left(\lambda(1-c^2)\int_0^t e^{r(s)}ds - \alpha_2'(0)\right),$$

then choosing

$$\alpha_2'(0) = \lambda(1-c^2)\int_0^{t_0} e^{r(s)}ds$$

for some $t_0 \geq 0$, it is clear that α_2 has a maximum at t_0.

Now, for any $t \in \mathbb{R}$,

$$\lim_{s \to +\infty} \frac{s\int_t^s e^{u^2}du}{e^{s^2}} = \lim_{s \to +\infty} \frac{se^{s^2} + \int_t^s e^{u^2}du}{2se^{s^2}} = \frac{1}{2}$$

implies that

$$\int_t^\infty e^{-s^2}\left(\int_t^s e^{u^2}du\right)ds = +\infty, \quad t \in \mathbb{R}.$$

Then, since

$$\frac{\alpha_2(t_0)-b}{\lambda(1-c^2)} = \left(\int_0^{t_0} e^{r(s)}ds\right)\left(\int_0^{t_0} e^{-r(s)}ds\right) - \int_0^{t_0} e^{-r(s)}\left(\int_0^s e^{r(u)}du\right)ds$$

$$= \int_0^{t_0} e^{r(s)}\left(\int_0^s e^{-r(u)}du\right)ds; \quad \text{integrating by parts}$$

$$= \int_0^{t_0} e^{\left(-(\sqrt{c/2}\ t-a/\sqrt{2c})^2\right)}\left(\int_0^s e^{\left((\sqrt{c/2}\ u-a/\sqrt{2c})^2\right)}du\right)ds$$

$$= \frac{2}{c}\int_{-a/\sqrt{2c}}^{\sqrt{c/2}\ t_0-a/\sqrt{2c}} e^{-s^2}\left(\int_{-a/\sqrt{2c}}^{\sqrt{c/2}\ s-a/\sqrt{2c}} e^{u^2}du\right)ds$$

it follows that $\lim_{t_0 \to +\infty} \alpha_2(t_0) = +\infty$. If $t_0 = 0$, then $\alpha_2(t_0) = b$, so $t_0 > 0$ can be chosen so that $0 < \alpha_2(t_0) < c$. In this case, on $[0,t_0)$, $\alpha_2'(t) > 0$ and $|\alpha_2(t)| < c$ so that

$$\begin{aligned}\alpha_2''(t) &= -(a-ct)\alpha_2'(t) - \lambda(1-c^2)\\ &\geq -\Big(a + \int_0^t \alpha_2(s)ds\Big)\alpha_2'(t) - \lambda\big(1 - \alpha^2(t)\big).\end{aligned}$$

Next, define α_3 similar to α in Case (i), by

$$\alpha_3(t) = \big(1 - \alpha_2(t_0)\big)\frac{\int_{t_0}^{t} \exp\Big(-\int_{t_0}^{s} q(u)du\Big)ds}{\int_{t_0}^{\infty} \exp\Big(-\int_{t_0}^{s} q(u)du\Big)ds} + \alpha_2(t_0), \quad t \in [t_0,\infty),$$

where $q(t) = a + \int_0^{t_0} \alpha_2(s)ds + (t-t_0)\alpha_2(t_0)$. Then,

$$\alpha_3(t_0) = \alpha_2(t_0), \quad \alpha_3(\infty) = 1, \quad \alpha_3'(t) > 0, \quad t \in (t_0,\infty),$$

and

$$\begin{aligned}\alpha_3''(t) &= -q(t)\alpha_3'(t)\\ &= -\Big(a + \int_0^{t_0} \alpha_2(s)ds + \int_{t_0}^{t} \alpha_2(t_0)ds\Big)\alpha_3'(t)\\ &\geq -\Big(a + \int_0^{t_0} \alpha_2(s)ds + \int_{t_0}^{t} \alpha_3(s)ds\Big)\alpha_3'(t), \quad t \in (t_0,\infty).\end{aligned}$$

Finally, define

$$\alpha_1(t) = \begin{cases} \alpha_2(t) \; ; & 0 \leq t \leq t_0 \\ \alpha_3(t) \; ; & t_0 < t < +\infty \end{cases}$$

$\beta_1(t) \equiv 1.$

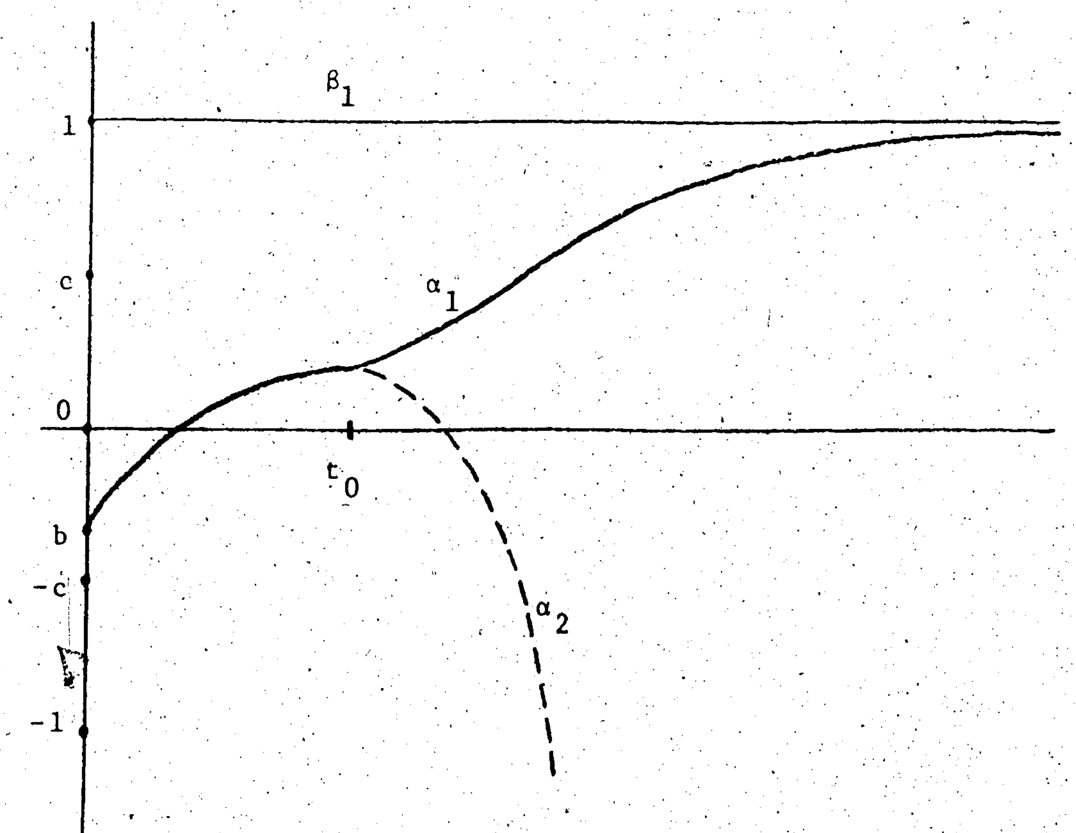

Figure 6

Then define $f:[0,\infty) \times \mathbb{R}^2 \to \mathbb{R}$ by

$$f(t,x,y) = -\left(a + \int_0^t \alpha_1(s)ds\right)y - \lambda(1-x^2).$$

It is easy to see that f satisfies (iv)-(vi) of II.8. Also, from the above, it is clear that $\alpha_1, \beta_1 \in \text{loc AC}[0,\infty)$ are piecewise loc $AC^1[0,\infty)$, and

$$\alpha_1(t) \leq \beta_1(t), \quad t \in [0,\infty),$$

$$\alpha_1'(t_0^-) = \alpha_2'(t_0^-) = 0 \leq \alpha_3'(t_0^+) = \alpha_1'(t_0^+),$$

$$\alpha_1''(t) \geq f\left(t,\alpha_1(t),\alpha_1'(t)\right),$$

$$\beta_1''(t) \leq f\left(t,\beta_1(t),\beta_1'(t)\right), \quad t \in [0,\infty), \quad t \neq t_0.$$

Thus, by II.8, there exists $\alpha \in \mathrm{loc}\ AC^1[0,\infty)$ satisfying

$$\alpha(0) = b, \quad \alpha_1(t) \le \alpha(t) \le \beta_1(t),$$

$$\alpha''(t) = -\left(a + \int_0^t \alpha_1(s)ds\right)\alpha'(t) - \lambda\left(1-\alpha^2(t)\right), \quad t \in [0,\infty).$$

Since

$$\alpha''(t) + \left(a + \int_0^t \alpha_1(s)ds\right)\alpha'(t) \le 0, \quad t \in [0,\infty),$$

then $\alpha'(t) \ge 0$ on $[0,\infty)$, and it follows that α satisfies (A), (B), and (C).

<u>Case (iii)</u>: Let $M = 2 - e^{1/2}$, $N = e^{1/2}-1$, and denote the well-known error function

$$\mathrm{erfc}(z) = \frac{2}{\sqrt{\pi}}\int_z^\infty e^{-s^2}ds.$$

For any $k > 0$,

$$-\frac{Ns^2}{2} + \left(k + \frac{N}{2k}\right)s = -\left(\sqrt{\frac{N}{2}}\,s - \frac{\left(k + \frac{N}{2k}\right)}{\sqrt{2N}}\right)^2 + \frac{k^2}{2N} + \frac{1}{2} + \frac{N}{8k^2}$$

so

$$\frac{M\exp\left(\frac{N}{8k^2} + \frac{1}{2}\right)}{\int_0^\infty \exp\left[\left(k + \frac{N}{2k}\right)s - \frac{Ns^2}{2}\right]ds} = \frac{M}{\sqrt{\frac{\pi}{2N}}\exp\left(\frac{k^2}{2N}\right)\mathrm{erfc}\left[\frac{-k - \frac{N}{2k}}{\sqrt{2N}}\right]}. \qquad (7)$$

Since $\lim_{z \to -\infty} \mathrm{erfc}(z) = 2$, then (7) has a finite positive limit as $k \to 0^+$. Thus, there exists $k_0 > 0$ such that

$$\frac{M \exp\left(\frac{N}{8k_0^2} + \frac{1}{2}\right)}{\int_0^\infty \exp\left(\left(k_0 + \frac{N}{2k_0}\right)s - \frac{Ns^2}{2}\right)ds} \geq k_0\, e^{1/2} . \tag{8}$$

For

$$-k_0 + \frac{1}{k_0}\bigl(b - \ln(b+1)\bigr) \leq a < +\infty ,$$

$$t_0 = -\frac{\ln(b+1)}{k_0} , \quad t_1 = t_0 + \frac{1}{2k_0} ,$$

define

$$\alpha_1(t) = \begin{cases} e^{k_0(t-t_0)} - 1 & ; \ 0 \leq t \leq t_1 \\[2ex] (2-e^{1/2})\dfrac{\int_{t_1}^{t} \exp\left(\left(k_0 + \frac{N}{2k_0}\right)(s-t_0) - \frac{N(s-t_0)^2}{2}\right)ds}{\int_{t_1}^{\infty} \exp\left(\left(k_0 + \frac{N}{2k_0}\right)(s-t_0) - \frac{N(s-t_0)^2}{2}\right)ds} + e^{\frac{1}{2}} - 1; & t_1 < t < +\infty . \end{cases}$$

$\beta_1(t) \equiv 1.$

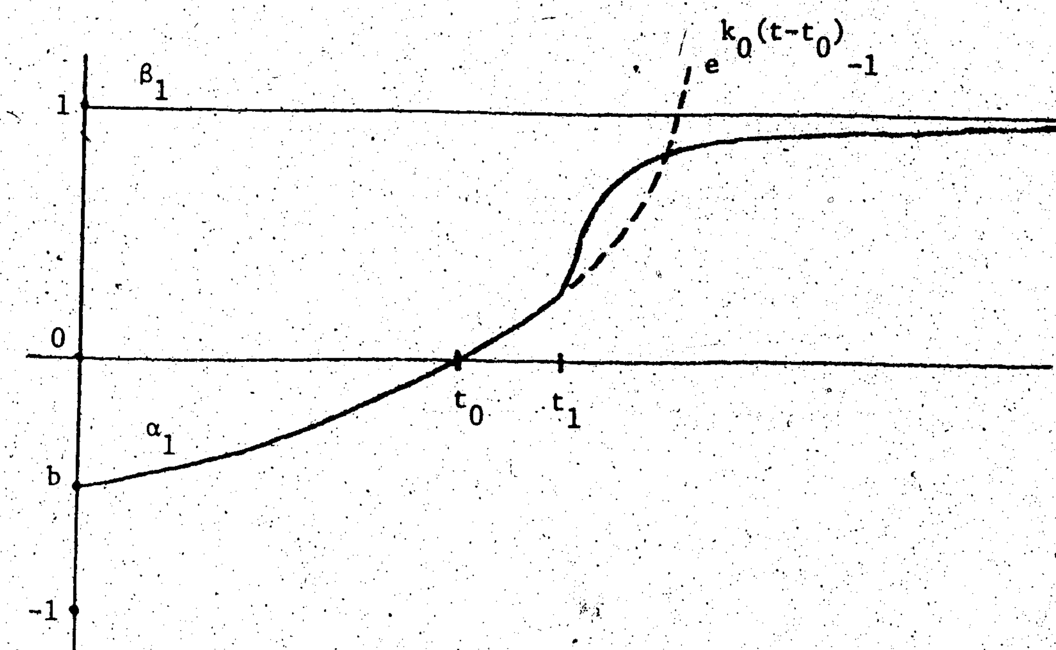

Figure 7

Then define $f:[0,\infty) \times \mathbb{R}^2 \to \mathbb{R}$ by

$$f(t,x,y) = -\left(a + \int_0^t \alpha_1(s)ds\right)y.$$

Again, it is easy to see that f satisfies (iv)-(vi) of II.8. Also, it can be shown that $\alpha_1, \beta_1 \in \text{loc } AC[0,\infty)$ satisfy the remaining hypotheses of II.8 as follows.

$$\text{(a)} \quad \alpha_1'(t) = \begin{cases} k_0 e^{k_0(t-t_0)}, & ;\ 0 \le t < t_1 \\[2ex] (2-e^{\frac{1}{2}}) \dfrac{\exp\left[\left(k_0 + \frac{N}{2k_0}\right)(t-t_0) - \frac{N(t-t_0)^2}{2}\right]}{\int_{t_1}^{\infty} \exp\left[\left(k_0 + \frac{N}{2k_0}\right)(s-t_0) - \frac{N(s-t_0)^2}{2}\right]ds} & ;\ t_1 < t < +\infty \end{cases}$$

$$\beta_1'(t) \equiv 0.$$

Clearly, α_1, β_1 are piecewise loc $AC^1[0,\infty)$, and by (8),

$$\alpha_1'(t_1^-) = k_0 e^{1/2} \le \alpha_1'(t_1^+)$$

since

$$\alpha_1'(t_1^+) = \frac{M \exp\left[\left(k_0 + \frac{N}{2k_0}\right)\frac{1}{2k_0} - \frac{N}{2}\left(\frac{1}{2k_0}\right)^2\right]}{\int_{t_1}^{\infty} \exp\left[\left(k_0 + \frac{N}{2k_0}\right)(s-t_0) - \frac{N(s-t_0)^2}{2}\right]ds}$$

$$\ge \frac{M \exp\left(\frac{N}{8k_0^2} + \frac{1}{2}\right)}{\int_0^{\infty} \exp\left[\left(k_0 + \frac{N}{2k_0}\right)s - \frac{Ns^2}{2}\right]ds}.$$

(b) Since

$$\alpha_1(0) = b, \quad \alpha_1(\infty) = 1, \quad \alpha_1'(t) > 0, \quad t \in [0,\infty),\ t \neq t_1,$$

then $\alpha_1(t) \leq \beta_1(t)$ on $[0,\infty)$.

(c) Since $\alpha_1(t_0) = 0$ and $\alpha_1'(t) > 0$ on $[0,t_1)$, then

$$\begin{aligned}
\alpha_1''(t) &= k_0\alpha_1'(t) \\
&\geq \left[-a + \frac{1}{k_0}\left(b - \ln(b+1)\right)\right]\alpha_1'(t) \\
&= -\left(a + \int_0^{t_0} \alpha_1(s)ds\right)\alpha_1'(t) \\
&\geq -\left(a + \int_0^{t} \alpha_1(s)ds\right)\alpha_1'(t) \\
&= f\left(t,\alpha_1(t),\alpha_1'(t)\right), \quad t \in [0,t_1).
\end{aligned}$$

Since $\alpha_1(t_1) = N$ and $\alpha_1'(t) > 0$ on (t_1,∞), then

$$\begin{aligned}
\alpha_1''(t) &= \left[\left(k_0 + \frac{N}{2k_0}\right) - N(t-t_0)\right]\alpha_1'(t) \\
&= \left[\left(k_0 - \int_{t_1}^{t} \alpha_1(t_1)ds\right)\right]\alpha_1'(t) \\
&\geq \left[-a + \frac{1}{k_0}\left(b - \ln(b+1)\right) - \int_{t_1}^{t} \alpha_1(s)ds\right]\alpha_1'(t) \\
&= -\left(a + \int_0^{t} \alpha_1(s)ds\right)\alpha_1'(t) \\
&= f\left(t,\alpha_1(t),\alpha_1'(t)\right), \quad t \in (t_1,\infty).
\end{aligned}$$

Since $\beta_1'(t) \equiv 0 \equiv \beta_1''(t)$, it is obvious that

$$\beta_1''(t) \leq f\left(t,\beta_1(t),\beta_1'(t)\right), \quad t \in [0,\infty).$$

Therefore, by II.8, there exists $\alpha \in \text{loc } AC^1[0,\infty)$ satisfying

$$\alpha(0) = b, \quad \alpha_1(t) \le \alpha(t) \le \beta_1(t),$$

$$\alpha''(t) = -\left(a + \int_0^t \alpha_1(s)ds\right)\alpha'(t), \quad t \in [0,\infty).$$

It follows that $\alpha'(t) \ge 0$ on $[0,\infty)$ so that α satisfies (A), (B), and (C).

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