University of Alberta

Designing a Neutral Elastic Inhomogeneity in the Case of a General Non-uniform Loading

BY

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Abstract

A general method is presented to design neutral elastic inhomogeneities in the presence of non-uniform stress fields in the matrix. The imperfect nature of the inhomogeneity-matrix interface is represented by the spring-layer interface model. Most of the works on neutral inhomogeneities so far have been based on the assumption of uniformity of the stress fields in both the matrix and the inhomogeneity. Of more practical importance is the case where the stress field is non-uniform. So in this thesis, a procedure is developed that will allow the design of a neutral inhomogeneity in any given non-uniform stress field in the matrix. Exact expressions for the interface parameters are developed with the use of conformal mapping techniques. The mathematical model is developed based on anti-plane elasticity and then extended to plane problems with specific examples. This concept of neutral inhomogeneities can be applied in fiber reinforced composite materials where fibers are used as inhomogeneities.

Acknowledgement

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Nomenclature

A	coefficient of the stress polynomial at the marix
B	coefficient of the stress polynomial inside the inhomogeneity
C, E, F	coefficients of polynomial
N	order of the stress polynomial
h	interface parameter
η	outward unit normal to the interface contour
p	order of stress polynomial for plane elasticity
R	Radius
S	region in space
$u,\ w$	displacement vector
v	complex conjugate of the vector u
x, y	Cartesian coordinates
z	complex coordinate
α_i	interface function for plane elasticity
β	interface function for antiplane elasticity
ξ	complex variable
δ_{ij}	Kronecker delta
ρ	Density
Γ	interface contour
μ_i	shear moduli
ϕ, ψ	complex functions
$\bar{\phi}, \overline{\psi}$	complex conjugate functions
σ_{ij}	stress tensor
λ, ς	Lamé constants
ν	Poisson's ratio
∇^2	Laplacian operator

1 Introduction

1.1 Background

A hole or an inhomogeneity introduced into an elastic body will change the body's original stress field and often lead to stress concentrations. Designing neutral holes and inhomogeneities can eliminate this stress concentration which is of great concern in composite materials. Ideally, it is believed that the bonding between the fibers and the matrix material in a composite is perfect or rigid. But in reality this bond is never perfect due to the presence of micro-cracks and voids. It is this very nature of the bonding interface that allows us to design neutral inhomogeneities. Based on the mathematical models that describe the imperfect nature of the interface, we can design neutral inhomogeneities that do not cause any stress disturbance in the surrounding elastic body. In this thesis the design of neutral inhomogeneities for general non-uniform loading is considered. It is expected that the concept of neutral inhomogeneities for non-uniform stress field will be useful in practical problems concerned with stress concentrations caused by material mismatch.

1.2 Composite Materials

In any field of research, the final limitation depends on materials. The demand for ever better performance is so great and diverse that conventional materials are not able to satisfy them. This has led to the resurgence of the ancient concept of combining different constituents to satisfy demands for high performance materials. Composites are composed of two or more constituents, arranged in specific internal configurations to obtain mechanical or other properties tailored for specific applications. The properties of the composite materials combine the best features of each constituent to maximize a given set of properties (stiffness, strength-to-weight ratio, tensile strength, etc.) and minimize others (e.g., weight and cost). We are now living in a more energy conscious society that has led to an increasing demand for lightweight yet strong and stiff structures. And composite materials are increasingly providing the answers.

Composite materials generally consist of a bulk material, called the matrix, and a stiff, strong reinforcing material in either long fiber, short fiber, whisker, or particulate form. The essence of the concept of composites is this: the bulk phase accepts the load over a large surface area, and transfers it to the reinforcement, which being stiffer, increases the strength of the composite. Binary composites are the most common, and typically consist of a dispersed (reinforcement) phase embedded in a continuous (matrix) phase. The reinforcement phase should be a material that is stiff. Compared with the dispersed phase, the matrix phase should be tough and ductile. Its purpose is to support and allow load to be transmitted to the dispersed phase. It prevents the propagation of brittle cracks from fiber to fiber, which means the matrix phase serves as a barrier to crack propagation.

Polymer-matrix composites (PMCs) and metal-matrix composites (MMCs) are the two broad categories of composite materials in terms of matrix classification. PMCs are typically used in low-temperature structural applications such as in civil-structures, biomedical implants, automobiles, and airframe structures. The fibers typically provide the stiffness and strength to the composite and can be made from a wide variety of materials, including glass, graphite, Kevlar, and boron, as examples. The fibers can be arranged in almost any fashion, ranging from totally random to highly structured and organized. In MMCs, the matrix phase consists of a continuous metallic material, such as aluminium, titanium, magnesium, copper, etc. The reinforcing constituent is normally a ceramic (silicon carbide, silicon nitride, alumina etc.). MMCs are used in the aerospace industry for airframe and spacecraft structures, as well as in the automotive, electronic, and even leisure industries. Particle-reinforced MMCs house electronics in Motorola's Iridium satellites. In automotive industry, Chevrolet Corvettes and GM pick-ups both use these kinds of lightweight materials as drive shafts, and the GM electric vehicles rely on long-lasting particle MMC brake rotors and drums.

Based on the form of reinforcements, composite materials can be classified as fibrous, particulate and laminate composites. In particulate composites, reinforcements are dispersed in particle form within the continuous matrix phase. Fiber reinforced composites are two-dimensional analogues of particulate composites. The fibers are arranged in a variety of orientations from unidirectional arrangements to fibers woven into fabrics. Laminate composites consist of thin, planar, unidirectional layers called lamina that have a preferred high-strength direction. The lamina are stacked and subsequently cemented together such that the orientation of the high strength reinforcing direction varies with each layer.

As mentioned earlier, composite materials are designed for high strength performances. The performance of composite materials depends not only on the properties of the constituent materials, but also on the quality of bonding that exists between the constituent phases. The presence of structural defects such as voids, impurities, imperfect adhesions and micro-cracks along the fiber matrix interface will compromise the effectiveness of load transfer between the matrix and the fiber.

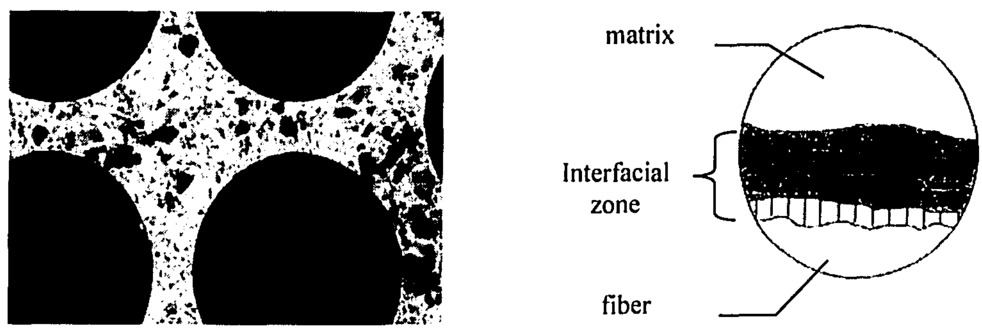

Figure 1.1: Cross-section of fiber-reinforced ceramic composite (left) and a schematic representation of the fiber-matrix interface (source MEIAF, UCSB)

1.3 Fiber-Matrix Interface

In a binary composite, fibers and the matrix are bonded by adhesives. In reality this adhesive layer creates an additional phase called the interface or interphase. The interface is a bounding surface or zone where a discontinuity occurs, whether physical, mechanical, chemical etc. To obtain desirable properties in a composite, the applied load should be effectively transferred from the matrix to the fibers via the interface. Failure at the interface (called debonding) is not desirable, which could lead to fracture propagation. This means that the interface must exhibit strong adhesion between fibers and matrix. The interphase layer may be the by-product of the chemical reactions that occurred between the constituent phases during the fabrication process, or a thin layer, such as in functionally graded materials, introduced deliberately by coating the fibers to improve the performance of the composite material by removing unfavorable residual and thermal stress along the fiber/matrix boundary. The layer may be of visco-elastic nature, which provides relaxation and damping characteristics to otherwise brittle elastic composite. Interfacial zones between the fibers and the matrix material affect the overall mechanical properties and the strength of composites.

1.4 Interface Models

The bonding between the fiber and the interface is actually imperfect due to the presence of micro voids and cracks. As a result the latest research has been directed towards models of imperfect interfaces. Various models have been proposed to study the physical properties of the interface layer. These models can be broadly classified as the *interphase* layer models and the imperfect *interface* models. The *interphase* layer models consider a distinct phase or layer between the fiber and the matrix having specified thickness and thermoelastic properties. It can be defined as a non-uniform, anisotropic region of finite thickness having significantly distinct chemical and mechanical composition as compared to the bulk fiber or matrix. Judicious choice of interphase thickness and interphase properties can be used to control the constituent stress components. The characterization of the interphase has to occur at several different levels (i.e. mechanical, chemical and microstructural) in order to completely analyze the nature of the bond along the fiber-matrix boundary. Because of the complexity of the thorough thermo-mechanical characterization of the interphase layer, simplified imperfect *interface* models were developed.

The imperfect interface model assumes a very thin interfacial zone of vanishing thickness existing between fiber and matrix. In other words, when the interphase becomes vanishingly thin it becomes an interface. This interface is defined as a two dimensional boundary separating distinct phases. Thus it can be assumed that the interface forms a transition zone from matrix material to fiber material through a distribution of discrete contacts. These contacts can transfer load directly, but they offer resistance to extension. In addition, the imperfect bonding model relaxes the classical condition of perfect bonding. The ability of load transfer between fiber and matrix depends on the degree of contact between them at the interface. Thus a decrease in load transfer, as a result of interfacial damage, gives rise to the concept of imperfect interface. In this context, one of the more common assumptions is the spring-layer interface, where the traction is continuous but the displacements are discontinuous with jumps proportional to their respective tractions. The proportionality constants characterize the stiffness and the strength of the interfacial zone. This model is much more tractable analytically than its interphase counterpart. In the current study this spring-layer interface model will be used.

1.5 Literature Review and Outline of the Study

The problem of neutral inhomogeneity can be traced back to Mansfield [1], who was among the first to recognize the feasibility of designing a 'neutral' hole which eliminates any stress concentrations introduced by the hole and hence does not disturb the original stress field in the uncut body. In [1] the neutral holes were referred to as reinforced holes, since the holes were reinforced with an inside liner in order to make them neutral. Mathematical formulae were developed to determine both the shape of a neutral hole and the variation along the whole boundary of the cross-sectional area of the reinforcement. Following this concept one could imagine the possibility of placing a neutral sphere in a matrix without disturbing the uniform field outside the inhomogeneity. Since these fields remain undisturbed, the displacements and loads at the boundary of the material also remain undisturbed and as a consequence the effective bulk modulus of the medium remains unchanged. Using this principle, Hashin [2, 3] obtained the exact expression for the effective bulk modulus of the sphere assemblage. His formula produced reasonable estimates of the effective bulk modulus of a matrix containing a suspension of well separated spheres.

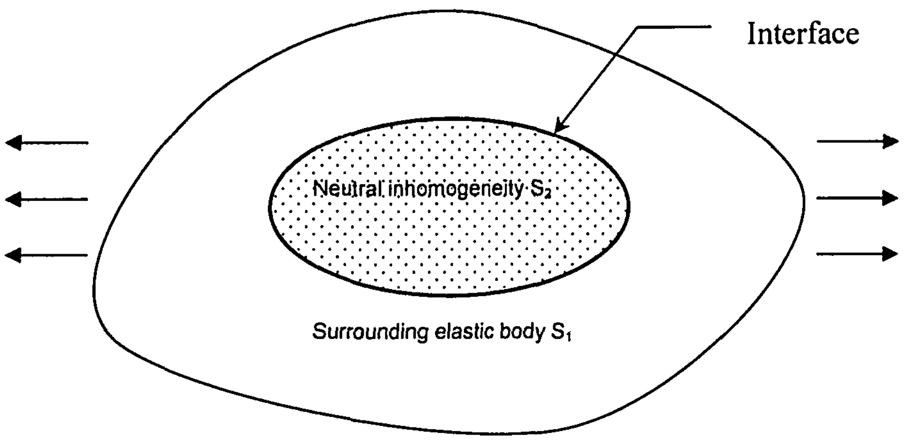

Figure 1.2: A neutral inhomogeneity embedded in the matrix.

The analogous problem of a neutral elastic inhomogeneity was studied by Ru [4]. Here it was shown that neutral elastic inhomogeneities cannot exist when a conventional perfectly bonded material interface is assumed to exist between the inhomogeneity and the surrounding elastic body. In addition, Ru [4] introduced a method for the design of neutral inhomogeneities based on the spring-layer model of imperfect interface [5-11]. The problem of three phase inclusion or the existence of an interphase instead of in *interface* has also been studied widely in literature. In [12], it was shown that the stress field within a circular inclusion with a homogeneous imperfect interface under remote uniform antiplane shearing is uniform. But the same is not true for an elliptic inclusion. Consequently, in the study of a three-phase elliptic inclusion in antiplane shear, since the imperfect interface is considered to be an adequate model for a thin interphase layer between the inclusion and its surrounding matrix, one would expect the internal stress field to be non-uniform. However, in [13] it was proved that the three phase elliptic inclusion in antiplane shear does indeed admit a uniform internal stress field provided its interfaces consist of two confocal ellipses. Furthermore this conclusion was proved to be valid even for an elliptic inclusion with a multilayer interface. An example of current interest is that related to passivated interconnect lines in large scale integrated circuits [14]. Here, the major cause of voiding and failure has been attributed to the residual stresses induced within the interconnect by thermal mismatch between the line and the surrounding passivation and substrate. Recently, Niwa et al. [15] have modeled the passivated line as an ellipsoid undergoing a uniform volume strain surrounded by an infinite homogenous matrix. These authors have found that at the solution derived by this simplified model is in good agreement with the known results. To reduce residual stresses, it would seem that one could apply a relatively compliant interphase layer between the interconnect line and the surrounding passivation and substrate. In doing so, residual stress analysis of the interconnect lines reduces to a three-phase elliptic inclusion problem. The concept of neutral inclusion has also been extended to other fields like electrical conductivity [16]. Such an inclusion when inserted in a matrix containing a uniform applied electric field does not disturb the field outside the inclusion. Consequently, assemblages of neutral inclusions have certain moduli of effective conductivity tensor that can be determined exactly. Provided that the inclusion shape satisfies a certain criteria for neutrality, the interface conductance is chosen in such a way that it shields the field perturbation caused by the inclusion. These criteria are easily extended to the case of a neutral inclusion in a fully anisotropic medium [17]. Ru [4] considered plane strain problems and found that certain inclusions could be made neutral by suitably choosing the interface stiffness along the boundary. Benveniste [18] constructed neutral inclusions for the torsion problem.

Our work is motivated by the investigation initiated in [4]. The neutral inhomogeneities designed in [4] however assume the existence of a uniform stress field in the surrounding matrix. Of more practical interest is the case where the stress field in the matrix is non-uniform. In [19], Van Vliet et al. extended the techniques in [4] to the case where the stress field in the matrix is non-uniform when the inhomogeneity is circular or elliptic and the composite is subjected to antiplane shear deformation. Here, however, the elliptic inhomogeneity is considered only for the simplest non-uniform (linear) stress field in the surrounding matrix, mainly because the complicated nature of the analysis involved precludes the extension of the method to the cases of higher order. To avoid this complexity of the analysis, Schiavone [20] used the Laurent's series expansion method to find the exact expression for the interface parameter in antiplane elasticity for a neutral elliptic inhomogeneity. In [20] it was proved that the method worked with both linear and quadratic stress fields in anti-plane elasticity. Mahboob [21] generalized the method to be used for higher order stress fields for elliptic inhomogeneities.

In the current work, a constructive method for designing neutral inhomogeneities for any given non-uniform stress field is developed based on [20, 21]. The advantage of using the series method used in the current study is that it can handle loading of a polynomial function. The interface parameter, defined as the proportionality constant between the displacements and their respective tractions, being itself proportional to the thickness of the interface layer provides the precondition that it must be real and positive. This precondition is applied in order to find physically permissible interface parameters. Using Laurent's theorem to express each complex potential as a power series in the regions, the stress and displacement boundary value problems are reduced to sets of simultaneous linear equations in the coefficients of the power series. The coefficients of the Laurent series can be found easily from the boundary conditions. Thus, the proposed new method can be applied to any non-uniform stress field for antiplane strain and can be extended to plane strain problems for special cases.

The purpose of this thesis is to provide a general model for designing a neutral inhomogeneity interface in general non-uniform stress field. The thesis is organized as follows. In chapter 1 the background and motivation for the work are introduced and also some of the concepts used in thesis are explained. In chapter 2 the basic kinematics related to the linearly elastic solid is discussed and subsequently the mathematical model in terms of the boundary value problem for the antiplane shear deformations is formulated. In chapter 3 a similar model is developed for plane strain cases with specific examples. Finally, suggestions for future work are outlined in chapter 4.

2 General Formulation of the Antiplane Problem

2.1 Antiplane deformation

Antiplane shear deformation is one of the simplest forms of deformations that a solid can undergo. With just a single scalar axial displacement, anti-plane shear can be modeled by a single second-order linear or quasi-linear partial differential equation, and is thus viewed as complementary to the plane strain deformation, with its two in-plane displacements. Despite the fact that the model of anti-plane shear deformations has the virtue of relative mathematical simplicity without loss of essential physical relevance, antiplane shear deformations have received comparatively little attention in the linear elasticity literature. The reason behind this may be that in the case of a homogeneous and isotropic, linearly elastic solid, the governing traction boundary value problem describing the antiplane shear deformations is simply the interior Neumann problem of Laplace's equation. It turns out, the analytical solutions to antiplane problems with imperfect interface can provide useful insight into the plane problems, which though more practical are difficult to solve.

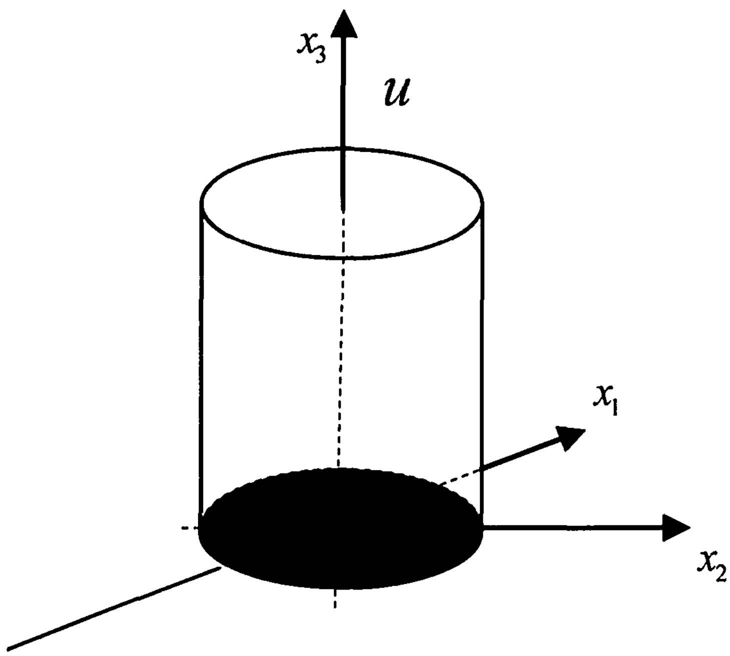

Figure 2.1: A linearly elastic cylinder in anti-plane deformation.

For an in-depth discussion of the theories of linear and non-linear antiplane shear deformations, the reader is referred to Horgan [22]. In antiplane shear, we consider the equilibrium of a deformable solid, which, in its unstressed undeformed state, occupies a cylindrical region whose generators are parallel to the X_3 axis of a rectangular Cartesian coordinate system. The cylinder is assumed to be sufficiently long so that the end effect in the axial direction is negligible. Here the out-of-plane displacement is a function of the cross-section of the cylinder.

2.2 Preliminaries

Consider a homogeneous and isotropic, linearly elastic body, finite or infinite in extent, simply or multiply-connected, which is subjected to a given state of stress under a prescribed loading system. Assume that the same elastic body is then cut over a number of simply connected sub-domains each of which is filled with a different homogeneous and isotropic linearly elastic material (each sub-domain is now referred to an inhomogeneity). Here, we are concerned with the design of the material interface between any single inhomogeneity and the elastic body such that the corresponding inhomogeneity is 'neutral' in the sense that it does not disturb the original pre-described field in the uncut elastic body.

With antiplane shear deformations in mind, consider then a domain in $\Re^2$ containing a single internal elastic inhomogeneity, with elastic properties different from the surrounding matrix. The linearly elastic materials occupying the matrix and the inhomogeneity are assumed to be homogeneous and isotropic with associated shear moduli $\mu_1(>0)$ and $\mu_2(>0)$, respectively. We represent the matrix by the domain S_1 and assume that the inhomogeneity occupies a region S_2. The inhomogeneity-matrix interface will be denoted by the curve Γ. Let (x,y) denote a generic point in $\Re^2$ and $z = x + iy = re^{i\theta}$ the complex coordinate. In what follows, the subscripts 1 and 2 will refer to the regions S_1 and S_2, respectively and $u_\alpha(x,y)$, $\alpha = 1,2$ will denote the elastic antiplane displacement at the point (x,y) in the S_α, respectively.

It is assumed that the inhomogeneity is imperfectly bonded to the matrix along Γ by the spring-layer type interface. The spring-layer type model physically represents a compliant interface. In practice, the interface is the adhesive layer between the fiber and the matrix (for example, an elastic interphase region [6]). This adhesive layer is generally softer that the matrix and inhomogeneity. In this context, it is more reasonable to use a soft spring layer model as compared to a rigid interface. A rigid interface is not able to represent the continuity of stress at the fiber-matrix interface. Another property of the compliant interface is the jump or discontinuity in displacement. Both of these properties can be adequately represented by the spring-layer model. The interface is considered to be an assembly of uniformly distributed springs. The spring constants are equivalent to the interface function. The thin interfacial layer can undergo a large difference in the displacement across its thickness. In a spring layer interface model the tractions are continuous and jumps in displacements are proportional to their respective traction components. The interface condition is given by [23],

$$\beta(x,y)[u_1 - u_2] = \mu_2 \frac{\partial u_2}{\partial \eta} = \mu_1 \frac{\partial u_1}{\partial \eta}. \tag{2.1}$$

Where η is the outward unit normal to the interface contour Γ and the interface function β is such that it maps the interface contour in a real and positive field i.e. $\beta(x,y): \Gamma\ (\subset \Re^2) \to \Re^+$. This condition (2.1) means the shear stress on an element of the boundary surface is continuous across the interface. In special cases, $\beta(x,y)=0$ represents the traction-free boundary, meaning no adhesion between fiber and matrix. On the other hand, $\beta(x,y)=\infty$ corresponds to a perfectly bonded interface. So the smaller the thickness of the interface, the more rigid is the adhesion between fiber and matrix. Density of the adhesive material also affects the property of the interface, making the joint stiffer with a high density adhesive. As such, β should be inversely proportional to the thickness and directly proportional to the density of the adhesive layer (see for example [3, 5, and 6]). So the interface function β can be chosen by varying the properties of the adhesive layer. The only restriction is that β must be non-negative everywhere. This precondition arises only for practical design constraints, not mathematical necessity. A negative interface parameter would necessitate an adhesive material with a negative shear modulus. It is this constraint that makes certain neutral inhomogeneity design problems unsolvable [19]. Consequently, the following boundary value problem describes the antiplane deformation of an inhomogeneity with imperfect interface:

$$\begin{aligned}
&\nabla^2 u_1 = 0 \text{ in } S_1,\\
&\nabla^2 u_2 = 0 \text{ in } S_2,\\
&\beta(x,y)(u_1 - u_2) = \mu_2 \frac{\partial u_2}{\partial \eta},\\
&\mu_2 \frac{\partial u_2}{\partial \eta} = \mu_1 \frac{\partial u_1}{\partial \eta} \text{ on } \Gamma,
\end{aligned} \tag{2.2}$$

where, ∇^2 is defined as the Laplacian operator as $\nabla^2 \equiv \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$.

2.3 Complex variable formulation:

For the sake of using complex functions to solve the governing equation (2.2), we want to express the stresses and the displacements by means of the analytic functions. $v_i(x, y)$ is denoted as the harmonic functions conjugate to $u_i(x, y)$. Since the external loading is self equilibrated, $v_i(x, y)$ are single valued and uniquely determined to within an integration constant and the corresponding potentials $\phi_1(z)$ and $\phi_2(z)$, with $z = x + iy$, are analytic within S_1 and S_2.

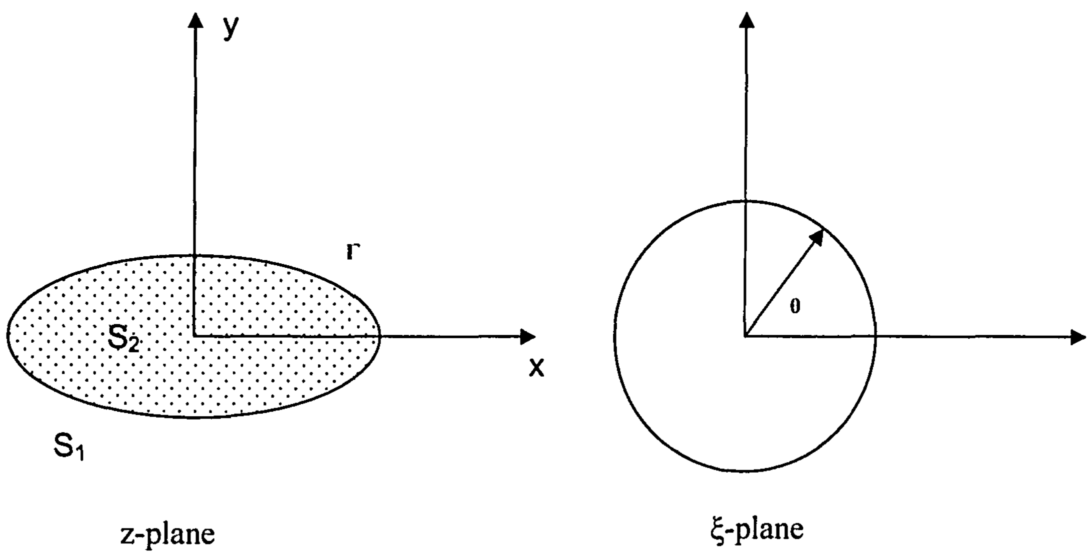

Figure 2.2: The conformal mapping from z-plane to ξ-plane

Thus, the values of the complex potentials are defined as,

$$\phi_i = u_i(x_1, x_2) + iv_i(x_1, x_2),$$
$$\text{where, } i = 1,2.$$

In view of the relations,

$$2u_i(z) = \phi_i(z) + \overline{\phi_i(z)}, \tag{2.3}$$

and

$$2iv_i(z) = \phi_i(z) - \overline{\phi_i(z)},$$

using displacement expressions for antiplane problems,

$$\begin{aligned} u_1 &= 0, \\ u_2 &= 0, \\ u_3 &= u_3(x_1, x_2), \end{aligned} \tag{2.4}$$

we find the expression for the stress-strain relations as,

and,

$$\begin{aligned} \sigma_{13} &= \mu u_{3,1}, \\ \sigma_{23} &= \mu u_{3,2}. \end{aligned} \tag{2.5}$$

This relation is modified to express the stress in terms of the complex potentials. In order to simplify the derivation, let us write, $u_3 = w$(since this is the only non-zero displacement component in this case), and its conjugate as $v_3 = v$,

$$\sigma_{13} - i\sigma_{23} = \mu[u_{3,1} - iu_{3,2}]$$
$$= \mu\left[\frac{\partial w}{\partial x_1} - i\frac{\partial w}{\partial x_2}\right]. \tag{2.6}$$

In the following section we will derive the complex differential operators as well as some other relations that will be used to convert the boundary value equations (2.2) to complex variable representation [23]. Now, let us define the operator $\frac{d}{dz}$ as for ordinary coordinate transformations by the relation,

$$\frac{d}{dz} = \frac{\partial}{\partial x_1}\frac{\partial x_1}{\partial z} + \frac{\partial}{\partial x_2}\frac{\partial x_2}{\partial z}$$
$$= \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right). \tag{2.7}$$

Using this complex differential operator on the definition $\phi_i = w(x_1, x_2) + iv(x_1, x_2)$,

$$\frac{d\phi}{dz} = \frac{1}{2}\left[\frac{\partial}{\partial x_1}(w + iv) - i\frac{\partial}{\partial x_2}(w + iv)\right]$$
$$= \frac{1}{2}\left[\frac{\partial w}{\partial x_1} + i\frac{\partial v}{\partial x_1} - i\frac{\partial w}{\partial x_2} + \frac{\partial v}{\partial x_2}\right]$$
$$= \left[\frac{\partial w}{\partial x_1} - i\frac{\partial w}{\partial x_2}\right]$$
$$= \frac{1}{\mu}[\sigma_{13} - i\sigma_{23}], \tag{2.8}$$

equation (2.8) can be written as a complex relation of the stress components,

$$\sigma_{13} - i\sigma_{23} = \mu_i \phi_i'(z), \quad z \in S_i (i = 1,2). \tag{2.9}$$

Differentiating with respect to η (vector normal to interface) [25] the equation (2.3) can be written as,

$$2\frac{\partial u_2}{\partial \eta} = \phi_i'(z)e^{i\eta(z)} + \overline{\phi_i(z)}e^{-i\eta(z)}, \tag{2.10}$$

where $e^{i\eta(z)}$ represents (in complex form) the outward unit normal to Γ.

So by using equation (2.10), the boundary value equation (2.2) can be written as,

$$(u_1 - u_2) = \frac{\mu_2}{2\beta}\left[\phi_2'(z)e^{i\eta(z)} + \overline{\phi_2(z)}e^{-i\eta(z)}\right] \tag{2.11}$$

Again from the continuity of stress at the interface we find,

$$\begin{aligned}
&\mu_1 \frac{\partial u_1}{\partial \eta} = \mu_2 \frac{\partial u_2}{\partial \eta} \\
&\Leftrightarrow \mu_1\left[\eta_1 \frac{\partial u_1}{\partial x_1} + \eta_2 \frac{\partial u_1}{\partial x_2}\right] = \mu_2\left[\eta_1 \frac{\partial u_2}{\partial x_1} + \eta_2 \frac{\partial u_2}{\partial x_2}\right] \\
&\Leftrightarrow \mu_1\left[\eta_1 \frac{\partial v_1}{\partial x_2} + \eta_2\left(-\frac{\partial v_1}{\partial x_1}\right)\right] = \mu_2\left[\eta_1 \frac{\partial v_2}{\partial x_2} + \eta_2\left(-\frac{\partial v_2}{\partial x_1}\right)\right],
\end{aligned} \tag{2.12}$$

where, components of the outward unit normal to the interface are:

$$\eta_1 = \frac{dx_2}{ds},$$
$$\eta_2 = -\frac{dx_1}{ds}. \tag{2.13}$$

As mentioned earlier ϕ_2 and ϕ_1 represent the complex potential inside the inhomogeneity and in the matrix respectively. Their correlation can be derived by further simplifying equation (2.12) as follows:

$$\begin{aligned}
&\mu_1\left[\frac{\partial x_2}{\partial s}\frac{\partial v_1}{\partial x_2}+\frac{\partial x_1}{\partial s}\frac{\partial v_1}{\partial x_1}\right]=\mu_2\left[\frac{\partial x_2}{\partial s}\frac{\partial v_2}{\partial x_2}+\frac{\partial x_1}{\partial s}\frac{\partial v_2}{\partial x_1}\right]\\
&\Leftrightarrow \mu_1\left[\frac{dv_1}{ds}\right]=\mu_2\left[\frac{dv_2}{ds}\right]\\
&\Leftrightarrow \mu_1 v_1 = \mu_2 v_2\\
&\Leftrightarrow \mu_1 \operatorname{Im}\phi_1 = \mu_2 \operatorname{Im}\phi_2\\
&\Leftrightarrow \operatorname{Im}\phi_1 = \frac{\mu_2}{\mu_1}\operatorname{Im}\phi_2\\
&\Leftrightarrow \operatorname{Im}\phi_1 = (2\delta - 1)\operatorname{Im}\phi_2 .
\end{aligned} \tag{2.14}$$

Now replacing the relations (2.8, 2.11, 2.14) into the basic definition of the complex potential,

$$
\begin{aligned}
\phi_1 &= u_1 + iv_1 \\
&= u_2 + i\frac{\mu_2}{\mu_1}v_2 + (u_1 - u_2) \\
&= \operatorname{Re}\phi_2 + i\frac{\mu_2}{\mu_1}\operatorname{Im}\phi_2 + (u_1 - u_2) \qquad (2.15) \\
&= \frac{1}{2}(\phi_2 + \bar{\phi}_2) + i\frac{\mu_2}{\mu_1}\frac{1}{2i}(\phi_2 - \bar{\phi}_2) + (u_1 - u_2) \\
&= \delta\phi_2 + (1-\delta)\bar{\phi}_2 + \frac{\mu_2}{2\beta}\left[\phi_2'(z)e^{i\eta(z)} + \overline{\phi_2(z)}e^{-i\eta(z)}\right]
\end{aligned}
$$

the boundary value problem (2.2) can be written in the following form,

$$
\phi_1(z) = \delta\phi_1(z) + (1-\delta)\overline{\phi_2(z)} + h(z)[\phi_1'(z)e^{i\eta(z)} + \overline{\phi_2(z)}e^{-i\eta(z)}], \quad z \in \Gamma, \qquad (2.16)
$$

where the interface parameter is defined by,

$$
h(z) \equiv \frac{\mu_2}{2\beta(z)} \geq 0,
$$

and, $$\delta \equiv \frac{\mu_1 + \mu_2}{2\mu_1} > \frac{1}{2}.$$

According to Weierstrass's approximation theorem any continuous function on a closed and bounded interval can be uniformly approximated on that interval by polynomials to any degree of accuracy. So any suitable boundary stress condition can be approximated by a polynomial form of the stress function. We use a polynomial to represent the general stress field of N^{th} order. Let the prescribed stress field be characterized by $\phi_1 = \sum_{n=0}^{N} A_n z^n$, where $A_n \in C$ are given and $N = 1,2,\ldots$. According to the definition of a neutral inhomogeneity, the original stress field in the uncut elastic body remains undisturbed when the neutral inhomogeneity is inserted, then we have $\phi_1 = \sum_{n=0}^{N} A_n z^n$ in S_1. Hence, for a neutral inhomogeneity, we require from (2.16),

$$\sum_{n=0}^{N} A_n z^n = \delta\phi_2(z) + (1-\delta)\overline{\phi_2(z)} + h(z)[\phi_2'(z)e^{i\eta(z)} + \overline{\phi_2(z)}e^{-i\eta(z)}], \quad z \in \Gamma. \tag{2.17}$$

Equation (2.17) has been solved analytically by Ru [4] and Van Vliet [19] for uniform and linearly varying stress fields for certain shapes, but proved to be too complicated for stress fields of any higher order [19]. This is due to the exponential terms present in the equation (2.17) that represents the unit normal to the inhomogeneity contour. For a circular inhomogeneity there is no need for mapping so equation (2.17) takes a simplified form. But for an elliptic shaped inhomogeneity $e^{i\eta(z)}$ takes the form,

$$e^{i\eta(z)} = \frac{\left(1+\frac{a^2}{b^2}\right)z + \left(1-\frac{a^2}{b^2}\right)\bar{z}}{2a\sqrt{1-\frac{a^2}{b^2}\left(\frac{1}{a^2}-\frac{1}{b^2}\right)y^2}},$$

where, a and b are the lengths of the two axis of the ellipse. As a result, finding the exact expression for $h(z)$ becomes too complicated. To avoid this difficulty, we use conformal mapping to transform (2.17) into the ξ plane as $z = w(\xi)$ and expand $h(z)[\phi_2'(z)e^{i\eta(z)} + \overline{\phi_2(z)}e^{-i\eta(z)}]$ in (2.17) with Laurent's series. Equation (2.17) in the ξ plane now takes the form,

$$\sum_{n=0}^{N} A_n (w(\xi))^n = \delta\phi_2(w(\xi)) + (1-\delta)\overline{\phi_2(w(\xi))} + h(w(\xi))[\phi_2'(w(\xi))e^{i\eta(w(\xi))} + \overline{\phi_2(w(\xi))}e^{-i\eta(w(\xi))}],$$
$$w(\xi) \in \Gamma.$$

(2.18)

The term $h(w(\xi))[\phi_2'(w(\xi))e^{i\eta(w(\xi))} + \overline{\phi_2(w(\xi))}e^{-i\eta(w(\xi))}]$ in (2.17) can now be represented by the Laurent's series expansion so that interface parameter is given by,

$$h(w(\xi)) = \frac{\sum_{n=-\infty}^{\infty} E_n \xi^n}{[\phi_2'(w(\xi))e^{i\eta(w(\xi))} + \overline{\phi_2(z)}e^{-i\eta(w(\xi))}]}, \tag{2.19}$$

where the Laurent's integral is given by,

$$E_n = \frac{1}{2\pi i} \int_{\partial\sigma} h(w(\xi))[\phi_2'(w(\xi))e^{i\eta(w(\xi))} + \overline{\phi_2(z)}e^{-i\eta(w(\xi))}] \frac{d\xi}{\xi^{n+1}}.$$

The benefit of this method is that both the numerator and the denominator in expression (2.19) can be expressed in terms of the single complex variable ξ. The requirement that $h(w(\xi))$ is a real and positive term is applied to (2.19) to provide necessary equations that define the interface parameter and the relations between the prescribed stress field in the matrix and the corresponding stress field inside the inhomogeneity.

Equation (2.19) is generalized for any shape of inhomogeneity that can be mapped onto the complex plane. In order to derive the complete general solution we shall use the example of an elliptic inhomogeneity. The elliptic shaped inhomogeneity is selected as a representative example.

2.4 Elliptic Inhomogeneity:

In this section we present a simple method for constructing a neutral elliptic inhomogeneity when the stress field in the matrix is characterized by $\phi_1 = \sum_{n=0}^{N} A_n z^n$, where $A_n \in C$ are given and $N = 1,2,\ldots$. Consider an elliptic inhomogeneity, centered at the origin, with axes of lengths a and b $(a \neq b)$, coincident with the x and y axes, respectively. Suppose that the region S_2 (in the z-plane) is mapped onto the region $\sigma = \{|\xi| \geq 1\}$ (in the ξ-plane) by the function [25],

$$z = w(\xi) = R\left(\xi + \frac{k^2}{\xi}\right), \quad k \in (0,1), \quad R > 0, \qquad (2.20)$$

where, for $k=0$ the ellipse is a circle of radius R, and in the limiting case when $k=1$ it degenerates to a straight slit or crack of length $4R$. Then this mapping function can be written by the power series,

$$z^m = R^m\left(\xi + \frac{k^2}{\xi}\right)^m = R^m \sum_{s=0}^{m} \binom{m}{s} \xi^{m-2s} k^{2s}, \quad \mathrm{m} = 1,2,3,.... \qquad (2.21)$$

Suppose the stress field inside the inhomogeneity is characterized by $\phi_2 = \sum_{n=0}^{N} B_n z^n$ where B_n are complex coefficients to be determined. In the ξ plane, the interface condition in (2.17) now becomes,

$$\sum_{n=0}^{N} A_n R^n \sum_{s=0}^{n} \binom{n}{s} \xi^{n-2s} k^{2s} - \delta \sum_{n=0}^{N} B_n R^n \sum_{s=0}^{m} \binom{n}{s} \xi^{n-2s} k^{2s} - (1-\delta) \sum_{n=0}^{N} \overline{B}_n R^n \sum_{s=0}^{n} \binom{n}{s} \xi^{2s-n} k^{2s}$$
$$= h(w(\xi))[\phi_2'(w(\xi))e^{i\eta(w(\xi))} + \overline{\phi_2(z)}e^{-i\eta(w(\xi))}],$$

where, $\xi \in \partial\sigma = \{|\xi| = 1\}$.

$$(2.22)$$

Next, we expand $h(w(\xi))[\phi_2'(w(\xi))e^{i\eta(w(\xi))} + \overline{\phi_2(z)}e^{-i\eta(w(\xi))}]$ in Laurent's series to obtain the interface condition from (2.11) as,

$$
\begin{aligned}
&\sum_{n=0}^{N} A_n R^n \sum_{s=0}^{n} \binom{n}{s} \xi^{n-2s} k^{2s} - \delta \sum_{n=0}^{N} B_n R^n \sum_{s=0}^{m} \binom{n}{s} \xi^{n-2s} k^{2s} \\
&\quad - (1-\delta) \sum_{n=0}^{N} \overline{B}_n R^n \sum_{s=0}^{n} \binom{n}{s} \xi^{2s-n} k^{2s} = \sum_{n=-\infty}^{\infty} E_n \xi^n, \quad \text{where } \xi \in \partial\sigma = \{|\xi| = 1\}.
\end{aligned}
\tag{2.23}
$$

Since the A_n are given and the E_n are fixed, we can equate coefficients of ξ^n in (2.22) and establish equations for the B_n in terms of the E_n and A_n.

Next, on the interface contour $\partial\sigma$ the unit normal is defined as [24],

$$
e^{i\eta(w(\xi))} = \xi \frac{w(\xi)}{|w'(\xi)|} = \frac{\xi - k^2 \xi^{-1}}{|1 - k^2 \xi^{-2}|}. \tag{2.24}
$$

Thus from (2.21) the interface function is given by,

$$
\begin{aligned}
h(w(\xi)) &= [\phi_2'(w(\xi)) e^{i\eta(w(\xi))} + \overline{\phi_2(z)} e^{-i\eta(w(\xi))}]^{-1} \sum_{n=-\infty}^{\infty} E_n \xi^n \\
&= \frac{|1 - k^2 \xi^{-2}| \sum_{n=-\infty}^{\infty} E_n \xi^n}{[\phi_2'(w(\xi))(\xi - k^2 \xi^{-1}) + \overline{\phi_2(z)(\xi^{-1} - k^2 \xi\ )}]}.
\end{aligned}
\tag{2.25}
$$

Finally, we must impose the conditions:

$$
\operatorname{Im} h(w(\xi)) = 0, \qquad \operatorname{Re} h(w(\xi)) > 0, \tag{2.26}
$$

in order to maintain the physical meaning of the interface function β. This precondition arises from the fact that the interface function is proportional to the thickness of the interface layer. So the interface parameter has to be real and positive.

The conditions (2.26) allow us to write the B_n entirely in terms of the A_n and the known material constants. Note that, complex potential ϕ_2 inside the inhomogeneity can be uniquely determined by the complex potential ϕ_1 at the matrix to within an arbitrary constant. For existence of the neutral inhomogeneity, this arbitrary constant has to be chosen to eliminate the possible rigid body translation (caused by the choice of the coordinates). If the inhomogeneity has two mutually orthogonal axes of symmetry, these axes can be taken as the coordinate axes and hence $A_0 = B_0 = 0$. We can then construct the interface function from (2.25).

2.5 Interface parameter

In this section the interface parameter is derived for the elliptic inhomogeneity. Equating coefficients of ξ on both sides of the interface condition (2.23) we obtain the coefficients E_n as follows,

$$E_n^N = \sum_{i=n,n+2,n+4,..}^{N} A_i R^i k^{i-n} \binom{i}{\frac{i-n}{2}} - \delta \sum_{i=n,n+2,n+4,..}^{N} B_i R^i k^{i-n} \binom{i}{\frac{i-n}{2}} - (1-\delta) \sum_{i=n,n+2,n+4,..}^{N} \overline{B}_i R^i k^{i+n} \binom{i}{\frac{i-n}{2}},$$

for $n = -N,...,N.$

$E_n^N = 0, \quad \text{for } n < -N, \text{and n} > \text{N}.$

(2.27)

This general expression is proved using mathematical induction (Appendix).

Let $h_0 = \dfrac{\mu_2}{2\beta} > 0$ corresponding to the case where $\beta > 0$ is uniform. The conditions (2.26) are satisfied if the coefficients take the form,

$$\begin{aligned} E_n^N &= h_0 C_n^N, \\ E_{-n}^N &= h_0 \overline{C}_n^N, \qquad N \geq n \geq -N, \end{aligned} \tag{2.28}$$

Where C_n^N and $\overline{C}_n^N$ are defined as,

$$C_n^N = nB_n R^{n-1} + \sum_{i=n+2,n+4,..}^{N} B_i R^{i-1} k^{i-1} \binom{i}{\frac{i-n}{2}} - n\overline{B}_n R^{n-1} k^{2N} + \sum_{i=n+2,n+4,..}^{N} \overline{B}_i R^{i-1} k^{i+1} \binom{i}{\frac{i-n}{2}},$$

(2.29)

and

$$\overline{C}_n^N = n\overline{B}_n R^{n-1} + \sum_{i=n+2,n+4,..}^{N} \overline{B}_i R^{i-1} k^{i-1} \binom{i}{\frac{i-n}{2}} - nB_n R^{n-1} k^{2N} - \sum_{i=n+2,n+4,..}^{N} \overline{B}_i R^{i+1} k^{i+1} \binom{i}{\frac{i-n}{2}}.$$

(2.30)

Using the above correlation, we can find the interface parameter. Putting n=N,

$$
\begin{aligned}
E_N^N &= A_N R^N - \delta B_N R^N - (1-\delta)\overline{B}_N R^N k^{2N}, \\
E_{-N}^N &= A_N R^N K^{2N} - \delta B_N R^N K^{2N} - (1-\delta)\overline{B}_N R^N, \\
C_N^N &= N B_N R^{N-1} - N\overline{B}_N R^{N-1} k^{2N}, \\
\overline{C}_N^N &= N\overline{B}_N R^{N-1} - N B_N R^{N-1} k^{2N}.
\end{aligned} \tag{2.31}
$$

Hence from (2.20),

$$A_N R^N K^{2N} - \delta B_N R^N K^{2N} - (1-\delta)\overline{B}_N R^N = h_0 \left[N\overline{B}_N R^{N-1} - N B_N R^{N-1} k^{2N} \right] \tag{2.32}$$

and

$$A_N R^N - \delta B_N R^N - (1-\delta)\overline{B}_N R^N k^{2N} = h_0 \left[N B_N R^{N-1} - N\overline{B}_N R^{N-1} k^{2N} \right] \tag{2.33}$$

Multiplying (2.34) by k^{2N} and subtract from (2.33),

$$h_0 N \left[\overline{B}_N (1+k^{4N}) - 2B_N k^{2N} \right] + (1-\delta)\overline{B}_N R \ (1-k^{4N}) = 0. \tag{2.34}$$

Taking the Imaginary part on both sides,

$$h_0 N \operatorname{Im} B_N \left[-(1+k^{4N}) - 2k^{2N} \right] - (1-\delta) R \ (1-k^{4N}) \operatorname{Im} B_N = 0. \tag{2.35}$$

Hence, the general formula for interface parameter is found as,

$$h_0 = \frac{R(1-k^{2N})(\delta-1)}{N(k^{2N}+1)}. \tag{2.36}$$

Note that $h_0 > 0$ when $\mu_2 > \mu_1$. This requires that, for neutrality, the inhomogeneity has to be harder than the surrounding matrix material. So for the elliptic inhomogeneity mapped onto a circular region has the interface parameter defined as,

$$h(w(\xi)) = h_0\left|1 - \frac{k^2}{\xi^2}\right| > 0, \tag{2.37}$$

and this parameter is related to the interface function by the relation $h(z) = \frac{\mu_2}{2\beta(z)}$.

Writing, $a = R(1 + k^2)$ and $b = R(1 - k^2)$, the interface function is,

$$\beta(x, y) = \frac{R\mu_2}{2h_0 b\left[1 + \frac{a^2}{b^2}\left(\frac{1}{b^2} - \frac{1}{a^2}\right)y^2\right]^{1/2}}. \tag{2.38}$$

The equation (2.22) cannot hold for any interface curve if $\beta(z)$ is infinite everywhere. This means that there is no neutral elastic inhomogeneity if a conventional perfectly bonded interface is presumed. This result suggests that the concept of imperfect interface plays an indispensable role in the design of neutral elastic inhomogeneities.

2.6 Examples

The solution method developed in the previous sections will be illustrated with three representative examples. First the case of a linearly varying stress field (i.e. $N = 2$) is shown. The second case is for a quadratic stress field $(N = 3)$. The results for linearly varying stress field and the quadratic stress field are already available in the literature [20, 21] and so provided here to compare for accuracy. The final example is for the case where N=5, which is a new result and shows that the method can be implemented to any general non-uniform stress field.

2.6.1. Case N=2

This example is for the linear stress field in the matrix, i.e. $\phi_1 = \sum_{n=0}^{2} A_n z^n$. Suppose the stress field inside the inhomogeneity is characterized by $\phi_2 = \sum_{n=0}^{2} B_n z^n$, from equation (2.28):

$$
\begin{aligned}
E_{-1}^2 &= A_1 R k^2 - \delta B_1 R k^2 - (1-\delta)\overline{B}_1 R, \\
E_{-2}^2 &= A_2 R^2 k^4 - \delta B_2 R^2 k^4 - (1-\delta)\overline{B}_2 R^2, \\
E_0^2 &= (A_0 + 2A_2 R^2 k^2) - \delta(B_0 + 2B_2 R^2 k^2) - (1-\delta)(\overline{B}_0 + \overline{B}_2 R^2 k^2), \\
E_1^2 &= A_1 R - \delta B_1 R - (1-\delta)\overline{B}_1 R k^2, \\
E_2^2 &= A_2 R^2 - \delta B_2 R^2 - (1-\delta)\overline{B}_2 R^2 k^4, \\
E_n^2 &= 0, \qquad n < -2, n > 2,
\end{aligned}
$$

and,

$$
\begin{aligned}
C_{-1}^2 &= \overline{B}_1 - B_1 k^2, \\
C_{-2}^2 &= 2\overline{B}_2 R - 2B_2 R k^4, \\
C_0^2 &= 0, \\
C_1^2 &= B_1 - \overline{B}_1 k^2, \\
C_2^2 &= 2B_2 R - 2\overline{B}_2 R k^4, \\
C_n^2 &= 0, \qquad n < -2, n > 2,
\end{aligned}
$$

then from relation (2.29),

$$
\begin{aligned}
E_2^2 &= h_0 C_2^2, \\
E_{-2}^2 &= h_0 \overline{C}_2^2,
\end{aligned}
$$

we find:

$$
B_2\left[\delta R^2 k^2 - 2h_0 R k^4\right] + \overline{B}_2\left[(1-\delta)R^2 + h_0 2R\right] = A_2 R^2 k^4 \tag{2.39}
$$

$$
B_2\left[\delta R^2 + 2h_0 R\right] + \overline{B}_2\left[(1-\delta)R^2 k^4 - 2h_0 R k^4\right] = A_2 R^2 \tag{2.40}
$$

Multiplying (2.40) by k^4 and subtracting from (2.39) we find:

$$
B_2\left[-4h_0 R k^4\right] + \overline{B}_2(1-\delta)R^2(1-k^8) + \overline{B}_2 h_0 2R(1+k^8) = 0.
$$

Taking Imaginary part on both sides gives,

$$
\left[4h_0 R k^4\right] + (1-\delta)R^2(1-k^8) + h_0 2R(1+k^8) = 0,
$$

and hence,

$$h_0 = \frac{R(1-k^4)(\delta-1)}{2(k^4+1)}.$$

The constants defining the complex potentials in the inhomogeneity can be found by using equation (2.29). Since the constants A_0 and B_0 do not affect the stresses, by setting them to zero we obtain,

$$\begin{aligned} A_1 &= B_1 = 0, \\ \operatorname{Re} A_2 &= 0, \\ B_2 &= \frac{i \operatorname{Im} A_2}{(2\delta - 1)}. \end{aligned}$$

In other words, if the linear stress in the matrix is characterized by $\phi_1(z) = A_2 z^2$ then the elliptic inhomogeneity is neutral with the interior stress described by $\phi_2(z) = \left[\frac{i \operatorname{Im} A_2}{(2\delta - 1)}\right] z^2$.

2.6.2. Case N=3

In this section the interface parameter for the quadratic stress field in the matrix i.e. the stress field at the matrix is given by $\phi_1 = \sum_{n=0}^{3} A_n z^n$. Similar to the previous section suppose the stress field in the inhomogeneity is characterized by $\phi_2 = \sum_{n=0}^{5} B_n z^n$,

$$
\begin{aligned}
E_1^3 &= (A_1R + 3A_3R^3k^2) - \delta(B_1R + 3B_3R^3k^2) \\
&\qquad - (1-\delta)(\overline{B}_1Rk^2 + 3\overline{B}_3R^3k^4), \\
E_{-1}^3 &= (A_1Rk^2 + 3A_3R^3k^4) - \delta(B_1Rk^2 + 3B_3R^3k^4) \\
&\qquad - (1-\delta)(\overline{B}_1R + 3\overline{B}_3R^3k^2), \\
E_2^3 &= A_2R^2 - \delta B_2R^2 - (1-\delta)\overline{B}_2R^2k^4, \\
E_{-2}^3 &= A_2R^2k^4 - \delta B_2R^2k^4 - (1-\delta)\overline{B}_2R^2, \\
E_3^3 &= A_3R^3 - \delta B_3R^3 - (1-\delta)k^6\overline{B}_3R^3, \\
E_{-3}^3 &= A_3R^3k^6 - \delta B_3R^3k^6 - (1-\delta)\overline{B}_3R^3, \\
E_0^3 &= 0, \\
E_n^3 &= 0, \qquad n < -3, n > 3
\end{aligned}
$$

and the corresponding expressions for C_n^3 are,

$$
\begin{aligned}
C_1^3 &= B_1 + 3B_3R^2k^2 - \overline{B}_1k^2 - 3\overline{B}_3R^2k^4, \\
\overline{C}_1^3 &= \overline{B}_1 + 3\overline{B}_3R^2k^2 - B_1k^2 - 3B_3R^2k^4, \\
C_2^3 &= 2B_2R - 2\overline{B}_2Rk^4, \\
\overline{C}_2^3 &= 2\overline{B}_2R - 2B_2Rk^4, \\
C_3^3 &= 3B_3R^2 - 3\overline{B}_3R^2k^6, \\
\overline{C}_3^3 &= 3\overline{B}_3R^2 - 3B_3R^2k^6, \\
C_0^3 &= 0, \\
C_n^3 &= 0, \qquad n < -3, n > 3.
\end{aligned}
$$

Then from,

$$
\begin{aligned}
E_3^3 &= h_0C_3^3, \\
E_{-3}^3 &= h_0\overline{C}_3^3,
\end{aligned}
$$

we find,

$$B_3\left[\delta R^3k^6 - 3h_0R^2k^6\right] + \overline{B}_3\left[(1-\delta)R^3 + 3h_0R^2\right] = A_3R^3k^6, \tag{2.41}$$

and
$$B_3\left[\delta R^3 + 3h_0R^2\right] + \overline{B}_3\left[(1-\delta)R^3k^6 - 3h_0R^2k^6\right] = A_3R., \tag{2.42}$$

Multiplying (2.42) by k^6 and subtracting from (2.41) we obtain:

$$B_3\left[-6h_0R^2k^6\right] + \overline{B}_3\left[(1-\delta)R^3(1-k^{12}) + 3h_0R^2(1+k^{12})\right] = 0.$$

Hence taking the imaginary part on both sides we find the interface parameter,

$$h_0 = \frac{R(1-k^6)(\delta-1)}{3(k^6+1)}.$$

And the constants defining the complex potential in the inhomogeneity are found to be,

$$\begin{aligned}
&A_0 = B_0 = A_2 = B_2 = 0,\\
&\operatorname{Re}A_1 = \operatorname{Re}B_1 = \operatorname{Re}A_3 = \operatorname{Re}B_3 = 0,\\
&B_1 = \frac{i\operatorname{Im}A_1}{(2\delta-1)},\\
&B_3 = \frac{i\operatorname{Im}A_3}{(2\delta-1)}.
\end{aligned}$$

In other words, if the quadratic stress in the matrix is characterized by $\phi_1(z) = A_1z + A_3z^3$ $(\operatorname{Re}(A_1, A_3) = 0)$ then the elliptic inhomogeneity is neutral with interior stress described by $\phi_2(z)=\left[\frac{i\operatorname{Im}A_1}{(2\delta-1)}\right]z+\left[\frac{i\operatorname{Im}A_3}{(2\delta-1)}\right]z^3$. This result concurs with [20,21].

2.6.3. Case N=5

Suppose the stress field inside the inhomogeneity is characterized by $\phi_2=\sum_{n=0}^{5}B_n z^n$,

$$
\begin{aligned}
E_1^5 &= (A_1R+3A_3R^3k^2+10A_5R^5k^4)-\delta(B_1R+3B_3R^3k^2+10B_5R^5k^4)\\
&\qquad -(1-\delta)(\overline{B}_1Rk^2+3\overline{B}_3R^3k^4+10\overline{B}_5R^5k^6),\\
E_{-1}^5 &= (A_1Rk^2+3A_3R^3k^4+10A_5R^5k^6)-\delta(B_1Rk^2+3B_3R^3k^4+10B_5R^5k^6)\\
&\qquad -(1-\delta)(\overline{B}_1R+3\overline{B}_3R^3k^2+10\overline{B}_5R^5k^4),\\
E_2^5 &= (A_2R^2+4A_4R^4k^2)-\delta(B_2R^2+4B_4R^4k^2)-(1-\delta)(\overline{B}_2R^2k^4+4\overline{B}_4R^6k^4),\\
E_{-2}^5 &= (A_2R^2k^4+4A_4R^4k^6)-\delta(B_2R^2k^4+4B_4R^4k^6)-(1-\delta)(\overline{B}_2R^2+4\overline{B}_4R^4k^2),\\
E_3^5 &= (A_3+5A_5R^5k^2)-\delta(B_3R^3+5B_5R^5k^2)-(1-\delta)(k^6\overline{B}_3R^3+5\overline{B}_5R^5k^8),\\
E_{-3}^5 &= (A_3R^3k^6+5A_5R^5k^8)-\delta(B_3R^3k^6+5B_5R^5k^8)-(1-\delta)(\overline{B}_3R^3+5\overline{B}_5R^5k^2),\\
E_4^5 &= A_4R^4-\delta B_4R^4-(1-\delta)\overline{B}_4R^4k^8,\\
E_{-4}^5 &= A_4R^4k^8-\delta B_4R^4k^8-(1-\delta)\overline{B}_4R^4,\\
E_5^5 &= A_5R^5-\delta B_5R^5-(1-\delta)\overline{B}_5R^5k^{10},\\
E_{-5}^5 &= A_5R^5k^{10}-\delta B_5R^5k^{10}-(1-\delta)\overline{B}_5R^5,\\
E_0^5 &= 0,\\
E_n^5 &= 0, \qquad n<-5, n>5
\end{aligned}
$$

and the corresponding coefficients C_n^5 and $\overline{C}_n^5$ are,

$$
\begin{aligned}
C_1^5 &= B_1 + 3B_3R^2k^2 + 10B_5R^4k^4 - \overline{B}_1k^2 - 3\overline{B}_3R^2k^4 - 10\overline{B}_5R^4k^6, \\
\overline{C}_1^5 &= \overline{B}_1 + 3\overline{B}_3R^2k^2 + 10\overline{B}_5R^4k^4 - B_1k^2 - 3B_3R^2k^4 - 10B_5R^4k^6, \\
C_2^5 &= 2B_2R + 8B_4R^3k^2 - 2\overline{B}_2Rk^4 - 8\overline{B}_4R^3k^6, \\
\overline{C}_2^5 &= 2\overline{B}_2R + 8\overline{B}_4R^3k^2 - 2B_2Rk^4 - 8B_4R^3k^6, \\
C_3^5 &= 3B_3R^2 + 15B_5R^4k^2 - 3\overline{B}_3R^2k^6 - 15\overline{B}_5R^4k^8, \\
\overline{C}_3^5 &= 3\overline{B}_3R^2 + 15\overline{B}_5R^4k^2 - 3B_3R^2k^6 - 15B_5R^4k^8, \\
C_4^5 &= 4B_4R^3 - 4\overline{B}_4R^3k^8, \\
\overline{C}_4^5 &= 4\overline{B}_4R^3 - 4B_4R^3k^8, \\
C_5^5 &= 5B_5R^4 - 5\overline{B}_5R^{10}k^4, \\
\overline{C}_5^5 &= 5\overline{B}_5R^4 - 5B_5R^{10}k^4, \\
C_0^5 &= 0, \\
C_n^5 &= 0, \qquad n < -5, n > 5.
\end{aligned}
$$

then from,

$$
\begin{aligned}
E_5^5 &= h_0 C_5^5, \\
E_{-5}^5 &= h_0 \overline{C}_5^5,
\end{aligned}
$$

we find,

$$
B_5\left[\delta R^5k^{10} - 5h_0R^4k^{10}\right] + \overline{B}_5\left[(1-\delta)R^5 + 5h_0R^4\right] = A_5R^5k^{10},
$$

and
$$
B_5\left[\delta R^5 + 5h_0R^4\right] + \overline{B}_5\left[(1-\delta)R^5k^{10} - 5h_0R^4k^{10}\right] = A_5R^5.
$$

therefore it can be found that,

$$
B_5\left[-10h_0R^4k^{10}\right] + \overline{B}_5\left[(1-\delta)R^5(1-k^{20}) + 5h_0R^4(1+k^{20})\right] = 0.
$$

Hence taking the imaginary part on both sides we find the interface parameter as follows,

$$h_0 = \frac{R(1-k^{10})(\delta-1)}{5(k^{10}+1)}.$$

3 Formulation of the Plane Strain Problem

3.1 Preliminaries

In this chapter the general equations for a neutral inhomogeneity will be derived for plane strain with a non-uniform stress field in the matrix. Then specific examples are illustrated for a circular inhomogeneity.

We need two analytic functions in plane elasticity as compared to the single function required for antiplane elasticity. The basic governing equations of two dimensional elasticity are the Navier's equations [15],

$$(\lambda+\varsigma)\nabla_1(\nabla_1 . u)+\mu\nabla_1^2 u+\rho F=0, \tag{3.1}$$

and the stress-displacements relations

$$\sigma_{\alpha\beta} = \mu(u_{\alpha,\beta}+u_{\beta,\alpha})+\lambda\delta_{\alpha\beta}u_{\gamma,\gamma}. \tag{3.2}$$

Where, F is the vector of known body forces (for example, gravity or centrifugal force), ρ is the initial mass density of the body, u is the displacement vector; λ,ς

are Lamé constants and δ_{ij} is the Kronecker delta: $\delta_{ij} = 0, i \neq j; \delta_{ij} = 1, i = j$ with the repeated index summation convention adopted. The stress tensor is denoted by σ_{ij} and it is assumed the body contains no stress couples so that $\sigma_{ij} = \sigma_{ji}$.

Several methods of solutions of these equations are available. We use the method of complex variable developed by Muskhelishvili [25]. The method is based on the construction of the Airy stress function and its subsequent representation in terms of the functions of a complex variable. The details of the solution method can be found in England [24]. The general solution is the expression of the displacements and stresses given in terms of the two analytic functions $\phi(z)$ and $\psi(z)$,

$$2\mu(u_x + u_y) = \left[\kappa\phi(z) - z\overline{\phi'(z)} - \overline{\psi(z)}\right] \tag{3.3}$$

$$\sigma_{xx} + \sigma_{yy} = 2\left[\phi'(z) + \overline{\phi'(z)}\right] \tag{3.4}$$

$$\sigma_{xx} - i\sigma_{xy} = \phi'(z) + \overline{\phi'(z)} - [\bar{z}\phi''(z) + \psi'(z)]. \tag{3.5}$$

Here κ is defined as,

$$\begin{aligned} &\kappa = 3 - 4\nu, \quad \text{(for plane strain)}, \\ &\kappa = \frac{3 - 4\nu}{1 + \nu}, \quad \text{(for plane stress)}. \end{aligned} \tag{3.6}$$

and ν is Poisson's ratio. The theory of isotropic elasticity allows Poisson's ratios in the range from -1 to 1/2. Most engineering materials have ν between 0.0 and 0.5. Cork is close to 0.0, steels are around 0.3, and rubber is almost 0.5. Some materials, like polymer foams, have a negative Poisson's ratio.

Consequently, the boundary traction and displacements are given by,

$$2\mu(u_n + iu_t) = e^{-i\eta(z)}\left[\kappa\phi(z) - z\overline{\phi'(z)} - \overline{\psi(z)}\right], \tag{3.7}$$

$$\sigma_{nn} - i\sigma_{nt} = \phi'(z) + \overline{\phi'(z)} - e^{2i\eta(z)}[\bar{z}\phi''(z) + \psi'(z)]. \tag{3.8}$$

Note that σ_{xx} and σ_{xy} are the stress components in any point (x, y) inside the body and σ_{nn} and σ_{nt} are the respective stress components at the boundary.

3.2 Interface Parameters

Across the interface Γ the inhomogeneity is assumed to be bonded to the cut elastic body by an imperfect interface, in terms of normal and tangential interface parameters α_1 and α_2 as follows,

$$[\![\sigma_{nn} - i\sigma_{nt}]\!] = 0, \tag{3.9}$$

$$\begin{aligned} \sigma_{nn} &= \alpha_1(z)[\![u_n]\!], \\ \sigma_{nt} &= \alpha_2(z)[\![u_t]\!], \end{aligned} \tag{3.10}$$

where [[*]]=(*)$_1$-(*)$_2$ denotes the jump across Γ.

Note that $\alpha_1 = \alpha_2 = 0$ represent the traction free boundary conditions and $\alpha_1 = \alpha_2 \to \infty$ correspond to a perfectly bonded interface. Similar to anti-plane shear, the interface model can be realized in practice using an adhesive layer. In doing so, any one of the two interface parameters, α_1 or α_2 , or a combination of them, can be specified at will by controlling the thickness or the density of adhesive layer. However, if only a single adhesive material is used, the ratio α_1/α_2 is usually a material constant, independent of the thickness of the adhesive layer [4]. For example, the ratio α_1/α_2 can be assumed to be unity for a layer of distributed springs (see [5]), or be a certain constant larger than one for an elastic interphase layer [3].

A general stress field in the matrix can be expressed by a polynomial function of N^{th} order. In order to simply the analysis, the prescribed stress field in the matrix is approximated by,

$$\begin{aligned} \phi_1(z) &= Az'', \\ \psi_1(z) &= Bz''. \end{aligned} \tag{3.11}$$

For a neutral inhomogeneity it turns out to be,

$$\phi_2 = \phi_1 + A_0,\\ \psi_2 = \psi_1 + B_0. \tag{3.12}$$

Here A_0 and B_0 are arbitrary complex numbers. For the existence of the neutral inhomogeneity, A_0 and B_0 have to be chosen to eliminate the possible rigid body displacement between the inhomogeneity and the matrix. In particular if the inhomogeneity is geometrically symmetric about two mutually orthogonal axes and the latter is chosen as the coordinate axes, we have $A_0 = B_0 = 0$.

Since the stresses are continuous then,

$$(\sigma_{nn})_1 = (\sigma_{nn})_2 = \sigma_{nn}(z),\\ (\sigma_{nt})_1 = (\sigma_{nt})_2 = \sigma_{nt}(z), \quad z \in \Gamma. \tag{3.13}$$

therefore from (3.10) we obtain:

$$\begin{aligned}(\sigma_{nn} - i\sigma_{nt})_1 &= [\alpha_1(u_n)_1 - \alpha_1(u_n)_2] - i[\alpha_2(u_t)_1 - \alpha_2(u_t)_2]\\ &= \left(\frac{\alpha_1 + \alpha_1}{2}\right)\{[(u_n)_1 - i(u_t)_1] - [(u_n)_2 - i(u_t)_2]\}\\ &\qquad + \left(\frac{\alpha_1 - \alpha_1}{2}\right)\{[(u_n)_1 + i(u_t)_1] - [(u_n)_2 + i(u_t)_2]\}\\ &= \left(\frac{\alpha_1 + \alpha_1}{2}\right)[\![u_n - iu_t]\!] + \left(\frac{\alpha_1 - \alpha_1}{2}\right)[\![u_n + iu_t]\!]\end{aligned}$$

(3.14)

Now from (3.8) we find:

$$(\sigma_{nn} - i\sigma_{nt})_1 = \phi_1'(z) + \overline{\phi_1'(z)} - e^{2i\eta(z)}[\bar{z}\phi_1''(z) + \psi_1'(z)],$$
$$(\sigma_{nn} - i\sigma_{nt})_2 = \phi_2'(z) + \overline{\phi_2'(z)} - e^{2i\eta(z)}[\bar{z}\phi_2''(z) + \psi_2'(z)], \tag{3.15}$$

and from (3.7):

$$(u_n + iu_t)_1 = \frac{e^{-i\eta(z)}}{2}\left[\frac{\kappa_1}{\mu_1}\phi_1(z) - z\frac{\overline{\phi_1'(z)}}{\mu_1} - \frac{\overline{\psi_1(z)}}{\mu_1}\right],$$
$$(u_n + iu_t)_2 = \frac{e^{-i\eta(z)}}{2}\left[\frac{\kappa_2}{\mu_2}\phi_2(z) - z\frac{\overline{\phi_2'(z)}}{\mu_2} - \frac{\overline{\psi_2(z)}}{\mu_2}\right]. \tag{3.16}$$

Therefore the expression for $[\![u_n + iu_t]\!]$ is,

$$[\![u_n + iu_t]\!] = \frac{e^{-i\eta(z)}}{2}\left[\frac{1}{\mu_1}(\kappa_1\phi_1(z) - z\overline{\phi_1'(z)} - \overline{\psi_1(z)}) - \frac{1}{\mu_2}(\kappa_2\phi_2(z) - z\phi_2'(z) - \overline{\psi_2(z)})\right], \tag{3.17}$$

and the conjugate is,

$$[\![u_n - iu_t]\!] = \frac{e^{i\eta(z)}}{2}\left[\frac{1}{\mu_1}(\kappa_1\overline{\phi_1(z)} - \bar{z}\phi_1' - \psi_1(z)) - \frac{1}{\mu_2}(\kappa_2\phi_2'(z) - \bar{z}\phi_2(z) - \psi_2(z))\right]. \tag{3.18}$$

Using these in (3.14) we find,

$$(\sigma_{nn}-i\sigma_{nt})_1=\left(\frac{\alpha_1+\alpha_2}{4}\right)e^{i\eta(z)}\left[\frac{1}{\mu_1}(\kappa_1\overline{\phi_1(z)}-\bar{z}\phi_1'-\psi_1(z))-\frac{1}{\mu_2}(\kappa_2\phi_2'(z)-\bar{z}\phi_2(z)-\psi_2(z))\right]$$
$$+\left(\frac{\alpha_1-\alpha_2}{4}\right)e^{-i\eta(z)}\left[\frac{1}{\mu_1}(\kappa_1\phi_1(z)-z\overline{\phi_1'(z)}-\overline{\psi_1(z)})-\frac{1}{\mu_2}(\kappa_2\phi_2(z)-z\phi_2'(z)-\overline{\psi_2(z)})\right]. \tag{3.19}$$

This leads to,

$$\begin{aligned}&\phi_1'(z)+\overline{\phi_1'(z)}-e^{2i\eta(z)}[\bar{z}\phi_1''(z)+\psi_1'(z)]\\&=Ie^{i\eta(z)}\left[\frac{1}{\mu_1}(\kappa_1\overline{\phi_1(z)}-\bar{z}\phi_1'-\psi_1(z))-\frac{1}{\mu_2}(\kappa_2\phi_2'(z)-\bar{z}\phi_2(z)-\psi_2(z))\right]\\&+Je^{-i\eta(z)}\left[\frac{1}{\mu_1}(\kappa_1\phi_1(z)-z\overline{\phi_1'(z)}-\overline{\psi_1(z)})-\frac{1}{\mu_2}(\kappa_2\phi_2(z)-z\phi_2'(z)-\overline{\psi_2(z)})\right],\end{aligned} \tag{3.20}$$

where,

$$I=\left(\frac{\alpha_1+\alpha_2}{4}\right),\qquad J=\left(\frac{\alpha_1-\alpha_2}{4}\right). \tag{3.21}$$

Now, using the Laurent's series expansion we can write,

$$I=\frac{\sum_{n=-\infty}^{\infty}E_nz^n}{e^{i\eta(z)}\left[\frac{1}{\mu_1}(\kappa_1\overline{\phi_1(z)}-\bar{z}\phi_1'-\psi_1(z))-\frac{1}{\mu_2}(\kappa_2\phi_2'(z)-\bar{z}\phi_2(z)-\psi_2(z))\right]}, \tag{3.22}$$

$$J = \frac{\sum_{n=-\infty}^{\infty} F_n z^n}{e^{-i\eta(z)}\left[\frac{1}{\mu_1}(\kappa_1\phi_1(z) - z\overline{\phi_1'(z)} - \overline{\psi_1(z)}) - \frac{1}{\mu_2}(\kappa_2\phi_2(z) - z\overline{\phi_2'(z)} - \overline{\psi_2(z)})\right]}, \quad (3.23)$$

and finally we find,

$$\phi_1'(z) + \overline{\phi_1'(z)} - e^{2i\eta(z)}[\bar{z}\phi_1''(z) + \psi_1'(z)] = \sum_{n=-\infty}^{\infty} E_n z^n + \sum_{n=-\infty}^{\infty} F_n z^n. \quad (3.24)$$

3.3 Examples

We solve (3.24) for I and J for the simplest case of a circular inhomogeneity. For a circular inhomogeneity we can say that,

$$z = e^{in}R = e^{in}, \qquad \text{(taking radius } R = 1\text{).}$$

setting the boundary condition as,

$$\phi_1 = 0, \psi_1 = Bz^p, \quad (3.25)$$

equation (3.28) becomes,

$$\begin{aligned} &\phi_1'(z) + \overline{\phi_1'(z)} - e^{2i\eta(z)}[\bar{z}\phi_1''(z) + \psi_1'(z)] = \sum_{n=-\infty}^{\infty} E_n z^n + \sum_{n=-\infty}^{\infty} F_n z^n \\ &\Leftrightarrow -Bpz^{p+1} = E_{p+1}z^{p+1} + F_{p+1}z^{p+1} \\ &\Leftrightarrow -Bp = E_{p+1} + F_{p+1}. \end{aligned} \quad (3.26)$$

Now putting the boundary conditions in the expressions for $I(z)$ and $J(z)$ we find,

$$\begin{aligned} I &= \frac{E_{p+1} z^{p+1}}{e^{i\eta(z)} \left[\frac{1}{\mu_1}(-Bz^p) - \frac{1}{\mu_2}(-Bz^p) \right]} \\ &= \frac{E_{p+1} z^{p+1}}{Bz^{p+1}(-\chi)}, \end{aligned} \tag{3.27}$$

and,

$$\begin{aligned} J &= \frac{F_{p+1} z^{p+1}}{e^{i\eta(z)} \left[\frac{1}{\mu_1}(-Bz^p) - \frac{1}{\mu_2}(-Bz^p) \right]} \\ &= \frac{E_{p+1} z^{p+1}}{Bz^{-(p+1)}(-\chi)}. \end{aligned} \tag{3.28}$$

where χ is defined as,

$$\chi = \frac{1}{\mu_1} - \frac{1}{\mu_2}$$

Now imposing $\operatorname{Im}\alpha_1 = \operatorname{Im}\alpha_2 = 0$ and $\operatorname{Re}\alpha_1 > 0$ and $\operatorname{Re}\alpha_2 > 0$, we find $F_{p+1} = 0$, therefore finally using (3.26) we find,

$$I(z) = \frac{p}{\chi} = \frac{\alpha_1 + \alpha_2}{4},$$

$$\text{and } J = 0 = \frac{\alpha_1 - \alpha_2}{4}.$$

$$\textit{i.e. } \alpha_1(z) = \alpha_2(z) = \frac{2p}{\chi}.$$

This is the interface function for a circular inhomogeneity with uniformly distributed stress in the tangential direction.

4 Conclusions and Recommendations for Future Research

The mechanical behavior of fiber-reinforced composites is significantly affected by the nature of the bond between the fibers and the matrix material. In most analytical and numerical works it has been assumed that the bond is perfect and can be modeled by the continuity of tractions and displacements across a discrete interface. This assumption is, however, not suitable in the presence of a thin interfacial zone or an interphase. In reality interfaces are imperfect due to the presence of micro-cracks and voids. Such an interfacial zone may have been created deliberately by coating the fibers or it may have developed during the manufacturing process due to chemical reactions between the contacting fiber and matrix materials. This concept of imperfect interfaces allows us to design neutral inhomogeneities, which is not possible in an ideal perfectly bonded interface.

In this study, a series method for finding interface parameter for neutral inhomogeneities has been developed for a general non-uniform stress field in antiplane elasticity. This new numerical method was motivated largely by the fact that analytical methods were unable to predict the neutral interface parameters for a non-uniform stress field of order higher than linear. The current method has provided results for an elliptic inhomogeneity, which is widely used in composite materials. It is expected that the problem can also be explicitly solved for the epitrochoid, the hypotrochoid and other shapes of inhomogeneities. An extension of the method to plane elasticity has been developed which can provide results for circular inhomogeneities. It is interesting to note that the conditions applied in solving for the interface parameter, are only necessary and not sufficient. So further study in this area is expected in order to find other necessary conditions that will lead to interface parameters for neutral inhomogeneities of any shape in plane elasticity.

The imperfect interface model used in the current study is based on the assumption of the uniformity or the homogeneous nature of the interface. However, a more comprehensive micromechanical model that takes into account the spatially non-uniform properties of the interface layer (i.e. the point-wise variation of the properties of the interface along its entire length) could be an interesting subject for future study. Kouris D. et al. [26] studied the interaction between two inhomogeneities with perfect interface condition to determine how the interaction affects the local stress state of the fiber-matrix interface. In [26], it was found that the distance between the fibers has a significant effect on the elastic stress field. This observation suggests that using the single fiber solution may be quite inaccurate in predicting the local stress fields. In a dilute fiber reinforced composite material (fiber volume fraction less than 40%) the interactions among nearby fibers can be ignored and the single inhomogeneity problem can be used as a representative volume. For composite materials of higher volume fraction further study can be modeled incorporating the effects of neighboring fibers in order to find out the effect of this problem in non-uniform stress fields for imperfect interfaces. The interface parameter is assumed to be a function of the thickness and density of the interface layer. The parameter may be functions of other variables too, which can be found through experimentations. An experimental study can validate the theoretical results developed in this thesis. Consequently, the results will provide valuable information for the design of radically inhomogeneous functionally graded interfaces in composite materials.

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Glossary

Analytic function

A complex function is said to be analytic on a region if it is complex differentiable at every point in the region.

Antiplane deformations

In anti-plane shear of a cylindrical elastic body, the displacement is parallel to the generators of the cylinder.

Inhomogeneity

An elastic material object inserted into another body of a different elastic material.

Neutral inhomogeneity

An inhomogeneity that does not change the original stress field of the uncut body.

Interface

A bounding surface or zone where a discontinuity occurs, whether physical, mechanical, chemical etc.

Interface parameter

A parameter relating the displacements and traction across the interface.

Neumann boundary conditions

Partial differential equation boundary conditions which give the normal derivative on a surface.

Conformal mapping

A conformal mapping, is a transformation that preserves local angles.

Harmonic functions

Any real function with continuous second partial derivatives which satisfies Laplace's equation is called a harmonic function.

Appendix

Proof by mathematical induction of equation (2.28):

The following steps are involved in proving the general formula for E_n. First the general expression for E_n is derived. The formula is then proved to hold for $n = 1$. Then the recurrence relation for consecutive terms of the series is formulated. Using the recurrence formula the expression for E_{n+1} is produced, which is then validated by the extension of the general term.

The proposed general expression for E_n is,

$$E_n^N = \sum_{i=n,n+2,n+4,..}^{N} A_i R^i k^{i-n} \binom{i}{\frac{i-n}{2}} - \delta \sum_{i=n,n+2,n+4,..}^{N} B_i R^i k^{i-n} \binom{i}{\frac{i-n}{2}} - (1-\delta) \sum_{i=n,n+2,n+4,..}^{N} \overline{B}_i R^i k^{i+n} \binom{i}{\frac{i-n}{2}},$$

for $n = -N,\ldots,N$,

$$E_n^N = 0, \quad \text{for } n < -N, \text{and n} > \text{N}. \tag{1}$$

Putting n=1,

$$E_1^3 = \sum_{i=1,3}^{3} A_i R^i k^{i-1} \binom{i}{\frac{i-1}{2}} - \delta \sum_{i=1,3}^{3} B_i R^i k^{i-1} \binom{i}{\frac{i-1}{2}} - (1-\delta) \sum_{i=1,3}^{3} \overline{B}_i R^i k^{i+n} \binom{i}{\frac{i-1}{2}}$$
$$= (A_1 R + 3A_3 R^3 k^2) - \delta(B_1 R + 3B_3 R^3 k^2) - (1-\delta)(\overline{B}_1 R + 3\overline{B}_3 R^3 k^4).$$

The recurrence formula is given by,

$$E_n^N + kE_n^N = A_n R^n + \sum_{i=n+2,n+4,..}^{N} A_i R^i k^{i-n} \binom{i+1}{\frac{i-n}{2}} - \delta \left[B_n R^n + \sum_{i=n+2,n+4,..}^{N} B_i R^i k^{i-n} \binom{i+1}{\frac{i-n}{2}} \right]$$
$$- (1-\delta) \left[\overline{B}_i R^n k^{2n} + \sum_{i=n+2,n+4,..}^{N} \overline{B}_i R^i k^{i+n} \left[\binom{i}{\frac{i-n}{2}} + k^4 \binom{i}{\frac{i-n-2}{2}} \right] \right], \quad \text{for } n = -N, ..., N. \tag{2}$$

Using equation (1) and putting $n = n + 2$ we find the expected expression for E_{n+2}^N,

$$E_{n+2}^N = \sum_{i=n+2,n+4,n+6..}^{N} A_i R^i k^{i-n-2} \binom{i}{\frac{i-n-2}{2}} - \delta \sum_{i=n+2,n+4,..}^{N} B_i R^i k^{i-n-2} \binom{i}{\frac{i-n-2}{2}} - (1-\delta) \sum_{i=n+2,n+4,..}^{N} \overline{B}_i R^i k^{i+n+2} \binom{i}{\frac{i-n-2}{2}}.$$

(3)

Now we put (1) in recurrence relation (2) and compare to see if expression for E_{n+2}^N is as expected in (3),

$$\sum_{i=n,n+2,n+4,..}^{N} A_i R^i k^{i-n} \binom{i}{\frac{i-n}{2}} - \delta \sum_{i=n,n+2,n+4,..}^{N} B_i R^i k^{i-n} \binom{i}{\frac{i-n}{2}} - (1-\delta) \sum_{i=n,n+2,n+4,..}^{N} \overline{B}_i R^i k^{i+n} \binom{i}{\frac{i-n}{2}} + k^2 E_{n+2}^N$$
$$= A_n R^n + \sum_{i=n+2,n+4,..}^{N} A_i R^i k^{i-n} \binom{i+1}{\frac{i-n+2}{2}} - \delta \left[B_n R^n + \sum_{i=n+2,n+4,..}^{N} B_i R^i k^{i-n} \binom{i+1}{\frac{i-n+2}{2}} \right]$$
$$- (1-\delta) \left[\overline{B}_i R^n k^{2n} + \sum_{i=n+2,n+4,..}^{N} \overline{B}_i R^i k^{i+n} \left[\binom{i}{\frac{i-n+2}{2}} + k^4 \binom{i}{\frac{i-n}{2}} \right] \right].$$

So,

$$A_nR^n+\sum_{i=n+2,n+4,..}^{N}A_iR^ik^{i-n}\binom{i}{\frac{i-n}{2}}-\delta\left[B_nR^n+\sum_{i=n+2,n+4,..}^{N}B_iR^ik^{i-n}\binom{i}{\frac{i-n}{2}}\right]$$
$$-(1-\delta)\left[\overline{B}_nR^nk^{2n}+\sum_{i=n+2,n+4,..}^{N}\overline{B}_iR^ik^{i+n}\left[\binom{i}{\frac{i-n}{2}}\right]\right]+k^2E_{n+2}^N$$
$$=A_nR^n+\sum_{i=n+2,n+4,..}^{N}A_iR^ik^{i-n}\left[\binom{i}{\frac{i-n}{2}}+\binom{i}{\frac{i-n-2}{2}}\right]-\delta\left[B_nR^n+\sum_{i=n+2,n+4,..}^{N}B_iR^ik^{i-n}\left[\binom{i-2}{\frac{i-n}{2}}+\binom{i-2}{\frac{i-n-2}{2}}\right]\right]$$
$$-(1-\delta)\left[\overline{B}_nR^nk^{2n}+\sum_{i=n+2,n+4,..}^{N}\overline{B}_iR^ik^{i+n}\left[\binom{i}{\frac{i-n}{2}}+k^4\binom{i}{\frac{i-n-2}{2}}\right]\right].$$

Using the relation $\binom{b+1}{a+1}=\binom{b}{a+1}+\binom{b}{a}$,

$$A_nR^n+\sum_{i=n+2,n+4,..}^{N}A_iR^ik^{i-n}\binom{i}{\frac{i-n}{2}}-\delta\left[B_nR^n+\sum_{i=n+2,n+4,..}^{N}B_iR^ik^{i-n}\binom{i}{\frac{i-n}{2}}\right]$$
$$-(1-\delta)\left[\overline{B}_nR^nk^{2n}+\sum_{i=n+2,n+4,..}^{N}\overline{B}_iR^ik^{i+n}\left[\binom{i}{\frac{i-n}{2}}\right]\right]+k^2E_{n+2}^N$$
$$=A_nR^n+\sum_{i=n+2,n+4,..}^{N}A_iR^ik^{i-n}\binom{i}{\frac{i-n}{2}}+\sum_{i=n+2,n+4,..}^{N}A_iR^ik^{i-n}\binom{i}{\frac{i-n-2}{2}}$$
$$-\delta\left[B_nR^n+\sum_{i=n+2,n+4,..}^{N}B_iR^ik^{i-n}\binom{i}{\frac{i-n}{2}}+\sum_{i=n+2,n+4,..}^{N}B_iR^ik^{i-n}\binom{i}{\frac{i-n-2}{2}}\right]$$
$$-(1-\delta)\left[\overline{B}_nR^nk^{2n}+\sum_{i=n+2,n+4,..}^{N}\overline{B}_iR^ik^{i+n}\binom{i}{\frac{i-n}{2}}+\sum_{i=n+2,n+4,..}^{N}\overline{B}_iR^ik^{i+n+4}\binom{i}{\frac{i-n-2}{2}}\right].$$

Collecting the terms on both sides,

$$k^2E_{n+2}^N=\sum_{i=n+2,n+4,n+6..}^{N}A_iR^ik^{i-n}\binom{i}{\frac{i-n-2}{2}}-\delta\sum_{i=n+2,n+4,..}^{N}B_iR^ik^{i-n}\binom{i}{\frac{i-n-2}{2}}-(1-\delta)\sum_{i=n+2,n+4,..}^{N}\overline{B}_iR^ik^{i+n+4}\binom{i}{\frac{i-n-2}{2}}.$$

Finally dividing both sides by k^2,

$$E_{n+2}^{N} = \sum_{i=n+2,n+4,n+6..}^{N} A_i R^i k^{i-n-2} \binom{i}{\frac{i-n-2}{2}} - \delta \sum_{i=n+2,n+4,..}^{N} B_i R^i k^{i-n-2} \binom{i}{\frac{i-n-2}{2}} - (1-\delta) \sum_{i=n+2,n+4,..}^{N} \overline{B}_i R^i k^{i+n+2} \binom{i}{\frac{i-n-2}{2}}.$$

This is the exact expression as predicted in (3). This concludes the proof of the general expression for E_n^N by method if mathematical induction.

Using similar methodology the general expression for C_n^N and $\overline{C}_n^N$ can be proved to be,

$$C_n^N = nB_n R^{n-1} + \sum_{i=n+2,n+4,..}^{N} B_i R^{i-1} k^{i-1} \binom{i}{\frac{i-n}{2}} - n\overline{B}_n R^{n-1} k^{2N} + \sum_{i=n+2,n+4,..}^{N} \overline{B}_i R^{i-1} k^{i+1} \binom{i}{\frac{i-n}{2}},$$

$$\overline{C}_n^N = n\overline{B}_n R^{n-1} + \sum_{i=n+2,n+4,..}^{N} \overline{B}_i R^{i-1} k^{i-1} \binom{i}{\frac{i-n}{2}} - nB_n R^{n-1} k^{2N} - \sum_{i=n+2,n+4,..}^{N} \overline{B}_i R^{i+1} k^{i+1} \binom{i}{\frac{i-n}{2}}.$$