The results for the current problem being investigated show that two equilibrium configurations in the form of helices were obtained for the same specified initial conditions of force, moment and pitch angle α. This ability of the segmental technique to analyse multiple equilibrium configurations for a given problem by varying initial conditions has been demonstrated by Tam [13] for planar elastica problems.

4.2 SHOOTING PROBLEM

In the previous sections the segmental technique was used to generate the deformed shaped of the rod as well as the forces and moments which act along the rod. The results obtained were compared to the exact analytical solution to test the validity of the solution delivered by the segmental technique. The solutions thus obtained showed excellent agreement with the analytical solutions provided a sufficient number of segments were used. These results, however, were based on a complete knowledge of the conditions at the start of the rod. There were no boundary conditions at the end of the rod which needed to be satisfied. This section presents the problem as one which has a single unknown at the start of the rod to demonstrate how the shooting procedure, presented in Section 3.4, is used to solve such problems.

Consider the previous problem where the rod is to be twisted into a helical spiral by a specified axial moment and force. This time, however, the pitch angle at the start of the rod is not specified but rather the deformed helix is specified to have a known axial length z_0. The major difference between this problem and the problem in the previous section is that previously the initial pitch angle α was specified *a priori*. This time, the pitch angle at the start of the rod is not known *a priori*, but must be chosen such that the resulting helix meets the specified boundary condition, namely $z(L) = z_0$. This can be considered as a problem with one unknown parameter since in this case the conditions (forces, moments and geometry) at the start of the rod are a function of the pitch angle only (with M_z and F_z specified).

It should be noted here that an exact analytical solution for this problem is available to calculate the appropriate pitch angle α (equations (4.2) and (4.5)). To demonstrate the shooting procedure, however, it will be assumed that such analytical information is not available, as would be the case in most practical problems. In what follows, however, it will be assumed that the deformed configuration of the rod is in all cases a helix with radius R and pitch angle α. The forces and moments at the start of the rod, which are functions of α only (with M_z and F_z specified), will be chosen such that the resulting shape is a helix in accordance with (2.81). In general, this *a priori* knowledge will not be available. As a result there would be more unknowns at the start of the rod since the helical deformed shaped would not be known beforehand and the relationships between the moments, forces, radius R and α would not be available. As well, the boundary conditions at the end of the segment would become correspondingly more complicated. However, for the purpose of demonstrating the shooting procedure, it will be assumed that such analytical information is available so that the problem can be presented as one with a single unknown variable (α) at the start of the rod and with a single corresponding known boundary condition at the end of the rod ($z(L) = z_0$). The only exception to this assumption is that the solution relating the pitch angle α and the axial length z is not known *a priori*.

Consider the previous problem where the same rod is to be twisted into a helix by an axial moment of 200 Nm and an axial tension of 100 N. As well the axial length of the deformed helix is specified to be 0.6m. To start the segmental procedure an initial estimate of the unknown pitch angle must be made. This initial guess is then used to calculate the conditions at the start of the rod as before. The segmental solution then solves the entire rod exactly as before. The axial length delivered by the segmental solution will be a function (not analytical) of the initial estimate of α. To illustrate this point, Figure 4.12 shows the axial length of the resulting helix as a function of the initial pitch angle α for the segmental technique using 500 segments.

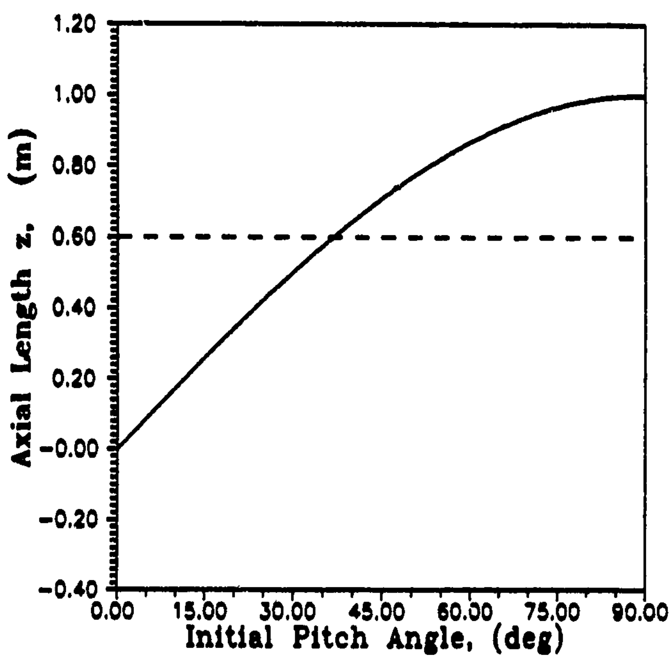

Figure 4.12: Axial Length z as a Function of Initial Pitch Angle α

This figure was generated by inputing initial values of α from 0 to 90° in increments of 0.5° into the segmental technique and calculating the resultant axial length. At this point no shooting has been involved. The purpose of this procedure is to demonstrate the functional relationship between axial length z and initial pitch angle α. From this figure it can be seen that the axial length is 0.6m when the initial pitch angle is approximately 37°. Also, since α can only take on values from 0 to 90° from physical considerations, this figure also shows that this is the only pitch angle which satisfies the required boundary condition. This value will not be known *a priori*, however, and the initial estimate may or may not be close to this value. For the purpose of illustration, assume that the initial estimate of the pitch angle is 10°. The segmental technique returns an axial length of 0.1664 m. This does not agree with the specified boundary condition of 0.6m. Therefore the segmental technique calculates an approximate derivative dz/dα in order to use the Newton Raphson false position method. This approximate derivative requires another initial guess close to the initial α to perform the backward substitution process. Hence two estimates of the initial pitch angle are required to start the technique. In general, if there are N unknowns at the start of the rod then 2N initial estimates of the unknowns are required. To determine the partial derivatives, N passes of the segmental approach are required. Thus each iteration requires N+1 passes of the segmental approach. Based on the axial length returned by the initial estimate of 10° and the approximated derivative at this point, the Newton Raphson method returns an improved estimate of 35.06°. This new estimate of α then returns an axial length of 0.5726 m. This procedure is repeated until the axial length returned by the segmental technique converges to the required boundary condition within some set tolerance. Table 4.6 shows the convergence behavior for this problem.

Table 4.6: Convergence Behavior of One Variable Shooting Pr[illegible]m

Iteration #	Pitch Angle α, (°)	Axial Length z, (m)	Convergence (z_0-z)
1	10	0.16646864915426	4.335314×10^{-1}
2	35.062553337435	0.57264143[illegible]	2.735857×10^{-2}
3	36.750690907168	0.596978917[illegible]	3.021082×10^{-3}
4	36.960244297435	0.59996[illegible]	3.571172×10^{-5}
5	36.962751025467	0.59999[illegible]	4.855431×10^{-8}
6	36.962754438301	0.59999999999922	7.842615×10^{-13}

Table 4.6 shows that segmental shooting technique converges quickly and monotonically for this problem. This is to be expected since Figure 4.12 shows that the function is well behaved. In some problems, however, these functions may be quite complex with steep gradients near the points of interest [13]. In such cases, especially for problems with two or more unknowns at the start of the rod, initial estimates close to the actual values are required. Relaxation factors may also be used to improve convergence.

CHAPTER

FIVE

CONCLUSIONS

5.1 SUMMARY

The objective of the present study was to develop a numerical procedure to analyse problems involving large three dimensional deflections of thin flexible rods. The equations that describe such deformations are highly nonlinear and difficult to solve analytically. This difficulty is further compounded by the wide variety of boundary conditions which may be involved. As a result, numerical procedures are usually required to obtain approximate solutions to such problems. This thesis presents a new approach to this problem.

The numerical procedure developed in the foregoing is based on the *segmental shooting technique*, a numerical method previously found to be successful for planar elastica problems. The major advantage of this technique is that the nonlinearity of the problem is avoided by dividing the rod under consideration into a large number of shorter segments, each of which undergoes only small deformations relative to neighboring segments. The equations describing the deformations of these individual segments can therefore be linearized and solved by standard techniques. The individual segments are assembled together, maintaining compatibility and continuity, to form the complete rod. In this way, the nonlinearity of the problem is avoided by solving a sequence of linear problems. In doing so, the original boundary value problem is converted into an initial value problem. A Newton Raphson false position technique is then used to ensure that the appropriate boundary conditions are satisfied.

The segmental shooting technique has several advantages over other techniques. Since each segment of the rod is analyzed separately, large computer memory space is not required as with the finite element method. As well, incremental loading is not required to handle large deflections, resulting in a much faster solution. For example, the solutions obtained in the previous chapter using 500 segments were all done on an IBM 80486/487 personal computer in under 5 seconds each. The segmental technique, as was noted previously [13], is well suited to finding multiple equilibrium configurations when they exist.

The major drawback with this technique is that the false position iterative procedure which allows the boundary conditions to be satisfied is susceptible to numerical problems due to the numerical differentiation involved in estimating partial derivatives of complex functions. As well, initial estimates of the unknown parameters close to the actual values are usually required to ensure convergence of the solution. It has been previous experience that when three boundary conditions (the maximum number for planar problems) were required for a given problem the convergence behavior of the solution became much slower than for problems with one or two boundary conditions. Three dimensional problems could easily have more than three boundary conditions which need to be satisfied. The convergence behavior for such problems will likely become worse. This requires further investigation.

The most significant difference between planar and three dimensional rod problems is that the twist along the rod couples the two transverse bending components. In planar problems no twisting occurs and one equation is sufficient to describe the deformation. For three dimensional problems, there are three differential equations, coupled by the twist, which describe the deformation of a rod segment. For the special case where the rod is initially straight and of circular cross section (the case investigated in this thesis) the bending components may be uncoupled from the twist resulting in two differential equations to be solved for the transverse deflections. The third equation for the twist can then be solved separately.

The computer program developed to implement this technique was used to solve the problem of an initially straight rod with circular cross section twisted into a helical spiral by axially applied end moments and forces. This problem has a well known analytical solution and served to verify the results obtained using the segmental technique. The resulting geometry, equilibrium results, a first integral of the equilibrium equations and a condition imposed on the solution from constitutive assumptions were all used to determine the validity of the numerical solution. In all cases it was found that there was close agreement between the analytical and segmental solutions provided that a sufficient number of segments were used. The possibility of obtaining multiple helices for specified applied loadings was noted and these multiple equilibrium configurations were found easily using the segmental technique.

5.2 FUTURE WORK

The present work considers only the special case of a rod with circular cross section which is initially straight, inextensible, and has a quadratic strain energy function. To allow this procedure to be applied to a wider variety of practical engineering problems the procedure should be modified to consider rods with different cross sections as well as rods with initial curvatures and pre-twisted rods. Extensibility should also be considered since this may be an important consideration in some problems. Major modifications to the technique may, in the future, allow it to be applied to materials with different constitutive relationships as well as to a dynamical analysis of rod structures.

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APPENDIX A

Dimensional Analysis of Equilibrium Equation

A dimensional analysis was performed on the equilibrium equation given in (3.18). This was done to ensure that the proper terms are retained when the assumption that only small deflections occur over a given segment was applied. The relevant terms were expressed relative to some reference values and ratios of small to large values (order ε) were obtained. The analysis was performed to determine which terms were more than order one in ε and these were subsequently neglected.

GOVERNING EQUATIONS

The pertinent equations from Chapter 3 are repeated here for convenience.

Equilibrium

$$\mathbf{F}' + \mathbf{b} = \mathbf{0} \tag{A.1}$$

$$\mathbf{M}' = \mathbf{F} \times \mathbf{t} \tag{A.2}$$

Constitutive Relationship

$$\mathbf{M} = EI\,\mathbf{t} \times \mathbf{t}' + GJ\kappa\,\mathbf{t} \tag{A.3}$$

Differentiating (A.3) with respect to arc length gives

$$\mathbf{M}' = EI\,\mathbf{t} \times \mathbf{t}'' + GJ\kappa'\,\mathbf{t} + GJ\kappa\,\mathbf{t}'. \tag{A.4}$$

Now define

$$\mathbf{f} = \frac{\mathbf{F}}{F_{ref}}, \quad \mathbf{m} = \frac{\mathbf{M}}{M_{ref}}, \quad \rho = \frac{s}{L}, \quad K = \frac{\kappa}{K_{ref}}. \tag{A.5}$$

Here F_{ref}, M_{ref} and K_{ref} are reference magnitudes of the force, moment and twist per unit length respectively. These will be used later to determine which quantities are "small". s is the arc length parameter and L is the length of a segment. ρ is therefore the dimensionless arc length parameter. Also, from equation (3.17),

$$\kappa = \Phi_0 + k \tag{A.6}$$

so

$$K = \frac{\Phi_0}{K_{ref}} + \frac{k_{ref}}{K_{ref}}\,\Omega \qquad \Omega = \frac{k}{k_{ref}}. \tag{A.7}$$

As well from (3.7),

$$\underline{t} = \underline{t}^0 + \underline{u}', \tag{A.8}$$

so that

$$\underline{t}' = \underline{u}'', \qquad \underline{t}'' = \underline{u}'''. \tag{A.9}$$

Also define

$$\underline{U} = \frac{\underline{u}}{U_{ref}}, \tag{A.10}$$

where U_{ref} is a reference deflection, then

$$\underline{u}' = \frac{U_{ref}}{L}\dot{\underline{U}}, \qquad \underline{u}'' = \frac{U_{ref}}{L^2}\ddot{\underline{U}}, \qquad \underline{u}''' = \frac{U_{ref}}{L^3}\dddot{\underline{U}}, \tag{A.11}$$

where the dot notation indicates differentiation with respect to ρ, the dimensionless arc length defined in (A.5).

Using the parameters defined above, (A.2) becomes

$$\frac{d\underline{M}}{ds} = \underline{F}\times\underline{t}^0 + \underline{F}\times\underline{u}', \tag{A.12}$$

or

$$\frac{M_{ref}}{L}\frac{d\underline{m}}{d\rho} = F_{ref}\underline{f}\times\underline{t}^0 + F_{ref}\underline{f}\times\frac{U_{ref}}{L}\dot{\underline{U}},$$

$$\frac{M_{ref}}{L}\dot{\underline{m}} = F_{ref}\underline{f}\times\underline{t}^0 + \frac{F_{ref}U_{ref}}{L}\underline{f}\times\dot{\underline{U}}. \tag{A.13}$$

Equation (A.4) becomes

$$\frac{M_{ref}}{L}\underline{\dot{m}} - EI\left(\underline{t}^0 + \underline{u}'\right)\times\underline{u}''' + GJk'\left(\underline{t}^0 + \underline{u}'\right) + GJ\left(\Phi_0 + k\right)\underline{u}''$$

$$- EI\left(\underline{t}^0 + \frac{U_{ref}}{L}\underline{\dot{U}}\right)\times\frac{U_{ref}}{L^3}\underline{\dddot{U}} + GJ\frac{k_{ref}}{L}\dot{\Omega}\left(\underline{t}^0 + \frac{U_{ref}}{L}\underline{\dot{U}}\right)$$

$$+ GJ\left(\Phi_0 + \Omega k_{ref}\right)\frac{U_{ref}}{L^2}\underline{\ddot{U}}$$

$$- \frac{EI}{L^2}\frac{U_{ref}}{L}\underline{t}^0\times\underline{\dddot{U}} + \frac{EI}{L^2}\left(\frac{U_{ref}}{L}\right)^2\underline{\dot{U}}\times\underline{\dddot{U}} + \frac{GJk_{ref}}{L}\dot{\Omega}\,\underline{t}^0$$

$$+ \frac{GJk_{ref}}{L}\frac{U_{ref}}{L}\dot{\Omega}\underline{\dot{U}} + \frac{GJ\Phi_0}{L}\frac{U_{ref}}{L}\underline{\ddot{U}} + \frac{GJk_{ref}}{L}\frac{U_{ref}}{L}\Omega\underline{\ddot{U}}. \qquad (A.14)$$

Equating (A.13) and (A.14) gives

$$F_{ref}\underline{f}\times\underline{t}^0 + \frac{F_{ref}U_{ref}}{L}\underline{f}\times\underline{\dot{U}} =$$

$$\frac{EI}{L^2}\frac{U_{ref}}{L}\underline{t}^0\times\underline{\dddot{U}} + \frac{EI}{L^2}\left(\frac{U_{ref}}{L}\right)^2\underline{\dot{U}}\times\underline{\dddot{U}} + \frac{GJk_{ref}}{L}\dot{\Omega}\,\underline{t}^0$$

$$+ \frac{GJk_{ref}}{L}\frac{U_{ref}}{L}\dot{\Omega}\underline{\dot{U}} + \frac{GJ\Phi_0}{L}\frac{U_{ref}}{L}\underline{\ddot{U}} + \frac{GJk_{ref}}{L}\frac{U_{ref}}{L}\Omega\underline{\ddot{U}},$$

and dividing through by F_{ref} gives

$$\underline{f}\times\underline{t}^0 + \frac{U_{ref}}{L}\underline{f}\times\underline{\dot{U}} =$$

$$\frac{EI}{F_{ref}L^2}\frac{U_{ref}}{L}\underline{t}^0\times\underline{\ddot{U}} + \frac{EI}{F_{ref}L^2}\left(\frac{U_{ref}}{L}\right)^2\underline{\dot{U}}\times\underline{\ddot{U}} + \frac{GJ}{F_{ref}L^2}k_{ref}L\dot{\Omega}\,\underline{t}^0 \tag{A.15}$$

$$+\frac{GJ}{F_{ref}L^2}k_{ref}L\frac{U_{ref}}{L}\dot{\Omega}\underline{U} + \frac{GJ}{F_{ref}L^2}\Phi_0 L\frac{U_{ref}}{L}\underline{\ddot{U}} + \frac{GJ}{F_{ref}L^2}k_{ref}L\frac{U_{ref}}{L}\Omega\underline{\ddot{U}}.$$

Consider equation (A.3) evaluated at s=0 (the start of the segment),

$$\underline{M}(0) = EI\,\underline{t}\times\underline{t}' + GJ\kappa\,\underline{t}. \tag{A.16}$$

At $s=0, \underline{t} = \underline{t}^0, \underline{t}' = \underline{u}''$ and $\kappa = \Phi_0$ so (A.16) becomes

$$\begin{aligned}\underline{M}(0) &= EI\,\underline{t}^0\times\underline{u}'' + GJ\Phi_0\underline{t}^0\\ &= EI\,\underline{t}^0\times\frac{U_{ref}}{L^2}\underline{\ddot{U}} + GJ\Phi_0\underline{t}^0\\ &= \frac{EI\,U_{ref}}{L^2}\underline{t}^0\times\underline{\ddot{U}} + GJ\Phi_0\underline{t}^0.\end{aligned} \tag{A.17}$$

The magnitude of $\underline{M}(0)$ is then

$$|\underline{M}(0)| = \sqrt{\left(\frac{EI\,U_{ref}}{L^2}\right) + (GJ\Phi_0)^2}\,. \tag{A.18}$$

Defining the quantities

$$M_{ref_1} = \frac{EI\,U_{ref}}{L^2}, \tag{A.19}$$

$$M_{ref_2} = GJ\Phi_0, \tag{A.20}$$

equation (A.15) becomes

$$\begin{aligned} \underline{f}\times\underline{t}^0 + \frac{U_{ref}}{L}\underline{f}\times\dot{\underline{U}} = \\ \frac{M_{ref_1}}{F_{ref}L}\underline{t}^0\times\ddot{\underline{U}} + \frac{M_{ref_1}}{F_{ref}L}\frac{U_{ref}}{L}\dot{\underline{U}}\times\ddot{\underline{U}} + \frac{M_{ref_2}}{F_{ref}L}\frac{k_{ref}}{\Phi_0}\dot{\Omega}\,\underline{t}^0 \\ + \frac{M_{ref_2}}{F_{ref}L}\frac{k_{ref}}{\Phi_0}\frac{U_{ref}}{L}\dot{\Omega}\dot{\underline{U}} + \frac{M_{ref_2}}{F_{ref}L}\frac{U_{ref}}{L}\ddot{\underline{U}} + \frac{M_{ref_2}}{F_{ref}L}\frac{k_{ref}}{\Phi_0}\frac{U_{ref}}{L}\Omega\ddot{\underline{U}}. \end{aligned} \tag{A.21}$$

Now consider equation (A.2) evaluated at s=0 so that

$$\underline{M}'(0) = \underline{F}\times\underline{t}^0$$

or

$$\frac{M_{ref}}{L}\dot{\underline{m}} = F_{ref}\underline{f}\times\underline{t}^0. \tag{A.22}$$

The magnitude is then

$$\left|\frac{dM}{ds}\right| = \left|\frac{M_{ref}}{L}\dot{\underline{m}}\right| = \left|F_{ref}\underline{f}\times\underline{t}^0\right| \tag{A.23}$$

and therefore,

$$\frac{M_{ref}}{L}|\dot{\underline{m}}| = F_{ref}|\underline{f}\times\underline{t}^0|. \tag{A.24}$$

$\underline{t}^0$ is a unit vector specifying the tangent direction along the rod. $\underline{f}$ is a unit vector in the direction of the internal contact force at arc length s=0. $\underline{f}$ can have any orientation (ie: independent of $\underline{t}^0$). The orientation of $\underline{f}$, which determines the magnitude of $\underline{f}\times\underline{t}^0$, will determine the magnitude of $(\underline{m})'$. Therefore we can say that

$$M_{ref} = F_{ref} L. \tag{A.25}$$

Using (A.25) and defining

$$\alpha = \frac{M_{ref_1}}{M_{ref}} = \frac{M_{ref_1}}{\sqrt{(M_{ref_1})^2 + (M_{ref_2})^2}}, \tag{A.26}$$

$$\beta = \frac{M_{ref_2}}{M_{ref}} = \frac{M_{ref_2}}{\sqrt{(M_{ref_1})^2 + (M_{ref_2})^2}}, \tag{A.27}$$

equation (A.21) becomes

$$\begin{aligned} \underline{f} \times \underline{t}^0 + \frac{U_{ref}}{L} \underline{f} \times \underline{\dot{U}} = \\ \alpha\, \underline{t}^0 \times \underline{\ddot{U}} + \alpha \frac{U_{ref}}{L} \underline{\dot{U}} \times \underline{\ddot{U}} + \beta \frac{k_{ref}}{\Phi_0} \dot{\Omega}\, \underline{t}^0 \\ + \beta \frac{k_{ref}}{\Phi_0} \frac{U_{ref}}{L} \dot{\Omega} \underline{\dot{U}} + \beta \frac{U_{ref}}{L} \underline{\ddot{U}} + \beta \frac{k_{ref}}{\Phi_0} \frac{U_{ref}}{L} \Omega \underline{\ddot{U}}. \end{aligned} \tag{A.28}$$

The terms (U_{ref}/L) and (k_{ref}/Φ_0) are assumed to be small (of order ε where $\varepsilon \ll 1$) when the segment being solved is sufficiently short. The other terms in (A.28) are approximately of order 1. Any terms with two or more order ε terms will be ignored as being higher order terms. The terms remaining are those shown in equation (3.18).

This same dimensional analysis was performed on the moment equation (3.30) to ensure the retention of the proper terms in equation (3.70). Again consider the constitutive relationship

$$\underline{M} = EI\,\underline{t}\times\underline{t}' + GJ\kappa\,\underline{t} \tag{A.29}$$

and substituting for $\underline{t}$, $\underline{t}'$ and κ as before gives

$$\begin{aligned}
\underline{M} &= EI\left(\underline{t}^0 + \underline{u}'\right)\times\underline{u}'' + GJ\left(\Phi_0 + \Omega k_{ref}\right)\left(\underline{t}^0 + \underline{u}'\right) \\
&= EI\left(\underline{t}^0 + \frac{U_{ref}}{L}\dot{\underline{U}}\right)\times\frac{U_{ref}}{L}\ddot{\underline{U}} + GJ\left(\Phi_0 + \Omega k_{ref}\right)\left(\underline{t}^0 + \frac{U_{ref}}{L}\dot{\underline{U}}\right) \\
&= \frac{EI\,U_{ref}}{L^2}\underline{t}^0\times\ddot{\underline{U}} + \frac{EI\,U_{ref}}{L^2}\frac{U_{ref}}{L}\dot{\underline{U}}\times\ddot{\underline{U}} + GJ\Phi_0\underline{t}^0 \\
&\quad + GJ\Phi_0\frac{U_{ref}}{L}\dot{\underline{U}} + GJ\Phi_0\frac{k_{ref}}{\Phi_0}\Omega\,\underline{t}^0 + GJ\Phi_0\frac{k_{ref}}{\Phi_0}\frac{U_{ref}}{L}\Omega\,\dot{\underline{U}}\,.
\end{aligned} \tag{A.30}$$

Using the results of (A.19) and (A.20) this becomes

$$\begin{aligned}
\underline{M} &= M^1_{ref}\underline{t}^0\times\ddot{\underline{U}} + M^1_{ref}\frac{U_{ref}}{L}\dot{\underline{U}}\times\ddot{\underline{U}} + M^2_{ref}\underline{t}^0 \\
&\quad + M^2_{ref}\frac{U_{ref}}{L}\dot{\underline{U}} + M^2_{ref}\frac{k_{ref}}{\Phi_0}\Omega\,\underline{t}^0 + M^2_{ref}\frac{k_{ref}}{\Phi_0}\frac{U_{ref}}{L}\Omega\,\dot{\underline{U}}\,.
\end{aligned} \tag{A.31}$$

As before the quantities (U_{ref}/L) and (k_{ref}/Φ_0) are considered to be of order ε. Any term in (A.31) with two or more order ε terms will be neglected as being a higher order term. Only the last term is therefore neglected and the remaining terms are those shown in equation (3.70).

APPENDIX B

Alternate Bases At End Of Segment

An important consideration in using the *segmental shooting technique* is to be able to determine a new orthonormal basis at the end of the current segment being solved. This new basis will then be used as the local basis in which to solve the next segment. There are several methods available to choose this new basis. This section considers some of the alternate choices available.

In determining the new basis at the end of the segment, the first item needed is the tangent direction. This can be obtained by the method outlined in Section 3.2.3. After the tangent direction, $\underline{e}_1$, has been determined, $\underline{e}_2$ and $\underline{e}_3$ which span the cross section must be determined. There are an infinite number of possible sets of $(\underline{e}_2, \underline{e}_3)$ which satisfy this requirement. In Section 3.2.3, $\underline{e}_2$ and $\underline{e}_3$ were chosen to be the normal and binormal vectors of the Frenet basis. Other choices are possible, however. For example, rather than using a Frenet basis, a material basis could be used. A material basis is one in which the vectors are embedded in the material and deform along with the rod. Thus the orientation of this basis at the end of the segment is not arbitrary. It depends upon the orientation of the basis initially and also upon the deformation. Recall that the kinematically admissible variations in the vectors $\underline{e}_i^0$ are given from equation (2.34) as $\underline{v}_i(s) = \underline{a}(s) \times \underline{e}_i^0$, where $\underline{e}_i(s)=\underline{e}_i^0 + \underline{v}_i(s)$. The vector $\underline{a}(s)$ determines how a material basis $\{\underline{e}_i^0\}$ changes orientation along the rod segment.

To determine the vector $\underline{a}$, recall from equation (2.39) that $\underline{u}'=\underline{a}\times\underline{t}$. Then with $\underline{t}$ interpreted as $\underline{t}^0$ and with $\underline{u}$ given by equation (3.12), this gives the result that

$$a_2 = -\frac{dw}{ds} = -\epsilon, \qquad a_3 = \frac{dv}{ds} = \delta, \tag{B.1}$$

where the components of $\underline{a}$ are expressed in terms of the basis $\{\underline{e}_i^0\}$. To determine the a_1 component consider the case of a straight rod undergoing pure torsion. In such a case the tangent direction remains unchanged so that $\underline{v}_1=0$. This implies that a_2 and a_3 are zero. The vectors $\underline{e}_2(s)$ and $\underline{e}_3(s)$ change orientation by rotating around the axis of the rod. The amount of this rotation is equal to the total angle of twist of the rod segment,

γ, between arc length 0 and s. Thus a_1 at the end of the rod can be determined in several ways. One method is by integrating the twist per unit length of the rod, given in equations (3.17) and (3.52), between arc lengths $\rho=0$ and $\rho=1$, which gives

$$\gamma = \Phi_0 L + \frac{EI}{GJ}\left[\nu_{20}\sum_{k=0}^{\infty}\frac{\beta_k}{(k+1)(k+2)} - \chi_2\sum_{k=0}^{\infty}\frac{\beta_k}{(k+2)(k+3)} - \nu_{30}\sum_{k=0}^{\infty}\frac{\alpha_k}{(k+1)(k+2)} + \chi_3\sum_{k=0}^{\infty}\frac{\alpha_k}{(k+2)(k+3)}\right] + \frac{EI}{GJ}\int_0^{\rho}\left(\epsilon\frac{d\delta}{d\rho} - \delta\frac{d\epsilon}{d\rho}\right)d\rho \,. \tag{B.2}$$

Here the last term involves a nontrivial integration of two power series multiplied together which requires a significant computational effort. Also it was mentioned in Section 3.2.3 that γ as expressed in (B.2) does not represent the physical twist per unit length but is a representation of the change in M_1 in the original $\underline{e}_1{}^0$ direction and does not give an accurate indication of the actual total twist.

Another method is to assume that the exact result of twist per unit length along the rod being constant applies and that the total twist is simply ΦL. However the results of the exact solution may or may not be applicable to the approximate solution. Thus this approach may not be justifiable although the errors introduce would be small and would be expected to be negligible as the number of segments is increased.

With the vector $\underline{a}$ thus determined, the new material basis at the end of the segment can be determined. However, the basis obtained will not be exactly orthonormal. Each time a new vector is calculated, a small amount is added to the original unit vector. Thus the resulting "unit" vectors become slightly larger in magnitude with each calculation and therefore do not remain unit vectors. The resulting vectors could be made unit vectors by normalizing them at the expense of increased computation. Further, the basis obtained in this manner will not be exactly orthonormal. There will be a slight deviation from orthonormality which could accumulate as the solution progresses. The basis could be orthonormalized, possibly using the Gram-Schmidt orthonormalization procedure for example, again at the further expense of increased computation.

Therefore the Frenet basis was chosen as being the best choice because it met all the requirements (orthonormality, $\underline{e}_1$ along the tangent direction) and was the simplest to compute. The instances when the Frenet basis is not uniquely defined (ie: rod remains straight) do not present any difficulty as mentioned in Section 3.2.3. This basis also has a further advantage in that the Frenet basis is used often in the literature and computing this basis at each segment may facilitate comparison with other problems and solutions.