54084 0-315-06164-2

# PERMISSION TO MICROFILM — AUTORISATION DE MICROFILMER

• Please print or type — Écrire en lettres moulées ou dactylographier

Full Name of Author — Nom complet de l'auteur
YOUSIF, MOHAMED FOUAD

Date of Birth — Date de naissance
June 16, 1955

Country of Birth — Lieu de naissance
EGYPT

Permanent Address — Résidence fixe
1708 G.H. Mickener PK., Edmonton, Alta T6H 5A2

Title of Thesis — Titre de la thèse
Isolators in Polycyclic Groups.

University — Université
Alberta

Degree for which thesis was presented — Grade pour lequel cette thèse fut présentée
M. Sc.

Year this degree conferred — Année d'obtention de ce grade
1981

Name of Supervisor — Nom du directeur de thèse
A. H. Rhemtulla



Date
Oct. 23, 81

Signature
M. Yousif

NL-91 (4/77)

THE UNIVERSITY OF ALBERTA

# ISOLATORS IN POLYCYCLIC GROUPS

by

© MOHAMED F. YOUSIF

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE
OF MASTER OF SCIENCE

DEPARTMENT OF MATHEMATICS

EDMONTON, ALBERTA

FALL, 1981

Micro

THE UNIVERSITY OF ALBERTA

RELEASE FORM

NAME OF AUTHOR: MOHAMED F. YOUSIF

TITLE OF THESIS: ISOLATORS IN POLYCYCLIC GROUPS.

DEGREE FOR WHICH THESIS WAS PRESENTED: M.Sc.

YEAR THIS DEGREE GRANTED: Fall 1981



M. Yousif

PERMANENT ADDRESS: 1708 - G. H.
Michener PK.
Edmonton.
T6H 5A2

DATED Oct. 23 1981

THE UNIVERSITY OF ALBERTA

Faculty of Graduate Studies and Research

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled "Isolators in Polycyclic Groups" submitted by Mohamed F. Yousif in partial fulfillment of the requirements for the degree of Master of Science.

..........................
Supervisor

..........................

..........................

Date Oct. 23 1981

# ABSTRACT

In this thesis we investigate the following class $\chi$ of groups: We say a group $G$ is in $\chi$ if $G$ is soluble of finite rank and there exists a subgroup $G_0$ of finite index in $G$ and a finite set of primes $\pi_1$ such that whenever $\pi_1 \subseteq \pi$ (a set of primes) and $K \leq G_0$ then the set $\{g \in G_0 : g^m \in K$, for some $\pi$-number $m\}$ is a subgroup, called the $\pi$-isolator of $K$ in $G_0$ denoted by $K_\pi^{G_0}$, and $|K_\pi^{G_0} : K|$ is a $\pi$-number.

In Chapter 1 a summary of known results concerning the theory of isolators developed by P. Hall [1] is presented. From his results it follows that if $G$ is finitely generated nilpotent by finite group then $G$ is always in $\chi$.

In Chaper 2, the class of Fitting Isolated groups is defined and we give a proof to one of Rhemtulla and Hartley's fruitful results, that every torsion free soluble group of finite rank has a Fitting Isolated subgroup of finite index, this result is used in the investigation of the class $\chi$.

Denote by $\sqrt[G]{H}$ to the set $\{g \in G : g^m \in H, m \geq 1\}$ where $H \leq G$. In Chapter 3 we show that, if $G$ is not nilpotent, there is no reason to expect $\sqrt[G]{H}$ to be always a subgroup. However it has been proved by Rhemtulla and Wehrfritz that every polycyclic group $G$ has a subgroup $G_0$ of finite index such that $\sqrt[G_0]{H}$ is a subgroup for every $H \leq G_0$.

In Chapter 4 the main result of this thesis is given. It is an example which shows that polycyclic metabelian groups already fail to be in $\chi$. As a result of this example and a discussion with A. Rhemtulla we conjecture that $G$ is in $\chi$ if and only if $G$ is finitely generated nilpotent by finite group.

## ACKNOWLEDGEMENTS

I am truly grateful to my supervisor, Professor A.H. Rhemtulla and to Professor A.R. Weiss, for suggesting the topic of the thesis. Their constant support and encouragement was a valuable inspiration to me during the stages of this thesis.

# TABLE OF CONTENTS

# Chapter 0

## Notations and Definitions

If $X$ and $Y$ are groups, we write $X \le Y$ to indicate that $X$ is a subgroup of $Y$. $X < Y$ means that $X$ is a proper subgroup of $Y$.

When $X$ is a subset of a group, $\langle X \rangle$ will denote the subgroup generated by the elements of $X$. If $X$ consists of the elements $x_1, x_2, \ldots$ we also write $\langle x_1, x_2, \ldots \rangle$.

For any subgroup $H$ of $G$, $N_G(H)$ denotes the normalizer of $H$ in $G$. $C_G(H)$ denotes the centraliser of $H$ in $G$. For any integer $n \neq 0$,

$$G^n = \{g^n : \ g \in G\} .$$

The commutator $[a,b]$ of any two elements $a$ and $b$ of $G$ is given by the equation

$$[a,b] = a^{-1}b^{-1}ab .$$

Let $H$ and $K$ be any subgroups of $G$. Then $[H,K]$ will denote the subgroup generated by the set of all commulators $[h,k]$ with $h \in H$ and $k \in K$.

If $H,K$ are subgroups of $G$ such that $H \le K$, then $|H:K|$ denotes the index of $H$ in $K$, i.e. the number of cosets of $H$ in $K$.

If $G$ is a group and we have a series of subgroups, $G = G_1 \ge G_2 \ge \ldots \ge G_n \ge G_{n+1} = \{1\}$, then the series is a subnormal series if $G_{i+1}$ is a normal subgroup of $G_i$, $1 \le i \le n$. The series is a normal series if each subgroup $G_i$ is a normal subgroup of $G$.

A group $G$ is soluble if it has a finite normal series,

$G = G_1 \geq G_2 \geq \ldots \geq G_{n+1} = \{1\}$, such that $G_i/G_{i+1}$ is Abelian $(1 \leq i \leq n)$.

A group $G$ is polcyclic if it has a finite subnormal series of subgroups $G = G_1 \geq G_2 \geq \ldots \geq G_{n+1} = \{1\}$, such that $G_i/G_{i+1}$ is cyclic $(1 \leq i \leq n)$.

A group $G$ is supersoluble if it has a finite normal series of subgroups $G = G_1 \geq G_2 \geq \ldots \geq G_{n+1} = \{1\}$ such that $G_i/G_{i+1}$ is cyclic $(1 \leq i \leq n)$.

A group $G$ is nilpotent if it is soluble, and the normal series $\{G_i\}$ $(1 \leq i \leq n)$ can be chosen so that the action of $G$ on each of the factors $G_i/G_{i+1}$ is trivial, i.e. every element of $G$ acts as the identity automorphism. That is to say $[G_i, G] \leq G_{i+1}$ $(1 \leq i \leq n)$ or equivalently, $G_i/G_{i+1}$ is in the center of $G/G_{i+1}$. For this reason, any series satisfying this condition is called a certral series of $G$.

A particular central series, defined for any group $G$, is the lower central series $\{\gamma_i\}$, defined as follows: $\gamma_1 = G$, and $\gamma_{i+1} = [\gamma_i, G]$. It is easy to show inductively that, $G$ is nilpotent if and only if for some positive integer $c$, $\gamma_{c+1} = \{1\}$. The number $c$ obtained in this way is called the class of nilpotency of the nilpotent group $G$.

Another central series, in some ways dual to the series $\gamma_i$, is the upper central series, $\{Z_i\}$, defined as follows: $Z_1$ is the center of $G$, and $Z_{i+1} = \{x \in G : xZ_i \text{ is in the center of } G/Z_i\}$. It is easy to verify that $G$ is nilpotent if and only if for some positive integer $n$, $Z_n = G$. The smallest such integer is the class, $c$, defined above.

For a complete reference, we refer the reader to P. Hall [1] and to B. Warfield [9].

Finally the following lemma will be used in the next chapter, for the proof see [1].

LEMMA (A): (i) A group $G$ is polycyclic if and only if it is soluble and all its subgroups are finitely generated.

(ii) Finitely generated nilpotent groups are supersoluble.

(iii) Every supersoluble group $G$ has a nilpotent subgroup $K$ such that $|G:K| < \infty$ and $K \geq G'$, where $G' = [G,G]$.

# Chapter 1

## Isolators in Nilpotent Groups

We study here the theory of isolators developed by P. Hall [1] and list the main definitions and results that will be used in this thesis.

Definition 1.1: Let $\pi$ be a set of prime numbers. A $\pi$-number is a positive integer whose prime divisors lie in $\pi$. A $\pi$-group is a finite group of order a $\pi$-nubmer. If $\pi$ contains just one prime p, we write p-number and p-group. Thus a p-group here always means a finite p-group.

Given a group G, an element $g \in G$ is called an m-element if its order divides a power of m. The $\pi$-elements of G are the elements whose orders are $\pi$-numbers.

Definition 1.2: Let P be a property of groups. A group G is called locally - P if every finitely generated subgroup of G has the property P.

The following lemma will be useful to prove the main result of this Chapter, for the proof see [1, page 12].

Lemma 1.3: Let $G = \{a_1, \ldots, a_r\}$ be a finitely generated nilpotent group. If $|G: HG'| = m$ is finite, where H is a subgroup of G, then $|G:H|$ is an m-number. In particular, if for each $i = 1,\ldots,r$ we have $a_i^{m_i} \in H$ where $m_i$ is a $\pi$-number, then $|G:H|$ is a $\pi$-number. If $|G:G'| = m$ is finite, then $|G|$ is an m-number. If $a_i^{m_i} = 1$, for $i = 1,2,\ldots r$, where $m_i$ is a $\pi$-number then $|G|$ is a $\pi$-number.

Definition 1.4: For any subgroup $H$ of a group $G$, and any set of primes $\pi$, let $H_\pi^G$ denote the set $\{g \in G: g^m \in H$, for some $\pi$-number $m\}$.

We say $H$ has a $\pi$-isolator in $G$, if $H_\pi^G$ is a subgroup of $G$, and that $H$ is $\pi$-isoloated in $G$, if $H = H_\pi^G$.

Theorem 1.5: Let $G$ be a locally nilpotent group, $H$ any subgroup, $\pi$ any set of primes. Then $H_\pi^G$ is a subgroup, called the $\pi$-isolator of $H$ in $G$. If $G$ is finitely generated and therefore nilpotent, then $|H_\pi^G:H|$ is a $\pi$-number.

Proof: Let $x^m, y^n$ belong to $H$, where $m \& n$ are $\pi$-numbers, and if $K = \{x,y\}$ and $L = H \cap K$, then $|K:LK'|$ is finite and divides $mn$. But $K$ is finitely generated and therefore nilpotent. By Lemma 1.3, $|K:L|$ is a $\pi$-number, and in particular $(xy)^\alpha \in L \leq H$ for some $\pi$-number $\alpha$. Thus $H_\pi$ is a group.

If $G$ is finitely generated, and therefore nilpotent, so is $H_\pi^G$ by lemma A(i). Hence $|H_\pi^G:H(H_\pi^G)'|$ is a $\pi$-number. Hence $|H_\pi^G:H|$ is a $\pi$-number by 1.3.

As an immediate result of this theorem we can state the following Corollary.

Corollary 1.6: The $\pi$-elements of a locally nilpotent group $G$ form a fully invariant subgroup of $G$. If $G$ is finitely generated, the $\pi$-elements form a finite subgroup.

Our goal in this thesis is to try to extend the result of 1.5 to a bigger class of groups. And that will be done in Chapters 3 and 4.

The next theorem plays a good role in the next chapters, but before that we need the following Lemma and for the proof see [1, page 12].

Lemma 1.7: Let $H = \{x,y\}$ be nilpotent and $x^m = y^m$, then $xy^{-1}$ is an m-element.

Theorem 1.8: P. Hall [1]

Let G be a locally nilpotent group without π-elements different from the identity. If C is any centralizer in G or any term of the upper central series of G, then $C = C_\pi^G$, i.e. C is π-isolated subgroup.

Proof: Suppose first that $C = C(A)$ is the centralizer of the set A of elements of G. Let $y \in C_\pi$, then $y^m \in C$, for some π-number m. Hence $(a^{-1}ya)^m = y^m$ for all $a \in A$. But $\{y, a^{-1}ya\}$ is nilpotent. By 1.7 $[y,a] = y^{-1}a^{-1}ya$ is a π-element. Hence $[y,a] = 1$, and $y \in C$, $C = C_\pi^G$. The case $C = Z_1 = Z(G)$ is induced by choosing $A = G$. Suppose the result proved for $C = Z_i$, where $Z_{i+1}/Z_i$ is the center of $G/Z_i$. Then $\bar{G} = G/Z_i$ has no π-element $\neq 1$ and is locally nilpotent. By the case $i = 1$, we have $\bar{C} = C_\pi^{\bar{G}}$ if $\bar{C} = Z(\bar{G}) = Z_{i+1}/Z_i$. Hence $(Z_{i+1})_\pi^G = Z_{i+1}$

As a result of this, the following is true

Corollary 1.9: In a torsion free locally nilpotent group, all centralizers and all terms of the upper central series are π-isolated, for all choices of π.

Now we need the following lemma to prove the last result of this chapter.

Lemma 1.10: P. Hall [1]

Let $G$ be a finitely generated nilpotent group and let $H \leq H_1$ and $K \leq K_1$ be subgroups of $G$ such that $|H_1:H|$ and $|K_1:K|$ are $\pi$-numbers. Then $|[H_1,K_1]:[H,K]|$ is a $\pi$-number.

As a result of this, if $H$ and $K$ are subgroups of a locally nilpotent group $G$, then

$$[H_\pi^G, K_\pi^G] \leq [H,K]_\pi^G .$$

Theorem 1.11: P. Hall [1]

Let $G$ be locally nilpotent, $H$ any subgroup of $G$ and $N$ its normalizer in $G$ then:

(i) $H_\pi^G \triangleleft N_\pi^G$ and if $H = H_\pi^G$, then $N = N_\pi^G$.

(ii) If $G$ is nilpotent, then

$$H = H_\pi^G \text{ if and only if } H = N \cap H_\pi^G .$$

(iii) If $G$ is finitely generated then $N_\pi^G$ is the normalizer of $H_\pi^G$ in $G$.

Proof: (i) $[H_\pi^G, N_\pi^G] \leq [H,N]_\pi^G$ by the preceding lemma and $[H,N] \leq H$. So $[H_\pi^G, N_\pi^G] \leq H_\pi^G$. Thus $H_\pi^G \triangleleft N_\pi^G$. Hence if $H = H_\pi^G$, we have $H \triangleleft N_\pi^G$ and so $N_\pi^G = N$.

(ii) Suppose $G$ is nilpotent and $H = H_\pi^G \cap N$. Let $N_1$ be the normalizer of $N$ in $G$. It will be sufficient to prove that $N = N_\pi^G \cap N_1$; for then, continuing the normalizer series $H, N, N_1, \ldots, N_r = G$, we obtain $H = H_\pi^G \cap N = H_\pi^G \cap N_1 = \ldots = H_\pi^G \cap G = H_\pi^G$. Thus we may take $G = N_1$ so that $N \triangleleft G$. Using a contradiction argument, we suppose $G = \{N,x\}$ where $x^m \in N$ for some $\pi$-number $m > 1$ and deduce the existence of an element $y$ in $(H_\pi^G \setminus H) \cap N$.

Let $K = \bigcap_{t \in G} H^t$ so that $K \triangleleft G$. Let $L/K$ be the center of $N/K$ since $G = N_\pi^G$, we have $[L,G] \le [L_\pi^G, N_\pi^G] \le [L,N]_\pi^G \le H_\pi^G$ by 1.10, since $[L,N] \le K \le H$. If $[L,G] \not\le H$, we can choose for $y$ any element of $[L,G]$ not in $H$. But $[L,G] \triangleleft G$ and so if $[L,G] \le H$, then $[L,G] \le K$, and therefore $L \triangleleft G$, $L \cap H = K$. But $H$ is not normal in $G$, and $K < H$. Define $H_0 = H$, $H_{i+1} = [H_i, N]$ for $i \ge 0$. Let $j$ be the first index for which $H_{j+1} \le K$. Then $H_j \le L$ and $L \cap H \ge H_j > K$, a contradiction.

(iii) Let $G$ be finitely generated and let $x$ be an element of $G$ such that $x^{-1} H_\pi^G x = H_\pi^G$. We have to prove $x \in N_\pi^G$. Take $G = \{N_\pi^G, x\}$. Then $H_\pi^G < G$ by (i), and the choice of $x$. By 1.6, $|H_\pi^G : H| = m$ is a $\pi$-number. Hence $(H_\pi^G)^m \le H$. But $(H_\pi^G)^m \triangleleft G$ since $H_\pi^G \triangleleft G$. Let $C$ be the centralizer in $G$ of the finite group $H_\pi^G/(H_\pi^G)^m$. Then $|G:C|$ is finite and since $G$ is nilpotent, $|G:C|$ is a $\pi$-number. We have $[H_\pi^G, C] \le (H_\pi^G)^m \le H$ so that $C \le N \le N$.

Hence $|G:N_\pi^G|$ is a $\pi$-number, $G = N_\pi^G$, $x \in N_\pi^G$ as required.

## Chapter 2

## Fitting Isolated Groups

Introduction 2.1: The work of this chapter has been done by A.H. Rhemtulla and B. Hartley, in order to develop enough isolator theory in polycyclic groups, extend the results of P. Hall which we already introduced in Chapter one and to give a proof to one of Roseblade's recent results on prime ideals in group rings of polycyclic groups. However we gave here a definition to the class of Fitting Isolated groups and a proof to one of their major results which will be used in the following chapter. Also we mention Roseblade's results and will prove it along the lines of Chapter 3.

Following terminology recently proposed by Roseblade [2] we shall call a subgroup H of a group G orbital if H has a finite orbit under the conjugation action of G on the set of subgroups of G, that is, if $|G:N_G(H)| < \infty$. A group G is called orbitally sound if $|H:H_G| < \infty$ for every orbital subgroup H of G, where $H_G = \bigcap_{x \in G} H^x$ is the core of H in G.

By a theorem of Ito and Szép [3, p.45], this is equivalent to saying that $|H^G|H_G| < \infty$, where $H^G$ is the normal closure of H in G.

In Roseblade's recent and fundamental work on prime ideals in group rings of polycyclic groups, a significant role is played by the following fact:

Proposition A: (Roseblade [2])

Every polycyclic group has an orbitally sound subgroup of finite index.

Since every polycyclic group has a torsion-free subgroup of finite index, this is a really a theorem about torsion free polycyclic groups. Now it is rather easy to see that every finitely generated torsion free nilpotent group is orbitally sound by using the theory of isolators developed by P. Hall in [1] which we introduced in chapter one.

<u>Definition 2.2</u>: For any subgroup H of a group G, let $\sqrt{H} = {}^{G}\sqrt{H}$ denote the set $\{g \in G: g^n \in H$ for some $n \geq 1\}$. We say that H has an isolator in G, if $\sqrt{H}$ is a subgroup of G, and that H is isolated in G, if $H = \sqrt{H}$.

When G is finitely generated torsion free nilpotent we know that every subgroup H of G has an isolator $\sqrt{H}$, and that $|\sqrt{H}:H| < \infty$. Also $N_G(\sqrt{H})$ is isloated in G. If H is orbital, then $N_G(H) \leq N_G(\sqrt{H})$, we find that $\sqrt{H} \triangleleft G$. Since $|\sqrt{H}:H| < \infty$, we see that H contains a characteristic subgroup of finite index of $\sqrt{H}$, and hence that $|H:H_G| < \infty$.

Instead of confining ourselves to torsion free polycyclic groups, we shall for the most part work within the somewhat wider class of torsion free soluble groups of finite rank. By the rank of a group we mean here its Mal'cev special or Prüfer rank. Thus, G has finite rank, if there exists an integer $n \geq 0$ such that every finitely generated subgroup of G can be generated by n elements. It is known that every torsion free soluble group of finite rank has a subgroup X of finite index whose derived subgroup X' is nilpotent, see [3,part2,p.138].

Also G has a finite series

$$1 = G_0 \triangleleft G_1 \triangleleft \dots \triangleleft G_n = G \cdot$$

in which each factor is either finite or a torsion free abelian group of rank 1. The number of torsion free factors in such a series is an invariant of $G$ which we call the Hirsch number of $G$ and denote by $h(G)$.

Finally, each torsion free soluble group $G$ of finite rank has a unique maximal normal nilpotent subgroup called the Fitting subgroup of $G$ and is denoted by $\underset{\sim}{F}(G)$.

Definition 2.3: Let $G$ be a torsion free soluble group of finite rank. We say that $G$ is FI (Fitting Isolated), if, whenever $H = K|L$ $(L<K\leq G)$ is a torsion free section of $G$, then $H|\underset{\sim}{F}(H)$ is torsion free abelian.

The class of FI-groups is clearly closed under taking subgroups and torsion free homomorphic images.

Theorem 2.4: [A.H. Rhemtulla and B. Hartley]

If $G$ is any torsion free soluble group of finite rank, then $G$ contains an FI-subgroup of finite index.

Proof: We may assume that $G|\underset{\sim}{F}(G)$ is abelian. Let $1 = Z_0 < Z_1 < \ldots < Z_c = \underset{\sim}{F}(G)$ be the upper central series of $\underset{\sim}{F}(G) = F$, and $C_i = C_G(Z_i/Z_{i-1})$, $1 \leq i \leq c$. Then $F = \bigcap_{i=1}^{c} C_i$, and each of the groups $G|C_i$ can be thought of as an abelian group of matrices over $Q$. The torsion subgroup of each $G/C_i$ is finite [4.9.33], and it follows that the torsion subgroup $T/F$ of $G/F$ is finite. Hence $G/F$ splits over $T/F$, and there exists a subgroup $G_1$ of finite index in $G$, such that $G_1/F$ is torsion free Since $F = \underset{\sim}{F}(G_1)$, we may even assume that $G/\underset{\sim}{F}(G)$ is torsion free abelian.

The proof now falls into two parts:

(1) There exists a subgroup $G_0$ of $G$ such that $G_0 \geq F(G)$, $|G:G_0| < \infty$, and $H/F(H)$ is torsion free, for every $H \leq G_0$.

(2) $G_0$ if F.I.

We prove (1) by induction on $h(G)$. If $h(G) = 0$ there is nothing to do, so assume $h(G) > 0$. Let $B$ be an abelian normal subgroup of minimal rank of $G$ contained in the centre $Z(F(G))$ of $F(G)$, and let $A = {}^G\sqrt{B}$. Then $A \leq F(G)$ since $G/F(G)$ is torsion free, and hence, by the theory of isolators in nilpotent groups, [1], A is a subgroup of $Z(F(G))$. By induction, there is a subgroup $G_1/A$ of $G/A$, containing $F(G/A)$ and hence $F(G)A/A$, such that $|G:G_1| < \infty$ and $G_1/A$ satisfies (1).

Let $C = C_G(A)$. Then $C \geq F(G)$, we have seen above that the torsion subgroup of $G/C$ is finite, so that $G/C$ contains a torsion free subgroup $G_2/C$ of finite index. Let $G_0 = G_1 \cap G_2$. Then $|G:G_0| < \infty$ and $G_0 \geq F(G)$.

Now let $H \leq G_0$. If $H \cap A = 1$, then $H \cong HA/A$, and the fact that $H/F(H)$ is torsion free follows from the properties of $G_1/A$. Suppose that $H \cap A \neq 1$, and let $h$ be an element of $H$ such that $h^n \in F(H)$, for some $n \geq 1$. Since $H \cap A \leq F(H)$, we have $[H \cap A, h^n, \ldots, h^n] = 1$, and hence $C_{H \cap A}(h^n) \neq 1$. Hence $C_A(h^n)$ is a non-trivial isolated subgroup of $A$, and is normal in $G$ since $G/C$ is abelian. Therefore $h^n \in C$. But $G_0/G_0 \cap C$ is torsion free. Hence $h \in C$. If $F/A \cap H = F(H/A \cap H)$, we also know by induction that $h \in F$. Hence $h \in C_F(A \cap H)$, a nilpotent normal subgroup of $H$. Therefore $h \in F(H)$, as required. •

Next we will prove (2). We have to show that if $G$ is a torsion free soluble group of finite rank and

(*) $\quad H/\underset{\sim}{F}(H)$ is torsion free abelian for every $H \le G$,

then $G/N$ has the property (*) whenever $N \triangleleft G$ and $G/N$ is torsion free. Suppose this is false, and let $r$ be the smallest integer for which there exists a counterexample $G$ with $h(G) = r$. Among all pairs $(G,N)$ which furnish a counterexample with $h(G) = r$, choose one with $h(N)$ minimal. Then $G/N$ contains a subgroup $H/N$ such that $(H/N)/\underset{\sim}{F}(H/M)$ is not torsion free. We may clearly assume that $H = G$.

Let $F/N = \underset{\sim}{F}(G/N)$, and choose an element $t \in G/F$ such that $t^m \in F$ for some $m > 0$. Let $G_1 = \underset{\sim}{F}(G)\langle t\rangle$, $N_1 = N \cap G_1$, $F_1 = F \cap G_1$. Then $G_1/N_1 \simeq G_1N/N$, under this isomorphism, $F_1/N_1$ corresponds to $(F \cap G_1)N/N = (F/N) \cap (G_1N/N) = \underset{\sim}{F}(G_1N/N)$, as $G_1N/N \triangleleft G/N$. We have $t \in G_1$, $t \notin F_1$, $t^m \in F_1$. Thus $(G_1,N_1)$ is a counterexample, so we may assume that $G = G_1$, that is $G = \underset{\sim}{F}(G)\langle t\rangle$.

Next we notice that, if $Z = \underset{\sim}{Z}(\underset{\sim}{F}(G))$, then

$$C_Z(t) = 1 \qquad (1)$$

clearly $C_Z(t) = \underset{\sim}{Z}(G)$. Let $Y$ denote this subgroup, which is isolated in $G$ since $Z$ is. By the minimality of $h(N)$, there is no normal subgroup $M$ of $G$ such that $1 < M < N$ and $G/M$ is torsion free. Clearly $N \cap \underset{\sim}{F}(G) \ne 1$, and so $N \cap Z \ne 1$. Since $Z$ is isolated in $\underset{\sim}{F}(G)$, [1] and $G/\underset{\sim}{F}(G)$ is torsion-free, $Z$ is isolated in $G$, so $N \le Z$. Hence $YN \le Z$, and the isolator ${}^G\sqrt{YN}$ of $YN$ in $G$ is an abelian normal subgroup of $G$. Also $\sqrt{YN}/N = V$ is a torsion free abelian normal subgroup of $G/N$, and $V/C_V(G)$ is periodic, since $C_V(G) \ge YN/N$. Hence $[V,G] = 1$, that is $\sqrt{YN}/N < Z(G/N)$.

Now trivial arguments show that $G/Y$ has the property (*). If $Y \neq 1$, then the minimality of $h(G)$ shows that $G/\sqrt{YN}$ has the property (*). But $\sqrt{YN}/N \leq F(G/N) = F/N$, and $F(G/\sqrt{YN}) = F/\sqrt{YN}$ as $\sqrt{YN}/N \leq Z(G/N)$. Hence $G/F$ is torsion free, a contradiction. This proves (1).

Clearly $Z/N \leq F/N$ , and since $t^m \in F$, we have

$$[Z,t^m,\dots,t^m] \leq N.$$

In particular, if $Z > N$, then $C_{Z/N}(t^m) = K/N \neq 1$. Now commutation with $t^m$ induces endomorphism $\xi$ of $K$ whose image lies in $N$ and so has smaller rank than $K$. Hence $\ker \xi = C_K(t^m) \neq 1$. Let $L = C_K(t^m)$. The $t^m \in \underset{\sim}{F}(L\langle t\rangle)$, and by (*); $t \in \underset{\sim}{F}(L\langle t\rangle)$, that is $L\langle t\rangle$ is nilpotent. Therfore $[L,t,\dots,t] = 1$, and $C_L(t) \neq 1$. This contradicts (1), We deduce that $Z = N$.

It clearly follows that $\underset{\sim}{F}'(G) \neq 1$. Let $U/\underset{\sim}{F}'(G)$ be the torsion subgroup of $\underset{\sim}{F}(G)/\underset{\sim}{F}'(G)$ . Then $N \leq U$, as $N \cap \underset{\sim}{F}'(G) \neq 1$. Since $\underset{\sim}{F}(G) \leq G$, we deduce that

$$[\underset{\sim}{F}(G),t^m,\dots,t^m] \leq U.$$

Let $C$ be the nilpotency class of $\underset{\sim}{F}(G)$. Then

$$1 \neq \gamma_c(\underset{\sim}{F}(G)) \leq N,$$

where $\{\gamma_i(X)\}$ is the lower central series of a group $X$.

If $X_1,\dots,X_c \in \underset{\sim}{F}(G)$, then since $\gamma_c(\underset{\sim}{F}(G))$ is torsion free, the value of $[x_1,\dots,x_c]$ only depends on the value of $x_1,\dots,x_c$ modulo $U$. We obtain a well-defined $\langle t\rangle$-module epimorphism of $\underset{\sim}{F}(G)/U \otimes \dots \otimes \underset{\sim}{F}(G)/U$ (with $c$ factors) onto $\gamma_c(\underset{\sim}{F}(G))$, namely

$$x_1U \otimes \ldots \otimes x_cU \to [x_1,\ldots,x_c] \; .$$

(cf. [3,part1,p.55]). The tensor product is to be viewed as a $\langle t\rangle$-module via the diagonal action, the action on the individual factors being by conjugation. Since $\underset{\sim}{F}(G)/U$ has a finite series with $t^m$-trivial factors, so does the tensor product, and hence also $\gamma_c(F(G))$ (cf. [3],part 1, p.56). Hence $C_N(t^m) \neq 1$. Arguing as in the previous paragraph, we deduce that $C_N(t) \neq 1$, and obtain a contradiction to (1). This concludes the proof.

# Chapter 3

## Isolators in Polycylic Groups

Introduction: The work of this Chapter is the first step in an attempt to expten P. Hall results in [1]. If $G$ is not nilpotent group, we show in 3.2.1 that, there is no reason to expect $\sqrt[G]{H}$ to be always a subgroup. In 3.2.2, we show that if $G$ is polycylic Fitting Isolated group then $G$ has a subgroup $G_0$ of finite index, such that $\sqrt[G]{H}$ is a subgroup for every $H \leq G_0$ and that $|\sqrt[G_0]{H}:H|$ is finite. Also throughout the lines of this Chapter we give a proof to proposition (A) of Chapter 2.

If $G$ is polycyclic group, Mal'cev in an earlier theorem, has proved that $G$ can be embedded in $GL(n,\mathbb{Z})$, the general linear group of degree $n$ over the integers. And since every linear group is a CZ-group, we need here to study those properties of CZ-groups that seem relevant to the theory of isolators.

I have therefore split this chapter into CZ-groups in section 1 and the isolator theory of polycyclic groups in section 2.

### Section 1

#### CZ-groups and the Zariski topology

Let $U$ be the space of n-row vectors over the field $F$ and $R = F[X_1,\ldots,X_n]$, the polynomial ring over $F$ in $n$ indeterminants. A subset $A$ of $U$ is said to be closed in $U$ if there exists a subset $S$ of $R$ such that $A$ is the set of zeros of $S$, that is, if

$$A = \{(a_1,\ldots,a_n) \in U: \ f(a_1,\ldots,a_n) = 0 \text{ for all } f \text{ in } S\} .$$

If $S$ is any subset of $R$ let $V(S)$ denote the set of zeors of $S$ (in $U$). Note that $V(S) = V(\text{ideal generated by } S)$

$$\bigcap_{\alpha} V(S_{\alpha}) = V(\bigcup_{\alpha} S_{\alpha}).$$

Suppose that $A = V(I)$ and $B = V(J)$ where $I$ and $J$ are ideals. We claim that $A \cup B = V(IJ)$. For $IJ \subseteq I$, so $A = V(I) \subseteq V(IJ)$, and similarly for $B$. Converslcy let $x \in V(IJ)\backslash A$. Since $A = V(I)$ there exists an element $f$ of $I$ such that $f(x) \neq 0$. If $g \in J$, then $fg \in IJ$, so $0 = fg(x) = f(x)g(x)$. Hence $g(x) = 0$ for all $g$ in $J$ and so $x \in V(J) = B$.

Thus the closed subsets of $U$ define a topology on $U$, called the Zariski topology. Note that $U = V(\{0\})$, $\phi = V(\{1\})$. If $(a_1,\ldots,a_n) \in U$ then $\{(a_1,\ldots,a_n)\} = V(\{x_1-a_1,\ldots,x_n-a_n\})$. Hence every one element subset of $U$ is closed (i.e. the topology is a $T_1$-topplogy). If $A \subseteq B$ are closed sets there exists ideals $I$ and $J$ of $R$ such that $A = V(I)$, $B = V(J)$ and $J \subseteq I$. For if $A = V(I')$ and $B = V(J)$, put $I = I' + J$. Thus the Hilbert basis theorem implies that the Zariski topology satisfies the descending chain condition on closed sets, or equivalently the ascending chain condition on open sets.

Definition 3.1.1: A Z-space is a topological space in which every one element set is closed (i.e. is a $T_1$-space) and which satsifies the descending chain condition on closed sets.

Every subspace of a Z-space is a Z-space.

Definition 3.1.2: A CZ-group is a Z-space $G$ whose underlying set carries a group structure such that for every $a$ in $G$ the four mappings given by

$$x \to ax \quad , \quad x \to x^{-1}$$
$$x \to xz \qquad x \to x^{-1}ax \quad , x \in G$$

are continuous.

Every linear group is a CZ-group, for if V is an arbitrary finite dimensional vector space over F, then V can be endowed with a Zariski topology after a suitable choice of basis.

Let G be a subgroup of $GL(n,F) \subseteq F_n$. $F_n$ is a vector space over F of dimension $n^2$ and so carries the Zariski topology. The four mappings above are continuous [4,5.1,page 72].

Since the topology induced on a given linear group is unaffected by ground field extension [4,page 73], we can unambiguously speak of its closed subsets without specifying the groung field.

<u>Theorem 3.1.3</u>: [4,page 75]

If G is CZ-group then the normalizer of a closed subset of G is closed and the centralizer of any subset of G is closed.

<u>Proof</u>: Let S be any closed subset of G and a any element of S. Denote by S(a) the inverse image of S in G under the continuous mapping given by $x \to x^{-1}ax$. Then $N_1 = \bigcap_{a \in S} S(a) = \{x \in G: \ x^{-1}Sx \subseteq S\}$ is closed in G.

If (a)S denotes the inverse image of S in G under the continuous mapping given by $x \to xax^{-1} = (x^{-1})^{-1}ax^{-1}$, then $N_2 = \bigcap_{a \in S} (a)S = \{x \in G: \ xSx^{-1} \subseteq S\}$ is closed. Therefore $N_G(S) = N_1 \cap N_2$ is closed in G.

Let T be any subset of G.

$$C_G(T) = \bigcap_{t\in T} C_G(t) = \bigcap_{t\in T} N_G(t)$$

and $N_G(t)$ is closed by the above. Consequently $C_G(T)$ is closed in G.

<u>Theorem 3.1.4</u>: [4,page 78]

In a CZ-group G the closure of a subgroup is a subgroup and the closure of a normal subgroup is a normal subgroup.

<u>Proof</u>: Let H be a subgroup of G and $\bar{H}$ its closure in G. Now $H = H^{-1}$. Since the mapping given by $x \to x^{-1}$ is a homeomorphism of G, $(\bar{H})^{-1} = \bar{H}$. Let $h \in H$. The inverse image of $\bar{H}$ in G under the mapping given by $x \to hx$ is closed and contains H. Therefore it contains $\bar{H}$; that is, $h\bar{H} \subseteq \bar{H}$ for all h in H. If $k \in \bar{H}$, the inverse image of $\bar{H}$ under the mapping given by $x \to xk$ contains H and so $\bar{H}$. Hence $\bar{H}\bar{H} \subseteq \bar{H}$ is a subgroup of G. Suppose that H is a normal subgroup of G. Since the mapping given by $x \to a^{-1}xa$ is continuous, $a\bar{H}a^{-1}$ is a closed subset of G containing H. Therefore $\bar{H} \subseteq a\bar{H}a^{-1}$ for all a in G; that is $\bar{H}$ is normal in G.

<u>Theorem 3.1.5</u>: [p,page 78]

Let A,B and C be sugroups of the CZ-group G and $\bar{A},\bar{B}$ and $\bar{C}$ their closures in G. If $[A,C] \le B$ then $[\bar{A},\bar{C}] \le \bar{B}$.

<u>Proof</u>: Let $c \in C$. The mapping given by $x \to [a,c]$ is continuous and the inverse image of $\bar{B}$ in G contains A. Therefore $[\bar{A},c] \le \bar{B}$ for all c in C.

If $a \in \bar{A}$ the inverse image of $\bar{B}$ under the mapping given by $x \to [a,x]$ contains C and hence $\bar{C}$. Thus $[\bar{A},\bar{C}] \le \bar{B}$

<u>Theorem 3.1.6:</u> [Wehrfritz]

R is finitely generated integral domain of characteristic $p \geq 0$ and $G = GL(n,R)$, then if $p = 0$, for all but finitely many primes $q$, then $G$ contains a subgroup $T_q$ of finite index in $G$ such that for all closed subgroups $H$ of $G$

$$(N_G(H) \cap T_q)H/H \text{ is residually finite } q\text{-group.}$$

For the proof see [5]. As a matter of fact we can take $T_q = G \cap (I + qM(n,\mathbb{Z}))$, $q$ any prime, where $I$ is the identity matrix of rank $n$, and $M(n,\mathbb{Z})$ is the set of $n$ by $n$ matrices over the integers $\mathbb{Z}$.

Then $T_q = \{g \in G: \; g \equiv 1 \pmod q\}$.

<u>Lemma 3.1.7</u> If $K$ is residually finite p-group and residually finite q-group then $K$ is torsion free.

<u>Proof</u>: Let $x \in K$, $x \neq e$, $x^n = e$, $n \geq 2$. Let $n = p^i q^j m$, where $(p,m) = (q,m) = 1$. and $q^j m \neq 1$ or $p^i m \neq 1$. Assume without loss of generality that $q^j m \neq 1$. $\therefore x^{p^i}$ has order $q^j m$, replace $x$ by $x^{p^i} = y$ so that $y \neq 1$ and order $y$ is $q^j m$, where $(p, q^j m) = 1$. Now, $y \in K$, which is residually finite p-group, implies $\exists H < K$ such that $y \notin H$ and $K/H$ is a finite p-group. So $Hy$ has order $p^\alpha$ (i.e. $y^{p^\alpha} \in H$, for some integer $\alpha$) and $(p^\alpha, q^j m) = 1$ hence $\exists$ integers r and s such that $rp^\alpha + sq^j m = 1$ i.e. $y = y^{rp^\alpha} \cdot y^{sq^j m} = y^{rp^\alpha} \in H$ **contradiction.**

So if we let $G_o = G \cap (I + 2M(n,\mathbb{Z})) \cap (I + 3M(n,\mathbb{Z}))$ then by 3.1.6, $N(H)/H$ is residually finite 2-group and residually finite 3-group, i.e. $N(H)/H$ is torsion free for all closed subgroups $H$ of $G_o$.

Corollary 3.1.8: If $H$ is a closed subgroup of $G_0$, then $H$ is isolated.

Proof: Suppose $H$ is not isolated in $G_0$, let $x^r \in H$ for some integer $r > 1$ and $x \notin H$. Let $J$ = the closure of $\langle x^r \rangle$ (the Zariski closure in $G_0$). Then $J \leq H$, and since $x$ normalizes $\langle x^r \rangle$ then $x \in N(J)$, and we have $x^r \in J$. But from the remark made above since $J$ is closed then $N(J)/J$ is torsion free. Then $x \in J$ i.e. $x \in H$ contradiction.

Definition 3.1.9: Let $R$ be a commutative ring and $M$ a finitely generated R-module. An element $g$ of $Aut_R(M)$ is called unipotent if $(g-1)$ is nilpotent, i.e. if some power of $(g-1)$ is the zero map. A subgroup $G$ of $Aut_R(M)$ is called unipotent if every element of $G$ is unipotent.

Theorem 3.1.10: [Wehrfritz]

If $H$ is a subgroup of $U(G)$, the set of unipotent elements of $G$, where $G = GL(n,Q)$. Then the Zariski colusre $\hat{H}$ of $H$ in $G$ is equal to the isolator of $H$ in $U(G)$.

For the proof see [6].

## 2. Isolators in Polycyclic Groups

There is no reason to expect $\sqrt[G]{H}$ to be always a subgroup, for

Example 3.2.1: Let $G = \langle a,b\colon a^b = a^{-1} \rangle$. Thus $[b^2,a] = 1$, $(ab)^2 = b^2$. Let $H = \langle b^2 \rangle$, then $\sqrt[G]{H} \geq \{ab,b\}$. But $G = \langle ab,b \rangle$ and $G \neq \sqrt[G]{H}$, for $a \notin \sqrt[G]{N}$ .

If $G$ is locally nilpotent group P. Hall, as we have seen in chapter 1, has proved the following:

i) ${}^{G}\sqrt{H}$ is always a subgroup

ii) If $H = {}^{G}\sqrt{H}$, then $N_G(H) = {}^{G}\sqrt{N_G(H)}$ i.e. if $H$ is islolated subgroup, so is $N_G(H)$.

iii) If $G$ is finitely generated, and therefore nilpotent, then $|{}^{G}\sqrt{H}:H| < \infty$.

iv) If $G$ is finitely generated, and therefore nilpotent, then ${}^{G}\sqrt{N_G(H)} = N_G({}^{G}\sqrt{H})$ .

<u>Theorem 3.2.2</u>: [Rhemtulla and Wehrfritz].

Every polycylic group $G$ has a subgroup $G_o$ of finite index in $G$ such that

i) $H \leq G_o$ , then ${}^{G_o}\sqrt{H}$ is a subgroup.

ii) If $H = {}^{G_o}\sqrt{H}$ , then $N_{G_o}(H) = {}^{G_o}\sqrt{N_{G_o}(H)}$

iii) $|{}^{G_o}\sqrt{H}:H| < \infty$ .

iv) ${}^{G_o}\sqrt{N_{G_o}(H)} = N_{G_o}({}^{G_o}\sqrt{H})$ .

<u>Corollary 3.2.3</u>: Roseblade's result in Chapter 2, follows from 3.2.2.

<u>Proof</u>: If $G_o$ if given as in 3.2.2 then $G_o$ is orbitally sound, for, let $H$ be a subgroup of $G_o$ and $N = N_{G_o}(H)$ with $|G_o:N| < \infty$ , then $G_o = {}^{G_o}\sqrt{N}$ . But from (ii), we get

$$G_o = N_{G_o}({}^{G_o}\sqrt{H}) \quad \text{i.e.} \quad {}^{G_o}\sqrt{H} \triangleleft G_o \quad ,$$

call $I = {}^{G_o}\sqrt{H}$, hence from (iii) we have $I^m \leq H$, $m \neq 0$. Then if $a \in I$ then $a^g \in I$ (since $I \triangleleft G$) , $g \in G_o$ . Hence $(a^g)^m = (a^m)^g \in H$ for every $g \in G_o$. Hence $a^m \in H^g$ , for every $g \in G_o$ i.e.

$a^m \in \bigcap_{g \in G_o} H^g = \text{core}_{G_o} H = H_{G_o}$ which implies $I/H_{G_o}$ is periodic, i.e. $H/H_{G_o}$ is periodic and since $G_o$ is polycyclic, then $H/H_{G_o}$ is finite, which gives the result.

<u>Proof of 3.2.4</u>: We shall [redacted] proof on steps.

<u>Step 1</u>. Mal'cev has proved that every polycyclic group $G$ can be embedded in $GL(n,\mathbb{Z})$, the general linear group of degree $n$ over the integers, also $G$ is nilpotent by abelian by finite, and the nilpotent by abelian part is traingularizabel over the algebraic closure of $Q$, see [4, Theorem 3.6, page 45].

Thus we pass from $G$ to this triangularizable subgroup and we call it $G$ again and let $U(G)$ be the set of unipotent matrices of $G$. By the definition of $U(G)$ it is easy to see that it is a normal subgroup of $G$ and by [2, Theorem 13.6, page 189], $U(G)$ is nilpotent. And by a theorem of Gruenberg [7] we may assume that $G' \leq U(G)$.

By 2.4, we may also assume that $G$ is Fittling isolated.

<u>Step 2</u>. If $G_o = G \cap (I+2M(n,\mathbb{Z})) \cap (I+3M(n,\mathbb{Z}))$ then by 3.1.6, $G_o$ is of finite index in $G$ and $G_o$ satisfies the following properties:

i) $N_{G_o}(H)/H$ is torsion free for every closed subgroup $H$ of $G$, by the remark made after Lemma 3.1.

ii) $U(G_o)$ is closed subgroup of $G_o$, by 3.5, with $U(G_o) \triangleleft G_o$, $U(G_o)$ nilpotent and $G_o' \leq U(G_o)$, from Step 1.

We shall try to show that $G_o$ satisfies i, ii, iii and iv mentioned in 3.2.2.

<u>Step 3</u>. $U(G_o)$ is isolated by Corollary 3.1.8. Let $U(H) = H \cap U(G_o)$, where $H \leq G_o$ and suppose $H$ is isolated subgroup. Then $U(H)$ is also isolated, since both $H$ and $U(G_o)$ are so. Then by 3.1.10,

$U(H)$ is closed, and since $G_o' \le U(G_o)$, then $[N(H),H] \le H \cap U(G_o) = U(H)$ (by N(H) we mean here $N_{G_o}(H)$). Applying 3.1.5, we get $[\overline{N(H)},\overline{H}] \le \overline{U(H)}$. But $U(H) = \overline{U(H)}$, then $[\overline{N(H)},H] \le U(H) \le H$. i.e. $[\overline{N(H)},H] \le H$, then $N(H) = \overline{N(H)}$ by the definition of the normalizer, i.e. N(H) is closed. Hence N(H) is isolated by Corollary 3.18.

Hence we conclude that if H is isolated subgroup of $G_o$ so is N(H), which proves (ii).

(*) We can also see from the preceding argument, that if H is any subgroup of $G_o$ and U(H) is isolated in $U(G_o)$ then $[\overline{N(H)},H] \le U(H) \le H$ .

Step 4. Now take $H \le G_o$, $U(H) = H \cap U(G_o)$ .

$$G_o\sqrt{U(H)} = U(G_o)\sqrt{U(H)} \text{ , since } G_o/U(G_o) \text{ is torsion-free}$$

$$= \text{The closure of } U(H) \text{ in } G_o, \text{ by 3.1.10}$$

$$= \overline{U(H)}$$

since $U(G_o)$ is nilpotent, then $\overline{U(H)}$ is a subgroup and $|\overline{U(H)}:U(H)| < \infty$, by 1.5. Since $[N(H),H] \le U(H)$, for every subgroup H of $G_o$ then $[N(U(H)),U(H)] \le U(H)$. But $N(H) \le N(U(H))$ , then $[N(H),U(H)] \le U(H)$ . i.e. $U(H) \triangleleft N(H)$ . Hence $\overline{U(H)} \triangleleft \overline{N(H)}$, for if $x \in \overline{U(H)}$, $g \in N(H)$, then $x^n \in U(H)$, $n \neq 0$ and

$g^{-1}x^n g = (g^{-1}xg)^n \in U(H)$, since $U(H) \triangleleft N(H)$ then $g^{-1}xg \in \overline{U(H)}$. i.e. $[N(H),\overline{U(H)}] \leq \overline{U(H)}$. Applying 2.1.5, then $[\overline{N(H)},\overline{U(H)}] \leq \overline{U(H)}$. Now let $H_1 = H\cdot\overline{U(H)}$ since $\frac{H\cdot\overline{U(H)}}{H} \cong \frac{\overline{U(H)}}{H \cap \overline{U(H)}}$ and $|\overline{U(H)}:U(H)| < \infty$, then $|H_1:H| < \infty$.

Since $U(H_1) = H_1 \cap U(G_o) = H\cdot\overline{U(H)} \cap U(G_o) = \overline{U(H)}\cdot(H\ U(G_o)$ $= \overline{U(H)}\cdot U(H) = \overline{U(H)}$ then $U(H_1)$ is closed, i.e. it is isolated. Then by (*) $[\overline{N(H_1)},H_1] \leq H_1$ i.e. $[\overline{N(H_1)},H_1] \leq H_1 \cap U(G_o) = U(H_1)$ and since $H_1 \leq N(H_1) \Rightarrow \overline{H_1} \leq \overline{N(H_1)}$, then $[\bar{H}_1,H_1] \leq U(H_1)$. Hence $H_1/U(H_1) \leq Z(\overline{H_1}/U(H_1)$ = the center of the group $\bar{H}_1/U(H_1)$. Now since $U(H_1)$ is isolated, then $\bar{H}_1/U(H_1)$ is torsion free, then it is Fitting isolated group. And since $Z(\bar{H}_1/U(H_1) \leq$ Fitting subgroup of $(\bar{H}_1/U(H_1))$, which is isloated in $\bar{H}_1/U(H_1)$, then $Z(\bar{H}_1/U(H_1)$ is isolated. Hence it is closed subgroup and since $H_1/U(H_1) \leq Z(\bar{H}_1/U(H_1))$ then $\bar{H}_1/U(H_1)$ $\leq Z(\bar{H}_1/U(H_1) \leq \bar{H}_1/U(H_1)$ i.e. $\bar{H}_1/U(H_1)$ is abelian, and since $U(H_1) \leq H_1$ then $\bar{H}_1/H_1$ is abelian which is periodic and finitely generated, then it is finite i.e. $|\bar{H}_1:H_1| < \infty$. But $|H_1:H| < \infty$, then $|\bar{H}_1:H| < \infty$ since $\bar{H} \leq \bar{H}_1$, then $|\bar{H}:H| < \infty$, and we know before, from 3.1.4, that $\hat{H}$ is a subgroup. And since $\hat{H}$ is closed then, by Corollary 3.1.8, it is isolated in $G_o$. i.e. $\hat{H} = {}^{G_o}\!\sqrt{\bar{H}}$ and since ${}^{G_o}\!\sqrt{\bar{H}} = {}^{G_o}\!\sqrt{H}$ then $\hat{H} = {}^{G_o}\!\sqrt{H}$, i.e. ${}^{G_o}\!\sqrt{H}$ is a subgroup of $G_o$ and $|{}^{G_o}\!\sqrt{H}:H| < \infty$, which proves (i) and (iii).

To see (iv), since $[N(\hat{H}),\hat{H}] \leq \hat{H}$, applying 3.1.5 then $[\overline{N(\hat{H})},\hat{H}] \leq \hat{H}$, i.e. $\overline{N(\hat{H})} = N(\hat{H})$, hence $N(\hat{H})$ is isolated. And since $N(H) \leq N(\hat{H})$, then $\overline{N(H)} \leq \overline{N(\hat{H})} = N(\hat{H})$ i.e. $\overline{N(H)} \leq N(\hat{H})$.

Now we want to prove $N(\hat{H}) \leq \overline{N(H)}$. Let $x \in G_o$ such that $x^{-1}\hat{H}x = \hat{H}$, we have to prove $x \in \overline{N(H)}$. Take $K = \{\overline{N(H)},x\}$. Then $\hat{H} \triangleleft K$ since $[N(H),H] \leq H$ and applying Theorem 3.1.5, we get $[\overline{N(H)},\hat{H}] \leq \hat{H}$, i.e. $\hat{H} \triangleleft \overline{N(H)}$ and by the choice of $x$, then $\hat{H} \triangleleft K$

By (iii), $|\bar{H}:H| = m$, where $m$ is a finite number. Hence $\bar{H}^m \leq H$. But $\bar{H}^m \triangleleft K$ since $\bar{H} \triangleleft K$. Let $C$ be the centralizer in $K$ of the finite group $\bar{H}/\bar{H}^m$. Then $|K:C|$ is finite. We have $[\bar{H},C] \leq \bar{H}^m \leq H$ so that $C \leq N(H) \leq \overline{N(H)}$. Hence $|K:\overline{N(H)}|$ is finite i.e. $K = \overline{N(H)}$ and $x \in \overline{N(H)}$ as required.

Now the following Corollary will, in some extent, generalize Theorem 3.1.10.

Corollary 3.2.5: Every polycyclic group $G$ has a subgroup $G_o$ of finite index, such that, if $H$ is a subgroup of $G_o$, then the Zariski closure $\bar{H}$ of $H$ in $G_o$ is equal to the isolator of $H$ in $G_o$.

Proof: If $G_o = G \cap (I+2M(n,\mathbb{Z})) \cap (I+3M(n,\mathbb{Z}))$ then by 3.1.6, $G_o$ is of finite index in $G$. And if $H \leq G_o$, then 3.1.4, the Zariski closure $\bar{H}$ of $H$ in $G_o$ is a subgroup and as we have seen throughout the proof of 3.2.4, $|\bar{H}:H|$ is finite. And since $\bar{H}$ is closed then by Corollary 3.1.8, it is isolated in $G_o$, i.e. $\bar{H} = {}^{G_o}\sqrt{\bar{H}}$. But since ${}^{G_o}\sqrt{\bar{H}} = {}^{G_o}\sqrt{H}$, then $\bar{H} = {}^{G_o}\sqrt{H}$.

# Chapter 4

## Π-Isolators and Polycyclic Groups

In this Chapter we present the main result of this thesis.

We define a class $\chi$ of groups as follows:

Definition 4.1: We say a group $G$ is in $\chi$ if $G$ is soluble of finite rank and there exists a subgroup $G_o$ of finite index in $G$ and a finite set of primes $\pi_1$ such that whenever $\pi_1 \subseteq \pi$ (a set of primes) and $K \leq G_o$ then the set $\{g \in G: g^m \in k$, for some $\pi$-number $m\}$ is a subgroup (called the $\pi$-isolator of $K$ in $G_o$ and denoted by $K_\pi^{G_o}$), and $|K_\pi^{G_o}:K|$ is a $\pi$-number.

Now the following theorem is an immediate consequence of the definition.

Theorem 4.2: i) If $G \in \chi$ and $H \leq G$, then $H \in \chi$.

ii) If $G \in \chi$ and $H \triangleleft G$, then $G/H \in \chi$.

iii) If $H \in \chi$ and $K \in \chi$, then $H \times K \in \chi$.

iv) If $G$ is finitely generated nilpotent by finite group then $G \in \chi$, see 1.5.

One could conceivably try to extend P. Hall result in 1.5 to some other class of grups by investigating the class $\chi$. The following example shows that some metabelian polycyclic groups already fail to be in $\chi$.

Theorem 4.3: If $G = \langle x,y,t : xy = yx, x^t = y, y^t = xy\rangle$ then $G \notin \chi$.

Proof: The subgroup $\langle x,y\rangle$ of $G$ represents the Fitting subgroup, $\underset{\sim}{F}(G)$, of $G$. And $G > \underset{\sim}{F}(G) > 1$ is a normal series of $G$ with $G/\underset{\sim}{F}(G) \simeq \langle t\rangle$ the infinite cyclic group and $\underset{\sim}{F}(G) = \langle x,y\rangle$ is torsion free abelian of rank 2. That is to say $G$ is a metabelian polycyclic group.

Now suppose $G \in \chi$. We may assume that $\pi_1$ has largest prime $p > 5$, by enlarging $\pi_1$ if necessary, we may also assume that $G_o$ contains $\underset{\sim}{F}(G)$. i.e. $G_o = \langle x,y,t^m\rangle$, for some $m > 0$.

Let $N = \langle x,y^q,t^{mp}\rangle$ be a subgroup of $G_o$, where $q$ is a prime, which will be chosen later.

<u>Question (*)</u>: If $N$ is given as above, is there a prime $q$ greater than $p$, such that $N$ is normalized by $t^{mp}$ but not by $t^m$?

If a positive answer were given to (*) then we are finished. For, let $t^m = \tau$ and suppose $\exists q > p$ such that $N = \langle x,y^q,\tau^p\rangle$ is normalized by $\tau^p$ but not by $\tau$. Let $L = N \cap \underset{\sim}{F}(G)$, $M = L^{G_o}$ (the normal closure of $L$ in $G_o$), then it is easy to see that $M = \underset{\sim}{F}(G)$ and $G_o = \langle L,\tau\rangle$. Since $N$ is not normal in $G_o$, then $N$ is a proper subgroup of FN. So if $q \notin \pi$, where $\pi$ is a set of primes containing $\pi_1$, then $N_\pi^{G_o}$ is not a subgroup. For, if $N_\pi^{G_o}$ is a subgroup, so it must be all of $G_o$, and $|G_o:N|$ is a finite $\pi$-number. But then $y^m \in N$, where $m$ is $\pi$-number, and so $y^q \in N$ implies $y \in N$, a contradiction.

And if we throw $q$ inside $\pi_1$ then by repeating the argument we will be able to find an increasing sequence of primes $q = q_1 < q_2 < \dots < q_i < \dots$ which will prevent us from keeping $\pi_1$ finite if we want ${}^{(i)}N_\pi^{G_o}$ to be a subgroup, where ${}^{(i)}N = \langle x,y^{q_{i+1}},\tau^{q_i}\rangle$, $i = 0,1,2,\dots,q_0 = p$.

Now by studying the action of the operator $t$ on $\underset{\sim}{F}(G)$ we can see that $t^2 = t + 1$ and $t^n = F_{n-1} + F_n t$, where $F_n$ are the Fibonacci numbers defined as follows:

$$F_0 = 0$$
$$F_1 = F_2 = 1$$
$$F_{n+2} = F_{n+1} + F_n \;, \quad n \geq 0.$$

Now (*) is equivalent to say:

<u>Question (**)</u>: Is there a prime $q > p$ such that $q | F_{mp}$ but $q \nmid F_m$?

For, if $t^{mp}$ normalizes N, then $x^{t^{mp}} \in N$, i.e., $x^{F_{mp-1}+F_{mp}t} \in N$, then $x^{F_{mp}t} \in N$, i.e. $y^{F_{mp}} \in N$ i.e. $q | F_{mp}$; also $t^m$ does not normalize N means $q \nmid F_m$.

Hence in order to give an answer to (**), we need to study the ring of integers $\mathbb{Z}[\theta]$, where $\theta^2 = \theta + 1$. Let $K = Q(\sqrt{5})$, i.e. K is a quadratic field. $\theta = \frac{1 \pm \sqrt{5}}{2}$, where the minimal polynomial of $\theta$ over K is $x^2 - x^2 - 1$. $R = \mathbb{Z}[\theta]$ is a Dedekind ring, for $\{1,\theta\}$ is a basis for R over $\mathbb{Z}$, and the discriminant of R over $\mathbb{Z}$ is equal to the discriminant of the basis $\{1,\theta\}$ which is given by:

$$\det\begin{bmatrix} Tr(1) & Tr(\theta) \\ Tr(\theta) & Tr(\theta^2) \end{bmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 5 \;.$$

Since disc.$(R/\mathbb{Z}) = 5$, is square free positive integer, then R is Dedekind, see [8]. And since 5 is the only prime dividing disc.$(R/\mathbb{Z})$, then 5 is the only prime which ramifies in K.

Let q be a rational prime number since R is a Dedekind ring then qR, as an ideal of R can be factorized as, $qR = Q_1^{e_1} \ldots Q_g^{e_q}$, a product of a distinct primes of R with powers $e_i \geq 1$ in $\mathbb{Z}$. The power $e_i = e(Q_i/q\mathbb{Z})$ is called the ramification index of $Q_i$ over $q\mathbb{Z}$.

Since $Q_i \cap \mathbb{Z} = q\mathbb{Z}$ the inclusion $\mathbb{Z} \to R$ induces a monomorphism

$\mathbb{Z}/q\mathbb{Z} \to R/Q_i$ of fields. Since $R$ is finitely generated as $\mathbb{Z}$-module the dimension: $f_i = \dim_{\mathbb{Z}/q\mathbb{Z}} R/Q_i$ is an integer $\geq 1$. The dimension $f_i = f(Q_i/q\mathbb{Z})$ is called the degree of interia of $Q_i$ over $q\mathbb{Z}$.

We denote by $N_K(I)$ to the cardinality of $R/I$ where $I$ is a non-zero ideal of $R$.

The integers $e_i$ and $f_i$ satisfy the following equation $2 = \sum_{i=1}^{g} e_i f_i$, see [8], and the later has only three solutions in positive integers.

1) $g = 1$, $e_1 = 2$, $f_1 = 1$. Then $qR = Q^2$, $N_k(Q) = q$ and we say $q$ ramifies in $K$.

2) $g = 1$, $e_1 = 1$, $f_1 = 2$. Then $qR = Q$, $N_K(Q) = q^2$ and we say $q$ is inert in $K$.

3) $g = 2$, $e_1 = e_2 = 1$, $f_1 = {}_2 = 1$. Then $qR = Q_1Q_2$, $N_k(Q_1) = N_k(Q_2) = q$, and we say $q$ splits in $K$.

Now, before we proceed with our proof we need the following:

<u>Lemma 4.4</u>: i) $F_{n+m} = F_{n-1}F_m + F_nF_{m+1}$.

ii) If $n$ is divisible by $m$, then $F_n$ is divisible by $F_m$.

iii) If $m > 0$ and $p > 5$, then $(F_{mp}/F_m) > P$.

**<u>Proof</u>**: i) We shall carry out the proof by induction on $m$. $m = 1$, $F_{n+1} = F_{n-1}F_1 + F_nF_2 = F_{n-1} + F_n$. Suppose it is true for $m = k$, and $m = k + 1$: we shall prove it is true when $m = k + 2$. Thus let $F_{n+k} = F_{n-1}F_k + F_nF_{k+1}$, $F_{n+k+1} = F_{n-1}F_{k+1} + F_nF_{k+2}$. Adding the last two equations term by term we obtain $F_{n+k+2} = F_{n-1}F_{k+2} + F_nF_{k+3}$. And this is the required result.

ii) Let $n = mm_1$. We shall prove by induction on $m_1$. For $m_1 = 1$, $n = m$ obvious. Suppose $F_{mm_1}$ is divisible by $F_m$, and consider $F_{m(m_1+1)}$. But $F_{m(m_1+1)} = F_{mm_1+m}$ from (i), we get $F_{m(m_1+1)} = F_{mm_1-1}F_m + F_{mm_1}F_{m+1}$. The right hand side of this equation is divisibel by $F_m$. Then $F_{m(m_1+1)}$ is divisible by $F_m$.

iii) We shall prove by induction on $m$. If $m = 1$, then $(F_{mp}/F_m) = F_p > p$. Suppose $m > 1$, and let $m = k + 1$, then

$$\frac{F_{(k+1)p}}{F_{k+1}} = \frac{F_{kp+p}}{F_{k+1}} = \frac{F_{kp-1}F_p + F_{kp}F_{p+1}}{F_{k+1}} > \frac{F_{kp-1}F_p + F_{kp}F_p}{F_{k+1}} = \frac{(F_{kp-1}+F_{kp})F_p}{F_{k+1}}.$$

$= \frac{F_{kp+1}F_p}{F_{k+1}}$. But since $kp+1 > k+1$, $p > 5$, then $F_{kp+1} > F_{k+1}$ that is $\frac{F_{kp+1}}{F_{k+1}} > 1$. Thus $\frac{F_{(k+1)p}}{F_{k+1}} > \frac{F_{kp+1}}{F_{k+1}} F_p > F_p > p$. i.e. $\frac{F_{mp}}{F_m} > p$, $\forall\, m > 0,\ p > 5$.

We now go back to our proof. We have $R = \mathbb{Z}[\Theta]$, $\Theta^2 = \Theta + 1$. The conjugate of $\Theta$ in $R$ is equal to $1 - \Theta$, since it satisfies the same minimal polynomial of $\Theta$. i.e. we have two automorphisms (and only two) on $R$. One of them is the identity and the other is given by $\sigma\colon R \overset{\text{auto.}}{\to} R$, $\sigma(\Theta) = 1 - \Theta$.

Now we have the following:

i) $\Theta^n = F_{n-1} + F_n\Theta$, $n > 0$

ii) $F_n = \frac{\sigma\Theta^n - \Theta^n}{\sigma\Theta - \Theta}$.

For (i), we shall prove by induction on $n$. $n = 1$, then $\Theta = F_0 + F_1\Theta = 0 + \Theta = \Theta$. Suppose $\Theta^k = F_{k-1} + F_k\Theta$. Then $\Theta^{k+1} = \Theta^k\Theta = (F_{k-1}+F_k\Theta)\Theta = F_{k-1}\Theta + F_k\Theta^2 = F_{k-1}\Theta + F_k(\Theta+1)$ $= F_k + (F_{k-1}+F_k)\Theta = F_k + F_{k+1}\Theta$. To see (ii), $\sigma\Theta^n - \Theta^n =$ $= (F_{n-1}+F_n-F_n\Theta) - (F_{n-1}+F_n\Theta) = F_n(1-2\Theta)$. Also $\sigma\Theta - \Theta = 1 - 2\Theta$.

i.e. $\frac{\sigma\Theta^n - \Theta^n}{\sigma\Theta - \Theta} = F_n$.

Now the proof will follow into two cases:

Case 1: Suppose $p \nmid F_m$ (p does not divide $F_m$). Since m divides mp, then by Lemma 4.4 (ii), $F_m$ divides $F_{mp}$. i.e. $(F_{mp}/F_m)$ is an integer, and by 4.4 (iii), it is greater than p. Hence there exists a prime q divides $(F_{mp}/F_m)$. Hence q divides $F_{mp}$, and we shall consider two cases:

i) Assume $q \nmid F_m$. Let $\varepsilon = \frac{\sigma\Theta}{\Theta}$, $\varepsilon$ is a unit of R, for its conjugate is $(\frac{\Theta}{\sigma\Theta})$, i.e. the norm of $\varepsilon$ is $\frac{\sigma\Theta}{\Theta}\cdot\frac{\Theta}{\sigma\Theta} = 1$. Then $F_n = \frac{\sigma\Theta^n - \Theta^n}{\sigma\Theta - \Theta} = \Theta^n \frac{\varepsilon^n - 1}{\sigma\Theta - \Theta}$, and since $q | F_{mp}$ and $q \nmid F_m$, then $\varepsilon^{mp} \equiv 1 \pmod{qR}$ and $\varepsilon^m \not\equiv 1 \pmod{qR}$. Hence p divides the order of $\bar{\varepsilon} = \varepsilon + qR$, in $(R/qR)^*$, the set of unit elements of the finite $F_q$-Algebra R/qR, hence p divides the cardinality of $(R/qR)^*$, and before we proceed we need the following, which can be found in [10,p.125].

Lemma 4.5: If I is a non zero ideal of R, then the number of units of the finite ring R/I is given by

$$\text{card}(R/I)^* = N(I) \prod_{P \supseteq I} \left(1 - \frac{1}{N(P)}\right) .$$

$$\text{Now, } \text{card}(R/qR)^* = \begin{cases} (q-1)(q+1) & \text{if } qR = Q \\ q(q-1) & \text{if } qR = Q^2 \\ (q-1)^2 & \text{if } qR = Q_1Q_2. \end{cases}$$

And if p divides q+1, then p divides $\frac{1}{2}(q+1)$, since q+1 is even integer, (unless q = 2 when $p|q+1$, contradicts $p > 5$). But $\frac{1}{2}(q+1) < q$, then $p < q$. Similarly if p divides q-1, we get $p < q$. And if p divides q, then $p = q$, and in this case q ramifies in K and since 5 is the only prime ramifies in K then

$p = q = 5$, a contradiction with $p > 5$. So this $q$ answers question (**).

ii) Assume $q | F_m$. And we have $q \left| \left(\frac{F_{mp}}{F_m}\right)\right.$. Since $F_{mp} = \theta^{mp} \frac{\varepsilon^{mp} - 1}{\sigma\theta - \theta}$, $F_m = \theta^m \frac{\varepsilon^m - 1}{\sigma\theta - \theta}$ then $\frac{F_{mp}}{F_m} = \theta^{mp-m} \frac{\varepsilon^{mp} - 1}{\varepsilon^m - 1} = \theta^{mp-m} (1+\varepsilon^m + \ldots + \varepsilon^{m(p-1)})$.

Since $q \left| \left(\frac{F_{mp}}{F_m}\right)\right.$, then $\theta^{mp-m}(1+\varepsilon^m + \ldots + \varepsilon^{m(p-1)}) \equiv 0 \pmod{qR}$, and since $q | F_m$, then $\varepsilon^m \equiv 1 \pmod{qR}$. From the last two equations, we get $\theta^{mp-m} p \equiv 0 \pmod{qR}$. Hence $p = q$, and since $q | F_m$, then $p | F_m$ a contradiction.

<u>Case 2</u>: Suppose $p | F_m$ i.e. $\varepsilon^m \equiv 1 \pmod{pR}$ since

$$\frac{F_{mp}}{F_m} = \frac{\theta^{mp}}{\theta^m} \cdot \frac{\varepsilon^{mp} - 1}{\varepsilon^m - 1}$$, so if we let $\varepsilon^m = 1 + \delta$. Hence

$$\theta^{m-mp} \frac{F_{mp}}{F_m} = \frac{(\varepsilon^m)^p - 1}{\varepsilon^m - 1} = \frac{(1+\delta)^p - 1}{\delta} \quad \text{i.e.} \quad \theta^{m-mp} \frac{F_{mp}}{F_m}$$

$$= p + \binom{p}{2}\delta + \binom{p}{3}\delta^2 + \ldots + \binom{p}{p-1}\delta^{p-1} \equiv p \pmod{p^2R}.$$ Hence $p^2 \nmid \left(\frac{F_{mp}}{F_m}\right)$.

But we know $\frac{F_{mp}}{F_m} > p$, where $p > 5$. Then there exists a prime $q$ different from $p$, such that $q | (F_{mp}/F_m)$.

Now, if $q | F_m$, then by (ii), $q = p$ a contradiction to the fact that $p$ is different from $q$. And if $q \nmid F_m$, then by (i), $q > p$, as required. This concludes the proof.

<u>Conjecture 4.6</u>: As a result of 4.3 and a discussion with A. Rhemtulla and A. Weiss we conjecture that $G$ is in $\chi$ if and only if $G$ is finitely generated nilpotent by finite group.

## BIBLIOGRAPHY


[1] P. Hall: The Edmonton notes on nilpotent groups, Queen Mary College, London (1969).

[2] J.E. Roseblade: Prime ideals in group rings of Polycyclic groups. Proc. of the London Math. Sco. Series 3, Vol. 36(1978), 385-477.

[3] D.J.S. Robinson: Finiteness conditions and generalized soluble groups Parts I and II, Springer Verlag (1972).

[4] B.A.F. Wehrfritz: Infinite linear groups, Springer Verlag (1973).

[5] B.A.F. Wehrfritz: Remarks on Centrality and cyclicity in Linear groups, J. Algebra 18(1971), 229-236.

[6] B.A.F. Wehrfritz: On the holomorphs of soluble groups of finite rank, J. of Pure & Applied Algebra 4(1974), 55-69.

[7] K.W. Gruenberg: The Engel Structure of linear groups, J. Algebra 8(1968), 34-40.

[8] P. Samuel: Algebraic Theory of Numbers, Published by Hermann, Paris, in 1967 in their Series Methods.

[9] R.B. Warfield: Nilpotent Groups, Lecture Notes in Mathematics, Springer-Verlag (1976).

[10] P. Ribenboim: Algebraic Numbers, New York, Wileyinterscience [1972], (Pure and Applied Mathematics, 27).