UNIVERSITY OF ALBERTA

A Study of Innovation Diffusion Using a Dynamic Fluid Model Analogy

A Thesis

BY

© David J. Warwaruk

FACULTY OF BUSINESS

Edmonton, Alberta

Fall, 1994

ISBN 0-315-94991-0

Name David J. Warwaruk

Dissertation Abstracts International is arranged by broad, general subject categories. Please select the one subject which most nearly describes the content of your dissertation. Enter the corresponding four-digit code in the spaces provided.

Business Administration - General | 0 3 1 0 | U·M·I

SUBJECT TERM | SUBJECT CODE

Subject Categories

THE HUMANITIES AND SOCIAL SCIENCES

COMMUNICATIONS AND THE ARTS
Architecture 0729
Art History 0377
Cinema 0900
Dance 0378
Fine Arts 0357
Information Science 0723
Journalism 0391
Library Science 0399
Mass Communications 0708
Music 0413
Speech Communication 0459
Theater 0465

EDUCATION
General 0515
Administration 0514
Adult and Continuing 0516
Agricultural 0517
Art 0273
Bilingual and Multicultural 0282
Business 0688
Community College 0275
Curriculum and Instruction 0727
Early Childhood 0518
Elementary 0524
Finance 0277
Guidance and Counseling 0519
Health 0680
Higher 0745
History of 0520
Home Economics 0278
Industrial 0521
Language and Literature 0279
Mathematics 0280
Music 0522
Philosophy of 0998
Physical 0523
Psychology 0525
Reading 0535
Religious 0527
Sciences 0714
Secondary 0533
Social Sciences 0534
Sociology of 0340
Special 0529
Teacher Training 0530
Technology 0710
Tests and Measurements 0288
Vocational 0747

LANGUAGE, LITERATURE AND LINGUISTICS
Language
General 0679
Ancient 0289
Linguistics 0290
Modern 0291
Literature
General 0401
Classical 0294
Comparative 0295
Medieval 0297
Modern 0298
African 0316
American 0591
Asian 0305
Canadian (English) 0352
Canadian (French) 0355
English 0593
Germanic 0311
Latin American 0312
Middle Eastern 0315
Romance 0313
Slavic and East European 0314

PHILOSOPHY, RELIGION AND THEOLOGY
Philosophy 0422
Religion
General 0318
Biblical Studies 0321
Clergy 0319
History of 0320
Philosophy of 0322
Theology 0469

SOCIAL SCIENCES
American Studies 0323
Anthropology
Archaeology 0324
Cultural 0326
Physical 0327
Business Administration
General 0310
Accounting 0272
Banking 0770
Management 0454
Marketing 0338
Canadian Studies 0385
Economics
General 0501
Agricultural 0503
Commerce-Business 0505
Finance 0508
History 0509
Labor 0510
Theory 0511
Folklore 0358
Geography 0366
Gerontology 0351
History
General 0578
Ancient 0579
Medieval 0581
Modern 0582
Black 0328
African 0331
Asia, Australia and Oceania 0332
Canadian 0334
European 0335
Latin American 0336
Middle Eastern 0333
United States 0337
History of Science 0585
Law 0398
Political Science
General 0615
International Law and Relations 0616
Public Administration 0617
Recreation 0814
Social Work 0452
Sociology
General 0626
Criminology and Penology 0627
Demography 0938
Ethnic and Racial Studies 0631
Individual and Family Studies 0628
Industrial and Labor Relations 0629
Public and Social Welfare 0630
Social Structure and Development 0700
Theory and Methods 0344
Transportation 0709
Urban and Regional Planning 0999
Women's Studies 0453

THE SCIENCES AND ENGINEERING

BIOLOGICAL SCIENCES
Agriculture
General 0473
Agronomy 0285
Animal Culture and Nutrition 0475
Animal Pathology 0476
Food Science and Technology 0359
Forestry and Wildlife 0478
Plant Culture 0479
Plant Pathology 0480
Plant Physiology 0817
Range Management 0777
Wood Technology 0746
Biology
General 0306
Anatomy 0287
Biostatistics 0308
Botany 0309
Cell 0379
Ecology 0329
Entomology 0353
Genetics 0369
Limnology 0793
Microbiology 0410
Molecular 0307
Neuroscience 0317
Oceanography 0416
Physiology 0433
Radiation 0821
Veterinary Science 0778
Zoology 0472
Biophysics
General 0786
Medical 0760

EARTH SCIENCES
Biogeochemistry 0425
Geochemistry 0996
Geodesy 0370
Geology 0372
Geophysics 0373
Hydrology 0388
Mineralogy 0411
Paleobotany 0345
Paleoecology 0426
Paleontology 0418
Paleozoology 0985
Palynology 0427
Physical Geography 0368
Physical Oceanography 0415

HEALTH AND ENVIRONMENTAL SCIENCES
Environmental Sciences 0768
Health Sciences
General 0566
Audiology 0300
Chemotherapy 0992
Dentistry 0567
Education 0350
Hospital Management 0769
Human Development 0758
Immunology 0982
Medicine and Surgery 0564
Mental Health 0347
Nursing 0569
Nutrition 0570
Obstetrics and Gynecology 0380
Occupational Health and Therapy 0354
Ophthalmology 0381
Pathology 0571
Pharmacology 0419
Pharmacy 0572
Physical Therapy 0382
Public Health 0573
Radiology 0574
Recreation 0575
Speech Pathology 0460
Toxicology 0383
Home Economics 0386

PHYSICAL SCIENCES

Pure Sciences
Chemistry
General 0485
Agricultural 0749
Analytical 0486
Biochemistry 0487
Inorganic 0488
Nuclear 0738
Organic 0490
Pharmaceutical 0491
Physical 0494
Polymer 0495
Radiation 0754
Mathematics 0405
Physics
General 0605
Acoustics 0986
Astronomy and Astrophysics 0606
Atmospheric Science 0608
Atomic 0748
Electronics and Electricity 0607
Elementary Particles and High Energy 0798
Fluid and Plasma 0759
Molecular 0609
Nuclear 0610
Optics 0752
Radiation 0756
Solid State 0611
Statistics 0463

Applied Sciences
Applied Mechanics 0346
Computer Science 0984

Engineering
General 0537
Aerospace 0538
Agricultural 0539
Automotive 0540
Biomedical 0541
Chemical 0542
Civil 0543
Electronics and Electrical 0544
Heat and Thermodynamics 0348
Hydraulic 0545
Industrial 0546
Marine 0547
Materials Science 0794
Mechanical 0548
Metallurgy 0743
Mining 0551
Nuclear 0552
Packaging 0549
Petroleum 0765
Sanitary and Municipal 0554
System Science 0790
Geotechnology 0428
Operations Research 0796
Plastics Technology 0795
Textile Technology 0994

PSYCHOLOGY
General 0621
Behavioral 0384
Clinical 0622
Developmental 0620
Experimental 0623
Industrial 0624
Personality 0625
Physiological 0989
Psychobiology 0349
Psychometrics 0632
Social 0451

UNIVERSITY OF ALBERTA

RELEASE FORM

NAME OF AUTHOR: David J. Warwaruk

TITLE OF THESIS: A Study of Innovation Diffusion Using a Dynamic Fluid Model Analogy

DEGREE: Masters of Business Administration

YEAR DEGREE GRANTED: Fall 1994

David J. Warwaruk
10835-38A Avenue
Edmonton, Alberta
T6J 0L6

October 4, 1994

UNIVERSITY OF ALBERTA

FACULTY OF GRADUATE STUDIES AND RESEARCH

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled A Study of Innovation Diffusion Using a Dynamic Fluid Model Analogy submitted by David J. Warwaruk in partial fulfilment of the requirements for the degree of Masters of Business Administration.

B. R. Nault

Research Supervisor

F. Riggins

Committee Member

S. P. Dozzi

Committee Member

September 30, 1994

Abstract

A model of the innovation diffusion process is developed using a process engineering approach and a fluid dynamic analogy. The fluid dynamic analogy provides a basis for explaining adoption at the aggregate level and allows an examination of the factors that influence the diffusion process. The process engineering approach to the model formulation permits the addition and deletion of the model components to best simulate the diffusion circumstances.

Our model has various parameter options that require the user to decide if they should be reduced to constants or increased to functions to allow better modelling of the diffusion process given certain characteristics of a population of potential adopters and the innovation being introduced to that population. As prior modelling of the diffusion process draws upon theories in the physical sciences (largely epidemiology), we used a physical science analogy to develop our model, but from a novel direction. We view the population as a fluid where the diffusion process obeys fluid dynamic properties.

We demonstrate that the principles governing ideal fluid dynamics are among those principles underlying the general diffusion process of the transmission of ideas. Through a fluid dynamic analogy we demonstrate that fluid dynamic theory is applicable to the study of a diffusion process.

We study the time dependent aspects of the innovation diffusion process, that is, the process by which an innovation is communicated through certain channels over time among members of a social system.

Acknowledgements

Appreciation is expressed to the committee members, Professors Peter Dozzi and Fred Riggins, who made time in their busy schedules to provide guidance and perspective in the development of this thesis.

To Professor Barrie Nault for his interminable enthusiasm and genuine interest in this undertaking.

To my parents, Dr. Joseph and Alexandra Warwaruk, who, through immense effort on their part, instilled in me a sincere appetite for learning and an eternally inquisitive spirit.

And finally to my wife, Corinne, whose patience, support and understanding was, and always is, invaluable beyond words.

Table of Contents

List of Tables

List of Figures

Figure | Page

List of Symbols

Symbol	Quantity	Dimensions* (MLT)
d	Diameter	L
F	Proportion of potential adopters that have adopted	
g	Acceleration of gravity	LT^{-2}
h	Depth, vertical distance	L
i	Reservoir properties	
j	Orifice/conveyance properties	
k_s	Spring constant	MT^{-2}
m	Mass	M
N	Cumulative number of adopters	
N^*	Finite population of adopters	
p	Probability of adoption	
P	Pressure	$ML^{-1}T^{-2}$
Q	Discharge	$L^{3}T^{-1}$
S_d	Spring deflection	L
t	Time	T
v	Velocity	LT^{-1}
V	Volume	L^{3}
w	Time varying effect of internal influence	
W	Force	MLT^{-2}
z	Vertical distance	L
z_d	Elevation difference	L
α	NUI model constant	
β	Sharif-Kabir model constant	
γ_d	Gamma distribution value	
γ	Unit gravity force	$ML^{-2}T^{-2}$
θ	Von Bertalanffy model constant	
A	Area	L^{2}
π	Constant of circumference	
ρ	Density	ML^{-3}
σ	External influence feature	
Φ	Internal influence feature	
ω	Nonuniform influence factor	

* **M** is mass, **L** is length, and **T** is time

1. Introduction

A diffusion process is a sequence of events which results in the transmission of information from one object to another (Mahajan and Peterson, 1985). These objects need not be human beings but might be idealized as fluid reservoirs, the idea itself a fluid, and the transmission of the idea obeying fluid principles.

> *"The task of a diffusion model is to produce a life-cycle sales curve based on usually a number of parameters, which may or may not have behavioral content. The presupposition is that these parameters may be estimated either by analogy to the histories of similar new products introduced in the past, by consumer pretests, or by early sales returns as the new product enters the market. In the past two decades a great deal of work on diffusion modelling has been done, which largely draws on the more well-developed theory of contagious diseases or the spread of epidemics." (Lilien and Kotler, 1983)*

It is our endeavour, through a physical science analogy, to develop a diffusion model with varying degrees of tractability depending on the analysis of the situation. Our model can be conceptualized as a vehicle that can adapt its performance to manoeuvre over varying topographies with a parameter adjustment, such as changing the drive mechanism of the front axle from a constant to some function of the torque applied by the drive train. Similarly, our model has various parameter options that require the user to decide if they should be reduced to constants or increased to functions to allow better modelling of the diffusion process given certain characteristics of a population of potential adopters and the innovation being introduced to that population. As prior modelling of the diffusion process draws upon theories in the physical sciences (largely epidemiology), we thought it appropriate to use a physical science analogy to develop our model, and as epidemiology has been vastly cited in diffusion studies it was our aim to approach the subject from a novel direction. Early in our research we noticed that frequently the literature refers to the diffusion process as a flow from a state of non-adoption to a state of adoption by a population. It struck us that perhaps we could view this population as a fluid and that the diffusion process obeys fluid dynamic properties. What follows is a result of that hypothesis and its investigation.

Information can be conceptualized as the knowledge of an innovation that is communicated from an external or internal source to a set of potential adopters within a social system. The difficulty in any scientific treatment of diffusion processes arises from the fact that the concept of information, although intuitively clear, can be neither formally defined nor precisely measured. If, however, we consider that diffusion processes are physical processes in which information is used to effect a certain result then the effectiveness of the information can be evaluated in terms of that result. Based on this consideration we can establish certain general properties of diffusion processes, i.e. properties possessed by such processes irrespective of their specific nature.

We shall demonstrate that the principles governing ideal fluid dynamics, as defined in by Pao (1967), Streeter and Wylie (1981), and others are among those principles underlying the general diffusion process of the transmission of ideas. We hypothesize that a diffusion process can be studied in terms of a fluid dynamic process. Through a fluid dynamic analogy we intend to demonstrate that fluid dynamic theory seems to be applicable to the study of a diffusion process rather than being restricted to the process of transmission of fluids.

In general, our fluid system (adoption system) can be characterized in terms of a volume *V* (a population) and a set of reservoirs *i* (non-adopter, adopter) or states which effects a partition of *V* at a given point in time. The transition (diffusion) of fluid from one reservoir to another is dependent upon the elevation differences between the fluid surfaces in the reservoirs.

A recent review of diffusion modelling by Mahajan, Muller, and Bass (1990) examines the development of the subject over the last two decades. In this paper we develop a physical framework for understanding the process of diffusion. As in any model, we do not attempt the difficult task of explicitly representing all of the factors, rather we focus on those we consider critical.

We study the time dependent aspects of the innovation diffusion process, that is, the process by which an innovation is communicated through certain channels over time among members of a social system.

In this paper we first present basic diffusion models as introduced by Bass (1969) and others. We then define some fluid characteristics and properties so the reader has a basic understanding of the principles governing fluid flow. Thirdly, through our fluid analogy of the innovation diffusion process, we describe the diffusion process and it's internal and external influences. We then compare the results of our model with existing flexible diffusion models. Finally we suggest areas of future research that our model compels.

1.2 References

Bass, F. M. (1969), "A New Product Growth Model for Consumer Durables", *Management Science*, 15, (5), 215-277.

Lilien, G. L. and Kotler, P. (1983), *Marketing Decision Making; A Model-Building Approach*, First Edition. Harper & Row, Publishers, Inc. (New York).

Mahajan, V., Muller, E., and Bass, F. (1990), "New Product Diffusion Models in Marketing: A Review and Direction for Research", *Journal of Marketing*, 54, (1), 1-26.

Mahajan, V. and Peterson R. A. (1985), "Models for Innovation Diffusion", *Quantitative Applications in the Social Sciences*, Series 7, Number 48. Sage Publications Inc. (London).

Pao, R. H. F. (1967), *Fluid Dynamics*, First Edition. Charles E. Merrill Books, Inc. (Columbus, Ohio).

Streeter, V. L. and Wylie, E. B. (1981), *Fluid Mechanics*, First SI Metric Edition. McGraw-Hill Ryerson Limited (Toronto).

2. Literature Review

If $N(t)$ is the cumulative number of adopters who have adopted an innovation to time t, then most of the innovation growth models give rise to either linear, concave, or S-shaped N versus t curves (Mahajan and Peterson, 1985). The curves could start at some initial value of adopters $N(0)$ ($N=N(0)$ at $t=0$), and each would have $N(t)=N^*$ as an asymptote, so that $N(t)\rightarrow N^*$ as $t\rightarrow\infty$. N^* is the ultimate or finite population level. The models differ in their respective rates of increase, $dN(t)/dt$. In some models, such as the S-shaped logistic curves, the rate of diffusion, $dN(t)/dt$, may increase very quickly at the beginning, reach a peak and then decrease slowly to fall to zero as $t\rightarrow\infty$ (Kumar and Kumar, 1992).

In this section we attempt to outline the major ideas of the theory of the adoption and diffusion of new ideas or new products by a social system.

Independent of others in a social system, some individuals will decide to adopt an innovation. They are commonly referred to as innovators. Rogers (1983) specifies the following classes of adopters: 1. Innovators; 2. Early Adopters; 3. Early Majority; 4. Late Majority; and 5. Laggards. The timing of the adoption by the various groups defines this classification. Rogers (1983) aggregates groups 2 through 5 and defines them as imitators. The imitators are influenced in their timing of adoption by the pressures of the social system which increases with time as the cumulative number of adopters increase. This classification of adopters has been referred to by some as the unbundling of adopters (Mahajan, Muller and Bass, 1990). The question is; are we able to classify adopters based on their propensity to adopt or is there simply one group of adopters that are affected by both the internal and external influences equally over time? This is an issue which should be addressed with the derivation of any innovation diffusion model. Another issue is the effectiveness of the model in capturing the communication structure between two assumed distinct groups of innovators and imitators. And finally can the model, which investigates innovation diffusion at the aggregate level, be linked to the adoption decisions at the individual level? These are three issues which we intend to investigate with our fluid model.

Before continuing it is important to define the earliest forms of diffusion modelling; the External-Influence model, and the Internal-Influence model.

The External-Influence model is mathematically represented by

$$\frac{dN(t)}{dt} = \sigma [N^{*} - N(t)] ,$$

where σ is defined as the coefficient of external influence emanating from outside the social system, i.e., the effect of mass media communications, the influence of government agencies, and the impact of salespeople on the diffusion process. Integration of the rate of diffusion equation results in a relationship for cumulative adoption to time,

$$N(t) = N^{*} [1 - e^{-\sigma t}] .$$

Over time the cumulative number of adopters increases, but at a constantly decreasing rate, (Fig. 2.1). The adopters are thought to be driven <u>only</u> by sources external to the social system.

The Internal-Influence model has a rate of change mathematically represented by

$$\frac{dN(t)}{dt} = \phi N(t) \; [N^{*} - N(t)] ,$$

where the constant ϕ is defined as the coefficient of internal influence as it reflects the efficiency of the interaction between prior adopters, $N(t)$, and potential adopters, $[N^{*}-N(t)]$. While the External-Influence model assumes that there is no communication between members of a social system, the Internal-Influence model is based on the assumption that diffusion occurs <u>only</u> through communication between prior adopters and potential adopters in the system. Integration of the rate of change equation results in the relationship for cumulative adoption,

$$N(t) = \frac{N^{*}}{1 + [\frac{N^{*} - N(0)}{N(0)}] \, e^{[-\phi N^{*} [t - t(0)]]}} ,$$

where $N(0)$ is the number of cumulative adopters at time $t=0$, and when $N(0)$ is 0, $N(t)$ is defined as 0. This correspond to a logistic, or S-shaped, diffusion curve, (Fig. 2.2).

The basic Mixed Influence innovation diffusion model was introduced into marketing by Bass (1969), and has become the standard for further development and modification. The Bass (1969) model explicitly combines the two previous models. Bass (1969) assumed that initial buyers are persuaded to adopt an innovation by factors outside their immediate environment or social system. These factors referred to external influences. Of importance to marketing, this external influence could be the innate value of the innovation or some kind of promotional message. Buyers are also influenced to adopt new products because they see those within their immediate environment using them, previously designated the internal influence.

The mixed influence diffusion model defines a propensity to adopt based on innovation and imitation,

$$\frac{dN(t)}{dt} = [\sigma + \phi N(t)]\,[N^{*} - N(t)]\,, \tag{2.1}$$

again where σ is the external influence and ϕ is the internal influence. Integration of Bass's (1969) model yields the following cumulative adoption distribution:

$$N(t) = \frac{N^{*} - \dfrac{\sigma[N^{*} - N(0)]}{[\sigma + \phi N(0)]} e^{-[\sigma + \phi N^{*}][t - t_0]}}{1 + \dfrac{\phi[N^{*} - N(0)]}{[\sigma + \phi N(0)]} e^{-[\sigma + \phi N^{*}][t - t_0]}}$$

where

$$N(t = t_0) = N(0)\,,$$

Plotting the cumulative adoption distribution results in a generalized logistic curve, the shape of which is determined by both σ and ϕ, and resembles the curve determined in the Internal-Influence model (Fig. 2.2).

In Eq. (2.1) each non-adopter has a probability, p, of adopting the innovation at time t,

$$p(t) = \sigma + \phi N(t)$$

where σ, the external influence, represents the information conveyed to the innovators relating to the product itself through alternative sources, ϕ, the internal influence, represents the effectiveness of experience-based information communication by previous adopters in convincing non-adopters to adopt, and $N(t)$ is the cumulative number of adopters at time t. As previously stated the finite number of potential adopters is N^*.

Several assumptions that underlie the fundamental diffusion model must be recognized (Mahajan and Peterson, 1985):

(i) The diffusion process is binary. Members of the social system either adopt the innovation or they do not adopt it. Thus, adoption is treated as a discrete rather than continuous event and therefore the stages of the adoption process, such as knowledge and awareness, are not accounted for, i.e., adoption takes place instantaneously upon the decision to adopt.

(ii) The fundamental diffusion model is based on the assumption that there is a finite number of potential adopters, N^*, in the social system and this number is known or can be estimated. Thus, the social system is not allowed to increase or decrease in size.

(iii) The fundamental diffusion model only permits one adoption by an adopting unit which can not be rescinded.

(iv) Market potential of the new product remains constant over time. Market potential of the product is determined at the time of introduction and remains unchanged over its entire life.

(v) The innovation does not change over the diffusion process and is assumed to be independent of other innovations.

(vi) Spatial diffusion, or geographic diffusion, of an innovation is not considered, i.e., the geographic boundaries of the social system remain constant throughout the diffusion process.

(vii) The model captures all pertinent information regarding the diffusion process.

2.1 Flexible Diffusion Models

If a model, such as the mixed influence model (Bass, 1969), has a fixed point of inflection (where the diffusion rate is at a maximum) and is symmetrical about that point of inflection, then it may well be limited in its flexibility to handle a wide variety of diffusion patterns (Mahajan and Peterson, 1985). Therefore, because of the lack of flexibility of the fundamental diffusion model, several attempts have been made to develop more flexible diffusion models.

Flexible diffusion models allow the generalized rate of diffusion, or $dF(t)/dt$ where $F(t)=N(t)/N^*$, curve to be symmetric as well as nonsymmetric, with the point of inflection of the cumulative adoption curve responding to the diffusion pattern instead of being determined a priori. In each of the following flexible diffusion models the addition of extra parameters allows for increased degrees of freedom.

2.1.1 Floyd Model

Floyd (1962) attempted to empirically fit certain observed diffusion patterns using

$$\ln\frac{F(t)}{1-F(t)}+\frac{1}{1-F(t)}=c+\phi t$$

where c is a constant. Differentiation of this equation reveals a cubic model

$$\frac{dF(t)}{dt}=\phi F(t)\,[1-F(t)]^2=\phi F(t)-2\phi F(t)^2+\phi F(t)^3.$$

The Floyd (1962) model is nonsymmetric with a point of inflection of $F(t)$=0.33 and behaves in a fashion similar to the Internal-Influence model (Mahajan and Peterson, 1985) with a single parameter, ϕ, representing the effect of the internal influence.

2.1.2 Sharif and Kabir Model

Sharif and Kabir (1976) combined the internal-influence logistic model and the Floyd model and models the diffusion process using two parameters ϕ and β,

$$[1-\beta]\,[\ln\frac{F(t)}{1-F(t)}]+\beta\,[\ln\frac{F(t)}{1-F(t)}+\frac{1}{1-F(t)}]=c+\phi t,$$

where β is a constant between 0 and 1 and represents a measure of the market potential of the product. Differentiation reveals

$$\frac{dF}{dt}=\frac{\phi F[1-F]^2}{1-F[1-\beta]}$$

The Sharif and Kabir (1976) model can be either symmetric or nonsymmetric and has a point of inflection between $F(t)$=0.33 and $F(t)$=0.50. When β=0 the model behaves like the Internal-Influence model, and when β=1 it becomes the Floyd (1962) model (Mahajan and Peterson, 1985).

2.1.3 Jeuland Model

Jeuland (1981) devised a model based on the following assumptions:

(i) External influence in the diffusion process relates to the potential adopter's propensity to adopt the innovation.

(ii) The population of potential adopters is heterogeneous with respect to propensity to adopt, which differed from Bass who assumed they were homogeneous.

(iii) Propensity to adopt varies with a gamma distribution, $\gamma_{/J}$, which is a probability distribution that approximates the normal distribution.

These assumptions led to the diffusion model

$$\frac{dF(t)}{dt} = [\sigma + \phi F(t)]\,[1 - F(t)]^{[1+\gamma_{/J}]}$$

that uses three parameters σ, ϕ and $\gamma_{/J}$.

The Jeuland (1981) model can be symmetric or nonsymmetric with a point of inflection ranging from $F(t)=0$ to $F(t)=0.50$. When $\gamma_{/J}=0$ the model reduces to the Bass model, and when $\sigma=0$ and $\gamma_{/J}=1$ the Jeuland model reduces to the Floyd model (Mahajan and Peterson, 1985).

2.1.4 Easingwood, Mahajan and Muller Model

Easingwood, Mahajan and Muller (1983) proposed two flexible versions of the fundamental diffusion model. They are respectively termed the Nonsymmetric Responding Logistic (NSRL) model and the Nonuniform Influence (NUI) model. These models represent the diffusion process by incorporating the use of four parameters σ, ϕ, τ and α. The purpose of these models was to overcome an inherent limitation of the fundamental diffusion model which was the assumption that the impact of the internal influence between adopters and potential adopters remain constant over time (i.e., the coefficient of internal influence, ϕ, is a time-invariant constant). For most innovations the impact of the internal influence is likely to change, either increasing or decreasing, as the diffusion process unfolds. The NUI model is

$$\frac{dF(t)}{dt} = [\sigma + \phi F(t)^{\tau}]\,[1 - F(t)]$$

and the impact of internal influence as a function of adoption level is represented through the relationship

$$w(t) = \phi F(t)^{\alpha}$$

where $w(t)$ is the time varying effect of the internal influence, α is a constant, and from the previous expression $\tau=(1+\alpha)$ and is referred to as the nonuniform influence factor. When $\tau=1$ (or $\alpha=0$), the model assumes a constant or uniform internal influence.The presence of a nonuniform influence effect in the diffusion process is indicated by $\tau \neq 1$. When $\sigma=0$, the NUI model reduces to the NSRL model (Easingwood, Mahajan and Muller, 1981);

$$\frac{dF(t)}{dt} = \phi F(t)^{\tau}[1-F(t)]$$

Both the NUI and NSRL model can be either symmetric or nonsymmetric with their point of inflection varying between $F(t)=0$ and $F(t)=1$ (Mahajan and Peterson, 1985).

2.1.5 Von Bertalanffy Model

Von Bertalanffy (1957), through work in metabolic analysis and biological growth, hypothesized that the diffusion rate behaves according to a two parameter model in ϕ and v;

$$\frac{dF(t)}{dt} = \frac{\phi}{1-\theta} F(t)^{\theta}[1-F(t)^{1-\theta}]$$

By examining various values of θ, the model can be shown to be flexible. When $\theta=0$ the model reduces to the External-Influence model, and when $\theta=1$ the model reduces to the Internal-Influence model (Mahajan and Peterson, 1985). The Von Bertalanffy (1957) model can be either symmetric or nonsymmetric, and has a point of inflection ranging from $F(t)=0$ to $F(t)=1$.

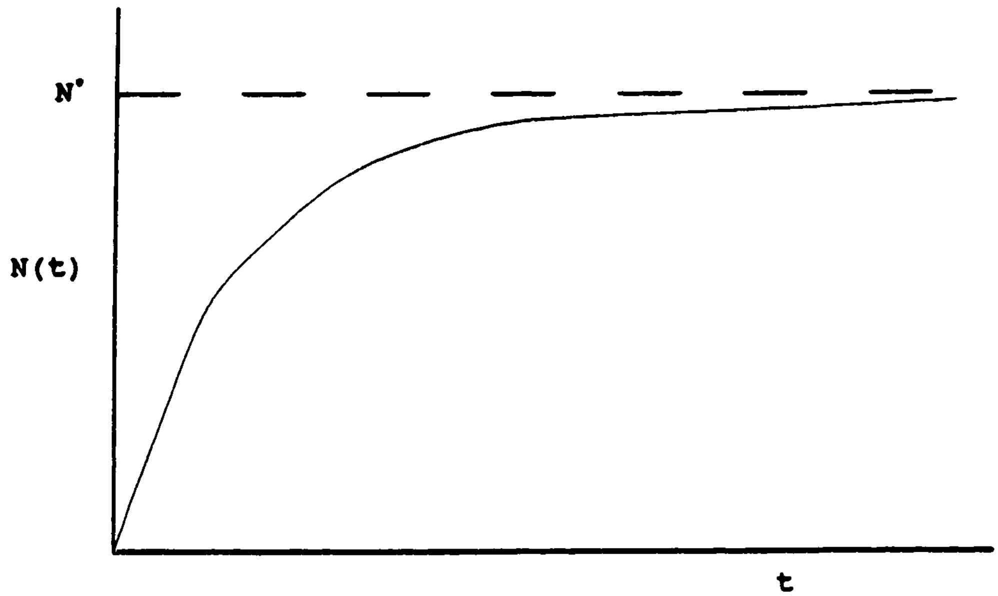

Figure 2.1 *External-Influence, cumulative adoption vs. time curve.*

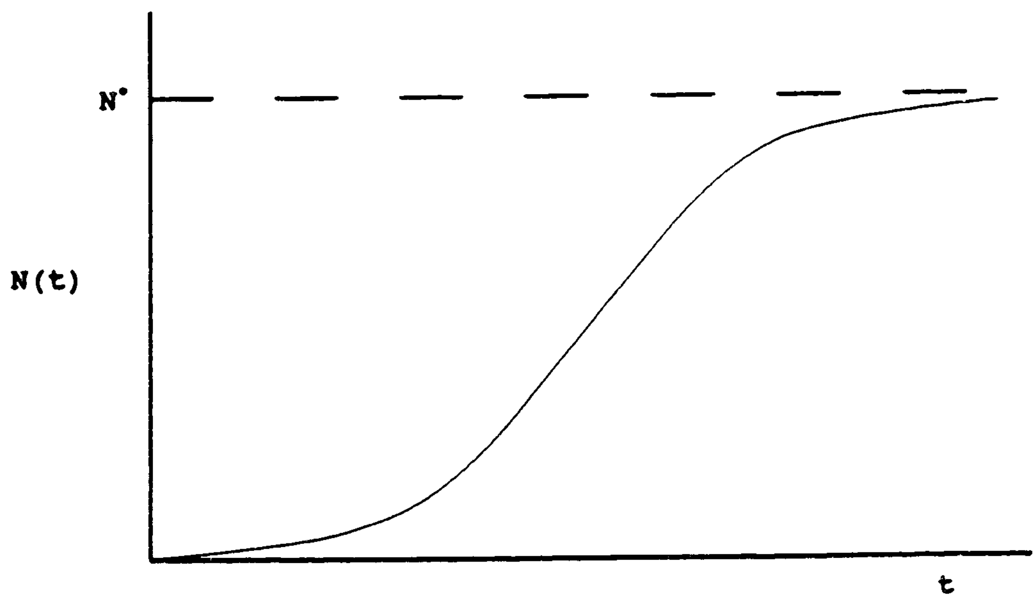

Figure 2.2 *Internal and Mixed Influence, cumulative adoption vs. time curve.*

2.2 References

Bass, F. M. (1969), "A New Product Growth Model for Consumer Durables", *Management Science*, 15, (5), 215-277.

Easingwood, C. J., Mahajan, V., and Muller, E. (1981), "A Nonsymmetric Responding Logistic Model for Technological Substitution", *Technological Forecasting and Social Change*, 20, 199-213.

Easingwood, C. J., Mahajan, V., and Muller, E. (1983), "A Nonuniform Influence Innovation Diffusion Model of New Product Acceptance", *Marketing Science*, 2, 273-296.

Floyd, A. (1962), "Trend Forecasting: A Methodology for Figure of Merit", in *Technological Forecasting for Industry and Government: Methods and Applications*, J. Bright, ed. Prentice-Hall, Inc. (NJ), 95-105.

Jeuland, A. P. (1981), "Parsimonious Models of Diffusion of Innovation, Part A: Derivations and Comparisons", Working Paper, University of Chicago.

Kumar, U. and Kumar, V. (1992), "Technological Innovation Diffusion: The Proliferation of Substitution Models and Easing the User's Dilemma", *IEEE Transactions on Engineering Management*, 39, (2), 158-168.

Mahajan, V., Muller, E., and Bass, F. (1990), "New Product Diffusion Models in Marketing: A Review and Direction for Research", *Journal of Marketing*, 54, (1), 1-26.

Mahajan, V. and Peterson R. A. (1985), "Models for Innovation Diffusion", *Quantitative Applications in the Social Sciences*, Series 7, Number 48. Sage Publications Inc. (London).

Rogers, E. M. (1983), *Diffusion of Innovations*, Third Edition. Collier Macmillan Publishers (London).

Sharif, M. N. and Kabir, C. (1976), "A Generalized Model for Forecasting Technological Substitution", *Technological Forecasting and Social Change*, 8, 353-364.

Von Bertalanffy, L. (1957), "Quantitative Laws in Metabolism and Growth", *The Quarterly Review of Biology*, 32, 217-231.

3. Development of Physical Model

If we were to draw a physical comparison to a diffusion process, it may be analogous to a fluid system where the adopters and non-adopters constitute the fluid volume and their movement from a non-adoption to adoption state is the flow. There may be many good analogies, but we found this one intuitive due to the ability to tangibly visualize the process at work. The use of a physical science model to analyze the diffusion of innovations is not unique to this paper, as discussed previously the science of epidemiology has been largely drawn upon for much of the prior literature. The use of fluid mechanics for such an analysis, to our knowledge, is novel. Applying fluid flow characteristics and principles, which will be detailed in later sections, we will specify a physical science model that behaves in a similar fashion to the innovation diffusion models. The first step is to build a simple model so we can easily relate the market actions to the fluid flow and then examine the dynamics of the system.

In this section we describe the properties of a fluid, which we later determine is a suitable medium for the description of the diffusion process, and the physical framework with which we shall model the diffusion of an innovation from a finite population of potential adoption to an adoption state. Our framework is developed in a piece-wise fashion so as to describe each component of the model and draw its analogy to a market situation at each step.

3.1 Fluid Properties and Characteristics

A fluid may be either a liquid or a gas. A gas is very compressible, and when all external pressure is removed, it tends to expand indefinitely. A liquid is <u>relatively</u> incompressible, and if all pressure, except that of its own vapour pressure, is removed, the cohesion between particles will hold them together so that the liquid will not expand indefinitely. Therefore a liquid may have a free surface, i.e., a surface from which all pressure is removed, except that of its own vapour. The notion of a free surface, i.e., river surface, lake surface, allows us to easily visualize the volume changes in the reservoirs and thus will also aid in the conceptual resemblance to the innovation diffusion process. The free surface, coupled with the flow characteristics described in the following section, make it apparent that our medium is in actuality a liquid, i.e., water, even though we refer to it as a fluid.

3.1.1 Continuum Theory

Mahajan, Muller and Bass (1990, pp. 6) state;

> *"A key aspect of the Bass model is that it addresses the market in the aggregate...The emphasis is on the total market response rather than an individual customer."*

This is a key aspect of not only the Bass (1969) model but the other flexible diffusion models which we have examined.

Similarly the study of fluid dynamics also addresses its medium in the aggregate. In spite of the molecular structure of matter, we consider fluids to be continuous and without voids. For our model to be an accurate representation of the diffusion of innovations this assumption must also apply to the market situation. The assumption is justified by the fact that we are not primarily interested in the behaviour of the individual fluid molecules, or adopters, and that under ordinary conditions the results of analysis of a continuous fluid, or population of adopters, agree fairly well with the observed behaviours of fluid motion, or innovation diffusion. This notion is often referred to as the concept of a continuum and is merely an idealization that simplifies the analysis of the problem. When we deal with engineering problems at the macroscopic level in which the dimensions are very large compared with molecular distances, we are concerned with volumes that are considerably larger than molecular dimensions, and therefore, contain many molecules. We are interested in the statistical average properties and behaviours of a large number of molecules and not in the properties and behaviours of the individual molecules. Because we disregard the action of individual molecules we can consider the fluid to be a continuous substance and adopt a continuum model of fluid. Similarly when evaluating the adoption behaviours of a finite population of potential adopters we are concerned, in our studies, with the patterns of adoptions of the aggregate not with the patterns of the individual.

3.1.2 Definitions

Before describing our physical model we define some of the characteristics and properties of any simple fluid system tc facilitate an understanding of the concepts affecting fluid flow.

Acceleration of gravity, g, or the local gravity constant varies slightly from one part of the earth to another depending on elevation relative to sea level (Streeter and Wylie, 1981) and is a necessary component in describing the forces and velocities acting in a fluid system. For the purpose of this analysis g will be a constant, however, in future research it may be necessary to examine a functional form of g that depends on elevation. We intend to show that g is analogous to the coefficient of external influence, σ, described by Bass (1969) and others as the influence of change agents on the diffusion process or any influence other than a prior adoption. Any analogy will be detailed further when we have established a simple physical framework to which we can refer.

Density of a fluid, ρ, is defined as its mass per unit volume.

$$\rho = \frac{m}{V}$$

The density of a given fluid is a function of temperature and pressure which act to expand or compress the volume of the given fluid. However, the density of liquids is only slightly affected by pressure. Frequently the density of a liquid may be assumed to be constant. We then speak of an incompressible fluid, and accept this to be the case with our analogy. The density of a fluid can be thought of as a characteristic of how "tightly packed" the fluid is. Similarly, a market will also have a density which describes, either tangibly or intangibly, the dispersion of the population. We could refer to the ease, or difficulty, with which communication takes place in the social systems as its density, i.e., if the population is tightly packed, information is conveyed much more easily, either through the mass media or through word of mouth. The density of a social system could also refer to the degree that technology speeds the communication process. In this case the population is not tightly packed in a physical sense, but technology can be thought of as a tool for increasing the density. In any case, ρ is a measure of the efficiency of the communication channels, either external or internal, or more specifically the acceptance of information by a characteristic population.

Unit gravity force, γ, is the force due to gravity per unit volume, or the force with which a gravitational body such as the earth attracts a unit volume of the fluid. The unit gravity force can be obtained by multiplying density by the local gravitational constant.

$$\gamma = \rho g$$

The unit gravity force varies with elevation, from sea level, and temperature, depending upon density and gravity.

Area, Λ, and cross-sectional area, will, unless stated otherwise, refer to the area normal or perpendicular to the direction of the velocity and/or force being considered. For example, in the case of the round orifice (i.e., a hole), or conveyance, Λ is a function of the orifice or conveyance diameter,

$$\Lambda = \frac{\pi d^2}{4}$$

where d is the diameter and π is a constant of circumference. We defer drawing an analogy for Λ until we have developed our framework, as cross-sectional area will refer to more than one characteristic of our model.

The depth of fluid in reservoir i, $h_i(t)$, is measured from the fluid surface to the centre of an exit or entrance orifice. The distance from the respective fluid surface to an arbitrary datum line is referred to as the elevation head, $Z_i(t)$. This term is convenient when we deal with two or more reservoirs at different elevations and allows us to analyze their dynamics relative to a fixed reference point. In this case the elevation head difference is described by $Z_{ij}(t)$. The term 'head' is recurrently used as a reference to the height of fluid from the datum line in the 'emptying' reservoir. The depth of the reservoir is a characteristic of the size of the social system, or more specifically the size of a population of adopters or non-adopters.

A force, $W(t)$, represents the action of one body on another. It is characterized by its point of application, its magnitude, and its direction; a force is represented by a vector (Beer and Johnston, 1984). In the next section we illustrate $W_i(t)$ as the force acting vertically on the bottom of reservoir i due to the mass of fluid in that reservoir (Fig. 3.1). We can idealize this force, in the market situation, to be analogous to the inertia of an increasing population of adopters, and a decreasing population of non-adopters, having an effect on the social system as a whole. In other words the force, $W_i(t)$, is a component of the internal influence of the diffusion process.

3.2 Single Reservoir System

We now establish a physical framework on which we can build a model of increasing complexity to aid in the analysis of the diffusion process. In Fig. (3.1) we illustrate a component of the system.

Reservoir i contains a fluid of volume $V_i(t)$ which is akin to a finite population of potential adopters of an innovation, N^*. The volume $V_i(t)$ is characterized by a depth $h_i(t)$ and a cross-sectional area $\Lambda_i(h_i(t))$, i.e., the area of the fluid surface. The volume in the reservoir is akin to the population size which is defined by the cross-sectional area and depth. The cross-sectional area is a characteristic of the population, i.e., is it constant over its depth (a cylinder or a cube), or does it vary over its depth (a funnel or a cone). We assume that $\Lambda_i(h_i(t))$ is constant over $h_i(t)$, Λ_i, which simplifies the analysis, i.e., the population characteristics invariant over time or the population is considered homogeneous. $V_i(t)$ exerts a force $W_i(t)$ on the reservoir bottom which is a function of fluid depth $h_i(t)$, the fluid density ρ, the local gravity constant g, and the area, Λ_i, upon which the force is acting. $W_i(t)$ is a force acting downwards on the bottom of the entire reservoir, therefore, the area with which we are concerned is the entire cross-section of the reservoir, Λ_i. Since $\gamma = g\rho$, $W_i(t) = \gamma h_i(t)\Lambda_i = \gamma V_i(t)$.

In Fig. (3.1) $v_i(t)$ represents the velocity of the fluid surface which is proportional to the exit velocity of the fluid given constant cross-sectional reservoir and orifice areas;

$$v_i(t) = v_j(t)\,\frac{\Lambda_j}{\Lambda_i}$$

When the orifice, j, of cross-sectional area Λ_j, in reservoir i is opened at time $t+\delta t$, Fig. (3.1), the fluid begins to exit the reservoir with a velocity $v_j(t)$. Where

$$v_j(t) = \sqrt{2gh_i(t)} = \sqrt{2g\frac{V_i(t)}{\Lambda_i}}$$

and the depth of the fluid in the reservoir, $h_i(t)$, is equal to the reservoir volume, $V_i(t)$, divided by its cross-sectional area, Λ_i.

This is a decelerating system,

$$\frac{dv_j(t)}{dt} = \frac{1}{2}\sqrt{\frac{1}{2gh_i(t)}}\,\frac{dh_i(t)}{dt} = \frac{1}{2}\sqrt{\frac{\Lambda_i}{2gV_i(t)}}\,\frac{dV_i(t)}{dt} < 0$$

as the depth $h_i(t)$ is decreasing over time so is the fluid velocity $v_j(t)$ through the orifice. The velocity for this and other cases of this nature is determined using Bernoulli's equation (Eq. 3.2.1) which is derived in

Streeter and Wylie (1981, pp. 98-103),

$$Z_1 - Z_2 + \frac{P_1 - P_2}{\gamma} + \frac{V_1^2 - V_2^2}{2g} = 0 \, . \tag{3.2.1}$$

Where *P* is pressure which we have not modelled, but discuss in section 5.3.

Having established our simple physical model, we are now able to better establish our analogy for gravity, *g*, as the coefficient of external influence, σ. If we examine Fig. (3.1) at time $t=0$, the reservoir is full of fluid and the orifice *j* has just been opened. Using our previous analogies for the market situation we are presented with a population of potential adopters, or non-adopters, to which a new product or innovation is introduced at time $t=0$. What causes the fluid to flow, or members of the population to adopt given no prior adoption or flow? According to Bass (1969) and others, the impetus to adopt is caused by the influence of change agents external to the social system which we have previously referred to as the coefficient of external influence, σ. In the case of the fluid, the only force acting upon the fluid volume to force it to flow is gravity, *g*. If we removed gravity no flow would occur. The single reservoir system is akin to the External-Influence model where the communication between members of the social system has no affect on the rate of adoption. Only the influence of the external change agents, i.e., mass media, or in the fluid case gravity, affects the rate of diffusion.

3.3 Two Reservoir System

Fig. (3.2) illustrates the addition of a second reservoir to the system. The two reservoirs are connected by a round conveyance, or pipe, of constant diameter over its length.

Examining reservoir *2* independently, as it is being filled with fluid it begins to develop the same conditions, forces and depth of fluid, as in the previous case. It is now important to discuss the depth of fluid being relative to a datum line. When the two reservoirs are joined, combined effects begin to occur. In this case where the reservoirs are on the same elevation, reservoir *1* will not be able to completely empty. The system will reach a point of equilibrium when Z_1 is equal to Z_2, at the same point in time where velocity in the conveyance will be equal to zero (Fig. 3.3). The question is; How are we be able to remove more fluid from reservoir *1* to reservoir *2*?

One solution is to reduce the elevation of reservoir *2*, Fig. (3.3), so that when Z_1 equals Z_2, or the elevation head difference is equal to zero (Z_d=0), and there is more fluid in *2* than in *1*. If we do not lower reservoir *2* the system will not allow all of the fluid to flow from reservoir *1* to reservoir *2*, i.e., a proportion of the population of non-adopters will never adopt. This may be a valid situation, however, we suggest that the potential for the process to reach completion should not be a boundary condition, but should be captured in the parameters of the model with regards to communication efficiencies and population characteristics. We observe that the system potential is defined by the relative position of reservoir *2* to reservoir *1* or $Z_d(t)$, i.e., the final cumulative number of adopters will occur where $Z_1(t)=Z_2(t)$ or where $Z_d(t)=0$, regardless of the size of N^*. N^* will only affect the rate of diffusion.

To this point we have developed a decelerating system that models the external influences. The system, to include the internal influences, requires a modification of the parameters to include a feedback system. Ideally we prefer a system where reservoir *2* has some influence on the rate of flow, thus, modelling the internal communication function that Mansfield (1961), Bass (1969) and Gray (1973) examined. As addressed earlier, the way to achieve this is by changing the elevation head difference, Z_d, over time through a dynamic, time and space variant, component which depends on the fluid volume in reservoir *2*. Reservoir *2* must, therefore, be moveable so as to increase the elevation head difference, Z_d, as the volume of reservoir *2* increases.

Fig. (3.4) shows the dynamic system. Reservoir *2* is placed on a spring. The spring, in turn, is not restricted from moving in the horizontal direction to avoid the added complication of lengthening the conveyance as the spring compresses downwards.

3.3.1 Spring Characteristics

The magnitude of a force W required to deflect (compress) a spring (Fig. 3.5) is proportional, in the linear case, to the deflection S_d of the spring, $W=k_sS_d$, or the deflection of the spring is the quotient of the force exerted on it and the spring constant, $S_d=W/k_s$, in the linear case, where k_s is the constant of proportionality.

The spring deflection may be a linear, Fig. (3.6a) or nonlinear function, Fig. (3.6b) and 3.6c) of force W. This spring function, $S_d(W(t))$, is significant in our analysis as it will control the diffusion rate of the innovation due to the internal influence (ϕ as illustrated in the Bass (1969) model) and k_i is a measure of the efficiency of the internal influence or the effectiveness of the word of mouth communication upon the characteristic population, i.e., the ability of the previous adopters to convince the non-adopters to adopt. It is therefore essential that the spring function be chosen to best represent the market situation. Intuitively, we may choose a spring function that is concave, Fig. (3.6b). This results in a spring that deflects substantially with initial loading and increasingly stiffens with further loading. Or, we may choose a spring function that is convex, Fig. (3.6c). This results in a spring that resists deflection initially with applied incremental force, then deflects at an increasing rate toward an asymptote. For the remainder of this thesis we examine the linear spring case, so that $S_{ii}(W(t)) = W(t)/k_i$, leaving the nonlinear instance for future research.

3.3.2 System Equations

Examining the forces at work in Fig. (3.4), at time $t=0$; reservoir *1* is full, $W_1(t)$ is at its maximum, reservoir *2* is empty, $W_2(t)$ is zero, the spring at *2* is completely uncompressed $S_{ii}(W_2(t))=0$, where $S_{ii}(W_2(t))$ is the deflection that the spring has undergone from its original uncompressed state, i.e., how far reservoir *2* has moved vertically from its relative position at time $t=0$. We model the conveyance diameter, at this level of analysis, as constant over its length, therefore $A_I = A_{II}$ which means that the velocity in the conveyance, $v_i(t)$, is uniform with respect to its length. Due to the uniform and frictionless nature of the flow, the conveyance can be neglected analytically altogether. It is required, for our purposes, as a boundary condition that only allows the fluid to flow from reservoir *1* to reservoir *2*, i.e., for our purposes the conveyance's only function is to ensure that fluid leaving reservoir *1* goes directly and instantaneously to reservoir *2*. This is analogous to assuming the adoption process is instantaneous in accordance with the same assumption underlying Bass's (1969) and the other models. Ideally, we can conceptualize our system as two reservoirs without a conveyance and with the condition that all fluid outflowing reservoir *1* at a velocity $v_i(t)$ and with a volumetric rate of flow $Q_i(t)$ (which is the volume of fluid per time period discharging through the orifice) through an orifice *j* of cross-sectional area A_i, inflows into reservoir *2* with the same velocity and discharge through an orifice of the same area.

At the instant that flow is allowed from reservoir *1* to reservoir *2* the velocity in the conveyance becomes

$$v(t=0)=\sqrt{2gh_1(t=0)}=\sqrt{2gZ_d(t=0)}\,.$$

This condition only holds at time $t=0$ because this is the only time at which $Z_d(t)$ equals $h_1(t)$.

At time $t+\delta t$, Fig. (3.7), a certain amount of fluid has been transferred from reservoir *1* to reservoir *2* through the conveyance. The mass of this fluid has caused the spring at reservoir *2* to compress and change the elevation head difference an order of magnitude dependent on the spring stiffness constant k_s.

As reservoir *2* fills or as $h_2(t)$ increases, $Z_1(t)$ is falling and $Z_2(t)$ is dependent on the spring function as to whether it is decreasing or increasing. As the fluid depth $h_2(t)$ in reservoir *2* increases, the vertical force $W_2(t)$ in reservoir *2* also increases, thus increasing the spring deflection $S_d(W_2(t))$ in a downward direction. We then can determine a number of equations for the system at any instant. The vertical force in reservoir *2* is

(3.3.1)

$$W_2(t)=\gamma\Lambda_2 h_2(t)=\gamma V_2(t)\,.$$

The velocity in the conveyance and entering reservoir *2*, assuming frictionless conditions, is

(3.3.2)

$$v_j(t)=\sqrt{2gZ_d(t)}$$

and the volumetric rate of flow, or discharge, is

(3.3.3)

$$Q_j(t)=[\sqrt{2gZ_d(t)}]\,\Lambda_j$$

where the discharge is equivalent to the product of the velocity through the section and its cross-sectional area. And based on the conservation of mass the discharge remains constant between any two sections in a system. The spring deflection is assumed to be a linear function of force

(3.3.4)

$$S_d(W_2(t)) = k_s W_2(t) .$$

The elevation difference can be expressed as

(3.3.5)

$$Z_1(t) - Z_2(t) = Z_d(t) = h_1(t) + S_d(W_2(t)) - h_2(t) .$$

Substituting Eq. (3.3.1) into Eq. (3.3.4) we can determine the linear spring deflection from the volume of fluid in reservoir 2.

(3.3.6)

$$S_d(W_2(t)) = k_s \gamma V_2(t)$$

Substituting Eq. (3.3.6) into Eq. (3.3.5) results in

(3.3.7)

$$h_1(t) = Z_d(t) + h_2(t) - k_s \gamma V_2(t)$$

and rearranging Eq. (3.3.2) to solve for the elevation head difference

(3.3.8)

$$Z_d(t) = \frac{v_j(t)^2}{2g} ,$$

then using Eq. (3.3.8) in Eq. (3.3.7)

(3.3.9)

$$h_1(t) = \frac{v_j(t)^2}{2g} + h_2(t) - k_s \gamma V_2(t)$$

and solving for velocity,

(3.3.10)

$$v_j(t) = \sqrt{2g[h_1(t) - h_2(t) + k_s \gamma V_2(t)]} .$$

3.4 Analysis of Two Reservoir System

3.4.1 Assumptions

We have made a number of assumptions in this model to simply the analysis;

1. There are no head losses due to friction, entrances and exits, bends, etc.
2. There are no temperature effects.
3. Fluids are incompressible.
4. Inherent in this system is a period of conveyance between reservoirs. In keeping with the assumption (i) (Section 2) underlying the fundamental diffusion model which partially states that in this and other analyses the adoption process is considered to be instantaneous. In the future directions research we will discuss how our model has the ability to supersede this assumption as there are many cases in the market system where the validity of the assumption is questionable. By reducing the system to a series of free body diagrams we dispose of the conveyance altogether and conceptualizing a velocity head that enters or exits a given reservoir instantaneously. This is only possible in the frictionless case where entrance and exit diameters were equivalent as was discussed in a previous section.
5. The concept of a continuum holds.

3.4.2 Model Solution and Properties

In section 3.3 we determined an equation for the velocity of the fluid influx into reservoir *2* using reservoir depths and a spring function. Ideally we want to determine an expression for the volume in reservoir *2* at time t, $V_2(t)$, and how it behaves over time, $dV_2(t)/dt$. To do so we are required to convert the depths into volumes and the influx velocity into the velocity of the free surface in reservoir *2*. The depth in reservoir *2* is

(3.4.1)

$$h_2(t) = \frac{V_2(t)}{\Lambda_2} .$$

We define the volume in the system to be V^*, where the volume in reservoir *1* at time $t=0$ is V^*. Therefore at any time t the volume in reservoir *1* can be determined by

(3.4.2)

$$V_1(t) = V^* - V_2(t) .$$

Using the same logic and defining that the cross-sectional area of reservoir *1* is equal to the cross-sectional area of reservoir *2*,

(3.4.3)

$$\Lambda_1 \equiv \Lambda_2 ,$$

we can determine the depth of reservoir *1* at any time t to be

(3.4.4)

$$h_1(t) = h^{*} - h_2(t) .$$

The velocity of the free surface in reservoir *2*, $v_2(t)$, is the ratio of the orifice area to the reservoir area times the influx velocity, $v_j(t)$, into reservoir 2, or

(3.4.5)

$$v_2(t) = v_j(t) \frac{\Lambda_j}{\Lambda_2} .$$

Rearranging Eq (3.4.5) to solve for $v_j(t)$ and substituting it into Eq. (3.3.10)

(3.4.6)

$$v_j(t)^2 = [v_2(t) \frac{\Lambda_2}{\Lambda_j}]^2 = 2g[h_1(t) + k_s W_2(t) - h_2(t)] .$$

Use Eq (3.4.4) in Eq. (3.4.6)

(3.4.7)

$$[v_2(t) \frac{\Lambda_2}{\Lambda_j}]^2 = 2g[h^{*} - h_2(t) + k_s(W_2(t) - h_2(t)] = 2g[h^{*} + k_s \gamma V_2(t) - 2h_2(t)]$$

Knowing that any depth is determined by the volume divided by the cross-sectional area we can state,

(3.4.8)

$$h^{*} = \frac{V^{*}}{\Lambda_1}$$

and using the relationship in Eq. (3.4.3) we can restate Eq. (3.4.8) as

$$h^{\bullet}=\frac{V^{\bullet}}{\Lambda_2} \tag{3.4.9}$$

Substituting Equations (3.4.1) and (3.4.9) into Eq. (3.4.7) results in

$$v_2(t)^2\left[\frac{\Lambda_2}{\Lambda_j}\right]^2=2g\left[\frac{V^{\bullet}}{\Lambda_2}+k_s\gamma V_2(t)-2\left[\frac{V_2(t)}{\Lambda_2}\right]\right]. \tag{3.4.10}$$

Rearranging Eq. (3.4.10)

$$v_2(t)=\sqrt{2g\left[\frac{\Lambda_j}{\Lambda_2}\right]^2\left[\frac{V^{\bullet}}{\Lambda_2}+\left[k_s\gamma-\frac{2}{\Lambda_2}\right]V_2(t)\right]}. \tag{3.4.11}$$

In Eq. (3.4.11) let

$$a=2g\left[\frac{\Lambda_j}{\Lambda_2}\right]^2\left[k_s\gamma-\frac{2}{\Lambda_2}\right]$$

and

$$b=2g\left[\frac{\Lambda_j}{\Lambda_2}\right]^2\left[\frac{V^{\bullet}}{\Lambda_2}\right].$$

Therefore Eq. (3.4.11) becomes

$$v_2(t)=\sqrt{aV_2(t)+b}. \tag{3.4.12}$$

The derivative with respect to time of the volume, $V_2(t)$, is the volumetric rate of flow in reservoir *2*,

$$\frac{dV_2(t)}{dt}=Q_2(t)=v_2(t)\Lambda_2$$

hence,

$$V_2(t)=[\frac{1}{\Lambda_2}]\frac{dV_2(t)}{dt}=\sqrt{aV_2(t)+b}$$

and rearranging to solve for dt

$$dt=\frac{dV_2(t)}{\Lambda_2\sqrt{aV_2(t)+b}}$$

to prepare to integrate

$$\int dt=t+C=\frac{1}{\Lambda_2}\int\frac{1}{\sqrt{aV_2(t)+b}}dV_2(t)=\frac{1}{\Lambda_2}\int[\sqrt{aV_2(t)+b}]^{-1}dV_2(t)\ .$$

Solving the integral results in

$$t+C=\frac{2}{a\Lambda_2}\sqrt{aV_2(t)+b},$$

and if we isolate $V_2(t)$ on the left hand side

$$V_2(t)=\frac{a\Lambda_2^2}{4}[t+C]^2-\frac{b}{a}\ .$$

Using our initial conditions, at time $t=0$, $V_2(t=0)=0$, we can determine the constant of integration C to be

$$C=\frac{2\sqrt{b}}{a\Lambda_2}\ .$$

Consequently our equation for $V_2(t)$ becomes

$$V_2(t)=\Lambda_2[\frac{a\Lambda_2}{4}t^2+\sqrt{b}t]\ .$$

Substituting for a and b results in

(3.4.13)

$$V_2(t)=\Lambda_j[\frac{g}{2}[k_s\gamma\Lambda_2-2][\frac{\Lambda_j}{\Lambda_2}]t^2+\sqrt{2g\frac{V^*}{\Lambda_2}}t]\ ,$$

which is a quadratic in time whose results are plotted in Figs. (3.8), (3.9) and (3.10).

Fig. (3.8) is concave and occurs when $k_s\gamma\Lambda_2<2$. Fig. (3.9) is linear and occurs when $k_s\gamma\Lambda_2=2$. Fig. (3.10) is convex and occurs when $k_s\gamma\Lambda_2>2$.

Differentiating Eq. (3.4.13) with respect to time results in a relation of volumetric rate of flow and time,

$$\frac{dV_2(t)}{dt}=Q_2(t)=\Lambda_j[g[k_s\gamma\Lambda_2-2]\,[\frac{\Lambda_j}{\Lambda_2}]\,t+\sqrt{2g\frac{V^*}{\Lambda_2}}]\,.$$

Dividing both sides by Λ_2 results in a relation of the velocity of increase of the free surface of reservoir 2, $v_2(t)$, with respect to time,

(3.4.14)

$$v_2(t)=\frac{\Lambda_j}{\Lambda_2}[g[k_s\gamma\Lambda_2-2]\,[\frac{\Lambda_j}{\Lambda_2}]\,t+\sqrt{2g\frac{V^*}{\Lambda_2}}]\,,$$

which is a linear function of time and is plotted in Figs. (3.11), (3.12) and (3.13).

The point of inflection of the $V_2(t)$ vs. t curve occurs where $v_2(t)$ reaches its maximum, or analogously where the maximum rate of diffusion occurs. The model is considered symmetric if the $V_2(t)$ vs. t curve after the point of inflection is the mirror image of the curve before the point of inflection. To determine the maximum value of $v_2(t)$ we take the derivative of Eq. (3.4.14) with respect to time and set it equal to zero and solve for t. The derivative of Eq. (3.4.14) is

$$\frac{dv_2(t)}{dt}=g[\frac{\Lambda_j}{\Lambda_2}]^2[k_s\gamma\Lambda_2-2]\,,$$

which is a constant. With this knowledge and by examining Figures (3.8) through (3.13) it is apparent that our model, in its present form, neither has a point of inflection nor is symmetric. It appears that the model has some of the same properties as the external-influence model and behaves in a like fashion from an output perspective. This is due to the simplification of the model components to either constants or monotonic linear functions. Adding a more complicated internal-influence function to our model would result in curves more characteristic of the flexible diffusion models that we outlined previously. We address the addition of a non-linear internal-influence function further in the section 5 on future directions and present a simulation analysis of the case in Appendix A.

In order to facilitate a comparison of the fluid model to existing flexible diffusion models we convert Eqs. (3.4.11) and (3.4.13) to the generalized form frequently utilised in the flexible diffusion model literature. Previously we defined $F(t)=N(t)/N^*$, in the fluid situation the cumulative population of adopters, $N(t)$, is represented by the volume of fluid in reservoir *2*, $V_2(t)$, and the finite population of the system, N^*, is embodied by the fluid volume in reservoir *1* at time $t=0$, V^*.

Therefore, using $F(t)=V_2(t)/V^*$, we determine the generalized diffusion model, where the proportion of adopters is

(3.4.15)

$$F(t)=\sqrt{\sigma}\,t-\frac{\sigma\phi}{4}t^2,$$

and the diffusion rate is

$$\frac{dF(t)}{dt}=\sqrt{\sigma[1-\phi F(t)]}.$$

Where the external influence is

$$\sigma=[2g\frac{\Lambda_j}{V^*}]\,[\frac{\Lambda_j}{\Lambda_2}]$$

and, in the case of a linear spring function, the internal influence is

$$\phi=[2-k_s\gamma\Lambda_2]\ .$$

At this point we shall again examine the fluid-market analogies of the parameters in the above equations and more importantly their characteristics when used in combination.

Parameters:

1. g In a physical system it represents the acceleration due to gravity, i.e., the rate at which any falling object

accelerates due to the action of the force of gravity. In our analogy we consider it to be the efficiency and/or effectiveness with which the forms of external communication (mass-media, salespersons, etc.) actually induce individuals within a population of potential adopters to purchase an innovation. In its present form, the fluid model has assumed g is constant, however, in future research we may determine it to vary over the diffusion process.

2. A_i Represents the cross-sectional area of the fluid exit/entrance/conveyance. To simplify this analysis we have held A_i constant, when in fact it could be varied over the length of the conveyance, in either a smooth or abrupt manner, in the real fluid condition. We presented A_i as an attribute of the product or innovation which is being introduced to the social system which either entices or discourages adoption. A fluid system with a conveyance of larger cross-sectional area will have a faster rate of flow than the same system with a conveyance of smaller cross-sectional area. Similarly, an innovation, introduced to a social system, that is inherently more desirable will have a greater rate of diffusion than an innovation of less desirability introduced to the same social system, all other things being equal. A_i can be envisioned as the coefficient of "product fit" for a specific social system. As the coefficient of product fit increases so does the rate of diffusion. In the future research section we discuss the modification of A_i from a constant to a function as controls are introduced into the process. These controls may encompass; supply and demand restrictions, product pricing, incentives, distribution, and the like.

3. V^* The volume in reservoir *1* at time $t=0$, or the finite population of the social system. Fundamental diffusion model assumption (ii) (section 2) states that there is a fixed ceiling on the number of potential adopters, thus V^* is finite and constant. In reality potential adopters frequently enter and exit the social system. Therefore, it may be valuable, in future analyses, to make V^* a function of time.

4. A_2 The cross-sectional area of reservoir *2*. We have specified that the cross-sectional areas are equal, $A_1=A_2$, and are currently constant. Intuitively we can compare the cross-sectional areas of the reservoirs to the characteristics of the population of the social system. When the cross-sectional

area is constant the population may be considered homogenous, and when A_i and A_p are functions of time or adoption level, i.e., cross-sectional area varies with depth, the population is heterogeneous. Whether the heterogeneity of the population has a positive or negative effect on the rate of diffusion depends on the function of the area. This was an area of research in diffusion theory that was examined by Chatterjee and Eliashberg (1990) who postulated that individual adoption times are an explicit function of the characteristics of potential adopters. Their resulting aggregate model consequently incorporated a micro-level behavioral basis to describe the innovation diffusion process in a heterogeneous population.

5. k_s The spring coefficient describes the spring stiffness, or it converts an imposed load to a magnitude of spring deflection. In a market sense, k_s can be visualized as the effectiveness and/or efficiency of internal or word-of-mouth communication in convincing a potential adopter to adopt a new product or innovation.

6. γ The unit gravity force is the product of the density of the fluid and the acceleration due to gravity, $\gamma=\rho g$. In a physical system it represents the force per unit volume that a fluid imparts on its surroundings. Previously, we deduced that the density of a fluid was analogous to the "packing" of a social system, either from a physical or technological perspective. If the social system is loosely packed the communication of word-of-mouth information, regardless of the effectiveness of the message, is more difficult. Conversely, if the social system is more densely packed the interaction of potential and prior adopters takes place more frequently and thus internal communication occurs more readily.

<u>Products and Ratios</u>:

Given the preceding analogies of the individual properties of the physical system, we proceed to rationalize their combined effects.

1. $k_s \gamma A_p$ The internal factor is the product of the effectiveness of word-of-mouth/internal communication, packing/density of the population, and the characteristics/homogeneity of the population.

2. A_i/A_p The product fit ratio is the quotient of the innovation properties to population characteristics. A_i could also include

some function representing market controls.

3. gA_i/V^* The potential energy of the physical system is akin the effectiveness of the mass media message multiplied by the ratio of innovation characteristic (combined with market controls) to finite population size.

Mahajan, Muller and Bass (1990) raise three questions regarding the Bass (1969) diffusion model which we described in a previous section as the three key issues. Does our model represent the three key issues of a diffusion process; unbundling of adopters, communication channels, and individual adoption decision?

The fluid model represents the unbundling of adopters through its structure. It was developed in a way such that its components were conceived with the primary adopter categories in mind. When we formulated the model in this paper we did so in a step-wise manner to illustrate that the system on its own was insufficient to adequately represent the characteristics of the diffusion process. To satisfactorily model the diffusion process we had to impose a specific set of boundary conditions coupled with a number of assumptions, i.e., a spring, frictionless flow, etc. We could not develop the paradigm without otherwise reverting to a probabilistic system. This is the reason that we chose fluid dynamics as the medium with which we investigated the innovation diffusion process, accepting Rogers' (1983) assertion of adopter categories as valid, fluid dynamics allows us to assemble an array of components to model a solution just as it does in the field of hydraulics. So to answer the question regarding the unbundling of adopters we return to the development of the fluid model in section 3. At the beginning of section 3.3 we described a two reservoir model without a spring. The flow from reservoir *1* to reservoir *2* was due to an elevation head difference, Z_{ij}, upon which the force of gravity, g, acted. This system had a constantly decelerating rate of flow to a point where the elevation head difference was equal to zero. This system behaves in a fashion very similar to that of the External Influence model. The fluid model without the spring does not permit reservoir 2 to move vertically. We state that the volume in reservoir *2*, when $Z_{ij}=0$, is analogous to the adoptions due only to external influence or innovation. In section 3.3, we added the spring, enabling the prior adopters to have an effect upon the system. Thus, modelling communication between prior and potential adopters, or imitation. The fluid model by the nature of its construction unbundles the adopters into innovators and imitators, perhaps not to the same degree as did Rogers (1983) but with the addition of a more complicated spring function and controls to the system the potential to do so is possible.

The next issue Mahajan, Muller and Bass (1990) considered was the communication between adopter categories, i.e., the communication between innovators and imitators. Tanny and Derzko (1988) proposed that both Innovators and Imitators are influenced by mass-media communications. The fluid model makes the same proposition, we could visualize the system composed of pure innovators, pure imitators, and a group of imitators that require both the internal and external influence to adopt. This suggests a multiplicative effect of the external and internal influence which is demonstrated by the fluid model. In Eq. (3.4.15) the first term is the pure innovation effect and the second term is the combined internal and external imitator effect. This second term varies from combined internal-external to a pure internal term as the spring function fluctuates. Therefore, the fluid model represents the similar communication channels to the Bass (1969) model and other flexible diffusion models with the addition of the combined external-internal influence.

The final issue is that of the individual adoption decision. Like the Bass (1969) model, the fluid model does not explicitly demonstrate the adoption patterns of the individual at the present level of analysis. To do so would require a molecular level analysis of the patterns of flow within the system, which is possible but beyond the scope of this thesis.

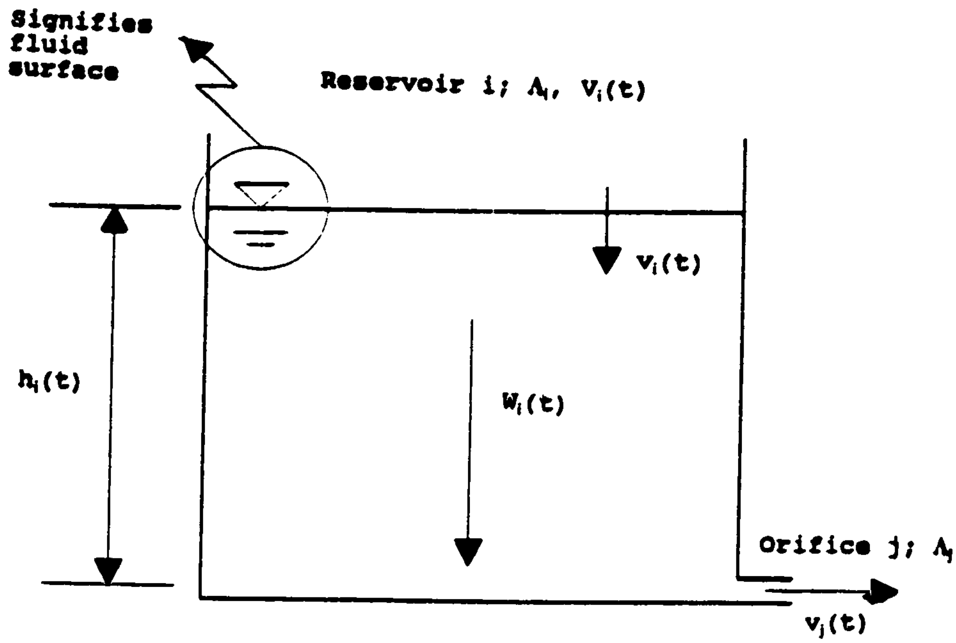

Figure 3.1 *Single reservoir.*

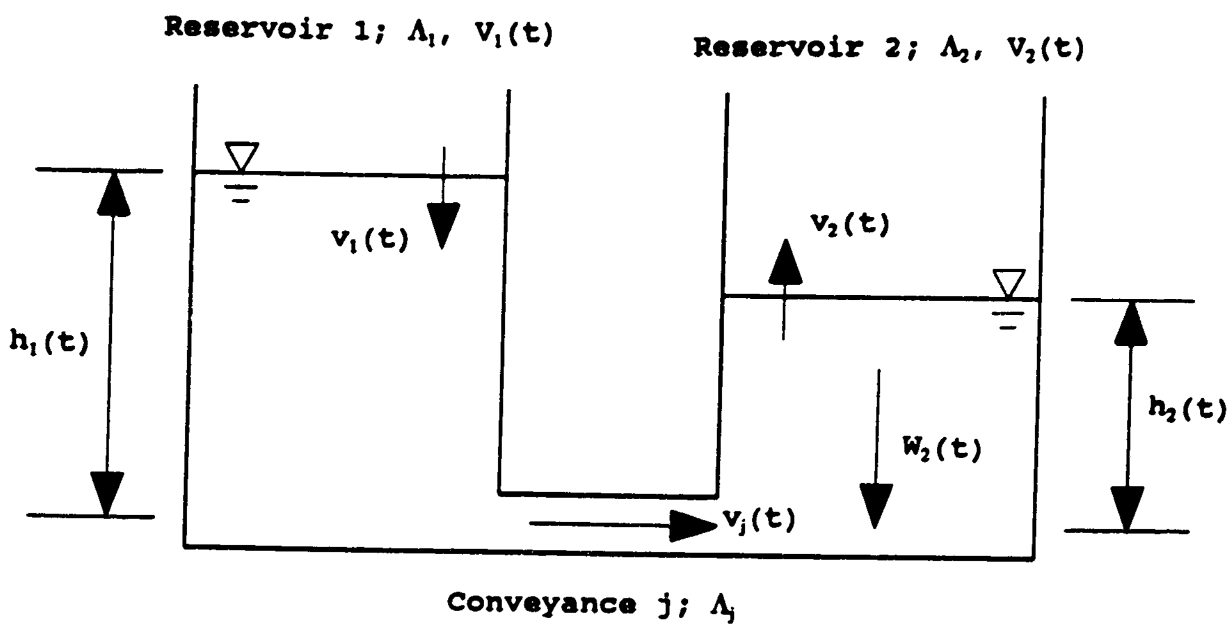

Figure 3.2 *Two reservoirs connected by a conveyance.*

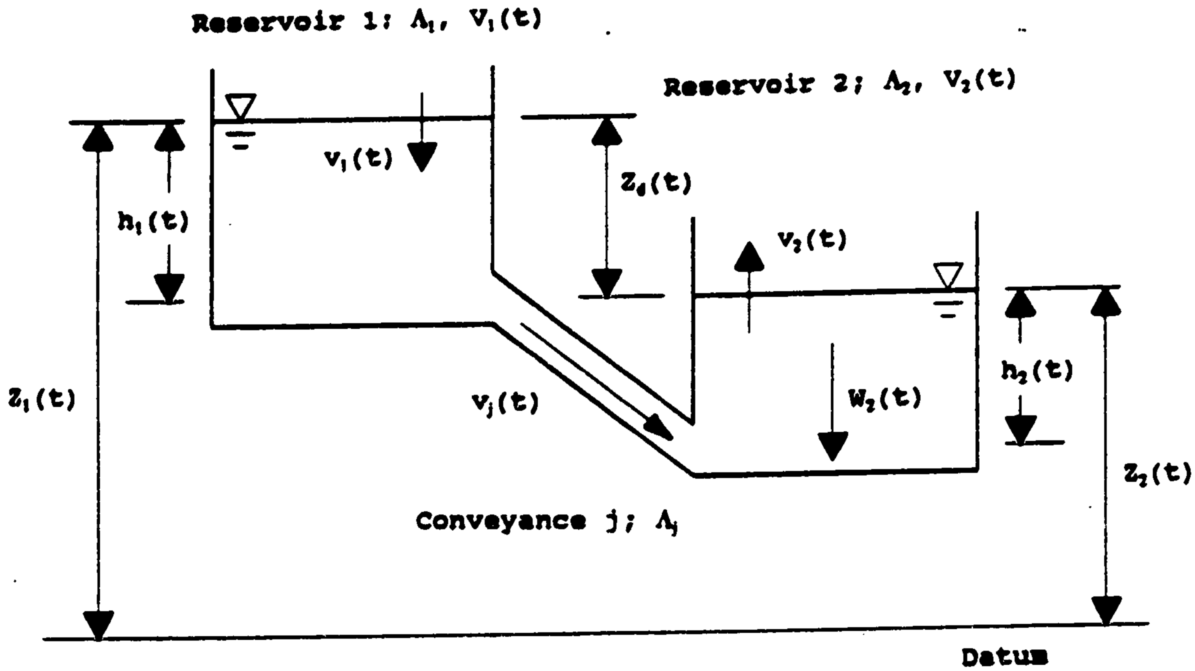

Figure 3.3 *Two reservoirs at different elevations.*

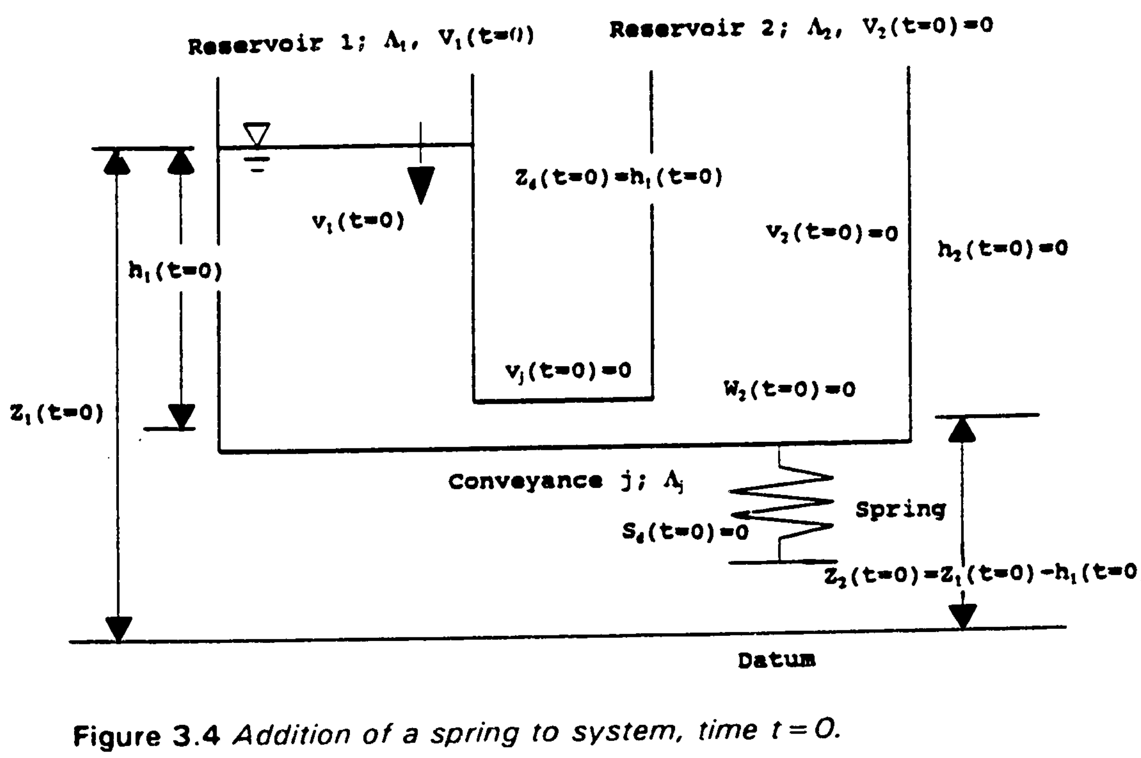

Figure 3.4 *Addition of a spring to system, time $t = 0$.*

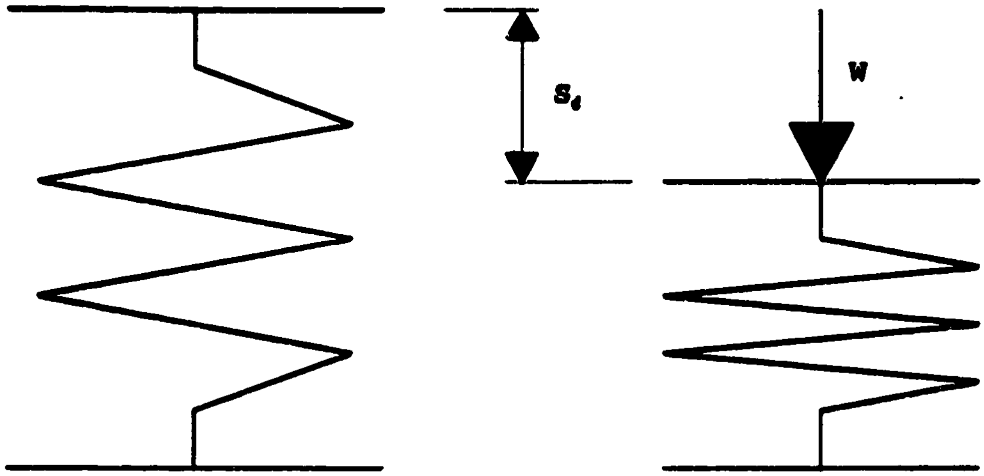

Figure 3.5 *Spring deflection.*

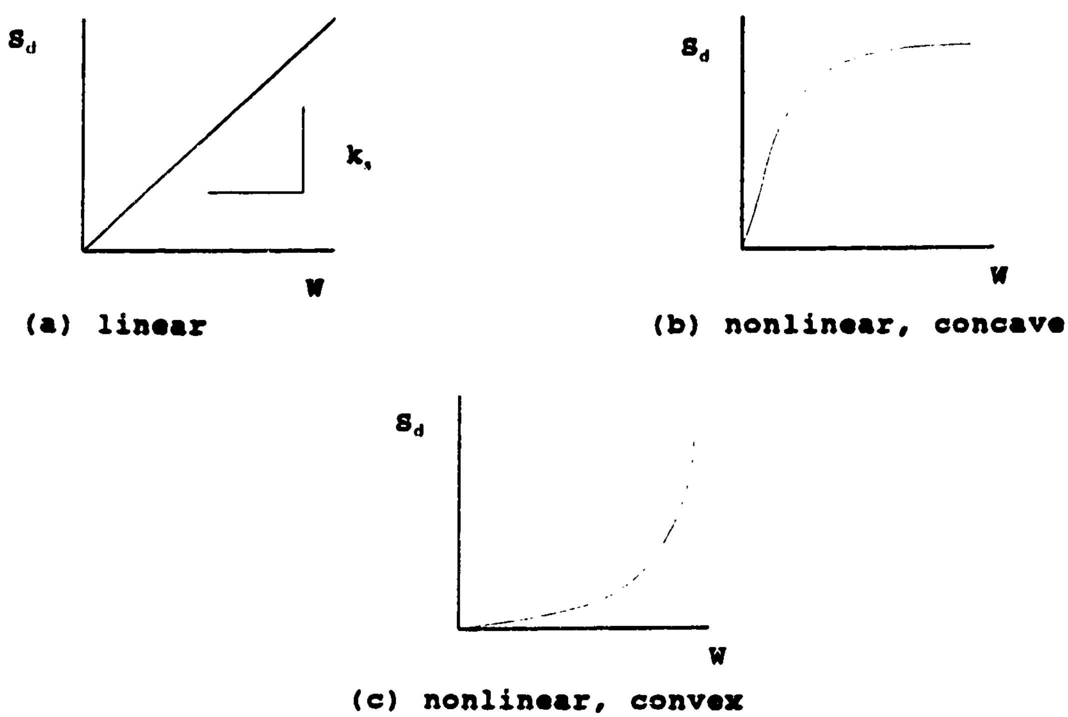

Figure 3.6 *Selected spring functions.*

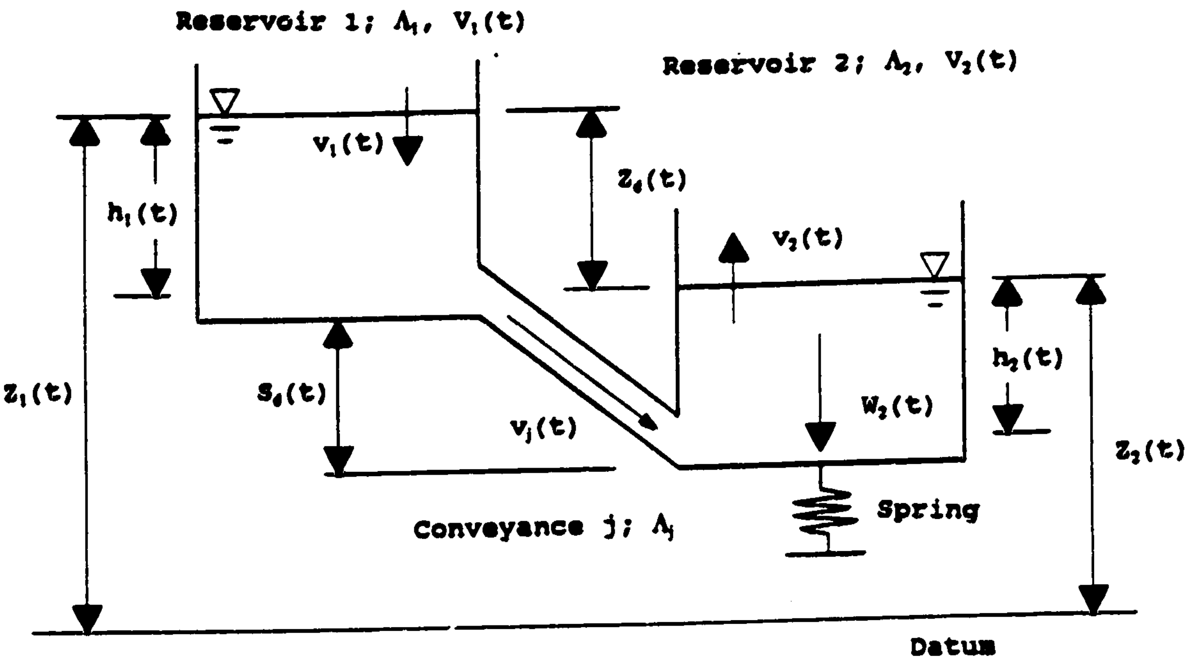

Figure 3.7 *Dynamic two reservoir system, time $t > 0$.*

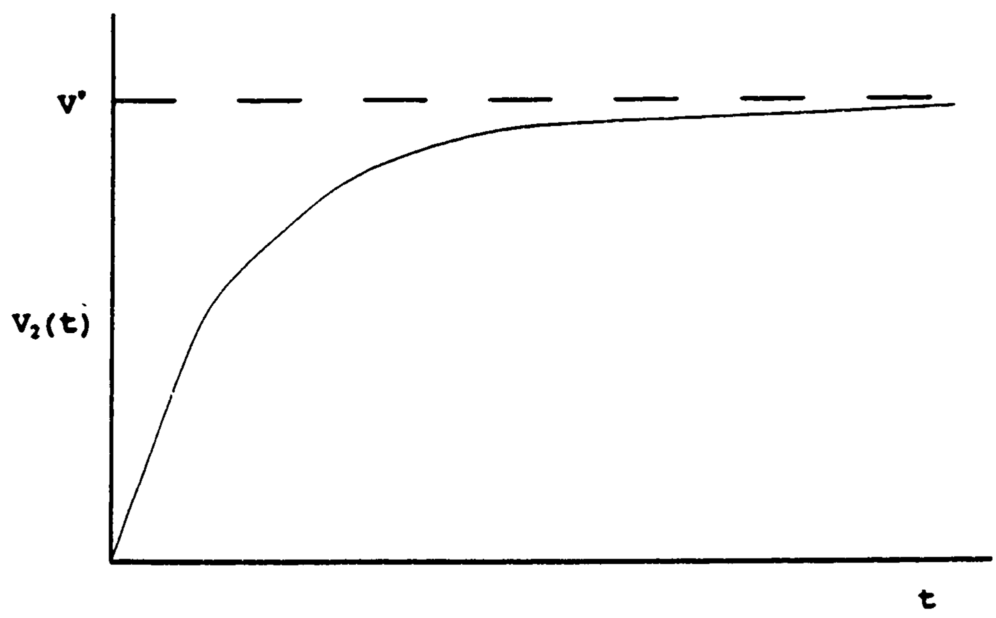

Figure 3.8 *Volume vs. time, concave.*

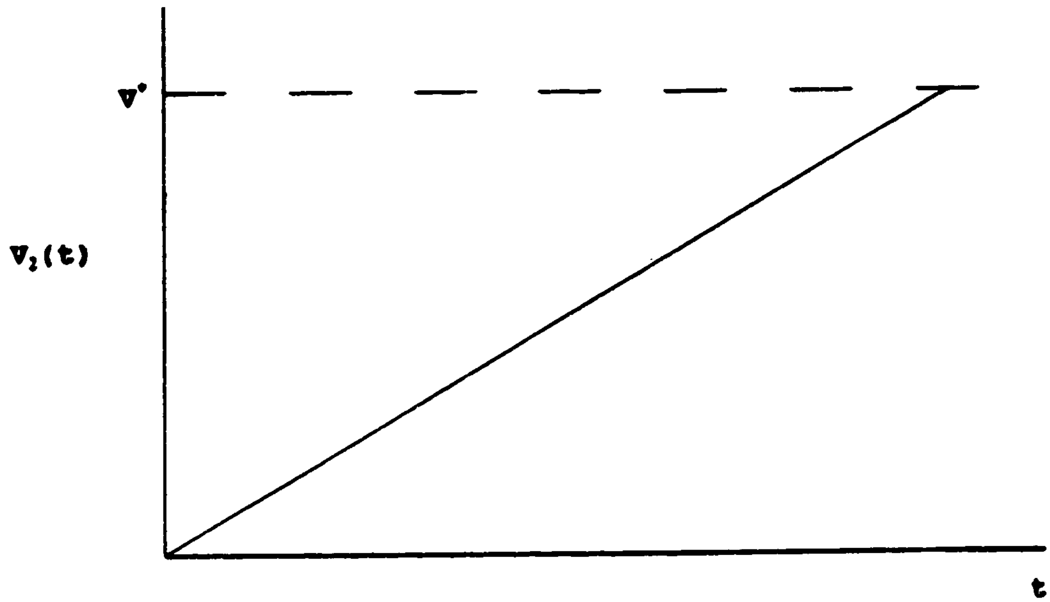

Figure 3.9 *Volume vs. time, linear.*

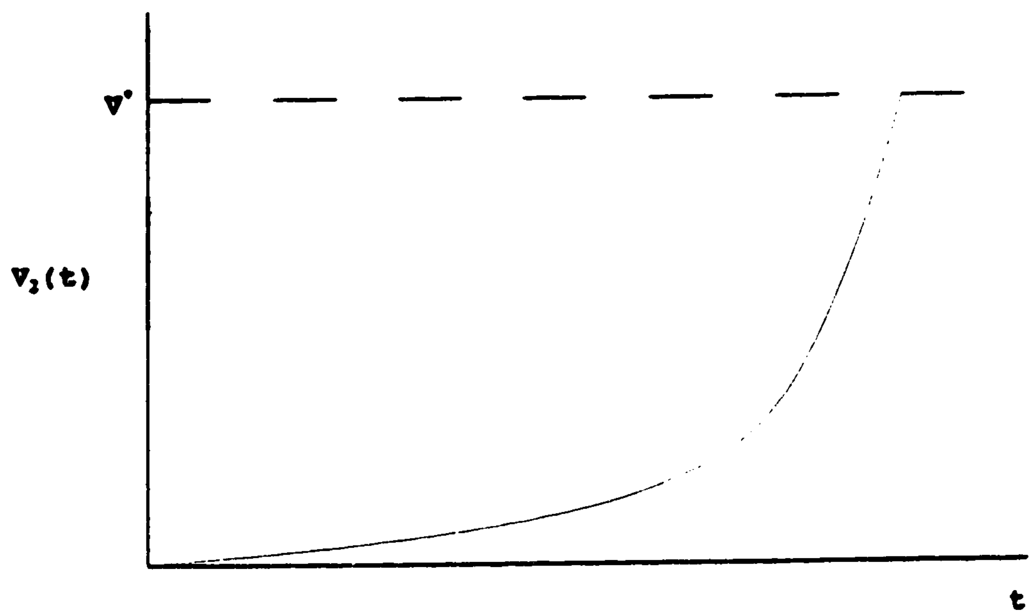

Figure 3.10 *Volume vs. time, convex.*

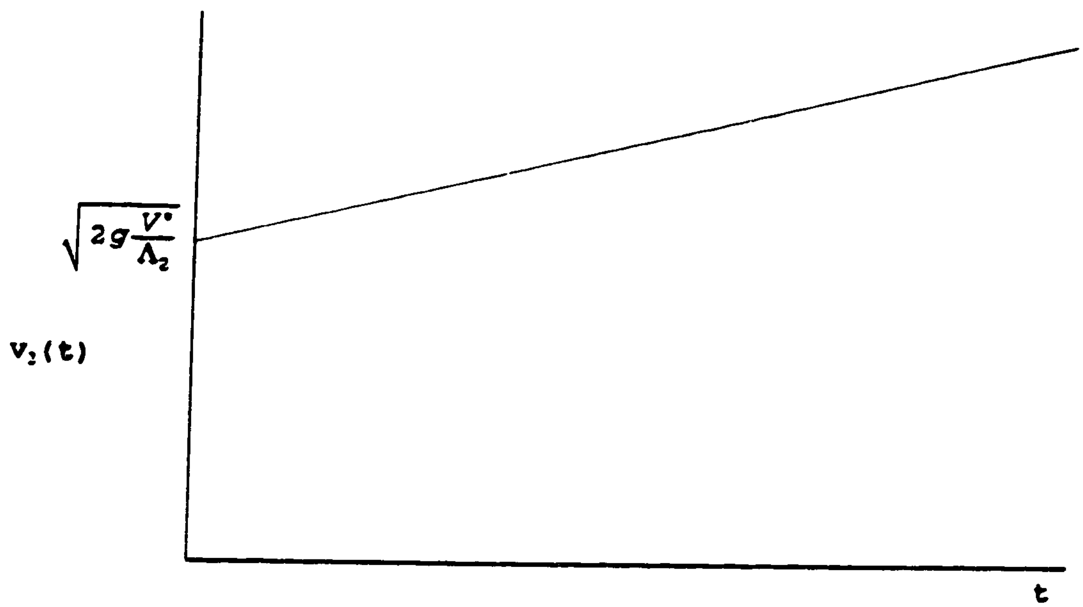

Figure 3.11 *Velocity vs. time, accelerating.*

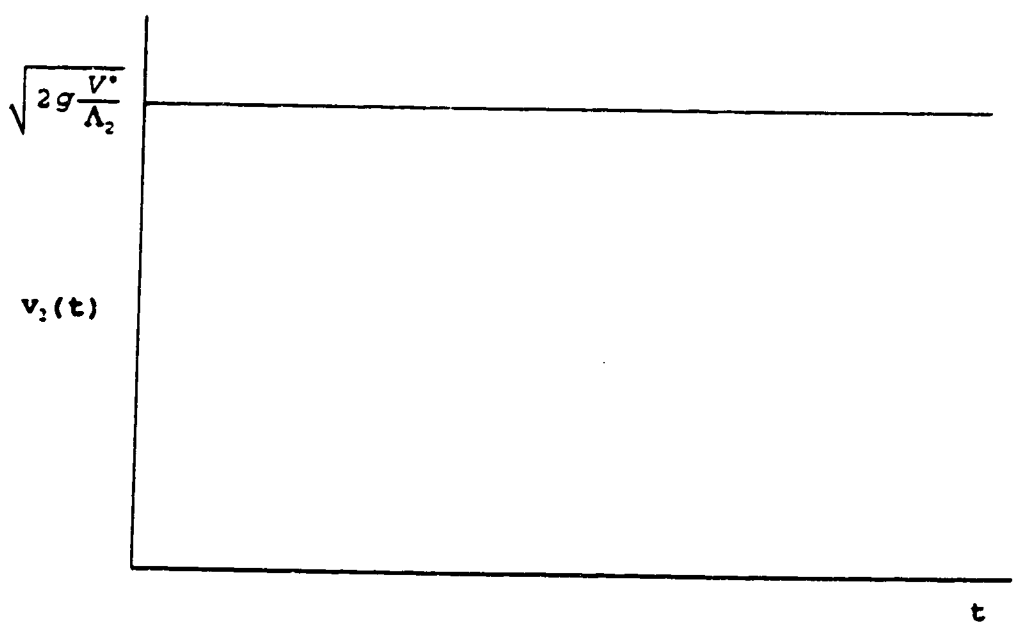

Figure 3.12 *Velocity vs. time, constant.*

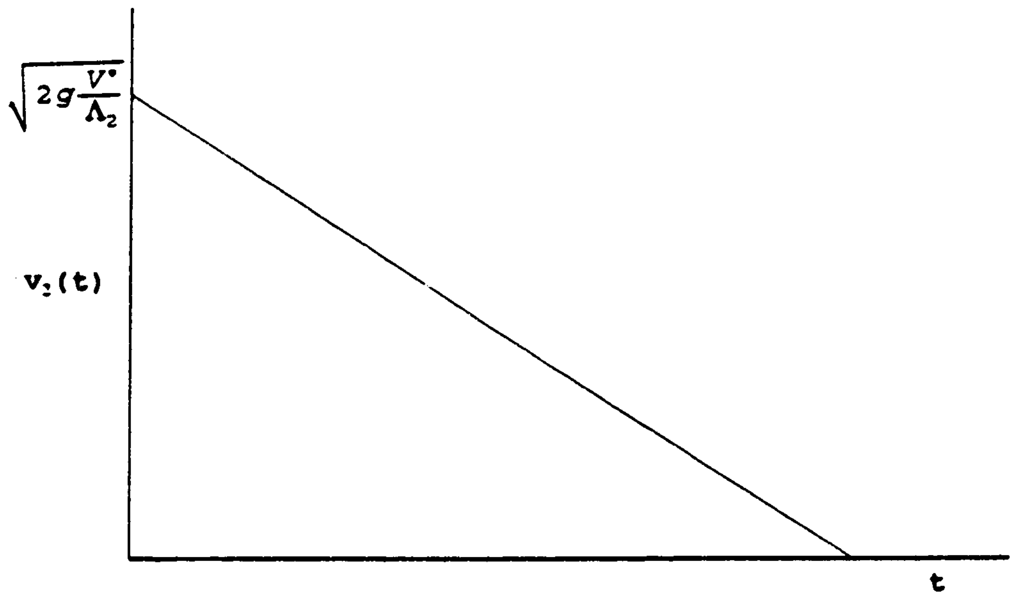

Figure 3.13 *Velocity vs. time, decelerating.*

3.5 References

Bass, F. M. (1969), "A New Product Growth Model for Consumer Durables", *Management Science*, 15, (5), 215-277.

Beer, F. P. and Johnston, E. R. (1984), *Vector Mechanics for Engineers*, Fourth Edition. McGraw-Hill Inc. (New York).

Chatterjee, R. and Eliashberg, J. (1990), "The Innovation Diffusion Process in a Heterogeneous Population: A Micromodeling Approach", *Management Science*, 36, (9), 1057-1079.

Gray, V. (1973), "Innovation in the States: A Diffusion Study", *The American Political Science Review*, 67, 1174-1182.

Mahajan, V., Muller, E., and Bass, F. (1990), "New Product Diffusion Models in Marketing: A Review and Direction for Research", *Journal of Marketing*, 54, (1), 1-26.

Mansfield, E. (1961), "Technical Change and the Rate of Imitation", *Econometrica*, 29, 741-766.

Rogers, E. M. (1983), *Diffusion of Innovations*, Third Edition. Collier Macmillan Publishers (London).

Streeter, V. L. and Wylie, E. B. (1981), *Fluid Mechanics*, First SI Metric Edition. McGraw-Hill Ryerson Limited (Toronto).

Tanny, S. M. and Derzko, N. A. (1988), "Innovators and Imitators in Innovation Diffusion Modelling", *Journal of Forecasting*, 7, (4), 225-234.

4. Flexible Diffusion Model Comparison

Flexible diffusion models have the advantage of capturing penetration patterns that are symmetric as well as nonsymmetric with no restrictions on the point of inflection. However, among all the models reviewed briefly in section 2.1; Floyd (1962), Sharif and Kabir (1976), Jeuland (1981), Easingwood et al. (1983), and Von Bertalanffy (1957), only the Von Bertalanffy (1957) model expresses the number of adopters as an explicit function of time, which is desirable for long-term forecasting.

Flexible diffusion models allow the generalized S-shaped, or logistic, diffusion curve to be symmetric as well as nonsymmetric, with the point of inflection responding to the diffusion pattern instead of being determined a priori. Although these models can be calibrated, they require the estimation of additional parameters. Hence, all flexible diffusion models achieve their flexibility by requiring estimation of additional parameters. As a consequence of their flexible nature, though, it is possible to develop a taxonomy of diffusion patterns because the models produce diffusion curves that mirror, rather than force, the shape of the underlying diffusion data. Despite the increased flexibility for capturing diffusion patterns, flexible diffusion models are also characterized by the same seven assumptions underlying the fundamental diffusion model.

Similarly, our model involves the estimation of a larger set of parameters than the fundamental diffusion model. The parameters at the highest level only involve the estimation of the external and internal influences, but in turn these parameters require the estimation of market and product characteristics that at present involve only constants. In the future the characteristics may be functional in form. We see a comparable form in the Easingwood et al. (1983) NSRL and NUI models in their treatment of the impact of the internal influence as a function of adoption level.

The major criticism of the fundamental and flexible diffusion models is that they are of little use to agencies interested in diffusing an innovation because they consider diffusion as a function of time only. The strategies employed by an agency to diffuse an innovation are not explicitly included in the models, thus inhibiting the evaluation of the effect of different strategies on innovation diffusion. Our model, through the nature of its development, has the capability of simulating many diverse influences through the addition of components to the physical model as we briefly discuss in the following section on future directions. For example the influence of diffusion strategies could be modelled as an external pressure function which could either "push" or "pull" fluid through the system varying with adoption level or time.

Comparisons of the fundamental and flexible diffusion models to this initial fluid paradigm reveal that the physical model, at this level of investigation, does not exhibit the degree of flexibility of the archetypes. Regarding the unbundling of adopters, the flexible diffusion models exhibit the normal distribution shape of the non-cumulative adoption curve as suggested by Rogers (1983) as a criterion for the indication of unbundling. However, as our model is unable to generate this form it does not meet the criterion. We could argue that the physical and component nature of our model allows us to describe the adopters in terms of those affected by only the external influence and those affected by the combined impetus of the external and internal influence, as we detailed in a previous section. Nonetheless, if we can not describe the unbundling of adopters neither mathematically nor numerically we can not support an assertion that our model represents such a claim. This is a serious and fundamental limitation of our model in its present form. To demonstrate the fluid model we present the following comparison of our physical paradigm to the Bass (1969) model. Sultan, Farley and Lehmann (1990) performed a meta-analysis of 213 applications of diffusion models from 15 articles and determined coefficients of innovation averaged .03 and coefficients of imitation averaged .38. For the purposes of this example we will assume that σ=.03 and ϕ=.38 for the following equations;

1. Fluid Model

$$F(t)=\sqrt{\sigma}\,t-\frac{\sigma\Phi}{4}t^{2},$$

and

$$\frac{dF(t)}{dt}=\sqrt{\sigma}\,[1-\Phi F(t)].$$

2. Bass (1969) Model

$$F(t)=\frac{1-e^{-[\sigma+\Phi]t}}{1+\frac{\Phi}{\sigma}e^{-[\sigma+\Phi]t}},$$

and

$$\frac{dF(t)}{dt} = [\sigma + \phi F(t)]\,[1 - F(t)]\,.$$

In the above equations *F(t)* represents the proportion of cumulative adoptions and can not be greater than 1. Table 4.1 is a compilation of the results of the equations when applied to a certain period of time using the influence coefficients determined by Sultan et al. (1990). Figures (4.1) and (4.2) are plots of the function of proportion of cumulative adoption over time for the fluid and Bass (1969) models respectively. Figure (4.3) is a plot of the function of the generalized diffusion rate over time for both the Bass (1969) and fluid models. It is clear from even a cursory inspection of Table 4.1 and Figures (4.1), (4.2) and (4.3) that the fluid model does not demonstrate the degree of flexibility of even the fundamental diffusion model.

With respect to the decision of the individual adopter versus that of the aggregate market, our model makes no claim as to a further understanding or explanation of this area or research. To do so would require an analysis of the model at the molecular level, violating our assumption of the fluid continuum. An analysis which is beyond the scope of this thesis. It is important to note that neither the fundamental nor the flexible diffusion models address the individual adoption decision process but emphasize the total market response.

Influence Coefficients:	
External	0.03
Internal	0.38

Time	Fluid		Bass (1969)	
t	F	dF/dt	F	dF/dt
0.00	0.00	0.173	0.000	0.030
0.15	0.03	0.172	0.005	0.032
0.30	0.05	0.171	0.009	0.033
0.45	0.08	0.171	0.015	0.035
0.60	0.10	0.170	0.020	0.037
0.75	0.13	0.169	0.026	0.039
0.90	0.15	0.168	0.032	0.041
1.05	0.18	0.167	0.038	0.043
1.20	0.20	0.166	0.044	0.045
1.35	0.23	0.166	0.051	0.047
1.50	0.25	0.165	0.059	0.049
1.65	0.28	0.164	0.066	0.051
1.80	0.30	0.163	0.074	0.054
1.95	0.33	0.162	0.082	0.056
2.10	0.35	0.161	0.091	0.059
2.25	0.38	0.160	0.100	0.061
2.40	0.40	0.160	0.109	0.064
2.55	0.42	0.159	0.119	0.066
2.70	0.45	0.158	0.129	0.069
2.85	0.47	0.157	0.140	0.071
3.00	0.49	0.156	0.151	0.074
3.15	0.52	0.155	0.162	0.077
3.30	0.54	0.154	0.174	0.079
3.45	0.56	0.154	0.186	0.082
3.60	0.59	0.153	0.198	0.084
3.75	0.61	0.152	0.211	0.087
3.90	0.63	0.151	0.224	0.089
4.05	0.65	0.150	0.238	0.092
4.20	0.68	0.149	0.252	0.094
4.35	0.70	0.148	0.266	0.096
4.50	0.72	0.148	0.281	0.098
4.65	0.74	0.147	0.295	0.100
4.80	0.77	0.146	0.311	0.102
4.95	0.79	0.145	0.326	0.104
5.10	0.81	0.144	0.342	0.105
5.25	0.83	0.143	0.358	0.107
5.40	0.85	0.142	0.374	0.108
5.55	0.87	0.142	0.390	0.109
5.70	0.89	0.141	0.406	0.109
5.85	0.92	0.140	0.423	0.110
6.00	0.94	0.139	0.439	0.110
6.15	0.96	0.138	0.456	0.111
6.30	0.98	0.137	0.472	0.111
6.45	1.00	0.136	0.489	0.110
7.00			0.549	0.108
7.50			0.602	0.103
8.00			0.652	0.097
8.50			0.698	0.089
9.00			0.741	0.081
9.50			0.779	0.072
10.00			0.813	0.063
10.50			0.842	0.055
11.00			0.868	0.047
11.50			0.890	0.040
12.00			0.909	0.034
12.50			0.924	0.029
13.00			0.938	0.024
13.50			0.949	0.020
14.00			0.958	0.017
14.50			0.965	0.014
15.00			0.972	0.011
15.50			0.977	0.009
16.00			0.981	0.008
16.50			0.984	0.006
17.00			0.987	0.005
17.50			0.990	0.004
18.00			0.992	0.003
18.50			0.993	0.003
19.00			0.994	0.002
19.50			0.995	0.002
20.00			0.996	0.002
20.50			0.997	0.001
21.00			0.998	0.001
21.50			0.998	0.001
22.00			0.998	0.001
22.50			0.999	0.001
23.00			0.999	0.000
23.50			0.999	0.000
24.00			0.999	0.000

Table 4.1 *Model Comparison.*

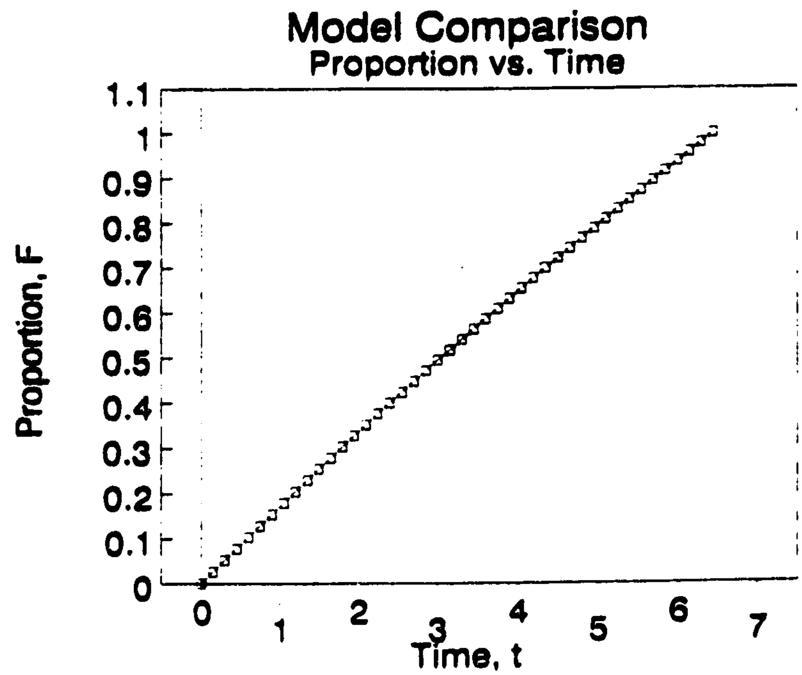

Figure 4.1 *Graph of fluid model, proportion vs. time, F(t) vs. t.*

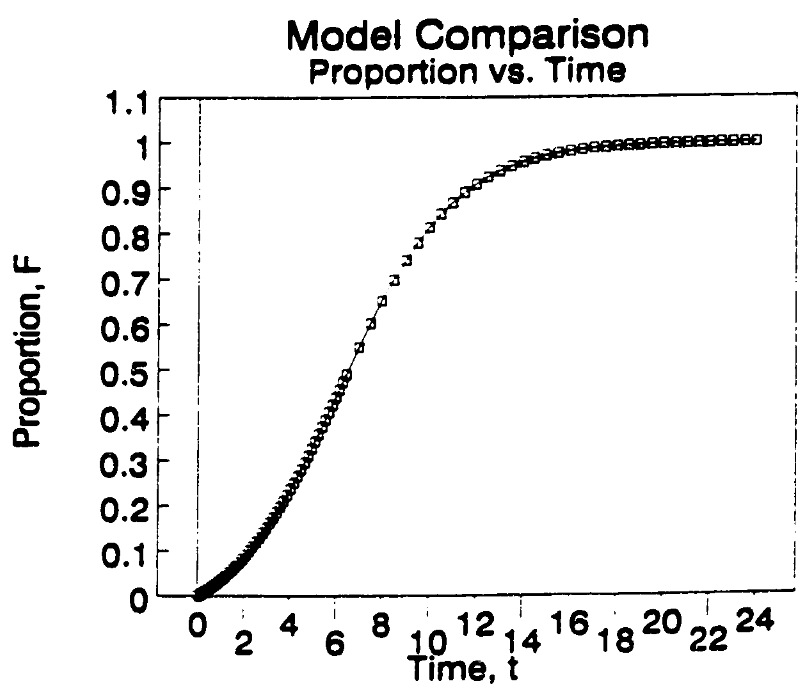

Figure 4.2 *Graph of Bass model, proportion vs. time, F(t) vs. t.*

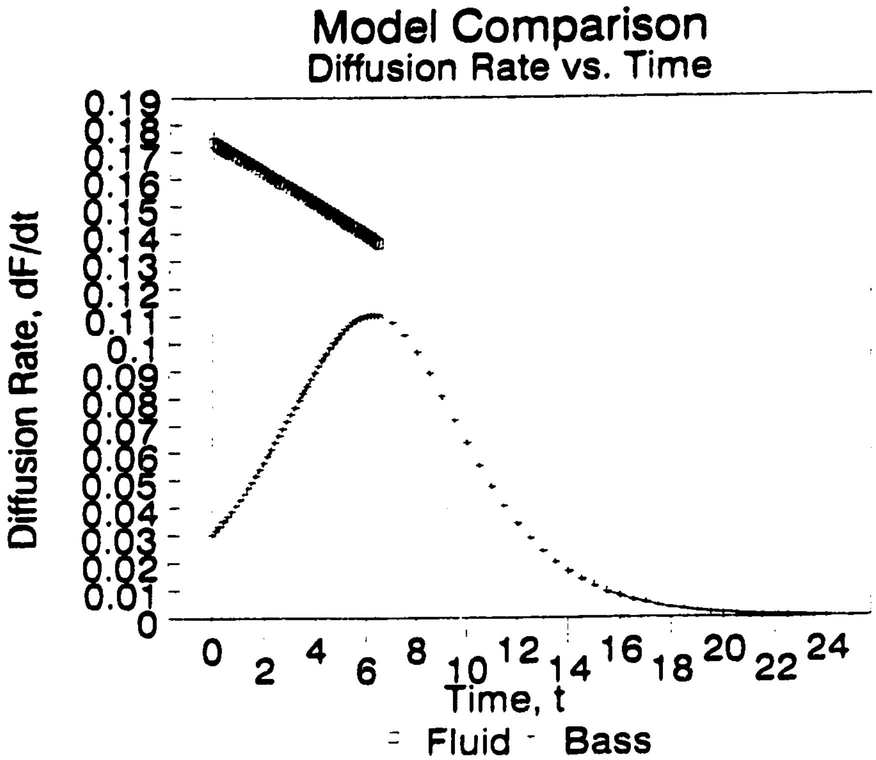

Figure 4.3 *Graph of diffusion rate vs. time, dF(t)/dt vs. t.*

4.1 References

Bass, F. M. (1969), "A New Product Growth Model for Consumer Durables", *Management Science*, 15, (5), 215-277.

Easingwood, C. J., Mahajan, V., and Muller, E. (1983), "A Nonuniform Influence Innovation Diffusion Model of New Product Acceptance", *Marketing Science*, 2, 273-296.

Floyd, A. (1962), "Trend Forecasting: A Methodology for Figure of Merit", in *Technological Forecasting for Industry and Government: Methods and Applications*, J. Bright, ed. Prentice-Hall, Inc. (NJ), 95-105.

Jeuland, A. P. (1981), "Parsimonious Models of Diffusion of Innovation, Part A: Derivations and Comparisons", Working Paper, University of Chicago.

Rogers, E. M. (1983), *Diffusion of Innovations*, Third Edition. Collier Macmillan Publishers (London).

Sharif, M. N. and Kabir, C. (1976), "A Generalized Model for Forecasting Technological Substitution", *Technological Forecasting and Social Change*, 8, 353-364.

Sultan, F., Farley J. U. and Lehmann, D. R. (1990), "A Meta-Analysis of Applications of Diffusion Models", *Journal of Marketing Research*, 27, (1), 70-77.

Von Bertalanffy, L. (1957), "Quantitative Laws in Metabolism and Growth", *The Quarterly Review of Biology*, 32, 217-231.

5. Future Directions

We briefly discuss four possibilities for further refinement and extension of our model.

5.1 Nonlinear Spring Function

As our research has indicated the degree of flexibility of our model in its present form is limited due to the monotonic form of the internal influence or, in the physical model, the spring function. Further analysis of this model with a nonlinear spring function is required to develop a model with a greater degree of flexibility. Appendix A uses a spreadsheet simulation model that replaces the spring with a mechanism that behaves according to the function;

$$S_d(W_2(t)) = k_s[[\gamma V^{*} W_2(t)] - W_2(t)^2]$$

For a more detailed examination see Appendix A.

The results of the analysis of this more complicated internal influence indicate that we can obtain a logistic shaped cumulative adoption curve and a diffusion rate curve that resembles a normal distribution. The difficulty rests in the complicated mathematical determination of the cumulative adoption and diffusion rate equations, so we created a spreadsheet simulation that models the diffusion process.

5.2 Substitution Model

Future research may include a model of the release of the next generation of an innovation, where we would have a three population model; non-adopters, incumbents and entrants. In this case the entrant innovation would be defined as a substitution of the incumbent innovation as investigated by Nault (1993). We would simulate this substitution model by adding a third reservoir to our physical representation. Due to analytical tractability, results of higher order differential equations, and difficulty in making comparisons between general solutions, Nault (1993) suggested an opening for numerical analysis or simulation using a physical science model.

5.3 Modelling Controls

A more in depth analysis of the physical representation would include the introduction of controls to the system. These controls would involve the introduction to the system of:

1. Pressure. As an external control, an induced pressure would enable

us to study the effect of attempting to push or pull more fluid through the system. We could accomplish this by sealing both reservoirs and imparting a positive or negative pressure head to the space between the seal and the fluid surface. This would allow us to push or pull fluid through the system from either, or both, reservoirs depending on the desired effect. We could use pressure to model a variety of pricing policies.

2. Valves, reducers, and expanders. These controls would allow us to vary the orifice/conveyance diameter by some function determined by the innovation and population characteristics. With these controls we could model supply and demand restrictions and innovation enticement or doubt characteristics.
3. Variance of the reservoir cross-sectional area with depth. Rather than utilising reservoirs of a constant cross-sectional area we suggest that future research could model the changing characteristics of a population over the duration of the diffusion process as a reservoir of a varying cross-sectional area over its depth such as a funnel or a cone. This control may allow us to examine the effect of the varying characteristics of the individuals adoption decision on the diffusion process for a given new product or innovation.

5.4 Real System

An ideal fluid is both incompressible and inviscid. No such fluid actually exists. An ideal fluid is a conceptual model which is adopted to simplify the mathematical treatment of fluid flow. In our model we have conceptualized the diffusion process as an ideal fluid system, frictionless and incompressible. Future research may involve developing a real fluid system by introducing head losses due to friction and compression. The introduction of friction to the model can be used to simulate a time interval for an adoption to transpire, i.e., a time to implementation. Friction forces would slow the rate of fluid transfer similar to the time to implement slowing the diffusion rate. Friction would manifest itself as a head loss term in the initial mathematical formulation of the model. The head loss term would include a coefficient of roughness for the various conveyance materials. The coefficient of roughness is analogous to the relative ease, or difficulty, with which an innovation is implemented.

5.5 References

Nault, B. R. (1993), "Release of the Next Generation of an Industry IOS", Working Paper, University of Alberta.

6. Summary

In this thesis a model representing the time pattern of durable purchases is developed which incorporates not only the internal and external influences but the components that constitute the influences. The basis for this model is that the innovation diffusion process behaves similar to a physical science system. In particular, we hypothesized that the innovation diffusion process is analogous to a fluid flow system. More important to the analysis is the nature of the model development which is rooted in process engineering. First, we assume that the population of the social system functions akin to a fluid, i.e., the behaviour of the aggregate population, in regards to its adoption patterns, exhibit properties coinciding with fluid characteristics. Second, given the previous research in the field that asserts that the diffusion process is composed of both an internal and external influence, we assembled a "process train" or model that exhibits these influences and allows us to study the relative effects of each influence and their components. This research, albeit based on literature that has been thoroughly investigated by others, is a departure from previous examinations of the subject and is in its initial stages of development.

The results of this research indicate that our model does not demonstrate the degree flexibility as the other diffusion models discussed earlier in this thesis. Our fluid model, in its present form with a linear feedback function, only has the ability to manifest the diffusion process as a convex to concave function. This is due to the simplified nature of the components of the fluid system which enabled us to develop the mathematical model with relative ease. As we demonstrate in Appendix A, the addition of a nonlinear feedback function allows the model to exhibit the various forms of the diffusion curves, from linear to logistic, like the other flexible diffusion models. While this model may not have realized the initial success we anticipated, it does motivate avenues of future research, discussed in the previous section. The research, through the physical analogies, demonstrates the components of the diffusion process. The use of the process train as a means of developing a physical model allows us to identify the components of the system through their interaction in the fluid model, i.e., we identified the influence of the product and market characteristics in the diffusion process through the analogy.

7. Bibliography

Bass, F. M. (1969), "A New Product Growth Model for Consumer Durables", *Management Science*, 15, (5), 215-277.

Beer, F. P. and Johnston, E. R. (1984), *Vector Mechanics for Engineers*, Fourth Edition. McGraw-Hill Inc. (New York).

Chatterjee, R. and Eliashberg, J. (1990), "The Innovation Diffusion Process in a Heterogeneous Population: A Micromodeling Approach", *Management Science*, 36, (9), 1057-1079.

Easingwood, C. J., Mahajan, V., and Muller, E. (1981), "A Nonsymmetric Responding Logistic Model for Technological Substitution", *Technological Forecasting and Social Change*, 20, 199-213.

Easingwood, C. J., Mahajan, V., and Muller, E. (1983), "A Nonuniform Influence Innovation Diffusion Model of New Product Acceptance", *Marketing Science*, 2, 273-296.

Floyd, A. (1962), "Trend Forecasting: A Methodology for Figure of Merit", in *Technological Forecasting for Industry and Government: Methods and Applications*, J. Bright, ed. Prentice-Hall, Inc. (NJ), 95-105.

Gray, V. (1973), "Innovation in the States: A Diffusion Study", *The American Political Science Review*, 67, 1174-1182.

Jeuland, A. P. (1981), "Parsimonious Models of Diffusion of Innovation, Part A: Derivations and Comparisons", Working Paper, University of Chicago.

Kumar, U. and Kumar, V. (1992), "Technological Innovation Diffusion: The Proliferation of Substitution Models and Easing the User's Dilemma", *IEEE Transactions on Engineering Management*, 39, (2), 158-168.

Lilien, G. L. and Kotler, P. (1983), *Marketing Decision Making; A Model-Building Approach*, First Edition. Harper & Row, Publishers, Inc. (New York).

Mahajan, V., Muller, E., and Bass, F. (1990), "New Product Diffusion Models in Marketing: A Review and Direction for Research", *Journal of*

Marketing, 54, (1), 1-26.

Mahajan, V. and Peterson R. A. (1985), "Models for Innovation Diffusion", *Quantitative Applications in the Social Sciences*, Series 7, Number 48. Sage Publications Inc. (London).

Mansfield, E. (1961), "Technical Change and the Rate of Imitation", *Econometrica*, 29, 741-766.

Nault, B. R. (1993), "Release of the Next Generation of an Industry IOS", Working Paper, University of Alberta.

Pao, R. H. F. (1967), *Fluid Dynamics*, First Edition. Charles E. Merrill Books, Inc. (Columbus, Ohio).

Rogers, E. M. (1983), *Diffusion of Innovations*, Third Edition. Collier Macmillan Publishers (London).

Sharif, M. N. and Kabir, C. (1976), "A Generalized Model for Forecasting Technological Substitution", *Technological Forecasting and Social Change*, 8, 353-364.

Streeter, V. L. and Wylie, E. B. (1981), *Fluid Mechanics*, First SI Metric Edition. McGraw-Hill Ryerson Limited (Toronto).

Sultan, F., Farley J. U. and Lehmann, D. R. (1990), "A Meta-Analysis of Applications of Diffusion Models", *Journal of Marketing Research*, 27, (1), 70-77.

Tanny, S. M. and Derzko, N. A. (1988), "Innovators and Imitators in Innovation Diffusion Modelling", *Journal of Forecasting*, 7, (4), 225-234.

Von Bertalanffy, L. (1957), "Quantitative Laws in Metabolism and Growth", *The Quarterly Review of Biology*, 32, 217-231.

APPENDIX A
Nonlinear Simulation

APPENDIX A

In this section we present the results of a fluid model with a spring that behaves in a nonlinear fashion. The spring function we have selected is

(A.1)

$$S_d(W_2(t)) = k_s[\gamma V^* W_2(t) - W_2(t)^2] ,$$

The results of which are plotted in Fig. (A.4) for the simulation which is detailed in this appendix.

Depending on the value of k_s this functional form of the feedback apparatus results in a final diffusion rate that behaves similar in form to the Bass and other flexible diffusion models. As k_s approaches *0*, the feedback system has a decreasing effect on the system, and the model will behave similar to the External Influence model. As k_s positively diverges from *0* the feedback system has an increasing influence on the model which generates the logistic or S curve which we have seen in the other flexible diffusion models. The ability to demonstrate various forms of diffusion patterns is an indication of the increased degree of flexibility that the addition of the nonlinear feedback system has achieved. We have referred to the spring function as the feedback system because of the complexity of the mathematical form does not in fact exist in the physical spring realm. Therefore, some more complicated physical feedback system would be required, i.e., a hydraulic cylinder with a computer actuator.

The mathematical development of our model now with a nonlinear feedback function follows, reiterating Eq. (3.4.6)

(A.2)

$$v_j(t)^2 = [v_2(t)\frac{\Lambda_2}{\Lambda_j}] = 2g[h_1(t) + S_d(W_2(t)) - h_2(t)] ,$$

and substituting Eq. (A.1) into Eq. (A.2) results in

(A.3)

$$V_2(t)^2 = [\frac{\Lambda_2}{\Lambda_j}]^2[-2g]\ [k_s\gamma^2 V_2(t)^2 + [\frac{2}{\Lambda_2} - k_s\gamma^2 V^*]\ V_2(t) - \frac{V^*}{\Lambda_2}] .$$

Rearranging Eq. (A.3) results in

(A.4)

$$v_2(t)=\sqrt{[\frac{\Lambda_j}{\Lambda_2}]^2[-2gk_s\gamma^2][V_2(t)^2+[\frac{2-k_s\gamma^2\Lambda_2V^*}{k_s\gamma^2\Lambda_2}]V_2(t)-\frac{V^*}{k_s\gamma^2\Lambda_2}]}.$$

Let

$$b=\sqrt{[\frac{\Lambda_j}{\Lambda_2}]^2[-2gk_s\gamma^2]}.$$

Rearranging Eq. (A.4) in anticipation of integrating

(A.5)

$$v_2(t)=[b]\sqrt{[V_2(t)+\frac{2-k_s\gamma^2\Lambda_2V^*}{2k_s\gamma^2\Lambda_2}]^2-[\frac{V^*}{k_s\gamma^2\Lambda_2}+[\frac{2-k_s\gamma^2\Lambda_2V^*}{2k_s\gamma^2\Lambda_2}]^2]}.$$

Set

$$u^2=[V_2(t)+\frac{2-k_s\gamma^2\Lambda_2V^*}{2k_s\gamma^2\Lambda_2}]^2$$

and

$$a^2=\frac{V^*}{k_s\gamma^2\Lambda_2}+[\frac{2-k_s\gamma^2\Lambda_2V^*}{2k_s\gamma^2\Lambda_2}]^2.$$

Therefore,

$$v_2(t)=b\sqrt{u^2-a^2}.$$

Since

$$v_2(t)=\frac{Q(t)}{\Lambda_2}=\frac{1}{\Lambda_2}\frac{dV_2(t)}{dt}=b\sqrt{u^2-a^2}.$$

Consequently

$$dt=\frac{dV_2(t)}{\Lambda_2 b\sqrt{u^2-a^2}},$$

and

$$\int dt=t+C=\frac{1}{\Lambda_2 b}\int[u^2-a^2]^{-\frac{1}{2}}dV_2(t)\ .$$

Integrating results in

$$t+C=\frac{1}{\Lambda_2 b}[\ln|u+\sqrt{u^2-a^2}|]\ .$$

Substituting back in for *b*, *u*, and *a* results in

$$t+C=\frac{1}{\Lambda_2\gamma\sqrt{-2gk_s}}[\ln|V_2(t)+\frac{2-k_s\gamma^2\Lambda_2 V^*}{2k_s\gamma^2\Lambda_2}$$

$$+\sqrt{[V_2(t)+\frac{2-k_s\gamma^2\Lambda_2 V^*}{2k_s\gamma^2\Lambda_2}]^2-[\frac{V^*}{k_s\gamma^2\Lambda_2}+[\frac{2-k_s\gamma^2\Lambda_2 V^*}{2k_s\gamma^2\Lambda_2}]^2]}\,|\ .$$

The isolation of $V_2(t)$ involves some complex mathematical manipulations beyond the scope of this thesis. We therefore developed a spreadsheet simulation that allows us to model the diffusion process with the specific nonlinear feedback function of Eq. (A.1). The program was created using the internal macro functions of LOTUS 1-2-3 version 2.2. The following pages contain; the program, a simulation run, and the graphs of cumulative adoption vs. time and the diffusion rate vs. time. The results indicate that future research in this area will result in a model with an increased degree of flexibility as demonstrated by the S-shaped curve of Fig. (A.2), and the bell-shaped diffusion rate curve of Fig. (A.3). Fig. (A.4) is the graph of the feedback function, Eq. (A.1), for this particular circumstance.

The equations used in the simulation are listed below:

Time,

$$t_n = t_{n-1} + \Delta t$$

Velocity in the conveyance,

$$v_{j_n} = \sqrt{2gZ_{d_n}}$$

Reservoir *1* datum height,

$$Z_{1_n} = Z_{1_{n-1}} - [h_{1_{n-1}} - h_{1_n}]$$

Depth of fluid in reservoir *1*,

$$h_{1_n} = h_{1_{n-1}} - [\frac{Q_{n-1}}{\Lambda_2} \Delta t]$$

Volume of fluid in reservoir *1*,

$$V_{1_n} = h_{1_n} \Lambda_2$$

Elevation head difference,

$$Z_{d_n} = h_{1_n} + S_{d_n} - h_{2_n}$$

Reservoir *2* datum height,

$$Z_{2_n} = Z_{1_n} - Z_{d_n}$$

Depth of fluid in reservoir *2*,

$$h_{2_n} = h_{2_{n-1}} + [h_{1_{n-1}} - h_{1_n}]$$

Volume of fluid in reservoir *2*,

$$V_{2_n} = h_{2_n} \Lambda_2$$

Downward force in reservoir *2*,

$$W_{2_n} = h_{2_n} \gamma \Lambda_2$$

Spring deflection,

$$S_{d_n}=k_s[\,[\gamma V^{\bullet}W_{2_n}]-[W_{2_n}^{\,2}]\,]$$

System discharge,

$$Q_n=v_{j_n}\Lambda_j$$

Reservoir *2* surface velocity,

$$v_{2_n}=\frac{Q_n}{\Lambda_2}$$

Proportion of finite fluid volume that has transferred to reservoir *2*,

$$F_n=\frac{V_{2_n}}{V^{\bullet}}$$

APPENDIX A

```
PROGRAM  {VAR}
         {TITLES}
         {INIT}
         {FOR CNTR,1,200,1,MAIN}

VAR      {HOME}
         /WGFF2~
         {GETLABEL "ENTER SIMULATION TITLE: ",A1}{D 2}{R}
         Gamma~{GETNUMBER "ENTER GAMMA VALUE: ",C3}/RNCGAM~C3~{D}
         /RFF3~C3~
         Lambda 2~{GETNUMBER "ENTER LAMBDA 2 VALUE: ",C4}/RNCLAM2~C4~{D}
         ks~{GETNUMBER "ENTER ks VALUE: ",C5}/RNCKS~C5~{D}
         /RFF6~C5~
         g~{GETNUMBER "ENTER g VALUE: ",C6}/RNCG~C6~{D}
         /RFF3~C6~
         Lambda j~{GETNUMBER "ENTER LAMBDA j VALUE: ",C7}/RNCLAMJ~C7~{D}
         Delta t~{GETNUMBER "ENTER TIME STEP VALUE: ",C8}/RNCDELT~C8~{D 2}{L}
         {RETURN}

TITLES   {GOTO}A10~
         Time~{R}Elev.~{R}Depth~{R}Vol.~{R}Elev.~{D}Diff.~{R}{U}Elev.~{R}Depth~{R}
         Vol.~{R}Force~{R}Spring~{D}Defl.~{R}{U}Vel.~{R}Disch.~{R}Vel.~{R}Prptn.~{D 2}{END}{L}
         t~{R}Z1~{R}h1~{R}V1~{R}Zd~{R}Z2~{R}h2~{R}V2~{R}W2~{R}Sd~{R}vj~{R}Q~{R}v2~{R}F~
         {U 2}{END}{L}/RLC{D 2}{END}{R}~
         {RETURN}

INIT     {GOTO}A13~
         0~{R}{GETNUMBER "ENTER INITIAL Z1 VALUE: ",B13}{R}+D13/LAM2~{R}
         {GETNUMBER "ENTER INITIAL V1 VALUE :",D13}{R}+C13-G13~{R}
         +B13-C13~{R}+H13/LAM2~{R}{GETNUMBER "ENTER INITIAL V2 VALUE: ",H13}{R}
         +G13*GAM*LAM2~{R}+KS*((GAM*$D$13*I13)-(I13^2))~{R}
         @SQRT(E13*2*G)~{R}+K13*LAMJ~{R}+K13*(LAMJ/LAM2)~{R}+H13/$D$13~{GOTO}A14~
         {RETURN}

MAIN     +{U}+$DELT~{R}+{U}-({U}{R}-{R})~{R}+{U}-(({U}{R 9}/$LAM2)*$DELT)~{R}+{L}*$LAM2~{R}
         +{L 2}+{R 5}-{R 2}~/RNCZD~~{R}+{L 4}-{L}~{R}+{U}+{U}{L 4}-{L 4}~{R}
         +{L}*$LAM2~{R}+{L 2}*$GAM*$LAM2~{R}+$KS*(($GAM*$D$13*{L})-({L}^2))~{R}
         @SQRT({L 6}*2*$G)~{R}+{L}*$LAMJ~{R}+{L}/$LAM2~{R}+{L 6}/$D$13~{D}{END}{L}
         {IF ZD<=0}{U}/RE{END}{R}~{FORBREAK}
         /RNDZD~
         {RETURN}
```

Figure A.1 *Simulation program.*

APPENDIX A

Nonlinear Simulation

Gamma	9.806
Lambda 2	10.00
ks	0.000078
g	9.806
Lambda j	0.30
Delta t	0.25

Time *t*	Elev. *Z1*	Depth *h1*	Vel. *V1*	Elev. Diff. *Zd*	Elev. *Z2*	Depth *h2*	Vel. *V2*	Force *W2*	Spring Defl. *Sd*	Vel. *vj*	Disch. *Q*	Vel. *v2*	Prptn. *F*
0.00	5000.00	25.00	250.00	25.00	4975.00	0.00	0.00	0.00	0.00	22.14	4.43	0.44	0.00
0.25	4999.89	24.89	248.88	40.01	4959.87	0.11	1.11	10.86	15.24	28.01	5.60	0.56	0.00
0.50	4999.75	24.75	247.40	58.82	4940.93	0.25	2.51	24.59	34.32	33.98	6.79	0.68	0.01
0.75	4999.59	24.59	245.79	81.32	4918.38	0.42	4.21	41.24	57.16	39.94	7.99	0.80	0.02
1.00	4999.38	24.38	243.80	107.37	4892.01	0.62	6.20	60.82	83.61	45.89	9.18	0.92	0.02
1.25	4999.15	24.15	241.50	136.76	4862.39	0.85	8.50	83.32	113.46	51.79	10.36	1.04	0.03
1.50	4998.89	23.89	238.91	169.23	4829.66	1.11	11.09	108.72	146.45	57.61	11.52	1.15	0.04
1.75	4998.60	23.60	236.06	204.48	4794.12	1.40	13.97	136.98	182.29	63.33	12.67	1.27	0.06
2.00	4998.29	23.29	232.87	242.17	4756.11	1.71	17.13	168.01	220.60	68.92	13.78	1.38	0.07
2.25	4997.94	22.94	229.42	281.93	4716.01	2.06	20.58	201.80	261.06	74.38	14.87	1.49	0.08
2.50	4997.57	22.57	225.70	323.35	4674.22	2.43	24.30	238.29	303.21	79.63	15.93	1.59	0.10
2.75	4997.17	22.17	221.72	366.02	4631.15	2.83	28.28	277.30	346.67	84.73	16.95	1.69	0.11
3.00	4996.75	21.75	217.48	409.49	4587.26	3.25	32.52	318.84	390.99	89.62	17.92	1.79	0.13
3.25	4996.30	21.30	213.00	453.31	4542.99	3.70	37.00	362.78	435.71	94.29	18.86	1.89	0.15
3.50	4995.83	20.83	208.29	497.01	4498.81	4.17	41.71	409.01	480.36	98.73	19.75	1.97	0.17
3.75	4995.34	20.34	203.35	540.15	4455.19	4.66	46.65	457.42	524.48	102.92	20.58	2.06	0.19
4.00	4994.82	19.82	198.21	582.24	4412.58	5.18	51.79	507.88	567.60	106.88	21.37	2.14	0.21
4.25	4994.29	19.29	192.86	622.85	4371.44	5.71	57.14	560.28	609.27	110.52	22.10	2.21	0.23
4.50	4993.73	18.73	187.34	661.52	4332.21	6.27	62.66	614.47	649.09	113.90	22.78	2.28	0.25
4.75	4993.16	18.16	181.64	697.85	4295.31	6.84	68.36	670.31	686.52	116.99	23.40	2.34	0.27
5.00	4992.58	17.58	175.79	731.43	4261.15	7.42	74.21	727.67	721.27	119.77	23.95	2.40	0.30
5.25	4991.98	16.98	169.80	761.88	4230.10	8.02	80.20	786.39	752.92	122.24	24.45	2.44	0.32
5.50	4991.37	16.37	163.69	788.88	4202.49	8.63	86.31	846.33	781.14	124.38	24.88	2.49	0.35
5.75	4990.75	15.75	157.47	812.10	4178.64	9.25	92.53	907.31	805.61	126.20	25.24	2.52	0.37
6.00	4990.12	15.12	151.16	831.30	4158.82	9.88	98.84	969.19	826.07	127.69	25.54	2.55	0.40
6.25	4989.48	14.48	144.78	846.24	4143.24	10.52	105.22	1031.70	842.29	128.83	25.77	2.58	0.42
6.50	4988.83	13.83	138.34	856.76	4132.09	11.17	111.66	1094.96	854.08	129.62	25.92	2.59	0.45
6.75	4988.19	13.19	131.86	862.69	4125.50	11.81	118.14	1158.51	861.32	130.07	26.01	2.60	0.47
7.00	4987.54	12.54	125.35	863.96	4123.58	12.46	124.65	1222.29	863.91	130.17	26.03	2.60	0.50
7.25	4986.88	11.88	118.84	860.50	4126.38	13.12	131.16	1286.11	861.82	129.91	25.98	2.60	0.52
7.50	4986.23	11.23	112.35	852.54	4133.70	13.77	137.65	1349.81	855.07	129.31	25.86	2.59	0.55
7.75	4985.59	10.59	105.88	839.89	4145.70	14.41	144.12	1413.21	843.71	128.34	25.67	2.57	0.58
8.00	4984.95	9.95	99.47	822.76	4162.18	15.05	150.53	1476.13	827.87	127.03	25.41	2.54	0.60
8.25	4984.31	9.31	93.11	801.33	4182.98	15.69	156.89	1538.41	807.70	125.36	25.07	2.51	0.63
8.50	4983.68	8.68	86.85	775.80	4207.88	16.32	163.15	1599.89	783.43	123.35	24.67	2.47	0.65
8.75	4983.07	8.07	80.68	746.44	4236.62	16.93	169.32	1660.38	755.31	120.99	24.20	2.42	0.68
9.00	4982.46	7.46	74.63	713.56	4268.90	17.54	175.37	1719.88	723.64	118.30	23.66	2.37	0.70
9.25	4981.87	6.87	68.71	677.50	4304.37	18.13	181.30	1777.68	688.75	115.27	23.05	2.31	0.73
9.50	4981.30	6.30	62.95	638.64	4342.66	18.70	187.06	1834.20	651.06	111.91	22.38	2.24	0.75
9.75	4980.74	5.74	57.36	597.39	4383.34	19.26	192.64	1889.07	610.92	108.24	21.65	2.16	0.77
10.00	4980.19	5.19	51.94	554.21	4425.98	19.81	198.06	1942.14	568.82	104.25	20.85	2.09	0.79
10.25	4979.67	4.67	46.73	509.55	4470.12	20.33	203.27	1993.28	525.20	99.97	19.99	2.00	0.81
10.50	4979.17	4.17	41.73	463.91	4515.27	20.83	208.27	2042.27	480.56	95.38	19.08	1.91	0.83
10.75	4978.70	3.70	36.96	417.78	4560.91	21.30	213.04	2089.04	435.38	90.52	18.10	1.81	0.85
11.00	4978.24	3.24	32.44	371.66	4606.58	21.76	217.56	2133.42	390.20	85.38	17.08	1.71	0.87
11.25	4977.82	2.82	28.17	326.13	4651.69	22.18	221.83	2175.28	345.49	79.97	15.99	1.60	0.89
11.50	4977.42	2.42	24.17	281.63	4695.79	22.58	225.83	2214.49	301.79	74.32	14.86	1.49	0.90
11.75	4977.05	2.05	20.45	238.69	4738.36	22.95	229.55	2250.93	259.80	68.42	13.68	1.37	0.92
12.00	4976.70	1.70	17.03	197.81	4778.90	23.30	232.97	2284.47	219.40	62.28	12.46	1.25	0.93
12.25	4976.39	1.39	13.92	159.47	4816.93	23.61	236.08	2315.01	181.68	55.92	11.18	1.12	0.94
12.50	4976.11	1.11	11.12	124.13	4851.98	23.89	238.88	2342.43	146.90	49.34	9.87	0.99	0.96
12.75	4975.87	0.87	8.66	92.23	4883.63	24.13	241.34	2366.62	115.50	42.53	8.51	0.85	0.97
13.00	4975.65	0.65	6.53	64.20	4911.46	24.35	243.47	2387.48	87.89	35.48	7.10	0.71	0.97
13.25	4975.48	0.48	4.75	40.43	4935.05	24.52	245.25	2404.87	64.48	28.16	5.63	0.56	0.98
13.50	4975.33	0.33	3.35	21.31	4954.02	24.67	246.66	2418.68	45.65	20.45	4.09	0.41	0.99
13.75	4975.23	0.23	2.32	7.30	4967.93	24.77	247.68	2428.70	31.84	11.97	2.39	0.24	0.99

Table A.1 *Simulation example.*

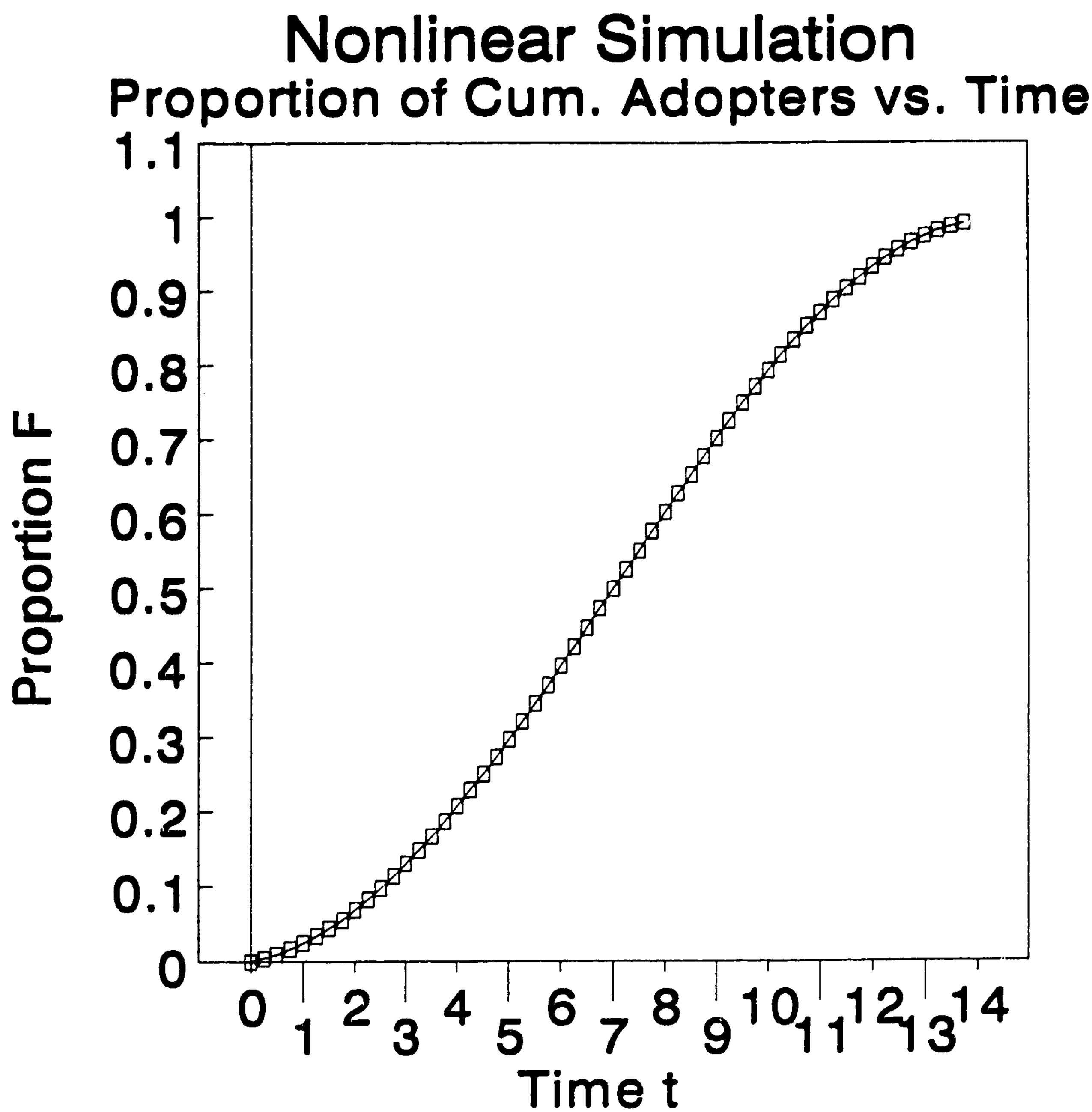

Figure A.2 *Graph of proportion vs. time, F(t) vs. t.*

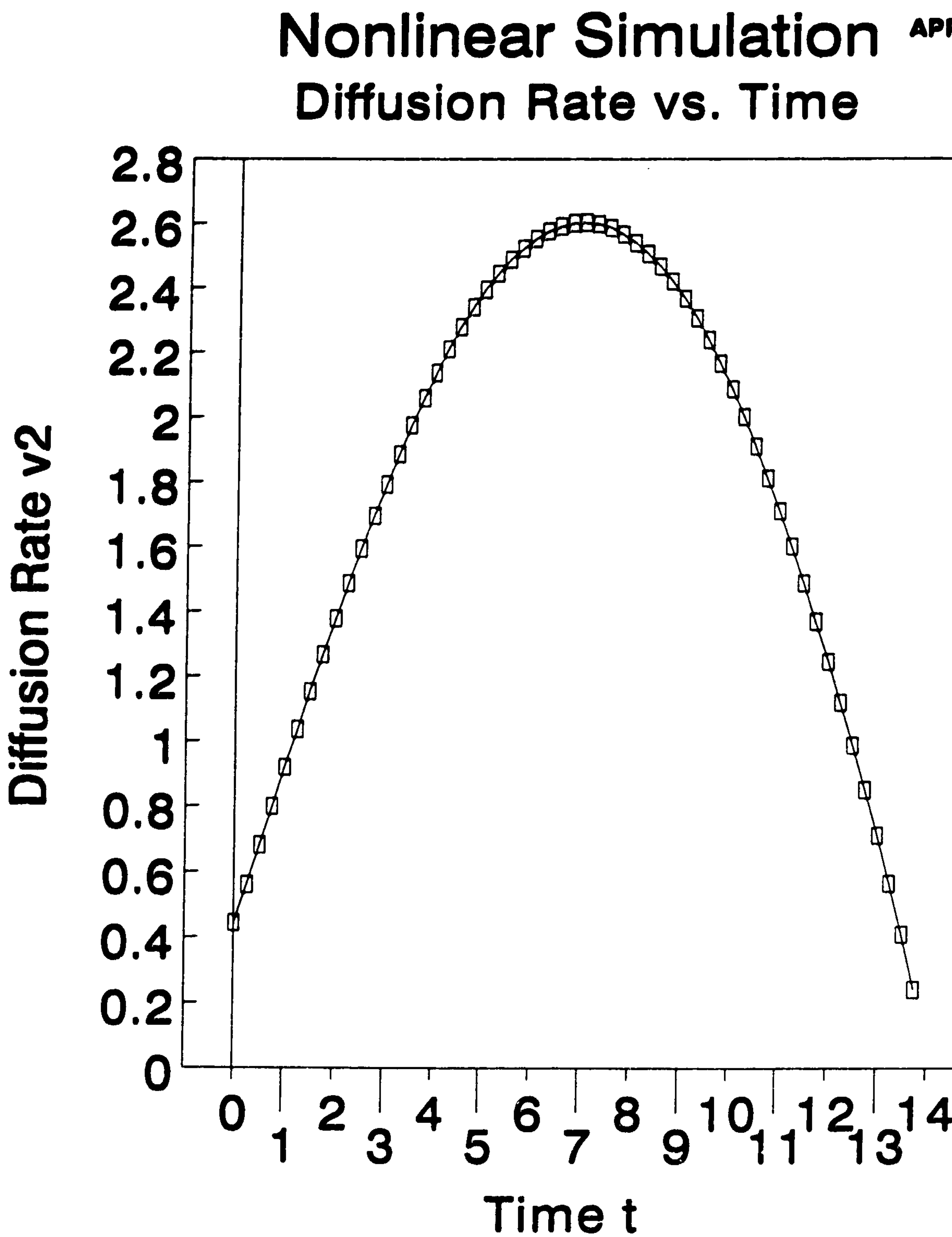

Figure A.3 *Graph of diffusion rate vs. time, $v_2(t)$ vs. t.*

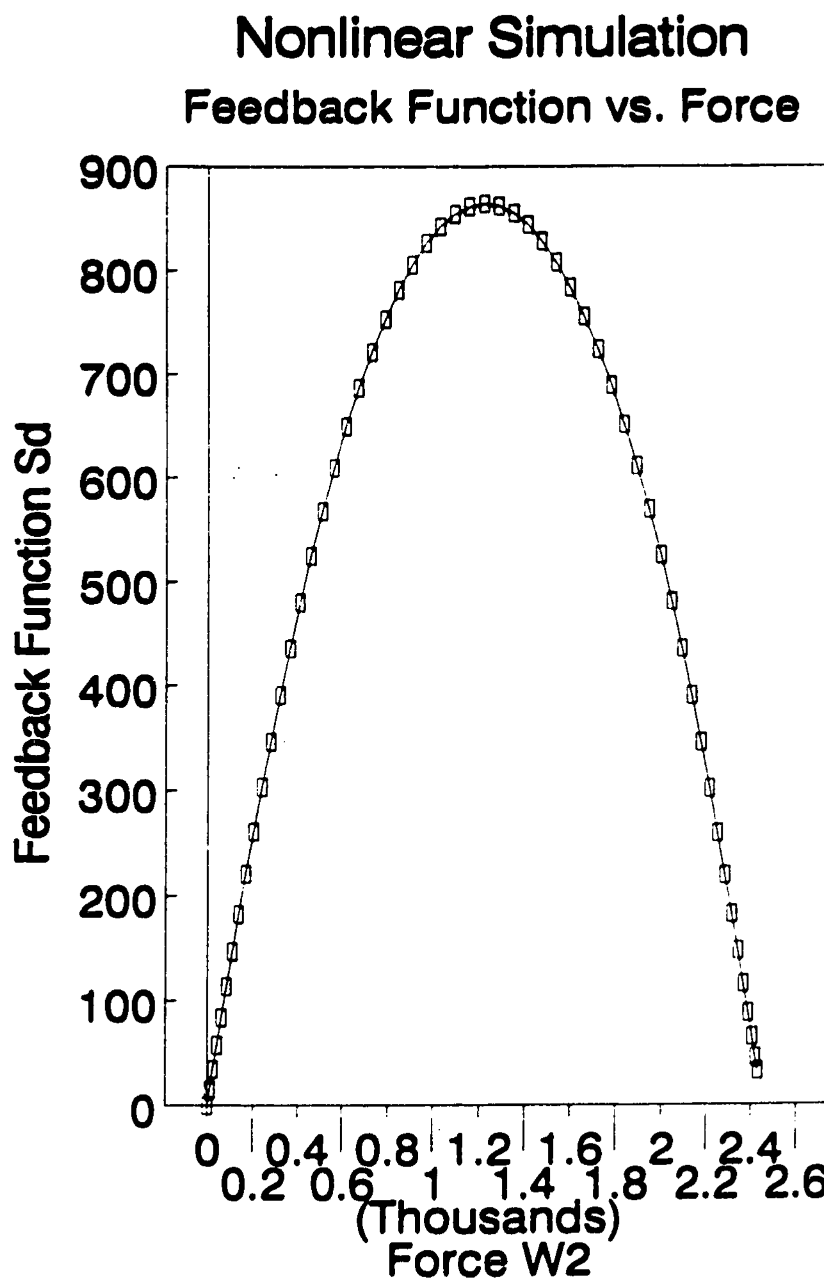

Figure A.4 *Graph of spring deflec. vs. force, S_d vs. W_2.*