Conditional Sentences in Belief Revision Systems

by

Ozkan Ozcevik

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Department of Philosophy
University of Alberta

Abstract

The first chapter of the thesis presents Frank P. Ramsey [1960]'s seminal treatment of "If ... , then ..." statements. We also explain how Stalnaker and Thomason [1970] picked up on Ramsey's idea and undertook the task of giving *truth conditions* for counterfactual conditionals in contrast to Ramsey's insistence on *rational acceptability conditions*. The second chapter is concerned with technicalities related to the proof theory of modern conditional logics. A Fitch-style natural deduction system for the Stalnaker/Thomason sentential conditional logic FCS already exists in the literature [Thomason, 1971]. Here we adjust FCS in a way to arrive at a proof system for Lewis's "official" conditional logic VC [Lewis, 1973]. We begin with expositions of Stalnaker/Thomason's CS/FCS and Lewis's VC. Next, we explain why FCS in its original form is incompatible with VC. Interestingly, it turns out that the strict reiteration rule corresponding to the Uniqueness Assumption ("Stalnaker's Assumption" in Lewis's terminology) underlies the incompatibility of FCS and VC. We observe that Stalnaker's Uniqueness Assumption becomes a very effective proof-theoretic device in natural deduction systems for conditional logics by virtue of allowing us to make use of "indirect conditional proofs". In FCS, those indirect proofs allow us to derive the VC axioms of *centering* and *rational monotony* without need of additional strict reiteration rules. However, the problem we face is that since the characteristic feature of VC is its rejection of Stalnaker's Assumption, we have no choice but to remove the *conditional excluded middle* strict reiteration rule and add two new strict reiteration rules (one for *centering* and one for *rational monotony*). After making the necessary adjustments to FCS, and thereby transforming it into FVC (that is, a Fitch-style natural deduction system for Lewis's VC), we prove that FVC and VC are equivalent systems. We remark that Stalnaker/Thomason/Lewis conditional connectives are to be treated as multi-modal connectives (interpreted as "relativized necessity" in the style of Chellas [1980]). We argue that our findings here suggest that int-elim style inferentialism about logical connectives can be problematic in view of multi-modal connectives; that is, introduction and elimination rules alone cannot uniquely determine the "meanings" of such logical connectives. The results seem to show that reiteration rules and restrictions on those reiteration rules also are extremely important for determinations of "meanings" of multi-modal logical connectives. The third chapter of the thesis is largely expository: we introduce the AGM theory of belief change and point out the theory's close connection with the analysis of conditional statements and the Ramsey Test. The fourth chapter is concerned with the question of whether an alternative *doxastic* semantics for VC is attainable. The answer is negative: belief change models (called "belief update") that can validate Lewis's VC are *ontic* models. Epistemic semantics for Stalnaker/Thomason/Lewis counterfactual conditional logics seems unattainable. The fifth chapter brings the thesis to a conclusion by summarizing our results.

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Contents

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CHAPTER 1

Introduction: The Ramsey Test

1.1 Ramsey's Insight

In this chapter we trace back the history of modern conditional logics to Frank P. Ramsey's original analysis of conditional idiom. Following our examination of Ramsey, we will then give an exposition of Stalnaker's ontic reinterpretation of Ramsey's analysis and explain to the reader in what respect this analysis differs from the original one. We will argue that Stalnaker's reinterpretation changes the scene by leaping from an epistemic/doxastic analysis to an ontic analysis which, we claim, is committed to a debatable analogy between a "belief set" and a "possible world" in doing so.

Ramsey's analysis is simple but ingenius. In *General Propositions and Causality*, Ramsey considers an event (say, a party) where a man believes that if he eats a certain cake, he will get sick. Formalize this as $p \rightarrow q$ where p is the antecedent (*protasis*) and q is the consequent (*apodosis*) of the conditional. Ramsey writes:

> " ... Before the event we do differ from him in a quite clear way: it is not that he believes p, we $\bar{p}$; but he has a different degree of belief in q given p from ours; and we can obviously try to convert him to our view. But after the event we both know that he did not eat the cake and that he was not ill; the difference between us is that he thinks that if he had eaten it he would have been ill, whereas we think he would not. But this is prima facie not a difference of degrees of belief in any proposition, for we both agree as to all the facts."

In order to understand what is at issue here two temporal segments should be considered: (1) the conditional statement *before* the event and (2) the conditional statement *after* the event. Before the event, what is in question is an *indicative conditional* the antecedent of which is *not* contrary-to-fact. After the event, what is in question is a *counterfactual conditional* the antecedent of which is contrary-to-fact. Because the party is now over and the man did not (actually) eat the cake (in other words, after the party, it is the case that *not-p*).

(1) For indicative conditionals we see Ramsey subscribing to what is now called a probabilistic account of conditionals according to which the assertibility of a conditional is identified with its subjective probability and its subjective probability in turn is identified with the conditional probability of the consequent given the antecedent. The probabilistic identification of probability of a conditional with its related conditional probability can be formalized as $P(\textit{If } A\textit{, then } B) = P(B|A)$. The probabilistic account gave rise to a research programme of its own (see (Adams, 1965)) and has recent supporters as well (see, for example, (McGee, 1994) and (Stalnaker and Jeffrey, 1994)). Despite its prima facie appeal, the probabilistic account of conditionals faces serious difficulties due to a well-known result by David Lewis (*triviality theorems* (Lewis, 1976)). For that reason, the probabilistic research programme will be largely ignored throughout this thesis. We will instead focus more on the qualitative and logical approaches to conditionals (as opposed to the quantitative probabilistic approaches).

(2) After the event, what we have is a counterfactual conditional and Ramsey seems to think that there is no propositional or factual disagreement between us (who believe that it is not the case that if he had eaten the cake, he would have been ill) and the man (who, in contrast, believes that

if he had eaten it, he would have been ill). Therefore, for Ramsey, there is no fact of the matter as to whether a counterfactual is true or false and counterfactuals are not truth-bearers (i.e., they are not propositional). In a footnote to the above quotation Ramsey writes:

> "If two people are arguing 'If p, then q?' and are both in doubt as to p, they are adding p hypothetically to their stock of knowledge and arguing on that basis about q; so that in a sense 'If p, q' and 'If p, $\bar{q}$' are contradictories. We can say that they are fixing their degree of belief in q given p. If p turns out false, these degrees of belief are rendered void. *If either party believes not-p for certain, the question ceases to mean anything to him except as a question about what follows from certain laws or hypotheses.*" (emphasis mine)

The emphasized section of the quote suggests that for Ramsey counterfactual idiom is concerned with argumentation and inference only and is not related to matters of fact or reality. If both parties agree that p is false, then the only disagreement that could exist between these two parties is a disagreement over which one of the following arguments fare better. To wit, *Argument 1* goes something like this: "Assume p as a premise and suppose others things are kept constant; from there we can infer that q; *Argument 2*, on the other hand, goes along the following lines: "Assume p as a premise and suppose other things are kept constant; from there it does *not* follow that q.[1]

The procedure given in the last quote came to be known as the *Ramsey Test* and was picked up on by Robert Stalnaker (1968). The next section is on Stalnaker's reinterpretation.

1.2 Stalnaker's Ontic Reinterpretation

Before we give an exposition of Stalnaker's reinterpretation of the Ramsey Test, let's reemphasize that the *original* Ramsey Test

- treats conditional statements as incapable of having truth values (i.e., treats them as non-propositional), and
- gives an epistemic/doxastic procedure for evaluating rational acceptability, but not truth, of conditional statements.

Stalnaker (1968) extends the original Ramsey Test by adding a consistency preservation requirement specifically to deal with the contrary-to-belief antecedents. Let's note that the consistency preservation requirement is a principle of minimal change since it requires the epistemic agent

[1] This helps explain why *If p had been the case, then q would have been the case* and *If p had been the case, then not-q would have been the case* are not real contradictories. Their surface form gives an impression of contrariety, but that impression is lost once one realizes that the type of conditional statements in question represent nothing more than abbreviated arguments and hence is not propositional. It is impossible for two arguments to contradict each other since contradiction is applicable to propositions only.

to make *only* those adjustments that are *necessary* to maintain consistency of her stock of beliefs. No change will be made in the stock of beliefs so long as there is no inconsistency with the antecedent and only a minimal change will be made in order to get rid of inconsistency if the added antecedent is inconsistent with the stock of beliefs. In Stalnaker's words:

> First, add the antecedent (hypothetically) to your stock of beliefs; second, make whatever adjustments are required to maintain consistency (without modifying the hypothetical belief in the antecedent), finally, consider whether or not the consequent is then true.

So far this is very much in keeping with the original Ramsey Test because, as we emphasized above, Ramsey thinks that the evaluation of conditionals with contrary-to-belief antecedents is a question of "what follows from certain laws and hypotheses". Indeed, we find Stalnaker asking us to consider whether or not the consequent follows after we make the necessary changes in the stock of beliefs as required to maintain consistency; this is a question of what follows from certain laws already present in the stock of beliefs after hypothetically adding the antecedent alongside them.

However, we observe that Stalnaker (1968) departs from the spirit of the original Ramsey Test when he insists on having truth conditions (as opposed to rational acceptability conditions) for conditional statements.

Remember that, if our reading of Ramsey is correct, conditional statements lack truth value because Ramsey treats indicative conditionals probabilistically, and counterfactual conditionals, he claims, have nothing to do with matters of fact, but are abbreviated arguments concerned with "what follows from certain laws and hypotheses". One may think that Stalnaker's departure rests on a questionable analogy between a stock of beliefs and a possible world. Stalnaker writes: "The concept of possible world is just what we need to make this transition, since a possible world is the ontological analogue of a stock of hypothetical beliefs." The "transition" Stalnaker has in mind is the transition from rational acceptability conditions to truth conditions. Let's examine Stalnaker's analogy.

We worry that Stalnaker's analogy between a stock of beliefs and a possible world may be misleading since a better candidate for the analogy could be a *set of possible worlds*. Our reasoning is as follows. As is well known, Stalnaker formulates a selection function semantics for his conditional logic on the basis of the above-mentioned analogy between a stock of beliefs and a possible world. Stalnaker's selection function is binary; there is one argument place for the world of evaluation and one argument place for the antecedent of the conditional to be evaluated. The function "selects" a unique world and if the consequent of the conditional in question is true in the selected world, then the conditional is evaluated to be true; otherwise the conditional is evaluated to be false. It seems that the analogy between a stock of beliefs and a possible world renders Stalnaker's reinterpretation radically incompatible with the original insight of Ramsey. The fact that Stalnaker's selection function *takes a single world of evaluation as an argument* shows that the ontic reinterpretation of the Ramsey Test is out of touch with Ramsey's original analysis. Accordingly, we have no reason to think that the intuitive power of the original Ramsey Test is in any sense conducive to the subsequent ontic analyses of conditional idiom. In particular, there is no reason to think that Ramsey's original insight lends itself to an analysis of *truth* conditions for conditional statements. The upshot here is that ontic conditional logics are in fact in conflict with Ramsey's analysis.

In this section we gave an exposition of Stalnaker's reinterpretation of the Ramsey Test and observed that Stalnaker's reinterpretation is in tension with Ramsey's ideas. Of course, it does not follow that Stalnaker's ontic analysis of conditional idiom is problematic; it is only indicated that the ontic conditional logics pioneered by Stalnaker are in fact discontinuous with Ramsey's analysis. In the next section we will consider Gärdenfors's project and observe that it is more in line with Ramsey's insight.

1.3 Gärdenfors's Project

In a moment we will introduce the epistemic approach to the semantic analysis of conditional statements, which is due to Peter Gärdenfors (1978). Before doing that, let's try to explain to the reader why someone would like to have an epistemic (rational acceptability conditions) account in contrast to an ontic (truth conditions) account. The primary motivation for the epistemic program comes from the intuitive force of the Ramsey Test. Other motivations, in our view, include an empiricist tendency to be sceptical of the metaphysically suspect "possible worlds" and of the ontic "similarity" relation postulated across those "worlds"; as well as some uneasy feelings about the intensional nature of counterfactual idiom. Quine (1960), for instance, writes:

> The subjunctive conditional depends, like indirect quotation and more so, on a dramatic projection: we feign belief in the antecedent and see how convincing we then find the consequent. What traits of the real world to suppose preserved in the feigned world of the contrary-to-fact antecedent can be guessed only from a sympathetic sense of the fabulists likely purpose in spinning his fable.

Another prominent philosopher expressing scepticism about the idea of giving truth conditions for counterfactual conditionals is Bas van Fraassen. After making explicit the extreme context-sensitivity of counterfactual conditional statements van Fraassen (1980) writes:

> ... we must conclude that there is nothing in science itself—nothing in the objective description of nature that science purports to give us—that corresponds to these counterfactual conditionals.

So much for vindicating Gärdenfors's motivations for the epistemic program. Now is the time to explain Gärdenfors's formalization of the Ramsey Test.

Gärdenfors formulates a formal version of the Ramsey Test on the basis of which he attempts to give rational acceptability conditions for conditional statements. First, the "stock of beliefs" of the original Ramsey Test is formalized as a *belief set*. A belief set is defined as a logically closed set of sentences. The logical closure indicates that we theorize in an idealized setting: we abstract from logical defects of real-life epistemic agents by assuming logical omniscience. Second, the requirement of making only a minimal change in order to maintain consistency is formalized by means of a revision operator $*$. The rational acceptability condition of a conditional is then given by the Formalized Ramsey Test:

$$(\phi \mathrel{\Box\!\!\rightarrow} \psi) \in T \quad \text{iff} \quad \psi \in (T * \phi)$$

where T is the belief set of the agent and $*$ is the minimal revision operator the details of which will be given in the *Belief Change* chapter. We use Lewis's connective $\Box\!\!\rightarrow$ because Gärdenfors's original project was to give an epistemic semantics specifically for Lewis's VC.

Gärdenfors's project came to a halt when he discovered that it is impossible to have a nontrivial belief change system which contains $\Box\!\!\rightarrow$-conditional sentences. More detail regarding the epistemic project and the impossibility result can be found in Chapter 4.

CHAPTER 2

A Fitch-Style Formulation of Conditional VC

In this chapter our goal is to build a Fitch-style natural deduction system for David Lewis's "official" conditional logic VC. To this end, we will first examine Thomason's FCS and show that there is no straightforward way to employ FCS for VC. We will see that certain proof-theoretic complications arise in connection with Stalnaker's much-debated assumption of *Uniqueness*. In order to avoid those complications we will need to adjust FCS by removing one reiteration rule and introducing two new reiteration rules. We will call the resultant system FVC. We will prove that FVC and VC are equivalent systems.

A Fitch-style natural deduction proof system for sentential conditional logic CS is given in (Thomason, 1971). The original axiomatic formulation of CS can be found in (Stalnaker and Thomason, 1970). Natural deduction systems differ from axiomatic systems as they mimic actual reasoning by allowing reasoning from arbitrary assumptions. This notable feature endows natural deduction with great intuitive power. As a stylistic variant, "Fitch-style" natural deduction was originally developed in (Fitch, 1952). For the history of natural deduction proof systems and the details of "the Fitch method" the reader is referred to (Pelletier and Hazen, 2012). We favor a natural deduction formulation of Lewis's VC because we believe that a formulation of that kind is the best candidate for an *intuitive* formalization of counterfactual reasoning.

2.1 Thomason's FCS

FCS has $\supset$, $\sim$, and the conditional connective $>$ as primitive connectives. $\wedge$, $\vee$, and $\equiv$ are defined as usual. $\Box$ and $\Diamond$ are defined as follows:

$$\Box\phi =_{df} \quad \sim\phi \; > \phi$$

$$\Diamond\phi =_{df} \quad \sim\Box\sim\phi$$

Rules for $\supset$, $\sim$, and for reiteration into ordinary derivations are just as usual Fitch-style natural deduction rules. In addition, FCS has an additional rule for *strict derivation*:

$$\begin{array}{r|l} \triangleright & \phi \\ \cline{2-2} & \vdots \\ & \vdots \end{array}$$

What is significant here is that, unlike ordinary derivations, strict derivations come with certain restrictions on reiteration rules. FCS has four restricted reiteration rules:

Reit 1

$\phi > \psi$

$$\begin{array}{r|l} \triangleright & \phi \\ \cline{2-2} & \vdots \\ & \psi \end{array}$$

Reit 2

$$\Box\psi$$

$$\begin{array}{r|l} \triangleright & \phi \\ \cline{2-2} & \vdots \\ & \psi \end{array}$$

Reit 3

$$\sim(\phi > \psi)$$

$$\begin{array}{r|l} \triangleright & \phi \\ \cline{2-2} & \vdots \\ & \sim\psi \end{array}$$

Reit 4

$$\phi > \psi$$

$$\psi > \phi$$

$$\psi > \chi$$

$$
\begin{array}{r|l}
\triangleright & \phi \\ \cline{2-2}
 & \vdots \\
 & \chi
\end{array}
$$

Here we assume that Thomason's strict reiteration rules are to be taken as "single-step" reiteration rules; i.e., they are only applicable into the immediate strict subproof which is at most one subproof deeper than the $>$-conditional licensing it. This means that a "multi-step" application of the strict reiteration rules is *not* allowed in Thomason's system even though his own formulation is not very explicit about this.

The introduction and elimination rules for the conditional connective $>$ are as follows:

Conditional Introduction Rule

$$
\begin{array}{l|l}
 & \begin{array}{l|l} \triangleright & \phi \\ \hline & \vdots \\ & \vdots \\ & \psi \end{array} \\
 & \phi > \psi
\end{array}
$$

Conditional Elimination Rule

$$
\begin{array}{|l}
\phi \\
\phi > \psi \\
\vdots \\
\vdots \\
\psi
\end{array}
$$

What follows is the original axiomatic formulation of CS (Stalnaker and Thomason, 1970):

$A1.$ $\Box(\phi \supset \psi) \supset (\Box\phi \supset \Box\psi)$

$A2.$ $\Box(\phi \supset \psi) \supset (\phi > \psi)$

$A3.$ $\Diamond\phi \supset ((\phi > \psi) \supset \sim(\phi > \sim\psi))$

$A4.$ $(\phi > (\psi \vee \chi)) \supset ((\phi > \psi) \vee (\phi > \chi))$

$A5.$ $(\phi > \psi) \supset (\phi \supset \psi)$

$A6.$ $(\phi > \psi) \wedge (\psi > \phi) \supset ((\phi > \chi) \supset (\psi > \chi))$

Rules of Inference: *Modus Ponens* and Necessitation

Thomason proves that the axiomatic system CS and the natural deduction system FCS are equivalent systems (Thomason, 1971). In the next section we will give an axiomatic formulation of Lewis's VC along with a brief explanation of each axiom.

2.2 Lewis's VC

In his first formulation of VC, Lewis prefers to use $\preccurlyeq$ because he thinks that $\preccurlyeq$-formulation is much simpler than the $\Box\!\!\rightarrow$-formulation (Lewis, 1973, p. 123). Later on in the book he decides that he owes us an axiomatic formulation using the primitive conditional connective $\Box\!\!\rightarrow$ as well (p. 132). The main reason behind Lewis's preference of $\preccurlyeq$-formulation to the primitive $\Box\!\!\rightarrow$-formulation is the "embarrassing" Axiom 5 of the $\Box\!\!\rightarrow$-formulation. The $\Box\!\!\rightarrow$-axiomatization of VC is as follows:

*R*1. *Modus Ponens*

*R*2. *Deduction within Conditionals*

$$\frac{\vdash\ (\chi_1 \wedge \ldots \wedge \chi_n) \supset \psi}{\vdash\ ((\phi \Box\!\!\rightarrow \chi_1) \wedge \ldots \wedge (\phi \Box\!\!\rightarrow \chi_n)) \supset (\phi \Box\!\!\rightarrow \psi)}$$

*R*3. *Interchange of Logical Equivalents*

*A*1. *Truth-functional tautologies*

*A*2. *Definitions of nonprimitive operators*

*A*3. $\phi \Box\!\!\rightarrow \phi$

*A*4. $(\sim \phi \Box\!\!\rightarrow \phi) \supset (\psi \Box\!\!\rightarrow \phi)$

*A*5. $(\phi \Box\!\!\rightarrow \sim \psi) \vee (((\phi \wedge \psi) \Box\!\!\rightarrow \chi) \equiv (\phi \Box\!\!\rightarrow (\psi \supset \chi)))$

*A*6. $(\phi \Box\!\!\rightarrow \psi) \supset (\phi \supset \psi)$

*A*7. $(\phi \wedge \psi) \supset (\phi \Box\!\!\rightarrow \psi)$

$\Box\!\!\rightarrow$ is interpreted as counterfactual-/would-conditional. The first axiom ensures that VC contains classical propositional logic. The second axiom takes care of the defined connectives employed by Lewis throughout the book. Intuitively, the third axiom states that had something been the case, it would have been indeed the case. The fourth axiom says that a necessary truth obtains in every counterfactual situation. The fifth axiom represents the principle of *rational monotony*; we will discuss it below in some detail.

The sixth axiom states that the would-conditional implies the material conditional, and hence is stronger. Finally, the seventh axiom says that if two statements are actually true, then one of them counterfactually implies the other.[1]

Note that $\Box\!\!\rightarrow$ is stronger than $\supset$, but weaker than $⥽$.[2] The idea is to have a *variably strict conditional* (hence the name VC) as distinct from the *strict conditional*. The invariably strict form $\phi ⥽ \psi$ says that it is logically necessary that either ϕ is false or ψ is true[3]; model-theoretically, there is no maximal-consistent set of sentences where ϕ is true and ψ is false. This is way too strict for our purposes. In our reasoning about counterfactual situations we do not take into account every possible situation; this is neither necessary nor feasible. We take into account only those situations which are sufficiently similar to the actual situation and where the antecedent of the counterfactual is imagined to be true.[4] Variability of strictness becomes apparent as soon as we realize that our conditional statements do not involve total logical strictness or strictness *simpliciter*, but involve what one might call *ceteris paribus* strictness, a kind of strictness that varies in

[1]Note that this particular intuitive translation requires the *Interchange of Logical Equivalents* rule R3.

[2]$⥽$ was introduced by C. I. Lewis, who was dissatisfied with the truth-functional material conditional of *Principia Mathematica*. One can include $⥽$ as a defined connective: $\phi ⥽ \psi =_{df} \Box(\phi \supset \psi)$.

[3]Note that, in the object language, we have $\phi ⥽ \psi \equiv \Box(\sim \phi \vee \psi)$

[4]Those theories of conditionals in which counterfactual reasoning is conceived as reasoning about a minimally different situation where the antecedent is entertained to be true are sometimes called *minimal change theories*.

accordance with the extent of the *ceteris paribus* enthymeme (see (Priest, 2008) for the enthymematic interpretation).

It can readily be seen that some of these axioms are very obviously derivable in FCS. The derivability of the first three axioms are evident. The fourth axiom is handled by FCS's reit 2. The fifth axiom of rational monotony is derivable in FCS using reit 3. The sixth axiom can be proved using reit 1 and $\supset$-introduction. The seventh axiom is derivable in FCS again by means of reit 3. Now notice that the difference between Lewis's VC and Stalnaker & Thomason's CS is the *Conditional Excluded Middle* axiom:

$A3.\ \diamond\phi \supset ((\phi > \psi) \supset \sim(\phi > \sim\psi))$

or alternatively,

$S.\ \ (\phi > \psi) \vee (\phi >\sim \psi)$

So, with the addition of the Conditional Excluded Middle, one arrives at a system that is equivalent to CS, a system Lewis calls VCS ("S" for "Stalnaker's Assumption"). Since our goal here is to adapt Thomason's natural deduction system for Lewis's VC, our immediate task now is to remove the reiteration rule representing the undesired principle of Conditional Excluded Middle. Rejection of the Conditional Excluded Middle is *the* characteristic feature of Lewis's conditional logic. It is easy to see that the rule that needs to be removed from FCS is reit 3. One might be deceived to think what needs to be done is just to remove the Conditional Excluded Middle rule (reit 3) from FCS, and we are good to go. Alas, things are not so simple: it turns out that Axiom 5 and Axiom 7 of VC are *not* derivable without reit 3. Since we know that reit 3 *must* go at all costs (otherwise what we get cannot be VC), we now have no choice but to replace reit 3 with an appropriate reiteration rule that can successfully handle Axiom 5 and (hopefully) Axiom 7 of VC.

2.3 Formulating FVC

2.3.1 Underivability of Axiom 5 and Axiom 7: The Proof-theoretic Method

First let's show that without reit 3, Axiom 5 and Axiom 7 of VC becomes underivable (i.e., FCS without reit 3 is semantically incomplete with respect to the Lewisean system of spheres models for VC).

Theorem 2.3.1. *If reit 3 is removed, Axiom 5 of VC becomes underivable in FCS.*

Now assume for reductio that reit 3 is removed from FCS but Axiom 5 is still derivable. If Axiom 5 is derivable, then there are subderivations in FCS of $((\phi \wedge \psi) \mathrel{\Box\!\!\rightarrow} \chi) \supset (\phi \mathrel{\Box\!\!\rightarrow} (\psi \supset \chi))$ and $(\phi \mathrel{\Box\!\!\rightarrow} (\psi \supset \chi)) \supset ((\phi \wedge \psi) \mathrel{\Box\!\!\rightarrow} \chi)$ under the assumption $\sim (\phi \mathrel{\Box\!\!\rightarrow} \sim \psi)$. These subderivations are either indirect proofs, or not. If they are indirect proofs, then we should be able to derive a contradiction from $((\phi \wedge \psi) \mathrel{\Box\!\!\rightarrow} \chi) \wedge \sim (\phi \mathrel{\Box\!\!\rightarrow} (\psi \supset \chi))$ and

$(\phi \mathrel{\Box\!\!\rightarrow} (\psi \supset \chi)) \wedge \sim ((\phi \wedge \psi) \mathrel{\Box\!\!\rightarrow} \chi)$. If a contradiction is derived from the former, then we are able to derive $\sim ((\phi \wedge \psi) \mathrel{\Box\!\!\rightarrow} \chi)$ or $(\phi \mathrel{\Box\!\!\rightarrow} (\psi \supset \chi))$; but such a derivation would have to involve reit 3 provided that the derivation in question is not a "devious" derivation. If contradiction is derived from the latter, then we are able to derive $\sim (\phi \mathrel{\Box\!\!\rightarrow} (\psi \supset \chi))$ or $((\phi \wedge \psi) \mathrel{\Box\!\!\rightarrow} \chi)$; again, such a derivation would have to involve reit 3 as long as it is not a "devious" derivation. Admittedly, what we offer here is not a rigorous underivability proof, but only a plausibility consideration, because we are unable to rule out the possibility that there is some devious and complicated way of getting such derivations even without reit 3[5]:

[5] A rigorous proof-theoretic proof here would depend on a *normalization* theorem for FCS and FVC. We leave the normalization theorem to future work.

$$\mid\; \underline{\mid\; \sim (\phi \mathrel{\Box\!\!\rightarrow} \sim \psi)}$$

$$\mid\;\mid\;\; \underline{\mid\; (\phi \wedge \psi) \mathrel{\Box\!\!\rightarrow} \chi}$$

$$\mid\;\mid\;\;\mid\; \vdots$$

$$\mid\;\mid\;\;\mid\; \triangleright \underline{\mid\; \phi}$$

$$\mid\;\mid\;\;\mid\;\;\mid\; \vdots$$

$$\mid\;\mid\;\;\mid\;\;\mid\; (\psi \supset \chi)$$

$$\mid\;\mid\;\; \underline{\mid\; (\phi \mathrel{\Box\!\!\rightarrow} (\psi \supset \chi)}$$

$$\mid\;\mid\;\;\mid\; \vdots$$

$$\mid\;\mid\;\;\mid\; \triangleright \underline{\mid\; \phi \wedge \psi}$$

$$\mid\;\mid\;\;\mid\;\;\mid\; \vdots$$

$$\mid\;\mid\;\;\mid\;\;\mid\; \chi$$

We need only examine one of the two inner-most subderivations. Take the latter innermost subproof: χ cannot have been reiterated by reit 1 since there exists no superordinate occurrence of any $\Box\!\!\rightarrow$-formula with the antecedent $\phi \wedge \psi$. Further, χ cannot have come from an application of reit 2 since, by hypothesis, there is neither a superordinate categorical derivation of χ nor an occurence of the $\Box$-connective. Likewise, χ cannot have been reiterated by virtue of reit 4 since there is no superordinate occurence of such formulas as $\phi \mathrel{\Box\!\!\rightarrow} D$ and $D \mathrel{\Box\!\!\rightarrow} \phi$. The only alternative left is that χ came by an application of reit 3. But this contradicts our initial assumption.

Theorem 2.3.2. *If reit 3 is removed, Axiom 7 becomes underivable in FCS.*

Again, what we offer here is a plausibility consideration, not a rigorous proof. Assume for reductio that reit 3 is removed but Axiom 7 is still derivable in FCS. If Axiom 7 is derivable, then there will be derivations of the following form (where $A \supset B$ is not a theorem):

$$
\begin{array}{|l}
\begin{array}{|l}
\phi \wedge \psi \\ \hline
\vdots \\
\triangleright \begin{array}{|l} \phi \\ \hline \vdots \\ \psi \end{array}
\end{array}
\end{array}
$$

We need examine where ψ comes from. ψ cannot have been classically derived from ϕ, since, *ex hypothesi*, $\phi \supset \psi$ is not a theorem. ψ cannot have come by reit 1 because $\phi \mathrel{\Box\!\!\rightarrow} \psi$ does not occur superordinately. ψ cannot have come by reit 2, because our hypothesis that $\phi \supset \psi$ is not a theorem implies that ψ is not a theorem, and hence, it cannot be the case that $\Box\psi$.

Reit 4 is not applicable because there is no superordinate occurrence of any third schematic letter χ. The only alternative source for ψ of the innermost supproof is then reit 3. But this contradicts our initial assumption given that the derivation in question is not a "devious" derivation.

2.3.2 Underivability of Axiom 5 and Axiom 7: The Model-theoretic Method

Since we have been unable to provide a rigorous proof, but only a plausibility consideration, using the proof-theoretic method, we now attempt to give a model-theoretic proof for the underivability of Axiom 5 and Axiom 7 without reit 3. We will make use of a "deviant" model: suppose that the model contains three worlds w_0, w_1, w_2, and is a two-sphere model. The first sphere is $\{w_0, w_1\}$, and the second one is $\{w_0, w_2\}$. The spheres are not nested, but this is not a problem since we are using the model merely as a technical device, *not* as real semantics. The truth conditions for $\Box\!\!\rightarrow$ is then given as follows:

- If $w_1 \models \phi$, then $w_0 \models \phi \Box\!\!\rightarrow \psi$ iff $\phi \supset \psi$ is true at all worlds of the first sphere (i.e., iff $w_0 \models \phi \supset \psi$ and $w_1 \models \phi \supset \psi$).
- If $w_1 \models \sim \phi$, then $w_0 \models \phi \Box\!\!\rightarrow \psi$ iff $\phi \supset \psi$ is true at all worlds of the second sphere (i.e., iff $w_0 \models \phi \supset \psi$ and $w_2 \models \phi \supset \psi$).

We are only insterested in truth values at w_0.

(1) Axiom 7 of VC is not valid in this model because neither sphere is a

singleton (i.e., the system of spheres is not centered).

(2) Axiom 5 of VC is not valid in this model.

Proof: Either $w_0 \models \phi \mathrel{\Box\!\!\rightarrow} \sim \psi$ or $w_0 \models \sim (\phi \mathrel{\Box\!\!\rightarrow} \sim \psi)$. Suppose the latter holds. Then if Axiom 5 is true at w_0, $((\phi \wedge \psi) \mathrel{\Box\!\!\rightarrow} \chi) \equiv (\phi \mathrel{\Box\!\!\rightarrow} (\psi \supset \chi))$ must also be true at w_0. Suppose $w_0 \models (\phi \wedge \psi) \mathrel{\Box\!\!\rightarrow} \chi$. Then it must be that *either* (a) $w_1 \models \phi \wedge \psi$ and $w_0 \models \phi \supset \psi$ and $w_1 \models \phi \supset \psi$ *or* (b) $w_1 \not\models \phi \wedge \psi$ and $w_0 \models \phi \supset \psi$ and $w_2 \models \phi \supset \psi$. Suppose (a) is true. By definition of $\equiv$, it must be the case that $w_0 \models \phi \mathrel{\Box\!\!\rightarrow} (\psi \supset \chi)$; by the truth conditions of $\mathrel{\Box\!\!\rightarrow}$ this means that if $w_1 \models \phi$, then $w_0 \models \phi \supset (\psi \supset \chi)$ and $w_1 \models \phi \supset (\psi \supset \chi)$. By (a), we have $w_1 \models \phi$, so it must be the case that $w_1 \models \psi \supset \chi$. Again by (a), we have $w_1 \models \psi$. For a counterexample, it will be sufficient to suppose $w_1 \not\models \chi$.

■

The "deviant" model here is devised to be a model of FCS without reit 3. Accordingly, we have shown that reit 3 axioms Axiom 5 and Axiom 7 are not validated by this model. Reit 1, the main form of strict reiteration, is validated because if $w_0 \models \phi \mathrel{\Box\!\!\rightarrow} \psi$, then by definition of truth-conditions of $\mathrel{\Box\!\!\rightarrow}$ it follows that $w_0 \models \phi \supset \psi$. But how about reit 2 and reit 4? Reit 2 is concerned with strict reiteration of $\Box$-formulas; reit 4 is concerned with strict reiteration of "counterfactual equivalents". These two reiteration rules are closely related to the defined connective $\Box$. As we explained before, we can choose to formulate $\Box$ as a primitive connective of the proof system, which, we believe, is more in line with its intended meaning. Here we argue that $\Box$ means logical necessity and is to be analyzed as truth at all possible worlds (even in "deviant" models!). In FCS, we can observe that $\Box$-introduction always depends on some categorical derivation of the relevant formula inside some strict subderivation; therefore, we believe that we are justified in taking $\Box$ as a primitive logical necessity connective. Similarly, reit 4 will be reinterpreted in terms of "logical equivalence". Reit 2 and reit 4 should not create any problems after this modification.

2.3.3 Derivation of Axiom 5

In order to handle Axiom 5 of VC we need to replace reit 3 with a new reiteration rule suitable for VC. The reiteration rule we
propose is as follows:

$$\phi \Box\!\!\rightarrow \chi$$

$$\phi \Diamond\!\!\rightarrow \psi$$

$$
\begin{array}{r|l}
\rhd & \phi \wedge \psi \\
\cline{2-2}
 & \vdots \\
 & \vdots \\
 & \chi
\end{array}
$$

Note that we are free to choose whether $\Diamond\!\!\rightarrow$ will be taken as a defined connective or a primitive connective of VC: $\phi \Diamond\!\!\rightarrow \psi =_{df} \sim(\phi \Box\!\!\rightarrow \sim\psi)$. In terms of its model theory, $\phi \Diamond\!\!\rightarrow \psi$ states that all ϕ-permitting spheres are $\phi \wedge \psi$-permitting. We believe that $\Diamond\!\!\rightarrow$ presents an intuitive advantage of Lewis's conditional logic VC over the Stalnaker/Thomason conditional logic CS. For this reason, we are inclined to suggest $\Diamond\!\!\rightarrow$ as a primitive connective. The natural language interpretation of $\Diamond\!\!\rightarrow$ is the might-conditional; that is, it is intended to capture natural language constructions of the form "If ϕ had been the case, ψ might have been the case". The new reiteration rule allows that we can reiterate χ in a strict derivation with antecedent $\phi \wedge \psi$ if the strict derivation is subordinate to occurrences of $\phi \Box\!\!\rightarrow \chi$ and $\phi \Diamond\!\!\rightarrow \psi$. Let's call the new rule *reit RM*. Reit RM allows us to accommodate the principle of rational monotony. The would-conditional shows monotonicity with a "rationality" constraint, in the sense that so long as what is being (conjunctively) added to the antecedent does not counterfactually imply the negation of what had already been contained within the antecedent, the truth value of the conditional does not decrease. This is in contrast with the material conditional since the material conditional has *total* monotonicity in the sense that its truth value never decreases whenever something new is added to its antecedent.

An extremely important point is that the addition of reit RM to the proof system makes it necessary to tweak the conditions for application of reit 1. As we noted before, Thomason's exposition of reit 1 is perhaps not as explicit as it should be; it is unclear whether "multi-step" applications of reit 1 are sanctioned or not. For Thomason's FCS we charitably assumed that only "single-step" applications of reit 1 are allowed because FCS does not need any "multi-step" application of reit 1 in order to be a complete proof system. In FVC, however, we need some "multi-step" applications of reit 1, and we formulate the condition for such "multi-step" applications as follows:

Reit 1*

$$\phi \mathrel{\Diamond\!\!\rightarrow} \psi$$

$$\phi \wedge \psi \mathrel{\Box\!\!\rightarrow} \chi$$

$$
\begin{array}{l|l}
\rhd & \underline{\;\phi\;} \\
 & \;\vdots \\
 & \begin{array}{l|l}
 \rhd & \underline{\;\phi \wedge \psi\;} \\
  & \;\vdots \\
  & \;\chi
 \end{array}
\end{array}
$$

So the "multi-step" application of reit 1 is possible if and only if the sentences are "counterfactually compatible". Again, we believe that this encapsulates the intuitive idea of rational monotony.

Rational monotony makes it possible to interpret the $\Box\!\!\rightarrow$-conditional as an object language counterpart of a default rule of (Reiter, 1980)'s default logic (see (Delgrande, 1988) for this interpretation). Reit RM accords well with the default rule interpretation of the $\Box\!\!\rightarrow$-conditional. In case we have the default rule *By default if A, then C* and we do not have the default rule *By default if A, then $\sim$ B*, we can infer the default rule *By default if A and B, then C*. To make this more intuitive consider the following example: It is reasonable to assume that an intelligent agent should have the following two default rules: *By default if something is a bird, it flies*. *By default, if something is a bird, it is not a penguin*. Hence according to reit RM we cannot infer *By default if something is a bird and a penguin, then it flies*. However, since we have no default rule to the effect that *By default if something is a bird, then it is not a mockingbird*, we can infer *By default if something is a bird and a mockingbird, then it flies*. The default rule interpretation of the $\Box\!\!\rightarrow$ allows us to reason about default rules. There are numerous technical problems in the area of default reasoning and most researchers believe that rational monotonicity is too permissive. So it is widely believed that a less permissive kind of monotonicity (suchs as *cautious monotonicity*) is required in order to appropriately handle default rules. We won't get into further details regarding the default rule interpretation of the $\Box\!\!\rightarrow$ in this thesis.

Now let's see reit RM in action:

1	$\mid\ \mid\ \phi \Diamond\!\!\rightarrow \psi$	(hyp)
2	$\mid\ \mid\ \mid\ \phi \Box\!\!\rightarrow (\psi \supset \chi)$	(hyp)
3	$\mid\ \mid\ \mid\ \triangleright \mid\ \phi \wedge \psi$	(hyp)
4	$\mid\ \mid\ \mid\ \quad \mid\ \psi \supset \chi$	(1-2, reit RM)
5	$\mid\ \mid\ \mid\ \quad \mid\ \psi$	(3-4, $\wedge$-elim, MP)
6	$\mid\ \mid\ \mid\ \quad \mid\ \chi$	(4-5, MP)
7	$\mid\ (\phi \Diamond\!\!\rightarrow \psi) \supset ((\phi \Box\!\!\rightarrow (\psi \supset \chi)) \supset ((\phi \wedge \psi) \Box\!\!\rightarrow \chi))$	(1-6, $\Box\!\!\rightarrow$-int, $\supset$-int)
8	$\mid\ \mid\ \phi \Diamond\!\!\rightarrow \psi$	(hyp)
9	$\mid\ \mid\ \mid\ (\phi \wedge \psi) \Box\!\!\rightarrow \chi$	(hyp)
10	$\mid\ \mid\ \mid\ \triangleright \mid\ \phi$	(hyp)
11	$\mid\ \mid\ \mid\ \quad \mid\ \mid\ \psi$	(hyp)
12	$\mid\ \mid\ \mid\ \quad \mid\ \mid\ \phi$	(10, reit)
13	$\mid\ \mid\ \mid\ \quad \mid\ \mid\ \phi \wedge \psi$	(11-12, $\wedge$-int)
14	$\mid\ \mid\ \mid\ \quad \mid\ \mid\ \triangleright \mid\ \phi \wedge \psi$	(hyp)
15	$\mid\ \mid\ \mid\ \quad \mid\ \mid\ \quad \mid\ \chi$	(8-9, reit 1*)
16	$\mid\ \mid\ \mid\ \quad \mid\ \mid\ (\phi \wedge \psi) \Box\!\!\rightarrow \chi$	(14-15, $\Box\!\!\rightarrow$-int)
17	$\mid\ \mid\ \mid\ \quad \mid\ \mid\ \chi$	(13-16, MP)
18	$\mid\ \mid\ \mid\ \quad \mid\ \psi \supset \chi$	(11-17, $\supset$-int)
19	$\mid\ \mid\ \mid\ \phi \Box\!\!\rightarrow (\psi \supset \chi)$	(10-18, $\Box\!\!\rightarrow$-int)
20	$\mid\ \mid\ ((\phi \wedge \psi) \Box\!\!\rightarrow \chi) \supset (\phi \Box\!\!\rightarrow (\psi \supset \chi))$	(9-19, $\supset$-int)
21	$\mid\ (\phi \Diamond\!\!\rightarrow \psi) \supset (((\phi \wedge \psi) \Box\!\!\rightarrow \chi) \supset (\phi \Box\!\!\rightarrow (\psi \supset \chi)))$	(8-20, $\supset$-int)
22	$\mid\ \mid\ \phi \Diamond\!\!\rightarrow \psi$	(hyp)
23	$\mid\ \mid\ (\phi \Diamond\!\!\rightarrow \psi) \supset ((\phi \Box\!\!\rightarrow (\psi \supset \chi)) \supset ((\phi \wedge \psi) \Box\!\!\rightarrow \chi))$	(7, reit)
24	$\mid\ \mid\ (\phi \Diamond\!\!\rightarrow \psi) \supset (((\phi \wedge \psi) \Box\!\!\rightarrow \chi) \supset (\phi \Box\!\!\rightarrow (\psi \supset \chi)))$	(21, reit)
25	$\mid\ \mid\ ((\phi \wedge \psi) \Box\!\!\rightarrow \chi) \supset (\phi \Box\!\!\rightarrow (\psi \supset \chi))$	(22-24, MP)
26	$\mid\ \mid\ (\phi \Box\!\!\rightarrow (\psi \supset \chi)) \supset ((\phi \wedge \psi) \Box\!\!\rightarrow \chi)$	(22-23, MP)
27	$\mid\ \mid\ (\phi \Box\!\!\rightarrow (\psi \supset \chi)) \equiv ((\phi \wedge \psi) \Box\!\!\rightarrow \chi)$	(25-26, $\equiv$-int)
28	$\mid\ (\phi \Diamond\!\!\rightarrow \psi) \supset ((\phi \Box\!\!\rightarrow (\psi \supset \chi)) \equiv ((\phi \wedge \psi) \Box\!\!\rightarrow \chi))$	(22-27, $\supset$-int)

2.3.4 Derivation of Axiom 7

Thus we have shown that Axiom 5 of VC is provable in the natural deduction system we propose. We can now move on to the other troublesome axiom, Axiom 7. Remember that Axiom 7 also required an application of the undesirable reit 3 of FCS. Since we removed reit 3 of FCS and introduced reit RM instead, our hope is that reit RM alone will be sufficient to provide a derivation of Axiom 7. Axiom 7 is linked to the *strong centering* condition on the system of spheres. The strong centering condition represents the metaphysically convincing idea that the actual world is the most similar world to itself. In terms of Lewis's sphere semantics, this translates to there being a smallest singleton sphere to which only the actual world belongs as a member. The weak centering condition on the other hand is weaker because it does not require that the smallest sphere be a singleton. In a system of spheres with only the weak centering condition, the actual world is one of those worlds which are equally most similar to the actual world.[6] Unfortunately, the new reiteration rule we have introduced cannot accommodate Axiom 7. In order to handle Axiom 7 we propose the following new reiteration rule (call it reit C):

[6] Strange though it may be from an ontic-conditional perspective, systems of spheres with weak centering condition are utilized in constructive models of belief revision systems.

$A \wedge B$

$$\begin{array}{r|l} \rhd & A \\ \cline{2-2} & \vdots \\ & B \end{array}$$

It is obvious that reit C handles Axiom 7. Call this modified system FVC.[7] Since *Modus Ponens* and *Interchange of Logical Equivalents* belong to Fitch-style natural deduction systems by default, we need only to show that *Deduction within Conditionals* is admissible in FVC:

Deduction within Conditionals:

If $\vdash (\chi_1 \wedge \ldots \wedge \chi_n) \supset \psi$, **then** $\vdash (\phi \;\Box\!\!\rightarrow \chi \;\wedge \ldots \wedge\; \phi \;\Box\!\!\rightarrow \chi_n) \supset (\phi \;\Box\!\!\rightarrow \psi)$.

In order to prove the admissiblity of this rule of inference, we will utilize the fact that the *Rule of Necessitation* is admissible in FCS.[8] Let's first show that *Necessitation* is admissible in FVC. *Ex hypothesi*, ψ is categorically derivable (i.e., a theorem), what we need then is to simply derive ψ inside

[7]In brief, *FVC = FCS - reit 3 + reit RM + reit C*.

[8]Note that $\Box$ can be taken as a defined connective: $\Box\phi =_{df} \sim \phi \;\Box\!\!\rightarrow \phi$.

the following strict subproof:

$$
\begin{array}{|ll}
 & \vdots \\
 \rhd & \begin{array}{|l} \sim \psi \\ \hline \vdots \\ \psi \end{array} \\
 & \Box\psi
\end{array}
$$

Now let's suppose that we already have a categorical derivation of the antecedent $(\chi_1 \wedge \ldots \wedge \chi_n) \supset \psi$. The following then is a proof of the admissibility of the *Deduction within Conditionals* rule of inference in FVC:

$$
\begin{array}{l}
\mid \triangleright \mid \underline{\ \sim ((\chi_1 \wedge \ldots \wedge \chi_n) \supset \psi)} \\
\mid \quad\ \mid \ \vdots \\
\mid \quad\ \mid \ (\chi_1 \wedge \ldots \wedge \chi_n) \supset \psi \\
\mid \quad \Box ((\chi_1 \wedge \ldots \wedge \chi_n) \supset \psi) \\
\mid \quad \mid \underline{\ (\phi \mathrel{\Box\!\!\rightarrow} \chi_1) \ \wedge \ldots \wedge \ (\phi \mathrel{\Box\!\!\rightarrow} \chi_n)} \\
\mid \quad \mid \ \phi \mathrel{\Box\!\!\rightarrow} \chi_1 \\
\mid \quad \mid \ \vdots \\
\mid \quad \mid \ \phi \mathrel{\Box\!\!\rightarrow} \chi_n \\
\mid \quad \mid \triangleright \mid \underline{\ \phi} \\
\mid \quad \mid \quad\ \mid \ \chi_1 \\
\mid \quad \mid \quad\ \mid \ \vdots \\
\mid \quad \mid \quad\ \mid \ \chi_n \\
\mid \quad \mid \quad\ \mid \ \chi_1 \wedge \ldots \wedge \chi_n \\
\mid \quad \mid \quad\ \mid \ (\chi_1 \wedge \ldots \wedge \chi_n) \supset \psi \\
\mid \quad \mid \quad\ \mid \ \psi \\
\mid \quad \mid \ \phi \mathrel{\Box\!\!\rightarrow} \psi \\
\mid \quad (\phi \mathrel{\Box\!\!\rightarrow} \chi \ \wedge \ldots \wedge \ \phi \mathrel{\Box\!\!\rightarrow} \chi_n) \supset (\phi \mathrel{\Box\!\!\rightarrow} \psi)
\end{array}
$$

This amounts to showing that the following theorem holds.

Theorem 2.3.3. *Everything provable in VC is derivable in FVC.*

Now what remains to be done is to prove that everything derivable in FVC is provable in VC. To do that, we need to show that every derivation of ψ from ϕ in FVC can be turned into a deduction from ϕ to ψ in VC.

Theorem 2.3.4. *If a formula is derivable in FVC, then there exists a deduction of that formula in VC.*

Proof: We will adapt (Thomason, 1971)'s method. Let's begin with a definition of *VC-derivation*: A VC-derivation is an array-like derivation in which the axioms of VC can be freely introduced anywhere in it. It is obvious that any VC-derivable formula is deducible in the axiomatic VC. In order to prove the theorem we must show that any strict subproof in FVC can be transformed into a VC-derivation. Let ϕ and ψ be two arbitrary formulas (including $\top$ and $\bot$, where a derivation from $\top$ is to be seen as a categorical derivation). *Case 1:* The derivation of ψ from ϕ is a pure classical derivation (i.e., no strict subproof is present): All classical derivations in FCV can be handled by VC in the style of (Thomason, 1970) since VC includes all classical tautologies as an axiom (Axiom 1) in addition to the rules *Interchange of Logical Equivalents* and *Modus Ponens*. *Case 2:* The derivation of ψ from ϕ involves a strict subproof: In this case we need to show that the FVC strict subproof can be transformed into a VC-derivation. The first step is to erase the line of the subproof to be eliminated and prefix C_1 to each item of the eliminated strict subproof.

$$
\begin{array}{l|l}
\vdots & \\
\begin{array}{l|l}
\triangleright & \underline{C_1} \\
 & C_2 \\
 & \vdots \\
 & \vdots \\
 & C_n \\
 &
\end{array}
&
\begin{array}{l}
\vdots \\
C_1 \mathrel{\Box\!\!\rightarrow} C_1 \\
C_1 \mathrel{\Box\!\!\rightarrow} C_2 \\
\vdots \\
C_1 \mathrel{\Box\!\!\rightarrow} C_n \\
\vdots
\end{array} \\
\vdots &
\end{array}
$$

Now let's show that $C_1 \mathrel{\Box\!\!\rightarrow} C_n$ is deducible in the VC-derivation without use of any strict subproofs. For this, let's examine the exhaustive set of possibilities for $C_1 \mathrel{\Box\!\!\rightarrow} C_i$ ($1 \leqslant i \leqslant n$) by considering the following cases: If C_i is the assumption C_1 of the strict subproof, we insert a proof of $C_1 \mathrel{\Box\!\!\rightarrow} C_1$ in VC. If C_i comes from D and $D \supset C_i$ by Modus Ponens, then we insert a deduction of $C_1 \mathrel{\Box\!\!\rightarrow} C_i$ from $C_1 \mathrel{\Box\!\!\rightarrow} D$ and $C_1 \mathrel{\Box\!\!\rightarrow} (D \supset C_i)$. If C_i is an axiom of VC, then we insert C_i and simply deduce $C_1 \mathrel{\Box\!\!\rightarrow} C_i$ using Necessitation and Axiom 4 of VC. If C_i comes by negation elimination, we deduce $C_1 \mathrel{\Box\!\!\rightarrow} C_i$ from $C_1 \mathrel{\Box\!\!\rightarrow} {\sim\sim}\, C_i$ by $R3$ of VC. If reit 1 was used to get

C_i from $C_1 \;\Box\!\!\rightarrow C_i$, we need only reiterate $C_1 \;\Box\!\!\rightarrow C_i$ by ordinary reiteration. If C_i comes from $\Box C_i$ by reit 2, we insert a deduction of $C_1 \;\Box\!\!\rightarrow C_i$ from $\Box C_i$ using Axiom 2 and Axiom 4 of VC. If reit RM was used to get C_i from $A \;\Box\!\!\rightarrow C_i$ and $B \;\Diamond\!\!\rightarrow C_i$, we deduce $A \wedge B \;\Box\!\!\rightarrow C_i$ in VC from $A \;\Box\!\!\rightarrow C_i$ and $B \;\Diamond\!\!\rightarrow C_i$ (where $C_1 \equiv A \wedge B$). If reit C was used to get C_i, we deduce $C_1 \;\Box\!\!\rightarrow C_i$ from $C_1 \wedge C_i$. If C_i comes from $D \;\Box\!\!\rightarrow C_1$, $C_1 \;\Box\!\!\rightarrow D$, and $D \;\Box\!\!\rightarrow C_i$ by reit 4, then we deduce in VC $C_1 \;\Box\!\!\rightarrow C_i$ from $D \;\Box\!\!\rightarrow C_1$, $C_1 \;\Box\!\!\rightarrow D$, and $D \;\Box\!\!\rightarrow C_i$. If C_i comes by conditional elimination from D and $D \;\Box\!\!\rightarrow C_i$, then we deduce $C_1 \;\Box\!\!\rightarrow C_i$ from $C_1 \;\Box\!\!\rightarrow D$ and $C_1 \;\Box\!\!\rightarrow (D \;\Box\!\!\rightarrow C_i)$.

■

Theorems 2.3.3 and 2.3.4 together imply the following equivalence theorem.

Theorem 2.3.5. *VC and FVC are equivalent systems.*

Everything deducible in VC are derivable in FVC and everything derivable in FVC are deducible in VC.

2.4 Some Remarks on the Proof Theory of Conditional Logics

It is well known that the main difference between the Stalnaker/Thomason conditional logic CS and Lewis's conditional logic VC is the much-debated

Limit and Uniqueness assumptions. The Limit Assumption requires that there always exist *at least* one closest world in the models. As an even stronger assumption, the Uniqueness Assumption requires that there always exist *exactly* one closest world in the models. In Lewisean systems of spheres the Limit and the Uniqueness assumptions happen to be satisfied only in certain peculiar models where the evaluated $\Box\!\!\rightarrow$-conditional is not in effect a counterfactual conditional. For example, evaluations of conditionals with *factual* (true) antecedents as in such cases as the evaluation of whether $@ \models \phi \Box\!\!\rightarrow \psi$ holds where $@ \models \phi$ holds.[9] In such cases, where the antecedent is actually true, the Limit and the Uniqueness assumptions happen to be satisfied; that is, there happens to be exactly one closest world (namely, the actual world @) in a degenerate sense, due to the strong centering condition on the system of spheres.

Prima facie, it is tempting to think that what needs to be done in order to give a natural deduction formulation of Lewis's VC is to simply remove from FCS the reiteration rule corresponding to the Limit and Uniqueness assumptions, and that we are done. We have shown that this is not the case. Our finding that there is in fact no uncomplicated way of adapting Thomason's Fitch-style proof system FCS for Lewis's VC is interesting because a surprising ramification of Stalnaker's assumptions is discovered. In FCS, Stalnaker's assumptions give us a very powerful proof-theoretic

[9]Here @ stands for the actual world, i.e., the index on which the system of spheres is centered.

tool by allowing us to incorporate a special sort of *reductio ad absurdum* in the proofs. Given Stalnaker's assumptions (represented by reit 3 of FCS), whenever a conditional $\phi \mathrel{\Box\!\!\rightarrow} \psi$ leads us into contradiction, we can proceed to derive the conditional with the contrary consequent but with the same antecedent, which is, according to our example, $\phi \mathrel{\Box\!\!\rightarrow} \sim \psi$. In FVC, however, things are not so simple. $\Diamond\!\!\rightarrow$-conditional (or, equivalently, $\sim(\phi \mathrel{\Box\!\!\rightarrow} \sim \psi)$) must be utilized in order to deal with Axiom 5. Besides, Axiom 7 needs an additional reiteration rule of its own (reit C).

All this perhaps suggests that something might be wrong with intelim-style inferentialism about logical connectives. Our examination of the proof theory of the two most popular conditional logics can be taken to provide evidence to the effect that the introduction and elimination rules *alone* fall short of uniquely determining the meanings of at least some logical connectives. Even though the introduction/elimination rules for the Thomason/Stalnaker conditional connective $>$, and the introduction/elimination rules for the Lewisean conditional $\Box\!\!\rightarrow$ connective are *exactly* the same, we have seen that (and immense literature concerning the Limit Assumption testifies to this) those two connectives and the related logics are significantly different from each other. The point being made here may be generalized to all kinds of multi-modal logics. In such logics, *reiteration rules* seem extremely relevant to the determination of the meanings of their associated logical connectives.

CHAPTER 3

Belief Change

3.1 The AGM Theory of Belief Revision

The study of belief revision began in the 1980s with the seminal work by Alchourrón, Gärdenfors and Makinson (Alchourron et al., 1985). The framework given in that paper has come to be known as the *AGM paradigm of belief revision* and is still relevant after years of subsequent research in knowledge representation. Our exposition of the AGM theory will be based on van Harmelen et al. (2008, pp. 317-329). Let's begin with defining the language and logic of belief revision.

Definition 3.1.1. *The Language and The Logic of Belief Revision:* Let L be the formal language of belief revision. L is governed by a logic $\vdash_L$. The following holds for $\vdash_L$:

(a) $\vdash_L$ contains all classical tautologies.

(b) $\vdash_L$ has *Modus Ponens* as a rule of inference.

(c) $\vdash_L$ is consistent.

(d) $\vdash_L$ is compact.

By $Cn_L(\Gamma)$ we denote the set of all $\vdash_L$-consequences of Γ where Γ is an arbitrary set of sentences of the language L. We call a set of sentences T of the language L a *theory* iff $T = Cn_L(T)$. T is a complete theory iff for any sentence φ either $\varphi \in T$ or $\sim \varphi \in T$. We denote by $[\![\varphi]\!]$ the set of all complete theories of L to which φ belongs as a member. For a theory T and a set of sentences Γ we denote by $T + \Gamma$ the $\vdash_L$-closure of $T \cup \Gamma$, i.e., $Cn(T \cup \Gamma)$. By $T + \varphi$ we abbreviate $T + \{\varphi\}$.

Definition 3.1.2. *AGM Belief Revision Function:* We represent beliefs as sentences of the language L, and belief sets as theories of L. Let $*$ be a binary function $\mathbb{T}_L \times L \mapsto \mathbb{T}_L$, where $\mathbb{T}_L$ is the set of all theories in L. We say that $*$ is an AGM revision function iff the following holds:

($*$ 1) $T * \varphi \in \mathbb{T}$.

($*$ 2) $\varphi \in T * \varphi$

($\ast$ 3) $T \ast \varphi \subseteq T + \varphi$

($\ast$ 4) If $\sim \varphi \notin T$, then $T + \varphi \subseteq T \ast \varphi$

($\ast$ 5) If ϕ is consistent, then $T \ast \phi$ is consistent.

($\ast$ 6) If $\vdash \varphi \leftrightarrow \psi$, then $T \ast \varphi = T \ast \psi$.

($\ast$ 7) $T \ast (\varphi \wedge \psi) \subseteq (T \ast \varphi) + \psi$.

($\ast$ 8) If $\sim \psi \notin T \ast \varphi$, then $T \ast \varphi + \psi \subseteq T \ast (\varphi \wedge \psi)$.

($\ast$ 1) says that $\mathbb{T}_L$ is closed under $\ast$. ($\ast$ 2) says that $\ast$ is always successful. ($\ast$ 3) and ($\ast$ 4) represent the principle of minimal change: There is no need to remove any beliefs if there is no inconsistency, and the posterior belief set will contain T and new belief φ, together with the logical consequences of these, and nothing more. ($\ast$ 5) says that consistency will be preserved as long as the newly acquired belief is consistent; otherwise inconsistency will occur due to ($\ast$ 2). ($\ast$ 6) is called the *irrelevance of syntax postulate*; logically equivalent sentences φ and ψ are equivalent from the point of view of AGM belief revision. ($\ast$ 7) and ($\ast$ 8) again represent the principle of minimal change: Revision by $(\varphi \wedge \psi)$ will not contract a smaller number of beliefs than revision by just φ does; and when inconsistency is not present, $\ast$ behaves in the same way as $+$.

It is important to notice that AGM postulates (1-8) do not characterize a unique revision function, but instead characterize a set of those. That is to say, the belief set $T * \varphi$ which results from the revision of T by φ is not uniquely determined. This is not to be seen as a weakness of the AGM paradigm as it can be taken to be a useful feature which captures the subjectivity of the belief revision process. Different epistemic agents may revise their belief sets in different ways due to certain *extra-logical* factors. The AGM postulates then are not all there is to represent the particular revision process of a certain agent. These extra-logical factors will be a central theme throughout this thesis.

Another interesting point is that $(* 2)$ (the success postulate) is extremely strong. Even in the most radical case of incoming inconsistent information, the success postulate prevails and as a result the belief set of the agent lapses into inconsistency (i.e., it becomes the case that $T = L = T_{\perp}$). So, according to the AGM postulates, any incoming information, be it consistent or inconsistent, will be treated as though it comes with absolute reliability. But how could some new piece of information that is inconsistent (and thereby absurd) be treated as totally reliable by an ideal epistemic agent? A fundamental assumption of the AGM theory is that more recent information is always treated as more reliable than earlier information. The strict success postulate is perhaps not very plausible in the limit case of most recent incoming information itself being contradictory. But we should keep in mind that incoming inconsistent information is highly unlikely; it might only happen in marginal cases, say, due to some faulty sensor or a sensory defect.

We will now go on with the second class of functions postulated by the AGM theory, *AGM contraction functions*. AGM contraction postulates are concerned with the question of how a rational epistemic agent should contract her prior belief set when she decides to give up some of her beliefs.

Definition 3.1.3. *AGM Belief Contraction:* We represent beliefs as sentences of the language L, and belief sets as theories of L. Let $\div$ be a binary function $\mathbb{T}_L \times L \mapsto \mathbb{T}_L$, where $\mathbb{T}_L$ is the set of all theories in L. We say that $\div$ is an AGM contraction function iff the following holds:

($\div$ 1) $T \div \varphi \in \mathbb{T}$.

($\div$ 2) $T \div \varphi \subseteq T$.

($\div$ 3) If $\varphi \notin T$, then $T \div \varphi = T$

($\div$ 4) If $\nvdash \varphi$, then $\varphi \notin T \div \varphi$.

($\div$ 5) If $\varphi \in T$, then $T \subseteq (T \div \varphi) + \varphi$.

($\div$ 6) If $\vdash \varphi \leftrightarrow \psi$, then $T \div \varphi = T \div \psi$.

($\div$ 7) $(T \div \varphi) \cap (T \div \psi) \subseteq T \div (\varphi \wedge \psi)$.

($\div$ 8) If $\varphi \notin T$, then $T \div (\varphi \wedge \psi) \subseteq T \div \varphi$.

($\div$ 1) says that $\mathbb{T}_L$ is closed under $\div$. ($\div$ 2) says that the contraction function never expands the belief set. ($\div$ 3) says that if the belief decided to be given up is not originally in the belief set, then the contraction function does nothing. ($\div$ 4) is the success postulate which says that $\div$ is successful as long as what is to be removed is not a theorem; only if it is a theorem the contraction function fails. ($\div$ 5) (*the recovery postulate*) says that contracting and expanding by the same belief gives us back the original belief set T. ($\div$ 6) is the contraction version of the *irrelevance of syntax* postulate; what matters is not the syntactic form of the belief, but is its content. ($\div$ 7) says that any belief not removed as a result of contraction by φ and as a result of contraction by ψ should not be removed by contraction by $(\varphi \wedge \psi)$. Finally, ($\div$ 8) says that if contraction by $(\varphi \wedge \psi)$ was not processed by removing only ψ, then contraction by $(\varphi \wedge \psi)$ cannot result in a larger set than contraction by only φ does.

As should be obvious, AGM contraction functions are definable in terms of AGM revision functions, and *vice versa*. This was formulated by Levi (1977) and therefore is called the *Levi Identity*:

- $T * \varphi = (T \div \sim \varphi) + \varphi$.

This should not be surprising because what we do when we revise a belief set T by a particular belief φ is just first remove the beliefs in the belief set which are contradictory to φ and then add φ into T by simple expansion. The interdefinability of $*$ and $\div$ and the significant parallel between the revision postulates and the contraction postulates give additional intuitive support to the AGM paradigm. Another identity defines the contraction function in terms of the revision function. It is called the *Harper Identity* (Gardenfors, 1988):

- $T \div \varphi = (T \ast \sim \varphi) \cap T$.

Intuitively, the Harper Identity states that whenever we contract a belief set T by a belief φ, what we do is equivalent to first revising T by $\sim \varphi$ and then cut off all the consequences that are not contained in the original belief set T. The reason for this cut-off should be clear if we emphasize that T is defined as a theory and is *not* a complete theory. A complete theory is identical to a maximally consistent set of sentences built through Lindenbaum construction and can be seen as a complete theory of a possible world. An incomplete theory such as T on the other hand should be seen merely as a partial representation.

3.2 Constructive Models for AGM Functions

As we mentioned above the AGM postulates determine not a unique revision function and a unique contraction function but instead classes of such functions, and extra-logical factors come into play when we need to decide which particular function is appropriate. Through constructive models one can accommodate those extra-logical factors and successfully determine a unique AGM function for each particular doxastic agent subject to the belief revision process. The most well-known constructions for AGM contraction functions are called *partial meet contractions* and we would like to begin with them.

Definition 3.2.1. *Remainder Set, AGM Selection Function:* Call any maximal subset of T that does not entail φ a φ*-remainder*. We denote the set of all φ-remainders of T as $T \parallel\!| \, \varphi$. An *AGM Selection Function* is a function γ from $X \subseteq \mathcal{P}(T)$ to X' such that $\varnothing \neq X' \subseteq X$.

Intuitively, the selection function γ selects the "best" φ*-remainders*; that is, $\gamma(T \parallel\!| \, \varphi)$ is the set of "best" φ-remainders of T. "Best" here is of course used in an epistemic sense. "Best" subsets are those subsets that are given epistemic priority by the agent when extra-logical factors are taken into consideration. The selection function defined therefore is a formal tool to represent the influence on the contraction process of those extra-logical factors. For each epistemic agent there must be a unique selection function γ which represents the extra-logical factors contributing to the belief change process. Once this unique γ is given, we are in a position to formally determine the contracted belief set $T \div \varphi$:

- $T \div \varphi = \bigcap \gamma(T \parallel\!| \, \varphi)$.

This equation says that the contracted belief set $T \div \varphi$ is the intersection of the collection of all "best" φ-remainders. However, it turns out that this equation does not satisfy all of the AGM postulates (Darwiche and

Pearl, 1997). In order to satisfy all of the AGM postulates for contraction, the selection function must be restricted to a *transitively relational selection function*. This is what we will do next.

Definition 3.2.2. *Transitively Relational Selection Function:* A selection function γ is a transitively relational selection function iff it can be defined in terms of a transitive binary relation $\gg$ on $\mathcal{P}(T)$.

It is helpful to think of a transitive binary relation $\gg$ on the powerset of T as an epistemic entrenchment ordering on the set of all contraction-candidate beliefs sets that are members of the powerset of T. Accordingly, $K_2 \gg K_1$ means that K_1 is at least as epistemically "valuable" as K_1. A transitively relational selection function then selects exactly the "best" or "most valuable" remainders in accordance with the associated epistemic entrenchment ordering.

Theorem 3.2.1. *(Alchourron et al., 1985) Let T be a theory of the language L and $\div$ be a function $\mathbb{T}_L \times L \mapsto \mathbb{T}_L$. $\div$ is a transitive relational partial meet contraction function iff it satisfies all of the AGM contraction postulates.*

Theorem 3.2.1 indicates that an AGM constructive model for AGM contraction functions should be given in terms of a transitively relational selection function based on a transitive binary relation $\gg$ on $\mathcal{P}(T)$, where $T \in \mathbb{T}_L$. Before going into the next section we should emphasize that the distinction between the axiomatic approach to belief revision (which is formulated in terms of postulates) and the constructive approach to belief revision (which is given in terms of selection functions, systems of spheres (Grove, 1988), and so on) are importantly different. As we saw above, the constructive approach has the advantage of being capable of accommodating easily the extra-logical factors that any satisfactory theory of belief revision needs to accommodate. The advantage of the axiomatic approach on the other hand lies in its intuitive strength and clarity. It is indeed difficult to reject any of the AGM postulates for revision or contraction; and any constructive theory of belief revision should answer to the AGM postulates if it purports to be a satisfactory theory of belief revision. In the next section we will compare the syntactical and semantical approaches to belief revision, and try to decide which one fares better.

3.3 (Ir)relevance of Syntax?

Recall that ($\ast$ 6) and ($\div$ 6) of the AGM paradigm state that the syntactic form of the sentences of the language L is irrelevant for the purposes of the belief revision process. This means that the AGM theory either assumes that syntactical variation in the incoming information is irrelevant, or tries to abstract away from the influence of those factors in the theory's idealized setting. Nevertheless, we should note that strong empirical evidence has been given showing that the syntactic form of the information has significant impact on the real-life revision processes of real epistemic agents. Humans are much better at using *Modus Ponens* as opposed to their typical failure in *Modus Tollens*, although both are objectively valid (Pelletier and Elio, 2005). If actual people make different revisions in view of syntactically different but semantically same information, can we afford to ignore syntactic form in theories of belief revision?

First, we need to explain the distinction between psychologism and objectivity in logic. Psychologism about logic refers to the view that principles of correct logic should answer to psychological facts about human reasoning. As we noted above, humans are very good at using *Modus Ponens* as a rule of inference in real-life situations whereas they fail badly at using *Modus Tollens*. A thoroughgoing psychologist about logic is perhaps committed to the position that *Modus Tollens* is not in fact a valid rule of inference since according to psychologism the principles of logic must be in accordance with the real-life reasoning of human beings. This means that the psychologist conceives of logic merely as a *descriptive* enterprise. Objective *normativity* of logical thinking is absent in this conception. However, we can observe that it is difficult to find psychologists about classical logic. It is widely held that the principles of classical logic hold regardless of whether human beings think in accordance with them or not, and that rational human beings *ought to* think in accordance with them. Classical logic is objectively valid and possesses normative power because it is not intended merely as a description of actual human reasoning. It is no wonder classical logic has been very successful in the realm of mathematics, especially when one realizes that mathematical truth seems to be the paradigm case of objective truth.

This is not to say that psychologism fails in all domains. It can be argued that psychologism about defeasible or common sense reasoning looks like a plausible position. When we, human beings, make defeasible inferences we do not thereby commit ourselves to the absolute certainty of the inference, or to the objective truth of the conclusion. What goes on is much more modest than that. Common sense reasoning involves extra logical factors such as our preferences, in addition to the every day necessity of dealing with uncertainty. We rarely find ourselves in situations in which we can secure absolute certainty. If we see that something looks like a dog, barks like dog, behaves like a dog etc., we will naturally infer that what we see is a dog. But this does not hold in the same manner as does a statement of mathematics. The difference is so fundamental that it looks as though we operate in a different fashion when we reason in common sense contexts. Therefore, the criteria for a theory of defeasible reasoning may be different from the objective criteria of classical logic. Perhaps we can start with recognizing that building a defeasible logic and a theory of belief revision is a descriptive enterprise. Our purpose is not to formulate the objective laws of logic, but we are instead interested in modeling the reasoning patterns of mortal humans in common sense contexts. Objective normativity, then, is out of the question; when we built defeasible logics or belief revision systems, we do not *prescribe* that a rational epistemic agent *ought to* infer this and that, or that she ought to give up a certain belief and endorse a certain other, and so on. At least part of the criteria in the context of defeasible reasoning seems to be descriptive and empirical, and can only be gathered through empirical/psychological experiments on human beings.

We think it is extremely important to notice that the postulate of irrelevance of syntax meant that the AGM theory was a semantic theory right from the beginning (Rott, 2011). Under the semantic interpretation of the AGM theory, AGM revision functions should take propositions, not sentences, as arguments. Yet another good reason for thinking that the AGM theory was a semantic theory right from the start is the indispensibility of extra-logical factors (in the form of an epistemic enthrenchment or otherwise). Semantic interpretation of the AGM theory should lead us to put more emphasis on the constructive model approach as opposed to the syntactic (postulate) approach. Accordingly, we will focus more on the constructive models in the remainder of this thesis.

3.4 The Problem of Iterated Revision

Let's briefly explain a well-known problem in belief change research called the *problem of iterated revision*. The original AGM theory comes with

AGM revision functions the values (outputs) of which are belief sets. Notice that the output belief sets are devoid of the extra-logical information that was contained in the input belief set. The result of this loss of extra-logical information is that the original AGM theory cannot handle iterated revisions. The output belief set does not contain the extra-logical information necessary for a repeated application of the revision function. The original AGM revision functions revise the belief sets but do not tell us anything as to what happens to the epistemic enthrenchment relation that was initially supplied with the input belief set of the agent. In order to handle iterated revisions, we need to extend the original AGM theory so that we can model and revise not only the belief set, but also the entire *belief state* of the idealized doxastic agent. By a belief state, we mean the entire doxastic content of the idealized agent, that is, the belief set, the doxastic entrenchment relation over the elements of that belief set, and the general doxastic outlook of the agent which will guide future revisions. A good way to model an entire doxastic state is to use constructive models. There are various ways to achieve such modeling, but our preference will be to use a variant of Lewisean system of spheres because, in our view, these systems of spheres provide a very intuitive representation of doxastic states.

CHAPTER 4

Alternative Semantics for Conditional Logic

4.1 Gärdenfors's Triviality Result

It is an open problem whether Stalnaker/Thomason/Lewis conditionals can be adequately handled within belief revision systems (Hansson, 2014). The motivation for including conditional statements in belief sets, of course, stems from the Ramsey Test (see Chapter 1 of this thesis). Also, it is obvious that we do possess some beliefs of the form $A \mathrel{\Box\!\!\rightarrow} B$ which indicate that in the event that we acquired the belief A, we would have inferred the belief B.

Gärdenfors (1986) proves that inclusion of Stalnaker/Thomason/Lewis conditionals is problematic for belief revision systems. The problems arising from the inclusion of such conditionals, Gärdenfors thinks, are related to a conflict between two otherwise desired principles: Formalized Ramsey Test (R) and Preservation (P).

- (R) $\psi \in T * \phi$ iff $\phi \mathrel{\Box\!\!\rightarrow} \psi \in T$
- (P) If $\sim \phi \notin T$ and $\psi \in T$, then $\psi \in T * \phi$

The left-to-right direction of (R) states that for every possible revision $T * \phi$ of T to which ψ belongs as a member, we must have in the belief set T the conditional $\phi \mathrel{\Box\!\!\rightarrow} \psi$. This implication requires too many conditionals in T, and is in general problematic. However, there is in fact a more serious problem since it turns out that the principles (R) and (P) are incompatible.

Definition 4.1.1. *Trivial Belief Revision System:* A belief revision system $(\mathbf{T}, *, +)$ is trivial if and only if there is no $T \in \mathbf{T}$ such that T is consistent with three pair-wise disjoint beliefs A, B, and C.

Theorem 4.1.1. *Gärdenfors's Theorem: There is no non-trivial belief revision system that satisfies the principles (R) and (P).*

Gärdenfors (1986) gives a proof of the theorem using the derived principle of *monotonicity*. We believe, as Rott (2011) notes, that using the monotonicity principle has been unproductive from the start because it distracted subsequent researchers from the fact that what is really salient here is the Ramsey Test and the behavior of conditionals. Instead of writing a formal proof we find it more intuitive to instantiate the problem with a relatively simple example: Let T be the belief set of a certain agent. Suppose $\sim P \notin T$ and $Q \in T$. Suppose the agent learns a scientific law $Q \mathrel{\Box\!\!\rightarrow} \sim P$, that is, by revision, it becomes the case that $Q \mathrel{\Box\!\!\rightarrow} \sim P \in T'$. It follows, by deductive closure, that $\sim P \in T'$. Now let's consider the revision of T' by P, i.e., T'_P. In T'_P one cannot keep both Q and $Q \mathrel{\Box\!\!\rightarrow} \sim P$ because Postulate 1 of AGM (success postulate) states that $P \in T'_P$, leading us into $P \in T'_P$ and $\sim P \in T'_P$, which would result in a contradiction. Considering that $Q \mathrel{\Box\!\!\rightarrow} \sim P$ is a scientific law, we might wish to retract Q. In that case, $Q \in T_P$ but $Q \notin T'_P$, which means that $T_P \nsubseteq T'_P$ even though $T \subseteq T'$. That is to say, there must be an R such that $P \mathrel{\Box\!\!\rightarrow} R \in T$ but $(Q \mathrel{\Box\!\!\rightarrow} \sim P) \mathrel{\Box\!\!\rightarrow} (P \mathrel{\Box\!\!\rightarrow} R) \notin T$ even though for all S, if $S \in T$, then $(Q \mathrel{\Box\!\!\rightarrow} \sim P) \mathrel{\Box\!\!\rightarrow} S \in T$. Substitute S for $P \mathrel{\Box\!\!\rightarrow} R$ in the former: Contradiction. Suppose, alternatively, that we choose to retract $Q \mathrel{\Box\!\!\rightarrow} \sim P$. Then, $Q \in T'_P$, $Q \mathrel{\Box\!\!\rightarrow} \sim P \notin T'_P$, but $Q \mathrel{\Box\!\!\rightarrow} \sim P \in T'$. However, since T' was defined as the output of the revision operation of T by $Q \mathrel{\Box\!\!\rightarrow} \sim P$, the statement $Q \mathrel{\Box\!\!\rightarrow} \sim P \notin T'_P$ (where T'_P stands for the output of the revision of T_P by $Q \mathrel{\Box\!\!\rightarrow} \sim P$) is in obvious conflict with the principle of success, i.e., the principle which states that for all A, $A \in T_A$.

Gärdenfors (1986) believes that we should part with Ramsey's principle (R). Ramsey's principle, Gärdenfors thinks, is a correct principle for the analysis of conditionals, but falls short of being an appropriate principle for belief revision systems, because its conflict with (P) reveals that (R) is *not* in fact a principle of *minimal change*. Any principle that purports to be a principle of minimal change must, on Gärdenfors' view, satisfy the preservation principle (P). He then considers a number of proposed solutions, the most promising of which appears to be (Levi, 1977)'s. Levi proposes that we treat conditional assertions not as object-language sentences, but as meta-linguistic assertions which are not themselves truth-bearers, and which can be seen as possibly accepted or rejected beliefs *about* the belief sets and their revisions.

We value highly Ramsey's insight so we reject the claim that the problem lies with the Ramsey Test. The Ramsey Test, we think, is the single most important idea for a correct analysis of conditional statements. The culprit, in our view, is the preservation principle. With the inclusion of Ramsey-conditionals in belief sets, the preservation principle, we argue, becomes obsolete. Of course, we allow that the factual (indicative, boolean) segment of belief sets should obey the preservation principle. But it seems obvious to us preservation becomes undesirable when our theory permits conditional beliefs. Consider the following. The factual segment of my belief set contains $A \vee B$. It is evident that the conditional segment of my belief set should contain $\sim A \;\Box\!\!\rightarrow B$. Assume that I learn that B; so my belief set is revised by B. The AGM preservation postulate rules that *noth-*

ing will be contracted from my belief set since no inconsistency is present (i.e., since $\sim B \notin T$). So, if preservation is applied to belief sets with conditional segments, my new belief set will be:

$$\{ A \vee B,\ B,\ A \mathrel{\Box\!\!\rightarrow} \sim B \}$$

This is of course unacceptable. Therefore, we defend that preservation is simply inapplicable when conditionals are included in belief sets. Rott (2011) seems to defend a similar position.

The main problem here, we believe, is an underappreciation of the fact that Ramsey conditionals belong in a different level and are to be distinguished from the indicative/factual beliefs of the epistemic agent. Ramsey conditionals, in our view, are meta-beliefs; i.e., they are beliefs of the agent about what she would infer had she believed the antecedent of the given Ramsey conditional. We think this analysis is favorable especially because it explains the extreme context-sensitivity (see (van Fraassen, 1980)) of some conditional idiom. What we do, in effect, is to try to formalize (at least some of) the context using the content of the belief set of the agent. We think this is in line with Levi (1977)'s position.

This meta-level conception of Ramsey conditionals leads us to treat the AGM theory as a potential model theory for conditional logics. In the next section we look into the task of formulating an alternative semantics for Lewis's conditional logic VC. We will investigate whether or not an epistemic/doxastic semantics is available for VC.

4.2 Can We Have an Epistemic/Doxastic Semantics for VC?

In order to address this question let's try to formulate a system of spheres model for the doxastic state of an agent with the belief set T. In contrast to the *ontic* system of spheres models, our doxastic system of spheres models will not have singleton innermost spheres. The innermost sphere will be used to represent the initial belief set T of the agent. In the doxastic context, the innermost sphere should not be a singleton because a singleton innermost sphere would misleadingly represent a *dogmatic* belief set where the doxastic agent always has a belief about the truth and falsity of every single proposition there is. In a nontrivial doxastic modeling we prefer to ascribe *nondogmatic* belief sets to the agents because we know that a realistic doxastic agent should never be completely opinionated about all matters of fact. An innermost sphere with more than one index provides, by contrast, a set of epistemically possible complete descriptions of the world relative to the agent's belief set. Given a belief set T, all indices will be T-indices where an index i is a T-index if and only if it is a model for T (i.e. $i \models T$). From the point of view of the agent, all indices within the innermost sphere are equally most plausible candidates for being the correct and complete description of "reality" since all of them satisfy the elements of her belief set equally, no more and no less. The second innermost sphere (in addition to containing the indices of the first sphere since the system of spheres is nested) contains slightly less plausible candidates for being the correct and complete description of reality because the indices which are members of the second innermost sphere but which do not belong to the innermost sphere represent alternative complete descriptions that are in slight conflict with the belief set T of the doxastic agent. An outer sphere will contain indices that represent less plausible candidates for being the correct and complete description of reality from the epistemic standpoint of the agent in question. Those "implausible" indices will contain varying amounts of conflicting content with respect to the belief set T. Note that whenever the belief set T of the agent is revised by a consistent belief, the innermost sphere $\$_0$ will shrink in size. This is intuitive because a consistent revision is nothing more than a belief expansion, and it makes the belief state of the agent *more opinionated* by reducing the number of the equally plausible candidates for being the complete and correct description of reality. The smaller the number of equally most plausible candidates, the smaller the size of the innermost sphere gets.

Let's formalize our intuitive characterization of doxastic states.

Definition 4.2.1. *Doxastic State:* The doxastic state E of an agent with a belief set T is a structure $\langle I, <_T \rangle$ where I is the set of all indices and $<_T$ is a total preorder (i.e., a reflexive and transitive binary relation) over I, and for all $i \in I$ and $i_0 \in [\![T]\!]$, $i_0 <_T i$.

Definition 4.2.2. *Doxastic System of Spheres:* A doxastic system of spheres is an assignment $\$_T \subseteq \mathcal{P}(I)$ to each belief set (theory) T of the belief revision system where the innermost (minimal) sphere is $[\![T]\!] = \$_0 \in \mathbf{\$}_T = \{i_0 \in I \mid \forall i \in I\ (i_0 < i)\}$. $\$_T$ is nested and closed under unions and (nonempty) intersections.

Intuitively, the $<$-relation is the "at least as plausible" relation over the set of all indices. The assignment $\$_T$ on the other hand is structurally equivalent to $\langle I, <_T \rangle$ and gives a system of spheres over the set of all indices. In light of these formulations, it is possible to define the interesting notion of consistency with a belief set.

Definition 4.2.3. *Consistency with a Belief State:* An input belief a is consistent with a given belief state T if and only if there is some index $i_0 \in \$_0$ such that $i_0 \in [\![a]\!]$.

Finally, we can attempt to define alternative models for conditional logic VC.

Definition 4.2.4. *Update Models for VC:* An update model for VC is a triple $\langle \mathbf{T}, *, + \rangle$ where $\mathbf{T}$ is a set of (conditional) theories, $*$ is a revision function and $+$ is an update function. $*$ obeys the following constraints (Costa and Levi, 1996):

($\Box\!\!\rightarrow$-1) For every sentence $a \in L_{\Box\!\rightarrow}$, and every conditional theory T, $T * a$ is a conditional theory.

($\Box\!\!\rightarrow$-2) $a \in (T * a)$.

($\Box\!\!\rightarrow$-3) If $a \in T$, then $T * a = T$.

($\Box\!\!\rightarrow$-4) If $a \in T * b$ and $b \in T * a$, then $T * a = T * b$.

($\Box\!\!\rightarrow$-5) $T * (a \wedge b) \subseteq (T * a) + b$.

($\Box\!\!\rightarrow$-6) If T is a dogmatic and consistent (maximal-consistent) conditional belief set and $\sim b \notin T * a$, then $(T * a) + b \subseteq T * (a \wedge b)$.

($\Box\!\!\rightarrow$-7) $T * a = \bigcap \{W * a \mid T \subseteq W$, and W is maximal-consistent$\}$.

($\Box\!\!\rightarrow$-1) ensures that $\boldsymbol{T}$ is closed under $*$. ($\Box\!\!\rightarrow$-2) represents the principle of success for $*$. ($\Box\!\!\rightarrow$-3) states that $*$ does nothing when input belief is already in the belief set. ($\Box\!\!\rightarrow$-4) says that for any belief set T and for arbitrary beliefs a and b, if the output of revision by a contains b and the output of revision by b contains a, then a and b are different syntactic representations of the same belief (proposition). ($\Box\!\!\rightarrow$-5) says that the output of T's revision by $a \wedge b$ cannot contain anything more than what is contained in the output of $(T * a)$'s expansion by b because either $\sim b \in (T * a)$ or $\sim b \notin (T * a)$. If $\sim b \in (T * a)$, the output will be the absurd belief set $T_{\perp}$, which contains every belief. Otherwise, $*$ will behave exactly like $+$. For ($\Box\!\!\rightarrow$-6), suppose that a dogmatic belief set T's revision by an arbitrary belief a does not rule out some belief b. By definition of dogmatic belief sets, T contains either b or $\sim b$. Then revision by a either removes $\sim b$ from T, or it must be the case that $b \in T$. If revision by a removes $\sim b$ from T, then $(a \wedge \sim b) \equiv \perp$, that is to say, $a \supset b$. Else, $b \in T$. If the former alternative is true, then $T * (a \wedge b) = (T * a)$ and $(T * a) + b = (T * a)$; so ($\Box\!\!\rightarrow$-6) is obviously satisfied. If the latter alternative is true, any expansion by b is simply effectless. ($\Box\!\!\rightarrow$-7) states that in order to get the output of the revision of T by a, we revise each $\$_0$-index by a and take the intersection of the outputs of those revisions.

We need to say more about ($\Box\!\!\rightarrow$-7). ($\Box\!\!\rightarrow$-7) means that the output of the revision function is given by the following procedure. Revise every single index that is compatible with the belief set T. Take the intersection of all of those revised indices. This intersection is the output of T's revision. Notice that, according to ($\Box\!\!\rightarrow$-7), the output of the revision of the belief set T depends solely on the revisions of individual indices, which are maximal-consistent sets of sentences or "possible worlds". The revision of an individual index (or "possible world"), in turn, depends on the extra-logical entrenchment relation defined for that particular index. The extra-logical entrenchment relation defined for a maximal-consistent set of sentences or a "possible world" is *nothing less than an ontic similarity relation postulated over the set of possible worlds*. That is, update functions give us ontic semantics. The notion of belief change appropriate to validate

Lewis's VC, therefore, is ontic.

4.2.1 Soundness Results

Now we show that the class of all update models is sufficient to validate Lewis's VC. Evaluation of a $\Box\!\!\rightarrow$-formula relative to a belief set T in an update model is given by the formalized Ramsey Test:

- *Ramsey Test:* $(A \Box\!\!\rightarrow B) \in T$ iff $B \in T * A$

Hence, $A \Box\!\!\rightarrow B$ is believed in a belief set T if and only if revision of T by A contains B. A very intuitive test can also be formulated for the $\supset$-conditional:

- *$\supset$-conditional Test:* $A \supset B \in T$ iff $B \in T + A$

So a material conditional $A \supset B$ is believed in a belief set if and only if the expansion of that belief set by A contains B.

We say that a formula is *valid in an update model* if and only if it is believed in all belief sets of that model. We say that a formula is *valid* if and only if it is valid in all update models. We say that a formula is *credible* if and only if it is believed in some belief set in some update model. For a soundness proof for VC with respect to its update semantics, we need to show that all axioms of VC are valid in all update models, and that the rules of inference of VC are soundness-preserving. Let's go step by step:

(A1) *Truth-functional tautologies*

All truth-functional tautologies are contained within all belief sets in all update models because the consequence relation governing all belief sets is classical.

(A2) *Definitions of nonprimitive operators*

The only nonprimitive connective we would like to use is $\Diamond\!\!\rightarrow$, which is defined as $\Diamond\!\!\rightarrow =_{df} \sim (\phi \;\Box\!\!\rightarrow\sim \psi)$. It will suffice to add the meta-clause: $\phi \Diamond\!\!\rightarrow \psi \in T \quad \text{iff} \quad \text{not } (\phi \;\Box\!\!\rightarrow\sim \psi) \in T$.

(A3) $\phi \;\Box\!\!\rightarrow \phi$

Since $\phi \;\Box\!\!\rightarrow \phi \in T \quad \text{iff} \quad \phi \in T * \phi$, and since ($\Box\!\!\rightarrow$-2) states that for any belief set T in all update models $a \in T * a$, it follows that for any belief set T in any update model $\phi \;\Box\!\!\rightarrow \phi \in T$.

(A4) $(\sim \phi \;\Box\!\!\rightarrow \phi) \supset (\psi \;\Box\!\!\rightarrow \phi)$

Assume for reductio that for some belief set T, $(\sim \phi \;\Box\!\!\rightarrow \phi) \in T$, but $(\psi \;\Box\!\!\rightarrow \phi) \notin T$. By the Ramsey Test, $\phi \in (T * \sim \phi)$ and $\phi \notin T * \psi$. From $\phi \in (T * \sim \phi)$, we infer (by ($\Box\!\!\rightarrow$-2)) $(T * \sim \phi) = T_\perp$. From $\phi \notin T * \psi$, we infer $T * \psi \neq T_\perp$. From $(T * \sim \phi) = T_\perp$ and $T * \psi \neq T_\perp$, it follows that $T \neq T_\perp$ and $\sim \phi \equiv \perp$. Equivalently, $\phi \equiv \top$. Since $\top$ is contained in all belief sets (by closure under classical consequence), this result contradicts our initial assumption.

(A5) $(\phi \;\Box\!\!\rightarrow\sim \psi) \vee (((\phi \wedge \psi) \;\Box\!\!\rightarrow \chi) \equiv (\phi \;\Box\!\!\rightarrow (\psi \supset \chi)))$

This is equivalent to $(\sim (\phi \;\Box\!\!\rightarrow\sim \psi) \supset (((\phi \wedge \psi) \;\Box\!\!\rightarrow \chi) \equiv (\phi \;\Box\!\!\rightarrow (\psi \supset$

$\chi)))) \in T$ for any belief set T of any update model $\mathscr{M}$. Assume for reductio $\sim(\phi \,\Box\!\!\rightarrow \sim\psi) \in T$, but $(((\phi \wedge \psi) \,\Box\!\!\rightarrow \chi) \equiv (\phi \,\Box\!\!\rightarrow (\psi \supset \chi))) \notin T$ (where T is an arbitrary belief set). The former implies $\sim\psi \notin T * \phi$. The latter implies either *(1)* $\chi \in T * (\phi \wedge \psi)$ and $\psi \supset \chi \notin T * \phi$ or *(2)* $\chi \notin T * (\phi \wedge \psi)$ and $\psi \supset \chi \in T * \phi$. If *(1)*, $\chi \notin (T * \phi) + \psi$; by ($\Box\!\!\rightarrow$-5), this contradicts $\chi \in T * (\phi \wedge \psi)$. Alternatively if *(2)*, $\chi \in (T * \phi) + \psi$ but $\chi \notin T * (\phi \wedge \psi)$. So it can't be the case that $\vdash_{CL} (\phi \wedge \psi) \supset \chi$. I.e., $\phi \wedge \psi$ is consistent. By ($\Box\!\!\rightarrow$-3), it can't be the case that T is inconsistent. However, if T is consistent, and since we know that $\phi \wedge \psi$ is also consistent, from the properties of AGM expansion functions it follows that $(T * \phi) + \psi = T * (\phi \wedge \psi)$, which is in contradiction with *(2)*.

(A6) $(\phi \,\Box\!\!\rightarrow \psi) \supset (\phi \supset \psi)$

Assume for reductio that there is an update model where a belief set T satisfies the antecedent but not the consequent. Then by the Ramsey Test we have $\psi \in T * \phi$, and by the $\supset$-conditional Test we have $\psi \notin T + \phi$. This is in obvious contradiction with property (* 3) of AGM revision functions.

(A7) $(\phi \wedge \psi) \supset (\phi \,\Box\!\!\rightarrow \psi)$

Again let's suppose for reductio that there is an update model where a belief set T satisfies the antecedent but not the consequent. Then the following holds: $\phi \wedge \psi \in T$ and $\psi \notin T * \phi$. From $\phi \wedge \psi \in T$ and

($\Box\!\!\rightarrow$-3) it follows that $T * \phi = T$. Then $\psi \in T * \phi$. Contradiction.

(R1) *Modus Ponens*

In order to show that *Modus Ponens* preserves soundness with respect to the update semantics for VC, let's consider the following: Assume that $\phi \supset \psi$ and ϕ are believed in a belief set T in an update model $\mathscr{M}$; that is, $\phi \in T$ and $\phi \supset \psi \in T$. Since, by definition, belief sets are closed under classical consequence, it immediately follows that $\psi \in T$.

(R2) *Deduction within Conditionals*

$$\frac{\vdash\ (\chi_1 \wedge \ldots \wedge \chi_n) \supset \psi}{\vdash\ ((\phi \Box\!\!\rightarrow \chi_1) \wedge \ldots \wedge (\phi \Box\!\!\rightarrow \chi_n)) \supset (\phi \Box\!\!\rightarrow \psi)}$$

This rule of inference is intended to preserve theoremhood so we will concern ourselves with preservation of update validity here. Let's assume that $(\chi_1 \wedge \ldots \wedge \chi_n) \supset \psi$ is update-valid. That is to say, for all models and for all belief sets T, we assume $((\chi_1 \wedge \ldots \wedge \chi_n) \supset \psi) \in T$. From the $\supset$-conditional Test we immediately have $\psi \in T + (\chi_1 \wedge \ldots \wedge \chi_n)$ for all T. Now assume for reductio that the conclusion of the rule of inference is invalid; i.e., in some model there is a belief set T_1 such that T_1 contains $((\phi \Box\!\!\rightarrow \chi_1) \wedge \ldots \wedge (\phi \Box\!\!\rightarrow \chi_n))$, but does not contain $\phi \Box\!\!\rightarrow \psi$. By the Ramsey Test, this means that $(\chi_1 \wedge \ldots \wedge \chi_n) \in T_1 * \phi$, but $\psi \notin T_1 * \phi$. But *ex hypothesi*, $((\chi_1 \wedge \ldots \wedge \chi_n) \supset \psi) \in T$ for all T in all update models. By universal instantiation, $((\chi_1 \wedge \ldots \wedge \chi_n) \supset \psi) \in T_1 * \phi$. Since, by definition, all belief sets are closed under classical consequence relation, it follows that $\psi \in T_1 * \phi$. Contradiction.

(R3) *Replacement of Classical Equivalents*

Due to the closure under classical consequence property of all belief sets, replacement of classical equivalents obviously holds.

CHAPTER 5

Conclusion

In this thesis, (1) we had a look at the history of conditional logics, (2) formulated a natural deduction system for David Lewis's conditional VC, (3) gave an exposition of the closely-related AGM theory of belief change, and (4) investigated whether an epistemic/doxastic semantics based on the AGM theory is attainable for VC.

As regards (1), we note that Frank P. Ramsey's test gives us a procedure for evaluation of rational acceptability of conditional statements. According to Ramsey, conditional statements do not get truth values because indicative conditionals, Ramsey thinks, should be handled in terms of subjective probability; whereas counterfactual statements are not concerned with matters of fact and are only related to what follows from given hypothetical assumptions along with certain laws the agent already endorses. We also remarked that Stalnaker's reinterpretation of Ramsey changed the scene significantly by presupposing that conditionals (be they indicative or counterfactual) are truth-bearers.

As regards (2), we note that the rejection of Stalnaker's assumption in VC creates certain complications in its proof theory. Stalnaker's assumption turns out to be a useful proof-theoretic device by permitting the use of "indirect conditional proofs" in the natural deduction system. Once we remove reit 3 of Thomason's system, we lose this useful device and it becomes necessary to introduce two new reiteration rules in order to ensure the semantic completeness of the proof system we propose. We name this system FVC. We also speculate that our results could suggest that intelim-style inferentialism about (at least some) logical connectives can be problematic.

Although (3) is by and large expository, we defended the position that the AGM theory of belief change was a semantic theory right from the start. We base this claim on the postulate of irrelevance of syntax.

For (4), let's emphasize that the only notion of belief change available to validate Lewis's VC is the notion of belief update (also called dynamic belief revision). We think that “belief update” and “dynamic belief revision” are misnomers. A more appropriate name would be “imaging”, a term which is due to Lewis. We observe that imaging semantics involves an ontic similarity relation postulated over the set of possible worlds, and therefore, is ontic. It is remarkable that VC does not seem to admit of any epistemic treatment. This indicates that a genuine epistemic conditional logic should have a different axiomatics.

What can we make of the apparent indispensability in VC's semantics of the ontic similarity relation defined over the set of possible worlds?

One way is to take the metaphysics seriously. Let's assume for a moment that Quine-Putnam indispensability thesis is applicable to the analysis of conditional idiom as well; indeed, one could make a strong case to the effect that conditional idiom (both indicative and counterfactual) plays an indispensable role in scientific theorizing. If, for all we know, the only adequate analysis of conditional idiom is committed to an ontic similarity relation over the set of possible worlds, then we should perhaps be prepared to endorse realism about possible worlds. This line of thinking is tenable only if the Quinean idea of “ontological commitment” is tenable. At any rate, it is tempting to become more “credulous” towards modal realism in view of the results in (4).

Another way would be to reject the axiomatics of VC on the basis of the claim that it depends on pre-theoretic intuitions such as the presupposition that counterfactual statements must have truth values. For my part, I think it is very difficult to say if a rational acceptability account is more intuitive than a truth-theoretic account, or vice versa.

In any event, research in the logic of conditionals has proved to be a significant success as a highly technical interdisciplinary programme with some actual and many potential "real-world" applications. I am of the persuasion that this appealing success has largely been due to a rigorous utilization of formal/mathematical methods. Study of conditionals will likely continue to be a central theme for future scientific philosophy.

Bibliography

Adams, E. (1965). The logic of conditionals. *Inquiry*, 8(1-4):166–197.

Alchourron, C. E., Gardenfors, P., and Makinson, D. (1985). On the logic of theory change: Partial meet contraction and revision functions. *The Journal of Symbolic Logic*, (2):510.

Costa, H. A. and Levi, I. (1996). Two notions of epistemic validity: Epistemic models for ramsey's conditionals. *Synthese*, (2):217.

Darwiche, A. and Pearl, J. (1997). On the logic of iterated belief revision. *Artificial Intelligence*, 89:1–29.

Delgrande, J. P. (1988). An approach to default reasoning based on a first-order conditional logic: Revised report. *Artificial Intelligence*, 36(1):63–90.

Fitch, F. B. (1952). *Symbolic logic : an introduction*. New York, Ronald Press Co. [1952].

Gardenfors, P. (1978). Conditionals and changes of belief. *Acta Philosophica Fennica*, 30:381–404.

Gardenfors, P. (1986). Belief revisions and the ramsey test for conditionals. *The Philosophical Review*, (1):81.

Gardenfors, P. (1988). *Knowledge in flux : modeling the dynamics of epistemic states*. Cambridge, Mass. : MIT Press, c1988.

Grove, A. (1988). Two modellings for theory change. *Journal of Philosophical Logic*, 17(2):pp. 157–170.

Hansson, S. O. (2014). Logic of belief revision. In Zalta, E. N., editor, *The Stanford Encyclopedia of Philosophy*. Winter 2014 edition.

Levi, I. (1977). Subjunctives, dispositions and chances. *Synthese*, (4):423.

Lewis, D. K. (1973). *Counterfactuals.* Cambridge, Harvard University Press, 1973.

Lewis, D. K. (1976). Probabilities of conditionals and conditional probabilities. *The Philosophical Review*, 85(3):pp. 297–315.

McGee, V. (1994). Learning the impossible. In Eells, E. and Skyrms, B., editors, *Probability and Conditionals: Belief Revision and Rational Decision*, pages 179–199. Cambridge University Press.

Pelletier, F. J. and Elio, R. (2005). The case for psychologism in default and inheritance reasoning. *Synthese*, 146(1-2):7–35.

Pelletier, F. J. and Hazen, A. P. (2012). A history of natural deduction. *Handbook of the History of Logic*, 11:341–414.

Priest, G. (2008). *An introduction to non-classical logic: from if to is*. Cambridge University Press.

Quine, W. V. (1960). *Word and Object*. MIT Press.

Reiter, R. (1980). A logic for default reasoning. *Artificial intelligence*, 13(1):81–132.

Rott, H. (2011). Reapproaching ramsey: Conditionals and iterated belief change in the spirit of agm. *Journal of Philosophical Logic*, (2):155.

Stalnaker, R. C. (1968). A theory of conditionals. *Americal Philosophical Quarterly*, pages 98–112.

Stalnaker, R. C. and Jeffrey, R. (1994). Conditionals as random variables. In Eells, E., Skyrms, B., and Adams, E. W., editors, *Probability and Conditionals: Belief Revision and Rational Decision*, page 31. Cambridge University Press.

Stalnaker, R. C. and Thomason, R. H. (1970). A semantic analysis of conditional logic. *Theoria: A Swedish Journal of Philosophy*, 36(1):23.

Thomason, R. H. (1970). *Symbolic logic : an introduction*. Macmillan.

Thomason, R. H. (1971). A fitch-style formulation of conditional logic. *Logique et Analyse*, 13:397 – 412.

van Fraassen, B. (1980). *The Scientific Image*. Oxford University Press.

van Harmelen, F., Lifschitz, V., and Porter, B. (2008). *Handbook of knowledge representation*. Foundations of artificial intelligence. Elsevier.