1. Introduction

For $i = 0, \ldots, k$, let

$$
{}_iH_{s,t} = \begin{bmatrix} {}_ih_0 & \cdots & {}_ih_t \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ {}_ih_s & \cdots & {}_ih_{s+t} \end{bmatrix} \tag{1}
$$

be a Hankel matrix with components ${}_ih_j$, $j = 0, \ldots, s+t$, from some field D. The objective of this report is to characterize the solutions of the block Hankel system

$$
{}_k\mathbf{H}_{s,t} \cdot Q = 0, \tag{2}
$$

where ${}_k\mathbf{H}_{s,t}$ is of the triangular form

$$
{}_k\mathbf{H}_{s,t} = \begin{bmatrix} & & & & {}_0H_{s,t} \\ & & & \cdot & \cdot \\ & & \cdot & & \cdot \\ & \cdot & & & \cdot \\ {}_0H_{s,t} & \cdot & \cdot & \cdot & {}_kH_{s,t} \end{bmatrix}. \tag{3}
$$

We are more interested in characterizing solutions of the system

$$
{}_k\mathbf{H}_{t-1,t} \cdot Q = 0. \tag{4}
$$

Consideration is given to the more general system (2) primarily to facilitate the development of results for the case $s = t - 1$. This characterization is established in section 2 through section 6. For applications of block Hankel matrices, the reader is referred to [1,4].

Solutions of (4) correspond directly to a certain type of Padé approximant for bivariate power series defined in section 7. Because of this correspondence, we are able to give a full characterization of this type of Padé approximants as well. The relevance of this work in the general context of multivariate Padé approximation is discussed in section 8.

We begin with some preliminary definitions and notational conventions. For $k, t \geq 0$, let

$$
{}_kV_t = \left\{ Q : Q = \begin{bmatrix} {}_kQ \\ \cdot \\ \cdot \\ \cdot \\ {}_0Q \end{bmatrix}, \ \ where \ \ {}_iQ = \begin{bmatrix} {}_iq_t \\ \cdot \\ \cdot \\ \cdot \\ {}_iq_0 \end{bmatrix} \ \text{for} \ \ i = 0, \ldots, k \right\} \tag{5}
$$

be a vector space of dimension $(k + 1)(t + 1)$. It is assumed that the components ${}_iq_j,\ i = 0, \ldots, k,\ j = 0, \ldots, t$ are elements from D.

Let l be a non-negative integer. For any $Q \in {}_kV_t$, define

$$
x^l \cdot Q = [\ {}_kQ, \cdots, {}_0Q, \mathbf{0}, \cdots, \mathbf{0}]^\dagger \in {}_{k+l}V_t \tag{6}
$$

and

$$
\underline{x}^l \cdot Q = [\ {}_{k-l}Q, \cdots, {}_0Q, \mathbf{0}, \cdots, \mathbf{0}]^\dagger \in {}_kV_{t,} \tag{7}
$$

where $\mathbf{0}$ is the zero vector of length $t + 1$. For any $Q \in {}_kV_t$, also define

$$
y^l \cdot Q = [y^l \cdot_k Q, \cdots, y^l \cdot {}_0Q]^\dagger \in {}_kV_{t+l}, \tag{8}
$$

where, for each $i = 0, \ldots, k$,

$$
y^l \cdot {}_iQ = [{}_iq_t, \cdots, {}_iq_0, 0, \cdots, 0]^\dagger \in {}_0V_{t+l}.
$$

Similarly, for any $Q \in {}_kV_t$, define

$$
\bar{y}^l \cdot Q = [\bar{y}^l \cdot {}_kQ, \cdots, \bar{y}^l \cdot {}_0Q]^\dagger \in {}_kV_{t+l}, \tag{9}
$$

where, for each $i = 0, \ldots, k$,

$$
\bar{y}^l \cdot {}_iQ = [0, \cdots, 0, {}_iq_t, \cdots, {}_iq_0]^\dagger \in {}_0V_{t+l}.
$$

Associated with definitions (6), (7), (8) and (9) are the following mappings defined in the obvious way:

$$
\begin{aligned}
&x^l : {}_kV_t \rightarrow {}_{k+l}V_t \\
&\underline{x}^l : {}_kV_t \rightarrow {}_kV_t \\
&y^l : {}_kV_t \rightarrow {}_kV_{t+l} \\
&\bar{y}^l : {}_kV_t \rightarrow {}_kV_{t+l}.
\end{aligned} \tag{10}
$$

It is easy to see that each of the mappings in (10) is linear, and that x^l, y^l and $\bar{y}^l$ are injective. For notational convenience, we also adopt the convention that $x = x^1$, $\underline{x} = \underline{x}^1$, $y = y^1$ and $\bar{y} = \bar{y}^1$. Let S be a subspace of ${}_kV_t$. Then the image of S by the transformation x^l is denoted by

$$x^l \cdot S = \{x^l \cdot Q : Q \in S\}. \tag{11}$$

Clearly, $x^l \cdot S$ is a subspace of ${}_{k+l}V_t$. A notation similar to (11) can be adopted for the other transformations in (10), but such is not required in what follows.

2. Existence of a Solution

Denote the space of solutions of (2) by

$${}_kS_{s,t} = \left\{ Q : {}_k\mathbf{H}_{s,t} \cdot Q = 0 \right\}. \tag{12}$$

Clearly, ${}_kS_{s,t}$ is a subspace of ${}_kV_t$. Denote the rank of ${}_k\mathbf{H}_{s,t}$ by ${}_kr_{s,t}$ and define ${}_{-1}r_{s,t} = 0$. A sufficient but not a necessary condition for ${}_kS_{s,t}$ to be non-trivial is given by

Theorem 1. If $s \le t - 1$, then a non-trivial solution Q to (2) always exists.

Proof: Since ${}_k\mathbf{H}_{s,t}$ has $(k+1)(s+1)$ rows, then

$${}_kr_{s,t} \le (k+1)(s+1).$$

Since there are $(k+1)(t+1)$ unknowns in (2), then

$$\begin{aligned} dim\ ({}_kS_{s,t}) &= (k+1)(t+1) - {}_kr_s \\ &\ge (k+1)(t-s) \\ &\ge k+1. \end{aligned} \tag{13}$$

■

The inequality (13) provides that for $t > s$ (and in particular for $s = t - 1$), (2) has at least $k + 1$ linearly independent solutions. The next few results are concerned with the nature of these solutions. Corresponding to (11), for $i = 0, 1, \ldots, k$, define

$$x^i \cdot {}_{k-i}\, S_{s,t} = \left\{ x^i \cdot Q : Q \in {}_{k-i}S_{s,t} \right\}. \tag{14}$$

Lemma 2. For $i = 0, \ldots, k$, $x^i \cdot {}_{k-i}S_{s,t}$ is a subspace of ${}_kS_{s,t}$.

Proof: Assume that $Q' \in x^i \cdot_{k-i} S_{s,t}$. Then, by the definition of the set $x^i \cdot_{k-i} S_{s,t}$, there exists

$$Q = [\,{}_{k-i}Q, \cdots, {}_0Q]^\dagger \in {}_{k-i}S_{s,t},$$

such that

$$Q' = x^i \cdot Q.$$

Thus,

$$Q' = [\,{}_{k-i}Q, \cdots, {}_0Q, \mathbf{0}, \cdots, \mathbf{0}]^\dagger$$

and clearly

$${}_k\mathbf{H}_{s,t} \cdot Q' = 0.$$

Therefore,

$$x^i \cdot_{k-i} S_{s,t} \subset {}_kS_{s,t}.$$

The result now follows since $x^i \cdot_{k-i} S_{s,t}$ is a subspace of ${}_kV_t$. ∎

Definition 3. Q is a **fundamental** solution of (2) if $Q \in {}_kS_{s,t}$ and $Q \notin x \cdot_{k-1} S_{s,t}$. ∎

Lemma 4. If

$$Q = [{}_kQ, \cdots, {}_0Q]^\dagger \in {}_kS_{s,t}\ ,$$

then Q is a fundamental solution in ${}_kS_{s,t}$ if and only if ${}_0Q \neq 0$.

Proof: Assume that Q is a fundamental solution in ${}_kS_{s,t}$, and suppose that ${}_0Q = 0$. Let

$$Q' = [{}_kQ, \cdots, {}_1Q]^\dagger.$$

Then $Q = x \cdot Q'$ and $Q' \in {}_{k-1}S_{s,t}$. Thus, $Q \in x \cdot_{k-1} S_{s,t}$. This contradicts the assumption that Q is a fundamental solution in ${}_kS_{s,t}$.

Conversely, assume that $Q \in {}_kS_{s,t}$ and that Q is not a fundamental solution. Then, by Definition 3,

$Q \in x \cdot {}_{k-1}S_{s,t}$, and consequently, there exists $Q' \in {}_{k-1}S_{s,t}$ such that $Q = x \cdot Q'$. Thus,

$$[{}_kQ, \cdots, {}_1Q, {}_0Q]^\dagger = [{}_{k-1}Q', \cdots, {}_0Q', \mathbf{0}]^\dagger,$$

from which it follows that ${}_0Q = 0$. Therefore, if ${}_0Q \neq 0$, then Q must be a fundamental solution in ${}_kS_{s,t}$. ■

Corollary 5. If

$$Q = [{}_kQ, \cdots, {}_0Q]^\dagger \in {}_kS_{s,t},$$

then Q is a fundamental solution in ${}_kS_{s,t}$ iff

$$Q' = [{}_{k-i}Q, \cdots, {}_0Q]^\dagger$$

is a fundamental solution in ${}_{k-i}S_{s,t}$.

Proof: Since $Q \in {}_kS_{s,t}$, then Q satisfies (2). From (2), it also follows that $Q' \in {}_{k-i}S_{s,t}$. Thus, by Lemma 4, ${}_0Q \neq 0$ iff Q is a fundamental solution in ${}_kS_{s,t}$ and Q' is a fundamental solution in ${}_{k-i}S_{s,t}$. ■

Theorem 6. If s≤t-1, then a fundamental solution in ${}_kS_{s,t}$ of (2) always exists.

Proof: We shall show that

$$dim\ ({}_kS_{s,t}) > dim\ (x \cdot {}_{k-1}S_{s,t}), \tag{15}$$

from which it then follows that there exists at least one $Q \in {}_kS_{s,t}$ such that $Q \notin x \cdot {}_{k-1}S_{s,t}$. Since the mapping x in (10) is injective, then

$$\begin{aligned} dim\ (x \cdot {}_{k-1}S_{s,t}) &= dim({}_{k-1}S_{s,t}) \\ &= k(t+1) - {}_{k-1}r_{s,t}. \end{aligned} \tag{16}$$

Thus,

$$\begin{aligned} dim\ ({}_kS_{s,t}) - dim(x \cdot {}_{k-1}S_{s,t}) &= [(k+1)(t+1) - {}_kr_{s,t}] - [k(t+1) - {}_{k-1}r_{s,t}] \\ &= t + 1 - ({}_kr_{s,t} - {}_{k-1}r_{s,t}) \\ &\geq t - s. \end{aligned} \tag{17}$$

In the last inequality, we have used the fact that

$$ {}_{k}r_{s,t} - {}_{k-1}r_{s,t} \leq s + 1, $$

since ${}_{k}\mathbf{H}_{s,t}$ has s+1 more rows than ${}_{k-1}\mathbf{H}_{s,t}$. ■

3. Quotient Spaces

Directly from the definition of a fundamental solution, it follows that the space ${}_{k}S_{s,t}$ of solutions to (2) is composed of (1) fundamental solutions in ${}_{k}S_{s,t}$ and (2) solutions contained in the subspace $x \cdot {}_{k-1} S_{s,t}$. The same observation can be made about the solution spaces ${}_{k-i}S_{s,t}$, $i = 1, \ldots, k$. Consequently, fundamental solutions in each ${}_{k-i}S_{s,t}$, $i = 0, \ldots, k$ describe the entire solution space ${}_{k}S_{s,t}$. Unfortunately, fundamental solutions are not a convenient concept to work with, because fundamental solutions in ${}_{k}S_{s,t}$ form a set and not a vector space.

As a remedy, we introduce the vector space of quotients ${}_{k}\mathbf{F}_{s,t}$ of ${}_{k}S_{s,t}$ with respect to the subspace $x \cdot {}_{k-1} S_{s,t}$, namely,

$$ \begin{aligned} {}_{k}\mathbf{F}_{s,t} &= {}_{k}S_{s,t} \,/\, x \cdot {}_{k-1} S_{s,t} \\ &= \{Q + x \cdot {}_{k-1} S_{s,t} : Q \in {}_{k}S_{s,t}\},\ k \geq 1. \end{aligned} \tag{18} $$

For $k = 0$, we define trivially

$$ {}_{0}\mathbf{F}_{s,t} = {}_{0}S_{s,t}. \tag{19} $$

Clearly, fundamental solutions in ${}_{k}S_{s,t}$ are representatives of non-zero cosets of ${}_{k}\mathbf{F}_{s,t}$, and conversely. Thus, representatives of the cosets of any basis for ${}_{k}\mathbf{F}_{s,t}$ together with a basis for $x \cdot {}_{k-1} S_{s,t}$ constitute a basis for ${}_{k}S_{s,t}$. Consequently, the problem of constructing a basis for ${}_{k}S_{s,t}$ (i.e., of characterizing the space of solutions of (2)) is reduced to the problem of constructing a basis for each of the quotient spaces ${}_{k-i}\mathbf{F}_{s,t}$, $i = 0, \ldots, k$. Since

$$ dim\ (x \cdot {}_{k-1} S_{s,t}) = dim\ ({}_{k-1}S_{s,t}), \tag{20} $$

then

$$ \begin{aligned} dim\ ({}_{k}\mathbf{F}_{s,t}) &= dim\ ({}_{k}S_{s,t}) - dim\ ({}_{k-1}S_{s,t}) \\ &= t + 1 - ({}_{k}r_{s,t} - {}_{k-1}r_{s,t}). \end{aligned} \tag{21} $$

The next lemma is crucial in the following sections.

Lemma 7. For $k \geq 1$, then

$$dim\ ({}_k\mathbf{F}_{s,t}) \leq dim\ ({}_{k-1}\mathbf{F}_{s,t}). \tag{22}$$

Proof: For

$$Q = [{}_kQ, \cdots, {}_0Q]^{\dagger} \in {}_kS_{s,t},$$

define

$$T\ (Q) = [{}_{k-1}Q, \cdots, {}_0Q]^{\dagger .}$$

Clearly, T is a linear transformation

$$T\ :\ {}_iS_{s,t} \rightarrow {}_{i-1}S_{s,t}.$$

By Corollary 5, T has the property that $Q + x \cdot_{k-1} S_{s,t}$ is the zero coset in ${}_k\mathbf{F}_{s,t}$ iff $T(Q) + x \cdot_{k-2} S_{s,t}$ is the zero coset in ${}_{k-1}\mathbf{F}_{s,t}$. Therefore (c.f., Marcus [13]), the mapping $\mathbf{T} : {}_k\mathbf{F}_{s,t} \rightarrow {}_{k-1}\mathbf{F}_{s,t}$ induced by T is a monomorphism, that is,

$$\mathbf{T}(Q + x \cdot_{k-1} S_{s,t}) = T(Q) + x \cdot_{k-2} S_{s,t}.$$

As a consequence,

$$dim\ ({}_k\mathbf{F}_{s,t}) = dim(\mathbf{T}({}_k\mathbf{F}_{s,t}))\ \leq dim\ ({}_{k-1}\mathbf{F}_{s,t}).$$

■

Corollary 8. For $k \geq 1$

$${}_kr_{s,t} - {}_{k-1}r_{s,t}\ \leq {}_{k+1}r_{s,t} - {}_kr_{s,t}. \tag{23}$$

Proof: From (21) and Lemma 7,

$$t + 1 - ({}_{k+1}r_{s,t} - {}_kr_{s,t})\ \leq t + 1 - ({}_kr_{s,t} - {}_{k-1}r_{s,t}),$$

and (23) now follows. ■

From (22), it follows that the dimensions of the quotient spaces ${}_k\mathbf{F}_{s,t}$ cannot increase with k, that is,

$$dim\ ({}_0\mathbf{F}_{s,t}) \geq dim\ ({}_1\mathbf{F}_{s,t}) \geq dim\ ({}_2\mathbf{F}_{s,t}) \geq \cdots .$$

Thus, if there are no fundamental solutions in ${}_iS_{s,t}$ (i.e., if $dim\ ({}_i\mathbf{F}_{s,t}) = 0$), then there are no fundamental solutions in ${}_jS_{s,t}$, $i \leq j \leq k$.

When fundamental solutions do exist, we wish to distinguish between two cases by means of

Definition 9. For any i, $0 \leq i \leq k$, the matrix ${}_k\mathbf{H}_{s,t}$ is **i-maximal** if $dim\ ({}_i\mathbf{F}_{s,t}) = 1$, and **i-nonmaximal** if $dim\ ({}_i\mathbf{F}_{s,t}) > 1$. ■

Thus, from (21), ${}_k\mathbf{H}_{s,t}$ is i-maximal if and only if

$${}_ir_{s,t} - {}_{i-1}r_{s,t} = t. \tag{24}$$

Corollary 10. If ${}_kH_{s,t}$ is i-nonmaximal for some i, $1 \leq i \leq k$, then it is $(i - 1)$-nonmaximal.

Proof: The result follows immediately from Theorem 7. ■

In the following sections, the notion of k-maximality is used to provide a condition for the uniqueness (in a certain sense) of solutions to (2). When ${}_k\mathbf{H}_{s,t}$ is k-nonmaximal, we show that solutions of (2) can be expressed in terms of "unique" solutions of other systems for which the Hankel matrix is k-maximal.

5. Uniqueness

If ${}_k\mathbf{H}_{s,t}$ is k-maximal, then by Definition 9 there exists a coset

$$\mathbf{Q}^* = Q^* + x \cdot {}_{k-1}S_{s,t}, \tag{25}$$

unique up to a multiplicative constant, which constitutes a basis for ${}_k\mathbf{F}_{s,t}$. The representative vector Q^* in (25) is a fundamental solution in ${}_kS_{s,t}$. In addition, if $Q \in {}_kS_{s,t}$ is any other solution of (2), then there exists a scalar α such that

$$Q = \alpha Q^* + Q',$$

where

$$Q' \in x \cdot {}_{k-1}S_{s,t}.$$

The same observation applies inductively to the solution spaces ${}_{k-i}S_{s,t}$, $i = 1, \ldots, k$, by which a basis for the entire space ${}_kS_{s,t}$ can be built. A stronger result is given by

Theorem 11. If ${}_k\mathbf{H}_{s,t}$ is i-maximal for $i = 0, \ldots, k$, then there exists $Q^* \in {}_kS_{s,t}$ such that

$$\{Q^*,\ \underline{x} \cdot Q^*, \cdots,\ \underline{x}^k \cdot Q^*\} \tag{26}$$

is a basis for ${}_kS_{s,t}$.

Proof: Let the unique non-zero coset in ${}_k\mathbf{F}_{s,t}$ be given by (25), where

$$Q^* = [{}_kQ^*, \cdots, {}_0Q^*]^\dagger$$

is a fundamental solution in ${}_kS_{s,t}$. Then by Corollary 5, $[{}_iQ^*, \cdots, {}_0Q^*]^\dagger$ is a fundamental solution in ${}_iS_{s,t}$, $i = 0, \ldots, k$. By induction, we now show that a basis for ${}_kS_{s,t}$ is given by

$$\{x^{k-j} \cdot [{}_jQ^*, \cdots, {}_0Q^*]^\dagger\},\quad j = 0, \ldots, k. \tag{27}$$

The theorem then follows from (27), because

$$\underline{x}^{k-j} \cdot Q^* = x^{k-j} \cdot [{}_jQ^*, \cdots, {}_0Q^*]^\dagger.$$

Since ${}_0Q^*$ is a fundamental solution in ${}_0S_{s,t}$, and

$$dim\ ({}_0S_{s,t}) = dim\ ({}_0\mathbf{F}_{s,t}) = 1,$$

then a basis for ${}_0S_{s,t}$ is composed of the single vector ${}_0Q^*$.

Now assume that a basis for ${}_iS_{s,t}$ is given by

$$\{x^{i-j} \cdot [{}_jQ^*, \cdots, {}_0Q^*]^\dagger\},\quad j = 0, \ldots, i. \tag{28}$$

Since ${}_k\mathbf{H}_{s,t}$ is $(i+1)$-maximal (i.e., $dim\ ({}_{i+1}\mathbf{F}_{s,t}) = 1$) and

$$[{}_{i+1}Q^*, \cdots, {}_0Q^*]^\dagger \tag{29}$$

is a fundamental solution in ${}_{i+1}S_{s,t}$, then the unique non-zero coset in ${}_{i+1}\mathbf{F}_{s,t}$ is

$$[{}_{i+1}Q^*, \cdots, {}_0Q^*]^\dagger + x \cdot {}_iS_{s,t}.$$

Therefore, a basis for ${}_{i+1}S_{s,t}$ is obtained by appending the representative vector (29) to a basis for $x \cdot {}_iS_{s,t}$. However, using the inductive hypothesis (28), a basis for $x \cdot {}_iS_{s,t}$ is given by

$$\{x^{i+1-j} \cdot [{}_jQ^*, \cdots, {}_0Q^*]^\dagger\}, \quad j = 0, \ldots, i. \tag{30}$$

The vector (29), together with (30), yield the required basis for ${}_{i+1}S_{s,t}$. ■

Corollary 12. Let ${}_k\mathbf{H}_{t-1,t}$ be 0-maximal. If Q is a solution of (4), then there exist scalars α_i, $i = 0, \ldots, k$, such that

$$Q = \sum_{i=0}^{k} \alpha_i \underline{x}^i \cdot Q^*, \tag{31}$$

where Q^* is a fundamental solution in ${}_kS_{t-1,t}$.

Proof: By Theorem 6, a fundamental solution in ${}_kS_{t-1,t}$ always exists, and consequently

$$dim\ ({}_k\mathbf{F}_{t-1,t}) \geq 1.$$

Furthermore,

$$dim\ ({}_0\mathbf{F}_{t-1,t}) = 1,$$

since ${}_kH_{t-1,t}$ is 0-maximal. Thus, by Lemma 7

$$dim\ ({}_i\mathbf{F}_{t-1,t}) = 1,\ i = 0, \ldots, k.$$

(31) now follows from Theorem 11. ■

6. Characterization of Solutions

The construction of a basis for ${}_k\mathbf{F}_{s,t}$ is significantly more complex when ${}_k\mathbf{H}_{s,t}$ is k-nonmaximal, since now $dim\ ({}_k\mathbf{F}_{s,t}) > 1$. The objective of this section is to construct a representative of a single coset in ${}_i\mathbf{F}_{s,t}$ (only under certain constraints on s and t) which generates a basis for ${}_i\mathbf{F}_{s,t}$. Then, the representatives of the $k+1$ generators of the quotient spaces ${}_i\mathbf{F}_{s,t}$, $i = 0, \ldots, k$, yield a basis for the solution space ${}_kS_{s,t}$ of (2). We begin with a number of preliminary lemmas.

Lemma 13. If $Q \in {}_kS_{s+1,t}$, then $y \cdot Q$, $\bar{y} \cdot Q \in {}_kS_{s,t+1}$.

Proof: The result follows from the definition of y and $\bar{y}$, and a very careful comparison of the matrices ${}_k\mathbf{H}_{s+1,t}$ and ${}_k\mathbf{H}_{s,t+1}$. ■

Lemma 14. Let $Q \in {}_kS_{s+1,t}$. Then the following statement are equivalent:

(1) Q is a fundamental solution in ${}_kS_{s+1,t}$.

(2) $y \cdot Q$ is a fundamental solution in ${}_kS_{s,t+1}$.

(3) $\bar{y} \cdot Q$ is a fundamental solution in ${}_kS_{s,t+1}$.

Proof: Since $Q \in {}_kS_{s+1,t}$, then by Lemma 13, $y \cdot Q \in {}_kS_{s,t+1}$. But, ${}_0Q = [{}_0q_t, \cdots, {}_0q_0]^\dagger = 0$ if and only if $y \cdot {}_0Q = [{}_0q_t, \cdots, {}_0q_0\ , 0]^\dagger = 0$. Thus, by

Lemma 4, (1) and (2) are equivalent.

Statements (1) and (3) can be shown to be equivalent in a similar fashion. ■

Lemma 15. Let $Q \in {}_kS_{s+1,t}$. Then the following statements are equivalent:

(1) $Q + x \cdot {}_{k-1}S_{s+1,t}$ is the zero coset in ${}_k\mathbf{F}_{s+1,t}$.

(2) $y \cdot Q + x \cdot {}_{k-1}S_{s,t+1}$ is the zero coset in ${}_k\mathbf{F}_{s,t+1}$.

(3) $\bar{y} \cdot Q + x \cdot {}_{k-1}S_{s,t+1}$ is the zero coset in ${}_k\mathbf{F}_{s,t+1}$.

Proof: The results are a direct consequence of Lemma 14 and the definition of a fundamental solution. ■

Define the mappings

$$\mathbf{y} : {}_k\mathbf{F}_{s+1,t} \ \rightarrow\ {}_k\mathbf{F}_{s,t+1} \tag{3}$$

$$\bar{\mathbf{y}} : {}_k\mathbf{F}_{s+1,t} \ \rightarrow\ {}_k\mathbf{F}_{s,t+1} \tag{33}$$

as follows: For an arbitrary coset

$$\mathbf{Q} = Q + x \cdot {}_{k-1}S_{s+1,t} \in {}_k\mathbf{F}_{s+1,t}, \tag{34}$$

where $Q \in {}_kS_{s+1,t}$ is a representative, define

$$\mathbf{y} \cdot \mathbf{Q} = y \cdot Q + x \cdot {}_{k-1}S_{s,t+1} \in {}_k\mathbf{F}_{s,t+1} \tag{35}$$

$$\bar{\mathbf{y}} \cdot \mathbf{Q} = \bar{y} \cdot Q + x \cdot {}_{k-1}S_{s,t+1} \in {}_k\mathbf{F}_{s,t+1}. \tag{36}$$

This definition is shown to be unambiguous in

Lemma 16. The mappings $\mathbf{y}$ and $\bar{\mathbf{y}}$ are monomorphisms of ${}_k\mathbf{F}_{s+1,t}$ into ${}_k\mathbf{F}_{s,t+1}$.

Proof: We give a proof for $\mathbf{y}$ only. The result for $\bar{\mathbf{y}}$ follows in a similar fashion.

We first show that $\mathbf{y}$ is well defined (i.e., $\mathbf{y}$ does not depend on the choice of representatives for cosets in ${}_k\mathbf{F}_{s+1,t}$). Suppose that $Q, Q' \in {}_kS_{s+1,t}$ are such that

$$Q + x \cdot {}_{k-1}S_{s+1,t} = Q' + x \cdot {}_{k-1}S_{s+1,t}. \tag{37}$$

Then, $Q - Q' \in x \cdot {}_{k-1}S_{s+1,t}$ and consequently $Q - Q' + x \cdot_{k-1} S_{s+1,t}$ is the zero coset in ${}_k\mathbf{F}_{s+1,t}$. Thus, by Corollary 15, $y \cdot Q - y \cdot Q' + x \cdot {}_{k-1}S_{s,t+1}$ is the zero coset in ${}_k\mathbf{F}_{s,t+1}$. That is,

$$y \cdot Q + x \cdot_{k-1} S_{s,t+1} = y \cdot Q' + x \cdot_{k-1} S_{s,t+1}.$$

Thus, we have shown that if (37) holds, then

$$\mathbf{y} \cdot (Q + x \cdot {}_{k-1}S_{s+1,t}) = \mathbf{y} \cdot (Q' + x \cdot {}_{k-1}S_{s+1,t}).$$

Next, we show that $\mathbf{y}$ is injective. For the coset

$$\mathbf{Q} = Q + x \cdot {}_{k-1}S_{s+1,t} \in {}_k\mathbf{F}_{s+1,t},$$

where $Q \in {}_kS_{s+1,t}$, suppose that $\mathbf{y} \cdot \mathbf{Q}$ is the zero coset in ${}_k\mathbf{F}_{s,t+1}$. That is, suppose that $y \cdot Q + x \cdot {}_{k-1}S_{s,t+1}$ is the zero coset in ${}_k\mathbf{F}_{s,t+1}$. Then, by Corollary 15, $\mathbf{Q} = Q + x \cdot {}_{k-1}S_{s,t}$ is the zero coset in ${}_k\mathbf{F}_{s+1,t}$.

Finally, we show that $\mathbf{y}$ is linear. Let $\mathbf{Q}, \mathbf{Q}' \in {}_k\mathbf{F}_{s+1,t}$. Then, there exist $Q, Q' \in {}_kS_{s+1,t}$ such that

$$\mathbf{Q} = Q + x \cdot {}_{k-1}S_{s+1,t}$$

and

$$\mathbf{Q}' = Q' + x \cdot {}_{k-1}S_{s+1,t}.$$

Then, for any scalars α and α’,

$$\mathbf{y} \cdot (\alpha\mathbf{Q} + \alpha'\mathbf{Q}') = \mathbf{y} \cdot (\alpha Q + x \cdot {}_{k-1}S_{s+1,t} + \alpha' Q' + x \cdot {}_{k-1}S_{s+1,t})$$

$$= \mathbf{y} \cdot (\alpha Q + \alpha' Q' + x \cdot {}_{k-1}S_{s+1,t})$$

$$= y \cdot (\alpha Q + \alpha' Q') + x \cdot {}_{k-1}S_{s,t+1}$$

$$= (\alpha y \cdot Q + x \cdot {}_{k-1}S_{s,t+1}) + (\alpha' y \cdot Q' + x \cdot {}_{k-1} S_{s,t+1})$$

$$= \alpha(y \cdot Q + x \cdot {}_{k-1}S_{s,t+1}) + \alpha'(y \cdot Q' + x \cdot {}_{k-1}S_{s,t+1})$$

$$= \alpha \mathbf{y} \cdot \mathbf{Q} + \alpha' \mathbf{y} \cdot \mathbf{Q}'.$$

■

Denote the images of ${}_k\mathbf{F}_{s+1,t}$ under the tranformations $\mathbf{y}$ and $\bar{\mathbf{y}}$ by $\mathbf{y} \cdot {}_k\mathbf{F}_{s+1,t}$ and $\bar{\mathbf{y}} \cdot {}_k\mathbf{F}_{s+1,t}$, respectively. We then have

Lemma 17. If $dim\ ({}_k\mathbf{F}_{s+1,t}) \geq 1$, then

$$\mathbf{y} \cdot {}_k\mathbf{F}_{s+1,t} \neq \bar{\mathbf{y}} \cdot {}_k\mathbf{F}_{s+1,t}. \tag{38}$$

Proof: Define

$$FS = \{Q : {}_0Q \neq 0 \text{ and } Q \in {}_kS_{s+1,t}\} \tag{39}$$

to be the set of fundamental solutions in ${}_kS_{s+1,t}$. Then, FS is not empty, because $dim\ ({}_k\mathbf{F}_{s+1,t}) > 0$. Clearly, by the definitions of y and $\bar{y}$ in (8) and (9), there exists $Q' \in FS$ such that for all $Q \in FS$

$$y \cdot {}_0Q' \neq \bar{y} \cdot {}_0Q. \tag{40}$$

We now show that the coset

$$y \cdot Q' + x \cdot {}_{k-1}S_{s,t+1} \notin \bar{\mathbf{y}} \cdot {}_k\mathbf{F}_{s+1,t}.$$

(41)

For suppose otherwise. Then, there exists $Q'' \in {}_kS_{s+1,t}$ such that

$$y \cdot Q' + x \cdot {}_{k-1}S_{s,t+1} = \bar{y} \cdot Q'' + x \cdot {}_{k-1}S_{s,t+1}.$$

Thus,

$$y \cdot Q' - \bar{y} \cdot Q'' \in x \cdot {}_{k-1}S_{s,t+1};$$

that is, $y \cdot Q' - \bar{y} \cdot Q''$ is not a fundamental solution in ${}_kS_{s,t+1}$. Consequently, by Lemma 4,

$$y \cdot {}_0Q' - \bar{y} \cdot {}_0Q'' = 0.$$

But, ${}_0Q' \neq 0$, because $Q' \in FS$. Thus, ${}_0Q'' \neq 0$, which implies Q'' is a fundamental solution in ${}_kS_{s+1,t}$. We have therefore found $Q'' \in FS$ such that

$$y \cdot {}_0Q' = \bar{y} \cdot {}_0Q'' = 0,$$

which violates the definition of Q' in (40). Thus, (41) is true.

On the other hand, by Lemma 14, $y \cdot Q'$ is a fundamental solution in ${}_kS_{s,t+1}$. Therefore,

$$y \cdot Q' + x \cdot {}_{k-1}S_{s,t+1} \in {}_k\mathbf{F}_{s,t+1} \tag{42}$$

is a nonzero coset in ${}_k\mathbf{F}_{s,t+1}$. ■

Corollary 18. If $dim\ ({}_k\mathbf{F}_{s+1,t}) \geq 1$, then

$$dim\ ({}_k\mathbf{F}_{s,t+1}) \geq dim\ ({}_k\mathbf{F}_{s+1,t}) + 1. \tag{43}$$

Proof: From (41) and (42), it follows that

$$dim\ (\bar{\mathbf{y}} \cdot {}_k\mathbf{F}_{s+1,t}) < dim\ ({}_k\mathbf{F}_{s,t+1}). \tag{44}$$

But, from lemma (16), $\bar{\mathbf{y}}$ is a monomorphism so that

$$dim\ ({}_k\mathbf{F}_{s+1,t}) = dim\ (\bar{\mathbf{y}} \cdot {}_k\mathbf{F}_{s+1,t}). \tag{45}$$

The result (43) now follows from (44) and (45) . ■

Corollary 19. If $dim\ ({}_k\mathbf{F}_{s+1,t}) \geq 1$, then

$${}_kr_{s,t+1} - {}_{k-1}r_{s,t+1} \leq {}_kr_{s+1,t} - {}_{k-1}r_{s+1,t}. \tag{46}$$

Proof: From (21),

$$dim\ ({}_kF_{s,t+1}) = t + - ({}_kr_{s,t+1} - {}_{k-1}r_{s,t+1}),$$

and

$$dim\ ({}_kF_{s+1,t}) = t + 1 - ({}_kr_{s+1,t} - {}_{k-1}r_{s+1,t}).$$

Thus, by Lemma 19,

$$t + 2 - ({}_k r_{s,t+1} - {}_{k-1}r_{s,t+1}) \geq t + 1 - ({}_k r_{s+1,t} - {}_{k-1}r_{s+1,t}) + 1$$

and (46) now follows. ∎

Corollary 20. If $dim\ ({}_k\mathbf{F}_{s,t}) \geq 1$, then

$${}_k r_{s-i,t+i} - {}_{k-1}r_{s-i,t+i} \leq {}_k r_{s,t} - {}_{k-1}r_{s,t},\ i = 0, \ldots, s \tag{47}$$

Proof: We proceed by induction to show that (47) is true and in addition that

$$dim\ ({}_k\mathbf{F}_{s-i,t+1}) \geq i + 1,\ i = 0, \ldots, s. \tag{48}$$

For $i = 0$, (47) and (48) hold true trivially. Now suppose (47) and (48) are valid for $i \geq 0$. Then from (47) and Corollary 19

$$\begin{aligned} {}_k r_{s-i-1,t+i+1} - {}_{k-1}r_{s-i-1,t+i+1} &\leq {}_k r_{s-i,t+i} - {}_{k-1}r_{s-i,t+i} \\ &\leq {}_k r_{s,t} - {}_{k-1}r_{s,t} \end{aligned}$$

and (47) is true at $i + 1$. Also, (48) is true at $(i + 1)$, because

$$\begin{aligned} dim\ ({}_k\mathbf{F}_{s-i-1,t+i+1}) &= t + i + 2 - ({}_k r_{s-i-1,t+i+1} - {}_{k-1}r_{s-i-1,t+i+1}) \\ &\geq t + i + 2 - ({}_k r_{s,t} - {}_{k-1}r_{s,t}) \\ &\geq i + 2. \end{aligned}$$

In the last inequality, we have used the fact that

$${}_k r_{s,t} - {}_{k-1}r_{s,t} \leq t,$$

which again follows from (21) because

$$1 \leq dim\ ({}_k\mathbf{F}_{s,t}) = t + 1 - ({}_k r_{s,t} - {}_{k-1}r_{s,t}).$$

∎

If ${}_k\mathbf{H}_{s,t}$ is k-maximal, then $dim\ ({}_k\mathbf{F}_{s,t}) = 1$. The quotient space ${}_k\mathbf{F}_{s,t}$ is then fully characterized by a single non-zero coset in ${}_k\mathbf{F}_{s,t}$. Characterization of ${}_k\mathbf{F}_{s,t}$ when ${}_k\mathbf{H}_{s,t}$ is k-nonmaximal is accomplished by means of

Theorem 21. Let

$$\gamma_k = t - ({}_k r_{s,t} - {}_{k-1} r_{s,t}). \tag{49}$$

If

$${}_k r_{s,t} - {}_{k-1} r_{s,t} \le \min\{s, t\}, \tag{50}$$

then there exists a fundamental solution $Q^{(k)} \in {}_k S_{s+\gamma_k, t-\gamma_k}$ such that

$$\{\bar{y}^{\gamma_k - j} \cdot y^j \cdot Q^{(k)} + x \cdot {}_{k-1} S_{s,t}\},\ j = 0, \ldots, \gamma_k \tag{51}$$

forms a basis for ${}_k \mathbf{F}_{s,t}$.

Proof: From (21) and (50)

$$dim\ ({}_k \mathbf{F}_{s,t}) = t + 1 - ({}_k r_{s,t} - {}_{k-1} r_{s,t}) \ge 1$$

Then, by Corollary 20 and (49), for $i = 0, \ldots, s$,

$${}_k r_{s-i,t+i} - {}_{k-1} r_{s-i,t+i} \le {}_k r_{s,t} - {}_{k-1} r_{s,t} = t - \gamma_k. \tag{52}$$

But, from (49) and (50),

$$0 \le t - \gamma_k \le s.$$

Therefore, in particular, for $i = s - t + \gamma_k$, inequality (52) becomes

$${}_k r_{t-\gamma_k, s+\gamma_k} - {}_{k-1} r_{t-\gamma_k, s+\gamma_k} \le t - \gamma_k. \tag{53}$$

Now consider

$${}_k \mathbf{H}_{s+\gamma_k, t-\gamma_k} = ({}_k \mathbf{H}_{t-\gamma_k, s+\gamma_k})^{\dagger}.$$

Clearly,

$$\begin{aligned} {}_k r_{s+\gamma_k, t-\gamma_k} - {}_{k-1} r_{s+\gamma_k, t-\gamma_k} &= {}_k r_{t-\gamma_k, s+\gamma_k} - {}_{k-1} r_{t-\gamma_k, s+\gamma_k} \\ &\le t - \gamma_k, \end{aligned} \tag{54}$$

using (53). Thus,

$$dim\ ({}_k\mathbf{F}_{s+\gamma_k, t-\gamma_k}) = t - \gamma_k + 1 - ({}_k r_{s+\gamma_k, t-\gamma_k} - {}_{k-1} r_{s+\gamma_k, t-\gamma_k}) \geq 1.$$

Corollary 20 can therefore be applied once again to yield

$$\begin{aligned} t - \gamma_k &= {}_k r_{s,t} - {}_{k-1} r_{s,t} \\ &\leq {}_k r_{s+\gamma_k, t-\gamma_k} - {}_{k-1} r_{s+\gamma_k, t-\gamma_k}. \end{aligned} \tag{55}$$

From (54), (55) and Corollary 20, it now follows that

$${}_k r_{s+i, t-i} - {}_{k-1} r_{s+i, t-i} = t - \gamma_k, \tag{56}$$

for all i such that $t - s - \gamma_k \leq i \leq \gamma_k$. Thus, from (21) and (56),

$$\begin{aligned} dim\ ({}_k\mathbf{F}_{s+i, t-i}) &= t - i + 1 - (t - \gamma_k) \\ &= \gamma_k - i + 1, \end{aligned} \tag{57}$$

i such that $t - s - \gamma_k \leq i \leq \gamma_k$.

In particular, observe that

$$dim\ ({}_k\mathbf{F}_{s+\gamma_k, t-\gamma_k}) = 1. \tag{58}$$

Therefore, in ${}_k\mathbf{F}_{s+\gamma_k, t-\gamma_k}$, there exists a unique non-zero coset

$$\mathbf{Q}^{(k)} = Q^{(k)} + x \cdot {}_{k-1} S_{s+\gamma_k, t-\gamma_k},$$

where $Q^{(k)}$ is a fundamental solution in ${}_k S_{s+\gamma_k, t-\gamma_k}$. We now show that $\mathbf{Q}^{(k)}$ generates a basis for ${}_k\mathbf{F}_{s+i, t-i}$, for $i = \gamma_k, \ldots, t - s - \gamma_k$. That is, we show that a basis for ${}_k\mathbf{F}_{s+i, t-i}$, $i = \gamma_k, \ldots, t - s - \gamma_k$, is given by

$$\{\bar{\mathbf{y}}^{\gamma_k - i - j} \cdot \mathbf{y}^j \cdot \mathbf{Q}^{(k)}\}, \quad j = 0, \ldots, \gamma_k - i. \tag{59}$$

We proceed by induction for decreasing values of i. For the initial step in the induction, $i = \gamma_k$, we have trivially that $\{\mathbf{Q}^{(k)}\}$ is a basis for ${}_k\mathbf{F}_{s+\gamma_k, t-\gamma_k}$. Assume now that (59) provides a basis for ${}_k\mathbf{F}_{s+\gamma_k, t-\gamma_k}, \cdots, {}_k\mathbf{F}_{s+i, t-i}$. It is required that we show (59) provides a basis for ${}_k\mathbf{F}_{s+i-1, t-i+1}$. Since

$$\{\bar{\mathbf{y}}^{\gamma_k - i - j} \cdot \mathbf{y}^j \cdot \mathbf{Q}^{(k)}\}, \quad j = 0, \ldots, \gamma_k - i,$$

is a basis for ${}_k\mathbf{F}_{s+i, t-i}$, by Lemma 16, a basis for each of $\mathbf{y} \cdot {}_k\mathbf{F}_{s+i, t-i}$ and $\bar{\mathbf{y}} \cdot {}_k\mathbf{F}_{s+i, t-i}$ are given by

$$\{\bar{\mathbf{y}}^{\gamma_k - i - j} \cdot \mathbf{y}^{j+1} \cdot \mathbf{Q}^{(k)}\}, \quad j = 0, \ldots, \gamma_k - i \tag{60}$$

and

$$\{\bar{\mathbf{y}}^{\gamma_k - i - j + 1} \cdot \mathbf{y}^{j} \cdot \mathbf{Q}^{(k)}\}, \quad j = 0, \ldots, \gamma_k - i, \tag{61}$$

respectively, since $\mathbf{y}$ and $\bar{\mathbf{y}}$ commute. The union of (60) and (61)

$$\{\mathbf{y}^{\gamma_k - i - j + 1} \cdot \mathbf{y}^{j} \cdot \mathbf{Q}^{(k)}\}, \quad j = 0, \ldots, \gamma_k - i + 1 \tag{62}$$

are all linearly independent in ${}_k\mathbf{F}_{s+i-1,t-i+1}$, according to Lemma 17. By (57),

$$dim\ ({}_k\mathbf{F}_{s+i-1,t-i+1}) = \gamma_k - i + 2,$$

and since there are $\gamma_k - i + 2$ cosets in (62), then (62) forms a basis for ${}_k\mathbf{F}_{s+i-1,t-i+1}$. The induction is therefore complete.

The theorem now follows by setting $i = 0$ in (59). ■

For a given matrix ${}_k\mathbf{H}_{s,t}$, let

$${}_k r_{s,t} - {}_{k-1} r_{s,t} \le \min\ \{s, t\}.$$

Then, by Corollary 8, it follows that

$${}_i r_{s,t} - {}_{i-1} r_{s,t} \le \min\ \{s, t\},\ i = 0, \ldots, k.$$

Thus, Theorem 21 is valid for each submatrix ${}_i\mathbf{H}_{s,t}$, $i = 0, \ldots, k$. We then obtain

Corollary 22. Let ${}_k\mathbf{H}_{s,t}$ be such that

$${}_k r_{s,t} - {}_{k-1} r_{s,t} \le \min\ \{s, t\}.$$

For $i = 0, \ldots, k$, define

$$\gamma_i = t - ({}_i r_{s,t} - {}_{i-1} r_{s,t}) \tag{63}$$

and let $Q^{(i)}$ be a fundamental solution in ${}_i S_{s+\gamma_i, t-\gamma_i}$. If $Q \in {}_k S_{s,t}$, then there exists scalars $\alpha_{i,j}$ such that

$$Q = \sum_{i=0}^{k} \sum_{j=0}^{\gamma_i} \alpha_{i,j} x^{k-i} \cdot \bar{y}^{\gamma_i - j} \cdot y^j \cdot Q^{(i)}. \tag{64}$$

Proof: From Theorem 21, a basis for ${}_kS_{s,t}$ is composed of the union of

$$\{\bar{y}^{\gamma_k - j} \cdot y^j \cdot Q^{(k)}\}, \quad j = 0, \ldots, \gamma_k$$

and an x-shift of a basis for ${}_{k-1}S_{s,t}$. A basis for ${}_{k-1}S_{s,t}$ (and iteratively for the subspaces ${}_{k-2}S_{s,t}, \cdots, {}_0S_{s,t}$) is obtained in a similar fashion by means of Theorem 21, and (61) now follows. ■

Theorem 23. Let $s \geq t - 1$, and define

$$\gamma_k = t - ({}_kr_{s,t} - {}_{k-1}r_{s,t}). \tag{65}$$

If $\gamma_k \geq 0$, then there exists a fundamental solution $Q^{(k)} \in {}_kS_{s+\gamma_k, t-\gamma_k}$ such that

$$\{\bar{y}^{\gamma_k - j} \cdot y^j \cdot Q^{(k)} + x \cdot {}_{k-1}S_{s,t}\}, \quad j = 0, \ldots, \gamma_{k,} \tag{66}$$

forms a basis for ${}_k\mathbf{F}_{s,t}$.

Proof: If $s \geq t$, then (65) and $\gamma_k \geq 0$ imply that

$${}_kr_{s,t} - {}_{k-1}r_{s,t} = t - \gamma_k \leq t = \min\{s, t\},$$

and the theorem follows from Theorem .

If $s = t - 1$, then ${}_k\mathbf{H}_{s,t}$ has t more rows than ${}_{k-1}\mathbf{H}_{s,t}$, and consequently

$${}_kr_{s,t} - {}_{k-1}r_{s,t} \leq t. \tag{67}$$

If, in addition,

$${}_kr_{s,t} - {}_{k-1}r_{s,t} \leq t - 1$$

in (67), then again condition (50) is satisfied and the theorem follows from Theorem 21. Finally, if $s = t - 1$ and

$${}_kr_{s,t} - {}_{k-1}r_{s,t} = t,$$

then ${}_k\mathbf{H}_{s,t}$ is k-maximal. Therefore, $\gamma_k = 0$ in (65) and (66) and $Q^{(k)}$ is a representative of the unique non-zero coset in ${}_k\mathbf{F}_{s,t}$. ■

Corollary 24. Let $s \geq t - 1$, and define

$$\gamma_i = t - ({}_i r_{s,t} - {}_{i-1} r_{s,t}),\ i = 0, \ldots, k. \tag{68}$$

If $\gamma_k \geq 0$, then for any $Q \in {}_k S_{s,t}$ there exists scalars $\alpha_{i,j}$ such that

$$Q = \sum_{i=0}^{k} \sum_{j=0}^{\gamma_i} \alpha_{i,j} x^{k-i} \cdot \bar{y}^{\gamma_i - j} \cdot y^j Q^{(i)}, \tag{69}$$

where, for $i = 0, \ldots, k$, $Q^{(i)}$ is a fundamental solution in ${}_i S_{s+\gamma_i, t-\gamma_i}$.

Proof: By Corollary 8,

$${}_i r_{s,t} - {}_{i-1} r_{s,t} \leq {}_k r_{s,t} - {}_{k-1} r_{s,t},\ i = 0, \ldots, k.$$

Thus, in (68) $\gamma_i \geq 0$ for $i = 0, \ldots, k$. From Theorem 3, a basis for ${}_i\mathbf{F}_{s,t}$, $i = 0, \ldots, k$ is given by (66) with k replaced by i. Then, Corollary 24 follows by arguments similar to those in the proof of Corollary 22. ■

7. Modular Padé Forms

In this section, we define modular Padé forms for a bivariate power series. It will be seen that the problem of obtaining a modular Padé form is equivalent to that of solving an associated triangular block Hankel system of the type defined in section 1. So, the results of the previous sections on the characterization of solutions for this Hankel system fully describe the nature of the Padeé forms. We begin with the introduction of a suitable notation.

A bivariate power series $A(x, y)$ is a formal power series in two variables x and y, i.e. a formal expression of the form

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} {}_i a_j x^i y^j, \tag{70}$$

where the coefficients ${}_i a_j$ are from D. For $i \geq 0$ and $j \geq 0$, an expression $O(x^i y^j)$ denotes an arbitrary bivariate power series $R(x, y)$ such that there exists a bivariate power series $R'(x, y)$ and $R(x, y) = x^i y^j R'(x, y)$. In this case it is said that $R(x, y)$ is of the order $x^i y^j$. Thus, for $i_0,\ j_0, \ldots, i_k,\ j_k \geq 0$, the expression

$$A(x, y) = O(x^{i_0} y^{j_0}) + \cdots + O(x^{i_k} y^{j_k})$$

indicates that there exist power series $R^{(0)}(x, y), \cdots, R^{(k)}(x, y)$ such that

$$A(x, y) = x^{i_0} y^{j_0} R^{(0)}(x, y) + \cdots + x^{i_k} y^{j_k} R^{(k)}(x, y).$$

Bivariate power series with a finite number of non-zero coefficients are bivariate polynomials. If $P(x, y) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} {}_i p_j x^i y^j$ is a polynomial, then a minimal $k \geq 0$ and a minimal $m \geq 0$ such that

$$P(x, y) = \sum_{i=0}^{k} \sum_{j=0}^{m} {}_i p_j x^i y^j$$

are called, respectively, the degree of $P(x, y)$ in x and the degree of $P(x, y)$ in y (in symbols, $\partial_x P(x, y) = k$ and $\partial_y P(x, y) = m$). A pair (k, m) is called simply the degree of $P(x, y)$ (in symbols, $\partial P(x, y) = (k, m)$). Also, the expression $\partial P(x, y) \leq (k, m)$ is used to indicate that $\partial_x P(x, y) \leq k$ and $\partial_y P(x, y) \leq m$.

A vector space of all bivariate polynomials $Q(x, y)$ with $\partial Q(x, y) \leq (k, t)$ is denoted by ${}_k BP_t$. It is an easy observation that the vector space ${}_k V_t$ given in (5) and vector space ${}_k BP_t$ are isomorphic in a natural way, i.e., if $Q \in {}_k V_t$, where $Q = [{}_k Q, \cdots, {}_0 Q]^\dagger$ and ${}_i Q = [{}_i q_t, \cdots, {}_i q_0]^\dagger$, $i = 0, \ldots, k$, then there exists a corresponding polynomial $Q(x, y) \in {}_k BP_t$, namely

$$Q(x, y) = \sum_{i=0}^{k} \sum_{j=0}^{t} {}_i q_j x^i y^j. \tag{71}$$

This isomorphism is denoted by $Pol_{k,t}$ and its inverse isomorphism by $Vec_{k,t}$. Thus,

$$Pol_{k,t} : {}_k V_t \rightarrow {}_k BP_t.$$

$$Vec_{k,t} : {}_k BP_t \rightarrow {}_k V_t,$$

and with Q and $Q(x, y)$ above, $Q(x, y) = Pol_{k,t}(Q)$ and $Q = Vec_{k,t}(Q(x, y))$. Moreover, the shift transformations x, $\underline{x}$, y and $\bar{y}$ from section 1 can be easily translated into operations on polynomials, which are given by the following

Lemma 25: Let $Q \in {}_k V_t$. Then

1. $Pol_{k+l,t}(x^l \cdot Q) = x^l \cdot Pol_{k,t}(Q)$,
2. $Pol_{k,t}(\underline{x}^l \cdot Q) = (x^l \cdot Pol_{k,l}(Q)) mod\ x^{k+1}$,
3. $Pol_{k,t+l}(y^l \cdot Q) = y^l \cdot Pol_{k,t}(Q)$, and

4. $Pol_{k,t+l}(\bar{y}^l \cdot Q) = Pol_{k,t}\,(Q)$.

Proof:

Only case 2 is not trivial, and the proof is given for this case only. Let $Q \in {}_kV_t$. Then

$$Pol_{k,t}\,(Q) = \sum_{i=0}^{k} \sum_{j=0}^{t} {}_iq_j x^i y^j .$$

Thus,

$$x^l \cdot Pol_{k,t}\,(Q) = \sum_{i=0}^{k} \sum_{j=0}^{t} {}_iq_j x^{i+l} y^j$$

and

$$(x^l \cdot Pol_{k,t}\,(Q)) mod x^{k+1} = \sum_{i=0}^{k-l} \sum_{j=0}^{t} {}_iq_j x^{i+l} y^j .$$

On the other hand, by the definition of the tranformation $\underline{x}$,

$$\underline{x}^l \cdot Q = [{}_{k-l}Q, \ldots, {}_0Q, \mathbf{0}, \ldots, \mathbf{0}]^\dagger = [{}_kQ', \ldots, {}_0Q']^\dagger ,$$

where ${}_iQ' = [{}_{i-l}q_t, \ldots, {}_{i-l}q_t]^\dagger$ for $i = l, \ldots, k$, and ${}_iQ' = 0$ for $i = 0, \ldots, l-1$. Thus,

$$Pol_{k,t}(\underline{x}^l \cdot Q) = \sum_{i=l}^{k} \sum_{j=0}^{t} {}_{i-l}q_j x^i y^j = \sum_{i=0}^{k-l} \sum_{j=0}^{t} {}_iq_j x^{i+l} y^j .$$

■

Let the bivariate power series $A(x, y)$ and non-negative integers k, m and n be given.

Definition 26. A bivariate rational expression $P(x, y)/Q(x, y)$ is called a modular Padé (k, m, n)-form for $A(x, y)$ if $\partial P(x, y) \le (k, m)$, $\partial Q(x, y) \le (k, n)$ and the following order condition is satisfied

$$A(x, y) \cdot Q(x, y) + P(x, y) = O(y^{m+n+1}) + O(x^{k+1}). \tag{72}$$

■

By equating appropriate powers of x and y, it is easy to see that the polynomials

$$P(x, y) = \sum_{i=0}^{k} \sum_{j=0}^{m} {}_ip_j x^i y^j \quad \text{and} \quad Q(x, y) = \sum_{i=0}^{k} \sum_{j=0}^{n} {}_iq_j x^i y^j$$

satisfy the order condition (72) if and only if its coefficients satisfy the following systems of linear equations:

$$\sum_{i=0}^{k}\sum_{j=0}^{n} {}_{s-i}a_{t-j}\,{}_{i}q_j = 0\ , \qquad 0 \le s \le k,\ m+1 \le t \le m+n\ , \tag{73}$$

and

$$\sum_{i=0}^{k}\sum_{j=0}^{n} {}_{s-i}a_{t-j}\,{}_{i}q_j + {}_{s}p_t = 0, \qquad 0 \le s \le k,\ 0 \le t \le m, \tag{74}$$

where ${}_{i}a_j = 0$ if $i < 0$ or $j < 0$.

The systems of equations (73) and (74) can be expressed in matrix form as follows. Let ${}_{k}\mathbf{H}_{n-1,n}$ be the triangular block Hankel matrix defined in (3), with components ${}_{i}h_j$ determined by the coefficients of the power series $A(x, y)$,namely, for $i = 0, \ldots, k$ and $j = 0, \ldots, 2n-1$,

$$ {}_{i}h_j = \begin{cases} {}_{i}a_{m-n+1+j}, & \text{if} \quad m-n+1+j \ge 0, \\ 0, & \textit{otherwise}. \end{cases} \tag{75}$$

Let ${}_{k}\mathbf{G}_{m,n}$ be a triangular block Toeplitz matrix, such that

$$ {}_{k}\mathbf{G}_{m,n} = \begin{bmatrix} {}_{0}G & \cdot & \cdot & \cdot & {}_{k}G \\ & \cdot & & & \cdot \\ & & \cdot & & \cdot \\ & & & \cdot & \cdot \\ & & & & {}_{0}G \end{bmatrix}, $$

where

$$ {}_{i}G = \begin{bmatrix} {}_{i}a_{m-n} & \cdot & \cdot & \cdot & {}_{i}a_m \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ {}_{i}a_{-n} & \cdot & \cdot & \cdot & {}_{i}a_0 \end{bmatrix}, \tag{76}$$

for $i = 0, \ldots, k$, where ${}_{i}a_j = 0$ if $j < 0$. Then, the system (73) is equivalent to

$$ {}_{k}\mathbf{H}_{n-1,n} \cdot Q = 0, \tag{77}$$

and the system (74) is equivalent to

$$ {}_k\mathbf{G}_{m,n} \cdot Q + P = 0, \tag{78} $$

where $Q \in {}_k V_n$ and $P \in {}_k V_m$.

Clearly, the polynomials $P(x, y) \in {}_k BP_m$ and $Q(x, y) \in {}_k BP_n$ satisfy the order condition (72) if and only if the corresponding vectors $P = Vec_{k,m}(P(x, y))$ and $Q = Vec_{k,n}(Q(x, y))$ satisfy equations (77) and (78). Thus, there is a one to one correspondence between modular Padé forms and solutions to a block Hankel systems which can be stated as

Lemma 27. $P(x, y)/Q(x, y)$ is a modular Padé (k, m, n)-form for $A(x, y)$ if and only if corresponding vectors P and Q are solutions of systems (77) and (78). ■

Solutions of (77) and (78) are uniquely determined by solutions of the system (77) alone (i.e. any solution of (77) can be substituted in (78) to calculate the vector P). This observation together with Lemma 26., gives a procedure for the characterization of all modular Padé (k, m, n)-forms for $A(x, y)$. First, the family of solutions Q to the system (77) is determined, and then the family of solutions P is given by solving system (78) for P. Then $P(x, y)/Q(x, y)$, such that $P(x, y) = Pol_{k,n}\,(P)$ and $Q(x, y) = Pol_{k,m}\,(Q)$, are all modular Padé (k, m, n)-forms for $A(x, y)$.

General results from the previous sections are applied to obtain solution of equation (77), which as it was shown in Corollary 24 can be expressed as a linear combination of shifts of solutions of a smaller system. It will be shown that the solution P corresponding to equation (78) can be expressed as the same linear combination of the same shifts of solutions P corresponding to solutions of these smaller systems.

It should be clear, from the definition of matrices ${}_k\mathbf{G}_{m,n}$, that if $Q \in {}_{k-i}V_n$, $0 \le i \le k$, then

$$ {}_k\mathbf{G}_{m,n} \cdot (x^i \cdot Q) = x^i \cdot ({}_{k-i}\mathbf{G}_{m,n} \cdot Q), \tag{79} $$

and also, that if $Q \in {}_k V_n$, then

$$ {}_k\mathbf{G}_{m,n} \cdot (\underline{x}^i \cdot Q) = \underline{x}^i \cdot ({}_k\mathbf{G}_{m,n} \cdot Q). \tag{80} $$

Solutions involving shifts y^i and $\bar{y}^i$ are more complex, and are addressed in

Lemma 28. If $Q \in {}_k V_{n-\gamma}$, for some γ, $0 \le \gamma \le n$, is such that $Q \in {}_k S_{n-1+\gamma, n-\gamma}$, then

$$ {}_k\mathbf{G}_{m,n} \cdot \bar{y}^{\gamma-l} \cdot y^l \cdot Q = \bar{y}^{\gamma-l} \cdot y^l \cdot {}_k\mathbf{G}_{m-\gamma, n-\gamma} Q, \tag{81} $$

for $0 \le l \le \gamma$.

Proof: Let

$$Q = [{}_kQ, \cdots, {}_0Q], \quad {}_iQ = [{}_iq_{n-\gamma}, \cdots, {}_iq_0], \; i = 0, \ldots, k,$$

and define

$$P = {}_k\mathbf{G}_{m-\gamma, n-\gamma} \cdot Q,$$

where

$$P = [{}_kP, \cdots, {}_0P], \quad {}_iP = [{}_ip_{m-\gamma}, \cdots, {}_ip_0], \; i = 0, \ldots, k.$$

If we set

$$P'' = \bar{y}^{\gamma-l} \cdot y^l \cdot P,$$

where

$$P'' = [{}_kP'', \cdots, {}_0P''], \; {}_iP'' = [{}_ip''_m, \cdots, {}_ip''_0], \; i = 0, \ldots, k,$$

then the r.h.s. of equality (81) is equal to P''.

Similarly, define

$$Q' = \bar{y}^{\gamma-l} \cdot y^l \cdot Q,$$

where

$$Q' = [{}_kQ', \cdots, {}_0Q], \quad {}_iQ = [{}_iq'_n, \cdots, {}_iq'_0], \; i = 0, \ldots, k.$$

Then the l.h.s. of equality (81) become

$$P' = {}_k\mathbf{G}_{m,n} \cdot Q',$$

where

$$P' = [{}_kP', \cdots, {}_0P'], \quad {}_iP' = [{}_ip'_m, \cdots, {}_ip'_0], \; i = 0, \ldots, k.$$

It will be shown that $P' = P''$, by proving that ${}_{i_0}p'_{j_0} = {}_{i_0}p''_{j_0}$, for $0 \le i_0 \le k$ and $0 \le j_0 \le m$. For $0 \le i_0 \le k$, $0 \le j_0 \le j - \gamma$

$$ {}_{i_0}p_{j_0} = \sum_{i=0}^{k} \sum_{j=0}^{n-\gamma} {}_{i_0-i}a_{j_o-j}\, {}_iq_j . $$

Thus, for $0 \le i_0 \le k$, $0 \le j_0 \le m$

$$ {}_{i_0}p''_{j_0} = \begin{cases} {}_{i_0}p_{j_0-l}, & l \le j_0 \le m - \gamma - l, \\ \\ 0, & otherwise. \end{cases} $$

Then, for $0 \le i_0 \le k$,

$$ {}_{i_0}p''_{j_0} = \begin{cases} \sum_{i=0}^{k} \sum_{j=l}^{n-\gamma+1} {}_{i_0-i}a_{j_0-j}\, {}_iq_{j-l}, & l \le j_0 \le m - \gamma + l, \\ \\ 0, & 0 \le j_0 < l, \quad or \quad m - \gamma + l + 1 \le j_0 \le m. \end{cases} $$

On the other hand, for $0 \le i_0 \le k$, $0 \le j_0 \le m$,

$$ {}_{i_0}p'_{j_0} = \sum_{i=0}^{k} \sum_{j=0}^{n} {}_{i_0-i}a_{j_0-j} \cdot {}_iq'_j , $$

where

$$ {}_iq'_j = \begin{cases} {}_iq_{j-l}, & l \le j \le n - \gamma + l, \\ \\ 0, & otherwise. \end{cases} $$

Therefore, for $0 \le i_0 \le k$, $0 \le j_0 \le m$,

$$ {}_{i_0}p'_{j_0} = \begin{cases} \sum_{i=0}^{k} \sum_{j=l}^{n-\gamma+l} {}_{i_0-i}a_{j_0-j}\, {}_iq_{j-l}, & l \le j_0 \le m, \\ \\ 0, & 0 \le j_0 < l. \end{cases} $$

It remains to show that for $0 \le i_0 \le k$, $m - \gamma + l + 1 \le j_0 \le m$

$$\sum_{i=0}^{k} \sum_{j=l}^{n-\gamma+l} {}_{i_0-i}a_{j_0-j\,i}\, q_{j-l} = 0.$$

From the assumption that $Q \in {}_kS_{n-1+\gamma,n-\gamma}$, it follows that for $0 \le i_0 \le k$, $m - \gamma + 1 \le j_0 \le m + n$,

$$\sum_{i=0}^{k} \sum_{j=0}^{n-\gamma} {}_{i_0-i}a_{j_0-j\,i}\, q_j = 0,$$

which is equivalent to

$$\sum_{i=0}^{k} \sum_{j=l}^{n-\gamma+l} {}_{i_0-i}a_{j_0-j\,i}\, q_{j+l} = 0,$$

for $0 \le i_0 \le k$ and $m - \gamma + 1 + l \le j_0 \le m + n + l$. ■

Definition 29. The power series $A(x, y)$ is (i, m, n)-maximal if the matrix ${}_k\mathbf{H}_{n-1,n}$ is i-maximal. ■

By definition, the matrix ${}_k\mathbf{H}_{n-1,n}$ for a $(0, m, n)$-maximal power series, is 0-maximal. Thus, by Corollary 12, there exists a single fundamental solution $Q*$. Let $P* = - {}_k\mathbf{G}_{m,n}\, Q*$ be the corresponding solution to the system (78), and let $P*(x, y)/Q*(x, y)$ be corresponding modular Padé (k, m, n)-form.

The next theorem shows that if $A(x, y)$ is $(0, m, n)$-maximal, then all modular Padé (k, m, n)-forms can be characterized in terms of a single modular Padé (k, m, n)-form.

Theorem 30. All modular Padé (k, m, n)-forms for a $(0, m, n)$-maximal power series $A(x, y)$ are of the form $P(x, y)$, $Q(x, y)$, where

$$P(x, y) = (U(x)P*(x, y)) \mathit{mod}\ x^{k+1}$$

$$Q(x, y) = (U(x)Q*(x, y)) \mathit{mod}\ x^{k+1},$$

and $U(x)$ is an arbitrary polynomial in x.

Proof: Let $P(x, y) = Pol_{k,m}(P)$ and $Q(x, y) = Pol_{k,n}(Q)$ where P and Q are solutions to (77) and (78). By Corollary 12

$$Q = \sum_{i=0}^{k} \alpha_i \underline{x}^i \cdot Q*,$$

for some $\alpha_0, \cdots, \alpha_k \in D$. Thus

$$P = -\,{}_k\mathbf{G}_{m,n} \cdot Q = \sum_{i=0}^{k} \alpha_i ({}_k\mathbf{G}_{m,n} \underline{x}^i \cdot Q^*)).$$

But, from (80),

$${}_k\mathbf{G}_{m,n} \cdot (\underline{x}^i \cdot Q^*) = \underline{x}^i \cdot ({}_k\mathbf{G}_{m,n} \cdot Q^*),$$

and therefore,

$$P = \sum_{i=0}^{k} \alpha_i \underline{x}^i \cdot P^*.$$

Let $U(x) = \sum_{i=0}^{k} \alpha_i x^i$. From Lemma 25, it follows that

$$Pol_{k,m}\,(P) = (U(x)\;P^*(x,y))\;mod\;\;x^{k+1} \text{ and}$$

$$Pol_{k,n}\,(Q) = (U(x)\;Q^*(x,y))\;mod\;\;x^{k+1}.$$

■

The above theorem characterizes modular Padé (k, m, n)-forms in a special case, when $A(x, y)$ is $(0, m, n)$-maximal. Full characterization is given below for the general case.

Given $A(x, y)$, k, m and n, let $Q^{(i)}$, $i = 0, \ldots, k$, be the fundamental solutions as given in Corollary 24. Thus, the $Q^{(i)}$'s are solutions to a system

$${}_i\mathbf{H}_{n-1+\gamma_i, n-\gamma_i} \cdot Q = 0.$$

Therefore, they are also solutions to a smaller system of the form (77), i.e.,

$${}_i\mathbf{H}_{n-1-\gamma_i, n-\gamma_i} \cdot Q = 0.$$

For $i = 0, \ldots, k$, let $P^{(i)}$ be defined as corresponding solution of the form (78), i.e.,

$$P^{(i)} = -\,{}_i\mathbf{G}_{m-\gamma_i, n-\gamma_i} \cdot Q.$$

For $i = 0, \ldots, k$, let

$$P^{(i)}(x, y) = Pol_{i, m-\gamma_i}(P^{(i)})$$

and

$$Q^{(i)}(x,y) = Pol_{i,n-\gamma_i}(Q^{(i)}).$$

Thus, by Lemma 27, $P^{(i)}(x,y)/Q^{(i)}(x,y)$ are modular Padé $(i, m-\gamma_i, n-\gamma_i)$-forms for $A(x,y)$ where $i = 0, \ldots, k$.

The following theorem shows that any modular Padé (k,m,n)-form can be expressed as a function of modular Padé $(i, m-\gamma_i, n-\gamma_i)$-forms, $i = 0, \ldots, k$.

Theorem 31. All modular Padé (k,m,n)-forms for $A(x,y)$ are of the form $P(x,y)/Q(x,y)$, where

$$P(x,y) = \sum_{i=0}^{k} \sum_{j=0}^{\gamma_i} \alpha_{i,j} x^{k-i} y^j P^{(i)}(x,y), \quad Q(x,y) = \sum_{i=0}^{k} \sum_{j=0}^{\gamma_i} \alpha_{i,j} x^{k-i} y^j \, Q^{(i)}(x,y)$$

and $\alpha_{i,j}$, $i = 0, \ldots, k$, $j = 0, \ldots, \gamma_i$ are arbitrary scalars.

Proof: Let $P(x,y) = Pol_{k,m}(P)$ and $Q(x,y) = Pol_{k,n}(Q)$, where P and Q are arbitrary solutions to (77) and (78). By Corollary 24,

$$Q = \sum_{i=0}^{k} \sum_{j=0}^{\gamma_i} \alpha_{i,j} x^{k-i} \cdot \bar{y}^{\gamma_i - j} \cdot y^j \cdot Q^{(i)}.$$

Thus,

$$P = -\sum_{i=0}^{k} \sum_{j=0}^{\gamma_i} \alpha_{i,j}\, {}_k\mathbf{G}_{m,n} \cdot (x^{k-i} \cdot \bar{y}^{\gamma_i - j} \cdot y^j \cdot Q^{(i)}).$$

From (79), it follows that

$${}_k\mathbf{G}_{m,n} \cdot (x^{k-i} \cdot \bar{y}^{\gamma_i - j} \cdot y^j \cdot Q^{(i)}) = x^{k-i} \cdot {}_i\mathbf{G}_{m,n} \cdot (\bar{y}^{\gamma_i - j} \cdot y^j \cdot Q^{(i)}).$$

Since $Q^{(i)} \in {}_kS_{n-1+\gamma_i, n-\gamma_i}$, $i = 0, \ldots, k$, by Lemma 28, it follows that

$${}_i\mathbf{G}_{m,n} \bar{y}^{\gamma_i - j} \cdot y^j \cdot Q^{(i)} = \bar{y}^{\gamma_i - j} \cdot y^j \cdot {}_i\mathbf{G}_{m-\gamma_i, n-\gamma_i}\, Q^{(i)}.$$

But, ${}_iG_{m-\gamma_i, n-\gamma_i}\, Q^{(i)} = -\,P^{(i)}$, and consequently,

$$P = \sum_{i=0}^{k} \sum_{j=0}^{\gamma_i} \alpha_{i,j} x^{k-i} \cdot \bar{y}^{\gamma_i - j} \cdot y^j \cdot P^{(i)}.$$

An application of Lemma 25 to the vectors P and Q gives

$$Pol_{k,m}(P) = \sum_{i=0}^{k} \sum_{j=0}^{\gamma_i} \alpha_{i,j} x^{k-i} y^j Pol_{i,m-\gamma_i}(P^{(i)})$$

and

$$Pol_{k,n}(Q) = \sum_{i=0}^{k} \sum_{j=0}^{\gamma_i} \alpha_{i,j} x^{k-i} y^j Pol_{i,n-\gamma_i}(Q^{(i)}).$$

Thus,

$$P(x, y) = \sum_{i=0}^{k} \sum_{j=0}^{\gamma_i} \alpha_{i,j} x^{k-i} y^j P^{(i)}(x, y)$$

and

$$Q(x, y) = \sum_{i=0}^{k} \sum_{j=0}^{\gamma_i} \alpha_{i,j} x^{k-i} y^j Q^{(i)}(x, y).$$

■

8. Concluding Remarks

The results presented in this report fully describe the nature of the solutions to the triangular block Hankel system (2). The existence and non-uniqeness of the solutions are discussed. Crucial to the results is the notion of fundamental solutions; fundamental solutions are characterized in terms of representatives of the cosets of a certain vector space of quotients. Theorem 11 describes the basis for the space of solutions when the matrix satisfies a certain condition of maximality. When the matrix is not maximal, the general format of the solution is given by Corollary 24.

The importance of this result comes from the relationship between the block Hankel system (2) and the modular Padé forms defined in section 7. It is established that there is a one-to-one correspondence between modular Padé forms and the solutions of a specific case of (2), namely the case where $s = t - 1$. The results presented provide a theortical framework for the effective computation of modular Padé forms.

In a more general setting, multivariate Padé approximants can be defined as follows. For a given integer d, let I be a set of multi-indices given by $I = \{(i_1, \ldots, i_d) \colon i_j \in Z^+,\ j = 1, \ldots, d\}$, where Z^+ denotes the non-negative integers. For $x \in R^d$ and $i \in I$, let x^i denote $x_1^{i_1} \ldots x_d^{i_d}$. The definition of multivariate Padé approximant involves choosing subsets I_P, I_Q and I_E of I such that, for a given multivariate power series $A(x) = \sum_{i \in I} a_i x^i$, two polynomials $P(x) = \sum_{i \in I_P} p_i x^i$ and $Q(x) = \sum_{i \in I_Q} q_i x^i$ can be found satisfying an order condition

$$A(x)Q(x) - P(x) = \sum_{i \in I \setminus I_E} r_i x^i. \tag{82}$$

The choice of the index sets I_P, I_Q and I_E of I is governed by many criteria. These criteria can be derived by postulating that approximants have some convenient properties (e.g., invariance under certain transformations), or can be imposed directly on the index sets (e.g., symmetry). One aspect, not of least importance, is the ease of effective computation of the approximants. For a specific choice of the index sets I_P, I_Q and I_E of I, the order condition (82) gives rise to a system of linear equations. Computation of all approximants satisfying the order condition is equivalent to finding all the solutions of this system of linear equations.

For the bivariate case (i.e., $d = 2$), Chisholm [5,6,7,12] approximants are defined by

$$I_P = I_Q = \{(i_1, i_2)\colon 0 \le i_1, i_2 \le m\}, \text{ and}$$

$$I_E = \{(i_1, i_2)\colon 0 \le i_1 + i_2 \le 2m\};$$

whereas, for Cuyt approximants [8,9,10,11]

$$I_P = \{(i_1, i_2)\colon mn \le i_1 + i_2 \le mn + m\},$$

$$I_Q = \{(i_1, i_2)\colon mn \le i_1 + i_2 \le mn + n\}, \text{ and}$$

$$I_E = \{(i_1, i_2)\colon mn \le i_1 + i_2 \le mn + m + n\}.$$

The index sets for modular Padé forms of section 7 are defined by

$$I_P = \{(i_1, i_2)\colon 0 \le i_1 \le k,\ 0 \le i_2 \le m\},$$

$$I_Q = \{(i_1, i_2)\colon 0 \le i_1 \le k,\ 0 \le i_2 \le n\}, \text{ and}$$

$$I_E = \{(i_1, i_2)\colon 0 \le i_1 \le k,\ 0 \le i_2 \le m + n\}.$$

The corresponding system of linear equations for modular Padé forms is the triangular block Hankel system ${}_k\mathbf{H}_{n-1,n}$ defined by (4).

The following question can be posed, which is a natural generalization of the results given in this report. Let a generalized (k,m,n)-modular Padé form for a d-variate power series be defined by the order condition (82), where the index sets are

$$I_P = \{(i_1, \ldots, i_d)\colon 0 \le i_j \le k,\ j = 1, \ldots, d-1,\ 0 \le i_d \le m\},$$

$$I_Q = \{(i_1, \ldots, i_d)\colon 0 \le i_j \le k,\ j = 1, \ldots, d-1,\ 0 \le i_d \le n\}, \text{ and}$$

$$I_E = \{(i_1, \ldots, i_d)\colon 0 \le i_j \le k,\ j = 1, \ldots, d-1,\ 0 \le i_d \le m + n\}.$$

The order condition (82), in this case, gives rise to a generalized block Hankel system. To what extent can the results of this report be replicated? It seems that the success of a such a generalization would depend on whether an equivalent of Lemma 13 can be formulated.

By rearranging rows and columns, the triangular block Hankel system (4) can be written as a full Hankel system with triangular matrices as components. From this perspective, the problem of solving this system becomes one of obtaining a (t-1,t)-Padé approximant for a univariate power series with triangular matrix coefficients. So, modular bivariate Padé approximants are a special case of univariate matrix Padé approximants. It remains to determine how the characterization of solutions of the matrix Padé problem [2,3,14] corresponds to the results in this report.

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