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THE TOPOLOGY OF SUBSTITUTION TILING DYNAMICAL SYSTEMS

by

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled **The Topology of Substitution System Dynamical Systems** submitted by **Stephen Patrick Sullivan** in partial fulfillment of the requirements for the degree of **Master of Science** in Mathematics.

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To my parents, Henry and Moira.

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ABSTRACT

In this thesis we look at the topology of substitution tiling dynamical systems $(\mathfrak{T}, \omega)$. We do this by looking at two systems that are topologically conjugate to $(\mathfrak{T}, \omega)$. One system consists of an inverse limit space that looks at each tiling $\mathcal{T}$ by considering how the origin lies in the tile containing it for each tiling $\omega^{-n}(\mathcal{T})$. The other system looks at $\mathcal{T}$ by considering the relative location of the origin in the tile containing it for $\mathcal{T}$, and then uses a sequence of choices to build up a tiling around the origin.

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0.1 Introduction

The concept of a tiling is something that is familiar to everyone. A pattern bricks forming a wall, or a design of tiles covering a floor, are a couple of common examples of tilings. They can range from simple designs, such as periodically repeating the same pattern, to very complicated designs, where there may be no obvious patterns. Recent discoveries in solid state materials and within mathematics have reawakened an interest in the study of tilings, especially in two and three dimensional aperiodic tilings. These aperiodic tilings can appear to very disorganised, while they will in fact demonstrate an amazing amount of symmetry and order.

One of the simplest ways to generate tilings that will have this aperiodic order is through use of a substition process, and this will be explained below. This method will start with a finite set of prototiles, then use a rule to inflate these prototiles and then subdivide into tiles. Iterating this process will create aperiodic tilings. This process has been used for a long time in the study of one and two dimensional tilings, and there are many well known, and interesting tilings that this process can create in one and two dimensions.

In studying tilings that are not strictly periodic (repeating in various spatial directions) new mathematical tools have had to be created. One of the most commonly used used ideas is to combine whole translation classes of tilings into a single topological space. More generally, we can combine classes of of locally indistinguishable tilings (meaning tilings that only display the same types of patterns) into a single topological space, see Radin and Wolff [5]. These spaces will often carry the stucture of a topological dynamical system, with the action of the dynamical system coming from the translation group, or in the case of substitutions, it may come from the substitution itself. The relevance of this connection is demonstrated by the relationships that emerge between fundamental concepts of dynamics and important geometrical concepts of tilings, Theorems 1.2 and 1.3 provide an example of this.

The topology that arises in tiling dynamics turns out to be surprisingly subtle. It is interesting enough to have attracted the attention of mathematicians in a variety of areas, such as non-commutative geometry, see Connes [9], operator algebras, see Putnam and Anderson [6], and diffraction theory, see Schlottmann [10]. What makes this topology interesting is the way that it combines two notions of closeness. Each point in the topological space consists of one tiling. The topology combines the standard notion of closeness, a small shift in the real space where the tilings live, with a non-standard notion of closeness which measures the extent to which the tilings match. In the case of substitution tilings this is further complicated by the fact that the substitution treats these notions of closeness differently, and will be expansive on the standard closeness, while being contractive on the the non-standard closeness.

It is well known that this topology will roughly look like a product space of a compact space and a totally disconnected space. However, discussions of these ideas are often widely scattered and hard to reconcile. It is the purpose of this thesis to look at some of the different ways of producing a topological dynamical system, given a substitution tiling, and show that these systems are essentially the same, i.e., show that they are topologically conjugate. Each of these systems has a slightly different look and feel than the others do, providing new insights into the nature of the topology. Presenting them together allows us to also examine the connections between these systems.

All of the ideas here work in any finite dimension. However, for simplicity, we have focused mainly on one dimensional tilings. Nothing is lost by doing this. The one dimensional situtation shows the full complexity of the topology, while avoiding the difficulties that can arise when trying to visualize exactly what these systems look like in higher dimensions. The discussion has been kept relatively informal. All of the key ideas are presented, but there are not too many truly formal proofs, which do not add anything in the way of understanding, but can even serve to obscure it. In the final section we give an idea of how the theory generalizes to higher dimensions.

Chapter 1

Substitution Tiling Dynamical Systems

1.1 Dynamical Systems

Let $\mathfrak{X}$ be a compact metric space and let $\phi : \mathfrak{X} \longrightarrow \mathfrak{X}$ be a continuous map. The pair $(\mathfrak{X}, \phi)$ is a *topological dynamical system.* For any $x \in \mathfrak{X}$ we have the *forward orbit* of x, which is the set $\{\phi^n(x) \mid n \in \mathbb{Z}_{\geq 0}\}$. If our map ϕ is a homeomorphism we will have the *(full) orbit* of x, which is the set $\{\phi^n(x) \mid n \in \mathbb{Z}\}$.

Let's consider an example. Let $\mathfrak{X}_1$ be $\mathbb{R}/\mathbb{Z}$ and give it the topology induced from $\mathbb{R}$. This space is homeomorphic to the space created when the interval $[0,1]$ is bent into a circle, with the points 0 and 1 identified. The topology on the circle will have a basis of all open intervals on the circle, and the space will be compact. See Figure 1.1. For the metric, let $d(x,y) = \min\{|x-y|, |1+x-y|, |1+y-x|\}$, which is just the shortest distance from x to y, travelling along the circle. For our mapping ϕ_1, fix any $z \in \mathbb{R}$ and let $\phi_1(x) \equiv z + x \pmod 1$ for all $x \in \mathfrak{X}_1$. The map ϕ_1 is a homeomorphism, so $(\mathfrak{X}_1, \phi_1)$ is a topological dynamical system and for any $x \in \mathfrak{X}_1$, the orbit is $\{nz + x \mid n \in \mathbb{Z}\}$.

An important subclass of topological dynamical systems is *symbolic dy-*

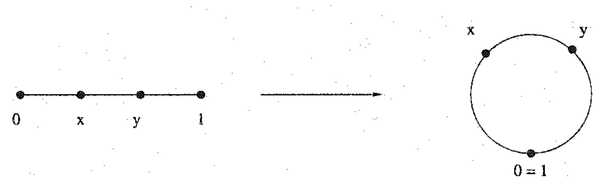

Figure 1.1: The transformation of the interval [0,1] into a space homeomorphic to the space $\mathfrak{X}_1$. On the circle, both the of intervals (x,y) and (y,x) will be open sets.

namical systems. These are spaces that are made up of symbol sequences from a finite alphabet of symbols. In this thesis we will devote much of our attention to this subclass of topological dynamical systems, and we will now describe their construction. A more thorough description of the space we are about to describe can be found in Lind and Marcus [1], pp. 5-8.

Let $\mathcal{A}$ be a nonempty, finite set of symbols, and give it the discrete topology. Let

$$\mathfrak{S} = \mathfrak{S}(\mathcal{A}) := \mathcal{A}^{\mathbb{Z}} = \{(x_n)_{n=-\infty}^{\infty} \mid x_n \in \mathcal{A}\ \forall n\} \ ,$$

and give $\mathfrak{S}$ the product topology. The cylinder sets

$$[a_m, \ldots, a_n]_{i=m}^{n} := \{(x_k)_{k=-\infty}^{\infty} \mid x_k = a_k, k = m, \ldots, n\}$$

form a basis for the topology on $\mathfrak{S}$. We define $\sigma : \mathfrak{S} \longrightarrow \mathfrak{S}$ by $(\sigma(x))_n = x_{n+1}$. In other words, if x is the sequence :

$$\ldots, x_{-1}, x_0, x_1, \ldots \ ,$$

then $\sigma(x)$ is the sequence x shifted left by one symbol :

$$\ldots, \sigma(x)_{-1} = x_0, \sigma(x)_0 = x_1, \sigma(x)_1 = x_2, \ldots \ .$$

Now $\sigma([a_m, \ldots, a_n]_{i=m}^{n}) = [a_m, \ldots, a_n]_{i=m-1}^{n-1}$, and so σ is a homeomorphism. We also have a metric on $\mathfrak{S}$. For $x, y \in \mathfrak{S}$, $x \neq y$, define

$$N(x, y) := \min\{n \geq 0 \mid x_n \neq y_n \text{ or } x_{-n} \neq y_{-n}\} \ ,$$

and let our metric be

$$d(x, y) := \begin{cases} (1/2)^{N(x,y)} & \text{if } x \neq y \\ 0 & \text{if } x = y \ . \end{cases}$$

Our dynamical system $(\mathfrak{S}, \sigma)$ is called the *full shift* on $\mathcal{A}$. We call σ the *shift map*, and it is the same map for all shift systems. For the remainder of this discussion the symbol σ will always refer to the shift map on whichever system we are dealing with. The elements $a \in \mathcal{A}$ are the *letters* of the *alphabet* $\mathcal{A}$. The set of *words* of the alphabet $\mathcal{A}$ is

$$\mathcal{A}^* := \bigcup_{n=0}^{\infty} \mathcal{A}^n \ .$$

We also have a map $\ell : \mathcal{A}^* \longrightarrow \mathbb{Z}_+$ given by

$$\ell|_{\mathcal{A}^n} = n \ ,$$

which represents the length of each word of $\mathcal{A}$. For a given word t of $\mathcal{A}$, the words of $\mathcal{A}$ that appear in t are the *subwords* of t. By definition, the words of $\mathcal{A}$ must be finite. However, we may also regard the sequences of $\mathfrak{S}$ as *biinfinite* words of $\mathcal{A}$. It will then make sense for us to the consider the subwords of any particular biinfinite word x.

For an example of the full shift on an alphabet $\mathcal{A}$, consider the case when $\mathcal{A} = \{A, B\}$. Then let $\mathfrak{S}_1$ be the set of all biinfinite sequences consisting only of A's and B's. If we let x be the element of $\mathfrak{S}_1$ defined by $x_{4n} = A, x_{4_n+1} = B, x_{4_n+2} = B, x_{4n+3} = B$, for all n, we see that

$$\begin{array}{rcl} & & \qquad\qquad\qquad\qquad\qquad\quad \downarrow^{0^{\text{th}}\text{ term}} \\ x & = & \ldots A\ B\ B\ B\ A\ B\ B\ B\ A\ B\ B\ B\ A\ B\ B\ B \ldots \\ \sigma(x) & = & \ldots B\ B\ B\ A\ B\ B\ B\ A\ B\ B\ B\ A\ B\ B\ B\ A \ldots \\ \sigma^2(x) & = & \ldots B\ B\ A\ B\ B\ B\ A\ B\ B\ B\ A\ B\ B\ B\ A\ B \ldots \\ \sigma^3(x) & = & \ldots B\ A\ B\ B\ B\ A\ B\ B\ B\ A\ B\ B\ B\ A\ B\ B \ldots . \end{array}$$

Now $\sigma^4(x) = x$, so x is periodic and we can see that the orbit of x is just made up of these four elements, x, $\sigma(x)$, $\sigma^2(x)$, and $\sigma^3(x)$. We also observe that $\{A, B\}$ is the set of all words of length one in $\mathcal{A}^*$, $\{AA, AB, BA, BB\}$ is the set of all words of length two in $\mathcal{A}^*$, and so on. Further, we can note that A is a subword of the word AA, but not of the word BB, and $ABBBA$ is a subword of x, while ABA is not.

Any $\phi^{\pm 1}$ invariant, closed subspace $\mathfrak{Y}$ of $\mathfrak{X}$ gives rise to a dynamical system $(\mathfrak{Y}, \phi)$. Such systems are called *subshifts* on $\mathcal{A}$. In our above example, the set of all elements of $\mathfrak{S}$ in which no two consecutive A's appear would be a subshift of $\mathfrak{S}$. This space is clearly $\sigma^{\pm 1}$ invariant and is easily seen to be closed due to the nature of the cylinder set basis on $\mathfrak{S}$. Here we have defined a subshift by choosing a set of words of $\mathfrak{S}$ that cannot occur in any word in that subshift. Any biinfinite word that does not contain any of these forbidden words will be contained in the subshift. It can be shown that any subshift of any shift space can be defined in this way.

Equivalently, we can create our subspaces of a shift space by defining a set of words that can appear in elements of the subspace. The subspace of $\mathfrak{S}_1$ that does not contain the word AA could alternatively be defined as the subspace of $\mathfrak{S}_1$ consisting of the elements that have only the words $\{A, B, AB, BA, BB, BBB, BBA, BAB, ABB, ABA, \ldots\}$ appearing in them. (All we have done to create this set is write down all of the words of each possible length that do not contain the word AA as a subword.) This is summed up in the following theorem.

Theorem 1.1. *1. Let $(\mathfrak{Y}, \phi)$ be a subshift of $(\mathfrak{X}, \phi)$. Then there is a set $\mathcal{F}$ of words of $\mathcal{A}$ so that $x = (x_n) \in \mathfrak{Y}$ if and only if no word of x lies in $\mathcal{F}$.*

2. Given any set $\mathcal{F}$ of words on $\mathcal{A}$, the set

$$\mathfrak{Y} = \mathfrak{Y}_{\mathcal{F}} := \{x \in \mathfrak{X} \mid \textit{no subword of } x \textit{ lies in } \mathcal{F}\}$$

is a subshift of $(\mathfrak{X}, \phi)$.

The proof of this is given by Lind and Marcus [1], pp. 9-11 and page 179.

A dynamical system is said to be *minimal* if for every $x \in \mathfrak{X}$, the orbit $\{\phi^n(x) \mid n \in \mathbb{Z}\}$ is dense. In our first example of a topological dynamical system, $(\mathfrak{X}_1, \phi_1)$ will be minimal if and only if the $z \in \mathbb{R}$ that is chosen to define ϕ_1 is irrational. Our example of a shift space is not minimal, since the element we looked at has only four elements in its orbit, and consequently its orbit is not dense. However, we shall see that there is a broad, and interesting, class of subshifts that is minimal.

A subset J of $\mathbb{Z}$ is *relatively dense* if there exists an $m \in \mathbb{Z}$ such that for any $n \in \mathbb{Z}$, there is a $j \in J$ such that $n < j < n + m$. This just means that there is a bound on how far we have to go in $\mathbb{Z}$ before we find an element of J. An element $x \in \mathfrak{X}$ will be said to be *almost periodic* if for all neighbourhoods V of x, the set $J(x, V) := \{n \in \mathbb{Z} \mid \phi^n(x) \in V\}$ is relatively dense. In other words, if we take any neighbourhood V of x, iterating our map ϕ on x will keep bringing us back into V, and there will be a bound on how many iterations are necessary to get back into this neighbourhood.

When we consider our shift spaces, an element $x \in \mathfrak{X}$ will be almost periodic if and only if every word of x repeats with bounded gaps. The element in our above example is clearly almost periodic, since it is in fact periodic, and every fourth iteration of σ returns us to the element x. So every word of x will repeat after four or fewer iterations of σ. The element $y = (y_n)$ in this same shift space, where $y_0 = A$, and $y_n = B$ for all $n \neq 0$, is not almost periodic, since the word A only occurs once.

We have a couple of theorems that relate minimality and almost periodicity.

Theorem 1.2. *If a space $\mathfrak{X}$ is minimal, then every $x \in \mathfrak{X}$ is almost periodic.*

Theorem 1.3. *If $x \in \mathfrak{X}$ is almost periodic, then the dynamical system $(\overline{\{\phi^n(x) \mid n \in \mathbb{Z}\}}, \phi)$ is minimal.*

These are proven by Furstenberg [2], pp. 29-30.

Two dynamical systems $(\mathfrak{X}, \phi)$ and $(\mathfrak{X}', \phi')$ are *topologically conjugate* if there exists a homeomorphism $\gamma : \mathfrak{X} \longrightarrow \mathfrak{X}'$ which satisfies

$$\gamma(\phi(x)) = \phi'(\gamma(x))$$

for all $x \in \mathfrak{X}$.

1.2 Substitution Systems

We start with an alphabet $\mathcal{A} = \{a_1, \dots, a_m\}$ and form the full shift $(\mathfrak{S}, \sigma)$ on $\mathcal{A}$, together with the usual product topology. A *substitution* on $\mathcal{A}$ is a mapping $\omega : \mathcal{A} \longrightarrow \mathcal{A}^*$ such that for at least one $a \in \mathcal{A}$, $\ell(\omega(a)) \geq 2$. The map ω induces a homomorphism (of semigroups) $\omega : \mathcal{A}^* \longrightarrow \mathcal{A}^*$ by

$$\omega(c_1 \dots c_k) = \omega(c_1) \dots \omega(c_k).$$

The *substitution matrix* M of ω is the $\text{card}(\mathcal{A}) \times \text{card}(\mathcal{A})$ matrix where M_{ab} is the number of a's in the word $\omega(b)$, for $a, b \in \mathcal{A}$. The matrix M is *positive* if $M_{ij} \geq 0$ for all i, j. The matrix M is *primitive* if there exists a $k > 0$ such that $(M^k)_{ij} > 0$ for all i, j. We will say that the substitution ω is *primitive* if the substitution matrix M of ω is a primitive matrix. We also observe that the substitution matrix for ω^k is M^k.

It is clear that ω being primitive is equivalent to saying that there exists an integer $k > 0$ such that for any $a \in \mathcal{A}$, every letter $b \in \mathcal{A}$ will occur in the word $\omega^k(a)$. Consequently, there is a $k' > 0$ and $a_0 \in \mathcal{A}$ such that a_0 occurs in the "interior" of $\omega^{k'}(a_0)$ (meaning this particular a_0 is not the first or last letter of $\omega^{k'}(a_0)$). So for some $c_{-n_1'}, \dots, c_{-1}, c_1, \dots, c_{n_1} \in \mathcal{A}$, we have

$$\omega^{k'}(a_0) = c_{-n_1'} \dots c_{-1} \; a_0 \; c_1 \dots c_{n_1} \; .$$

We replace ω with $\omega^{k'}$ and M with $M^{k'}$, so now

$$\omega(a_0) = c_{-n_1'} \dots c_{-1} \; a_0 \; c_1 \dots c_{n_1} \; .$$

We may now use ω and a_0 to form a biinfinite word u of $\mathfrak{S}$ that is stable under the substitution ω. (By stable we mean that we will have some way of defining ω on u so that $\omega(u) = u$. In general, we will not define a map $\omega : \mathfrak{S} \longrightarrow \mathfrak{S}$, we will only define an action for ω on the word u.) We start with the word a_0, and we create the larger word $\omega(a_0)$. We then fix one of the copies of a_0 that occurs in the interior of $\omega(a_0)$ (there may be more than one). Then we build a biinfinite word around this a_0, by continuing to fix its location in the same spot through infinitely many iterations of ω. The biinfinite word of $\mathcal{A}$ that results from this procedure is u, and u is not changed when ω acts on it in this way. So, for some $c_{-n_2'}, \ldots, c_{-n_1'-1}, c_{n_1+1}, \ldots, c_{n_2} \in \mathcal{A}$, we have

$$\begin{array}{rcc} a_0 & = & a_0 \\ \omega(a_0) & = & c_{-n_1'} \ldots c_{-1}\; a_0\; c_1 \ldots c_{n_1} \\ \omega^2(a_0) & = & c_{-n_2'} \ldots c_{-n_1'} \ldots c_{-1}\; a_0\; c_1 \ldots c_{n_1} \ldots c_{n_2} \\ \vdots & & \vdots \end{array}.$$

We see that each word $\omega^n(a_0)$ contains $\omega^{n-1}(a_0)$ as a subword with the generating a_0 fixed in the same location. So this process will generate a stable biinfinite word $u \in \mathfrak{S}$. It follows from the primitivity of the system that it is also almost periodic (since this will put a lower bound on how often each letter, and ultimately each word, will appear) and as a consequence, we have the following result.

Theorem 1.4. *$(\overline{\{\sigma^k(u)|k \in \mathbb{Z}\}}, \sigma)$ is a minimal dynamical system.*

Proven by Queffélec [3], pp. 71-72.

Let's consider an example. Let $\mathcal{A}_1 = \{A, B\}$ and let

$$\begin{array}{rrcl} \omega_1 : & \mathcal{A} & \longrightarrow & \mathcal{A}^* \\ & A & \longrightarrow & ABA \\ & B & \longrightarrow & A \end{array}.$$

This is known as the silver mean substitution. Our substituion matrix is $M_1 = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$. We see that our substitution is primtive since $M_1^2 = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$. We also see that primitivity ensures that each letter of $\mathcal{A}_1$ appears in both $\omega_1^2(A) = ABAAABA$, and $\omega_1^2(B) = ABA$. To construct a biinfinite word that is stable under our substitution, we use the fact that B appears in the interior of $\omega_1^2(B)$. So we *replace* ω_1 with ω_1^2, and we construct our biinfinite word

$$\begin{array}{rcc} B & = & B \\ \omega_1(B) & = & ABA \\ \omega_1^2(B) & = & ABAAABAABAABAAABA \\ \vdots & & \vdots \end{array}.$$

Let's call this word u_1, so $(\overline{\{\sigma^k(u_1)|k \in \mathbb{Z}\}}, \sigma)$ is a minimal dynamical system.

We notice here that we could have chosen to build our biinfinite word in several differnt ways. For instance, we could have noticed that, using our original ω_1, $\omega_1^3(A) = ABAAABAABAABAAABA$, and chosen to fix the third A in this word, replacing ω_1 with ω_1^3. Constructing the word that results from these choices give us

$$\begin{array}{rcl} A & = & A \\ \omega_1(A) & = & ABAAABAABAABAAABA \\ \omega_1^2(A) & = & ABAA\ldots ABAAABAABAABAAABAABA\ldots ABA \\ \vdots & & \vdots \end{array}$$

and we can call this word u_1', so we have a minimal dynamical system $(\overline{\{\sigma^k(u_1')|k \in \mathbb{Z}\}}, \sigma)$. We can see that u_1 and u_1' are not the same word, but we do have the same types of patterns emerging inside of these words. Since our system is primitive, every letter of our alphabet appears in both words, and consequently they will generate the same dynamical system. This result applies to all dynamical systems that are constructed in this way, not just to this particular example. The dynamical systems generated by this process depend only upon the alphabet and the substitution on that alphabet.

Another example is given when $\mathcal{A}_2 = \{C, D, E, F\}$ and

$$\begin{array}{rccl} \omega_2: & \mathcal{A}_2 & \longrightarrow & \mathcal{A}_2^* \\ & C & \longrightarrow & DF \\ & D & \longrightarrow & CF \\ & E & \longrightarrow & CF \\ & F & \longrightarrow & E \end{array}$$

Then the substitution matrix is

$$M_2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \text{ and } M_2^5 = \begin{bmatrix} 3 & 4 & 4 & 1 \\ 2 & 1 & 1 & 2 \\ 3 & 3 & 3 & 2 \\ 5 & 5 & 5 & 3 \end{bmatrix},$$

so we know that our substitution is primitive. We notice that $\omega_2^3(E) = CFECF$, so if we replace ω_2 with ω_2^3 we can produce a bininfinite word by fixing this E, and we get

$$\begin{array}{rcl} E & = & E \\ \omega_2(E) & = & CFECF \\ \omega_2^2(E) & = & DFECFDFECFECFDFECFDFE \\ \vdots & & \vdots \end{array}$$

Call this word u_2, and $(\overline{\{\sigma^k(u_2)|k \in \mathbb{Z}\}}, \sigma)$ is a dynamical system.

This method of constructing these dynamical systems doesn't give much insight into what the elements in these dynamical systems look like. Fortunately, we can visualize these dynamical systems in a different way. These dynamical systems are subshifts of the full shift on their alphabet. Each of these subshifts can be defined as the set of elements in $\mathfrak{S}$ that only contain subwords of u. Equivalently, they could be defined as the elements that do not contain a certain word if and only if u does not contain that word.

Let's consider what this means for our two examples. First consider $(\overline{\{\sigma^k(u_1)|k \in \mathbb{Z}\}}, \sigma)$. The words A, and B both appear in u_1. We also see that AB, BA, and AA all appear in u_1. However, the word BB will never occur in u_1. We can see this because if it was to appear, it would have to be generated by some other word that does occur in u_1. But $\omega_1(B) = ABA$ and $\omega_1(A) = ABAAABA$, and no possible ordering of these words will produce the subword BB. Similarily, we see that the words AAA, AAB, ABA, and BAA can occur in u_1, but the words BBB, BBA, BAB, and ABB can not occur in u_1. We continue in this way, deciding which words of every possible length are and are not allowed to occur in u_1. Then, $\overline{\{\sigma^k(u_1)|k \in \mathbb{Z}\}}$ is the set of biinfinite words of $\mathcal{A}_1$ that only contain the subwords $\{A, B, AB, BA, AA, AAA, AAB, ABA, BAA, \dots\}$, (just list all of the legal words of length one, then all of the legal words of length two, and so on), or equivalently, is the set of biinfinite words of $\mathcal{A}_1$ that do not contain any of the words $\{BB, BBB, BBA, BAB, ABB, \dots\}$ (just list all of the illegal words of length one, then all of the illegal words of length two, and so on). Similarily, we see that $\overline{\{\sigma^k(u_2)|k \in \mathbb{Z}\}}$ is the set of all biinfinite words of $\mathcal{A}_2$ that only contain the subwords $\{C, D, E, F, CF, DF, EC, FE, FD, \dots\}$, or equivalently, as the set of biinfinite words of $\mathcal{A}_2$ that do not contain any of the words $\{CC, CD, CE, DC, DD, DE, ED, EE, EF, FC, FF, \dots\}$. (We can note here that the word ED will be a legal subword of u_2 if and only if FC is also a legal subword. Further, we will only get the word ED appearing by letting ω_2 act on the word FC, since $\omega_2(FC) = EDF$, and we only get FC from $\omega_2(ED) = CFCF$. However, u_2 is generated by letting ω_2 repeatedly act on the words generated by just the letter E. So we will never generate either of the words ED or FC, which is why these two are not legal subwords of u_2.) We note that it is the almost periodicity of these systems that ensures that we can always construct these sets of words. If we want to know which words of a given length can and cannot occur in the word u, we only need to check if it occurs in any sufficiently large subword of u. How large of a subword we must look at will depend on the particular system and the length of word we are concerned about, see Queffélec [3] pp. 98-104 for details..

At this point we may also explain why it is that we look at the space $(\overline{\{\sigma^k(\alpha)|k \in \mathbb{Z}\}}, \sigma)$ instead of the space $(\{\sigma^k(\alpha)|k \in \mathbb{Z}\}, \sigma)$. It turns out that the closure of the shift space is a much larger space than just the shift space. To get an idea of how an element can belong to the closure, but not to the shift space itself, consider the following. Let's form the spaces $(\{\sigma^k(\alpha)|k \in \mathbb{Z}\}, \sigma)$ and $(\overline{\{\sigma^k(\alpha)|k \in \mathbb{Z}\}}, \sigma)$ for some almost periodic biinifinite word α. Now let $\beta(0)$ be any biinifinite word that is made up only of subwords of α, such that

$$\beta(0)_0 = \alpha_0,$$

and $\beta(0) \neq \alpha$. Now take the smallest $|k_1|$ so that

$$\sigma^{k_1}(\alpha)_{-1} = \beta(0)_{-1},$$

$$\sigma^{k_1}(\alpha)_0 = \beta(0)_0,$$

$$\sigma^{k_1}(\alpha)_1 = \beta(0)_1.$$

The almost periodicity of α ensures that we can find such a k_1. Now let $\beta(1)$ be any biinfinite word made up only of subwords of α so that

$$\beta(1)_{-1} = \sigma^{k_1}(\alpha)_{-1} = \beta(0)_{-1}$$

$$\beta(1)_0 = \sigma^{k_1}(\alpha)_0 = \beta(0)_0,$$

$$\beta(1)_1 = \sigma^{k_1}(\alpha)_1 = \beta(0)_1,$$

but $\beta(1) \neq \sigma^{k_1}(\alpha)$. (It is very likely that $\beta(0)$ will be an adequate choice for $\beta(1)$.) Now take the smallest $|k_2|$ so that

$$\sigma^{k_2}(\alpha)_{-2} = \beta(1)_{-2},$$

$$\sigma^{k_2}(\alpha)_{-1} = \beta(1)_{-1},$$

$$\sigma^{k_2}(\alpha)_0 = \beta(1)_0,$$

$$\sigma^{k_2}(\alpha)_1 = \beta(1)_1,$$

$$\sigma^{k_2}(\alpha)_2 = \beta(1)_2.$$

It is not hard to see that $|k_1| \leq |k_2|$. Inductively we will define a sequence of biinfinite words $(\beta(n))_{n=0}^{\infty}$, and an increasing monotonic sequence $(|k_n|)_{n=1}^{\infty}$ where

$$\beta(n)_{-n} = \sigma^{k_n}(\alpha)_{-n} = \beta(n-1)_{-n}$$

$$\vdots$$

$$\beta(n)_n = \sigma^{k_n}(\alpha)_n = \beta(n-1)_n$$

and $\beta(n) \neq \sigma^{k_n}(\alpha)$. So we have the situation that is depicted below.

$\beta(n-1)$	$\dots$	$\beta(n-1)_{-n-1}$	$\beta(n-1)_{-n}$	$\dots$	$\beta(n-1)_0$	$\dots$	$\beta(n-1)_n$	$\beta(n-1)_{n+1}$	$\dots$
$\sigma^{k_n}(\alpha)$	$\dots$	$\sigma^{k_n}(\alpha)_{-n-1}$	$\sigma^{k_n}(\alpha)_{-n}$	$\dots$	$\sigma^{k_n}(\alpha)_0$	$\dots$	$\sigma^{k_n}(\alpha)_n$	$\sigma^{k_n}(\alpha)_{n+1}$	$\dots$
$\beta(n)$	$\dots$	$\beta(n)_{-n-1}$	$\beta(n)_{-n}$	$\dots$	$\beta(n)_0$	$\dots$	$\beta(n)_n$	$\beta(n)_{n+1}$	$\dots$

may differ | will agree | may differ

The fact that α is almost periodic but not periodic ensures that we can always find such $\beta(n)$ and k_n. Further, we can see that $(\beta(n))$ will converge to some biinfinite word β (since $\beta(N)$ and $\beta(N+K)$ will agree on entries $\beta(N)_{-N}, \dots, \beta(N)_N$ and $\beta(N+K)_{-N}, \dots, \beta(N+K)_N$), and that $|k_n| \to \infty$ as $n \to \infty$ (since we always choose $\beta(n) \neq \sigma^{k_n}(\alpha)$). We can also see that the sequence $\sigma^{k_n}(\alpha) \to \beta$ as $n \to \infty$, so we will have $\beta \in \overline{\{\sigma^k(\alpha) | k \in \mathbb{Z}\}}$, but $\beta \notin \{\sigma^k(\alpha) | k \in \mathbb{Z}\}$.

1.3 Substitution Tilings

We will start with a substitution system as above and we will define a collection of tilings of the line that are related to this system.

A *tile* is a subset of $\mathbb{R}^d$ that is homeomorphic to a closed ball in $\mathbb{R}^d$. A *partial tiling* is a collection of tiles with pairwise disjoint interiors, and its *support* is the union of its tiles. A *tiling* is a partial tiling with support $\mathbb{R}^d$. Sometimes it is desirable to have different tiles that look alike, and we can associate a label with each tile to distinguish them. Now suppose that we have a finite set of tiles $\{P_1, \dots, P_n\}$, and we wish to consider tilings that only contain translations of these tiles. Then we will call these tiles $\{P_1, \dots, P_n\}$ the *prototiles* of the tilings which contain only translates of these tiles. We will often refer to tiles as translates of prototiles. If we have a prototile P and a real number x (or δ), then we will let $x + P$ denote the tile that we get when we shift the prototile P by x (or δ). In our considerations, our tilings will be one dimensional, and each of our tiles will consist of a closed interval in $\mathbb{R}$. So if we have a prototile P consisting of the interval $[a, b]$, then the tile $x + P$ will be the interval $[x + a, x + b]$. The first thing that we need to make our connection between substitution systems and tilings is the Perron-Frobenius (PF) theorem, which is proven in Queffélec [3], pp. 91-93.

Theorem 1.5. *Let M be a primitive, positive matrix. Then*

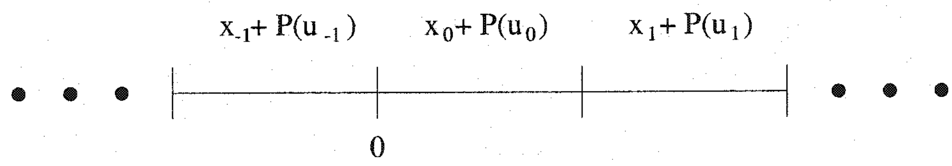

Figure 1.2: The tiling $\mathcal{T}(u)$

1. *M admits a strictly positive eigenvalue λ, such that $\lambda > |\theta|$ for any other eigenvalue θ of M,*

2. *There exists a strictly positive right eigenvector corresponding to λ,*

3. *λ has multiplicity 1.*

Now suppose that we have our alphabet $\mathcal{A} = \{a_1, \ldots, a_m\}$, with a substitution ω, and substitution matrix M. Let λ be our Perron-Frobenius eigenvalue, and let $v = [\ell_1, \ldots, \ell_m]^T$ be our PF eigenvector. Now let $\{P_1, \ldots, P_m\}$ be a set of closed intervals, of lengths $\ell_1, \ldots, \ell_m$ respectively, where ℓ_s and ℓ_l represent the shortest and longest lengths, respectively. These will be our prototiles, and any tile in the tilings and partial tilings we construct will just be a translate of one of these prototiles. If two of these prototiles have the same length, making them translates of each other, then we will associate labels with them to distinguish these prototiles and the tiles they create. We will let ${}^\bullet P_i$ and $P_i^\bullet$ denote the left and right endpoints of these intervals, respectively. We will also have a map $P : \mathcal{A} \longrightarrow \{P_1, \ldots, P_m\}$ defined by $P(a_i) = P_i$.

Now let $(\mathfrak{S}, \sigma)$ be the dynamical system generated by the procedure in section 2, and let $u = (u_i)$ be the biinfinite word that was fixed under ω. Then we can construct a tiling $\mathcal{T}(u)$ of the line using u as a guide, as in Figure 1.2.

Here we have shifted the prototile $P(u_0)$ by some x_0, so that the left endpoint of the tile $x_0 + P(u_0)$ lies on the origin. We have then shifted the prototile $P(u_1)$ by some x_1 so that its left endpoint matches up with the right endpoint of $x_0 + P(u_0)$, and we have shifted $P(u_{-1})$ by some x_{-1} so that its right endpoint matches up with the left endpoint of $x_0 + P(u_0)$. In this way we shift all of the prototiles $P(u_n)$ to form a tiling of the line.

We will want to simplify our notation somewhat. Instead of denoting our tiles as $x_i + P(u_i)$, we will often denote them as $T(u_i)$, where $T(u_i) := x_i + P(u_i)$. So $T(u_i)$ will symbolize a tile of type u_i which has been shifted to the appropriate location in a tiling or partial tiling of the line. If the exact position of one of these tiles is important, it will be made clear. However, it is often the patterns of tiles that appear that has the most importance for our

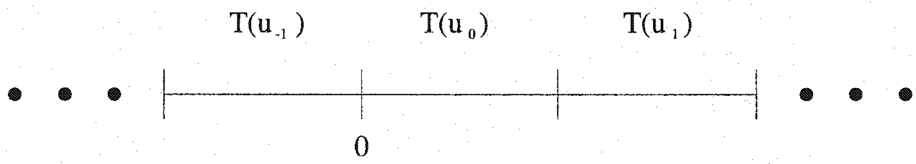

Figure 1.3: Alternate labeling of the tiling $\mathcal{T}(u)$

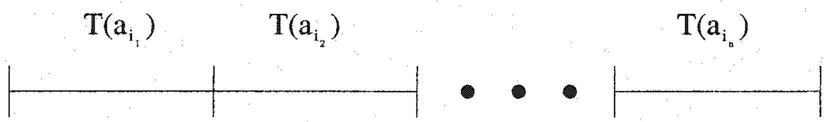

Figure 1.4: A partial tiling of type $a_{i_1} \dots a_{i_n}$.

considerations, so we can adopt this notation without losing any important information. So we may also represent the tiling $\mathcal{T}(u)$ as in Figure 1.3.

We will say that a tile $T(a_i)$ is a *tile of type* a_i. We will say that a sequence of tiles $T(a_{i_1}), \dots, T(a_{i_n})$, all connected continuously as in Figure 1.4, is a *partial tiling of type* $a_{i_1} \dots a_{i_n}$.

Now consider the space $(\mathcal{T}(u), \mathbb{R})$. This space consists of all possible shifts of the tiling $\mathcal{T}(u)$. So the elements of this space are all of the form $\mathcal{T} = x + \mathcal{T}(u)$, for some $x \in \mathbb{R}$, and this tiling will be similar to $\mathcal{T}(u)$, the difference being that each tile has been shifted by x to the right.

We can put a metric on this space as follows. For any $\mathcal{T}, \mathcal{T}' \in (\mathcal{T}(\alpha), \mathbb{R})$, let

$$d(\mathcal{T}, \mathcal{T}') := \inf(\{1/\sqrt{2}\} \cup \{\epsilon \mid \mathcal{B}_{1/\epsilon}(0) \cap (-v + \mathcal{T}) = \mathcal{B}_{1/\epsilon}(0) \cap (-v' + \mathcal{T}') \text{ for some } |v|, |v'| < \epsilon\}).$$

It is easy to verify that this is a metric, with the only tricky part being the verification of the triangle inequality, and we will follow the proof of a similar problem given in Lee, Moody and Solomyak [4]. Suppose that we have tilings $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \in (\mathcal{T}(\alpha), \mathbb{R})$, with $d(\mathcal{T}_1, \mathcal{T}_2) \leq \epsilon_1$ and $d(\mathcal{T}_2, \mathcal{T}_3) \leq \epsilon_2$. We need to show that $d(\mathcal{T}_1, \mathcal{T}_3) \leq \epsilon_1 + \epsilon_2$. We can assume that $\epsilon_1, \epsilon_2 < 1/\sqrt{2}$, otherwise the claim will be obvious. Then

$$\mathcal{B}_{1/\epsilon_1}(0) \cap (-v_1 + \mathcal{T}_1) = \mathcal{B}_{1/\epsilon_1}(0) \cap (-v_2 + \mathcal{T}_2) \text{ for some } v_1, v_2 \in \mathcal{B}_{1/\epsilon_1}(0),$$

$$\mathcal{B}_{1/\epsilon_2}(0) \cap (-v_2' + \mathcal{T}_2) = \mathcal{B}_{1/\epsilon_2}(0) \cap (-v_3' + \mathcal{T}_3) \text{ for some } v_2', v_3' \in \mathcal{B}_{1/\epsilon_2}(0).$$

It follows that

$$\mathcal{B}_{1/\epsilon_1}(-v_2') \cap (-v_1 - v_2' + \mathcal{T}_1) = \mathcal{B}_{1/\epsilon_1}(-v_2') \cap (-v_2 - v_2' + \mathcal{T}_2).$$

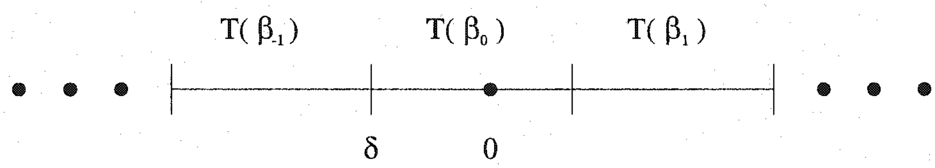

Figure 1.5: The tiling $\mathcal{T} = \delta + \mathcal{T}(\beta)$.

Since $\mathcal{B}_{(1/\epsilon_1)-\epsilon_2)}(0) \subset \mathcal{B}_{1/\epsilon_1}(0)$, we have

$$\mathcal{B}_{(1/\epsilon_1)-\epsilon_2}(0) \cap (-v_1 - v_2' + \mathcal{T}_1) = \mathcal{B}_{(1/\epsilon_1)-\epsilon_2}(0) \cap (-v_2 - v_2' + \mathcal{T}_2).$$

Similarily,

$$\mathcal{B}_{(1/\epsilon_2)-\epsilon_1}(0) \cap (-v_2 - v_2' + \mathcal{T}_2) = \mathcal{B}_{(1/\epsilon_2)-\epsilon_1}(0) \cap (-v_2 - v_3' + \mathcal{T}_3).$$

It is easily computed that $1/\epsilon_1 - \epsilon_2 \geq 1/(\epsilon_1 + \epsilon_2)$ and $1/\epsilon_2 - \epsilon_1 \geq 1/(\epsilon_1 + \epsilon_2)$ when $\epsilon_1, \epsilon_2 < 1/\sqrt{2}$, so we have

$$\mathcal{B}_{1/(\epsilon_1+\epsilon_2)}(0) \cap (-v_1 - v_2' + \mathcal{T}_1) = \mathcal{B}_{1/(\epsilon_1+\epsilon_2)}(0) \cap (-v_2 - v_3' + \mathcal{T}_3),$$

so $d(\mathcal{T}_1, \mathcal{T}_3) \leq \epsilon_1 + \epsilon_2$.

This metric will induce a topology on $(\mathcal{T}(\alpha), \mathbb{R})$, which is given by letting all

$$\mathcal{U}_\epsilon(\mathcal{T}) := \{\mathcal{T}' \mid d(\mathcal{T}, \mathcal{T}') < \epsilon\}$$

form a basis. Basically, this topology and metric will consider two tilings to be close if they agree on a large patch around the origin, after a small shift of each tiling.

Now let $\mathfrak{T} = \overline{(\mathcal{T}(u), \mathbb{R})}$. This is a compact metric space (see Radin and Wolff [5] pp. 357-358 for proof). Further, every element $\mathcal{T} \in \mathfrak{T}$ can be uniquely written as

$$\mathcal{T} = \delta + \mathcal{T}(\beta)$$

for some unique $\beta \in \mathfrak{S}$ and $\delta \in (-\ell(P(\beta_0)), 0]$. The tiling $\mathcal{T}$ is shown in Figure 1.5.

We notice here that $T(\beta_0)$ is the tile created when $P(\beta_0)$ is shifted to have its left endpoint lie at δ. The tiles $T(\beta_i)$ are copies of $P(\beta_i)$ shifted appropriately to create a tiling. We also notice that our above restriction that $\delta \in (-\ell(P(\beta_0)), 0]$ will allow us to represent a tiling with $\delta = 0$, but we will not be able to represent the same tiling with $\delta = -\ell(P(\beta_0))$..

Now our substitution ω will act on $\mathfrak{T}$ in a natural way. First, we will consider what ω does to any lone tile. Suppose that $\omega(a_i) = c_1 \ldots c_k$ for some

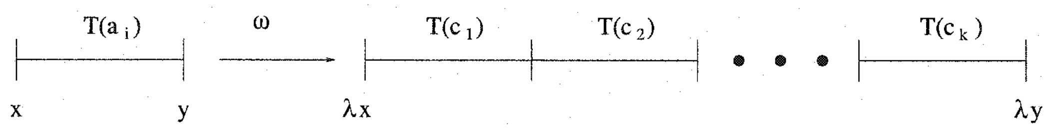

Figure 1.6: The action of ω on $\mathcal{T}(a_i)$

$c_1, \ldots, c_k \in \mathcal{A}$. The tile $T(a_i)$ has length ℓ_i, and by the PF theorem, we know that

$$M \begin{bmatrix} \vdots \\ \ell_i \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ M_{i1}\ell_1 + \cdots + M_{im}\ell_m \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ \lambda\ell_i \\ \vdots \end{bmatrix}.$$

But each M_{ij} is just the number of times the letter a_j appears in the word $\omega(a_i)$, so

$$\lambda\ell_i = M_{i1}\ell_1 + \cdots + M_{im}\ell_m = \ell(P(c_1)) + \cdots + \ell(P(c_k)).$$

If we inflate the tile $T(a_i)$ by a factor of λ, we can continuously place the tiles $T(c_1), \ldots, T(c_k)$, in order, inside of $\lambda T(a_i)$, with no overlaps or gaps, as in Figure 1.6.

So for any tiling $\mathcal{T}$, we do this to every tile in $\mathcal{T}$, and we are left with a new tiling of the line, $\omega(\mathcal{T})$.

We will say that a tiling satisfies the *finite pattern condition* if for any given tile, it's boundary can be covered by other tiles, with no overlapping, in only a finite number of ways. It is trivial to see that all of the one dimensional tilings that we study here will satisfy this condition. (However, it is possible to create tiling spaces that do not satisfy this condition, see Radin and Wolff [5] page 355 for an example.) We can also consider this condition in a slightly different way: for any positive radius r, there will be only finitely many partial tilings up to translation that are subsets of tilings in $\mathfrak{T}$ and have support of diameter less than r. (This is trivial to see for all of the examples we consider here, since there are only a finite number of ways we can line up our tiles before their support becomes greater than or equal to some fixed r.)

Now we have already assumed that ω is primitive, and we know that our space has the finite pattern condition. If we now assume that ω is one-one on $\mathfrak{T}$, then we can also conclude that ω is both onto and bicontinuous, see Putnam and Anderson [6] pp. 512-513. Consequently, ω will have an inverse, ω^{-1}, and we will have a well defined map $\omega^n : \mathfrak{T} \longrightarrow \mathfrak{T}$ for every $n \in \mathbb{Z}$. So $(\mathfrak{T}, \omega)$ is a topological dynamical system.

We will say that ω *forces the border* if there is an integer $B_F \geq 0$ such that for any tiling $\mathcal{T}$, and any tile $T \in \mathcal{T}$, the tiles that come immediately before and after the partial tiling $\omega^{B_F}(T)$ in the tiling $\omega^{B_F}(\mathcal{T})$ are completely determined by what type of tile T is.

1.3.1 Examples of Substitution Tilings

Let's look at a couple of examples. First we will consider the silver mean substitution. Recall that $\mathcal{A}_1 = \{A, B\}$, $\omega_1 : \begin{array}{ccc} A & \longrightarrow & ABA \\ B & \longrightarrow & A \end{array}$, and our substitution matrix is $M_1 = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$. We calculate our PF eigenvalue $\lambda_1 = 1 + \sqrt{2}$, and our PF eigenvector is $[\lambda_1, 1]^T$. We can choose the prototiles $P(A) = [0, \lambda_1]$, and $P(B) = [0, 1]$. Recall that we have also created the word u_1, which is stable under ω_1, and so we can form the tiling $\mathcal{T}(u_1)$, and we have a compact metric space $\mathfrak{T}_1 = \overline{(\mathcal{T}(u_1), \mathbb{R})}$.

We can see that ω_1 forces the border since $\omega_1^2 : \begin{array}{ccc} A & \longrightarrow & ABAAABA \\ B & \longrightarrow & ABA \end{array}$, and BAB is a forbidden word in $\mathfrak{S}_1$, so if we take any tile of type A in a tiling $\mathcal{T}$, we must have a tile of type A come immediately before $\omega_1^2(T(A))$ in $\omega_1^2(\mathcal{T})$, and we must have a tile of type A come immediately after $\omega_1^2(T(A))$ in $\omega_1^2(\mathcal{T})$. Similarily, if we take a tile of type B in $\mathcal{T}$, we must have a tile of type A come immediately before $\omega_1^2(T(B))$ in $\omega_1^2(\mathcal{T})$, and we must have a tile of type A come immediately after $\omega_1^2(T(B))$ in $\omega_1^2(\mathcal{T})$. This is depicted in Figure 1.7.

We can also see that $\omega_1 : \mathfrak{T}_1 \longrightarrow \mathfrak{T}_1$ is one to one. Take any $\mathcal{T} \in \mathfrak{T}_1$, and consider the possible candidates for $\omega_1^{-1}(\mathcal{T})$ in $\mathfrak{T}_1$. For every tile of type B in $\mathfrak{T}_1$, the tiles to the immediate left and right must both be of type A. So we have a partial tiling of type ABA. But this must have come from a tile of type A in $\omega_1^{-1}(\mathcal{T})$ (if such a tiling exists). See Figure 1.8 a.

The only tiles that this situation will not account for are tiles of type A in $\mathcal{T}$, that are neighboured by tiles of type A on both sides. These tiles must come from a tile of type B in any candidate for $\omega_1^{-1}(\mathcal{T})$. See Figure 1.8 b.

This leaves us with only one possible candidate for $\omega_1^{-1}(\mathcal{T})$, since we have accounted for where each tile of $\mathcal{T}$ must have come from, so ω_1 must be one to one. Consequently, ω_1 will also be onto, invertible and bicontinuous. So we have a topological dynamical system $(\mathfrak{T}_1, \omega_1)$.

For another example lets consider the Fibonnacci substitution. Here we will again use the alphabet $\mathcal{A}_1 = \{A, B\}$, but now we will let our substitution

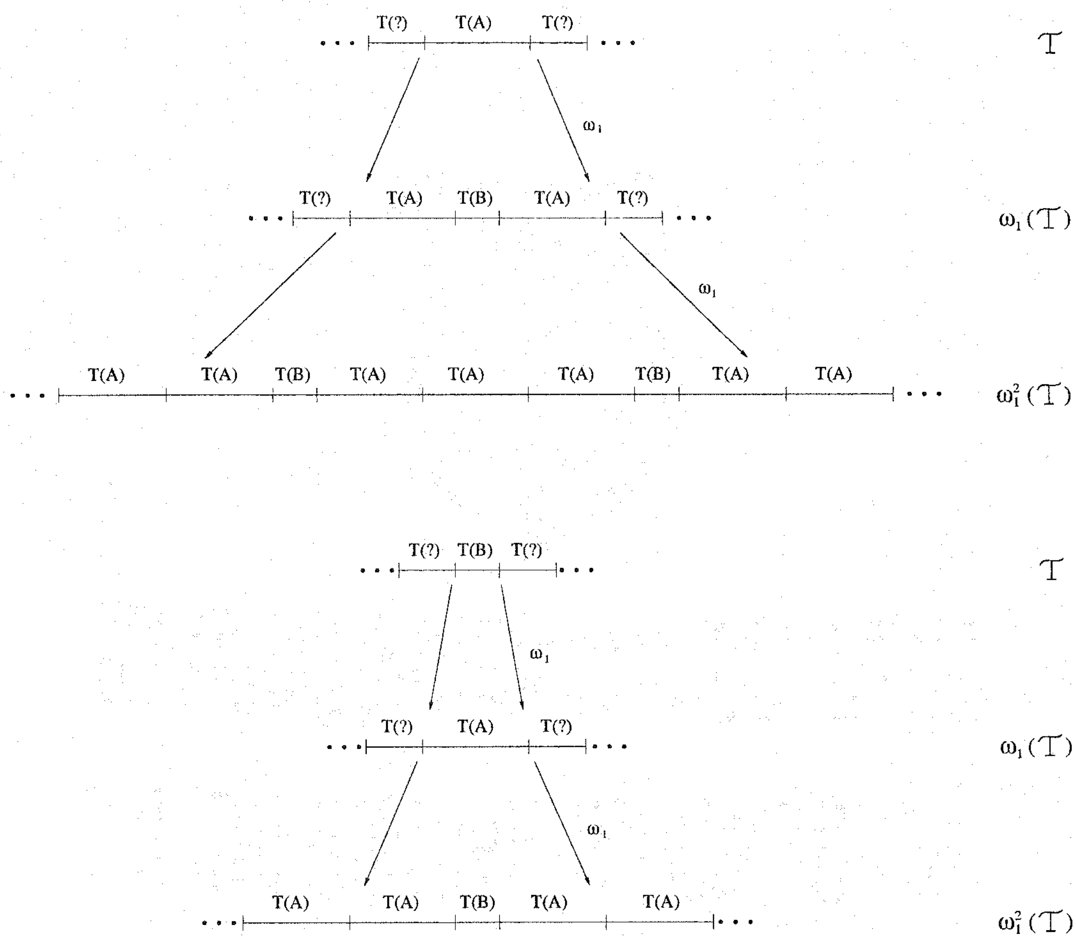

Figure 1.7: Forcing the border for ω_1.

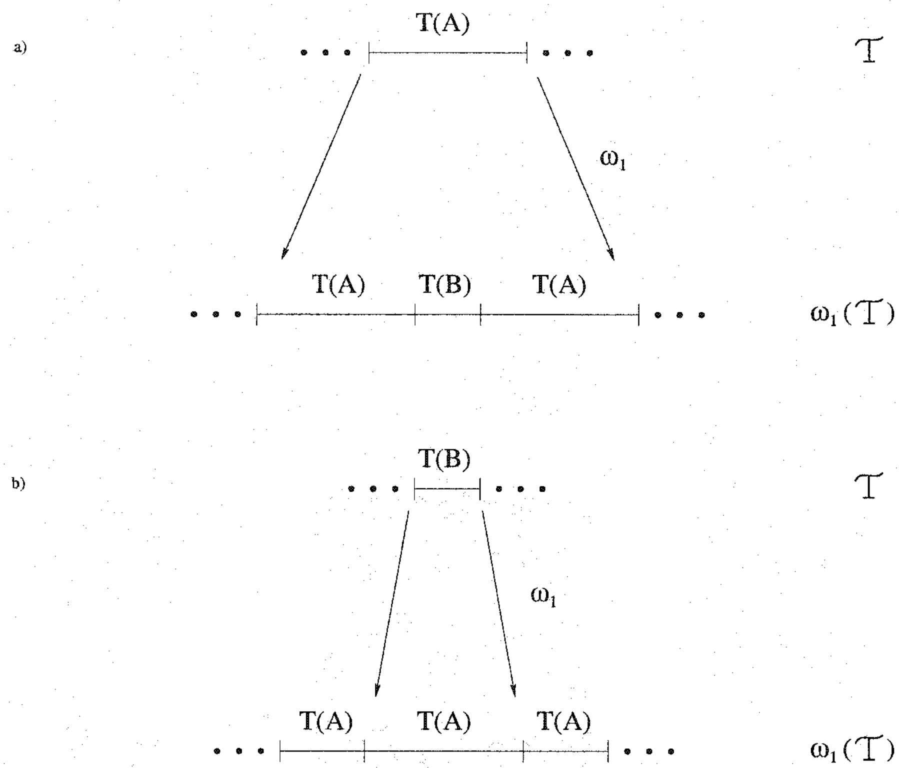

Figure 1.8: The map ω_1 is one to one.

be

$$\begin{aligned} \omega_2 : \ \mathcal{A} &\longrightarrow \mathcal{A}^* \\ A &\longrightarrow AB \\ B &\longrightarrow A \end{aligned}$$

As above we can create a topological dynamical system from this substitution, say $(\mathfrak{T}'_1, \omega_2)$. However, this substitution does not force the border. Let's consider the ways in which a tile of type A can be collared in this system. A *collaring* of a tile is any one of the ways a tile can share its boundary with other tiles in a tiling. In this substitution, a tile of type A can be collared three ways, it can have a tile of type B to its immediate left and a tile of type A to its immediate right, as in BAA, or it can have a tile of type A to its immediate left and a tile of type B to its immediate right, as in AAB, or it can have a tile of type B to its immediate left and a tile of type B to its immediate right, as in BAB. (A tile of type B can only be collared one way, as in ABA. We note that the ways a tile can be collared depend on the particular substitution system.) Let's consider the ways a partial tiling $\omega_2^k(T(A))$ can be collared in this system. If ω_2 forces the border there will have to be some $k > 0$ such that $\omega_2^k(T(A))$ can only be collared in one way. However, we can see from

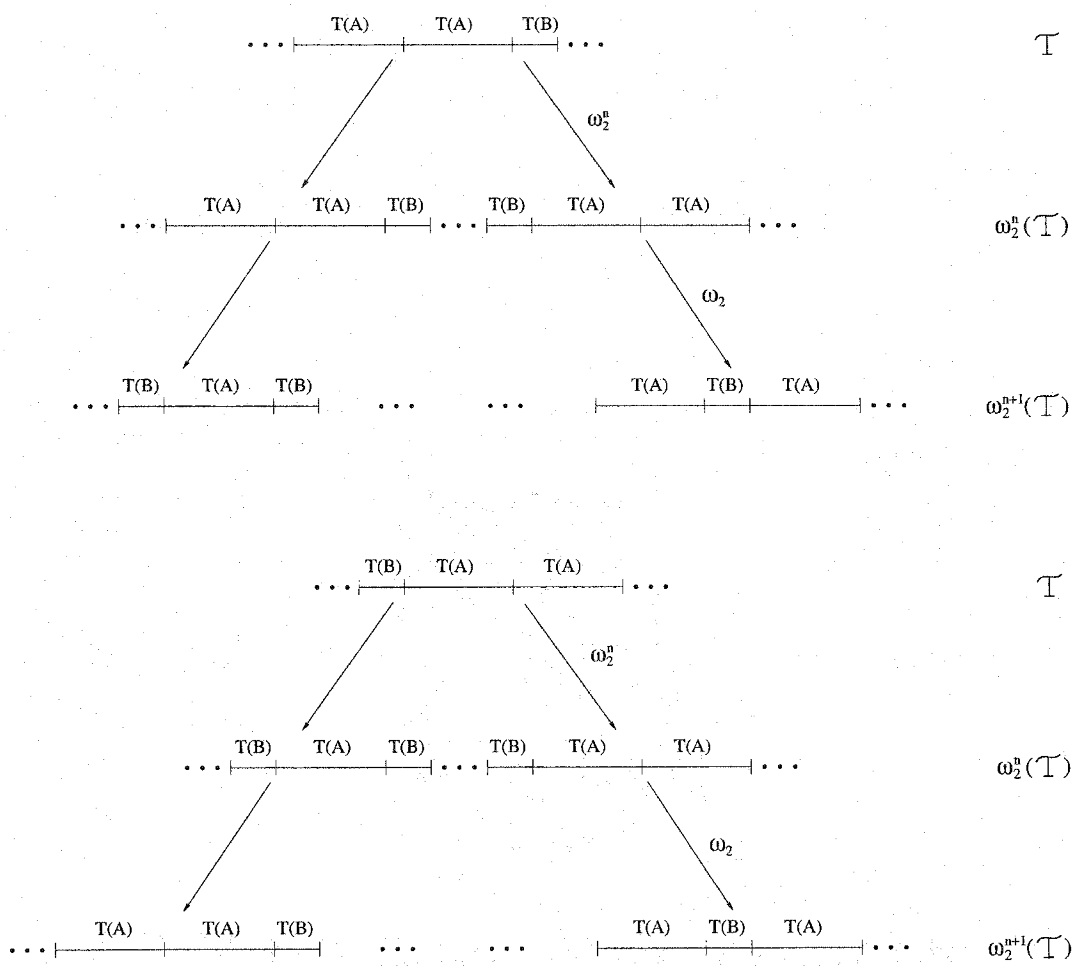

Figure 1.9: The map ω_2 does not force the border.

Figure 1.9, that the partial tiling $\omega_2^k(T(A))$ can be bordered on the left by either a tile of type A, or a tile of type B, for any k. So ω_2 does not force the border.

We will replace this system with a topologically conjugate system that does force the border. We create an alphabet consisting of one symbol for each way our existing symbols can be collared in our existing substitution system. The symbol A can be collared three ways (one each for BAA, AAB, and BAB), and B can only be collared one way (ABA). So we create a new substitution system where our alphabet $\mathcal{A}_2$ contains four letters, say C,D,E, and F. Here we let C correspond to the A in the middle of AAB, D corresponds to the A in BAB, E corresponds to the A in the middle of BAA, and F corresponds to the B in ABA. To see what our new substitution rule does we look at what it did to the original alphabet. We will still call this new substitution ω_2, but we will understand that it now acts on the alphabet $\mathcal{A}_2$. Our C corresponds to the second A in AAB, and $\omega_2(AAB) = AB\ AB\ A$. (Here we have inserted spaces to make it easier to recognize what ω_2 does to each letter in AAB.) So the second A in AAB, the one that corresponds to C, is mapped to an A, which is collared BAB, then a B, which is collared ABA. But the A in BAB corresponds to D, and the B in ABA corresponds to F. So our new substitution maps C to DF. When we look at the rest of the possiblities, we see that

$$\left.\begin{array}{rrcl} \omega_2: & AAB & \longrightarrow & AB\ AB\ A \\ & BAB & \longrightarrow & A\ AB\ A \\ & BAA & \longrightarrow & A\ AB\ AB \\ & ABA & \longrightarrow & AB\ A\ AB \end{array}\right\} \Longrightarrow \left\{\begin{array}{rrcl} \omega_2: & C & \longrightarrow & DF \\ & D & \longrightarrow & CF \\ & E & \longrightarrow & CF \\ & F & \longrightarrow & E \end{array}\right.$$

We have already looked at this substitution in the last section. It has substitution matrix M_2, which has PF eigenvalue $\lambda_2 = 1 + \sqrt{5}/2$ and PF eigenvector $[\lambda_2, \lambda_2, \lambda_2, 1]^T$. So we can create four prototiles $P(C) = [0, \lambda_2]$, $P(D) = [0, \lambda_2]$, $P(E) = [0, \lambda_2]$, and $P(F) = [0, 1]$, and we have the tiling $\mathcal{T}(u_2)$ and the compact metric space $\mathfrak{T}_2 = \overline{(\mathcal{T}(u_2), \mathbb{R})}$.

We can also see why this substitution is one to one. We must look at the partial tilings of length three. The words of length three that can occur in $\mathfrak{S}_2$ are CFD, CFE, DFE, ECF, FEC, and FDF.

We can see immediately that a partial tiling of type DFE must always come from a partial tiling of type CF. This is the case because our tile of type E must come from a tile of type F, and the tile of type D must come from a tile of type C. See Figure 1.10 a.

We see that a partial tiling of type CFE must come from a partial tiling of type DF. This is because the tiles of type CFE must be contained in a larger partial tiling of type $ECFE$. Both of the tiles of type E must come from tiles of type F, and the tiles CF between these tiles of type E must come from either a tile of type D or a tile of type E. However, the word FEF is illegal in $\mathfrak{S}_2$, while FDF is legal, so the tiles CF must come from a tile of type D. See Figure 1.10 b.

A partial tiling of type CFD must be part of a larger partial tiling of type $ECFDFE$. The tiles of type E must come from a tile of type F, the tiles DF must come from a tile of type C, and consequently, the tiles CF must come from a tile of type E (since FDC is not a legal word of $\mathfrak{S}_2$). See Figure 1.10 c.

A partial tiling of type FDF must be part of a larger partial tiling of type $ECFDFE$, and we have seen where this must come from in our considerations of the partial tiling CFD. See Figure 1.10 d.

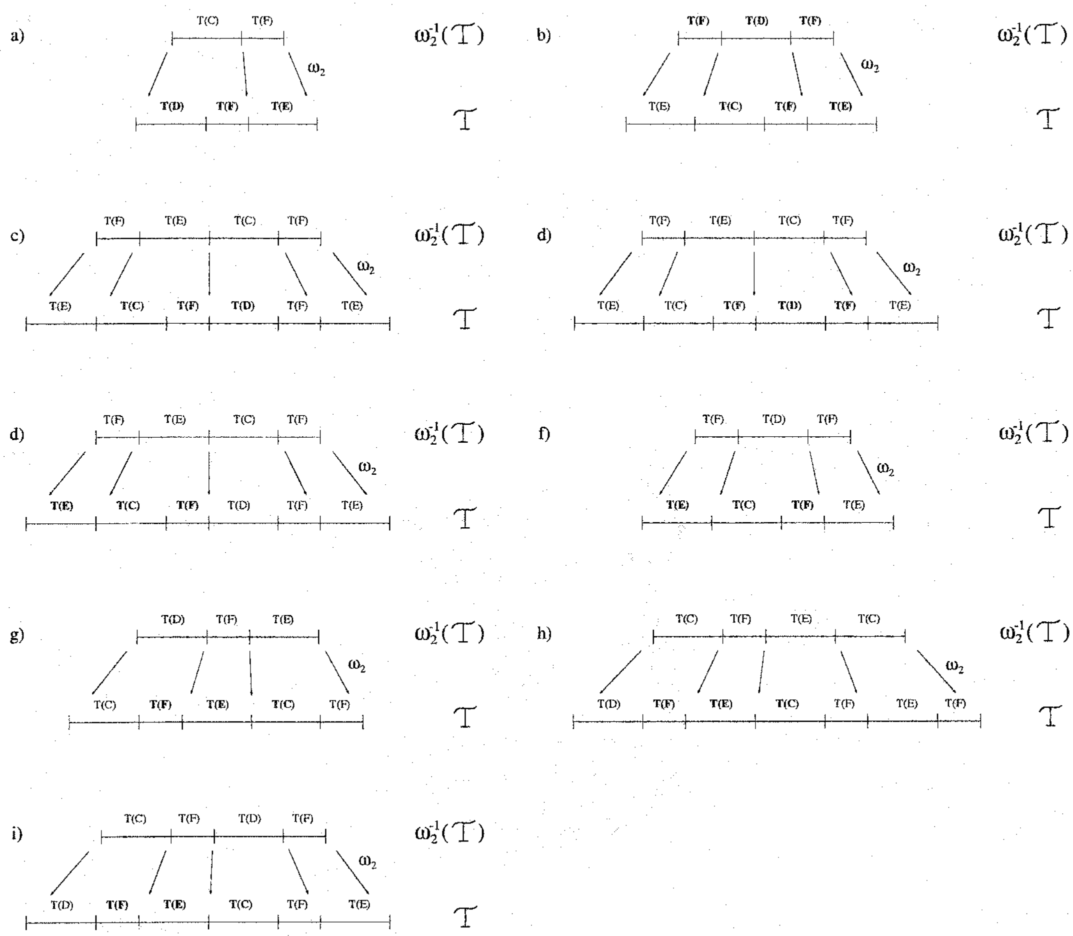

Figure 1.10: The map ω_2 is one to one.

For the partial tiling of type ECF we must also look at the tile bordering it on the immediate right. This tile can be of either type D or E. If it is of type D, then the partial tiling $ECFD$ must be part of the larger partial tiling of type $ECFDFE$, and this can only come from a partial tiling of type $FECF$. See Figure 1.10 e. If our partial tiling ECF is part of the partial tiling $ECFE$, on the other hand, then it must come from a partial tiling of type FDF. See Figure 1.10 f.

All that remains is the partial tiling FEC. We can see that this tiling must be in one of the larger partial tilings of type $CFECF$, $DFECFDF$, or $DFECFE$, and as above, these partial tilings come from the partial tilings of type DFE, $CFEC$, or $CFDF$, respectively. See Figure 1.10 g,h,i.

Putting all of this together, we can decompose each tiling $\mathcal{T}$ in $\mathfrak{T}_2$ into only one possible candidate for $\omega_2^{-1}(\mathcal{T})$, and we see that ω_2 is one to one. So we have a topological dynamical system $(\mathfrak{T}_2, \omega_2)$. Also, it is easy to show that for

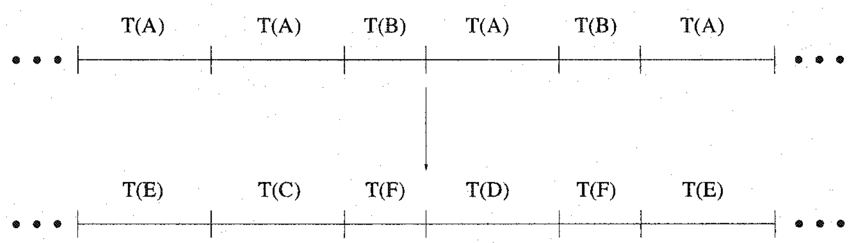

Figure 1.11: The bijection from $\mathfrak{T}_1'$ to $\mathfrak{T}_2$.

any $a \in \mathcal{A}_2$, the tiles that border $\omega(T(a))$ will be completely determined by the symbol a, so ω_2 forces the border (in this system). It is a trivial matter to verify that $(\mathfrak{T}_1', \omega_2)$ and $(\mathfrak{T}_2, \omega_2)$ are topologically conjugate. The spaces are clearly bijective since for each tiling in $\mathfrak{T}_1'$ there will be a tiling in $\mathfrak{T}$ that is identical, except for the fact each tile has been labelled to correspond to a prototile consistent with the alphabet in that system. See Figure 1.11 for an example. This bijection is certainly bicontinuous, and the way ω_2 was defined on $\mathcal{A}_2$ ensures that ω_2 and the bijection from $\mathfrak{T}_1'$ to $\mathfrak{T}_2$ intertwine.

We will note that this process of creating a topologically conjugate substitution tiling dynamical system that forces the border is possible for any substitution tiling dynamical system, and the process above will always create a suitable system. We merely create an alphabet containing one symbol for each way an existing symbol can be collared, then we redefine our substitution in the natural way that respects the collarings.

Chapter 2

Other Dynamical Systems

2.1 The dynamical sytem $(\mathfrak{L}, \omega)$

We will create a dynamical system that is topologically conjugate to $(\mathfrak{T}, \omega)$ by considering each tiling in $\mathfrak{T}$ to be related to a sequence of points from the prototiles.

2.1.1 The space $\mathfrak{U}$

We will define a relation Ψ from $\mathfrak{T}$ into the space of all sequences of points from the prototiles, and we will define our space as $\mathfrak{U} := \Psi(\mathfrak{T})$. It is easy to define the map we want by considering an example, then generalizing the process.

Let's consider the silver mean tiling space $\mathfrak{T}_1$ (as defined in Section 1.3.1), and let $\mathcal{T} \in \mathfrak{T}_1$, as in Figure 2.1. We note that for the tiling $\mathcal{T}$, the origin is contained in a tile of type A, and the left endpoint of this tile is at the point $\delta_0 \in \mathbb{R}$. We let $x_0 = -\delta_0 + {}^{\bullet}P(A)$, and we note that x_0 is just a point in the interval $P(A)$. (Recall that ${}^{\bullet}P(A)$ denotes the left endpoint of the interval $P(A)$, so adding $-\delta_0$ and ${}^{\bullet}P(A)$ makes sense.) If we define a point x_n in this way for each of tilings $\omega^{-n}(\mathcal{T})$, we will form a sequence $(x_n)_{n=0}^{\infty}$ where each point x_n is in one of our prototiles (either $P(A)$ or $P(B)$. We will let $\Psi_1(\mathcal{T}) = \{(x_n)_{n=0}^{\infty}\}$, where

$$x_0 = -\delta_0 + {}^{\bullet}P(A)$$

$$x_1 = -\delta_1 + {}^{\bullet}P(B)$$

$$x_2 = -\delta_2 + {}^{\bullet}P(A)$$

$$x_3 = -\delta_3 + {}^\bullet P(A)$$
$$x_4 = -\delta_4 + {}^\bullet P(A)$$
$$\vdots$$

Now consider the tiling $\mathcal{T}'$ in Figure 2.2. We will let $\Psi_1(\mathcal{T}') = \{(x'_n)_{n=0}^\infty, (x''_n)_{n=0}^\infty\}$, where

$$x'_0 = {}^\bullet P(A)$$
$$x'_1 = {}^\bullet P(A)$$
$$x'_2 = {}^\bullet P(B)$$
$$x'_3 = -\delta'_3 + {}^\bullet P(A)$$
$$x'_4 = -\delta'_4 + {}^\bullet P(B)$$
$$\vdots$$

and

$$x''_0 = P(A)^\bullet$$
$$x''_1 = P(A)^\bullet$$
$$x''_2 = P(A)^\bullet$$
$$x''_3 = -\delta'_3 + {}^\bullet P(A)$$
$$x''_4 = -\delta'_4 + {}^\bullet P(B)$$
$$\vdots$$

(In this situation we have two elements in $\Psi_1(\mathcal{T}')$ because the origin of $\mathcal{T}'$ lies on the boundary between two tiles. It will be the quotient space that is formed when we identify these two sequences that is of interest to us.)

It is easy to see how to generalize this relation to the rest of the space $\mathfrak{T}_1$, and to any general tiling space $\mathfrak{T}$. Suppose that we have a tiling $\mathcal{T} \in \mathfrak{T}$, where $\omega^{-n}(\mathcal{T}) = \delta(n) + \mathcal{T}(\beta(n))$, for all n. We will let the sequence $(x_n)_{n=0}^\infty$, where $x_n = -\delta(n) + {}^\bullet P(\beta(n)_0)$, be in $\Psi(\mathcal{T})$. If the origin of $\mathcal{T}$ does not lie on the boundary of a tile, i.e., if $\delta(0) \neq 0$, then $(x_n)_{n=0}^\infty$ will be the only sequence in $\Psi(\mathcal{T})$. However, if the origin of $\mathcal{T}$ does lie on the boundary of a tile, i.e., if $\delta(0) = 0$, then we will add one other sequence to $\Psi(\mathcal{T})$. If $\delta(n) = 0$ for all $n < N$, but $\delta(N) \neq 0$, then define the sequence $(x'_n)_{n=0}^\infty$ where $x'_n = \begin{cases} P(\beta(n)_{-1})^\bullet & \text{for } n < N \\ x_n & \text{for } n \geq N \end{cases}$. If $\delta(n) = 0$ for all n, then we define $(x'_n)_{n=0}^\infty$ by $x'_n = P(\beta(n)_{-1})^\bullet$ for all n. Then we let $\Psi(\mathcal{T}) = \{(x_n)_{n=0}^\infty, (x'_n)_{n=0}^\infty\}$. All

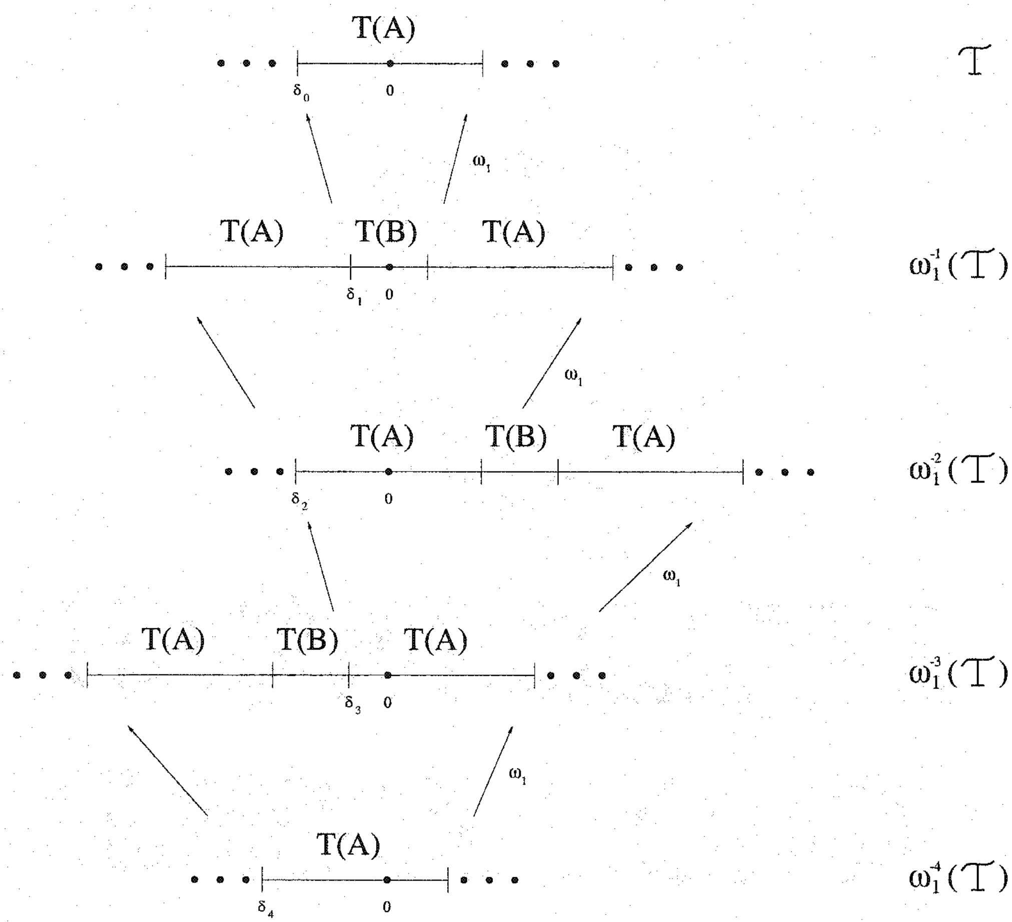

Figure 2.1: A tiling in $\mathfrak{T}_1$

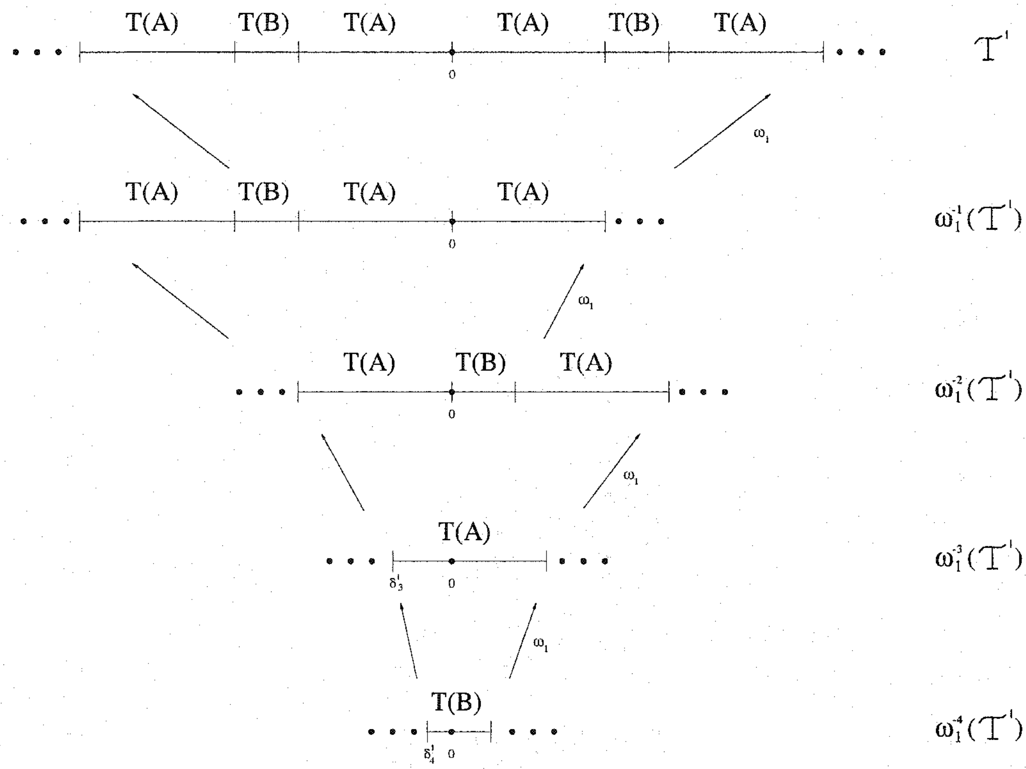

Figure 2.2: Another tiling in $\mathfrak{T}_1$

we have done is rigorously defined the process used in defining Ψ_1 above, for a general tiling system. The following lemma ensures that this map Ψ is well defined.

Lemma 2.1. *If $\mathcal{T} \in \mathfrak{T}$, where $\omega^{-n}(\mathcal{T}) = \delta(n) + \mathcal{T}(\beta(n))$, and $\delta(n) \neq 0$, then $\delta(n+1) \neq 0$.*

Proof This is clear by the contrapositive. It is easy to see that if $\delta(n+1) = 0$, then $\delta(n) = 0$. □

We let $\mathfrak{U} := \Psi(\mathfrak{T})$, and it is clear that there is exactly one element in $\mathfrak{U}$ for every tiling in $\mathfrak{T}$ where the origin does not lie on the boundary of a tile, and exactly two elements in $\mathfrak{U}$ for every tiling in $\mathfrak{T}$ where the origin lies on the boundary between two tiles.

We can put a topology on $\mathfrak{U}$ as follows. First we note that each prototile $P(a)$ has the subspace topology from the usual topology on $\mathbb{R}$. We will let $\mathfrak{P}$ be the disjoint topological union of the prototiles. We can form the product space $\Pi_{i=0}^{\infty}\mathfrak{P}$, and we let $\mathfrak{U}$ have the subspace topology from this space.

Let's consider what this topology will look like for the space $\mathfrak{U}_1 := \Psi_1(\mathfrak{T}_1)$. First, the space $\mathfrak{P}_1$ is the disjoint topological union of the prototiles $P(A)$ and

$P(B)$. So all intervals of the form

$$(a,b), (a, P(A)^\bullet], [^\bullet P(A), b) \subset P(A),$$

or of the form

$$(a,b), (a, P(B)^\bullet], [^\bullet P(B), b) \subset P(B),$$

will form a basis for the topology on $\mathfrak{P}_1$. The product space $\Pi_{i=0}^{\infty}\mathfrak{P}_1$ has a basis consisting of all sets of the form $\Pi_{i=0}^{\infty}V_i$ such that each V_i is open in $\mathfrak{P}_1$, and all but finitely many $V_i = \mathfrak{P}_1$. So the set

$$W_1 := (a_0, b_0) \times (a_1, P(B)^\bullet] \times \Pi_{i=2}^{\infty}\mathfrak{P}_1,$$

where $(a_0, b_0) \subset P(A)$ and $(a_1, P(B)^\bullet] \subset P(B)$ will be a typical basis element for the topology on $\Pi_{i=0}^{\infty}\mathfrak{P}_1$. Now $\mathfrak{U}_1$ has the subspace topology from the topology on $\Pi_{i=0}^{\infty}\mathfrak{P}_1$. So $U \subset \mathfrak{U}_1$ will be open if and only if there is some open set $W \subset \Pi_{i=0}^{\infty}\mathfrak{P}_1$ so that $W \cap U = U$ and $W \cap (\mathfrak{U}_1 \backslash U) = \emptyset$. So the set

$$\mathfrak{U}_1 \cap W_1 = \{(x_i)_{i=0}^{\infty} \in \mathfrak{U}_1 \mid x_0 \in (a_0, b_0) \text{ and } x_1 \in (a_1, P(B)^\bullet]\}$$

will be a typical example of an open set of $\mathfrak{U}_1$.

2.1.2 The map $\Psi^{-1} : \mathfrak{U} \longrightarrow \mathfrak{T}$

We can define a map $\Psi^{-1} : \mathfrak{U} \longrightarrow \mathcal{T}$ such that if $(x_n)_{n=0}^{\infty} \in \Psi(\mathcal{T})$, then $\Psi^{-1}((x_n)_{n=0}^{\infty}) = \mathcal{T}$.

Theorem 2.1. *For each $(x_n)_{n=0}^{\infty} \in \mathfrak{U}$, there is a unique $\mathcal{T} \in \mathfrak{T}$ such that $(x_n)_{n=0}^{\infty} \in \Psi(\mathcal{T})$. So there is a well defined map $\Psi^{-1} : \mathfrak{U} \longrightarrow \mathfrak{T}$ where $\Psi^{-1}((x_n)_{n=0}^{\infty}) = \mathcal{T}$ if and only if $(x_n)_{n=0}^{\infty} \in \Psi(\mathcal{T})$.*

Proof. Let $(x_n)_{n=0}^{\infty} \in \mathfrak{U}$, and suppose that each $x_n \in P(c_n)$, for some $c_n \in \mathcal{A}$. For each n, let $\mathcal{T}_n$ be any tiling where $\omega^{-n}(\mathcal{T}_n)$ contains the tile $-x_n + P(c_n)$. Now $(x_n)_{n=0}^{\infty} \in \mathfrak{U}$, so there must be some $\mathcal{T} \in \mathfrak{T}$ such that $(x_n)_{n=0}^{\infty} \in \Psi(\mathcal{T})$. Further, for every n, the tile $-x_n + P(c_n)$ must be in the tiling $\omega^{-n}(\mathcal{T})$. But the tile $-x_n + P(c_n)$ contains the origin, so the tilings $\omega^{-n}(\mathcal{T})$ and $\omega^{-n}(\mathcal{T}_n)$ will agree on some interval containing the origin. Now our substition ω forces the border, so for some integer $B_F > 0$, the tiles to the immediate left and right of the partial tiling $\omega^{B_F}(-x_n + P(c_n))$ will be completely determined by the symbol c_n. Let's label these tiles T_L and T_R, respectively, and they are as shown in Figure 2.3. These tiles will have length greater than or equal to ℓ_s (the length of the shortest prototile), and the partial tiling $\omega^{B_F}(-x_n + P(c_n))$

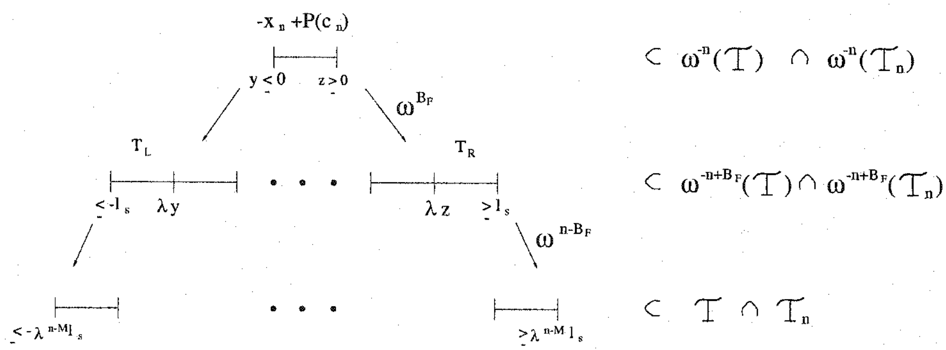

Figure 2.3: Common tiles in $\mathcal{T}_n$ and $\mathcal{T}$.

will contain the origin, and be contained in both $\omega^{B_F-n}(\mathcal{T})$ and $\omega^{B_F-n}(\mathcal{T}_n)$. So the tilings $\omega^{B_F}(\omega^{-n}(\mathcal{T}))$ and $\omega^{B_F}(\omega^{-n}(\mathcal{T}_n))$ will both contain the tiles T_L and T_R, so they will agree on the interval $[-\ell_s, \ell_s]$. Further, if $n > B_F$, then the tilings $\omega^{n-B_F}(\omega^{B_F-n}(\mathcal{T})) = \mathcal{T}$ and $\omega^{n-B_F}(\omega^{B_F-n}(\mathcal{T}_n)) = \mathcal{T}_n$ will agree on the interval $[-\lambda^{n-B_F}\ell_s, \lambda^{n-B_F}\ell_s]$. As we let n approach ∞, this interval will become $\mathbb{R}$. So the sequence $(\mathcal{T}_n)$ will converge to $\mathcal{T}$. We let $\Psi^{-1}((x_n)_{n=0}^{\infty}) = \mathcal{T}$, and we have established the theorem. □

Let's consider an example of how this map works. Take the sequence $(x_n)_{n=0}^{\infty} \in \mathfrak{U}_1$ given by

$$
\begin{aligned}
x_0 &= -\delta_0 + {}^\bullet P(A) \\
x_1 &= -\delta_1 + {}^\bullet P(B) \\
x_2 &= -\delta_2 + {}^\bullet P(A) \\
x_3 &= -\delta_3 + {}^\bullet P(A) \\
x_4 &= -\delta_4 + {}^\bullet P(A) \\
&\vdots
\end{aligned}
$$

and let's find $\Psi_1^{-1}((x_n)_{n=0}^{\infty})$. We can let $\mathcal{T}_0$ be any tiling with the tile $-x_0 + P(A)$, as in Figure 2.4 a. The tiling $\mathcal{T}_1$ can be any tiling where $\omega_1^{-1}(\mathcal{T}_1)$ contains the tile $-x_1 + P(B)$, as in Figure 2.4 b. The tiling $\mathcal{T}_2$ can be any tiling where $\omega_1^{-2}(\mathcal{T}_2)$ contains the tile $-x_2 + P(A)$, as in Figure 2.4 c. We can see that these tilings are agreeing on larger areas as n increases, and we will let $\Psi^{-1}(((x_n)_{n=0}^{\infty}))$ be the tiling that they converge to. We can note that this sequence is the one created when Ψ_1 acts on the tiling in Figure 2.1, and we can see that Ψ_1^{-1} appears to be mapping $(x_n)_{n=0}^{\infty}$ back to this tiling.

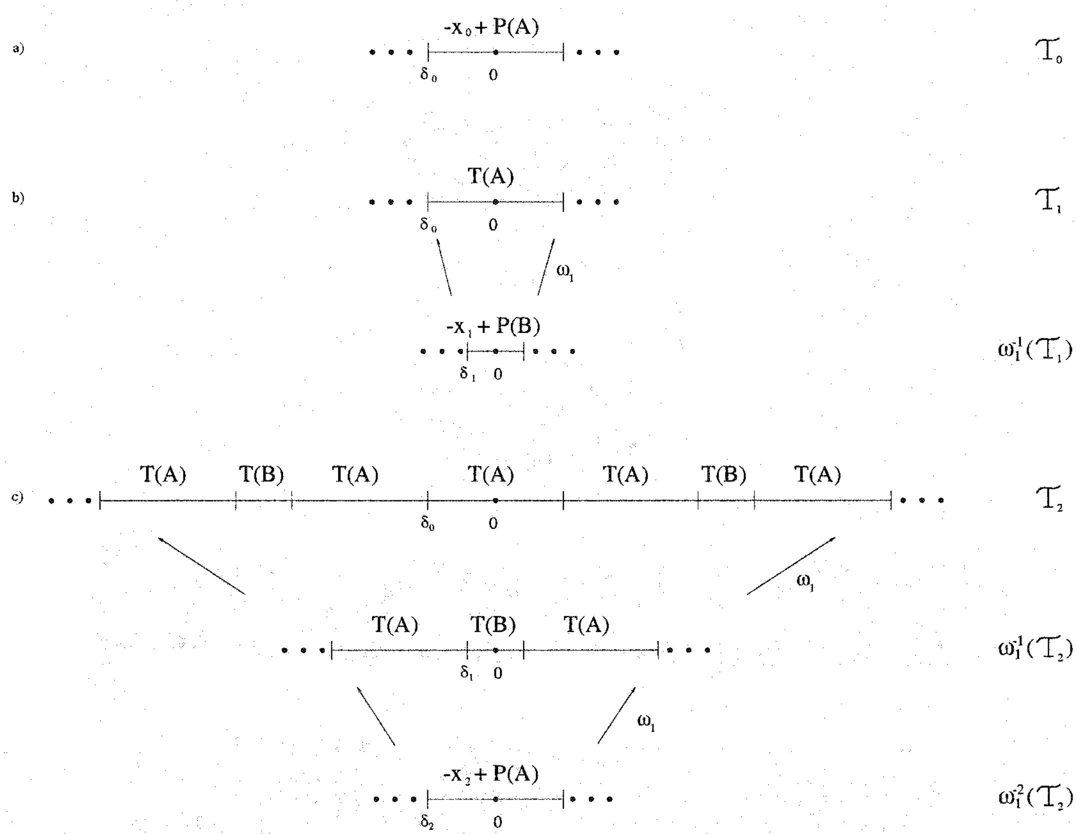

Figure 2.4: The map Ψ_1^{-1} for the siver mean substitution.

2.1.3 The space $\tilde{\mathfrak{U}}$

We can now define the space $\tilde{\mathfrak{U}}$. If $(x_n)_{n=0}^{\infty}, (x'_n)_{n=0}^{\infty} \in \Psi(\mathcal{T})$, and $(x_n)_{n=0}^{\infty} \neq (x'_n)_{n=0}^{\infty}$, then identify them. Let $\tilde{\mathfrak{U}}$ be the quotient space under this identification, and let $\tilde{}: \mathfrak{U} \longrightarrow \tilde{\mathfrak{U}}$ be our identification map. Also let $\tilde{\Psi}$ and $\tilde{\Psi}^{-1}$ be the naturally induced maps between $\mathfrak{T}$ and $\tilde{\mathfrak{U}}$. The following theorem is clear.

Theorem 2.2. *The maps $\tilde{\Psi} : \mathfrak{T} \longrightarrow \tilde{\mathfrak{U}}$ and $\tilde{\Psi}^{-1} : \tilde{\mathfrak{U}} \longrightarrow \mathfrak{T}$ are both bijections.*

Let's consider what this identification will look like. When we consider $\mathcal{T}'$ from Figure 2.2, we see that $(x'_n)_{n=0}^{\infty}$ and $(x''_n)_{n=0}^{\infty}$, will be identified. This example illustrates how our identification will typically look. Suppose we take two sequences $(y_n)_{n=0}^{\infty}$ and $(y'_n)_{n=0}^{\infty}$ from some space $\mathfrak{U}$. If either of y_0 or y'_0 is not an endpoint of the prototile containing it, then these sequences will not be identified. The only way an identification might be possible is if, without loss of generality, $y_0 = P(c_0)^{\bullet}$ and $y'_0 = {}^{\bullet}P(c'_0)$, for some $c_0, c'_0 \in \mathcal{A}$. Further, it will be necessary that the word $c_0c'_0$ can appear in $\mathfrak{S}$. If y_0 and y'_0 satisfy these conditions, then an identification might be possible. We need to look at the rest of the sequence. If we see that $y_n = y'_n$ for all other n, then we will identify the sequences. Or, if $y_1 = P(c_1)^{\bullet}$ and $y'_1 = {}^{\bullet}P(c'_1)$, for some $c_1, c'_1 \in \mathcal{A}$, where $c_1c'_1$ is a legal word of $\mathfrak{S}$, then an identification might still be possible, and we look at the points y_2 and y'_2. We continue on in this way until our sequences agree on the remaining points, in which case we will identify them, or until we get two points y_n and y'_n which will be inconsistent with an identification. The only other possibility would be when for all n, we have $y_n = P(c_n)^{\bullet}$ and $y'_n = {}^{\bullet}P(c'_n)$, and the word $c_nc'_n$ is a legal word of $\mathfrak{S}$. In this case we will identify the sequences. For an example of this situation, consider the tiling in Figure 2.5. Both of the sequences $(y_n)_{n=0}^{\infty}$ and $(y'_n)_{n=0}^{\infty}$, where $y_n = P(A)^{\bullet}$ and $y'_n = {}^{\bullet}P(A)$, for all n, will be in $\Psi_1(\mathcal{T})$, and since AA is a legal word of $\mathfrak{S}_1$, they meet the above criteria for identification.

2.1.4 The Space $\mathfrak{L}$

All of the non trivial results in this section have been verified by Anderson and Putnam [6].

We can consider this space in slightly different way. Let's consider the space $\mathfrak{P}$ again, the disjoint topological union of the prototiles. Now, if it is possible for a partial tiling of type aa' to appear in our tiling space, we will identify the points $P(a)^{\bullet}$ and ${}^{\bullet}P(a')$. Performing all possible identifications of this form will give us a quotient space of $\mathfrak{P}$, and we will call this space $\mathfrak{M}$.

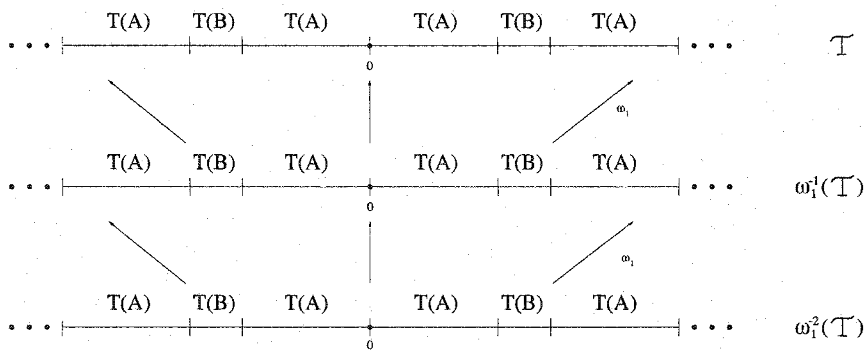

Figure 2.5: A tiling in $\mathfrak{T}_1$

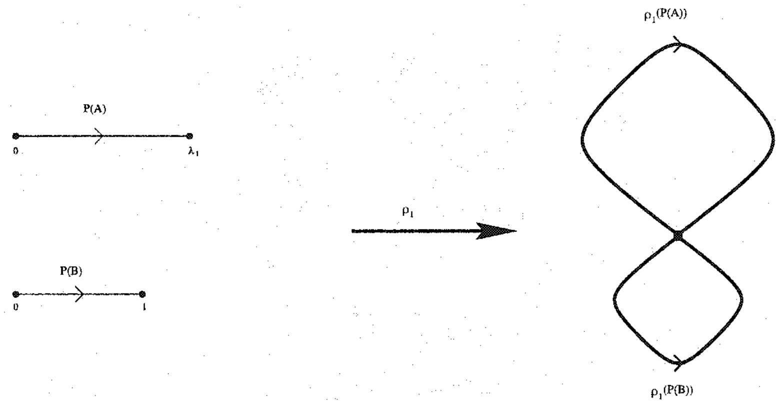

Figure 2.6: Formation of the space $\mathfrak{M}_1$.

Let $\rho : \mathfrak{P} \longrightarrow \mathfrak{M}$ be the identification map. We may also put a metric on this space in a natural way, and our first example of a topological dynamical system, given in Section 1.1, illustrates this metric for a simple example.

Let's consider what this space is for our examples. For the silver mean substitution we have our two prototiles $P(A)$ and $P(B)$, and the legal words of length two in this system are $\{AA, AB, BA\}$. So we will identify the endpoints $P(A)^\bullet$ and ${}^\bullet P(A)$, as well as the points $P(A)^\bullet$ and ${}^\bullet P(B)$, and the points $P(B)^\bullet$ and ${}^\bullet P(A)$. When we identify these points and apply the quotient topology, we form a new topological space $\mathfrak{M}_1$, which is a compact metric space, and is depicted in Figure 2.6. (Note that all the identification ρ_1 does is "bend" $P(A)$ and $P(B)$ so that their endpoints become identified. The arrows in this figure, as well as the arrows in Figure 2.7 are only present to help illustrate that the only deformation to these intervals occurs in "bending" them. If $0 < a < b < \lambda_1$, then if we would trace $\rho_1(P(A))$, we would start at $\rho_1(0)$, then come to $\rho_1(a)$, then $\rho_1(b)$, then finally to $\rho_1(\lambda_1)$, which is actually the same point as $\rho_1(0)$.)

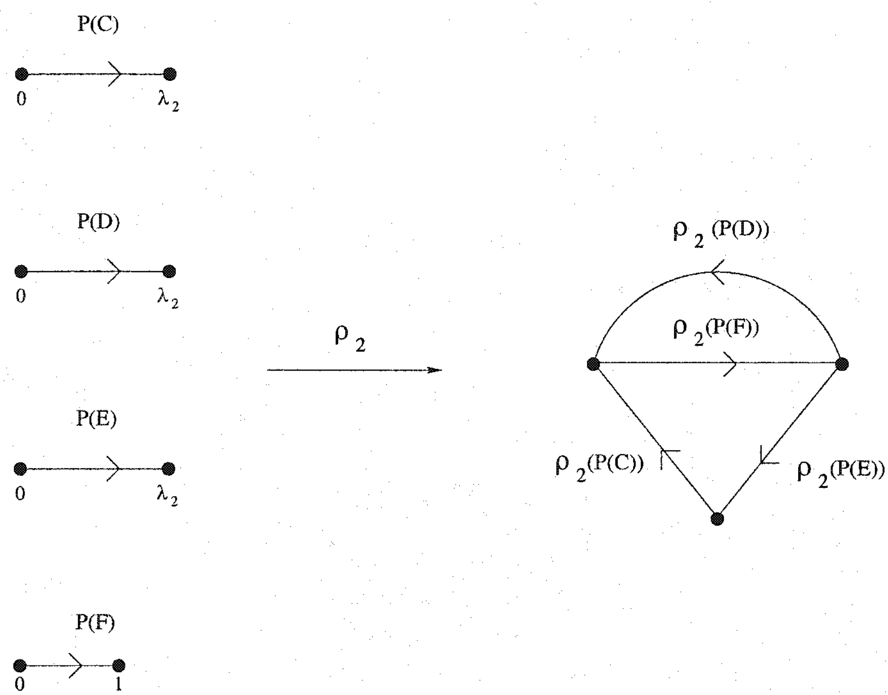

Figure 2.7: Formation of the space $\mathfrak{M}_2$

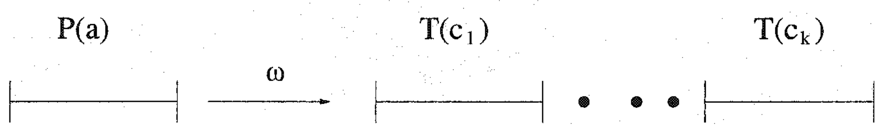

Figure 2.8: The action of ω on $P(a)$

For our new Fibonacci substitution we have the prototiles $P(C)$, $P(D)$, $P(E)$, and $P(F)$. The legal words of length two are $\{CF, FD, FE, DF, EC\}$, and when we make the corresponding identifications we get a compact metric space $\mathfrak{M}_2$, as in Figure 2.7.

Now ω will induce a continuous map from $\mathfrak{M}$ to $\mathfrak{M}$. Suppose that $\omega(a) = c_1 \ldots c_k$, for some $a, c_1, \ldots, c_k \in \mathcal{A}$. Then we know that ω will map the prototile $P(a)$ to a partial tiling of type $c_1 \ldots c_k$, as in Figure 2.8, where each $T(c_i)$ is just a translate of some prototile $P(c_i)$, and for each i, we will suppose that $T(c_i) = x_i + P(c_i)$ for some x_i. So we may subdivide $P(a)$ into k sections, say $P(a)_1, \ldots P(a)_k$, where ω maps each $P(a)_i$ exactly onto the tile $T(c_i)$. (We note here that these sections of $P(a)$ may share boundary points.) Now for each section, we will let $\omega(\rho(P(a)_i)) = \rho(P(c_i))$, and due to the fact that $\rho(P(c_i)^\bullet) = \rho(^\bullet P(c_{i+1}))$ for all $i = 1, \ldots, k-1$, this map will be continuous.

To get a better idea of how this mapping works, let's consider the silver mean example. We have already constructed our space $\mathfrak{M}_1$, and we will now

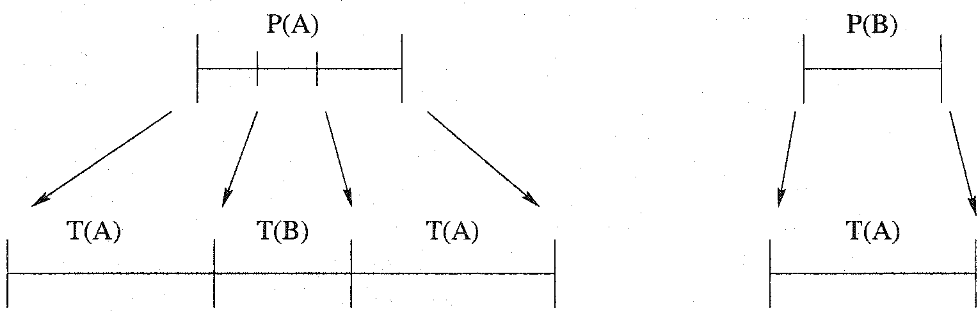

Figure 2.9: The action of ω_1 on the prototiles $P(A)$ and $P(B)$. Notice that $P(A)$ has been divided into three sections, corresponding to $P(A)_1$, $P(A)_2$, and $P(A)_3$.

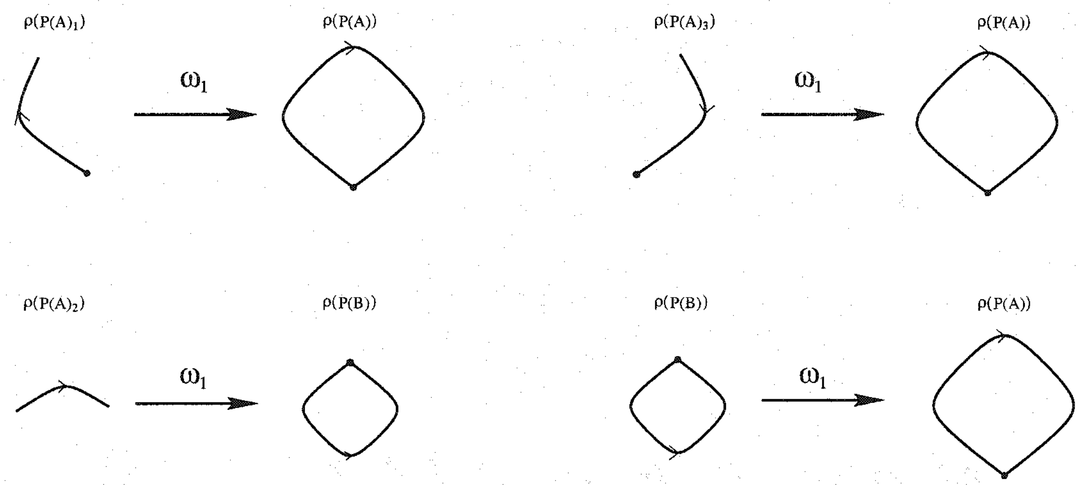

Figure 2.10: The action of ω_1 on $\mathfrak{M}_1$

consider how ω_1 will act on $\mathfrak{M}_1$. We know that ω_1 maps $P(A)$ to a partial tiling of type ABA, and we can subdivide $P(A)$ into three sections, $P(A)_1$ which is mapped to the first tile in $\omega(P(A))$, $P(A)_2$ which is mapped to the second tile in $\omega(P(A))$, and $P(A)_3$ which is mapped to the third tile in $\omega(P(A))$. This is shown in Figure 2.9.

So in $\mathfrak{M}_1$, we let $\omega(\rho(P(A)_1)) = \rho(P(A))$, $\omega(\rho(P(A)_2)) = \rho(P(B))$, and $\omega(\rho(P(A)_3)) = \rho(P(A))$. Now we also know that ω_1 maps $P(B)$ to a tile of type A, so we can let $\omega(\rho(P(B))) = \rho(P(A))$. This is depicted in Figure 2.10, and since $\rho(P(A)^\bullet) = \rho(^\bullet P(A)) = \rho(P(B)^\bullet) = \rho(^\bullet P(B))$, we can easily see why ω_1 is continuous.

We will now define an inverse limit space using $\mathfrak{M}$ and ω. We let $\mathfrak{L}$ be the inverse limit of $\mathfrak{M}$ relative to the map ω. Our space $\mathfrak{L}$ will consist of all infinite sequences $(x_i)_{i=0}^{\infty}$ of points in $\mathfrak{M}$ such that $\omega(x_{i+1}) = x_i$ for all $i = 0, 1, 2, \ldots$.

If we give the space $\Pi_{i=0}^{\infty}\mathfrak{M}$ the product topology, then we will let $\mathfrak{L}$ have the subspace topology in this space. A basis for this topology will consist of all sets of the form

$$\mathcal{B}^{\mathfrak{L}}_{U,n} := \{x \in \mathfrak{L} \mid x_i \in \omega^{n-i}(U) \text{ for } i = 0, 1, \ldots, n\}.$$

The substitution ω will also provide a natural map on $\mathfrak{L}$ given by $\omega(x)_i = \omega(x_i)$, and this map will have an inverse given by $\omega^{-1}(x)_i = x_{i+1}$.

At this point it is clear that when forming the space $\tilde{\mathfrak{U}}$ above, we would identify the sequences $(x_n)_{n=0}^{\infty}$ and $(x'_n)_{n=0}^{\infty}$ if and only if $\rho(x_n) = \rho(x'_n)$ for all n. Consequently, we have the following theorem.

Theorem 2.3. *The spaces $\tilde{\mathfrak{U}}$ and $\mathfrak{L}$ are homeomorphic.*

We may also establish the following theorem.

Theorem 2.4. *The dynamical systems $(\mathfrak{T}, \omega)$ and $(\mathfrak{L}, \omega)$ are topologically conjugate.*

Proof. We have a homeomorphism $\Gamma : \mathfrak{T} \longrightarrow \mathfrak{L}$. Let $\mathcal{T} \in \mathfrak{T}$, where $\omega^{-n}(\mathcal{T}) = \delta(n) + \mathcal{T}(\beta(n))$ for all $n \geq 0$. Then for every n, ${}^{\bullet}P(\beta(n)_0) - \delta(n) \in P(\beta(n)_0)$, and this point will be related to the location of the origin in $\mathcal{T}(\beta(n)_0)$. We let $x_n = \rho({}^{\bullet}P(\beta(n)_0) - \delta(n))$, for all n, and $\Gamma(\mathcal{T}) = (x_n)_{n=0}^{\infty}$. □

2.2 The Dynamical System $(\mathfrak{B}^*, \omega)$

Above we created a space $\mathfrak{U}$ and then identified some of its elements to form a space homeomorphic to $\mathfrak{T}$. We will again start with the space $\mathfrak{U}$, then we will create a space bijective to $\mathfrak{U}$, and then make the same identification in this new space. We use this approach to get a different look at the nature of the topology on $\mathfrak{T}$. We will use Bratteli diagrams in defining this space, so we will develop the needed theory below.

2.2.1 Bratteli Diagrams

We develop the theory by following Durand, Host, and Skau [7]. A *Bratteli diagram* is an infinite directed graph $(\mathcal{V}, \mathcal{E})$ such that the vertex set $\mathcal{V}$ and the edge set $\mathcal{E}$ can be partitioned into finite sets

$$\mathcal{V} = \mathcal{V}_0 \cup \mathcal{V}_1 \cup \mathcal{V}_2 \cup \ldots \text{ and } \mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2 \cup \ldots$$

with the following properties

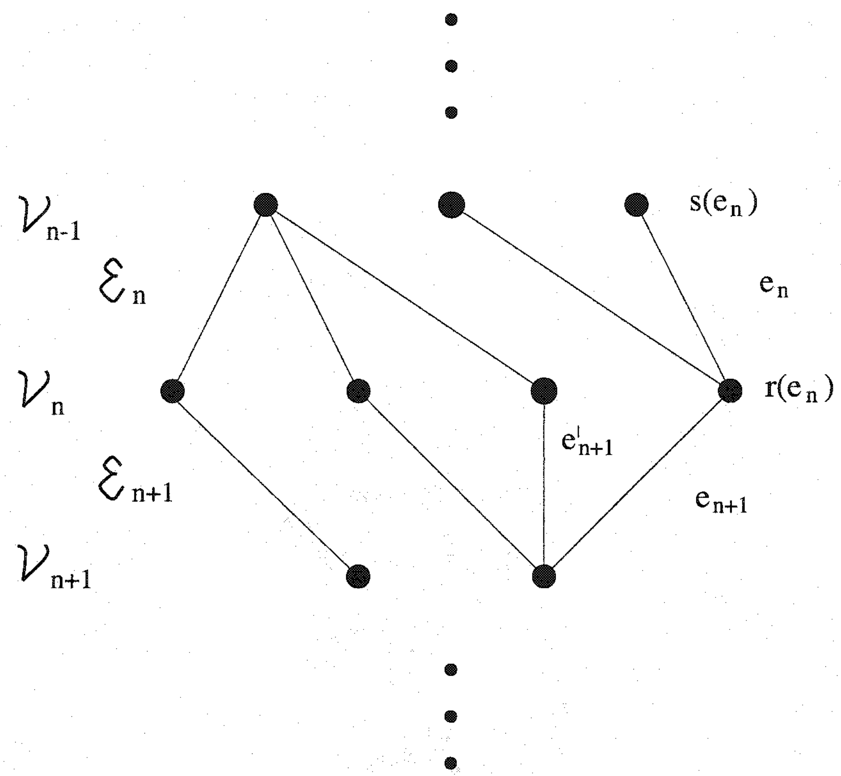

Figure 2.11: An example of a Bratteli diagram.

1. $\mathcal{V}_0 = \{v_0\}$ is a one point set.

2. $r(\mathcal{E}_n) \subset \mathcal{V}_n$, $s(\mathcal{E}_n) \subset \mathcal{V}_{n-1}$, $n = 1, 2, \ldots$, where r is the associated range map and s is the associated source map. Also, $s^{-1}(v) \neq \emptyset$ for all $v \in \mathcal{V}$ and $r^{-1}(v) \neq \emptyset$ for all $v \in \mathcal{V}\backslash\mathcal{V}_0$

It is convenient to represent a Bratteli diagram as a diagram with the vertices $\mathcal{V}_n$ at a horizontal level and with $\mathcal{E}_n$ as the downward edges connecting $\mathcal{V}_{n-1}$ to $\mathcal{V}_n$. Also, if $|\mathcal{V}_{n-1}| = t_{n-1}$ and $|\mathcal{V}_n| = t_n$, then $\mathcal{E}_n$ determines a $t_n \times t_{n-1}$ incidence matrix. See Figure 2.11 for an example. In this example we will note that $\mathcal{E}_n$ has incidence matrix $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ and $\mathcal{E}_{n+1}$ has incidence matrix $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$.

A Bratteli diagram is stationary if $k = |\mathcal{V}_1| = |\mathcal{V}_2| = \ldots$, and if, by an appropriate labelling of the vertices, the incidence matrix between level n and level $n+1$ is the same $k \times k$ matrix C for $n = 1, 2, \ldots$. In other words, beyond

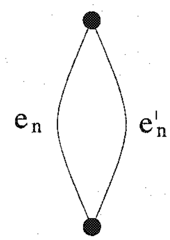

Figure 2.12: An illegal set of edges for a Bratteli diagram.

level one, the diagram repeats.

Throughout our discussion we will be dealing with stationary Bratteli diagrams with the property that there are no multiple edges between any two vertices, i.e., if $e_n, e'_n \in \mathcal{E}_n$, with $s(e_n) = s(e'_n)$, and $r(e_n) = r(e'_n)$, then $e_n = e'_n$. (All we have done here is prohibit the situation in Figure 2.12 from happening.)

Let $\mathcal{B} = (\mathcal{V}, \mathcal{E})$ be a Bratteli diagram. We will let $X_\mathcal{B}$ denote the associated *infinite path space* of $\mathcal{B}$,

$$X_\mathcal{B} := \{(e_1, e_2, \dots) \mid e_i \in \mathcal{E}_i, r(e_i) = s(e_{i+1}), i = 1, 2, \dots\}$$

Each element in this space corresponds to one of the ways you can trace an infinite path along the edges of the diagram. In Figure 2.11, we could have a path looking something like $(\dots, e_n, e_{n+1}, \dots)$, but we can not have a path like $(\dots, e_n, e'_{n+1}, \dots)$, since $r(e_n) \neq s(e'_{n+1})$. We let the cylinder sets

$$[e_m, \dots, e_n]_{i=m}^n := \{(f_1, f_2, \dots) \in X_\mathcal{B} \mid f_i = e_i \text{ for } m \leq i \leq n\}$$

form a basis for the topology on $X_\mathcal{B}$.

2.2.2 The Space $\mathfrak{B}_1$ for the silver mean substitution

Let's consider our space $\mathfrak{U}_1$ again. We will use it to define a space $\mathfrak{B}_1$, which is bijective to $\mathfrak{U}_1$. Then we will generalize the construction.

Let's suppose that $(x_n)_{n=0}^\infty \in \mathfrak{U}_1$, and $x_0 \in P(A)$. Any tiling $\mathcal{T}$ containing the tile $-x_0 + P(A)$ will have x_0 as the first point in one of the sequences in $\Psi(\mathcal{T})$. Now let's consider what the point x_1 might be, if all we know is x_0. Suppose that $x_1 \in P(c_1)$, where c_1 is either A or B. Then there must be some tiling $\mathcal{T}$ so that $\omega_1^{-1}(\mathcal{T})$ contains the tile $-x_1 + P(c_1)$ and $\mathcal{T}$ contains the tile $-x_0 + P(A)$. So the tile $-x_0 + P(A)$ must be contained in the partial tiling $\omega_1(-x_1 + P(c_1))$. In this substitution system there are only three ways that this can happen, and they are shown in Figure 2.13 a. The substitution ω_1

will produce two tiles of type A when it acts on a tile of type A, and it will produce one tile of type A when it acts on a tile of type B. When we further require that this tile of type A is shifted to match the tile $-x_0 + P(A)$, we are left with only three possible choices for x_1.

Similarily, if we had originally supposed that x_0 was in $P(B)$, we would only have one possible choice for the point x_1, and this is illustrated in Figure 2.13 b.

Suppose now that x_0 and x_1 have both been specified. Then, by the same argument as above, if x_1 is in a tile of type A, we will have three choices for what x_2 can be. If x_1 is in a tile of type B, we will have only one choice for x_2. Inductively, if x_n is in a tile of type A, there will be three choices for x_{n+1}, and if x_n is in a tile of type B, there will be one choice for what x_{n+1} can be.

Let's label these choices. If $x_n \in P(A)$, and we make the choice shown in Figure 2.13 a i, let's call this choice A_1. If $x_n \in P(A)$, and we make the choice shown in Figure 2.13 a ii, let's call this choice A_2. If $x_n \in P(A)$, and we make the choice shown in Figure 2.13 a iii, let's call this choice A_3. If $x_n \in P(B)$, then the only choice we can make is as in Figure 2.13 b i, and let's call this choice B_1. It's clear that we can represent every sequence $(x_n)_{n=0}^{\infty} \in \mathfrak{U}_1$ as the point x_0, along with the sequence of choices we would make in defining the remaining $(x_n)_{n=0}^{\infty}$. For example, consider the sequence $(x_n)_{n=0}^{\infty}$ that we created by looking at the tiling in Figure 2.1. We can consider this sequence as the sequence where $x_0 = -\delta_0 + {}^{\bullet}P(A)$, x_1 is the point we get from making choice A_3, x_2 is the point we get from making choice B_1, x_3 is the point we get from making choice A_1, x_4 is the point we get from making choice A_2, and so on. We can consider the sequence $(x'_n)_{n=0}^{\infty}$ that we created from the tiling Figure 2.2 to be the sequence created when we let $x'_0 = {}^{\bullet}P(A)$, x'_1 is the point we get from making choice A_1, x'_2 is the point we get from making choice A_3, x'_3 is the point we get from making choice B_1, x'_4 is the point we get from making choice A_3, and so on. We can consider the sequence $(x''_n)_{n=0}^{\infty}$ that we created from the tiling Figure 2.2 to be the sequence created when we let $x''_0 = P(A)^{\bullet}$, x''_1 is the point we get from making choice A_2, x''_2 is the point we get from making choice A_2, x''_3 is the point we get from making choice A_1, x''_4 is the point we get from making choice A_3, and so on.

We can use a Bratteli diagram to keep track of the legal sequences of choices. We will define a Bratteli diagram $\mathcal{B}_1$, which we will associate with the silver mean substitution. First we define the vertex sets. Let $\mathcal{V}_0 = \{v_0\}$. For all other n, let $\mathcal{V}_n = \{A_1^n, A_2^n, A_3^n, B_1^n\}$. Our intention is to let each vertex A_i^n (or B_1^n) signify making choice A_i (or B_1) when defining the point x_n. Then, if we choose our edges properly, each path through the diagram will define one

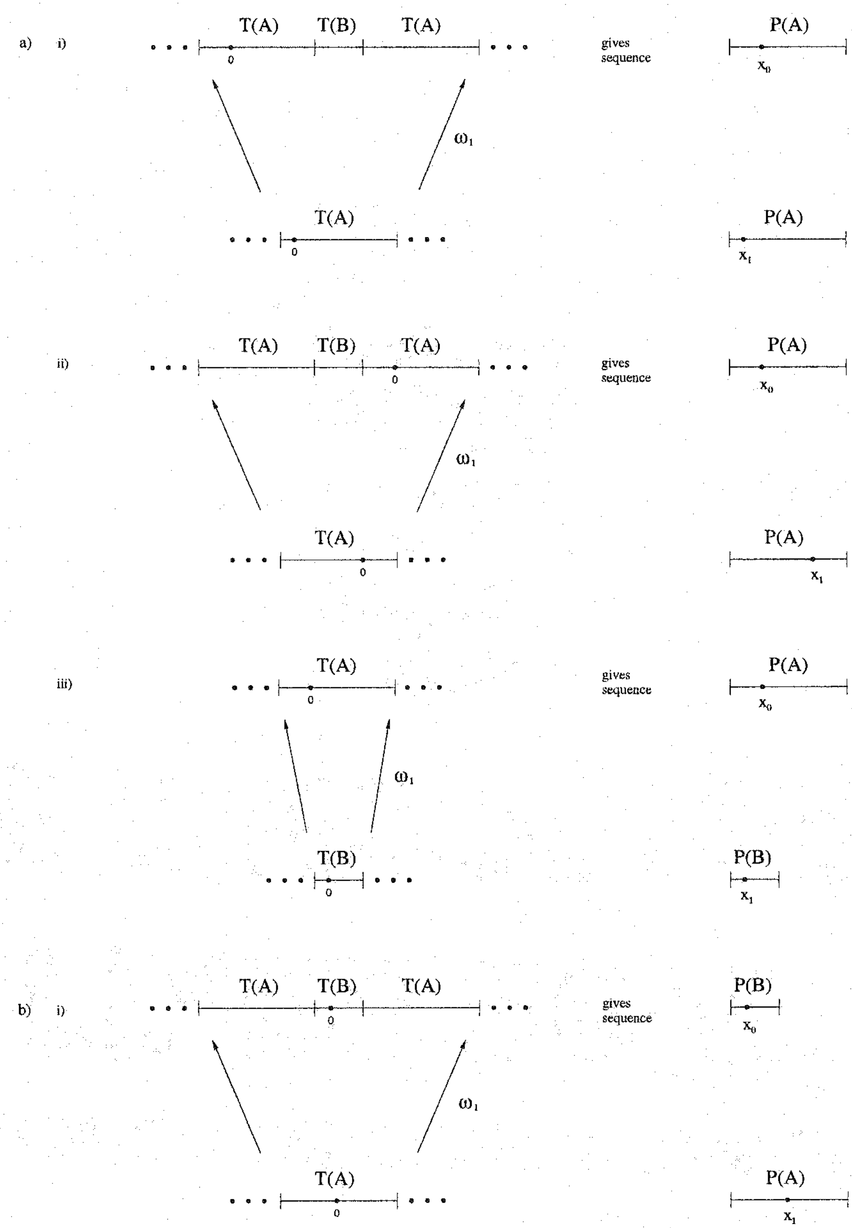

Figure 2.13: Possible ways of generating tiles in $\mathfrak{T}_1$.

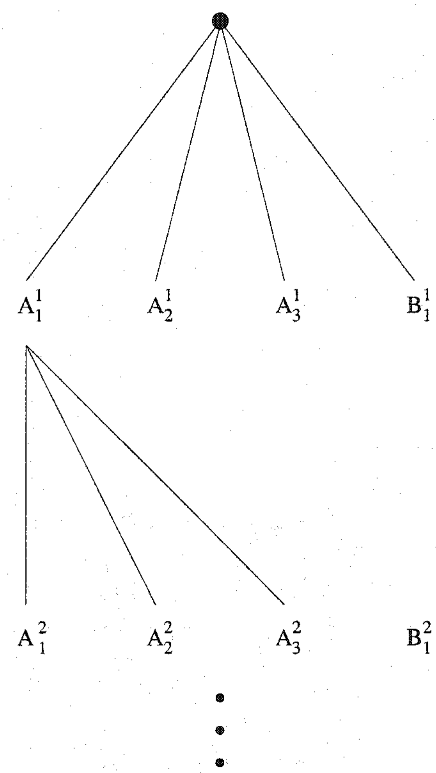

Figure 2.14: The set $\mathcal{E}_1$ and part of $\mathcal{E}_2$ for $\mathcal{B}_1$.

legal sequence of choices. We will let $\mathcal{E}_1$ consist of one edge from v_0 to each of A_1^1, A_2^1, A_3^1, and B_1^1. Now let's consider what the set $\mathcal{E}_2$ must look like. If our first choice is A_1, then x_1 will be in a tile of type A. So we need to be able to make one of the choices A_1, A_2, or A_3. So we will draw one edge from A_1^1 to each of A_1^2, A_2^2, and A_3^2. This is shown in Figure 2.14. Similarily, we will need one edge from A_2^1 to each of A_1^2, A_2^2, and A_3^2 and one edge from B_1^1 to each of A_1^2, A_2^2, and A_3^2. We will also need one edge from A_3^1 to B_1^2. By the same reasoning we can inductively define all other sets $\mathcal{E}_n$. We will need one edge from A_1^{n-1} to each of A_1^n, A_2^n, and A_3^n, one edge from A_2^{n-1} to each of A_1^n, A_2^n, and A_3^n, one edge from B_1^{n-1} to each of A_1^n, A_2^n, and A_3^n, and one edge from A_3^{n-1} to B_1^n. The resulting Bratteli diagram $\mathcal{B}_1$ is shown in Figure 2.15.

Now we can see that if $x_0 \in P(A)$, then every path that starts with an edge to one of A_1^1, A_1^2, or A_1^3 will define an element in $\mathfrak{U}_1$. A path that starts with B_1^1 will not make sense, because it would require x_0 to be in $P(B)$. So we define

$$X_{\mathcal{B}_1}(A) := \{\xi \in \mathcal{B}_1 \mid \text{the first edge in } \xi \text{ is to one of } A_1^1, A_1^2, \text{ or } A_1^3\}.$$

Similarily, if $x_0 \in P(B)$, then every path that starts with B_1^1 will define an

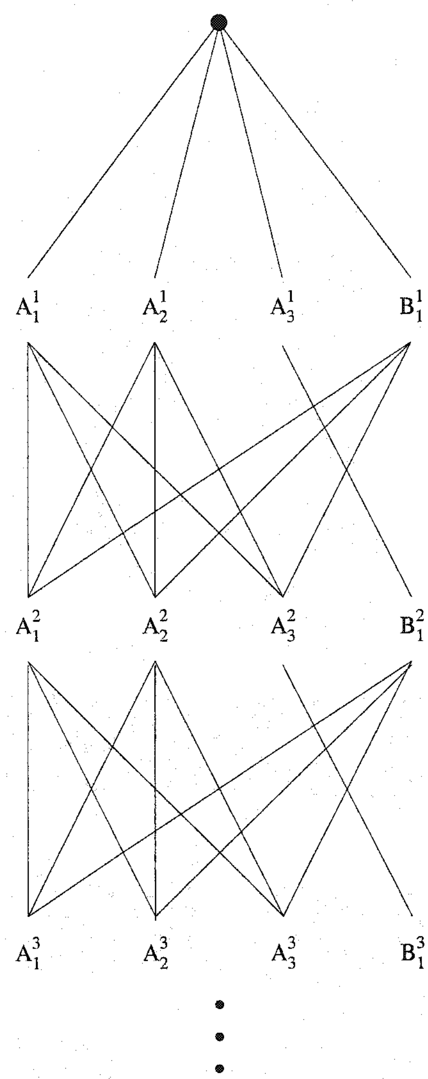

Figure 2.15: The Bratteli diagram $\mathcal{B}_1$.

element in $\mathfrak{U}_1$, while any other path will not make sense. We define

$$X_{\mathcal{B}_1}(B) := \{\xi \in \mathcal{B}_1 \mid \text{the first edge in } \xi \text{ is to } B_1^1\}.$$

Now we can define the space $\mathfrak{B}_1$,

$$\mathfrak{B}_1 := \{P(A) \times X_{\mathcal{B}_1}(A)\} \cup \{P(B) \times X_{\mathcal{B}_1}(B)\}.$$

It is clear from the construction of $\mathfrak{B}_1$ that $\mathfrak{B}_1$ and $\mathfrak{U}_1$ are bijective. If $(x_n)_{n=0}^{\infty} \in \mathfrak{U}_1$, we define $\Upsilon_1 : \mathfrak{U}_1 \longrightarrow \mathfrak{B}_1$ by $\Upsilon_1((x_n)_{n=0}^{\infty}) = (x, \xi)$, where $x = x_0$, and ξ is the path that corresponds to the choices made in selecting each point x_n. Let $\Upsilon_1^{-1} : \mathfrak{B}_1 \longrightarrow \mathfrak{U}_1$ as follows. If $(x, \xi) \in \mathfrak{B}$, where ξ corresponds to making choice C_1, then choice C_2, and so on, then let $\Upsilon_1((x, \xi)) = (x_n)_{n=0}^{\infty}$, where the point $x_0 = x$, the point x_1 corresponds to making choice C_1, the point x_2 corresponds to making choice C_2, and so on. By using this bijection we can now define a relation

$$\Phi_1 := \Upsilon_1 \circ \Psi_1 : \mathfrak{T}_1 \longrightarrow \mathfrak{B}_1$$

and a map

$$\Phi^{-1} := \Psi_1^{-1} \circ \Upsilon_1^{-1} : \mathfrak{B}_1 \longrightarrow \mathfrak{T}_1.$$

2.2.3 The space $\mathfrak{B}$ in general

It is easy to generalize the process we used in constructing $\mathfrak{B}_1$. First we will form the Bratteli diagram $\mathcal{B}$. We let $\mathcal{V}_0 = \{v_0\}$. Now consider any $c \in \mathcal{A}$, and suppose that our substitution gives us m ways to generate a tile of type c. Let's label these choices of how to generate c as $c_1, \ldots, c_m$. Then for every $n \geq 1$, we let $c_i^n \in \mathcal{V}_n$, for $1 \leq i \leq m$. For the edge sets, first we will let $\mathcal{E}_1$ consist of one edge from v_0 to each element in $\mathcal{V}_1$. We define all other $\mathcal{E}_n$ as follows. Suppose that the choice c_i corresponds to a situation where ω generates a tile of type c by acting on a tile of type d. Then, for every $n \geq 1$, we need to draw an edge from $c_i^n \in \mathcal{V}_n$ to every element of the form $d_j^{n+1} \in \mathcal{V}_{n+1}$. This process will define all edge sets $\mathcal{E}_n$, and will give us a stationary Bratteli diagram. We let

$$X_{\mathcal{B}}(c) := \{\xi \in X_{\mathcal{B}} \mid \text{the first edge of } \xi \text{ goes to } c_i^1 \text{ for some } 1 \leq i \leq m\}.$$

Then we have the space

$$\mathfrak{B} := \bigcup_{c \in \mathcal{A}} P(c) \times X_{\mathcal{B}}(c).$$

Now let $(x,\xi) \in \mathfrak{B}$, where $x \in P(a)$ and the path ξ corresponds to the choice C_1, then the choice C_2, and so on. Then, similarily to how we defined Υ_1, we can define $\Upsilon : \mathfrak{U} \longrightarrow \mathfrak{B}$ by $\Upsilon((x_n)_{n=0}^{\infty}) = (x,\xi)$, where $x = x_0$, and ξ is the path determined by the choices made in defining each other x_n. Due to the similarities between the spaces $\mathfrak{B}$ and $\mathfrak{U}$, the following theorem is now clear.

Theorem 2.5. *The map $\Upsilon : \mathfrak{U} \longrightarrow \mathfrak{B}$ is a bijection. Further, we can define a relation $\Phi := \Upsilon \circ \Psi : \mathfrak{T} \longrightarrow \mathfrak{B}$, and $\Phi(\mathcal{T})$ will contain exactly one element if the origin of $\mathcal{T}$ does not lie on the boundary of a tile, and $\Phi(\mathcal{T})$ will contain exactly two elements if the origin of $\mathcal{T}$ does lie on the boundary of a tile. We also have a well defined map $\Phi^{-1} := \Psi^{-1} \circ \Upsilon^{-1} : \mathfrak{B} \longrightarrow \mathfrak{T}$ where $\Phi^{-1}((x,\xi)) = \mathcal{T}$ if and only if $(x,\xi) \in \Phi(\mathcal{T})$.*

The map Φ^{-1} is easy to understand. For any $(x,\xi) \in \mathfrak{B}$, form the sequence $\Upsilon^{-1}((x,\xi))$, and then create the tiling associated with this sequence.

We can put a topology on $\mathfrak{B}$. Recall that the space $X_{\mathcal{B}}$ has the topology given when the cylinder sets form a basis. We can put the subspace topology on each of the spaces $X_{\mathcal{B}}(c)$. So the set of all cylinder sets that are contained in $X_{\mathcal{B}}(c)$ will form a basis for the topology on $X_{\mathcal{B}}(c)$. Each space $P(c)$ has the subspace topology from the usual topology on $\mathbb{R}$. So we can give each space $P(c) \times X_{\mathcal{B}}(c)$ the product topology from these two topologies. Then we can let $\mathfrak{B}$ be the disjoint topological union of these spaces. If the point $(x,\xi) \in \mathfrak{B}$, where $x \in ({}^{\bullet}P(a), P(a)^{\bullet})$ and $\xi \in X_{\mathcal{B}}(a)$, then the set of all basis elements

$$(y,z) \times [e_1,\ldots,e_N]_{i=1}^{N},$$

where $x \in (y,z) \subset P(a)$ and $\xi \in [e_1,\ldots,e_N]_{i=1}^{N}$, will form a fundamental system of neighbourhoods for (x,ξ). If the point $(x,\xi) \in \mathfrak{B}$, where $x = P(a)^{\bullet}$ and $\xi \in X_{\mathcal{B}}(a)$, then the set of all basis elements

$$(y, P(a)^{\bullet}] \times [e_1,\ldots,e_N]_{i=1}^{N},$$

where $(y, P(a)^{\bullet}] \subset P(a)$ and $\xi \in [e_1,\ldots,e_N]_{i=1}^{N}$, will form a fundamental system of neighbourhoods for (x,ξ). If the point $(x,\xi) \in \mathfrak{B}$, where $x = {}^{\bullet}P(a)$ and $\xi \in X_{\mathcal{B}}(a)$, then the set of all basis elements

$$[{}^{\bullet}P(a), z) \times [e_1,\ldots,e_N]_{i=1}^{N},$$

where $[{}^{\bullet}P(a), z) \subset P(a)$ and $\xi \in [e_1,\ldots,e_N]_{i=1}^{N}$, will form a fundamental system of neighbourhoods for (x,ξ).

2.2.4 The space $\mathfrak{B}^*$

We can now define the space $\mathfrak{B}^*$. We start with the space $\mathfrak{B}$, and if $\Phi^{-1}((x,\xi)) = \Phi^{-1}((x',\xi'))$, then we will identify (x,ξ) and (x',ξ'). We call the resulting space $\mathfrak{B}^*$, and we will let $^* : \mathfrak{B} \longrightarrow \mathfrak{B}^*$ be our identification map. We let Φ^* and $\Phi^{*^{-1}}$ be the naturally induced maps between $\mathfrak{B}^*$ and $\mathfrak{T}$. The following theorem follows from the relationship between the spaces $\mathfrak{B}$ and $\mathfrak{U}$, and Theorem 2.2.

Theorem 2.6. *The maps $\Phi^* : \mathfrak{T} \longrightarrow \mathfrak{B}^*$ and $\Phi^{*^{-1}} : \mathfrak{B}^* \longrightarrow \mathfrak{T}$ are both bijections.*

We have also established the following.

Theorem 2.7. *The spaces $\mathfrak{T}$, $\mathfrak{L}$, and $\mathfrak{B}^*$ are all bijective to each other.*

We give $\mathfrak{B}^*$ the quotient topology under the map *. So, if $(x,\xi)^* \in \mathfrak{B}^*$, where $x \in ({}^\bullet P(a), P(a)^\bullet)$ and $\xi \in X_{\mathcal{B}}(a)$, then the set of all basis elements

$$((y,z) \times [e_1,\ldots,e_N]_{i=1}^N)^*,$$

where $x \in (y,z) \subset P(a)$ and $\xi \in [e_1,\ldots,e_N]_{i=1}^N$, will form a fundamental system of neighbourhoods for (x,ξ). On the other hand, if $(x,\xi)^* \in \mathfrak{B}^*$, where $x = {}^\bullet P(a)$ and $\xi \in X_{\mathcal{B}}(a)$, then there will be exactly one other $(x',\xi') \in \mathfrak{B}$ such that $(x,\xi)^* = (x',\xi')^*$, where $x' = P(a')^\bullet$ and $\xi' \in X_{\mathcal{B}}(a')$, for some $a' \in \mathcal{A}$. Then the set of all basis elements

$$(\{(y, P(a)^\bullet] \times [e_1,\ldots,e_N]_{i=1}^N\}$$
$$\cup\{[{}^\bullet P(a'), z) \times [e'_1,\ldots,e'_{N'}]_{i=1}^{N'}\})^*,$$

where $(y, P(a)^\bullet] \subset P(a)$, $[{}^\bullet P(a'), z) \subset P(a')$, $\xi \in [e_1,\ldots,e_N]_{i=1}^N$, and $\xi' \in [e'_1,\ldots,e'_{N'}]_{i=1}^{N'}$ will form a fundamental system of neighbourhoods for $(x,\xi)^*$.

Theorem 2.8. *The maps $\Phi^* : \mathfrak{T} \longrightarrow \mathfrak{B}^*$ and $\Phi^{*^{-1}} : \mathfrak{B}^* \longrightarrow \mathfrak{T}$ are both continuous. Consequently, the spaces $\mathfrak{T}$, $\mathfrak{L}$, and $\mathfrak{B}^*$ are all homeomorphic.*

Proof. Let B_F be the integer that is associated with the property of forcing the border for this substitution system.

Let $U \subset \mathfrak{T}$ be open, and let $\mathcal{T} \in U$, where $\omega^{-n}(\mathcal{T}) = \delta(n) + \mathcal{T}(\beta(n))$, for all n. Then there is a basis element $\mathcal{U}_\epsilon(\mathcal{T}) \subset U$. Let $\Phi^*(\mathcal{T}) = (x,\xi)^* \in \mathfrak{B}^*$, where $x \in P(a)$ and $\xi \in X_{\mathcal{B}}(a)$, for some $a \in \mathcal{A}$. We have two cases to consider.

The first case is when x is not an endpoint of $P(a)$.

Then we can choose some $\delta < \epsilon$ so that $(-\delta + x, \delta + x) \subset P(a)$. We can also choose an integer $N > 0$ such that $\lambda^{N-B_F}\ell_s > 1/\epsilon + \delta$. Now take any

$(x', \xi') \in V := (-\delta + x, \delta + x) \times [e_1, \ldots, e_N]_{i=1}^N$. The point x' will not be an endpoint of $P(a)$, so $\Phi^{*^{-1}}((x', \xi')^*) = \Phi^{-1}((x', \xi')) = \mathcal{T}'$, for some $\mathcal{T}' \in \mathfrak{T}$. Now the origin of $\mathcal{T}$ and the origin of $\mathcal{T}'$ will be in the same type of tile, since they are both in $\Phi^{*^{-1}}(V)$, and if we shift $\mathcal{T}'$ by some v with $|v| < \delta$, then $\mathcal{T}$ and $-v + \mathcal{T}'$ will agree on the tile containing the origin, which will be $\delta(0) + \mathcal{T}(\beta(0)_0)$. Also, since the paths ξ and ξ' will agree on the first N edges, the tiling $\omega^{-N}(\mathcal{T})$ will generate the tile containing the origin in $\mathcal{T}$ in the same way as the tiling $\omega^{-N}(-v + \mathcal{T}')$ will generate the origin in $-v + \mathcal{T}$. So $\omega^{-N}(\mathcal{T})$ and $\omega^{-N}(-v + \mathcal{T}')$ will agree on the tile containing the origin, and by forcing the border, the tilings $\mathcal{T}$ and $-t + \mathcal{T}'$ will agree on the interval $[-\lambda^{N-B_F}\ell_s, \lambda^{N-B_F}\ell_s]$, which contains the interval $[-1/\epsilon - \delta, 1/\epsilon + \delta]$. So $\mathcal{T}' \in \mathcal{U}_\epsilon(\mathcal{T})$, and $\Phi^{*^{-1}}(V) \subset U$. So $\Phi^*(U)$ is open.

The other case to consider is when x is an endpoint of $P(a)$, and without loss of generality, assume $x = P(a)^\bullet$. Then there will be exactly one other $(x', \xi') \in \mathfrak{B}$, such that $(x, \xi) \neq (x', \xi')$, but $(x, \xi)^* = (x', \xi')^*$. We will know that $x' = {}^\bullet P(a')$, and $\xi' \in X_\mathcal{B}(a')$ for some $a' \in \mathcal{A}$. By a similar argument as above, we can find some $\delta < \epsilon$, and some integer $N > 0$, with $\lambda^{N-B_F}\ell_s \geq 1/\epsilon + \delta$ so that the basis element

$$V = (\{(-\delta + P(a)^\bullet, P(a)^\bullet] \times [e_1, \ldots, e_N]_{i=1}^N\}$$

$$\cup\{[{}^\bullet P(a'), \delta + {}^\bullet P(a')) \times [e_1', \ldots, e_N']_{i=1}^N\})^*,$$

will satisfy $\Phi^{*^{-1}}(V) \subset U$. So $\Phi^*(U)$ is open.

So the map $\Phi^{*^{-1}}$ is continuous.

Now lets take any open set $V \subset \mathfrak{B}^*$, and let $(x, \xi)^* \in V$, where $x \in P(a)$ and $\xi \in X_\mathcal{B}(a)$. Let $\Phi^{*^{-1}}((x, \xi)^*) = \mathcal{T} \in \mathfrak{T}$. First we will suppose that x is not an endpoint of $P(a)$.

So we have some basis element

$$W := ((-\delta + x, \delta + x) \times [e_1, \ldots, e_N]_{i=1}^N)^* \subset V.$$

Now since $\omega : \mathfrak{T} \longrightarrow \mathfrak{T}$ is invertible, if we look at a sufficiently large patch around the origin in the tiling $\mathcal{T}$, we can determine what type of tile contains the origin in $\omega^{-1}(\mathcal{T})$, and how much it has been shifted. Further, we can choose a $R > 0$, so that a ball of radius R around the origin in any tiling of $\mathfrak{T}$ will determine the tile containing the origin in the inverse of that tiling. It follows that a ball of radius $\lambda R + R = (\lambda + 1)R$ around the origin in $\mathcal{T}$ will determine which tiles lie in a ball of radius R around the origin in $\omega^{-1}(\mathcal{T})$, which will determine which tile contains the origin in $\omega^{-2}(\mathcal{T})$. Inductively, a ball of radius $(\lambda+1)^{N-1}R$ will determine which tile contains the origin in $\omega^{-N}(\mathcal{T})$.

Choose an $\epsilon > 0$ such that $\epsilon < \delta/2$, and $1/\epsilon - 2\epsilon > (\lambda+1)^{N-1}R$. Then choose any $\mathcal{T}' \in \mathcal{B}_\epsilon(\mathcal{T})$. There is some v with $|v| < \epsilon < \delta$ so that $\mathcal{T}$ and $-t+\mathcal{T}'$ agree on a ball of radius $(\lambda+1)^{N-1}R$ around the origin. Further, for each $n \leq N$, $\omega^{-n}(\mathcal{T})$ and $\omega^{-n}(-t+\mathcal{T}')$ will agree on a ball of radius $(\lambda+1)^{N-n-1}R$ around the origin. This forces the tile containing the origin in $\mathcal{T}$ and the tile containing the origin in $-t+\mathcal{T}'$ to be generated in the same way by the tilings $\omega^{-N}(\mathcal{T})$ and $\omega^{-N}(-t+\mathcal{T})$, respectivley. So $\Phi^*(\mathcal{T}) \in W$, and $\Phi^{*^{-1}}(V)$ is open.

The other case is when x is an endpoint of $P(a)$. Without loss of generality, assume that $x = P(a)^\bullet$. Then there is some other $(x', \xi') \in \mathfrak{B}$, with $x' = {}^\bullet P(a')$, and $\xi' \in X_{\mathcal{B}}(a')$, such that $(x,\xi)^* = (x',\xi')^*$. There will be some basis element

$$W := (\{(-\delta + P(a)^\bullet, P(a)^\bullet] \times [e_1, \ldots, e_N]_{i=1}^N\}$$
$$\cup\{[{}^\bullet P(a'), \delta + {}^\bullet P(a')) \times [e'_1, \ldots, e'_N]_{i=1}^N\})^* \subset V.$$

By a similar argument as above, if we choose an $\epsilon > 0$ such that $\epsilon < \delta/2$, and $1/\epsilon - 2\epsilon > (\lambda+1)^{N-1}R$, then $\Phi^*(\mathcal{B}_\epsilon(\mathcal{T})) \subset W$. So $\Phi^{*^{-1}}(V)$ is open

So the map Φ^* is continuous.

□

2.2.5 The map ω on $\mathfrak{B}^*$

Since the maps $\Phi^* : \mathfrak{T} \longrightarrow \mathfrak{B}^*$ and $\omega : \mathfrak{T} \longrightarrow \mathfrak{T}$ are both continuous and invertible, we may define

$$\omega := \Phi^* \circ \omega \circ \Phi^{*^{-1}} : \mathfrak{B}^* \longrightarrow \mathfrak{B}^*.$$

Further, this map will be both continuous and invertible, and by definition, it intertwines with Φ^*. This gives us the following result.

Theorem 2.9. *The topological dynamical systems $(\mathfrak{T}, \omega)$, $(\mathfrak{L}, \omega)$, and $(\mathfrak{B}^*, \omega)$ are all topologically conjugate dynamical systems.*

To get a better idea of how the substitution ω will act on $\mathfrak{B}^*$, let's consider an example. Let $(x,\xi)^* \in \mathfrak{B}_1^*$, where $\Phi^{*^{-1}}((x,\xi)^*) = \omega_1^{-1}(\mathcal{T})$, where $\mathcal{T}$ is the tiling in Figure 2.1. So $x = -\delta_1 + {}^\bullet P(B)$, and ξ is the path corresponding choice B_1, then choice A_1, then choice A_2, and so on. We will have $\omega((x,\xi)^*) = (x',\xi')^*$ where $x' = -\delta_0 + {}^\bullet P(A)$, and ξ' is the path corresponding to choice A_3, then B_1, then choice A_1, then choice A_2, and so on. We have $\omega^{-1}((x,\xi)^*) = (x'',\xi'')^*$ where $x'' = -\delta_2 + {}^\bullet P(A)$, and ξ'' is the path corresponding to choice A_1, then choice A_2, and so on.

Chapter 3

Higher Dimensions and The Cantor Set

3.1 Higher Dimensions

So far, for simplicity, we have limited our discussion to one dimensional tilings. However, all of the ideas involved also work in higher dimensions. Given a substitution tiling dynamical system $\mathfrak{T}$, in any dimension, we can always find a topologically conjugate system that forces the border. Then, once we are in the system that forces the border, we can always form the the topologically conjugate inverse limit system $(\mathfrak{L}, \omega)$, and the topologically conjugate Bratteli diagram system $(\mathfrak{B}^*, \omega)$. We will briefly look at how to do this for the chair tiling substitution system in $\mathbb{R}^2$. For a more rigorous and general approach to the spaces $\mathfrak{T}$ and $\mathfrak{L}$ in higher dimensions, refer to [6].

We will start with a set of prototiles and a substitution rule. The substitution will inflate each tile by some constant λ, and exactly subdivide this larger tile into a set of translates of our prototiles. For the chair tiling, the prototiles $P(A)$, $P(B)$, $P(C)$, and $P(D)$, the substitution ω_3, are shown in Figure 3.1, and the inflation constant is 2.

Any way that we can arrange translates of these prototiles to cover all of $\mathbb{R}^2$ will be a tiling of $\mathbb{R}^2$. If $\mathcal{T}$ is one of these tilings, and if any partial tiling $\mathcal{P} \subset \mathcal{T}$ will be a subset of $\omega_3^N(-x + P)$, for some $x \in \mathbb{R}^2$, and $N > 0$, where P is one of our prototiles, then $\mathcal{T}$ is a substitution tiling, under ω_3. We let $\mathfrak{T}_3$ be the set of all substitution tilings for ω_3. A section from a tiling of $\mathfrak{T}_3$ is shown in Figure 3.2.

We can put a metric and topology on $\mathfrak{T}_3$ in the same way as we did for a

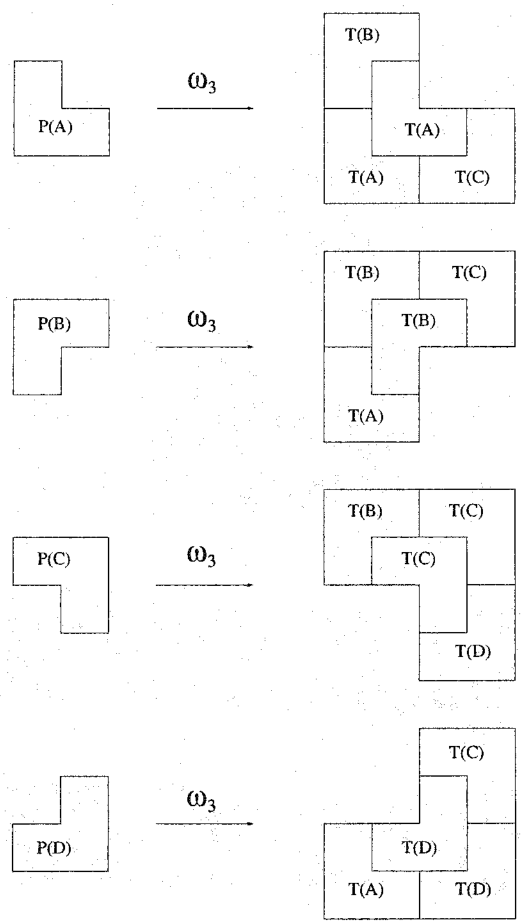

Figure 3.1: The prototiles and substitution rule for the chair tiling.

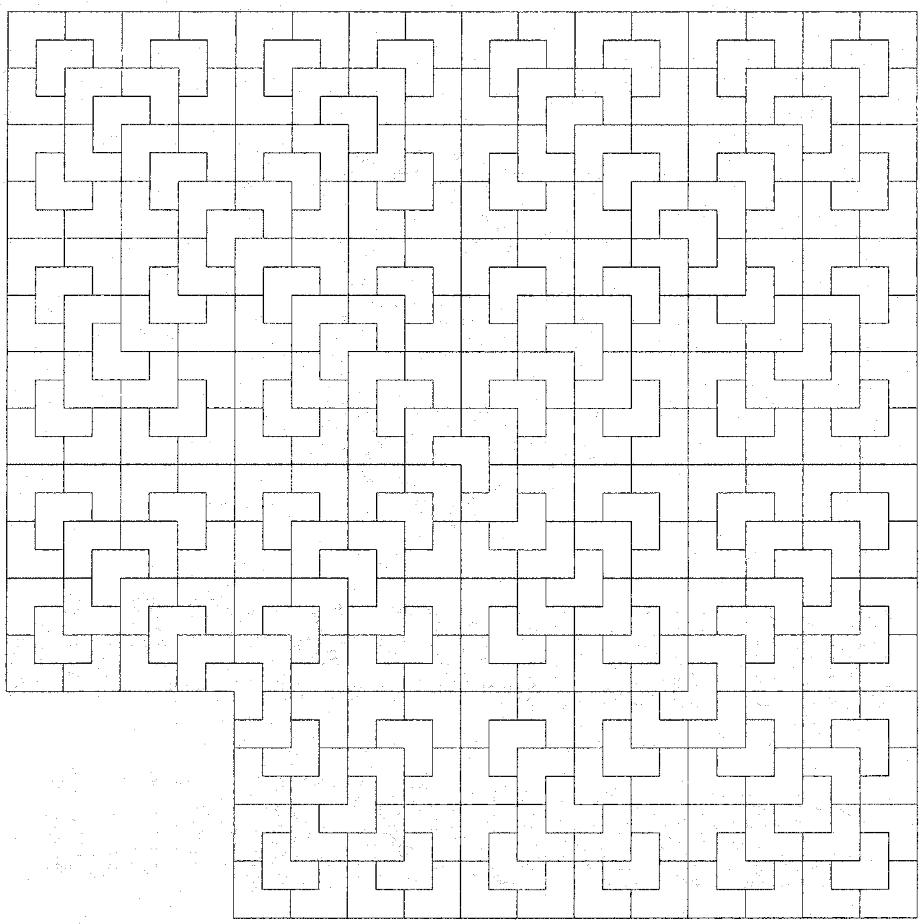

Figure 3.2: A section of a tiling in $\mathfrak{T}_3$.

one dimensional system. We put a metric on $\mathfrak{T}_3$ by

$$d(\mathcal{T},\mathcal{T}') := \inf(\{1/\sqrt{2}\}\cup\{\epsilon \mid \mathcal{B}_{1/\epsilon}(-v+\mathcal{T}) = \mathcal{B}_{1/\epsilon}(-v'+\mathcal{T}') \text{ for some } ||v||, ||v'|| < \epsilon\}),$$

and define a topology where the sets

$$\mathcal{U}_\epsilon(\mathcal{T}) := \{\mathcal{T}' \mid d(\mathcal{T},\mathcal{T}') < \epsilon\}$$

form a basis. The substitution ω_3 extends to an invertible and continuous map from $\mathfrak{T}_3$ to $\mathfrak{T}_3$,, and the space $(\mathfrak{T}_3, \omega_3)$ is a topological dynamical system.

This system does not force the border. In higher dimensions, forcing the border will require that for some B_F, and for any tile T in a tiling $\mathcal{T}$, all of the tiles that border the partial tiling $\omega_3^{B_F}(T)$, and how they share boundary points with $\omega_3^{B_F}(T)$, will be completely determined by what type of tile T is. It is straightforward to see that each of the four types of tiles in $\mathfrak{T}_3$ can be collared (surrounded) in 14 ways, so we can create a new substitution tiling with the prototiles $P(A_1), \dots, P(A_{14})$, $P(B_1), \dots, P(B_{14})$, $P(C_1), \dots, P(C_{14})$, $P(D_1), \dots, P(D_{14})$ (one for each type of collared tile). For example, we can let $P(A_1)$ correspond to the situation where a tile of type A is collared as in Figure 3.3 a, we can let $P(A_2)$ correspond to the situation where a tile of type A is collared as in Figure 3.3 b, we can let $P(B_3)$ correspond to the situation where a tile of type B is collared as in Figure 3.3 c, and so on. We will replace $\mathfrak{T}_3$ with this new space (so $\mathfrak{T}_3$ now refers to the space with 56 prototiles), and redefine ω_3 accordingly. We are now in a system that forces the border.

We can let $\mathfrak{P}_3$ be the disjoint topological union of the prototiles, and we will identify the points $x \in P$ and $x' \in P'$ if there is some tiling in $\mathfrak{T}_3$ that contains the tiles $-t+P$ and $-t'+P'$, and $-t+x = -t'+x'$. This identification will form the space $\mathfrak{M}_3$. So if we have a partial tiling as in Figure 3.4 appear in some tiling in $\mathfrak{T}_3$, then we will have to make many identifications. A couple of the identifications that will be necessary are to identify the points x, x', and x'', and also to identify the points y, y', where these points are shown in Figure 3.5. (We notice that in higher dimensions the identifications we will make in defining the spaces $\mathfrak{M}$, and $\mathfrak{B}^*$ may contain more than two points. However, there will always be a finite bound, depending on the system we are considering, on how many points of $\mathfrak{P}$ (or $\mathfrak{B}$) will be identified to any particular point of $\mathfrak{M}$ (or $\mathfrak{B}^*$).) Let $\rho_3 : \mathfrak{P}_3 \longrightarrow \mathfrak{M}_3$ be our identification map.

We can define ω_3 on $\mathfrak{M}_3$ in the same way that we did in one dimension. If ω_3 maps the point x in the prototile P to the point $-t+x'$ in the tile $-t+P'$, then we let ω_3 map the point $\rho(x)$ to $\rho(x')$. If we look at Figure 3.6, then we see that the point x in the prototile $P(A_i)$ is mapped to the point x' in the tile

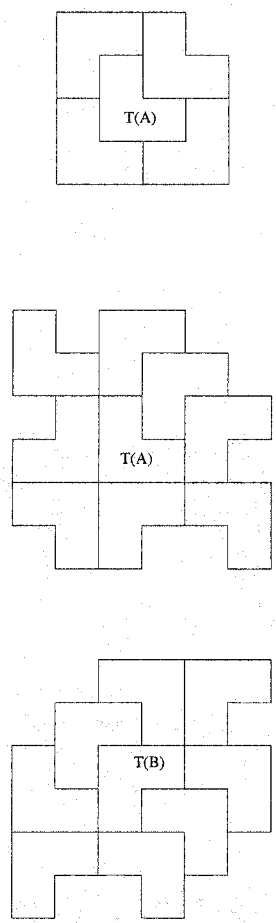

Figure 3.3: Three ways of collaring a tile in $\mathfrak{T}_3$.

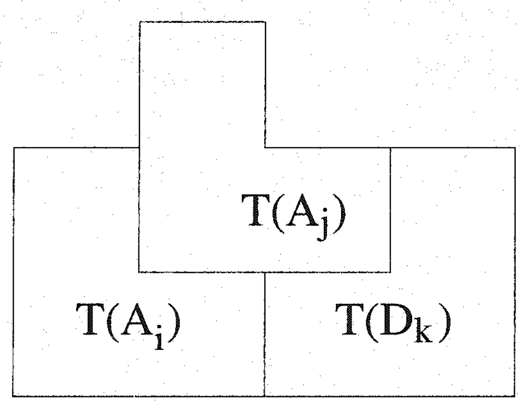

Figure 3.4: A partial tiling in $\mathfrak{T}_3$.

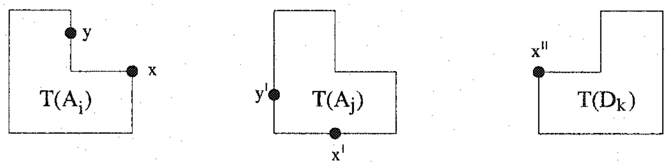

Figure 3.5: Points identified by ρ_3.

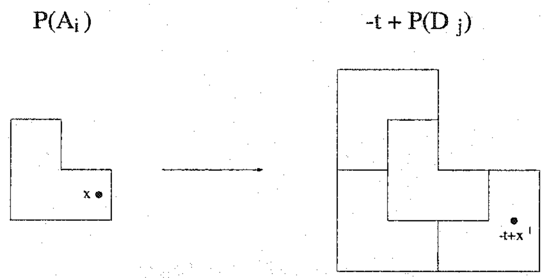

Figure 3.6: The action of ω_3 on $P(A_i)$.

$-t + P(D_j)$. (Note that A_i can be any one of $A_1, \ldots, A_{14}$, but D_j will depend on which prototile A_i is.) So we will let $\omega_3(\rho(x)) = \rho(x')$.

Now we define the space $\mathfrak{L}_3$ to be the inverse limit space of $\mathfrak{M}_3$ relative to the map ω_3. Our space $\mathfrak{L}_3$ will consist of all infinite sequences $(x_i)_{i=0}^{\infty}$ of points in $\mathfrak{M}_3$ such that $\omega_3(x_{i+1}) = x_i$ for all $i = 0, 1, 2, \ldots$. We give $\mathfrak{L}_3$ the subspace topology from the product space $\Pi_{i=0}^{\infty}$, and it will have a basis consisting of all sets of the form

$$\mathcal{B}_{U,n}^{\mathfrak{L}} := \{x \in \mathfrak{L} | x_i \in \omega_3^{n-i}(U) \text{ for } i = 0, 1, \ldots, n\}.$$

Also, ω_3 will provide a bijection on $\mathfrak{L}_3$ given by $\omega_3(x)_i = \omega_3(x_i)$, and the inverse will be given by $\omega_3^{-1}(x)_i = x_{i+1}$.

The dynamical systems $(\mathfrak{T}_3, \omega_3)$ and $(\mathfrak{L}_3, \omega_3)$ are topologically conjugate.

Now let's consider how to form the appropriate Bratteli diagram for this system. We let ω_3 act on each prototile, and count the number of times a translate of each prototile appears. When we do this, fourteen tiles of type A_1 will appear. Each type of tile $A_1, \ldots, A_{14}$ will produce one tile of type A_1. We'll let the symbols $(A_1)_1, \ldots, (A_1)_{14}$ denote the choice made when generating a tile of type A_1 by using a tile of type $A_1, \ldots, A_{14}$, respectively. Only one tile of type A_2 will appear, and it will be produced by a tile of type B_3. We will let the symbol $(A_2)_1$ denote the choice of generating a tile of type A_2 in this way. So when we form our Bratteli diagram, we will need to have $\mathcal{V}_0 = \{v_0\}$, and every other $\mathcal{V}_n$ must contain the subset $\{(A_1)_1^n, \ldots, (A_1)_{14}^n, (A_2)_1^n\}$. It will also contain elements corresponding each way that each of the other prototiles can be generated.

Now for the edge sets. The set $\mathcal{E}_1$ will consist of one edge from v_0 to each vertex in $\mathcal{V}_1$. For every other $\mathcal{E}_n$, we construct the edge set by considering what sequences of choices will be legal. If we make the choice $(A_1)_1$ at level n, then this will mean that our tile of type A_1 was generated by a tile of type A_1, so we must be able to make all of the same choices at the next level. So we need edges from $(A_1)_1^n$ to each of $(A_1)_1^{n+1}, \ldots, (A_1)_{14}^{n+1}$. If we make the choice $(A_1)_2$ at level n, then this will mean that our tile of type A_1 was generated by a tile of type A_2, so we will need an edge from $(A_1)_2^n$ to $(A_2)_1^{n+1}$. We continue using this process until all of our edges have been defined.

We will let $X_{\mathcal{B}_3}$ denote the infinite path space of this diagram. If we let $\mathcal{A}_3 = \{A_1, \ldots, A_{14}, B_1, \ldots, B_{14}, C_1, \ldots, C_{14}, D_1, \ldots, D_{14}\}$, then for each $a \in \mathcal{A}_3$ we can define the path space

$$X_{\mathcal{B}_3}(a) := \{\xi \in X_{\mathcal{B}_3} \mid \text{the first edge of } \xi \text{ goes to a vertex } a_i^1\}.$$

We let $\mathfrak{B}_3 := \bigcup_{a \in \mathcal{A}_3} P(a) \times X_{\mathcal{B}_3}(a)$.

As before, we can define a map $\Phi_3 : \mathfrak{B}_3 \longrightarrow \mathfrak{T}_3$, and this map constructs a tiling $\mathcal{T}$ of $\mathfrak{T}_3$ by using the point $(x, \xi) \in P(a) \times X_{\mathcal{B}_3}(a) \subset \mathfrak{B}_3$ as a rule. It will place the origin of $\mathcal{T}$ in a tile of type a, with its exact location in that tile determined by the point x. The sequence ξ will then dictate how to build a tiling around this tile, and it will do it by specifying how the tile containing the origin in each tiling $\omega_3^{-n}(\mathcal{T})$ is generated by the tiling $\omega_3^{-n-1}(\mathcal{T})$.

Now we let $\mathfrak{B}^*$ be the space that is formed when we identify $(x, \xi), (x', \xi') \in \mathfrak{B}$ if $\Phi^{-1}((x, \xi)) = \Phi^{-1}((x', \xi'))$, and we let $\Phi^* : \mathfrak{B}^* \longrightarrow \mathfrak{T}$ be the naturally induced bijection between the two spaces. Then we can define our substitution

$$\omega_3 := \Phi_3 \circ \omega \circ \Phi_3^{-1} : \mathfrak{B}_3^* \longrightarrow \mathfrak{B}_3^*.$$

The dynamical system $(\mathfrak{B}_3^*, \omega_3)$ is topologically conjugate to $(\mathfrak{T}_3, \omega_3)$.

3.2 Comparisons to the Cantor Set

The Cantor set is a subset of the interval $[0, 1]$. We start with the interval $[0, 1]$, and remove its middle third, the interval $(1/3, 2/3)$. So we are left with two intervals, $[0, 1/3]$ and $[2/3, 1]$. Then we remove the middle third of each of these intervals, leaving us with the four intervals $[0, 1/9]$, $[2/9, 1/3]$, $[2/3, 7/9]$, and $[8/9, 1]$. We continue in this way, successively removing the middle third from each remaining interval. Repeating this process infinitely often we are left with an infinite set of points from this interval. This set, with the induced topology from the real line, is the Cantor set. Some points that are obviously in the Cantor set are $\{0, 1, 1/3, 2/3, 1/9, 2/9, 7/9, 8/9, \dots\}$. However, the Cantor set is uncountable.

A space S is *totally disconnected* if for any distinct $x, y \in S$, there exist disjoint open sets U and V such that $x \in U$, $y \in V$, and $U \cup V = S$. It is not hard to see that for any Bratteli diagram $\mathcal{B}$, the infinite path space $X_{\mathcal{B}}$ is totally disconnected. Each cylinder set is both open and closed, and it is clear that for any distinct $\xi, \xi' \in X_{\mathcal{B}}$, we can find a cylinder set containing ξ but not containing ξ', so the space $X_{\mathcal{B}}$ is totally disconnected.

A space S is *perfect* if every point of S is a limit point of S. In general, the space $X_{\mathcal{B}}$ does not need to be perfect. However, the path space for any of the diagrams that we create from a substitution system will be. Suppose $\mathcal{B}$ was created using a substitution system, and let $[e_1, \dots, e_n]_{i=1}^n$ be any basis element in $X_{\mathcal{B}}$. Since our substitution system is primitive, any type of tile will eventually produce every other type tile. So no matter what type of tile the vertex $r(e_n)$ corresponds to, there will be at least two ways to produce it through iterating our substitution on other tiles. So the basis element $[e_1, \dots, e_n]_{i=1}^n$ will contain at least two elements. So $X_{\mathcal{B}}$ will be perfect.

It is also clear that the space $X_{\mathcal{B}}$ is compact. (Each edge set is finite, hence compact, so we can regard $X_{\mathcal{B}}$ as a subspace of a product space of compact spaces.) We can also put a metric on $X_{\mathcal{B}}$ in the same way that we put a metric on the cylinder sets in Section 1.1.

The following theorem about the Cantor set is verified in Hocking and Young [8] pp. 97-100.

Theorem 3.1. *Any compact totally disconnected perfect metric space is homeomorphic to the Cantor set.*

So each of the Bratteli diagrams that we create from a substitution system will be homeomorphic to the Cantor set. Further, if we are considering a tiling space in $\mathbb{R}^n$, then each of our prototiles will be homeomorphic to the unit ball in $\mathbb{R}^n$, which we will denote as I_n. Now let $\mathfrak{B}$ be the Bratteli diagram space constructed from a tiling space in $\mathbb{R}^n$, and suppose that $\mathfrak{B} = \bigcup_{a \in \mathcal{A}} P(a) \times X_{\mathcal{B}}(a)$. If we use these homeomorphisms on $(x, \xi) \in X_{\mathcal{B}}$, then we can map x to the corresponding point in I_n, and ξ to corresponding point in the Cantor set. It follows that $\mathfrak{B}$ and $I_n \times \mathfrak{C}$ (where we are letting $\mathfrak{C}$ denote the Cantor set) are homeomorphic spaces. So we have the following theorem.

Theorem 3.2. *Let I_n be the unit interval and let $\mathfrak{C}$ be the Cantor set. Then any substitution tiling space $\mathfrak{T}$ is homeomorphic to a quotient space of $I_n \times \mathfrak{C}$.*

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