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REGULATOR CURRENTS ON COMPACT COMPLEX MANIFOLDS

by

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Abstract

This thesis pursues the study of non-algebraic and non-Kähler compact complex manifolds by traditionally algebraic methods involving sheaves, cohomology and K-theory. To that end, Bott-Chern cohomology is developed to complement De Rham and Dolbeault cohomology. The first substantial chapter is devoted to the construction of Bott-Chern cohomology, including products. The next chapter is an investigation of $\mathrm{Pic}^0(X)$ for non-Kähler complex manifolds. The next chapter uses line bundles represented by classes in this $\mathrm{Pic}^0(X)$, along with Cartier divisors, to define a group of twisted cycle classes, generalizing a previous algebraic definition. On this group of twisted cycle classes, we use currents to construct a regulator map into Bott-Chern cohomology. Finally, in a chapter of conjectural statements and future directions, we explore the possibility of an alternate regulator using a cone complex of currents. We also conjecturally define a height pairing for certain kinds of compatible twisted cycle classes.

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Chapter 1

Introduction

1.1 Cohomology Theory

Modern abstract algebra and algebraic geometry begin with classification problems. Complex analytic geometry can and has often been treated the same way, by asking the following questions: What are all the complex manifolds? How do we classify their structure? How do we differentiate between them?

The goal of differentiation between objects leads to the idea of homology and cohomology. Very broadly speaking, one can interpret the idea of a cohomology theory as any sufficiently powerful and robust construction which serves to differentiate between algebraic or topological objects.

The creation of a wide variety of interesting cohomology theories to understand and differentiate between objects has been a major advancement in the mathematics of the 20th century. However, the most powerful and subtle of these theories, such as the higher Chow Groups, algebraic K-theory, or the conjectural theory of motivic cohomology, are exceedingly difficult to calculate.

This leads to the idea of understanding cohomology theories by investigating the relationships between them. Maps between cohomologies theories are an excellent tool to better understand both the domain and target theories. Even when such a map fails to be injective and loses information, its existence and the nature of its image can shed a great deal of light on the component pieces of its domain. One of the most famous instances of such a map is the cycle class map on Chow groups; its range in Dolbeault cohomology is the subject of the celebrated Hodge conjecture and remains a central problem in complex algebraic geometry.

Regulators have their origins in algebraic number theory, where they originally were used to recover arithmetic information. In algebraic geometry, the term is used to refer to certain functions from K-theoretic cohomology to more tangible theories. Regulators exists for the higher Chow groups, Milnor K-theory, and conjecturally on some theory of motivic cohomology, usually taking values in some version of Deligne cohomology. Such regulator maps and their success in providing interesting information on algebraic manifolds are motivating examples for this thesis.

1.2 Algebraic Methods

The whole realm of cohomology theories and regulator maps is, essentially, part of algebra. The domains of regulators are algebraically constructed theories over objects of algebraic geometry, whether they be complex varieties or schemes of finite type over an arbitrary ring or field. However, there is little reason why such constructions should not apply to objects which, classically, are excluded from the realm of algebra, provided we have appropriate sheaf theoretic tools at our disposal. One such arena is that of complex spaces and analytic complex varieties, where a theory of coherent sheaves is well developed. This thesis considers that arena by starting with a potentially non-algebraic and non-Kähler complex manifold and using analytic subvarieties and a sheaf-theoretic approach to build invariants and regulators for these complex manifolds.

Such an idea is certainly not new. One could argue that the original source of the ideas of sheaves and cohomology came from the study of analytic objects, in the tradition of Henri Cartan. However, particularly since the revolutionary idea of schemes over arbitrary rings was championed by Alexandre Grothendieck, these sheaf-theoretic methods have mostly been used to study algebraic varieties. In that arena, enormous progress has been made, particularly in the use of complicated algebraic invariants such as Chow groups and K-theory. The general question of how much of this progress can be recovered for analytic objects is extremely interesting to us.

This thesis aims to be a part of the project of adapting algebraic geometric results to analytic objects. It focuses on a small number of algebraic constructions which can be generalized to analytic varieties. We will investigate Bott-Chern cohomology, which generalizes Dolbeault cohomology for algebraic varieties. We will look at line bundles over analytic varieties, and investigate what becomes of the Picard variety. Using line bundles, we will build analytic twisted cycle classes, generalizing the algebraic construction in [Lew04] and mimicking the structure of Milnor K-theory. Finally, we shall put this all together by constructing regulator maps on these twisted cycle classes into Bott-Chern cohomology.

Thoughout the thesis, we will keep careful track of how the new material is related to the original algebraic constructions and where the results of algebraic geometry start to fail. In particular, we will ensure that the new definitions coincide with the old when specializing back to algebraic manifolds.

Chapter 2

Setting and Definitions

2.1 Assumed Material

We hope for this thesis to be readable by other graduate students with backgrounds in algebra and geometry. The complex analytic content is limited, but the thesis relies heavily on definitions and ideas from many parts of algebra, topology and algebraic geometry.

Therefore, we assume the reader has a sufficent background in abstract algebra: groups, rings, fields and modules. In addition, we assume a knowledge of the definitions and techniques of homological algebra: complexes, exactness, homology and cohomology of complexes, commutative diagrams and quasi-isomorphisms. Specifically, the homological algebra that constructs sheaf cohomology by way of resolutions and derived functors is important for the majority of Chapter 3.

We also assume the reader is familiar with the basics of real and complex analytic geometry: analytic functions and their properties; holomorphic and meromorphic functions; poles and residues; manifolds (real and complex); singular homology and cohomology; De Rham cohomology and Dolbeault cohomology.

Line bundles and their metrics are central to this thesis. We expect that the reader, being familiar with the topics already listed, is most likely also familiar with vector bundle constructions. However, since line bundles are so central, we will review those definitions further along in this chapter. This review also serves to clarify upon which version of the various definitions of bundles we will rely.

From topology, in addition to the basics of point-set topology, we rely on standard homological constructions. In particular, we make use of singular homology and cohomology and the duality theorems between those theories.

Knowledge of sheaf theory, which forms the core of modern algebraic geometry, is also necessary. The reader should be familiar with the idea of a sheaf, its sections, maps, germs and stalks. The calculation of sheaf cohomology by derived functors has already been mentioned; calculation by the Čech resolution is also important.

As indicated in the introduction, the major parts of this thesis are adaptations of techniques from algebraic geometry to non-algebraic complex manifolds. Therefore, the last major area of necessary knowledge involves algebraic geometry. Though we will not make heavy use of the very powerful machinery of these areas, the reader would be well served by a familiarity with the basics of the following subjects: schemes, Hodge theory, Chow groups, Algebraic K-theory and Milnor K-theory.

Major references for this background material are [Huy05], [GH94], [Har77], [Voi02], [Lew99], [Dem07] and [GR84].

2.2 Definitions and Preliminary Results

Though we will be comparing our results with those of algebraic geometry, the setting for this thesis is the world of complex spaces and analytic varieties. In that world, smooth complex manifolds, singularities, differential forms, currents, and coherent sheaves are important. Since we do not assume a general knowledge of this setting, we start with the following definitions and propositions.

2.2.1 Definitions concerning Complex Spaces

The first definitions come almost exactly from Chapter 1 of [GR84]. We will proceed in brief, and we encourage the reader to look to that reference for a more substantial exposition.

Definition 2.2.1: A *ringed space* is a topological space X with a sheaf of rings $\mathcal{F}$. A *locally ringed space* is a ringed space where all stalks are local rings. A *sheaf of local* $\mathbb{C}$*-algebras* is a sheaf of local rings where each ring of sections over an open set U is a $\mathbb{C}$-algebra. Equivalently, it is a sheaf of modules over the constant sheaf $\mathbb{C}$. In addition, the definition imposes that no stalks are zero rings, which guarantees that the morphism from the constant sheaf $\mathbb{C}$ into the sheaf of local $\mathbb{C}$-algebras is an injection. A locally ringed space is called a $\mathbb{C}$-*ringed space* if its sheaf of rings is a sheaf of local $\mathbb{C}$-algebras. A *morphism of ringed spaces* $(X, \mathcal{F}) \to (Y, \mathcal{G})$ consists of a topological map $f : X \to Y$ and a sheaf map $\tilde{f} : \mathcal{G} \to f_*\mathcal{F}$. An *isomorphism of ringed spaces* is a morphism which has a two sided inverse.

Definition 2.2.2: A *domain* in $\mathbb{C}^n$ is an open, connected subset. A *complex* *model space* is a locally ringed space given by a domain D in $\mathbb{C}^n$ and an ideal sheaf $\mathcal{I}$ in $\mathcal{O}_D$ such that $\mathcal{I}$ is locally described by a finite number of generators. The complex model space is topologically given by the zeros set of the ideal sheaf, with the sheaf of rings $\mathcal{O}_D/\mathcal{I}$ restricted to this topological space. Standard results, as in [GR84], show that this space is a $\mathbb{C}$-ringed space.

Definition 2.2.3: A *complex space* is a $\mathbb{C}$-ringed space X which is locally isomorphic, as a ringed space, to a complex model space. Its sheaf of rings is called its *structure sheaf* and is usually written $\mathcal{O}_X$.

Definition 2.2.4: A complex space is *reduced* if all stalks are reduced rings, *i.e.* the stalks have no nilpotent elements. A complex space is *locally irreducible* if all stalks are integral domains. A complex space is *irreducible* if is cannot be written $X = A \cup B$ for A, B proper analytic subvarieties. (There are several other equivalent definition of irreducibility in Theorem 9.1.2 of [GR84].) A complex space is *normal* if all stalks are integrally closed in their quotient rings. A complex space is *smooth* if it is locally isomorphic to an open set in $\mathbb{C}^n$. Since all these defintions are local, the same statements hold for the definitions of a smooth point, normal point, reduced point or irreducible point. A *singular point* is any point which is not a smooth point.

Definition 2.2.5: An *analytic variety* is a reduced complex space. (Note the use of the word variety is not necessarily consistent throughout the literature.) We will use the terms irreducible variety, smooth variety and normal variety to refer to irreducible, smooth, or normal complex spaces, since all three notions imply that the complex space is reduced.

Definition 2.2.6: A *holomorphic map* or *morphism of analytic varieties* is a morphism of $\mathbb{C}$-ringed spaces between analytic varieties. A *biholomorphic* *map* is an isomorphism of $\mathbb{C}$-ringed spaces between analytic varieties. A *closed map* is a holomorphic map where the image of a closed set remains closed. A *finite map* is a closed map where all fibres are finite sets. A *proper map* is a holomorphic map where the inverse image of any compact set if compact.

It is easy to see, in the local description, that the notion of a holomorphic map recovers the ordinary definition of a holomorphic function between subsets of $\mathbb{C}^n$. The map of ringed spaces requires that holomorphic functions pullback to holomorphic functions, which is only locally possible for functions which satisfy the conventional definition of holomorphicity.

The notion of coherence is particularly important in the developments of complex spaces. The formal definition is given in [GR84], particularly in the Annex at the end of their book.

Definition 2.2.7: Roughly speaking, a sheaf of modules is *of finite type* if it is locally generated by a finite number of sections. A sheaf of modules is *of relation finite type* if the sheaf of relations between a generating set of sections is of finite type. A sheaf of modules is *coherent* if both are true. We also refer to this as finitely generated and finitely presented (a fairly standard algebraic term). It is a standard result in Section 2 of [GR84] that structure sheaves and ideal sheaves of complex spaces are coherent.

Though coherence is important to much of what we consider in this thesis, we will usually supress notions of coherence and make use of the standard results that guarantee coherence.

2.2.2 Definitions concerning Currents

These definitions and results are taken from Chapter 1, Section 2 of [Dem07], as well as from the paper [Kin71].

Definition 2.2.8: There is a *topology on the space of differential forms* $\mathcal{A}^r_X(X)$, defined in Chapter I, Definition 2.2 of [Dem07], and a similar topology on the space of differential forms with compact support $\mathcal{A}^r_c$.

In Section 2.1 of [Kin71], we are given the useful characterization of limits in this topology: a sequence of global forms converges to a limit if the local descriptions of those forms, along with all higher derivatives, converge uniformly in compact subsets of the local neighbourhood.

Definition 2.2.9: The *sheaf of currents* on X, written $\mathcal{D}_r$, is the topological dual of $\mathcal{A}^r_c$. With raised indices, this is written $\mathcal{D}^{2d-r} = \mathcal{D}_r$. This sheaf decomposes by types: $\mathcal{D}^n = \oplus_{p+q=n}\mathcal{D}^{p,q}$. The currents in $\mathcal{D}^{p,q}$ are those which are supported on $\mathcal{A}^{n-p,n-q}$.

There are two important examples of currents. First, given a form $\eta \in \mathcal{A}^{p,q}$, the current $\sigma \mapsto \int_X \eta \wedge \sigma$ is a current in $\mathcal{D}^{p,q}$. This provides an injective map $\mathcal{A}^{p,q} \hookrightarrow \mathcal{D}^{p,q}$. The notation δ_η for the current associated with the form η is relatively standard. However, we will often abuse notation and simply write η for both the form and the current associated with that form.

Second, for any subvariety Z in X of codimension r (even singular analytic subvarieties), the integration over Z acts on $2n-2r$ forms, giving a current in $\mathcal{D}^r$. We call these, unsurprisingly, analytic currents of integration.

Definition 2.2.10: By duality, currents act very much like differential forms and we can define the *exterior derivative* d, as well as its components ∂ and

$\overline{\partial}$, in the natural way on currents. If ω is a test form, then $d\sigma(\omega) := \pm\sigma(d\omega)$, and similarly for ∂ and $\overline{\partial}$.

Definition 2.2.11: Let $f : X \to Y$ be a proper holomorphic map of reduced complex spaces. If τ is a current on X and ω a test form on Y, then the *pushforward* is defined as follows.

$$f_*\tau(\omega) := \tau(f^*\omega) \tag{2.2.1}$$

Pullbacks, in general, do not exist for currents.

2.2.3 Definitions concerning Line Bundles

There are a number of ways to define bundles over manifolds and varieties. The following is a fairly standard definition of line bundles.

Definition 2.2.12: A *holomorphic line bundle* L over a complex manifold X is a manifold E with a holomorphic map $\pi : E \to X$ such that the fibres of the map are isomorphic, as complex manifolds, to $\mathbb{C}$. In addition, there must be an open cover U_i of X and isomorphisms of complex manifolds $r_i : \pi^{-1}(U_i) \cong U_i \times \mathbb{C}$, which are refered to as local trivializations. Moreover, these isomorphisms must be linear maps, *i.e.* multiplication by a non-zero constant, when restricted to each of the fibres.

Definition 2.2.13: A *map of line bundles* is a map of manifolds which commutes with the projections to X and induces linear maps on the trivializations. An *isomorphism* is an invertible map which induces isomorphisms on trivializations.

On an intersection $U_i \cap U_j$, this definition gives a nowhere vanishing function $l_{ij} := r_i/r_j$ which, fibrewise, describes the linear map between the trivialization over U_i and that over U_j. The collection of such functions clearly satisfies $l_{ij} = (l_{ji})^{-1}$, as well as the cocycle condition $l_{ij}l_{jk} = l_{ik}$. The line bundle can be completely described by these transition functions: the manifold E can be recovered by considering the disjoint union of the sets $U_i \times \mathbb{C}$ and making identifications on the intersections $U_i \cap U_J$ *via* the functions l_{ij}. The holomorphic nature of the line bundle is captured by the fact that the functions l_{ij} are holomorphic functions. We essentially use the following alternate definition throughout the thesis.

Definition 2.2.14: A *holomorphic line bundle* L over a complex manifold X is given by an open cover $\mathcal{U} = U_i$ of X along with nowhere vanishing holomorphic functions l_{ij} on $U_i \cap U_j$ which satisfy a cocycle condition on triple intersections: $l_{ij}l_{jk} = l_{ik}$. We call the collection $\{l_{ij}\}$ the cocycle of transition functions.

Definition 2.2.15: The *tensor product of line bundles* of the line bundles $\{l_{ij}\}$ and $\{k_{ij}\}$ is the line bundle given by the cocycle $\{l_{ij}k_{ij}\}$. The *inverse of a line bundle* $\{l_{ij}\}$ is the line bundle given by the cocycle $\{(l_{ij})^{-1}\}$.

Definition 2.2.16: The *Picard Group* of X, written $\mathrm{Pic}(X)$, is the group of isomorphism classes of line bundles. It is a group by the previously stated product operation, with the trivial line bundle $\mathbb{C} \times X$ as the identity element.

Proposition 2.2.17: *There is an isomorphism* $\mathrm{Pic}(X) \cong H^1(X, \mathcal{O}_X^\times)$

Proof. This is a standard result, found, for example, in Corollary 2.2.10 in [Huy05]. The cocycle of transition functions is a Čech cocyle with coefficients in $\mathcal{O}_X^\times$, and this map simply takes that cocycle to its Čech cohomology class. □

So far, we've considered line bundles over complex manifolds. However, line bundles can be defined *via* transition functions even over arbitrary analytic varieties. The following definition follows from page 32 of [GR84].

Definition 2.2.18: A line bundle over an analytic complex variety is given by an open cover $\mathcal{U}$ and a cocycle $\{l_{ij}\}$ of sections of $\mathcal{O}_X^\times$ over the open sets of $\mathcal{U}$. The l_{ij} are still called transition functions of the line bundle, even though the sheaf $\mathcal{O}_X^\times$ is no longer *aprioric* a sheaf of functions.

Since these line bundles are defined by transition functions, any constructions *via* transition functions are preserved, including products and inverses as previously stated in Definition 2.2.15.

Line bundles are often accessed by studying divisors. In this thesis, we work almost exclusively with Cartier divisors.

Definition 2.2.19: A *Cartier divisor* on X is a global section of the quotient sheaf $\mathcal{M}_X^\times/\mathcal{O}_X^\times$. Equivalently, it is given by an open cover $\mathcal{U}$ that is acyclic for the sheaf $\mathcal{O}^\times$ and meromorphic functions $c_i \in \mathcal{M}_X^\times(U_i)$ such that $c_i/c_j \in \mathcal{O}_X^\times(U_i \cap U_j)$.

There is a straightforward identification between Cartier divisors and meromorphic sections of line bundles. A Cartier divisor described by c_i gives rise to a line bundle by the fact that c_i/c_j define a cocycle of transition functions. In the line bundle described by those transition functions, the Cartier divisor c_i is nothing more than a meromorphic section.

2.2.4 Standard Results

There are some standard results which we review in this section. We do this both for the sake of readers who might not be familiar with them, and for the sake of stating them for easy reference.

Proposition 2.2.20: *Differential forms pullback under proper maps. The pushforward of currents is defined to be the current acting on the pullback of a form, i.e.* $f_*\eta(\omega) := \eta(f^*\omega)$*, so currents pushfoward under proper maps.*

This is standard for forms, and from Section 2.C.1 of Chapter 1 of [Dem07] for currents.

Proposition 2.2.21: *Direct products of complex spaces exist, and the topological space of the direct product is the direct product of the topological spaces.*

This is a theorem on page 26 of [GR84].

The various versions of the Poincaré Lemma are important. We state it here for $\mathbb{C}$-valued forms and currents, even though it is mostly a result of real differential geometry.

Proposition 2.2.22: *Let* Δ *be an open polydisc in* $\mathbb{C}^n$*. If* ω *is a differential form on* Δ *and* T *is a current on* Δ*, then the following are true.*

- *If* $d\omega = 0$*, then there exists a form* η *with* $d\eta = \omega$*.*
- *If* $\partial\omega = 0$*, then there exists a form* η *with* $\partial\eta = \omega$*.*
- *If* $\overline{\partial}\omega = 0$*, then there exists a form* η *with* $\overline{\partial}\eta = \omega$*.*
- *If* $dT = 0$*, then there exists a current* S *with* $dS = T$*.*

- *If $\partial T = 0$, then there exists a current S with $\partial S = T$.*
- *If $\overline{\partial} T = 0$, then there exists a current S with $\overline{\partial} S = T$.*

The first three results are found in [Huy05], particularly Proposition 1.3.9. The fourth and fifth are found in [Dem07], Chapter I, Proposition 2.2.4, and the sixth follows easily from those.

The following corollary of the Poincaré Lemma will be used frequently. We refer to this as the Poincaré Corollary throughout the thesis.

Corollary 2.2.23: *The Poincaré Lemma shows that the sequence of sheaves $\mathcal{A}^{r-2} \to \mathcal{A}^{r-1} \to \mathcal{A}^{r} \to \mathcal{A}^{r+1}$ is exact for $r \geq 2$. Tracing the isomorphisms of images and kernels provided by this exact sequence gives the following isomorphisms.*

$$\frac{\mathcal{A}^{r-1}}{d\mathcal{A}^{r-2}} = \frac{\mathcal{A}^{r-1}}{\operatorname{Ker}(d : \mathcal{A}^{r-1} \to \mathcal{A}^{r})} \cong d\mathcal{A}^{r-1} = \operatorname{Ker}(d : \mathcal{A}^{r} \to \mathcal{A}^{r+1}) \tag{2.2.2}$$

The isomorphism in the middle is realized by the differential $d : \mathcal{A}^{r-1} \to \mathcal{A}^{r}$.

The same is true for the ∂ and $\overline{\partial}$ operators. Restricted to acting on the sheaves Ω and $\overline{\Omega}$, respectively, we have the following isomorphisms.

$$\frac{\Omega^{r-1}}{\partial\Omega^{r-2}} = \frac{\Omega^{r-1}}{\operatorname{Ker}(\partial : \Omega^{r-1} \to \Omega^{r})} \cong \partial\Omega^{r-1} = \operatorname{Ker}(\partial : \Omega^{r} \to \Omega^{r+1}) \tag{2.2.3}$$

$$\frac{\overline{\Omega}^{r-1}}{\overline{\partial\Omega}^{r-2}} = \frac{\overline{\Omega}^{r-1}}{\operatorname{Ker}(\overline{\partial} : \overline{\Omega}^{r-1} \to \overline{\Omega}^{r})} \cong \overline{\partial\Omega}^{r-1} = \operatorname{Ker}(\overline{\partial} : \overline{\Omega}^{r} \to \overline{\Omega}^{r+1}) \tag{2.2.4}$$

Again, the isomorhpisms in the middle are realized, respectively, by the differ-

entials ∂ and $\overline{\partial}$.

Similar results hold for the equivalent sequences defined in terms of currents instead of forms.

Another very useful result is the $\partial\overline{\partial}$ lemma:

Proposition 2.2.24: *If ω is a d-closed differential form of type (p, q), either on a polydisc Δ in $\mathbb{C}^n$ or on an arbitrary Kähler manifold X, then the following are equivalent.*

- *ω is d-exact.*
- *ω is ∂-exact.*
- *ω is $\overline{\partial}$-exact.*
- *ω is $\partial\overline{\partial}$-exact.*

This can be found in [Huy05] Proposition 3.2.10.

An important result about currents is the following, which is known as the regularity theorem for currents. This is from Chapter 3, Section 1 of [GH94].

Proposition 2.2.25: *If T is a holomorphic current on a complex manifold X, then there exists a holomorphic form η such that T is of the form*

$$\omega \mapsto \int_X \omega \wedge \eta \tag{2.2.5}$$

When working with sheaf cohomology and considering local sections, the following result becomes very useful.

Definition 2.2.26: An *acyclic open cover* of a complex space with regards to a sheaf $\mathcal{F}$ is a countable open cover $\mathcal{U} = \{U_i | i \in I\}$ such that for any subset $J \subset I$, and for any $q > 0$, the following statement holds.

$$H^q(\cap_{i\in J} U_i, \mathcal{F}) = 0 \tag{2.2.6}$$

Proposition 2.2.27: *If $\mathcal{U}$ is an acyclic open cover of X with regards to the sheaf $\mathcal{F}$, then it calculates Čech cohomology.*

$$H^p(X, \mathcal{F}) = \check{H}(\mathcal{U}, \mathcal{F}) \tag{2.2.7}$$

This is from [Voi02] Theorem 4.41. For complex manifolds, open covers by polydiscs are acyclic for the structure sheaf and other coherent sheaves. For analytic varieties, we have the following two propositions.

Proposition 2.2.28: *Coverings of analytic varieties by Stein spaces are acyclic for any coherent analytic sheaf.*

Proposition 2.2.29: *Any analytic variety admits a Stein covering.*

These are found on page 35 of [GR84]. They allow us to freely use acyclic covers for coherent sheaves, which is extremely convenient. For those familiar with schemes theory, Stein spaces act, in some ways, like affine schemes.

Frequently in the thesis, we consider integrals with polar coefficients. Though we could deal with convergence arguments for those integrals as we come to them, it is convenient to collect those arguments together at this point.

Proposition 2.2.30: *The integrals in Equations 6.1.3 and 6.2.2 and the four integrals in Equation 7.1.2 are convergent.*

Proof. We establish this result by appealing to the literature, where the convergence of these integrals are established over projective algebraic manifolds. For example, convergence of one of these integrals is explicitly considered in Lemma 3.1 in [Gon95], and the convergence of two of the others is implicit in the papers [Lew01] and [Lew04]. The style of these arguments is to use local coordinates and resolution of singularities to reduce to integrals on a polydisc in $\mathbb{C}^n$ which are polar along a normal crossing divisor. Then standard arguments of several complex variables ensure that such integrals take finite values.

These arguments are entirely complex-analytic in nature and are not dependent on the assumption of a projective algebraic structure. Therefore, they continue to hold in our non-algebraic environment. □

Finally, the ability to make modifications to remove singularities of analytic varieties is extremely important. The classic reference for this is the paper [Hir64], which establishes resolution of singularities for both algebraic and analytic varieties. This definition of modifications is from II.10.1 in [Dem07].

Definition 2.2.31: A *proper modification* of an analytic variety Z is a proper map $f : \tilde{Z} \to Z$ such that f is proper and there exists a nowhere dense closed analytic subset $B \subset Z$ such that the restriction $f : \tilde{Z}\backslash f^{-1}(B) \to Z\backslash B$ is an isomorphism.

Proposition 2.2.32: *If Z is a singular reduced analytic variety, there exists a proper modification of Z to a smooth analytic variety.*

Though this is an important issue, we generally supress the idea of desingularizations and proper modifications throughout the thesis. When necessary, modifications will be implicit in our constructions of objections and integrals.

2.3 Notations

Here is a list of notation used in this thesis. In this table, X is a topological space, analytic variety or complex manifold as necessary. The subscripts refering to the space X are often dropped when the space is understood.

Notation	Object
$\mathcal{A}^p_{X,R}$	Sheaf of C^∞ R-valued differential forms on X
$\mathcal{A}^p_X$	Sheaf of C^∞ $\mathbb{C}$-valued differential forms on X
$\mathcal{A}^{p,q}_X$	Sheaf of forms of type p, q
$\mathcal{O}_X$	Structure sheaf of a complex space X
$\mathcal{O}_X$	Sheaf of holomorphic functions on a complex manifold X
$\overline{\mathcal{O}}_X$	Sheaf of anti-holomorphic functions on X
$\mathcal{O}^\times_X$	Sheaf of nowhere vanishing holomorphic functions on X
$\overline{\mathcal{O}}^\times_X$	Sheaf of nowhere vanishing anti-holomorphic functions
$\mathcal{M}_X$	Sheaf of meromorphic functions on X
$\mathcal{M}^\times_X$	Sheaf of non-zero meromorphic functions on X
Ω^p_X	Sheaf of holomorphic $0, p$-forms
$\Omega^p_{\mathcal{D}}$	Sheaf of holomorphic $0, p$-currents
$\overline{\Omega}^q_X$	Sheaf of anti-holomorphic $0, q$-forms
$\overline{\Omega}^q_{X,\mathcal{D}}$	Sheaf of anti-holomorphic $0, q$-currents
$\mathcal{D}_r$	Compactly supported currents of type r on X,
$= \mathcal{D}^{2n-r}$	topological dual to $\mathcal{A}^r$

Notation	Object
$\mathcal{D}_{p,q}$	Compactly supported currents of type p, q on X,
$= \mathcal{D}^{n-p,n-q}$	topological dual to $\mathcal{A}^{p,q}$
$H^n(X, R)$	Singular cohomology of X with coefficients in R
$\check{H}^n(\mathcal{U}, R)$	Čech cohomology of X with cover $\mathcal{U}$ and coeff. R
$\mathcal{H}^n(\mathcal{F}^\bullet)$	Cohomology of a complex of sheaves
$\mathbb{H}^n(X, \mathcal{F}^\bullet)$	Hypercohomology of a complex of sheaves $\mathcal{F}^\bullet$ on X
$H_{\mathrm{DR}}(X, R)$	De Rham cohomology of X with coefficients in R
$H^{p,q}(X)$	Dolbeault cohomology of X
$H_{\mathrm{BC}}(X, R)$	Bott-Chern cohomology of X with coefficients in R
$H_{\mathrm{Ap}}(X, R)$	Aeppli cohomology of X with coefficients in R
$H_{\mathcal{D}}(X, R)$	Deligne cohomology of X with coefficients in R
$\mathrm{Pic}(X)$	Picard Group of X
$\mathrm{Pic}^0(X)$	Picard Variety of X
δ_ω	Current of integration associated with a form ω
$\pi, \quad \pi_r \quad \pi_{p,q}$	Various projections as defined in the text
$\underline{d}$	Ordinary differential of forms or currents d followed
	by projection on an appropriate degree
$\mathrm{Ker}(\mathcal{F} \xrightarrow{f} \mathcal{G})$	Kernel sheaf of a map of sheaves f
$\mathrm{Im}(\mathcal{F} \xrightarrow{f} \mathcal{G})$	Image sheaf of a map of sheaves f

Notation	Object
$\mathcal{L}^{\bullet}_{p,q}$	Complex of sheaves defining B-C cohomologies
$\mathcal{M}^{\bullet}_{p,q}$	Complex of sheaves defining B-C cohomologies
$\mathcal{S}^{\bullet}_{p,q}$	Complex of sheaves defining B-C cohomologies
$\mathcal{B}^{\bullet}_{p,q}$	Complex of sheaves defining B-C cohomologies
$\mathcal{F}^{\bullet}[n]$	Complex of sheaves shifted n to the right
$\mathcal{F}^{\bullet}[-n]$	Complex of sheaves shifted n to the left
$R(p)$	Tate twist on a subring of $\mathbb{R}$
$R(p)^{\bullet}_{\mathcal{D}}$	Deligne complex for $R(p)$
$\overline{R(p)}^{\bullet}_{\mathcal{D}}$	Anti-holomorphic Deligne complex for R and subring of $\mathbb{C}$
$\epsilon_{\mathcal{D}}$	Map from Bott-Chern to Deligne cohomology
$\overline{\epsilon_{\mathcal{D}}}$	Map from Bott-Chern to Anti-Holomorphic Deligne
$F^s\mathcal{F}^{\bullet}$	Hodge filtration on a complex of sheaves
$\mathcal{Z}^{r,s}_{\mathcal{D}}$	Sheaf Kernel of $\overline{\partial} : \mathcal{D}^{p,q} \to \mathcal{D}^{p,q+1}$
δ	Čech differential
ϕ_q	Projection onto antiholomorphic degree $< q$
$\mathcal{D}^{\bullet}_{<q}$	Sheaves of currents with anti-holomorphic degree $< q$
$\mathcal{A}^{\bullet}_{<q}$	Sheaves of forms with anti-holomorphic degree $< q$
$\mathrm{Cone}(A^{\bullet} \to B^{\bullet})$	Cone complex of a map of complexes of sheaves
$\circ$	Product of cone complexes
$\mathcal{L} = \{l_{ij}\}$	Line bundle by transition functions
FLB	Group of flat line bundles in $\mathrm{Pic}^0(X)$
FLS	Group of FLB with non-zero meromorphic sections

Notation	Object
$\lvert\sigma\rvert$ or $\lvert\sigma\rvert_L$	Metric on a line bundle L applied to a section σ
DST	Cartier divisors corresponding to densely stably trivial bundles
$\hat{\Omega}^p$	Forms in Ω^p which are d-closed, equivalantly ∂-closed
$\check{C}^r(\mathcal{U}, X)$	Group of Čech cocycles
$H^n_{\mathrm{tor}}(X, \mathbb{Z}(r))$	Torsion subgroup of singular cohomology
$\tilde{Z} \to Z$	Desingularization of a singular analytic variety Z
A^n_Z	Twisted cycles on a subvariety Z
$T(R)$	Tensor algebra of a module R
$\underline{z}^k(X, m)$	Analytic twisted cycles of codim k and rank m on X
$\{\sigma_1, \ldots, \sigma_m\}_Z$	Basic element of $\underline{z}^k(X, m)$
T^m_D	Tame symbol for a given subvariety D
T^m	Tame symbol
ν_D	Discrete valuation given by a subvariety D
$k(\nu_D)$	Residue field
Π	Special basis element for κ-algebra
∂_ϕ	A κ-algebra map
∂_{ν_D}	A κ-algebra map
$V^k(X, m)$	Twisted cycle classes of rank m and codimension $k - m$
$\star$	Product on Čech complexes
$\cap$	Various intersections and similar products
$\langle \cdot, \cdot \rangle$	Height pairings

2.4 Global Assumptions

Throughout the thesis, the following assumptions will hold. X will always be a smooth, compact complex manifold of dimension d. Z will always be an irreducible analytic subvariety. We do not assume that X is projective algebraic or Kähler. We work exclusively with the strong topology of X.

Chapter 3

Bott-Chern Cohomology

3.1 $\mathbb{C}$-Valued Bott-Chern Cohomology

The constructions for this section on Bott-Chern cohomology are taken mostly from the online book [Dem07] and the unpublished thesis by Demailly's former student [Sch07]. The former is a good reference, but the latter is the major source of inspiration for this section of the thesis. For the latter, as an unpublished work, we try to be very careful to check the calculations and provide proofs when necessary; however, that should not distract from the intellectual debt we owe to that work.

We start with the basic definition. In a non-algebraic context, Bott-Chern cohomology is a variant of the standard De Rham and Dolbeault cohomologies and is calculated by differential forms. As a note, we will often carelessly use the term 'Bott-Chern cohomology' to include both the Bott-Chern and Aeppli versions, since they are so closely connected.

Definition 3.1.1: The *Bott-Chern cohomology* of X, with complex coefficients, is defined as follows.

$$H^{p,q}_{\mathrm{BC}}(X,\mathbb{C}) := \frac{\mathrm{Ker}(d:\ \mathcal{A}^{p,q}(X)\to\mathcal{A}^{p+q+1}(X))}{\partial\overline{\partial}\mathcal{A}^{p-1,q-1}(X)} \tag{3.1.1}$$

Sometimes the numerator here is written $\mathrm{Ker}\partial\cap\mathrm{Ker}\overline{\partial}$, which gives the same thing given that $d=\partial+\overline{\partial}$.

Note that the denominator is trivial if either $p=0$ or $q=0$. If, for example $q=0$, then Bott-Chern cohomology simply captures the (finite dimensional) vector space of global closed holomorphic p-forms.

Definition 3.1.2: The *Aeppli cohomology* is similarly defined:

$$H^{p,q}_{\mathrm{Ap}}(X,\mathbb{C}) := \frac{\mathrm{Ker}(\partial\overline{\partial}:\ \mathcal{A}^{p,q}(X)\to\mathcal{A}^{p+1,q+1}(X))}{\partial\mathcal{A}^{p-1,q}(X)+\overline{\partial}\mathcal{A}^{p,q-1}(X)} \tag{3.1.2}$$

Definition 3.1.3: There are maps from these cohomologies to the De Rham groups as follows.

$$H^{p,q}_{\mathrm{BC}}(X,\mathbb{C})\to H^{p+q}(X,\mathbb{C}) \qquad H^{p,q}_{\mathrm{Ap}}(X,\mathbb{C})\to H^{p+q+1}(X,\mathbb{C}) \tag{3.1.3}$$

Assume the De Rham groups are given as the quotients of d-closed forms by d-exact forms. Then the map on Bott-Chern is given by the identity on forms, since Bott-Chern classes are given by d-closed forms and $\partial\overline{\partial}$-exact forms are also d-exact. The map on Aeppli cohomology is given by ∂ acting on forms, since ∂ maps $\partial\overline{\partial}$-closed forms to d-exact forms and ∂ or $\overline{\partial}$-exact forms to d-exact forms.

3.1.1 Products on $\mathbb{C}$-Valued Bott-Chern Cohomology

It is straightfoward to define two products on Bott-Chern cohomologies with complex coefficients.

Proposition 3.1.4: *The exterior derivative of forms is well defined on the classes defining Bott-Chern and Aeppli cohomologies, giving two products:*

$$\wedge : H^{p,q}_{\mathrm{BC}}(X,\mathbb{C}) \times H^{r,s}_{\mathrm{BC}}(X,\mathbb{C}) \to H^{r+p,s+q}_{\mathrm{BC}}(X,\mathbb{C}) \tag{3.1.4}$$

$$\wedge : H^{p,q}_{\mathrm{BC}}(X,\mathbb{C}) \times H^{r,s}_{\mathrm{Ap}}(X,\mathbb{C}) \to H^{r+p,s+q}_{\mathrm{Ap}}(X,\mathbb{C})$$

Proof. This is almost immediate: it is only necessary to check that the product preserves the relations defining these groups. In the numerators: the product of d-closed forms remains d-closed, and the product of a d-closed form and a $\partial\overline{\partial}$-closed form is $\partial\overline{\partial}$-closed. In the denominators: the product of $\partial\overline{\partial}$-exact forms remains $\partial\overline{\partial}$-exact, and the product of a $\partial\overline{\partial}$-exact form with a ∂-exact or $\overline{\partial}$-exact forms remains, respectively, ∂-exact or $\overline{\partial}$-exact. □

The first product gives a ring structure on the direct product of all bidegrees of Bott-Chern cohomology. However, unlike the Dolbeault cohomology on smooth manifolds, this ring does not give a Serre duality result. The second product is required for a duality. In degree (n,n), the denominator is equivalent to $d\mathcal{A}^{2n-1}(X)$, which shows that $H^{n,n}_{\mathrm{Ap}}(X,\mathbb{C}) \cong H^{n}_{\mathrm{DR}}(X,\mathbb{C}) \cong \mathbb{C}$. This leads to the following result.

Proposition 3.1.5: *The product in Proposition 3.1.4 gives a perfect pairing.*

$$\wedge : H^{p,q}_{\mathrm{BC}}(X, \mathbb{C}) \times H^{n-p,n-q}_{\mathrm{Ap}}(X, \mathbb{C}) \to \mathbb{C} \tag{3.1.5}$$

Proof. This proof is from [Sch07]. It relies on the theory of harmonic forms describing Bott-Chern and Aeppli cohomology, which we do not investigate in this thesis. In summary, there are elliptic operators $\tilde{\Delta}_{\mathrm{BC}}$ and $\tilde{\Delta}_{\mathrm{Ap}}$ such that Bott-Chern and Aeppli cohomology are realized as the harmonic forms under these operators, respectively. Then the proof of duality is reduced to a simple calculation.

$$\begin{aligned} u \in \mathcal{H}^{p,q}_{\tilde{\Delta}_{\mathrm{BC}}}(X) &\Rightarrow \partial u = 0, \quad \overline{\partial} u = 0, \quad (\partial\overline{\partial})^* u = 0 \\ &\Rightarrow \overline{\partial}^*(*u) = 0, \quad \partial^*(*u) = 0, \quad \partial\overline{\partial}(*u) = 0 \\ &\Rightarrow *u \in \mathcal{H}^{n-p,n-q}_{\tilde{\Delta}_{\mathrm{Ap}}}(X) \end{aligned} \tag{3.1.6}$$

□

Note that there is no product pairing classes from Aeppli cohomology with classes from Aeppli cohomology. The exterior derivative preserves neither the numerator defining Aeppli cohomology, nor the denominator. This is an important difference when working with Bott-Chern and Aeppli cohomology, as compared to De Rham and Dolbeault cohomology, and it complicates products in many situations, as shall be seen in Section 3.9.

3.2 Bott-Chern Hypercohomology

The definition by differential forms is convenient for its simplicity, but lacking in flexibility. Therefore, following the constructions suggested in [Dem07], which are elaborated on in [Sch07], we define Bott-Chern cohomology as the hypercohomology of a complex of sheaves.

Before starting the hypercohomology constructions, the following result must be established. This is essentially a version of the Poincaré Lemma for Aeppli cohomology, identifying that Aeppli classes are locally trivial. Since our use concerns complexes of sheaves, we prove a sheaf-theoretic version of the result.

Lemma 3.2.1: *Let $0 \leq p < n$ and $0 \leq q < n$. Then the kernel sheaf of $\partial\overline{\partial}$ acting on forms of fixed bidgree (p, q) is:*

$$\operatorname{Ker}(\partial\overline{\partial} : \mathcal{A}^{p,q} \to \mathcal{A}^{p+1,q+1}) = \mathcal{A}^{p,q}_{\partial-\text{closed}} + \mathcal{A}^{p,q}_{\overline{\partial}-\text{closed}} \tag{3.2.1}$$

Proof. The inclusion of the right-hand side in the $\partial\overline{\partial}$-kernel is immediate, so it is only necessary to prove the opposite inclusion. To do so, we work with germs of forms, considering only a small polydisc neighbourhood of the stalk where we can apply the Poincaré Lemma.

Assume that η is a $\partial\overline{\partial}$-closed locally defined (p, q)-form. The Poincaré Lemma (Proposition 2.2.22) implies that there exists a form τ such that $\overline{\partial}\eta = \partial\tau$, *i.e.* $\overline{\partial}\eta$ is ∂-exact. By Proposition 2.2.24, since $\overline{\partial}\eta$ is of a single bidegree, $\overline{\partial}\eta$ is also $\partial\overline{\partial}$-exact. Therefore, there exists a form ν such that $\overline{\partial}\eta = \overline{\partial}\partial\nu$.

Then $\overline{\partial}(\eta - \partial\nu) = 0$, so there exists a form σ with $(\eta - \partial\nu) = \overline{\partial}\sigma$, which is $\eta = \overline{\partial}\sigma + \partial\nu$, completing the proof. □

In order to define the complex of sheaves which will eventually calculate Bott-Chern cohomology, we must proceed in two cases based on the bidegree of $H^{p,q}_{\mathrm{BC}}(X, \mathbb{C})$. The two cases are the case where $p \geq 1$ and $q \geq 1$ (which we call the strictly positive case), and the case where p, q, or both are 0 (which we call the case allowing degree zero). Those cases determined the necessary definitions; inside those cases, in order to do proofs and calculations, we also consider a variety of subcases.

3.2.1 Strictly Positive Bidegrees

Definition 3.2.2: For each strictly positive bidegree (p, q), we define the complex $\mathcal{L}_{p,q}$ and its differentials as follows.

$$\begin{aligned}
\mathcal{L}^k_{p,q} &= \bigoplus_{\substack{r+s=k \\ r<p,\ s<q}} \mathcal{A}^{r,s} \text{ if } k \leq p+q-2 \\
\mathcal{L}^k_{p,q} &= \bigoplus_{\substack{r+s=k+1 \\ r\geq p\ \ s\geq q}} \mathcal{A}^{r,s} \text{ if } k \geq p+q-1 \\
d^k_{p,q} &= \underline{d} \text{ if } k < p+q-2 \\
d^{p+q-2}_{p,q} &= \partial\overline{\partial} \\
d^k_{p,q} &= d \text{ if } k > p+q-2
\end{aligned} \tag{3.2.2}$$

Recall that we write $\pi_{p,q}$ for the projection onto a given bidegree of forms or currents and that we use $\underline{d}$ to mean the normal differential d followed by projection onto whatever bidegree(s) are necessary to give a form in the target space. To clarify this implied projection, consider two examples. If $p = 3$ and

$q = 4$, then $\mathcal{L}^1_{3,4} = \mathcal{A}^{1,0} \oplus \mathcal{A}^{0,1}$ and $\mathcal{L}^2_{3,4} = \mathcal{A}^{2,0} \oplus \mathcal{A}^{1,1} \oplus \mathcal{A}^{0,2}$. In this degree, no projection is necessary, since the restrictions $r < 3$ and $s < 4$ are trivially satisfied. Here $\underline{d}$ is simply d. Later on, we have $\mathcal{L}^3_{3,4} = \mathcal{A}^{2,1} \oplus \mathcal{A}^{1,2} \oplus \mathcal{A}^{0,3}$ and $\mathcal{L}^4_{3,4} = \mathcal{A}^{2,2} \oplus \mathcal{A}^{1,3}$. Here the image of d on $\mathcal{L}^3_{3,4}$ gives a form in $\mathcal{A}^{3,1} \oplus \mathcal{A}^{2,2} \oplus \mathcal{A}^{1,3} \oplus \mathcal{A}^{0,4}$. To satisfy the restrictions and give a form in $\mathcal{L}^4$, the projection here is projection onto the middle two components, so $\underline{d} = (\pi_{2,2} + \pi_{1,3}) \circ d$.

The following proposition concerning the cohomology of the $\mathcal{L}^\bullet_{p,q}$ complex is from [Dem07]. The proof is done in brief in that reference, so we give a detailed version here.

A caution should be noted concerning the calculation of the cohomology of the $\mathcal{L}^\bullet_{p,q}$ complex. By cohomology here, we simply mean the quotients of kernels modulo images in the complex; these quotients are still sheaves. This is very different from the eventual calculation of sheaf cohomology, where we calculate hypercohomology by taking resolutions and using the derived functors associated with the global section function. Even though both constructions use the term cohomology liberally, it is important not to confuse them. In addition, a quasi-isomorphism of complexes is, by definition, a map which induces an isomorphism on the level of the cohomology of the complexes. A quasi-isomorphism also gives rise to an isomorphism on hypercohomology, but that is an important theorem, not part of the definition.

Proposition 3.2.3: *The cohomology of the complex $\mathcal{L}^\bullet_{p,q}$ vanishes in all degrees except 0, $p-1$ and $q-1$. In those degrees, cohomology is calculated in cases.*

Case 1: If $p \geq 2$, $q \geq 2$, and $p \neq q$ then all three degrees are distinct.

$$\mathcal{H}^0(\mathcal{L}^\bullet_{p,q}) = \mathbb{C} \quad \mathcal{H}^{p-1}(\mathcal{L}^\bullet_{p,q}) = \frac{\Omega^{p-1}}{\partial\Omega^{p-2}} \quad \mathcal{H}^{q-1}(\mathcal{L}^\bullet_{p,q}) = \frac{\overline{\Omega}^{q-1}}{\overline{\partial\Omega}^{q-2}} \tag{3.2.3}$$

Case 2: If $p = q \geq 2$, then only degrees 0 and $p-1$ are nonvanishing.

$$\mathcal{H}^0(\mathcal{L}^\bullet_{p,p}) = \mathbb{C} \quad \mathcal{H}^{p-1}(\mathcal{L}^\bullet_{p,p}) = \frac{\Omega^{p-1}}{\partial\Omega^{p-2}} \oplus \frac{\overline{\Omega}^{p-1}}{\overline{\partial\Omega}^{p-2}} \tag{3.2.4}$$

Case 3: If $p = 1$ and $q \geq 2$, then only degrees 0 and $q-1$ are nonvanishing.

$$\mathcal{H}^0(\mathcal{L}^\bullet_{p,q}) = \mathcal{O} \quad \mathcal{H}^{q-1}(\mathcal{L}^\bullet_{p,q}) = \frac{\overline{\Omega}^{q-1}}{\overline{\partial\Omega}^{q-2}} \tag{3.2.5}$$

Case 4: If $p \geq 2$ and $q = 1$, then only degrees 0 and $p-1$ are nonvanishing.

$$\mathcal{H}^0(\mathcal{L}^\bullet_{p,q}) = \overline{\mathcal{O}} \quad \mathcal{H}^{p-1}(\mathcal{L}^\bullet_{p,q}) = \frac{\Omega^{p-1}}{\partial\Omega^{p-2}} \tag{3.2.6}$$

Case 5: If $p = q = 1$, then only degree 0 is nonvanishing.

$$\mathcal{H}^0(\mathcal{L}^\bullet_{p,q}) = \mathcal{O} + \overline{\mathcal{O}} \tag{3.2.7}$$

Proof. The proof proceeds in the same cases as the statement of the proposition.

Case 1: $(p \geq 2,\ q \geq 2,\ p \neq q)$

Assume, without loss of generality, that $p < q$. (The case $q < p$ proceeds precisely in parallel.) Isolating degree zero gives this sequence.

$$0 \to \mathcal{A}^{0,0} \xrightarrow{d} \mathcal{A}^{1,0} \oplus \mathcal{A}^{0,1} \tag{3.2.8}$$

The 0th cohomology of $\mathcal{L}^\bullet_{p,q}$ is the sheaf $\mathrm{Ker}\{\mathcal{A}^{0,0} \xrightarrow{d} \mathcal{A}^{1,0} \oplus \mathcal{A}^{0,1}\}$. Since $\mathcal{A}^{0,0}$ is simply the sheaf of functions and since functions which have zero differentials are locally constant, this kernel is simply the constant sheaf $\mathbb{C}$. This gives the conclusion that $\mathcal{H}^0(\mathcal{L}^\bullet_{p,q}) = \mathbb{C}$.

For degrees $0 < i < p-1$, the complex $\mathcal{L}^\bullet_{p,q}$ has these terms.

$$\begin{aligned} &\left[\mathcal{A}^{i-1,0} \oplus \ldots \oplus \mathcal{A}^{0,i-1}\right] \xrightarrow{d} \\ &\left[\mathcal{A}^{i,0} \oplus \ldots \oplus \mathcal{A}^{0,i}\right] \xrightarrow{d} \\ &\left[\mathcal{A}^{i+1,0} \oplus \ldots \oplus \mathcal{A}^{0,i+1}\right] \end{aligned} \tag{3.2.9}$$

These degrees are exact on the stalks by the Poincaré Lemma (Proposition 2.2.22). Exactness on the stalk implies exactness on the sheaf level, so the cohomology sheaves in these degrees vanish.

The next term is at degree $p-1$.

$$\begin{aligned} &\left[\mathcal{A}^{p-2,0} \oplus \ldots \oplus \mathcal{A}^{0,p-2}\right] \xrightarrow{d} \\ &\left[\mathcal{A}^{p-1,0} \oplus \ldots \oplus \mathcal{A}^{0,p-1}\right] \xrightarrow{d} \\ &\left[\mathcal{A}^{p-1,1} \oplus \ldots \oplus \mathcal{A}^{0,p}\right] \end{aligned} \tag{3.2.10}$$

The only difference between this degree and the previous is that the target space lacks the component $\mathcal{A}^{p,0}$, which affects the kernel of the differential. Restricted to the sheaves $\mathcal{A}^{p-2,1} \oplus \ldots \oplus \mathcal{A}^{0,p-1}$, the kernel is the set of d-closed forms. By the Poincaré Lemma (Proposition 2.2.22), the complex is exact on all these component sheaves, since locally d-closed forms are d-exact. This leaves only the kernel restricted to $\mathcal{A}^{p-1,0}$. Here the image of the ∂ portion of the differential would land in $\mathcal{A}^{p,0}$, which is excluded from the target space. Therefore, the differential is only the $\overline{\partial}$ part of d, and the kernel consists of forms in $\mathcal{A}^{p-1,0}$ which are $\overline{\partial}$-closed, *i.e.* Ω^{p-1}.

The image of the previous differential, restricted to the summand $\mathcal{A}^{p-1,0}$ is $\partial\mathcal{A}^{p-2,0}$. Since $\mathcal{L}^{\bullet}_{p,q}$ is a complex, this is necessarily a subset of Ω^{p-1}, so these ∂-exact forms are $\overline{\partial}$-closed. Since ∂ and $\overline{\partial}$ commute, up to a sign, this image is the same as $\partial\Omega^{p-2}$. We conclude the the cohomology in degree $p-1$ is described by the following quotient sheaf.

$$\mathcal{H}^{p-1}(\mathcal{L}^{\bullet}_{p,q}) = \frac{\Omega^{p-1}}{\partial\Omega^{p-2}} \tag{3.2.11}$$

Then, for degrees $p-1 < i < q-1$, we have these terms.

$$\begin{aligned} &\left[\mathcal{A}^{p-1,i-p} \oplus \ldots \oplus \mathcal{A}^{0,i-1}\right] \xrightarrow{d} \\ &\left[\mathcal{A}^{p-1,i-p+1} \oplus \ldots \oplus \mathcal{A}^{0,i}\right] \xrightarrow{d} \\ &\left[\mathcal{A}^{p-1,i-p+2} \oplus \ldots \oplus \mathcal{A}^{0,i+1}\right] \end{aligned} \tag{3.2.12}$$

Again, by the Poincaré Lemma (Proposition 2.2.22), using both the ∂ and $\overline{\partial}$ version of that result, this is exact on the stalks, so the cohomology of the complex is trivial.

At degree $q-1$ we have these terms.

$$\begin{aligned}&\left[\mathcal{A}^{p-1,q-p-1} \oplus \ldots \oplus \mathcal{A}^{0,q-2}\right] \xrightarrow{d} \\ &\left[\mathcal{A}^{p-1,q-p} \oplus \ldots \oplus \mathcal{A}^{0,q-1}\right] \xrightarrow{d} \\ &\left[\mathcal{A}^{p-1,q-p+1} \oplus \ldots \oplus \mathcal{A}^{1,q-1}\right]\end{aligned} \tag{3.2.13}$$

As in degree $p-1$, the only part of this which contributes is the following sequence.

$$\mathcal{A}^{0,q-2} \xrightarrow{d} \mathcal{A}^{0,q-1} \xrightarrow{d} \mathcal{A}^{1,q-1} \tag{3.2.14}$$

By the same reasoning as in the degree $p-1$ case, the kernel at this degree is $\overline{\Omega}^{q-1}$ and the image is $\overline{\partial\Omega}^{q-2}$, so the cohomology is the quotient sheaf.

$$\mathcal{H}^{q-1}(\mathcal{L}^{\bullet}_{p,q}) = \frac{\overline{\Omega}^{q-1}}{\overline{\partial\Omega}^{q-2}} \tag{3.2.15}$$

For degrees $q-1 < i < p+q-3$, we have these terms.

$$\begin{aligned}&\left[\mathcal{A}^{p-1,i-p} \oplus \ldots \oplus \mathcal{A}^{i-q,q-1}\right] \xrightarrow{d} \\ &\left[\mathcal{A}^{p-1,i-p+1} \oplus \ldots \oplus \mathcal{A}^{i-q+1,q-1}\right] \xrightarrow{d} \\ &\left[\mathcal{A}^{p-1,i-p+2} \oplus \ldots \oplus \mathcal{A}^{i-q+2,q-1}\right]\end{aligned} \tag{3.2.16}$$

For degrees $p+q-3$, $p+q-2$ and $p+q-1$, we have these terms.

$$\begin{aligned}&\left[\mathcal{A}^{p-1,q-3}\oplus\mathcal{A}^{p-2,q-2}\oplus\mathcal{A}^{p-3,q-1}\right]\xrightarrow{d}\\&\left[\mathcal{A}^{p-1,q-2}\oplus\mathcal{A}^{p-2,q-1}\right]\xrightarrow{d}\mathcal{A}^{p-1,q-1}\xrightarrow{\partial\overline{\partial}}\\&\mathcal{A}^{p,q}\xrightarrow{d}\left[\mathcal{A}^{p+1,q}\oplus\mathcal{A}^{p,q+1}\right]\end{aligned}\tag{3.2.17}$$

For all higher degrees, we have the the complex of forms $\mathcal{A}^{r,s}$ with $r \geq p$ and $s \geq q$ and with the differential simply d.

In all these last three cases, the Poincaré Lemma (Proposition 2.2.22) shows that everything is exact on the stalks, so all cohomology vanishes. We make use of the $\partial\overline{\partial}$-lemma (Proposition 2.2.24) in the cases which involve $\partial\overline{\partial}$ instead of the normal differential d. This is justified since, at this point, the forms in $\mathcal{A}^{p,q}$ or $\mathcal{A}^{p-1,q-1}$ are of a single bidegree, and the $\partial\overline{\partial}$-lemma applies.

Case 2: $p \geq 2$, $q \geq 2$, $p = q$.

In this case, everything is precisely the same as Case 1 except in degree $p-1 = q-1$. At that degree the complex is as follows.

$$\begin{aligned}&\left[\mathcal{A}^{p-2,0}\oplus\ldots\oplus\mathcal{A}^{0,p-2}\right]\xrightarrow{d}\\&\left[\mathcal{A}^{p-1,0}\oplus\ldots\oplus\mathcal{A}^{0,p-1}\right]\xrightarrow{d}\\&\left[\mathcal{A}^{p-1,1}\oplus\ldots\oplus\mathcal{A}^{1,p-1}\right]\end{aligned}\tag{3.2.18}$$

We have contributing terms on both ends of the direct sum. These terms are calculated as in Case 1 in degree $p-1$ and $q-1$, and the calculations are independent of each other since $p \geq 2$ and $q \geq 2$. This gives the following cohomology.

$$\mathcal{H}^{p-1}(\mathcal{L}^{\bullet}_{p,q}) = \frac{\Omega^{p-1}}{\partial\Omega^{p-2}} \oplus \frac{\overline{\Omega}^{p-1}}{\overline{\partial\Omega}^{p-2}} \tag{3.2.19}$$

Case 3: $p = 1$ and $q \geq 2$.

All degrees other than 0 and $q-1$ are the same as Case 1. At degree 0, we have these terms.

$$0 \to \mathcal{A}^{0,0} \xrightarrow{\overline{\partial}} \mathcal{A}^{1,0} \tag{3.2.20}$$

The kernel of $\overline{\partial}$ is the sheaf of holomorphic functions $\mathcal{O}$.

At degree $q-1$ we have the following complex.

$$\mathcal{A}^{0,q-2} \xrightarrow{\overline{\partial}} \mathcal{A}^{0,q-1} \xrightarrow{\partial\overline{\partial}} \mathcal{A}^{1,q} \tag{3.2.21}$$

The kernel, by Lemma 3.2.1, is $\mathcal{A}^{0,q-1}_{\partial-closed} + \mathcal{A}^{0,q-1}_{\overline{\partial}-\text{closed}}$. The image of the previous map is $\overline{\partial}\mathcal{A}^{0,q-2}$. In the second component, the quotient is trivial by the $\overline{\partial}$-version of the Poincareé Lemma (Proposition 2.2.22). In the first component, the numerator is $\overline{\Omega}^{q-1}$ and the denominator is $\overline{\partial\Omega}^{q-2}$, which gives the desired cohomology.

Case 4: $q = 1$ and $p \geq 2$.

This is precisely in parallel with the argument given in the Case 3, with holomorphic and anti-holomorphic forms and differentials reversed.

Case 5:

Past degree 0, the complex is exact by the arguments given in Case 1. At degree zero, we have these terms.

$$0 \to \mathcal{A}^{0,0} \stackrel{\partial\overline{\partial}}{\to} \mathcal{A}^{1,1} \tag{3.2.22}$$

By Lemma 3.2.1, the kernel here is $\mathcal{O} + \overline{\mathcal{O}}$. This is the desired cohomology sheaf, which completes the last of the five cases.

□

Now that $\mathcal{L}^\bullet_{p,q}$ and its cohomology are known, we want to construct new sequences of sheaves which will be easier to work with but recover the same hypercohomology. To that end, we use the following definitions from [Sch07].

Definition 3.2.4: The $\mathcal{S}^\bullet_{p,q}$ complex of sheaves is defined in two cases. In the first case, assume $p = q$.

$$\mathcal{S}^\bullet_{p,p} := 0 \to \mathcal{O} + \overline{\mathcal{O}} \to \Omega^1 \oplus \overline{\Omega}^1 \to \ldots \to \Omega^{p-1} \oplus \overline{\Omega}^{p-1} \to 0 \tag{3.2.23}$$

The sheaf $\mathcal{O} + \overline{\mathcal{O}}$ is in degree 0. Note here that the sum in degree 0 is *not* direct. The differentials are $(\partial, \overline{\partial})$, both acting on the one element in degree 0 and acting componentwise in positive degrees. In the second case, first assume $p > q$.

$$\begin{aligned}\mathcal{S}^\bullet_{p,q} :=& 0 \to \mathcal{O} + \overline{\mathcal{O}} \to \Omega^1 \oplus \overline{\Omega}^1 \to \\ & \ldots \to \Omega^{q-1} \oplus \overline{\Omega}^{q-1} \to \Omega^q \to \ldots \to \Omega^{p-1} \to 0\end{aligned} \tag{3.2.24}$$

The differentials are similarly $(\partial, \overline{\partial})$ up to term $q-1$ and ∂ from $q-1$ onward.

If instead we assume $q > p$, then the definition is a mirror of this, with a trail of purely antiholomorphic forms and a differential of $\overline{\partial}$ from $p-1$ onward.

The complex $\mathcal{S}^{\bullet}_{p,q}$ is a subcomplex of $\mathcal{L}^{\bullet}_{p,q}$, so the inclusion provides a map $\mathcal{S}^{\bullet}_{p,q} \to \mathcal{L}^{\bullet}_{p,q}$.

Definition 3.2.5: We define $\mathcal{B}^{\bullet}_{p,q}$ very similarly.

$$\mathcal{B}^{\bullet}_{p,q} = 0 \to \mathbb{C} \to \mathcal{O} \oplus \overline{\mathcal{O}} \to \Omega^1 \oplus \overline{\Omega}^1 \to \dots \tag{3.2.25}$$

The definition assumes that $\mathcal{B}^{\bullet}_{p,q}$ continues in the same pattern as $\mathcal{S}^{\bullet}_{p,q}$, depending on p and q as in the cases above. The map $\mathbb{C} \to \mathcal{O} \oplus \overline{\mathcal{O}}$ is $z \mapsto (z, -z)$. Unlike the $\mathcal{S}^{\bullet}_{p,q}$ complex, the product $\mathcal{O} \oplus \overline{\mathcal{O}}$ is direct.

We have a choice on the indexing of this $\mathcal{B}^{\bullet}_{p,q}$ complex. We can match the indexing of the $\mathcal{L}^{\bullet}_{p,q}$ and $\mathcal{S}^{\bullet}_{p,q}$ complexes, where the term $\mathcal{O} \oplus \overline{\mathcal{O}}$ is in degree zero, or we can make the more natural choice of $\mathbb{C}$ in degree zero. Another consideration is that the definition of Deligne cohomology involves a similar complex, with $\mathbb{C}$ in degree 0. In the interest of matching up well with the Deligne complex, we make that choice and set the term $\mathbb{C}$ at index 0.

There is a simple map $\mathcal{B}^{\bullet}_{p,q} \to \mathcal{S}^{\bullet}_{p,q}[-1]$. In degree 0, it is the zero map. In degree 1, it is simply addition of the two direct product components: $\mathcal{O} \oplus \overline{\mathcal{O}} \to \mathcal{O} + \overline{\mathcal{O}}$. In higher degrees, it is the identity map. It is easy to see that this is a map of complexes: we only need to check compatibility at degrees 0 and 1. In degree 0, there is very little to check: one side factors through 0, so is a zero map, and the other maps $z \to (-z, z) \to z + (-z) = 0$. In degree 1, if we take the differential first, we get $(f, g) \mapsto (\partial f, \overline{\partial} g) \mapsto (\partial f, \overline{\partial} g)$. Otherwise we have $(f, g) \mapsto f + g \mapsto (\partial(f+g), \overline{\partial}(f+g)) = (\partial f, \overline{\partial} g)$, which agrees.

The importance of the $\mathcal{B}^\bullet_{p,q}$ and $\mathcal{S}^\bullet_{p,q}$ complexes is made clear by the following proposition.

Proposition 3.2.6: *The three complexes of sheaves $\mathcal{L}^\bullet_{p,q}[-1], \mathcal{S}^\bullet_{p,q}[-1]$ and $\mathcal{B}^\bullet_{p,q}$ are quasi isomorphic by the map $\mathcal{B}^\bullet_{p,q} \to \mathcal{S}^\bullet_{p,q}[-1]$ and the inclusion $\mathcal{S}^\bullet_{p,q}[-1] \to \mathcal{L}^\bullet_{p,q}[-1]$.*

Proof. We've already explicitly constructed the cohomology of $\mathcal{L}^\bullet_{p,q}$, so all we must do is check that $\mathcal{S}^\bullet_{p,q}$ and $\mathcal{B}^\bullet_{p,q}$ have the same cohomology and that the induced maps on cohomology are isomorphisms. For positive degrees, in each of the five cases, this is essentially by construction and simple application of the ∂ and $\overline{\partial}$ versions of the Poincaré Lemma (Proposition 2.2.22. Therefore, the proof is only concerned with degree 1. We follow the same cases as in the calculation of the cohomology of the $\mathcal{L}^\bullet_{p,q}$ complex in Proposition 3.2.3.

Case 1 and Case 2: $p \geq 2$ and $q \geq 2$ (the $p = q$ distinction is not important here).

At degree 1, for $S^\bullet_{p,q}[-1]$ we have this sequence.

$$0 \to \mathcal{O} + \overline{\mathcal{O}} \to \Omega^1 \oplus \overline{\Omega}^1 \to \tag{3.2.26}$$

The differential in degree 1 is just d, split into components. Only constant functions have zero differential everywhere, so the kernel in degree 1 is $\mathbb{C}$. The map from $\mathcal{S}^\bullet_{p,q}[-1]$ into $\mathcal{L}^\bullet_{p,q}[-1]$ is simply the inclusion. This maps $\mathbb{C}$ to $\mathbb{C}$, giving an isomorphism of cohomology.

At degree 1, the sequence $B^\bullet_{p,q}$ is as follows.

$$\mathcal{B}^\bullet_{p,q} = 0 \to \mathbb{C} \to \mathcal{O} \oplus \overline{\mathcal{O}} \to \Omega^1 \oplus \overline{\Omega}^1 \to \tag{3.2.27}$$

(Remember that $\mathbb{C}$ is in degree 0.) The differential at degree 0 is an injection, $z \mapsto (z, -z)$, so the 0th cohomology vanishes.

The kernel of the first differential is $\mathbb{C} \oplus \mathbb{C}$. The image of the injection from the previous sheaf is the constant sheaf described by $\{(z, -z) | z \in \mathbb{C}\}$. This is the kernel of the addition map $+ : \mathbb{C} \oplus \mathbb{C} \to \mathbb{C}$, so the quotient, by the addition map, is isomorphic to $\mathbb{C}$. Moreover, the addition map is precisely the map of complexes in degree 1 into $\mathcal{S}^\bullet_{p,q}[-1]$, so this induces an isomorphism on cohomology, mapping to the constants $\mathbb{C}$ in $\mathcal{O} + \overline{\mathcal{O}}$.

Case 3: $p = 1$ and $q \geq 2$.

At degree zero, the sequence $S^\bullet_{1,q}[-1]$ is as follows.

$$\mathcal{S}^\bullet_{1,q}[-1] = 0 \to 0 \to \mathcal{O} + \overline{\mathcal{O}} \stackrel{(0,\overline{\partial})}{\to} \overline{\Omega}^1 \to \tag{3.2.28}$$

The 0th cohomology is the entire sheaf $\mathcal{O}$ plus the kernel of $\overline{\partial}$ on $\overline{\mathcal{O}}$. That kernel is the constants, which are already included in $\mathcal{O}$. (Recall the sum is not direct here). Therefore, the cohomology is $\mathcal{O}$, which matches with the $\mathcal{L}^\bullet_{1,q}$ complex, and the inclusion of $\mathcal{O}$ in $\mathcal{A}^0$ is the appropriate isomorphism on cohomology.

At degree zero, the differential on $B^\bullet_{1,q}$ is inclusion and there is no kernel. At degree 1, we have:

$$\mathcal{B}^\bullet_{1,q} = 0 \to \mathbb{C} \to \mathcal{O} \oplus \overline{\mathcal{O}} \stackrel{(0,\overline{\partial})}{\to} \oplus \overline{\Omega}^1 \to \tag{3.2.29}$$

The arguments here are the same as in Case 1, where the elements $(z, -z)$ vanish under addition of components. The inclusion of the constants into the kernel $\mathcal{O} \oplus \mathbb{C}$ gives a cohomology isomorphic to $\mathcal{O}$, which is identified with $\mathcal{O}$ in $\mathcal{S}^{\bullet}_{p,q}[-1]$ by the addition map.

Case 4: As before, this is a direct parallel with Case 3, switching holomorphic and anti-holomorphic terms.

Case 5: $p = q = 1$.

At degree zero, the sequence $S^{\bullet}_{1,1}[-1]$ consists of these terms.

$$\mathcal{S}^{\bullet}_{1,1}[-1] = 0 \to 0 \to \mathcal{O} + \overline{\mathcal{O}} \to 0 \to \tag{3.2.30}$$

This is trivially the correct cohomology: $\mathcal{O} + \overline{\mathcal{O}}$. It injects into $\mathcal{L}^{\bullet}_{1,1}[-1]$, which is the isomorphism on cohomology.

At degree zero, the sequence $B^{\bullet}_{1,1}$ is as follows.

$$\mathcal{B}^{\bullet}_{1,1} = 0 \to \mathbb{C} \to \mathcal{O} \oplus \overline{\mathcal{O}} \to 0 \to\to \tag{3.2.31}$$

This argument here is similar to the previous cases. The cohomology of $\mathcal{B}^{\bullet}_{1,1}$ is $\mathcal{O} \oplus \overline{\mathcal{O}}$ modulo $\{(z, -z) | z \in \mathbb{C}\}$. The addition map is well defined on this quotient, sending it to $\mathcal{O} + \overline{\mathcal{O}}$, moreover, the kernel of the addition map $\mathcal{O} \oplus \overline{\mathcal{O}} \to \mathcal{O} + \overline{\mathcal{O}}$ is precisely the set $\{(z, -z) | z \in \mathbb{C}\}$. Therefore, the addition map provides an isomorphism on the cohomology of the complexes. □

Bott-Chern cohomology is now calculated according to this proposition.

Proposition 3.2.7:

$$H^{p,q}_{BC}(X, \mathbb{C}) = \mathbb{H}^{p+q-1}(\mathcal{L}^\bullet_{p,q}) \cong \mathbb{H}^{p+q-1}(\mathcal{S}^\bullet_{p,q}) \cong \mathbb{H}^{p+q}(\mathcal{B}^\bullet_{p,q}) \tag{3.2.32}$$

Proof. The second and third isomorphisms follow directly as a corollary of the previous proposition, since all the complexes are quasi-isomorphic.

The first equality (which is an actual equality, not an isomorhpism) is essentially by construction. The sheaves in the $\mathcal{L}^\bullet_{p,q}$ complex are acyclic for the global section functor, since they are sheaves of C^∞ forms which admit partitions of unity. Therefore, the hypercohomology is the cohomology of the sequence of global sections.

The global section term $\mathcal{L}^{p+q-1}_{p,q}(X)$ is $\mathcal{A}^{p,q}(X)$, with the differential d. The previous global section term $\mathcal{L}^{p+q-2}_{p,q}(X)$ is $\mathcal{A}^{p+q-2}(X)$ with the differential $\partial\overline{\partial}$. The cohomology is the kernel of the first term modulo the image of the second term, which is precisely the definition of Bott-Chern cohomology as in Definition 3.1.1. □

This finishes the construction of Bott-Chern cohomology as the hypercohomology of complexes of sheaves for all subcases in the strictly positive case $1 \leq p, q \leq n$.

3.2.2 Cases with Degree Zero

Now we consider the cases where at least one of the degrees p and q is 0. We consider the case where $q = 0$ and $p \neq 0$; the $p = 0$ case will follow in parallel.

In the cases with degree zero, there are new definitions of the relevent complexes.

Definition 3.2.8: The complex $\mathcal{L}^\bullet_{p,0}$ is defined as follows.

$$\mathcal{L}^k_{p,0} = 0 \text{ if } k \leq p-2 \tag{3.2.33}$$

$$\mathcal{L}^k_{p,0} = \bigoplus_{\substack{r+s=k+1 \\ r\geq p}} \mathcal{A}^{r,s} \text{ if } k \geq p-1$$

The differential is the exterior derivative at all non-zero terms, with projections as necessary (we still refer to this operator as $\underline{d}$).

Proposition 3.2.9: *The cohomology of $\mathcal{L}^\bullet_{p,0}$ in this case is only supported in degree $p-1$, where it is the sheaf $\hat{\Omega}^p = \mathrm{Ker}(\partial : \Omega^p \to \Omega^{p+1})$.*

Proof. Away from degree $p-1$, repeated use of the Poincaré Lemma (Proposition 2.2.22) shows that the complex has trivial cohomology. Isolating around degree $p-1$ gives the following terms.

$$\ldots \to 0 \to \mathcal{A}^{p,0} \stackrel{d}{\to} \mathcal{A}^{p+1,0} \oplus \mathcal{A}^{p,1} \stackrel{d}{\to} \ldots \tag{3.2.34}$$

The cohomology of the complex in degree $p-1$ is the kernel of $d\mathcal{A}^{p,0} \to \mathcal{A}^{p+1,0} \oplus \mathcal{A}^{p,1}$. By the decomposition of the derivative, we can write this as the kernel of $\partial : \Omega^{p-1} \to \Omega^p$. □

In this case, a complex like $\mathcal{S}^\bullet_{p,0}$ is not particularly helpful, but the $\mathcal{B}^\bullet_{p,0}$ complex is still useful.

Definition 3.2.10: $\mathcal{B}^\bullet_{p,0}$ is the following complex.

$$0 \to \mathbb{C} \to \mathcal{O} \to \Omega^1 \to \ldots \to \Omega^{p-1} \to 0 \tag{3.2.35}$$

All differentials in this complex are ∂ except the inclusion map $\mathbb{C} \to \mathcal{O}$. $\mathbb{C}$ is in degree 0, as it was in the definition of $\mathcal{B}^\bullet_{p,q}$ in the case with non-zero degrees.

Proposition 3.2.11: *The complexes $\mathcal{B}^\bullet_{p,0}$ and $\mathcal{L}^\bullet_{p,0}[-1]$ are quasi-isomorphic by the following map:*

$$\begin{array}{ccccccccccccc}
0 & \to & \mathbb{C} & \to & \mathcal{O} & \to & \ldots & \to & \Omega^{p-1} & \to & 0 & \to & \ldots \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow \partial & & \downarrow & & \\
0 & \to & 0 & \to & 0 & \to & \ldots & \to & \mathcal{A}^{p,0} & \to & \mathcal{A}^{p+1,0} \oplus \mathcal{A}^{p,1} & \to & \ldots
\end{array}$$

Proof. The complex $\mathcal{B}^\bullet_{p,0}$ has cohomology only supported in degree $p-1$. In that degree, it has the cohomology sheaf $\Omega^{p-1}/\partial\Omega^{p-2}$. By Corollary 2.2.23, $\Omega^{p-1}/\partial\Omega^{p-2} \cong \mathrm{Ker}(\partial : \Omega^p \to \Omega^{p+1})$.

However, Ω^p is the same as $\mathrm{Ker}(\overline{\partial} : \mathcal{A}^{p,0} \to \mathcal{A}^{p,1})$, so we can make this identification.

$$\mathrm{Ker}(\partial : \Omega^p \to \Omega^{p+1}) = \mathrm{Ker}(d : \mathcal{A}^{p,0} \to \mathcal{A}^{p+1,0} \oplus \mathcal{A}^{p,1}) = \mathcal{H}^p(\mathcal{L}^\bullet_{p,0}) \tag{3.2.36}$$

All of the maps involved here are actual equalities, excepting the isomorphism given by Corollary 2.2.23; that isomorphism is accomplished by the map ∂. Since the map between the complexes at this degree is ∂, the induced map on cohomology is the desired isomorphism. $\square$

The equivalent calculation to Proposition 3.2.7, calculating Bott-Chern cohomology in this case, is the following proposition.

Proposition 3.2.12:

$$H^{p,0}_{BC}(X, \mathbb{C}) = \mathbb{H}^{p-1}(\mathcal{L}^\bullet_{p,0}) \cong \mathbb{H}^p(\mathcal{B}^\bullet_{p,0}) \tag{3.2.37}$$

Proof. Like the previous case, the first equality is by construction and the second isomorphism follows from the quasi-isomorphism. □

The case for $p = 0$ is directly in parallel. We define a $\mathcal{B}^\bullet_{0,q}$ complex that only has anti-holomorphic terms and the quasi-isomorphism with $\mathcal{L}^\bullet_{0,q}[-1]$ in degree q is realized by the map $\overline{\partial}$ instead of ∂. The final result is a similar isomorphism.

$$H^{0,q}_{BC}(X, \mathbb{C}) = \mathbb{H}^{q-1}(\mathcal{L}^\bullet_{0,q}) \cong \mathbb{H}^q(\mathcal{B}^\bullet_{0,q}) \tag{3.2.38}$$

Finally, if both $p = q = 0$, then $H^{0,0}_{\mathrm{BC}}(X, \mathbb{C}) = H^0_{\mathrm{DR}}(X, \mathbb{C})$ is equivalent to the De Rham cohomology in degree zero. Is this context, $\mathcal{L}^\bullet_{0,0}[-1] = \mathcal{A}^\bullet$ is the normal De Rham resolution, and $\mathcal{B}^\bullet_{0,0}$ is $\mathbb{C}$ in degree 0 and trivial everywhere else.

3.3 Bott-Chern with Finer Coefficients

The advantage of working with the hypercohomology of $\mathcal{B}^\bullet_{p,q}$ is that we can alter the constant term to vary the coefficients. Before starting that construction, we need the definition of Tate twists.

Definition 3.3.1: If R is a subring of $\mathbb{R}$, the pth *Tate twist* of R, written $R(p)$, is defined to be $(2\pi\imath)^p R$.

Note that the Tate twists remain subrings of $\mathbb{C}$ only when p is even; otherwise, they are simply abelian subgroups.

The Tate twists are not present in the description of Bott-Chern cohomology in [Dem07] and [Sch07], but introducing them is only a minor adjustment to those definitions. The early introduction of Tate twists and the total inclusion of them in our development simplifies the relationship with Deligne cohomology in future sections.

Recall that Bott-Chern cohomology is defined in two cases in Section 3.2. We work with the same cases here.

3.3.1 Strictly Positive Degrees

Definition 3.3.2: If R is a subring of $\mathbb{R}$, we define the complex of sheaves $\mathcal{B}^{\bullet}_{p,q}(R(r))$ to be the same as the complex $\mathcal{B}^{\bullet}_{p,q}$ in all degrees excepting that $\mathbb{C}$ in degree 0 is replaced by the subgroup $R(r)$.

For example, if $p, q > 1$ the complex starts as follows.

$$\mathcal{B}^{\bullet}_{p,q}(R(r)) := 0 \to R(r) \to \mathcal{O} \oplus \overline{\mathcal{O}} \to \Omega^1 \oplus \overline{\Omega}^1 \to \dots \tag{3.3.1}$$

The map $R(r) \to \mathcal{O} \oplus \overline{\mathcal{O}}$ is $z \to (z, -z)$, as was the case with $\mathbb{C}$ coefficients.

The definition is given for arbitrary r, but in practice we are usually concerned with $r = p$ or $r = q$, matching with one of the two bidgrees of $\mathcal{B}^{\bullet}_{p,q}$.

Similarly, though the subring $R \subset \mathbb{R}$ in the definition is arbitrary, we limit ourselves to considering subrings $\mathbb{Z}$ and $\mathbb{R}$, leading to the following definitions.

Definition 3.3.3: The *Tate-twisted integral and real Bott-Chern Cohomologies* are defined, respectively, as follows.

$$\begin{aligned} H^{p,q}_{\mathrm{BC}}(X, \mathbb{Z}(p)) &:= \mathbb{H}^{p+q}(\mathcal{B}^\bullet_{p,q}(\mathbb{Z}(p)) \\ H^{p,q}_{\mathrm{BC}}(X, \mathbb{R}(p)) &:= \mathbb{H}^{p+q}(\mathcal{B}^\bullet_{p,q}(\mathbb{R}(p)) \end{aligned} \tag{3.3.2}$$

The holomorphic-antiholomorphic symmetry is broken here: we insist that the Tate twists match with the holomorphic index of Bott-Chern cohomology. We could alternatively choose to match the anti-holomorphic index, leading to a parallel theory of Tate-twisted integral and real anti-holomorphic Bott-Chern cohomologies. (Note that $\overline{R(q)} = R(q)$, but the distinction is useful for notational purposes.) These would be the definitions with the antiholomorphic matching.

$$\begin{aligned} H^{p,q}_{\mathrm{BC}}(X, \overline{\mathbb{Z}(q)}) &= \mathbb{H}^{p+q}(\mathcal{B}^\bullet_{p,q}(\mathbb{Z}(q)) \\ H^{p,q}_{\mathrm{BC}}(X, \overline{\mathbb{R}(q)}) &= \mathbb{H}^{p+q}(\mathcal{B}^\bullet_{p,q}(\mathbb{R}(q)) \end{aligned} \tag{3.3.3}$$

An important property of the Tate twisted Bott-Chern cohomologies is that they relate to well-known constructions, particularly Analytic Deligne cohomology, which is defined in [EV86] and [Jan88]. We repeat the definition here.

Definition 3.3.4: The *Deligne complex*, with coefficients in a subring $R \subset \mathbb{R}$, is defined as the following complex of sheaves.

$$R(p)^\bullet_{\mathcal{D}} := 0 \to R(p) \to \mathcal{O} \to \Omega^1 \to \ldots \to \Omega^{p-1} \to 0. \tag{3.3.4}$$

The $R(p)$ term in degree 0.

Definition 3.3.5: The *Deligne cohomology* is the hypercohomology of the Deligne complex.

$$H^k_{\mathcal{D}}(X, R(p)) := \mathbb{H}^k(R(p)^\bullet_{\mathcal{D}}) \tag{3.3.5}$$

Using definitions from [Sch07], there is a map of complexes of sheaves $\epsilon_{\mathcal{D}}$: $\mathcal{B}^\bullet_{p,q}(R(p)) \to R(p)^\bullet_{\mathcal{D}}$ defined as follows.

$$\begin{array}{ccccccccc}
0 & \to & R(p) & \to & \mathcal{O} \oplus \overline{\mathcal{O}} & \to & \Omega^1 \oplus \overline{\Omega}^1 & \to & \ldots \\
\downarrow & & \downarrow \mathrm{Id} & & \downarrow \pi_1 & & \downarrow \pi_1 & & \ldots \\
0 & \to & R(p) & \to & \mathcal{O} & \to & \Omega^1 & \to & \ldots
\end{array}$$

(The notation π_1 stands for projection onto the first component in the direct sum, not projection on degrees or bidegrees of forms.)

Definition 3.3.6: As a map of complexes of sheaves, $\epsilon_{\mathcal{D}}$ induces a map on hypercohomolgoy, which is expressed with the same notation.

$$\epsilon_{\mathcal{D}} : H^{p,q}_{\mathrm{BC}}(X, R(p)) \to H^{p+q}_{\mathcal{D}}(X, R(p)) \tag{3.3.6}$$

One reason we prefer the holomorphic matching defining integral and real Bott-Chern cohomology is that it mirrors the definition of Deligne cohomology. However, there is also a similar, anti-holomorphic version of Deligne cohomology which cooperates with our anti-holomorphic version of Bott-Chern.

Definition 3.3.7: The *anti-holomorphic Deligne complex* is the following complex.

$$\overline{R(q)}^{\bullet}_{\mathcal{D}} := 0 \to R(q) \to \overline{\mathcal{O}} \to \overline{\Omega}^{1} \to \ldots \to \overline{\Omega}^{q-1} \to 0 \tag{3.3.7}$$

The anti-holomorphic version of Deligne cohomology is the hypercohomology of the anti-holomorphic Deligne complex.

$$H^k_{\mathcal{D}}(X, \overline{R(q)}) := \mathbb{H}^k(\overline{R(q)}^{\bullet}_{\mathcal{D}}) \tag{3.3.8}$$

Definition 3.3.8: There is a similar map $\overline{\epsilon_{\mathcal{D}}} : \mathcal{B}^{\bullet}_{p,q}(R(q)) \to \overline{R(q)}^{\bullet}_{\mathcal{D}}$, defined by the following diagram.

$$\begin{array}{ccccccccc} 0 & \to & R(q) & \to & \mathcal{O} \oplus \overline{\mathcal{O}} & \to & \Omega^1 \oplus \overline{\Omega}^1 & \to & \ldots \\ \downarrow & & \downarrow \mathrm{Id} & & \downarrow \pi_2 & & \downarrow \pi_2 & & \\ 0 & \to & R(q) & \to & \overline{\mathcal{O}} & \to & \overline{\Omega}^1 & \to & \ldots \end{array}$$

This similarly induces a map in hypercohomology.

$$\overline{\epsilon_{\mathcal{D}}} : H^{p,q}_{\mathrm{BC}}(X, \overline{R(q)}) \to H^{p+q}_{\mathcal{D}}(X, \overline{R(q)}) \tag{3.3.9}$$

In the remainder of this thesis, we will ignore this parallel anti-holomorphic construction in favour of the holomorphic version, both for Bott-Chern and Deligne cohomologies.

3.3.2 Cases with Degree Zero

In the case where $q = 0$, the definition of Bott-Chern cohomology with coefficients $R \subset \mathbb{R}$ is similar to the previous case. The $\mathcal{B}^{\bullet}_{p,0}$ complex has the constant sheaf $\mathbb{C}$ in degree 0. We replace $\mathbb{C}$ with $R(p)$ to define $\mathcal{B}^{\bullet}_{p,0}(R(p))$. The Bott-Chern cohomology with R coefficients is calculated as the hypercohomology of this new complex. We similarly construct the maps $\epsilon_{\mathcal{D}}$ and $\overline{\epsilon_{\mathcal{D}}}$. In this case, by looking at the definition of $\mathcal{B}_{p,0}(R(p))$, we see that this sequence is exactly the analytic Deligne complex, and $\epsilon_{\mathcal{D}}$ is the identity map.

The case where $p = 0$ is in parallel to the $q = 0$ case and the induced map $\overline{\epsilon_{\mathcal{D}}}$ is the identity map between anti-holomorphic versions of cohomology.

In the case where both $p = q = 0$, $H^{0,0}_{BC}(X, \mathbb{C}) = H^0(X, \mathbb{C})$ and altering the coefficients simply calculates $H^0(X, \mathbb{R})$. The $\mathcal{B}^{\bullet}_{0,0}(\mathbb{R})$ conmplex is simply $\mathbb{R}$ in degree 0 and trivial everywhere else.

3.4 Bott-Chern Cohomology by Currents

Recall from our global assumptions in Section 2.4 that X is a *compact* complex manifold. Compactness and finite volume are important for the following work with currents and allow us to work with global currents without worrying about compact support.

We work in the same two cases: strictly positive and allowing degree zero. These definitions are from [Dem07].

3.4.1 Strictly Positive Degrees

If $p > 0$ and $q > 0$, the definition of $\mathcal{M}^{\bullet}_{p,q}$ is parallel to $\mathcal{L}^{\bullet}_{p,q}$, but in terms of currents instead of forms:

$$\begin{aligned}
\mathcal{M}^k_{p,q} &= \bigoplus_{\substack{r+s=k \\ r<p,\ s<q}} \mathcal{D}^{r,s} \text{ if } k \leq p+q-2 \\
\mathcal{M}^k_{p,q} &= \bigoplus_{\substack{r+s=k+1 \\ r\geq p\ \ s\geq q}} \mathcal{D}^{r,s} \text{ if } k \geq p+q-1 \\
d^k_{p,q} &= \underline{d} \text{ if } k < p+q-2 \\
d^{p+q-2}_{p,q} &= \partial\overline{\partial} \\
d^k_{p,q} &= d \text{ if } k > p+q-2
\end{aligned} \tag{3.4.1}$$

Using the map which sends forms to their associated currents, there is an injective map of complexes $\mathcal{L}^{\bullet}_{p,q} \hookrightarrow \mathcal{M}^{\bullet}_{p,q}$. Therefore, there are also injective maps from the subcomplexes of $\mathcal{L}^{\bullet}_{p,q}$, in particular $\mathcal{S}^{\bullet}_{p,q}$, into $\mathcal{M}^{\bullet}_{p,q}$. We want to argue that the map $\mathcal{S}^{\bullet}_{p,q} \hookrightarrow \mathcal{M}^{\bullet}_{p,q}$ is quasi-isomorphism. We can do this directly as in Section 3.2 by calculating the cohomology of the sequence of sheaves of currents. However, following [Dem07], a spectral sequence argument is open to us. The following proposition and proof are sketched in that reference and provided here in full detail.

After the spectral sequence argument shows a quasi-isomorphism between $\mathcal{M}^{\bullet}_{p,q}$ and $\mathcal{S}^{\bullet}_{p,q}$, the compatibility of the maps and the transitivity of quasi-isomorphism will show that $\mathcal{M}^{\bullet}_{p,q}$ is quasi isomorphic to $\mathcal{L}^{\bullet}_{p,q}$ and $\mathcal{B}^{\bullet}_{p,q}[1]$ as well.

Proposition 3.4.1: *The injection $\mathcal{S}^\bullet_{p,q} \to \mathcal{M}^\bullet_{p,q}$ is a quasi-isomorphism.*

Proof. We work with the same cases as Proposition 3.2.3, except that we don't need to separate $p = q$ and $p \neq q$.

Case $p \geq 2$ and $q \geq 2$:

We have the standard filtration on forms and currents by bidgree as follows.

$$F^s\mathcal{A}^r = \bigoplus_{\substack{i \geq s \\ i+j=r}} \mathcal{A}^{i,j} \tag{3.4.2}$$
$$F^s\mathcal{D}^r = \bigoplus_{\substack{i \geq s \\ i+j=r}} \mathcal{D}^{i,j}.$$

We apply this filtration to $\mathcal{M}^\bullet_{p,q}$.

$$F^r\mathcal{M}^k_{p,q} = \mathcal{M}^k_{p,q} \cap F^r\mathcal{D}^\bullet \tag{3.4.3}$$

Since $\mathcal{S}^\bullet_{p,q}$ is a subcomplex, it gets filtered similarly (though that filtration is nearly trivial).

A filtered complex naturally gives rise to an associated spectral sequence, written $E_t^{r,s}$. The 0th sheet is

$$E_0^{r,s} = \mathrm{Gr}_r^F \mathcal{M}^{r+s}_{p,q} = \frac{F^r\mathcal{M}^{r+s}_{p,q}}{F^{r+1}\mathcal{M}^{r+s}_{p,q}} \tag{3.4.4}$$

Explicitly doing the calculation gives the following quotients.

$$E_0^{r,s} = \mathcal{D}^{r,s} \text{ if } r \leq p-1 \text{ and } s \leq q-1 \tag{3.4.5}$$
$$E_0^{r,s} = 0 \text{ if } r \leq p-1 \text{ and } s > q-1$$
$$E_0^{r,s} = 0 \text{ if } r > p-1 \text{ and } s < q-1$$
$$E_0^{r,s} = \mathcal{D}^{r,s+1} \text{ if } r \geq p-1 \text{ and } s \geq q-1$$

The 0th sheet has differentials $E_0^{r,s} \to E_0^{r,s+1}$, which are $\overline{\partial}$ or zero maps in all cases. The columns of the 0th sheet, where they are non-zero, come in two cases. If $r < p$, the following sequence is the upward proceeding column.

$$0 \to \mathcal{D}^{r,0} \xrightarrow{\overline{\partial}} \mathcal{D}^{r,1} \xrightarrow{\overline{\partial}} \ldots \xrightarrow{\overline{\partial}} \mathcal{D}^{r,q-1} \to 0 \tag{3.4.6}$$

If $r \geq p$, instead we have this sequence as a column.

$$0 \to \mathcal{D}^{r,q} \xrightarrow{\overline{\partial}} \mathcal{D}^{r,q+1} \xrightarrow{\overline{\partial}} \ldots \xrightarrow{\overline{\partial}} \mathcal{D}^{r,n} \to 0 \tag{3.4.7}$$

Since the first sheet is the cohomology of the 0th sheet, it is calculated by the cohomology of these truncated pieces of the $\overline{\partial}$-complex of currents. By the Poincaré Corollary (Corollary 2.2.23), the internal pieces of the $\overline{\partial}$-complex are exact; only the truncations give rise to non-zero cohomology sheaves. We can explicitly calculate these non-zero terms.

$$
\begin{aligned}
E_1^{r,0} &= \Omega_{\mathcal{D}}^r := \text{ if } 0 \leq r < p \\
E_1^{r,q-1} &= \frac{\mathcal{D}^{r,q-1}}{\overline{\partial}\mathcal{D}^{r,q-2}} \text{ if } 0 \leq r < p \\
E_1^{r,q-1} &= \operatorname{Ker}\{\overline{\partial} : \mathcal{D}^{r,q} \to \mathcal{D}^{r,q+1}\} \text{ if } p \leq r \leq n \\
E_1^{r,s} &= 0 \text{ otherwise}
\end{aligned} \tag{3.4.8}
$$

Recall that the notation $\Omega_{\mathcal{D}}^r$ stands for holomorphic currents of bidegree $(0, r)$. By regularity, the sheaf $\Omega_{\mathcal{D}}^r$ can be identified with Ω^r. However, for the current exposition, it is convenient to preserve the distinction.

For $E_1^{r,q-1}$, the Poincaré Corollary (Corollary 2.2.23) gives the following calculations.

$$
\overline{\partial}\mathcal{D}^{r,q-1} \cong \operatorname{Ker}\{\overline{\partial} : \mathcal{D}^{r,q} \to \mathcal{D}^{r,q+1}\} \tag{3.4.9}
$$

We write $\mathcal{Z}_{\mathcal{D}}^{r,q}$ for $\operatorname{Ker}\{\overline{\partial} : \mathcal{D}^{r,q} \to \mathcal{D}^{r,q+1}\}$. With these identifications and definitions, the first sheet can give a concise description of the first sheet of the spectral sequence.

$$
\begin{aligned}
E_1^{r,0} &= \Omega_{\mathcal{D}}^r \text{ if } 0 \leq r < p \\
E_1^{r,q-1} &\cong \mathcal{Z}_{\mathcal{D}}^{r,q} \text{ if } 0 \leq r < p \\
E_1^{r,q-1} &= \mathcal{Z}_{\mathcal{D}}^{r,q} \text{ if } p \leq r \leq n \\
E_1^{r,s} &= 0 \text{ otherwise.}
\end{aligned} \tag{3.4.10}
$$

The first sheet has support only when $s = 0$ or $s = q - 1$. The differentials are $\partial : E_1^{r,s} \to E_1^{r+1,s}$, so there are only two non-zero sequences in the first sheet.

$$\begin{aligned} E_1^{\bullet,0} &= 0 \to \Omega_{\mathcal{D}}^0 \xrightarrow{\partial} \Omega_{\mathcal{D}}^1 \xrightarrow{\partial} \ldots \xrightarrow{\partial} \Omega_{\mathcal{D}}^{p-1} \to 0 \\ E_1^{\bullet,q-1} &= 0 \to \mathcal{Z}_{\mathcal{D}}^{0,q} \xrightarrow{\partial} \mathcal{Z}_{\mathcal{D}}^{1,q} \xrightarrow{\partial} \ldots \xrightarrow{\partial} \mathcal{Z}_{\mathcal{D}}^{n,q} \to 0 \end{aligned} \tag{3.4.11}$$

These sequences are exact at all intermediate terms, so only the initial or final terms can contribute to cohomology. Moreover, the end of the second sequence is the exact as well. Therefore, there are only three non-zero terms in the second sheet.

$$\begin{aligned} E_2^{0,0} &= \mathbb{C} \\ E_2^{p-1,0} &= \frac{\Omega_{\mathcal{D}}^{p-1}}{\partial \Omega_{\mathcal{D}}^{p-2}} \\ E_2^{0,q-1} &= \operatorname{Ker}\{\partial : \mathcal{Z}_{\mathcal{D}}^{0,q-1} \to \mathcal{Z}_{\mathcal{D}}^{1,q-1}\} \end{aligned} \tag{3.4.12}$$

The calculations of first term is clear, since the only holomorphic currents which are ∂-closed are constant. The second term is isomorphic to $\partial \Omega_{\mathcal{D}}^{p-1}$ by the Poincaré Corollary (Corollary 2.2.23). The third term is $\operatorname{Ker}(\partial : \mathcal{Z}_{\mathcal{D}}^{0,q} \to \mathcal{Z}_{\mathcal{D}}^{1,q})$. By the anticommutative nature of ∂ and $\overline{\partial}$, this can be realized as $\operatorname{Ker}(\overline{\partial} : \overline{\Omega}_{\mathcal{D}}^{q} \to \overline{\Omega}_{\mathcal{D}}^{q+1})$, which is isomorphic to $\partial \overline{\Omega}_{\mathcal{D}}^{p-1}$. Therefore, the non-zero terms of the second sheet can be simplified.

$$
\begin{aligned}
E_2^{0,0} &= \mathbb{C} \\
E_2^{p-1,0} &= \partial\Omega_{\mathcal{D}}^{p-1} \\
E_2^{0,q-1} &= \overline{\partial\Omega}_{\mathcal{D}}^{q-1}
\end{aligned} \tag{3.4.13}
$$

The spectral sequence stabilizes here, so these sheaves calculate a graded piece of the cohomology of the complex. However, this already matches up with the cohomology groups of $\mathcal{L}^{\bullet}_{p,q}$, indicating that not only do we have the right cohomology, but the grading on cohomology is trivial.

For the sake of completeness and parallel structure, we do the same spectral sequence construction for $\mathcal{S}^{\bullet}_{p,q}$, even though we already know the cohomology. It is useful to see how the spectral sequence for $\mathcal{S}^{\bullet}_{p,q}$ constructs the same E_2 terms. The 0th sheet of that spectral sequence has the following terms.

$$
E_0^{r,s} = \mathrm{Gr}_r^F \mathcal{S}^{r+s}_{p,q} = \frac{F^r \mathcal{S}^{r+s}_{p,q}}{F^{r+1}\mathcal{S}^{r+s}_{p,q}} \tag{3.4.14}
$$

Because the complex $\mathcal{S}^{\bullet}_{p,q}$ is only supported in holomorphic and anti-holmorphic forms, this 0th sheet has non-zero terms only when $r = 0$ or $s = 0$.

$$
\begin{aligned}
E_0^{0,s} &= \overline{\Omega}^s \text{ if } 0 \leq s < q \\
E_0^{r,0} &= \Omega^r \text{ if } 0 < r < p
\end{aligned} \tag{3.4.15}
$$

For $r > 0$, the terms $E^{r,0}$ are isolated in the 0th sheet, so they descend to the

1st sheet of cohomology. The other terms form a sequence in the 0th sheet.

$$E_0^{0,\bullet} = 0 \to \mathcal{O} + \overline{\mathcal{O}} \xrightarrow{\bar\partial} \overline{\Omega}^1 \xrightarrow{\bar\partial} \dots \xrightarrow{\bar\partial} \overline{\Omega}^{q-1} \to 0 \tag{3.4.16}$$

Preserving the isolated terms and calculating the cohomology of this sequence gives a 1st sheet with the following non-zero terms.

$$\begin{aligned} E_1^{r,0} &= \Omega^r \text{ if } 0 < r < p \\ E_1^{0,0} &= \mathcal{O} \\ E_1^{0,q-1} &= \frac{\overline{\Omega}^{q-1}}{\overline{\partial \Omega}^{q-2}} \end{aligned} \tag{3.4.17}$$

The term $E_1^{0,q-1}$ is isolated, and the remaining terms form a sequence.

$$E_1^{\bullet,0} = 0 \to \mathcal{O} \xrightarrow{\partial} \Omega^1 \xrightarrow{\partial} \dots \xrightarrow{\partial} \Omega^{p-1} \to 0 \tag{3.4.18}$$

Taking cohomology of this sequence and remembering the isolated term, the second sheet has only three non-zero entries.

$$\begin{aligned} E_2^{0,0} &= \mathbb{C} \\ E_2^{p-1,0} &= \frac{\Omega^{p-1}}{\partial \Omega^{p-2}} \\ E_2^{0,q-1} &= \frac{\overline{\Omega}^{q-1}}{\overline{\partial \Omega}^{q-2}} \end{aligned} \tag{3.4.19}$$

We simplify these terms by applying Corollary 2.2.23. This gives the following

2nd sheet.

$$
\begin{aligned}
E_2^{0,0} &= \mathbb{C} \\
E_2^{p-1,0} &\cong \partial\Omega^{p-1} \\
E_2^{0,q-1} &\cong \overline{\partial\Omega}^{q-1}
\end{aligned}
\tag{3.4.20}
$$

Now the two second sheets have the same form, the only difference is that one is calculated in terms of currents and one in terms of forms. So the conclusion rests on $\partial\Omega^{p-1} \cong \partial\Omega_{\mathcal{D}}^{p-1}$, which is regularity of holomorphic currents as stated in Proposition 2.2.25.

The matching of the E^2 terms ensures that the two complexes calculate the same cohomology, completing the required quasi-isomorphism. The map realizing this quasi-isomorphisms is either the identity after identifying forms and currents, or the differential ∂.

Case $p = 1$ *and* $q \geq 2$

Now assume that $p = 1$, while $q \geq 2$. We can follow the same general proof as before, making adjustments as we go along.

The calculations of the 0th sheet are the same as in Equation 3.4.8. The cohomology calculation in Equation 3.4.8 is also unaltered in this case, which gives the 1st Sheet as in Equation 3.4.10. Note that the case $0 \leq r < p$ is simply one case: $r = 0$, since $p = 1$.

There are two sequences that make up the first sheet.

$$E_1^{\bullet,0} = 0 \to \Omega^0_{\mathcal{D}} \to 0 \tag{3.4.21}$$
$$E_1^{\bullet,q-1} = 0 \to \mathcal{Z}^{0,q}_{\mathcal{D}} \xrightarrow{\partial} \mathcal{Z}^{1,q}_{\mathcal{D}} \xrightarrow{\partial} \ldots \xrightarrow{\partial} \mathcal{Z}^{n,q}_{\mathcal{D}} \to 0$$

Using the same arguments and identifications as before, which led to Equation 3.4.13, we calculate the cohomology of the first sheet to get the second sheet.

$$E_2^{0,0} = \Omega^0_{\mathcal{D}} \tag{3.4.22}$$
$$E_2^{0,q-1} = \overline{\partial \Omega}^{q-1}_{\mathcal{D}}$$

For the spectral sequence associated with the $\mathcal{S}^{\bullet}_{p,q}$ complex in this case, a very similar adjustment takes place. In the first sheet in the general case, we had the sequence in Equation 3.4.18. Since $p = 1$ in this case, this sequence only consists of the isolated term $\Omega^0_{\mathcal{D}}$. Therefore, the first sheet has only these two isolated terms, and the second sheet is as follows.

$$E_2^{0,0} = \Omega^0_{\mathcal{D}} \tag{3.4.23}$$
$$E_2^{0,q-1} = \overline{\partial \Omega}^{q-1}_{\mathcal{D}}$$

This agrees with the $\mathcal{M}^{\bullet}_{p,q}$ spectral sequence.

Case $p > 1$ *and* $q = 1$

The calculation in Equation 3.4.5 still holds in this case. However, the fact that

$q = 1$ means that for $r < p$ the terms $E_0^{r,0} = \mathcal{D}^{r,0}$ are isolated terms instead of being the first terms of a sequence. Taking cohomology, and treating $r \geq p$ as in the general case gives the following first sheet.

$$\begin{aligned} E_1^{r,0} &= \mathcal{D}^{r,0} \text{ if } 0 \leq r < p \\ E_1^{r,0} &= \mathcal{Z}_{\mathcal{D}}^{r,1} \text{ if } p \leq r \leq n \\ E_1^{r,s} &= 0 \text{ otherwise.} \end{aligned} \tag{3.4.24}$$

We cannot make the identifications that worked in the general case. However, we have only one non-zero sequence in this first sheet.

$$E_1^{\bullet,0} = 0 \to \mathcal{D}^{0,0} \xrightarrow{\partial} \dots \xrightarrow{\partial} \mathcal{D}^{p-1,0} \xrightarrow{\partial\overline{\partial}} \mathcal{D}^{p,1} \xrightarrow{\partial} \dots \xrightarrow{\partial} \mathcal{D}^{n,1} \to 0 \tag{3.4.25}$$

Using previous calculations and the $\partial\overline{\partial}$-lemma (Proposition 2.2.24), the cohomology of this sequence is only supported in degrees 0 and $p-1$. Calculating those cohomologies gives us the second sheet.

$$\begin{aligned} E_2^{0,0} &= \overline{\Omega_{\mathcal{D}}^0} \\ E_2^{p-1,0} &= \frac{\text{Ker}\partial\overline{\partial} : \mathcal{D}^{p-1,0} \to \mathcal{D}^{p,1}}{\partial\mathcal{D}^{p-2,0}} \end{aligned} \tag{3.4.26}$$

By regularity (Proposition 2.2.25 and Corollary 2.2.23), we can simplify this.

$$E_2^{0,0} = \overline{\mathcal{O}} \tag{3.4.27}$$
$$E_2^{p-1,0} = \frac{\Omega^{p-1}}{\partial\Omega^{p-2}}$$

For the $\mathcal{S}^\bullet_{p,q}$ complex in this case, all terms in the first sheet are isolated, so they descend directly to the second sheet. There we have only one non-zero sequence.

$$E_1^{\bullet,0} = 0 \to \mathcal{O} + \overline{\mathcal{O}} \to \Omega^1 \to \ldots \to \Omega^{p-1} \to 0 \tag{3.4.28}$$

The cohomology of this sequences gives the second sheet.

$$E_2^{0,0} = \overline{\mathcal{O}} \tag{3.4.29}$$
$$E_2^{p-1,0} = \frac{\Omega^{p-1}}{\partial\Omega^{p-2}}$$

This matches up, and finishes this case.

Case $p = 1$ and $q = 1$

This case proceeds similarly to the previous case. For $\mathcal{M}^\bullet_{p,q}$, the 1st sheet of the spectral sequence is only one sequence.

$$E_1^{\bullet,0} = 0 \to \mathcal{D}^{0,0} \xrightarrow{\partial\overline{\partial}} \mathcal{D}^{1,1} \xrightarrow{\partial} \ldots \xrightarrow{\partial} \mathcal{D}^{n,1} \to 0 \tag{3.4.30}$$

The only cohomology here is in degree 0, so we have the second sheet only supported in degree $(0,0)$.

$$E_2^{0,0} = \text{Ker}\partial\overline{\partial} : \mathcal{D}^{0,0} \to \mathcal{D}^{1,1} \tag{3.4.31}$$

For $\mathcal{S}^\bullet_{p,q}$, the spectral sequence only has one term, which is $\mathcal{O} + \overline{\mathcal{O}}$ at degree $(0,0)$. Regularity identifies this sheaf with the kernel in the above equation, which finishes the case. □

3.4.2 Cases with Degree Zero

For cases where $p = 0$, $q = 0$ or both, we use the alternate definitions of $\mathcal{L}^\bullet_{p,q}$ and $\mathcal{B}^\bullet_{p,q}$ as in Definitions 3.2.8 and 3.2.10. This is done for $p = 0$ and $q > 0$, but the other case is precisely in parallel. This inspires a complex of currents:

$$\begin{aligned} \mathcal{M}^k_{p,0} &= 0 \text{ if } k \leq p-2 \\ \mathcal{M}^k_{p,0} &= \bigoplus_{\substack{r+s=k+1 \\ r \geq p}} \mathcal{D}^{r,s} \text{ if } k \geq p-1 \end{aligned} \tag{3.4.32}$$

There is no need for a spectral sequence here to prove that this is still quasi-isomorphic to the $\mathcal{L}^\bullet_{p,q}$ and $\mathcal{B}^\bullet_{p,q}$ complexes. The only sheaf cohomology is supported in degree $p-1$, where it is the currents in $\mathcal{D}^{p,0}$ which are d-closed. By regularity, this is $\hat{\Omega}^p$, which is the cohomology of $\mathcal{B}^\bullet_{p,q}$ in degree p.

The case for $q = 0$ is, as noted before, simply in parallel. If both $p = q = 0$, then we recover the 0th De Rham group when calculating Bott-Chern, which we know can be calculated by currents as well as forms. The $\mathcal{B}^\bullet_{0,0}$ complex is just $\mathbb{C}$ in degree zero, and the complex of currents $\mathcal{M}^{\bullet}_{0,0}[-1] = \mathcal{D}^{\bullet}$ is a resolution of $\mathbb{C}$.

3.5 Aeppli Hypercohomology

The previous three sections on hypercohomology, altered coefficients and currents dealt exclusively with the Bott-Chern cohomology, but similar results hold for the Aeppli groups. This is indicated in [Sch07], but we give more explicit details here. With Aeppli cohomology, unlike Bott-Chern, we do not need to separate cases where $p, q > 0$ and where one or both may be 0.

Recall the definition of $\mathcal{L}^{\bullet}_{p,q}$ in Definition 3.2.2. By comparing with the definition of Aeppli cohomology in 3.1.2, and using the fact that $\mathcal{L}^{\bullet}_{p,q}$ is a complex of fine sheaves which calculate hypercohomology by taking the cohomology of the complex of global sections, we see that we can use this complex to calculate Aeppli cohomology.

$$H^{p,q}_{\mathrm{Ap}}(X, \mathbb{C}) = \mathbb{H}^{p+q}(\mathcal{L}^{\bullet}_{p+1,q+1}) \tag{3.5.1}$$

Unlike Bott-Chern cohomology, this still holds for cases where $p = 0$ or $q = 0$. This is due to the shift in degree in the complex: Aeppli in degree (p, q) is defined by the $\mathcal{L}^{\bullet}_{p+1,q+1}$ complex, not the $\mathcal{L}^{\bullet}_{p,q}$ complex.

It is convenient that we can make use of the same complex, since that gives us access to all of the quasi-isomorphisms from previous sections. With those in place, we have the following isomorphisms.

$$H^{p,q}_{\mathrm{Ap}}(X,\mathbb{C}) \cong \mathbb{H}^{p+q}(\mathcal{M}^\bullet_{p+1,q+1}) \cong \mathbb{H}^{p+q}(\mathcal{S}^\bullet_{p+1,q+1}) \cong \mathbb{H}^{p+q+1}(\mathcal{B}^\bullet_{p+1,q+1}) \quad (3.5.2)$$

3.6 Aeppli with Finer Coefficients

We define the Aeppli cohomology with coefficients in $R(p)$, (for R a subring of $\mathbb{R}$), similarly to Bott-Chern, by replacing $\mathbb{C}$ with $R(k)$ in the complex $\mathcal{B}^\bullet_{p,q}$. The notation is the same as before: $\mathcal{B}^\bullet_{p,q}(R(k))$, and the definition is in parallel with Definition 3.3.3.

$$H^{p,q}_{\mathrm{Ap}}(X,R(k)) := \mathbb{H}^{p+q+1}(\mathcal{B}^\bullet_{p+1,q+1}(R(k))) \quad (3.6.1)$$

As with Bott-Chern cohomology, we have a choice to match this to holomorphic or anti-holomorphic indices. The conventional choice is to match with holomorphic indices in order to easily match with Deligne cohomology. However, for Aeppli cohomology, we will need to shift by one in maps to Deligne cohomology. Therefore, our standard indexing is as follows.

Definition 3.6.1:

$$H^{p,q}_{\mathrm{Ap}}(X,\mathbb{R}(p+1)) := \mathbb{H}^{p+q+1}(\mathcal{B}^\bullet_{p+1,q+1}(\mathbb{R}(p+1))) \quad (3.6.2)$$

Now consider the projection map to the Deligne complex, parallel to Definition 3.3.6. Aeppli cohomology in degree (p,q) is related to the Deligne complex in degree $p+1$, and projection onto the first component gives a map of complexes $\mathcal{B}^\bullet_{p+1,q+1}(\mathbb{R}(p+1)) \to \mathbb{R}(p+1)_{\mathcal{D}}$. (For this diagram, assume $q \geq p$. The parallel case is similar.)

$$\begin{array}{ccccccccccccc}
0 & \to & \mathbb{R}(p+1) & \to & \mathcal{O} \oplus \overline{\mathcal{O}} & \to & \dots & \to & \Omega^p \oplus \overline{\Omega}^p & \to & \overline{\Omega}^{p+1} & \to & \dots \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
0 & \to & \mathbb{R}(p+1) & \to & \mathcal{O} & \to & \dots & \to & \Omega^p & \to & 0 & \to & \dots
\end{array}$$

Then the map on hypercohomology of these complexes defines the following map. We use the same notation for this map as for the equivalent map on Bott-Chern cohomology.

$$\epsilon_{\mathcal{D}} : H^{p,q}_{\mathrm{Ap}}(X, \mathbb{R}(p+1)) \to H^{p+q+1}_{\mathcal{D}}(X, \mathbb{R}(p+1)) \tag{3.6.3}$$

If we wished to adapt this to the anti-holomorphic version of analytic Deligne cohomology, we could follow the parallel construction to build a similar map

$$\overline{\epsilon_{\mathcal{D}}} : H^{p,q}_{\mathrm{Ap}}(X, \mathbb{R}(q+1)) \to H^{p+q+1}_{\mathcal{D}}(X, \overline{\mathbb{R}(p+1)}) \tag{3.6.4}$$

3.7 Identifying Real Bott-Chern Classes

Assume that ω is current defining a class in Bott-Chern or Aeppli cohomology, according to the original description in Definition 3.1.1. We would like to determine, from the properties of that current, whether it defines a class in Bott-Chern or Aeppli cohomology with $\mathbb{R}(r)$ coefficients. The following construction answers that question for Aeppli cohomology, but the case for Bott-Chern is similar.

Consider a class in Aeppli cohomology $H^{p,q}_{\mathrm{Ap}}(X, \mathbb{C})$ identified by a current $\zeta \in \mathcal{D}^{p,q}$, according to the description that $H^{p,q}_{\mathrm{Ap}}(X, \mathbb{C}) = \mathbb{H}^{p+q}(\mathcal{M}^{\bullet}_{p+1,q+1})$. (Alternatively, we could work with a form, coming from the description *via* the $\mathcal{L}^{\bullet}_{p+1,q+1}$ complex – the following construction would hold just the same, replacing currents with forms.)

Let π_p be the projection onto the first component in $\mathbb{C} = \mathbb{R}(p) \oplus \mathbb{R}(p+1)$. With a common abuse of notation, we also use π_p to be the induced map on sheaves of forms and currents, instead of $\pi_{p,*}$ or some other similar notation.

If $(p, q) = (0, 0)$, then Aeppli cohomology is equivalent to $H^0(X, \mathbb{C}) \cong \mathbb{C}$. The classes with $\mathbb{R}(1)$ coefficients are cleary those which vanish under π_0, the projection to $\mathbb{R}$. If $(p, q) \neq (0, 0)$, then we have the following proposition.

Proposition 3.7.1: *The class $[\zeta]$ defines an element of the cohomology quotient $H^{p,q}_{\mathrm{Ap}}(X, \mathbb{R}(p+1))/\delta(H^{p+q}(X, \mathbb{R}(p)))$ if the current ζ satifies $\pi_p \partial \zeta = 0$. In particular, it is satisfied if either ζ is a $\mathbb{R}(p+1)$-valued current or if ζ is a $\overline{\partial}$-closed current. The map $\delta : H^{p+q}(X, \mathbb{R}(p)) \to H^{p,q}_{\mathrm{Ap}}(X, \mathbb{R}(p+1))$ will be defined in the proof.*

Proof. We move back to the description by the complexes $\mathcal{B}^{\bullet}_{p+1,q+1}$. Without loss of generality, assume that $p < q$. (The other cases are very minor adjustments of this proof). Then we have a short exact sequence of complexes $\mathcal{B}^{\bullet}_{p+1,q+1}(\mathbb{R}(p+1)) \to \mathcal{B}^{\bullet}_{p+1,q+1} \to \mathbb{R}(p)$.

$$
\begin{array}{cccccccccccc}
0 & \to & \mathbb{R}(p+1) & \to & \mathcal{O}\oplus\overline{\mathcal{O}} & \to & \dots & \to & \Omega^p\oplus\overline{\Omega}^p & \to & \overline{\Omega}^{p+1} & \dots \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & \\
0 & \to & \mathbb{C} & \to & \mathcal{O}\oplus\overline{\mathcal{O}} & \to & \dots & \to & \Omega^p\oplus\overline{\Omega}^p & \to & \overline{\Omega}^{p+1} & \dots \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & \\
0 & \to & \mathbb{R}(p) & \to & 0 & \to & \dots & \to & 0 & \to & 0 & \dots
\end{array}
$$

This gives a long exact sequence in hypercohomology. In particular, we have the following exact sequence, where δ is the connecting morphism. (Recall that Aeppli cohomology is the $(p+q+1)$th hypercohomology of $\mathcal{B}^\bullet_{p+1,q+1}$.)

$$
H^{p+q}(X,\mathbb{R}(p)) \xrightarrow{\delta} H^{p,q}_{\mathrm{Ap}}(X,\mathbb{R}(p+1)) \to H^{p,q}_{\mathrm{Ap}}(X,\mathbb{C}) \xrightarrow{\phi} H^{p+q+1}(X,\mathbb{R}(p)) \tag{3.7.1}
$$

The first and last terms here are the hypercohomology of the constant sheaf $\mathbb{R}(p)$, which is isomorphic to ordinary (singular) cohomology with coefficients in $\mathbb{R}(p)$.

In this exact sequence, we can identify classes that come from Aeppli cohomology with $\mathbb{R}(p+1)$ coefficients by finding those that map to zero in $H^{p+q+1}(X,\mathbb{R}(p))$, *i.e.* $\phi[\zeta]=0$. However, we can only identify such classes up to their image in $H^{p,q}_{\mathrm{Ap}}(X,\mathbb{C})$. Since the sequence is exact, that image is isomorpic to $H^{p,q}_{\mathrm{Ap}}(X,\mathbb{R}(p+1))$ modulo $\delta(H^{p+q}(X,\mathbb{R}(p)))$. Then the proof reduces to the following lemma. □

Lemma 3.7.2: *This map ϕ is the same as the map $\pi_p\partial$.*

Proof. The map of complexes factors as follows.

$$\begin{array}{ccccccccccccc}
0 & \to & \mathbb{C} & \to & \mathcal{O} \oplus \overline{\mathcal{O}} & \to & \dots & \to & \Omega^{p-1} \oplus \overline{\Omega}^{p-1} & \to & \overline{\Omega}^{p} & \dots \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & \\
0 & \to & \mathbb{C} & \to & 0 & \to & \dots & \to & 0 & \to & 0 & \dots \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & \\
0 & \to & \mathbb{R}(p) & \to & 0 & \to & \dots & \to & 0 & \to & 0 & \dots
\end{array}$$

The second map in this factorization is just π_p. To understand how the first map influences hypercohomology, we can replace the complex $\mathbb{C}$ by the quasi-isomorphic complex $(\Omega^\bullet, \partial)$.

$$\begin{array}{ccccccccccccc}
0 & \to & \mathbb{C} & \to & \mathcal{O} \oplus \overline{\mathcal{O}} & \to & \dots & \to & \Omega^{p-1} \oplus \overline{\Omega}^{p-1} & \to & \overline{\Omega}^{p} & \dots \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & \\
0 & \to & \mathcal{O} & \to & \Omega^1 & \to & \dots & \to & \Omega^p & \to & \Omega^{p+1} & \dots
\end{array}$$

The vertical maps here are inclusion in degree 0 and ∂ in all higher degrees. It is an easy check of all the squares to see that these maps give a map of complexes. (Actually, all routes around squares compose to zero maps. Past degree $p-1$, since the domains are anti-holomorphic, the vertical maps are themselves zero maps.)

The complex of currents $\mathcal{M}^\bullet_{p+1,q+1}$ serves as a fine resolution of the complex $\mathcal{B}^\bullet_{p+1,q+1}$, by the quasi-isomorphism in constructed in Proposition 3.4.1 and the fine nature of sheaves of C^∞ currents. The map between complexes is the map of forms to their associated currents: $\eta \mapsto \delta_\eta = \int_X \eta \wedge \_$.

Using $\mathcal{M}_{p+1,q+1}[-1]$ as a fine resolution gives us the definition of Aeppli in terms of classes of $\partial\overline{\partial}$-closed currents. Using $\mathcal{D}^\bullet$ as a resolution of $\mathbb{C}$ also gives classes of currents. The induced map on hypercohomology, at least until degree $p+q+2$ where $\mathcal{M}_{p+q+1}[-1]$ has a differential $\partial\overline{\partial}$, must be ∂ to agree with the original complex. In degree $p+q+1$, which is the relevant degree to calculate Aeppli cohomology, this ∂ sends the $\partial\overline{\partial}$-closed current η to $\partial\eta$, which is a d-closed current, defining a class in $H^{p+q+1}(X, \mathbb{C})$. (Note that this recovers the map defined in Definition 3.1.3). Then we reach $H^{p+q+1}(X, \mathbb{R}(p))$ by appliying π_p, finishing the lemma. □

As noted at the start of the section, this proof can be adopted for Bott-Chern cohomology as well, giving the following similar proposition. Assume that ζ is a current or a form identifying a classs in $H^{p,q}_{\mathrm{BC}}(X, \mathbb{C})$.

Proposition 3.7.3: *The class* $[\zeta]$ *defines an element of the cohomology quotient* $H^{p,q}_{\mathrm{BC}}(X, \mathbb{R}(p))/\delta(H^{p+q-1}(X, \mathbb{R}(p-1)))$ *if the current* ζ *satifies* $\pi_{p-1}\zeta = 0$. *In particular, it is satisfied if* ζ *is a* $\mathbb{R}(p)$ *valued current.*

Proof. We only need to adjust slightly the previous proof. Since the current defining the Bott-Chern class is from degree $p+q$ in $\mathcal{M}^{\bullet}_{p,q}$, which is past the $\partial\overline{\partial}$-differential in degree $p+q-1$, the map on this level to $H^{p+q}(X, \mathbb{C})$ is (up to a sign) the identify map on the current instead of ∂ as in the Aeppli case. The adjustment here changing ∂ to the identity is necessary because the $\mathcal{M}^{\bullet}_{p,q}$ complex changes and we need the following square to commute, where the first vertical map is ∂.

$$\begin{array}{ccc} \mathcal{D}^{p,q} & \stackrel{\partial\overline{\partial}}{\rightarrow} & \mathcal{D}^{p+1,q+1} \\ \downarrow & & \downarrow \\ \mathcal{D}^{p+q+1} & \stackrel{d}{\rightarrow} & \mathcal{D}^{p+q+2} \end{array}$$

Up to a sign, the identity map in the second vertical map allows the square to commute. Then to get to $\mathbb{R}(p)$ coefficients is just the projection π_p, as in the Aeppli proof.

However, this argument only holds for degrees p, q larger than zero, since a different resolution $\mathcal{M}^\bullet_{p,q}$ is defined if either are zero. The adjustments for the degree zero cases are simple and we omit them here. However, as in the Aeppli case, the $(p, q) = (0, 0)$ cohomology is just $\mathbb{C}$ and the conclusion follows immediately. $\square$

3.8 Resolutions of the Bott-Chern complexes

3.8.1 Čech Resolution

The material in this section is mostly taken from [Voi02], particularly Theorem 4.4.1 and its proof. In that reference, Voisin is following the original constructions of [God58].

For Bott-Chern and Aeppli cohomologies with coefficients in $\mathbb{C}$, the $\mathcal{L}^\bullet_{p,q}$ complex is a complex of fine sheaves; therefore, hypercohomology is calculated by cohomology of the complex of global sections $\mathcal{L}^\bullet_{p,q}(X)$. The Bott-Chern and Aeppli cohomologies with finer coefficients are the hypercohomology of $\mathcal{B}^\bullet_{p,q}(R(k))$, which is not a complex of fine sheaves. To get explicit representatives for these cohomology classes, we need to build a resolution of $\mathcal{B}^\bullet_{p,q}(R(k))$ which consists of fine sheaves or other sheaves which are acyclic for the global section functor.

One such resolution is the Čech resolution of a complex of sheaves. This is the resolution pursued in [Sch07] for calculations and products. We review the construction here.

Let $\mathcal{F}^\bullet$ be a complex of sheaves of $\mathcal{O}_X$-modules (or abelian groups) on a reasonably well behaved topological space X with an appropriate structure sheaf $\mathcal{O}_X$. For an appropriately fine open cover $\mathcal{U}$, each sheaf $\mathcal{F}^r$ has a Čech resolution $\check{C}^\bullet(\mathcal{U}, \mathcal{F}^r)$, with a map $\mathcal{F}^r \to \check{C}^0(\mathcal{U}, \mathcal{F}^r)$. The Čech resolutions are functorial; that is, the maps $d_\mathcal{F}$ of the complex $\mathcal{F}^\bullet$ give rise to maps on the Čech groups. These maps commute with the Čech differential δ, specifically: $d_\mathcal{F}\delta = \delta d_\mathcal{F}$.

Definition 3.8.1: The *Čech resolution* of a complex of sheaves $\mathcal{F}^\bullet$ is the double complex formed out of the individual Čech resolutions.

$$\bigoplus_{j+k=\bullet} \check{C}^j(\mathcal{U}, \mathcal{F}^k) \tag{3.8.1}$$

The differential of this complex, acting on an element in $\check{C}^j(\mathcal{U}, \mathcal{F}^k)$, is $d_D := d_\mathcal{F} + (-1)^k\delta$. The alternating sum is necessary to ensure that $d_D^2 = 0$. (Note there is a choice of sign in this process. There is an alternate description where the differentials δ and $d_\mathcal{F}$anti-commute and the differential of the double complex is an ordinary sum.)

Since this is a compatible collection of resolutions, for a sufficiently fine open cover $\mathcal{U}$, the arguments in the proof of Theorem 4.4.1 in [Voi02] show that the cohomology of the associated single complex calculates the hypercohomology of $\mathcal{F}^\bullet$.

$$\mathbb{H}^p(\mathcal{F}^\bullet) \cong H^p\left(\bigoplus_{j+k=\bullet} \check{C}^j(\mathcal{U}, \mathcal{F}^k)\right) \tag{3.8.2}$$

Now we return to the complex $\mathcal{B}^\bullet_{p,q}(\mathbb{R}(p))$. (We could work with other coefficients, but our need for these results only includes $\mathbb{R}(p)$ coefficients. For simplicity, we consider only these coefficients.) Let η be an element of the rth degree of the associated diagonal single complex.

$$\begin{aligned} \eta \in & \bigoplus_{j+k=r} \check{C}^j(\mathcal{U}, \mathcal{B}^k_{p,q}(\mathbb{R}(p))) \\ & = \check{C}^r(\mathcal{U}, \mathcal{B}^0_{p,q}(\mathbb{R}(p))) \oplus \ldots \oplus \check{C}^0(\mathcal{U}, \mathcal{B}^r_{p,q}(\mathbb{R}(p))) \end{aligned} \tag{3.8.3}$$

In terms of that decomposition, we can write η in a particular form.

$$\eta = \eta^c_{i_0,i_1,\ldots,i_r} + (\eta^{h,0} + \eta^{a,0})_{i_0,i_1,\ldots,i_{r-1}} + \ldots + (\eta^{h,r} + \eta^{a,r})_{i_0,i_1,\ldots,i_{r-1}} \tag{3.8.4}$$

This notation requires some explanantion. The superscripts on η stand for the position in the complex $\mathcal{B}^\bullet_{p,q}$. The c superscript refers to $\mathcal{B}^0_{p,q} = \mathbb{R}(p)$ and represents the constant terms. The h and a superscripts represent holomorphic or antiholomorphic pieces in the $\mathcal{B}^\bullet_{p,q}$ complex, since past degree 0, the terms of that complex have the form $\Omega^m \oplus \overline{\Omega}^m$. The holomorphic and antiholomorphic pieces truncate at p and q, respectively, so $\eta^{h,m} = 0$ for $m \geq p$ and $\eta^{a,m} = 0$ for $m \geq q$.

The subscripts $i_0, \ldots, i_m$ indicate the Čech cocycle degree and index the open sets of the cover. From now on, we entirely supress these Čech indices.

Following [Sch07] , there is a convenient adjustments to this notation for element η. (Note that there are no holomorphic terms if $p = 0$, and no antiholomorphic terms if $q = 0$.)

$$\eta = \eta^c; \eta^{h,0}, \eta^{h,1}, \ldots, \eta^{h,r}; \eta^{a,0}, \eta^{a,1}, \ldots, \eta^{a,r} \tag{3.8.5}$$

The only difference between Equation 3.8.4 and Equation 3.8.5 is that we group the antiholomorphic and holomorphic pieces together. We will refer to these superscripts as the degree of the piece of η, *i.e.* η in degree $(a, 3)$ is $\eta^{a,3}$. Note that though the indices go all the way to $r = p + q$, the truncations still hold: $\eta^{h,i} = 0$ for $i > p$ and $\eta^{a,i} = 0$ for $i > q$.

This notation is convenient because the differentials of the $\mathcal{B}^\bullet_{p,q}$ respect the holomorphic and antiholomorphic decomposition. In this notation, those differentials $d_{\mathcal{B}}$ restricted to the various degrees act as described in the following chart.

$d_{\mathcal{B}}\eta^c = (\eta_c, -\eta_c)$	
$d_{\mathcal{B}}\eta^{h,0} = \partial\eta^{h,0}$	$d_{\mathcal{B}}\eta^{a,0} = \overline{\partial}\eta^{a,0}$
$d_{\mathcal{B}}\eta^{h,1} = \partial\eta^{h,1}$	$d_{\mathcal{B}}\eta^{a,1} = \overline{\partial}\eta^{a,1}$
$\vdots$	$\vdots$
$d_{\mathcal{B}}\eta^{h,p} = 0$	$d_{\mathcal{B}}\eta^{a,q} = 0$

Then using the fact that the differential of the simple complex associated with the double complex is $d_D : d_{\mathcal{B}} + (-1)^{\deg \mathcal{B}}\delta$, we can write explicitly how the differential of the double complex acts. For a form η as presented in this new notation, the differential $d_D\eta$ is calculated as follows. For this chart, assume that $r \geq \max\{p, q\}$; this is a fair assumption since it is the only case we require.

degree	term	degree	term
c	$\delta\eta^c$		
$h, 0$	$\eta^c - \delta\eta^{h,0}$	$a, 0$	$-\eta^c - \delta\eta^{a,0}$
$h, 1$	$\partial\eta^{h,0} + \delta\eta^{h,1}$	$a, 1$	$\overline{\partial}\eta^{a,0} + \delta\eta^{a,1}$
$h, 2$	$\partial\eta^{h,1} - \delta\eta^{h,2}$	$a, 2$	$\overline{\partial}\eta^{a,1} - \delta\eta^{a,2}$
$\ldots$			
h, j	$\partial\eta^{h,j-1} + (-1)^{j-1}\delta\eta^{h,j}$	a, j	$\overline{\partial}\eta^{a,j-1} + (-1)^{j-1}\delta\eta^{a,j}$
$\ldots$			
$h, p-2$	$\partial\eta^{h,p-3} + (-1)^{p-3}\delta\eta^{h,p-2}$	$a, q-2$	$\overline{\partial}\eta^{h,q-3} + (-1)^{q-3}\delta\eta^{h,q-2}$
$h, p-1$	$\partial\eta^{h,p-2} + (-1)^{p-2}\delta\eta^{h,p-1}$	$a, q-1$	$\overline{\partial}\eta^{h,q-2} + (-1)^{q-2}\delta\eta^{h,q-1}$
h, p	$\partial\eta^{h,p-1}$	a, q	$\overline{\partial}\eta^{a,q-1}$
$h, p+1$	0	$a, q+1$	0
$\ldots$		$\ldots$	
h, r	0	a, r	0

This explicit calculation of the differential completes our look at the Čech resolution, until we return to it for the calculation of products.

3.8.2 Cone Complex Resolution

A common tool in cohomology is resolution by a cone complex. Though the machinery of cone complexes is very general, we restrict to $\mathbb{R}$-coefficients. The main advantage of $\mathbb{R}$-coefficients is the decomposition $\mathbb{C} = \mathbb{R}(p-1) \oplus \mathbb{R}(p)$, which is not available in other coefficient systems.

This approach is following [Jan88], where cone complex constructions are described in detail for Deligne cohomology. We take inspiration from that source both in substance and style: both for the constructions and the ideas in the proofs.

It is useful to recall some notation. In this section, the projections π_p will refer to projections in $\mathbb{C} = \mathbb{R}(p) \oplus \mathbb{R}(p-1)$, and the induced maps between cohomology theories with $\mathbb{C}$ and $\mathbb{R}(p)$ coefficients.

We use the notation ϕ_q for projection onto bidgrees (r, s) where $s < q$. This is a unusual, non-standard notation, but extremely useful in describing the following construction.

We also use the notation $\mathcal{D}^\bullet_{<q}$ to refer to the sheaf of currents whose anti-holomorphic bidgree is $< q$, and similarly for $\mathcal{A}^\bullet_{<q}$.

Recall the shorthand $\underline{d}$ to refer to the differential of forms or currents d followed by whatever projection onto bidegree is necessary to fit the target space. We continue to use this convenient notation.

We start with the definition of a cone complex.

Definition 3.8.2: If $f : \mathcal{A}^\bullet \to \mathcal{B}^\bullet$ is a map of complexes, the *cone complex*, $\mathrm{Cone}(f)^\bullet$, is the direct product complex $(\mathcal{A}[1] \oplus \mathcal{B})^\bullet$ with the differential $(-d_{\mathcal{A}}, f + d_{\mathcal{B}})$.

Our inspiration comes from Deligne cohomology, and we want this construction to work well with Deligne cohomology and the map $\epsilon_{\mathcal{D}}$ in Definition 3.3.6. Therefore, we recall the specific cone complex results for Deligne cohomology from [Jan88] and [EV86].

Proposition 3.8.3: *Deligne cohomology is isomorphic to the cohomology of the following as a cone complex.*

$$H^r_{\mathcal{D}}(X, \mathbb{R}(p)) = H^{r-1}(\mathrm{Cone}(F^p\mathcal{D}^\bullet_X(X) \overset{-\pi_{p-1}}{\to} \mathcal{D}^\bullet_{X,\mathbb{R}(p-1)}(X)) \qquad (3.8.6)$$

Proof. From [Jan88], [EV86], and [Lew01]. The main details of this proof are also used in the proof of the next proposition. □

The proof constructs a quasi-isomorphism between the Deligne complex $\mathbb{R}(p)^\bullet_{\mathcal{D}}$ and the above cone complex. We follow the same construction, adapted to $\mathcal{B}^\bullet_{p,q}(\mathbb{R}(p)$ instead of the Deligne complex. The added information in $\mathcal{B}^\bullet_{p,q}(\mathbb{R}(p))$ is the trail of anti-holomorphic terms $\overline{\mathcal{O}} \to \overline{\Omega}^1 \to \ldots \to \overline{\Omega}^{p-1}$. In order to account for these, we add a second term to the target of the map defining the cone complex.

Proposition 3.8.4: *The sequence $\mathcal{B}^\bullet_{p,q}(\mathbb{R}(p))$ is quasi-isomorphic to the following cone complex.*

$$\mathrm{Cone}(F^p\mathcal{D}^\bullet_X \overset{-\pi_{p-1},\phi_q}{\to} \mathcal{D}^\bullet_{X,\mathbb{R}(p-1)} \oplus \mathcal{D}^\bullet_{<q})[-1] \qquad (3.8.7)$$

The $[-1]$ shift is necessary to match up with the indexing in $\mathcal{B}^\bullet_{p,q}(\mathbb{R}(p)$, we place the constants $\mathbb{R}(p)$ in degree 0. This is consistent with the Deligne complex in the literature. The maps realizing this quasi-isomorphism will be constructed piece by piece in the proof.

Proof. In this proof, currents are either holomorphic or antiholomorphic currents which can be identified with forms by regularity. Throughout the proof, we refer to forms and their associated currents by the same symbols. This is an abuse of notation, but makes the construction easier to follow.

This cone complex in degree r is a direct product sheaf.

$$F^p\mathcal{D}^r_X \oplus \mathcal{D}^{r-1}_{\mathbb{R}(p-1)} \oplus \mathcal{D}^{r-1}_{<q} \tag{3.8.8}$$

If (a, b, c) is a triple with respect to this decomposition, the differential is calculated as follows.

$$d_C(a, b, c) = (-da, -\pi_{p-1}(a) + db, -\phi_q(a) + \underline{d}c). \tag{3.8.9}$$

By the frequent use of the Poincaré Lemma (Proposition 2.2.22) on the stalks, the cone complex is exact except at degrees $0, p-1, q-1$. We must work in cases, though all of the substantial calculations occur in the first case.

Case $p > 0$, $q > 0$ and $(p, q) \neq (1, 1)$.

Near degree 0, the $\mathcal{B}^\bullet_{p.q}(\mathbb{R}(p))$ complex and this cone complex are:

$$\begin{array}{ccccccccc}
0 & \to & \mathbb{R}(p) & \to & \mathcal{O} \oplus \overline{\mathcal{O}} & \to & \Omega^1 \oplus \overline{\Omega}^1 & \to \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \to & 0 & \to & \mathcal{D}^0_{X,\mathbb{R}(p-1)} \oplus \mathcal{D}^{0,0} & \to & \mathcal{D}^1_{X,\mathbb{R}(p-1)} \oplus (\mathcal{D}^{1,0} \oplus \mathcal{D}^{0,1}) & \to
\end{array}$$

The degree 0 vertical map must obviously be a zero map. On an element $(f, g) \in \mathcal{O} \oplus \overline{\mathcal{O}}$, we define the degree 1 map to be $(f, g) \mapsto (\pi_{p-1}f, (f + g))$. The following vertical maps are similarly defined: $(f, g) \mapsto (\pi_{p-1}f, (f + g))$.

It is not difficult to show that these maps commute with the differentials at this degree. In degree 0, and element $r \in \mathbb{R}(p)$ maps to $(r, -r)$ in $\mathcal{O} \oplus \mathcal{O}$, then to $(-\pi_{p-1}(r), r + (-r)) = (0, 0)$, which commutes with the clearly zero map of the other route. In degree 1, let $(f, g) \in \mathcal{O} \oplus \overline{\mathcal{O}}$. Taking the differential first gives $(\partial f, \overline{\partial} g)$ which maps to $(\pi_{p-1}(\partial f), \partial f + \overline{\partial} g)$. If we take the vertical map first, we get $(\pi_{p-1}(f), g)$, which maps to $(\pi_{p-1}(df), d(f + g))$. These two agree because $\overline{\partial} f = 0$ and $\partial g = 0$. The same argument holds for higher degrees, until degree $p - 1$.

The kernel at degree 1 of $\mathcal{B}^{\bullet}_{p.q}(\mathbb{R}(p))$ is the following sheaf.

$$\frac{\mathbb{C} \oplus \mathbb{C}}{\{(r, -r) | r \in \mathbb{R}(p)\}} \tag{3.8.10}$$

By the complex linear change of variables given by $(a, b) \to (a, a+b)$ on $\mathbb{C} \oplus \mathbb{C}$, this quotient is isomorphic to $\mathbb{R}(p - 1) \oplus \mathbb{C}$.

In the cone complex, we can construct the same kernel in degree 1 by working locally with the stalks. The closed $\mathbb{R}(p - 1)$-valued currents, are simply the constants $\mathbb{R}(p-1)$, by regularity. Similarly, $\mathbb{C}$ is the group of local, closed $(0, 0)$ currents. Together we have a kernel of $\mathbb{R}(p) \oplus \mathbb{C}$. The map of complexes is of the form $(f, f + g)$. This is precisely the change of variables map suggested in the previous paragraph, so the map of complexes realizes isomorphism on cohomology.

Assuming $p, q > 1$, we look at subcases as follows.

Subcase $p < q$

In this case, the complex is exact with the given vertical maps until we get to degree $p - 1$. At that degree, the two complexes are as follows.

$$
\begin{array}{ccccc}
\Omega^{p-2} \oplus \overline{\Omega}^{p-2} & \to & \Omega^{p-1} \oplus \overline{\Omega}^{p-1} & \to \\
\downarrow & & \downarrow & \\
\mathcal{D}^{p-2}_{X,\mathbb{R}(p-1)} \oplus \mathcal{D}^{p-2} & \to & \mathcal{D}^{p,0}_{X,\mathbb{C}} \oplus \mathcal{D}^{p-1}_{X,\mathbb{R}(p-1)} \oplus \mathcal{D}^{p-1} & \to
\end{array}
$$

$$
\begin{array}{ccc}
\to & \overline{\Omega}^{p} & \to \\
& \downarrow & \\
\to & (\mathcal{D}^{p+1,0}_{X,\mathbb{C}} \oplus \mathcal{D}^{p,1}_{X,\mathbb{C}}) \oplus \mathcal{D}^{p}_{X,\mathbb{R}(p-1)} \oplus \mathcal{D}^{p} & \to
\end{array}
$$

The first vertical map is as before: $(f, g) \mapsto (\pi_{p-1}f, (f+g))$. The second vertical map is defined to be $(f, g) \mapsto (\partial f, \pi_{p-1}f, (f+g))$. The third vertical map, and those continuing past this point, are simply $g \mapsto (0, 0, g)$.

We can check that these maps commute with the differentials. Taking the differential first applied to (f, g) gives $\overline{\partial} g$, which maps to $(0, 0, \overline{\partial} g)$. (The assumption $p < q$ is important at this point.) The other direction is more involved: (f, g) maps to $(\partial f, \pi_{p-1}(f), f + g)$. Taking the cone complex differential gives $(-d(\partial f), -\pi_{p-1}(\partial f) + \pi_{p-1}(\partial f), -\phi_q(\partial f) + \underline{d}(f) + \overline{\partial} g)$. Using the fact that $\overline{\partial} f = 0$ and that $\underline{d} f$ and $\phi_q \partial f$ are equivalent in this situation, this expression reduces to $(0, 0, \overline{\partial} g)$.

The cohomology of $\mathcal{B}^{\bullet}_{p,q}(\mathbb{R}(p))$ in degree $p-1$ is $\Omega^{p-1}/\partial\Omega^{p-2}$. We need to make sure the cone complex agrees, and the induced map on cohomology is an isomorphism. Therefore, consider the kernel in degree $p-1$ in the cone complex.

$$\left\{ (a,b,c) \in \mathcal{D}^{p,0}_{X,\mathbb{C}} \oplus \mathcal{D}^{p-1}_{X,\mathbb{R}(p-1)} \oplus \mathcal{D}^{p-1} \;\middle|\; \begin{array}{l} da = 0 \\ \pi_{p-1}(a) = db \\ \phi_q(a) = dc \end{array} \right\} \tag{3.8.11}$$

In the first summand, we have the differential d, so we get a kernel of $\hat{\Omega}^p_{\mathcal{D}}$. By regularity, this is the group of currents associated with the holomorphic forms $\hat{\Omega}^p$.

Then consider the equation $\pi_{p-1}(a) = db$. Up to a constant, this projection π_{p-1} is $a + \overline{a}$, which is a $(p, 0)$ form plus a $(0, p)$ form. Provided that $p \geq 2$, the equation $c(a + \overline{a}) = db$ implies that b decomposes as $b^{p-1,0} + b^{0,p-1}$, with $\partial b^{p-1,0} = ca$ and $\overline{\partial} b^{0,p-1} = c\overline{a}$. However, then we can use the Poincaré Lemma (Proposition 2.2.22), to show that b is uniquely determined by a in both bidegrees. Therefore, this second component is fixed by the choice of a in the first component.

If $p = 1$, we have that b is a $(1, 1)$ form with $\partial b = ca$ and $\overline{\partial} b = c\overline{a}$, which is similarly uniquely determined by the Poincaré Lemma (Proposition 2.2.22).

The third component of the image of (a, b, c) is $-\phi_q(a) + dc$. In the kernel, this imposes the relations $dc = a$ on the original triple. But we are working locally and $dc = a$ implies that c is a $(p - 1, 0)$ current which satisfies $\overline{\partial} c = 0$ and $\partial c = a$. By regularity and the Poincaré Lemma, this means that the form c which satisfies $dc = a$ is unique.

These observations allow us to conclude that both the second and third components are determined uniquely by the first. Projection onto the first summand preserves the kernel entirely, and is an isomorphism with the sheaf $\hat{\Omega}^p$. There is no image in the first component, so the cohomology of the cone complex in degree $p-1$ is $\hat{\Omega}^p$.

Lastly, the induced map between the kernels is the map on the first component, which is ∂. By Corollary 2.2.23, the map ∂ is an isomorphism from $\Omega^{p-1}/\partial\Omega^{p-2}$ to $\hat{\Omega}^p$. This proves the quasi-isomorphism in degree $p-1$.

Still working in the subcase $p<q$,the complexes continue exactly until $q-1$, where we have the following terms.

$$\begin{array}{ccc} \overline{\Omega}^{q-2} & \to & \overline{\Omega}^{q-1} \\ \downarrow & & \downarrow \\ \mathcal{F}^p\mathcal{D}^{q-2}_{X,\mathbb{C}} \oplus \mathcal{D}^{q-2}_{X,\mathbb{R}(p-1)} \oplus \mathcal{D}^{q-2} & \to & \mathcal{F}^p\mathcal{D}^{q-1}_{X,\mathbb{C}} \oplus \mathcal{D}^{q-1}_{X,\mathbb{R}(p-1)} \oplus \mathcal{D}^{q-1} \end{array}$$

$$\begin{array}{cc} \to & 0 \\ & \downarrow \\ \to & \mathcal{F}^p\mathcal{D}^{q}_{X,\mathbb{C}} \oplus \mathcal{D}^{q}_{X,\mathbb{R}(p-1)} \oplus (\mathcal{D}^{q,0} \oplus \ldots \oplus \mathcal{D}^{1,q-1}) \end{array}$$

The vertical maps are $g \mapsto (0,0,g)$, which still form commutative squares. The cohomology of $\mathcal{B}^\bullet_{p,q}(\mathbb{R}(p))$ in degree $q-1$ is $\overline{\Omega}^{q-1}/\overline{\partial\Omega}^{q-2}$. In the cone complex, the first two summands are exact and the third summand starts to truncate at this degree. This gives the cohomology of precisely $\overline{\Omega}^{q-1}/\overline{\partial\Omega}^{q-2}$, agreeing with the $\mathcal{B}^\bullet_{p,q}(\mathbb{R}(p))$. The map between the two cohomologies is the identity map after identifying forms with currents, by regularity of holomorphic currents.

The rest of $\mathcal{B}^\bullet_{p,q}(\mathbb{R}(p))$ is zero, and the cone complex continues exactly by the Poincaré Lemma (Proposition 2.2.22), which completes the proof in this case.

Subcase $p>q$

For this case, the anti-holomorphic part of the complex ends first. At degree $q-1$, we have the following terms.

$$
\begin{array}{ccccc}
\Omega^{q-2} \oplus \overline{\Omega}^{q-2} & \to & \Omega^{q-1} \oplus \overline{\Omega}^{q-1} & \to \\
\downarrow & & \downarrow & \\
\mathcal{D}^{q-2}_{X,\mathbb{R}(p-1)} \oplus \mathcal{D}^{q-2} & \to & \mathcal{D}^{q-1}_{X,\mathbb{R}(p-1)} \oplus \mathcal{D}^{q-1} & \to
\end{array}
$$

$$
\begin{array}{ccc}
\to & \Omega^{q} & \to \\
& \downarrow & \\
\to & \mathcal{D}^{q}_{X,\mathbb{R}(p-1)} \oplus (\mathcal{D}^{q,0} \oplus \ldots \oplus \mathcal{D}^{1,q-1}) & \to
\end{array}
$$

The vertical maps are still defined to be $(f, g) \mapsto (-\pi_{p-1}(f), f + g)$ and they still form commutative squares. The cohomology in the top complex is only supported in the second component, which is $\overline{\Omega}^{q-1}/\overline{\partial\Omega}^{q-2}$. The calculation for the cohomology of the cone complex, in this case, is like degree $q - 1$ above: we only have non-trivial cohomology because of the start of the truncation in the second component. This cohomology is precisely the same, $\overline{\Omega}^{q-1}/\overline{\partial\Omega}^{q-2}$, and the map in the second degree restricted to cohomology is the identity (up to identification of anti-holomorphic forms with currents).

The sequences continue exactly until degree $p-1$. We have to adjust the maps slightly, taking $\phi_q(f)$ in the second component instead of just (f), to account for the truncation. These maps still commute with the differentials.

At degree $p - 1$, we have the next possible contribution.

$$\begin{array}{ccccc}
\Omega^{p-2} & \to & \Omega^{p-1} & \to \\
\downarrow & & \downarrow & \\
\mathcal{D}^{p-2}_{X,\mathbb{R}(p-1)} \oplus \mathcal{D}^{p-2} & \to & \mathcal{D}^{p,0}_{X,\mathbb{C}} \oplus \mathcal{D}^{p-1}_{X,\mathbb{R}(p-1)} \oplus \mathcal{D}^{p-1} & \to
\end{array}$$

$$\begin{array}{ccc}
\to & 0 & \to \\
 & \downarrow & \\
\to & (\mathcal{D}^{p+1,0}_{X,\mathbb{C}} \oplus \mathcal{D}^{p,1}_{X,\mathbb{C}}) \oplus \mathcal{D}^{p}_{X,\mathbb{R}(p-1)} \oplus \mathcal{D}^{p} & \to
\end{array}$$

The vertical map in degree $p-1$ is $f \mapsto (\partial f, -\pi_{p-1} f, \phi_q f)$. Past this, the vertical maps are zero. These maps still form commutative squares.

The kernel calculation here is just a simplification of the calculation in degree $p-1$ in the previous subcase, where cohomology is realized by restricting to the first component and ∂ induces an isomorphism in cohomology.

Subcase $p = q$

Here the two cohomology calculations happen as the same time, which is in degree $p-1 = q-1$.

$$\begin{array}{ccccc}
\Omega^{p-2} \oplus \overline{\Omega}^{p-2} & \to & \Omega^{p-1} \oplus \overline{\Omega}^{p-1} & \to \\
\downarrow & & \downarrow & \\
\mathcal{D}^{p-2}_{X,\mathbb{R}(p-1)} \oplus \mathcal{D}^{p-2} & \to & \mathcal{D}^{p,0}_{X,\mathbb{C}} \oplus \mathcal{D}^{p-1}_{X,\mathbb{R}(p-1)} \oplus \mathcal{D}^{p-1} & \to
\end{array}$$

$$\begin{array}{ccc}
\to & 0 & \to \\
 & \downarrow & \\
\to & (\mathcal{D}^{p+1,0}_{X,\mathbb{C}} \oplus \mathcal{D}^{p,1}_{X,\mathbb{C}}) \oplus \mathcal{D}^{p}_{X,\mathbb{R}(p-1)} \oplus \mathcal{D}^{p} & \to
\end{array}$$

The cohomology of $\mathcal{B}^\bullet_{p,q}(\mathbb{R}(p))$ here is a direct sum: $\Omega^{p-1}/\partial\Omega^{p-2} \oplus \overline{\Omega}^{p-1}/\overline{\partial\Omega}^{p-2}$. The cohomology of the cone complex is found by the same methods as the calculations in degree $p-1$ and $q-1$ in the previous subcases; it works out to a direct sum support in the first and third components of $\Omega^p_{d-\text{closed}} \oplus \overline{\Omega}^{p-1}/\overline{\partial\Omega}^{p-2}$. The induced map is ∂ on the first component, which is an isomorphism, and the identity on the second.

Other Cases

The first case covered all degrees where $p, q > 1$. There are other cases with non-zero p and q, following cases describing cohomology calculations in the proof of Proposition 3.2.3, as well as cases where p and q may be zero. These cases required some slight adjustment to the calculations, since various pieces of the calculation interfere with degree 0. Though the details vary slightly, the calculations are similar to what we have already done, and are excluded here. As a particular note, the case $q = 0$ recovers the Deligne cohomology cone complex construction, since $\mathcal{D}_{<0}$ is trivial. □

A pleasant fact about this quasi-isomorphism is that all sheaves involved are fine, so the map into the cone complex acts as a resolution of $\mathcal{B}^\bullet_{p,q}(\mathbb{R}(p))$. Therefore, we can calculate hypercohomology simply by taking the cohomology of the global sections of the cone complex.

$$H^{p,q}_{\text{BC}}(X, \mathbb{R}(p) = \mathbb{H}^{p+q}(\mathcal{B}^\bullet_{p,q}(\mathbb{R}(p)) =$$
$$H^{p+q-1}(\text{Cone}(F^p\mathcal{D}^\bullet_X(X) \overset{-\pi_{p-1},-\phi_q}{\to} \mathcal{D}^\bullet_{X,\mathbb{R}(p-1)}(X) \oplus \mathcal{D}^\bullet_{<q}(X))) \qquad (3.8.12)$$

$$H^{p,q}_{\mathrm{Ap}}(X, \mathbb{R}(p+1) = \mathbb{H}^{p+q+1}(\mathcal{B}^\bullet_{p+1,q+1}(\mathbb{R}(p+1)) =$$
$$H^{p+q}(\mathrm{Cone}(F^{p+1}\mathcal{D}^\bullet_X(X) \stackrel{-\pi_p, -\phi_q}{\rightarrow} \mathcal{D}^\bullet_{X,\mathbb{R}(p)}(X) \oplus \mathcal{D}^\bullet_{<q}(X))) \quad (3.8.13)$$

The cone complex construction allows for new definitions of the maps between Bott-Chern or Aeppli cohomology and Deligne cohomology.

Definition 3.8.5: There are maps ϵ_C.

$$\epsilon_C : H^{p,q}_{\mathrm{BC}}(X, \mathbb{R}(p)) \to H^{p+q}_{\mathcal{D}}(X, \mathbb{R}(p)) \quad (3.8.14)$$
$$\epsilon_C : H^{p,q}_{\mathrm{Ap}}(X, \mathbb{R}(p+1)) \to H^{p+q+1}_{\mathcal{D}}(X, \mathbb{R}(p+1))$$

Assume that all cohomologies are described by cone complexes as in Equations 3.8.6, 3.8.12 and 3.8.13, and assume that a class in Bott-Chern or Aeppli cohomology is represented by a triple as follows.

$$(a, b, c) \in F^p\mathcal{D}^r_X \oplus \mathcal{D}^{r-1}_{\mathbb{R}(p-1)} \oplus \mathcal{D}^{r-1}_{<q} \quad (3.8.15)$$

Then the maps ϵ_C are the maps induced by the projection $(a, b, c) \mapsto (a, b)$.

Proposition 3.8.6: *The maps ϵ_C are well defined.*

Proof. The result is essentially by construction. To give well defined maps on hypercohomology, the maps need to commute with the differentials on the levels of complexes. However, the differentials in the cone complex defining

Deligne cohomology are precisely the same as those in the cone complex defining Bott-Chern, simply ignoring the third component. Since the differentials are the same, and the map is the identity on the first two components, the commutativity is trivial. □

This also clarifies the claim earlier that we recover all of the necessary calculations for Deligne cohomology. The nature of this map, simply forgetting a component, shows that all the information in the Deligne complex is already part of these Bott-Chern cone complex constructions.

Proposition 3.8.7: *The following diagram is commutative.*

$$\begin{array}{ccc} \mathcal{B}^\bullet_{p,q} & \to & \mathrm{Cone}(F^p\mathcal{D}^\bullet_X(X) \overset{-\pi_{p-1},-\phi_q}{\to} \mathcal{D}^\bullet_{X,\mathbb{R}(p-1)}(X) \oplus \mathcal{D}^\bullet_{<q}(X)) \\ \downarrow & & \downarrow \\ \mathbb{R}(p)^\bullet_{\mathcal{D}} & \to & \mathrm{Cone}(F^p\mathcal{D}^\bullet_X(X) \overset{-\pi_{p-1}}{\to} \mathcal{D}^\bullet_{X,\mathbb{R}(p-1)}(X)) \end{array}$$

Here the horizontal maps are the quasi-isomorphisms constructed in this section. The first vertical map is the map on complexes inducing $\epsilon_{\mathcal{D}}$: projection away from the anti-holomorphic terms. The second vertical map is the map just described defining ϵ_C: the projection onto the first two components in the cone complex.

Proof. The quasi-isomorphisms are defined in cases, as in the proof of Proposition 3.8.4. In each case, we can explicitly calculate the two paths around the diagram. We assume $p, q > 1$ and use the maps defined in Proposition 3.8.4

In all cases, the degree zero situation is simply this square, for $z \in \mathbb{C}$.

$$\begin{array}{ccc} z & \to & 0 \\ \downarrow & & \downarrow \\ z & \to & 0 \end{array}$$

Case $p < q$

For degree $1 \leq d \leq p-2$, we have this square.

$$\begin{array}{ccc} (f,g) & \to & (-\pi_{p-1}(f), f+g) \\ \downarrow & & \downarrow \\ f & \to & -\pi_{p-1}(f) \end{array}$$

For degree $d = p-1$ we have this square.

$$\begin{array}{ccc} (f,g) & \to & (\partial f, -\pi_{p-1}(f), f+g) \\ \downarrow & & \downarrow \\ f & \to & (\partial f, -\pi_{p-1}(f)) \end{array}$$

For degrees $d \geq p$ we have this square.

$$\begin{array}{ccc} (0,g) & \to & (0,0,g) \\ \downarrow & & \downarrow \\ 0 & \to & (0,0) \end{array}$$

Case $p > q$

For degrees $1 \leq d \leq p-2$ we have this square.

$$\begin{array}{ccc}
(f,g) & \to & (-\pi_{p-1}(f), f+g) \\
\downarrow & & \downarrow \\
f & \to & -\pi_{p-1}(f)
\end{array}$$

For degree $d = p-1$ we have this square.

$$\begin{array}{ccc}
(f,g) & \to & (\partial f, -\pi_{p-1}(f), \phi_q(f+g)) \\
\downarrow & & \downarrow \\
f & \to & (\partial f, -\pi_{p-1}(f))
\end{array}$$

For higher degrees, the initial elements in the top left of the square are just zero, so the square commutes automatically.

Case $p = q$

For degrees $1 \leq d \leq p-1$, this is the same as the previous two cases. For degree $p-1$ this is the same as the case $p > q$. Finally, for higher degrees, the initial top left elements are zero and commutativity is trivial.

As with the proof of Proposition 3.8.4, the other cases from Proposition 3.2.3 for $p, q > 0$ and cases involving $p = 0$ or $q = 0$ are similar and the calclulations are excluded here. □

The commutativity of the maps of complexes gives this corollary.

Corollary 3.8.8: *The maps* ϵ_D *and* ϵ_C *are the same map, once the the Bott-Chern and Deligne cohomologies have been identified with their cone complex descriptions.*

3.9 Bott-Chern and Aeppli Products

The first Bott-Chern and Aeppli products were defined in Proposition 3.1.4, working with $\mathbb{C}$-coefficients. These products were defined by the wedge product of forms, but that option is no longer available when discussing products for Bott-Chern cohomology with finer coefficients. To investigate products of cohomology with finer coefficients, we use the two resolutions established in Section 3.8.

3.9.1 Products using the Čech resolution

First we return to the Čech resolution. These constructions and calculations are following [Sch07]. Elaborating on that source, we have checked the work in detail and made the signs much more explicit, as well as adapted for formula to include Aeppli cohomology. Though the Čech resolution machinery is more general regarding coefficients, we restrict to $\mathbb{R}(p)$ coefficients here as we have done before.

In the following, we have repressed the Čech indices and the cup product notation in Čech cohomology. When we write consecutive terms, the implicit product is always the cup product of Čech cocycles.

Many of the product calculations for the Čech resolution are lengthy and technical, so we have relegated most of them to Appendix A.

Assume that $u \geq \max\{p-1, q-1\}$ and $v \geq \max\{r-1, s-1\}$, and assume that η and ν are elements, of degrees u and v respectively, in two different Čech double complexes as follows, using the notation from Definition 3.8.5.

$$\eta \in \bigoplus_{j+k=u} \check{C}^j(\mathcal{U}, \mathcal{B}^k_{p,q}(\mathbb{R}(p))) \tag{3.9.1}$$

$$\eta = \eta^c; \eta^{h,0}, \eta^{h,1}, \ldots, \eta^{h,u}; \eta^{a,0}, \eta^{a,1}, \ldots, \eta^{a,u}$$

$$\nu \in \bigoplus_{j+k=v} \check{C}^j(\mathcal{U}, \mathcal{B}^k_{r,s}(\mathbb{R}(r)))$$

$$\nu = \nu^c; \nu^{h,0}, \nu^{h,1}, \ldots, \nu^{h,v}; \nu^{a,0}, \nu^{a,1}, \ldots, \nu^{a,v}$$

Then there is a product formula which defines the following element.

$$\eta \star \nu \in \bigoplus_{j+k=u+v-1} \check{C}^j(\mathcal{U}, \mathcal{B}^k_{p+r,q+s}(\mathbb{R}(p+r))) \tag{3.9.2}$$

The term by term definition of this product is given in Appendix A. Note that the product maps degree u and degree v to degree $u+v-1$.

Proposition 3.9.1: *The formula presented in Appendix A is a well defined map of complexes and induces a product structure on cohomology.*

Proof. The proof involves checking the Leibniz rule. This is a long, detailed and technical calculation which is also found in Appendix A. □

This product formula can be used to define two Bott-Chern cohomology products:

$$H^{p,q}_{\mathrm{BC}}(X, \mathbb{R}(p)) \times H^{r,s}_{\mathrm{BC}}(X, \mathbb{R}(r)) \to H^{p+r,q+s}_{\mathrm{BC}}(X, \mathbb{R}(p+r)) \tag{3.9.3}$$

$$H^{p,q}_{\mathrm{BC}}(X, \mathbb{R}(p)) \times H^{r,s}_{\mathrm{Ap}}(X, \mathbb{R}(r+1)) \to H^{p+r,q+s}_{\mathrm{BC}}(X, \mathbb{R}(p+r))$$

The details and the adjustment in degrees and calculations are listed in Appendix A.

In a remark in [Sch07], it is noted that this product is compatible with the map $\epsilon_{\mathcal{D}}$ defined in 3.6.3 and the product on Deligne cohomology defined in [EV86]. That reference does not give the details, nor have we worked them out.

3.9.2 Products using the Cone Complex Resolution

The details of a product in Deligne cohomology *via* the cone description are from [EV86]. We repeat those details, adjusting them only in notation.

Deligne cohomology is given by the cohomology of cone complex.

$$H^r_{\mathcal{D}}(X, \mathbb{R}(p)) = H^{r-1}(\mathrm{Cone}(F^p\mathcal{D}^\bullet_X(X) \stackrel{-\pi_{p-1}}{\rightarrow} \mathcal{D}^\bullet_{X,\mathbb{R}(p-1)}(X)) \tag{3.9.4}$$

The $r-1$th term of the cone complex is $F^p\mathcal{D}^r_X(X) \oplus \mathcal{D}^{r-1}_{X,\mathbb{R}(p)}(X)$. With respect to this direct sum, an element of Deligne cohomology is represented by a pair (f, g).

Proposition 3.9.2: *If* (f_r, g_r) *and* (f_s, g_s) *are representatives of Deligne cohomology classes in* $H^r_{\mathcal{D}}(X, \mathbb{R}(p))$ *and* $H^s_{\mathcal{D}}(X, \mathbb{R}(q))$ *respectively, then the product in* $H^{r+s}_{\mathcal{D}}(X, \mathbb{R}(p+q))$ *is represented by the following expression.*

$$(f_r \wedge f_s, g_r \wedge \pi_q f_s + (-1)^{\deg f_r}\pi_p f_r \wedge g_s) \tag{3.9.5}$$

Proof. In order to ensure that this leads to a product, it is necessary to check that it respects the Leibniz rule, as is done in [EV86] Lemma 3.11. That calculation is straightforward, using the observation that projection and the wedge product interact as follows.

$$\pi_{p+q-1}f_r \wedge f_s = \pi_{p-1}f_r \wedge \pi_q f_s + \pi_p f_r \wedge \pi_{q-1} f_s \tag{3.9.6}$$

□

Recall that the Bott-Chern cohomology is given, in a cone complex, by this formula.

$$\begin{aligned} &H^{p,q}_{\mathrm{BC}}(X, \mathbb{R}(p) = \mathbb{H}^{p+q}(\mathcal{B}^{\bullet}_{p,q}(\mathbb{R}(p)) = \\ &H^{p+q-1}(\mathrm{Cone}(F^p\mathcal{D}^{\bullet}_X(X) \stackrel{-\pi_{p-1},0}{\to} \mathcal{D}^{\bullet}_{X,\mathbb{R}(p-1)}(X) \oplus \mathcal{D}^{\bullet}_{<q}(X))) \end{aligned} \tag{3.9.7}$$

The difference between this and the Deligne cohomology is the extra $\mathcal{D}^{\bullet}_{<q}$ component. We need to incorporate the extra component into the product.

Proposition 3.9.3: *There is a product on Bott-Chern cohomology with* $\mathbb{R}(p)$*-coefficients. In terms of the cone complex, assume that classes in Bott-Chern cohomology* $H^{p,q}_{\mathrm{BC}}(X, \mathbb{R}(p))$ *and* $H^{r,s}_{\mathrm{BC}}(X, \mathbb{R}(r))$ *are represented, respectively, by the following elements.*

$$\begin{aligned} (\eta, \sigma, \omega) &\in F^p D^{p+q}_X(X) \oplus D^{p+q-1}_{X,\mathbb{R}(p-1)}(X) \oplus \mathcal{D}^{p+q-1}_{<q} \\ (\widetilde{\eta}, \widetilde{\sigma}, \widetilde{\omega}) &\in F^r D^{r+s}_X(X) \oplus D^{r+s-1}_{X,\mathbb{R}(r-1)}(X) \oplus \mathcal{D}^{r+s-1}_{<s} \end{aligned} \tag{3.9.8}$$

Then the product in $H^{p+r,q+s}_{\mathrm{BC}}(X, \mathbb{R}(p+r))$ *is represented by the following element.*

$$\left(\eta \wedge \widetilde{\eta},\ \sigma \wedge \pi_r \widetilde{\eta} + (-1)^p \pi_p \eta \wedge \widetilde{\sigma},\ \frac{1}{2}\left(\omega \wedge \phi_s \widetilde{\eta} + (-1)^p \phi_q \eta \wedge \widetilde{\omega}\right)\right) \tag{3.9.9}$$

Proof. The product on complexes needs to be stated slightly more generally in order to check that the Leibniz rule applies. Therefore, we take two elements similar to those in the statement of the proposition, but in arbitrary degrees n and m respectively.

$$\begin{aligned} (\eta, \sigma, \omega) &\in F^p \mathcal{D}_X^n(X) \oplus \mathcal{D}^{n-1}_{\mathbb{R}(p-1}(X) \oplus \mathcal{D}^{n-1}_{<q}(X) \\ (\widetilde{\eta}, \widetilde{\sigma}, \widetilde{\omega}) &\in F^r \mathcal{D}_X^m(X) \oplus \mathcal{D}^{m-1}_{\mathbb{R}(r-1}(X) \oplus \mathcal{D}^{m-1}_{<s}(X) \end{aligned} \tag{3.9.10}$$

Note that these terms have degrees n and m, respectively, in the $[-1]$ shifted cone complexes. The $[-1]$ here is very necessary to make sure that we have the right sign in the Leibniz rule. Also note that the numbers p, q, r, s are fixed, since they define the original complexes, but the degrees n, m are variable.

The product gives an element of order $m+n$ in the new shifted cone complex.

$$(\eta, \sigma, \omega) \circ (\widetilde{\eta}, \widetilde{\sigma}, \widetilde{\omega}) \in F^{p+r} \mathcal{D}_X^{m+n}(X) \oplus \mathcal{D}^{n+m-1}_{\mathbb{R}(p+r-1)}(X) \oplus \mathcal{D}^{n+m-1}_{<q+s}(X) \tag{3.9.11}$$

This is the general product formula.

$$\left(\eta \wedge \widetilde{\eta},\ \sigma \wedge \pi_r(\widetilde{\eta}) + (-1)^n \pi_p(\eta) \wedge \widetilde{\sigma},\ \frac{1}{2}\left(\omega \wedge \phi_s(\widetilde{\eta}) + (-1)^n \phi_q(\eta) \wedge \widetilde{\omega}\right)\right) \tag{3.9.12}$$

With this formula in place, it sufficies to prove the Leibniz rule. We first calculate the differential on the product.

$$\begin{aligned} &d_C\left[(\eta, \sigma, \omega) \circ (\widetilde{\eta}, \widetilde{\sigma}, \widetilde{\omega})\right] = \\ &d_C\left[\left(\eta \wedge \widetilde{\eta},\ \sigma \wedge \pi_r(\widetilde{\eta}) + (-1)^n \pi_p(\eta) \wedge \widetilde{\sigma},\ \frac{1}{2}\left(\omega \wedge \phi_s(\widetilde{\eta}) + (-1)^n \phi_q(\eta) \wedge \widetilde{\omega}\right)\right)\right] \end{aligned} \tag{3.9.13}$$

We work by components. The first component has the following form.

$$-d(\eta \wedge \widetilde{\eta}) = -d\eta \wedge \widetilde{\eta} + (-1)^{n+1} \eta \wedge d\widetilde{\eta} \tag{3.9.14}$$

The second component is more complicated.

$$\begin{aligned} &-\pi_{p+r-1}(\eta \wedge \widetilde{\eta}) + d\left(\sigma \wedge \pi_r(\widetilde{\eta}) + (-1)^n \pi_p(\eta) \wedge \widetilde{\sigma}\right) \\ &= \pi_{p+r-1}(\eta \wedge \widetilde{\eta}) + d\sigma \wedge \pi_r(\widetilde{\eta}) + (-1)^{n-1} \sigma \wedge d\pi_r(\widetilde{\eta}) + \\ &\quad (-1)^n d\pi_p(\eta) \wedge \widetilde{\sigma} + (-1)^n (-1)^n \pi_p(\eta) \wedge d\widetilde{\sigma} \\ &= \pi_{p+r-1}(\eta \wedge \widetilde{\eta}) + d\sigma \wedge \pi_r(\widetilde{\eta}) + (-1)^{n-1} \sigma \wedge \pi_r(d\widetilde{\eta}) + \\ &\quad (-1)^n \pi_p(d\eta) \wedge \widetilde{\sigma} + \pi_p(\eta) \wedge d\widetilde{\sigma} \end{aligned} \tag{3.9.15}$$

The third component is similar to the second.

$$
\begin{aligned}
&- \phi_{q+s}(\eta \wedge \widetilde{\eta}) + \frac{1}{2}\underline{d}\left(\omega \wedge \phi_s(\widetilde{\eta}) + (-1)^n \phi_q(\eta) \wedge \widetilde{\omega}\right) \\
&= \phi_{q+s}(\eta \wedge \widetilde{\eta}) + \frac{1}{2}\underline{d}\omega \wedge \phi_s(\widetilde{\eta}) + (-1)^{n-1}\frac{1}{2}\omega \wedge \underline{d}\phi_s(\widetilde{\eta}) + \\
&\quad (-1)^n \frac{1}{2}\underline{d}\phi_q(\eta) \wedge \widetilde{\omega} + (-1)^n(-1)^n \frac{1}{2}\phi_q(\eta) \wedge \underline{d}\widetilde{\omega} \\
&= \phi_{q+s}(\eta \wedge \widetilde{\eta}) + \frac{1}{2}\underline{d}\omega \wedge \phi_s(\widetilde{\eta}) + (-1)^{n-1}\frac{1}{2}\omega \wedge \phi_s(d\widetilde{\eta}) + \\
&\quad (-1)^n \frac{1}{2}\phi_q(d\eta) \wedge \widetilde{\omega} + (-1)^n(-1)^n \frac{1}{2}\phi_q(\eta) \wedge \underline{d}\widetilde{\omega}
\end{aligned} \tag{3.9.16}
$$

We must calculate the other half of the Leibniz formula and compare to these espressions. The other half of the formula is the product of the differentials.

$$
d_c(\eta, \sigma, \omega) \circ (\widetilde{\eta}, \widetilde{\sigma}, \widetilde{\omega}) + (-1)^n (\eta, \sigma, \omega) \circ d_c(\widetilde{\eta}, \widetilde{\sigma}, \widetilde{\omega}) \tag{3.9.17}
$$

Note that the sign is $(-1)^n$. According to the shift $[-1]$ in the cone complex, the first element has degree n.

Then we expand the differentials.

$$
\begin{aligned}
&(-d\eta, -\pi_{p-1}(\eta) + d\sigma, -\phi_q(\eta) + \underline{d}\omega) \circ (\widetilde{\eta}, \widetilde{\sigma}, \widetilde{\omega}) + \\
&\quad (-1)^n (\eta, \sigma, \omega) \circ (-d\widetilde{\eta}, -\pi_{r-1}(\widetilde{\eta}) + d\widetilde{\sigma}, -\phi_s(\eta) + \underline{d}\widetilde{\omega})
\end{aligned} \tag{3.9.18}
$$

Again, we work by components. The first component is straightforward.

$$
-d\eta \wedge \widetilde{\eta} + (-1)^{n+1} \eta \wedge d\widetilde{\eta} \tag{3.9.19}
$$

This agrees with the previous construction. The second component is, again, more involved.

$$
\begin{aligned}
&(-\pi_{p-1}(\eta) + d\sigma) \wedge \pi_r(\widetilde{\eta}) + (-1)^n + 1\pi_p(-d\eta) \wedge \widetilde{\sigma} + \\
&\quad (-1)^n \left[\sigma \wedge \pi_r(-d\widetilde{\eta}) + (-1)^n \pi_p(\eta) \wedge (-\pi_{r-1}(\widetilde{\eta}) + d\widetilde{\sigma})\right] \\
&= -\pi_{p-1}(\eta) \wedge \pi_r(\widetilde{\eta}) + d\sigma \wedge \pi_r(\widetilde{\eta}) + (-1)^n + 1\pi_p(-d\eta) \wedge \widetilde{\sigma} + \\
&\qquad (-1)^n \left[\sigma \wedge \pi_r(-d\widetilde{\eta}) + (-1)^n \pi_p(\eta) \wedge (-\pi_{r-1}(\widetilde{\eta})) + (-1)^n \pi_p(\eta) \wedge d\widetilde{\sigma}\right] \\
&= -\pi_{p-1}(\eta) \wedge \pi_r(\widetilde{\eta}) - \pi_p(\eta) \wedge \pi_{r-1}(\widetilde{\eta}) + d\sigma \wedge \pi_r(\widetilde{\eta}) + \\
&\qquad (-1)^{n+1} \sigma \wedge \pi_r(d\widetilde{\eta}) + (-1)^n \pi_p(d\eta) \wedge \widetilde{\sigma} + \pi_p(\eta) \wedge d\widetilde{\sigma}
\end{aligned}
\tag{3.9.20}
$$

This agrees with the previous second component using the relation between the projections and the exterior derivative: $\pi_{p+r-1}(\eta \wedge \widetilde{\eta}) = \pi_{p-1}(\eta) \wedge \pi_r(\widetilde{\eta}) + \pi_p(\eta) \wedge \pi_{r-1}(\widetilde{\eta})$.

This leaves the third component.

$$
\begin{aligned}
&\frac{1}{2}\left[(-\phi_q(\eta) + \underline{d}\omega) \wedge \phi_s(\widetilde{\eta}) + (-1)^{n+1} \phi_q(-d\eta) \wedge \widetilde{\omega}\right] + \\
&\quad (-1)^n \frac{1}{2}\left[\omega \wedge \phi_s(-d\widetilde{\eta}) + (-1)^n \phi_q(\eta) \wedge (-\phi_s(\widetilde{\eta}) + \underline{d}\widetilde{\omega})\right] \\
&= \frac{1}{2}\left[-\phi_q(\eta) \wedge \phi_s(\widetilde{\eta}) + \underline{d}\omega \wedge \phi_s(\widetilde{\eta}) + (-1)^{n+1} \phi_q(-d\eta) \wedge \widetilde{\omega} + \right] \\
&\qquad (-1)^n \frac{1}{2}\left[\omega \wedge \phi_s(-d\widetilde{\eta}) + (-1)^n \phi_q(\eta) \wedge (-\phi_s(\widetilde{\eta})) + (-1)^n \phi_q(\eta) \wedge \underline{d}\widetilde{\omega}\right] \\
&= -\phi_q(\eta) \wedge \phi_s(\widetilde{\eta}) + \frac{1}{2}\underline{d}\omega \wedge \phi_s(\widetilde{\eta}) + (-1)^{n+1} \frac{1}{2}\omega \wedge \phi_s(d\widetilde{\eta}) + \\
&\qquad (-1)^n \frac{1}{2}\phi_q(d\eta) \wedge \widetilde{\omega} + \frac{1}{2}\phi_q(\eta) \wedge \underline{d}\widetilde{\omega}
\end{aligned}
\tag{3.9.21}
$$

This agrees with the previous third component. We use the fact that $\underline{d}\phi_q = \phi_q d$, since the implied projections in $\underline{d}$ are precisely the projections imposed by ϕ_q. This finishes the argument, since the establishment of the Leibniz rule ensures that the product is well defined on cohomology. We specialize to the case that $n = p$ and $m = q$ to get the desired product relation on Bott-Chern cohomology. $\square$

Corollary 3.9.4: *We also have a product* $H^{p,q}_{\mathrm{BC}}(X, \mathbb{R}(p) \times H^{r,s}_{\mathrm{Ap}}(X, \mathbb{R}(r+1)) \to H^{p+r,q+s}_{\mathrm{Ap}}(X, \mathbb{R}(p+r+1)$*.*

Proof. This product is given by the formula established in the previous proof. It suffices to check that the degrees for Bott-Chern $\times$ Aeppli $\to$ Aeppli match with this product construction, which is immediate. $\square$

We do not have a product on Aeppli cohomology itself, since the degrees in this calculation do not match up with the degrees calculating Aeppli cohomology. This mirrors the original scenario in Section 3.1.1, where we lacked an Aeppli-only product.

By construction, we developed this product to mimic the product on Deligne cohomology in 3.9.5 defined previously. The explicit compatibility we want is expressed in the following proposition.

Proposition 3.9.5: *The following diagrams are commutative, where the hortizontal maps are the products just defined, and the vertical maps, componentwise, are* ϵ_C*.*

$$\begin{array}{ccc} H^{p,q}_{\mathrm{BC}}(X,\mathbb{R}(p)) \times H^{r,s}_{\mathrm{BC}}(X,\mathbb{R}(r)) & \rightarrow & H^{p+r,q+s}_{\mathrm{BC}}(X,\mathbb{R}(p+r)) \\ \downarrow & & \downarrow \\ H^{p+q}_{\mathcal{D}}(X,\mathbb{R}(p)) \times H^{r+s}_{\mathcal{D}}(X,\mathbb{R}(r)) & \rightarrow & H^{p+r+q+s}_{\mathcal{D}}(X,\mathbb{R}(p+r)) \end{array}$$

$$\begin{array}{ccc} H^{p,q}_{\mathrm{BC}}(X,\mathbb{R}(p)) \times H^{r,s}_{\mathrm{Ap}}(X,\mathbb{R}(r+1)) & \rightarrow & H^{p+r,q+s}_{\mathrm{Ap}}(X,\mathbb{R}(p+r+1)) \\ \downarrow & & \downarrow \\ H^{p+q}_{\mathcal{D}}(X,\mathbb{R}(p)) \times H^{r+s+1}_{\mathcal{D}}(X,\mathbb{R}(r+1)) & \rightarrow & H^{p+r+q+s+1}_{\mathcal{D}}(X,\mathbb{R}(p+r+1)) \end{array}$$

Proof. These compatibilities are almost automatic. In the above product calculations, we simply drop the third terms and recover the calculations for the product on Deligne cohomology. $\square$

Chapter 4

Flat Bundles and $\mathrm{Pic}^0(X)$

4.1 The Exponential Sequence

We begin this section with some exposition on bundles, Picard groups and Picard varieties. This material is standard, following [Huy05] and others. The only notable difference between the main references and our exposition is the inclusion of Tate-twisted coefficients.

The exponential map on functions gives a sheaf map $\mathcal{O}_X \to \mathcal{O}_X^\times$. This leads to the exponential short exact sequences of sheaves.

$$0 \to \mathbb{Z}(1) \to \mathcal{O}_X \stackrel{\exp}{\to} \mathcal{O}_X^\times \to 0 \tag{4.1.1}$$

A short exact sequence of sheaves produces a long exact sequence in sheaf cohomology.

$$
\begin{aligned}
0 \to & H^0(X, \mathbb{Z}(1)) \to H^0(X, \mathcal{O}_X) \to H^0(X, \mathcal{O}_X^\times) \\
\to & H^1(X, \mathbb{Z}(1)) \to H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X^\times) \\
\to & H^2(X, \mathbb{Z}(1)) \to \dots
\end{aligned} \tag{4.1.2}
$$

Definition 4.1.1: The connecting homomorphism from the exponential sequence is called the (integral) *Chern class*.

$$H^1(X, \mathcal{O}_X^\times) \xrightarrow{c_1} H^2(X, \mathbb{Z}(1)) \tag{4.1.3}$$

The classical description of this has $\mathbb{Z}$ coefficients. Our definition agrees up to multiplication by $(2\pi\imath)$.

Definition 4.1.2: The *Picard Variety*, written $\mathrm{Pic}^0(X)$, is the kernel of the Chern class map $c_1 : H^1(X, \mathcal{O}_X^\times) \to H^2(X, \mathbb{Z}(1))$.

Proposition 4.1.3: *The logarithm allows the following identification.*

$$\mathrm{Pic}^0(X) \cong \mathrm{Coker}(H^1(X, \mathbb{Z}(1)) \to H^1(X, \mathcal{O}_X)) \tag{4.1.4}$$

Proof. Since Equation 4.1.2 is a long exact sequence, we can identify $\mathrm{Pic}^0(X)$ with the image of $H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X^\times)$ which in turn is identified with the cokernel of the map $H^1(X, \mathbb{Z}(1)) \to H^1(X, \mathcal{O}_X)$. This identification is a logarithm by definition, since it is the inverse of an exponential operation. The logarithm is multivalued, which explains why the value of the logarithm is only defined in the quotient by $H^1(X, \mathbb{Z}(1))$. □

For projective algebraic manifolds, the structure of $\mathrm{Pic}^0(X)$ is well understood. The map $H^1(X, \mathbb{Z}(1)) \to H^1(X, \mathcal{O}_X)$ is the embedding of a lattice into a $\mathbb{C}$-vector space of half the lattice dimension, and the quotient is a $\mathbb{C}$-torus. This torus admits a polarization, making it an abelian variety. This fact is usually argued by way of the Hodge decomposition where we can identify $\mathrm{Pic}^0(X)$ as the quotient of $H^{0,1}(X)$ by the lattice $H^1(X, \mathbb{Z})$. The kernel of the Chern class map into $H^2(X, \mathbb{Z}(1))$ identifies which line bundles admit flat metrics, so that the abelian variety $\mathrm{Pic}^0(X)$ is exactly the group of isomorphism classes of flat line bundles. Finally, all line bundles over an projective algebraic manifold admit non-zero meromorphic sections.

It is not clear, for non-algebraic manifolds, which of the previous lists of results fail and how they fail. The remainder of this chapter tries to answer that question by understanding the structure of $\mathrm{Pic}^0(X)$ for non-algebraic manifolds.

4.2 Flat Bundles

The following definition is a specialized version of more general definitions of metrics on vector bundles, following [Lew04].

Definition 4.2.1: A *metric* on a line bundle $L = \{l_{ij}\}$ with respect to an open cover $\mathcal{U} = \{U_i\}$ is defined to be a collection of C^∞ functions $\rho_i : U_i \to (0, \infty)$ such that $\rho_i \, |l_{ij}|^2 = \rho_j$ on intersections $U_i \cap U_j$.

This metric gives an absolute value on meromorphic sections. Locally, this has the form $|f_i| = \overline{f_i} \rho_i f_i$. An easy calculation shows that this absolute value does not depend on the local description.

Definition 4.2.2: A metric ρ_i on L is called *flat* if the $\partial\overline{\partial}\log\rho_i = 0$. The differential form $\partial\overline{\partial}\log\rho_i$ is called the *curvature form*.

Definition 4.2.3: If $L = \{l_{ij}\}$ is a line bundle over X, then any metric ρ_i on L defines a class $[2\pi\imath\partial\overline{\partial}\log\rho_i] \in H^{1,1}_{\mathrm{BC}}(X, \mathbb{C})$. This defines a map $H^1(X, \mathcal{O}_X) \to H^{1,1}_{\mathrm{BC}}(X, \mathbb{C})$ which is called (somewhat redundantly) the *Bott-Chern Chern class.*

Proposition 4.2.4: *The Bott-Chern Chern class is well defined, i.e. this class is independent of the metric chosen. Furthermore, L admits a flat metric if and only if this class vanishes.*

Proof. If σ_i is another metric on $L = \{l_{ij}\}$, then we can calculate the difference between the two Bott-Chern Chern classes.

$$2\pi\imath\partial\overline{\partial}\log\rho_i - 2\pi\imath\partial\overline{\partial}\sigma_i = 2\pi\imath\partial\overline{\partial}\log\frac{\rho_i}{\sigma_i} \tag{4.2.1}$$

But $\log\frac{\rho_i}{\sigma_i}$ is a C^∞ function on X. Therefore, the difference due to a change in metrics is a $\partial\overline{\partial}$-exact form. Since Bott-Chern cohomology is defined modulo $\partial\overline{\partial}\mathcal{A}^0(X)$, we conclude the Bott-Chern Chern class is well-defined.

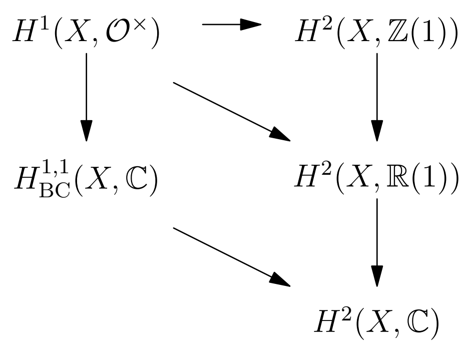
If ρ_i is a flat metric, then by definition $2\pi\imath\partial\overline{\partial}\log\rho_i = 0$ so the class is zero. Conversely, if the class $[2\pi\imath\partial\overline{\partial}\log\rho_i] = 0$, then $2\pi\imath\partial\overline{\partial}\log\rho_i = 2\pi\imath\partial\overline{\partial}f$ for some global C^∞ $\mathbb{R}$-valued function f. It is easy to see that ρ_i/e^f is also a metric on L. The following calculation shows that ρ_i/e^f is a flat metric.

$$2\pi\imath\partial\overline{\partial}\log\rho_i/e^f = 2\pi\imath\partial\overline{\partial}\log\rho_i - 2\pi\imath\partial\overline{\partial}\log e^f = 0 \tag{4.2.2}$$

□

In addition to the Chern class maps into $H^2(X, \mathbb{Z}(1)$ and $H_B^{1,1}C(X, \mathbb{C})$, there is also the Chern class $H^1(X, \mathcal{O}_X^\times) \to H^2(X, \mathbb{R}(1))$ defined by the curvature of a connection. All three target spaces admit maps to $H^2(X, \mathbb{C})$, so we can consider what compatibilities exists among the Chern classes.

Proposition 4.2.5: *Up to a sign, the following is a commutative diagram involving all three Chern classes. The map $H^{1,1}_{\mathrm{BC}}(X, \mathbb{C}) \to H^2(X, \mathbb{C})$ is the map defined in Definition 3.1.3.*



Proof. The compatibility of the upper right triangle is found in Proposition 4.4.12 of [Huy05], up to a sign or multiplication by $2\pi\imath$ where necessary. The compatibility of the lower part of the diagram follows from the fact that both Chern characters can be calculated by the class $[2\pi\imath\partial\overline{\partial}\log\rho_i]$, and the inclusion into $H^2(X, \mathbb{C})$ preserves this class. □

Proposition 4.2.6: *Up to a quotient by the image of $H^1(X, \mathbb{R}(1))$, the Bott-Chern Chern class factors through $H^{1,1}_{\mathrm{BC}}(X, \mathbb{R}(1))$.*

Proof. This is just an application of Proposition 3.7.3, since π_0 of the the curvature form vanishes. □

We can make an addition the commutative diagram based on this proposition.

$$\begin{array}{ccc}
H^1(X,\mathcal{O}^\times) & \longrightarrow & H^2(X,\mathbb{Z}(1)) \\
\downarrow & \searrow & \downarrow \\
\frac{H^{1,1}(X,\mathbb{R}(1))}{H^1(X,\mathbb{R})} & & H^2(X,\mathbb{R}(1)) \\
\downarrow & & \downarrow \\
H^{1,1}_{\mathrm{BC}}(X,\mathbb{C}) & \longrightarrow & H^2(X,\mathbb{C})
\end{array}$$

Having formed some understanding of Chern classes, we want to understand how they relate to flat line bundles.

Definition 4.2.7: FLB' is the group of flat line bundles. That is, $\mathrm{FLB}' := \mathrm{Ker}(H^1(X,\mathcal{O}_X^\times) \to H^{1,1}_{\mathrm{BC}}(X,\mathbb{C}))$.

For projective algebraic manifolds, it is well known that $\mathrm{FLB}' = \mathrm{Pic}^0(X)$. However, we have no guarantee that the kernels of the Bott-Chern and integral Chern classes correspond. Since both kernels are important to our development, we impose both restrictions.

Definition 4.2.8: $\mathrm{FLB} := \mathrm{FLB}' \cap \mathrm{Pic}^0(X)$.

Our constructions in the following chapters will depend on meromorphic sections of line bundles. For non-algebraic manifolds, line bundles without any non-zero meromorphic sections may exist. We wish to exclude these.

Definition 4.2.9: FLS is the subgroup of FLB containing those line bundles which admit at least one non-trivial meromorphic section.

The fact that FLS is a subgroup is easy to see. The group structure comes from the tensor product of line bundles, described by multiplication of the transition functions. Since meromorphic sections are equivalent to Cartier divisors, we can take their product as Cartier divisors. That product is a meromorphic section over the tensor product of the line bundles.

Using the identification of meromorphic sections and Cartier divisors, we will often treat FLS as a group of Cartier divisors whose corresponding line bundles are flat with vanishing integral Chern classes. This only caution in doing so is that FLS is defined to be a group of isomorphism classes, and the group of Cartier divisors has many elements corresponding to the same isomorphism class.

4.3 The Structure of $\mathrm{Pic}^0(X)$

In order to investigate the structure of $\mathrm{Pic}^0(X)$, recall we made the identification $\mathrm{Pic}^0(X) = \mathrm{Coker}(H^1(X, \mathbb{Z}(1)) \to H^1(X, \mathcal{O}_X))$ in Proposition 4.1.3. The first three terms in the long exact sequence associated with the exponential sequence are as follows.

$$0 \to \mathbb{Z}(1) \to \mathbb{C} \to \mathbb{C}^\times \to \tag{4.3.1}$$

This is an exact sequence, so the next map factors through 0. Therefore, the map $H^1(X, \mathbb{Z}(1)) \to H^1(X, \mathcal{O}_X)$ is injective (and that $H^1(X, \mathbb{Z}(1))$ is a torsion-free abelian group). Since this map is injective, we identify $H^1(X, \mathbb{Z}(1))$ with its image, allowing a new description of $\mathrm{Pic}^0(X)$.

$$\mathrm{Pic}^0(X) \cong \frac{H^1(X, \mathcal{O}_X)}{H^1(X, \mathbb{Z}(1))} \tag{4.3.2}$$

$H^1(X, \mathcal{O}_X)$ is a complex vector space and $H^1(X, \mathbb{Z}(1))$ is a finite rank free abelian group. We need to understand how the latter is embedded in the former.

Proposition 4.3.1: *$H^1(X, \mathbb{Z})$ is discrete in $H^1(X, \mathcal{O}_X)$. More precisely, any basis for $H^1(X, \mathbb{Z}(1))$ is $\mathbb{R}-$linearly independent and $H^1(X, \mathbb{Z}(1))$ forms a lattice in $H^1(X, \mathcal{O}_X)$.*

Proof. Consider the following short exact sequence of sheaves.

$$0 \to \mathbb{C} \to \mathcal{O}_X \xrightarrow{d} \hat{\Omega}^1 \to 0 \tag{4.3.3}$$

The start of the corresponding long exact sequence on cohomology is as follows.

$$0 \to H^0(X, \mathbb{C}) \to H^0(X, \mathcal{O}_X) \to H^0(X, \hat{\Omega}^1) \to H^1(X, \mathbb{C}) \to H^1(X, \mathcal{O}_X) \tag{4.3.4}$$

The first two groups are both isomorphic to $\mathbb{C}$, so the third map factors through zero. Therefore, $H^0(X, \hat{\Omega}^1) \hookrightarrow H^1(X, \mathbb{C})$ is an injective map which identifies $H^0(X, \hat{\Omega}^1)$ with its image. (This connecting morphism is nothing more than the map which sends a global closed holomorphic 1-form to its De Rham class.) Then the map from the quotient $H^1(X, \mathbb{C})/H^0(X, \hat{\Omega}^1)$ into $H^1(X, \mathcal{O}_X)$ is injective. We conclude that any subgroup of $H^1(X, \mathbb{C})$ injects into $H^1(X, \mathcal{O}_X)$ if it intersects $H^0(X, \hat{\Omega}^1)$ trivially.

Elements of the subgroup $H^1(X, \mathbb{R}(1))$ are classes of $\mathbb{R}$-valued closed forms, according to the De Rham description. Let ω be a form in $H^0(X, \hat{\Omega}^1)$ and assume that its class $[\omega]$ in $H^1(X, \mathbb{C})$ is equal to the class of a Tate twisted real form $[\eta]$. Equality in $H^1(X, \mathbb{C})$ implies that there exists a C^∞ function

$f : X \to \mathbb{C}$ such that $\omega = \eta + df$.

This is an equation of forms, so it preserves type. The form ω is of pure type $(1, 0)$, which gives these relations.

$$\omega = \eta^{1,0} + \partial f, \qquad 0 = \eta^{0,1} + \overline{\partial} f \tag{4.3.5}$$

We assumed that η is a real form, so $\eta^{0,1} = \overline{\eta^{1,0}}$. We substitute $\eta^{0,1} = -\overline{\partial} f$ and $\eta^{1,0} = \overline{-\overline{\partial} f}$ into the previous equation.

$$\omega = -\overline{\overline{\partial} f} + \partial f = \partial f - \partial \overline{f} = \partial \left(f - \overline{f} \right) \tag{4.3.6}$$

This calculation shows that ω is ∂-exact. Since ω is holomorphic ($\overline{\partial}\omega = 0$), we have $\overline{\partial}\partial(f - \overline{f}) = 0$, *i.e.* the function $f - \overline{f}$ is a $\partial\overline{\partial}$-closed function.

Any such function g with $\partial\overline{\partial} g = 0$ is an element in $H^{0,0}_{\mathrm{Ap}}(X, \mathbb{C})$, which is dual to $H^{n,n}_{\mathrm{BC}}(X, \mathbb{C})$. But this Bott-Chern cohomology is just $\mathbb{C}$ (relying on our assumption that X is compact). Therefore, we have only a one dimensional complex space of functions satisfying $\partial\overline{\partial} g = 0$, and we conclude that g must be constant.

Returning to ω, we see that $f - \overline{f}$ is a constant function, and $\omega = \partial(f - \overline{f}) = 0$. The class of ω was assumed to be in the intersection of $H^0(X, \hat{\Omega}^1)$ and $H^1(X, \mathbb{R}(1))$. By proving that ω vanishes, we have established that the two vector spaces intersect trivially. We conclude that $H^1(X, \mathbb{R}(1))$ injects into $H^1(X, \mathcal{O}_X)$.

The map $H^1(X, \mathbb{Z}(1)) \to H^1(X, \mathcal{O})$ is induced by inclusion, so it factors through $H^1(X, \mathbb{R}(1))$. Therefore, a basis for $H^1(X, \mathbb{Z}(1))$ must be $\mathbb{R}$-linearly independent in $H^1(X,\mathcal{O})$, *i.e.* $H^1(X,\mathbb{Z}(1))$ forms a lattice in $H^1(X,\mathcal{O})$. □

For projective algebraic manifolds, we can argue that this lattice has full rank. Without an algebraic assumption, we lose this result; the following proposition is as much as we can say.

Corollary 4.3.2: *The group* $\mathrm{Pic}^0(X)$ *is the product of a real torus and a finite dimensional real vector space.*

$$\mathrm{Pic}^0(X) \cong \mathbb{R}^r \oplus T^s \tag{4.3.7}$$

Where $T^s \cong H^1(X,\mathbb{R}(1))/H^1(X,\mathbb{Z}(1))$ *and* r, s *are the real dimensions of the pieces. This isomorphism is non-canonical.*

We might have hoped for this to be a decomposition into a complex torus and a complex vector space, but this expectation is not reasonable. In particular, if the first Betti number, b_1, is odd, then the lattice has odd rank and the quotient can never have a complex structure. We know that for Kähler manifolds, b_1 is always even, but we are interested in non-Kähler examples. Hopf manifolds, for examples, have odd first Betti numbers.

4.4 Constant Transition Functions

Take $L = \{l_{ij}\} \in$ FLS with a meromorphic section (or Cartier divisor) σ_i. Recall that the local functions σ_i satisfy $\sigma_i = \sigma_j l_{ij}$ on intersections of the open cover. L has a flat metric is given by ρ_i and we write $|\sigma_i|$ for the absolute value of σ given by the metric. The following idea is from [Lew04].

Proposition 4.4.1: *For $L \in \mathrm{FLS}$, we can choose the transition functions l_{ij} to be constants in $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$.*

Proof. Lemma 5.3 in [Lew04] tells us that $|\sigma_i|$ is locally of the form $h_i\overline{h_i}$ where h_i is a meromorphic function on the open set U_i. (We refine the open cover $\mathcal{U} = \{U_i\}$ if necessary, so that U_i and $U_i \cap U_j$ are isomorphic to simply connected open sets in $\mathbb{C}^n$). Then, on intersections $U_i \cap U_j$, we consider the meromorphic function h_i/h_j. As the quotient of meromorphic functions, this is $\overline{\partial}$-closed. However, by the identity $|\sigma_i| = |\sigma_j|$, the norm is preserved on the intersection. This implies that $h_i\overline{h_i} = h_j\overline{h_j}$. We can write this equality as follows.

$$\frac{h_i}{h_j} = \overline{\left(\frac{h_j}{h_i}\right)} = \overline{\left(\frac{h_i}{h_j}\right)^{-1}} \tag{4.4.1}$$

The middle term is antiholomorphic, so it is ∂-closed; we conclude that h_i/h_j is ∂-closed as well. This implies that h_i/h_j must be locally constant. Write $c_{ij} = h_i/h_j$ for this constant. Then the left hand side of the above equation implies that $c_{ij} = \overline{c_{ij}^{-1}}$, which translates to $|c_{ij}| = 1$, so the c_{ij} are constant complex numbers of norm 1.

The c_{ij} satisfy $c_{ij}c_{jk} = c_{ik}$ on triple intersections, so they form a Čech cocycle in $C^1(\mathcal{U}, S^1)$ and a class in $H^1(X, S^1)$. By the inclusion of $S^1 \hookrightarrow \mathcal{O}_X^\times$, the c_{ij} define a class in $H^1(X, \mathcal{O}_X^\times)$, hence the transition functions for a line bundle.

What line bundle is this? The meromorphic functions h_i, by definition, satisfy the conditions of a meromorphic section of this line bundle. But $|\sigma_i| = h_i\overline{h_i}$ implies that the associated Weil divisor satisfies $\mathrm{div}(\sigma_i) = \mathrm{div}(h_i)$. Therefore, the line bundle defined by the c_{ij} is the line bundle of the divisor $\mathrm{div}(\sigma_i)$. That is just the original line bundle L, since σ is a Cartier divisor associated with

L, which completes the proof. □

The following proposition establishes an important property of line bundles with constant transition functions.

Proposition 4.4.2: *In terms of the decomposition* $\mathrm{Pic}^0(X) \cong \mathbb{R}^r \bigoplus T^s$ *into a vector space and a torus from 4.3.2, line bundles with constant transition functions lie in the torus, i.e.* $\mathrm{FLS} \subset T^s$.

Proof. We have a short exact sequence of constant sheaves.

$$0 \to \mathbb{Z}(1) \to \mathbb{R}(1) \to \mathbb{R}(1)/\mathbb{Z}(1) \to 0 \tag{4.4.2}$$

By the logarithm, we identify S^1 with $\mathbb{R}(1)/\mathbb{Z}(1)$. This gives a long exact sequence in sheaf cohomology.

$$\begin{aligned} 0 \to H^1(X, \mathbb{Z}(1)) \to H^1(X, \mathbb{R}(1)) \to H^1(X, S^1) \to \\ \to H^2(X, \mathbb{Z}(1)) \to H^2(X, \mathbb{R}(1)) \to \dots \end{aligned} \tag{4.4.3}$$

The kernel of this last map is torsion, so we truncate to get a short exact sequenece of cohomology groups.

$$0 \to \frac{H^1(X, \mathbb{R}(1))}{H^1(X, \mathbb{Z}(1))} \to H^1(X, S^1) \to H^2_{\mathrm{tor}}(X, \mathbb{Z}(1)) \to 0 \tag{4.4.4}$$

This is totally compatible with the exponential sequence, *i.e.* the following diagram is commutative.

$$\begin{array}{ccccccccc}
0 & \to & \frac{H^1(X,\mathbb{R}(1))}{H^1(X,\mathbb{Z}(1))} & \overset{\exp}{\hookrightarrow} & H^1(X,S^1) & \to & H^2_{\mathrm{tor}}(X,\mathbb{Z}(1)) & \to \\
 & & \downarrow & & \downarrow & & \downarrow & \\
0 & \to & \mathrm{Pic}^0(X) = \frac{H^1(X,\mathcal{O}_X)}{H^1(X,\mathbb{Z}(1))} & \overset{\exp}{\hookrightarrow} & H^1(X,\mathcal{O}_X^\times) & \to & H^2(X,\mathbb{Z}(1)) & \to
\end{array}$$

The commutativity of this diagram follows from the observation that the map $H^1(X, S^1) \to H^2_{\mathrm{tor}}(X, \mathbb{Z}(1))$ is the classical Chern class map. By assumption, the line bundles in FLB have zero Chern class. Therefore, by the short exact sequence in Equation 4.4.4, they come from classes in $H^1(X, \mathbb{R}(1))/H^1(X, \mathbb{Z}(1))$. We conclude that line bundles with transition functions in $H^1(X, S^1)$ come from the map $H^1(X, \mathbb{R}(1))/H^1(X, \mathbb{Z}(1)) \hookrightarrow H^1(X, \mathcal{O}_X)/H^1(X, \mathbb{Z}(1))$. That target space is the torus part of $\mathrm{Pic}^0(X)$, completing the proof. □

Since all line bundles in FLS have constant transition functions by Proposition 4.4.2, we conclude the FLS $\subset T^s$, where T^s is the torus part of $\mathrm{Pic}^0(X)$.

As a curious aside, the constant S^1-valued transition functions satisfy $|l_{ij}|^2 = 1$. This implies that the metric ρ for L satisfies $\rho_i |l_{ij}|^2 = \rho_i$, which is $\rho_i = \rho_j$. Therefore, the ρ_i actually define a global C^∞ function $\rho : X \to (0, \infty)$. Moreover, any such function which is closed under $\partial\overline{\partial}\log$ satisfies the metric condition.

Though we have FLS $\subset T^s$, it is not clear what relationship FLS has to the torus. This question is intriguing for the parallel with the projective algebraic case; we are curious how close we can get to a non-algebraic analogue of the Poincaré bundle. The following is a conjecture in that direction.

Conjecture 4.4.3: FLS is a real sub-torus of T^s.

Though the conjecture is open, the following argument might be the start of a proof. We can consider a projection map onto the real torus.

$$H^1(X,\mathbb{R}) \xrightarrow{\pi} \frac{H^1(X,\mathbb{R})}{H^1(X,\mathbb{Z})} \tag{4.4.5}$$

Then the inverse image $\pi^{-1}(\text{FLS})$ is a subgroup of $H^1(X,\mathbb{R})$. A necessary condition for FLS to be a subtorus is that this subgroup is a subvector space, *i.e.* $\pi^{-1}\text{FLS}$ is preserved under scaling by real numbers.

If we take a class $[r_{ij}]$ in $H^1(X,\mathbb{R})$, the transition functions for the corresponding line bundle are $e^{2\pi\imath r_{ij}}$. If $\alpha \in \mathbb{R}$ is a scalar, then α acts on $[r_{ij}]$ by multiplication. The action on the transition function is then $e^{2\pi\imath\alpha r_{ij}}$ takes l_{ij} to $(l_{ij})^\alpha$. This is still a constant non-zero function, hence defines a class in $H^1(X,\mathcal{O}_X^\times)$, so these are the transition functions of a line bundle. Moreover, this line bundle still admits a flat metric.

At this point, the next step would be to argue that $\pi^{-1}\text{FLS}$ is a lattice-invariant subspace. We are not familiar with any technique that immediately addresses this step, so the statement is left as a conjecture.

A similar conjecture would be to consider the subset of FLS consisting of densely stably trivial bundles, which will be defined shortly in Section 4.6. These line bundles more closely mimic the Poincaré bundle; perhaps they are more likely to have a sub-torus structure.

4.5 Families of Flat Line Bundles

Proposition 2.2.21 ensures that direct products of complex spaces exist. Moreover, Section 10.5 of [GR84] uses the ideas expressed in [Kod86] to construct proper families of complex spaces. We can make use of those constructions to build families of flat line bundles. Though families of complex spaces are defined as the fibres of proper maps, we can use a more direct method for understanding a family of line bundles on a fixed complex space. Let X be a complex space and Δ an open polydisc about 0 in $\mathbb{C}^n$. We define families of line bundles by working over the trivial family of complex spaces formed by the product $X \times \Delta$, with the projection onto Δ.

Definition 4.5.1: A *local analytic family of line bundles with sections* is a global Cartier Divisor $A \in \mathcal{M}^{\times}_{X\times\Delta}/\mathcal{O}^{\times}_{X\times\Delta}(X \times \Delta)$ which restricts to a Cartier Divisor in each of the fibres X_t.

We only care about a small neighbourhood of $0 \in \Delta$; to that end, we shrink Δ as necessary to preserve local properties. We have the following useful description of this Cartier divisor. Given a suitably fine open cover of X, $\mathcal{U} = \{U_i\}$, and assuming that we shrink Δ as necessary to a smaller open polydisc, the open sets $U_i \times \Delta$ form a suitably fine open cover for the trivial family. Then the Cartier divisor A is described by non-zero meromorphic functions A_i on $U_i \times \Delta$ such that the quotients A_i/A_j are nowhere vanishing holomorphic functions on the intersections $U_i \cap U_j \times \Delta$.

This description of the Cartier divisor leads to a more intuitive notion of families of line bundles. On each fibre X_t, the meromorphic functions A_i restrict to meromorphic functions on $U_i \times \{t\}$, which we identify with U_i. Since we assumed that the global Cartier divisor restricted to a Cartier divisor on each fibres X_t, this restriction is neither the zero function nor everywhere a pole, so it defines a non-zero meromorphic function. The agreements between the A_i are preserved, so this gives a well defined Cartier divisor on $X_t \cong X$. The line bundle over this fibre is the line bundle with transition functions given by A_i/A_j restricted to the open intersections of the fibre. The Cartier divisor itself naturally restricts to a non-zero section of this line bundle.

Definition 4.5.2: A *family of metrics* on a family of line bundles defined by a Cartier Divisor A is a collection of C^∞ functions $\rho_i : U_i \times \Delta \to (0, \infty)$ which satisfy the compatibility condition of a metric on the global line bundle associated with A and restrict to metrics on the line bundles over each fibre.

Having families of metrics, we can talk about flatness and Chern classes. In the construction of families of line bundles, we can limit our scope by insisting that all fibres satisfy particular conditions. In particular, we have this definition.

Definition 4.5.3: A *family of* FLS *line bundles* on X is a family of line bundles where all fibres are Cartier divisors corresponding to flat line bundles with zero Chern class, *i.e.* bundles in FLS.

We need the following important result.

Proposition 4.5.4: *Assume we have a family of line bundles over $X \times \Delta$ described by a Cartier divisor A. If σ is a meromorphic section of the line bundle over* 0*, then there exists a Cartier divisor $\tilde{A}$ over $X \times \Delta$ such that $\tilde{A}$ defines the same family of line bundles and $\tilde{A}$ restricts to the section σ in the fibre over* 0.

Proof. The Cartier divisor A is defined by A_i, which are meromorphic functions on $U_i \times \Delta$. We can write $A_i = A_i(x, z)$ in terms of the variables in this direct product. Then we can make this definition.

$$\tilde{A}_i(x, z) := \frac{A_i(x, z)\sigma_i(z)}{A_i(x, 0)} \tag{4.5.1}$$

$\tilde{A}_i$ is still a meromorphic function on $U_i \times \Delta$. On the intersections $U_i \cap U_j \times \Delta$ the following is true.

$$\frac{\frac{A_i(x,z)\sigma_i(z)}{A_i(x,0)}}{\frac{A_j(x,z)\sigma_j(z)}{A_j(x,0)}} = \frac{\sigma_i(x)}{\sigma_j(x)} \frac{A_j(x, 0)}{A_i(x, 0)} \frac{A_i(x, z)}{A_j(x, z)} \tag{4.5.2}$$

This is still a nowhere vanishing holomorphic function. Restricted to the fibre over z, this product of three transition functions gives the line bundle $L_0 \otimes L_0^{-1} \otimes L_z = L_z$, so $\tilde{A}$ defines the same family of line bundles as A.

By construction, the $\mathcal{A}(x, 0)$ terms cancel when restricted to the fibre over 0, giving the section σ_i. □

If we have a family of line bundles, we can realize any section over the central fibre as the restriction of a section over the whole family. In particular, we can use limits; we can realize a central section as the limit of sections over a sequence of points converging to 0 in the polydisc Δ.

Lastly, we wish to construct the product of a family of line bundles.

Definition 4.5.5: Consider two families defined by Cartier divisors A and B on trivial families $X \times \Delta_A$ and $X \times \Delta_B$, respectively. The product $\Delta_A \times \Delta_B$ is a polydisc in a larger dimensional complex vector space. We can form the trivial family $X \times \Delta_A \times \Delta_B$. Let z_A and z_B be local coordinates of Δ_A and Δ_B

respectively, and define a Cartier divisor AB over this family by the following formula.

$$AB(x, z_A, z_B) = A(x, z_A)B(x, z_B) \tag{4.5.3}$$

The *product of the two families of line bundles* is the family of line bundles associated with the Cartier divisor AB.

Since A and B are Cartier divisors, AB is also a Cartier divisor. Moreover, AB can only have zeros matching the fibres of the trivial family if either A or B vanish along the fibres of their trivial families. The same is true for poles. Since neither A nor B are zero or polar along any fibre, the divisor AB defines a valid family of line bundles. It is easy to see that the line bundle in the central fibre of AB is the tensor product of the central fibres of each of the two factors.

4.6 Stably Trivial Families

Though the term 'stably trivial' does not appear in the paper, the following property is important in [Lew04] and it behooves us to give it a name.

Definition 4.6.1: A line bundle L over an analytic variety Z is *stably trivial* if there exists a finite covering $\widetilde{Z} \xrightarrow{\phi} Z$ such that ϕ^*L is the trivial line bundle over $\tilde{Z}$.

Definition 4.6.2: A local family of line bundles given by a Cartier divisor A over $X \times \Delta$ is called *densely stably trivial* if there exists a dense subset

$D \subset \Delta$ such that the line bundles over all fibres X_d, $d \in D$ are stably trivial line bundles. We write DST for the set of such bundles.

The following proposition is necessary for the development in the next chapter.

Proposition 4.6.3: DST *is a subgroup of* FLS*.*

Proof. Consider two stably trivial families defined by Cartier divisors A and B on trivial families $X \times \Delta_A$ and $X \times \Delta_B$, respectively. The product construction in Definition 4.5.5. gives the product of these families, with Cartier divisor AB. The product of dense subsets S_A of Δ_A and S_B of Δ_B is dense in the product, and the line bundles over the dense set $S_A \times S_B$ are the products of stably trivial bundles, which remain stably trivial. Therefore, this is a densely stably trivial family. □

We will often think of DST as a group of Cartier divisors. The definitions and constructions here are compatible with such a perspective, and the group structure is very easily realized by the product of Cartier divisors. However, this is a fundamentally different definition, since many Cartier divisors correspond to the same isomorphism class of line bundles. To recover the original description of DST, it is necessary to quotient by the relation that identifies two divisors if they correspond to isomorphic line bundles.

Chapter 5

Analytic Twisted Cycle Classes

5.1 Analytic Twisted Cycles

The constructions in this chapter are generalizations of the ideas in [Lew01] and [Lew04] on Milnor K-Theory and related invariant theories on projective algebraic manifolds. We frequently look to those papers, as well as original sources on Milnor K-Theory, for definitions and inspiration.

Definition 5.1.1: Let Z be an irreducible analytic subvariety of a compact complex manifold X. The *group of analytic twisted 1-cycles over* Z is the following.

$$A^1_Z = \bigoplus_L \mathcal{M}^\times_L \tag{5.1.1}$$

The sum runs over all line bundles $L \in \text{DST}$ over Z. If we interpret DST as a group of Cartier divisors, then the identification of Cartier divisors and meromorphic sections implies that $A^1_Z = \text{DST}$.

We can consider the tensor algebra associated with this group.

$$T(A_Z^1) := \sum_{i=0}^{\infty} (A_Z^1)^{\otimes j} \tag{5.1.2}$$

If we think of working with sections of line bundles, we must take note that there are two tensor products. The tensor product of line bundles and their sections defines the abelian group structure of A_Z^1. The tensor product of abelian groups is the multiplication in the tensor algebra. We must be careful not to confuse the two tensor products. However, working with Cartier divisors avoids this confusion, since the tensor products of the underlying line bundles become implicit.

$T(A_Z^1)$ is a graded ring. In analogue with the Milnor K-theory of a field, define R to be the two-sided graded ideal generated by all elements of the form $\sigma \otimes (-\sigma)$ for $\sigma \in A_Z^1$ and $(1-f) \otimes f$ for $f \in \mathcal{M}(Z)^\times$. We treat meromorphic functions as Cartier divisors corresponding to the trivial line bundles.

Then $T(A_Z^1)/R$ is a graded quotient ring.

$$A_Z^\bullet = \frac{\left(\sum_{j=0}^{\infty} (A_Z^1)^{\otimes j}\right)}{R} \tag{5.1.3}$$

The bullet indicates the grading, and we write A_Z^m for the module of elements of degree m. Note that this definition agrees with A_Z^1, since the relations start in degree two. Also note that $A_Z^0 = \mathbb{Z}$.

Definition 5.1.2: The *group of analytic twisted m-cycles* is defined as follows.

$$\underline{z}^k(X,m) := \bigoplus_{\substack{\text{codim}\\ Z=k}} A_Z^m \tag{5.1.4}$$

Taking inspiration from Milnor K-theory, we can clarify a notation for elements of $\underline{z}^k(X,m)$. Such an element can be written as a sum of basic elements, which have the following notation.

$$(Z, \{L_1, \rho_1, \sigma_1; L_2, \rho_2, \sigma_2; \ldots; L_m, \rho_m, \sigma_m\}) \tag{5.1.5}$$

In this notation, Z is some fixed subvariety of codimension k, each L_i is a line bundle over Z_i with flat metric ρ_i and meromorphic section σ_i. To avoid confusion with notation from the previous chapters, note that the subscripts on the σ_i are the indices of the iterated tensor product, not the indices of local sections over an open cover.

However, we prefer to work with a Cartier divisor description, which leads to this simplified notation.

$$\{\sigma_1, \sigma_2, \ldots, \sigma_m\}_Z \tag{5.1.6}$$

This notation is appropriate from the perspective of Cartier divisors: the σ_i are Cartier divisors and the only extra information required is the subvariety Z. This makes the supression of the line bundles and metrics more natural, particularly since the Cartier divisor uniquely determines the line bundle and we shall argue that all future constructions are independent of the choice of a flat metric.

5.2 Analytic Twisted Milnor Complex

The details of the Tame symbols on Milnor complexes originally come from [BT73] and are explored in detail in [Lew04]. Following the latter reference, since our version of a real regulator will also ignore torsion, we simplify our exposition and work modulo 2-torsion.

Definition 5.2.1: Let D be an irreducible analytic subvariety of X of codimension $k+1$. We define the *D-Tame symbol* by the following formula.

$$\begin{aligned} T_D^m : \underline{z}^k(X,m) &\to \underline{z}^{k+1}(X,m-1) \\ \{\sigma_1,\dots,\sigma_m\}_Z &\mapsto \sum_{j=1}^{m}(-1)^{(m-j)}\nu_D(\sigma_j)\{\left.\sigma_1\right|_D,\dots,\left.\widehat{\sigma_j}\right|_D,\dots,\left.\sigma_m\right|_D\}_D \end{aligned} \tag{5.2.1}$$

This formula holds whenever $D \subset Z$. If $D \not\subset Z$, then $T_D^m(\{\sigma_1,\dots,\sigma_m\}_Z) = 0$. After being defined on symbols, the formula is extended by linearity to all of $\underline{z}^k(X,m)$.

Definition 5.2.2: The complete *Tame symbol* is the sum of all these D-Tame symbols.

$$\begin{aligned} T^m : \underline{z}^k(X,m) &\to \underline{z}^{k+1}(X,m-1) \\ T^m &:= \bigoplus_D T_D^k \end{aligned} \tag{5.2.2}$$

The sum is taken over all irreducible analytic subvarieties of codimension 1 in Z.

Proposition 5.2.3: *This Tame symbol is well defined.*

Proof. The proof follows [BT73] and [Lew04].

The approach of the proof is to restrict to suitable open sets where we can apply the results of [BT73] on the Milnor K-Theory of fields. To that end, let $\mathcal{U} = \{U_\alpha\}$ be a suitable open cover, as per Proposition 2.2.28. We restrict our attention to a specific subvariety Z, with D of codimension 1 in Z, and consider T_D acting on $A_Z^\bullet$. Let $\{f_1, \ldots, f_m\}_Z \in A_Z^m$ be a symbol. (Note that throughout this proof we use Latin indices for the symbol index and Greek indices for the open cover index.)

The suitable open cover allows us to treat Cartier divisors f_j over Z as local meromorphic functions in $\mathbb{C}(U_\alpha)$. The advantage of dealing with the problem locally is that $\mathbb{C}(U_\alpha)$ is a field, and the results in [BT73] on fields apply.

On the field $\mathbb{C}(U_\alpha)$, the order of vanishing along $D \cap U_\alpha$ is a discrete valuation, which we will write as $\nu_{D,\alpha}$. Choose a function $\phi \in \mathbb{C}(U_\alpha)$ defining $D \cap U_\alpha$ such that $\nu_{D,\alpha}(\phi) = 1$. According to [BT73], this discrete valuation gives rise to a map on Milnor K-theory.

$$\partial_\phi : K_M^\bullet \mathbb{C}(Z) \to K_M^\bullet k(\nu_{D,\alpha}) \langle \Pi \rangle \tag{5.2.3}$$

Π is a variable that satisfies $\Pi^2 = (-1)\Pi$, and $A\langle \Pi \rangle$ is the free module over A with basis $1, \Pi$. (This can be described more carefully using κ-algebras, but we omit those details.)

The map ∂_ϕ is a homomorphism. It is induced by the map defined on K_M^1 and the product in the κ-algebra. More precisely, ∂_ϕ restricted to $K_M^1 = \mathbb{C}(Z)^\times$ can be described as follows. If $f \in \mathbb{C}(Z)^\times$, then the discrete valuation implies that there exists unique $u \in \mathbb{C}(Z)^\times$ and $r \in \mathbb{Z}$ such that $f = u\phi^r$ and $\nu_{D,\alpha}(u) = 0$. Then the following formula gives the map ∂_ϕ.

$$\partial_\phi(f) = l(\overline{u}) + r\Pi \tag{5.2.4}$$

The notation $\overline{u}$ refers to the restriction of u to the residue field of $\nu_{D,\alpha}$, namely $k(\nu_{D,\alpha})$, and $l(\overline{u})$ is the element in Milnor K-theory associated with $\overline{u} \in k(\nu_{D,\alpha})^\times$.

If we write $f_i = u_i\phi^{r_i}$, we can calculate the value for ∂_ϕ on higher rank symbols using the κ-algebra product.

$$\partial_\phi\{f_1, \ldots, f_m\} = \prod_{i-1}^{m}(l(\overline{u}_i) + r_i\Pi) \tag{5.2.5}$$

This product is calculated using the fact that $\Pi^2 = l(-1)\Pi$. However, to simplify our situation, we impose the condition $\Pi^2 = 0$ by taking everything modulo 2-torsion, *i.e.* imposing the relation $l(-1) = 0$. With this simplification, we can directly determine the product formula.

$$\partial_\phi\{f_1, \ldots, f_m\} = \{\overline{u}_1, \ldots \overline{u}_m\} + \left[\sum_{j=1}^{m}(-1)^{m-j} r_j\{\overline{u}_1, \ldots, \widehat{\overline{u}_j}, \ldots, \overline{u}_m\}\right]\Pi \tag{5.2.6}$$

The coefficient of Π in the previous formula is an element of $K_M^{m-1}k(\nu_{D,\alpha})$, which is denoted $\partial_{\nu_{D,\alpha}}(\{f_1, \ldots, f_m\})$. This is precisely the same formula that defined our Tame symbol. Therefore, when restricting to the open set U_α we conclude that $T\{f_1, \ldots, f_m\} = \partial_{\nu_{D,\alpha}}(\{f_1, \ldots, f_m\})$.

Describing the Tame symbol by this local process using discrete valuations allows us to use the results in [BT73] concerning such maps. Proposition 4.5 in that paper gives the two results we need.

1. The map ∂_ν associated with a discrete valuation ν is independent of the choice of the element π with $\nu(\pi) = 1$.

2. The map ∂_ν is well defined on symbols in Milnor K-theory, *i.e.* it respects all relations defining such symbols.

In our context, the first result implies that this construction of the Tame symbol doesn't depend on the choice of the function ϕ defining the subvariety D. The second result ensures that the Tame symbol is well defined modulo the relations in the ideal R defining $A_Z^\bullet$.

The argument so far is local; in order to complete the proposition, we must ensure that the local descriptions of the Tame symbol patch together into a consistent global description. However, this is almost immediate. For any Cartier divisor $f = f_\alpha$ used locally in this construction, the compatibility condition implies that $f_\alpha / f_\beta = l_{\alpha\beta}$. Then a description $f_\alpha = u_\alpha \phi_\alpha^{r_\alpha}$ in terms of the discrete valuation allows $\overline{u}_\alpha / \overline{u}_\beta = l_{\alpha\beta}|_D$. Finally, this works across symbols, and the order of vanishing along D is globally consistent. □

Proposition 5.2.4: *For any m, $T^{m-1} \circ T^m = 0$.*

Proof. This is a technical calculation from [Lew04]. It starts with the T^1 case and works inductively. There are no adjusments necessary to adopt that proof.

□

Let k be a fixed non-negative integer. Since the Tame symbols compose to 0, they furnish us with the maps defining the following complex.

$$\underline{z}^0(X,k) \stackrel{T^k}{\to} \underline{z}^1(X,k-1) \stackrel{T^{k-1}}{\to} \dots \stackrel{T^{k-m+1}}{\to} \underline{z}^m(X,k-m) \stackrel{T^{k-m}}{\to} \dots \underline{z}^{k-1}(X,1) \stackrel{T^1}{\to} \underline{z}^k(X,0) \stackrel{T^0}{\to} 0 \tag{5.2.7}$$

We set the final term in degree 0, so T^m acts on degree $-m$.

5.3 Analytic Twisted Cycle Classes

The results of the previous section on the Tame symbol allow the following definitions. These parallel the definitions from [Lew04], in that we build a complex that resembles a Gersten resolution and then work on the cohomology of that complex.

Definition 5.3.1: For a fixed $k > 0$ and some $0 \leq m \leq k$, the *group of analytic twisted m-cycle classes* is defined as follows.

$$V^k(X,m) := \frac{\mathrm{Ker}\left(T^m : \ \underline{z}^{k-m}(X,m) \to \underline{z}^{k-m+1}(X,m-1)\right)}{T^{m+1}\left(\underline{z}^{k-m-1}(X,m+1)\right)} \tag{5.3.1}$$

Note the shift in the superscript: $V^k(X,m)$ represents classes from $z^{k-m}(X,m)$, not from $z^k(X,m)$. In this situation, m is the symbol rank, $k-m$ is the codimension; therefore the upper index in $V^k(X,m)$ stands for the sum of the codimension and symbol rank. This is a natural choice, as explained in Section 5.5. It is also a convenient choice for the degrees of the target spaces of regulator maps.

Proposition 5.3.2: *Our construction $V^k(X, m)$ generalizes the construction of twisted cycle classes in [Lew04] on projective algebraic manifolds.*

Proof. The definition of $V^k(X, m)$ is similar to $H^{k-m}(X\underline{K}^M_{k-\bullet}), X)$ from [Lew04]. The pieces of the definiton are the same; the major difference is that the later construction makes use of the Zariski topology over algebraic varieties. The question of compatibility comes down to moving between the two topologies: strong and Zariski. However, by use of Serre's GAGA result, we know that the sets of algebraic and analytic subvarieties of a projectdive algebraic manifold are equivalent. Moreover, there is a matching on sheaf cohomology between $\mathcal{O}_X$ and its equivalent Zariski sheaf. Therefore, the isomorphism classes of line bundles do not depend on the choice of Zariski or strong topology. With that established, the same line bundles are flat, and all line bundles over projective algebraic manifolds admit non-zero sections. Again, GAGA proves that the sets of sections over strong or Zariski line bundles correspond, since such sections can be calculated cohomologically. This implies that both constructions give rise to the same twisted cycles. Finally, the Tame symbol has precisely the same form in either case, so the equivalent descends to twisted cycle classes. We conclude that $V^k(X, m)$ is independent of the choice of topology over projective algebraic manifolds. □

5.4 Twisted Cycle Classes Product

On the level of twisted cycle groups, $\underline{z}^{k-m}(X, m)$, there is a natural candidate for a product.

Definition 5.4.1: Assume we have two analytic subvarieties Z_1 and Z_2 of codimensions $k_1 - m_1$ and $k_2 - m_2$ respectively. Consider two twisted cycles written as follows.

$$\sigma := \{\sigma_1, \ldots, \sigma_{m_1}\}_{Z_1} \in \underline{z}^{k_1-m_1}(X, m_1) \tag{5.4.1}$$
$$\tau := \{\tau_1, \ldots, \tau_{m_2}\}_{Z_2} \in \underline{z}^{k_2-m_2}(X, m_2)$$

If the two twisted cycles are in a sufficiently general position (a vague notion to be clarified in the proof of the next proposition), we can consider the following combination.

$$\sigma \cap \tau = \{\sigma_1|_{Z_1\cap Z_2}, \ldots \sigma_{m_1}|_{Z_1\cap Z_2}, \tau_1|_{Z_1\cap Z_2}, \ldots \tau_{m_2}|_{Z_1\cap Z_2}\}_{Z_1\cap Z_2}$$
$$\in \underline{z}^{k_1+k_2-m_1-m_2}(X, m_1+m_2) \tag{5.4.2}$$

The $\cap$ notation is used because this structure is reminiscent of intersection products in other theories.

Proposition 5.4.2: *For a sufficiently powerful notion of cycles in general position, which is defined in the proof, the formula in Equation 5.4.2 gives a well defined product on the level of twisted cycle classes in* $V^k(X, m)$.

Proof. Before considering cycle classes, we must note that in order for the product to make sense, it must be well defined on symbols. This is almost immediate, since any symbol relations in σ or τ that would give the trivial cycle are preserved in the product.

Then we have the following calculation to ensure that the Leibniz rule holds for the Tame symbol.

$$T^{m_1+m_2}(\sigma\cap\tau)=\sum_{D\subset Z_1\cap Z_2} T_D^{m_1+m_2}(\sigma\cap\tau) \tag{5.4.3}$$
$$=\sum_{D\subset Z_1\cap Z_2}\Bigg[\sum_{i=1}^{m_1}(-1)^i\nu_D(\sigma_i)\{\bar\sigma_1,\ldots,\hat{\bar\sigma}_i,\ldots,\bar\sigma_{m_1},\bar\tau_1,\ldots,\bar\tau_{m_2}\}_D+$$
$$\sum_{j=1}^{m_2}(-1)^{j+m_1}\nu_D(\tau_j)\{\bar\sigma_1,\ldots,\bar\sigma_{m_1},\bar\tau_1,\ldots,\hat{\bar\tau}_j,\ldots,\bar\tau_{m_2}\}_D\Bigg]$$
$$=\sum_{D\subset Z_1\cap Z_2}\left[\left(\sum_{i=1}^{m_1}(-1)^i\nu_D(\sigma_i)\{\bar\sigma_1,\ldots,\hat{\bar\sigma}_i,\ldots,\bar\sigma_{m_1}\}_D\right)\cap\tau|_D\right]+$$
$$(-1)^{m_1}\sum_{D\subset Z_1\cap Z_2}\left[\sigma|_D\cap\left(\sum_{j=1}^{m_2}(-1)^j\nu_D(\tau_j)\{\bar\tau_1,\ldots,\hat{\bar\tau}_j,\ldots,\bar\tau_{m_2}\}_D\right)\right]$$
$$=\left[\sum_{D\subset Z_1}\sum_{i=1}^{m_1}(-1)^i\nu_D(\sigma_i)\{\bar\sigma_1,\ldots,\hat{\bar\sigma}_i,\ldots,\bar\sigma_{m_1}\}_D\right]\cap\tau+$$
$$(-1)^{m_1}\sigma\cap\left[\sum_{D\subset Z_2}\sum_{j=1}^{m_2}(-1)^j\nu_D(\tau_j)\{\bar\tau_1,\ldots,\hat{\bar\tau}_j,\ldots,\bar\tau_{m_2}\}_D\right]$$
$$=(T^{m_1}\sigma)\cap\tau+(-1)^{m_1}\sigma\cap(T^{m_2}\tau)$$

In this calculation, $D\subset Z$ implies that D is irreducible of codimension 1 in the larger subvariety and the bars over Cartier divisors indicate restriction to the appropriate subvariety.

In order for the above calculation to make sense, every $D\subset Z_1\cap Z_2$ must be realized as both $D=D'\cap Z_2$ where D' is irreducible codimension 1 in Z_1 and $D=D''\cap Z_1$ where D'' is irreducible codimension 1 in Z_2. This is the condition of general position that we define and apply to Z_1 and Z_2 in order to allow the definition.

Finally, we extend the product over sums of cycles which are pairwise in general position with each other, to give a partial product on $V^k(X,m)$. □

Note that only a partial product is possible. While we can use the symbols and the Steinberg relations to adjust poles and zeros of the meromorphic sections, it is not as easy to adjust the subvarieties that support the cycles. Unlike the Chow groups, we do not have a moving lemma to adjust the subvarieties into general position.

5.5 Comparison with Other Invariants

Very roughly speaking, there are partial comparisons between $V^k(X,m)$ and the projective algebraic version of twisted cycles, Milnor K-theory and higher Chow groups. These identifications are very far from perfect and they are only defined for particular degrees and situations. Nonetheless, it is useful for inspiration, if nothing else, to have an idea of how the degrees match up.

$$V^k(X,m) \sim H^{k-m}(\underline{K}^M_{k-\bullet}, X) \sim H^{k-m}_{\text{Zar}}(X, \underline{K}^M_{k,X}) \sim CH^k(X,m) \qquad (5.5.1)$$

Again very roughly speaking, the relations in the above list are as follows. The first relation is an identification when reducing the analytic to the projective algebraic case, as argued in Proposition 5.3.2. The second identification comes from considering only functions (sections of the trivial line bundle) instead of sections of arbitrary line bundles. The functions are then sheafified, and the third element is, in fact, a sheaf cohomology group. Lastly, the relationship between Milnor K-theory and a certain coniveau graded part of the higher Chow groups is well known and given by a graph map.

Chapter 6

Regulators

In algebraic geometry, regulators refer to various maps from K-theoretic invariants (such as Milnor K-theory, higher Chow group, and our twisted cycle classes) to simpler cohomologies theories (such as Deligne cohomology and Bott-Chern cohomology). Our inspiration to consider regulators on twisted cycle classes comes from the previous definition on Milnor K-theory and Chow groups.

Even though we identify Deligne and Bott-Chern cohomology as target spaces for regulator maps, it is often true that such maps are only defined modulo some relations. Quotients of Deligne and Bott-Chern cohomology become the actual target spaces. This is consistent with classically defined regulators.

6.1 Regulators on Twisted 1-Cycles

The regulator constructions in this section are adaptations of the real regulators constructed for projective algebraic complex manifolds in [Lew01] and

[Lew04].

We briefly recall the details of Aeppli cohomology: $H^{p,q}_{\mathrm{Ap}}(X, \mathbb{R}(p+1))$. Aeppli cohomology with coefficients in $\mathbb{C}$ was defined as hypercohomology in terms of the complex of currents $\mathcal{M}^{\bullet}_{p+1,q+1}$. It is the hypercohomology in degree $2k-2$ of this complex.

$$\frac{\mathrm{Ker}\left(\partial\overline{\partial} : \mathcal{D}^{p,q}(X) \to \mathcal{D}^{p+1,q+1}(X)\right)}{\partial\mathcal{D}^{p-1,q}(X) \oplus \overline{\partial}\mathcal{D}^{p,q-1}(X)} \tag{6.1.1}$$

The currents in the numerator act on differential forms of bidegree $(n-p, n-q)$, that is, complex valued forms in $\mathcal{A}^{n-p,n-q}(X)$. Then Proposition 3.7.1 proves that classes with coefficients in $\mathbb{R}(p+1)$ can be identified, up to a quotient, as classes of currents which are either entirely real or entirely imaginary.

Now we can proceed to define the regulator current in the simplest degree, $m = 1$. Recall the group of twisted 1-cycle classes.

$$V^k(X,1) := \frac{\mathrm{Ker}\left(T^1 : \ \underline{z}^{k-1}(X,1) \to \underline{z}^k(X,0)\right)}{T^2\left(\underline{z}^{k-2}(X,2)\right)} \tag{6.1.2}$$

Definition 6.1.1: We define the *Bott-Chern regulator* on analytic twisted 1-cycle classes as follows.

$$\begin{aligned} r_{\mathrm{BC}} : V^k(X,1) &\to H^{k-1,k-1}_{\mathrm{Ap}}(X, \mathbb{R}(k))/Q \\ \sum_i (Z_i, L_i, \rho_i, \sigma_i) &\mapsto \left(\omega \mapsto (2\pi\imath)^k \sum_i \int_{\widetilde{Z_i}} \omega \log|\sigma_i|\right) \end{aligned} \tag{6.1.3}$$

Q is a subspace consististing of two parts: $Q_1 + Q_2$.

Q_1 is defined to be the image of $H^{2k-2}(X, \mathbb{R}(k-1))$ in $H_{\text{Ap}}^{k-1,k-1}(X, \mathbb{R}(k))$. The quotient by Q_1 is necessary to have well defined classes in Aeppli cohomology with $\mathbb{R}(k)$ coefficients according Proposition 3.7.1.

Q_2 is defined to be the subspace of twisted real currents generated by integration over subvarieties:

$$Q_2 := \left\langle \omega \mapsto (2\pi\imath)^k \int_{\widetilde{Z}} \omega \;\middle|\; \text{codim}_X Z = k-1 \right\rangle \tag{6.1.4}$$

We refer to Q_2 as the group of analytic currents or currents of integration over analytic subvarieties. It defines classes in the quotient $H_{\text{Ap}}^{k-1,k-1}(X, \mathbb{R}(k))/Q_1$ according to Proposition 3.7.1 since it takes values in $\mathbb{R}(k)$.

Proposition 6.1.2: *The regulator on twisted 1-cycle classes is well defined.*

The proof of this proposition is broken up into several lemmas. First, in order to get an element of $H_{\text{Ap}}^{k-1,k-1}(X, \mathbb{C})$ we must ensure the current is $\partial\overline{\partial}$-closed. Second, the formula must be independent of the metric on the line bundle. Third, we must check that the current satisfies $\pi_{k-1} r_{\text{BC}}(\sigma)) = 0$, so that by Proposition 3.7.1 the regulator take values in Aeppli Cohomology with $\mathbb{R}(k)$ coefficients. These three give a regulator defined on twisted cycles in the kernel of the Tame symbol; to get a map on twisted cycle classes, we need to show that the regulator vanishes on the image of the previous Tame symbol.

The first lemma is only a very minor adaption of the calculation in [Lew04].

Lemma 6.1.3: *Let η be a differential form of type $n-k, n-k$ on X. The regulator vanishes on $\partial\overline{\partial}\eta$.*

$$r_{\mathrm{BC}}\left(\sum(Z_i, L_i, \rho_i, \sigma_i)\right)(\partial\overline{\partial}\eta) = \sum_i \int_{Z_i} \partial\overline{\partial}\eta \log|\sigma_i| = 0 \tag{6.1.5}$$

Proof. We apply the chain rule to the integrand.

$$\partial\overline{\partial}\eta\log|\sigma_i| = \partial\left(\overline{\partial}\eta\log|\sigma_i|\right) + (-1)^{n-k+1}\overline{\partial}\eta \wedge \partial\log|\sigma_i| \tag{6.1.6}$$

This expression is substituted back into the integral.

$$\begin{aligned} r_{\mathrm{BC}}\left(\sum(Z_i, L_i, \rho_i, \sigma_i)\right)(\partial\overline{\partial}\eta) = \sum_i \Big[\int_{Z_i} \partial\left(\overline{\partial}\eta\log|\sigma_i|\right) \\ +(-1)^{n-k+1}\int_{Z_i} \overline{\partial}\eta \wedge \partial\log|\sigma_i| \Big] \end{aligned} \tag{6.1.7}$$

Consider the first of these two terms. Note that Z is of dimension $n-k+1$, so it can only support holomophic or anti-holomorphic forms of type at most $n-k+1$. This means that the form $\partial(\overline{\partial}\eta\log\rho_i(\sigma_i))$ on Z is equivalent to $d(\overline{\partial}\eta\log\rho_i(\sigma_i))$, since the $\overline{\partial}$-component is anti-holomorphic of type $n-k+2$. We may exclude the divisor set of σ_i from the integral, since it is of measure zero. Applying these changes gives the following integral.

$$\sum_i \int_{Z_i} d\left(\overline{\partial}\eta\log|\sigma_i|\right) = \sum_i \int_{Z_i - \mathrm{div}(\sigma_i)} d\left(\overline{\partial}\eta\log|\sigma_i|\right) = \lim_{\epsilon\to 0}\sum_i \int_{Z_i - B_\epsilon} d\left(\overline{\partial}\eta\log|\sigma_i|\right) \tag{6.1.8}$$

Here B_ϵ is a solid tube of radius ϵ around the divisor set of σ_i. T_ϵ is the boundary of B_ϵ, *i.e.* the hollow tube of radius ϵ around the divisor of σ_i. Now we can apply Stokes theorem.

$$\lim_{\epsilon\to 0}\sum_i \int_{Z_i - B_\epsilon} d\left(\overline{\partial}\eta \log|\sigma_i|\right) = \lim_{\epsilon\to 0}\sum_i \int_{T_\epsilon} \overline{\partial}\eta \log|\sigma_i| \qquad (6.1.9)$$

The last expression here is a residue calculation. Since $\overline{\partial}\eta$ is a C^∞ form on B_ϵ, we can simplify the residue calculation.

$$\lim_{\epsilon\to 0}\sum_i \int_{T_\epsilon} \overline{\partial}\eta \log|\sigma_i| = \sum_i \int_{\text{div}(\sigma_i)} \text{Res}(\overline{\partial}\eta \log|\sigma_i|) \qquad (6.1.10)$$

If σ tends to zero along a particular divisor, this residue integral clearly vanishes. If σ is polar along a divisor, since $\overline{\partial}\eta$ is a C^∞ form defined on the whole variety, it does not contribute to the polar behaviour along the divisor. Therefore, in appropriate local coordinates, this integral behaves like a standard (1-dimensional case) line integral.

$$\int_{|z|=\epsilon} \log|z| dz = \int_{|z|=\epsilon} \epsilon \log\epsilon d\theta = 2\pi\epsilon\log\epsilon \qquad (6.1.11)$$

It is easy to see that this integral vanishes in the limit $\epsilon \to 0$. Therefore, the first term on the right-hand side in Equation 6.1.7 is 0.

For the second term on the right-hand side in Equation 6.1.7, we use another chain rule calculation.

$$\overline{\partial}\eta \wedge \partial\log|\sigma_i| = \overline{\partial}\left(\eta \wedge \partial\log|\sigma_i|\right) + (-1)^{n-k+1}\eta \wedge \overline{\partial}\partial\log|\sigma_i| \qquad (6.1.12)$$

By assumption, the metric ρ_i is flat, so the second term vanishes. For reasons of type and holomorphic support, we can replace $\overline{\partial}$ with the simple differential d in the first term. Then we apply a similar argument as we did for the first term around the divisor set.

$$\sum_i \int_{Z_i} d\left(\eta \wedge \partial \log|\sigma_i|\right) = \sum_i \int_{Z_i - \mathrm{div}(\sigma_i)} d\left(\eta \wedge \partial \log|\sigma_i|\right) \tag{6.1.13}$$
$$= \lim_{\epsilon \to 0} \sum_i \int_{Z_i - B_\epsilon} d\left(\eta \wedge \partial \log|\sigma_i|\right)$$

Here we have Stokes theorem again.

$$\lim_{\epsilon \to 0} \sum_i \int_{Z_i - B_\epsilon} d\left(\eta \wedge \partial \log|\sigma_i|\right) = \lim_{\epsilon \to 0} \sum_i \int_{T_\epsilon} \eta \wedge \partial \log|\sigma_i| \tag{6.1.14}$$

This is now a residue calculation, and the $d$log term returns the order of vanishing. This order of vanishing is unaffected by taking the absolute value of the section.

$$\lim_{\epsilon \to 0} \sum_i \int_{T_\epsilon} \eta \wedge \partial \log|\sigma_i| = \sum_i \sum_{D \in \Sigma_i} \nu_D(\sigma_i) \int_D \eta \tag{6.1.15}$$

This gives the integral of η over all of the various divisors of the σ_i. However, the condition of T^1 vanishing is precisely that the sums of the divisors of the σ_i vanish. Therefore, this is an integral over an empty set and we conclude that the second term in Equation 6.1.7 vanishes. □

We need the regulator formula to be independent of the metric ρ_i chosen. The second lemma establishes this fact, and is also very similar to the calculation in [Lew04].

Lemma 6.1.4: *Let ρ and τ be flat metrics on the line bundle L over Z of codimension k. Let σ be a non-zero Cartier divisor which corresponds to a* *meromorphic section of L over Z. Then for a differential form ω of type $n-k, n-k$, the following equality is true modulo the subspace Q_1.*

$$\int_Z \omega \log|\sigma|_\rho = \int_Z \omega \log|\sigma|_\tau \tag{6.1.16}$$

Proof. The difference of the two integrals is the following expression.

$$\int_Z \omega \left(\log|\sigma|_\rho - \log|\sigma|_\tau\right) = \int_Z \omega \log\frac{|\sigma|_\rho}{|\sigma|_\tau} \tag{6.1.17}$$

If ρ and τ are given by local positive functions ρ_i and τ_i, these local functions agree with the cocycle l_{ij} defining the line bundle, that is, $\rho_j = \rho_i \left|l_{ij}\right|^2$ and likewise for τ. Then, locally, the integrand in question has this form.

$$\omega \log\frac{|\sigma|_\rho}{|\sigma|_\tau}\bigg|_{U_i} = \omega \log\frac{\sigma_i \rho_i \overline{\sigma_i}}{\sigma_i \tau_i \overline{\sigma_i}} == \omega \log\frac{\rho_i}{\tau_i} \tag{6.1.18}$$

The local descriptions for the quotient ρ/τ, which is also a positive function, agree on intersection $U_i \cap U_j$.

$$\frac{\rho_j}{\tau_j} = \frac{\rho_i \left|l_{ij}\right|^2}{\tau_i \left|l_{ij}\right|^2} = \frac{\rho_i}{\tau_i} \tag{6.1.19}$$

Therefore, the quotient glues into a global *function* instead of a global section. Since $\partial\overline{\partial}\log\rho_i/\tau_i = 0$, the function $\log\rho_i/\tau_i$ is a global, $\partial\overline{\partial}$-closed function, therefore a real constant c.

This means that the integral in Equation 6.1.17 now takes the following form.

$$2\pi\imath c \int_Z \omega \tag{6.1.20}$$

This current acting on ω belongs to the group Q_2, completing the proof. $\square$

At this point, the calculation differs from that of [Lew04]. That paper used Hodge theory directly to get a current on the appropriate twisted real cohomology. Here, we instead make use of Proposition 3.7.1 to find the right coefficients.

Lemma 6.1.5: *The current $r_{\mathrm{BC}}(\sigma)$ satisfies the relation $\pi_{k-1} r_{\mathrm{BC}}(\sigma) = 0$. Therefore, it gives a class in Aeppli cohomology with $\mathbb{R}(k)$ coefficients, modulo the subspace Q_1.*

Proof. This is nearly trivial. Since the integrand is a real-valued form, the current is a real-valued current. Then the twisting term of $(2\pi\imath)^k$ before the integral corrects the parity and ensures that projection of the current to $\mathbb{R}(k-1)$ vanishes. By Proposition 3.7.1, the lemma follows. $\square$

Finally, we investigate how the regulator interacts with the Tame symbol. Consider the tail of the complex defining $V^k(X, m)$.

$$\underline{z}^2(X, k-2) \xrightarrow{T^2} \underline{z}^1(X, k-1) \xrightarrow{T^1} \underline{z}^0(X, k) \to 0 \tag{6.1.21}$$

The above three lemmas show that the regulator is defined on the subgroup $\mathrm{Ker}(T^1)$ of $\underline{z}^1(X, k-1)$. To descend to $V^1(X, m)$, the regulator must vanish on the image of T^2. We prove this in three lemmas, following [Lev88] and [Lew04].

Lemma 6.1.6: *For any holomorphic functions $f, g : Z \to \mathbb{P}^1$, the regulator on the Tame symbol $\{f, g\}$ vanishes.*

$$r_{\mathrm{BC}}(T^2(\{f, g\})) = 0 \tag{6.1.22}$$

Proof. This proof is due to Levine in [Lev88]. Choose a desingularization $\mu : \widetilde{Z} \to Z$ according to Proposition 2.2.32. Then $F := \mu^*(f \times g) : \widetilde{Z} \to \mathbb{P}^1 \times \mathbb{P}^1$ is a holomorphic map of complex manifolds.

On $\mathbb{P}^1 \times \mathbb{P}^1$, let (s, t) be affine coordinates on each summand. We can interpret s and t as meromorphic functions on $\mathbb{P}^1 \times \mathbb{P}^1$. The class of the symbol $\{s, t\}$ naturally lives in $V^2(\mathbb{P}^1 \times \mathbb{P}^1, 1)$. We can consider the regulator current $r_{\mathrm{BC}}(T^2(\{s, t\}))$, which acts on $1, 1$-forms on $\mathbb{P}^1 \times \mathbb{P}^1$. Explicitly using the Tame symbol formula and the fact that the coordinate functions divisors are easy to determine, we get the following current.

$$\omega \mapsto \int_{\infty \times \mathbb{P}^1} \ln |t| \omega - \int_{0 \times \mathbb{P}^1} \ln |t| \omega + \int_{\mathbb{P}^1 \times 0} \ln |s| \omega - \int_{\mathbb{P}^1 \times \infty} \ln |s| \omega \qquad (6.1.23)$$

Levine claims that this current vanishes on cohomology, and we fill out that argument. The Künneth decomposition for $\mathbb{P}^1 \times \mathbb{P}^1$ means that a form representing a cohomology class in $H^2(\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{C})$ is either a constant times ω_s, a 2-form in the variable s, or a constant times ω_t, a 2-form in the variable t. The current applied to forms of the first type is only supported on the first two integrals. However, the first two integrals are equal on such forms, since the t dependence is only a constant, meaning the difference vanishes. The same is true of the forms of the second type and we conclude that the $r_{\mathrm{BC}}(T^2(\{s, t\}))$ represents the trivial cohomology class.

Since F is a smooth map, we can pull back the current $r_{\mathrm{BC}}(T^2(\{s, t\}))$. Since s and t are coordinate functions on $\mathbb{P}^1$, $F^* r_{\mathrm{BC}}(T^2(\{s, t\})) = r_{\mathrm{BC}}(T^2(\{f, g\}))$. Since the pullback is a morphism in cohomology, it preserves the zero cohomology class. Finally, pushing forward by the desingularization μ back to the original Z also preserves the zero class in cohomology and we conclude that the regulator on the image of the Tame symbol of functions is trivial. □

Using the result of Levine, the next two lemmas are adaptations of those in [Lew04], bringing in the language of stably trivial bundles. Lacking a Poincaré bundle, which is used to construct families of stably trivial line bundles in [Lew04], we use Definition 4.5.1 to understand such families.

Lemma 6.1.7: *For any elements $\{f, g\} \in \underline{z}^2(X, k-2)$ such that f and g are Cartier divisors corresponding to stably trivial line bundles, the regulator vanishes on the Tame symbol image of $\{f, g\}$.*

$$r_{\mathrm{BC}}(T^2(\{f, g\}_Z)) = 0 \tag{6.1.24}$$

Proof. Assume we have a map $\phi : \widetilde{Z} \to Z$ which is a finite covering map of complex spaces, and under which both the line bundles pullback to trivial line bundles. The assumption that both f and g correspond to stably trivial bundles implies such a map exists.

The current defining $r(T\{f, g\})$ pulls back to a zero current under ϕ^* since, on a trivial line bundle, we are dealing with functions and the Lemma 6.1.6 applies. We can consider forms ω on $\tilde{Z}$ and currents η on Z with the definition of pullback of currents.

$$\eta(\phi_* \omega) = (\phi^* \eta)\omega \tag{6.1.25}$$

Since our current pulls back to zero, this identity shows that our current vanishes on all forms in the image of the pushforward. Lastly, the pushforward map in homology is surjective; since the regulator is defined modulo closed forms (or $\partial\overline{\partial}$-closed forms), the regulator is zero. □

Lemma 6.1.8: *For any element* $\{f, g\}_Z \in \underline{z}^2(X, k-2)$, *such that* f *and* g *are Cartier divisors corresponding to line bundles* L *and* M, *the following is true.*

$$r_{\mathrm{BC}}(T^2(\{f, g\})) = 0 \tag{6.1.26}$$

Proof. These Cartier divisors are limits of Cartier divisors corresponding to stably trivial line bundes, as per Propositions 4.5.4 and 4.6.3.

Explicitly, let F and G be Cartier divisors over the trivial family $X \times \Delta$ such that F_0 gives the Cartier divisor f of the line bundle L and G_0 gives the Cartier divisor g of the line bundle M. Then there exists sequences of points x_i and y_i in Δ such that $\lim_{i\to\infty} x_i = \lim_{i\to\infty} y_i = 0$, $\lim_{i\to\infty} F(x_i) = f$ and $\lim_{i\to\infty} G(y_i) = g$ and the line bundles defined by the divisors over x_i and y_i are all stably trivial.

The Tame map and the regulator are all continuous operations on Cartier divisors. This allows us to conclude the following.

$$r(T(\{f, g\})) = \lim_{i\to\infty} r(T(\{F_{x_i}, G_{y_i}\})) \tag{6.1.27}$$

All terms in this limit are symbols of stably trivial bundles. On such bundles, the regulator vanishes on the image of the Tame symbols by Lemma 6.1.7. So this is a limit of zero values, and we conclude that $r(T(\{f, g\})) = 0$. □

6.2 Regulators on Twisted m-cycles

Having completed the $m = 1$ case in the previous section, we have a very good idea of what form a general regulator should take. This section addresses other values of m, and defines the regulator in general.

For $m = k$, the domain of the regulator we wish to construst is a piece of the Milnor K-theory of the function field of X (for X connected), so this regulator can be defined by the same formula as the regulator in [Lew01] on Milnor K-theory into Deligne cohomology. For $m = 0$ the domain of the regulator is the analytic cycle group, and the regulator is nothing more than a cycle class map into Aeppli cohomology.

With those cases accounted for, we assume $1 < m < k$. We want to construct a regulator with the following form.

$$r_{\mathrm{BC}} : V^k(X, m) \to \frac{H_{\mathrm{Ap}}^{k-1,k-m}(X, \mathbb{R}(k))}{\delta H^{2k-m-1}(X, \mathbb{R}(k-1)} \tag{6.2.1}$$

We could also consider the regulator acting on $H_{\mathrm{Ap}}^{k-m,k-1}(X, \mathbb{R}(k-m+1))$. However, no new information is provided by this calculation: such a regulator would simply be the complex conjugation of what we are about to construct.

Definition 6.2.1: Consider a form $\omega \in \mathcal{A}^{n-k+m,n-k+1}$. Then the action of the regulator for an element $\{\sigma_1, \ldots, \sigma_m\}_Z$, where all the σ_i are defined over Z of codimension $k - m$, is given by the following formula.

$$r_{\mathrm{BC}}(\{\sigma_1, \dots, \sigma_m\}_Z)(\omega) := \tag{6.2.2}$$
$$(2\pi\imath)^k \sum_{i=1}^{k-m} (-1)^{i+1} \int_Z \log|\sigma_i| d\log|\sigma_1| \wedge \dots \widehat{d\log|\sigma_i|} \wedge \dots d\log|\sigma_m| \wedge \omega$$

Proposition 6.2.2: *This regulator is well defined on $V^k(X, m)$ and defines a class in $H^{k-1,k-m}_{\mathrm{Ap}}(X, \mathbb{R}(k))/\delta H^{2k-m-1}(X, \mathbb{R}(k-1)$*

The proof proceeds according to several lemmas. The proof is similar to the $m = 1$ case in Section 6.1, but some small adjustments are necessary.

Lemma 6.2.3: *The formula in Equation 6.2.2 is well defined on symbols.*

Proof. For this to be true, the formula has to be independent of the relations defining symbols. Those relations are twofold. First, the symbols represent tensors in a tensor product, so the relations implicit in the tensor product must be respected. This isn't an issue here, since any expression involving logarithms satisfies those relations.

The are two other relations defining symbols, originally stated in Equation 5.1.3. Specifically, symbols $\{\sigma, -\sigma\}$ for Cartier divisors σ and $\{f, 1-f\}$ for meromorphic functions f are trivial, and similar relations in symbols of higher degree cause the symbol to vanish.

First consider $\{\sigma, -\sigma\}$. All such Cartier divisors $\sigma \in$ DST are the limits of stably trivial Cartier divisors. Stably trivial Cartier divisors pullback to meromorphic functions over a finite cover, by definition. That reduces us to the situation in [Lew01], where this result is argued for rational functions. The argument is only algebraic in its use of Hodge theory to describe the target cohomology. If we replace the Hodge-theoretic quotient in that arugment with the quotient of $H_{\text{Ap}}^{k-1,k-m}(X, \mathbb{R}(p)$ in our regulator image, we can make use of that argument. The regulator is continuous, so we can make use of the limit argument to conclude that $\{\sigma, -\sigma\}$ vanishes under the regulator for general Cartier divisors (still in DST, of course).

The same argument also applies to the meromorphic functions $\{f, 1-f\}$.

If we have $\sigma_i = -\sigma_j$ or $f_i = 1 - f_i$ in a symbol of degree higher than 2, these same arguments still show that the regulator vanishes on such symbols, completing the proof. □

Lemma 6.2.4: *For any component piece T_D of the Tame symbol, the regulator vanishes on the image.*

$$r_{\text{BC}}(T_D(\{\sigma_1, \ldots, \sigma_{k-m-1}\}))(\omega) = 0 \tag{6.2.3}$$

Proof. The projective algebraic version of this result is proved in the papers [Lew01] and [Lew04], and follows a similar form to Lemmas 6.1.6, 6.1.7 and 6.1.8. The crux of those arguments is the reduction to the case of meromorphic functions, realized as morphisms to $\mathbb{P}^1$. Then, by pulling back and pushing foward currents to products of $\mathbb{P}^1$, the regulator currents on the Tame symbol of such functions are shown to vanish.

Then the argument considers Cartier divisors corresponding to stably trivial line bundles. By combining the various covering maps, we pull those Cartier divisors back to meromorphic functions on the cover. The regulator on the level of the cover is trivial on meromorphic functions, and it is shown that the original current depends on the current pulled back to the cover.

Finally, in the projective algebraic case, all Cartier divisors are limits of stably trivial Cartier divisors by a Poincaré bundle argument. By the continuity of the regulator, the final result is established.

In the analytic case, we specifically restrict our attention to those Cartier divisors which are limits of stably trivial Cartier divisors, thus getting around the problem of lacking a Poincaré bundle. By using limits of stably trivial bundles and pullbacks of sections to meromorphic function, we can also reduce to the case of considering Tame symbols of functions.

We can consider a set of meromorphic functions defining an analytic morphism to a product of $\mathbb{P}^1$. Then, a proper modification to a desingularization allows us to assume that this analytic morphism is a smooth map of manifolds. Under a smooth map of manifolds, we can use the same argument as in the projective algebraic case. The relations on $\mathbb{P}^1$ are the same, and the currents pullback *via* the smooth map. Then we proceed back though the desingularization, covering maps, and limits, to show that the result holds for Tame images of arbitrary symbols in $\underline{z}^{k-m}(X, m)$. □

Lemma 6.2.5: *For any form η of type $n-k+m-1, n-k$, the regulator is trivial on $\partial\overline{\partial}\eta$.*

$$r_{\mathrm{BC}}(\{\sigma_1, \ldots, \sigma_{k-m}\})(\partial\overline{\partial}\eta) = 0 \tag{6.2.4}$$

Proof. Recall the regulator formula.

$$\sum_{i=1}^{k-m} (-1)^{i+1} \int_Z \log|\sigma_i| d\log|\sigma_1| \wedge \ldots \widehat{d\log|\sigma_i|} \wedge \ldots d\log|\sigma_m| \wedge \overline{\partial}\partial\eta \tag{6.2.5}$$

Since $\overline{\partial}\partial\eta$ is already $n-k+m$ times holomorphic and Z is of dimension $n-k+m$, any holomorphic differential involved in the $d$log terms leads to a form which cannot be supported. Therefore, this expression simplifies.

$$\sum_{i=1}^{k-m}(-1)^{i+1}\int_Z \log|\sigma_i|\overline{\partial}\log|\sigma_1|\wedge\ldots\widehat{\overline{\partial}\log|\sigma_i}|\wedge\ldots\overline{\partial}\log|\sigma_m|\wedge\overline{\partial}\partial\eta \tag{6.2.6}$$

We apply the chain rule.

$$\begin{aligned}\overline{\partial}\left(\log|\sigma_i|\overline{\partial}\log|\sigma_1|\wedge\ldots\widehat{\overline{\partial}\log|\sigma_i}|\wedge\ldots\overline{\partial}\log|\sigma_m|\wedge\overline{\partial}\eta\right)&= \\ \overline{\partial}\log|\sigma_1|\wedge\ldots\wedge\overline{\partial}\log|\sigma_m|\wedge\partial\eta& \\ +(-1)^{m-1}\log|\sigma_i|\overline{\partial}\log|\sigma_1|\wedge\ldots\widehat{\overline{\partial}\log|\sigma_i}|\wedge\ldots\overline{\partial}\log|\sigma_m|\wedge\overline{\partial}\partial\eta&\end{aligned} \tag{6.2.7}$$

The original formula can be realzed by integrating the left hand side less the second term. Consider the left hand side of Equation 6.2.7.

$$\sum_{i=1}^{k-m}(-1)^{i+1}\int_Z \overline{\partial}\left(\log|\sigma_i|\overline{\partial}\log|\sigma_1|\wedge\ldots\widehat{\overline{\partial}\log|\sigma_i}|\wedge\ldots\overline{\partial}\log|\sigma_m|\wedge\overline{\partial}\eta\right) \tag{6.2.8}$$

The steps here are precisely the same as in the proof for $m=1$, where we remove an ϵ-tube around the divisor set and integrate over Z less this tube. Letting $\epsilon\to 0$ gives a residue calculation. Since we can assume the σ_i have disjoint divisors by the symbol relations, we are left with a finite C^∞ term and a term which acts like $\oint \epsilon\log\epsilon$, which vanishes in the limit, as in Equation

6.1.11.

Therefore, we are left with the second term in Equation 6.2.7. Again we apply the chain rule.

$$\partial\left(\overline{\partial}\log|\sigma_1|\wedge\ldots\wedge\overline{\partial}\log|\sigma_m|\wedge\eta\right)= \tag{6.2.9}$$

$$\partial\left(\overline{\partial}\log|\sigma_1|\wedge\ldots\wedge\overline{\partial}\log|\sigma_m|\right)\wedge\eta+(-1)^m\overline{\partial}\log|\sigma_1|\wedge\ldots\wedge\overline{\partial}\log|\sigma_m|\wedge\partial\eta$$

We are interested in the second term; the first term must vanish because each of the metrics is flat and locally satisfies $\partial\overline{\partial}\log|\sigma|=0$. The regulator calculation reduces to the following formula.

$$\int_Z \partial\left(\overline{\partial}\log|\sigma_1|\wedge\ldots\wedge\overline{\partial}\log|\sigma_m|\wedge\eta\right) \tag{6.2.10}$$

We can remove the divisor set of σ from the integral, since it is of measure zero. Also we can revert to the complete differential $d\log$ here for type reasons. Then, letting T_ϵ be an ϵ-tube around the divisors of the σ_i, we apply Stokes theorem.

$$\int_{T_\epsilon} d\log|\sigma_1|\wedge\ldots d\log|\sigma_m|\wedge\eta \tag{6.2.11}$$

This is a residue calculation with $d\log$ arguments, so it recovers the order of vanishing along the divisors. Since the divisors of the σ_i are disjoint, we get the following expression, letting Σ be the union of the divisors and D the specific divisor.

$$\int_{\Sigma} \sum_{D \subset \Sigma} \sum_{j=1}^{m} (-1)^{m-j} \nu_D(\sigma_i) \overline{\partial} \log|\sigma_1| \wedge \ldots \widehat{\overline{\partial} \log|\sigma_i|} \wedge \ldots \overline{\partial} \log|\sigma_m| \wedge \eta \qquad (6.2.12)$$

This expression in the integrand is precisely the Tame symbol of a twisted cycle class. Since the definition of $V^k(X, m)$ involves the kernel of the Tame symbol, we can conclude this integrand vanishes, completing the proof. □

Lemma 6.2.6: *The regulator satisfies* $\pi_{k-1} r = 0$*, allowing it to take values in Aeppli cohomology with* $\mathbb{R}(k)$ *coefficients, up to a quotient.*

Proof. As in the $m = 1$ case, this is essentially by construction. The current is integration against a $\mathbb{R}$-valued form, and is twisted to take values in $\mathbb{R}(k)$. Thus, it vanishes under the $\mathbb{R}(k-1)$ projection. By Proposition 3.7.1, the lemma follows. □

Lemma 6.2.7: *The regulator is independent of the choices of flat metrics on the line bundles.*

Proof. The elements of this calculation are very similar to the calculation in the proof of Lemma 6.1.4. There we argued, in the $m = 1$ case, that the difference caused by changing the metric on a line bundle was, up to a constant, an integration current over the subvariety Z.

In this case, we apply the same calculation in two pieces. For the $d\log$ terms in the integral, the same calculation as in Lemma 6.1.4 implies that the difference caused by a variation of flat metrics is the differential applied to a constant, which is zero. For the log term, the difference caused by a variation of flat metrics is a constant. However, the remaining terms are all closed forms. The integration of a closed form over a compact subvariety is trivial in cohomology, so those terms vanish as well.

We conclude that any adjustments to the metric in any of the involved line bundles make no difference to the regulator current. □

This last lemma is the final result needed to show that the regulator was well defined.

We have constructed this regulator with compatibility with previous constructions in mind, which leads to the following expected result.

Proposition 6.2.8: *This regulator is equivalent to the construction in [Lew04] when restricting to projective algebraic manifolds.*

Proof. For projective algebraic manifolds, the natural equivalent to $V^k(X,m)$ is written $H^{k-m}(\underline{K}^M_{k-\bullet}, X)$. This object is constructed precisely in the same way, and as we argued in Proposition 5.3.2.

The target, in the projective algebraic setting, is this cohomology group.

$$\left\{H^{k-1,k-m}(X) \oplus H^{k-m,k-1}(X)\right\} \cap H^{2k-m-1}(X,\mathbb{R}(k)) \tag{6.2.13}$$

The target injects into the following cohomology product.

$$\left\{H^{k-1,k-m}(X) \oplus \ldots \oplus H^{k-m,k-1}(X)\right\} \cap H^{2k-m-1}(X,\mathbb{R}(k)) \tag{6.2.14}$$

Since this is an projective algebraic situation, we can make use of the Hodge filtration to identify this group as the following quotient.

$$\frac{H^{2k-m-1}(X,\mathbb{R}(k))}{\pi_k F^k H^{2k-m-1}(X,\mathbb{C})} \tag{6.2.15}$$

This quotient can be identified with a quotient of $H_{\mathcal{D}}^{2k-m01}(X,\mathbb{R}(k))$, as in [Lew04]. Let Q be the implicitly defined subspace of Deligne cohomology that realizes such a quotient.

Our regulator lands in a quotient of $H_{\mathrm{Ap}}^{k-1,k-m}(X,\mathbb{R}(k))$ by $\delta H^{2k-m}(X,\mathbb{R}(k-1)$. The numerator maps, according to Equation 3.6.3, to $H_{\mathcal{D}}^{2k-m-1}(X,\mathbb{R}(k))$. However, the quotient in Equation 6.2.15 involves a restriction to $\mathbb{R}(k)$ coefficients. Therefore, the map between Aeppli and Deligne descends to a map on our Aeppli quotient to the group described in Equation 6.2.15.

Then consider the following diagram.

$$\begin{array}{ccc}
V^k(X,m) & = & H^{k-m}(\underline{K}^M_{k-\bullet,X}) \\
\downarrow & & \downarrow \\
\frac{H_{\mathrm{Ap}}^{k-1,k-m}(X,\mathbb{R}(k))}{H^{2k-m-1}(X,\mathbb{R}(k-1))} & \longrightarrow & \frac{H_{\mathcal{D}}^{2k-m-1}(X,\mathbb{R}(k)}{Q}
\end{array}$$

Since the calculation of the regulators in each case is given by precisely the same formula and the identifications in cohomology are compatible, this diagram commutes and our construction generalizes the algebraic construction. Note that if $m = 1$, this diagram must be adjusted by taking a quotient by the group of analytic integration currents in the target spaces. $\square$

Chapter 7

Conjectural Regulators and Height Pairings

This chapter considers alternate approaches to regulators, based on the cone complex construction. It then proceeds to look at constructions of height pairings on twisted cycle classes, adapting the idea of height pairings on Chow groups.

The material in this chapter is labeled as conjectural since, at the time of completing this thesis, the proofs and details remain unfinished. We have good confidence that most statements in this chapter are realistic, and look forward to producing rigorous proofs as future work.

7.1 Standard Currents and Relations

If σ is a Cartier divisor over Z, then $\log|\sigma|$ is a well-defined $\mathbb{R}$-valued section, excepting the pole set of the divisor. As a coefficient function in an integrand, when integrating against an L^1 form, both $\log|\sigma|$ and $d\log|\sigma$ are well defined and lead to convergent integrals.

Hoever, the notion of $\log\sigma$ and $\arg\sigma$ are not necessary defined, particularly since σ is not a global function and branch cuts are necessary. Taking inspiration from identities that can be proved in simpler situations, we make the following definitions.

Definition 7.1.1:

$$d\log\sigma = 2\partial\log|\sigma| \tag{7.1.1}$$

$$d\arg\sigma = \mathrm{Im}(d\log\sigma) = \mathrm{Im}(2\partial\log|\sigma|)$$

Similar to the second definition, we have the following observation.

$$d\log|\sigma| = \mathrm{Re}(d\log\sigma) = \mathrm{Re}(2\partial\log|\sigma|)$$

These construction give us access to the language of standard currents in the literature, which are often defined using $d\log\sigma$ and $d\arg\sigma$.

Definition 7.1.2: Let $\sigma = \{\sigma_i, \ldots, \sigma_m\}_Z$ be a twisted analytic cycle on Z of codimension $k - m$ in X. We define the following currents, where ω is an appropriate test form.

$$\begin{aligned}
\Xi_{\log}(\sigma) :=& \int_Z d\log|\sigma_1| \wedge \ldots \wedge d\log|\sigma_m| \wedge \omega \\
\Omega_B(\sigma) :=&(2\pi\imath)^{-m} \int_Z d\log\sigma_1 \wedge \ldots \wedge d\log\sigma_m \wedge \omega \\
R_{\log}(\sigma) :=& \int_Z \sum_{j=1}^{m}(-1)^{j-1}\log|\sigma_j|d\log|\sigma_1| \wedge \ldots \wedge \widehat{d\log|\sigma_j}| \wedge \ldots \wedge d\log|\sigma_m| \wedge \omega \\
R_B(\sigma) :=&(2\pi\imath)^{-m}\mathrm{Alt}_m \sum_{j\geq 0} \frac{\imath^{m-2j-1}}{(2j+1)!(m-2j-1)!} \\
& \log|\sigma_1|d\log|\sigma_2| \wedge \ldots \wedge d\log|\sigma_{2j+1}| \wedge d\arg\sigma_{2j+2} \wedge \ldots \wedge d\arg\sigma_m
\end{aligned} \tag{7.1.2}$$

In this definition, Alt_m refers to an alternating sum over all permutations of $\{1, 2, \ldots, m\}$, where the sign of each summand is given by the sign of the associated permutaion.

Using the relations from Definition 7.1.1, we can make the following adjustments.

$$\begin{aligned}
\Omega_B(\sigma) :=&(\pi\imath)^{-m} d\int_Z \partial\log|\sigma_1| \wedge \ldots \wedge \partial\log|\sigma_m| \wedge \omega \\
R_B(\sigma) :=&(2\pi\imath)^{-m}\mathrm{Alt}_m \sum_{j\geq 0} \cdot \frac{(2\imath)^{m-2j-1}}{(2j+1)!(m-2j-1)!} \\
& \log|\sigma_1|d\log|\sigma_2| \wedge \ldots \wedge d\log|\sigma_{2j+1}| \wedge \\
& \wedge \mathrm{Im}(\partial\log|\sigma_{2j+2}|) \wedge \ldots \wedge \mathrm{Im}(\partial\log|\sigma_m|)
\end{aligned} \tag{7.1.3}$$

These are, in some ways, the archetypical current formulae that show up in various regulator calculations. The $R_{\log}$ formula has been the basis for our own calculations in Chapter 6.

The currents themselves are not well-defined on symbols, twisted cycles nor twisted cycle classes. However, following the pattern in the previous chapter, it can be shown that they can be used as representatives for well defined classes in quotients of Aeppli or Deligne cohomology.

However, simply working on the current level, we have the following relations, for σ as before. We expect that these are true.

Statement 7.1.3:

$$
\begin{aligned}
d(\Xi_{\log}(\sigma)) &= 0 \\
d(R_{\log}(\sigma)) &= m\Xi_{\log}(\sigma) \\
d(\Omega(\sigma)) &= \Omega(T^m(\sigma)) \\
d(R_B(\sigma)) &= \mathrm{Im}\Omega(\sigma) + R_B(T^m(\sigma))
\end{aligned}
\tag{7.1.4}
$$

7.2 Cone Regulators

We can use the currents in 7.1.2 and the relations in 7.1.1 to give a new regulator construction. In the Chapter 6, we defined regulators following the main constructions of [Lew01] and [Lew04]. We used the complex $\mathcal{M}^{\bullet}_{p,q}$ to define Bott-Chern or Aeppli cohomology, then identified classes with $\mathbb{R}(p)$ coefficients. The target of the regulator is a quotient of Aeppli cohomology, which maps to a corresponding quotient of Deligne cohomology, under the map ϵ_D of Equation (3.6.3). This map between quotients was a part of the proof of Proposition 6.2.8.

However, we can also look at regulators defined *via* the cone complex definitions of Deligne or Bott-Chern cohomology from Section 3.8. We refer to these constructions as cone regulators. Our guiding sources here are the second section of [Lew01] discussing the Beilinson regulator, as well as the regulator calculations in [Gon95].

Conjecture 7.2.1: We can define a map as follows.

$$\begin{aligned} \underline{z}^{k-m}(X,m) &\to \mathrm{Cone}(F^k\mathcal{D}^\bullet_X(X) \to \mathcal{D}^\bullet_{X,\mathbb{R}(k-1)}(X))^{2k-m-1} \\ \sigma &\mapsto (\Omega(\sigma), R_B(\sigma)) \end{aligned} \tag{7.2.1}$$

This map is well defined up to a quotient of this cone complex, and descends to a regulator map.

$$r_B : V^k(X,m) \to \frac{H^{2k-m}_{\mathcal{D}}(X,\mathbb{R}(k))}{Q} \tag{7.2.2}$$

Here Q is a subspace of Deligne cohomology which is not yet determined.

Proof. This proof, once constructioned, should involved well-defined arguments on symbols, on twisted cycles and on twisted cycle classes, including independence of choices of metrics. These independences will necessarily define the subspace Q. □

The following is an insight into the necessary quotient.

Conjecture 7.2.2: Assume the previous conjecture holds and we have a regulator map into a quotient of Aeppli cohomology described by a cone complex.

Then the image of this quotient under the map ϵ_C to Deligne cohomology is well defined and equal to the Deligne quotient defined in [Lew04]. That quotient was defined to be a quotient by a subspace Q such that the following is an isomorphism.

$$\frac{H_{\mathcal{D}}^{2k-m-1}(X, \mathbb{R}(k))}{Q'} \overset{\cong}{\to} \big(H^{k-m,k-1}(X) \oplus H^{k-1,k-m}(X)\big) \cap H^{2k-m-1}(X, \mathbb{R}(k) \tag{7.2.3}$$

This conjecture implies, among other things, an approach which involves currents of bidegree $(k-1, k-m)$ and $(k-m, k-1)$. (The potential currents involved in Deligne cohomology *aprioric* could have any bidgree between those two extremes.) We refer to this as 'restriction to currents of outside Hodge type', since these are the two bidgrees which have the highest holomorphic or anti-holomorphic degrees which can be supported.

In order to understand and prove this conjecture, some calculations must be undertaken on the cone complex defining Deligne cohomology. The algebraic construction must be generalized to the analytic areas, where the Hodge decomposition of cohomology is no longer available.

The cone complexes, as in Section 3.9, have well defined products for Deligne cohomology. A potential advantage of the cone regulator is compatibility with this product.

Conjecture 7.2.3: If $\sigma \in V^k(X, m)$ and $\tau \in V^r(X, s)$ represent classes in general position, then $r_B(\sigma \cap \tau) = r_B(\sigma) \circ r_B(\tau)$.

When regarding a product calculation, we could alternatively consider products working with the regulators defined by $R_{\log}$ as in the previous chapter. However, there is no intrisic product on Aeppli cohomology, so we would have to identify some classes which lift to Bott-Chern cohomology and work with the Bott-Chern / Aeppli product as defined in Section 3.9. Since the cone structure isn't natural here, the Čech product might be useful in this situation.

7.3 Height Pairing

Using both the $R_{\log}$ regulator and the conjectural cone regulator, it may be possible to construct something like a height pairing for certain twisted cycles. This is a generalization of a similar construction for Chow groups in the appendix to the paper [LC11]. Assume we have twisted cycle defined as follows.

$$\begin{aligned} \xi_1 \in \underline{z}^{k-m}(X, m) \quad \xi_2 \in \underline{z}^{d-k+1}(X, m) \\ \xi_1 = T^{m+1}\tau_1 \quad \tau_1 \in \underline{z}^{k-m-1}(X, m+1) \\ \xi_2 = T^{m+1}\tau_2 \quad \tau_2 \in \underline{z}^{d-k}(X, m+1) \end{aligned} \tag{7.3.1}$$

We want to consider products of cycles, matching one cycle with the Tame pre-image of the other.

$$\xi_i \cap \tau_2 \in \underline{z}^{d-m}(X, 2m+1) \quad \tau_1 \cap \xi_2 \in \underline{z}^{d-m}(X, 2m+1) \tag{7.3.2}$$

In order to even consider this idea, we have to ensure that these intersections are well defined, invoking the notion of general position from the proof of Proposition 5.4.2. Collecting all this data together leads to the following definition.

Definition 7.3.1: *Compatible cycles* $\xi_1 \in \underline{z}^{k-m}(X, m)$ and $\xi_2 \in \underline{z}^{d-k+1}(X, m)$ are cyles which satisfy the following properties.

- Both are in the image of the Tame symbol.
- The cycles intersect in general position.
- If $\xi_i = T^{m+1}\tau_i$, then ξ_1 intersects τ_2 in general position and ξ_2 intersects τ_1 in general position.

Using the $R_{\log}$ regulator from the Chapter 6, we can make a direct definition of the pairing. The motivation here is to use one term in the pairing to define the regulator, and use the other term in the pairing to define a form acted upon by the regulator current. Such a form might be the integrand to the Ω current on the other side of the pairing. (With an abuse of notation, write $\Omega(\sigma)$ for that form as well as the current defined in Definition 7.1.2.) Explicitly, this gives the following definition.

Definition 7.3.2: The height pairing $\langle \xi_1, \xi_2 \rangle$ is calculated by $R_{\log}(\tau_1)\Omega(\xi_2)$.

In this notation, ξ_1 is the twisted cycle defining the current $R_{\log}(\tau_1)$, which then acts on the form $\Omega(\xi_2)$. Such a formula gives a real number, since the involved currents are real currents. Moreover, such a formula allows easy access to the following result.

Conjecture 7.3.3: The height pairing is independent of the choice of Tame pre-image τ_1 of ξ_1.

This proof is yet to come, but should make use of the structure of $R_{\log}$ from Chapter 6. Unlike independence from the choice of Tame pre-images, reciprocity is difficult with this approach. The parallel construction is the pairing calculated by $R_{\log}(\tau_2)(\Omega(\xi_1)$; and proving that the two formulae differ by a sign seems challenging.

Instead, if we consider the cone regulator, we can take a different approach. For the cone regulator, the conjectural agreement with the product in Conjecture 7.2.3 is of great use. Instead of defining the current $R_{\log}$, which then acts on another form, we can calculate the regulator directly acting on the product of cycles $\xi_i \cap \tau_j$. Such a product falls in $V^{d+m+1}(X, 2m+1)$. The regulator image from that group of twisted cycles is in $H^{2d+1}_{\mathcal{D}}(X, \mathbb{R}(d+m+1))$.

Recall the definition of the relevant cone complex.

$$\mathrm{Cone}(F^{d+m+1}\mathcal{D}^\bullet_X \overset{-\pi_{d+m}}{\to} \mathcal{D}^\bullet_{X,\mathbb{R}(d+m)}) \tag{7.3.3}$$

The first term in the cone complex at degree $2d$ cannot be supported, since the filtration degree exceeds the dimension d. Therefore, this cone complex is simply the complex of $\mathbb{R}(d+m)$-valued currents. With the shift defining Deligne cohomology, the $2d$th cohomology of this complex gives the desired group. That cohomology (since X is compact) is simply $\mathbb{R}(d+m)$. Therefore, the value of the regulator on the product twisted cycle $\xi_i \cap \tau_j$ defines a (twisted) real number, and hence a pairing.

We want to be more explicit with this regulator calculation. If Deligne cohomology is calculated by the cone complex, the regulator is given by the pair $(\Omega_B(\xi_1 \cap \tau_2), R_B(\xi_1 \cap \tau_2))$.

Recall the product on the cone complex describing Deligne cohomology in Equation 3.9.5. Applying the product formula to the current situation and using the compatibility in Corollary 7.2.3 gives the following.

$$r_B(\xi_1 \cap \tau_2) = r_B(\xi_1) \cap r_C(\tau_2) = \Big(\Omega_m(\xi_1) \wedge \Omega_B(\tau_2), R_B(\xi_1) \wedge \pi_m \Omega_B(\tau_2) + (-1)^{m+1} \pi_m \Omega_B(\xi_1) \wedge R_B(\tau_2)\Big) \tag{7.3.4}$$

However, since $\Omega_m(\xi_2)$ is a current that cannot be supported, having holomorphic degree $m+1$ over a subvariety of, at most, dimension m, we are left with only one term defining this regulator: $R_B(\tau_2) \wedge \Omega_B(\xi_1)$. If we abuse notation and use Ω_B and R_B to refer to the integrands in the currents as well as the currents themselves (*i.e.* we drop the $\int_Z$ in the definition and use the same notation), we can explicitly write this integral as follows.

$$r_B(\xi_1 \cap \tau_2) = \sum_{Z_1, Z_2} \int_{Z_1 \cap Z_2} R_B(\tau_2) \wedge \pi_m \Omega_B(\xi_1) \tag{7.3.5}$$

Returning to the other intersection, $\tau_1 \cap \xi_2$, we can repeat the same arguments by simply switching components. We end up with a very similar formula.

$$r_B(\tau_1 \cap \xi_2) = \sum_{Z_1, Z_2} \int_{Z_1 \cap Z_2} \pi_m \Omega_B(\xi_2) \wedge R_B(\tau_1) \tag{7.3.6}$$

Definition 7.3.4: The two *height pairings* on a duple (ξ_1, ξ_2) of compatible analytic twisted coboundaries are the values in $\mathbb{R}(d+m)$ given by the two previous formulae. We use the following notation for these pairings, respectively.

$$\langle \xi_1, \xi_2 \rangle_m^+ = r_B(\xi_1 \cap \tau_2) \in H_{\mathcal{D}}^{2d+1}(X, \mathbb{R}(d+m)) = \mathbb{R}(d+m) \tag{7.3.7}$$
$$\langle \xi_1, \xi_2 \rangle_m^- = r_B(\tau_1 \cap \xi_2) \in H_{\mathcal{D}}^{2d+1}(X, \mathbb{R}(d+m)) = \mathbb{R}(d+m)$$

A disadvantage of this approach is that it seems the independence of pre-images may be more difficult to establish.

Conjecture 7.3.5: This height pairing is independent of the choice of Tame pre-image of ξ_1 and ξ_2

The cone version, however, gives easy access to a reciprocity result.

Proposition 7.3.6: *The two conjectural height pairings in 7.3.5 have a reciprocity result:*

$$\langle \xi_1, \xi_2 \rangle_m^+ = (-1)^m \langle \xi_1, \xi_2 \rangle_m^- \tag{7.3.8}$$

Proof. The Tame symbol satisfies the Leibniz rule.

$$T^{2m+2}(\tau_1 \cap \tau_2) = T^{m+1}(\tau_1) \cap \tau_2 + (-1)^{m+1} \tau_1 \cap T^{m+1}(\tau_2) \tag{7.3.9}$$
$$= \xi_1 \cap \tau_2 + (-1)^{m+1} \tau_1 \cap \xi_2$$

We simply apply the regulator r_B to this equation. The left hand side vanishes, since the regulator vanishes on the image of the Tame symbol. The right side, rearranged, is this relation:

$$r_B(\xi_1 \cap \tau_2) = (-1)^m r_B(\tau_1 \cap \xi_2) \tag{7.3.10}$$

By definition, this is the required reciprocity relation. □

Finally, we expect that at least for $m \leq 2$, the two definitions of the height pairing should coincide. As well, for $m = 0$ we should recover a height pairing on Milnor K-theory.

Chapter 8

Conclusion

In many ways, this thesis feels like a work only just begun. While we are very pleased with the adaptability of various regulator calculations on analytic varieties, only a small portion of the possible questions have been answered.

The regulator of Chapter 6 was the first goal when the thesis was originally suggested, and we are pleased that goal has been achieved. We are intrigued that the goal led to investigations of the properties of Bott-Chern and Aeppli cohomology and the nature of $\mathrm{Pic}^0(X)$ over non-algebraic complex manifolds, both of which seem like interesting topics in their own right.

We have constructed a reasonable collection of target spaces for regulators in Bott-Chern and Aeppli cohomologies. We have seen, particularly in the penultimate chapter, that Bott-Chern and Aeppli cohomology should serve as additions to Deligne cohomology, not replacements. Having a variety of such target cohomology spaces which are relatively easily constructed by forms, currents and cone complexes makes the business of regulator maps more flexible.

While we have given it a good definition, this thesis has only started to consider the properties of $V^k(x, m)$, and how much K-theoretic information might be contained in such a group. The details of products and the possibility of constructing some kind of moving lemma is a question we have only barely had time to consider, let alone address. We have not considered the larger question of what K-theory means over non-algebraic object. It is interesting to wonder how much can be recovered from Milnor K-theory on functions fields, or how an analytic equivalent to the Chow groups would be defined. All these questions make for interesting directions and possibilities for the future.

Appendix A

Čech resolution calculations

The differential on the double complex is calculated as follows.

degree	term	degree	term
c	$\delta\eta^c$		
$h,0$	$\eta^c - \delta\eta^{h,0}$	$a,0$	$-\eta^c - \delta\eta^{a,0}$
$h,1$	$\partial\eta^{h,0} + \delta\eta^{h,1}$	$a,1$	$\overline{\partial}\eta^{a,0} + \delta\eta^{a,1}$
$h,2$	$\partial\eta^{h,1} - \delta\eta^{h,2}$	$a,2$	$\overline{\partial}\eta^{a,1} - \delta\eta^{a,2}$
$\dots$			
h,j	$\partial\eta^{h,j-1} + (-1)^{j+1}\delta\eta^{h,j}$	a,j	$\overline{\partial}\eta^{a,j-1} + (-1)^{j+1}\delta\eta^{a,j}$
$\dots$			
$h,p-1$	$\partial\eta^{h,p-2} + (-1)^{p}\delta\eta^{h,p-1}$	$a,q-1$	$\overline{\partial}\eta^{h,q-2} + (-1)^{q}\delta\eta^{h,q-1}$
h,p	$\partial\eta^{h,p-1}$	a,q	$\overline{\partial}\eta^{a,q-1}$
$h,p+1$	0	$a,q+1$	0
$\dots$		$\dots$	
h,r	0	a,r	0

1.1 Calculations for Čech Resolution Products in Bott-Chern and Aeppli Cohomology

Assume that $p, q, r, s > 0$. The cases involving $p = 0$ or $q = 0$ are similar calculations adjusted for the overlap of some calculations with degree 0. They are excluded from this appendix. The use of $p-2$ or $q-2$ at some points in these calculations can be excluded if $p = 1$ or $q = 1$ respectively, since they will correspond to indices already considered. The same is true for r and s.

Assume that $u \geq \max\{p-1, q-1\}$ and $v \geq \max\{r-1, s-1\}$, and assume that η and ν are elements of degree u and v respectively.

$$\eta \in \bigoplus_{j+k=u} \check{C}^j(\mathcal{U}, \mathcal{B}^k_{p,q}(\mathbb{R}(p))) \tag{1.1.1}$$

$$\eta = \eta^c; \eta^{h,0}, \eta^{h,1}, \ldots, \eta^{h,u}; \eta^{a,0}, \eta^{a,1}, \ldots, \eta^{a,u}$$

$$\nu \in \bigoplus_{j+k=v} \check{C}^j(\mathcal{U}, \mathcal{B}^k_{r,s}(\mathbb{R}(r)))$$

$$\nu = \nu^c; \nu^{h,0}, \nu^{h,1}, \ldots, \nu^{h,v}; \nu^{a,0}, \nu^{a,1}, \ldots, \nu^{a,v}$$

Then the the product is an element of this group.

$$\eta * \nu \in \bigoplus_{j+k=u+v-1} \check{C}^j(\mathcal{U}, \mathcal{B}^k_{p+r,q+s}(\mathbb{R}(p+r))) \tag{1.1.2}$$

$$\eta * \nu = (\eta * \nu)^c; (\eta * \nu)^{h,0}, (\eta * \nu)^{h,1}, \ldots, (\eta * \nu)^{h,u+v-1};$$
$$(\eta * \nu)^{a,0}, (\eta * \nu)^{a,1}, \ldots, (\eta * \nu)^{a,u+v-1}$$

We identify each term by its degree. Depending on the parity of u, these terms are giving by the following table. The last column is the term itself. It is multiplied by 1 or (-1) according to the middle columns.

degree	u even	u odd	term
c	1	1	$\eta^c\nu^c$
$h,0$	1	(-1)	$\eta^c\nu^{h,0}$
$h,1$	1	1	$\eta^c\nu^{h,1}$
$h,2$	1	(-1)	$\eta^c\nu^{h,1}$
$\vdots$			$\vdots$
h,m_1	1	$(-1)^{m_1+1}$	$\eta^c\nu^{h,m_1}$
$\vdots$			$\vdots$
$h,r-1$	1	$(-1)^r$	$\eta^c\nu^{h,r-1}$
h,r	$(-1)^r$	1	$\eta^{h,0}\partial\nu^{h,r-1}$
$h,r+1$	1	$(-1)^r$	$\eta^{h,1}\partial\nu^{h,r-1}$
$\vdots$			$\vdots$
$h,r+m_2$	$(-1)^{rm_2+r}$	$(-1)^{rm_2}$	$\eta^{h,m_2}\partial\nu^{h,r-1}$
$\vdots$			$\vdots$
$h,r+p-1$	$(-1)^{rp}$	$(-1)^{rp+r}$	$\eta^{h,p-1}\partial\nu^{h,r-1}$
$h,r+p$	1	1	0
$\vdots$			$\vdots$
$h,u+v-1$	1	1	0

degree	u even	u odd	term
$a, 0$	1	1	$\eta^{a,0}\nu^{c}$
$a, 1$	1	1	$\eta^{a,1}\nu^{c}$
$\vdots$			$\vdots$
a, m_3	1	1	$\eta^{a,m_3}\nu^{c}$
$\vdots$			$\vdots$
$a, q-1$	1	1	$\eta^{a,q-1}\nu^{c}$
a, q	(-1)	1	$\overline{\partial}\eta^{a,q-1}\nu^{a,0}$
$a, q+1$	$(-1)^{q+1}$	$(-1)^{q+1}$	$\overline{\partial}\eta^{a,q-1}\nu^{a,1}$
$\vdots$			$\vdots$
$a, q+m_4$	$(-1)^{m_4 q+1}$	$(-1)^{m_4 q+m_4}$	$\overline{\partial}\eta^{a,q-1}\nu^{a,m_4}$
$\vdots$			$\vdots$
$a, q+s-1$	$(-1)^{sq+s+1}$	$(-1)^{sq+s+q+1}$	$\overline{\partial}\eta^{a,q-1}\nu^{a,s-1}$
$a, q+s$	1	1	0
$\vdots$			$\vdots$
$a, u+v-1$	1	1	0

1.1.1 Leibniz Rule Calculations

To prove that the product is well defined, we need to check the Leibniz rule. For η and ν as above, the formula takes the following form.

$$d_D(\eta * \nu) = (d_D\eta) * \nu + (-1)^u \eta * (d_D\nu) \tag{1.1.3}$$

Recall d_D is the differential of the double complex, and δ is the Čech differential. Recall as well that the wedge form is explicitly written $\wedge$, and the Čech cup product is implicit between any two adjacent terms. Lastly, d, ∂ and $\overline{\partial}$ remain the normal differentials of forms or currents.

This section makes heavy abuse of the "…" notation between steps. The formulae are given for m_1 and the other indices in the middle, since the patterns are difficult to pull out immediately.

We calculate each side independently in degree, then compare. We also do this in cases, one for odd u and one for even u.

The first four pages are the left hand side in degrees, assuming u is even. The next four pages are the right hand side with the same assumption. The calculation can be repeated with minor variation of signs assuming degree u odd, but those calculation are ommited here.

As a convention regarding exponents of (-1), I have expanded multiplication whenever possible and simplified mod 2.

In the final steps of the right hand side calculation, I frequently use the fact that the differentials commute: $\delta\partial = \partial\delta$ and $\delta\overline{\partial} = \overline{\partial}\delta$.

	Holomorphic LHS chart for deg(η) even.
Degree	Expression
c	$\delta(\eta * \nu)^c$
	$= \delta(\eta^c \nu^c)$
	$= \delta\eta^c\nu^c + \eta^c\delta\nu^c$
$h,0$	$(\eta * \nu)^c - \delta(\eta * \nu)^{h,0}$
	$= \eta^c\nu^c - \delta(\eta^c\nu^{h,0})$
	$= \eta^c\nu^c - \delta\eta^c\nu^{h,0} - \eta^c\delta\nu^{h,0}$
$h,1$	$\partial(\eta * \nu)^{h,0} + \delta(\eta * \nu)^{h,1}$
	$= \partial(\eta^c\nu^{h,0}) + \delta(\eta^c\nu^{h,1})$
	$= \partial\eta^c\nu^{h,0} + \eta^c\partial\nu^{h,0} + \delta\eta^c\nu^{h,1} + \eta^c\delta\nu^{h,1}$
	$= \eta^c\partial\nu^{h,0} + \delta\eta^c\nu^{h,1} + \eta^c\delta\nu^{h,1}$
$\dots$	$\dots$
h,m_1	$\partial(\eta * \nu)^{h,m_1-1} + (-1)^{m_1+1}\delta(\eta * \nu)^{h,m_1}$
	$= \partial(\eta^c\nu^{h,m_1-1}) + (-1)^{m_1+1}\delta(\eta^c\nu^{h,m_1})$
	$= \partial\eta^c\nu^{h,m_1-1} + \eta^c\partial\nu^{h,m_1-1}$
	$+(-1)^{m_1+1}\delta\eta^c\nu^{h,m_1} + (-1)^{m_1+1}\eta^c\delta\nu^{h,m_1}$
	$= \eta^c\partial\nu^{h,m_1-1} + (-1)^{m_1+1}\delta\eta^c\nu^{h,m_1} + (-1)^{m_1+1}\eta^c\delta\nu^{h,m_1}$
$\dots$	$\dots$
$h,r-1$	$\partial(\eta * \nu)^{h,r-2} + (-1)^r\delta(\eta * \nu)^{h,r-1}$
	$= \partial(\eta^c\nu^{h,r-2}) + (-1)^r\delta(\eta^c\nu^{h,r-1})$
	$= \partial\eta^c\nu^{h,r-2} + \eta^c\partial\nu^{h,r-2} + (-1)^r\delta\eta^c\nu^{h,r-1} + (-1)^r\eta^c\partial\nu^{h,r-1}$
	$= \eta^c\partial\nu^{h,r-2} + (-1)^r\delta\eta^c\nu^{h,r-1} + (-1)^r\eta^c\partial\nu^{h,r-1}$

	Holomorphic LHS chart for deg(η) even.
Degree	Expression
h, r	$\partial(\eta * \nu)^{h,r-1} + (-1)^{r+1}\delta(\eta * \nu)^{h,r}$
	$= \partial(\eta^c \nu^{h,r-1}) + (-1)^{r+1}(-1)^r \delta(\eta^{h,0}\partial\nu^{h,r-1})$
	$= \partial\eta^c \nu^{h,r-1} + \eta^c \partial\nu^{h,r-1} - \delta\eta^{h,0}\partial\nu^{h,r-1} + \eta^{h,0}\partial\delta\partial\nu^{h,r-1}$
	$= \eta^c \partial\nu^{h,r-2} - \delta\eta^{h,0}\partial\nu^{h,r-1} + \eta^{h,0}\delta\partial\nu^{h,r-1}$
$\dots$	$\dots$
$h, r + m_2$	$\partial(\eta * \nu)^{h,r+m_2-1} + (-1)^{r+m_2+1}\delta(\eta * \nu)^{h,r+m_2}$
	$= \partial((-1)^{rm_2}\eta^{h,m_2-1}\partial\nu^{h,r-1}) + (-1)^{r+m_2+1}\delta((-1)^{r(m_2+1)}\eta^{h,m_2}\partial\nu^{h,r-1})$
	$= (-1)^{rm_2}\partial\eta^{h,m_2-1}\partial\nu^{h,r-1} + (-1)^{rm_2+m_2+1}\eta^{h,m_2-1}\partial\partial\nu^{h,r-2}$
	$+(-1)^{rm_2+m_2+1}\delta\eta^{h,m_2}\partial\nu^{h,r-1} + (-1)^{rm_2}\eta^{h,m_2}\delta\partial\nu^{h,r-1}$
	$= (-1)^{rm_2}\partial\eta^{h,m_2-1}\partial\nu^{h,r-1} + (-1)^{rm_2+m_2+1}\delta\eta^{h,m_2}\partial\nu^{h,r-1}$
	$+(-1)^{rm_2}\eta^{h,m_2}\delta\partial\nu^{h,r-1}$
$\dots$	$\dots$
$h, r + p - 1$	$\partial(\eta * \nu)^{h,p-2} + (-1)^{r+p}\delta(\eta * \nu)^{h,p-2}$
	$= \partial((-1)^{rp-r}\eta^{h,p-2}\partial\nu^{h,r-1}) + (-1)^{r+p}\delta((-1)^{rp}\eta^{h,p-1}\partial\nu^{h,r-1})$
	$= (-1)^{rp+r}\partial\eta^{h,p-2}\partial\nu^{h,r-1} + (-1)^{rp+r+p}\eta^{h,p-2}\partial\partial\nu^{h,r-2}$
	$+(-1)^{rp+r+p}\delta\eta^{h,p-1}\partial\nu^{h,r-1} + (-1)^{rp+r}\eta^{h,p-1}\delta\partial\nu^{h,r-1}$
	$= (-1)^{rp+r}\partial\eta^{h,p-2}\partial\nu^{h,r-1} + (-1)^{rp+r+p}\delta\eta^{h,p-1}\partial\nu^{h,r-1}$
	$+(-1)^{rp+r}\eta^{h,p-1}\delta\partial\nu^{h,r-1}$

	Anti-holomorphic LHS chart for deg(η) even.
Degree	Expression
$a,0$	$-(\eta * \nu)^c - \delta(\eta * \nu)^{a,0}$
	$= -\eta^c\nu^c - \delta(\eta^{a,0}\nu^c)$
	$= -\eta^c\nu^c - \delta\eta^{a,0}\nu^c + \eta^{a,0}\delta\nu^c$
$a,1$	$\overline{\partial}(\eta * \nu)^{a,0} + \delta(\eta * \nu)^{a,1}$
	$= \overline{\partial}(\eta^{a,0}\nu^c) + \delta(\eta^{a,1}\nu^c)$
	$= \overline{\partial}\eta^{a,0}\nu^c - \eta^{a,0}\overline{\partial}\nu^c + \delta\eta^{a,1}\nu^c + \eta^{a,1}\delta\nu^c$
	$= \overline{\partial}\eta^{a,0}\nu^c + \delta\eta^{a,1}\nu^c + \eta^{a,1}\delta\nu^c$
$\dots$	$\dots$
a,m_3	$\overline{\partial}(\eta * \nu)^{a,m_3-1} + (-1)^{m_3+1}\delta(\eta * \nu)^{a,m_3}$
	$= \overline{\partial}(\eta^{a,m_3-1}\nu^c) + (-1)^{m_3+1}\delta(\eta^{a,m_3}\nu^c)$
	$= \overline{\partial}\eta^{a,m_3-1}\nu^c + (-1)^{m_3+1}\eta^{a,m_3-1}\overline{\partial}\nu^c$
	$+(-1)^{m_3+1}\delta\eta^{a,m_3}\nu^c + \eta^{a,m_3}\delta\nu^c$
	$= \overline{\partial}\eta^{a,m_3-1}\nu^c + (-1)^{m_3+1}\delta\eta^{a,m_3}\nu^c + \eta^{a,m_3}\delta\nu^c$
$\dots$	$\dots$
$a,q-1$	$\overline{\partial}(\eta * \nu)^{a,q-2} + (-1)^q\delta(\eta * \nu)^{a,q-1}$
	$= \overline{\partial}(\eta^{a,q-2}\nu^c) + (-1)^q\delta(\eta^{a,q-1}\nu^c)$
	$= \overline{\partial}\eta^{a,q-2}\nu^c + (-1)^q\eta^{a,q-2}\overline{\partial}\nu^c$
	$+(-1)^q\delta\eta^{a,q-1}\nu^c + \eta^{a,q-1}\delta\nu^c$
	$= \overline{\partial}\eta^{a,q-2}\nu^c + (-1)^q\delta\eta^{a,q-1}\nu^c + \eta^{a,q-1}\delta\nu^c$

	Anti-holomorphic LHS chart for $\deg(\eta)$ even.
Degree	Expression
$\ldots$	$\ldots$
a, q	$\overline{\partial}(\eta * \nu)^{a,q-1} + (-1)^{q+1}\delta(\eta * \nu)^{a,q}$
	$= \overline{\partial}(\eta^{a,q-1}\nu^{c}) + (-1)^{q+1}\delta(-\partial\eta^{a,q-1}\nu^{a,0})$
	$= \overline{\partial}\eta^{a,q-1}\nu^{c} + (-1)^{q+1}\eta^{a,q-1}\overline{\partial}\nu^{c}$
	$+(-1)^{q}\delta\partial\eta^{a,q-1}\nu^{a,0} + \partial\eta^{a,q-1}\delta\nu^{a,0}$
	$= \overline{\partial}\eta^{a,q-1}\nu^{c} + (-1)^{q}\delta\partial\eta^{a,q-1}\nu^{a,0} + \partial\eta^{a,q-1}\delta\nu^{a,0}$
$\ldots$	$\ldots$
$a, q+m_4$	$\overline{\partial}(\eta * \nu)^{a,q+m_4-1} + (-1)^{q+m_4+1}\delta(\eta * \nu)^{a,q+m_4}$
	$= \overline{\partial}((-1)^{m_4q+q+1}\overline{\partial}\eta^{a,q-1}\nu^{a,m_4-1}$
	$+(-1)^{q+m_4+1}\delta((-1)^{m_4q+1}\overline{\partial}\eta^{a,q-1}\nu^{a,m_4})$
	$= (-1)^{m_4q+q+1}\overline{\partial}\overline{\partial}\eta^{a,q-1}\nu^{a,m_4-1} + (-1)^{m_4q+1}\overline{\partial}\eta^{a,q-1}\overline{\partial}\nu^{a,m_4-1}$
	$+(-1)^{m_4q+m_4+q}\delta\overline{\partial}\eta^{a,q-1}\nu^{a,m_4} + (-1)^{m_4q+m_4}\overline{\partial}\eta^{a,q-1}\delta\nu^{a,m_4}$
	$= (-1)^{m_4q+1}\overline{\partial}\eta^{a,q-1}\overline{\partial}\nu^{a,m_4-1} + (-1)^{m_4q+m_4+q}\delta\overline{\partial}\eta^{a,q-1}\nu^{a,m_4}$
	$+(-1)^{m_4q+m_4}\overline{\partial}\eta^{a,q-1}\delta\nu^{a,m_4}$
$\ldots$	$\ldots$
$a, q+s-1$	$\overline{\partial}(\eta * \nu)^{a,q+s-2} + (-1)^{q+s}\delta(\eta * \nu)^{a,q+s-1}$
	$= \overline{\partial}((-1)^{sq+1}\overline{\partial}\eta^{a,q-1}\nu^{a,s-2}$
	$+(-1)^{q+s}\delta((-1)^{sq+s+1}\overline{\partial}\eta^{a,q-1}\nu^{a,s-1})$
	$= (-1)^{sq+1}\overline{\partial}\overline{\partial}\eta^{a,q-1}\nu^{a,s-2} + (-1)^{sq+q+1}\overline{\partial}\eta^{a,q-1}\overline{\partial}\nu^{a,s-2}$
	$+(-1)^{sq+s+1}\delta\overline{\partial}\eta^{a,q-1}\nu^{a,s-1} + (-1)^{sq+q+s+1}\overline{\partial}\eta^{a,q-1}\delta\nu^{a,s-1}$
	$= (-1)^{sq+q+1}\overline{\partial}\eta^{a,q-1}\overline{\partial}\nu^{a,s-2} + (-1)^{sq+s+1}\delta\overline{\partial}\eta^{a,q-1}\nu^{a,s-1}$
	$+(-1)^{sq+q+s+1}\overline{\partial}\eta^{a,q-1}\delta\nu^{a,s-1}$

	Holomorphic RHS chart for $\deg(\eta)$ even.
Degree	Expression
c	$((d_D\eta) * \nu)^c + (\eta * (d_D\nu))^c$
	$= (d_D\eta)^c\nu^c + \eta^c(d_D\nu)^c$
	$= \delta\eta^c\nu^c + \eta^c\delta\nu^c$
$h, 0$	$((d_D\eta) * \nu)^{h,0} + (\eta * (d_D\nu))^{h,0}$
	$= -(d_D\eta)^c\nu^{h,0} + \eta^c(d_D\nu)^{h,0}$
	$= -\delta\eta^c\nu^{h,0} + \eta^c[\nu^c - \delta\nu^{h,0}]$
	$= \eta^c\nu^c - \delta\eta^c\nu^{h,0} - \eta^c\delta\nu^{h,0}$
$h, 1$	$((d_D\eta) * \nu)^{h,1} + (\eta * (d_D\nu))^{h,1}$
	$= (d_D\eta)^c\nu^{h,1} + \eta^c(d_D\nu)^{h,1}$
	$= \delta\eta^c\nu^{h,1} + \eta^c[\partial\nu^{h,0} + \delta\nu^{h,1}]$
	$= \eta^c\partial\nu^{h,0} + \delta\eta^c\nu^{h,1} + \eta^c\delta\nu^{h,1}$
$\dots$	$\dots$
h, m_1	$((d_D\eta) * \nu)^{h,m_1} + (\eta * (d_D\nu))^{h,m_1}$
	$= (-1)^{m_1+1}(d_D\eta)^c\nu^{h,m_1} + \eta^c(d_D\nu)^{h,m_1}$
	$= (-1)^{m_1+1}\delta\eta^c\nu^{h,m_1} + \eta^c[\partial\nu^{h,m_1-1} + (-1)^{m_1+1}\delta\nu^{h,m_1}]$
	$= (-1)^{m_1+1}\delta\eta^c\nu^{h,m_1} + \eta^c\partial\nu^{h,m_1-1} + (-1)^{m_1+1}\eta^c\delta\nu^{h,m_1}$
$\dots$	$\dots$
$h, r-1$	$((d_D\eta) * \nu)^{h,r-1} + (\eta * (d_D\nu))^{h,r-1}$
	$= (-1)^r(d_D\eta)^c\nu^{h,r-1} + \eta^c(d_D\nu)^{h,r-1}$
	$= (-1)^r\delta\eta^c\nu^{h,r-1} + \eta^c[\partial\nu^{h,r-2} + (-1)^r\delta\nu^{h,r-1}]$
	$= \eta^c\partial\nu^{h,r-2} + (-1)^r\delta\eta^c\nu^{h,r-1} + (-1)^r\eta^c\delta\nu^{h,r-1}$

	Holomorphic RHS chart for deg(η) even.
Degree	Expression
h, r	$((d_D\eta) * \nu)^{h,r} + (\eta * (d_D\nu))^{h,r}$
	$= (d_D\eta)^{h,0}\partial\nu^{h,r-1} + (-1)^r\eta^{h,0}\partial(d_D\nu)^{h,r-1}$
	$= [\eta^c - \delta\eta^{h,0}]\partial\nu^{h,r-1} + (-1)^r\eta^{h,0}\partial[\partial\nu^{h,r-2} + (-1)^r\delta\nu^{h,r-1}]$
	$= \eta^c\partial\nu^{h,r-1} - \delta\eta^{h,0}\partial\nu^{h,r-1} + (-1)^r\eta^{h,0}\partial\partial\nu^{h,r-2} + \eta^{h,0}\partial\delta\nu^{h,r-1}$
	$= \eta^c\partial\nu^{h,r-1} - \delta\eta^{h,0}\partial\nu^{h,r-1} + \eta^{h,0}\delta\partial\nu^{h,r-1}$
$\dots$	$\dots$
$h, r + m_2$	$((d_D\eta) * \nu)^{h,r+m_2} + (\eta * (d_D\nu))^{h,r+m_2}$
	$= (-1)^{rm_2}(d_D\eta)^{h,m_2}\partial\nu^{h,r-1} + (-1)^{rm_2+r}\eta^{h,m_2}\partial(d_D\nu)^{h,r-1}$
	$= (-1)^{rm_2}[\partial\eta^{h,m_2-1} + (-1)^{m_2+1}\delta\eta^{h,m_2}]\partial\nu^{h,r-1}$
	$+(-1)^{rm_2+r}\eta^{h,m_2}\partial[\partial\nu^{h,r-2} + (-1)^r\delta\nu^{r-1}]$
	$= (-1)^{rm_2}\partial\eta^{h,m_2-1}\partial\nu^{h,r-1} + (-1)^{rm_2+m_2+1}\delta\eta^{h,m_2}\partial\nu^{h,r-1}$
	$+(-1)^{rm_2+r}\eta^{h,m_2}\partial\partial\nu^{h,r-2} + (-1)^{rm_2}\eta^{h,m_2}\partial\delta\nu^{r-1}$
	$= (-1)^{rm_2}\partial\eta^{h,m_2-1}\partial\nu^{h,r-1} + (-1)^{rm_2+m_2+1}\delta\eta^{h,m_2}\partial\nu^{h,r-1}$
	$+(-1)^{rm_2}\eta^{h,m_2}\delta\partial\nu^{r-1}$
$\dots$	$\dots$
$h, r + p - 1$	$((d_D\eta) * \nu)^{h,r+p-1} + (\eta * (d_D\nu))^{h,r+p-1}$
	$= (-1)^{rp+r}(d_D\eta)^{h,p-1}\partial\nu^{h,r-1} + (-1)^{rp}\eta^{h,p-1}\partial(d_D\nu)^{h,r-1}$
	$= (-1)^{rp+r}[\partial\eta^{h,p-2} + (-1)^p\delta\eta^{h,p-1}]\partial\nu^{h,r-1}$
	$+(-1)^{rp}\eta^{h,p-1}\partial[\partial\nu^{h,r-2} + (-1)^r\delta\nu^{r-1}]$
	$= (-1)^{rp+r}\partial\eta^{h,p-2}\partial\nu^{h,r-1} + (-1)^{rp+r+p}\delta\eta^{h,p-1}\partial\nu^{h,r-1}$
	$+(-1)^{rp}\eta^{h,p-1}\partial\partial\nu^{h,r-2} + (-1)^{rp+r}\eta^{h,p-1}\partial\delta\nu^{r-1}$
	$= (-1)^{rp+r}\partial\eta^{h,p-2}\partial\nu^{h,r-1} + (-1)^{rp+r+p}\delta\eta^{h,p-1}\partial\nu^{h,r-1}$
	$+(-1)^{rp+r}\eta^{h,p-1}\delta\partial\nu^{r-1}$

	Anti-holomorphic RHS chart for deg(η) even.
Degree	Expression
$a, 0$	$((d_D\eta) * \nu)^{a,0} + (\eta * (d_D\nu))^{a,0}$
	$= (d_D\eta)^{a,0}\nu^c + \eta^{a,0}(d_D\nu)^c$
	$= [-\eta^c - \delta\eta^{a,0}]\nu^c + \eta^{a,0}\delta\nu^c$
	$= -\eta^c\nu^c - \delta\eta^{a,0}\nu^c + \eta^{a,0}\delta\nu^c$
$a, 1$	$((d_D\eta) * \nu)^{a,1} + (\eta * (d_D\nu))^{a,1}$
	$= (d_D\eta)^{a,1}\nu^c + \eta^{a,1}(d_D\nu)^c$
	$= [\overline{\partial}\eta^{a,0} + \delta\eta^{a,1}]\nu^c + \eta^{a,1}\delta\nu^c$
	$= \overline{\partial}\eta^{a,0}\nu^c + \delta\eta^{a,1}\nu^c + \eta^{a,1}\delta\nu^c$
$\ldots$	$\ldots$
a, m_3	$((d_D\eta) * \nu)^{a,m_3} + (\eta * (d_D\nu))^{a,m_3}$
	$= (d_D\eta)^{a,m_3}\nu^c + \eta^{a,m_3}(d_D\nu)^c$
	$= [\overline{\partial}\eta^{a,m_3-1} + (-1)^{m_3+1}\delta\eta^{a,m_3}]\nu^c + \eta^{a,m_3}\delta\nu^c$
	$= \overline{\partial}\eta^{a,m_3-1}\nu^c + (-1)^{m_3+1}\delta\eta^{a,m_3}\nu^c + \eta^{a,m_3}\delta\nu^c$
$\ldots$	$\ldots$
$a, q-1$	$((d_D\eta) * \nu)^{a,q-1} + (\eta * (d_D\nu))^{a,q-1}$
	$= (d_D\eta)^{a,q-1}\nu^c + \eta^{a,q-1}(d_D\nu)^c$
	$= [\overline{\partial}\eta^{a,q-2} + (-1)^q\delta\eta^{a,q-1}]\nu^c + \eta^{a,q-1}\delta\nu^c$
	$= \overline{\partial}\eta^{a,q-2}\nu^c + (-1)^q\delta\eta^{a,q-1}\nu^c + \eta^{a,q-1}\delta\nu^c$

	Anti-holomorphic RHS chart for deg(η) even.
Degree	Expression
a,q	$((d_D\eta)*\nu)^{a,q}+(\eta*(d_D\nu))^{a,q}$ $=\overline{\partial}(d_D\eta)^{a,q-1}\nu^{a,0}-\overline{\partial}\eta^{a,q-1}(d_D\nu)^{a,0}$ $=\overline{\partial}[\overline{\partial}\eta^{a,q-2}+(-1)^q\delta\eta^{q-1}]\nu^{a,0}-\overline{\partial}\eta^{a,q-1}[-\nu^c-\delta\nu^{a,0}]$ $=\overline{\partial}\overline{\partial}\eta^{a,q-2}\nu^{a,0}+(-1)^q\overline{\partial}\delta\eta^{q-1}\nu^{a,0}+\overline{\partial}\eta^{a,q-1}\nu^c+\overline{\partial}\eta^{a,q-1}\delta\nu^{a,0}$ $=\overline{\partial}\eta^{a,q-1}\nu^c+(-1)^q\delta\overline{\partial}\eta^{q-1}\nu^{a,0}+\overline{\partial}\eta^{a,q-1}\delta\nu^{a,0}$
$\dots$ $a,q+m_4$	$\dots$ $((d_D\eta)*\nu)^{a,q+m_4}+(\eta*(d_D\nu))^{a,q+m_4}$ $=(-1)^{m_4q+m_4}\overline{\partial}(d_D\eta)^{a,q-1}\nu^{a,m_4}+(-1)^{m_4q+1}\overline{\partial}\eta^{a,q-1}(d_D\nu)^{a,m_4}$ $=(-1)^{m_4q+m_4}\overline{\partial}[\overline{\partial}\eta^{a,q-2}+(-1)^q\delta\eta^{a,q-1}]\nu^{a,m_4}$ $+(-1)^{m_4q+1}\overline{\partial}\eta^{a,q-1}[\overline{\partial}\nu^{a,m_4-1}+(-1)^{m_4-1}\nu^{a,m_4}]$ $=(-1)^{m_4q+m_4}\overline{\partial}\overline{\partial}\eta^{a,q-2}\nu^{a,m_4}+(-1)^{m_4q+m_4+q}\overline{\partial}\delta\eta^{a,q-1}\nu^{a,m_4}$ $+(-1)^{m_4q+1}\overline{\partial}\eta^{a,q-1}\overline{\partial}\nu^{a,m_4-1}+(-1)^{m_4q+m_4}\overline{\partial}\eta^{a,q-1}\delta\nu^{a,m_4}$ $=(-1)^{m_4q+1}\overline{\partial}\eta^{a,q-1}\overline{\partial}\nu^{a,m_4-1}+(-1)^{m_4q+m_4+q}\delta\overline{\partial}\eta^{a,q-1}\nu^{a,m_4}$ $+(-1)^{m_4q+m_4}\overline{\partial}\eta^{a,q-1}\delta\nu^{a,m_4}$
$\dots$ $a,q+s-1$	$\dots$ $((d_D\eta)*\nu)^{a,q+s-1}+(\eta*(d_D\nu))^{a,q+s-1}$ $=(-1)^{sq+q+s+1}\overline{\partial}(d_D\eta)^{a,q-1}\nu^{a,s-1}+(-1)^{sq+q+1}\overline{\partial}\eta^{a,q-1}(d_D\nu)^{a,s-1}$ $=(-1)^{sq+q+s+1}\overline{\partial}[\overline{\partial}\eta^{a,q-2}+(-1)^q\delta\eta^{a,q-1}]\nu^{a,s-1}$ $+(-1)^{sq+q+1}\overline{\partial}\eta^{a,q-1}[\overline{\partial}\nu^{a,s-2}+(-1)^s\nu^{a,s-1}]$ $=(-1)^{sq+q+s+1}\overline{\partial}\overline{\partial}\eta^{a,q-2}\nu^{a,s-1}+(-1)^{sq+s+1}\overline{\partial}\delta\eta^{a,q-1}\nu^{a,s-1}$ $+(-1)^{sq+q+1}\overline{\partial}\eta^{a,q-1}\overline{\partial}\nu^{a,s-2}+(-1)^{sq+q+s+1}\overline{\partial}\eta^{a,q-1}\delta\nu^{a,s-1}$ $=(-1)^{sq+q+1}\overline{\partial}\eta^{a,q-1}\overline{\partial}\nu^{a,s-2}+(-1)^{sq+s+1}\delta\overline{\partial}\eta^{a,q-1}\nu^{a,s-1}$ $+(-1)^{sq+q+s+1}\overline{\partial}\eta^{a,q-1}\delta\nu^{a,s-1}$

1.1.2 Specific Formula for Bott-Chern, Bott-Chern

Let $u = p + q - 1$ and $v = r + s - 1$, and η and ν as before. The Bott-Chern, Bott-Chern product is given by this table.

degree	p+q-1 even	p+q-1 odd	term
c	1	1	$\eta^c \nu^c$
$h, 0$	1	(-1)	$\eta^c \nu^{h,0}$
$h, 1$	1	1	$\eta^c \nu^{h,1}$
$h, 2$	1	(-1)	$\eta^c \nu^{h,1}$
$\vdots$			$\vdots$
h, m_1	1	$(-1)^{m_1+1}$	$\eta^c \nu^{h,m_1}$
$\vdots$			$\vdots$
$h, r-1$	1	$(-1)^r$	$\eta^c \nu^{h,r-1}$
h, r	$(-1)^r$	1	$\eta^{h,0} \partial \nu^{h,r-1}$
$h, r+1$	1	$(-1)^r$	$\eta^{h,1} \partial \nu^{h,r-1}$
$\vdots$			$\vdots$
$h, r+m_2$	$(-1)^{r(m_2+1)}$	$(-1)^{rm_2}$	$\eta^{h,m_2} \partial \nu^{h,r-1}$
$\vdots$			$\vdots$
$h, r+p-1$	$(-1)^{rp}$	$(-1)^{r(p+1)}$	$\eta^{h,p-1} \partial \nu^{h,r-1}$
$h, r+p$	1	1	0
$\vdots$			$\vdots$
$h, p+r+q+s-1$	1	1	0

degree	p+q-1 even	p+q-1 odd	term
$a, 0$	1	1	$\eta^{a,0}\nu^c$
$a, 1$	1	1	$\eta^{a,1}\nu^c$
$\vdots$			$\vdots$
a, m_3	1	1	$\eta^{a,m_3}\nu^c$
$\vdots$			$\vdots$
$a, q-1$	1	1	$\eta^{a,q-1}\nu^c$
a, q	(-1)	1	$\partial\eta^{a,q-1}\nu^{a,0}$
$a, q+1$	$(-1)^{q+1}$	$(-1)^{q+1}$	$\partial\eta^{a,q-1}\nu^{a,1}$
$\vdots$			$\vdots$
$a, q+m_4$	$(-1)^{m_4 q+1}$	$(-1)^{m_4(q+1)}$	$\partial\eta^{a,q-1}\nu^{a,m_4-1}$
$\vdots$			$\vdots$
$a, q+s-1$	$(-1)^{(s+1)q+1}$	$(-1)^{(q+1)(s+1)}$	$\partial\eta^{a,q-1}\nu^{a,s-1}$
$a, q+s$	1	1	0
$\vdots$			$\vdots$
$a, p+r+q+s-1$	1	1	0

1.1.3 Specific Formula for Bott-Chern, Aeppli

Let η and ν be slightly redefined as follows:

$$\eta \in \bigoplus_{j+k=p+q-1} \check{C}^j(\mathcal{U}, \mathcal{B}^k_{p,q}(\mathbb{R}(p))) \tag{1.1.4}$$

$$\eta = \eta^c; \eta^{h,0}, \eta^{h,1}, \ldots, \eta^{h,p+q-1}; \eta^{a,0}, \eta^{a,1}, \ldots, \eta^{a,p+q-1}$$

$$\nu \in \bigoplus_{j+k=r+s} \check{C}^j(\mathcal{U}, \mathcal{B}^k_{r+1,s+1}(\mathbb{R}(r)))$$

$$\nu = \nu^c; \nu^{h,0}, \nu^{h,1}, \ldots, \nu^{h,r+s}; \nu^{a,0}, \nu^{a,1}, \ldots, \nu^{a,r+s}$$

Then the following table gives the product formula for the element

$$\eta * \nu \in \bigoplus_{j+k=p+r+q+s} \check{C}^j(\mathcal{U}, \mathcal{B}^k_{p+r+1,q+s+1}(\mathbb{R}(p+r))) \tag{1.1.5}$$

$$\eta * \nu = (\eta * \nu)^c; (\eta * \nu)^{h,0}, (\eta * \nu)^{h,1}, \ldots, (\eta * \nu)^{h,p+r+q+s};$$

$$(\eta * \nu)^{a,0}, (\eta * \nu)^{a,1}, \ldots, (\eta * \nu)^{a,p+r+q+s}$$

degree	p+q-1 even	p+q-1 odd	term
c	1	1	$\eta^c\nu^c$
$h,0$	1	(-1)	$\eta^c\nu^{h,0}$
$h,1$	1	1	$\eta^c\nu^{h,1}$
$h,2$	1	(-1)	$\eta^c\nu^{h,1}$
$\vdots$			$\vdots$
h,m_1	1	$(-1)^{m_1+1}$	$\eta^c\nu^{h,m_1}$
$\vdots$			$\vdots$
h,r	1	$(-1)^{r+1}$	$\eta^c\nu^{h,r}$
$h,r+1$	$(-1)^{r+1}$	1	$\eta^{h,0}\partial\nu^{h,r}$
$h,r+2$	1	$(-1)^{r+1}$	$\eta^{h,1}\partial\nu^{h,r}$
$\vdots$			$\vdots$
$h,r+m_2$	$(-1)^{(r+1)m_2}$	$(-1)^{(r+1)(m_2+1)}$	$\eta^{h,m_2-1}\partial\nu^{h,r}$
$\vdots$			$\vdots$
$h,r+p$	$(-1)^{(r+1)p}$	$(-1)^{(r+1)(p+1)}$	$\eta^{h,p-1}\partial\nu^{h,r}$
$h,r+p+1$	1	1	0
$\vdots$			$\vdots$
$h,p+r+q+s$	1	1	0

degree	p+q-1 even	p+q-1 odd	term
$a, 0$	1	1	$\eta^{a,0}\nu^c$
$a, 1$	1	1	$\eta^{a,1}\nu^c$
$\vdots$			$\vdots$
a, m_3	1	1	$\eta^{a,m_3}\nu^c$
$\vdots$			$\vdots$
$a, q-1$	1	1	$\eta^{a,q-1}\nu^c$
a, q	(-1)	1	$\partial\eta^{a,q-1}\nu^{a,0}$
$a, q+1$	$(-1)^{q+1}$	$(-1)^{q+1}$	$\partial\eta^{a,q-1}\nu^{a,1}$
$\vdots$			$\vdots$
$a, q+m_4$	$(-1)^{m_4q+1}$	$(-1)^{m_4(q+1)}$	$\partial\eta^{a,q-1}\nu^{a,m_4-1}$
$\vdots$			$\vdots$
$a, q+s$	$(-1)^{(s)q+1}$	$(-1)^{(q+1)(s)}$	$\partial\eta^{a,q-1}\nu^{a,s}$
$a, q+s+1$	1	1	0
$\vdots$			$\vdots$
$a, p+r+q+s$	1	1	0

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