The Status of Mathematical Induction in an Axiomatic System

by

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Abstract

This thesis investigates the status of Mathematical Induction (MI) in an axiomatic system. It first reviews and analyses the status of MI in the works of Gotlob Frege and Richard Dedekind, the pioneers of logicism who, in providing foundations for arithmetic, attempted to reduce MI to what they considered logic to be. These analyses reveal that their accounts of MI have the same structure and produce the same result. This is true even though the two thinkers used different components as fundamental logical elements and went through different routes to eventually prove (on the basis of more fundamental logical axioms and rules of inference and definitions) what they considered MI to be. Based on these analyses, we infer a formulation, i.e., U-MI, that presents both Frege's and Dedekind's formulations of MI.

We then evaluate the possible proof- and model-theoretic problems that such a formulation of MI faces. These problems include the problem of impredicativity and the unattainability of the infinitary nature of MI in a finitary logic. We then introduce and defend our own account of the status of MI in an axiomatic system, in which MI is axiomatizable/derivable in an infinitary many-sorted logic. The final part of the study investigates concerns with the metatheoretical use of MI – in particular the circularity problem in such a use. Within this last part, we also explicate and elaborate on one of the advantages of our account of the status of MI in an axiomatic system in comparison to the rival accounts.

To my wife Maryam, with love.

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Introduction

One of the most important principles or methods of proof in mathematics is Mathematical Induction (henceforth MI), variants of which also apply to other well-ordered or recursively defined collections of items or entities (I use the word “items” or “entities” instead of “objects” to avoid any unnecessary ontological commitment). Historically, an implicit proof by MI can be traced back to Euclid's proof of the infinitude of primes, and perhaps even to one of the arguments proposed in Plato’s Parmenides[1]. However, it is a generally accepted opinion that the first explicit formulation of MI is contained in the *Traité du triangle arithmétique* (1653) by Blaise Pascal. Since Jacob Bernoulli’s use of it, this method of proof (in its complete form, namely the proof from n to $n + 1$) has become more or less well known. (In fact, Frege, in his *Begriffsschrift*[2], considered Bernoulli to be one of the originators of MI.) However, the systematic treatment of MI came only in the 19th and early 20th centuries, with

[1] The argument occurs “in the discussion of the consequences of the second hypothesis, [when] Parmenides envisages in his inquiry about the nature of the ‘one’” (Acerbi, 2000, p. 65).
[2] Frege, G. (1879); translated in Frege, G., & Bynum, T. W. (1972).

mathematicians, logicians, and philosophers such as Augustus De Morgan (who introduced the term “Mathematical Induction”),[3] George Boole, Gottlob Frege, Charles Sanders Pierce, Richard Dedekind, Giuseppe Peano and Bertrand Russell.

In fact, the vigorous development of mathematical logic, together with the development of the rigorous axiomatic method in the foundations of mathematics, during this period motivated mathematicians, logicians and philosophers to work on the foundations of *arithmetic* as well, focusing in particular on the axiomatization of arithmetic. The important role of MI in this context was recognized, and MI came to be considered one of the most important axioms of an axiomatic system of arithmetic. In addition, just as the importance of MI in axiomatic systems was acknowledged, its importance in metamathematical and metalinguistic proofs was also recognized.

In providing the foundation for arithmetic, however, some mathematicians, logicians, and philosophers have tried to go one step further than merely taking MI as an axiom in an axiomatic system. They have sought to reduce MI to more fundamental axioms and prove it as a theorem. Frege and Dedekind – as pioneers and the most prominent thinkers among those who have provided foundations for arithmetic – tried, in particular, to justify and prove MI on the basis of more fundamental logical axioms and rules of inference and definitions.

[3] See Cajori (1918), p. 200, and Burton (2011), p. 466, in which they refer to the article "Induction (Mathematics)" (1838), in Long (1833-1843), Volume XII, pp. 465-466, written by Augustus de Morgan.

Therefore, in seeking justification and proof for MI, we are directed to the foundations of arithmetic. Logicism, among the doctrines of the foundations of arithmetic, defends the reduction of arithmetic to logic. It proves, first, that the axioms/theorems of arithmetic (including MI, which is our main concern from the proof-theoretic viewpoint) are fully derived from basic truths (axioms) and definitions of logic by its rules of inference; and second, that the concepts involved in such theorems, and the objects whose existence they might imply, are of a purely logical nature. Frege, Russell (in collaboration with Whitehead), and Dedekind are pioneers of this view. For the sake of brevity, due to the similarity between Frege's and Russell's works in what we are concerned about – namely the proof-theoretic status of Mathematical Induction – and due to Frege's pioneering works on this issue, we concentrate on Frege's works on MI on the one hand, and Dedekind's on the other, as representatives of two types of approaches in logicism.

There are dissimilarities between what these thinkers include within their conception of logic. At the same time, what they accept as the constituents of logic are different from what is generally accepted today; and this is one of the reasons that full-fledged logicism is problematic. For example, while Dedekind explicitly defends logicism, he, as one of the earliest founders of rigorous axiomatic set theory, uses "classes" and relation of "belonging to a class as an element" in his structure as logical foundational stones[4], although these items are not commonly accepted as elements of logic today. By contrast, Frege uses

[4] As Quine mentioned in Quine (1970), p. 65, "pioneers in modern logic [explicitly or implicitly and directly or indirectly] viewed set theory as logic."

“concepts” and “logical relations” as his logical foundational stones – items that are, in a sense, more compatible with modern views of the constituents of logic. However, Frege’s logical system includes second-order logic, about which there are debates; scholars are divided over whether it should be accepted in a logical system in addition to first-order logic. An example of such a debate is whether to consider second-order logic as a part of set theory, or set theory in disguise, or “set theory in sheep's clothing” as Quine calls it.[5] Frege also makes use of additional principles, such as his Axiom V (or Basic Law V), that turned out on one hand to be inconsistent, and on the other not to be a part of logic.

Subsequent attempts – most notably by Whitehead and Russell, and later, by Neo-Fregeans – to repair Frege’s system have also had to appeal to principles that are not considered logical. That is, in order to provide foundations for arithmetic it is necessary to add to logic other things such as set theory, as generally accepted in the literature, or second-order logic accompanied by Hume’s principle, as advocated by some neo-logicists. Therefore, based on what generally is accepted as logic, full-fledged logicism has failed. This conclusion is further reinforced by Gödel’s incompleteness theorems, which likewise reveal the problematic nature of the full-fledged logicist project. As Hellman argues[6], according to Gödel’s second incompleteness theorem, we cannot formalize any finitely axiomatizable logicist system that includes elementary arithmetic, and although the non-finitely axiomatizable systems may exist we are not able to know of any particular system of this kind.

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[5] See Quine (1970), p. 66.
[6] See Hellman (1981).

The main goal of this research, however, is not a defence or critique of logicism or revised versions of it (though as a subsequent and a secondary result, it does end up, in a sense, to advocate a side of the debate). The focus is rather on the status of MI, from the proof-theoretic view, within an axiomatic system. However, since logicists aim to prove all the axioms and theorems of arithmetic, including MI, based on logic, their works are important in our investigation, and we will analyze them as far as they are related to our goal. Hence, we first analyze the proof of MI within the works of the pioneers of logicism, Frege and Dedekind. We then evaluate their proofs, and investigate possible proof- and model-theoretic problems. Finally, we introduce and defend our account. We also analyze and evaluate possible concerns in regard to the metateoretical use of MI.

Accordingly, the first two chapters of the study are dedicated to the explication and analysis of Frege's and Dedekind's works in proving MI as a theorem within an axiomatic system. These works are the earliest, and at the same time, among the best available in the literature. They are also referred to by logicians and philosophers who defend the plausibility of the existence of such a justification of MI based on axioms of logic supplemented by some other necessary axioms. It is noteworthy that as a result of the dissimilarities between the logicist foundations of arithmetic introduced by Frege and Dedekind, their justifications and proofs for MI are constructed in different conceptual frameworks; however, as we will find in our investigation, they have the same structure and end up with the same result. Therefore, in the third chapter of the thesis, we analyze and evaluate Frege's and Dedekind's works *together*. In that chapter we begin to evaluate the plausibility of

Frege's and Dedekind's proofs of MI as a theorem based on a set of axioms, definitions, and rules of inference (supplemented by other required extra axioms), and we investigate the variety of problems that might be raised in their approach. These problems include misrepresentation of MI, the impredicativity problem, and the unattainability of infinitary nature of MI in a finitary logic. Finally, we introduce and defend our account of the status of MI in an axiomatic system in which MI is axiomatizable/derivable in an infinitary many-sorted logic. That is, we take MI as a fundamental axiom *independent* of axioms of classical logic, or we derive MI as a theorem from a set of axioms that includes a fundamental axiom *independent* of axioms of classical logic. At the end we investigate concerns with the metatheoretical use of MI – in particular the circularity problem in the metatheoretical use of MI. Within this part of the last chapter, we also explicate and elaborate on one of the advantages of our account of the status of MI in an axiomatic system in comparison to its rival accounts.

Chapter 1

Analysis of Frege's Works on Mathematical Induction

In this chapter we analyze Frege's works on mathematical induction from a proof-theoretic viewpoint. These works include *Begriffsschrift* (1879)*, Grundlagen*[7] (1884), and *Grundgesetze*[8] (vol. 1, 1893; vol. 2, 1903), although his other writings have been investigated as well. Our focus is primarily on *Begriffsschrift*, and, when necessary, on *Grundgesetze*.

As Frege remarks in the preface to *Begriffsschrift*, arithmetic "was the starting point of the train of thoughts that led"[9] him to write *Begriffsschrift* and his later works. That was to make the fundamental concepts and basic assumptions upon which arithmetic is built absolutely clear, and eventually to prove the basic laws

[7] Frege, G. (1884); translated in Frege, G., & Austin, J. L. (1980).
[8] Frege, G. (1893), and Frege, G. (1903); translated in Frege, G., Ebert, P. A., Rossberg, M., & Wright, C. (2013), and partly translated in Frege, G., & Furth, M (1964).
[9] See Frege, G., & Bynum, T. W. (1972), p. 107.

of arithmetic. Confronted with the latter task, he had to decide what would constitute a proof. In the preface to *Begriffsschrift*, he tells us that “we divide all truths that require a proof into two kinds: those whose proof can be given purely logically, and those whose proof must be grounded on empirical facts.”[10] In his later book, *Grundlagen*, Frege argues that not only are the laws of arithmetic not synthetic *a posteriori* truths, as Mill had thought, but they are also not synthetic *a priori* truths, as Kant maintained, which leaves only the possibility that they are analytic *a priori* truths. Therefore, the laws of arithmetic must proceed purely logically.

In the explanation of the course he took to investigate “how far one could get in arithmetic by means of logical deduction alone,”[11] Frege points out that he first sought to reduce the concept of “ordering in a sequence” to that of “logical ordering” or “logical consequence.” In striving to fulfil this goal in the strictest way, he found ordinary language inadequate: its words and phrases are often ambiguous and imprecise, having many different meanings. In ordinary discourse, assumptions are not explicitly and clearly stated. The modes of inference are numerous and loose, and Frege believed that they must be syntactically defined to ensure correctness of reasoning. Finally, he thought that two-dimensional writing must be exploited for the sake of perspicuity. Thus, Frege devised his symbolic language, with its definitions, axioms and inference rules, in his book *Begriffsschrift*, and further developed it in his book *Grundgesetze*. In what

[10] Frege, G., & Beaney, M. (1997), p. 48.
[11] See Frege, G., & Bynum, T. W. (1972), p. 104.

follows, we focus on those parts of *Begriffsschrift*, (and *Grundgesetze*, when necessary) that are required for our present purpose.

In the first part of his *Begriffsschrift*, 'Definitions of the Symbols,' Frege introduces his notation for his primitive connectives, and using ordinary language he provides us with pre-constructive or elucidative explanations for them. He also explains what the counterpart of these connectives are in ordinary language, and at the same time he presents the semantics of these connectives – a crucial step toward the invention or discovery of the truth tables[12, 13] we have today.

Frege chooses symbols for: (1) assertion, (2) negation, and (3) conditionalization (implication) of propositions; and then he uses negation and implication to define conjunction and disjunction. Furthermore, to state the fact that two formulae express the same conceptual content, he adds a sign indicating identity of content. Using these tools, he was able to express logical relations among judgeable (assertible) contents. To express relations *within* such judgeable contents, Frege "regard[s] sentences as functions of the names occurring within them, treating property-expressions as functions of one argument, and relation-expressions as functions of two *or* more arguments, and adding what would later be called 'variable-binding quantifiers,' "[14] and he introduces new symbols for property-expressions and relation-expressions, and adds a sign indicating universal quantifiers.

[12] See Kneale, W. C., & Kneale, M. (1962), pp. 420, 531.
[13] See Church, A. (1996), pp. 161-2.
[14] See Frege, G., & Bynum, T. W. (1972), p. 13.

Furthermore, he explicitly introduces and labels Modus Ponens as his only mode of inference, “at least in all cases where a new judgment is derived from more than one single judgment.”[15] He was apparently aware that he was using other modes of inference, in particular the rule of substitution, which is non-derivable from the rule of Modus Ponens, to derive a new judgment from a single given judgement. He also uses other rules such as universal generalization or universal introduction (as a rule of inference specific to predicate logic) without assigning a specific name to them as rules of inference. It is noteworthy that he introduces Modus Ponens (and the universal introduction rule) in Part I, ‘Definition of the Symbols,’ as a result of (or more precisely, in connection with) the definition and meaning/semantics of the conditionals (and universal quantifiers), and not in Part II, where he presents his axioms (that in principal, to some degree, are interchangeable with inference rules). This shows the close connection between conditionals and Modus Ponens. In fact, Frege explains that he chooses implication as his basic sign because it simplifies the formulation of his inferences, the main rule of which is Modus Ponens. (A similar argument might be given for the case of universal quantifiers and the universal introduction rule.) These preliminary steps enable Frege to develop the first system of predicate logic.

In Part II of *Begriffsschrift*, entitled ‘Representation and Derivation of Some Judgements of Pure Thought,’ Frege lays down nine axioms through which (accompanied by the rules of inference) he shows how complex judgements can

[15] See Frege, G., & Bynum, T. W. (1972), p. 119.

be represented and derived in his axiomatic system. These axioms, presented in modern notation (along with their numbers in *Begriffsschrift*), are as follows:

(1) $a \rightarrow (b \rightarrow a)$

(2) $[c \rightarrow (b \rightarrow a)] \rightarrow [(c \rightarrow b) \rightarrow (c \rightarrow a)]$

(8) $[d \rightarrow (b \rightarrow a)] \rightarrow [b \rightarrow (d \rightarrow a)]$

(28) $(b \rightarrow a) \rightarrow (\sim a \rightarrow \sim b)$

(31) $\sim\sim a \rightarrow a$

(41) $a \rightarrow \sim\sim a$

(52) $c = d \rightarrow f(c) \rightarrow f(d)$ or $c = d \rightarrow F_c \rightarrow F_b$

(54) $a = a$

(58) $(\forall \mathfrak{a}) f(\mathfrak{a}) \rightarrow f(c)$ or $(\forall \mathfrak{a})\, F_{\mathfrak{a}} \rightarrow F_c$

Axioms (1), (2), (28), (31), and (41) can form a complete set of axioms for propositional logic (although, using negation and implication, we can form a complete set of axioms with fewer axioms). Axiom (8) can be derived (using inference rules Modus Ponens and substitution) from Axioms (1) and (2). Axioms (52) and (54) are concerned with identity of content, and Axiom (58) is the axiom for predicate logic (the counterpart of the inference rule universal elimination in a natural deduction system).

Several developments in Frege's philosophical views emerged between the publication of *Begriffsschrift* and that of *Grundgesetze* that necessitate some changes in and additions to his logical theory. In *Grundgesetze*, Frege makes two main additions to his notation: a new symbol, '$\dot{\varepsilon}\Phi(\varepsilon)$', to indicate the extension of a concept Φ (or course-of-value or value-range of the function $\Phi(\xi)$), and a further new symbol, ‘ $\backslash\xi$ ’, representing the function to be used for replacing a definite article or definite description in ordinary language. Furthermore, he introduces certain additions to the axioms presented in *Begriffsschrift*, as well as a certain amount of reorganization and reformulation of axioms and rules of inference. In *Grundgesetze*, Axiom V (or the famous Basic Law V), the one responsible for the contradiction discovered by Russell, and Axiom VI, the one illustrating Frege’s theory of description, are *new* axioms[16]; and from a proof-theoretic perspective, we are not concerned about them.

Nine axioms and one explicit inference rule, as well as three implicit inference rules, from *Begriffsschrift* are condensed into the first four axioms[17] and expanded into eighteen rules in *Grundgesetze.* Axioms (1) and (58) in *Begriffsschrift* are retained unchanged as Axiom I and IIa in *Grundgesetze*. Axioms (2), (8), and (28) become provable by means of Rules 4, 2, and 3, respectively, in *Grundgesetze*. Furthermore, Axioms (31), (41), (52) and (54) in *Begriffsschrift* become derivable from Axioms IV, IV, III and III, respectively, in *Grundgesetze*. In fact, in *Grundgesetze*, for convenience and to ensure the brevity of inferences, Frege replaces some of the axioms and theorems presented in *Begriffsschrift* with new inference rules (that is, Rule 1 as a formation rule for horizontal stroke, Rules 2 to 8 as inference rules of propositional and predicate logic, Rules 9 to 12 as rules of

[16] The new symbols ‘ $\dot{\varepsilon}\Phi(\varepsilon)$ ’ and ‘ $\backslash\xi$ ’are used in these axioms as follows: ‘ $\dot{\varepsilon}\Phi(\varepsilon)$ ’ in Axiom V, and both ‘$\dot{\varepsilon}\Phi(\varepsilon)$’ and ‘ $\backslash\xi$ ’ are used in Axiom VI.
[17] These four axioms exclude the two aforementioned new axioms proposed by Frege in *Grundgesetze*.

substitution, and Rules 13 to 18 for the use of brackets).[18]

Axiom IIb of *Grundgesetze*, presented in modern notation,

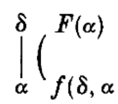
$(\forall \mathfrak{f}\,) M_\beta (\mathfrak{f}(\beta)) \rightarrow M_\beta (f (\beta))$ or $(\forall \mathfrak{F}\,) M_\beta (\mathfrak{F}_\beta) \rightarrow M_\beta (F_\beta)$

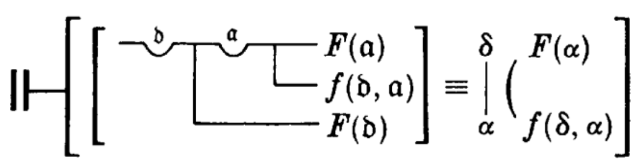
is a second-order formulation of Axiom (58) in *Begriffsschrift*. In fact, in proving Formula (81), i.e. his formulation of MI, in Part III of *Begriffsschrift*, Frege uses Axiom (58) and the derivable theorems from it (in particular Formula (68)). However, to be able to prove Formula (81), he needs Axiom IIb, and the derivable second-order theorems from it (in particular, a second-order theorem analogous to Theorem (68)), which allows quantification over functions or properties. Although he does not yet separate first- and second-order axioms in *Begriffsschrift*, and hence uses the first-order axioms when he needs their analogous second-order ones, this problem can easily be resolved through the addition of the second-order formulation of Axiom (58). Therefore, with that formulation available, from a proof-theoretic perspective his proof of Formula (81) in *Begriffsschrift* is unproblematic.

By the end of Part II of *Begriffsschrift*, Frege has devised the tools necessary to undertake the first phase of his Logicism. As he mentions in the preface to the text, the course he took was first to seek to reduce the concept of "ordering in a sequence" to that of "logical ordering" or "logical consequence." The crucial importance of this reduction, he says, was to provide the strictest possible logical

[18] Frege, G., & Beaney, M. (1997), p. 382.

base for the concept of "number" so that nothing intuitive could intrude here unnoticed, since he believed that any intuitive idea of "sequence," at most, would have validity only in the domain of particular intuition upon which it was founded. In fact, it seems that one of the central ideas that Frege had in mind was that MI must be proven purely logically. Since MI essentially involves sequential ordering, it was a very appropriate choice to provide a logical base for the concept of "ordering in a sequence."

Therefore, in Part III (the final part) of *Begriffsschrift*, entitled 'Some Topics from a General Theory of Sequences,' he pays attention to propositions about sequences. In this part, Frege, using his formal language (i.e. his logic, devised in the first and second parts), and the primitive notion of function or relation f (as a two-place function or relation), starts by providing Definition (69), of *a hereditary property in a sequence*. He denotes this concepts as  (we express it as H^{f}_{F}). The definition is as follows:



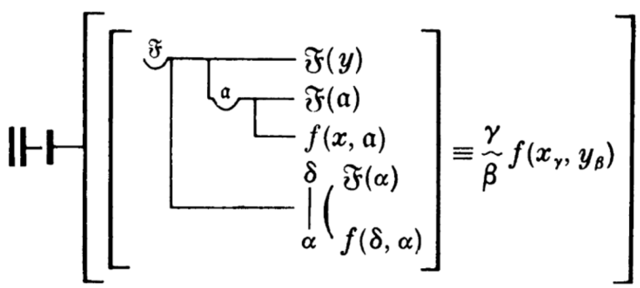
(69)

which can be translated into modern notation as:

$$(\forall \mathfrak{d})\big(F\mathfrak{d} \rightarrow (\forall \mathfrak{a})(f(\mathfrak{d},\mathfrak{a}) \rightarrow F\mathfrak{a})\big) \equiv H^{f}_{F} \qquad (69)$$

in which $\overset{\delta}{\underset{\alpha}{\Big|}}\Big(\begin{smallmatrix}F(\alpha)\\ \\ f(\delta,\alpha\end{smallmatrix}$ (or H^{f}_{F}) is translated into ordinary language as *'(the circumstance that) the property* F *is hereditary in the f-sequence'*. Within the explication of this definition he introduces the idea of a sequence based on the concept of logical ordering, and formalizes it using a two-place function or a logical relation *f*.

Later, in this part of *Begriffsschrift*, he introduces his most innovative definition. This is Definition (76), of *ancestral relation in a sequence* or *ancestral of a relation*. He denotes this concept as $\overset{\gamma}{\underset{\beta}{\sim}} f(x_{\gamma}, y_{\beta})$ (we express it as $P^{x,f}_{y}$). The definition is as follows:



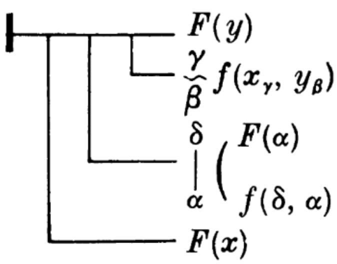
(76)

which can be translated into modern notation as:

$$(\forall \mathfrak{F})\big([\, H^{f}_{\mathfrak{F}} \;\&\; (\forall \mathfrak{a})(f(x, \mathfrak{a}) \rightarrow \mathfrak{F}\mathfrak{a})\,] \rightarrow \mathfrak{F}_y\big) \equiv P^{x,f}_{y} \qquad (76)$$

in which $\overset{\gamma}{\underset{\beta}{\sim}} f(x_{\gamma}, y_{\beta})$ (or $P^{x,\,f}_{y}$) is translated into ordinary language as '*y follows x in the f-sequence' or '**x P**recedes **y** in the **f**-sequence'*. In fact, this definition is a logical analysis of the concept of *ancestral relation in a sequence* or *ancestral of a relation*.

Using his axioms (including the required second-order axioms) and rules of inference (including those which he implicitly uses), along with Definitions (69) and (76), Frege manages, straightforwardly and without any problems, to prove Theorem (81):



(81)

which can be translated into modern notation as:

$$\left(\underset{\text{Basis Clause}}{F_x} \quad \& \quad \underset{\text{Inductive Step}}{H^{f}_{F}} \right) \to \underset{\text{Conclusion}}{\left(P^{x,f}_{y} \to F_y \right)} \qquad (81)$$

upon which, he claims, "Bernoullian induction" or Mathematical Induction (MI) is constructed.[19]

Although Definition (76) is Frege's logical analysis of the concept '*y follows x in the f-sequence*,' from the proof-theoretic point of view, Definitions (69) and (76) are abbreviatory and stipulative definitions, without which one can also prove a

[19] See Frege, G., & Bynum, T. W. (1972), p. 177, footnote.

formula equivalent to Theorem (81) without any problem. We can express such an equivalent formula or theorem as:

$$\Big(F_x \;\&\; \big(\forall\mathfrak{d}\big)\big(F_{\mathfrak{d}} \to (\forall\mathfrak{a})(f(\mathfrak{d},\mathfrak{a}) \to F_{\mathfrak{a}})\big)\Big) \to$$
$$\Big(\Big[\big(\forall\mathfrak{F}\big)\big(\big[(\forall\mathfrak{d})(\mathfrak{F}_{\mathfrak{d}} \to (\forall\mathfrak{a})(f(\mathfrak{d},\mathfrak{a}) \to \mathfrak{F}_{\mathfrak{a}}))\;\&\;(\forall\mathfrak{a})(f(x,\mathfrak{a}) \to \mathfrak{F}_{\mathfrak{a}})\big] \to \mathfrak{F}_y\big)\Big] \to F_y\Big)$$

If we use the uncontroversial abbreviatory Definition (69) to shorten this formula, we can derive Formula (81a) as follows:

$$\Big(F_x \;\&\; H^f{}_F\Big) \to \Big(\Big[\big(\forall\mathfrak{F}\big)\big(\big[H^f{}_{\mathfrak{F}} \;\&\; (\forall\mathfrak{a})(f(x,\mathfrak{a}) \to \mathfrak{F}_{\mathfrak{a}})\big] \to \mathfrak{F}_y\big)\Big] \to F_y\Big) \qquad (81a)$$

For our purpose, we can simplify Formula (81a) as follows:

$$\Big(F_x \;\&\; H^f{}_F\Big) \to \Big(\big(\forall\mathfrak{F}\big)\big([\mathfrak{F}_x \;\&\; H^f{}_{\mathfrak{F}}] \to \mathfrak{F}_y\big) \to F_y\Big) \qquad (81b)$$

where $(\forall\mathfrak{a})(f(x,\mathfrak{a}) \to \mathfrak{F}_{\mathfrak{a}})$ is replaced by $\mathfrak{F}_x$. That is, since $\mathfrak{a}$ immediately follows x in the *f*-sequence (namely it is in the *f*-relation with x, or it is its immediate successor), and since we have hereditary property $H^f{}_F$ (or $H^f{}_{\mathfrak{F}}$) in the *f*-sequence, appearing in both the antecedent and the consequent of the main conditional,[20] whatever is true of x is also true of its immediate successor $\mathfrak{a}$, and we can replace/transform each instance of x with/to its immediate successor $\mathfrak{a}$, such that

[20] In the latter case, in fact, it appears in the antecedent of the consequent of the main conditional.

we can consider $\mathfrak{a}$ as the first member of the sequence. (In other words, in the *f*-sequence the initial element of the sequence shifts from *x* to its immediate successor $\mathfrak{a}$.) Then, for convenience, we can rename $\mathfrak{a}$ as *x*. The only change in the new formulation, (81b), is that not only does *y* follow *x* the in the *f*-sequence, but it can also be equal to *x*. In other words, *y* belongs to the *f*-sequence beginning with *x*, or *x* bears the weak ancestral of the relation *f* to *y*. However, it is noteworthy that for our purpose, whether *x* bears the strong or weak ancestral of the relation *f* to *y* does not matter, and as far as our arguments and conclusions in the following chapters are concerned, Formula (81a) is as adequate as Formula (81b), and we use Formula (81b) for the sake of simplicity and convenience.

Finally, from (81b) we can derive Formula (81c) as follows:

$$(\forall \mathfrak{F})\left([\mathfrak{F}_x \mathbin{\&} H^f{}_{\mathfrak{F}}] \rightarrow \mathfrak{F}_y \right) \rightarrow \left([F_x \mathbin{\&} H^f{}_F] \rightarrow F_y \right) \qquad (81c)$$

In this formula the antecedent of the main conditional is the second-order formulation of MI for an object *y* following an object *x* in an *f*-sequence, which we denote as $\mathbf{MI2}^{x, f}{}_{y}$; and the consequent of the main conditional is the first-order formulation of MI (which can be considered as an schema) for a property F and an object *y* following an object *x* in an *f*-sequence, which we denote as $\mathbf{MI1}^{x, f}{}_{F,\, y}$. Therefore, we can summarize the Formula (81c), as U-MI_{F}, as follows:

$$\mathbf{MI2}^{x, f}{}_{y} \rightarrow \mathbf{MI1}^{x, f}{}_{F,\, y} \qquad \text{U-MI}_{\mathrm{F}}$$

or simply as U-MI, as follows:

$$\text{MI2} \rightarrow \text{MI1} \qquad\qquad \text{U-MI}$$

By the end of Chapter 2, which presents an analysis of Dedekind's works on MI, we are also able to arrive at a formula, which we call U-MI_D, derived from Dedekind's formulation of MI. As we will see, U-MI_D has the same structure as U-MI_F. Therefore, in the Chapter 3, we analyze and evaluate Frege's and Dedekind's works *together*.

Chapter 2

Analysis of Dedekind's Works on Mathematical Induction

In this chapter, we concentrate on Dedekind's main works on the foundations of arithmetic (from which Peano's axioms were adopted), namely *Was Sind Und Was Sollen Die Zahlen?* (*The Nature and Meaning of Numbers*, or more literally, *What are the numbers and what are they for?*) (1888).[21] The text, henceforth referred to as *Was Sind Zahlen*, also offers a pioneering contribution to set theory (although in its initial and early steps).

In Section I of the essay, Dedekind sets out the basic principles of sets (which he calls *systeme*, meaning *systems*). He begins by stating what he means by the term *dinge* (*things or objects*), denoted with lowercase letters such as a, b, c, and s; and he describes the conditions under which two *things* are equal. Then he explicates the concept of *sets*, denoted with uppercase letters such as A, B, C, S, and T,

[21] Dedekind, R. (1888); translated in Dedekind, R. & Beman, W. W. (1909).

observing that they consist of *elements* (the *things* explicated before). Dedekind also defines the condition under which two sets are equal. He considers a set as a thing and hence allows for a set of sets.

Based on his view, a set that contains only one element (namely a singleton set $\{a\}$), should not be considered the same as the element itself (namely an urelement a). However, he uses the same notation for a singleton set $\{a\}$ and an urelement a. In fact, he does not use curly brackets to indicate sets. Later, when he defines the subset relation, he mentions that since every element s of a set S can be regarded as a set (a singleton), he employs the notation ‘ ɜ ’ for both the membership relation, i.e. s ɜ S, and the subset relation, i.e. A ɜ S. For the sake of convenience, however, we use modern notations, namely $s \in S$ for the membership relation, and $A \subset S$ for the subset relation, in this study. Interestingly, he mentions that “we intend here for certain reasons wholly to exclude the empty system [set] which contains no element at all, although for other investigations it may be appropriate to imagine such a system.”[22] Therefore, when he later discusses the intersection of sets, he states that if some sets do not have a common element, their intersection is meaningless.

Dedekind then defines subset (*part*), proper subset (*proper part*), union (*compounded* out), and intersection (*community*), and presents and proves their typical properties.

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[22] Dedekind, R. & Beman, W. W. (1909), pp. 45-46.

In Section II, Dedekind deals with mappings (*transformations*, or functions) Φ of a set S, $S'=\Phi(S)$, the so-called *transform* of its members (*elements*) $s'=\Phi(s)$, and the composition of two or more mappings. First he provides their definitions, and then he presents and proves principles governing them.

In Section III, he develops the idea of one-to-one (*similar*) mappings, *similar* sets (which means sets that are in one-to-one correspondence), and the *class* of sets that are *similar* to a determinate set – the *representative* of the class. He defines these concepts, and presents and proves their fundamental properties.

The core of our analysis is on Section IV of *Was Sind Zahlen*. This section starts with Dedekind's Definition (36), of *a mapping* Φ *of a set in itself*. Then Dedekind introduces his innovative idea of *a chain* K *in respect to mapping* Φ, in Definition (37). This definition goes as follows: a set A is a chain in respect to a mapping Φ, when $K' \subset K$, or $\Phi(K) \subset K$ (or K is closed under Φ). Based on Dedekind's definitions, $K' \subset K$ is equivalent to $(\forall x)(x \in K \rightarrow \Phi(x) \in K)$.

The Definition (37), of a *chain* K *in respect to mapping* Φ corresponds with Definition (69), of *a hereditary property F in an f-sequence*, in Frege's *Begriffsschrift*.

However, the main innovative idea that enables him to demonstrate MI is expressed in Definition (44). There, he defines the *chain* ***of*** *set* A *in respect to mapping* Φ, or simply *chain of* A (distinguished from *chain* A), as the intersection of all those chains (in respect to mapping Φ) of which A is a subset. He denotes it by $\Phi_0(A)$ or simply A_0.

Before we present Definition (44) in modern notation, it is worth noting that in *Was Sind Zahlen*, Dedekind, in contrast to Frege, does not obligate himself to use a purely formal language. In particular, he does not use logical notation to present his definitions and the proofs of his theorems. Moreover, he does not explicitly provide the axioms and inference rules of logic required in the proof of his theorems.) In this study, however, we present Dedekind's definitions and theorems in the formal language of logic and set theory in order to discover and demonstrate the fundamental structure of his definitions and theorems.

As we mentioned above, Dedekind, in Definition (44), defines *chain of* A (*in respect to mapping* Φ), denoted as A_0, as the intersection of all those chains (in respect to mapping Φ) of which A is a subset. We can present this definition in modern notation of logic and set theory as follows:

$$y \in A_0 \equiv (\forall K)\Big([(K' \subset K) \;\&\; (A \subset K)] \to (y \in K)\Big) \qquad \text{(44a)}$$

or:

$$y \in A_0 \equiv (\forall K)\Big([(\forall t)(t \in K \to \Phi(t) \in K) \;\&\; (\forall x)(x \in A \to x \in K)] \to (y \in K)\Big) \qquad \text{(44b)}$$

or:

$$y \in A_0 \equiv \Big(\forall K\Big)\Big(\big[(\forall x)(x \in A \to x \in K) \;\&\; (\forall t)(t \in K \to \Phi(t) \in K)\big] \to (y \in K)\Big) \qquad (44c)$$

Dedekind's Definition (44), of *chain of* A *in respect to mapping* Φ, is closely related to, and corresponds with Frege's Definition (76) of *following x in f-sequence* (or *ancestral relation*).

In Frege's Definition (76), where we have universal quantification ranging over all properties as variable, we used Gothic letters to denote these properties. Likewise, for the sake of convenience we use Gothic letters where we have universal quantification ranging over all sets as variable, as follows:

$$y \in A_0 \equiv \Big(\forall \mathfrak{K}\Big)\Big(\big[(\forall x)(x \in A \to x \in \mathfrak{K}) \;\&\; (\forall t)(t \in \mathfrak{K} \to \Phi(t) \in \mathfrak{K})\big] \to (y \in \mathfrak{K})\Big) \qquad (44d)$$

After developing and proving all the necessary properties about chains, using his other set theoretic definitions and theorems, Dedekind manages to prove, without any problem, what he calls the *theorem of complete induction* (we call it MI). This is represented in Theorem (59) as:

> *"In order to show that chain* A_0 *is part of system* Σ *– be this latter part of* S *– it is sufficient to show,*
>
> *ρ. that* $A \ni \Sigma$*, and*
> *σ. that the transform of every common element of* A_0 *and* Σ *is likewise element of* Σ*."*

which can be presented in modern notation (with partial use of Dedekind's notation) as follows:

$$\Big((A \subset \Sigma) \;\&\; [(A_0 \cap \Sigma)' \subset \Sigma]\Big) \rightarrow (A_0 \subset \Sigma) \qquad \text{(59a)}$$

We can proceed through the following steps to arrive at Formula (59c):

$$\Big((\forall x)(x \in A \rightarrow x \in \Sigma) \;\&\; (\forall t')\big(t' \in (A_0 \cap \Sigma)' \rightarrow t' \in \Sigma\big)\Big) \rightarrow (\forall y)(y \in A_0 \rightarrow y \in \Sigma)$$

$$\Big((\forall x)(x \in A \rightarrow x \in \Sigma) \;\&\; (\forall t)\big(\Phi(t) \in \Phi(A_0 \cap \Sigma) \rightarrow \Phi(t) \in \Sigma\big)\Big) \rightarrow (\forall y)(y \in A_0 \rightarrow y \in \Sigma)$$

$$\Big((\forall x)(x \in A \rightarrow x \in \Sigma) \;\&\; (\forall t)\big(t \in (A_0 \cap \Sigma) \rightarrow \Phi(t) \in \Sigma\big)\Big) \rightarrow (\forall y)(y \in A_0 \rightarrow y \in \Sigma)$$

$$\Big((\forall x)(x \in A \rightarrow x \in \Sigma) \;\&\; (\forall t)\big((t \in A_0 \;\&\; t \in \Sigma) \rightarrow \Phi(t) \in \Sigma\big)\Big) \rightarrow (\forall y)(y \in A_0 \rightarrow y \in \Sigma) \qquad \text{(59b)}$$

Basic Clause Inductive Step Conclusion

which is Mathematical Induction, MI.

In Paragraph (60) of the essay, Dedekind restates MI, "known by the name of complete induction (the inference from n to n+1),"[23] in two alternative forms. In the first case, he states that we can replace Σ with a certain property $\mathfrak{C}$ to be possessed by all elements of the chain A_0. This can be formalized in modern notation as follows:

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[23] Dedekind, R. & Beman, W. W. (1909), p. 60.

$$\Big((\forall a)(a \in A \to \mathfrak{E}_a) \ \& \ (\forall n)\big((n \in A_0 \ \& \ \mathfrak{E}_n) \to \mathfrak{E}_{n'}\big)\Big) \to (\forall n)(n \in A_0 \to \mathfrak{E}_n) \qquad (60a)$$

Basic Clause Inductive Step Conclusion

In the second case, Dedekind states that we can replace *Σ* with a certain theorem $\mathfrak{S}$ which deals "with an undetermined thing *n*" that holds for all elements *n* of the chain A_0.

This can be formalized in modern notation as follows:

$$\Big((\forall a)(a \in A \to \mathfrak{S}_a) \ \& \ (\forall n)\big((n \in A_0 \ \& \ \mathfrak{S}_n) \to \mathfrak{S}_{n'}\big)\Big) \to (\forall n)(n \in A_0 \to \mathfrak{S}_n) \qquad (60b)$$

This formulation of MI is subsequently used in Theorem (80) at the end of Section VI of Dedekind's essay, in which he restates his *theorem of complete induction* (*inference from n to n'*.) This move is based partly on the steps he takes in earlier sections of his essay. In Section V, he introduces his famous definition of infinite sets, and provides a few theorems concerning finite and infinite sets. By the end of Section V, he has completed his general theory of chains. In Section VI, he starts by defining a simply infinite set, *N*, as a one-to-one mapping "*Φ* of *N* in itself such that *N* appears as chain … of an element not contained in *Φ*(*N*),"[24] which is the chain of its initial element, denoted by symbol 1. Later in the essay, he shows that *N* can be considered to be *the* set of natural numbers. In

[24] Dedekind, R. & Beman, W. W. (1909), p.67

Theorem (80), he introduces MI for N (as the number-series or number-chain). It can be presented in formal notation as follows:

$$\Big((\forall m)(m\in\{m\}\rightarrow \mathfrak{S}_m)\ \&\ (\forall n)\big((n\in m_0\ \&\ \mathfrak{S}_n)\rightarrow \mathfrak{S}_{n'}\big)\Big)\rightarrow (\forall n)(n\in m_0\rightarrow \mathfrak{S}_n) \qquad (80)$$

in which set A in formula (60) becomes a singleton $\{m\}$, and the chain $\{m\}$ is denoted as m_0 (which we can also denote $\{m\}_0$). Dedekind notes that "the most frequently occurring case is where m=1 and therefore m_0 is the complete number-series N." Hence Theorem (80) can be rephrased as follows:

$$\Big(\mathfrak{S}_1\ \&\ (\forall n)\big((n\in 1_0\ \&\ \mathfrak{S}_n)\rightarrow \mathfrak{S}_{\Phi(n)}\big)\Big)\rightarrow (\forall n)(n\in 1_0\rightarrow \mathfrak{S}_n) \qquad (80b)$$

and since chain 1_0 is N, then:

$$\Big(\mathfrak{S}_1\ \&\ (\forall n)\big((n\in N\ \&\ \mathfrak{S}_n)\rightarrow \mathfrak{S}_{\Phi(n)}\big)\Big)\rightarrow (\forall n)(n\in N\rightarrow \mathfrak{S}_n)$$

For our purpose, we take Formula (59b) as Dedekind's general formulation of MI. That is:

$$\Big((\forall x)(x\in A\rightarrow x\in\Sigma)\ \&\ (\forall t)\big((t\in A_0\ \&\ t\in\Sigma)\rightarrow \Phi(t)\in\Sigma\big)\Big)\rightarrow (\forall y)(y\in A_0\rightarrow y\in\Sigma) \qquad (59b)$$

in which A_0 (or $y\in A_0$) is defined by Definition (44d):

$$y \in A_0 \equiv \Big(\forall \mathcal{K}\Big)\Big(\big[(\forall x)(x \in A \rightarrow x \in \mathcal{K}) \;\&\; (\forall t)(t \in \mathcal{K} \rightarrow \Phi(t) \in \mathcal{K})\big] \rightarrow (y \in \mathcal{K})\Big) \qquad (44d)$$

For the sake of convenience, and to match the letters used in Formula (59b) with those used in Definition (44d), instead of the letter Σ in Formula (59b), we use the letter K when it is a free variable, and the letter $\mathcal{K}$ when it is a universally quantified variable, as follows:

$$\Big((\forall x)(x \in A \rightarrow x \in K) \;\&\; (\forall t)\big((t \in A_0 \;\&\; t \in K) \rightarrow \Phi(t) \in K\big)\Big) \rightarrow (\forall y)(y \in A_0 \rightarrow y \in K) \qquad (59c)$$

If we substitute the equivalent of $y \in A_0$ (from the Definition (44d)) in the consequent of the main conditional in Formula (59c), by using axioms and inference rules of logic, we can infer Formula (59d) as follows:

$$\Big(\forall y\Big)\Big(\Big(\forall \mathcal{K}\Big)\Big(\big[(\forall x)(x \in A \rightarrow x \in \mathcal{K}) \;\&\; (\forall t)(t \in \mathcal{K} \rightarrow \Phi(t) \in \mathcal{K})\big] \rightarrow y \in \mathcal{K}\Big) \rightarrow$$
$$\Big(\big[(\forall x)(x \in A \rightarrow x \in K) \;\&\; (\forall t)\big((t \in A_0 \;\&\; t \in K) \rightarrow \Phi(t) \in K\big)\big] \rightarrow y \in K\Big)\Big) \qquad (59d)$$

In Theorem (59d), for the sake of simplicity, we can take y as a free variable and reformulate this theorem in a schematic form, which is presented in (59f):

$$\Big(\forall \mathcal{K}\Big)\Big(\big[(\forall x)(x \in A \rightarrow x \in \mathcal{K}) \;\&\; (\forall t)(t \in \mathcal{K} \rightarrow \Phi(t) \in \mathcal{K})\big] \rightarrow y \in \mathcal{K}\Big) \rightarrow$$
$$\Big(\big[(\forall x)(x \in A \rightarrow x \in K) \;\&\; (\forall t)\big[(t \in A_0 \;\&\; t \in K) \rightarrow \Phi(t) \in K\big)\big] \rightarrow y \in K\Big) \qquad (59f)$$

Here, in the analysis of Dedekind's works, A,Φ, $\mathfrak{K}$, K, and y, correspond with x, f, $\mathfrak{F}$, F, and y, respectively, as presented in the analysis of Frege's works in Chapter 1.

In this formula, as in our analysis of Frege's formulation of MI, the antecedent of the main conditional, which we denote it as $\mathbf{MI2}^{\boldsymbol{A,\Phi}}_{y}$, is a second-order formulation of MI; and the consequent of the main conditional, which we denote as $\mathbf{MI1}^{\boldsymbol{A,\Phi}}_{K,\,y}$, is a first-order formulation of MI. Therefore, as in the previous chapter, we can summarize Formula (59f), as U-$\mathrm{MI_D}$, as follows:

$$\mathbf{MI2}^{\boldsymbol{A,\Phi}}_{y} \rightarrow \mathbf{MI1}^{\boldsymbol{A,\Phi}}_{K,\,y} \qquad \text{U-MI}_{\mathrm{D}}$$

or simply as U-MI, as follows:

$$\mathrm{MI2} \rightarrow \mathrm{MI1} \qquad \text{U-MI}$$

It is worth noting that in the Formula (59f), the presence of $t \in A_0$ (which is equal to $\mathbf{MI2}^{\boldsymbol{A,\Phi}}_{y}$) in the antecedent of the consequent of the main conditional, that is in $\mathbf{MI1}^{\boldsymbol{A,\Phi}}_{K,\,y}$, as an additional condition, in fact repeats the antecedent of the whole conditional. As we will see, this does not affect the validity of our analysis concerning this formulation of MI, since this additional condition duplicates and reinforces those assumptions and conditions that are already present in the antecedent of the main conditional, and based on this formulation of MI, are needed in order to use MI in any domain of entities.

U-MI_F (from the first chapter) and U-MI_D (from this chapter), or simply U-MI, together with Frege's Definition (76) and Dedekind's Definition (44), will be used, in the third chapter, for our evaluation of Frege's and Dedekind's works on MI.

Chapter 3

A Proof- and Model-Theoretic Analysis of the Status of Mathematical Induction (MI) in an Axiomatic System

In this chapter, we evaluate Frege's and Dedekind's formulations and proofs of mathematical induction. As we showed in Chapters 1 and 2, the theorems U-MI_F and U-MI_D, which the two authors claimed to represent mathematical induction, are not problematic from the proof-theoretic viewpoint other than requiring some amendments and corrections. We will, however, address three major problems and issues in Frege's and Dedekind's formulations of MI, and we will present our account of the status of MI in an axiomatic system.

The initial concern is that U-MI or MI2 $\rightarrow$ MI1 is not the *principle of mathematical induction*, MI, accepted as the central principle in arithmetic and also as an important principle in mathematical and metatheoretical and other realms of reasoning. In fact, MI2 $\rightarrow$ MI1 is a formulation of an axiom (or in its alternative formulation, a rule of inference) of second-order logic, namely the universal instantiation axiom, and that is why we call it U-MI. We defend the view in which MI is expressed either in its second-order formulation, i.e. as MI2, or in its first-order formulation (that is, in schematic form), i.e. as MI1, such that either formulation is true of any collection of entities – abstract or concrete – which are *recursively defined, constructed or ordered*. To be sure, one can take or define a statement of universally quantified form (of first-order or second-order level), and using Axiom (58) of *Begriffsschrift*, or Axiom IIb of *Grundgesetze* (which are analogous to the universal instantiation rule of first- and second-order logic, respectively) one can prove a theorem by instantiation of the first-order or the second-order quantified variable. In the case of U-MI or MI2 → MI1, we have a statement of a universally quantified form of second-order level, i.e. MI2, in which the second-order variable is instantiated, which results in MI1; and based on axioms of logic (in particular, Axiom IIb), we can show that MI2 → MI1 is a theorem of logic, whereas, in principle, MI2 or MI1 can be true or false. Hence, disregarding the fact that MI2 might independently be shown to have a model, U-MI by itself is devoid of any content as far as the content of MI2 or MI1 is concerned, and in this sense, from the proof-theoretic view, U-MI, standing alone, is vacuous and uninformative. Therefore, from the proof-theoretic view, by proving U-MI or MI2 → MI1, one cannot claim that s/he has proven MI2 or MI1 as a theorem of logic.

It seems that the only way to use U-MI or MI2 → MI1 in any proof, such as a proof in arithmetic or a metatheoretical proof, is to provide a model that satisfies

MI2.[25] Hence, the question of the status of MI, in a sense, shifts from the proof-theoretic level to the model-theoretic level. Here there seems to be three options: one can prove MI2 at the model-theoretic level (to be able to have a model for it), postulate it as an axiom at the model-theoretic level (again to be able to have a model for it), or define a model containing a set of entities by using MI2 as *definiens*.[26] In the first option, in fact, the initial problem at the proof-theoretic level is shifted to the model-theoretic level, and we face the same issue as we did at the proof-theoretic level, and obviously the solution cannot be another (or a higher level) U-MI account at the model-theoretic level. The second option is, in a sense, close to the account that we defend in the following sections, in which we postulate MI, but at the proof-theoretic level in an infinitary many-sorted logic. The advantage of our account is that it is more faithful to the actual proof-theoretic status of MI – that is, taking MI to be a principle independent of the axioms and inference rules of classical logic. Furthermore, it has the advantage of saving the model-theoretic level for dictating stronger (or alternatively weaker) restrictions than those MI2 dictates in the model, depending on the realm of reasoning in which we use MI. (This will be discussed in later sections.) The third option, which is more commonly taken into the consideration in the literature, also has some problems. The first problem is the impredicativity of such a definition, which will be discussed in the next section. Furthermore, there is a problem in re-defining entities that are already defined or constructed by

[25] It worth noting that a model that falsifies MI2, or a model with an empty domain, makes U-MI or MI2 $\rightarrow$ MI1 vacuously and uninformatively true.

[26] Alternatively one might define a special predicate at the proof-theoretic level by using MI2 as *definiens*.

independent criteria. This problem is more serious in the metatheoretical use of MI, as we will discuss in the last section. Finally, any proof concerning the properties of the entities in such a model, which is defined by using MI2 as definiens, provides only a circular argument. For such a definiens, which is used to define the model (or alternatively is used to define a special predicate as mentioned in footnote 26), is a stronger assumption[27] than that which can be achieved by the truth of the consequent, MI1, in the theorem MI2 $\rightarrow$ MI1, since the content of MI1 is contained in the content of MI2, which is, in turn, assumed by definition at the model-theoretic level. In other words, U-MI or MI2 $\rightarrow$ MI1 as an axiom at the proof-theoretic level of a theory cannot prove anything other than what is already assumed at the model-theoretic level.

3.1 The Problem of Impredicativity

From model-theoretic point of view, the main problem with U-MI, or MI2 $\rightarrow$ MI1, as a formulation of the principle of mathematical induction, MI, is related to an obligation imposed at the model-theoretic level[28]: predicativity. That is, as we have explained, the model consists of those items that have to satisfy, and in fact have to be defined by using MI2 as definiens.[29] However, such a definition is impredicative.

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[27] At best, it is an identical assumption, in the case in which we take MI1 as a schema equal to MI2 itself, namely MI2 $\rightarrow$ MI2.

[28] In the work of some philosophers, such an obligation is imposed at the proof-theoretic level.

[29] In the case the obligation imposed at the proof-theoretic level, some philosophers impredicatively define a specific predicate by using MI2 as definiens; for example in the case of arithmetic, the predicate Natural Number, "N", is defined by using MI2 as definiens.

Frege's Definition (76) in *Begriffsschrift*, which is the logical analysis of the concept *y following x in f-sequence*, $\overset{\gamma}{\underset{\beta}{\smile}} f(x_{\gamma}, y_{\beta})$, or $\boldsymbol{P}^{x,f}{}_{y}$ as we present it, is:

$$\overset{\gamma}{\underset{\beta}{\smile}} f(x_{\gamma}, y_{\beta}) \equiv (\forall \mathfrak{F})([\mathfrak{F}_x \,\&\, H^{f}{}_{\mathfrak{F}}] \rightarrow \mathfrak{F}_y)$$

Furthermore, in the analysis of Frege's proof of U-MI or MI2 $\rightarrow$ MI1, we saw that, in MI2 $\rightarrow$ MI1, or $\mathbf{MI2}^{x,f}{}_{y} \rightarrow \mathbf{MI1}^{x,f}{}_{F,y}$, in fact, MI2 or $\mathbf{MI2}^{x,f}{}_{y}$ is the definition of *y following x in f-sequence*, or $\boldsymbol{P}^{x,f}{}_{y}$.

The counterpart of this definition in Dedekind's *Was Sind Zahlen* is Definition (44) of *the chain of set A in respect to mapping Φ*, $\Phi_0(A)$, or simply *chain of A* or A_0, which is defined as:

$$y \in \Phi_0(A) \equiv y \in A_0 \equiv (\forall \mathscr{K})\Big([(\forall x)(x \in A \rightarrow x \in \mathscr{K}) \,\&\, (\forall v)(v \in \mathscr{K} \rightarrow \Phi(v) \in \mathscr{K})] \rightarrow y \in \mathscr{K}\Big)$$

Likewise, in the analysis of Dedekind's proof of U-MI, or MI2 $\rightarrow$ MI1, we saw that, in MI2 $\rightarrow$ MI1, or $\mathbf{MI2}^{A,\Phi}{}_{y} \rightarrow \mathbf{MI1}^{A,\Phi}{}_{K,y}$, in fact, MI2 or $\mathbf{MI2}^{A,\Phi}{}_{y}$ is the definition of *membership of y in the chain of set A in respect to mapping Φ*, or the definition of *the chain of set A in respect to mapping Φ*, $\Phi_0(A)$.

As we have argued, to be able to have any model for U-MI, or MI2 $\rightarrow$ MI1, we are forced to define our model by using MI2 (or $\mathbf{MI2}^{x,f}{}_{y}$, or $\boldsymbol{P}^{x,f}{}_{y}$ which is Frege's definition of the property *following x in f-sequence*; or $\mathbf{MI2}^{A,\Phi}{}_{y}$ or $\Phi_0(A)$ which is

Dedekind's definition of [membership in] *the chain of set A in respect to a mapping* Φ) as definiens. But these definitions are impredicative. For they invokes or range over (that is, they consist of universal quantification over) a set of properties/sets containing the property/set being defined, i.e. $P^{x,f}{}_{y}$, or $\Phi_0\,(A)$.

Hence, as a result of deriving MI2 $\rightarrow$ MI1 as a presentation of MI we are forced to adopt an impredicative definition at the model-theoretic level or at the proof-theoretic level as a new predicate. Our analysis is a comprehensive approach to the analysis of MI for any discourse in which MI is required as a theorem or axiom, such as arithmetic, mathematics (in general), metatheoretical discourse, or other realms of reasoning. In the literature on the foundations of arithmetic, some philosophers impredicatively define special predicates for natural numbers at the proof-theoretic level. However, we have tried to analyse MI in as broad as possible a framework, and not just in arithmetic. Hence, we prefer not to define such a predicate at the proof-theoretic level, and consequently we separate the proof-theoretic realm from the model-theoretic realm to gain a more general account of MI. However, if we define a predicate, impredicatively, at the proof-theoretic level, we will have the same problem of impredicativity.

In the philosophical literature in general, and in particular in the foundations of arithmetic, there are views that reject and views that accept impredicative definitions in which an entity of a certain type is defined in terms of entities of the same or a higher type which contains the entity being defined. Some of these circular and self-referencing definitions or constructions end up in paradox, and in this case, there is more agreement that we should avoid such constructions or find some way out of them. Other constructions lead either to circularity or to infinite regress. Either we argue that since an entity is defined/constructed partly by itself, it is circularly defined/constructed, or we argue that to avoid circularity, in the definiens we substitute the entity being defined by its equivalent, and we know that this will lead to infinite regress. Putting it differently, if one defines an entity in terms of entities of the same or a higher type than that which contains the entity being defined, s/he implicitly presupposes the entity being defined. Several philosophers, logicians and mathematicians claim that this is a vicious circle. Impredicative definitions are similar to implicit equations (or functions, or definitions) in practical mathematics, but the difference is that in practical mathematics, we are able to change the implicit definitions or equations to explicit ones, which is to solve an equation to find the *definiendum* in an explicit presentation. However, in many cases this is not possible and we use a numerical method, which is not applicable in philosophical and foundational discourse.

One of the main reasons to accept impredicative definitions and constructions in mathematics is a concern about how much of mathematics would be constructible solely by using predicative constructions and definitions. Since for example, in classical mathematics, analysis is claimed to be constructed based on impredicative constructions and definitions, several philosophers and mathematicians, such as Ramsey, Bernays and Gödel, accept at least some form of impredicativity. They argue that if an entity can be *specified independently* of the totality to which it belongs, and in terms of which it is defined, or if it *exists* *independently* of our construction and definition, then an impredicative definition is allowed, and reference to this totality is permissible as in the famous example the "tallest person in the room." It is noteworthy that the view that requires an entity being constructed or defined to exist independently of our construction and definition is committed to a realist metaphysical view of the entities being constructed or defined. As we will explain, we prefer not to defend a view that forces us to accept such a metaphysical commitment with respect to numbers.

On the other hand, there are several philosophers, such as Poincaré, Russell and Whitehead, Weyl, and more recently Solomon Feferman, who defend predicativism. It has turned out that a large part of mathematics, and in particular the part that is required for scientific purposes (including analysis), can be achieved with predicative constructions given natural numbers.

Our concern in this thesis is MI in general, the particular model of which is natural numbers that might or might not require impredicative definitions. The *independent existence* of entities being defined or constructed is too strong a restriction, and we may not be willing to accept it, since it *restricts* the *nature of entities* in our domain that we would like to accept in the model, either in the case of arithmetic or in other discourses in which we would like to have MI as an axiom or theorem.

For example, in some versions of structuralism, which gives a plausible account of *sequences*, about which MI holds in general (and in particular about natural numbers), one might not want to be committed to a full-blown realist account.

Furthermore, if the entities are obviously fictional, there are well known difficulties with a realist account. However, even if one accepts the *independent specifiability* of entities being defined or constructed, there is no doubt that this also introduces a new restriction that one might want to avoid, if s/he can achieve the same result without the use of impredicative definitions in constructions. Furthermore, although impredicative definitions or constructions of entities that are *independently specifiable* might not be paradoxical, they also might be unusable due to their self-referential nature (analogously to an unsolvable implicit equation or function in practical mathematics in the absence of numerical methods). Moreover, the use of impredicative definitions forces us to have *independently specifiable* entities, and this is a restriction that we might want to avoid (if we accept such an impredicative definition for *independently specifiable* entities at all); and furthermore, if one *can* specify or characterize an entity or set of entities, further definitions (especially impredicative ones) might not even be needed. In fact, as we will discuss later, in the metatheoretic use of MI, such a definition (re-specifying) of something that already exists or is specified might cause some difficulties.

In our account of MI, which is not limited to arithmetic, we do not need impredicative definitions either in constructing/specifying our model, or in the proof-theoretic realm, and hence we avoid the potential problems of impredicative definitions. In fact, by avoiding the first problem (explained above) concerning the U-MI or MI2 → MI1 formulation of MI, we automatically avoid an impredicative definition of the model, since we do not need to define the entities in the model (or in the axiomatic system itself) by using MI2 as definiens, which is required for any use of MI2 → MI1. Our solution is to postulate an infinitary axiom for such specifiable/constructible entities.

3.2 The Unattainability of the Infinitary Nature of MI in a Finitary Logic, and the Axiomatizability/Derivabilityof MI in an Infinitary Many-Sorted Logic

The next issue with U-MI, or MI2 → MI1, as a formulation of the principle of mathematical induction, MI, is that it lacks a part of the nature of MI, i.e., its infinitary nature. Our account of the status of MI, just as it does not have the previously mentioned problems, captures this fundamental characteristic of MI, which is absent in the alternative accounts of MI. In this section, as we introduce our account of MI, we will examine this third problem with the alternative accounts.

MI is a unique type of axiom or inference rule that can also be derived from a similar type of axiom or inference rule of infinitary nature such as the ω-rule, or any other axiom or inference rule that allows one to prove claims about an infinite number of items (phrases or premises). In fact, MI in its standard form, as it is used in different realms of discourse, has a potentially infinite number of phrases (or premises, in its inference rule form), since the "inductive step", i.e. hereditariness $F_x \rightarrow F_{s(x)}$ [30], can be expanded as potentially infinitely iterated

[30] Here, for the sake of simplicity, instead of relation *f* or a function with two arguments $f(x, y)$, we use a function with one argument $s(x)$ such that y which is f-related to x is shown as $s(x)$.

conjoined conditionals (or a potentially infinite hypothetical syllogism, in its inference rule form), that is:

$$(F_0 \ \& \ (\forall x)(F_x \rightarrow F_{s(x)}) \ ) \rightarrow (\forall n)(F_n)$$

or:

$$(F_0 \ \& \ F_a \rightarrow F_{s(a)} \ \& \ F_{s(a)} \rightarrow F_{s(s(a))} \ \& \ F_{s(s(a))} \rightarrow F_{s(s(s(a)))} \ \& \ \ldots \ ) \rightarrow (\forall n)(F_n)$$

or:

$$(F_0 \ \& \ F_a \rightarrow F_{a'} \ \& \ F_{a'} \rightarrow F_{a''} \ \& \ F_{a''} \rightarrow F_{a'''} \ \& \ \ldots) \rightarrow (\forall n)(F_n)$$

in which F is a predicate in a schema formulation of MI[31] that is true of a set of linguistic items,[32] namely individual constants and variables (which are sorted-constant or sorted-variables, 0, or a, or n, in a many-sorted logic). It consists of a sequence (in the case of taking MI as a rule of inference) or sentence (in the case of taking MI as an axiom) of infinite length constructed through a recursive application of function s by applying function s recursively to an item. If we substitute 0 – of which the "basis clause" is true, namely F_0 – in a, in the expanded conjoined conditionals mentioned above, then:

[31] Alternatively, it can also be a predicate variable $\mathfrak{F}$ ranging over all predicates in second-order form.
[32] We say "linguistic" to make a minimal metaphysical claim about these entities.

$$(F_0 \ \& \ F_0 \rightarrow F_{0'} \ \& \ F_{0'} \rightarrow F_{0''} \ \& \ F_{0''} \rightarrow F_{0'''} \ \& \ F_{0''} \rightarrow F_{0'''} \ \& \ \ldots.) \rightarrow (\forall n)(F_n)$$

and by axioms and rules of logic we can infer:

$$(F_0 \ \& \ F_{0'} \ \& \ F_{0''} \ \& \ F_{0'''} \ \& \ F_{0''''} \ \& \ \ldots.) \rightarrow (\forall n)(F_n)$$

Likewise, if we take MI as an inference rule, a form of a potentially infinitely iterated instances of Modus Ponens (or Modi Ponentes) can be inferred, as follows:

$$
\begin{array}{l}
F_0 \\
F_0 \rightarrow F_{0'} \\
\text{----------} \\
\qquad F_{0'} \\
\qquad F_{0'} \rightarrow F_{0''} \\
\qquad \text{-----------} \\
\qquad\qquad F_{0''} \\
\qquad\qquad F_{0''} \rightarrow F_{0'''} \\
\qquad\qquad \text{------------} \\
\qquad\qquad\quad F_{0'''} \\
\qquad\qquad\quad F_{0'''} \rightarrow F_{0''''} \\
\qquad\qquad\quad \text{------------} \\
\qquad\qquad\qquad F_{0''''} \\
\qquad\qquad\qquad\qquad \ldots \\
\qquad\qquad\qquad\qquad\quad \ldots \\
\qquad\qquad\qquad\qquad\qquad \ldots \\
\text{---} \\
\qquad\qquad (\forall n)(F_n)
\end{array}
$$

and hence:

$$
\begin{array}{l}
F_0 , \ F_{0'} , \ F_{0''} , \ F_{0'''} , \ F_{0''''} , \ \ldots \\
\text{---} \\
\qquad\qquad (\forall n)(F_n)
\end{array}
$$

In classical first-order logic, a well-formed formula cannot contain an infinite number of symbols, and a deduction cannot be of infinite length; hence we do not have any axiom or rule of inference that can accommodate what can be proven by MI, and that can prove a result holding for an infinite number of items.

As we can see above, MI in its axiomatic form has an infinite number of symbols; in its rule-of-inference (or deduction) form, it is infinitely long. This feature enables us to prove results for an infinite number of items. This is a unique feature of MI in comparison to other *deductive* rules of inference or axiom; it makes MI irreducible and hence independent of other axioms and rules of inference of classical logic, unless we claim that an axiom or a rule of inference of classical logic is reducible to MI. In fact, this is a more radical claim than the claim we are defending, and it can be investigated separately. However, one might defend the view that Modus Ponens is a special case or an instance of MI, and hence MI is a mode of reasoning even more fundamental or general than Modus Ponens in classical logic. In this sense an axiomatic system with MI, instead of MP, is a more general deductive system.

It is worth noting that our claim about the status of MI is not particularly concerned with its axiom-hood or theorem-hood – statuses that are usually interchangeable in any axiomatic system. It is about the *fundamentality* and *independence* of MI, or any MI-type infinitary principle, from the axioms or rules of inference of classical logic. That is, in our account we can take MI either as a *fundamental* and *independent* axiom (or a rule of inference) which is irreducible and un-derivable from axioms and rules of inference of classical logic, or as a derived theorem (or a derived rule of inference) from another *fundamental* and *independent* MI-type infinitary axiom (or rule of inference) which is irreducible and un-derivable from the axioms and rules of inference of classical logic. For, in the latter case, we can derive MI from the ω-rule, which is:

$$F_0 \ \& \ F_{0'} \ \& \ F_{0''} \ \& \ F_{0'''} \ \& \ F_{0''''} \ \& \ \ldots \rightarrow (\forall n)(F_n)$$

or:

$$\frac{F_0\,,\ F_{0'}\,,\ F_{0''}\,,\ F_{0'''}\,,\ F_{0''''}\,,\ \ldots}{(\forall n)(F_n)}$$

Hence, MI is a counterpart of the ω-rule, which is accepted, in the literature, as a semi-formal inference rule (or axiom), that cannot be captured by classical logic. Therefore, we can consider MI as an axiom or theorem (or alternatively a rule of inference) of infinitary logic. Later, we will argue that since MI is true of specific domains of items or entities (that is, in its most general formulation, in addition to being true of numbers in arithmetic, it is true of any infinitely recursively defined or constructed or ordered entities), we should use a many-sorted infinitary logic such that we can assign sorted-variables to these recursively defined or constructed or ordered entities.

Since MI is not reducible to classical logic, one might consider MI (and hence arithmetic) to be synthetic. In other words, MI, as an axiom, is a truth about infinite items; as an inference rule, it is a method of reasoning about infinite premises. In neither case is it derived from classical logic. However, from the point of view that it is a part of a generalized deductive system, one might argue in defence of its analyticity. In fact, the idea that classical logic requires the length of the sentences and number of premises to be finite is based on the fact that logic has to simulate or formulate the finitude of the human mind. But since the dependence of logic on the human mind and psychology has been criticized by many philosophers and logicians (including Frege in his arguments against psychologism), there have been several attempts to remove finitude restrictions on logic (for example in works of Löwenheim or Tarski, who allow conjunctive or disjunctive infinitely long formulae, or formulae having an infinite number of quantifiers). As well, results from research about infinitary logic or ω-logic allow us to include infinitary axioms or rules of inference within a broader definition of logic. In this sense, although we have defended the view that MI is not reducible to classical logic and that it is a fundamental and independent infinitary axiom or rule of inference, if we are to decide whether MI is synthetic or analytic, we take the latter position. The only reason that MI is irreducible to classical logic is because of its infinitary nature (and the infinite number of application of axioms or rules of inference within it). Therefore, in light of several 20th century studies in mathematical logic that expand our understanding of logic, we can take MI as a generalized deductive rule or axiom. This account of MI is in contrast to that of synthetic knowledge or of a synthetic truth, for which other sources of knowledge or truth are required.

It is noteworthy that Gödel's incompleteness theorems prove that no consistent formally axiomatizable theory that includes an elementary fragment of arithmetic can prove all truths of arithmetic, and such a theory cannot demonstrate its own consistency. It has been shown that the extra resource or axiom that enables us to prove the theory's own consistency is an infinitary axiom or inference rule, which is higher-level induction (or more specifically, transfinite induction up to ε_0). This might show the fundamentality of infinitary axioms or inference rules. In regard to the fundamentality of MI and its independence from other axioms and inference rules of logic, based on Gödel's incompleteness theorems we might also argue as follows: Gödel's incompleteness theorems are only true of those theories that include an elementary fragment of arithmetic, and the essential part of this fragment is MI; therefore, the presence of MI in a theory makes the proof of some truths, including the consistency, of the theory impossible. Hence MI must be an axiom (or theorem) independent of axioms of classical logic.

In regard to a model that can satisfy MI in its full strength (namely an account of MI which is not a finite number of iterated Modus Ponens inferences but an infinite number), we need a model that consists of infinite entities that can be defined, constructed, or ordered recursively.

As noted above, we do not define this model using MI2 as definiens. As we argued, such a definition is impredicative; it also removes the specific content and information from an MI axiom at the proof-theoretic level and as a result any proof based on it becomes vacuous and circular. On the other hand, it seems wrong to follow a strategy that forces us to postulate MI2 at the model-theoretic level, or locate MI2 as a defining condition or restriction in the model, since in general, the restriction that is required for entities to satisfy MI might turn out to be weaker or stronger than MI2 upon investigation. A better strategy, therefore, is to include MI or MI2 itself at the proof-theoretic level and leave the required restrictions on the entities in the model as an open question. (For example, in the case of MI in arithmetic, such stronger [or alternatively weaker] assumptions or restrictions might be needed to avoid non-standard models.)

In regard to our model theoretic account and possible related concerns, the question of whether we can have *entities* that correspond with our syntax in proof-theoretic discourse depends on our metaphysical commitments. In fact, the individual constants and variables play the role of *placeholders*; the sequence is a relational structure, and the *places* in this structure have a specific relation to each other. That is, they are recursively constructed by a function or relation f (or function s). In such a minimal syntactic account, the entities in the model do not have any intrinsic or internal properties. Therefore, all of their properties are relational, meaning that they are based on a relation that a place (or a set of places) might have with another place (or set of places). For example, a place (or a place-holder) in the sequence, which is constructed by relation f, is in a complex relation with a reference (or initial) place or placeholder x, and these new complex relations and their consequent properties are ultimately derived from relation f. In the case of arithmetic, the relation f is the *successor,* and the relational properties that we prove for these places or placeholders are constructed based on some recursively defined operations on these placeholders (that are themselves recursively defined). These recursively defined operations can all be reduced to the operation *addition*, which can also be reduced to the successor function or relation *f*, and the initial placeholder. Therefore, all properties that are attributed to these recursively defined places, or placeholders, or entities, can ultimately be reduced to the recursively defined operation *addition*, and in turn to the relation *f* and the initial place or placeholder. Therefore, these non-intrinsic relational properties are complex functions of the relation *f* on places or placeholders or entities that are themselves recursively constructed based on the relation *f*. It is worth noting that hereditariness is based on the fact that the sequence is recursively constructed by the relation *f*, and that all hereditary properties are complex functions of this relation *f* within/among complex combinations of places or placeholders in the sequence. (In cases in which entities in the sequence are concrete physical objects, such as the case of the domino effect, the hereditary property will still be necessitated by a physical relation among the objects of the sequence).

Based on this account of MI in which we only need places or placeholders that are recursively constructed, we defend a structuralist view of MI, the model of which takes the most minimal, abstract and general form, and it enforces a minimal or no metaphysical commitment. However, these places or placeholders or individual constants also can be filled or replaced or interpreted by abstract entities or items such as linguistic items (in particular, in meta-linguistic discourse, in which we are not necessarily interested in the semantics of these linguistic items) or by non-abstract or concrete (or physical) ordered objects. In the latter case, the relation f becomes concrete (or physical) such that it can be related to internal or intrinsic properties of objects. However, even in this case, the properties which are to be proven true of these objects can be considered independent of the intrinsic or internal properties of objects, as far as a proof by MI is concerned. (An example of this might be found, again, in the domino effect.) Although I have defended a minimal ontological account in regard to MI and its model (which is based on a structuralist view), the debate about the metaphysical account of a structure and the places in it is as complex as the metaphysical account of universals; hence all of those epistemic and semantic concerns might play a role in accepting an account. In referring to a *minimal* account, I am suggesting a view that takes the minimal requirement that is needed for establishing the status of MI in an axiomatic system, disregarding epistemological and semantic concerns. If we consider these concerns, however, we might accept more ontological commitments, to be more accountable to these concerns.

In arithmetic, we are dealing with the most abstract case. If we disregard philosophical concerns, we need only the *places* or positions in a structure, and the relations among these places in the structure. In fact, in the case of arithmetic, the entities in the model, which are natural numbers, have no intrinsic properties but only relational properties. In this sense we defend a structural and ordinal, rather than a cardinal, conception of natural numbers. However, in order to provide a plausible account so as to be accountable for epistemic and semantic concerns, we might accept more metaphysical commitments, and accept a model which consists of abstract entities that are recursively defined or constructed, such as those defined by Zermelo or von Neumann. For example the numeral 2 (as a singular term), which is the Arabic-number-name for the second place in the structure, can refer to an entity that is recursively constructed by von Neumann as $\{\Phi, \{\Phi\}\}$ or by Zermelo as $\{\{\Phi\}\}$. Therefore, in a sense, the ontology of these entities that fills these places in the structure is arbitrary, and they need only be recursively constructed or defined. In other words, places or positions in the structure can be considered to be a generalization or abstraction from a set of ontologically defined entities that might fill these places.

An important point is that these recursively defined or constructed entities (or places) have to be infinite in number to capture the unique infinitary characteristic of MI. Therefore, we need a form of the axiom of infinity, such as Zermelo-Fraenkel's axiom of infinity or Neumann-Bernays- Gödel's axiom of infinity, that guarantees the existence of at least one infinite set.

It seems that the axiom of infinity can be understood in terms of MI – that is, as an instance of the use of the principle of mathematical induction in which, in the place of a property (predicate) to be held by existing entities (name), we instantiate *existence* (which in a metaphoric and analogical sense, should be a property/predicate in metaphysical/linguistic realm), which guarantees the existence of infinite entities or items.[33] In this sense, one might think of the axiom of infinity as the ontological basis that might be needed for a structuralist account

[33] This is the case if there exists a first entity or item and the existence of any entity or item guarantees the existence of the next one.

of MI. Interestingly, similar to the fact that MI is *independent* of the other axioms of classical logic, the axiom of infinity is also independent of other axioms of set theory, and in a sense they are counterparts of each other in the proof- and model-theoretic realm. In fact, as we move toward a minimal ontological account of the model, the axiom of infinity becomes more similar to MI, in the sense that places in the sequence continue to infinity and can be filled with anything, and we are just interested in the relational properties of these places that are provable by MI. Note that the axiom of infinity is needed for the problem of impredicativity, although one might still not accept that it solves the problem, as was discussed earlier.

3.3 Concerns with the Metatheoretical Use of MI

The last concern that might affect the status of MI in an axiomatic system is the role of MI in other realms of reasoning, a particular case of which is in the metatheoretical and metalinguistic realms – that is, the use of MI as an axiom or theorem or inference rule at the metatheoretical and metalinguistic level, either in proving the required properties of the syntax and semantics of a recursively defined or constructed formal language (or system or theory) in which there are entities with infinite length, or in justifying the metatheoretical properties of a system or theory. One of the most important examples of the latter is consistency. An example of the former is the syntactic property according to which the left and right parentheses in sentential logic are equinumerous. Note that in this example, which is an example of the use of MI in metatheoretical discourse, we *do not* define the objects of the domain by MI or MI2 as definiens, but we define or construct them recursively, and we accept MI or MI2 as an axiom of many-sorted infinitary logic postulated at the proof-theoretic or syntax level.

This raises an important question. Given that we use MI in the meta-language, ML, or metatheory to prove metatheoretical claims such as the above examples involving the construction of the syntax and semantics of a formal system or a theory, or to prove such metatheoretical properties of an axiomatic system as its consistency, is it plausible to claim that we have proven or justified MI as a theorem *in* the theory, that is, in the Object Language (OL)? In other words, if MI is a derived theorem in the axiomatic system OL, can it play such an essential role in the construction of the OL, or in proving its essential metatheoretical properties such as consistency that show the legitimacy or acceptability of the system? Does this involve any circularity?

We will address these questions in the pages that follow. When necessary, we will, for the sake of simplicity, focus on two examples of metatheoretical issues as representative of others, namely the use of axioms and inference rules of a system that are needed, first in proving some syntactic properties of the system (when we are constructing a system), and second in proving some metatheoretical properties of the system the most important of which is a consistency proof.

First of all, it seems that in the construction of a system and in a consistency proof, we legitimately use many resources of the OL, such as the axioms and inference rules of classical logic and MI, without any hesitation. In fact, historically, when there was no sharp distinction between object language and meta-language, not only was it not a defect to use the resources of a system including its axioms and inference rules, in proving a metateoretical properties of a system, but also it was a desired goal to use *only* the axioms and inference rules of the system under scrutiny. For example, in the case of the consistency proof of a system, not only was there not any hesitation to use the axioms and inference rules of the system itself in proving the consistency of the system, but the *goal* was to prove consistency using *only* the system's own axioms and inference rules. (Interest in such a goal is reflected in Gödel's works on completeness; he eventually proved that such a goal is not always attainable. That is, Gödel's second incompleteness theorem proved that a formal system containing arithmetic cannot prove its own consistency.)

Therefore, historically, the use of the resources of a system for metatheoretical purposes was at least permissible, and not problematic. A reason might be that, in general, we are inclined to use the underlying logic of ordinary language in any intellectual activities, in particular in formal metatheoretical investigations, since that is the way we naturally reason. Furthermore, since classical logic is at least one of the best formal languages that closely and straightforwardly capture the fundamental structure of the underlying logic of ordinary language, we are inclined to use its resources (including its axioms and inference rules) in formal metatheoretical investigation. Likewise, in order to reason about infinite sequences at the metatheoretical level, we need MI in addition to axioms and inference rules of classical logic. In this respect, we may either take it as an independent axiom/inference rule or as an axiom/inference rule that is reducible to the axioms/inference rules of classical logic.

A separate investigation would be required in order to determine in what categories of cases it is desired, possible, or necessary – and under what conditions – to use a metalanguage that does not use *any* resources of an OL (including whatever axioms and inference rules such an OL has) to prove its metatheoretical properties such as its consistency. The answer to these questions depends in part on what we mean by *consistency*, and whether we look for an internal or an external conception of consistency. We will address this issue at the end.

The second issue is whether being an axiom or a theorem in an OL makes their use in metatheoretical proof more or less legitimate. In principle, we have some degree of freedom to replace the set of fundamental axioms of a system with another set of fundamental axioms, while maintaining equivalence among the old and new systems.[34] As a result, some axioms in the old system become theorems in the new system, and some theorems in the old system become axioms in the new system.

Likewise, in our account of MI, in which we take MI in its standard form (in contrast to MI2 $\rightarrow$ MI1 or U-MI), and accept it as an infinitary axiom of a

[34] In addition, there is a trade-off between the axioms and inference rules of a system.

many-sorted logic, we do not believe that we necessarily have to take it as an axiom. Our claim is that MI is logically independent of the axioms of classical logic. It can be inferred as a theorem from another axiom or inference rule of the same nature, namely another infinitary axiom or inference rule, such as the ω-rule; or alternatively, the ω-rule can be taken as a theorem and MI as an axiom.

Therefore, at first glance, it seems that it does not make a difference whether we use axioms or derived theorems for metatheoretical proofs. This is because, neither axioms nor theorems precede each other chronologically; if one is to be given priority, it should be on the basis of the fundamentality or justificatory status of the axioms and theorems of that system. Since they are, in principle, interchangeable, there is no difference, from the aforementioned perspective, whether we use axioms or theorems of a system in metatheoretical proofs. Therefore, the axiom-hood or theorem-hood, *per se*, does not legitimize or illegitimize the use of an axiom or a theorem in metatheoretical proofs.

Considering these points, and given our account of the status of MI, it follows that the use of MI in metatheoretical proofs has the same status whether we take it as a theorem or as an axiom[35], and from this perspective the same judgement should be true of the alternative account of the status of MI, i.e. the MI2 $\rightarrow$ MI1 account, or the U-MI account.

[35] Note that, as we have mentioned, there is always a trade-off between axioms and inference rules of a system too, and for the sake of brevity we do not always express it.

However, if an axiom and set of theorems derived from it are independent of the other axioms and theorems of a system, then there is no possibility of interchanging a member of the former with a member of the latter. Such independence shows the fundamentality of the former set, or at least the fundamentality of a *member* of the former set, which is taken as its representative (and as an axiom), and it shows its irreducibility to or unjustifiability by the axioms or theorems of the latter set.

Therefore, in our account, since MI is a fundamental axiom or inference rule independent of other axioms and inference rules of classical logic due to its infinitary nature, it cannot be replaced by other axioms or inference rules of classical logic, although it can be replaced by one of its counterpart axioms or inference rules which are of an infinitary nature.

On the other hand, since in metatheoretical proofs we need to prove the desired results for an infinite number of items or entities, we need MI in metatheoretical proofs, no matter which account of MI we accept or adopt. But due to the differences between the use of MI in an OL proof and its use in an ML proof, the problems of the U-MI account described in the context of an OL become more serious in the context of an ML.

As we have explained, an advantage of our account of MI is that it is more general. It can be used in any realm of reasoning, with a recursively defined or constructed model, in contrast to the U-MI account, which requires, in each realm, a specifically **MI2**-defined model, and this in turn might cause further problems. (For the sake of clarity I use boldface and larger fonts to show the use of **U-MI**, **MI2** $\rightarrow$ **MI1**, **MI2**, or **MI** in metatheoretical proofs.)

Before elaborating on the aforementioned problems, it is important to point out the unique role that **MI** plays in a ML in comparison with the role of MI in an OL. Let us consider again the example of consistency proof. A formal system can imply a contradiction or absurdity, and if a system is inconsistent this can appear somewhere in a derivation or in an inference within the system. From the meta-judgemental viewpoint, we want to have a formal system free of such contradictions, and hence we would like to make sure such contradictions do not happen anywhere in the derivations and inferences. When there are a finite number of steps in the inferences within an OL, in which axioms and inference rules of classical logic and MI (of OL level) are used, these axioms, inference rules and MI, *by themselves* (or in the worst via an exact copy of them in a ML, using a different notation) *show* (in a Wittgensteinian sense) the presence or absence of a contradiction or absurdity, although it might be tedious work to go through all of these derivations and inferences to make sure no contradiction appears. However, when the number of steps (each step of which might use axioms and inference rules of classical logic and MI) and hence the number of formulae that are produced is infinite, the OL inferences (or a copy of them in a ML) are unable to *show, by themselves*, that the presence or absence of a contradiction is guaranteed. Hence we require mathematical induction, **MI**, at the metatheoretical level. However, this instance of mathematical induction does not replicate an MI of the OL level, since at the metatheoretical level, it ranges over totally different entities, namely formulae of the OL. That is, its basis clause is about all the axioms, inference rules and MI of an OL, and its inductive step is likewise concerned with these axioms, inference rules and MI of an OL. Therefore, **MI** (at the metatheoretical level) plays a unique and genuine metatheoretical role in ML that cannot be *shown* in an OL. Hence, in this sense the use of **MI** in metatheoretical proofs is different from the use of other axioms and rules of classical logic at the metatheoretical level. Nevertheless, we do not believe that this by itself makes the use of MI at the metatheoretical level problematic.

However, the **U-MI** account of mathematical induction in metatheoretical proofs is problematic. To explain the problem, let us again use the consistency proof as an example. In a consistency proof at the metatheoretical level, to be able to use **U-MI**, or **MI2** $\rightarrow$ **MI1** one should *define* a model *using* (or *by*) **MI2** as definiens on the level of metatheory. We know that the entities for which **U-MI** should be used are formulae (theorems) of the OL, and the variable ranges over these formulae. That is, we would like to show that at any step of inferences and derivations in the OL, if there is no contradiction (that is, no absurdity), then there is no contradiction or absurdity in the next step either. As we know, the transition from one step of derivation in the OL to the next involves the use of axioms and inference rules of the OL (including U-MI itself in the OL). The problem is that although the derivation of formulae or theorems of the OL can be recursively constructed, unlike what the defenders of the U-MI account are required to do in using U-MI in an OL, one cannot antecedently *define* the sequence of formulae or theorems *using* **MI2** (as definiens) in the model of metatheory (in order to be able to use **U-MI**, or **MI2** $\rightarrow$ **MI1** in metatheory and eventually to prove that there is no contradiction or absurdity in any step of derivation in the OL, and hence no contradiction at all). For, just as we explained in the context of OL in previous sections, this is too strong an assumption, in providing a model at the metatheoretical level, to permit a proof to use **MI2** $\rightarrow$ **MI1**. That is, in this case, at the model-theoretic level of the metatheory, we have to presuppose what we are going to prove about sequences of derivations in the OL. In other words, in the model-theoretic level of the metatheory, we have to define a model consisting of linguistic entities that constitute the sequence of formulae in the OL such that they satisfy **MI2**. But this is what we want to prove, and we do not want to presuppose it as the defining condition – a case of *petitio principipia*. Note that the problem with the metatheoretical use of **U-MI**, in comparison to its OL use, is more serious due to the nature of the entities for which it is used. For although these entities are abstract (that is, they are linguistic entities), they are determinately defined by independent restrictions – in this case, the structure of the derivations of the formulae in the OL. This is, in a sense, unlike the case of arithmetic, in which one might argue in defence of the view that numbers can be defined based on the rules by which they are governed. (Even in that case – in which, in answering the impredicativity problem, the defender of the U-MI account presents the independent specifiability or independent existence argument – s/he has to show how these independently specified or existed model can be re-specified or redefined using MI2 as definiens.)

Since our account of MI does not suffer from such a problem, and only requires a set of recursively defined items – in this case linguistic entities which are sequence of formulae – it can unproblematicallly serve as **MI** in metatheoretical proofs. Therefore, from the metatheoretical viewpoint, our account does not face the problem that the U-MI account does.

Furthermore, our account requires a minimal ontological commitment; at most, it requires the axiom of infinity for the items or entities that are recursively defined.

The last part of this section addresses the question posed earlier in this section about possible conceptions of the consistency of a system. In so doing, it also re-examines the question of the legitimacy of using the axioms and inference rules of the system itself in evaluating and proving its consistency. (Likewise, similar analyses can be proposed for other metatheoretical properties of a system).

If a system is inconsistent, there should be one or more axioms that cause such an inconsistency. Suppose we manage to prove, in a metatheoretical proof using MI and other axioms and rules of inference, that a formal system is consistent. Based on Gödel's second incompleteness theorem, we know that if the formal system contains arithmetic, it cannot prove its own consistency, and it requires external resources. Let's assume that we use the formal system's own axioms and rules of inference, including MI, and an external axiom to prove the consistency of the system. The question is this: is it possible that one of the axioms of the system, for example MI, is inconsistent with others, but that due to the use of this very axiom in proving consistency, its inconsistency is covered up?

That is, given that without a consistency proof, we are not sure that all axioms of our OL are consistent, and supposing that they are not and we do not know which axiom is the source of inconsistency, how can we use one of these suspicious axioms, such as MI, in showing that they are consistent?

It seems that a more robust and self-contained conception of consistency is that of *internal* consistency. A consistency proof, in such a conception, uses the system's own axioms and inference rules, or to put it differently, its own rules of the game, to show that there is no contradiction. In other words, at least one kind of legitimate conception of consistency is one in which a system with a set of axioms (and inference rules) is considered to be consistent based on using its own axioms and inference rules in the process of proving consistency, and not based on using external axioms and inference rules within that process. Gödel showed that for those formal systems expressive enough to model arithmetic, we need external resources to prove the consistency of the system. However, if such an internal consistency proof is not entirely possible (that is, in the case of those theories that meet the hypotheses/assumptions of Gödel's second incompleteness theorem), it is nonetheless not a disadvantage to use the system's own axioms and inference rules in its consistency proof to the greatest extent possible. Furthermore, we know that there are many self-verifying first-order systems of arithmetic that are weaker than Peano arithmetic, and they are capable of proving their own consistency. That is, they are capable of expressing the provability but not of formalizing diagonalization.[36]

One response to objections about internal consistency is that it has the advantage of being a self-contained and self-verifying attribute of a system. In seeking to prove such a consistency, one uses those axioms and inference rules that are being investigated. If some *inappropriate* set of axioms (that is, a set of axioms deemed inconsistent based on an external inference machinery, and inference rules) is proven to be consistent by using these *inappropriate* axioms themselves (which should be accompanied by an external axiom, when the required hypotheses/assumptions of Gödel's second incompleteness theorem are met), then this inappropriateness is *consistently* held, and we do not necessarily need to reject such a system, since it has the virtue of self-contained or internal consistency. We might use another system to check the consistency, but still we can say that the system is consistent based on its own principles.

It appears that this conception of consistency is also plausible; in proving it, one uses the axioms and inference rules of a system itself, and uses as few external resources as possible. Such an apparent circularity is considered to be part of the concept of consistency itself (in contrast to, for example, the concept of truth *simpliciter*).

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[36] See Willard, D. (2001)

In other words, to adopt a system in which the axioms and principles are non-contradictory is to adopt these axioms and principles themselves as judging axioms and principles that are used in the evaluation of their own consistency (given, of course, that the axiomatic system *can* provide such principles and evaluation tools). That is, in proving this type of consistency, not only is the use of the axioms and theorems of the system not illegitimate, but it provides more evidence for the internal or absolute consistency of the system. This is due to the fact that if one always follows the axioms and inference rules of the system – even in a metatheoretical consistency proof – the system still proves to be consistent based on its own axioms and inference rules. This is in accordance with the way we use the rules and principles of the ordinary language, when using that language metalinguistically to make assertions about the language itself.

We can reformulate this problem as follows. This conception or definition of consistency, and in particular its corresponding consistency proof, is impredicative, in the sense that the evaluating axioms and inference rules invoke (or are identical to) axioms or inference rules that are evaluated (analogous to impredicative definitions in which the definiens invokes or appeals to the *definiendum* itself or an entity of higher type that contains the definiendum). However, this type of impredicativity, in contrast to others, can be considered as a virtue, an advantage, and a desirable feature of a system of axioms and inference rules. This is because a full-fledged understanding of consistency views it as a self-contained property that does not need an external reference point or criteria. It presents a mutual-referential (analogous to self-referential) relation among a set of axioms and inference rules that should not contradict each other according to these same axioms and inference rules as evaluating tools.

The impossibility of such a project in those cases in which the Gödel's second incompleteness theorem is valid (namely its required hypotheses/assumptions are satisfied) does not make the project undesirable, and we know that if the required assumptions are not satisfied, it is not impossible, as noted earlier.

As an example, if we have an axiomatic system in three-valued logic, we might prefer to prove its consistency based on its own axioms and rules of inference. That is, for those people (or for a reasoning machine) who have such an axiomatic system, it might be preferable to have a consistency proof based on their (or its) own axiomatic system, and if such a proof can be provided, then the system is consistent based on its own axioms and inference rule. (An analogy might be Neurath's example of the situation in which one is in a boat on the sea, and does not have any choice to repair the boat except to do so while one is using it on the sea.)

If we prove the consistency of a system using the axioms and inference rules of another system at the ML level, it shows that the latter system, which we use to argue about the former system (or sub-system) under evaluation, is preferable. Therefore, we might defend the view that this is a matter of preference (or application).

To evaluate a system's consistency based solely on the axioms and rules of inference of another system is a kind of consistency proof that does not necessarily validate the use of the system's internal rules in judging itself. In this sense, it is an external or relative consistency proof, which is in fact relative to a more authentic or reliable system. The impossibility of absolute internal consistency proof for those theories that meet the hypotheses/assumptions of Gödel's second incompleteness theorem shows that at least for such theories a kind of self-referentiality does not allow for this sort of absolute self-consistency proof.

A separate investigation may be required to determine in which situations a metalanguage might have more, or alternatively fewer, axioms and inference rules in comparison with the object language for which it is to be used in metatheoretical proofs.

In general, depending on the claims we need to prove in the metalanguage, we need to add to or remove axioms from an object language in the metatheoretical proofs. However, **MI** is one of those axioms that are always needed for proofs about an infinite number of terms or items.

These observations show that there is no problem in principle in using an axiom or theorem of a formal system in its own consistency proof. In fact, the necessity of the use of a type of **MI** principle in such a proof shows its importance.[37] The other uses of **MI** in metatheoretical proofs, such as proofs related to the construction of the syntax or semantics of a language, also show the fundamentality of MI, even in the construction of a system of which it is going to be a part.

However, the metatheoretical use of MI reveals an advantage of our account of MI in comparison to rival accounts, due to the particular use of MI in the specific domain of entities (which are determined by independent restrictions), in metatheoretical proofs.

[37] It is worth noting that in metatheoretical proofs, and in particular in the consistency proof of those theories that meet the hypotheses/assumptions of satisfy Gödel's incompleteness theorems, we need to deal with a larger infinity, of higher type ordinals; that is, we need higher-level induction or transfinite MI.

Bibliography

Acerbi, F. (2000). Plato: Parmenides 149a7-c3. A Proof by Complete Induction? *Archive for History of Exact Sciences, 55*, 57-76.

Beaney, M. (1996). *Frege: Making sense*. London: Duckworth.

Beaney, M., & Reck, E. H. (Eds.). (2005). *Gottlob Freg,Vols. 1-4*. London: Routledge.

Benacerraf, P. (1981). Frege: The Last Logicist. *Midwest Studies in Philosophy, 6*, 17-35.

Benacerraf, P., & Putnam, H. (Eds.). (1964). *Philosophy of mathematics; Selected readings*. Oxford: Blackwell.

Bernays, P., & Müller, G. H. (Eds.). (1976). *Sets and classes: On the work by Paul Bernays*. Amsterdam; New York: North-Holland Pub. Co.

Blanchette, P. A. (1994). Frege's Reduction. *History and Philosophy of Logic, 15*(1), 85-103.

Blanchette, P. A. (2007). Frege on Consistency and Conceptual Analysis; *Philosophia Mathematica, 15*(3), 321-346.

Blanchette, P. A. (2012). *Frege's conception of logic [electronic resource]*. Oxford: Oxford University Press.

Boolos, G. (1995). Frege's Theorem and the Peano Postulates. *The Bulletin of Symbolic Logic, 1*(3), pp. 317-326.

Boolos, G., Burgess, J. P., & Jeffrey, R. C. (2007). *Computability and logic* (5th ed.). Cambridge; New York: Cambridge University Press.

Boolos, G., & Jeffrey, R. C. (Eds.). (1998). *Logic, logic, and logic*. Cambridge, Mass: Harvard University Press.

Burge, T. (2005). *Truth, thought, reason: Essays on Frege*. Oxford; New York: Clarendon; Oxford.

Burgess, J. P. (1984). Frege's Conception of Numbers as Objects [book review]. *The Philosophical Review, 93*(4), 638-640.

Burgess, J. P. (2005). *Fixing Frege*. Princeton, N.J: Princeton University Press.

Burton, D. M. (2011). *The history of mathematics: An introduction* (7th ed.). New York: McGraw-Hill.

Bussey, W. H. (1917). The Origin of Mathematical Induction. *American Mathematical Monthly, 24*(5), 199-207.

Cajori, F. (1918). Origin of the Name "Mathematical Induction." *American Mathematical Monthly, 25*(5), 197-201.

Church, A. (1996). *Introduction to mathematical logic*. Princeton: Princeton University Press.

Craig, E. (Ed.). *Routledge encyclopedia of philosophy* (1998). London; New York: Routledge.

Currie, G. (1982). *Frege, an introduction to his philosophy*. Brighton: The Harvester Press.

Currie, G. (1982). Frege, Sense and Mathematical Knowledge. *Australasian Journal of Philosophy, 60*(1), 5-19.

Dedekind, R. (1888). *Was sind und was sollen die zahlen?* Braunschweig: F. Vieweg.

Dedekind, R. & Beman, W. W. (1909). *Essays on the theory of numbers, I. Continuity and irrational numbers, II. The nature and meaning of numbers*. Chicago: Open Court Publ. Co.

Detlefsen, M. (1992). (Ed.). *Proof and knowledge in mathematics [electronic resource]*. London; New York: Routledge.

Detlefsen, M. (1992). (Ed.). *Proof, logic, and formalization [electronic resource]*. London; New York: Routledge.

Detlefsen, M. (1998). Gödel's theorems. In E. Craig (Ed.), *Routledge Encyclopedia of Philosophy*. London: Routledge.

Demopoulos, W. (1994). Frege and the Rigorization of Analysis. *Journal of Philosophical Logic, 23*(3), 225-245.

Demopoulos, W. (Ed.). (1995). *Frege's philosophy of mathematics*. Cambridge, Mass: Harvard University Press.

Demopoulos, W. (2000). On the Origin and Status of Our Conception of Number. *Notre Dame Journal of Formal Logic, 41*(3), 210-226.

Dummett, M. A. E. (1978). *Truth and other enigmas*. London: Duckworth.

Dummett, M. A. E. (1981). *Frege: Philosophy of language* (2nd ed.). London: Duckworth.

Dummett, M. A. E. (1981). *The interpretation of Frege's philosophy*. Cambridge: Harvard University Press.

Dummett, M. A. E. (1991). *Frege: Philosophy of mathematics*. London: Duckworth.

Dummett, M. A. E. (1991). *Frege and other philosophers*. Oxford; New York: Clarendon Press; Oxford University Press.

Enderton, H. B. (1977). *Elements of set theory*. New York: Academic Press.

Ewald, W. B. (1999). *From Kant to Hilbert [electronic resource]: A source book in the foundations of mathematics,Volume 1*. Oxford; New York: Clarendon Press; Oxford University Press.

Ewald, W. B. (2005). *From Kant to Hilbert [electronic resource]: A source book in the foundations of mathematics, Volume 2*. Oxford; New York: Clarendon Press; Oxford University Press.

Feferman, S., & Hellman, G. (1995). Predicative Foundations of Arithmetic. *Journal of Philosophical Logic, 24*(1), 1-17.

Ferreira, F. (1999). A Note on Finiteness in the Predicative Foundations of Arithmetic. *Journal of Philosophical Logic, 28*(2), 165-174.

Frege, G. (1879). *Begriffsschrift: Eine der arithmetischen nachgebildete formelsprache des reinen denkens*. Halle a/S: L. Nebert.

Frege, G. (1884). *Die grundlagen der arithmetik: Eine logisch mathematische untersuchung über den begriff der zahl*. Breslau: W. Koebner.

Frege, G. (1893). *Grundgesetze der arithmetik: Begriffsschriftlich abgeleitet, vol. 1*. Jena: Pohle; reprinted in 1966 (Hildesheim: G. Olms)

Frege, G. (1903). *Grundgesetze der arithmetik: Begriffsschriftlich abgeleitet, vol. 2*. Jena: Pohle; reprinted in 1966 (Hildesheim: G. Olms)

Frege, G., Ebert, P. A., Rossberg, M., & Wright, C. (2013). *Basic laws of arithmetic, derived using concept-script: Volumes I & II*. Oxford, U.K: Oxford University Press.

Frege, G., & Kluge, E.H.W. (1971). *On the foundations of geometry and formal theories of arithmetic*. New Haven, CT: Yale University Press.

Frege, G., & Geach, P. T. (1977). *Logical investigations*. Oxford: Basil Blackwell.

Frege, G., & Austin, J. L. (1980). *The foundations of arithmetic: A logico-mathematical enquiry into the concept of number* (2nd rev; English language only ed.). Evanston, Ill: Northwestern University Press.

Frege, G., Geach, P. T., & Black, M. (Eds.). (1980). *Translations from the philosophical writings of Gottlob Frege* (3rd ed.). Oxford: Blackwell.

Frege, G., & Beaney, M. (1997). *The Frege reader*. Oxford; Malden, MA: Blackwell Publishers.

Frege, G., & Bynum, T. W. (1972). *Conceptual notation and related articles translated [from the German] and edited with a biography and introd. by Terrell Ward Bynum*. Oxford: Clarendon Press.

Frege, G., & Furth, M (1964). *The basic laws of arithmetic; exposition of the system. Translated and edited, with an introd. by Montgomery Furth*. Berkeley: University of California Press.

Frege, G., Hermes, H., Kambartel, F., & Kaulbach, F. (Eds.). (1979). *Posthumous writings*. Chicago: University of Chicago Press.

Frege, G., & McGuinness, B. (Eds.). (1984). *Collected papers on mathematics, logic, and philosophy*. Oxford, UK; New York, NY, USA: B. Blackwell.

Gödel, K., Feferman, S., & Association for Symbolic Logic (Eds.). (1986). *Collected works*. Oxford Oxfordshire; New York: Clarendon Press; Oxford University Press.

Gabbay, D. M. (2011). *Handbook of philosophical logic [electronic resource]* (2nd ed.), *Vols. 1, 11*. Dordrecht; New York: Springer.

Gabbay, D. M., & Guenthner, F. (Eds.). (1983). *Handbook of philosophical logic*

George, A. (Ed.). (1994). *Mathematics and mind*. New York: Oxford University Press.

Grattan-Guinness, I. (2000). *The search for mathematical roots, 1870-1940 [electronic resource]: Logics, set theories and the foundations of mathematics from Cantor through Russell to Gödel*. Princeton, N.J: Princeton University Press.

Griffin, N. (2003). *The Cambridge companion to Bertrand Russell*. Cambridge, U.K; New York: Cambridge University Press.

Hale, B., & Wright, C. (2001). *The reason's proper study: Essays towards a neo-Fregean philosophy of mathematics*. Oxford; New York: Clarendon Press; Oxford University Press.

Hellman, G. (1981). How to Gödel a Frege-Russell – Gödel’s Incompleteness Theorems and Logicism. *Nous, 15*(4), 451-468.

Hendricks, V. F. (2004). *First order logic revisited*. Berlin: Logos Verlag

Horty, J. F. (2007). *Frege on definitions: A case study of semantic content*. Oxford; New York: Oxford University Press.

Kenny, A. J. P. (2000). *Frege: An introduction to the founder of modern analytic philosophy*. Oxford, UK; Malden, MA: Blackwell Publishers.

Kneale, W. C., & Kneale, M. (1962). *The development of logic*. Oxford: Clarendon Press.

Long, G. (Ed.). (1833-1843). *Penny cyclopaedia of the society for the diffusion of useful knowledge*, *Volume XII*. London: C. Knight.

Macbeth, D. (2005). *Frege's logic*. Cambridge, Mass: Harvard University Press.

Mancosu, P. (Ed.). (1998). *From Brouwer to Hilbert: The debate on the foundations of mathematics in the 1920s*. Oxford; New York: Oxford University Press

Mendelson, E. (2010). *Introduction to mathematical logic* (5th ed.). Boca Raton: CRC Pressed.

Parsons, C. (1983). *Mathematics in philosophy: Selected essays*. Ithaca, N.Y.; London: Cornell University Press.

Parsons, C., Sher, G., & Tieszen, R. L. (Eds.). (2000). *Between logic and intuition: essays in honor of Charles Parsons*. Cambridge, U.K; New York: Cambridge University Press.

Poincare, H. & Maitland F. (1914). *Science and method*. New York: Dover Publications.

Putnam, H., & Benacerraf, P. (1964). *Philosophy of mathematics: selected readings*. Englewood Cliffs, N.J: Prentice-Hall.

Putnam, H. (1979). *Mathematics, matter, and method* (2nd ed.). Cambridge; New York: Cambridge University Press.

Quine, W. V. (1970). *Philosophy of logic*. Englewood Cliffs, N.J: Prentice-Hall.

Rabinovitch, N. L. (1970). Rabbi Levi Ben Gershon and the Origins of Mathematical Induction. *Archive for History of Exact Sciences, 6*(3), 237-248.

Ricketts, T., & Potter, M. D. (2010). *The Cambridge companion to Frege*. Cambridge; New York: Cambridge University Press.

Russell, B. (2008). *Introduction to mathematical philosophy*. Nottingham, England: Spokesman.

Russell, G. K. (2008). *Truth in virtue of meaning*. Oxford; New York: Oxford University Press.

Shapiro, S. (Ed.). (2005). *The oxford handbook of philosophy of mathematics and logic*. Oxford; New York: Oxford University Press.

Sluga, H. D. (1993). *Logic and foundations of mathematics in Frege's philosophy*. New York: Garland Pub.

Steiner, M. (1975). *Mathematical knowledge*. Ithaca: Cornell University Press.

Suppes, P. (1960). *Axiomatic set theory*. Princeton, N.J: Van Nostrand.

Takeuti, G. (1987). *Proof theory* (2nd ed.). Amsterdam; New York, N.Y.: North-Holland.

Van Heijenoort, J. (Ed.). (1967). *From Frege to Godel: A source book in mathematical logic, 1879-1931*. Cambridge, Mass: Harvard University Press.

Wang, H. (1957). The Axiomatization of Arithmetic. *Journal of Symbolic Logic, 22*, 145-158.

Weiner, J. (2004). *Frege explained: from arithmetic to analytic philosophy*. Chicago: Open Court.

Willard, D. (2001). Self-Verifying Axiom Systems, the Incompleteness Theorem and Related Reflection Principles. *Journal of Symbolic Logic,* 66, pp. 536-596.

Wright, C. (1983). *Frege's conception of numbers as objects*. Aberdeen: Aberdeen University Press.

Zalta, E. N. (Ed.). (1997). *Stanford encyclopedia of philosophy [electronic resource]*. Stanford, Calif.: Stanford University.