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St. Alasdair on lattices everywhere

Katalin Bimbó and J. Michael Dunn

Abstract Urquhart works in several areas of logic where he has proved important results. Our paper outlines his topological lattice representation and attempts to relate it to other lattice representations. We show that there are different ways to generalize Priestley's representation of distributive lattices—Urquhart's being one of them, which tries to keep prime filters (or their generalizations) in the representation. Along the way, we also mention how semi-lattices and lattices figured into Urquhart's work.

Keywords: Galois connection • Lattice • Relational semantics • Semilattice • Topological frame

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1 Introduction

Alasdair Urquhart has the title of "St. Alasdair" in the Logicians Liberation League.[1] We have never known why. Is it because he works miracles, or because he is very nice? We think both. We have each known Urquhart for many years, and indeed one

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[1] The "LLL" was founded by Robert K. Meyer in 1969. The LLL's manifesto, group pictures of some of the members and more may be found at the URL (as of June 2020): aal.ltumathstats.com/curios/logicians-liberation-league.

of us (JMD) has known him for over 50 years. JMD once thought of Alasdair as a kind of younger brother, arriving in the Anderson–Belnap family of relevantists just shortly after his two older brothers, Bob Meyer and JMD left the nest. Indeed, JMD might have overlapped with Alasdair in an alternative possible world. The last chapter of JMD's dissertation (Dunn, 1966) was intended to show the decidability of the two relevance logics **E** and **R**. Fortunately, his dissertation director Nuel Belnap insisted that the dissertation was complete as it was, and JMD did not need to spend another year on it. We say fortunately, because as almost every reader knows, Urquhart (1984) showed the undecidability of these logics.

Urquhart has been very helpful to both of us (JMD and KB) in various ways, including through reading our work and giving us helpful suggestions and criticisms (almost all of which we have agreed with).

Urquhart has made considerable contributions to logic and the philosophy of logic, including non-classical logics (particularly, relevance logic), lattice theory, foundations of mathematics, history of logic, theory of computation, and computational complexity theory. In this chapter, we focus on just one of these, his *topological representation of lattices*. However, we will mention some other contexts where lattices, modular lattices and semilattices appear in Urquhart's work—seemingly, everywhere. Lattices had been given a number of different representations since Birkhoff and Frink (1948) (using sets of subsets of the elements), but Urquhart's was the first one using topological structures.

In Section 2, we illustrate the idea of emulating abstract algebras by concrete objects, namely, groups by permutations. Then, we turn to lattices in Section 3, where we present Urquhart's lattice representation, and we briefly compare it to Priestley's representation. Section 4 explores the confluence of two trains of thought, one coming from a representation of orthocomplemented lattices and the other originating in Galois theory. This leads to another generalization of Priestley's representation. In Section 5, we explain the importance of the topologies on frames. Among the various lattice representations, Urquhart's is the first one to provide all components for duality. In the concluding Section 6, we quickly point out the importance of all the lattice representations for the model theory of substructural logics.

2 Representations of abstract algebras

As any schoolchild knows from personal experience, algebra is abstract, though they may not know this word. There is the term "abstract algebra" to cover algebras that do not just give you laws for manipulating numbers, but give you laws for various structures that abstract out properties of various structures beyond numbers. A good example is a *monoid*, and we shall quickly examine representations of monoids to give a kind of introduction and paradigm for representations of algebras. We start by defining a *semigroup* as a set S together with an associative binary operation $\cdot$ on S. "Associative," of course, means that $x\cdot(y\cdot z) = (x\cdot y)\cdot z$. Additional axioms are the usual ones for identity, $x = x$ (reflexivity), if $x = y$, then $y = x$ (symmetry), and if $x = y$ and $y = z$, then $x = z$ (transitivity). And, of course, we must not forget the substitution of identicals, which in this case can be stated as if $x' = x$ then $x' \cdot y = x \cdot y$, and if $y' = y$ then $x \cdot y' = x \cdot y$. A monoid is a semigroup with an *identity element* e satisfying $e \cdot x = x = x \cdot e$ (identity). A *group* is a monoid with a unary *inverse* operation $^{-1}$ satisfying $x \cdot x^{-1} = e = x^{-1} \cdot x$.

Suppose you are creating a "butterfly zoo."[2] The ideal would not just to have a couple of butterflies, but in fact, to have examples of every species in Lepidoptera. This may be unrealistic for butterflies, but it is obtainable for groups. This may seem surprising because the number of different kinds of groups is obviously infinite. However, it turns out that we can construct groups that are representatives of every kind of group in the sense that every group is isomorphic to one of these representatives. This gives what is called the "Cayley Representation Theorem" for groups.

A standard example of a group is a *permutation group*, i.e., a collection of 1–1 functions from a set onto itself that is closed under composition. But this is much more than just an example. Any group is isomorphic to a subgroup of a permutation group of some set. Cayley showed that every group is isomorphic to a subgroup of the collection of 1–1 functions on some set closed under composition, where e is an identity function, and $^{-1}$ is the converse forming operation, i.e., if $f(x) = y$, then $f(y) = x$.[3] Important and beautiful as this theorem is, we can give the idea of a "representation" with its simpler monoid version. A *transformation monoid* is just like a permutation group except the functions are not required to be 1–1, nor are they required to be onto.

Given a monoid $\langle S, \cdot \rangle$, each element $a \in S$ determines a function f_a that maps each element x onto the element $a \cdot x$. Consider the set $F = \{ f_a : a \in S \}$ of all such functions. Note that F is clearly closed under composition since $f_a(f_b(x)) = a \cdot (b \cdot x) = (a \cdot b) \cdot x = f_{a \cdot b}(x)$. Thus we can map S onto F in a way that carries each $a \in S$ to f_a, namely, $h(a) = f_a$. Moreover, h is 1–1. Now suppose $a \neq b$ but $f_a = f_b$. Then $a \cdot e = b \cdot e$, and so $a = b$ contrary to our assumption. So, h is an isomorphism of the monoid to a submonoid of a transformation monoid.

We can easily expand the above to link groups to permutation groups. We only need to show that f_a is 1–1 and onto. But we will not expand on this here. Another expansion would be to consider all the subgroups of a group, which form a lattice. Indeed, Whitman (1946) showed that every lattice can be viewed as a lattice of subgroups of some group. We will not expand on this either.

On the other hand, we can use transformation monoids to provide a kind of semantics for a very simple logic. Define $\mathcal{A} \vDash_a \mathcal{B}$ iff $\forall x \in S (\text{if } x \vDash \mathcal{A} \text{ then } a \cdot x \vDash \mathcal{B})$. Informally, sentence $\mathcal{A}$ has sentence $\mathcal{B}$ as a consequence according to state a iff every state x where $\mathcal{A}$ holds is such that when viewing a as a function, a transforms x into a state $a(x)$ where $\mathcal{B}$ holds. Where e is the identity element, note that $\mathcal{A} \vDash_e \mathcal{B}$ iff, $\forall a \in S (\text{if } e \vDash \mathcal{A} \text{ then } a \cdot e = a \vDash \mathcal{B})$. We can define $\mathcal{A} \vDash_S \mathcal{B}$ iff $\mathcal{A} \vDash_e \mathcal{B}$.

[2] We will call it *Lepidopterary* to attract scientifically minded visitors. :-)

[3] In other words, every group is a subgroup of the automorphism group on some set; "automorphism" is the taxonomical name for a permutation in the scheme of morphisms.

We have not yet infused our sentences with any logical structure such as connectives. Nonetheless, it is obvious that $\mathcal{A} \vDash_S \mathcal{A}$. It is also clear that $\vDash_S$ is transitive, that is, if $\mathcal{A} \vDash_S \mathcal{B}$ and $\mathcal{B} \vDash_S \mathcal{C}$ then $\mathcal{A} \vDash_S \mathcal{C}$. Now, we may consider adding a pair of naturally emerging connectives, which resemble the familiar conjunction and conditional. $\mathcal{A} \circ \mathcal{B} = \{a \cdot b : a \in \mathcal{A} \text{ and } b \in \mathcal{B}\}$, $\mathcal{A} \to \mathcal{B} = \{x : \forall a(\text{if } x \in \mathcal{A} \text{ then } x \cdot a \in \mathcal{B})\}$. It is worth pointing out that the definition of $\mathcal{A} \to \mathcal{B}$ parallels the valuation clause for implications in Urquhart's (1972b), where he gave a semantics for relevant implication. The latter is called *semilattice semantics*, because the operation $\cdot$ is not functional composition, rather, it is set union, which has the additional properties of commutativity and idempotence. This semantics fits precisely the implicational fragment of **R**, as he showed in Urquhart (1972a).

Having found the first use of a semilattice in Urquhart's work—while illustrating the idea of a representation—we go on to lattices.

3 Representations of lattices

A slightly different approach to the semantics of a logic than what we have already mentioned may be sketched as follows. We start with sentences. Sets of sentences describe a situation, and in turn, sets of situations characterize propositions. However, sentences are often too delicate, and they make too many distinctions. If a logic cannot distinguish between $\mathcal{A}$ and $\mathcal{B}$ with respect to their role in reasoning, then there is no need to distinguish $\mathcal{A}$ and $\mathcal{B}$ in their interpretations. Then it is just as good (or better) to start with the Lindenbaum algebra of a logic as with all the sentences.

Definition 3.1. A *lattice logic* ($\mathfrak{Lat}$) has two binary connectives $\wedge$ (conjunction) and $\vee$ (disjunction) with a denumerable set of sentence letters. The formulas (wff's) are defined as usual; they are abbreviated by $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$. The consequence relation ($\vdash$) satisfies the following axioms and rules. (The two-way turnstile $\dashv\vdash$ indicates that $\vdash$ holds in both directions.)

(1) $\mathcal{A} \vdash \mathcal{A}$, $\quad$ $\mathcal{A} \vdash \mathcal{B}$ and $\mathcal{B} \vdash \mathcal{C}$ imply $\mathcal{A} \vdash \mathcal{C}$;
(2) $\mathcal{A} \wedge \mathcal{B} \vdash \mathcal{A}$, $\quad$ $\mathcal{A} \wedge \mathcal{B} \vdash \mathcal{B}$, $\quad$ $\mathcal{A} \vdash \mathcal{A} \vee \mathcal{B}$, $\quad$ $\mathcal{A} \vdash \mathcal{B} \vee \mathcal{A}$;
(3) $(\mathcal{A} \wedge (\mathcal{B} \wedge \mathcal{C})) \dashv\vdash ((\mathcal{A} \wedge \mathcal{B}) \wedge \mathcal{C})$, $\quad$ $(\mathcal{A} \vee (\mathcal{B} \vee \mathcal{C})) \dashv\vdash ((\mathcal{A} \vee \mathcal{B}) \vee \mathcal{C})$;
(4) $\mathcal{A} \vdash \mathcal{C}$ and $\mathcal{B} \vdash \mathcal{C}$ imply $\mathcal{A} \vee \mathcal{B} \vdash \mathcal{C}$, $\quad$ $\mathcal{A} \vdash \mathcal{B}$ and $\mathcal{A} \vdash \mathcal{C}$ imply $\mathcal{A} \vdash \mathcal{B} \wedge \mathcal{C}$.

A *lattice logic with limits* ($\mathfrak{LatL}$) additionally includes two zero-ary connectives T (triviality or constant truth) and F (absurdity or constant falsity). The next axioms hold for T and F.

(5) $\mathcal{A} \vdash T$; $\quad$ $F \vdash \mathcal{A}$.

REMARK 3.1. Often, it is useful to think of a lattice as two semilattices glued together. Indeed, if we would exclude $\wedge$ or $\vee$ from the vocabulary, then the leftovers would be semilattice logics. The addition of T and F is technically motivated, and it is, by and large, harmless. T is triviality, or in a more favorable tone of voice, T is a formula implied by all formulas, thus, in a sense a theorem. T and F are the limits of what a user of $\mathfrak{LatL}$ can say—to use a Wittgensteinian metaphor.

If a logic extends $\mathfrak{Lat}$ (or $\mathfrak{LatL}$), then we can define an equivalence relation, which we denote by $\equiv$, on the set of wff's by $\mathcal{A} \equiv \mathcal{B} := \mathcal{A} \dashv\vdash \mathcal{B}$. Then the Lindenbaum algebra of the logic contains a *lattice*, in which the elements are $[\mathcal{A}]$, where $[\mathcal{A}] := \{\mathcal{B} : \mathcal{A} \dashv\vdash \mathcal{B}\}$. The algebra of $\mathfrak{LatL}$ is *bounded*, which is advantageous if we want to obtain the algebra from a topology.[4]

Definition 3.2. A *lattice* $\mathbf{L} = \langle A; \wedge, \vee \rangle$ is an algebra where $\wedge$ and $\vee$ are binary operations on the set A, and the following equations hold.

(1) $a \wedge a = a, \qquad a \wedge b = b \wedge a, \qquad a \wedge (b \wedge c) = (a \wedge b) \wedge c;$
(2) $a \vee a = a, \qquad a \vee b = b \vee a, \qquad a \vee (b \vee c) = (a \vee b) \vee c;$
(3) $a \wedge (b \vee a) = a, \qquad a \vee (b \wedge a) = a.$

A *bounded lattice* $\mathbf{L} = \langle A; \wedge, \vee, \top, \bot \rangle$ is a lattice with two distinguished elements of A satisfying the equations in (4).

(4) $\bot \vee a = a, \qquad \top \wedge a = a.$

NOTATION 3.2. We have assumed some commonly used notational conventions. For instance, $a, b, c, \ldots$ are elements of A, and an equation holds in a structure when its universal closure does. Other symbols for the least and greatest elements of an algebraic structure that supports an order relation are 1 and 0. We do not introduce a special label for bounded lattices—even though not all lattices are bounded—because we almost always mean bounded lattices.

Lattices, with or without bounds, have a rich theory. For our purposes, it is interesting to carve out two equational subclasses of lattices. The lattices in which (m) holds are *modular*, and those in which (d) holds are *distributive*.

(m) $a \wedge (b \vee (a \wedge c)) = (a \wedge b) \vee (a \wedge c);$
(d) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c).$

The equation (d) implies (m) in the context of a lattice, but not the other way around. From the point of view of the semantics of substructural logics, a dividing line that is useful to draw is between lattices in which (d) holds, and the rest of lattices. Every particular lattice is distributive or not, and in fact, the Lindenbaum algebras of many logics include a lattice that is not distributive. However, we will not rely on a lattice not being distributive; rather, we will not assume that it is distributive. Sometimes, we may call a lattice for which we have not stipulated distributivity a *general lattice* for emphasis.

[4] Sometimes the Lindenbaum algebra is called Lindenbaum–Tarski algebra.

3.1 Urquhart's lattice representation

We recall Urquhart's lattice representation from (Urquhart, 1978). Urquhart saw his own lattice representation as a generalization of Priestley's representation of distributive lattices that she published in 1970. (See Priestley, 1970, 1972.)

Definition 3.3. A *doubly ordered space* $\mathfrak{F} = \langle U; \sqsubseteq_1, \sqsubseteq_2 \rangle$ satisfies the conditions listed in (f1)–(f3). (The u's range over U.)

(f1) $U \neq \varnothing, \quad \sqsubseteq_1 \subseteq U \times U, \quad \sqsubseteq_2 \subseteq U \times U;$
(f2) for $n \in \{1,2\}$: $\quad u \sqsubseteq_n u, \quad u_1 \sqsubseteq_n u_2$ and $u_2 \sqsubseteq_n u_3$ imply $u_1 \sqsubseteq_n u_3$;
(f3) $u_1 \sqsubseteq_1 u_2$ and $u_1 \sqsubseteq_2 u_2$ imply $u_1 = u_2$.

REMARK 3.3. A distinctive feature of Priestley's representation, especially, in comparison to Stone's in (Stone, 1937–38), is that the space from which a distributive lattice is defined is partially ordered. Then, Urquhart goes a step further by adding another order relation. We may also note that the omission of anti-symmetry from both relations is not essential, because both $\sqsubseteq_1$ and $\sqsubseteq_2$ are (weak) partial orders in the doubly ordered space of a lattice. It is also useful to note that the complements of $\sqsubseteq_1$ and $\sqsubseteq_2$ are *irreflexive*.

Definition 3.4. The *left image* of V (a subset of U) in a doubly ordered space is defined by (fl); similarly, the *right image* of V is given by (fr).

(fl) $l(V) = \{ u \in U : \forall v (u \sqsubseteq_1 v \Rightarrow v \notin V) \};$
(fr) $r(V) = \{ u \in U : \forall v (u \sqsubseteq_2 v \Rightarrow v \notin V) \}.$

A subset of a doubly ordered space V is *stable* when $lr(V) = V$. The set of all stable subsets of U is denoted by $\mathcal{P}(U)^{\dagger}$.

REMARK 3.4. Subsets with the property $rl(V) = V$ would do just as well as stable sets. Universal instantiation in the defining conditions in (fl) and (fr) yields that $V \cap l(V) = \varnothing$ and $V \cap r(V) = \varnothing$.

Proposition 3.5. *If* $\mathfrak{F} = \langle U; \sqsubseteq_1, \sqsubseteq_2 \rangle$ *is a doubly ordered space, then the set of stable subsets of* U *is a* lattice *with meet and join defined as* $\cap$ *and* $\Cup$, *where the latter is*

(f4) $V_1 \Cup V_2 = l(r(V_1) \cap r(V_2)).$

Proof. First, we note that l's type is $l : \mathcal{P}(U) \longrightarrow \mathcal{C}_1$ and r's type is $r : \mathcal{P}(U) \longrightarrow \mathcal{C}_2$.[5] To see this, let us assume that $V \subseteq U$, $u_1 \in lV$ and $u_1 \sqsubseteq_1 u_2$ but $u_2 \notin lV$. From the latter, it follows that there is a u_3 such that $u_2 \sqsubseteq_1 u_3$ while $u_3 \in V$. However, this contradicts $u_1 \in lV$ via $u_1 \sqsubseteq_1 u_2$ and $u_2 \sqsubseteq_1 u_3$, which imply $u_1 \sqsubseteq_1 u_3$. The two orders are alike, hence showing r's type is alike too.

l and r form a Galois connection between $\mathcal{C}_2$ and $\mathcal{C}_1$, that is, if $V \in \mathcal{C}_1$ and $W \in \mathcal{C}_2$, then $V \subseteq lW$ iff $W \subseteq rV$. We show that the "only-if" conditional holds. Let us assume

[5] We use the notation $\mathcal{C}$ as in Bimbó and Dunn (2008), that is, $C \in \mathcal{C}$ iff C is a cone (or an upset, or an increasing set—to use other terms). Then, $\mathcal{C}_1$ and $\mathcal{C}_2$ are the sets of cones with respect to $\sqsubseteq_1$ and $\sqsubseteq_2$, respectively. We may omit parentheses—for readability—from $r(V)$ and $l(V)$.

that $V \subseteq lW$ and $u_1 \in W$. Toward a contradiction, let $u_1 \notin rV$ be stipulated. Then, for some u_2, $u_1 \sqsubseteq_2 u_2$ and $u_2 \in V$. But then both $u_2 \in W$ and $u_2 \in lW$, which is impossible. The defining property of a Galois connection is symmetric in l and r; hence, the proof is complete. ∴

REMARK 3.5. The proposition and its proof is not a literal quote from (Urquhart, 1978), but it is essentially in Urquhart's paper. Our notation for the doubly ordered space intends to suggest that such a space may be viewed as a frame, and the logic $\mathfrak{LatL}$ can be interpreted by mapping sentences of this logic into stable subsets of the space.

It is pleasing that a doubly ordered frame carries a lattice, indeed, a *complete* one. However, from an algebraic point of view, it is more interesting to know if every lattice can be viewed as a set of certain subsets of a doubly ordered frame. Since lattices come in all sizes and shapes, the usual strategy to establish that an *isomorphic set representation* exists is to define a doubly ordered frame from an arbitrary lattice and then to show that there is a suitable isomorphism.

Definition 3.6. A *filter* F in a lattice $\mathbf{L}$ is subset of the carrier set (i.e., $F \subseteq A$) with properties (1)–(2).

(1) $a, b \in F$ implies $a \wedge b \in F$;
(2) $a \in F$ and $a \wedge b = a$ imply $b \in F$.

A filter is *proper* when $F \neq A$; a filter is *non-empty* when $F \neq \emptyset$.

Ideals, *proper* ideals and *non-empty* ideals are duals of respective filters. In particular, (3) and (4) define ideals.

(3) $a, b \in I$ implies $a \vee b \in I$;
(4) $b \in I$ and $a \vee b = b$ imply $a \in I$.

REMARK 3.6. Filters (in the algebra of a logic) correspond to theories (in the logic). They are sublattices, therefore, the set of filters is closed under intersection. The intersection of a pair of filters includes the joins of the elements in those filters, that is, intersection can represent join. However, the union of a pair of filters does not need to be a filter. The intersection of a pair of cones of filters is a cone of filters, and it can represent meet. Turning a lattice around, we can see that intersection on ideals can stand for meet, and intersection on cones of ideals can represent join. To improve on these matches, special filters and ideals may be used.

Definition 3.7. Let I be an ideal on the lattice $\mathbf{L}$.

(1) I is *principal* when there is a $b \in I$ such that $a \in I$ iff $a \vee b = b$.
(2) I is *prime* when $a \wedge b \in I$ implies $a \in I$ or $b \in I$.
(3) I is *meet-irreducible* when for no I_1, I_2 distinct from I, $I_1 \cap I_2 = I$.

Principal, prime and *join-irreducible* filters are defined dually.

Stone (1936) used cones of prime ideals to represent Boolean algebras. The context of a semantic interpretation for a logic motivates the equivalent view of a representation by certain sets of ultrafilters. ("Ultrafilter" is an alternative name for a maximal filter, and in a Boolean algebra all of these are prime.) Unfortunately, in a lattice that is not distributive, there are too few prime filters to anchor an isomorphic representation. Meet-irreducible ideals and join-irreducible filters generalize their prime counterparts, and as Birkhoff and Frink (1948) proved, cones of join-irreducible filters provide an isomorphic representation of a lattice with intersection standing in for meet. They called such a representation a *meet-representation*, perhaps, because they did not define an operation to represent joins.

REMARK 3.7. Prime filters are really special. Upward closed sets of prime ideals provide a meet- and a join-representation at the same time in a distributive lattice (including a Boolean algebra, where the upward closure trivializes). Furthermore, the complement of a prime filter is a prime ideal in any lattice. Prime filters are *relatively maximal*, that is, they are maximal with respect to not containing a particular element of a distributive lattice. Join-irreducible filters are similarly relatively maximal in lattices. However, cones of join-irreducible filters do not provide a join-representation (in modular but non-distributive lattices), nor is it true that the complement of a join-irreducible filter is a meet-irreducible ideal. (The complement of any filter is a prime co-cone in any lattice—see Bimbó and Dunn (2008, Ch. 4).) The ingenuity of Urquhart's representation relies on the observation that the complement of a join-irreducible filter contains at least one meet-irreducible ideal such that the filter and the ideal are relatively maximal with respect to each other, even though they may not exhaust the carrier set of the lattice.

Definition 3.8. The pair $\langle F, I \rangle$ is a *maximal disjoint filter–ideal pair* (MDFIP, for short) when (1)–(2) are satisfied.

(1) F is a non-empty, proper filter, and I is a non-empty, proper ideal;
(2) for any F', $F \subsetneq F'$ implies $F' \cap I \neq \varnothing$, and for any I', $I \subsetneq I'$ implies $F \cap I' \neq \varnothing$.

Proposition 3.9. *If* $\langle F, I \rangle$ *is a* MDFIP*, then* F *is a* join-irreducible filter *and* I *is a* meet-irreducible ideal.

The proof of this proposition can be pieced together from Birkhoff and Frink (1948); see also Urquhart (1978, Lemma 3). To put it quickly, if F were the intersection of two *other* filters, then F either would not be maximal or it would have a common element with I. It is also true that if a filter F and an ideal I are disjoint, then they can be extended into a MDFIP $\langle F', I' \rangle$ such that $F \subseteq F'$ and $I \subseteq I'$. (In general, this is a non-trivial claim that is usually proved using Zorn's lemma. We discuss this on page 339.)

Definition 3.10. If $\mathbf{L}$ is a lattice, then the *doubly ordered space of* $\mathbf{L}$ is $\mathfrak{F}_{\mathbf{L}} = \langle \mathbb{U}, \subseteq_1, \subseteq_2 \rangle$, where (1) and (2) describe the components.

(1) $\mathbb{U}$ is the set of MDFIP's on $\mathbf{L}$;
(2) $\langle F_1, I_1 \rangle \subseteq_1 \langle F_2, I_2 \rangle$ iff $F_1 \subseteq F_2$, and $\langle F_1, I_1 \rangle \subseteq_2 \langle F_2, I_2 \rangle$ iff $I_1 \subseteq I_2$.

REMARK 3.8. Obviously, $\subseteq$ is a partial order, hence, $\subseteq_1$ and $\subseteq_2$ are pre-orders. If both $\langle F_1, I_1 \rangle \subseteq_1 \langle F_2, I_2 \rangle$ and $\langle F_1, I_1 \rangle \subseteq_2 \langle F_2, I_2 \rangle$ hold, then both the filter and the ideal in the first pair is, respectively, a subset of the filter and the ideal in the second pair. But the pairs are maximally disjoint, hence, they are the same.

Despite all the duality between operations and sets of elements in a lattice, a lattice does not need to be self-dual. Further, the MDFIP's have to be linked to elements of a lattice, which suggests that we have to choose between filters and ideals.

Proposition 3.11. *A lattice* **L** *is* isomorphic *to a subset of stable sets on* $\mathbb{U}$.

Proof (sketch). By favoring filters, an $a \in A$ is mapped by h into elements of $\mathbb{U}$ as $h(a) = \{\langle F, I\rangle \in \mathbb{U} : a \in F\}$. It can be shown that $lrh(a) = h(a)$, that is, $h(a)$ is a stable set. Then, $h(a \wedge b) = h(a) \cap h(b)$ is immediate. And also, $h(a \vee b) = l(rh(a) \cap rh(b))$. So far, h is a lattice homomorphism. The fact that h is injective follows from separation; if $a \nleq b$, then there is a $\langle F, I\rangle \in \mathbb{U}$ such that $a \in F$ but $b \notin F$. ⸫

3.2 Priestley's representation generalized

Priestley's representation of distributive lattices was motivated by a certain dissatisfaction she had with Stone's representation in (Stone, 1937–38), in particular, with features of the topological characterization of the prime ideal space.[6] The set of prime filters in a Boolean algebra forms an anti-chain, but in other distributive lattices it is easy to find prime filters (or prime ideals) that are distinct, yet one is a subset of the other. Priestley's invention is the addition of an order relation to a topology, which concretely will be realized by the subset relation.

Definition 3.12. An *ordered space* is $\mathfrak{F} = \langle U, \leq\rangle$, where $\leq$ is a partial order on U.

Proposition 3.13. *The set of cones on* $\mathfrak{F}$ *is a* distributive lattice *with* $\cap$ *and* $\cup$ *as the lattice operations.*

The proof of this claim is practically obvious, hence, we do not even sketch it.

Definition 3.14. If **L** is a distributive lattice, then the *ordered space of* **L** is $\mathfrak{F}_{\mathbf{L}} = \langle \mathbb{U}, \subseteq\rangle$, where $\mathbb{U}$ is the set of prime filters, which is ordered by set inclusion.

Proposition 3.15. *A distributive lattice* **L** *is* isomorphic *to a subset of the set of cones on* $\mathbb{U}$.

Proof (sketch). First of all, $h(a)$ for $a \in A$, is $\{F \in \mathbb{U} : a \in F\}$. Cones of filters provide a meet-representation via intersection; since all the elements in the cones are prime filters, the union of such cones is a join-representation. The injectivity of h follows by an old and well-known result of Birkhoff, stating that distinct elements of a distributive lattice are separable by a prime filter. ⸫

[6] A Stone space for a Boolean algebra is a compact totally disconnected topology. But for a distributive lattice, Stone gave a *more complicated* characterization. Namely, the topology is T_0 with a basis comprising relatively bicompact sets with a further property linking intersections of basic sets with a closed set.

REMARK 3.9. To see Urquhart's representation as a generalization of Priestley's, we may compare the spaces first. Priestley's partial order could be weakened to a pre-order, or $\sqsubseteq_1$ and $\sqsubseteq_2$ could be strengthened to partial orders. To handle the one vs two orders, we could set $\sqsubseteq_1$ to be $\leq$ and $\sqsubseteq_2$ to be $\leq^{-1}$ (the inverse of the $\leq$ relation). Moving into the other direction, $\sqsubseteq_2$ could be simply omitted.

We have already pointed out that the complement of a prime filter is a prime ideal, that is, the MDFIP's are uniquely determined by either element of the pair in a distributive lattice. This means that there is a 1–1 map between prime filters and MDFIP's in a distributive lattice. It may be useful to glance at $rh(a)$. $F_1 \sqsubseteq_2 F_2$ iff $F_2 \subseteq F_1$, and the latter, iff $F_2 \sqsubseteq_1 F_1$. Thus, $F' \in rh(a)$ iff for all F, $F \subseteq F'$ implies $F \notin h(a)$.

Example 3.10. Let us consider some easy lattices. $\mathbb{Z}$ is a distributive lattice with min and max (as binary operations). Every principal filter is prime, and $h(n) = \{\,[m) : m \leq n\,\}$, where $[m) = \{\,n \in \mathbb{Z} : m \leq n\,\}$. $rh(n) = \{\,[\,i\,) : n < i\,\}$, in other words, $rh(n)$ is the complement of $h(n)$ in the set of prime filters on $\mathbb{Z}$. If we take $\mathbb{Q}$ in place of $\mathbb{Z}$, then the definition of $rh(n)$ looks as before; a difference between those sets in the filter spaces of $\mathbb{Z}$ and $\mathbb{Q}$ is that the latter principal cocone is not generated by a principal cone. Finally, if we take **4** (the four-element Boolean algebra) with a and b the labels for the non-extremal elements, then $h(a) = \{\,[a)\,\}$, and $rh(a) = h(b)$, that is, $\{\,[b)\,\}$. In each case, $rh(a) = \mathbb{U} - h(a)$, and by a similar argument, $l(rh(a) \cap rh(b)) = \mathbb{U} - ((\mathbb{U} - h(a)) \cap (\mathbb{U} - h(b))) = h(a) \cup h(b)$.

4 One or two binary relations

The work of De Morgan, Boole and Frege led to a logic that was new in its time—in the 19th century. However, challenges to 2-valued logic popped up soon after its first formulation in linear notation. In the early 20th century, practicing logicians found the 2-valued conditional too weak, which inspired the invention of strict implication and modal logic by C. I. Lewis. A serious challenge from physics produced the first example of a logic that questions the distributivity of $\wedge$ and $\vee$.[7]

Birkhoff and von Neumann in (1936) explain certain differences between the views of reality derived from classical mechanics and those derived from quantum mechanics. Aspects of quantum mechanics that often attract attention are properties of its phase-spaces, namely, their principal incompleteness in description and in computable dependence. In other words, many observations are mutually exclusive and predictions of the position and momentum at the same time are necessarily imprecise. Birkhoff and von Neumann focus on the differences between reasoning in classical mechanics and quantum mechanics.

The classical view of a phase-space allows one to consider arbitrary subsets as experimental propositions, that is, propositions stating position and momentum of

[7] At least, it is one of the earliest and best-known examples of a non-distributive logic.

a certain kind. This classical view is *classical* in the sense of classical (Newtonian) mechanics and classical in the sense of 2-valued (Boolean) logic. Propositions are subsets of a phase-space and the operations on them correspond to intersection, union and complementation. Birkhoff and von Neumann argue that, in contrast, experimental propositions in quantum mechanics correspond to closed linear subspaces of Hilbert space. And the operations on these propositions are intersection, linear sum and orthogonal complementation.

The algebraic characterization of the experimental propositions in quantum mechanics leads to a bounded modular lattice with orthocomplementation. Birkhoff and von Neumann (*ibid.*, §10) pinpoint the failure of distributivity as the central difference between the calculi of classical and quantum propositions.[8]

Interlude: Modular lattices and KR frames. Modularity is a tricky property. Every distributive lattice is modular, and modularity is readily definable by a side condition on a prototypical equation expressing distributivity. We repeat (d) from above, which is to be compared with (m′).

(d) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
(m′) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$, provided that $c \leq a$.

Dedekind was the first to characterize non-modular lattices as lattices that have a sublattice isomorphic to a five-element lattice (which is often labeled as $\mathbf{N}_5$). Despite this elegant algebraic description, modularity has not been characterized in terms of sequent calculus rules. The proof of the sequent $\mathcal{A} \wedge (\mathcal{B} \vee (\mathcal{A} \wedge \mathcal{C})) \Vdash (\mathcal{A} \wedge \mathcal{B}) \vee (\mathcal{A} \wedge \mathcal{C})$ (assuming more or less standard rules for $\wedge$ and $\vee$) appears to require the same structural rules on the left-hand side as $\mathcal{A} \wedge (\mathcal{B} \vee \mathcal{C}) \Vdash (\mathcal{A} \wedge \mathcal{B}) \vee (\mathcal{A} \wedge \mathcal{C})$, that is, distributivity does.

Urquhart used modular lattices in a crucial way in (Urquhart, 1984) to prove the undecidability of some of the major relevance logics such as **T** (ticket entailment), **E** (entailment) and **R** (relevant implication).[9] He also constructed a representation of modular lattices. We give an overview of the representation in a nutshell.

Definition 4.1. If **L** is a modular lattice with least element $\perp$, then its **KR**-*frame* is $\mathfrak{F}_{\mathbf{L}} = \langle A; R, \perp \rangle$, where $R \subseteq A^3$ such that

(1) $R(a,b,c)$ iff $a \vee b = c \vee b$ and $a \vee b = a \vee c$.

KR is a crypto-relevance logic (cf. Routley et al., 1982, Ch. 5, §5) or perhaps, a corrupted one (cf. Anderson et al., 1992, §54 & §65). To put it quickly, **KR** adds $(\mathcal{A} \wedge {\sim}\mathcal{A}) \to \mathcal{B}$ to **R**, and its relevant character is hidden in its positive fragment, which is corrupted by negation. That is, from the point of view of relevance logic **KR** degrades **R**, because **KR** lacks the variable sharing property. **KR** may be given a Meyer–Routley style semantics with a ternary accessibility relation. In the previous definition, we predicted that $\mathfrak{F}_{\mathbf{L}}$ has suitable properties to be called a **KR**-frame. (We do not prove here that it does.)

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[8] Orthocomplemented modular lattices should not be confused with orthomodular lattices. The set of lattices in the latter category is a proper subset of those in the former.

[9] We cannot go into the details here, but we mention Anderson et al. (1992, §65) too.

Definition 4.2. If $\mathfrak{F} = \langle U; R, \perp \rangle$ is a **KR**-frame, then its *modular lattice* with least element is defined as $\mathcal{L}(\mathfrak{F}) = \langle \mathcal{P}(U)^{\text{lin}}; \cap, \circ, \varnothing \rangle$, where (1)–(3) hold.

(1) $V \in \mathcal{P}(U)^{\text{lin}}$ iff $V \in \mathcal{P}(U)$ and $\forall u, v, w\, ((R(u, v, w) \land u, v \in V) \Rightarrow w \in V)$;
(2) if $V, W \subseteq U$, then $V \circ W = \{ u : \exists v, w\, (v \in V \text{ and } w \in W \text{ and } R(v, w, u)) \}$;
(3) $\cap$ is intersection, and $\varnothing$ is the empty set.

The superscript $^{\text{lin}}$ abbreviates "linear." It is true for all subsets of U that $V \subseteq V \circ V$, but the other inclusion does not hold, in general; on $\mathcal{P}(U)^{\text{lin}}$, $\circ$ is idempotent.

Proposition 4.3. *If* $\mathbf{L}$ *is a modular lattice with least element, then* $\mathbf{L}$ *is* isomorphic *to a sublattice of* $\mathcal{L}(\mathcal{F}(\mathbf{L}))$.

The claim is proved in (Urquhart, 2017). We note that this representation uses *subsets* of the carrier set rather than *sets of subsets*. To this extent, it does not fit the paradigm that we sketched at the beginning of Section 3, and it is not a derivative of the lattice representation in Section 3.1. The isomorphism establishing $\mathbf{L} \cong \mathcal{L}(\mathcal{F}(\mathbf{L}))$ maps $a \in A$ into $(a]$ (the principal ideal generated by a). Another way to look at the essence of this representation is to say that the ideal space of a modular lattice can be delineated precisely as the space of linear subsets with respect to $\circ$.[10]

Let us return to the logic of quantum mechanics. At the time of the writing of (Birkhoff and von Neumann, 1936), the equivalence of fields of sets and Boolean algebras was already known. Indeed, it is mentioned inter alia (on p. 831) about the logic of classical mechanics. Orthocomplemented distributive lattices are Boolean algebras, thus, the non-distributive modular lattices should be bunched together with non-modular lattices from the point of view of their representation by sets. The algebraic equations stipulated by Birkhoff and von Neumann give an *ortholattice* if modularity is omitted. These lattices are of interest in themselves, but they also played a fascinating role in the discovery of lattice representations on relational frames—including Urquhart's. Birkhoff and von Neumann (1936) mentions various models of modular lattices with orthocomplementation, including projective geometries and skew fields. Thus, it should not be surprising that Birkhoff abstracted out the idea of a *polarity* by 1940 or so.

Definition 4.4. A *polarity* is a triple $\langle X, Y, R \rangle$, where X and Y are sets and $R \subseteq X \times Y$. For $V \subseteq X$ and $W \subseteq Y$, their respective *polars* are defined by (1) and (2).

(1) $r(V) = \{ y \in Y : \forall x\, (x \in V \Rightarrow R(x, y)) \}$
(2) $l(W) = \{ x \in X : \forall y\, (y \in W \Rightarrow R(x, y)) \}$

Birkhoff (1967, V.7) also proved that the composition of r and l are closure operations (lr on subsets of X, rl on subsets of Y.) Furthermore, $lr[\mathcal{P}(X)]$ is a complete lattice that is dually isomorphic to $rl[\mathcal{P}(Y)]$. Birkhoff observed that if R is symmetric and irreflexive on a set (i.e., $X = Y$), then the set of closed subsets is an ortholattice. One of his examples is Cartesian n-space with R being $\perp$, the orthogonality relation.

[10] Another representation of modular lattices was obtained by Jónsson (1953). He proved that every lattice that can be represented with join being $R_1;R_2;R_1$ (where R_1 and R_2 are two equivalence relations on a set) is modular.

REMARK 4.1. Birkhoff and von Neumann in their paper of 1936 were concerned with logic, though they invoked many algebraic and geometric ideas too. It appears that Birkhoff went on to pursue the development of lattice theory (on which he published a book in 1940), whereas von Neumann, after developing what he called continuous geometry, focused on more practical areas such as computing and physics, and even game theory.

The abstraction of polarities is very useful, but in logical terms, it is the "easy direction" in giving a semantics for a logic. That is, it shows that a set X with an appropriate binary relation could serve as a frame for orthologic. Goldblatt (1974, 1975) showed that an *isomorphic representation* of ortholattices can be obtained along the lines of orthoframes. We, in effect, already defined an *orthoframe* above as a set with an irreflexive symmetric relation on it.

Definition 4.5. A lattice $\mathbf{L} = \langle A; \wedge, \vee, ', \bot, \top \rangle$ is an *ortholattice* when $\mathbf{L}$ is a lattice in which the following quasi-equations hold.

(1) $a'' = a, \qquad a \wedge a' = \bot, \qquad a \vee a' = \top, \qquad a \wedge b = a$ implies $a' \vee b' = a'$.

An ortholattice is quite like a Boolean algebra—except that it does not need to be distributive. Every Boolean algebra is an ortholattice, but not vice versa.

Definition 4.6. If $\mathbf{L}$ is an ortholattice, then its *orthoframe* is $\mathfrak{F} = \langle \mathbb{X}, \bot \rangle$, where (1) and (2) specify the components.

(1) $\mathbb{X}$ is the set of proper filters on A;
(2) $F_1 \perp F_2$ iff $\exists a \in A$ such that $a' \in F_1$ and $a \in F_2$.

Ortholattices are interesting in themselves. However, we wish to emphasize their generalization to polarities. We labeled the two functions as r and l in (1) and (2) in Definition 4.4 to point at certain similarities with Urquhart's functions r and l.[11] (fl) and (fr) have a similar form as the definitions in (1) and (2), except that there are two relations $\not\leq_1$ and $\not\leq_2$. Both relations are irreflexive, but not much else seems to be true of them. The functions r and l form a Galois pair in both cases.[12]

We recall some results about Galois pairs to show how the ideas about complementation, orthogonality and negation led to the lattice representation by Hartonas and Dunn (1993, 1997).

Definition 4.7. Let $\mathbf{U} = \langle U; \leq_1 \rangle$ and $\mathbf{W} = \langle W; \leq_2 \rangle$ be two posets and let f be a function form U to W, and let g be a function from W to U. Then the pair $\langle f, g \rangle$ is a *Galois connection* between $\mathbf{U}$ and $\mathbf{W}$ iff $\forall x \in U\, \forall y \in W$, $x \leq_1 g(y)$ iff $y \leq_2 f(x)$.

REMARK 4.2. It is easy to show that—equivalently—we can require $x_1 \leq_1 x_2$ implies $f(x_2) \leq_2 f(x_1)$, $y_1 \leq_2 y_2$ implies $g(y_2) \leq_1 g(y_1)$, $x \leq_1 g(f(x))$ and $y \leq_2 f(g(y))$.

[11] "Right" and "left" are, obviously, at hand, in particular, they are used in the theory of fields.

[12] Such functions, in an abstract setting, i.e., outside of Galois theory, have been studied by Everett (1944) and Ore (1944, 1962). Since the power set (or a set of special subsets of a set) has a natural ordering on it, namely, the subset relation, it is immediate that Galois connections on a collection of subsets induce a lattice (cf. Birkhoff, 1967, V.8).

If we view the members of U as propositions, $x_1 \leq_1 x_2$ may be interpreted as "x_1 implies x_2," and similarly, with W and $y_1 \leq_2 y_2$. You may be puzzled as to why we have two possibly disjoint sets of propositions, but try to set that aside. It is natural to view f and g as *negations*. Writing them as $\backsim$ and $\sim$, we have that x implies $\backsim y$ iff y implies $\sim x$. If you squint a bit (so as not to be able to distinguish the two different negations), this looks like a familiar principle of contraposition. And the last two inequations are double negation introductions: $x \leq_1 \backsim\sim x$ and $y \leq_2 \sim\backsim y$.

There is a straightforward way to construct a Galois connection between all subsets of a set X (the powerset of X) and all subsets of a set Y, where $\leq_1$ is set inclusion (the subset relation) restricted to $\mathcal{P}(X)$ and $\leq_2 = \subseteq \upharpoonright \mathcal{P}(Y)$. First, let us think of the members of X and Y as information states, and so their subsets can be viewed as "U.C.L.A. propositions." We could pick any relation R between the two sets. However, for reasons that pertain to seeing a Galois connection as involving a pair of negations, it is common to use the symbol $\perp$—like in an orthoframe. $\perp$ may be thought of as *orthogonality* or *perp* (for perpendicularity), or more generally as a kind of *incompatibility*, which may go one way but not the other.

Definition 4.8. Let X and Y be connected with $\perp$. For any $V \subseteq X$, $V^\perp = \{y \in Y : \forall x \in V\, x \perp y\}$. Dually, for any $W \subseteq Y$, ${}^\perp W = \{x \in X : \forall y \in W\, x \perp y\}$.

REMARK 4.3. $V^\perp$ can be thought of as a kind of negation of V, i.e., the set of states $y \in Y$ such that every state x that verifies V is incompatible with y. And symmetrically, with ${}^\perp W$. It is worth noting that when $\perp$ is a symmetric relation, that is, $x \perp y$ implies $y \perp x$ and $X = Y$—like in the orthoframe of an ortholattice in Definition 4.6—then $V^\perp = {}^\perp V$. Conversely, $\perp$ is symmetric if it comes from orthonegation.

Theorem 4.9. *$V \subseteq {}^\perp W$ iff $W \subseteq V^\perp$; that is, $\langle \cdot^\perp, {}^\perp\cdot \rangle$ are a* Galois connection *between $\mathcal{P}(X)$ and $\mathcal{P}(Y)$.*

Proof. ($\Rightarrow$) Suppose that $V \subseteq {}^\perp W$ and that $y \in W$, to show that $y \in V^\perp$, which means $\forall x \in V\, x \perp y$. So let us suppose $x \in V$; then $x \in {}^\perp W$. Since $y \in W$, $x \perp y$. ($\Leftarrow$) It is proved similarly. ∴

Thus perp allows us to define a "concrete" Galois connection between all the subsets of a set X and all the subsets of a set Y. But there are other "concrete" Galois connections that hold just between some subsets of a set X and Y—as we saw in Proposition 3.5. Not only does perp allow us to define a Galois connection between subsets of X and subsets of Y, but this is fully general way to obtain Galois connections.

Theorem 4.10. *Every Galois connection is generated—up to isomorphism—as in Definition 4.8, for some sets X, Y and $\perp$.*

Proof. Let us assume that $\mathbf{U}$ and $\mathbf{W}$ are Galois connected with $\langle f, g \rangle$ as in Definition 4.7. We consider the sets of cones on U and W, which we denote by $\mathcal{C}_1$ and $\mathcal{C}_2$. We define a perp relation between two cones ($C \in \mathcal{C}_1$ and $D \in \mathcal{C}_2$) so that $C \perp D$ iff $\exists x \in C$ such that $f(x) \in D$. Note that we can define the dual perp relation $\perp'$ so that

$D \perp' C$ iff $\exists y \in D$ such that $g(y) \in C$. It is easy to see that $\perp'$ is the *converse* of $\perp$. Thus if $C \perp D$ then $\exists x \in C\, f(x) \in D$. We invoke $x \leq_1 g(f(x))$ to obtain $g(f(x)) \in C$. So $\exists y \in D$, namely, $y = f(x)$, such that $g(y) \in C$; therefore, $D \perp' C$. The other direction is proven "dually."

We now define the embedding $h(x) = \{C \in \mathcal{C}_1 : x \in C\}$ and $h(y) = \{D \in \mathcal{C}_2 : y \in D\}$. The trick then is to show that $h(f(x)) = h(x)^{\perp}$, i.e., $f(x) \in D$ iff $\forall C$ (if $x \in C$ then $C \perp D$). The left-to-right direction is immediate.

For right to left, we contrapose. Thus, assume that $f(x) \notin D$. We need to show that it is not the case that $\forall C$ (if $x \in C$ then $C \perp D$), i.e., we need to find some cone C so that $x \in C$, and yet not $C \perp D$. Let C be the principal cone $[x) = \{x' : x \leq_1 x'\}$. Then $x \in C$, and yet it is not the case that $C \perp D$, for otherwise, $\exists x' \in C$ and $f(x') \in D$. That is, $x \leq_1 x'$ and $f(x') \in D$, but $f(x') \leq_2 f(x)$ and $f(x') \in D$; hence, $f(x) \in D$. Contradiction!

That $h(g(y)) = {}^{\perp}h(y)$ may be shown similarly. ⁘

This theorem is a *representation of Galois connections* between partially ordered sets, which can readily be extended to a Galois connection between two *semilattices*. A meet semilattice is a partially ordered set, where every pair of elements a and b has a greatest lower bound $a \wedge b$. A join semilattice is defined dually, requiring that every pair of elements a and b has a least upper bound $a \vee b$. This can be done quite elegantly by requiring the set X to be a meet semilattice $\langle S; \wedge \rangle$ and the set Y to be a join semilattice $\langle S'; \vee \rangle$. But a more casual way is just to let the right-hand set also be a meet semilattice (with the order inverted). Then all one needs to do is to extend the definition of a cone to a filter F. An advantage of this track is that one can avoid the apparatus of "dual filters" altogether. This naturally leads to a representation of lattices once one realizes that a lattice is just two semilattices "glued together," one up and the other down.

REMARK 4.4. In the case of Urquhart's representation, the functions r and l have to be restricted to upward closed subsets (with respect to one or the other order relation) on the frame. That is, the Galois connected posets are $\langle \mathcal{C}_1(U), \subseteq \rangle$ and $\langle \mathcal{C}_2(U), \subseteq \rangle$. Then, taking cones on each set, we can find $\perp$ using Theorem 4.10. For example, in the doubly ordered space of a lattice, if $C \in \mathcal{C}(\mathcal{C}_1(\mathbb{U}))$ such that $h(a) \in C$, then $C \perp D$ holds when $D \in \mathcal{C}(\mathcal{C}_2(\mathbb{U}))$ and $rh(a) \in D$. Of course, not all C's and D's are of this form, but $h(a)$ and $rh(a)$ are the lr and rl stable sets in $\mathcal{C}_1(\mathbb{U})$ and $\mathcal{C}_2(\mathbb{U})$, respectively. For any $\langle F, I \rangle \in h(a)$, $a \in F$ and for any $\langle F, I \rangle \in rh(a)$, $a \in I$.

In comparison with ortholattices, the two orders $\sqsubseteq_1$ and $\sqsubseteq_2$ seem to be complicated. But the ortholattices have an additional component—the orthocomplement—that is used in the definition of $\perp$ in the isomorphic representation. Now, we have seen that from a Galois connection on a lattice (chopped into two semilattices), one can find a polarity. Concretely, Urquhart's representation contains a Galois connection between two posets, which are constructed from the two orders. So, we might wonder how these observations may be put to use to construct a lattice representation. Indeed, this can be accomplished in more than one way. Next, we outline a lattice representation due to Hartonas and Dunn (1997, 1993).

Definition 4.11. A *frame* is $\mathfrak{F} = \langle X, Y, \perp \rangle$, where $X, Y \neq \varnothing$ and $\perp \subseteq X \times Y$.

This definition simply takes a polarity for a frame. (Continuing the idea of the previous theorem, we use the $\perp$ notation.)

Definition 4.12. If $\mathbf{L}$ is a lattice, then its *lattice frame* is $\mathfrak{F}_{\mathbf{L}} = \langle \mathbb{F}, \mathbb{I}, \between \rangle$, where the components are specified in (1)–(3).

(1) $\mathbb{F}$ is the set of filters on A;
(2) $\mathbb{I}$ is the set of ideals on A;
(3) $F \between I$ iff for some $a \in A$, $a \in F$ and $a \in I$.

REMARK 4.5. The definitions of the $\perp$ relation in Definition 4.6 and in the proof of Theorem 4.10 and the definition of $\between$ above are remarkably similar. A lattice does not need to be complemented, and so there is no unary operation that could be applied to a. However, a lattice with its natural order relation $\leq$ and its converse $\leq^{-1}$ can be seen to have a Galois connection $\langle \mathrm{Id}, \mathrm{Id} \rangle$, where $\mathrm{Id}(x) = x$. Of course, $F_1 \cap F_2 \neq \varnothing$ would give the total relation on non-empty filters. But $\leq^{-1}$ turns filters into ideals and vice versa. It seems to us that using the above frame is also in the spirit of Birkhoff and Frink (1948), since cones of filters give a meet-representation and cones of ideals give a join-representation—both with intersection. The role of $\between$ is to create a dual isomorphism between the complete lattices of closed subsets of the set of filters and closed subsets of the set of ideals.

The remaining piece is to ensure that the elements of a lattice can be mapped into appropriate collections of filters (or ideals). Of course, if we take as a hint the usual definition of h that uses $\in$, then we get that $h(a) = [[a))$. That is, each lattice element is mapped into the principal cone of filters generated by the principal filter generated by the lattice element.

REMARK 4.6. Triples $\langle G, M, I \rangle$, which have the structure of a polarity or of a frame in the sense of Definition 4.11 have been termed *contexts* by Wille (1982). A subset of G is an *extension* of a concept, whereas a subset of M is its *intension* (in Church's terminology). A hierarchy of concepts (in the Aristotelian sense) pertaining to a context can be constructed based on the observation that "The more specific a concept is, the fewer exemplars it has." Thus, a pair of an extension and intension fits into a lattice of concepts given a context.[13] Although Wille *does construct* a lattice from a formal context, his interest lies with concepts and their relationships (cf. Wille, 1985). His representation of a lattice diverges markedly from that of (Hartonas and Dunn, 1997 and Hartonas and Dunn, 1993), because his context of a lattice $\mathbf{L} = \langle L; \wedge, \vee \rangle$ is $\langle L, L, \leq \rangle$. Hartung (1992) uses topological contexts to give an isomorphic representation for formal concept lattices.

We may quickly compare (or even contrast) Urquhart's and Hartonas and Dunn's representations. The most striking difference is the disjointness of the MDFIP's and

[13] Wille's notions of (formal) context and (formal) concept, which he designed for computer science applications should not be confused with philosophical investigations of concepts following Wittgenstein or with the use of the term "concept" in cognitive science.

the overlap that defines $\perp$. In the spirit of reverse mathematics (or of concerns about uses of equivalents of the axiom of choice), we should point out that Hartonas and Dunn do not rely on prime filters, prime ideals, join-irreducible filters or meet-irreducible ideals. To prove that such objects exist, or that a disjoint pair of a filter and an ideal can be extended into a MDFIP (Urquhart, 1978, Lemma 3), one apparently has to appeal to the axiom of choice (or to Zorn's lemma, etc.). The two representations appear not to be equivalent in the sense that Hartonas and Dunn's resides in ZF, but Urquhart's seems to require ZFC. On the side of similarities, we may point out that in both representations the ideals play second fiddle to filters (of a certain kind).

Three more lattice representations. The differences may inspire us to consider variations on either representation that would make them more similar.

1. For example, it may seem that the "dual" of maximally disjoint filter–ideal pairs should be *minimally overlapping filter–ideal pairs* (MOFIP's, for short). Since there are no "subatomic particles" in a lattice, minimal overlap implies that there is exactly one element that is common to the filter and the ideal in the pair. Furthermore, the shared element must be the least element of the filter and the greatest element of the ideal. Then, we are talking about filter–ideal pairs, in which both the filter and the ideal are principal, and they are generated by the same element. Principal filters have some pleasant properties. For example, a meet-representation of the lattice by sets of filters that contain a particular element preserves arbitrary meets iff all the filters are complete filters (which principal filters are).[14] However, not all lattices are complete, and in a lattice that is not complete, there are non-principal filters. Perhaps, the best-known example of a non-complete distributive lattice is $\mathbb{Q}$, the set of rationals with min and max for meet and join. If we set aside the problems caused by non-complete lattices, then the minimally overlapping filter–ideal pairs provide a relatively simple representation.[15] If $h(a) = \{\langle F,I\rangle : a \in F\}$ is the embedding of the lattice into its MOFIP space, then set of the first projection is generated by $[a)$. Using the same relation as in the Hartonas–Dunn representation, $rh(a) = \{\langle F,I\rangle : a \in I\}$. This obviously suffices to model $\vee$.

2. Allwein and Dunn (1993, p. 522) point out that a representation may be had without insisting upon maximality in MDFIP's. Indeed, this improves the representation in the sense of duality theory, which we briefly touch upon in the next section. This representation has been worked out in Allwein and Hartonas (1993), where the focus is on duality theory, and in Gehrke and Harding (2001), with an emphasis on canonical extensions. Dually, a representation can be constructed from arbitrary overlapping filter–ideal pairs. This representation is worked out in Bimbó and Dunn (2008, Ch. 9), where we called the frames *centered spaces*. We could argue that this is the most balanced representation in the sense that there is no need to choose between a filter and an ideal as the preferred object. A pair of a filter and ideal both of which contain an element a, carve out a sublattice of a lattice (by their common

[14] See Birkhoff and Frink (1948, §11).

[15] Bimbó (1999, 2001) used such a representation of lattices as a component of a semantics.

elements). So a is mapped into the sublattices that contain a with two "tentacles," so to speak, which are the rest of the filter and that of the ideal.

3. We started this section by considering the number of binary relations that each representation stipulates. Priestley added order to the space, and the general lattice representations all added further elements—so far. The preference for filters suggests that we may consider another generalization of the Priestley space. $\mathfrak{F} = \langle U; \leq \rangle$ is an *inclusion space* when $U \neq \varnothing$ and $\leq$ is a partial order on U. Of course, $\cup$ cannot stand for $\vee$, if $\wedge$ is $\cap$. However, r and l can be defined as usual in a polarity, and $\Cup$ can represent $\vee$. For an isomorphic copy of a lattice $\mathbf{L}$, we simply take the filter space of $\mathbf{L}$ with set inclusion as the partial order. This representation does not coincide with Priestley's on a distributive lattice if the filters are restricted to prime filters, because the concrete $\Cup$ which is a closure of $\cup$ relies on filters that are not join-irreducible. On the other hand, this representation coincides with Priestley's in the definition of the topology on $\langle \mathbb{U}, \subseteq \rangle$. Namely, the subbasis is defined as $\{ h(a), -h(a) : a \in A \}$, where $h(a) = \{ F \in \mathbb{U} : a \in F \}$. For more details, see Bimbó and Dunn (2008, Ch. 9).

REMARK 4.7. For certain purposes, it is satisfactory to have a lattice that emerges from a frame. Indeed, in the area of modal logic, some researchers prefer to weaken a topological frame to a general frame. (Retaining a set of propositions confers benefits without the burden of imposing additional conditions on a Stone space with a binary relation.) We mention a couple of recent papers, Orłowska and Rewitzky (2005), Hartonas (2019), and Düntsch and Orłowska (2019), which advocate for lattice representations without topologies within the "discrete duality" program.

5 Topological structures

The previous sections showed how to obtain a lattice from a relational structure, moreover, how to define a relational structure from any lattice, so that an isomorphic copy of the lattice can be found in the set algebra on its relational structure. However, in general, it would be unreasonable to expect that the isomorphism is *surjective*. For instance, every element of a lattice generates a filter, but an infinite lattice may have non-principal filters too. And the power set of the set of filters has a *strictly greater cardinality*, which may be inherited by a subset of the power set, which perhaps, allows only for cones.

Stone's representation of Boolean algebras by sets in (Stone, 1936) is acclaimed, because he found an elegant way to characterize the *image of a Boolean algebra* under the intended isomorphism. A Stone space is simply a compact totally disconnected topology, in which a Boolean algebra emerges as the set of clopen sets. That is, the elements of the Boolean algebra are those open sets in the topology that are also closed (i.e., their complements are open). Sets of prime filters of the form $h(a)$ constitute a basis for the Stone space of a Boolean algebra. Indeed, $h[A]$ is the set of clopen sets of the topology in which the basis comprises the sets $h(a)$, for $a \in A$.

Priestley's representation added an order, hence, her space is a compact, totally order disconnected topology. A distributive lattice arises as the clopen cones of the topology, that is, increasing sets that are both open and closed. Given a distributive lattice, sets of the form $h(a)$ together with their complements form a subbasis of a topology, which is the Priestley space of a distributive lattice.

Urquhart's representation requires additions in its topological component, because it has two order relations on a set.

Definition 5.1. A *doubly ordered topological space* is $\mathfrak{F}$ as in Definition 3.3 with a compact topology on U that satisfies (1)–(5).

(1) $\textcircled{C} \subseteq \mathcal{P}(U)$ such that, if $X \in \textcircled{C}$ then both $-X$ and $-rX$ are open;
(2) $x \nleq_1 y$ implies that for some $X \in \textcircled{C}$ such that $X \in \mathcal{P}(U)^\dagger$ both $x \in X$ and $y \notin X$;
(3) $x \nleq_2 y$ implies that for some $X \in \textcircled{C}$ such that $X \in \mathcal{P}(U)^\dagger$ both $x \in rX$ and $y \notin rX$;
(4) if $X, Y \in \textcircled{C}$, then both $-r(X \cap Y)$ and $-l(rX \cap rY)$ are open;
(5) the set $\{ -X : X \in \textcircled{C} \wedge X \in \mathcal{P}(U)^\dagger \} \cup \{ -rX : X \in \textcircled{C} \wedge X \in \mathcal{P}(U)^\dagger \}$ is a subbasis for the topology.

Given a doubly ordered topological space $\mathfrak{F}$, $\{ X : X \in \textcircled{C} \wedge X \in \mathcal{P}(U)^\dagger \}$ is a lattice.

Definition 5.2. If $\mathbf{L}$ is a lattice, then the *subbasis of the topology* of its doubly ordered topological space is $\{ h(a) : a \in A \} \cup \{ -rh(a) : a \in A \}$.

The definition of the subbasis is quite simple and it closely resembles the definition of the subbasis in the Priestley representation.

Dualities. Having outlined examples of topological frames for lattices (and ortholattices), now we mention another role that topologies can play in a lattice representation. Topologies may lead to dualities between a class of algebras and a class of relational structures. We fix both the class of algebras (as lattices) and the class of relational structures (e.g., doubly ordered topological spaces). Then, given any $\mathfrak{F}$, we can construct a lattice, that is, $\mathcal{L}(\mathfrak{F})$ is a lattice, where $\mathcal{L}$ is a map from frames to lattices. This step gives a semantics for our $\mathfrak{LatL}$ with *soundness* guaranteed. In order to get an isomorphic copy of a lattice $\mathbf{L}$, we define its frame $\mathcal{F}(\mathbf{L})$ and show that $\mathbf{L} \cong \mathcal{L}(\mathcal{F}(\mathbf{L}))$. Here $\mathcal{F}$ is a map from lattices to frames. This step gives *completeness* for $\mathfrak{LatL}$ with respect to the class of relational structures.

Isomorphisms are special homomorphisms, and the latter are maps that are natural companions of algebras. If $\mathfrak{F}$ is a relational structure, then the counterpart notion is relational isomorphism between a pair of frames. If, in addition, $\mathfrak{F}$ is equipped with a topology, then relational isomorphisms should be homeomorphisms (in both direction). In other words, we would like to have that $\mathfrak{F} \rightleftharpoons \mathcal{F}(\mathcal{L}(\mathfrak{F}))$. Once we have both correspondences, we have *object duality*, because we can match frames and lattices to each other.

However, we can go a step or two further. Homomorphisms between algebras are of interest in themselves. They are the maps between algebras the properties of which tell us a lot about the particular algebras in question. We can define maps between frames too so that the map turns a frame of a certain kind into a frame of the same kind. If the homomorphisms (on the side of algebras) and the frame morphisms (on the side of frames) compose and certain maps can function as identities for composition, then we may talk about a duality between categories of lattices and frames (cf. Awodey 2010). Pulling back from full categorical duality (i.e., functorial duality), we could simply consider full duality, that is, a 1–1 correspondence between frame morphisms and homomorphisms on top of object duality.

For the sake of comparison, we outline two representations that we already mentioned—now outfitted with topologies.

Definition 5.3. An *ordered topological orthospace* is $\mathfrak{F} = \langle X; \leq, \perp, \mathcal{O} \rangle$, where $\langle X, \leq \rangle$ with $\mathcal{O}$ is a poset with a compact topology, and $\perp \subseteq X^2$ is an orthogonality relation (irreflexive and symmetric). Also, (1)–(4) hold. ($\mathfrak{C}$ is the set of clopens and $\mathfrak{C}^\dagger$ denotes the set of clopen stable sets.)

(1) $x \nleq y$ implies $\exists O \in \mathfrak{C}^\dagger (x \in O \wedge y \notin O)$;
(2) $x \perp y$ and $x \leq z$ imply $z \perp y$;
(3) $O \in \mathfrak{C}^\dagger$ implies $O^\perp \in \mathfrak{C}$;
(4) $x \perp y$ implies $\exists O \in \mathfrak{C}^\dagger (x \in O \wedge y \in O^\perp)$.

A *frame morphism* f is a continuous function with properties (5)–(6).

(5) $fx \perp fy$ implies $x \perp y$;
(6) $\neg z \perp fy$ implies $\exists x (\neg x \perp y \wedge z \leq fx)$.

The above definition (from Bimbó, 2007), which enriches an orthoframe with not only a topology but also with an order relation, allows us to prove *full duality* (i.e., duality for both objects and maps) between ortholattices and orthoscapes.

We argued that inclusion spaces provide an alternative generalization of Priestley spaces.

Definition 5.4. A *topological inclusion space* is $\mathfrak{F} = \langle X; \leq, \mathcal{O} \rangle$, where $\langle X; \leq \rangle$ is a poset, and $\mathcal{O}$ is a compact topology with (1)–(2) true.

(1) $x \nleq y$ implies $\exists O \in \mathfrak{C}^\dagger (x \in O \wedge y \notin O)$;
(2) $U, V \in \mathfrak{C}^\dagger$ implies $U \uplus V \in \mathfrak{C}$, that is, $l(rU \cap rV)$ is clopen.

A *frame morphism* f is a continuous order preserving map satisfying (3).

(3) $O \in \mathfrak{C}^\dagger$ implies $f^{-1}[O] \in \mathcal{P}(X)^\dagger$, that is, the inverse image of a clopen stable set is stable.

The enriched inclusion spaces support *full duality*, including a duality between homomorphisms and frame morphisms (see Bimbó and Dunn, 2008, Ch. 9).

REMARK 5.1. We may note that both of these representations rely on mere filters, and do not require the use of Zorn's lemma or some other equivalent of the axiom of choice to prove maximality of any kind. Also, both frames include an order, which is the only relation in an inclusion space, and it makes orthospaces smoother. Allwein and Hartonas (1993) discuss the question of full duality in Urquhart's representation, and in its relaxed version—where maximality is omitted.

6 Conclusions

The logic of quantum logic isolated by Birkhoff and von Neumann may be the first but surely not the last example of a logic that does not stipulate distributivity for $\wedge$ and $\vee$. Such logics were developed for technical reasons (like lattice-**R**, by Meyer, 1966), for the sake of simplicity (like full Lambek calculus, by Ono, 2003) and to capture resource-minded reasoning (like linear logic, by Girard, 1987). Non-distributive logics arise naturally as "substructural logics," since Gentzen's structural rules of permutation, thinning and contraction are essential for a proof of distribution.[16] The various lattice representations differ on how easily they can be extended to semantics for logics. We only mention Allwein and Dunn (1993), Bimbó and Dunn (2008) and Düntsch et al. (2004) as examples that provide semantics for a wide range of logics starting from Urquhart's lattice representation.

We close by mentioning once more Alasdair Urquhart's Sainthood. He was the first to give a *topological representation for lattices*. Thus, we think that he should become the "Patron Saint of Lattices" for doing this, and for all of the miracles he has performed with all kinds of (semi)lattices.

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[16] "Substructural logics" is often used as an honorific to include relevance logics such as **T**, **E** and **R** too, even though these logics have a distributive lattice reduct in their Lindenbaum algebra.

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