Characterizing Benford's Law in Linear Systems

by

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Abstract

We study the widespread logarithmic distribution of first significant digits and significands of data sets (referred to as *Benford's Law*) in the context of dynamical systems. Using recent tools and conditions under which a recursively defined sequence is *Benford* via the classical theory of uniform distribution modulo one, this study derives a necessary and sufficient condition ("*nonresonant spectrum*") on $A \in \mathbb{R}^{d\times d}$ for every sequence $(y^\top A^n x)_{n\in\mathbb{N}}$, with arbitrary $x, y \in \mathbb{R}^d$, emanating from the difference equation $x_n = Ax_{n-1}$, to be *Benford* or terminating. This result in turn is used to also show that the function $t \longmapsto y^\top e^{tA} x$ arising from the differential equation $\dot{x}(t) = Ax(t)$ is either *Benford* or identically zero for $t \geq 0$. The results generalize and unify already known facts for one- and higher-dimensional systems.

To Daa, Maa, Victoria, Siblings and You

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Introduction

1.1 Background

Throughout this work, the symbols $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}^+$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of natural, non-negative integer, integer, rational, positive real, real, and complex numbers, respectively. For any $x \in \mathbb{R}$, let $\lfloor x \rfloor$ denote the integral part of x. That is, $\lfloor x \rfloor$ is the largest integer not larger than x. Also, let $\langle x \rangle$ be the fractional part of x, i.e. $\langle x \rangle = x - \lfloor x \rfloor$. Every $x \in \mathbb{R}^+$ can be written uniquely as $x = \mathcal{S}(x) \times 10^l$, for some appropriate (unique) integer l and $1 \leq \mathcal{S}(x) < 10$; if $x = 0$ then, for convenience, $\mathcal{S}(0) := 0$. The function $\mathcal{S} : \mathbb{R}^+ \to [1, 10)$ is referred to as the (base-10) *significand function*. The *first (decimal) significant digit* of x is the integer $D_1 = \lfloor \mathcal{S}(x) \rfloor \in \{1, \ldots, 9\}$. For example, let $x = \sqrt{200} = 10\sqrt{2}$. Then $\mathcal{S}(x) = \sqrt{2}$, $D_1(x) = 1$ and $\langle x \rangle = \sqrt{200} - 14$.

The occurrence of the first significant digit (also known as *leading digit*) of randomly generated data can easily be presumed to be approximately uniformly distributed on $1, \ldots, 9$. However, that is not always the case in the real world. Instead, nature seems to favour a digit distribution that is heavily skewed towards the smaller digits, according to a certain logarithmic distribution called *Benford's Law*. Thus, for a collection of data, numbers with leading digit one (i.e. with $D_1 = 1$) appear more often than numbers with leading digit two (i.e. with $D_1 = 2$), numbers with leading digit two appear more often than those with leading digit three and so on. Specifically, numbers with first significant digit 1 occur approximately 30 percent of the time and about seven times more often than numbers with 9 as first digit, the latter accounting for only about 4.6 percent. More specifically, Benford's Law asserts that digits are distributed according to the formula

$$\operatorname{Prob}(D_1 = k) = \log(k+1) - \log k = \log\left(1 + \frac{1}{k}\right), \quad \forall k = 1, 2, \ldots, 9, \tag{1.1}$$

which is a probability distribution, where k is the possible leading digit. Here and henceforth log denotes the logarithm base 10. The appropriate interpretation of the term "probability" used here, i.e. the precise meaning of Prob in (1.1) can take several forms. For sequences of real numbers (ζ_n), for instance, probability usually refers to the proportion of times n for which an event such as $D_1 = 2$ occurs. Thus $\text{Prob}(D_1 = 2)$ is the limiting proportion, as $N \to \infty$, of times $n \leq N$ that the first significant digit of ζ_n equals 2 (see also Definition 2.1). Similarly, for real-valued functions $f : [0, +\infty) \longrightarrow \mathbb{R}$, $\text{Prob}(D_1 = 2)$ refers to the limiting proportion, as $T \longrightarrow +\infty$, of the total length of time $\tau < T$ for which the first significant digit of $f(\tau)$ is 2 (see also Definition 2.2). Here an underlying but crucial assumption is that all limiting proportions exist. Usually, (1.1) is referred to as the *first digit law*. We state a concise and more general form of (1.1) in terms of the significand function as

$$\text{Prob}(\mathcal{S} \leq t) = \log t, \quad \forall t \in [1, 10). \tag{1.2}$$

Although astronomer/mathematician Simon Newcomb first documented the phenomenon in 1881 by examining tables of logarithms and realizing that earlier pages were much more worn and grubbier than later pages [17], the phenomenon is named after physicist Frank Benford whose work in 1938 contained considerable empirical evidence of the distribution in different tables of data [3]. Following Benford's article, many real-life data including physical constant [8], stock market indices [14], widths of hadrons [22], stochastic processes [5], and deterministic sequences like $(n!)$ and the Fibonacci numbers (F_n) [9] have been established to follow Benford's Law in some sense.

It is worth noting that some data such as lottery, telephone numbers and prime numbers (see Tables 1.2 and 1.3) do not obey Benford's Law, and an objective of research is to establish conditions under which data conforms to Benford's Law. In practice, the law is applicable in detecting fraud in tax, accounting and election data [2, 10, 18] as well as in speeding up calculation and optimizing computer data storage [1, 20]. Another useful application of this law is in the area of testing mathematical models. This is based on the observation that if current data is a good fit to Benford's Law, then the predicted data from any new model should also obey the law. Benford's Law is the only scale-invariant digit law. The law does not distinguish between numbers 400 and 40000 since both have first significant digit 4. Thus, Benford's Law does not depend on any particular choice of unit. It is also known that Benford's Law is base-invariant. In the framework of probability theory, it is proven that scale-invariance implies base-invariance, and base-invariance implies Benford's Law [11, 12].

1.2 Scope and Goals of this Thesis

Dynamical systems are objects in mathematics for modeling or describing physical phenomena with time changing states. The state $x \in \mathbb{R}^d$ of a system can continuously depend on time (that is, $x = x(t)$ with $t \in \mathbb{R}$) or time may take only integer values (that is, $x = x_n$

1	20365011074	573147844013817084101	16130531424904581415797907386349
1	32951280099	927372692193078999176	26099748102093884802012313146549
2	53316291173	1500520536206896083277	42230279526998466217810220532898
3	86267571272	2427893228399975082453	68330027629092351019822533679447
5	139583862445	3928413764606871165730	110560307156090817237632754212345
8	225851433717	6356306993006846248183	178890334785183168257455287891792
13	365435296162	10284720757613717413913	289450641941273985495088042104137
21	591286729879	16641027750620563662096	468340976726457153752543329995929
34	956722026041	26925748508234281076009	757791618667731139247631372100066
55	1548008755920	43566776258854844738105	1226132595394188293000174702095995
89	2504730781961	70492524767089125814114	1983924214061919432247806074196061
144	4052739537881	114059301025943970552219	3210056809456107725247980776292056
233	6557470319842	184551825793033096366333	5193981023518027157495786850488117
377	10610209857723	298611126818977066918552	8404037832974134882743767626780173
610	17167680177565	483162952612010163284885	13598018856492162040239554477268290
987	27777890035288	781774079430987230203437	220002056689466296922983322104048463
1597	44945570212853	1264937032042997393488322	35600075545958458963222876581316753
2584	72723460248141	2046711111473984623691759	57602132235424755886206198685365216
4181	117669030460994	3311648143516982017180081	93202207781383214849429075266681969
6765	190392490709135	5358359254990966640871840	150804340016807970735635273952047185
10946	308061521170129	8670007398507948658051921	244006547798191185585064349218729154
17711	498454011879264	14028366653498915298923761	394810887814999156320699623170776339
28657	806515533049393	22698374052006863956975682	638817435613190341905763972389505493
46368	1304969544928657	36726740705505779255899443	1033628323428189498226463595560281832
75025	2111485077978050	59425114757512643212875125	1672445759041379840132227567949787325
121393	3416454622906707	96151855463018422468774568	2706074082469569338358691163510069157
196418	5527939700884757	155576970220531065681649693	4378519841510949178490918731459856482
317811	8944394323791464	251728825683549488150424261	7084593923980518516849609894969925639
514229	14472334024676221	407305795904080553832073954	11463113765491467695340528626429782121
832040	23416728348467685	659034621587630041982498215	18547707689471986212190138521399707760
1346269	37889062373143906	1066340417491710595814572169	30010821454963453907530667147829489881
2178309	61305790721611591	1725375039079340637797070384	48558529144435440119720805669229197641
3524578	99194853094755497	2791715456571051233611642553	78569350599398894027251472817058687522
5702887	160500643816367088	4517090495650391871408712937	127127879743834334146972278486287885163
9227465	259695496911122585	7308805952221443105020355490	205697230343233228174223751303346572685
14930352	420196140727489673	11825896447871834976429068427	332825110087067562321196029789634457848
24157817	679891637638612258	19134702400093278081449423917	538522340430300790495419781092981030533
39088169	1100087778366101931	30960598847965113057878492344	871347450517368352816615810882615488381
63245986	1779979416004714189	50095301248058391139327916261	1409869790947669143312035591975596518914
102334155	2880067194370816120	81055900096023504197206408605	2281217241465037496128651402858212007295
165580141	4660046610375530309	131151201344081895336534324866	3691087032412706639440686994833808526209
267914296	7540113804746346429	212207101440105399533740733471	5972304273877744135569338397692020533504
433494437	12200160415121876738	343358302784187294870275058337	9663391306290450775010025392525829059713
701408733	19740274219868223167	555565404224292694404015791808	15635695580168194910579363790217849593217
1134903170	31940434634990099905	898923707008479989274290850145	25299086886458645685589389182743678652930
1836311903	51680708854858323072	1454489111232772683678306641953	40934782466626840596168752972961528246147
2971215073	83621143489848422977	2353412818241252672952597492098	66233869353085486281758142155705206899077
4807526976	135301852344706746049	3807901929474025356630904134051	107168651819712326877926895128666735145224
7778742049	218922995834555169026	6161314747715278029583501626149	173402521172797813159685037284371942044301
12586269025	354224848179261915075	9969216677189303386214405760200	280571172992510140037611932413038677189525

Table 1.1: List of the first 200 Fibonacci numbers, $F_1, F_2, \ldots, F_{200}$. From this list, it is established that F_n starts with a 1 much more often than with a 9: $\#\{1 \le n \le 200 : D_1(F_n) = 1\} = 60$ and $\#\{1 \le n \le 200 : D_1(F_n) = 9\} = 9$.

2	73	179	283	419	547	661	811	947	1087
3	79	181	293	421	557	673	821	953	1091
5	83	191	307	431	563	677	823	967	1093
7	89	193	311	433	569	683	827	971	1097
11	97	197	313	439	571	691	829	977	1103
13	101	199	317	443	577	701	839	983	1109
17	103	211	331	449	587	709	853	991	1117
19	107	223	337	457	593	719	857	997	1123
23	109	227	347	461	599	727	859	1009	1129
29	113	229	349	463	601	733	863	1013	1151
31	127	233	353	467	607	739	877	1019	1153
37	131	239	359	479	613	743	881	1021	1163
41	137	241	367	487	617	751	883	1031	1171
43	139	251	373	491	619	757	887	1033	1181
47	149	257	379	499	631	761	907	1039	1187
53	151	263	383	503	641	769	911	1049	1193
59	157	269	389	509	643	773	919	1051	1201
61	163	271	397	521	647	787	929	1061	1213
67	167	277	401	523	653	797	937	1063	1217
71	173	281	409	541	659	809	941	1069	1223

Table 1.2: List of the first 200 prime numbers, $P_1, P_2, \ldots, P_{200}$, which clearly do not conform to the first digit law: $\#\{1 \le n \le 200 : D_1(P_n) = 1\} = 57$ and $\#\{1 \le n \le 200 : D_1(P_n) = 9\} = 15$.

digit (d)	Fibonacci Numbers	Prime Numbers	$\log\left(1+\frac{1}{k}\right)$
1	60 (30.0%)	57 (28.5%)	30.1%
2	36 (18.0%)	19 (9.5%)	17.6%
3	25 (12.5%)	19 (9.5%)	12.5%
4	18 (9.0%)	20 (10.0%)	9.7%
5	17 (8.5%)	17 (8.5%)	7.9%
6	12 (6.0%)	18 (9.0%)	6.7%
7	11 (5.5%)	18 (9.0%)	5.8%
8	12 (6.0%)	17 (8.5%)	5.1%
9	9 (4.5%)	15 (7.5%)	4.6%

Table 1.3: The first 200 Fibonacci numbers conform well to the first digit law, while the first 200 prime numbers do not.

with $n \in \mathbb{Z}$ or $n \in \mathbb{N}$). These give rise to *continuous-time* and *discrete-time* dynamical systems, respectively. In this work, (autonomous) *linear dynamical systems* will be studied both in continuous and discrete time. Suppose $x \in \mathbb{R}^d$ and $A \in \mathbb{R}^{d\times d}$, with $d \in \mathbb{N}$ being the dimension of the system. In this context, a linear continuous-time dynamical system is of the form

$$\dot{x}(t) = Ax(t), \qquad t \in \mathbb{R},$$

and a linear discrete-time system takes the form

$$x_n = Ax_{n-1}, \qquad n \in \mathbb{N}.$$

It is justifiable to consider linear (autonomous) systems in this study for the basic reason that we can always write down their explicit solutions. Also many non-linear systems can be approximated by linear systems or studied via a linearisation process.

Since not all data obey Benford's Law, the focus of most research is to establish conditions under which given data will be a good fit to Benford's distribution (1.1), in one way or the other. Hence to study Benford's phenomenon in the area of dynamical systems, it stands to reason that not all data generated from the solution of, say, a recurrence relation (difference equation or discrete-time system) would exhibit Benford behaviour. However, empirical evidence is established in [7] that data from numerical simulation related to dynamical systems are Benford by examining the frequency of the first digit coordinates of the generated trajectories. For instance, the Lorenz system generates trajectories that follow Benford's Law to some extent.

For linear systems of arbitrary dimension, so far no condition is known that is both necessary and sufficient for Benford behaviour. The main goal of this research is to provide such a condition. The study builds on earlier work, notably [4, 6, 16, 21]. The main result of this work can be motivated by looking at the difference equation (general Fibonacci recursion)

$$\zeta_n = \zeta_{n-1} + \zeta_{n-2}, \qquad n \geq 3, \tag{1.3}$$

which we rewrite as $\begin{bmatrix} \zeta_n \\ \zeta_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \zeta_{n-1} \\ \zeta_{n-2} \end{bmatrix}$. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ be the associated matrix with spectrum $\sigma(A) = \left\{ \frac{1}{2}\left(1 \pm \sqrt{5}\right) \right\}$. Every solution of (1.3) is of the form

$$\zeta_n = c_1 \left(\frac{1+\sqrt{5}}{2} \right)^n + c_2 \left(\frac{1-\sqrt{5}}{2} \right)^n, \tag{1.4}$$

where c_1, c_2 are arbitrary real constants determined by the initial values ζ_1, ζ_2. Take for instance $\zeta_1 = \zeta_2 = 1$.Then $(\zeta_n) = (F_n)$ and thus Tables 1.1 and 1.3 suggest that

$$\lim_{N\to\infty} \frac{\#\{1 \leq n \leq N : D_1(F_n) = k\}}{N} = \log\left(1 + \frac{1}{k}\right), \quad \forall k = 1, 2, \ldots, 9, \tag{1.5}$$

i.e. (F_n) conforms to (1.1) in some sense. Here and henceforth, the symbol $\#$ denotes the number of elements in a finite set. Note that, for all $n \in \mathbb{N}$, $\zeta_n = y^\top A^n x$ with $x = \begin{bmatrix} \zeta_2 - \zeta_1 \\ 2\zeta_1 - \zeta_2 \end{bmatrix}$, $y = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. How do we decide whether (ζ_n), or more generally $(y^\top A^n x)$ satisfies (1.5)? In fact, the main result of this work (Theorem 3.20) implies that, for all $x, y \in \mathbb{R}^2$ either $y^\top A^n x \equiv 0$ or

$$\lim_{N\to\infty} \frac{\#\{1 \leq n \leq N : \mathcal{S}(y^\top A^n x) \leq t\}}{N} = \log t, \quad \forall t \in [1, 10). \tag{1.6}$$

Note that (1.5) follows from (1.6). It turns out that in order to establish (1.6), the spectrum $\sigma(A)$ of A has to be analyzed carefully. The key property is that $\sigma(A)$ is *nonresonant* (see Definition 3.11). Specifically for $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ from (1.3), this means that

$$\log\left|\frac{1}{2}\left(1 \pm \sqrt{5}\right)\right| = \pm \log\left|\frac{1}{2}\left(1 + \sqrt{5}\right)\right| \notin \mathbb{Q}.$$

In general, Theorem 3.20 asserts that a solution to a difference equation is either terminating or displays Benford's phenomenon in the sense of (1.6) if and only if the spectrum of the associated matrix is nonresonant. As illustrated by means of numerous examples, it is often quite simple to identify the matrix associated with a linear difference equation as having nonresonant spectrum. Thus, the main result of this work paves the way to making conclusions regarding conformance to Benford's Law using hardly any calculations at all.

The theory for the continuous-time case (differential equation) is also considered in this research. Consider for instance the differential equation

$$\ddot{\zeta}(t) = \zeta(t), \qquad t \in \mathbb{R}, \tag{1.7}$$

which has the general solution

$$\zeta(t) = c_1 e^t + c_2 e^{-t},$$

where c_1, c_2 are any real constants. Choose for instance $c_1 = 1$ and $c_2 = 0$, then the solution becomes $\zeta(t) = e^t$. It can be shown that

$$\lim_{T\to+\infty} \frac{\text{length}\ \{0 \le \tau < T : \mathcal{S}(e^\tau) \le t\}}{T} = \log t, \quad \forall t \in [1, 10), \tag{1.8}$$

and hence ζ conforms to the continuous time analogue of (1.6). To see this, realize $\mathcal{S}(e^\tau) = 10^{\langle \tau \log e\rangle} < t$ whenever $\langle \tau \log e \rangle < \log t$. This means

$$\tau \in \bigcup_{n=0}^{\infty} \left[\frac{n}{\log e}, \frac{n + \log t}{\log e}\right] \cap [0, T),$$

and with $\dfrac{n}{\log e} \le T < \dfrac{n+1}{\log e}$ then

$$0 \le \frac{\text{length}\ \{0 \le \tau < T : \mathcal{S}(e^\tau) \le t\}}{T} - \frac{n}{T}\frac{\log t}{\log e} \le \frac{\log t}{T \log e},$$

and hence

$$\lim_{T\to+\infty} \frac{\text{length}\ \{0 \le \tau < T : \mathcal{S}(e^\tau) \le t\}}{T} = \lim_{T\to+\infty} \frac{\lfloor T \log e \rfloor}{T} \cdot \frac{\log t}{\log e} = \log t.$$

We will see later that Theorem 3.35 gives a more direct answer to why the solutions of (1.7) satisfy a form of (1.8).

As the analysis of the continuous-time case depends heavily on the discrete-time case, more attention is given to the latter. The Benford analysis used in this study proceeds via the classical theory of *uniform distribution modulo one* due to the strong correspondence between (1.6) and that theory (see Chapter 2).

Chapter 2

Basic Definitions and Tools

This chapter gives the formal definitions for most of the terms used in this thesis. Basic facts from the theory of uniform distribution which are useful for this work will be reviewed. Some facts pertaining to Benford sequences and Benford functions relevant to this work will be discussed. Throughout, let (ζ_n) and (ξ_n) be sequences of real numbers. The symbols $\#$ and λ_d will denote the number of elements in a finite set (cardinality) and d-dimensional Lebesgue measure, respectively; for simplicity write $\lambda := \lambda_1$. Remember that log means the logarithm base 10, and ln is the natural logarithm (base e). For convenience, set $\log 0 := \ln 0 := 0$.

Benford Theory and Uniform Distribution

We state the formal definition of Benford sequence and Benford function respectively as follows:

Definition 2.1. A sequence (ζ_n) of real numbers is *Benford* if

$$\lim_{N\to\infty} \frac{\#\{n \le N : \mathcal{S}(|\zeta_n|) \le t\}}{N} = \log t, \quad \forall t \in [1, 10).$$

Definition 2.2. A (measurable) function $f : [0, +\infty) \longrightarrow \mathbb{R}$ is *Benford* if

$$\lim_{T\to+\infty} \frac{\lambda\left(\{\tau \in [0, T) : \mathcal{S}(f(\tau)) \le t\}\right)}{T} = \log t, \quad \forall t \in [1, 10).$$

The tool employed in the analysis in this work is uniform distribution modulo one. Basically, the theory of uniform distribution modulo one is concerned with the distribution of fractional parts of real numbers in the unit interval $[0, 1)$. Thus for a sequence (ζ_n) in $\mathbb{R}$ and a given interval $I \subset [0, 1)$, we look at the proportion of the elements of the sequence $(\langle\zeta_n\rangle)$ that lie in I, and compare it to the length of I. If, for every interval I, this proportion converges to the length of the interval I as $N \longrightarrow \infty$ then the sequence is said to be *uniformly distributed modulo one*, henceforth abbreviated as *u.d.* mod 1.

Definition 2.3. A sequence (ζ_n) is *uniformly distributed modulo one* (u.d. mod 1) if

$$\lim_{N\to\infty} \frac{\#\{1 \le n \le N : \langle \zeta_n \rangle \le s\}}{N} = s, \quad \forall s \in [0,1).$$

Definition 2.4. A (measurable) function $f : [0,+\infty) \longrightarrow \mathbb{R}$ is *continuously uniformly distributed modulo one* (c.u.d. mod 1) if

$$\lim_{T\to+\infty} \frac{\lambda\left(\{\tau \in [0,T) : \langle f(\tau)\rangle \le s\}\right)}{T} = s, \quad \forall s \in [0,1).$$

The next proposition states the strong correspondence between the Benford property and the uniform distribution modulo one of sequence. This is the key tool used in the analysis in this work.

Proposition 2.5. [9, Thm.1] For every sequence (ζ_n), the following statements are equivalent:

(i) (ζ_n) is Benford;

(ii) $(\log|\zeta_n|)$ is u.d. mod 1.

Remark 2.6. By replacing the decimal significand $\mathcal{S}(x)$ with base-b significand $\mathcal{S}_b(x)$, that is, $\mathcal{S}_b(x) : \mathbb{R}^+ \to [1,b)$ in Definitions 2.1 and 2.2, the Benford property can be studied w.r.t. any integer base $b \ge 2$. In this case, we instead take all logarithms w.r.t. base b. However, for simplicity, only the most common base $b = 10$ is considered from now on.

In view of Proposition 2.5, we discuss some basic results about u.d. mod 1 (see [13] for a comprehensive account on uniform distribution of sequences).

Lemma 2.7. The following statements are equivalent for any sequence (ζ_n) in $\mathbb{R}$:

(i) (ζ_n) is u.d. mod 1;

(ii) For every $\epsilon > 0$ there exists a uniformly distributed sequence (ξ_n) with

$$\limsup_{N\to\infty} \frac{\#\{1 \le n \le N : |\zeta_n - \xi_n| > \epsilon\}}{N} < \epsilon;$$

(iii) Whenever (ξ_n) converges in $\mathbb{R}$ then $(\zeta_n + \xi_n)$ is u.d. mod 1;

(iv) The sequence $(k\zeta_n)$ is u.d. mod 1 for every $k \in \mathbb{Z} \setminus \{0\}$;

(v) The sequence $(\zeta_n + \alpha \log n)$ is u.d. mod 1 for every $\alpha \in \mathbb{R}$.

Proof. Clearly, (i) implies (ii), and the converse is [4, Lem.2.3]. Also (iii) implies (i) and the reverse implication is [13, Thm.I.1.2]. To show that (iv) and (i) are equivalent, assume (iv) is true and choose $k = 1$. Then clearly $(k\zeta_n) = (\zeta_n)$ is u.d. mod 1. Conversely, assume (ζ_n) is u.d. mod 1. Then for any $0 < s < 1$,

$$\{\zeta_n : \langle k\zeta_n \rangle \leq s\} = \begin{cases} \left\{ \zeta_n : \langle \zeta_n \rangle \in \bigcup_{i=0}^{k-1} \left[\frac{i}{k}, \frac{i+s}{k} \right] \right\} & \text{for} \quad k > 0, \\ \left\{ \zeta_n : \langle \zeta_n \rangle \in \bigcup_{i=0}^{|k|-1} \left[\frac{i+1-s}{|k|}, \frac{i+1}{|k|} \right] \right\} & \text{for} \quad k < 0, \end{cases}$$

and hence

$$\begin{aligned} \lim_{N\to\infty} \frac{\#\{1 \leq n \leq N : \langle k\zeta_n \rangle \leq s\}}{N} &= \begin{cases} \lambda \left(\bigcup_{i=0}^{k-1} \left[\frac{i}{k}, \frac{i+s}{k} \right] \right) & \text{for} \quad k > 0, \\ \lambda \left(\bigcup_{i=0}^{|k|-1} \left[\frac{i+1-s}{|k|}, \frac{i+1}{|k|} \right] \right) & \text{for} \quad k < 0, \end{cases} \\ &= \begin{cases} k \cdot \frac{s}{k} & \text{for} \quad k > 0, \\ |k| \cdot \frac{s}{|k|} & \text{for} \quad k < 0, \end{cases} \\ &= s. \end{aligned}$$

Hence $(k\zeta_n)$ is u.d. mod 1. To show that (i) and (v) are equivalent, note that clearly (v) implies (i), simply choose $\alpha = 0$. Now assume (ζ_n) is u.d. mod 1. Let $f(x) = \alpha \log x$. Then, $f \in C^1(\mathbb{R}^+)$ and $\lim_{x\to+\infty} xf'(x) = \frac{\alpha}{\ln 10}$. It follows from [19, Lem.6] that $(\zeta_n + \alpha \log n)$ is u.d. mod 1. □

Proposition 2.8. [6, Prop.4.8(ii)] Let (ζ_n) be a sequence of real numbers. If (ζ_n) is periodic, thus $\zeta_{n+k} = \zeta_n$ for some $k \in \mathbb{N}$ and all n, then $(n\vartheta + \zeta_n)$ is u.d. mod 1 if and only if ϑ is irrational.

A direct application of the above proposition to Benford's Law is stated in the following lemma.

Lemma 2.9. If a, b, α, β are real numbers with $a \neq 0$ and $|\alpha| > |\beta|$ then the following two statements are equivalent:

(i) The sequence $(\alpha^n a + \beta^n b)$ is Benford;

(ii) $\log |\alpha|$ is irrational.

Proof. Since $a \neq 0$, $\log|\alpha^n a + \beta^n b| = n\log|\alpha| + \log\left|a + b\left(\frac{\beta}{\alpha}\right)^n\right|$ for all sufficiently large n. Using the fact that $|\alpha| > |\beta|$, $\log\left|a + b\left(\frac{\beta}{\alpha}\right)^n\right| \xrightarrow{n\to\infty} \log|a|$. Thus $(\log|\alpha^n a + \beta^n b|)$ is u.d. mod 1 if and only if $(n\log|\alpha| + \log|a|)$ is. By letting $\zeta_n = \log|a|$, it follows from Propositions 2.5 and 2.8 that $(n\log|\alpha| + \log|a|)$ is u.d. mod 1 if and only if $\log|\alpha|$ is irrational. □

Lemma 2.10. Let (ζ_n) be a sequence in $\mathbb{R}$, and $L \in \mathbb{N}$. If (ζ_{nL+k}) is u.d. mod 1 for every $k \in \{1,\ldots,L\}$ then (ζ_n) is u.d. mod 1 as well.

Proof. Using Weyl's criterion [13, Thm.I.2.1], for every $p \in \mathbb{Z}\setminus\{0\}$,

$$\begin{aligned}\left|\frac{1}{N}\sum_{n=1}^{N} e^{2\pi\imath p\zeta_n}\right| &\le \left|\frac{1}{N}\sum_{n=1}^{L\lfloor N/L\rfloor} e^{2\pi\imath p\zeta_n}\right| + \left|\frac{1}{N}\sum_{n=L\lfloor N/L\rfloor+1}^{N} e^{2\pi\imath p\zeta_n}\right| \\ &\le \left|\frac{1}{N}\sum_{k=1}^{L}\sum_{n=0}^{\lfloor N/L\rfloor-1} e^{2\pi\imath p\zeta_{nL+k}}\right| + \frac{L}{N} \\ &\le \frac{1}{L}\sum_{k=1}^{L}\left|\frac{1}{\lfloor N/L\rfloor}\sum_{n=0}^{\lfloor N/L\rfloor-1} e^{2\pi\imath p\zeta_{nL+k}}\right| + \frac{L}{N} \xrightarrow{N\to\infty} 0,\end{aligned}$$

because $\lim\limits_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1} e^{2\pi\imath p\zeta_{nL+k}} = 0$ for every $k = 1, 2, \ldots, L$. □

Combining the well-known fact that $(n\vartheta)$ is u.d. mod 1 precisely if $\vartheta \in \mathbb{R}$ is irrational, which follows from Proposition 2.8, with Lemmas 2.7 and 2.10, we arrive at the following corollary.

Corollary 2.11. Let α, $\vartheta \in \mathbb{R}$, $L \in \mathbb{N}$, and assume the sequence (ξ_n) in $\mathbb{R}$ has the property that (ξ_{nL+k}) converges for every $k \in \{1,\ldots,L\}$. Then $(n\vartheta + \alpha\log n + \xi_n)$ is u.d. mod 1 if and only if ϑ is irrational.

Denote by λ_d the (normalized) Lebesgue measure on $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, the d-dimensional torus. Now fix $d \in \mathbb{N}$, and let the real numbers $1, \vartheta_0, \vartheta_1, \ldots, \vartheta_d$ be $\mathbb{Q}$-independent (or rationally independent). Recall that $1, \vartheta_0, \ldots, \vartheta_d$ are *$\mathbb{Q}$-independent* if $\rho_{-1} + \rho_0\vartheta_0 + \cdots + \rho_d\vartheta_d = 0$ with $\rho_j \in \mathbb{Q}$ implies $\rho_j = 0$ for all $j \in \{-1, 0, 1, \ldots, d\}$; otherwise, $1, \vartheta_0, \ldots, \vartheta_d$ are said to be *$\mathbb{Q}$-dependent* (or rationally dependent).

Proposition 2.12. [4, Cor.2.6] Let $f : \mathbb{T}^d \to \mathbb{R}$ be continuous on a set of full λ_d-measure. Then the sequence

$$\Big(n\vartheta_0 + f\left(\langle n\vartheta_1\rangle, \ldots, \langle n\vartheta_d\rangle\right)\Big)$$

is u.d. mod 1.

Lemma 2.13. Assume $f : \mathbb{T}^d \to \mathbb{C}$ is continuous, and non-zero (Lebesgue) almost everywhere. Then the sequence (ζ_n) with

$$\zeta_n := n\vartheta_0 + \alpha \log n + \beta \log |f(\langle n\vartheta_1\rangle, \dots, \langle n\vartheta_d\rangle) + z_n|$$

is u.d. mod 1 for every $\alpha, \beta \in \mathbb{R}$ and every sequence (z_n) in $\mathbb{C}$ with $\lim_{n\to\infty} z_n = 0$.

Proof. The function $g : \mathbb{T}^d \to \mathbb{R}$ defined as

$$g(x) := \beta \log |f(x)|, \ \ \forall x \in \mathbb{T}^d,$$

is continuous on a set of full λ_d-measure, and so Proposition 2.12 together with Lemma 2.7(v) shows that the sequence (ξ_n) with

$$\xi_n := n\vartheta_0 + \alpha \log n + \beta \log |f(\langle n\vartheta_1\rangle, \dots, \langle n\vartheta_d\rangle)|, \quad \forall n \in \mathbb{N},$$

is u.d. mod 1 for every $\alpha, \beta \in \mathbb{R}$. Given $0 < \epsilon \le 1$, choose $0 < \delta < \frac{\epsilon}{2+|\beta|}$ so small that $\lambda_d(B_\delta) < \epsilon$, where $B_\delta := \{x : |f(x)| \le \delta\}$. There exists $\widetilde{B_\delta} \supset B_\delta$ such that $\widetilde{B_\delta}$ is a finite union of open balls, and $\lambda_d(\widetilde{B_\delta}) < \epsilon$. Observe that if $(\langle n\vartheta_1\rangle, \dots, \langle n\vartheta_d\rangle) \notin \widetilde{B_\delta}$ and $|z_n| < \delta^2$ then

$$|\zeta_n - \xi_n| = |\beta| \left| \log \left| 1 + \frac{z_n}{f(\langle n\vartheta_1\rangle, \dots, \langle n\vartheta_d\rangle)} \right| \right| \le |\beta|\delta < \epsilon.$$

By the $\mathbb{Q}$-independence of $1, \vartheta_1, \dots, \vartheta_d$, the sequence $(\langle n\vartheta_1\rangle, \dots, \langle n\vartheta_d\rangle)$ is uniformly distributed on $\mathbb{T}^d$, and so

$$\lim_{N\to\infty} \frac{\#\{1 \le n \le N : (\langle n\vartheta_1\rangle, \dots, \langle n\vartheta_d\rangle) \in \widetilde{B_\delta}\}}{N} = \lambda_d(\widetilde{B_\delta}) < \epsilon.$$

With this and $\lim_{n\to\infty} z_n = 0$, it follows that

$$\begin{aligned}
&\limsup_{N\to\infty} \frac{\#\{1 \le n \le N : |\zeta_n - \xi_n| > \epsilon\}}{N} \\
&\le \limsup_{N\to\infty} \frac{\#\{1 \le n \le N : (\langle n\vartheta_1\rangle, \dots, \langle n\vartheta_d\rangle) \in \widetilde{B_\delta} \text{ or } |z_n| \ge \delta^2\}}{N} \\
&\le \limsup_{N\to\infty} \frac{\#\{1 \le n \le N : (\langle n\vartheta_1\rangle, \dots, \langle n\vartheta_d\rangle) \in \widetilde{B_\delta}\}}{N} \ + \\
&\qquad\qquad\qquad\qquad \limsup_{N\to\infty} \frac{\#\{1 \le n \le N : |z_n| \ge \delta^2\}}{N} \\
&= \lambda_d(\widetilde{B_\delta}) + 0 < \epsilon,
\end{aligned}$$

and an application of Lemma 2.7(ii) completes the proof. ∎

Lemma 2.14. Given any integers $p_1, \ldots, p_d$, and $\alpha \in \mathbb{R} \setminus \{0\}$, there exists $\xi \in \mathbb{R}^d$ such that the sequence

$$\Big(p_1 n\vartheta_1 + \cdots + p_d n\vartheta_d + \alpha \ln |\xi_1 \cos(2\pi n\vartheta_1) + \cdots + \xi_d \cos(2\pi n\vartheta_d)|\Big)$$

is not u.d. mod 1 whenever $1, \vartheta_1, \ldots, \vartheta_d$ are $\mathbb{Q}$-independent.

Proof. See Appendix A. $\square$

Chapter 3

Benford Theory for Linear Systems

3.1 Introduction

Dynamical systems are objects in mathematics for modeling or describing physical phenomena with time-changing states. When the state $x \in \mathbb{R}^d$ of a system is continuously dependent on time (that is, $x = x(t)$ with $t \in \mathbb{R}$), we categorize it as a *continuous-time dynamical system.* In other cases, time may take only integer values (that is, $x = x_n$ with $n \in \mathbb{Z}$ or $n \in \mathbb{N}$), such is called a *discrete-time dynamical system.*

Throughout this section, let $A \in \mathbb{R}^{d\times d}$ and $x, y \in \mathbb{R}^d$ be a constant real $d \times d$-matrix and d-dimensional vectors, respectively, with $d \in \mathbb{N}$. A linear continuous-time dynamical system is of the form

$$\dot{x}(t) = Ax(t), \qquad t \in \mathbb{R}, \tag{3.1}$$

and a linear discrete-time system takes the form

$$x_n = Ax_{n-1}, \qquad n \in \mathbb{N}. \tag{3.2}$$

The solutions to the above equations are $x(t) = e^{tA}x_0$ and $x_n = A^n x_0$, respectively, where $x_0 \in \mathbb{R}^d$ is a given initial value, with e^{0A} and A^0 understood as I_d, the $d \times d$-identity matrix.

Example 3.1. Some basic examples of linear dynamical system are

$$\ddot{\zeta}(t) = \zeta(t), \qquad t \in \mathbb{R}, \tag{3.3}$$

which can be put in the form (3.1) as $\frac{\mathrm{d}}{\mathrm{dt}}\begin{bmatrix} \dot{\zeta} \\ \zeta \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \dot{\zeta} \\ \zeta \end{bmatrix}$ and

$$\zeta_n = \zeta_{n-1} + \zeta_{n-2}, \qquad n \geq 3, \tag{3.4}$$

which can also be written in the form (3.2) as $\begin{bmatrix} \zeta_n \\ \zeta_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \zeta_{n-1} \\ \zeta_{n-2} \end{bmatrix}$.

As already encountered in Section 1.2, evidence suggests that solutions to (3.3) and (3.4) are Benford functions and sequences, respectively.

Henceforth let $\sigma(A) \subset \mathbb{C}$ denote the spectrum of the matrix A, i.e. $\sigma(A)$ contains at most d numbers, namely the eigenvalues of A. Recall that the characteristic polynomial of every matrix A is $P_A(\mu) = \det(A - \mu I_d)$. For any complex number z, denote its complex conjugate, real part, imaginary part and modulus (or absolute value) by $\overline{z}$, $\Re z$, $\Im z$ and $|z|$, respectively. For $z \neq 0$, define $\arg z$ as the unique number in $(-\pi, \pi]$ that satisfies $z = |z|e^{\imath \arg z}$. Let $\mathbb{S} := \{z \in \mathbb{C} : |z| = 1\}$. For any set $Z \subset \mathbb{C}$, denote by $\text{span}_{\mathbb{Q}} Z$ the smallest subspace of $\mathbb{C}$ (over $\mathbb{Q}$) containing Z; equivalently, if $Z \neq \varnothing$ then $\text{span}_{\mathbb{Q}} Z$ is the set of all *finite* rational linear combinations of elements of Z, i.e.

$$\text{span}_{\mathbb{Q}} Z = \{\rho_1 z_1 + \rho_2 z_2 + \cdots + \rho_n z_n : n \in \mathbb{N}, \rho_1, \rho_2, \ldots, \rho_n \in \mathbb{Q}, z_1, z_2, \ldots, z_n \in Z\}\,;$$

note that $\text{span}_{\mathbb{Q}} \varnothing = \{0\}$. Recall that $z_1, \ldots, z_n$ are $\mathbb{Q}$-independent precisely if the dimension of $\text{span}_{\mathbb{Q}} Z$ is n.

The goal of this chapter is to characterize Benford's Law in linear dynamical systems. More precisely, the central question is whether there is a necessary and sufficient condition for Benford behaviour in solutions of dynamical systems that is independent on the choice of initial conditions. To investigate this, much attention will be devoted to answer this question for discrete-time dynamical system since the continuous-time case will employ concepts from the discrete-time case.

3.2 Discrete-Time Dynamical System

Consider a general linear difference equation

$$\zeta_n = a_1\zeta_{n-1} + a_2\zeta_{n-2} + \cdots + a_{d-1}\zeta_{n-d+1} + a_d\zeta_{n-d}, \qquad n \geq d+1, \tag{3.5}$$

where $d \in \mathbb{N}$ is the *order* of (3.5), and $a_1, a_2, \ldots, a_d \in \mathbb{R}$ are given numbers with $a_d \neq 0$. For instance, (3.4) is a second-order equation, i.e. $d = 2$, with $a_1 = a_2 = 1$. Once the initial values $\zeta_1, \zeta_2, \ldots, \zeta_d \in \mathbb{R}$ are specified, (3.5) defines a unique sequence (ζ_n), referred to as a *solution* of (3.5). The central question in this work is: Under which conditions on $a_1, a_2, \ldots, a_d$, and presumably also on $\zeta_1, \zeta_2, \ldots, \zeta_d$ is (ζ_n) Benford? Instead of studying directly the Benford property of sequences (ζ_n) generated by (3.5), the analysis used in this work uses a matrix approach which is more general and transparent. We can re-write (3.5) using matrix-vector notation by inserting $d-1$ redundant rows and thus, for every $n \ge d+1$,

$$\begin{bmatrix} \zeta_n \\ \zeta_{n-1} \\ \vdots \\ \zeta_{n-d+1} \end{bmatrix} = \begin{bmatrix} a_1\zeta_{n-1} + \cdots + a_d\zeta_{n-d} \\ \zeta_{n-1} \\ \vdots \\ \zeta_{n-d+1} \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & \cdots & a_{d-1} & a_d \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \zeta_{n-1} \\ \zeta_{n-2} \\ \vdots \\ \zeta_{n-d} \end{bmatrix}.$$

Let $A \in \mathbb{R}^{d\times d}$ be the associated matrix, that is,

$$A = \begin{bmatrix} a_1 & a_2 & \cdots & a_{d-1} & a_d \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}, \tag{3.6}$$

and note that A is invertible since $a_d \neq 0$. Represent the standard basis of $\mathbb{R}^d$ by $e_1, e_2, \ldots, e_d$. Then, for every $n \ge 2$,

$$\zeta_n = e_1^\top A^{n-d} \begin{bmatrix} \zeta_d \\ \zeta_{d-1} \\ \vdots \\ \zeta_1 \end{bmatrix} = e_1^\top A^n (A^{-1})^d \begin{bmatrix} \zeta_d \\ \zeta_{d-1} \\ \vdots \\ \zeta_1 \end{bmatrix}.$$

Thus, $\zeta_n = y^\top A^n x$ with $y = e_1$ and $x = (A^{-1})^d \begin{bmatrix} \zeta_d & \zeta_{d-1} & \cdots & \zeta_1 \end{bmatrix}^\top$. Here and henceforth $y^\top$ denotes the transpose of $y \in \mathbb{R}^d$, and an expression $y^\top x$ is understood as the real number $\sum_{j=1}^{d} y_j x_j$. Note that conversely *every* sequence $(y^\top A^n x)$, with arbitrary $x, y \in \mathbb{R}^d$ and A given by (3.6), is a solution of (3.5). Hence we address the central question asked earlier by considering a more general sequence of the form $(y^\top A^n x)$. Our main question of interest therefore becomes this: Under which conditions is $(y^\top A^n x)$ Benford, where A is any fixed real $d \times d$-matrix and $x, y \in \mathbb{R}^d$ are given vectors?

To develop intuition about the Benford property for linear difference equations, we will first consider a couple of examples. Call a sequence (ζ_n) *terminating* if $\zeta_n = 0$ for all sufficiently large n. A sequence (ζ_n) is said to be *k-periodic* whenever $\zeta_{n+k} = \zeta_n$ for some $k \in \mathbb{N}$ and all $n \in \mathbb{N}$. Also recall that log denotes the logarithm base 10, and ln is the natural logarithm (base e).

Example 3.2. For $a \in \mathbb{R} \setminus \{0\}$, consider a simple first-order difference equation

$$\zeta_n = a\zeta_{n-1}, \qquad n \ge 2. \tag{3.7}$$

This is a scalar equation with a general solution

$$\zeta_n = a^{n-1}\zeta_1, \qquad n \in \mathbb{N}, \tag{3.8}$$

where $\zeta_1 \in \mathbb{R}$ is an arbitrary constant. In other words, the associated matrix according to (3.6) is 1×1; that is, $A = [a]$. Suppose $\zeta_1 = 0$, then the solution $\zeta_n = a^{n-1}\zeta_1 = 0$ and hence the sequence is terminating (in fact, zero). For $\zeta_1 \neq 0$, simply use Lemma 2.9: For the sequence (ζ_n) defined by (3.8) with $\zeta_1 \neq 0$ to be Benford, it is necessary and sufficient that $\log|a|$ be irrational. For instance the sequences (e^n) and (2^n) are Benford, while the sequence (10^n) is not.

It is natural to investigate what happens if we consider systems with dimension $d \geq 2$. Thus, is there any analogous necessary and sufficient condition under which solutions of higher dimensional system are Benford? It makes sense to require that if such a condition exists then it must generalize the condition $\log|a| \notin \mathbb{Q}$ for $d = 1$. To characterize the Benford property in sequences of the form $(y^\top A^n x)$ with $A \in \mathbb{R}^{d\times d}$ and $x, y \in \mathbb{R}^d$, we will first identify cases that cause problems in that they lead to sequences $(y^\top A^n x)$ that are neither Benford nor terminating.

Example 3.3. Consider the equation

$$\zeta_n = 12\zeta_{n-1} - 20\zeta_{n-2}, \qquad n \geq 3, \tag{3.9}$$

which has associated matrix $A = \begin{bmatrix} 12 & -20 \\ 1 & 0 \end{bmatrix}$ with $\sigma(A) = \{2, 10\}$. Any solution of (3.9) has the form

$$\zeta_n = c_1 10^n + c_2 2^n, \qquad n \in \mathbb{N}, \tag{3.10}$$

where c_1, c_2 are arbitrary real constants. Realize that if $c_1 = 0$, then the sequence $(c_2 2^n)$ is either Benford or zero (see Example 3.2). Now suppose $c_1 \neq 0$ then by Lemma 2.9, (ζ_n) is not Benford. An explanation for this problem is that there exists an eigenvalue μ of the matrix associated with (3.9) with $\log|\mu|$ rational; more specifically, for $\mu = 10$, $\log\mu = \log 10 = 1$ is rational.

Example 3.4. Another instance where some solutions (ζ_n) may be neither Benford nor terminating is when the matrix A has eigenvalues of the same modulus but opposite signs. Consider the linear two-step recursion

$$\zeta_n = 4\zeta_{n-2}, \qquad n \geq 3, \tag{3.11}$$

and using (3.6), we have $A = \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix}$. The spectrum of A is $\sigma(A) = \{-2, 2\}$. Every solution of (3.11) is given by

$$\zeta_n = c_1 2^n + c_2(-2)^n, \qquad n \in \mathbb{N}, \tag{3.12}$$

where $c_1, c_2 \in \mathbb{R}$ are arbitrary constants. Is the sequence (ζ_n) given by (3.12) Benford for every choice of real constants c_1, c_2? For example, consider the case where $c_1 = c_2 = c \neq 0$, then

$$\zeta_n = c2^n(1 + (-1)^n) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ c2^{n+1} & \text{if } n \text{ is even.} \end{cases}$$

Clearly, for our choice of constants, (ζ_n) is not Benford. This phenomenon of half of the time zero and other half non-zero generally occurs whenever $|c_1| = |c_2| \neq 0$. The oscillatory behaviour of (ζ_n) is due to the characteristic equation $\mu^2 - 4 = 0$ of A having two roots (i.e. $\mu_1 = -2, \mu_2 = 2$) of opposite signs but of the same modulus. Note, however, that (ζ_n) is Benford whenever $|c_1| \neq |c_2|$ because $\log|\zeta_n| = n \log 2 + \log|c_1 + c_2(-1)^n|$, and since the sequence $(\log|c_1 + c_2(-1)^n|)$ is 2-periodic, Proposition 2.8 applies.

Example 3.5. The problem of half of the time zero and other half non-zero can also occur when the spectrum contains non-real eigenvalues. To see this, consider the recurrence equation

$$\zeta_n = -\zeta_{n-2}, \qquad n \geq 3. \tag{3.13}$$

The associated matrix is $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, and $\sigma(A) = \{-\imath, \imath\}$. We write down the general solution to (3.13) as

$$\zeta_n = c_1 \cos\left(\frac{1}{2}\pi n\right) + c_2 \sin\left(\frac{1}{2}\pi n\right), \qquad n \in \mathbb{N}, \tag{3.14}$$

with $c_1, c_2 \in \mathbb{R}$. For simplicity choose $c_1 = 1, c_2 = 0$, then

$$\zeta_n = \cos\left(\frac{1}{2}\pi n\right) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ (-1)^{n/2} & \text{if } n \text{ is even.} \end{cases}$$

The issue of half of the time zero and other half non-zero makes the sequence (ζ_n) not Benford at least for our choice of constants. Generally, the sequence (ζ_n) defined by (3.14) is 4-periodic (oscillatory), that is, $\zeta_n = \zeta_{n+4}$ and thus not Benford for any choice of c_1, c_2. This oscillatory behaviour of (ζ_n) corresponds to the fact that the roots of the characteristic equation $\mu^2 + 1 = 0$ associated with (3.13) are $\mu = \pm\imath$ and hence lie on the unit circle and their arguments are rational multiples of π.

Example 3.6. Now consider the sequence

$$\zeta_n = (\sqrt{2})^n \left[c_1 \cos\left(\frac{1}{4}\pi n\right) + c_2 \sin\left(\frac{1}{4}\pi n\right)\right], \qquad n \in \mathbb{N},$$

which is the general solution of the recurrence relation

$$\zeta_n = 2\zeta_{n-1} - 2\zeta_{n-2}, \qquad n \geq 3. \tag{3.15}$$

Contrary to the previous Example 3.5, the sequence (ζ_n) is not periodic, and so may be Benford or not. Suppose $c_1 = 0$ and $c_2 \neq 0$ then $\zeta_n = (\sqrt{2})^n \left[c_2 \sin\left(\frac{1}{4}\pi n\right)\right]$. Clearly $(\zeta_n) = 0$ for $n = 4m$, $(m \in \mathbb{N}_0)$. Hence (ζ_n) fails to be Benford in this case. Similarly, (ζ_n) is not Benford for $c_2 = 0$ and $c_1 \neq 0$ (since $\zeta_{4m-2} = 0$, for any $m \in \mathbb{N}$). However if $c_1, c_2 \neq 0$ and $|c_1| \neq |c_2|$, then $\log|\zeta_n| = n \log\sqrt{2} + \log\left|c_1 \cos\left(\frac{1}{4}\pi n\right) + c_2 \sin\left(\frac{1}{4}\pi n\right)\right|$, and the sequence $(\log|\zeta_n|)$ is uniformly distributed modulo 1 by Proposition 2.8. We conclude that (ζ_n) is Benford for our choice of constants c_1 and c_2.

Remark 3.7. In Example 3.6 the irrationality of $\log\sqrt{2}$ is crucial for Benford behaviour. Generally, the irrationality of $\log|\mu|$ is critical in the analysis of the Benford phenomenon of sequences of real numbers (see Example 3.3). One similarity between Examples 3.5 and 3.6 is that in each case $\frac{1}{2\pi}\arg\mu$ is rational making both sequences a candidate to be zero periodically (and hence not Benford). However, with Example 3.5 the spectrum of the associated matrix, $\{\pm\imath\}$, has $\log|\pm\imath| = \log 1 = 0$, which is rational. In other words and with this particular example, both eigenvalues, $\mu = e^{\pm\frac{\pi}{2}\imath} = \pm\imath$, lie on the unit circle. This completely eliminated the possibility of Benford behaviour in the solution generated, no matter the choice of constants.

Example 3.8. In the light of the previous examples, it is natural to investigate what could happen when both $\log|\mu|$ and $\frac{1}{2\pi}\arg\mu$ are irrational. This example demonstrates that a problem can arise even in this case. For the two-step recursion

$$\zeta_n = 2\gamma\zeta_{n-1} - 10^{2\sqrt{2}}\zeta_{n-2}, \quad n \geq 3, \tag{3.16}$$

where $\gamma = 10^{\sqrt{2}}\cos(\pi\sqrt{2})$, the associated matrix is $A = \begin{bmatrix} 2\gamma & -10^{2\sqrt{2}} \\ 1 & 0 \end{bmatrix}$, and the spectrum of A is $\sigma(A) = \{10^{\sqrt{2}}e^{\pm\imath\pi\sqrt{2}}\}$. With arbitrary constants $c_1, c_2 \in \mathbb{R}$, the solution of (3.16) is given by

$$\zeta_n = 10^{n\sqrt{2}}\left(c_1 \cos\left(\pi n\sqrt{2}\right) + c_2 \sin\left(\pi n\sqrt{2}\right)\right), \quad n \in \mathbb{N}, \tag{3.17}$$

and $\log|\zeta_n| = n\sqrt{2} + \log|c_1\cos(\pi n\sqrt{2}) + c_2\sin(\pi n\sqrt{2})|$. Note that now $\log|\mu| = \sqrt{2}$ and $\frac{1}{2\pi}\arg\mu = 1 - \frac{\sqrt{2}}{2}$ are both irrational. This makes the sequence $\Big(c_1\cos\left(\pi n\sqrt{2}\right) + c_2\sin\left(\pi n\sqrt{2}\right)\Big)$ non-periodic for all choices of constants c_1 and c_2, except for $(c_1, c_2) = (0, 0)$. We claim the sequence (ζ_n) is neither Benford nor terminating for some choice of constants $c_1, c_2 \in \mathbb{R}$. For example, let $c_1 = 0, c_2 = 1$, then $\log|\zeta_n| = n\sqrt{2} + \log|\sin(\pi n\sqrt{2})|$. Let $s = \langle n\sqrt{2}\rangle$, the fractional part of $n\sqrt{2}$, and define $f(s) = \langle s + \log|\sin(\pi s)|\rangle$, thus $f : [0,1) \to [0,1)$ and $\langle\log|\zeta_n|\rangle = f(\langle n\sqrt{2}\rangle)$ for every $n \in \mathbb{N}$. Recall that $(n\sqrt{2})$ is u.d. mod 1. However, $\big(f(\langle n\sqrt{2}\rangle)\big)$ is not u.d. mod 1.

To see this, observe that f is piecewise smooth and has a local maximum at some $0 < s_0 < 1$, precisely $s_0 = 1 - \frac{1}{\pi}\arctan\frac{\pi}{\ln 10}$. Given that $(n\sqrt{2})$ is u.d. mod 1, if $\big(f(\langle n\sqrt{2}\rangle)\big)$ were also u.d. mod 1 then the length of each interval $J \subset [0,1)$ would be the same as the total length of the preimage $f^{-1}(J)$. However this is not the case (see also Figure 3.1), as can be seen for instance by considering $J_\epsilon = [f(s_0 - \epsilon), f(s_0)]$. For all sufficiently small $\epsilon > 0$,

$$\frac{f(s_0) - f(s_0 - \epsilon)}{\epsilon} = \frac{\lambda(J_\epsilon)}{\epsilon} = \frac{\lambda(f^{-1}(J_\epsilon))}{\epsilon} \geq \frac{\lambda([s_0 - \epsilon, s_0))}{\epsilon} = 1,$$

which is impossible since $f'(s_0) = 0$. Hence (ζ_n) is not Benford. An explanation for this is that while $\log|\mu| = \sqrt{2}$ is irrational for the characteristic roots $\mu = 10^{\sqrt{2}} e^{\pm \imath \pi \sqrt{2}}$ associated with (3.16), the real numbers $1, \log|\mu|, \frac{1}{2\pi}\arg\mu$ are rationally dependent, as $\rho_{-1} \cdot 1 + \rho_0\bigl(\sqrt{2}\bigr) + \rho_1\bigl(1 - \frac{\sqrt{2}}{2}\bigr) = 0$ with $\rho_{-1} = 1$, $\rho_0 = -\frac{1}{2}$ and $\rho_1 = -1$.

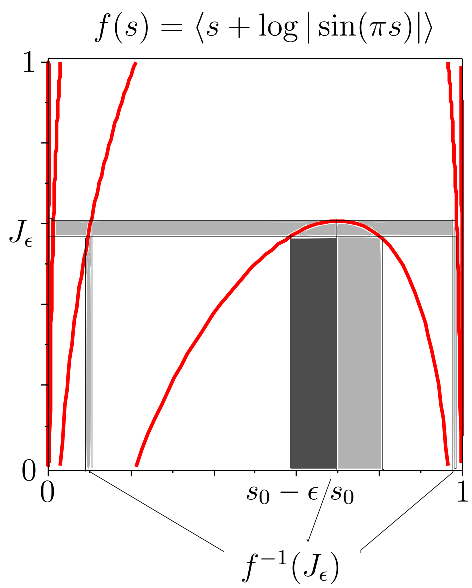

Figure 3.1: The length of J_ϵ is not equal to that of $f^{-1}(J_\epsilon)$.

Example 3.9. In this example we will realize that $(y^\top A^n x)$ may be Benford even when the real numbers $1, \log|\mu|$ and $\frac{1}{2\pi}\arg\mu$ are rationally dependent. Consider the matrix

$$A = \sqrt{3}\begin{bmatrix} \cos\bigl(3\pi\sqrt{2}\bigr) & -\sin\bigl(3\pi\sqrt{2}\bigr) & 0 & 0 \\ \sin\bigl(3\pi\sqrt{2}\bigr) & \cos\bigl(3\pi\sqrt{2}\bigr) & 0 & 0 \\ 0 & 0 & \cos\bigl(6\pi\sqrt{2}\bigr) & -\sin\bigl(6\pi\sqrt{2}\bigr) \\ 0 & 0 & \sin\bigl(6\pi\sqrt{2}\bigr) & \cos\bigl(6\pi\sqrt{2}\bigr) \end{bmatrix},$$

with $\sigma(A) = \left\{\sqrt{3}e^{\pm 3\pi\imath\sqrt{2}}, \sqrt{3}e^{\pm 6\pi\imath\sqrt{2}}\right\}$. Realize that $\rho_{-1} \cdot \frac{1}{2\pi}\arg\mu_1 + \rho_1 \cdot \frac{1}{2\pi}\arg\mu_2 = 0$, with $\rho_{-1} = 2, \rho_1 = 1$ (and $\mu_1 = \sqrt{3}e^{3\pi\imath\sqrt{2}}$, $\mu_2 = \sqrt{3}e^{-6\pi\imath\sqrt{2}}$). Hence $1, \log|\mu|, \frac{1}{2\pi}\arg\mu_1 = \frac{3}{2}\sqrt{2} - 2$ and $\frac{1}{2\pi}\arg\mu_2 = 4 - 3\sqrt{2}$ are rationally dependent. Given any $x, y \in \mathbb{R}^4$, let

$c_1 = y_1x_1 + y_2x_2$, $c_2 = y_2x_1 - y_1x_2$, $c_3 = y_3x_3 + y_4x_4$, $c_4 = y_4x_3 - y_3x_4$, and note that

$$\begin{aligned}\zeta_n &= y^\top A^n x \\ &= \sqrt{3}^n\Big[c_1\cos(3\pi n\sqrt{2}) + c_2\sin(3\pi n\sqrt{2}) + c_3\cos(6\pi n\sqrt{2}) + c_4\sin(6\pi n\sqrt{2})\Big].\end{aligned} \tag{3.18}$$

Thus $\log|\zeta_n| = n\log\sqrt{3} + f(n\sqrt{2})$, with the function $f : [0,1) \to \mathbb{R}$ given by

$$f(s) = \log\Big|c_1\cos(3\pi s) + c_2\sin(3\pi s) + c_3\cos(6\pi s) + c_4\sin(6\pi s)\Big|.$$

The function f has at most finitely many discontinuities, and since $1, \log\sqrt{3}$ and $\sqrt{2}$ are $\mathbb{Q}$-independent, according to Proposition 2.12 the sequence $(\log|\zeta_n|)$ defined as above is u.d. mod 1 and hence the sequence (ζ_n) is either Benford or terminating.

Remark 3.10. We realize that for the sequence $(y^\top A^n x)$ to be either Benford or terminating for every $x, y \in \mathbb{R}^d$, it is necessary to avoid $\log|\mu| \in \mathbb{Q}$ and also to rule out real eigenvalues with $\mu_1 = -\mu_2$. In addition, $\frac{1}{2\pi}\arg\mu \notin \mathbb{Q}$ must hold for every non-real eigenvalue μ. On the other hand, with $\frac{1}{2\pi}\arg\sigma(A) := \{\frac{1}{2\pi}\arg\mu : \mu \in \sigma(A)\}$, we know from Example 3.9 that rational independence of the numbers $1, \log|\mu|$ and the elements of $\frac{1}{2\pi}\arg\sigma(A)$ is not a necessary condition for $(y^\top A^n x)$ to be either Benford or terminating. However, it can be shown that for systems with dimension $d \le 3$, having $1, \log|\mu|$ and the elements of $\frac{1}{2\pi}\arg\sigma(A)$ rationally independent is necessary for $(y^\top A^n x)$ to be either Benford or terminating, see [6, Thm.5.37].

Having studied all the examples above, we seek a condition that takes into consideration or eliminates all possible problems encountered. As it will turn out, the right condition to impose on the spectrum of A to characterize Benford behaviour in sequences of the form $(y^\top A^n x)$ rests on the following definition.

Definition 3.11. A non-empty set $Z \subset \mathbb{C}$ with $|z| = r$ for some $r > 0$ and all $z \in Z$, i.e. $Z \subset r\mathbb{S}$ (and hence Z is contained in the periphery of a disc with radius r), is *nonresonant* if its associated set $\Delta_Z \subset \mathbb{R}$, defined as

$$\Delta_Z := \left\{1 + \frac{\arg z - \arg w}{2\pi} : z, w \in Z\right\}, \tag{3.19}$$

satisfies the following two conditions:

(i) $\Delta_Z \cap \mathbb{Q} = \{1\}$;

(ii) $\log r \notin \mathrm{span}_{\mathbb{Q}}\Delta_Z$.

An arbitrary set $Z \subset \mathbb{C}$ is *nonresonant* if, for every $r > 0$, the intersection of the Z and the circle of radius r, i.e. the set $Z \cap r\mathbb{S}$, is either nonresonant or empty; the set Z is *resonant* otherwise.

Note that the set Δ_Z according to (3.19) satisfies $1 \in \Delta_Z \subset (0,2)$ and is symmetric w.r.t. the point 1. The empty set $\varnothing$ is nonresonant, and so is the singleton $\{0\}$, as $\{0\} \cap r\mathbb{S} = \varnothing$ for every $r > 0$.

Example 3.12. The singleton $\{u\}$ is nonresonant if and only if either $u = 0$ or $\log|u| \notin \mathbb{Q}$. That is, if $u \neq 0$, $\Delta_Z = \{1 + \frac{0}{2\pi}\} = \{1\}$ so condition (i) is trivially satisfied, and $\log|u| \notin \operatorname{span}_{\mathbb{Q}}\{1\}$ means $\log|u|$ is irrational.

Example 3.13. Let $u \in \mathbb{C} \setminus \mathbb{R}$. Then the set $Z = \{u, \bar{u}\}$ is nonresonant if and only if $1, \log|u|$ and $\frac{1}{2\pi}\arg u$ are $\mathbb{Q}$-independent. To see this, realize $\Delta_Z = \{1 - \frac{2\arg u}{2\pi}, 1, 1 + \frac{2\arg u}{2\pi}\}$ and $\Delta_Z \cap \mathbb{Q} = \{1\}$ if and only if $1 - \frac{2\arg u}{2\pi}$ and $1 + \frac{2\arg u}{2\pi}$ are irrational, i.e. $\frac{1}{2\pi}\arg u$ is irrational. Moreover $\log|u| = \log|\bar{u}| \notin \operatorname{span}_{\mathbb{Q}}\{1 - \frac{2\arg u}{2\pi}, 1, 1 + \frac{2\arg u}{2\pi}\} = \operatorname{span}_{\mathbb{Q}}\{1, \frac{1}{2\pi}\arg u\}$ means $1, \log|u|$ and $\frac{1}{2\pi}\arg u$ are $\mathbb{Q}$-independent.

Remark 3.14. If $Z \subset r\mathbb{S}$ is symmetric with respect to the real axis, i.e if $\overline{Z} = Z$ where $\overline{Z} = \{\bar{z} : z \in Z\}$, then condition (ii) in Definition 3.11 is equivalent to $\log r \notin \operatorname{span}_{\mathbb{Q}}(\{1\} \cup \{\frac{1}{2\pi}\arg z : z \in Z\})$.

Example 3.15. For $Z_1 = \{10\}, Z_2 = \{-2, 2\}$ and $Z_3 = \{-\imath, \imath\}$, we have that $\Delta_{Z_1} = \{1\}, \Delta_{Z_2} = \Delta_{Z_3} = \left\{\frac{1}{2}, 1, \frac{3}{2}\right\}$. Since $\log 10 = 1 \in \mathbb{Q}, \Delta_{Z_2} \cap \mathbb{Q} \neq \{1\}$ and $\Delta_{Z_3} \cap \mathbb{Q} \neq \{1\}$, the sets Z_1, Z_2, Z_3 are resonant.

Example 3.16. For $Z = \left\{\sqrt{3}e^{\pm 3\pi\imath\sqrt{2}}, \sqrt{3}e^{\pm 6\pi\imath\sqrt{2}}\right\}$ with $\Delta_Z = \frac{1}{2}\{18 - 12\sqrt{2}, 14 - 9\sqrt{2}, 10 - 6\sqrt{2}, 6 - 3\sqrt{2}, 2, -2 + 3\sqrt{2}, -6 + 6\sqrt{2}, -10 + 9\sqrt{2}, -14 + 12\sqrt{2}\}$ and $\operatorname{span}_{\mathbb{Q}}\Delta_Z = \operatorname{span}_{\mathbb{Q}}\{1, \sqrt{2}\}$. Since $\log r = \log\sqrt{3} \notin \operatorname{span}_{\mathbb{Q}}\Delta_Z$ and $\Delta_Z \cap \mathbb{Q} = \{1\}$, Z is nonresonant.

The following observations are useful in our investigation of Benford behaviour in sequences of the form $(y^\top A^n x)$.

Lemma 3.17. Assume $x_1, ..., x_L \in \mathbb{R}^d$ are linearly independent. Then, given any $\xi \in \mathbb{R}^L$, there exists $y \in \mathbb{R}^d$ such that $y^\top x_\ell = \xi_\ell$ for every $1 \leq \ell \leq L$.

Proof. Define a function

$$\Phi : \begin{cases} \mathbb{R}^d & \rightarrow \quad \mathbb{R}^L, \\ x & \mapsto \quad \displaystyle\sum_{\ell=1}^{L} (x_\ell^\top x) e_\ell, \end{cases}$$

then Φ is linear. Since $x_1, \ldots, x_L \in \mathbb{R}^d$ are linearly independent,

$$\det[\Phi(x_1), \ldots, \Phi(x_L)] = \begin{vmatrix} x_1^\top x_1 & x_2^\top x_1 & \cdots & x_L^\top x_1 \\ x_1^\top x_2 & x_2^\top x_2 & \cdots & x_L^\top x_2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^\top x_L & x_2^\top x_L & \cdots & x_L^\top x_L \end{vmatrix} = \det[x_k^\top x_\ell]_{\ell,k=1}^L \neq 0,$$

showing that Φ is onto. $\square$

Recall that a set $S \subset \mathbb{N}$ *has density* if the relative proportion of elements of S among $\mathbb{N}$ from 1 to N converges to a limit as N approaches infinity, that is, if

$$\rho(S) := \lim_{N\to\infty} \frac{\#\{1 \le n \le N : n \in S\}}{N}$$

exists. Call $\rho(S)$ the *density* of S and note that $\rho(S) \in [0,1]$ whenever S has density.

Lemma 3.18. For every $A \in \mathbb{R}^{d\times d}$ and $x, y \in \mathbb{R}^d$, let

$$S_{A,x,y} := \{n \in \mathbb{N} : y^\top A^n x = 0\}. \tag{3.20}$$

Then $S_{A,x,y}$ has density, and $\rho(S_{A,x,y}) \in \mathbb{Q} \cap [0,1]$.

Proof. According to the Cayley-Hamilton Theorem, there exist $a_1, a_2, \ldots, a_d \in \mathbb{R}$ such that

$$A^d = a_1 A^{d-1} + a_2 A^{d-2} + \cdots + a_{d-1} A + a_d I_d.$$

Thus for every $n \in \mathbb{N}$ and $x, y \in \mathbb{R}^d$,

$$\begin{aligned} y^\top A^{n+d} x &= y^\top (a_1 A^{n+d-1} + a_2 A^{n+d-2} + \cdots + a_{d-1} A^{n+1} + a_d A^n) x \\ &= a_1 y^\top A^{n+d-1} x + a_2 A^{n+d-2} + \cdots + a_{d-1} y^\top A^{n+1} x + a_d y^\top A^n x, \end{aligned}$$

showing that $(y^\top A^n x)$ satisfies a linear d-step recursion relation with constant coefficients. By the Skolem-Mahler-Lech Theorem [15, Thm.A], the set $S_{A,x,y}$ is the union of a finite (possibly empty) set S_0 and a finite (possibly zero) number of lattices, i.e.

$$S_{A,x,y} = S_0 \cup \bigcup_{\ell=1}^{L} \{nM_\ell + N_\ell : n \in \mathbb{N}\}, \tag{3.21}$$

where L is a non-negative integer, and $M_\ell, N_\ell \in \mathbb{N}$ for $1 \le \ell \le L$. From (3.21) it is clear that $S_{A,x,y}$ has density, and $\rho(S_{A,x,y})$ is a rational number, in fact $\rho(S_{A,x,y}) \cdot \operatorname{lcm}\{M_1, \ldots, M_L\}$ is a non-negative integer. $\square$

For a concise formulation of the following observation, call a set $S \subset \mathbb{N}$ *co-finite* if $\mathbb{N} \setminus S$ is finite. With this, $(y^\top A^n x)$ is terminating precisely if $S_{A,x,y}$ is co-finite.

Lemma 3.19. For every $A \in \mathbb{R}^{d\times d}$ the following statements are equivalent:

(i) For every $x, y \in \mathbb{R}^d$, the set $S_{A,x,y}$ according to (3.20) is either finite or co-finite;

(ii) $\rho(S_{A,x,y}) \in \{0,1\}$ for every $x, y \in \mathbb{R}^d$;

(iii) For every $r > 0$, either $\Delta_{\sigma(A)\cap r\mathbb{S}} \cap \mathbb{Q} = \{1\}$ or $\sigma(A) \cap r\mathbb{S} = \varnothing$.

Proof. Clearly $\boxed{\text{(i)}\Rightarrow\text{(ii)}}$, because $\rho(S) = 0$ or $\rho(S) = 1$ whenever S is, respectively, finite or co-finite. Next, the implication $\boxed{\text{(ii)}\Rightarrow\text{(iii)}}$ will be established by showing that (ii) fails whenever (iii) fails. Assume, therefore, that (iii) does not hold. (Note that this is possible only if $d \geq 2$.) Thus $\#(\Delta_{\sigma(A)\cap r\mathbb{S}} \cap \mathbb{Q}) \geq 2$ for some $r > 0$, which in turn entails one of the following three possibilities: Either

$$\text{both } -r \text{ and } r \text{ are eigenvalues of } A, \tag{3.22}$$

or

$$A \text{ has an eigenvalue } \mu \in \mathbb{C}\setminus\mathbb{R} \text{ with } |\mu| = r \text{ and } \frac{\arg\mu}{2\pi} > 0 \text{ rational}, \tag{3.23}$$

or

$$\begin{aligned} &A \text{ has two eigenvalues } \mu_1, \mu_2 \in \mathbb{C}\setminus\mathbb{R} \text{ with } |\mu_1| = |\mu_2| = r \text{ and} \\ &\arg\mu_1 > \arg\mu_2 > 0 \text{ such that at least one of the two numbers} \\ &\frac{\arg\mu_1 \pm \arg\mu_2}{2\pi} \text{ is rational.} \end{aligned} \tag{3.24}$$

Note that these cases are not mutually exclusive, and (3.24) can occur only for $d \geq 4$. In case (3.22), let $u, v \in \mathbb{R}^d$ be eigenvectors of A corresponding, respectively, to the eigenvalues $-r, r$. Let $x := u + v$ and pick $y \in \mathbb{R}^d$ such that $y^\top u = y^\top v = 1$. This is possible because u, v are linearly independent, see Lemma 3.17. Then

$$y^\top A^n x = y^\top((-r)^n u + r^n v) = r^n\left((-1)^n + 1\right), \quad \forall n \in \mathbb{N},$$

showing that $S_{A,x,y} = \{2n - 1 : n \in \mathbb{N}\}$. Thus $\rho(S_{A,x,y}) = \frac{1}{2} \notin \{0, 1\}$, and (ii) does not hold. In case (3.23), let $z \in \mathbb{C}^d$ be an eigenvector of A corresponding to the eigenvalue μ, and observe that, for every $n \in \mathbb{N}$,

$$\begin{aligned} A^n \Re z &= r^n\left(\cos(n\arg\mu)\Re z - \sin(n\arg\mu)\Im z\right), \\ A^n \Im z &= r^n\left(\sin(n\arg\mu)\Re z + \cos(n\arg\mu)\Im z\right). \end{aligned} \tag{3.25}$$

Again, since $\Re z, \Im z \in \mathbb{R}^d$ are linearly independent, it is possible to choose $y \in \mathbb{R}^d$ such that $y^\top \Re z = 1$ and $y^\top \Im z = 0$. With $x := \Im z$, therefore,

$$y^\top A^n x = r^n \sin(n\arg\mu), \quad \forall n \in \mathbb{N}.$$

Since $\frac{1}{2\pi}\arg\mu$ is rational and strictly between 0 and $\frac{1}{2}$, the set $S_{A,x,y}$ equals $L\mathbb{N}$ for some integer $L \geq 2$. Thus $0 < \rho(S_{A,x,y}) = \frac{1}{L} < 1$ contradicting (ii). Lastly, in case (3.24) let $z, w \in \mathbb{C}^d$ be eigenvalues of A corresponding to the eigenvalues μ_1 and μ_2, respectively. As seen in (3.25) above, for every $n \in \mathbb{N}$,

$$\begin{aligned} A^n(\Re z + \Re w) = r^n\big(&\cos(n\arg\mu_1)\Re z - \sin(n\arg\mu_1)\Im z + \\ &+ \cos(n\arg\mu_2)\Re w - \sin(n\arg\mu_2)\Im w\big). \end{aligned}$$

Again, $\Re z, \Im z, \Re w, \Im w \in \mathbb{R}^d$ are linearly independent, and so by Lemma 3.17 it is possible to choose $y \in \mathbb{R}^d$ such that $y^\top \Re z = -1$, $y^\top \Im z = y^\top \Im w = 0$, and $y^\top \Re w = 1$. Then, with $x := \Re z + \Re w$,

$$\begin{aligned} y^\top A^n x &= r^n \left(\cos(n \arg \mu_2) - \cos(n \arg \mu_1)\right) \\ &= 2r^n \sin\left(\pi n \frac{\arg \mu_1 - \arg \mu_2}{2\pi}\right) \sin\left(\pi n \frac{\arg \mu_1 + \arg \mu_2}{2\pi}\right). \end{aligned}$$

Since both numbers $\frac{1}{2\pi}(\arg \mu_1 \pm \arg \mu_2)$ are strictly between 0 and 1 and at least one of them is rational, the set $S_{A,x,y}$ once more has a rational density that equals neither 0 nor 1: From $S_{A,x,y} = L\mathbb{N} \cup M\mathbb{N}$ with two (not necessarily different) integers $L, M \geq 2$, it follows that

$$0 < \frac{1}{\min\{L, M\}} \leq \rho(S_{A,x,y}) \leq 1 - \frac{1}{\operatorname{lcm}\{L, M\}} < 1.$$

Again this contradicts (ii) and hence completes the proof that indeed (ii) $\Rightarrow$ (iii).

Finally, to show that $\boxed{\text{(iii)}\Rightarrow\text{(i)}}$, denote the different non-zero eigenvalues of A in the upper half-plane $\{z \in \mathbb{C} : \Im z \geq 0\}$ by $\mu_1, \ldots, \mu_L$ with $L \leq d$, thus $\Im \mu_1, \ldots, \Im \mu_L \geq 0$ and $\sigma(A) \setminus \{0\} = \{\mu_1, \ldots, \mu_L\} \cup \{\overline{\mu_1}, \ldots, \overline{\mu_L}\}$. Assume w.l.o.g. that $|\mu_1| \geq \cdots \geq |\mu_L| > 0$. Given $x, y \in \mathbb{R}^d$, recall that as a consequence of, for instance, the Jordan Normal Form Theorem, the representation

$$y^\top A^n x = \Re\left(P_1(n)\mu_1^n + \cdots + P_L(n)\mu_L^n\right), \quad \forall n \geq d, \tag{3.26}$$

holds, where $P_1, \ldots, P_L$ are complex polynomials of degree at most $d-1$, determined by A and x, y. Assume now (iii) holds, and using (3.26) let $L_1 := 1 + \max\{1 \leq \ell \leq L : P_\ell = 0\}$, with $\max \varnothing := 0$. If $L_1 = 1 + L$ then $y^\top A^n x = 0$ for all $n \geq d$, and $S_{A,x,y}$ is co-finite. On the other hand, as will be shown next, $S_{A,x,y}$ is actually finite whenever $L_1 \leq L$. Clearly this will demonstrate that (iii) $\Rightarrow$ (i) and conclude the overall proof.

Assume, therefore, that $L_1 \leq L$, i.e., at least one polynomial P_ℓ in (3.26) does not vanish identically, and suppose $S_{A,x,y}$ was not finite. Then, by (3.21),

$$S_{A,x,y} \supset \{nM + N : n \in \mathbb{N}\}, \tag{3.27}$$

with the appropriate $M, N \in \mathbb{N}$. Using (3.26) again, let $L_2 := \max\{\ell \geq L_1 : |\mu_\ell| = |\mu_{L_1}|\}$ and denote by p the maximal degree of the polynomials $P_{L_1}, \ldots, P_{L_2}$. By permuting $\mu_{L_1}, \ldots, \mu_{L_2}$ and decreasing L_2, if necessary, for the purpose of the following argument it can be assumed w.l.o.g. that $\deg P_\ell = p$ for every $L_1 \leq \ell \leq L_2$. With this, rewrite (3.26) as

$$y^\top A^n x = n^p |\mu_{L_1}|^n \Re\left(c_{L_1} e^{\imath n \arg \mu_{L_1}} + \cdots + c_{L_2} e^{\imath n \arg \mu_{L_2}} + \zeta_n\right), \tag{3.28}$$

where $c_\ell := \lim_{n\to\infty} P_\ell(n)n^{-p} \neq 0$ for all $L_1 \le \ell \le L_2$, and (ζ_n) is a sequence in $\mathbb{C}$ for which $(n\zeta_n)$ is bounded, hence $\lim_{n\to\infty} \zeta_n = 0$. In view of (3.27), it follows from (3.28) that, for every $n \in \mathbb{N}$,

$$\Re\left(c_{L_1}e^{\imath N\arg\mu_{L_1}}\left(e^{\imath M\arg\mu_{L_1}}\right)^n + \cdots + c_{L_2}e^{\imath N\arg\mu_{L_2}}\left(e^{\imath M\arg\mu_{L_2}}\right)^n\right) = -\Re\zeta_{nM+N},$$

and so, with $z_\ell := e^{\imath M\arg\mu_\ell} \in \mathbb{S}$ for $L_1 \le \ell \le L_2$,

$$\lim_{n\to\infty}\Re\left(c_{L_1}e^{\imath N\arg\mu_{L_1}}z_{L_1}^n + \cdots + c_{L_2}e^{\imath N\arg\mu_{L_2}}z_{L_2}^n\right) = 0. \tag{3.29}$$

Since $c_\ell e^{\imath N\arg\mu_\ell} \neq 0$ for all ℓ, Lemma A.2 implies that either $z_\ell \in \{z_k, \overline{z_k}\}$ for some $\ell \neq k$ with $L_1 \le \ell, k \le L_2$, or $z_\ell^2 = 1$ for some $L_1 \le \ell \le L_2$. In the former case, at least one of the two numbers $\frac{M}{2\pi}(\arg\mu_\ell \pm \arg\mu_k)$ is a non-zero integer, which in turn shows that $\#(\Delta_{\sigma(A)\cap|\mu_{L_1}|\mathbb{S}} \cap \mathbb{Q}) \ge 2$ and hence contradicts the assumed validity of (iii). In the latter case, assume w.l.o.g. that $z_{L_1}^2 = 1$ and deduce from (3.29) that

$$\lim_{n\to\infty}\Re\left(c_{L_1+1}e^{\imath N\arg\mu_{L_1+1}}z_{L_1+1}^{2n} + \cdots + c_{L_2}e^{\imath N\arg\mu_{L_2}}z_{L_2}^{2n}\right) = -\Re\left(c_{L_1}e^{\imath N\arg\mu_{L_1}}\right).$$

According to Lemma A.2, this is possible only if either $z_\ell^2 \in \{z_k^2, \overline{z_k}^2\}$ for some $\ell \neq k$ with $L_1 + 1 \le \ell, k \le L_2$, or $z_\ell^4 = 1$ for some $L_1 + 1 \le \ell \le L_2$. In the former case, as before, at least one of the two numbers $\frac{2M}{2\pi}(\arg\mu_\ell \pm \arg\mu_k)$ is a non-zero integer. In the latter case, w.l.o.g. let $z_{L_1+1}^4 = 1$. But then $\frac{2M}{2\pi}\arg\mu_{L_1}$ and $\frac{4M}{2\pi}\arg\mu_{L_1+1}$ are both integers, hence $\frac{1}{2\pi}(\arg\mu_{L_1} - \arg\mu_{L_1+1})$ is a non-zero rational number. In either case, this again contradicts (iii). Overall, as claimed, the set $S_{A,x,y}$ is necessarily finite whenever $L_1 \le L$. Thus (iii)$\Rightarrow$(i), and the proof is complete. ☐

The main result of this work is stated in the following theorem.

Theorem 3.20. *Given any $A \in \mathbb{R}^{d\times d}$, the following two statements are equivalent:*

(i) *For every $x, y \in \mathbb{R}^d$, the sequence $(y^\top A^n x)$ is either Benford or terminating;*

(ii) *The set $\sigma(A)$ is nonresonant.*

Proof. To prove $\boxed{\text{(i)}\Rightarrow\text{(ii)}}$, assume by way of contradiction that $\sigma(A)$ is resonant. The goal, then, is to deduce that (i) cannot hold. If $\sigma(A)$ is resonant then, for some $r > 0$, either $\#(\Delta_{\sigma(A)\cap r\mathbb{S}} \cap \mathbb{Q}) \ge 2$ or $\log r \in \operatorname{span}_{\mathbb{Q}}\Delta_{\sigma(A)\cap r\mathbb{S}}$, or both. In the former case, Lemma 3.19 guarantees the existence of $x, y \in \mathbb{R}^d$ for which $0 < \rho(S_{A,x,y}) < 1$ and hence $(y^\top A^n x)$ is neither Benford nor terminating.

It remains to consider the case where $\#(\Delta_{\sigma(A)\cap r\mathbb{S}} \cap \mathbb{Q}) \le 1$ for every $r > 0$ yet $\log r_0 \in \operatorname{span}_{\mathbb{Q}}\Delta_{\sigma(A)\cap r_0\mathbb{S}}$ for some $r_0 > 0$. Label the elements of $\sigma(A) \cap r_0\mathbb{S}$ as $\mu_1, \ldots, \mu_L$. Since $\overline{\sigma(A)} = \sigma(A)$,

$$\log r_0 \in \operatorname{span}_{\mathbb{Q}}\Delta_{\sigma(A)\cap r_0\mathbb{S}} = \operatorname{span}_{\mathbb{Q}}\left(\{1\} \cup \left\{\frac{\arg\mu_\ell}{2\pi} : 1 \le \ell \le L\right\}\right).$$

Let L_0+1 be the dimension (over $\mathbb{Q}$) of $\mathrm{span}_{\mathbb{Q}}\Delta_{\sigma(A)\cap r_0\mathbb{S}}$. Hence $L_0 \le L$, and $L_0\in\mathbb{N}$ unless $\frac{1}{2\pi}\arg\mu_\ell$ is rational for every $1\le \ell\le L$, in which case $L_0=0$. (For instance, the latter inevitably occurs if $d=1$.)

Consider first the case that $L_0=0$. Here, $\log r_0$ is rational, and if $\mu_1\in\mathbb{R}$ then taking x to be any eigenvector of A corresponding to the eigenvalue μ_1 yields

$$\log|(x^\top A^n x)| = \log(r_0^n|x|^2) = n\log r_0 + 2\log|x|,$$

which is periodic modulo one. Hence $(x^\top A^n x)$ is neither Benford nor terminating. If, on the other hand, $\mu_1\in\mathbb{C}\setminus\mathbb{R}$ then let again $z\in\mathbb{C}^d$ be an eigenvector of A corresponding to the eigenvalue μ_1. Given any $y\in\mathbb{R}^d$, it follows from (3.25) that

$$y^\top A^n \Re z = r_0^n\left(y^\top\Re z\cos(n\arg\mu_1) - y^\top\Im z\sin(n\arg\mu_1)\right), \quad \forall n\in\mathbb{N}.$$

With $x:=\Re z$ and $y\in\mathbb{R}^d$ chosen such that $y^\top\Re z=0$ and $y^\top\Im z=-1$, a choice possible due to the linear independence of $\Re z,\Im z$ and Lemma 3.17, and with $\mathbb{N}\cap\frac{\pi}{|\arg\mu_1|}\mathbb{N}=M\mathbb{N}$ for the appropriate integer $M\ge 2$,

$$\log|y^\top A^n x| = \begin{cases} n\log r_0 + \log|\sin\left(2\pi n\frac{\arg\mu_1}{2\pi}\right)|, & \text{if } n\notin M\mathbb{N},\\ 0, & \text{if } n\in M\mathbb{N},\end{cases}$$

again is periodic modulo one, and the sequence $(y^\top A^n x)$ is neither Benford nor terminating.

Assume from now on that $L_0\ge 1$. In this case, by re-labelling the numbers $\mu_1,\ldots,\mu_L$, it can be assumed that $1,\frac{1}{2\pi}\arg\mu_1,\ldots,\frac{1}{2\pi}\arg\mu_{L_0}$ are $\mathbb{Q}$-independent, and consequently

$$\log r_0 = \frac{p_0}{q} + \frac{p_1}{q}\frac{\arg\mu_1}{2\pi} + \cdots + \frac{p_{L_0}}{q}\frac{\arg\mu_{L_0}}{2\pi}, \tag{3.30}$$

with the appropriate $p_0,p_1,\ldots,p_{L_0}\in\mathbb{Z}$ and $q\in\mathbb{N}$. Let $w_1,\ldots,w_{L_0}\in\mathbb{C}^d$ be eigenvectors of A corresponding to the eigenvalues $\mu_1,\ldots,\mu_{L_0}$, respectively. Note that $\mu_1,\ldots,\mu_{L_0}$ are all non-real, and consequently the $2L_0$ vectors $\Re w_1,\Im w_1,\ldots,\Re w_{L_0},\Im w_{L_0}$ are linearly independent. Lemma 3.17 guarantees that, given any $\xi\in\mathbb{R}^{L_0}$, it is possible to choose $y\in\mathbb{R}^d$ such that $y^\top\Re w_\ell=\xi_\ell$ and $y^\top\Im w_\ell=0$ for all $1\le\ell\le L_0$. With $x:=\Re(w_1+\cdots+w_{L_0})$ then

$$y^\top A^n x = r_0^n\Big(\xi_1\cos(n\arg\mu_1)+\cdots+\xi_{L_0}\cos(n\arg\mu_{L_0})\Big).$$

If $(y^\top A^n x)$ is not terminating then, by Lemma 3.19, $y^\top A^n x\ne 0$ for all sufficiently large n, and (3.30) leads to

$$\begin{aligned} q\log|y^\top A^n x| = p_0 n + p_1 n\frac{\arg\mu_1}{2\pi} + \cdots + p_{L_0} n\frac{\arg\mu_{L_0}}{2\pi} + \\ + \frac{q}{\ln 10}\ln\left|\xi_1\cos\left(2\pi n\frac{\arg\mu_1}{2\pi}\right)+\cdots+\xi_{L_0}\cos\left(2\pi n\frac{\arg\mu_{L_0}}{2\pi}\right)\right|. \end{aligned}$$

Since $1, \frac{1}{2\pi}\arg\mu_1, \ldots, \frac{1}{2\pi}\arg\mu_{L_0}$ are $\mathbb{Q}$-independent, according to Lemma 2.14 one can choose $\xi \in \mathbb{R}^{L_0}$ such that $(q\log|y^\top A^n x|)$ is not u.d. mod 1, and hence $(\log|y^\top A^n x|)$ is not u.d. mod 1 either, by Lemma 2.7(iv). Thus $(y^\top A^n x)$ is neither Benford nor terminating. Overall, as claimed, (i) fails whenever (ii) fails.

To prove $\boxed{\text{(ii)}\Rightarrow\text{(i)}}$, assume $\sigma(A)$ is nonresonant. Given $x, y \in \mathbb{R}^d$, recall from (3.28) that

$$y^\top A^n x = n^p|\mu_1|^n \Re\Big(c_1 e^{\imath n\arg\mu_1} + \cdots + c_L e^{\imath n \arg\mu_L} + \zeta_n\Big), \quad \forall n \in \mathbb{N} \tag{3.31}$$

where $p \in \mathbb{N}_0$ and $L \in \mathbb{N}$, the numbers $\mu_1, \ldots, \mu_L$ are appropriate (different) eigenvalues of A with $|\mu_1| = \cdots = |\mu_L| > 0$ and $\Im\mu_\ell \geq 0$ for all $1 \leq \ell \leq L$, the numbers $c_1, \ldots, c_L \in \mathbb{C}$ are all non-zero, and $(n\zeta_n)$ is a bounded sequence in $\mathbb{C}$. By assumption,

$$\log|\mu_1| \notin \operatorname{span}_{\mathbb{Q}}\Delta_{\sigma(A)\cap|\mu_1|\mathbb{S}} \supset \operatorname{span}_{\mathbb{Q}}\left(\{1\} \cup \left\{\frac{\arg\mu_\ell}{2\pi} : 1 \leq \ell \leq L\right\}\right).$$

As before, let $L_0 + 1$ be the dimension (over $\mathbb{Q}$) of $\operatorname{span}_{\mathbb{Q}}\left(\{1\} \cup \left\{\frac{1}{2\pi}\arg\mu_\ell : 1 \leq \ell \leq L\right\}\right)$, and consider first the case $L_0 = 0$, that is, $\frac{1}{2\pi}\arg\mu_\ell$ is rational for every $1 \leq \ell \leq L$. As $\sigma(A)$ would be resonant otherwise, this implies that $L = 1$ and $\mu_1 \in \mathbb{R}$. Since μ_1 is real, so is c_1, and

$$|y^\top A^n x| = n^p|\mu_1|^n \left|\Re\Big(c_1 e^{\imath\arg\mu_1} + \zeta_n\Big)\right| = n^p|\mu_1|^n|c_1||1 + \eta_n|,$$

where the real sequence (η_n) is given by $\eta_n := \frac{1}{c_1}e^{-\imath n\arg\mu_1}\Re\zeta_n \to 0$. As $\log|\mu_1|$ is irrational, it follows from

$$\log|y^\top A^n x| = n\log|\mu_1| + \frac{p}{\ln 10}\ln n + \log|c_1| + \log|1 + \eta_n|$$

and Lemma 2.7 that $(y^\top A^n x)$ is Benford.

It remains to consider the case $L_0 \geq 1$. Assume w.l.o.g. that $1, \frac{1}{2\pi}\arg\mu_1, \ldots, \frac{1}{2\pi}\arg\mu_{L_0}$ are $\mathbb{Q}$-independent. Hence there exists $q \in \mathbb{N}$ and, for every $\ell \in \{L_0 + 1, \ldots, L\}$, an integer $p_{0\ell}$ as well as a vector $p^{(\ell)} \in \mathbb{Z}^{L_0}$ such that

$$\frac{\arg\mu_\ell}{2\pi} = \frac{p_{0\ell}}{q} + \frac{p_1^{(\ell)}}{q}\frac{\arg\mu_1}{2\pi} + \cdots + \frac{p_{L_0}^{(\ell)}}{q}\frac{\arg\mu_{L_0}}{2\pi}, \quad \forall L_0 + 1 \leq \ell \leq L.$$

Note that $p^{(\ell)} = 0 \in \mathbb{Z}^{L_0}$ for at most one value of ℓ, and the $2L - L_0$ vectors

$$qe_1, \ldots, qe_{L_0}, \pm p^{(L_0+1)}, \ldots, \pm p^{(L)} \in \mathbb{Z}^{L_0}$$

are all different because otherwise $\sigma(A)$ would be resonant. As a consequence, for every $\xi, \eta \in \mathbb{R}^L$ the (multi-variable trigonometric) function $\Psi_{\xi,\eta} : \mathbb{R}^{L_0} \to \mathbb{R}$ given by

$$\Psi_{\xi,\eta}(x) := \sum_{\ell=1}^{L_0} \xi_\ell \Re\left(e^{2\pi\imath(qx_\ell + \eta_\ell)}\right) + \sum_{\ell=L_0+1}^{L} \xi_\ell \Re\left(e^{2\pi\imath(x^\top p^{(\ell)} + \eta_\ell)}\right)$$

is non-constant unless $\xi_1 = \cdots = \xi_L = 0$. Fix now any $m \in \{1,\ldots,q\}$ and deduce from (3.31) that

$$\begin{aligned}
y^\top A^{nq+m}x &= (nq+m)^p|\mu_1|^{nq+m}\Re\left(\sum_{\ell=1}^{L} c_\ell e^{\imath(nq+m)\arg\mu_\ell} + \zeta_{nq+m}\right)\\
&= |\mu_1|^{nq}n^p|\mu_1|^m\left(q+\frac{m}{n}\right)^p\Re\Bigg(\sum_{\ell=0}^{L_0} c_\ell e^{\imath m\arg\mu_\ell}e^{\imath nq\arg\mu_\ell}+\\
&\qquad\qquad + \sum_{\ell=L_0+1}^{L} c_\ell e^{\imath m\arg\mu_\ell}\prod_{\ell=1}^{L_0} e^{\imath np_k^{(\ell)}\arg\mu_k} + \zeta_{nq+m}\Bigg)\\
&= |\mu_1|^{nq}n^p|\mu_1|^m\left(q+\frac{m}{n}\right)^p\left(\Psi_{\xi,\eta}\left(n\frac{\arg\mu_1}{2\pi},\ldots,n\frac{\arg\mu_{L_0}}{2\pi}\right)+\Re\zeta_{nq+m}\right),
\end{aligned}$$

where $\xi,\eta \in \mathbb{R}^L$ are given by

$$\xi_\ell = |c_\ell|, \qquad \eta_\ell = \frac{m\arg\mu_\ell + \arg c_\ell}{2\pi}, \quad \forall 1 \le \ell \le L.$$

Recall that by assumption the L_0+2 numbers $1, q\log|\mu_1|, \frac{1}{2\pi}\arg\mu_1,\ldots,\frac{1}{2\pi}\arg\mu_{L_0}$ are $\mathbb{Q}$-independent. Since $\lim_{n\to\infty}\zeta_{nq+m} = 0$ as well, Lemma 2.7 and 2.13, applied to

$$\begin{aligned}
\log|y^\top A^{nq+m}x| = nq\log|\mu_1| &+ \frac{p}{\ln 10}\ln n + m\log|\mu_1| + p\log\left|q+\frac{m}{n}\right| +\\
&+ \frac{1}{\ln 10}\ln\left|\Psi_{\xi,\eta}\left(n\frac{\arg\mu_1}{2\pi},\ldots,n\frac{\arg\mu_{L_0}}{2\pi}\right)+\Re\zeta_{nq+m}\right|,
\end{aligned}$$

show that $(\log|y^\top A^{nq+m}x|)$ is u.d. mod 1. As $m \in \{1,\ldots,q\}$ was arbitrary, the sequence $(\log|y^\top A^n x|)$ is u.d. mod 1, by Lemma 2.10. Thus $(y^\top A^n x)$ is Benford, and the proof is complete. □

Remark 3.21. If the matrix A is invertible, i.e. whenever $0 \notin \sigma(A)$ then we can replace the term "Benford or terminating" in Theorem 3.20(i) by "Benford or identically zero". The reason for this is that (3.26) holds for all $n \in \mathbb{N}$ in this case, and so if $\sigma(A)$ is nonresonant then either $y^\top A^n x = 0$ for all n, or else $(y^\top A^n x)$ is Benford.

Example 3.22. The spectrum associated with the matrix in Example 3.9 is nonresonant (see Example 3.16) and hence, for every $x, y \in \mathbb{R}^4$, the sequence $(y^\top A^n x)$, explicitly given by

$$y^\top A^n x = \sqrt{3}^n\left[\Re\Big((x_1+\imath x_2)(y_1-\imath y_2)e^{-\imath 3\pi n\sqrt{2}} + (x_3+\imath x_4)(y_3-\imath y_4)e^{-\imath 6\pi n\sqrt{2}}\Big)\right]$$

is either Benford or identically zero. (Realize that the matrix A in Example 3.9 is invertible with $\det A = 9$.)

Example 3.23. The matrices

$$A = \begin{bmatrix} 12 & -20 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

associated with Examples 3.3, 3.4 and 3.5, respectively, have resonant spectra (see Example 3.15). Thus, for some $x, y \in \mathbb{R}^2$, the sequence $(y^\top A^n x)$ is neither Benford nor identically zero, and similar for B and C. To see this, choose $x = [10 \;\; 1]^\top$ and $y = e_2$, then $\left(y^\top A^n x\right) = (10^n)$ is not Benford or identically zero. Also, let $y = e_2$ and $x = [0 \;\; 2]^\top$ then $\left(y^\top B^n x\right) = (2^n\left(1 + (-1)^n\right))$ which is clearly neither Benford nor identically zero. Finally, let x, y be e_1, e_2, respectively, then $\left(y^\top C^n x\right) = \left(\sin\left(\frac{\pi}{2}n\right)\right)$ and hence is not Benford or identically zero.

Example 3.24. The spectrum of the matrix associated with Fibonacci recursion (3.4), $\sigma(A) = \left\{\frac{1}{2}\left(1 \pm \sqrt{5}\right)\right\}$ is nonresonant and thus the sequence $(y^\top A^n x)$ with $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ is either Benford or identically zero for every $x, y \in \mathbb{R}^2$. The latter happens if x and y are proportional to, respectively, the eigenvectors $\frac{1}{2}\left(1 + \sqrt{5}\right) e_2 - e_1$, corresponding to the eigenvalue $\frac{1}{2}\left(1 - \sqrt{5}\right)$ of A, and to the eigenvector $\frac{1}{2}\left(1 + \sqrt{5}\right) e_1 + e_2$, corresponding to the eigenvalue $\frac{1}{2}\left(1 + \sqrt{5}\right)$, or vice versa.

Example 3.25. The 3×3-matrix $A = \begin{bmatrix} 3 & 20 & -3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ has nonresonant spectrum. This is because it has three real eigenvalues, as its discriminant is $18(-3)(-20)(3) - 4(-3)^3(3) + (-3)^2(-20)^2 - 4(-20)^3 - 27(3)^2 = 38921 > 0$. Also, the eigenvalues have different absolute values, and none is of the form $\pm 10^{p/q}$ with $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. To see this, first realize that the characteristic polynomial of A is $P_A(z) = z^3 - 3z^2 - 20z + 3$. Suppose two roots of P_A have the same absolute value r, let $z = c$ be the third root, then $(z^2 - r^2)(z - c) = z^3 - 3z^2 - 20z + 3$. Expanding the expression on the left and comparing coefficients, we have that $r = \sqrt{20}, c = 3$, but $cr^2 = 60 \neq 3$. Hence the three real roots of P_A have different absolute values. Suppose now that $P_A(z) = 0$ and $|z| = 10^{p/q}$. If $p = 0$ then $|z| = 1$, yet $P_A(\pm 1) = \mp 19 \neq 0$. On the other hand, if $p > 0$ then one of the two numbers $\pm 10^p$ is an eigenvalue of the (integer) matrix A^q and hence divides $|\det A^q| = |\det \mathrm{A}|^q = 3^q$. Clearly, this is impossible. Similarly, if $p < 0$ then one of the two numbers $\pm 3^q 10^{|p|}$ is an eigenvalue of the (integer) matrix $(3A^{-1})^q$ and hence divides $|\det(3A^{-1})^q| = 3^{2q}$. Again, this is impossible as $|p| > 0$. Overall, $\log|\lambda| \notin \mathbb{Q}$ for every eigenvalue λ of A, i.e. $\sigma(A)$ is nonresonant. It follows that every sequence $(y^\top A^n x)$ is either Benford or identically zero.

The following corollary of Theorem 3.20 is applicable to difference equations (3.5).

Corollary 3.26. *Let $a_1, \ldots, a_d$ be real numbers with $a_d \neq 0$. Then the following statements are equivalent:*

(i) *Every solution* (ζ_n) *of* (3.5) *is Benford, unless* $\zeta_1 = \cdots = \zeta_d = 0$;

(ii) The set $\left\{z \in \mathbb{C} : z^d = a_1 z^{d-1} + a_2 z^{d-2} + \cdots + a_{d-1} z + a_d\right\}$ is nonresonant.

Proof. Realize that, as seen earlier, (3.5) can be put into matrix vector equation by inserting $d-1$ redundant rows. In this case let A be the associated matrix as in (3.6). Note that A is invertible since $a_d \neq 0$ and also realize that $P_A(z) = (-1)^d(z^d - a_1 z^{d-1} - a_2 z^{d-2} - \cdots - a_{d-1}z - a_d)$. Hence $\sigma(A)$ precisely equals $\{z \in \mathbb{C} : z^d = a_1 z^{d-1} + a_2 z^{d-2} + \cdots + a_{d-1} z + a_d\}$.

To show that $\boxed{\text{(ii)}\Rightarrow\text{(i)}}$, let $e_1, \ldots, e_d$ be the standard basis of $\mathbb{R}^d$, then for every $n \geq 2$,

$$\zeta_n = e_1^\top A^n (A^{-1})^d \begin{bmatrix} \zeta_d \\ \zeta_{d-1} \\ \vdots \\ \zeta_1 \end{bmatrix},$$

i.e. $\zeta_n = y^\top A^n x$ with $y = e_1$ and $x = (A^{-1})^d \begin{bmatrix} \zeta_d & \zeta_{d-1} & \cdots & \zeta_1 \end{bmatrix}^\top$. Thus, if $\sigma(A)$ is nonresonant then $(\zeta_n) = (y^\top A^n x)$ is Benford or identically zero by Theorem 3.20 and Remark 3.21.

To show that $\boxed{\text{(i)}\Rightarrow\text{(ii)}}$, we assume that (ii) is not true and deduce that (i) cannot hold either. To this end, realize that every sequence $(y^\top A^n x)$ with arbitrary $x, y \in \mathbb{R}^d$ and A given by (3.6) is a solution of (3.5), as seen in the proof of Lemma 3.19. Assuming (ii) does not hold then by Theorem 3.20, there exists $x, y \in \mathbb{R}^d$ such that $(\zeta_n) = (y^\top A^n x)$ is a solution of (3.5) that is neither Benford nor identically zero. $\square$

Example 3.27. Some solutions of the difference equation (3.15) with $(\zeta_1, \zeta_2) \neq (0,0)$ are not Benford as the set $\{z \in \mathbb{C} : z^2 = 2z - 2\} = \{1 \pm \imath\}$ is resonant, with $\Delta_{\{1\pm\imath\}} = \{\frac{3}{4}, 1, \frac{5}{4}\}$ and hence $\Delta_{\{1\pm\imath\}} \cap \mathbb{Q} \neq \{1\}$. For instance let $\zeta_1 = 1$ and $\zeta_2 = 0$, then clearly $(\zeta_n) = \left(\sqrt{2}^n \cos\left(\frac{1}{4}\pi n\right)\right)$ is not Benford.

Example 3.28. The set $\{z \in \mathbb{C} : z^3 = z^2 + 4z - 1\}$ associated with the difference equation $\zeta_n = \zeta_{n-1} + 4\zeta_{n-2} - \zeta_{n-3}$ is nonresonant. This can be seen similarly as in Example 3.25. It follows that every solution (ζ_n), except for the trivial $\zeta_n \equiv 0$, is Benford.

Example 3.29. The set $\left\{z \in \mathbb{C} : z^2 = 2\left(10^{\sqrt{2}} \cos(\pi\sqrt{2})\right) z - 10^{2\sqrt{2}}\right\}$ associated with (3.16) is resonant, with $\Delta_{\{10^{\sqrt{2}} e^{\pm\imath\pi\sqrt{2}}\}} = \{3 - \sqrt{2}, 1, \sqrt{2} - 1\}$ and hence $\log|10^{\sqrt{2}} e^{\pm\imath\pi\sqrt{2}}| = \sqrt{2} \in \operatorname{span}_{\mathbb{Q}} \Delta_{\{10^{\sqrt{2}} e^{\pm\imath\pi\sqrt{2}}\}}$. Consequently, there exist solutions (ζ_n) of (3.16) with $(\zeta_1, \zeta_2) \neq (0,0)$ which are not Benford. To see a concrete example, choose for instance $\zeta_1 = 10^{\sqrt{2}} \sin(\pi\sqrt{2})$ and $\zeta_2 = 10^{2\sqrt{2}} \sin(2\pi\sqrt{2})$. Then $(\zeta_n) = 10^{n\sqrt{2}} \sin(n\pi\sqrt{2})$ is not Benford, as seen in Example 3.8.

3.3 Continuous-Time Dynamical System

Recall that a linear (autonomous) continuous-time dynamical system (differential equation) is of the form $\dot{x}(t) = Ax(t)$, $t \in \mathbb{R}$, with solutions $x(t) = e^{tA}x_0$, where $x_0 = x(0)$ is a given initial value. Here the matrix exponential e^{tA} is defined as

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k, \qquad t \in \mathbb{R}.$$

We ask an analogous question as in the discrete-time case: Given $A \in \mathbb{R}^{d \times d}$, and any $x, y \in \mathbb{R}^d$, is the function $f(t) = y^\top e^{tA} x$ either Benford or zero? We investigate this question by imposing an appropriate condition on the spectrum of the matrix A.

Definition 3.30. A set $Z \subset \mathbb{C}$ is *exponentially nonresonant* if the set $e^{tZ} := \{e^{tz} : z \in Z\}$ is nonresonant for some $t \in \mathbb{R}$; otherwise Z is *exponentially resonant.*

Remark 3.31. The empty set is exponentially nonresonant, and the singleton $\{z\}$ is exponentially nonresonant if and only if $\Re z \neq 0$. For $A \in \mathbb{R}^{d \times d}$, the set $\sigma(e^{tA}) = \{e^{t\mu} : \mu \in \sigma(A)\}$ is resonant for every $t \in \mathbb{R}$ if and only if

$$\frac{\Re \mu_1}{\ln 10} \in \operatorname{span}_{\mathbb{Q}} \left\{ \frac{\Im \mu_1}{2\pi}, \ldots, \frac{\Im \mu_L}{2\pi} \right\}, \tag{3.32}$$

for some $L \in \mathbb{N}$ and the appropriate different eigenvalues $\mu_1, \ldots, \mu_L$ of A with $\Re \mu_1 = \cdots = \Re \mu_L$ and $\Im \mu_\ell \geq 0$ for all $1 \leq \ell \leq L$.

Example 3.32. Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ which is associated with the differential equation $\ddot{\zeta}(t) = \zeta(t)$ in (3.3). Then $\sigma(A) = \{\pm 1\}$ and thus $\sigma(e^{tA}) = \{e^{\pm t}\}$. Choose $t = 1$ and then $\sigma(e^A) = \{e^{\pm 1}\}$ which is clearly nonresonant since $\log e^{\pm 1} = \pm \log e$ is irrational.

Example 3.33. Consider the matrix $A = \begin{bmatrix} 2 & \frac{2\pi}{\ln 10} \\ -\frac{2\pi}{\ln 10} & 2 \end{bmatrix}$ which has spectrum $\sigma(A) = \left\{2 \pm \frac{2\pi\imath}{\ln 10}\right\}$ and thus $\sigma(e^{tA}) = \left\{e^{t(2 \pm \frac{2\pi\imath}{\ln 10})}\right\}$. Realize $\log r = \log e^{2t} = \frac{2t}{\ln 10}$ and $\Delta_{\sigma(e^{tA})} = \left\{1, 1 \pm \frac{2t}{\ln 10}\right\}$. Clearly, $\frac{2t}{\ln 10} \in \operatorname{span}_{\mathbb{Q}} \left\{1, 1 \pm \frac{2t}{\ln 10}\right\}$ for every $t \in \mathbb{R}$. Note that (3.32) is equivalent to the condition that $\log r \in \operatorname{span}_{\mathbb{Q}} \Delta_{\sigma(e^{tA})}$. Overall, $\sigma(A)$ is exponentially resonant.

The following lemma facilitates the proof of the main result of this section.

Lemma 3.34. For every $A \in \mathbb{R}^{d \times d}$ the set $\{t \in \mathbb{R} : \sigma(e^{tA}) \text{ is resonant}\}$ either equals $\mathbb{R}$ or else is countable.

Proof. Assume $\sigma(e^{t_0A})$ is nonresonant for some $t_0 \in \mathbb{R}$ but $\sigma(e^{tA})$ is resonant for some non-zero $t \neq t_0$. Then there exists $r > 0$ such that either $\#\left(\Delta_{\sigma(e^{tA})\cap r\mathbb{S}} \cap \mathbb{Q}\right) \geq 2$ or $\log r \in \operatorname{span}_{\mathbb{Q}}\Delta_{\sigma(e^{tA})\cap r\mathbb{S}}$, or both. Labeling the elements of $\sigma(e^{tA}) \cap r\mathbb{S}$ as $e^{t\mu_1}, \dots, e^{t\mu_L}$ with $L \leq d$ and $\Re\mu_1 = \cdots = \Re\mu_L = t^{-1}\ln r$, in the first case $\frac{1}{2\pi}\left(\arg e^{t\mu_\ell} - \arg e^{t\mu_k}\right)$ is a non-zero rational number for some $1 \leq \ell, k \leq L$. As $\arg e^z$ and $\Im z$ differ, for any $z \in \mathbb{C}$, be an integer multiple of 2π, it follows that $\frac{1}{2\pi}(\Im\mu_\ell - \Im\mu_k) \in \mathbb{Q} \setminus \{0\}$, and consequently

$$t \in \frac{2\pi}{\Im\mu_\ell - \Im\mu_k}\mathbb{Q} \subset \bigcup_{\mu\in\sigma(A)} \bigcup_{\lambda\in\sigma(A):\lambda\neq\mu,\Re\lambda=\Re\mu} \frac{2\pi}{\Im\lambda - \Im\mu}\mathbb{Q} =: \Sigma_1.$$

If, on the other hand,

$$\log r = \frac{t\Re\mu_1}{\ln 10} \in \operatorname{span}_{\mathbb{Q}}\Delta_{\sigma(e^{tA})\cap r\mathbb{S}} = \operatorname{span}_{\mathbb{Q}}\left(\{1\} \cup \left\{\frac{t\Im\mu_\ell}{2\pi} : 1 \leq \ell \leq L\right\}\right),$$

then, with the appropriate $p_0, p_1, \dots, p_L \in \mathbb{Z}$ and $q \in \mathbb{N}$,

$$t\left(q\frac{\Re\mu_1}{\ln 10} - \sum_{\ell=1}^{L} p_\ell \frac{\Im\mu_\ell}{2\pi}\right) = p_0.$$

Note that $\tilde{q}\frac{\Re\mu_1}{\ln 10} - \frac{1}{2\pi}\sum_{\ell=1}^{L} \tilde{p}_\ell \Im\mu_\ell \neq 0$ for every $\tilde{p}_1, \dots, \tilde{p}_L \in \mathbb{Z}$ and $\tilde{q} \in \mathbb{N}$, since otherwise $\sigma(e^{t_0A})$ would be resonant as well. Hence $p_0 \neq 0$, and

$$\begin{aligned} t^{-1} &\in \operatorname{span}_{\mathbb{Q}}\left\{\frac{\Re\mu_1}{\ln 10}, \frac{\Im\mu_1}{2\pi}, \dots, \frac{\Im\mu_L}{2\pi}\right\} \\ &\subset \bigcup_{\mu\in\sigma(A)} \operatorname{span}_{\mathbb{Q}}\left(\left\{\frac{\Re\mu}{\ln 10}\right\} \cup \left\{\frac{\Im\lambda}{2\pi} : \lambda \in \sigma(A), \Re\lambda = \Re\mu\right\}\right) =: \Sigma_2. \end{aligned}$$

Overall, $t \in \Sigma_1$ or $t^{-1} \in \Sigma_2$, and both sets $\Sigma_1, \Sigma_2 \subset \mathbb{R}$ are countable (and independent of t). $\square$

The main result of this section is stated in the theorem below.

Theorem 3.35. *Given any $A \in \mathbb{R}^{d\times d}$, the following two statements are equivalent:*

(i) *For every $x, y \in \mathbb{R}$, the function $t \mapsto y^\top e^{tA}x$ is either Benford or identically zero;*

(ii) *The set $\sigma(A)$ is exponentially nonresonant.*

Proof. Given $x, y \in \mathbb{R}$, for convenience let $f(t) := y^\top e^{tA}x$ for all $t \geq 0$. To prove that $\boxed{\text{(i)}\Rightarrow\text{(ii)}}$, as in the proof of Theorem 3.20, assume that $\sigma(A)$ is exponentially resonant, and hence $\sigma(e^{tA})$ is resonant for every $t \in \mathbb{R}$. As seen in the proof of Lemma 3.34 above, this is only possible if

$$\frac{\Re\mu_1}{\ln 10} \in \operatorname{span}_{\mathbb{Q}}\left\{\frac{\Im\mu_1}{2\pi}, \ldots, \frac{\Im\mu_L}{2\pi}\right\},$$

for some $L \in \mathbb{N}$ and the appropriate different eigenvalues $\mu_1, \ldots, \mu_L$ of A with $\Re\mu_1 = \cdots = \Re\mu_L$ and $\Im\mu_\ell \geq 0$ for all $1 \leq \ell \leq L$. Let L_0 be the dimension of $\operatorname{span}_{\mathbb{Q}}\left\{\frac{1}{2\pi}\Im\mu_\ell : 1 \leq \ell \leq L\right\}$. If $L_0 = 0$ then necessarily $L = 1$ and $\mu_1 = 0$. In this case, picking any $x \neq 0$ with $Ax = 0$ yields $x^\top e^{tA}x = |x|^2 \neq 0$ for every $t \geq 0$, hence (i) does not hold.

Consider in turn the case $L_0 \geq 1$, and assume w.l.o.g. that $\frac{1}{2\pi}\Im\mu_1, \ldots, \frac{1}{2\pi}\Im\mu_{L_0}$ are $\mathbb{Q}$-independent, and so

$$\frac{\Re\mu_1}{\ln 10} = \frac{p_1}{q}\frac{\Im\mu_1}{2\pi} + \cdots + \frac{p_{L_0}}{q}\frac{\Im\mu_{L_0}}{2\pi},$$

with the appropriate $p_1, \ldots, p_{L_0} \in \mathbb{Z}$ and $q \in \mathbb{N}$. Let $w_1, \ldots, w_{L_0} \in \mathbb{C}^d$ be eigenvectors of A corresponding to the eigenvalues $\mu_1, \ldots, \mu_{L_0}$, respectively. Note that $\mu_1, \ldots, \mu_{L_0}$ are non-real, and hence the $2L_0$ vectors $\Re w_1, \Im w_1, \ldots, \Re w_{L_0}, \Im w_{L_0}$ are linearly independent. Given any $\xi \in \mathbb{R}^{L_0}$, use Lemma 3.17 to choose $y \in \mathbb{R}^d$ such that $y^\top \Re w_\ell = \xi_\ell$ and $y^\top \Im w_\ell = 0$ for all $1 \leq \ell \leq L_0$. With $x := \Re(w_1 + \cdots + w_{L_0})$ then, for all $t \geq 0$,

$$\begin{aligned} f(t) = y^\top e^{tA}\Re(w_1 + \cdots + w_{L_0}) &= \sum_{\ell=1}^{L_0} y^\top \Re(e^{tA}w_\ell) \\ &= \sum_{\ell=1}^{L_0} y^\top \Re(e^{t\mu_\ell}w_\ell) = e^{t\Re\mu_1}\sum_{\ell=1}^{L_0} \xi_\ell \cos(t\Im\mu_\ell). \end{aligned}$$

It follows that for all but countably many $t \geq 0$,

$$\begin{aligned} g(t) &:= q\log|f(t)| \\ &= tp_1\frac{\Im\mu_1}{2\pi} + \cdots + tp_{L_0}\frac{\Im\mu_{L_0}}{2\pi} + \frac{q}{\ln 10}\ln|\xi_1\cos(t\Im\mu_1) + \cdots + \xi_{L_0}\cos(t\Im\mu_{L_0})|. \end{aligned}$$

Observe that $1, \frac{\delta}{2\pi}\Im\mu_1, \ldots, \frac{\delta}{2\pi}\Im\mu_{L_0}$ are $\mathbb{Q}$-independent for all but countably many $\delta > 0$. According to Theorem A.4, it is possible to choose $\xi \in \mathbb{R}^{L_0}$ in such a way that, for almost all $\delta > 0$, the sequence $(\langle g(n\delta)\rangle)$ is distributed according to a probability measure μ on $\mathbb{T}$ with $\mu \neq \lambda_1$. This means that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\psi\left(\langle g(n\delta)\rangle\right) = \int_{[0,1)}\psi(x)\,\mathrm{d}\mu(x)$$

holds for every continuous function $\psi : \mathbb{T} \to \mathbb{C}$. Since $\mu \neq \lambda_1$, there exists a non-zero integer

p with $\int_{\mathbb{T}} e^{2\pi\imath p x}\,\mathrm{d}\mu(x) \neq 0$, and so, by the Dominated Convergence Theorem,

$$\begin{aligned}
0 \neq \int_{\mathbb{T}} e^{2\pi\imath p x}\,\mathrm{d}\mu(x) &= \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} e^{2\pi\imath p g(n\delta)} \\
&= \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \int_0^1 e^{2\pi\imath p g(n\delta)}\mathrm{d}\delta \\
&= \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \frac{1}{n}\int_0^n e^{2\pi\imath p g(u)}\mathrm{d}u \\
&= \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \frac{1}{n}\sum_{\ell=1}^{n}\int_{\ell-1}^{\ell} e^{2\pi\imath p g(u)}\mathrm{d}u.
\end{aligned}$$

Since $\left|\int_{\ell-1}^{\ell} e^{2\pi\imath p g(u)}\mathrm{d}u\right| \leq 1$ for all $\ell \in \mathbb{N}$,

$$\begin{aligned}
0 \neq \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \frac{1}{n}\sum_{\ell=1}^{n}\int_{\ell-1}^{\ell} e^{2\pi\imath p g(u)}\mathrm{d}u &= \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \frac{1}{n}\int_{n-1}^{n} e^{2\pi\imath p g(u)}\mathrm{d}u \\
&= \lim_{N\to\infty} \frac{1}{N}\int_0^N e^{2\pi\imath p g(u)}\mathrm{d}u.
\end{aligned}$$

As in [13, Thm.I.9.6], it follows that $t \mapsto g(t)$ is not c.u.d. mod 1. But then $t \mapsto \log|f(t)|$ is not c.u.d. either by [13, Exc.I.9.6]. With the particular choice of ξ, x, y, therefore, the function $t \mapsto f(t)$ is neither Benford nor does it vanish identically. This contradicts (i) and shows that (i)$\Rightarrow$(ii) as claimed.

Conversely, to demonstrate $\boxed{\text{(ii)}\Rightarrow\text{(i)}}$, first recall that if f does not vanish identically then, as a continuous-time analogue of (3.31),

$$f(t) = t^p e^{t\Re\mu_1}\Re\left(c_1 e^{\imath t\Im\mu_1} + \cdots + c_L e^{\imath t\Im\mu_L} + h(t)\right), \quad \forall t \geq 0, \tag{3.33}$$

where $p \in \mathbb{N}_0$ and $L \in \mathbb{N}$, the numbers $\mu_1, \ldots, \mu_L$ are appropriate different eigenvalues of A with $\Re\mu_1 = \cdots = \Re\mu_L$ and $\Im\mu_\ell \geq 0$ for all $1 \leq \ell \leq L$, the numbers $c_1, \ldots, c_L \in \mathbb{C}$ are all non-zero, and $h : [0, +\infty) \to \mathbb{C}$ is a smooth function with $\lim_{t\to+\infty} h(t) = 0$. Next deduce from Lemma A.2 that there exists $\delta_0 > 0$ with the property that $(f(n\delta))$ is not terminating for any $0 < \delta \leq \delta_0$: Indeed, if $(f(n\delta_N))$ were terminating for every N with some strictly decreasing real sequence (δ_N) satisfying $\lim_{N\to\infty}\delta_N = 0$ then (3.33) would imply that

$$\lim_{n\to\infty} \Re\left(c_1\left(e^{\imath\delta_N\Im\mu_1}\right)^n + \cdots + c_L\left(e^{\imath\delta_N\Im\mu_L}\right)^n\right) = 0$$

holds for every N. According to Lemma A.2, however, this would require that $\Im\mu_\ell = \Im\mu_k$ for some $\ell \neq k$, which is not the case. For all $0 < \delta \leq \delta_0$, therefore, the sequence $(f(n\delta))$

is not terminating, and for all but countably many such δ the set $\sigma(e^{\delta A})$ is nonresonant, by Lemma 3.34. Thus for almost every $\tau \in [0,1]$ the sequence $(\log|f(n\tau\delta_0)|)$ is u.d. mod 1 by Theorem 3.20, and [13, Thm.I.9.6] implies that $t \mapsto \log|f(t)|$ is c.u.d. mod 1, i.e. $t \mapsto f(t)$ is Benford. $\square$

Example 3.36. For $x, y \in \mathbb{R}^2$, the solution associated with the differential equation in Example 3.32,

$$\zeta(t) = y^\top e^{tA}x = \frac{1}{2}(y_1x_1 + y_1x_2 + y_2x_1 + y_2x_2)e^t + \frac{1}{2}(y_1x_1 - y_1x_2 - y_2x_1 + y_2x_2)e^{-t}$$

is either Benford or identically zero. The latter is the case if and only if $y_1x_1 + y_2x_2 = 0$ and $y_1x_2 + y_2x_1 = 0$. Recall that the spectrum of the matrix A in Example 3.32 is exponentially nonresonant.

Example 3.37. Consider the matrix A in Example 3.33 which has exponentially resonant spectrum. We have

$$e^{tA} = e^{2t}\begin{bmatrix} \cos\left(\frac{2\pi t}{\ln 10}\right) & \sin\left(\frac{2\pi t}{\ln 10}\right) \\ -\sin\left(\frac{2\pi t}{\ln 10}\right) & \cos\left(\frac{2\pi t}{\ln 10}\right) \end{bmatrix}.$$

Choose $x = \begin{bmatrix} 1 & 0 \end{bmatrix}^\top$ and $y = \begin{bmatrix} 0 & 1 \end{bmatrix}^\top$, then $y^\top e^{tA}x = -e^{2t}\sin\left(\frac{2\pi t}{\ln 10}\right)$ and thus $\log|y^\top e^{tA}x| = \frac{2t}{\ln 10} + \log\left|\sin\left(\frac{2\pi t}{\ln 10}\right)\right| = h\left(\frac{2t}{\ln 10}\right)$, where $h(s) = s + \log|\sin(\pi s)|$. The function h is not c.u.d. mod 1 (see Example 3.8) and hence $t \mapsto y^\top e^{tA}x$ is not Benford.

Example 3.38. Consider the matrix $A = \begin{bmatrix} -2 & -5 \\ 1 & 0 \end{bmatrix}$ with $\sigma(A) = \{-1 \pm 2\imath\}$. Since $-\frac{1}{\ln 10} \notin \operatorname{span}_{\mathbb{Q}}\left\{\frac{1}{\pi}\right\}$, $\sigma(A)$ is exponentially nonresonant. We have that

$$e^{tA} = \frac{1}{2}e^{-t}\begin{bmatrix} 2\cos(2t) - \sin(2t) & -5\sin(2t) \\ \sin(2t) & 2\cos(2t) + \sin(2t) \end{bmatrix},$$

and the function

$$y^\top e^{tA}x = \frac{1}{2}e^{-t}\left[(2y_1x_1 + 2y_2x_2)\cos(2t) + (y_2x_1 + y_2x_2 - y_1x_1 - 5y_1x_2)\sin(2t)\right]$$

is either Benford or identically zero.

Finally, consider the linear (autonomous) differential equation

$$\zeta^{(d)}(t) = a_1\zeta^{(d-1)}(t) + a_2\zeta^{(d-2)}(t) + \cdots + a_{d-1}\dot{\zeta}(t) + a_d\zeta(t). \tag{3.34}$$

The following corollary of Theorem 3.35 is applicable to (3.34).

Corollary 3.39. *Let $a_1, \dots, a_d$ be real numbers with $a_d \neq 0$. The following statements are equivalent:*

(i) *Every solution $\zeta = \zeta(t)$ of (3.34) is Benford, unless $\zeta(t) \equiv 0$;*

(ii) *The set $\left\{z \in \mathbb{C} : z^d = a_1 z^{d-1} + a_2 z^{d-2} + \cdots + a_{d-1} z + a_d\right\}$ is exponentially nonresonant.*

Proof. Realize that (3.34) can be re-stated as a matrix-vector equation by inserting $d-1$ redundant rows, thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} \zeta^{(d-1)} \\ \zeta^{(d-2)} \\ \vdots \\ \zeta \end{bmatrix} = \begin{bmatrix} a_1\zeta^{(d-1)} + \cdots + a_d\zeta \\ \zeta^{(d-1)} \\ \vdots \\ \dot{\zeta} \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & \cdots & a_{d-1} & a_d \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} \zeta^{(d-1)} \\ \zeta^{(d-2)} \\ \vdots \\ \zeta \end{bmatrix}.$$

With this set-up, (3.34) takes the form $\dot{V}(t) = AV(t)$, where A is the associated matrix as in (3.6) and $V = \begin{bmatrix}\zeta^{(d-1)} & \zeta^{(d-2)} & \cdots & \zeta\end{bmatrix}^\top$. Also realize that $P_A(z) = (-1)^d(z^d - a_1 z^{d-1} - a_2 z^{d-2} - \cdots - a_{d-1}z - a_d)$ and hence the set $\sigma(A)$ precisely equals $\{z \in \mathbb{C} : z^d = a_1 z^{d-1} + a_2 z^{d-2} + \cdots + a_{d-1} z + a_d\}$.

To show that $\boxed{\text{(ii)}\Rightarrow\text{(i)}}$, realize that the solution to the system above is given by $V(t) = e^{tA}V(0)$ where $V(0) = \begin{bmatrix}\zeta^{(d-1)}(0) & \cdots & \zeta(0)\end{bmatrix}^\top$. Thus $\zeta(t) = e_d^\top e^{tA}V(0) = y^\top e^{tA}x$, with $y = e_d$ and $x = V(0)$. Thus, if (ii) holds then the function $\zeta(t) = y^\top e^{tA}x$ is Benford or identically zero, by Theorem 3.35.

To show that $\boxed{\text{(i)}\Rightarrow\text{(ii)}}$, we will assume that (ii) does not hold and deduce that (i) cannot hold either. To this end, note that every function $\zeta(t) := y^\top e^{tA}x$ with arbitrary $x, y \in \mathbb{R}^d$, and with A given by (3.6), is a solution of (3.34). Assuming (ii) does not hold, then by Theorem 3.35, there exists $x, y \in \mathbb{R}^d$ such that $t \mapsto y^\top e^{tA}x$ is neither Benford nor identically zero. $\square$

Example 3.40. The set $Z = \{z \in \mathbb{C} : z^2 = z + 1\} = \{-\varphi^{-1}, \varphi\}$ with $\varphi = \frac{1}{2}(1+\sqrt{5})$, associated with the differential equation $\ddot{\zeta} = \dot{\zeta} + \zeta$ is exponentially nonresonant. To see this, realize that $e^{tZ} = \left\{e^{t\varphi}, e^{-t\varphi^{-1}}\right\}$, and with $t = 1$, both $\log e^{\varphi}$ and $\log e^{-\varphi^{-1}}$ are irrational and hence the set $e^Z = \{e^{\varphi}, e^{-\varphi^{-1}}\}$ is nonresonant. It follows that every solution $\zeta = \zeta(t)$ of $\ddot{\zeta} = \dot{\zeta} + \zeta$ is Benford, except for $\zeta(t) \equiv 0$.

Example 3.41. Consider the differential equation $\ddot{\zeta} + \zeta = 0$. The associated set $Z = \{z \in \mathbb{C} : z^2 = -1\} = \{\pm\imath\}$ is exponentially resonant. Consequently, no solution ζ is Benford. This can also be seen directly, as

$$\zeta(t) = c_1 \cos t + c_2 \sin t = \sqrt{c_1^2 + c_2^2}\cos(t - t_0),$$

with the appropriate $c_1, c_2, t_0 \in \mathbb{R}$, and Lemma A.7 with $p_1 = 0$, $\alpha = \frac{1}{\ln 10}$, $\xi = \sqrt{c_1^2 + c_2^2}$ shows that ζ is not Benford.

3.4 Discussion of Work

Although the tailor-made Definition 3.11 is for arbitrary subsets of complex numbers, it was not studied how the Benford analysis would look for sequences of the form $(y^\top A^n x)$ for which the matrix $A \in \mathbb{C}^{d\times d}$ and/or $x, y \in \mathbb{C}^d$ are non-real. Thus, the main Theorems 3.20 and 3.35 as stated are for *real* $d \times d$-matrices only. Even with this, difficulties can arise in their practical usage. For instance, consider the second order recurrence relation

$$\zeta_n = 2\zeta_{n-1} - 5\zeta_{n-2}, \qquad n \geq 3. \tag{3.35}$$

The associated matrix is $A = \begin{bmatrix} 2 & -5 \\ 1 & 0 \end{bmatrix}$ with $\sigma(A) = \{1 \pm 2\imath\}$ and thus $|1 \pm 2\imath| = \sqrt{5}$. For $\sigma(A)$ to be nonresonant both conditions of Definition 3.11 must be satisfied. Realize that $\Delta_{\sigma(A)} = \left\{1 - \frac{1}{\pi}\arctan(2),\ 1,\ 1 + \frac{1}{\pi}\arctan(2)\right\}$, so condition (i) of Definition 3.11 is clearly satisfied. Hence nonresonance of $\sigma(A)$ is equivalent to $\log\sqrt{5} \notin \operatorname{span}_{\mathbb{Q}}\Delta_{\sigma(A)}$. While both $\log\sqrt{5}$ and $\frac{1}{\pi}\arctan(2)$ are irrational, it is an open problem in number theory whether or not 1, $\log\sqrt{5}$, $\frac{1}{\pi}\arctan(2)$ are rationally independent [24]. Hence, it is unknown whether the set $\sigma(A)$ is nonresonant. In this case, we become limited in making theoretical conclusions as to whether the solution of (3.35),

$$\zeta_n = \sqrt{5}^n \left[c_1 \cos\left(n \arctan(2)\right) + c_2 \sin\left(n \arctan(2)\right)\right], \;\; n \in \mathbb{N}, \tag{3.36}$$

is Benford or not. Experimental evidence suggests that except for the trivial case $\zeta_1 = \zeta_2 = 0$, every solution (ζ_n) of (3.35), as explicitly given by (3.36) is Benford (see Table 3.1).

digit (d)	ζ_n	$\log\left(1 + \frac{1}{k}\right)$
1	29.99	30.10
2	17.23	17.60
3	12.78	12.49
4	9.51	9.69
5	7.92	7.91
6	6.61	6.67
7	6.01	5.79
8	5.19	5.11
9	4.76	4.57

Table 3.1: Leading digit distribution for the first 10000 terms of the solution (ζ_n) of (3.35), with $\zeta_1 = \zeta_2 = 1$. The data suggests that (ζ_n) is Benford.

In this work, we considered real matrices for their simplicity and since Benford's Law is often stated in terms of sequences of real numbers. One may be curious to know how the results obtained in this work could be extended to complex matrices and vectors. Given $A \in \mathbb{C}^{d \times d}$, denote by $A_{\mathbb{R}}$ its *realification*, i.e. the real matrix

$$A_{\mathbb{R}} := \begin{bmatrix} \Re A & -\Im A \\ \Im A & \Re A \end{bmatrix} \in \mathbb{R}^{2d \times 2d}.$$

In analogy to Definition 3.11, a possible definition could be that $A \in \mathbb{C}^{d \times d}$ has nonresonant spectrum if and only if the spectrum of the realification $A_{\mathbb{R}}$ is nonresonant. Realize that since A is a complex matrix, eigenvalues do not occur in conjugate pairs and thus $\sigma(A)$ may not be symmetric; however, $\sigma(A_{\mathbb{R}}) = \sigma(A) \cup \overline{\sigma(A)}$ is symmetric. For $x, y \in \mathbb{C}^d$, $(y^\top A^n x)$ is a sequence of complex numbers. We may require that for the sequence $(y^\top A^n x)$ to be Benford both the real part and the imaginary part, that is $\Re(y^\top A^n x)$ and $\Im(y^\top A^n x)$, are to be Benford. Do analogues of Theorems 3.20 and 3.35 hold in this context?

Most systems, e.g. biological models, are *nonlinear* in nature. It is of interest to know if their digit distribution would be Benford. The level of completeness achieved in this work for linear (autonomous) systems cannot be expected for the vast class of nonlinear systems. To extend the work done in this thesis to some nonlinear systems, one would like to do so via the process of *linearisation*. Consider the nonlinear system

$$\dot{x} = f(x), \tag{3.37}$$

where $x \in \mathbb{R}^d$ and f is a smooth vector function. Could we claim that solutions $x = x(t)$ of (3.37) would lead to Benford functions or sequences if the matrix $A \in \mathbb{R}^{d \times d}$ associated with (3.37) via linearisation has $\sigma(A)$ nonresonant?

3.5 Summary and Conclusions

Linear dynamical systems provide important models for many areas of applied science. The study of digits generated by these (and other) systems is a classical and rather wide subject that continues to attract interest from many disciplines. Building on earlier work, notably [4, 6, 16, 21], this work provides a generalization and unification of already known facts about the relationship between Benford's Law and solutions of dynamical systems. Even though the implication (i)⇒(ii) of Theorems 3.20 and 3.35 seems to have been addressed in the past only for systems with dimension $d \leq 3$ (see [6, Thm.5.37]), the results of this work fully characterize Benford's behaviour in dynamical systems of arbitrary dimension.

This work theoretically characterizes Benford's Law in linear dynamical systems by providing a necessary and sufficient condition under which solutions of these systems are Benford. This condition takes into account that if the spectrum of the arbitrary real $d \times d$-matrix

A associated with the system is (exponentially) nonresonant, then for a sequence $(y^\top A^n x)$ (or function $t \mapsto y^\top e^{tA} x$) only two outcomes are possible irrespective of the choice of real d-dimensional vectors x, y. The sequence $(y^\top A^n x)$ (or function $t \mapsto y^\top e^{tA} x$) is either Benford or identically zero. However, it is worth noting that even in the case of resonant spectrum *some* solutions of the system may be Benford nevertheless. The characterization presented here is based on the review of basic facts about Benford sequences (and functions) and uniform distribution modulo one.

Even though the analysis employed in this work for simplicity considers only integer base 10, the Benford property can be studied with respect to any integer base $b \geq 2$ by requiring the second condition of Definition 3.11 to be $\log_b r \notin \operatorname{span}_{\mathbb{Q}} \Delta_Z$. Hence the main theorems can be extended to, and proved in a similar way for, any base $b \in \mathbb{N} \setminus \{1\}$.

Appendix A

Some auxiliary results

The purpose of this appendix is to provide proofs for several analytical facts that have been used in establishing the main results of this work. Throughout, let $d \in \mathbb{N}$ be fixed.

Lemma A.1. Given any $z_1, \ldots, z_d \in \mathbb{S} = \{z \in \mathbb{C} : |z| = 1\}$, the following statements are equivalent:

(i) If $\lim_{n\to\infty} (c_1 z_1^n + \cdots + c_d z_d^n)$ exists with $c_1, \ldots, c_d \in \mathbb{C}$ then $c_1 = \cdots = c_d = 0$;

(ii) $z_j \notin \{1\} \cup \{z_k : k \neq j\}$ for every $1 \le j \le d$.

Proof. Clearly (i)⇒(ii) because if $z_j = 1$ for some j simply let $c_j = 1$ and $c_\ell = 0$ for all $\ell \neq j$, whereas if $z_j = z_k$ for some $j \neq k$ take $c_j = 1, c_k = -1$, and $c_\ell = 0$ for all $\ell \in \{1, \ldots, d\} \setminus \{j, k\}$. To show that (ii)⇒(i) as well, proceed by induction. Trivially, if $d = 1$ then $(c_1 z_1^n)$ with $z_1 \in \mathbb{S}$ converges only if $c_1 = 0$ or $z_1 = 1$. Assume now that (ii)⇒(i) has been established already for $1 \le d \le D$, let $d = D + 1$, and assume that $z_j \notin \{1\} \cup \{z_k : k \neq j\}$ for every $1 \le j \le D+1$. If $\lim_{n\to\infty} \left(c_1 z_1^n + \cdots + c_{D+1} z_{D+1}^n\right)$ exists then, as $z_{D+1} \neq 1$,

$$\begin{aligned} &\left\{ c_1 \left(\frac{z_1}{z_{D+1}}\right)^n \frac{z_1 - 1}{z_{D+1} - 1} + \cdots + c_D \left(\frac{z_D}{z_{D+1}}\right)^n \frac{z_D - 1}{z_{D+1} - 1} + c_{D+1} \right\} z_{D+1}^n (z_{D+1} - 1) \\ &\qquad = c_1 z_1^n (z_1 - 1) + \cdots + c_{D+1} z_{D+1}^n (z_{D+1} - 1) \\ &\qquad = c_1 z_1^{n+1} + \cdots + c_{D+1} z_{D+1}^{n+1} - \left(c_1 z_1^n + \cdots + c_{D+1} z_{D+1}^n\right) \xrightarrow{n\to\infty} 0, \end{aligned}$$

which in turn yields

$$\lim_{n\to\infty} \left\{ c_1 \left(\frac{z_1}{z_{D+1}}\right)^n \frac{z_1 - 1}{z_{D+1} - 1} + \cdots + c_D \left(\frac{z_D}{z_{D+1}}\right)^n \frac{z_D - 1}{z_{D+1} - 1} \right\} = -c_{D+1}.$$

Note that $\frac{z_j}{z_{D+1}} \notin \{1\} \cup \left\{ \frac{z_k}{z_{D+1}} : k \neq j \right\}$ for every $1 \le j \le D$. Hence by induction assumption, $c_j \frac{z_j - 1}{z_{D+1} - 1} = 0$ for all $1 \le j \le D$, and so $c_1 = \cdots = c_D = 0$. As $z_{D+1} \neq 1$, $c_{D+1} = 0$ as well. $\square$

Lemma A.2. The following statements are equivalent for any $z_1, \ldots, z_d \in \mathbb{S}$:

(i) If $\lim_{n\to\infty} \Re\left(c_1 z_1^n + \cdots + c_d z_d^n\right)$ exists with $c_1, \ldots, c_d \in \mathbb{C}$ then $c_1 = \cdots = c_d = 0$;

(ii) $z_j \notin \{-1, 1\} \cup \{z_k, \overline{z_k} : k \neq j\}$ for every $1 \leq j \leq d$.

Proof. Clearly (i)$\Rightarrow$(ii) because if $z_j \in \{-1, 1\}$ for some $1 \leq j \leq d$ simply let $c_j = \imath$ and $c_\ell = 0$ for all $\ell \neq j$, whereas if $z_j \in \{z_k, \overline{z_k}\}$ for some $j \neq k$, take $c_j = 1$, $c_k = -1$, and $c_\ell = 0$ for all $\ell \in \{1, \ldots, d\} \setminus \{j, k\}$. Conversely, if

$$\lim_{n\to\infty} \Re\left(c_1 z_1^n + \cdots + c_d z_d^n\right) = \frac{1}{2} \lim_{n\to\infty} \left(c_1 z_1^n + \overline{c_1}\ \overline{z_1}^n + \cdots + c_d z_d^n + \overline{c_d}\ \overline{z_d}^n\right)$$

exists then, by Lemma A.1, $c_1 = \cdots = c_d = 0$ unless either $z_j = 1$ or $z_j = \overline{z_j}$ (and hence $z_j \in \{-1, 1\}$) for some j, or $z_j \in \{z_k, \overline{z_k}\}$ for some $j \neq k$. Overall, $c_1 = \cdots = c_d = 0$ unless $z_j \in \{-1, 1, z_k, \overline{z_k}\}$ for some $j \neq k$. Thus (ii)$\Rightarrow$(i), as claimed. $\square$

Let $\vartheta_1, \ldots, \vartheta_d$ be real numbers such that $1, \vartheta_1, \ldots, \vartheta_d$ are $\mathbb{Q}$-independent. Furthermore, let $p_1, \ldots, p_d$ be arbitrary integers, and $\alpha \in \mathbb{R} \setminus \{0\}$. With these ingredients, given any $\xi \in \mathbb{R}^d$, consider the sequence (a_n) of real numbers defined as

$$a_n = p_1 n\vartheta_1 + \cdots + p_d n\vartheta_d + \alpha \ln\left|\xi_1 \cos(2\pi n\vartheta_1) + \cdots + \xi_d \cos(2\pi n\vartheta_d)\right|, \quad \forall n \in \mathbb{N}, \qquad \text{(A.1)}$$

where ln denotes the natural logarithm, and $\ln 0 := 0$ for convenience.

Lemma A.3. Given $d \in \mathbb{N}$, $\vartheta_1, \ldots, \vartheta_d \in \mathbb{R}$, $p_1, \ldots, p_d \in \mathbb{Z}$ and $\alpha \in \mathbb{R} \setminus \{0\}$ as above, there exists $\xi \in \mathbb{R}^d$ such that the sequence (a_n) according to (A.1) is not u.d. mod 1.

The remainder of this appendix is devoted to establishing Lemma A.3 which has been instrumental in the proof of Theorem 3.20. To prepare for the arguments, denote by $\mathbb{T}^d$ the d-dimensional torus, i.e. $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, together with the σ-algebra $\mathcal{B}\left(\mathbb{T}^d\right)$ of its Borel sets, and let $\mathcal{P}\left(\mathbb{T}^d\right)$ be the set of all probability measures on $\left(\mathbb{T}^d, \mathcal{B}\left(\mathbb{T}^d\right)\right)$. Given any $\mu \in \mathcal{P}\left(\mathbb{T}^d\right)$, associate with it the family $\left(\widehat{\mu}(k)\right)_{k\in\mathbb{Z}^d}$ of its *Fourier coefficients*, defined as

$$\widehat{\mu}(k) = \int_{\mathbb{T}^d} e^{2\pi\imath k^\top x}\, \mathrm{d}\mu(x) = \int_{\mathbb{T}^d} e^{2\pi\imath(k_1x_1+\cdots+k_dx_d)}\, \mathrm{d}\mu(x_1, \ldots, x_d), \quad \forall k \in \mathbb{Z}^d.$$

Recall that $\mu \mapsto \left(\widehat{\mu}(k)\right)_{k\in\mathbb{Z}^d}$ is one-to-one, i.e. the Fourier coefficients determine μ uniquely. Arguably the most prominent element in $\mathcal{P}\left(\mathbb{T}^d\right)$ is the *uniform distribution* (or *Lebesgue measure*) on $\mathbb{T}^d$, henceforth denoted λ_d, for which, with $\mathrm{d}\lambda_d(x)$ abbreviated $\mathrm{d}x$ as usual,

$$\widehat{\lambda_d}(k) = \int_{\mathbb{T}^d} e^{2\pi\imath(k_1x_1+\cdots+k_dx_d)}\, \mathrm{d}x = \prod_{j=1}^{d} \int_{\mathbb{T}} e^{2\pi\imath k_j x}\, \mathrm{d}x = \left\{ \begin{array}{ll} 1 & \text{if } k = 0 \in \mathbb{Z}^d, \\ 0 & \text{if } k \neq 0; \end{array} \right.$$

here and throughout, write $\mathbb{T}^1$ simply as $\mathbb{T}$. Given $\mu \in \mathcal{P}\left(\mathbb{T}^d\right)$, therefore, to show that $\mu \neq \lambda_d$ it is (necessary and) sufficient to find at least one $k \in \mathbb{Z}^d \setminus \{0\}$ for which $\widehat{\mu}(k) \neq 0$. Recall also that, given a measurable map $T : \mathbb{T}^d \to \mathbb{T}$, each $\mu \in \mathcal{P}\left(\mathbb{T}^d\right)$ induces a unique $\mu \circ T^{-1} \in \mathcal{P}(\mathbb{T})$ via

$$\mu \circ T^{-1}(B) = \mu\left(T^{-1}(B)\right), \quad \forall B \in \mathcal{B}(\mathbb{T}).$$

The Fourier coefficients of $\mu \circ T^{-1}$ are readily computed as

$$\widehat{\mu \circ T^{-1}}(k) = \int_{\mathbb{T}} e^{2\pi\imath kx}\, \mathrm{d}\left(\mu \circ T^{-1}\right)(x) = \int_{\mathbb{T}^d} e^{2\pi\imath kT(x)}\, \mathrm{d}\mu(x), \quad k \in \mathbb{Z}.$$

If in particular $d = 1$ and $\mu \circ T^{-1} = \mu$, then μ is said to be *invariant* under T (and T is *μ-preserving*).

With a view towards Lemma A.3, it will be useful to consider, for any given $d \in \mathbb{N}$, $p_1, \ldots, p_d \in \mathbb{Z}$ and $\alpha \in \mathbb{R} \setminus \{0\}$, the family of maps

$$T_\xi : \left\{ \begin{array}{ccl} \mathbb{T}^d & \to & \mathbb{T}, \\ x & \mapsto & \langle p_1 x_1 + \cdots + p_d x_d + \alpha \ln |\xi_1 \cos(2\pi x_1) + \cdots + \xi_d \cos(2\pi x_d)| \rangle ; \end{array} \right. \tag{A.2}$$

here $\xi \in \mathbb{R}^d$ may be thought of as a parameter. Note that each map T_ξ is (Borel) measurable, in fact differentiable outside a set of λ_d-measure zero. For every $\mu \in \mathcal{P}(\mathbb{T}^d)$, therefore, the measure $\mu \circ T_\xi^{-1}$ is a well-defined element of $\mathcal{P}(\mathbb{T})$. Lemma A.3 is a consequence of the following fact.

Theorem A.4. For every $p_1, \ldots, p_d \in \mathbb{Z}$ and $\alpha \in \mathbb{R} \setminus \{0\}$, there exists $\xi \in \mathbb{R}^d$ such that $\lambda_d \circ T_\xi^{-1} \neq \lambda_1$, with T_ξ given by (A.2).

To see that Theorem A.4 does indeed imply Lemma A.3, let $p_1, \ldots, p_d \in \mathbb{Z}$ and $\alpha \in \mathbb{R} \setminus \{0\}$ be given, and pick $\xi \in \mathbb{R}^d$ such that $\lambda_d \circ T_\xi^{-1} \neq \lambda_1$. Consequently, there exists a continuous function $f : \mathbb{T} \to \mathbb{C}$ for which $\int_{[0,1)} f \,\mathrm{d}(\lambda_d \circ T_\xi^{-1}) \neq \int_{[0,1)} f \,\mathrm{d}\lambda_d$. Note that $f \circ T_\xi : \mathbb{T}^d \to \mathbb{C}$ is continuous λ_d-almost everywhere as well as bounded, and hence Riemann integrable. Also recall that the sequence $((n\vartheta_1, \ldots, n\vartheta_d))$ is u.d. mod 1 in $\mathbb{R}^d$ whenever $1, \vartheta_1, \ldots, \vartheta_d$ are $\mathbb{Q}$-independent. In the latter case, therefore,

$$\begin{aligned} \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^N f(\langle a_n \rangle) &= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^N f \circ T_\xi \left(\langle (n\vartheta_1, \ldots, n\vartheta_d) \rangle\right) \\ &= \int_{\mathbb{T}^d} f \circ T_\xi \mathrm{d}\lambda_d = \int_{\mathbb{T}} f \mathrm{d}\left(\lambda_d \circ T_\xi^{-1}\right) \neq \int_{\mathbb{T}} f \mathrm{d}\lambda_1, \end{aligned}$$

showing that (a_n) is *not* u.d. mod 1.

Thus it remains to prove Theorem A.4. The proof presented here is computational and proceeds in essentially two steps: First the case $d = 1$ is analyzed in detail. Specifically, it is shown that $\lambda_1 \circ T_\xi^{-1} \neq \lambda_1$ unless $p_1 \neq 0$ and $\alpha\xi_1 = 0$. For itself, this could be seen directly by noticing that the map $T_\xi : \mathbb{T} \to \mathbb{T}$ has a non-degenerate critical point whenever $\alpha\xi_1 \neq 0$, and hence cannot possibly preserve λ_1, see e.g. [6, Ex.5.27(iii)]. The more elaborate calculation given here, however, is useful also in the second step of the proof, i.e. the analysis for $d \geq 2$. As it turns out, the case $d \geq 2$ can, in essence, be reduced to calculations already done for $d = 1$.

To concisely formulate the subsequent results, recall that the Euler Gamma function, denoted $\Gamma = \Gamma(z)$ as usual, is a meromorphic function with poles precisely at $z = 0, -1, -2, \ldots$, and $\Gamma(z+1) = z\Gamma(z) \neq 0$ for every $z \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$. Also, for convenience every "empty sum" is understood to equal zero, e.g., $\sum_{2 \leq j \leq 1} j^2 = 0$, whereas every "empty product" is understood to equal 1, e.g., $\prod_{2 \leq j \leq 1} j^2 = 1$. Finally, the standard (ascending) Pochhammer symbol $(z)_n$ will be used where, given any $z \in \mathbb{C}$,

$$(z)_n := z(z+1)\ldots(z+n-1) = \prod_{l=0}^{n-1}(z+l), \quad \forall n \in \mathbb{N}$$

and $(z)_0 := 1$, in accordance with the convention on empty products.

Lemma A.5. Given any $q \in \mathbb{Z}$ and $\beta \in \mathbb{R} \setminus \{0\}$, let

$$J(q, \beta) := \int_{\mathbb{T}} e^{4\pi\imath q x\imath + 2\imath\beta \ln|\cos(2\pi x)|} \, \mathrm{d}x. \tag{A.3}$$

Then

$$\begin{aligned} e^{\imath\beta \ln 4} J(q, \beta) &= (-1)^q \frac{2\imath\beta\Gamma(2\imath\beta)}{(\imath\beta\Gamma(\imath\beta))^2} \cdot \frac{(-\imath\beta)_{|q|}}{(1+\imath\beta)_{|q|}} \\ &= \frac{\sin(\pi|q| - \pi\imath\beta)}{\pi|q| - \pi\imath\beta} \Gamma(1 + 2\imath\beta) \frac{\Gamma(1 + |q| - \imath\beta)}{\Gamma(1 + |q| + \imath\beta)} \end{aligned} \tag{A.4}$$

and hence in particular

$$|J(q, \beta)|^2 = \frac{\beta \tanh(\pi\beta)}{\pi(q^2 + \beta^2)}. \tag{A.5}$$

Proof. Substituting $-x$ for x in (A.3) shows that $J(q, \beta) = J(|q|, \beta)$, and a straightforward calculation, with T_l denoting the l-th Chebyshev polynomial ($l \in \mathbb{N}_0$), yields

$$\begin{aligned} J(q, \beta) &= \int_{\mathbb{T}} e^{4\pi\imath|q|x + 2\imath\beta \ln|\cos(2\pi x)|} \, \mathrm{d}x = \int_0^1 e^{2\pi\imath|q|x + 2\imath\beta \ln|\cos(\pi x)|} \, \mathrm{d}x \\ &= \int_0^{\frac{1}{2}} 2\cos(\pi|q|x) e^{2\imath\beta \ln|\cos(\pi x)|} \, \mathrm{d}x = 2\int_0^{\frac{1}{2}} T_{2|q|}\left(\cos(\pi x)\right) e^{2\imath\beta \ln|\cos(\pi x)|} \, \mathrm{d}x \\ &= \frac{2}{\pi} \int_0^1 \frac{T_{2|q|}(s)}{\sqrt{1 - s^2}} e^{2\imath\beta \ln s} \, \mathrm{d}s = \frac{2}{\pi} \int_0^{+\infty} T_{2|q|}\left(\frac{1}{\sqrt{1+t^2}}\right) \frac{e^{-\imath\beta \ln(1+t^2)}}{1+t^2} \, \mathrm{d}t. \end{aligned}$$

As the polynomial $T_{2|q|}$ can, for every $q \in \mathbb{Z}$ and $u \neq 0$, be written as

$$T_{2|q|}(u) = u^{2|q|} \sum_{l=0}^{|q|} \binom{2|q|}{2l} (1 - u^{-2})^l,$$

it follows that

$$\begin{aligned} J(q, \beta) &= \frac{2}{\pi} \sum_{l=0}^{|q|} (-1)^l \binom{2|q|}{2l} \int_0^{+\infty} \frac{t^{2l}}{(1+t^2)^{1+|q|+\imath\beta}} \, \mathrm{d}t \\ &= \frac{1}{\pi} \sum_{l=0}^{|q|} (-1)^l \binom{2|q|}{2l} \int_0^{+\infty} \frac{u^{l-\frac{1}{2}}}{(1+u)^{1+|q|+\imath\beta}} \, \mathrm{d}u \\ &= \frac{1}{\pi \Gamma(1+|q|+\imath\beta)} \sum_{l=0}^{|q|} (-1)^l \binom{2|q|}{2l} \Gamma\left(\frac{1}{2} + l\right) \Gamma\left(\frac{1}{2} + |q| - l + \imath\beta\right). \end{aligned}$$

Note that Γ is finite and non-zero for each argument appearing in this sum. Recall that

$$\Gamma\left(\frac{1}{2} + l\right) = \frac{(2l)!\sqrt{\pi}}{l! 2^{2l}}, \quad \forall l \in \mathbb{N}_0,$$

and so

$$\begin{aligned} J(q, \beta) &= \frac{(-1)^q (2|q|)!}{\sqrt{\pi} 2^{2|q|} \Gamma(1+|q|+\imath\beta)} \sum_{l=0}^{|q|} \left\{ (-1)^l \frac{2^{2l} \Gamma\left(\frac{1}{2} + l + \imath\beta\right)}{(2l)!(|q|-l)!} \right\} \\ &= \frac{(-1)^q \Gamma\left(\frac{1}{2} + |q|\right) \Gamma\left(\frac{1}{2} + \imath\beta\right)}{\pi \Gamma(1+|q|+\imath\beta)} \sum_{l=0}^{|q|} \left\{ (-1)^l \binom{|q|}{l} \prod_{k=1}^{l} \frac{2k-1+2\imath\beta}{2k-1} \right\} \\ &= \frac{(-1)^q \Gamma\left(\frac{1}{2} + \imath\beta\right)}{\sqrt{\pi} 2^{2|q|} \Gamma(1+|q|+\imath\beta)} \sum_{l=0}^{|q|} \left\{ (-1)^l \binom{|q|}{l} \prod_{k=1}^{l} (2k-1+2\imath\beta) \prod_{k=l+1}^{|q|} (2k-1) \right\} \\ &= \frac{(-1)^q \Gamma\left(\frac{1}{2} + \imath\beta\right)}{\sqrt{\pi} 2^{2|q|} \Gamma(1+|q|+\imath\beta)} Q_{|q|}(2\imath\beta), \end{aligned}$$

where, for every $m \in \mathbb{N}_0$, the polynomial Q_m is given by

$$Q_m(z) = \sum_{l=0}^{m} \left\{ (-1)^l \binom{m}{l} \prod_{k=1}^{l} (2k-1+z) \prod_{k=l+1}^{m} (2k-1) \right\}. \tag{A.6}$$

Thus for example $Q_0(z) \equiv 1, Q_1(z) = -z, Q_2(z) = -2z + z^2$. Note that the degree of Q_m

equals m, and for every $j \in \{0, 1, \ldots, m-1\}$,

$$\begin{aligned} Q_m(2j) &= \sum_{l=0}^{m} \left\{ (-1)^l \binom{m}{l} \prod_{k=1}^{l} (2k-1+2j) \prod_{k=l+1}^{m} (2k-1) \right\} \\ &= \sum_{l=0}^{m} \left\{ (-1)^l \binom{m}{l} \prod_{k=j}^{m} (2k-1) \prod_{k=l+1}^{l+j} (2k-1) \right\} \\ &= \left\{ \prod_{k=j+1}^{m} (2k-1) \right\} \sum_{l=0}^{m} \left\{ (-1)^l \binom{m}{l} \prod_{k=1}^{j} (2l+2k-1) \right\} = 0. \end{aligned}$$

Here the elementary fact has been used that $\sum_{l=0}^{m} (-1)^l \binom{m}{l} P(l) = 0$ holds for every polynomial P of degree less than m. As the polynomial Q_m has degree m, it cannot have any further roots besides $0, 2, 4, \ldots, 2m-2$, and so

$$Q_m(z) = c_m \prod_{l=0}^{m-1} (z - 2l),$$

with a constant c_m yet to be determined. The correct value of c_m is readily found by observing that

$$Q_m(-1) = c_m \prod_{l=0}^{m-1} (-1-2l) = c_m (-1)^m \cdot 1 \cdot 3 \cdot \ldots \cdot (2m-1),$$

whereas on the other hand, by the very definition (A.6) of Q_m,

$$Q_m(-1) = \sum_{l=0}^{m} \left\{ (-1)^l \binom{m}{l} \prod_{k=1}^{l} (2k-2) \prod_{k=l+1}^{m} (2k-1) \right\} = \prod_{k=1}^{m} (2k-1).$$

Thus $c_m = (-1)^m$, and overall

$$Q_m(z) = (-1)^m \prod_{l=0}^{m-1} (z-2l) = \prod_{l=0}^{m-1} (2l - z) = 2^m \Big(-\frac{1}{2} z \Big)_m .$$

With this, one obtains

$$\begin{aligned} J(q, \beta) &= \frac{(-1)^q \Gamma\left(\frac{1}{2} + \imath\beta\right)}{\sqrt{\pi} 2^{|q|} \Gamma(1 + |q| + \imath\beta)} \prod_{l=0}^{|q|-1} (2l - 2\imath\beta) = \frac{(-1)^q \Gamma\left(\frac{1}{2} + \imath\beta\right)}{\sqrt{\pi} \Gamma(\imath\beta)} \cdot \frac{1}{|q| + \imath\beta} \prod_{l=0}^{|q|-1} \frac{l - \imath\beta}{l + \imath\beta} \\ &= 2(-1)^{q+1} e^{-\imath\beta \ln 4} \frac{\Gamma(2\imath\beta)}{\Gamma(\imath\beta)^2} \cdot \frac{1}{|q| - \imath\beta} \prod_{l=1}^{|q|} \frac{l - \imath\beta}{l + \imath\beta}, \end{aligned}$$

where the so-called Legendre duplication formula for the Γ-function has been used in the form

$$\Gamma(\imath\beta) \Gamma\Big(\frac{1}{2} + \imath\beta \Big) = 2^{1 - 2\imath\beta} \sqrt{\pi} \Gamma(2\imath\beta), \quad \forall \beta \in \mathbb{R} \setminus \{0\}.$$

Minor re-arrangements of the above expression for $J(q, \beta)$, using the Euler reflection formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}, \quad \forall z \in \mathbb{C} \setminus \mathbb{Z}, \tag{A.7}$$

lead to

$$\begin{aligned} e^{\imath\beta \ln 4} J(q, \beta) &= (-1)^q \frac{2\imath\beta\Gamma(2\imath\beta)}{\left(\imath\beta\Gamma(\imath\beta)\right)^2} \cdot \frac{(-\imath\beta)_{|q|}}{(1+\imath\beta)_{|q|}} \\ &= \frac{\sin(\pi|q| - \pi\imath\beta)}{\pi|q| - \pi\imath\beta} \Gamma(1+2\imath\beta) \frac{\Gamma(1+|q|-\imath\beta)}{\Gamma(1+|q|+\imath\beta)}, \end{aligned}$$

as claimed. Using the standard fact that

$$|\Gamma(\imath\beta)|^2 = \frac{\pi}{\beta \sinh(\pi\beta)}, \quad \forall \beta \in \mathbb{R} \setminus \{0\},$$

which follows for instance from (A.7) together with $\Gamma(\bar{z}) = \overline{\Gamma(z)}$, claim (A.5) now is immediate:

$$|J(q, \beta)|^2 = \left| \frac{\sin(\pi|q| - \pi\imath\beta)}{\pi|q| - \pi\imath\beta} \right|^2 |\Gamma(1+2\imath\beta)|^2 = \frac{\sinh^2(\pi\beta)}{\pi^2(q^2+\beta^2)} \cdot \frac{4\beta^4\pi}{2\beta\sinh(2\pi\beta)} = \frac{\beta\tanh(\pi\beta)}{\pi(q^2+\beta^2)}.$$

□

Remark A.6. Relations (A.4) and (A.5) are valid for $\beta = 0$ also, in the sense that

$$J(q, 0) = \lim_{\beta \to 0} J(q, \beta) = \lim_{\beta \to 0} \frac{\sin(\pi|q| - \pi\imath\beta)}{\pi|q| - \pi\imath\beta} = \begin{cases} 1 & \text{if } q = 0, \\ 0 & \text{if } q \neq 0; \end{cases}$$

here the fact $\lim_{z \to 0} z^{-1} \sin z = 1$ has been used.

An immediate consequence of Lemma A.5 is that for $d = 1$, and any $p_1 \in \mathbb{Z}$, $\alpha \in \mathbb{R} \setminus \{0\}$, the map T_ξ according to (A.2) does typically not preserve λ_1.

Lemma A.7. For every $p_1 \in \mathbb{Z}$ and $\alpha \in \mathbb{R} \setminus \{0\}$,

$$\lambda_1 \circ T_\xi^{-1} = \lambda_1 \iff \xi_1 = 0 \text{ and } p_1 \neq 0$$

Proof. Simply note that for every non-zero $\xi = [\xi_1] \in \mathbb{R}^1$,

$$\begin{aligned} \widehat{\lambda_1 \circ T_\xi^{-1}}(2) &= \int_{\mathbb{T}} e^{4\pi\imath x} \, \mathrm{d}\left(\lambda_1 \circ T_\xi^{-1}\right)(x) = \int_{\mathbb{T}} e^{4\pi\imath(p_1 x + \alpha \ln|\xi_1 \cos(2\pi x)|)} \, \mathrm{d}x \\ &= e^{4\pi\imath\alpha \ln|\xi_1|} J(p_1, 2\pi\alpha) \neq 0, \end{aligned}$$

showing that $\lambda_1 \circ T_\xi^{-1} \neq \lambda_1$. On the other hand if $\xi_1 = 0$ then $T_\xi(x) = \langle p_1 x \rangle$, and

$$\widehat{\lambda_1 \circ T_\xi}(k) = \int_{\mathbb{T}} e^{2\pi\imath k p_1 x} \mathrm{d}x = \begin{cases} 1 & \text{if } kp_1 = 0, \\ 0 & \text{if } kp_1 \neq 0, \end{cases}$$

shows that $\lambda_1 \circ T_\xi \neq \lambda_1$ if and only if $p_1 \neq 0$. □

As mentioned earlier, the case $d \geq 2$ will now be studied and, in a way, reduced to the case $d = 1$. To this end, let again $q \in \mathbb{Z}$ and $\beta \in \mathbb{R} \setminus \{0\}$ be given, and consider the function $\iota_{q,\beta} : \mathbb{R} \to \mathbb{C}$ defined as

$$\iota_{q,\beta}(t) := \int_{\mathbb{T}} e^{4\pi \imath q x + 2\imath\beta \ln|t+\cos(2\pi x)|} \, \mathrm{d}x, \qquad \forall t \in \mathbb{R}.$$

A few elementary properties of $\iota_{q,\beta}$ are contained in

Lemma A.8. For every $q \in \mathbb{Z}$ and $\beta \in \mathbb{R}\setminus\{0\}$, the function $\iota_{q,\beta}$ is continuous and even, with $|\iota_{q,\beta}(t)| \leq 1$ for all $t \in \mathbb{R}$. Moreover, $\iota_{q,\beta}(0) = J(q,\beta)$ and $\iota_{q,\beta}(1) = e^{\imath\beta \ln 4} J(2q, 2\beta) \neq \iota_{q,\beta}(0)$.

Proof. Since for every $t^* \in \mathbb{R}$,

$$\lim_{t \to t^*} \ln|t + \cos(2\pi x)| = \ln|t^* + \cos(2\pi x)|$$

holds for all but (at most) two $x \in \mathbb{T}$, the continuity of $\iota_{q,\beta}$ follows from the Dominated Convergence Theorem. Clearly, $\iota_{q,\beta}$ is even, with $|\iota_{q,\beta}(t)| \leq \int_{\mathbb{T}} \mathrm{d}\lambda_1 = 1$. Finally, it follows with

$$\begin{aligned}\iota_{q,\beta}(1) &= \int_{\mathbb{T}} e^{4\pi\imath q x + 2\imath\beta \ln|2\cos^2(\pi x)|} d\lambda_1(x) \\ &= e^{\imath\beta\ln 4} \int_{\mathbb{T}} e^{4\pi\imath q x + 4\imath\beta \ln|\cos(\pi x)|} d\lambda_1(x) = e^{\imath\beta \ln 4} J(2q, 2\beta) \neq 0,\end{aligned}$$

that, for every $q \in \mathbb{Z}$ and $\beta \in \mathbb{R} \setminus \{0\}$,

$$\left| \frac{\iota_{q,\beta}(1)}{\iota_{q,\beta}(0)} \right|^2 = \frac{2\beta \tanh(2\pi\beta)}{4q^2 + 4\beta^2} \cdot \frac{q^2 + \beta^2}{\beta \tanh(\pi\beta)} = \frac{1}{2}\left(1 + \frac{1}{\cos h(2\pi\beta)}\right) < 1,$$

and hence $\iota_{q,\beta}(1) \neq \iota_{q,\beta}(0)$. $\square$

For the subsequent arguments, it will be crucial that $\iota_{q,\beta}$ is actually much smoother than Lemma A.8 seems to suggest. Recall that a function $f : \mathbb{R}^m \to \mathbb{C}$ is *real-analytic* on an open set $U \subset \mathbb{R}^m$ if f can, in a neighbourhood of each point in U, be represented as a convergent power series. The ultimate proof of Lemma A.3 will rely heavily on the following refinement of Lemma A.8.

Lemma A.9. For every $q \in \mathbb{Z}$ and $\beta \in \mathbb{R} \setminus \{0\}$, the function $\iota_{q,\beta}$ is real-analytic on the interval $(-1, 1)$.

Proof. Given $q \in \mathbb{Z}$ and $\beta \in \mathbb{R} \setminus \{0\}$, by Lemma A.8, the function $\tau \mapsto \iota_{q,\beta}(\cos(\pi\tau))$ is continuous and 1-periodic. Hence it can be represented, at least in the $L^2(\lambda_1)$-sense, as a

Fourier series $\iota_{q,\beta}(\cos(\pi\tau)) \sim \sum_{k\in\mathbb{Z}} c_k e^{2\pi\imath k\tau}$ where, for every $k \in \mathbb{Z}$,

$$\begin{aligned}
c_k &= \int_{\mathbb{T}} \iota_{q,\beta}(\cos(\pi\tau))\, e^{-2\pi\imath k\tau}\,\mathrm{d}\tau \\
&= \int_{\mathbb{T}\times\mathbb{T}} e^{-2\pi\imath k\tau} e^{4\pi\imath qx+2\imath\beta\ln|\cos(\pi\tau)+\cos(2\pi x)|}\,\mathrm{d}x\,\mathrm{d}\tau \\
&= \int_{\mathbb{T}^2} e^{2\pi\imath(2qx-k\tau)+2\imath\beta\ln|\cos(\pi\tau)+\cos(2\pi x)|}\,\mathrm{d}\lambda_2(x,\tau) \\
&= \int_{\mathbb{T}^2} e^{4\pi\imath q(y+\sigma)-4\pi\imath k(y-\sigma)+2\imath\beta\ln|2\cos(2\pi y)\cos(2\pi\sigma)|}\,\mathrm{d}\lambda_2(y,\sigma) \\
&= e^{\imath\beta\ln 4}\int_{\mathbb{T}} e^{4\pi\imath(q-k)y+2\imath\beta\ln|\cos(2\pi y)|}\,\mathrm{d}\lambda_1(y)\int_{\mathbb{T}} e^{4\pi\imath(q+k)\sigma+2\imath\beta\ln|\cos(2\pi\sigma)|}\,\mathrm{d}\lambda_1(\sigma) \\
&= e^{\imath\beta\ln 4} J(q-k,\beta)J(q+k,\beta).
\end{aligned}$$

Since $c_{-k} = c_k$, the Fourier series of $\tau \mapsto \iota_{q,\beta}(\cos(\pi\tau))$ is $c_0 + 2\sum_{n\in\mathbb{N}} c_n \cos(2\pi n\tau) = c_0 + 2\sum_{n=1}^{\infty} c_n T_{2n}(\cos(\pi\tau))$, and since furthermore

$$|c_n| = |J(n-q,\beta)J(n+q,\beta)| = \frac{\beta\tanh(\tau\beta)}{\pi\sqrt{(n^2+q^2+\beta^2)^2-4n^2q^2}} = \mathcal{O}(n^{-2}), \qquad \text{as } n\to\infty,$$

the Weierstrass M-test implies that, uniformly in $t \in [-1,1]$,

$$\iota_{q,\beta}(t) = c_0 + 2\sum_{n=1}^{\infty} c_n T_{2n}(t).$$

For every $\kappa \in \mathbb{R}$ with $|\kappa| < 1$, consider now the auxiliary function

$$h(t,\kappa) := 2\sum_{n=1+|q|}^{\infty} c_n T_{2n}(t)\kappa^n.$$

Note that, uniformly in $t \in [-1,1]$,

$$\iota_{q,\beta}(t) = c_0 + 2\sum_{n=1}^{|q|} c_n T_{2n}(t) + \lim_{\kappa\uparrow 1} h(t,\kappa).$$

In addition, introduce an analytic function on the open unit disc as

$$H(z) := \sum_{n=1+|q|}^{\infty} c_n z^n, \quad \forall z \in \mathbb{C} : |z| < 1, \tag{A.8}$$

and observe that

$$\begin{aligned}
H(z) &= z^{1+|q|}\sum_{n=0}^{\infty} c_{n+1+|q|} z^n = e^{\imath\beta\ln 4} z^{1+|q|}\sum_{n=0}^{\infty} J(n+1,\beta)J(n+1+2|q|,\beta) z^n \\
&= e^{-\imath\beta\ln 4} z^{1+|q|}\frac{\left(2\imath\beta\Gamma(2\imath\beta)\right)^2}{\left(\imath\beta\Gamma(\imath\beta)\right)^4}\sum_{n=0}^{\infty}\frac{(-\imath\beta)_{n+1}\,(-\imath\beta)_{n+1+2|q|}}{(1+\imath\beta)_{n+1}\,(1+\imath\beta)_{n+1+2|q|}} z^n \\
&= e^{-\imath\beta\ln 4} z^{1+|q|}\frac{\left(2\imath\beta\Gamma(2\imath\beta)\right)^2}{\left(\imath\beta\Gamma(\imath\beta)\right)^4}\cdot\frac{(\imath\beta)^2}{(1+\imath\beta)^2}\cdot \\
&\qquad\qquad \cdot\frac{(1-\imath\beta)_{2|q|}}{(2+\imath\beta)_{2|q|}}\sum_{n=0}^{\infty}\frac{(1-\imath\beta)_n(1+2|q|-\imath\beta)_n}{(2-\imath\beta)_n(2+2|q|+\imath\beta)_n} z^n \\
&= \frac{4e^{-\imath\beta\ln 4}\Gamma(2\imath\beta)^2(1-\imath\beta)_{2|q|}}{(1+\imath\beta)^2\Gamma(\imath\beta)^4(2+\imath\beta)_{2|q|}}\ {}_3F_2(1-\imath\beta, 1+2|q|-\imath\beta, 1; 2+ \\
&\qquad\qquad + \imath\beta, 2+2|q|+\imath\beta; z) z^{1+|q|};
\end{aligned}$$

here the standard notation for (generalized) hypergeometric functions has been used, see e.g. [23, Ch.II]. Recall that ${}_3F_2$ really is an analytic function on $\mathbb{C}\setminus\{z : \Re z \geq 1, \Im z = 0\}$, that is, on the entire complex plane minus a cut from 1 to ∞ along the positive real axis. Hence H as given by (A.8) can be extended analytically to $\mathbb{C}\setminus\{z : \Re z \geq 1, \Im z = 0\}$ as well. Observe now that

$$\begin{aligned}
H(e^{2\pi i\tau}\kappa) + H(e^{-2\pi i\tau}\kappa) &= 2\sum_{n=1+|q|}^{\infty} c_n \cos(2\pi n\tau)\kappa^n = 2\sum_{n=1+|q|}^{\infty} c_n T_{2n}\left(\cos(\pi\tau)\right)\kappa^n \\
&= h\left(\cos(\pi\tau), \kappa\right), \qquad \forall\tau\in\mathbb{R}, |\kappa|<1.
\end{aligned}$$

It follows that, for all $t\in[-1,1]$,

$$\begin{aligned}
\iota_{q,\beta}(t) &= c_0 + 2\sum_{n=1}^{|q|} c_n T_{2n}(t) + \lim_{\kappa\uparrow 1}\Big\{H\left((2t^2-1+2\imath t\sqrt{1-t^2})\kappa\right) + \\
&\qquad\qquad + H\left((2t^2-1-2\imath t\sqrt{1-t^2})k\right)\Big\} \\
&= c_0 + 2\sum_{n=1}^{|q|} c_n T_{2n}(t) + H\left(2t^2-1+2\imath t\sqrt{1-t^2}\right) + \\
&\qquad\qquad + H\left(2t^2-1-2\imath t\sqrt{1-t^2}\right).
\end{aligned}$$

This in turn shows that $t\mapsto\iota_{q,\beta}(t)$ is real-analytic on $(-1,1)$ because $t\mapsto 2t^2-1\pm 2\imath t\sqrt{1-t^2}$ is real-analytic on $(-1,1)$, and $2t^2-1\pm 2\imath t\sqrt{1-t^2} = 1$ only if $|t|=1$. In fact, $\iota_{q,\beta}(t) = \sum_{n=0}^{\infty}\iota_{q,\beta}^{(n)}(0)t^n/n!$ for all $t\in(-1,1)$. □

For every $d \in \mathbb{N}$, define a non-empty open subset of $\mathbb{R}^d$ as

$$E_d := \left\{ \xi \in \mathbb{R}^d : \exists j_0 \in \{1, ..., d\} \text{ s.t. } |\xi_{j_0}| > \sum_{j \neq j_0} |\xi_j| \right\}.$$

In order to utilize Lemma A.9 for a proof of Theorem A.4, given any $d \in \mathbb{N}, p_1, \ldots, p_d \in \mathbb{Z}$ and $\alpha \in \mathbb{R} \setminus \{0\}$, recall the map T_ξ from (A.2) and consider the function $I : \mathbb{R}^d \to \mathbb{C}$ defined as

$$\begin{aligned} I(\xi) &= \widehat{\lambda_d \circ T_\xi^{-1}}(2) = \int_{\mathbb{T}} e^{4\pi i x} \mathrm{d}\lambda_d \circ T_\xi^{-1}(x) \\ &= \int_{\mathbb{T}^d} e^{4\pi \imath (p_1 x_1 + \ldots p_d x_d + \alpha \ln |\xi_1 \cos(2\pi x_1) + \ldots + \xi_d \cos(2\pi x_d)|)} \mathrm{d}\lambda_d(x_1, ..., x_d). \end{aligned} \tag{A.9}$$

An important consequence of Lemma A.9 is

Lemma A.10. For every $p_1, ..., p_d \in \mathbb{Z}$ and $\alpha \in \mathbb{R} \setminus \{0\}$, the function $I = I(\xi)$ given by (A.9) is real-analytic and non-constant on E_d.

Proof. If $d = 1$ then, as seen already in the proof of Lemma A.7,

$$I(\xi) = \int_{\mathbb{T}} e^{4\pi \imath p_1 x + 4\pi \imath \alpha \ln |\xi_1 \cos(2\pi x)|} \, \mathrm{d}x = e^{4\pi \imath \alpha \ln |\xi_1|} J(p_1, 2\pi\alpha)$$

is clearly real-analytic and non-constant on $\mathbb{R}^1 \setminus \{0\} = E_1$.

Assume in turn that $d \geq 2$. As the roles of $\xi_1, ..., \xi_d$ can be interchanged in (A.9), assume w.l.o.g. that $\xi_d \neq 0$. Since $I(\pm\xi_1, ..., \pm\xi_d) = I(\xi_1, ..., \xi_d)$ for all $\xi \in \mathbb{R}^d$ and every possible combination of $+$ and $-$ signs, and since also

$$I(\xi) = e^{4\pi \imath \alpha \ln |\xi_d|} I\left(\frac{\xi_1}{\xi_d}, ..., \frac{\xi_{d-1}}{\xi_d}, 1 \right),$$

it suffices to show that $\xi \mapsto I(\xi_1, ..., \xi_{d-1}, 1)$ is real-analytic on $\widetilde{E}_{d-1} := \{\xi \in \mathbb{R}^{d-1} : |\xi_1| + ... + |\xi_{d-1}| < 1\}$. To this end, recall that $\iota_{p_d, 2\pi\alpha}$ is real-analytic on $(-1, 1)$ by Lemma A.9 and note that

$$\widetilde{I}(\xi) = \int_{\mathbb{T}^{d-1}} e^{4\pi \imath (p_1 x_1 + \cdots + p_{d-1} x_{d-1})} \iota_{p_d, 2\pi\alpha} \left(\xi_1 \cos(2\pi x_1) + \cdots + \xi_{d-1} \cos(2\pi x_{d-1}) \right) \mathrm{d}x.$$

With Lemma A.8 and the Dominated Convergence Theorem, it is clear that $\widetilde{I}$ is continuous on $\mathbb{R}^{d-1}$. Recall from the proof of Lemma A.9 that $\iota_{p,\alpha}(t) = \sum_{n=0}^{\infty} \iota_{p,\alpha}^{(n)}(0) t^n / n!$ for all $p \in \mathbb{Z}$, $\alpha \in \mathbb{R}$ and $|t| < 1$. For every $\xi \in \widetilde{E}_{d-1}$, therefore,

$$\begin{aligned} \widetilde{I}(\xi) &= \int_{\mathbb{T}^{d-1}} e^{4\pi \imath (p_1 x_1 + \cdots + p_{d-1} x_{d-1})} \sum_{n=0}^{\infty} \frac{\iota_{p_d, 2\pi\alpha}^{(n)}(0)}{n!} (\xi_1 \cos(2\pi x_1) + \cdots + \xi_{d-1} \cos(2\pi x_{d-1}))^n \, \mathrm{d}x \\ &= \sum_{n=0}^{\infty} 2^{-2n} \iota_{p_d, 2\pi\alpha}^{(2n)}(0) \sum_{|\nu| = n} \left\{ \prod_{j=1}^{d-1} \frac{\xi_j^{2\nu_j}}{(2\nu_j)!} \binom{2\nu_j}{\nu_j + |p_j|} \right\}; \end{aligned} \tag{A.10}$$

here the standard notation for multiindices $\nu = (\nu_1, ..., \nu_{d-1}) \in (\mathbb{N}_0)^{d-1}$ has been used.

It remains to demonstrate that $\widetilde{I}$ is not constant (on $\widetilde{E}_{d-1}$). Consider first the case $d = 2$, for which (A.10) takes the form

$$\widetilde{I}(\xi_1, 1) = \sum_{n=|p_1|}^{\infty} 2^{-2n} \iota^{(2n)}_{p_d, 2\pi\alpha}(0) \binom{2n}{n + |p_1|} \frac{\xi_1^{2n}}{(2n)!}. \tag{A.11}$$

As has been shown above, $\xi_1 \mapsto \widetilde{I}(\xi_1, 1)$ is real-analytic for $|\xi_1| < 1$. If $p_1 \neq 0$ then $\widetilde{I}(0, 1) = 0$, whereas

$$\begin{aligned} \widetilde{I}(1, 1) &= \int_{\mathbb{T}^2} e^{4\pi\imath(p_1 x_1 + p_2 x_2 + \alpha \ln|\cos(2\pi x_1) + \cos(2\pi x_2)|)} \, \mathrm{d}x \\ &= \int_{\mathbb{T}^2} e^{4\pi\imath(p_1(x_1 - x_2) + p2(x_1 + x_2) + \alpha \ln|2\cos(2\pi x_1)\cos(2\pi x_2)|)} \, \mathrm{d}x \\ &= e^{4\pi\imath\alpha \ln 2} J(p_1 + p_2, 2\pi\alpha) J(p_1 - p_2, 2\pi\alpha) \neq 0, \end{aligned}$$

because $\alpha \neq 0$. If, on the other hand, $p_1 = 0$ then $\widetilde{I}(0, 1) = J(p_2, 2\pi\alpha)$, while $\widetilde{I}(1, 1) = e^{4\pi\imath\alpha \log 2} J(p_2, 2\pi\alpha)^2 \neq \widetilde{I}(0, 1)$. In either case, therefore, $\xi_1 \mapsto \widetilde{I}(\xi_1, 1)$ is not constant. This concludes the proof for $d = 2$.

Finally, to deal with the case $d \geq 3$ note first that the above argument for $d = 2$ really shows that, given any $q \in \mathbb{Z}$ and $\alpha \in \mathbb{R} \setminus \{0\}$, the number $\iota^{(2n)}_{p_2, \pi\alpha}(0)$ is non-zero for infinitely many $n \in \mathbb{N}_0$. (Otherwise, by (A.11), the function $\xi_1 \mapsto I(\xi_1, 1)$ would be constant for all $|p_1|$ sufficiently large, which has just been shown not to be the case.) But then

$$\widetilde{I}(\xi) = \sum_{n=|p_1|+\ldots+|p_{d-1}|}^{\infty} 2^{-2n} \iota^{(2n)}_{p_d, 2\pi\alpha}(0) \sum_{|\nu|=n} \left\{ \prod_{j=1}^{d-1} \frac{\xi_j^{2\nu_j}}{(2\nu_j)!} \binom{2\nu_j}{\nu_j + |p_j|} \right\},$$

is obviously not constant on $\widetilde{E}_{d-1}$. □

Given $p_1, ..., p_d \in \mathbb{Z}$ and $\alpha \in \mathbb{R} \setminus \{0\}$, denote by D_d the set of all ξ for which $\lambda_d \circ T_\xi^{-1}$ coincides with λ_1, i.e.

$$D_d = \{\xi \in \mathbb{R}^d : \lambda_d \circ T_\xi^{-1} = \lambda_1\}.$$

An immediate consequence of Lemma A.10 is

Lemma A.11. For every $p_1, ..., p_d \in \mathbb{Z}$ and $\alpha \in \mathbb{R} \setminus \{0\}$ the set $D_d \cap E_d$ is nowhere dense and has Lebesgue measure zero.

Proof. This is clear from the fact that $D_d \cap E_d \subset \{\xi \in E_d : I(\xi) = 0\}$, and the real-analytic function I is not constant on any component of E_d. Hence the zero-locus of I on E_d is nowhere dense and has Lebesgue measure zero, see e.g. [5, Lem.19]. □

At long last, it is now easy to give the *Proof of Theorem A.4*: Since $D_d \cap E_d$ is nowhere dense, $E_d \setminus D_d \neq \varnothing$, and $\lambda_d \circ T_\xi^{-1} \neq \lambda_d$ for every $\xi \in E_d \setminus D_d$, by the definition of D_d. □

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