Hecke operators on vector-valued modular forms of the Weil representation

by

Aniket Joshi

A thesis submitted in partial fulfillment of the requirements for the degree of

Master of Science

in

Mathematics

Department of Mathematical and Statistical Sciences
University of Alberta

Abstract

Vector-valued modular forms of the Weil representation are an indispensable tool in diverse areas of mathematics such as enumerative geometry of Calabi-Yau manifolds and rational conformal field theory. In this thesis, we study Hecke operators on vector-valued modular forms of the Weil representation ρ_L of a lattice L. We first construct Hecke operators $\mathcal{T}_r$ that map vector-valued modular forms of type ρ_L into vector-valued modular forms of type $\rho_{L(r)}$, where $L(r)$ is the lattice L with rescaled bilinear form $(\cdot,\cdot)_r = r(\cdot,\cdot)$, by lifting standard Hecke operators for scalar-valued modular forms using Siegel theta functions. We also get a set of algebraic relations satisfied by the Hecke operators $\mathcal{T}_r$ similar to the scalar-valued case. In the particular case when $r = n^2$ for some positive integer n, the Weil representation of the rescaled lattice $\rho_{L(n^2)}$ carries a subrepresentation of the Weil representation of the original lattice ρ_L and we can compose $\mathcal{T}_{n^2}$ with a projection operator to construct new Hecke operators $\mathcal{H}_{n^2}$ that map vector-valued modular forms of type ρ_L into vector-valued modular forms of the same type. We study algebraic relations satisfied by the operators $\mathcal{H}_{n^2}$, and compare our operators with the alternative construction of Bruinier and Stein in [BS, St]. The original results of this thesis have appeared as a preprint in [BCJ].

Acknowledgements

I would like to express my sincere gratitude to my supervisors Dr. Vincent Bouchard and Dr. Thomas Creutzig for their patience, encouragement and immense knowledge without which this thesis would not have been possible. I also extend my gratitude to the faculty and staff of the University of Alberta directly and indirectly involved in my studies. Finally, I would like to thank my parents and my brother for all their love and support.

Contents

Chapter 1

Introduction

The subject of modular forms first arose as a way to study elliptic curves and other arithmetic objects but now spans almost all of pure mathematics with connections to number theory, algebraic geometry, string theory and representation theory. In particular, vector-valued modular forms were first proposed as a tool to better understand properties of classical objects such as weakly holomorphic modular forms on non-congruence subgroups and Jacobi forms. Vector-valued forms of the Weil representation of a lattice L are intimately linked to Jacobi forms of lattice index L. Jacobi forms make an appearance in diverse areas of mathematical physics such as vertex operator algebras, mirror symmetry and enumerative geometry of Calabi-Yau manifolds, for example as denominator identities of BKM Lie superalgebras. Hence, vector-valued modular forms offer ways to probe into these deep and inter-connected areas. In addition vector-valued modular forms have also been a catalyst in the development of other modular objects such as modular forms for orthogonal groups through the work of Borcherds and harmonic maass and mock modular forms through the work of Kudla-Milson, Bruinier-Ono and others (For example see [BrO1] and [BrO]).

An important tool in the theory of classical modular and Jacobi forms are Hecke operators that preserve spaces of modular forms and cusp forms of a certain weight. For example, they form a set of commuting Hermitian operators on the space of cusp forms of the full modular groups with respect to the Peterson inner product and hence give us a method to compute basis of spaces of cusp forms. Hecke operators also have geometric interpretation in many cases and are a link between the geometry and arithmetic of modular forms. For example, Hecke operators can be thought of as a sum over isogenies of elliptic curves in the case of classical modular forms. The main goal of this thesis is to construct Hecke operators for vector-valued modular forms of the Weil representation using Siegel theta functions and give their algebraic properties. This is then compared to Hecke operators for vector-valued modular forms appearing previously in [BS]. In [BS] the authors first define an extension of the Weil representation to a suitable double coset of the metaplectic group. However, our construction is simpler and more general and could have broader applications to other fields such as enumerative geometry and rational conformal field theories (RCFTS). We then discuss a few applications in Chapter 4 to other areas of mathematics including rational vertex operator algebras (VOAs) and Donaldson-Thomas invariants. The construction of Hecke operators in section 3.2 and the comparison of section 3.4 form the majority of [BCJ] of which I am one of the co-authors. We now briefly summarize the original results that will be presented in this thesis and that appear in [BCJ].

1.1 Summary of original results

We summarize below some of the original results obtained in sections 3.4 and 3.2 and appearing in [BCJ]. Let L be an even non-degenerate integral lattice of signature (b^+, b^-) with bilinear form $(.,.)$, and $A = L'/L$ be the associated discriminant form with $\mathbb{Q}/\mathbb{Z}$-valued quadratic form $q(\cdot) = \frac{1}{2}(\cdot,\cdot)$. We denote by $L(r)$ the lattice L with the rescaled bilinear form $(\cdot,\cdot)_r = r(\cdot,\cdot)$, and by $A(r) = L'(r)/L(r)$ its associated discriminant form, with $\mathbb{Q}/\mathbb{Z}$-valued rescaled quadratic form $q_r(\cdot) = \frac{1}{2}(\cdot,\cdot)_r$.

1.1.1 Hecke operators for vector-valued modular forms

Let $\{e_\lambda\}_{\lambda \in A}$ be the standard basis for the vector space $\mathbb{C}[A]$, and $\psi(\tau) = \sum_{\lambda\in A} \psi_\lambda(\tau)\, e_\lambda$ be a vector-valued modular of weight $(v, \bar{v})$ for the Weil representation ρ_L associated to L. Our first result is the construction of a Hecke operator $\mathcal{T}_r$ that maps vector-valued modular forms[1] of type ρ_L to vector-valued modular forms of type $\rho_{L(r)}$. This Hecke operator is defined by (Definition 3.2.1):

$$\mathcal{T}_r[\psi](\tau) = r^{w+\bar{w}-1} \sum_{\mu\in A(r)} \left(\sum_{\substack{k,l>0\\ kl=r}} \frac{1}{l^{w+\bar{w}}} \sum_{s=0}^{l-1} \Delta_r(\mu,k) \boldsymbol{e}\left(-\frac{s}{k} q_r(\mu)\right) \psi_{l\mu}\left(\frac{k\tau+s}{l}\right) \right) e_\mu, \tag{1.1}$$

[1] By vector-valued modular forms here and in the rest of the introduction we simply mean $\mathbb{C}[A]$-valued real analytic functions that transform as vector-valued modular forms under the Weil representation, see Definition 2.2.22. As explained in Remark 2.2.23, we do not impose a growth condition, or holomorphicity (meromorphicity) at the cusps, or some condition involving the Laplacian. We also include "Jacobi-like" variables in the definition – see Remark 2.2.24.

where $(w, \bar{w}) = \left(v + \frac{b^+}{2}, \bar{v} + \frac{b^-}{2}\right)$, $\boldsymbol{e}(x) = \exp(2\pi i x)$, and

$$\Delta_r(\mu, k) = \begin{cases} 1 & \text{if } \mu \in A(l) \subseteq A(r), \\ 0 & \text{otherwise.} \end{cases} \tag{1.2}$$

The idea behind the construction is to pair the components of the vector-valued modular form $\psi(\tau)$ with the components of Siegel theta functions to construct a scalar-valued modular form, and then apply the standard Hecke operator for scalar-valued modular forms to define our Hecke operator on vector-valued modular forms appropriately. More precisely, let us define an inner product $\langle e_\lambda, e_\gamma \rangle = \delta_{\lambda\gamma}$ where $\delta_{\lambda\gamma}$ is the Kronecker delta. We then prove that (Theorem 3.2.2)

$$T_r \left[\langle \psi, \Theta_L \rangle \right] (\tau, \alpha, \beta) = \left\langle \mathcal{T}_r[\psi], \Theta_{L(r)} \right\rangle (\tau, \alpha, \beta), \tag{1.3}$$

where $\Theta_L(\tau, \alpha, \beta)$ is the $\mathbb{C}[A]$-valued Siegel theta function of the lattice L, T_r are the usual Hecke operators for scalar-valued modular forms and $\langle ., . \rangle$ is the inner product on $\mathbb{C}[A]$ defined by,

$$\left\langle \sum_{\lambda \in A} f_\lambda e_\lambda, \sum_{\delta \in A} g_\delta e_\delta \right\rangle = \sum_{\gamma \in A} f_\gamma \overline{g}_\gamma. \tag{1.4}$$

From this relation it follows that, indeed, $\mathcal{T}_r[\psi](\tau)$ is a vector-valued modular form of type $\rho_{L(r)}$ and weight $(v, \bar{v})$.

Our next step is to study algebraic relations satisfied by the operators $\mathcal{T}_r$. To this end we define a scaling operator $\mathcal{U}_{n^2}$ on vector-valued modular forms of type ρ_L (Definition 3.2.4):

$$\mathcal{U}_{n^2}[\psi](\tau, \alpha, \beta) = \sum_{\nu \in A(n^2)} \Delta_{n^2}(\nu, n) \psi_{n\nu}(\tau) e_\nu. \tag{1.5}$$

This is an appropriate scaling operator since (Lemma 3.2.5):

$$U_{n^2}\left[\langle\psi,\Theta_L\rangle\right](\tau,\alpha,\beta)=\left\langle\mathcal{U}_{n^2}[\psi],\Theta_{L(n^2)}\right\rangle(\tau,\alpha,\beta), \tag{1.6}$$

where $U_{n^2}[f](\tau,\alpha,\beta) = f(\tau,n\alpha,n\beta)$ is the standard scaling operator for scalar-valued modular forms. Then we show that (Theorem 3.2.7):

- For m and n such that $\gcd(m,n)=1$,

$$\mathcal{T}_m\circ\mathcal{T}_n=\mathcal{T}_{mn}; \tag{1.7}$$

- For $l\geq 2$ and p prime,

$$\mathcal{T}_{p^l}=\mathcal{T}_p\circ\mathcal{T}_{p^{l-1}}-p^{w+\bar{w}-1}\mathcal{U}_{p^2}\circ\mathcal{T}_{p^{l-2}}. \tag{1.8}$$

Those properties are analogous to the algebraic relations satisfied by the scalar-valued Hecke operator T_r.

1.1.2 The special case when $r=n^2$

We then focus on the special case when $r=n^2$ for some integer n. In this case, we show (Lemma 3.2.8) that ρ_L is a sub-representation of the Weil representation $\rho_{L(n^2)}$ for the rescaled lattice $L(n^2)$. This allows us to define a projection operator $\mathcal{P}_{n^2}$ (Definition 3.2.9), which takes vector-valued modular forms of type $\rho_{L(n^2)}$ into vector-valued modular forms of type ρ_L of the same weight. This projection operator acts as a left inverse of the scaling operator (Lemma 3.2.11):

$$\mathcal{P}_{n^2}\circ\mathcal{U}_{n^2}=\mathcal{I}. \tag{1.9}$$

This projection operator allows us to define new Hecke operators

$\mathcal{H}_{n^2}$ which map vector-valued modular forms of type ρ_L into vector-valued modular forms of the same type and weight (Definition 3.2.12):

$$\mathcal{H}_{n^2} = \mathcal{P}_{n^2} \circ \mathcal{T}_{n^2}. \tag{1.10}$$

The explicit expression for $\mathcal{H}_{n^2}$ is given by (Lemma 3.2.13):

$$\mathcal{H}_{n^2}[\psi](\tau) = n^{2(v+\bar{v}-1)}$$
$$\times \sum_{\lambda \in A} \left(\sum_{\substack{\gamma \in A(n^2) \\ n\gamma = \lambda}} \sum_{\substack{k,l>0 \\ kl=n^2}} \frac{1}{l^{v+\bar{v}+\frac{1}{2}\dim L}} \sum_{s=0}^{l-1} \Delta_{n^2}(\gamma, n)\Delta_{n^2}(\gamma, k)\boldsymbol{e}\left(-\frac{s}{k} q_{n^2}(\gamma)\right) \psi_{l\gamma}\left(\frac{k\tau + s}{l}\right)\right) e_\lambda. \tag{1.11}$$

As for $\mathcal{T}_{n^2}$, we study algebraic relations satisfied by the $\mathcal{H}_{n^2}$. We obtain (Theorem 3.2.17):

- For m and n such that $\gcd(m, n) = 1$,

$$\mathcal{H}_{m^2} \circ \mathcal{H}_{n^2} = \mathcal{H}_{m^2 n^2}; \tag{1.12}$$

- Tor $l \geq 2$ and p prime,

$$\mathcal{H}_{p^{2l}} = \mathcal{P}_{p^{2l-2}} \circ \mathcal{H}_{p^2} \circ \mathcal{H}_{p^{2l-2}} \circ \mathcal{U}_{p^{2l-2}} - p^{w+\bar{w}-1}\mathcal{H}_{p^{2l-2}} - p^{2(w+\bar{w}-1)}\mathcal{H}_{p^{2l-4}}. \tag{1.13}$$

The recursion relation is slightly different from the standard one for scalar-valued Hecke operators. This is due to two reasons: first, $\mathcal{H}_r$ is only defined when $r = n^2$, and second, the projection operator $\mathcal{P}_{n^2}$ and Hecke operator $\mathcal{T}_{m^2}$ only commute when m and n are coprime (Lemma 3.2.16).

1.1.3 Comparision to Bruinier-Stein

Hecke operators that map vector-valued modular forms of type ρ_L into vector-valued modular forms of the same type and weight were also constructed by Bruinier and Stein in [BS, St]. The approach however is quite different. In [BS] the authors first construct Hecke operators $T_{m^2}^{(BS)}$ where m is a positive integer that is coprime with the level N of the lattice L. They do so by extending the Weil representation of $Mp_2(\mathbb{Z})$ to some appropriate subgroup of $\widetilde{\mathrm{GL}}_2^+(\mathbb{Q})$. They then extend their construction to Hecke operators $T_{m^2}^{(BS)}$ for all positive integers m. However, explicit formulae are only given when m is coprime with the level of the lattice. Stein generalizes this in [St] by providing the explicit action of their Hecke operators $T_{p^{2l}}^{(BS)}$ for any odd prime p and positive number l.

Given that the construction of Bruinier and Stein is *a priori* quite different from ours, it is interesting to compare the two and investigate whether the resulting Hecke operators $T_{p^{2l}}^{(BS)}$ and $\mathcal{H}_{p^{2l}}$ are the same. In Section 3.4, we prove a precise match between our Hecke operators and the Bruinier-Stein Hecke operators. There is a mistake in the statement and proof of Theorem 5.2 of [St] that provides explicit formulae for their extension of the Weil representation. However we redid the computations in order to address some of their errors and compared our results to the formulas thus obtained. These results are presented as Proposition 3.3.3. We get a precise match between the two constructions if we use the formulae derived in this proposition.

We note however that our Hecke construction is fairly straightforward and more general. For instance, our Hecke operators are constructed for any r. But perhaps more interestingly, our construction should generalize beyond the Weil representation (for example to sub-representations of the Weil representation that arise from rational conformal field theories): it should apply whenever one has a pairing of two vector-valued modular forms that yield a scalar-valued modular form, to which one can apply standard Hecke operators. The key is to choose one of the two vector-valued modular forms carefully so that we know how it transforms under the action of $GL_2^+(\mathbb{Q})$. In the case of the Weil representation, this was accomplished by using Siegel theta functions for the pairing.

1.2 Overview of the Thesis

In Chapter 1 we give a brief summary of the original results on Hecke operators on vector-valued modular forms of the Weil representation. These results will be worked out in detail in sections 3.2 and 3.4 and form much of the contents of [BCJ]. In Chapter 2 we give a brief survey of background on some modular objects including modular forms, Jacobi forms, vector-valued modular forms and Siegel theta functions and these definitions and facts will be used in the rest of the thesis. In Chapter 3 we first introduce Hecke operators for classical modular and Jacobi forms as appearing widely in classical literature. We then give a construction of Hecke operators for vector-valued modular forms by lifting Hecke operators on classical modular forms using the Siegel theta functions of Borcherds and study their algebraic properties. This new construction is then compared to the Hecke operators of Bruinier-Stein. In Chapter 4 we discuss some applications of vector-valued modular forms and Hecke operators to contemporary problems. In particular we survey the theory of Borcherds' products, Donaldson-Thomas invariants and rational VOAs. Finally we comment on Harvey and Wu's work in [HW] where they define Hecke operators acting on vector-valued modular forms that are characters of rational conformal field theories.

Chapter 2

Scalar and vector-valued modular forms

In this chapter, we will give a brief survey of the various modular objects mentioned in Chapter 1. In particular we will first define the classical modular forms for $\mathrm{SL}_2(\mathbb{Z})$ and its various congruence subgroups. This will be followed by an overview of the theory of discriminant forms and vector-valued modular forms of the Weil representation. These are intimately connected to the theory of Jacobi forms and this connection will be discussed. Finally, we will introduce the Siegel theta functions of Borcherds. These are real analytic functions that transform like vector-valued modular forms and will be essential to the development of the rest of the thesis.

2.1 Modular forms

2.1.1 Modular forms and their q- expansion

Modular forms are (typically holomorphic) functions on the upper half complex plane that satisfy certain transformation properties with respect to an action of the group $\mathrm{SL}_2(\mathbb{Z})$ or its subgroups alongside a growth condition. However, their major utility lie in the properties of their Fourier coefficients. Modular forms typically arise as generating functions of counting problems in number theory, string theory and enumerative geometry or as characters of representations of vertex operator algebras. The main references used for this section are [B–Z], [Stw] and [J].

Let $\mathbb{H} = \{\tau \in \mathbb{C} | \mathrm{Im}(\tau) > 0\}$ denote the upper half plane. The group $\mathrm{SL}_2(\mathbb{R}) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : a, b, c, d \in \mathbb{R}, ad - bc = 1\}$ has a left group action on $\mathbb{H}$ called the Mobius transformation. For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ it is given by,

$$\gamma : \tau \to \gamma\tau = \frac{a\tau + b}{c\tau + d}. \tag{2.1}$$

Remark 2.1.1. The group action (2.1) is a biholomorphic automorphism of $\mathbb{H}$. In fact it can be shown that the projective linear group $\mathrm{SL}_2(\mathbb{R})/\{-I_2\}$ is isomorphic to the group of biholomorphic automorphisms of $\mathbb{H}$ denoted by $\mathrm{Aut}(\mathbb{H})$.

It can be seen through a quick computation that the above group action restricts to a group action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$.

Definition 2.1.2. The group $\Gamma_1 := \mathrm{SL}_2(\mathrm{Z})$ is called the modular group. It is generated by $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ modulo the relations $S^2 = 1, (ST)^3 = -1$.

The group action (2.1) be extended to an action on $\mathbb{H}^* := \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$ by setting $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\infty = \frac{a}{c}$. The set of points $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$ are called the cusps. It can be show that all the cusps are Γ_1 equivalent to ∞. However for subgroups Γ of Γ_1 the cusps don't all lie in the same orbit, but the number of orbits are bounded by the index of the subgroup. Thus for finite index subgroups (in particular congruence subgroups) the cusps lie in a finite number of Γ-orbits.

A fundamental domain is an open set $\mathfrak{F} \subset \mathbb{H}$ such that no two points lie on the same orbit of $\Gamma \leq \Gamma_1$, and the closure $\bar{\mathfrak{F}}$ has at least one element from all the orbits on $\mathfrak{H}$. It can be shown that (See [DS]) $\mathfrak{F}_1 = \{z \in \mathbb{H} | \quad |z| > 1, |\mathrm{Re}(z)| < \frac{1}{2}\}$ is a fundamental domain for the modular group Γ_1.

We first define the Petersson slash operator in a more general form than what is required at the moment for convenience.

Definition 2.1.3. Let f be function on $\mathbb{H}$ and $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{R})$. The slash operator of weight (k, k') is defined by,

$$f|_{(k,k')}\alpha(\tau) = \det(\alpha)^{\frac{k+k'}{2}}(c\tau + d)^{-k}(c\bar{\tau} + d)^{-k'} f(\frac{a\tau + b}{c\tau + d}). \tag{2.2}$$

Remark 2.1.4. We will usually deal with slash operators of weight $(k, 0)$ and in this case we omit the second argument in the notation entirely.

Definition 2.1.5. A *meromorphic modular form* of weight k (and *level* 1) is a function $f : \mathbb{H} \to \mathbb{C}$ satisfying the following conditions:

1. It transforms as

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau) \qquad \forall \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1 \tag{2.3}$$

where k is an integer or in terms of the slash operator

$$f|_k\alpha(\tau) = f(\tau) \qquad \forall\alpha \in \Gamma_1. \tag{2.4}$$

In particular $f(\tau)$ is periodic with period 1.

2. $f(\tau)$ is meromorphic in $\mathbb{H}$ and has a Fourier expansion

$$f(\tau) = \sum_{n=-m}^{\infty} a(n)q^n \qquad (q := e^{2\pi i\tau}) \tag{2.5}$$

for a positive integer m. If this condition is true, $f(\tau)$ is said to be meromorphic at $\{\infty\}$.

We are typically interested in examining functions satisfying stricter conditions such as holomorphicity and thus we introduce the following notation:

1. If $f(\tau)$ is holomorphic on $\mathbb{H}\cup\{\infty\}$ then $f(\tau)$ is bounded as $\mathrm{Im}(\tau) \to \infty$ so that the Fourier expansion is truncated to

$$f(\tau) = \sum_{n=0}^{\infty} a(n)q^n \qquad (q := e^{2\pi i\tau}), \tag{2.6}$$

then $f(\tau)$ is called a holomorphic modular form. We denote the $\mathbb{C}$-vector space of holomorphic modular forms by $M_k(\Gamma_1)$.

2. Holomorphic modular forms that satisfy $a(0) = 0$ are called cusp forms and we denote this subspace by $S_k(\Gamma_1)$.

3. If $f(\tau)$ is holomorphic in the open upper half plane $\mathbb{H}$ but the growth condition is weakened to $f(\tau) = O(q^{-N})$, then $f(\tau)$ has a q-expansion with $a(n) = 0$ for $n < -N$. Such functions are called weakly holomorphic modular forms and denoted by $M_k^!(\Gamma_1)$.

Remark 2.1.6. There is a weaker notion of modular forms and functions with character. Let χ be a character of the modular group Γ_1. Then $f(\tau)$ is a modular form of integer weight k and character χ with respect to Γ_1 if,

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = \chi\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right)(c\tau+d)^k f(\tau) \qquad \forall \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z}) \tag{2.7}$$

and we can impose conditions on the Fourier expansion as above.

Two important examples of modular forms are the Eisenstein series, and the discriminant function, which we describe below.

Example 2.1.7. The Eisenstein series $E_{2k}(\tau)$ given by,

$$E_{2k}(\tau) = \frac{1}{2} \sum_{\substack{m,n\in\mathbb{Z} \\ \gcd(m,n)=1}} \frac{1}{(m\tau+n)^{2k}} \tag{2.8}$$

is a cusp form of weight $2k$. This series converges absolutely only for $k \geq 2$. However $E_2(\tau)$ is not a modular form and transforms as

$$(c\tau+d)^{-2} E_2\left(\frac{a\tau+b}{c\tau+d}\right) = E_2(\tau) + \frac{6}{\pi}\frac{c}{c\tau+d}$$

where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and such functions are said to quasi-modular.

An alternate way to define the Eisenstein series is:

$$G_{2k}(\tau) = \sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)\neq(0,0)}} \frac{1}{(m\tau+n)^{2k}} \tag{2.9}$$

and the two are related by $G_{2k}(\tau) = 2\zeta(2k)E_{2k}(\tau)$, where ζ is the Riemann zeta function evaluated at k. The Eisenstein series G_{2k} generates the divisor function $\sigma_{2k-1}(n)$ and has the following Fourier expansion:

$$G_{2k}(z) = 2\zeta(2k) + \frac{2(2\pi i)^{2k}}{(2k-1)!}\sum_{n=1}^{\infty}\sigma_{2k-1}(n)\cdot q^n, \tag{2.10}$$

where

$$\sigma_l(n) := \sum_{0<d|n} d^l. \tag{2.11}$$

Example 2.1.8. The generating function of Ramanujan's τ-function (called the modular discriminant) $\Delta(\tau)$ is a modular form of weight 12 and has a product formula given by

$$\Delta(\tau) = \sum_{n\geq 1}\tau(n)q^n = q\prod_{n=-1}^{\infty}(1-q^n)^{24} \tag{2.12}$$

The 24-th root of $\Delta(\tau)$ called the Dedekind eta function is a modular form of weight 1/2 with a multiplier $\zeta_{24} = e^{\frac{\pi i}{12}}$ the primitive 24-th root of unity,

$$\eta(\tau) := \Delta(\tau)^{\frac{1}{24}}. \tag{2.13}$$

It satisfies the transformation law,

$$\eta(\tau+1) = \zeta_{24}\eta(\tau) \tag{2.14}$$

$$\eta(-\frac{1}{\tau}) = \sqrt{-i\tau}\eta(\tau) \tag{2.15}$$

Modular forms are very powerful tools for doing computations because the spaces $M_k(\Gamma_1)$ have small dimensions and are generated by the Eisenstein series of weight 4 and 6. Hence, it is often possible to compute modular forms satisfying certain constraints easily. The following two theorems are taken from [B–Z].

Theorem 2.1.9. *The dimension of $M_k(\Gamma_1)$ is 0 for $k<0$ or k odd, and for* *even* $k \geq 0$

$$\dim M_k(\Gamma_1) \leq \begin{cases} \lfloor k/12 \rfloor + 1 & \text{if } k \not\equiv 2 \ (mod\ 12) \\ \lfloor k/12 \rfloor & \text{if } k \equiv 2 \ (mod\ 12) \end{cases}$$

Denote $M_*(\Gamma_1) = \bigoplus_k M_k(\Gamma_1)$ the space of modular forms of all integral weights.

Theorem 2.1.10. *The ring $M_*(\Gamma_1)$ is freely generated by the modular forms E_4 and E_6, i.e. any modular form of weight k can be written as sum of monomials $E_4^\alpha E_6^\beta$ with $4\alpha + 6\beta = k$. In addition we also have, $M_k(\Gamma_1) = \mathbb{C} \cdot E_k \oplus S_k(\Gamma_1)$ and $S_k(\Gamma) = \Delta \cdot M_{k-12}(\Gamma_1)$, where $S_k(\Gamma_1)$ is the space of cusp forms of weight k for the modular group.*

2.1.2 Modular forms for congruence subgroups

Modular forms for congruence subgroups are a natural generalization of the modular forms for $\mathrm{SL}_2(\mathbb{Z})$ discussed in the last section. Moreover these are also very important in arithmetic geometry. Riemann surfaces constructed from $\Gamma\backslash\mathbb{H}$ or their compactifications are called modular curves and modular curves for congruence subgroups have algebraic interpretations as moduli spaces of certain elliptic curves over various rational number fields. For example the moduli space of elliptic curves over $\mathbb{C}$ is isomorphic to the fundamental domain $X(1) = \mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}$ and its function field is generated by the j-invariant of 4.1.

Definition 2.1.11. A congruence subgroup Γ is any subgroup of $SL(2, \mathbb{Z})$ that contains

$$\Gamma(N) := \ker(f : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}))$$

for some positive integer N called the level of Γ and the map f is just entrywise reduction modulo N. $\Gamma(N)$ is called the principal congruence subgroup of level N.

Example 2.1.12. Two important examples of congruence subgroups are,

$$\Gamma_1(N) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}| \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right) \mod N\}$$

$$\Gamma_0(N) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}| \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \mod N\},$$

where $*$ stands for any entry.

Modular forms for congruence groups are defined in the same way as for the modular group.

Definition 2.1.13. Let $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup. A (meromorphic) modular form for Γ integer weight k for a congruence subgroup Γ is a function $f : \mathbb{H} \to \mathbb{C}$ satisfying the following conditions:

1. $$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau) \qquad \forall \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma. \tag{2.16}$$

2. It is meromorphic in $\mathbb{H}$ and at the cusps $\mathbb{P}^1(\mathbb{Q})$. Meromorphicity at $\{\infty\}$ means that $f(\tau)$ has a Fourier expansion

$$f(\tau) = \sum_{n=-m}^{\infty} a_n q^{\frac{n}{h}}, \quad q := e^{2\pi i \tau}, \tag{2.17}$$

for some positive integers m and h.

If $f(\tau)$ is holomorphic in $\mathbb{H}$ it is called a weakly modular form, and if it is holomorphic in $\mathbb{H}$ and at all the cusps it is called a holomorphic modular form.

Remark 2.1.14. We clarify here what it means to be holomorphic/meromorphic at the cusps. Let $\alpha \in \mathbb{P}^1 \cup \{\infty\}$ be a cusp. As Γ_1 acts transitively on

$\mathbb{P}^1 \cup \{\infty\}$ there exists a $\gamma \in \Gamma_1$ such that $\gamma(\alpha) = \infty$. We say that f is holomorphic/meromorphic at a cusp α if the modular function $(c\tau + d)^{-k} f(\gamma(\tau))$ is holomorphic/meromorphic at ∞. In fact a weakly modular function f is holomorphic/meromorphic at all the cups if it is holomorphic/meromorphic at representative elements of each of the finite number of Γ-orbits.

Example 2.1.15. If L is an even positive definite lattice of even rank m and level N with a bilinear form $(.)$ then the theta function,

$$\theta_L(q) = \sum_{\lambda \in L} q^{\frac{(\lambda,\lambda)}{2}} = \sum_{n=0}^{\infty} R(n) q^n \tag{2.18}$$

is a modular form of weight $m/2$ for $\Gamma_0(N)$ and character $\chi : \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \rightarrow \left(\frac{(-1)^{\frac{m}{2}} \det(G)}{d}\right)$ where $(\frac{\cdot}{\cdot})$ is an extension of the Legendre symbol called the Kronecker symbol (See [Str], [CS]) and $\det(G)$ is the determinant of the Gram matrix and will be defined in 2.2 (See [O, Chapter 6]). Here $R(n)$ denotes the number of vectors v in L such that $Q(v) = n$. Theta functions are some of the most important examples of modular forms and will be essential to the rest of the thesis. The second equality is just the statement that $\theta_L(q)$ is the generating function of the number of representations of an integer by $Q(.)$. It is useful to note the following transformation proved in [CS] as an application of the Poisson resummation formula,

$$\theta_L(-\frac{1}{\tau}) = \frac{\sqrt{-i\tau}}{\sqrt{L'/L}} \theta_{L'}(\tau). \tag{2.19}$$

Example 2.1.16. A special case of Example 2.1.15 is the theta function given by

$$\theta(\tau, k) = (\sum_{n=-\infty}^{\infty} q^{n^2})^k = \sum_{n=0} r(n,k) q^n, \tag{2.20}$$

where $r(n,k)$ is the number of ways of writing n as a sum of k squares. It can be shown that

$$\theta(\frac{\tau}{4\tau+1}, k) = (4\tau+1)^{k/2}\theta(\tau, k). \tag{2.21}$$

Thus $\theta(\tau, 4k)$ is a modular form of weight $2k$ and trivial character for the congruence subgroup $\Gamma_0(4)$.

We make a couple of remarks here before proceeding to the next section. Firstly, modular forms for congruence subgroups have dimension formulae and and their algebra is generated by a small set of generators similar to Theorem 2.1.9 and 2.1.10 (See for example [DS, Theorem 3.5.1, Theorem 3.6.1]). Secondly, modular forms for discrete subgroups of $SL_2(\mathbb{Z})$ that are not congruent subgroups exist but haven't been studied as much due to their weaker links to arithmetic and geometry.

2.2 Vector-valued modular forms

The study of vector-valued modular forms was initiated by Selberg as a way to study growth of Fourier coefficients of scalar-valued modular forms for congruence and non-congruence subgroups (See for example [S]). In this section, we define vector-valued modular forms for the Weil representation in particular and then study the connections of these with classical objects such as modular forms for congruence subgroups and the Jacobi forms of [EZ]. The references used for this section are [Sch], [Ni], [Sch1], [Sch2], [Sch3], [Sk], [Str], [CS] and the interested reader should refer to these for further details.

2.2.1 Lattices and discriminant forms

In this section we introduce the two closely related notions of a quadratic module and a discriminant form. We also briefly explain, how a quadratic module can be decomposed into its Jordan components.

Definition 2.2.1. A finite quadratic module is a finite abelian group A equipped with a $\mathbb{Q}/\mathbb{Z}$-valued quadratic form $q(.) : A \to \mathbb{Q}/\mathbb{Z}$ so that $q(rx) = r^2 q(x)$ for any $r \in \mathbb{Z}, x \in A$. The associated bilinear form is given by,

$$(.,.) : A \times A \to \mathbb{Q}/\mathbb{Z} : (x, y) \to q(x + y) - q(x) - q(y). \tag{2.22}$$

Let A be generated freely by $(e_1, e_2, \ldots, e_m)$, then the symmetric matrix $G = ((e_i, e_j))$ is called the Gram matrix. The bilinear form is called non-degenerate if $\det(G) \neq 0$ and we will assume this to be the case from now. The quadratic form $q(.,.)$ is said to be of type (b^+, b^-) if G has b^+ and b^- positive and negative eigenvalues respectively.

It is important to define the notion of isomorphic and indecomposible quadratic modules in order to classify them.

Definition 2.2.2. Two quadratic modules $(A, q_A(.))$ and $(B, q_B(.))$ are isomorphic if and only if there exists a group isomorphism $\phi : A \to B$ such that $q_A(.) = q_B \circ \phi(.)$. A is said to be indecomposable if it cannot be written as an orthogonal direct sum of two smaller quadratic modules.

From the fundamental theorem of abelian groups it follows that every quadratic module A is an orthogonal sum of its p-components $A_p \cong A \otimes_{\mathbb{Z}} \mathbb{Z}_p$. Nikulin classified all the indecomposible quadratic modules which we list below. In addition he also showed that every finite quadratic module (and hence its p-components) can be written as a direct sum of the indecomposable modules listed below. Such a orthogonal decomposition is called the Jordan decomposition.

Theorem 2.2.3 (Proposition 1.8.1, [Ni])**.** *Every indecomposible finite quadratic module is isomorphic to one of the following modules:*

$$\mathcal{A}^t_{p^k} = \left(\mathbb{Z}/p^k\mathbb{Z}, \frac{tx^2}{p^k}\right) \quad p > 2 \tag{2.23}$$

$$\mathcal{A}^t_{2^k} = \left(\mathbb{Z}/2^k\mathbb{Z}, \frac{tx^2}{2^{k+1}}\right) \tag{2.24}$$

$$\mathcal{B}_{2^k} = \left(\mathbb{Z}/2^k\mathbb{Z} \oplus \mathbb{Z}/2^k\mathbb{Z}, \frac{x^2 + 2xy + y^2}{2^k}\right) \tag{2.25}$$

$$\mathcal{C}_{2^k} = \left(\mathbb{Z}/2^k\mathbb{Z} \oplus \mathbb{Z}/2^k\mathbb{Z}, \frac{xy}{2^k}\right) \tag{2.26}$$

where p is a prime and t an integer not dividing p.

Remark 2.2.4. Not all the quadratic modules defined by $\mathcal{A}^t_{p^k}, \mathcal{A}^t_{2^k}$ above for different values of t are non-isomorphic. Isomorphic indecomposible modules can be classified by comparing their "Jacobi-Legendre" symbol. See [Ni], [CS], [Sch] for more details regarding this.

An important way of getting quadratic modules is through the discriminant group of lattices.

Definition 2.2.5. A lattice is a free, finitely-generated $\mathbb{Z}$-module equipped with a symmetric bilinear form $(\cdot, \cdot) : L \times L \to \mathbb{Z}$. We denote the dimension or rank of the lattice L by $\dim L$. It is *integral* if $(x, x) \in \mathbb{Z}$ for all $x \in L$. In particular it is also said to be *even* if $(x, x) \in 2\mathbb{Z}$ for all $x \in L$. We denote by (b^+, b^-) the signature of L, and let

$$\mathrm{sgn}(L) = b^+ - b^-, \qquad \dim(L) = b^+ + b^-. \tag{2.27}$$

Remark 2.2.6. From now on we will assume that all our lattices L are even and non-degenerate.

We can extend $(\cdot,\cdot)$ $\mathbb{Q}$-linearly to $L \otimes_{\mathbb{Z}} \mathbb{Q}$. We then define the *dual lattice* L' of L as

$$L' = \{x \in L \otimes_{\mathbb{Z}} \mathbb{Q} \mid (x,y) \in \mathbb{Z} \text{ for all } y \in L\}. \tag{2.28}$$

Since L is integral we have $L \subseteq L'$. The *discriminant group* of L is the finite abelian group $A = L'/L$. If $L' = L$ then $A = \{0\}$ and L is said to be unimodular or self-dual. When L is even we define the *discriminant form* of L as A equipped with the the $\mathbb{Q}/\mathbb{Z}$-valued quadratic form

$$q : A \to \mathbb{Q}/\mathbb{Z} \tag{2.29}$$

$$x + L \mapsto \frac{1}{2}(x,x) \bmod \mathbb{Z}. \tag{2.30}$$

The associated bilinear form $A \times A \to \mathbb{Q}/\mathbb{Z}$ is of course $(x+L, y+L) \mapsto (x,y) \bmod \mathbb{Z}$. The *level* N of L is the smallest positive integer N such that $N(x,y) \in \mathbb{Z}$ for all $x, y \in L'$. The discriminant group has size $|L'/L| = \det(G)$ where G is the Gram matrix of pairing for a chosen basis.

Example 2.2.7.

1. The rank 2 lattice $(\mathbb{Z}^2, (.,.))$ with the Gram matrix $\left(\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right)$ is the A_2 root lattice and has type $(2,0)$ and level 3. The discriminant group $A_2'/A_2 = \mathbb{Z}/3\mathbb{Z}$.
2. The Leech lattice Λ_{24} is the unique unimodular (i.e. self-dual) even lattice in R^{24} such that $(x,x) > 2$ for all $x \in \Lambda_{24}$.

Clearly by definition the pair $(L'/L, q(.))$ is a finite quadratic module. In fact it was shown in [Ni] that every finite quadratic module can be obtained as a discriminant form of an even non-degenerate lattice.

Theorem 2.2.8 (Theorem 1.3.2 of [Ni]). *Let $(A, q(.))$ be a finite quadratic module. Then there is an even lattice $(L, q(.))$ such that $L'/L \cong A$ as quadratic modules.*

Remark 2.2.9. It is important to note that this correspondence is not bijective. Multiple lattices of a given type can give rise to the same quadratic module. For a quadratic module A, the set of all even lattices of type (b^+, b^-) such that $L'/L \cong A$ is denoted by $II_{b^+,b^-}(A)$ and are said to be of the same genus. The above theorem states that for every finite quadratic module A, $II_{b^+,b^-}(A)$ is non-empty.

Example 2.2.10. The discriminant form of the lattice $(\mathbb{Z}_p, \frac{p^k x^2}{4t})$ is isomorphic to the indecomposable module $\mathcal{A}^t_{p^k}$ defined in Theorem 2.2.3. Similarly $\left(\mathbb{Z}_2, \frac{2^{k-1}x^2}{4t}\right)$ gives rise to $\mathcal{A}^t_{2^k}$ and the discriminant forms associated to $\mathcal{B}_{2^k}$ and $\mathcal{C}_{2^k}$ are $\left(\mathbb{Z}_2 \times \mathbb{Z}_2, 2^k(x^2 + xy + y^2)\right)$ and $\left(\mathbb{Z}_2 \times \mathbb{Z}_2, 2^k xy\right)$ respectively.

Let r be a positive number. We denote by $L(r)$ the lattice L with rescaled bilinear form $(\cdot, \cdot)_r := r(\cdot, \cdot)$. Its dual lattice $L(r)'$ is defined as usual by

$$L(r)' = \{x \in L \otimes \mathbb{Q} \mid (x, y)_r \in \mathbb{Z} \text{ for all } y \in L\}. \tag{2.31}$$

We remark here that by definition, $L(r)' = \frac{1}{r}L'$, and thus $L' \subseteq L(r)'$. We denote the rescaled discriminant group by $A(r) = L(r)'/L(r) \cong \frac{1}{r}L'/L$. If L is even, then $L(r)$ is also even, and we can make $A(r)$ into a discriminant form as above:

$$q_r : A(r) \to \mathbb{Q}/\mathbb{Z}, \qquad x + L \mapsto \frac{1}{2}(x, x)_r \bmod \mathbb{Z}.$$

Rescaled lattices will play an important role in section 3.2 in the construction of Hecke operators for vector-valued modular forms.

2.2.2 Vector-valued modular forms for the Weil representation

In this section we define the Weil representation and vector-valued forms for the Weil representation. The Weil representation is a representation of the double cover of $\mathrm{SL}_2(\mathbb{Z})$ acting on the group algebra $\mathbb{C}[A]$ of a discriminant form (or a quadratic module) $(A, q(.))$. The double cover of $\mathrm{SL}_2(\mathbb{R})$ also called the metaplectic group is denoted by $\mathrm{Mp}_2(\mathbb{R})$ and consist of the pairs $(M, \phi(\tau))$ where $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{R})$ and ϕ is a holomorphic function on $\mathbb{H} = \{\tau \in \mathbb{C} | \mathrm{Im}(z) > 0\}$ such that $\phi(\tau)^2 = c\tau + d$. The group multiplication law is given by,

$$(M_1, \phi_1(\tau)) \cdot (M_2, \phi_2(\tau)) = (M_1 M_2, \phi_1(M_2\tau)\phi_2(\tau)).$$

Let p be the covering map $p : \mathrm{Mp}_2(\mathbb{R}) \to SL_2(\mathbb{R})$. Then $\mathrm{Mp}_2(\mathbb{Z}) := p^{-1}(\mathrm{SL}_2(\mathbb{Z}))$ is generated by,

$$T = \left(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), 1\right) \qquad S = \left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \sqrt{\tau}\right),$$

and has the relations $S^2 = (ST)^3 = Z$, where Z is the generator of the center of $Mp_2(\mathbb{Z})$ given by,

$$Z = \left(\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right), i\right).$$

Every discriminant form $(A, q(.))$ defines a unitary representation of the metaplectic group $Mp_2(\mathbb{Z})$ on the group algebra $\mathbb{C}[A]$. Let $\{e_\gamma\}_{\gamma \in A}$ be the standard basis for the vector space $\mathbb{C}[A]$ with $e_\gamma e_\lambda = e_{\gamma+\lambda}$. We define an inner product on $\mathbb{C}[A]$ by

$$\left\langle \sum_{\lambda \in A} f_\lambda e_\lambda, \sum_{\delta \in A} g_\delta e_\delta \right\rangle = \sum_{\lambda \in A} f_\lambda \bar{g}_\lambda. \tag{2.32}$$

that is linear in the first and anti-linear in the second argument.

Definition 2.2.11. The Weil representation ρ on the generators S and T is defined by,

$$\rho_L(T)(e_\lambda) = \boldsymbol{e}(q(\lambda))\, e_\lambda \tag{2.33}$$

$$\rho_L(S)(e_\lambda) = \frac{\boldsymbol{e}(-\mathrm{sgn(L)}/8)}{\sqrt{|A|}} \sum_{\mu \in A} \boldsymbol{e}(-(\lambda,\mu))\, e_\mu. \tag{2.34}$$

Here, we have used the abbreviation $\boldsymbol{e}(x) = \exp(2\pi i x)$ which will be used throughout the thesis. In addition the representation on the center Z is given by,

$$\rho_L(Z) = \mathbf{e}(-\mathrm{sgn}(L)/4)\, e_\lambda. \tag{2.35}$$

Remark 2.2.12. The Weil representation is a unitary representation on $\mathbb{C}[A]$ so that,

$$\langle \rho_L(M)e_\alpha, \rho_L(M)e_\beta \rangle = \langle e_\alpha, e_\beta \rangle\,. \tag{2.36}$$

This can be checked easily for the generators T and S and that is sufficient. This implies that the dual representation ρ_L^* is isomorphic to the complex conjugate representation $\bar{\rho}$. In other words the dual representation ρ_L^* is the Weil representation for the discriminant form $(L'/L, -q(.))$ where $q(.)$ is the quadratic form defining ρ_L.

Another important fact is that the principal congruence subgroup $\Gamma(N)$ is in the kernel of the Weil representation when the signature is even. As a consequence the Weil representation factors through $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ for lattices of level N when the signature is even. For odd signature the Weil representation factors through a double cover of $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$. This fact will be important in Section 3.3 where we survey the construction of the Hecke operators for vector-valued modular forms of Bruinier-Stein.

We are now ready to define vector-valued modular forms of the Weil representation. We first extend the definition of the slash operator in (2.2) to the metaplectic group.

Definition 2.2.13. Let $f(\tau,\bar{\tau})$ be a scalar-valued function on $\mathbb{H}$. The Peterson slash operator of weight $(v,\bar{v})$ defined by

$$f|_{v,\bar{v}}(M,\phi)(\tau,\bar{\tau}) = \phi(\tau)^{-2v}\overline{\phi(\tau)}^{-2\bar{v}} f(M\tau, M\bar{\tau}), \tag{2.37}$$

gives a right action for $(M,\phi(\tau)) \in \mathrm{Mp}_2(\mathbb{Z})$ on scalar-valued functions. Similarly let $\psi(\tau,\bar{\tau}) = \sum_{\lambda\in A}\psi_\lambda(\tau,\bar{\tau})\, e_\lambda$ be a $\mathbb{C}[A]$-valued function on $\mathbb{H}$. We refer to $\psi_\lambda(\tau,\bar{\tau})$ as the components of $\psi(\tau)$. Then the Peterson slash operator on vector-valued functions is,

$$\begin{aligned}\psi|_{v,\bar{v}}(M,\phi)(\tau,\bar{\tau}) &= \phi(\tau)^{-2v}\overline{\phi(\tau)}^{-2\bar{v}} \rho_L^{-1}(M,\phi) f(M\tau, M\bar{\tau}) \\ &= \sum_{\lambda\in A} \psi_\lambda|_{v,\bar{v}}(M,\phi)\rho_L^{-1}(M,\phi)\, e_\lambda.\end{aligned}$$

Definition 2.2.14. For $v \in \mathbb{Z}/2$, a holomorphic function $\psi : \mathbb{H} \to \mathbb{C}[A]$ is called a weakly holomorphic vector-valued modular form of weight k and type ρ_L if,

$$\psi|_v(M,\phi)(\tau) = \psi(\tau) \qquad \forall (M,\phi) \in \mathrm{Mp}_2(\mathbb{Z}) \tag{2.38}$$

and $\psi(\tau)$ has a Fourier expansion of the form

$$\psi(\tau) = \sum_{\lambda\in A} \boldsymbol{e}(-\tau q(\lambda)) \sum_{n=-m_\lambda}^{\infty} c_\lambda(n)\mathbf{e}(n\tau)\, e_\lambda$$

for some non-negative integers m_λ. Note that $\boldsymbol{e}(\tau q(\lambda))\psi_\lambda(\tau)$ is periodic with period 1. In particular, $\psi(\tau)$ is called holomorphic if $c_\lambda(n) = 0$ for $n < 0$ and all $\lambda \in A$.

Remark 2.2.15. This definition can be easily generalized to vector-valued modular forms of any representation. Let $\rho : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}_n(\mathbb{C})$ be a representation of the modular group such that $\rho(T)$ is a diagonal matrix. Then a holomorphic function $\psi : \mathbb{H} \to \mathbb{C}^n$ is called a weakly holomorphic vector-valued modular form of weight k and type ρ if,

$$\psi|_k M(\tau) = \psi(\tau) \qquad \forall M \in \mathrm{SL}_2(\mathbb{Z}) \tag{2.39}$$

and each component has a Fourier expansion,

$$\psi_i(\tau) = q^{h_i} \sum_{n=-m_i}^{\infty} a_{ni} q^n \quad (q := e^{2\pi i \tau}) \tag{2.40}$$

for some real numbers h_i and positive integers m_i. We will only be concerned with vector-valued modular forms of the Weil representation in the rest of the thesis except for a brief mention in section 4.3.3.

Example 2.2.16. Let L be a positive definite even lattice of even rank $2k$ and quadratic form $q(.)$ and we define,

$$\theta_\gamma(\tau) = \sum_{\alpha \in \gamma + L} \boldsymbol{e}(q(\alpha)). \tag{2.41}$$

Then

$$\theta(\tau) = \sum_{\gamma \in L'/L} \theta_\gamma e_\gamma \tag{2.42}$$

is a holormorphic vector-valued modular form for ρ_L of weight k.

Example 2.2.17. A natural way to construct vector-valued modular forms is by lifting modular forms for congruence subgroups. In [Sch] and [Sch2], the author constructs vector-valued modular forms of ρ_L from modular forms of the congruence subgroup $\Gamma_1(N), \Gamma_0(N)$ and $\Gamma(N)$. We state here one of the results as an example. Let L be an even lattice with even signature and level dividing N with $\gamma \in A$ and $f(\tau)$ be a scalar-valued modular form for $\Gamma_1(N)$ of weight k and character $\chi_\gamma\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \boldsymbol{e}(bq(\gamma))$. Then,

$$F_{\Gamma_1(N), f, \gamma} = \sum_{M \in \Gamma_1(N) \backslash \Gamma} f|_M \rho_L(M^{-1}) e_\gamma \tag{2.43}$$

is a vector-valued modular form for ρ_L of weight k.

Conversely given a vector-valued modular form, it is easy to obtain examples of scalar-valued modular forms from its components. As mentioned earlier for a lattice of level N and even signature, the principal congruence subgroup $\Gamma(N)$ acts trivially on $\mathbb{C}[A]$ so that $\rho_L(M)e_\gamma = e_\gamma$. Hence, if we have a vector-valued modular form $F(\tau) = \sum_{\gamma\in A} f_\gamma(\tau)e_\gamma$ then each of the components $f_\gamma(\tau)$ is a scalar-valued modular form for $\Gamma(N)$. We also have the following fact from [Sch].

Theorem 2.2.18 ([Sch])**.** *Let L be an even lattice with even signature and level diving N with $\gamma \in A, M \in \Gamma_1(N)$. Then for $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1(N)$*

$$\rho_L(M)e_\gamma = \boldsymbol{e}(-bq(\gamma))e_\gamma. \tag{2.44}$$

From the theorem above it follows that if $F(\tau) = \sum_{\gamma\in A} f_\gamma(\tau)e_\gamma$ is a vector-valued modular form of type ρ_L, then the components f_γ are scalar-valued modular forms for $\Gamma_1(N)$ and character $\chi_\gamma\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \boldsymbol{e}(bq(\gamma))$.

The above examples hints at a connection between vector-valued modular forms and modular forms for congruence subgroups through the Weil representation. The exact relation between the two can be summarized by the following conjecture due to Atkin, Swinnerton and Dyer. As stated by Mason it reads:

Conjecture 2.2.19. [M] The following two statements are equivalent:

1. $F(\tau) = \{f_i(\tau)\}$ is a vector-valued modular form associated to a representation ρ with rational Fourier coefficients and bounded denominators.

2. Each $f_i(\tau)$ is a modular form for a congruence subgroup of Γ_1.

Another way to get scalar-valued modular forms is by pairing two vector-valued modular forms,

Lemma 2.2.20. *Let* $F(\tau) = \sum_{\lambda \in A} f_\lambda(\tau) e_\lambda$ *and* $G(\tau) = \sum_{\lambda \in A} g_\lambda(\tau) e_\lambda$ *be weakly holomorphic vector-valued modular forms of type* ρ_L *and weight* k *and* k' *respectively. Then,*

$$\langle F(\tau), G(\tau) \rangle := \sum_{\lambda \in A} f_\lambda(\tau) \overline{g_\lambda}(\tau) \tag{2.45}$$

is a scalar-valued weakly holomorphic modular form of weight $k + k'$.

Proof. This follows directly from the unitarity of ρ_L for the pairing $\langle e_\alpha, e_\beta \rangle = \delta_{\alpha+\beta}$ and it can be checked by a direct computation that the pairing has the correct q-expansion. □

The Lemma above will be used in section 3.2 to lift Hecke operators of scalar-valued modular forms to vector-valued ones.

Example 2.2.21. Characters of rational vertex operator algebras (VOAs) are vector-valued modular forms for some representation ρ of $\mathrm{SL}_2(\mathbb{Z})$ with a congruence subgroup as a kernel as discussed in [HW]. This will be outlined in detail in section 4.3.3.

Vector-valued modular forms as defined in definition 2.2.14 commonly appear in the literature. However in section 3.2 we will work with modular forms that are just real analytic and non necessarily holomorphic. We will also need a slightly more general definition that includes "Jacobi-like" variables.

Let ρ be a representation of $\mathrm{Mp}_2(\mathbb{Z})$ on some vector space V, and let W be a $\mathbb{R}$-vector space.

Definition 2.2.22. For $v, \bar{v} \in \frac{1}{2}\mathbb{Z}$, we say that a V-valued real analytic function $\psi(\tau, \alpha, \beta)$ on $\mathbb{H} \times W \times W$ is *vector-valued modular of weight* $(v, \bar{v})$ *and type* ρ if:

$$\psi(M\tau, a\alpha + b\beta, c\alpha + d\beta) = \phi_M(\tau)^{2v}\overline{\phi_M(\tau)}^{2\bar{v}}\rho(M, \phi)\psi(\tau, \alpha, \beta), \qquad (2.46)$$

for all $(M, \phi_M) \in \mathrm{Mp}_2(\mathbb{Z})$. We say that it is *scalar-valued modular* if V is one-dimensional, $v, \bar{v} \in \mathbb{Z}$ and ρ is trivial. We denote by $M_{v,\bar{v},\rho}$ the space of V-valued real analytic functions on $\mathbb{H} \times W \times W$ that are vector-valued modular of weight $(v, \bar{v})$ and type ρ.

Remark 2.2.23. In Definition 2.2.22 we do not impose a growth condition, or holomorphicity (meromorphicity) at the cusps, or that the functions satisfy a condition involving the Laplacian.

Remark 2.2.24. Note that in Definition 2.2.22 we include "Jacobi-like" variables; these are needed for our construction. But for $\alpha = \beta = 0$ we recover the standard transformation property of vector-valued modular forms. For clarity we will drop the dependence on α and β when we consider objects that transform as vector-valued modular forms.

An important example of these are the Siegel Theta functions of Borcherds and we dedicate section 2.4 to describe them in detail.

2.3 Jacobi forms

In this section, we give a brief introduction to Jacobi forms and their connection to vector-valued modular forms of the Weil representation. Jacobi forms of rank 1 were introduced by Eichler and Zagier in [EZ] and have since found applications to diverse areas such as elliptic genera, moonshine, and construction of BKM Lie super-algebras. We now define their generalization to Jacobi forms of lattice index. The main references for these are [AJ] and [Mo]. Let $(L, q(.))$ be an even lattice as before. The Heisenberg group associated to L is,

$$H_L(\mathbb{Z}) := \{(x, y) : x, y \in L\}, \tag{2.47}$$

with the group composition law as component-wise addition. The Jacobi group is the semi-direct product,

$$J_L(\mathbb{Z}) := \mathrm{SL}_2(\mathbb{Z}) \ltimes H_L(\mathbb{Z}). \tag{2.48}$$

Let $A, A' = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $h, h' \in H_L(\mathbb{Z})$. Then the group composition is given by,

$$(A, (x, y))(A', (x', y')) = (AA', (\alpha x + \gamma y + x', \beta x + \delta y + y')). \tag{2.49}$$

Definition 2.3.1 ([AJ]). If $k \in \mathbb{Z}$ and $(L, q(.))$ a positive-definite even lattice, a Jacobi form of weight k and index L is a holomorphic function $\varphi : \mathbb{H} \times (L \otimes_{\mathbb{Z}} \mathbb{C}) \to \mathbb{C}$ with the following properties,

$$\varphi\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) = (c\tau + d)^k \boldsymbol{e}\left(\tfrac{cq(z)}{c\tau+d}\right)\varphi(\tau, z) \quad \forall \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \tag{2.50}$$

$$\varphi(\tau, z + x\tau + y) = \boldsymbol{e}(-\tau q(x) - (x, z))\varphi(\tau, z) \quad \forall (x, y) \in H_L(\mathbb{Z}) \tag{2.51}$$

We denote this space of functions by $J_{k,L}$

Jacobi forms have a Fourier expansion of the form,

$$\varphi(\tau, z) = \sum_{\substack{n\in\mathbb{Z}, r\in L' \\ n \geq q(r)}} c(n, r)\boldsymbol{e}(n\tau + (r, z)). \tag{2.52}$$

By letting $(L, q(.)) = (\mathbb{Z}, q(x) = mx^2)$ where $m \in \mathbb{N}$, we get the space of Jacobi forms of [EZ] of rank 1 with index an integer m.

Example 2.3.2. An example of a Jacobi form of rank 1 is the unique Jacobi form of weight 0 and index 1,

$$\phi_{0,1}(\tau,z) = 4\left(\frac{\vartheta_2^2(\tau,z)}{\vartheta_2^2(\tau,0)} + \frac{\vartheta_3^2(\tau,z)}{\vartheta_3^2(\tau,0)} + \frac{\vartheta_4^2(\tau,z)}{\vartheta_4^2(\tau,0)}\right), \tag{2.53}$$

where

$$\vartheta_2(\tau,z) = \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2} e^{(2n+1)\pi iz} \tag{2.54}$$

$$\vartheta_3(\tau,z) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{(2n\pi iz)} \tag{2.55}$$

$$\vartheta_4(\tau,z) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{(2n\pi iz)} \tag{2.56}$$

are the "auxiliary" Jacobi theta functions. We remark here that $2\phi_{0,1}(\tau)$ is in fact the elliptic genus of a $K3$ surface.

A rather remarkable property of Jacobi forms is their connection to vector-valued modular forms of the Weil representation through Jacobi's theta functions. For $\gamma \in A = L'/L$, the Jacobi theta function is given by

$$\vartheta_{L,\gamma}(\tau,z) = \sum_{\lambda \in L+\gamma} \boldsymbol{e}(\tau q(\lambda) + (\lambda, z)). \tag{2.57}$$

This is a generalization of the classical "unary" theta functions introduced by Jacobi,

$$\vartheta_{m,l}(\tau,z) = \sum_{r\in\mathbb{Z}, r\equiv l \bmod 2\mathrm{m}} q^{r^2/4m} y^r = \sum_{n\in\mathbb{Z}} q^{\frac{(l+2mn)^2}{4m}} y^{l+2mn} \quad q := \boldsymbol{e}(\tau), y := \boldsymbol{e}(z). \tag{2.58}$$

We state below the following important theorem from [AJ, Section 2.4].

Theorem 2.3.3. *Let $\varphi(\tau,z)$ is a Jacobi form of weight k, then $\varphi(\tau,z)$ can* *be written as,*

$$\varphi(\tau,z)=\sum_{\lambda\in A}\psi_{\varphi,\lambda}(\tau)\vartheta_{L,\lambda}(\tau,z) \tag{2.59}$$

where $\{\psi_{\varphi,\lambda}\}_{\lambda\in A}$ *is a vector-valued modular form of weight* $k-\frac{\dim L}{2}$ *and type* ρ_L^*.

In fact this correspondence goes both ways,so that if $\psi(\tau)=\sum_{\lambda\in A}\psi_\lambda(\tau)e_\lambda$ is a vector-valued modular form of type ρ_L^* and weight k, then the pairing,

$$\varphi(\tau,z)=\sum_{\lambda\in A}\psi_\lambda(\tau)\vartheta_{L,\lambda}(\tau,z) \tag{2.60}$$

is a Jacobi form of weight $k+\frac{\dim L}{2}$ and index L. This is because the Jacobi theta functions transform as "vector-valued Jacobi forms". Let

$$\vartheta_L(\tau,z)=\sum_{\lambda\in A}\vartheta_{L,\lambda}(\tau,z)e_\lambda \tag{2.61}$$

and $((\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}),\phi(\tau))\in \mathrm{Mp}_2(\mathbb{Z})$, then Boylan proved in [Boy] that $\vartheta_L(\tau,z)$ satisfies the following transformation law,

$$\vartheta_{L,}(\frac{a\tau+b}{c\tau+d},\frac{z}{\phi(\tau)^2})=\phi(\tau)^{\dim L}\boldsymbol{e}(-\frac{c\beta(z)}{\phi(\tau)^2})\rho_L((\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}),\phi(\tau))\vartheta_L(\tau,z). \tag{2.62}$$

Hence, the correspondence of theorem 2.3.3 is bijective, and Jacobi forms of lattice index are in one-to-one correspondence with vector-valued modular forms of the Weil representation and vice-versa.

2.4 Siegel Theta functions

In this section we introduce the Siegel theta functions introduced by Borcherds in [Bo1]. These are generalizations of the vector-valued theta functions defined in Example 2.2.16. In particular Borcherds proved that the Siegel theta function $\Theta_L(\tau, v)$ for a lattice L of type (b^+, b^-) transforms as a vector-valued modular form for the Weil representation of weight $(\frac{b^+}{2}, \frac{b^-}{2})$. This property allows us to pair it with a vector-valued modular form and generate a scalar-valued modular form of weight 0. Such a pairing gives the 'singular theta correspondence' through the theta integral in (4.6) and this will be discussed in section 4.1.

Let L be an even non-degenerate lattice of signature (b^+, b^-) and A the associated discriminant form with quadratic form $q(x) : A \to \mathbb{Q}/\mathbb{Z}$. The bilinear form on L induces a bilinear form on $V := L \otimes \mathbb{R}$. Let $\mathrm{Gr}(L)$ denote the Grassmannian of L that is the set of b^+ positive definite subspaces of V. For a chosen $v \in \mathrm{Gr}(L)$ we denote the orthogonal complement of v in V by v^- and thus we have a decomposition $V = v \oplus^{\perp} v^-$. For $\lambda \in L'/L \subset V$, let λ_{v^+} and λ_{v^-} be the projection onto the chosen spaces v^+ and v^- respectively. In particular, note that $q(\lambda) = q(\lambda_{v^+}) + q(\lambda_{v^-})$

The Siegel theta function (following Borcherd [Bo]) of a coset $L + \gamma$ of A is a *real analytic* function in τ and defined by

$$\theta_{L+\gamma}(\tau, v) = \sum_{\lambda \in L+\gamma} e\left(\tau q(\lambda_{v^+}) + \bar{\tau} q(\lambda_{v^-})\right). \tag{2.63}$$

Note that this definition is valid for any discriminant form, not just $A = L'/L$. For instance, for the rescaled discriminant form $A(r) = L(r)'/L(r) \cong \frac{1}{r}L'/L$, the Siegel theta function of a coset $L(r) + \gamma$ of $A(r)$ is given by

$$\theta_{L(r)+\gamma}(\tau, v) = \sum_{\lambda \in L+\gamma} e\left(\tau q_r(\lambda_{v^+}) + \bar{\tau} q_r(\lambda_{v^-})\right), \tag{2.64}$$

where $\gamma \in \frac{1}{r}L'/L$. If we let $\{e_\gamma\}$, $\gamma \in A$ be a basis for the group algebra

$\mathbb{C}[A]$, define

$$\Theta_L(\tau, v) = \sum_{\gamma \in A} \theta_{L+\gamma}(\tau, v) e_\gamma \tag{2.65}$$

and this is vector-valued modular of weight (b^+, b^-) of type ρ_L. As in [Bo], we define the more general Siegel theta function $\theta_L(\tau, \alpha, \beta, v)$ on $\mathbb{H} \times V \times V$,

$$\theta_{L+\lambda}(\tau, \alpha, \beta, v) = \sum_{\lambda \in L+\gamma} e\left(\tau q((\lambda+\beta)_{v^+}) + \bar{\tau} q((\lambda+\beta)_{v^-}) - \left(\lambda + \frac{\beta}{2}, \alpha\right)\right), \tag{2.66}$$

and

$$\Theta_L(\tau, \alpha, \beta, v) = \sum_{\gamma \in A} \theta_{L+\gamma}(\tau, \alpha, \beta, v)\, e_\gamma. \tag{2.67}$$

Theorem 2.4.1 (Theorem 4.1 of [Bo])**.** *The function $\Theta_L(\tau, \alpha, \beta, v)$ is vector-valued modular of weight (b^+, b^-) for the Weil representation ρ_L. Thus, it has the following transformation law for $(M, \phi) \in \mathrm{Mp}_2(\mathbb{Z})$, $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$,*

$$\Theta_L(M\tau, a\alpha + b\beta, c\alpha + d\beta, v) = \phi(\tau)^{b^+} \overline{\phi(\tau)}^{b^-} \rho_L(M, \phi) \Theta_L(\tau, \alpha, \beta, v) \tag{2.68}$$

In the rest of the thesis we suppress the argument v while writing theta functions for simplicity. So we will write,

$$\theta_{L+\lambda}(\tau, \alpha, \beta) = \sum_{\lambda \in L+\gamma} e\left(\tau q((\lambda+\beta)_+) + \bar{\tau} q((\lambda+\beta)_-) - \left(\lambda + \frac{\beta}{2}, \alpha\right)\right) \tag{2.69}$$

and

$$\Theta_L(\tau, \alpha, \beta) = \sum_{\gamma \in A} \theta_{L+\gamma}(\tau, \alpha, \beta)\, e_\gamma, \tag{2.70}$$

where it is assumed that λ_+ and λ_- are projections onto v and v^- for some fixed choice of $v \in \mathrm{Gr}(L)$.

In the following we will often make use of the following result.

Lemma 2.4.2. *$\psi(\tau)$ is a vector-valued modular form of type ρ_L and weight $(v, \bar{v})$ if and only if*

$$\langle \psi, \Theta_L \rangle (\tau, \alpha, \beta) = \sum_{\lambda \in A} \psi_\lambda(\tau) \overline{\theta_{L+\lambda}}(\tau, \alpha, \beta) \tag{2.71}$$

is a scalar-valued modular form of weight $(w, \bar{w}) = \left(v + \frac{1}{2}b^+, \bar{v} + \frac{1}{2}b^-\right)$.

Proof. On the one hand, if $\psi(\tau)$ is vector-valued of type ρ_L and weight $(v, \bar{v})$, then it follows directly that $\langle \psi, \Theta_L \rangle (\tau, \alpha, \beta)$ is scalar-valued of weight $\left(v + \frac{1}{2}b^+, \bar{v} + \frac{1}{2}b^-\right)$, since $\Theta_L(\tau, \alpha, \beta)$ is vector-valued of type ρ_L and weight $\left(\frac{1}{2}b^+, \frac{1}{2}b^-\right)$ and the Weil representation is unitary with respect to the inner product.

On the other hand, if $\langle \psi, \Theta_L \rangle (\tau, \alpha, \beta)$ is scalar-valued of weight $(w, \bar{w})$, then $\psi(\tau)$ must be vector-valued of type ρ_L and weight $(v, \bar{v}) = \left(w - \frac{1}{2}b^+, \bar{w} - \frac{1}{2}b^-\right)$. This follows again from unitary of the Weil representation, but also from the fact that the components $\bar{\theta}_{L+\lambda}(\tau, \alpha, \beta)$ of the Siegel theta functions are non-zero and linearly independent, which is crucial. This is why we need to include Jacobi-like variables α and β; otherwise the components of the Siegel theta functions would not be linearly independent in general, and we would not be able to deduce vector-valued modularity for $\psi(\tau)$ directly. □

We also introduce the following notation, which will be useful later on:

Definition 2.4.3. For any $\mu \in A(r)$, and positive integers k and l such that $kl = r$, we define $\Delta_r(\mu, k)$ by:

$$\Delta_r(\mu, k) = \begin{cases} 1 & \text{if } \mu \in A(l) \subseteq A(r), \\ 0 & \text{otherwise.} \end{cases} \tag{2.72}$$

We now prove a lemma relating Siegel theta functions of L and $L(r)$. This Lemma will be essential in the next section for formulating Hecke operators for vector-valued modular forms.

Lemma 2.4.4. *Let k, l, r be positive integers such that $kl = r$, and let $s \in \{0, 1, \ldots, l-1\}$. Let $L + \gamma$ be a coset of L in L', with $\gamma \in A$. Then:*

$$\theta_{L+\gamma}\left(\frac{k\tau + s}{l}, k\alpha + s\beta, l\beta\right) = \sum_{\substack{\nu \in A(r) \\ l\nu = \gamma}} \Delta_r(\nu, k)\boldsymbol{e}\left(\frac{s}{k}q_r(\nu)\right)\theta_{L(r)+\nu}(\tau, \alpha, \beta), \tag{2.73}$$

where $\nu + L(r)$ is a coset of $L(r)$ in $L(r)'$, with $\nu \in A(r)$, and $\Delta_r(\mu, k)$ defined in Definition 2.4.3.

Proof. Let $\lambda \in L + \gamma$, with $\gamma \in A$. First we compute that

$$\begin{aligned}
&\theta_{L+\gamma}\left(\frac{k\tau + s}{l}, k\alpha + s\beta, l\beta\right) \\
&\quad= \sum_{\lambda \in L+\gamma} \boldsymbol{e}\left(\frac{k\tau + s}{l}q\left((\lambda + l\beta)_+\right) + \frac{k\bar{\tau} + s}{l}q\left((\lambda + l\beta)_-\right) - \left(\lambda + \frac{l\beta}{2}, k\alpha + s\beta\right)\right) \\
&\quad= \sum_{\lambda \in L+\gamma} \boldsymbol{e}\left(\frac{k\tau}{l}q\left((\lambda + l\beta)_+\right) + \frac{k\bar{\tau}}{l}q\left((\lambda + l\beta)_-\right) - k\left(\lambda + \frac{l\beta}{2}, \alpha\right)\right)\boldsymbol{e}\left(\frac{s}{l}q(\lambda)\right).
\end{aligned} \tag{2.74}$$

Now there is a bijection between elements λ of the coset $L + \gamma$ and elements δ of the cosets $L + \nu$, with $\nu \in A(l)$ and such that $l\nu = \gamma$. The bijection is given by lattice rescaling, that is, $\lambda \mapsto \delta = \frac{1}{l}\lambda$. We use this to rewrite the sum as follows:

$$\begin{aligned}
&\theta_{L+\gamma}\left(\frac{k\tau+s}{l}, k\alpha+s\beta, l\beta\right) \\
&\quad= \sum_{\substack{\nu\in A(l)\\ l\nu=\gamma}} \sum_{\delta\in L+\nu} \boldsymbol{e}\left(\tau q_r\left((\delta+\beta)_+\right)+\bar{\tau} q_r\left((\delta+\beta)_-\right)-\left(\delta+\frac{\beta}{2},\alpha\right)_r\right)\boldsymbol{e}\left(sq_l\left(\delta\right)\right) \\
&\quad= \sum_{\substack{\nu\in A(l)\\ l\nu=\gamma}} \boldsymbol{e}\left(sq_l\left(\nu\right)\right) \sum_{\delta\in L+\nu} \boldsymbol{e}\left(\tau q_r\left((\delta+\beta)_+\right)+\bar{\tau} q_r\left((\delta+\beta)_-\right)-\left(\delta+\frac{\beta}{2},\alpha\right)_r\right),
\end{aligned} \tag{2.75}$$

where in the last line we used the fact that $q_l(\delta) = q_l(\nu) \bmod \mathbb{Z}$, since $\nu \in A(l)$.

We now extend the sum over $\nu \in A(l) \subseteq A(r)$ to a sum over all elements $\nu \in A(r)$, using the Delta function from Definition 2.4.3. We get:

$$\theta_{L+\gamma}\left(\frac{k\tau+s}{l}, k\alpha+s\beta, l\beta\right) = \sum_{\substack{\nu\in A(r)\\ l\nu=\gamma}} \Delta_r(\nu,k)\boldsymbol{e}\left(\frac{s}{k}q_r\left(\nu\right)\right)\theta_{L(r)+\nu}(\tau,\alpha,\beta), \tag{2.76}$$

where we introduced the Siegel theta functions of the rescaled lattice $L(r)$:

$$\theta_{L(r)+\nu}(\tau,\alpha,\beta) = \sum_{\delta\in L+\nu} \boldsymbol{e}\left(\tau q_r\left((\delta+\beta)_+\right)+\bar{\tau} q_r\left((\delta+\beta)_-\right)-\left(\delta+\frac{\beta}{2},\alpha\right)_r\right). \tag{2.77}$$

□

Chapter 3

Hecke operators

Hecke operators for classical modular forms were first introduced by Mordell and Hecke to study arithmetic properties of modular forms such as the multiplicativity of the Ramanujan τ function. These are operators that map spaces of modular forms $M_k(\Gamma_1)$ and cusp forms $S_k(\Gamma_1)$ of a certain weight k to modular forms and cusp forms of the same weight. Hecke operators for Jacobi forms of rank 1 lattices were developed and Eicher and Zagier in [EZ] and generalizations to higher rank have been given in [AJ]. In this chapter, we first give a quick review of Hecke operators for modular forms of the full modular group Γ_1. In section 3.2 we give Hecke operators for vector-valued modular forms of the Weil representation obtained as lifts of Hecke operators on scalar-valued modular forms using various properties of the Siegel theta functions as presented in [BCJ]. Subsequently Hecke operators of Bruinier-Stein are introduced in section 3.3 as a sum over double coset representatives. Finally we compare the two different constructions of Hecke operators in section 3.4.

3.1 Hecke operators for classical modular forms

Hecke operators for the full modular group are operators defined on the space of modular forms,

$$T_n : M_k(\Gamma_1) \to M_k(\Gamma_1) \tag{3.1}$$

and these satisfy nice algebraic relations such as,

$$T_m \circ T_n = T_{mn} \quad \gcd(m,n) = 1 \tag{3.2}$$

$$T_p \circ T_{p^n} = T_{p^{n+1}} + p^{k-1} T_{p^{n-1}} \quad p \text{ prime.} \tag{3.3}$$

In addition T_n also preserves the decomposition $M_k = S_k \oplus \mathbb{C}E_k$ i.e. it preserves the space of cusp forms and all E_k are eigenvectors of T_n. The Hecke operators T_n are Hermitian operators on $S_k(\Gamma_1)$. Thus for $f, g \in S_k(\Gamma_1)$,

$$\langle T_n f, g \rangle_P = \langle f, T_n g \rangle_P \tag{3.4}$$

where $\langle ., . \rangle_P$ is the Petersson inner product on the space of cusp forms defined as,

$$\langle ., . \rangle_P : S_k(\Gamma_1) \times S_k(\Gamma_1) \to \mathbb{C}, \qquad \langle f, g \rangle_P = \int \int_F f \cdot \bar{g}\, y^{2k-2} \cdot dxdy \tag{3.5}$$

where $\tau = x + iy$ and F is any fundamental domain. As a consequence of the spectral theorem, this means that the space of cusp forms of weight k have a basis of simultaneous eigenvectors for the set of commuting Hecke operators.

Example 3.1.1. An example of the effectiveness of Hecke operators is the multiplicativity of the Ramanujan τ-function. The discriminant modular form is the unique cusp form of weight 12 and hence the multiplicativity of its Fourier coefficient $\tau(n)$ follows directly from (3.2).

Hecke operators can be defined through several different but equivalent ways each of which has its own merits. Below, we present some of the definitions. These definitions have generalizations to Hecke operators on modular forms of level greater than 1. We will not discuss them here, but the interested reader could refer to [DS] for generalizations to higher level. We will closely follow [Stw] and [Mil] as references for this section.

3.1.1 Lattice interpretation

There is a one-to-one correspondence between modular forms f on $\mathbb{H}$ of weight k and homogeneous functions F on the set of all latices $\mathcal{L}$ of weight k. Let $L \in \mathcal{L}$ with integral generators ω_1/ω_2 such that $\omega_1/\omega_2 \in \mathbb{H}$. We denote this by $L(\omega_1, \omega_2)$. This correspondence is given by,

$$f \to F_f(L) = \omega_2^{-k} f(\frac{\omega_1}{\omega_2}) \tag{3.6}$$

$$F \to f_F(\tau) = F(L(1, \tau)). \tag{3.7}$$

Remark 3.1.2. Hecke operators preserve growth conditions on modular forms. In the following formulation, the set of functions F should be appropriately restricted so that the correspondence $F \to f_F(\tau)$ gives the correct growth conditions for $f_F(\tau)$.

Definition 3.1.3. The Hecke operator T_n acting on the space of homogeneous functions of weight k on $\mathcal{L}$ is defined by:

$$(T_n F)(L) = n^{k-1} \sum_{\substack{L' \subseteq L \\ [L:L']=n}} F(L'). \tag{3.8}$$

This gives rise to the following definition for Hecke operators on modular forms.

Definition 3.1.4. Let $f(\tau)$ be a modular form of weight k and F_f the corresponding homogeneous function on $\mathcal{L}$. The Hecke transformed modular form $T_n f(\tau)$ is defined through the following equation:

$$F_{T_n f} = T_n F_f. \tag{3.9}$$

More explicitly,

$$T_n f(\tau) = (T_n \cdot F_f)(L(1, \tau)). \tag{3.10}$$

The set of all sublattices of index n is in one-to-one correspondence with the set of matrices,

$$X_n := \{(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}), a, b, d \in \mathbb{Z}, ad = n, a \geq 1, 0 \leq b < d - 1\} \tag{3.11}$$

and is given by $L(a\omega_1 + b\omega_2, d\omega_2)$ where a, b, d are as above. In addition X_n give a complete set of representatives of the orbits of $\Gamma_1 \backslash M_n(\mathbb{Z})$ where $M_n(\mathbb{Z})$ is the set of matrices of determinant n with integer-valued entries. This gives us an explicit formula for the Hecke operator on modular forms of weight k,

$$T_n f(\tau) = n^{k/2-1} \sum_{\gamma \in \Gamma_1 \backslash M_n(\mathbb{Z})} f|_\gamma(\tau) \tag{3.12}$$

$$= n^{k-1} \sum_{\substack{a,b,d \in \mathbb{Z}, ad = n \\ a \geq 1, 0 \leq b < d-1}} d^{-k} f(\frac{a\tau + b}{d}) \tag{3.13}$$

where $|_\gamma$ is the Petersson slash operator as defined in (2.2).

3.1.2 Action of double cosets

There is another more abstract definition of Hecke operators using an action of double cosets. This generalizes easily to congruence subgroups, and will also be used in section 3.3 to define Hecke operators on vector-valued modular forms. Let Γ be a congruence subgroup of Γ_1 and let Δ be a set of real matrices with positive determinant closed under multiplication (usually taken to be rational valued). The Hecke algebra $\mathcal{H}(\Gamma, \Delta)$ is the free abelian group generated by the double cosets $\Gamma\alpha\Gamma, \alpha \in \Delta$ and we abbreviate $[\alpha] = \Gamma\alpha\Gamma$. In addition we assume that each double coset has a decomposition into *finite* disjoint unions,

$$\Gamma\alpha\Gamma = \bigsqcup_{i=i}^{n} \Gamma\alpha_i \tag{3.14}$$

as is the case when $\Delta = \mathrm{GL}_2^+(\mathbb{Q})$.

Definition 3.1.5. Multiplication on $\mathcal{H}(\Gamma, \Delta)$ is defined as

$$[\alpha] \cdot [\beta] = \sum c_{\alpha,\beta}^{\gamma} \cdot [\gamma] \tag{3.15}$$

where the sum is over $\gamma \in \Delta$ such that $\Gamma\gamma\Gamma \subset \Gamma\alpha\Gamma\beta\Gamma = \bigcup_{i,j} \Gamma\alpha_i\beta_j$ and $c_{\alpha,\beta}^{\gamma}$ is the number of pairs (i, j) with $\Gamma\alpha_i\beta_j = \Gamma\gamma$.

We now define Hecke operators as an action of the Hecke algebra $\mathcal{H}(\Gamma, \Delta)$ on modular forms. In the case of modular forms of the full modular group, take $\Gamma = \Gamma_1$ and $\Delta = \mathrm{GL}_2^+(\mathbb{Z})$. For $[\alpha] = \bigcup_i \Gamma_1 \cdot \alpha_i$, the action of the Hecke algebra on $f \in M_k(\Gamma_1)$ is defined by,

$$[\alpha] \cdot f(\tau) = \sum f|_k\alpha_i(\tau). \tag{3.16}$$

The Hecke operator can now be defined.

Definition 3.1.6. For $n \geq 1$,

$$T_n f(\tau) = \sum_{\substack{a,d \in \mathbb{Z} \\ a|d, ad=n \\ a \geq 1}} [\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)] \cdot f(\tau). \tag{3.17}$$

It can be shown that the definition 3.1.6 is equivalent to the definition of the Hecke operator in the previous section. Every term of the sum above is a disjoint union of right cosets. Hence the sum above can be also be written as a sum over right coset actions given by,

$$T_n f(\tau) = n^{k-1} \sum_{\substack{a,b,d\in\mathbb{Z}, ad=n \\ a\geq 1, 0\leq b<d-1}} d^{-2k} f(\frac{a\tau + b}{d}) \tag{3.18}$$

and this is exactly (3.12).

3.1.3 Hecke operators for modular forms with Jacobi-like variables

We defined a slight generalization of modular forms in Definition 2.2.22 to real analytic functions that also have two extra arguments in the form of vectors of $L \otimes \mathbb{R}$. Hecke operators on modular forms have natural generalizations to these generalized functions that also include scaling of the vector-valued arguments. Such operators will prove useful in the rest of the section while dealing with Siegel theta functions.

Definition 3.1.7. Let r be a positive integer and $f(\tau, \alpha, \beta)$ a scalar-valued function on $\mathbb{H} \times W \times W$ of weight $(w.\bar{w})$, as defined in Definition 2.2.22. We define the Hecke operator on $f(\tau, \alpha, \beta)$ by:

$$T_r[f](\tau, \alpha, \beta) = r^{w+\bar{w}-1} \sum_{\substack{k,l>0 \\ kl=r}} l^{-w-\bar{w}} \sum_{s=0}^{l-1} f\left(\frac{k\tau + s}{l}, k\alpha + s\beta, l\beta\right). \tag{3.19}$$

Lemma 3.1.8. *$T_r[f](\tau, \alpha, \beta)$ is scalar-valued modular on $\mathbb{H} \times W \times W$ of weight $(w.\bar{w})$.*

Proof. The argument is word by word the same as for scalar-valued modular forms (see for example [Stw, Proposition 2.28]). □

To study algebraic relations satisfied by Hecke operators, we define a scaling operator:

Definition 3.1.9. Let r be a positive integer and $f(\tau, \alpha, \beta)$ be a scalar-valued modular of weight $(w, \bar{w})$. We define the scaling operator U_{r^2} by:

$$U_{r^2}[f](\tau, \alpha, \beta) = f(\tau, r\alpha, r\beta). \tag{3.20}$$

It is clear that:

Lemma 3.1.10. *$U_{r^2}[f](\tau, \alpha, \beta)$ is scalar-valued modular of weight $(w, \bar{w})$.*

Hecke operators satisfy algebraic relations summarized in the following lemma.

Lemma 3.1.11. *For m and n such that $\gcd(m, n) = 1$,*

$$T_m \circ T_n = T_{mn}, \tag{3.21}$$

and for $l \geq 2$ and p prime,

$$T_{p^l} = T_p \circ T_{p^{l-1}} - p^{w+\bar{w}-1} U_{p^2} \circ T_{p^{l-2}}. \tag{3.22}$$

Note that (3.21) and (3.22) can be proved following the exact same steps as the proof of the respective relations for scalar-valued modular forms presented for instance in Propositions 2.28 and 2.29 of [Stw].

3.2 Hecke operators on vector-valued modular forms

Let L be an even lattice of type (b^+, b^-) and ρ_L the associated Weil representation. In this section we define Hecke operators on functions that are vector-valued modular of type ρ_L as defined in Definition 2.2.22 and derive some algebraic relations between them. The content of this section is much of the same as [BCJ]. We first define a Hecke operator that takes a function that is vector-valued modular of type ρ_L to a a function that is vector-valued modular for the Weil representation of the rescaled lattice $\rho_{L(r)}$. Later on we state an important theorem on the relation between the representations ρ_L and $\rho_{L(n^2)}$ and also give a Hecke operator that maps functions that are vector-valued modular of type ρ_L to functions of the same type. All that we impose in this section is the vector-valued modular transformation property. However, our construction could potentially restrict to various classes of modular objects, such as holomorphic modular forms, weakly holomorphic modular forms, Mass forms, etc.

Definition 3.2.1. Let $\psi(\tau) = \sum_{\lambda \in A} \psi_\lambda(\tau)\, e_\lambda$ be vector-valued modular of weight $(v, \bar{v})$ and type ρ_L. Let $(w, \bar{w}) = \left(v + \frac{b^+}{2}, \bar{v} + \frac{b^-}{2}\right)$. We define the operator $\mathcal{T}_r$ by:

$$\mathcal{T}_r[\psi](\tau) = r^{w+\bar{w}-1} \sum_{\mu \in A(r)} \left(\sum_{\substack{k,l>0 \\ kl=r}} \frac{1}{l^{w+\bar{w}}} \sum_{s=0}^{l-1} \Delta_r(\mu, k) \boldsymbol{e}\left(-\frac{s}{k} q_r(\mu)\right) \psi_{l\mu}\left(\frac{k\tau + s}{l}\right) \right) e_\mu, \tag{3.23}$$

with $\Delta_r(\mu, k)$ defined in Definition 2.4.3.

The main result is:

Theorem 3.2.2. *For any positive integer r,*

$$T_r\left[\langle\psi,\Theta_L\rangle\right](\tau,\alpha,\beta)=\left\langle\mathcal{T}_r[\psi],\Theta_{L(r)}\right\rangle(\tau,\alpha,\beta). \tag{3.24}$$

In other words, the standard Hecke transform of the scalar-valued $\langle\psi,\Theta_L\rangle(\tau,\alpha,\beta)$ is equal to the scalar-valued $\left\langle\mathcal{T}_r[\psi],\Theta_{L(r)}\right\rangle(\tau,\alpha,\beta)$ obtained by pairing $\mathcal{T}_r[\psi](\tau)$ with the Siegel theta functions of the rescaled lattice $L(r)$.

An immediate corollary, using Lemmas 2.4.2 and 3.1.8, is:

Corollary 3.2.3. *If $\psi(\tau)$ is vector-valued modular of weight $(v,\bar{v})$ and type ρ_L, then $\mathcal{T}_r[\psi](\tau)$ is vector-valued modular of type $\rho_{L(r)}$ of the same weight. In other words, Definition 3.2.1 gives a Hecke operator*

$$\mathcal{T}_r:\ M_{v,\bar{v},\rho_L}\to M_{v,\bar{v},\rho_{L(r)}}. \tag{3.25}$$

This is the main reason for Definition 3.2.1. Let us now prove Theorem 3.2.2.

Proof. We have:

$$\begin{aligned}T_r\left[\langle\psi,\Theta_L\rangle\right](\tau,\alpha,\beta)=&T_r\left[\sum_{\lambda\in A}\psi_\lambda(\tau)\overline{\theta}_{L+\lambda}(\tau,\alpha,\beta)\right]\\=&r^{w+\bar{w}-1}\sum_{\substack{k,l>0\\kl=r}}\frac{1}{l^{w+\bar{w}}}\sum_{s=0}^{l-1}\sum_{\lambda\in A}\psi_\lambda\left(\frac{k\tau+s}{l}\right)\overline{\theta}_{L+\lambda}\left(\frac{k\tau+s}{l},k\alpha+l\beta,l\beta\right).\end{aligned} \tag{3.26}$$

By Lemma 2.4.4, we know that

$$\overline{\theta_{L+\lambda}}\left(\frac{k\tau+s}{l},k\alpha+l\beta,l\beta\right)=\sum_{\substack{\nu\in A(r)\\ l\nu=\lambda}}\Delta_r(\nu,k)\boldsymbol{e}\left(-\frac{s}{k}q_r\left(\nu\right)\right)\overline{\theta}_{L(r)+\nu}(\tau,\alpha,\beta). \tag{3.27}$$

Substituting, we get

$$\begin{aligned}
&T_r\left[\langle\psi,\Theta_L\rangle\right](\tau,\alpha,\beta)\\
=&r^{w+\bar{w}-1}\sum_{\substack{k,l>0\\ kl=r}}\frac{1}{l^{w+\bar{w}}}\sum_{s=0}^{l-1}\sum_{\lambda\in A}\sum_{\substack{\nu\in A(r)\\ l\nu=\lambda}}\Delta_r(\nu,k)\boldsymbol{e}\left(-\frac{s}{k}q_r\left(\nu\right)\right)\psi_\lambda\left(\frac{k\tau+s}{l}\right)\overline{\theta}_{L(r)+\nu}(\tau,\alpha,\beta).\\
=&r^{w+\bar{w}-1}\sum_{\nu\in A(r)}\sum_{\substack{k,l>0\\ kl=r}}\frac{1}{l^{w+\bar{w}}}\sum_{s=0}^{l-1}\Delta_r(\nu,k)\boldsymbol{e}\left(-\frac{s}{k}q_r\left(\nu\right)\right)\psi_{l\nu}\left(\frac{k\tau+s}{l}\right)\overline{\theta}_{L(r)+\nu}(\tau,\alpha,\beta)\\
=&\left\langle\mathcal{T}_r[\psi],\Theta_{L(r)}\right\rangle(\tau,\alpha,\beta),
\end{aligned} \tag{3.28}$$

where we used Definition 3.2.1. □

3.2.1 Algebraic relations satisfied by the operators $\mathcal{T}_r$

In this section we study algebraic relations satisfied by the operators $\mathcal{T}_r$. Those trickle down from the corresponding relations stated in Lemma 3.1.11 for the standard Hecke operators T_r.

Recall the scaling operator U_{n^2} for scalar-valued functions from Definition 3.1.9. We now define a scaling operator for $M_{v,\bar{v},L}$

Definition 3.2.4. Let $\psi(\tau)=\sum_{\lambda\in A}\psi_\lambda(\tau)\,e_\lambda$ be vector-valued modular of type ρ_L. We define the scaling operator $\mathcal{U}_{n^2}$ by:

$$\mathcal{U}_{n^2}[\psi](\tau,\alpha,\beta)=\sum_{\nu\in A(n^2)}\Delta_{n^2}(\nu,n)\psi_{n\nu}(\tau)e_\nu. \tag{3.29}$$

Then we have:

Lemma 3.2.5. *For any positive integer n,*

$$U_{n^2}\left[\langle\psi,\Theta_L\rangle\right](\tau,\alpha,\beta)=\left\langle\mathcal{U}_{n^2}[\psi],\Theta_{L(n^2)}\right\rangle(\tau,\alpha,\beta). \tag{3.30}$$

Proof. We have:

$$\begin{aligned}U_{n^2}\left[\langle\psi,\Theta_L\rangle\right](\tau,\alpha,\beta)=&U_{n^2}\left[\sum_{\lambda\in A}\psi_\lambda(\tau)\bar{\theta}_{L+\lambda}(\tau,\alpha,\beta)\right]\\=&\sum_{\lambda\in A}\psi_\lambda(\tau)\bar{\theta}_{L+\lambda}(\tau,n\alpha,n\beta).\end{aligned} \tag{3.31}$$

But Lemma 2.4.4, with $k=n$, $l=n$ and $s=0$, states that

$$\bar{\theta}_{L+\lambda}(\tau,n\alpha,n\beta)=\sum_{\substack{\nu\in A(n^2)\\ n\nu=\lambda}}\Delta_{n^2}(\nu,n)\bar{\theta}_{L(n^2)+\nu}\left(\tau,\alpha,\beta\right). \tag{3.32}$$

Thus

$$\begin{aligned}U_{n^2}\left[\langle\psi,\Theta_L\rangle\right](\tau,\alpha,\beta)=&\sum_{\lambda\in A}\psi_\lambda(\tau)\sum_{\substack{\nu\in A(n^2)\\ n\nu=\lambda}}\Delta_{n^2}(\nu,n)\bar{\theta}_{L(n^2)+\nu}\left(\tau,\alpha,\beta\right)\\=&\sum_{\nu\in A(n^2)}\Delta_{n^2}(\nu,n)\psi_{n\nu}(\tau)\bar{\theta}_{L(n^2)+\nu}\left(\tau,\alpha,\beta\right)\\=&\left\langle\mathcal{U}_{n^2}[\psi],\Theta_{L(n^2)}\right\rangle(\tau,\alpha,\beta).\end{aligned} \tag{3.33}$$

□

It immediately follows from Lemmas 2.4.2 and 3.1.8 that:

Corollary 3.2.6. *Let $\psi(\tau)$ be vector-valued modular of type ρ_L. Then $\mathcal{U}_{n^2}[\psi](\tau,\alpha,\beta)$ is vector-valued modular of type $\rho_{L(n^2)}$ of the same weight. In other words,* *Definition 3.2.4 gives a scaling operator*

$$\mathcal{U}_{n^2}: \quad M_{v,\bar{v},\rho_L} \to M_{v,\bar{v},\rho_{L(n^2)}}. \tag{3.34}$$

With this definition, we obtain the following theorem, analogous to Lemma 3.1.11.

Theorem 3.2.7. *For m and n such that* $\gcd(m,n) = 1$,

$$\mathcal{T}_m \circ \mathcal{T}_n = \mathcal{T}_{mn}, \tag{3.35}$$

while for $l \geq 2$ and p prime,

$$\mathcal{T}_{p^l} = \mathcal{T}_p \circ \mathcal{T}_{p^{l-1}} - p^{w+\bar{w}-1}\mathcal{U}_{p^2} \circ \mathcal{T}_{p^{l-2}}. \tag{3.36}$$

Proof. These two statements follow directly by applying the analogous statements from Lemma 3.1.11 to the scalar-valued $\langle \psi, \Theta_L \rangle (\tau, \alpha, \beta)$ and then using the definition of our operators $\mathcal{T}_n$ and $\mathcal{U}_{n^2}$. □

3.2.2 The $r = n^2$ case

We now specialize to Hecke operators $\mathcal{T}_r$ with $r = n^2$ for some positive integer n. What is special in this case is the existence of a sub-representation ρ_L of the Weil representation $\rho_{L(n^2)}$ for the rescaled lattice $L(n^2)$. This allows us to define a projection operator $\mathcal{P}_{n^2}$, which is a left inverse of the scaling operator $\mathcal{U}_{n^2}$. We can use this projection operator to define a new Hecke operator $\mathcal{H}_{n^2} = \mathcal{P}_{n^2} \circ \mathcal{T}_{n^2}$, which takes functions that are vector-valued modular type ρ_L to functions of the same type.

3.2.3 Weil sub-representation

Let us start by proving the existence of a sub-representation ρ_L of the Weil representation $\rho_{L(n^2)}$ for the rescaled lattice $L(n^2)$. The Weil representation was defined in (2.33) by its action on the basis elements of the group algebra $\mathbb{C}[L'/L]$. The Weil representation $\rho_{L(n^2)}$ for the rescaled lattice is a representation of $\mathrm{Mp}_2(\mathbb{Z})$ on $\mathbb{C}[A(n^2)]$ and is defined by

$$\rho_{L(n^2)}(T)e_\nu = \boldsymbol{e}(q_{n^2}(\nu))\, e_\nu, \tag{3.37}$$

$$\rho_{L(n^2)}(S)e_\nu = \frac{\boldsymbol{e}(-\mathrm{sgn(L)}/8)}{\sqrt{|A(n^2)|}} \sum_{\mu \in A(n^2)} \boldsymbol{e}(-(\nu,\mu)_{n^2})\, e_\mu, \tag{3.38}$$

where $\{e_\nu\}_{\nu \in A(n^2)}$ is the standard basis for the vector space $\mathbb{C}[A(n^2)]$, and S and T are the generators of $\mathrm{Mp}_2(\mathbb{Z})$.

Consider the subspace $\mathbb{C}[A] \subseteq \mathbb{C}[A(n^2)]$ spanned by the basis vectors $\{f_\lambda\}_{\lambda \in A}$ defined by

$$f_\lambda = \frac{1}{n^{\dim L}} \sum_{\substack{\nu \in A(n) \subseteq A(n^2) \\ n\nu = \lambda}} e_\nu. \tag{3.39}$$

The $\{f_\lambda\}_{\lambda \in A}$ form the standard basis for $\mathbb{C}[A]$. Indeed, one sees that $f_\lambda f_\delta =$

$f_{\lambda+\delta}$:

$$\begin{aligned}
f_\lambda f_\delta =& \frac{1}{n^{2\dim L}} \sum_{\substack{\nu\in A(n)\\ n\nu=\lambda}} \sum_{\substack{\mu\in A(n)\\ n\mu=\delta}} e_\mu e_\nu \\
=& \frac{1}{n^{2\dim L}} \sum_{\substack{\nu\in A(n)\\ n\nu=\lambda}} \sum_{\substack{\mu\in A(n)\\ n\mu=\delta}} e_{\mu+\nu} \\
=& \frac{1}{n^{2\dim L}} \sum_{\substack{\alpha\in A(n)\\ n\alpha=\lambda+\delta}} e_\alpha \left(\sum_{\substack{\mu\in A(n)\\ n\mu=\delta}} 1 \right) \\
=& \frac{1}{n^{\dim L}} \sum_{\substack{\alpha\in A(n)\\ n\alpha=\lambda+\delta}} e_\alpha \\
=& f_{\lambda+\delta}, \qquad (3.40)
\end{aligned}$$

since

$$\sum_{\substack{\mu\in A(n)\\ n\mu=\delta}} 1 = \left| \frac{1}{n} L/L \right| = n^{\dim L}. \qquad (3.41)$$

We prove the following important lemma.

Lemma 3.2.8. *The restriction of $\rho_{L(n^2)}$ to the subspace $\mathbb{C}[A]$ is the Weil representation ρ_L:*

$$\rho_{L(n^2)}\big|_{\mathbb{C}[A]} = \rho_L. \qquad (3.42)$$

In other words,

$$\rho_{L(n^2)}(T) f_\lambda = \boldsymbol{e}(q(\lambda)) f_\lambda = \rho_L(T)(f_\lambda), \qquad (3.43)$$

$$\rho_{L(n^2)}(S) f_\lambda = \frac{\boldsymbol{e}(-\mathrm{sgn(L)}/8)}{\sqrt{|A|}} \sum_{\gamma\in A} \boldsymbol{e}(-(\lambda,\gamma))\, f_\lambda = \rho_L(S)(f_\lambda). \qquad (3.44)$$

Proof. Let us begin with the T transformation:

$$\begin{aligned}\rho_{L(n^2)}(T)f_\lambda =& \frac{1}{n^{\dim L}} \sum_{\substack{\nu\in A(n)\\ n\nu=\lambda}} \rho_{L(n^2)}(T)(e_\nu)\\ =& \frac{1}{n^{\dim L}} \sum_{\substack{\nu\in A(n)\\ n\nu=\lambda}} \boldsymbol{e}(q_{n^2}(\nu))\, e_\nu\\ =& \frac{1}{n^{\dim L}} \boldsymbol{e}(q(\lambda)) \sum_{\substack{\nu\in A(n)\\ n\nu=\lambda}} e_\nu\\ =& \boldsymbol{e}(q(\lambda)) f_\lambda. \end{aligned} \tag{3.45}$$

As for the S transformation,

$$\begin{aligned}\rho_{L(n^2)}(S)f_\lambda =& \frac{1}{n^{\dim L}} \sum_{\substack{\nu\in A(n)\\ n\nu=\lambda}} \rho_{L(n^2)}(S)(e_\nu)\\ =& \frac{1}{n^{\dim L}} \frac{\boldsymbol{e}(-\mathrm{sgn(L)}/8)}{\sqrt{|A(n^2)|}} \sum_{\substack{\nu\in A(n)\\ n\nu=\lambda}} \sum_{\mu\in A(n^2)} \boldsymbol{e}(-(\nu,\mu)_{n^2})\, e_\mu. \end{aligned} \tag{3.46}$$

Now consider the sum $\sum_{\substack{\nu\in A(n)\\ n\nu=\lambda}} \boldsymbol{e}(-(\nu,\mu)_{n^2})$. We can do a shift $\nu \mapsto \nu+\beta$ for any $\beta \in \frac{1}{n}L/L$. It should not change the sum, since if $n\nu=\lambda$, then $n(\nu+\beta)=\lambda$, and hence it only amounts to relabeling the summands. Thus for all $\beta \in \frac{1}{n}L/L$, we must have:

$$\sum_{\substack{\nu\in A(n)\\ n\nu=\lambda}} \boldsymbol{e}(-(\nu,\mu)_{n^2}) = \boldsymbol{e}\left(-(\beta,\mu)_{n^2}\right) \sum_{\substack{\nu\in A(n)\\ n\nu=\lambda}} \boldsymbol{e}(-(\nu,\mu)_{n^2}). \tag{3.47}$$

This implies that either the summation over ν is zero, or $\boldsymbol{e}\left(-(\beta,\mu)_{n^2}\right)=1$ for all $\beta \in \frac{1}{n}L/L$, which will be the case if $\mu \in A(n) \subseteq A(n^2)$. Thus we conclude that the summation over ν is zero whenever $\mu \notin A(n) \subseteq A(n^2)$. As a result, we get

$$
\begin{aligned}
\rho_{L(n^2)}(S)f_\lambda =& \frac{1}{n^{\dim L}}\frac{\boldsymbol{e}(-\mathrm{sgn(L)}/8)}{\sqrt{|A(n^2)|}} \sum_{\substack{\nu\in A(n)\\ n\nu=\lambda}} \sum_{\mu\in A(n)} \boldsymbol{e}(-(\nu,\mu)_{n^2})\, e_\mu \\
=& \frac{1}{n^{\dim L}}\frac{\boldsymbol{e}(-\mathrm{sgn(L)}/8)}{\sqrt{|A(n^2)|}} \sum_{\substack{\nu\in A(n)\\ n\nu=\lambda}} \sum_{\mu\in A(n)} \boldsymbol{e}(-(n\nu,n\mu))\, e_\mu \\
=& \frac{1}{n^{\dim L}}\frac{\boldsymbol{e}(-\mathrm{sgn(L)}/8)}{\sqrt{|A(n^2)|}} \left|\frac{1}{n}L/L\right| \sum_{\mu\in A(n)} \boldsymbol{e}(-(\lambda,n\mu))\, e_\mu \\
=& \frac{1}{n^{\dim L}}\frac{\boldsymbol{e}(-\mathrm{sgn(L)}/8)}{\sqrt{|A|}} \sum_{\delta\in A} \boldsymbol{e}(-(\lambda,\delta)) \sum_{\substack{\mu\in A(n)\\ n\mu=\delta}} e_\mu \\
=& \frac{\boldsymbol{e}(-\mathrm{sgn(L)}/8)}{\sqrt{|A|}} \sum_{\delta\in A} \boldsymbol{e}(-(\lambda,\delta)) f_\delta. \qquad (3.48)
\end{aligned}
$$

□

3.2.4 Projection operator

The existence of the sub-representation given in Lemma 3.2.8 allows us to define a projection operator $\mathcal{P}_{n^2} : \rho_{v,\bar{v},L(n^2)} \to \rho_{v,v,L}$.

Definition 3.2.9. Let $\psi(\tau) = \sum_{\nu\in A(n^2)} \psi_\nu(\tau)\, e_\nu$ be vector-valued modular of type $\rho_{L(n^2)}$. We define the projection operator $\mathcal{P}_{n^2}$ by:

$$\begin{aligned}\mathcal{P}_{n^2}[\psi](\tau) =& \frac{1}{n^{\dim L}} \sum_{\lambda\in A} \left(\sum_{\substack{\gamma\in A(n)\\ n\gamma=\lambda}} \psi_\gamma(\tau) \right) e_\lambda \\ =& \frac{1}{n^{\dim L}} \sum_{\lambda\in A} \left(\sum_{\substack{\gamma\in A(n^2)\\ n\gamma=\lambda}} \Delta_{n^2}(\gamma, n)\psi_\gamma(\tau) \right) e_\lambda, \end{aligned} \tag{3.49}$$

with $\Delta_{n^2}(\gamma, n)$ defined in Definition 2.4.3.

As a direct corollary of Lemma 3.2.8 we get:

Corollary 3.2.10. *Let $\psi(\tau) = \sum_{\nu\in A(n^2)} \psi_\nu(\tau)\, e_\nu$ be vector-valued modular of type $\rho_{L(n^2)}$. Then $\mathcal{P}_{n^2}[\psi](\tau)$ is vector-valued modular of type ρ_L of the same weight. In other words, Definition 3.2.9 gives a projection operator*

$$\mathcal{P}_{n^2}: \quad M_{v,\bar{v},L(n^2)} \to M_{v,\bar{v},L}. \tag{3.50}$$

We now show that the projection operator $\mathcal{P}_{n^2}$ is a left inverse of the scaling operator $\mathcal{U}_{n^2}$.

Lemma 3.2.11.

$$\mathcal{P}_{n^2} \circ \mathcal{U}_{n^2} = \mathcal{I}, \tag{3.51}$$

where $\mathcal{I}$ is the identity operator.

Proof. Let $\psi(\tau)$ be vector-valued modular of type ρ_L. We have:

$$\begin{aligned}
\mathcal{P}_{n^2} \circ \mathcal{U}_{n^2}[\psi](\tau) =& \mathcal{P}_{n^2} \left(\sum_{\nu \in A(n^2)} \Delta_{n^2}(\nu, n) \psi_{n\nu}(\tau) e_\nu \right) \\
=& \frac{1}{n^{\dim L}} \sum_{\lambda \in A} \left(\sum_{\substack{\gamma \in A(n) \\ n\gamma = \lambda}} \psi_{n\gamma}(\tau) \right) e_\lambda \\
=& \frac{1}{n^{\dim L}} \sum_{\lambda \in A} \left(\sum_{\substack{\gamma \in A(n) \\ n\gamma = \lambda}} 1 \right) \psi_\lambda(\tau) e_\lambda
\end{aligned} \tag{3.52}$$

The sum in bracket was evaluated in (3.41), and is equal to $n^{\dim L}$. Thus we get

$$\mathcal{P}_{n^2} \circ \mathcal{U}_{n^2}[\psi](\tau) = \sum_{\lambda \in A} \psi_\lambda(\tau) e_\lambda. \tag{3.53}$$

□

3.2.5 New Hecke operator $\mathcal{H}_{n^2}$

We can now compose our Hecke operator $\mathcal{T}_{n^2}$ with the projection operator $\mathcal{P}_{n^2}$ to get a new Hecke operator $\mathcal{H}_{n^2}$ which maps functions of type ρ_L to functions of the same type.

Definition 3.2.12. We define the Hecke operator:

$$\mathcal{H}_{n^2} := \mathcal{P}_{n^2} \circ \mathcal{T}_{n^2} : \quad M_{v,\bar{v},L} \to M_{v,\bar{v},L}. \tag{3.54}$$

We can give an explicit formula for the components of $\mathcal{H}_{n^2}[\psi](\tau)$.

Lemma 3.2.13. *Let $\psi(\tau) = \sum_{\lambda \in A} \psi_\lambda(\tau)\, e_\lambda$ be vector-valued modular of type*

ρ_L and weight $(v, \bar{v})$. Then $\mathcal{H}_{n^2}[\psi](\tau)$ is also vector-valued of type ρ_L, and can be written as:

$$\mathcal{H}_{n^2}[\psi](\tau) = n^{2(v+\bar{v}-1)}$$
$$\times \sum_{\lambda \in A} \left(\sum_{\substack{\gamma \in A(n^2) \\ n\gamma = \lambda}} \sum_{\substack{k,l>0 \\ kl=n^2}} \frac{1}{l^{v+\bar{v}+\frac{1}{2}\dim L}} \sum_{s=0}^{l-1} \Delta_{n^2}(\gamma, n) \Delta_{n^2}(\gamma, k) \boldsymbol{e}\left(-\frac{s}{k} q_{n^2}(\gamma)\right) \psi_{l\gamma}\left(\frac{k\tau + s}{l}\right)\right) e_\lambda. \tag{3.55}$$

Proof. This follows directly from Definitions 3.2.1 and 3.2.9. □

3.2.6 Algebraic relations satisfied by the Hecke operators $\mathcal{H}_{n^2}$

In the previous section, we proved Theorem 3.2.7 for the Hecke operators $\mathcal{T}_r$. We now study the analogous result for the operators $\mathcal{H}_{n^2}$.

We first need the following lemmas.

Lemma 3.2.14. *For any positive integers m and n,*

$$\mathcal{U}_{n^2} \circ \mathcal{T}_{m^2} = \mathcal{T}_{m^2} \circ \mathcal{U}_{n^2}. \tag{3.56}$$

Proof. This follows directly by applying the analogous statement for U_{n^2} and T_{m^2} on the scalar-valued function $\langle \psi, \Theta_L \rangle (\tau, \alpha, \beta)$ and then using the definition of our operators $\mathcal{T}_{m^2}$ and $\mathcal{U}_{n^2}$. □

Lemma 3.2.15. *For any positive integers m and n,*

$$\mathcal{P}_{m^2} \circ \mathcal{P}_{n^2} = \mathcal{P}_{m^2 n^2}. \tag{3.57}$$

Proof. Let $\psi(\tau) = \sum_{\nu \in A(m^2n^2)} \psi_\nu(\tau) e_\nu$ be vector-valued modular of type $\rho_{L(m^2n^2)}$. Then

$$\begin{aligned}\mathcal{P}_{m^2} \circ \mathcal{P}_{n^2}[\psi](\tau) =& \frac{1}{n^{\dim L}} \mathcal{P}_{m^2} \left[\sum_{\alpha \in A(m^2)} \sum_{\substack{\gamma \in A(m^2n^2) \\ n\gamma = \alpha}} \Delta_{m^2n^2}(\gamma, n) \psi_\gamma(\tau) e_\alpha \right] \\ =& \frac{1}{(mn)^{\dim L}} \sum_{\lambda \in A} \sum_{\substack{\beta \in A(m^2) \\ m\beta = \lambda}} \sum_{\substack{\gamma \in A(m^2n^2) \\ n\gamma = \beta}} \Delta_{m^2}(\beta, m) \Delta_{m^2n^2}(\gamma, n) \psi_\gamma(\tau) e_\lambda.\end{aligned} \tag{3.58}$$

The two delta conditions imply that $\gamma \in A(mn)$. We can then rewrite the sums as

$$\begin{aligned}\mathcal{P}_{m^2} \circ \mathcal{P}_{n^2}[\psi](\tau) =& \frac{1}{(mn)^{\dim L}} \sum_{\lambda \in A} \sum_{\substack{\beta \in A(m) \\ m\beta = \lambda}} \sum_{\substack{\gamma \in A(mn) \\ n\gamma = \beta}} \psi_\gamma(\tau) e_\lambda \\ =& \frac{1}{(mn)^{\dim L}} \sum_{\lambda \in A} \sum_{\substack{\gamma \in A(mn) \\ mn\gamma = \lambda}} \psi_\gamma(\tau) e_\lambda \\ =& \mathcal{P}_{m^2n^2}[\psi](\tau).\end{aligned} \tag{3.59}$$

□

However, the projection and Hecke operators only commute when $\gcd(m, n) = 1$:

Lemma 3.2.16. *For m and n such that* $\gcd(m, n) = 1$,

$$\mathcal{P}_{n^2} \circ \mathcal{T}_{m^2} = \mathcal{T}_{m^2} \circ \mathcal{P}_{n^2}. \tag{3.60}$$

Proof. We start with the left-hand-side. Let $\psi(\tau)$ be vector-valued modular of weight $(v, \bar{v})$ and type $\rho_{L(n^2)}$. We have:

$$\mathcal{P}_{n^2} \circ \mathcal{T}_{m^2}[\psi](\tau) = m^{2(w+\bar{w}-1)} \times$$
$$\mathcal{P}_{n^2} \left[\sum_{\mu \in A(m^2 n^2)} \left(\sum_{\substack{k,l>0 \\ kl=m^2}} \frac{1}{l^{w+\bar{w}}} \sum_{s=0}^{l-1} \Delta_{m^2 n^2}(\mu, k) \boldsymbol{e}\left(-\frac{s}{k} q_{m^2 n^2}(\mu)\right) \psi_{l\mu}\left(\frac{k\tau + s}{l}\right)\right) e_\mu \right] \tag{3.61}$$

$$= \frac{m^{2(w+\bar{w}-1)}}{n^{\dim L}} \sum_{\lambda \in A(m^2)} \left(\sum_{\substack{k,l>0 \\ kl=m^2}} \frac{1}{l^{w+\bar{w}}} \sum_{s=0}^{l-1} \boldsymbol{e}\left(-\frac{s}{k} q_{m^2}(\lambda)\right) \right.$$
$$\left. \times \sum_{\substack{\gamma \in A(m^2 n^2) \\ n\gamma = \lambda}} \Delta_{m^2 n^2}(\gamma, n) \Delta_{m^2 n^2}(\gamma, k) \psi_{l\gamma}\left(\frac{k\tau + s}{l}\right)\right) e_\lambda. \tag{3.62}$$

On the right-hand-side, we get:

$$\mathcal{T}_{m^2} \circ \mathcal{P}_{n^2}[\psi](\tau) = \frac{1}{n^{\dim L}} \mathcal{T}_{m^2} \left[\sum_{\lambda \in A} \left(\sum_{\substack{\mu \in A(n^2) \\ n\mu = \lambda}} \Delta_{n^2}(\mu, n) \psi_\mu(\tau) \right) e_\lambda \right]$$
$$= \frac{m^{2(w+\bar{w}-1)}}{n^{\dim L}} \sum_{\lambda \in A(m^2)} \left(\sum_{\substack{k,l>0 \\ kl=m^2}} \frac{1}{l^{w+\bar{w}}} \sum_{s=0}^{l-1} \boldsymbol{e}\left(-\frac{s}{k} q_{m^2}(\lambda)\right) \right.$$
$$\left. \times \sum_{\substack{\mu \in A(n^2) \\ n\mu = l\lambda}} \Delta_{m^2}(\lambda, k) \Delta_{n^2}(\mu, n) \psi_\mu\left(\frac{k\tau + s}{l}\right)\right) e_\lambda. \tag{3.63}$$

To prove equality between the two sides we need to show that

$$\sum_{\substack{\gamma \in A(m^2n^2) \\ n\gamma = \lambda}} \Delta_{m^2n^2}(\gamma, n)\Delta_{m^2n^2}(\gamma, k)\psi_{l\gamma}\left(\frac{k\tau + s}{l}\right)$$
$$= \sum_{\substack{\mu \in A(n^2) \\ n\mu = l\lambda}} \Delta_{m^2}(\lambda, k)\Delta_{n^2}(\mu, n)\psi_{\mu}\left(\frac{k\tau + s}{l}\right) \qquad (3.64)$$

for all $k, l > 0$ such that $kl = m^2$, $s \in \{0, \ldots, l-1\}$, and $\lambda \in A(m^2)$.

On the right-hand-side, the two delta functions impose that $\mu \in A(n)$ and $\lambda \in A(l)$, so we can write the right-hand-side as

$$\sum_{\substack{\mu \in A(n) \\ n\mu = l\lambda}} \psi_{\mu}\left(\frac{k\tau + s}{l}\right) \qquad (3.65)$$

when $\lambda \in A(l)$ and zero otherwise.

On the left-hand-side, the first delta function imposes that $\gamma \in A(m^2n)$, while the second imposes that $\gamma \in A(ln^2)$. Together those impose that $\gamma \in A(s)$, where $s = \gcd(m^2n, ln^2)$. Assuming that $\gcd(m, n) = 1$, we have $s = ln$, hence $\gamma \in A(ln)$. Since $n\gamma = \lambda$, this imposes that $\lambda \in A(l) \subseteq A(m^2)$. So the left-hand-side can be written as

$$\sum_{\substack{\gamma \in A(ln) \\ n\gamma = \lambda}} \psi_{l\gamma}\left(\frac{k\tau + s}{l}\right), \qquad (3.66)$$

when $\lambda \in A(l)$ and zero otherwise. We note that knowing $n\gamma$ and $l\gamma$ completely fixes $\gamma \in A(ln)$ by the Euclidean algorithm. Thus if we define

$\mu = l\gamma$, we can rewrite the sum as

$$\sum_{\substack{\mu \in A(n) \\ n\mu = l\lambda}} \psi_\mu \left(\frac{k\tau + s}{l} \right), \tag{3.67}$$

and (3.64) is satisfied. $\square$

We then prove the following algebraic relations.

Theorem 3.2.17. *For m and n such that* $\gcd(m, n) = 1$,

$$\mathcal{H}_{m^2} \circ \mathcal{H}_{n^2} = \mathcal{H}_{m^2 n^2}, \tag{3.68}$$

while for $l \geq 2$ and p prime,

$$\mathcal{H}_{p^{2l}} = \mathcal{P}_{p^{2l-2}} \circ \mathcal{H}_{p^2} \circ \mathcal{H}_{p^{2l-2}} \circ \mathcal{U}_{p^{2l-2}} - p^{w+\bar{w}-1} \mathcal{H}_{p^{2l-2}} - p^{2(w+\bar{w}-1)} \mathcal{H}_{p^{2l-4}}. \tag{3.69}$$

Proof. To prove the first statement, we start with

$$\mathcal{T}_{m^2} \circ \mathcal{T}_{n^2} = \mathcal{T}_{m^2 n^2}, \tag{3.70}$$

and apply the projection operator $\mathcal{P}_{m^2 n^2} = \mathcal{P}_{m^2} \circ \mathcal{P}_{n^2}$ (using Lemma 3.2.15) on both sides of the equation. The right-hand-side becomes $\mathcal{H}_{m^2 n^2}$, while the left-hand-side becomes $\mathcal{H}_{m^2} \circ \mathcal{H}_{n^2}$ after using Lemma 3.2.16.

For the second statement, we start with

$$\mathcal{T}_{p^m} = \mathcal{T}_p \circ \mathcal{T}_{p^{m-1}} - p^{w+\bar{w}-1} \mathcal{U}_{p^2} \circ \mathcal{T}_{p^{m-2}}, \tag{3.71}$$

for $m \geq 2$ and p prime. Consider the three cases $m = 2l$, $m = 2l - 1$ and

$m = 2l - 2$, with $l \geq 2$:

$$\mathcal{T}_{p^{2l}} = \mathcal{T}_p \circ \mathcal{T}_{p^{2l-1}} - p^{w+\bar{w}-1} \mathcal{U}_{p^2} \circ \mathcal{T}_{p^{2l-2}}, \tag{3.72}$$

$$\mathcal{T}_{p^{2l-1}} = \mathcal{T}_p \circ \mathcal{T}_{p^{2l-2}} - p^{w+\bar{w}-1} \mathcal{U}_{p^2} \circ \mathcal{T}_{p^{2l-3}}, \tag{3.73}$$

$$\mathcal{T}_{p^{2l-2}} = \mathcal{T}_p \circ \mathcal{T}_{p^{2l-3}} - p^{w+\bar{w}-1} \mathcal{U}_{p^2} \circ \mathcal{T}_{p^{2l-4}}. \tag{3.74}$$

Inserting the second equation into the first, and using Lemma 3.2.14, we get

$$\mathcal{T}_{p^{2l}} = \mathcal{T}_p \circ \mathcal{T}_p \circ \mathcal{T}_{p^{2l-2}} - p^{w+\bar{w}-1} \mathcal{U}_{p^2} \circ \mathcal{T}_p \circ \mathcal{T}_{p^{2l-3}} - p^{w+\bar{w}-1} \mathcal{U}_{p^2} \circ \mathcal{T}_{p^{2l-2}}. \tag{3.75}$$

Then inserting the third equation, using Lemma 3.2.14 again, we get

$$\mathcal{T}_{p^{2l}} = \mathcal{T}_p \circ \mathcal{T}_p \circ \mathcal{T}_{p^{2l-2}} - p^{w+\bar{w}-1} \mathcal{U}_{p^2} \circ \mathcal{T}_{p^{2l-2}} - p^{2(w+\bar{w}-1)} \mathcal{U}_{p^2} \circ \mathcal{U}_{p^2} \circ \mathcal{T}_{p^{2l-4}} - p^{w+\bar{w}-1} \mathcal{U}_{p^2} \circ \mathcal{T}_{p^{2l-2}}. \tag{3.76}$$

But

$$\mathcal{T}_p \circ \mathcal{T}_p = \mathcal{T}_{p^2} + p^{w+\bar{w}-1} \mathcal{U}_{p^2}, \tag{3.77}$$

hence we get

$$\mathcal{T}_{p^{2l}} = \mathcal{T}_{p^2} \circ \mathcal{T}_{p^{2l-2}} - p^{w+\bar{w}-1} \mathcal{U}_{p^2} \circ \mathcal{T}_{p^{2l-2}} - p^{2(w+\bar{w}-1)} \mathcal{U}_{p^2} \circ \mathcal{U}_{p^2} \circ \mathcal{T}_{p^{2l-4}}. \tag{3.78}$$

We now apply the projection operator $\mathcal{P}_{p^{2l}}$ on both sides of the equation. The left-hand-side becomes $\mathcal{H}_{p^{2l}}$, and the last two terms on the right-hand-side become

$$- p^{w+\bar{w}-1} \mathcal{H}_{p^{2l-2}} - p^{2(w+\bar{w}-1)} \mathcal{H}_{p^{2l-4}}, \tag{3.79}$$

using Lemma 3.2.11. For the first term on the right-hand-side, we get:

$$\begin{aligned} \mathcal{P}_{p^{2l}} \circ \mathcal{T}_{p^2} \circ \mathcal{T}_{p^{2l-2}} =& \mathcal{P}_{p^{2l-2}} \circ \mathcal{H}_{p^2} \circ \left(\mathcal{P}_{p^{2l-2}} \circ \mathcal{U}_{p^{2l-2}} \right) \circ \mathcal{T}_{p^{2l-2}} \\ =& \mathcal{P}_{p^{2l-2}} \circ \mathcal{H}_{p^2} \circ \mathcal{H}_{p^{2l-2}} \circ \mathcal{U}_{p^{2l-2}}, \end{aligned} \tag{3.80}$$

where we used Lemmas 3.2.11 and 3.2.14. □

3.3 Hecke operators of Bruinier-Stein

Let L be an even non-degenerate lattice of level N as before. Bruinier-Stein in [St] and [BS] define Hecke operators $T_{p^{2l}}$ on holomorphic vector-valued modular forms of type ρ_L where p is an odd prime and $l \geq 1$ and give formulas for the action of these operators on the basis elements $\{e_\lambda\}$, for $\lambda \in A$. In [BS], Hecke operators T_n on modular forms of type ρ_L are defined for $\gcd(n, N) = 1$ and $n \equiv m^2 \mod n$ by extending the Weil representation of $SL_2(\mathbb{Z})$ and $\mathrm{Mp}_2(\mathbb{Z})$ to subgroups of $\mathrm{GL}_2^+(\mathbb{Q})$ for even and odd signature respectively, where $\mathrm{GL}_2^+(\mathbb{Q})$ denotes matrices in $\mathbb{Q}$ with positive determinant. They use this definition to give an explicit formula for the action of T_{p^2} on the Fourier coefficients of a modular form of type ρ_L. The case of Hecke operators for powers of all odd primes $T_{p^{2l}}$ is dealt separately in [St]. These operators are defined as a sum over left coset representatives of the double coset $\mathrm{Mp}_2(\mathbb{Z})\left(\begin{smallmatrix} p^{2l} & 0 \\ 0 & 1 \end{smallmatrix}\right)\mathrm{Mp}_2(\mathbb{Z})$. An explicit action on $\mathbb{C}[A]$ is computed through an application of Shintani's formula for the Weil representation of [Shin]. Below, we review the definition of operators T_{n^2} of [BS] for the odd and even signature cases respectively, and the more general case of [St]. In this section we adopt the notation $\Gamma_1 = \mathrm{SL}_2(\mathbb{Z})$ and $\tilde{\Gamma_1} = \mathrm{Mp}_2(\mathbb{Z})$. We remark here that Bruinier-Stein only consider holomorphic vector-valued modular forms and hence only consider weights of the form $(v, \bar{v}) = (k, 0)$. A good reference for a detailed study of this construction and related issues can be found in [W].

3.3.1 Hecke operators T_n for $(\gcd(n, N)) = 1$

In this subsection, we review the formulation of Hecke operators in [BS] by an action of a Hecke algebra that is an extension of Γ_1 used in classical modular forms. This Hecke algebra depends on the parity of the signature, and we will outline the two cases separately. This is a more abstract definition for the Hecke operators, and an alternate and more general formulation will be given in Section 3.3.2 and for comparison with our results. While defining the operator T_n we assume that $\gcd(n, N) = 1$ and furthermore,

$$\begin{aligned} n &\equiv r^2 \mod N, && \text{if sgn(L) is even} \\ n &= m^2, && \text{if sgn(L) is odd.} \end{aligned} \tag{3.81}$$

The Hecke operator T_n is defined in [St] in terms of the Hecke algebra given by the pair of groups $(\mathcal{Q}(N), \Gamma)$ for even signature and $(\mathcal{Q}_2(N), \Gamma)$ for odd signature where Γ are the following subgroups,

$$\Gamma = \begin{cases} \Gamma_1 \times \{1\} \subset \mathcal{Q}(N), & \text{if sgn(L) is even} \\ L(\widetilde{\Gamma_1}) \subseteq \mathcal{Q}_2(N), & \text{if sgn(L) is odd.} \end{cases} \tag{3.82}$$

We will now define the group $\mathcal{Q}(N)$, $\mathcal{Q}_2(N)$ and $L(\widetilde{\Gamma_1})$ and the unitary Weil representations of these groups for the even and odd signatures respectively.

Even signature

In Section 2.2 it was mentioned that the Weil representation factors through the group $S(N) := \mathrm{SL}_2(\mathbb{Z}) = \Gamma_1/\Gamma(N)$, where $\Gamma(N)$ is the principal congruence group of level N. Let s be a section $s : S(N) \to \Gamma_1$ so that $\pi_N \circ s = \mathrm{id}_{S(N)}$ where π_N is the entry-wise reduction modulo N, then one can define the Weil representation on $S(N)$,

$$\rho_L(A)\, e_\lambda = \rho_L(s(A))\, e_\lambda, \qquad A \in S(N). \tag{3.83}$$

Let $Q(N)$ be the group,

$$Q(N) = \{(M,r) \in \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}) \times (\mathbb{Z}/N\mathbb{Z})^* : \det(M) \equiv r^2 \pmod N\}, \tag{3.84}$$

with the the point-wise group multiplication. Note that we have an embedding of $S(N)$ into $Q(N)$, $M \to (M,1), M \in S(N)$. We also have a converse correspondence through the map $(M,r) \to \left(M\left(\begin{smallmatrix} r & 0 \\ 0 & r \end{smallmatrix}\right)^{-1}, r\right)$ that defines an isomorphism $Q(N) \cong S(N) \times (\mathbb{Z}/N\mathbb{Z})^*$. To extend the Weil representation to $Q(N)$, we make the following definition for $(\mathbb{Z}/N\mathbb{Z})^*$,

$$\rho_L\left(\left(\begin{smallmatrix} r & 0 \\ 0 & r \end{smallmatrix}\right), r\right) e_\lambda = \frac{g_1(L)}{g_r(L)}\, e_\lambda, \tag{3.85}$$

where

$$g_r(L) = \sum_{\lambda \in A} \boldsymbol{e}\left(rq(\lambda)\right), \tag{3.86}$$

is a character of $(\mathbb{Z}/N\mathbb{Z})^*$. We now define the Weil representation of $Q(N)$ by the composition of the actions for $Q(N)$ and $(\mathbb{Z}/N\mathbb{Z})^*$,

$$\rho_L(M,r)\, e_\lambda = \rho_L\left(M\left(\begin{smallmatrix} r & 0 \\ 0 & r \end{smallmatrix}\right)^{-1}, 1\right) \circ \rho_L\left(\left(\begin{smallmatrix} r & 0 \\ 0 & r \end{smallmatrix}\right), r\right) e_\lambda. \tag{3.87}$$

We are now ready to define $\mathcal{Q}(N)$ appearing in equation ((3.82)). In order to get a factorization via $Q(N)$ we make the following definitions,

$$\begin{aligned} \mathcal{G}(N) = \{M \in \mathrm{GL}_2^+(\mathbb{Q}), \exists n \in \mathbb{Z} \text{ with } \gcd(n,N) = 1 \text{ such that } nM \in M_2(\mathbb{Z}) \\ \text{and } \gcd(\det(nM), n) = 1\} \end{aligned} \tag{3.88}$$

$$\mathcal{Q}(N) = \{(M,r) \in \mathcal{G}(N) \times (\mathbb{Z}/N\mathbb{Z})^* : \det(M) \equiv r^2 \pmod N\}. \tag{3.89}$$

Note that we have an embedding of the modular group $\gamma \in \mathrm{SL}_2(\mathbb{Z}) := \Gamma_1 \to (\gamma, 1) \in \mathcal{Q}(N)$. Now we extend the Weil representation to $\mathcal{Q}(N)$ by noting that the entry-wise reduction π_N maps the group $\mathcal{Q}(N)$ to $Q(N)$,

$$\rho_L : \mathcal{Q}(N) \to \mathrm{GL}(\mathbb{C}[A]), \quad (M, r) \mapsto \rho_L(\pi_N(M), r), \tag{3.90}$$

where ρ_L is the Weil representation on $Q(N)$ as defined in equation (3.87). Finally, we have the following action of $\mathcal{Q}(N)$ on a vector-valued modular form $f(\tau) = \sum_{\lambda \in A} f_\lambda(\tau)\, e_\lambda$ of weight k by an extension of the Petersson slash operator,

$$f|_{k,L}(M, r) = \sum_{\lambda \in A} (f_\lambda|_k M) \rho_L^{-1}(M, r)\, e_\lambda, \tag{3.91}$$

where $f_\lambda|_k M$ is the usual Petersson slash operation on the scalar-valued components $f_\lambda(\tau)$ as defined in (2.2).

Odd signature

In the case of odd signature, we have half integer-weights and (3.90) defines only a projective representation of the group $\mathcal{Q}(N)$,

$$\rho_L = \mathcal{Q}(N) \to \mathrm{GL}(\mathbb{C}[A])/\{\pm 1\}, \quad g \mapsto \rho_L(g). \tag{3.92}$$

To get an honest representation, we need to make a two-fold central extension of $\mathcal{Q}(N)$. We get two 2-cocycles, one from the choice of a section s : $\mathrm{GL}(\mathbb{C}[A])/\{\pm 1\}$ and another one determined by the sign of the square root of the automorphy factor. These two cocycles are not cohomologous on the whole group $\mathcal{Q}(N)$, however they are identical on $\widetilde{\Gamma_1}$. We define a twofold central group extension of $\mathcal{Q}(\mathcal{N})$ by $\mathbb{Z}_2$,

$$\mathcal{Q}_2(N) = \{(M, \phi(M, \tau), r, t) : M \in \mathcal{G}(N), r \in (\mathbb{Z}/N\mathbb{Z})^*, \det(M) \equiv r^2 \mod N, t \in \{\pm 1\}. \tag{3.93}$$

There is a natural embedding of $\widetilde{\Gamma_1}$ with the sign of the automorphy factor being defined by the cocycle t,

$$L : \widetilde{\Gamma_1} \longrightarrow \mathcal{Q}_2(N), \quad (\gamma, \pm j(\gamma, \tau)) \mapsto (\gamma, \pm j(\gamma, \tau), 1, \pm 1), \tag{3.94}$$

where $j(((\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}), \tau)) = \sqrt{c\tau + d}$ is the automorphy factor. For $(M, \phi(M, \tau), r, t) \in \mathcal{Q}_2(N)$ the Weil representation of $\mathcal{Q}_2(N)$ can be defined as,

$$\rho_L(M, \phi(M, \tau), r, t) = t\rho_L(M, r), \tag{3.95}$$

as an extension of the Weil representation $\rho_L(M, r)$ given in equation (3.90). This is a natural extension of the Weil representation of $\widetilde{\Gamma_1}$ through the embedding L,

$$\rho_L(L(\gamma, \phi(\gamma, \tau))) = \rho_L(\gamma, \phi(\gamma, t)). \tag{3.96}$$

Finally, the action of $\mathcal{Q}_2(N)$ on a modular form $f(\tau) = \sum_{\lambda \in A} f_\lambda(\tau)\, e_\lambda$ of weight k is given by,

$$f|_{k,L}(M, \phi, r, t) = \sum_{\lambda \in A} (f_\lambda|_k(M, \phi))\, \rho^{-1}(M, r, t)\, e_\lambda, \tag{3.97}$$

with the Petersson slash operator on the scalar-valued components f_λ for $(M, \phi) \in \widetilde{\mathrm{GL}}_2(\mathbb{R})$ is given by

$$f_\lambda|_k(M, \phi) = \det(M)^{\frac{k}{2}} \phi(M, \tau)^{-2k} f_\lambda(M\tau). \tag{3.98}$$

Definition of Hecke operators

We now define the Hecke operator T_n for $\gcd(n, N) = 1$ and satisfying the conditions in equation (3.81). We denote by Γ the following subgroups as in equation (3.82)

$$\Gamma = \begin{cases} \Gamma_1 \times \{1\} \subset \mathcal{Q}(N), & \text{if sgn(L) is even} \\ L(\widetilde{\Gamma_1}) \subseteq \mathcal{Q}_2(N), & \text{if sgn(L) is odd.} \end{cases} \tag{3.99}$$

and let n satisfying the conditions of (3.81), we define double cosets

$$\mathcal{M}(n) = \begin{cases} \Gamma\left(\left(\begin{smallmatrix} n & 0 \\ 0 & 1 \end{smallmatrix}\right), r\right)\Gamma, & \text{if sgn(L) is even} \\ \Gamma\left(\left(\begin{smallmatrix} n & 0 \\ 0 & 1 \end{smallmatrix}\right), 1, m, 1\right)\Gamma, & \text{if sgn(L) is odd.} \end{cases} \tag{3.100}$$

The Hecke operator is then defined as a sum over left coset representatives of this double coset $\mathcal{M}(n)$, that is

$$f|_{k,L}T_n = n^{\frac{k}{2}-1} \sum_{\tilde{M} \in \Gamma \backslash \mathcal{M}(n)} f|_{k,L}\tilde{M}, \tag{3.101}$$

where the slash operator is given by equation (3.91) or equation (3.97) in the case of even or odd signature respectively.

3.3.2 Generalized Hecke operators

The case of T_n for $\gcd(n, N) > 1$ needs to be dealt separately because the reduction modulo N of a matrix $M \in M_2(\mathbb{Z})$ doesn't belong to $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$. Hence the factorization of the Weil representation through $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ or its double cover cannot be used. The action via the Weil representation is denoted by,

$$e_\lambda|_L M = \rho_L^{-1}(M)e_\lambda. \tag{3.102}$$

Like the $\gcd(n,N)=1$ case, we make the following definition

$$e_\lambda|_L\left(\left(\begin{smallmatrix} n^2 & 0\\ 0 & 1\end{smallmatrix}\right)\right) = e_{n\lambda}. \tag{3.103}$$

Let $\alpha = \left(\begin{smallmatrix} n^2 & 0\\ 0 & 1\end{smallmatrix}\right)$ and $\delta = \gamma\alpha\gamma' \in \tilde{\Gamma_1}\alpha\tilde{\Gamma_1}$. An action of the double coset is defined by an extension of this action,

$$e_\lambda|_L\delta = e_\lambda|_L\,\gamma|_L\,\alpha|_L\,\gamma'. \tag{3.104}$$

Finally, the Hecke operator T_{n^2} can be defined in the following manner,

Definition 3.3.1. Let $n \in \mathbb{Z}_{>0}$ and $\alpha = \left(\left(\begin{smallmatrix} n^2 & 0\\ 0 & 1\end{smallmatrix}\right), 1\right) \in \widetilde{\mathrm{GL}}_2^+(\mathbb{Q})$ be the left coset decomposition given by,

$$\tilde{\Gamma_1}\cdot\alpha\cdot\tilde{\Gamma_1} = \bigcup_i \tilde{\Gamma_1}\cdot\delta_i.$$

For a vector-valued modular form $f(\tau)$ of weight k and type ρ_L, the Hecke operator T_{n^2} on $f(\tau)$ is defined by

$$T_{n^2}[f](\tau) = n^{k-2}\sum_i\sum_{\lambda\in A}\left(f_\lambda|_k\delta_i\right)\left(e_\lambda|_L\delta_i\right), \tag{3.105}$$

where $f_\lambda|_k\delta_i$ is the Petersson slash operator as defined in (3.98).

Consider the following left coset decomposition of $\widetilde{\Gamma_1}\alpha\widetilde{\Gamma_1}$,

$$\widetilde{\Gamma_1}\alpha\widetilde{\Gamma_1} = \widetilde{\Gamma_1}\alpha \cup \bigcup_{s=1}^{2l-1}\bigcup_{h\in(\mathbb{Z}/p^s\mathbb{Z})^*}\widetilde{\Gamma_1}\beta_{h,s} \cup \bigcup_{b\in\mathbb{Z}/p^{2l}\mathbb{Z}}\widetilde{\Gamma_1}\gamma_b \tag{3.106}$$

where

$$\beta_{b,a} = \left(\begin{pmatrix} p^{2l-a} & b\\ 0 & p^a\end{pmatrix}, \sqrt{p^a}\right), \qquad \gamma_b = \left(\begin{pmatrix} 1 & b\\ 0 & p^{2l}\end{pmatrix}, p^l\right). \tag{3.107}$$

Shintani's formula [Shin, Proposition 1.6] and some Gauss sum computations were used in [St] to get explicit formulas for the extension of Weil representation for these left coset representatives. First, we get the following result for the α and γ_b cosets:

Proposition 3.3.2 (Proposition 5.1 of [St]). *Let p be an odd prime, a, l positive integers with $a < 2l$ and $b \in (\mathbb{Z}/p^a\mathbb{Z})^*$. Then*

$$\rho_L^{-1}(\alpha)e_\lambda = e_{p^l\lambda}, \tag{3.108}$$

$$\rho_L^{-1}(\gamma_b)e_\lambda = \sum_{\substack{\nu \in A \\ p^l\nu=\lambda}} \boldsymbol{e}(-bq(\nu))e_\nu. \tag{3.109}$$

We also need the extension of the Weil representation for the $\beta_{b,a}$ cosets. Theorem 5.2 of [St] presents explicit formulae for the cases $l \geq a$ and $l < a$ separately. However, we claim that there is a mistake in the calculation leading to the formula for the case $l < a$ presented in Theorem 5.2 of [St]. As such, we provide here new formulae for the extension of the Weil representation studied in [BS, St].

Proposition 3.3.3. *Let p be an odd prime, a, l positive integers with $a < 2l$ and $b \in (\mathbb{Z}/p^a\mathbb{Z})^*$. Then*

$$\rho_L^{-1}(\beta_{b,a})e_\lambda = \begin{cases} p^{-\frac{a}{2}\dim L} \displaystyle\sum_{\substack{\delta \in A(p^a) \\ p^a\delta=\lambda}} \boldsymbol{e}\left(-bq_{p^a}(\delta)\right) e_{p^{l-a}\lambda} & \text{if } l \geq a, \\ p^{-\frac{a}{2}\dim L} \displaystyle\sum_{\substack{\mu \in A \\ p^{a-l}\mu=\lambda}} \sum_{\substack{\delta \in A(p^l) \\ p^l\delta=\mu}} \boldsymbol{e}\left(-bp^{a-l}q_{p^l}(\delta)\right) e_\mu & \text{if } l < a. \end{cases} \tag{3.110}$$

Proof. Our starting point is the beginning of the proof of Theorem 5.2 in [St]. Note that h and s in [St] are denoted by b and a respectively in this thesis. Stein notes that $\beta_{b,a}$ has the following decomposition,

$$\beta_{b,a} = \left(\left(\begin{smallmatrix} r & b \\ t & p^a \end{smallmatrix}\right), \sqrt{t\tau + p^a}\right) \alpha \left(\left(\begin{smallmatrix} 1 & 0 \\ -p^{2l-a} & 1 \end{smallmatrix}\right), \sqrt{-p^{2l-a}t\tau + 1}\right) \tag{3.111}$$

where $rp^a - bt = 1$ and $\alpha = \left(\left(\begin{smallmatrix} p^{2l} & 0 \\ 0 & 1 \end{smallmatrix}\right), 1\right)$ as before. The action of $\beta_{b,a}$ on the group algebra as defined by (3.104) is given by,

$$e_\lambda|_{\beta_{b,a}} = e_\lambda|_L \left(\left(\begin{smallmatrix} r & b \\ t & p^a \end{smallmatrix}\right), \sqrt{t\tau + p^a}\right) |_{:}\alpha|_L \left(\left(\begin{smallmatrix} 1 & 0 \\ -p^{2l-a} & 1 \end{smallmatrix}\right), \sqrt{-p^{2l-a}t\tau + 1}\right) \tag{3.112}$$

An application of Shintani's formula [Shin, Prop 1.6] and a reciprocity relation is used to obtain the following formula:

$$\rho_L^{-1}(\beta_{b,a})e_\lambda = \frac{1}{\sqrt{|A|^3}\sqrt{|A(p^a)|}} \sum_{\nu,\rho,\mu \in A} \boldsymbol{e}\left(brq(\lambda) - p^{2l-a}tq(\nu) - (\nu,\rho) - b(\mu,\lambda) + p^l(\mu,\nu)\right) e_\rho \times \sum_{\delta \in A(p^a)} \boldsymbol{e}\left(tq_{p^a}(\delta) - r(\delta,\lambda)_{p^a} + (\mu,\delta)_{p^a}\right). \tag{3.113}$$

Since

$$\frac{\sqrt{|A|}}{\sqrt{|A(p^a)|}} = p^{-\frac{a}{2}\dim L}, \tag{3.114}$$

we can rewrite this equation as

$$\rho_L^{-1}(\beta_{b,a})e_\lambda = \frac{p^{-\frac{a}{2}\dim L}}{|A|^2} \sum_{\nu,\rho,\mu \in A} \boldsymbol{e}\left(brq(\lambda) - p^{2l-a}tq(\nu) - (\nu,\rho) - b(\mu,\lambda) + p^l(\mu,\nu)\right) e_\rho \times \sum_{\delta \in A(p^a)} \boldsymbol{e}\left(tq_{p^a}(\delta) - r(\delta,\lambda)_{p^a} + (\mu,\delta)_{p^a}\right). \tag{3.115}$$

Note that the integers r, p^a, b and t are related by

$$rp^a - bt = 1. \tag{3.116}$$

In particular, b and p^a are coprime.

$\boxed{l \geq a}$

We first consider the case $l \geq a$. Let us do a shift $\delta \mapsto \delta - p^{l-a}\nu$. Since $p^{l-a}\nu \in A \subseteq A(p^a)$, the shift does not change the sum over δ since it is just relabeling. We get:

$$\begin{aligned} \rho_L^{-1}(\beta_{b,a})e_\lambda = \frac{p^{-\frac{a}{2}\dim L}}{|A|^2} \sum_{\nu,\rho,\mu \in A} \boldsymbol{e}\left(brq(\lambda) - (\nu,\rho) - b(\mu,\lambda) + rp^l(\nu,\lambda)\right) e_\rho \\ \times \sum_{\delta \in A(p^a)} \boldsymbol{e}\left(tq_{p^a}(\delta) - (p^a\delta, tp^{l-a}\nu + r\lambda - \mu)\right). \end{aligned} \tag{3.117}$$

Let us do a further shift $\mu \mapsto \mu + r\lambda + tp^{l-a}\nu$, which also does not change the sum:

$$\begin{aligned} \rho_L^{-1}(\beta_{b,a})e_\lambda = \frac{p^{-\frac{a}{2}\dim L}}{|A|^2} \sum_{\nu,\rho,\mu \in A} \boldsymbol{e}\left(-brq(\lambda) - (\nu,\rho) - b(\mu,\lambda) + p^{l-a}(\nu,\lambda)\right) e_\rho \\ \times \sum_{\delta \in A(p^a)} \boldsymbol{e}\left(tq_{p^a}(\delta) + (p^a\delta, \mu)\right). \end{aligned} \tag{3.118}$$

The sum over $\nu \in A$ is non-zero and equal to $|A|$ if and only if $\rho = p^{l-a}\lambda$

mod L. Thus we get:

$$\rho_L^{-1}(\beta_{b,a})e_\lambda = \frac{p^{-\frac{a}{2}\dim L}}{|A|} \times \sum_{\mu\in A} \boldsymbol{e}\left(-brq(\lambda) - b(\mu,\lambda)\right) e_{p^{l-a}\lambda} \sum_{\delta\in A(p^a)} \boldsymbol{e}\left(tq_{p^a}(\delta) + (p^a\delta,\mu)\right). \quad (3.119)$$

The sum over μ is then non-zero and equal to $|A|$ if and only if $b\lambda = p^a\delta$ mod L. Thus we get:

$$\rho_L^{-1}(\beta_{b,a})e_\lambda = p^{-\frac{a}{2}\dim L}\boldsymbol{e}\left(-brq(\lambda)\right) e_{p^{l-a}\lambda} \sum_{\substack{\delta\in A(p^a)\\ p^a\delta=b\lambda}} \boldsymbol{e}\left(tq_{p^a}(\delta)\right) \quad (3.120)$$

Now let S be the set of $\delta \in A(p^a)$ such that $p^a\delta = b\lambda$ for some fixed $\lambda \in A$, and S' be the set of $\delta' \in A(p^a)$ such that $p^a\delta' = \lambda$. We claim that there is a bijection $f : S' \to S$ given by $f : \delta' \mapsto \delta = b\delta'$.

First, let us show that it is injective. Any two $\delta'_1, \delta'_2 \in S'$ must differ by an element of $\frac{1}{p^a}L/L \subseteq A(p^a)$, that is, $\delta'_1 = \delta'_2 + \mu$ for some $\mu \in \frac{1}{p^a}L/L$. But then, $b\delta'_1 = b\delta'_2 + b\mu$, and $b\mu = 0$ mod L if and only if $\mu = 0$ mod L, since b and p^a are coprime. Therefore $b\delta'_1 = b\delta'_2$ if and only if $\delta'_1 = \delta'_2$ mod L.

Second, we show that f is surjective. We need to show that any $\delta \in S$ can be written as $\delta = b\delta'$ for some $\delta' \in S'$. Pick a $\delta' \in S'$. δ can be written as $\delta = b\delta' + \mu$ for some $\mu \in A(p^a)$. But then

$$b\lambda = p^a\delta = bp^a\delta' + p^a\mu = b\lambda + p^a\mu. \quad (3.121)$$

and hence $p^a\mu = 0$ mod L, that is, $\mu \in \frac{1}{p^a}L/L \subseteq A(p^a)$. Now, since b is coprime with p^a, we can always write $\mu = b\nu$ for some $\nu \in \frac{1}{p^a}L/L$. Thus we get

$$\delta = b\delta' + b\nu = b(\delta' + \nu) = b\delta'', \quad (3.122)$$

where $\delta'' = \delta' + \nu \in A(p^a)$, and $p^a\delta'' = p^a\delta' + p^a\nu = p^a\delta' = \lambda$. Thus we conclude that $\delta = h\delta''$, with $\delta'' \in S'$.

As a result, the bijection $f : S' \to S$ allows us to substitute $\delta = b\delta'$ in (3.120) and replace the sum over $\delta \in A(p^a)$ such that $p^a\delta = b\lambda$ by a sum over $\delta' \in A(p^a)$ such that $p^a\delta' = \lambda$. We get:

$$\rho_L^{-1}(\beta_{b,a})e_\lambda = p^{-\frac{a}{2}\dim L}\boldsymbol{e}\left(-brq(\lambda)\right)e_{p^{l-a}\lambda}\sum_{\substack{\delta'\in A(p^a)\\ p^a\delta'=\lambda}}\boldsymbol{e}\left(tb^2q_{p^a}(\delta')\right). \tag{3.123}$$

Now using $bt = rp^a - 1$,

$$\rho_L^{-1}(\beta_{b,a})e_\lambda = p^{-\frac{a}{2}\dim L}\boldsymbol{e}\left(-brq(\lambda)\right)e_{p^{l-a}\lambda}\sum_{\substack{\delta'\in A(p^a)\\ p^a\delta'=\lambda}}\boldsymbol{e}\left(-bq_{p^a}(\delta')\right)\boldsymbol{e}\left(brq(p^a\delta')\right) \tag{3.124}$$

$$= p^{-\frac{a}{2}\dim L}\sum_{\substack{\delta'\in A(p^a)\\ p^a\delta'=\lambda}}\boldsymbol{e}\left(-bq_{p^a}(\delta')\right)e_{p^{l-a}\lambda}. \tag{3.125}$$

$\boxed{l < a}$

Let us start again with

$$\rho_L^{-1}(\beta_{b,a})e_\lambda = \frac{p^{-\frac{a}{2}\dim L}}{|A|^2} \times \sum_{\nu,\rho,\mu\in A}\boldsymbol{e}\left(brq(\lambda) - p^{2l-a}tq(\nu) - (\nu,\rho) - b(\mu,\lambda) + p^l(\mu,\nu)\right)e_\rho \times \sum_{\delta\in A(p^a)}\boldsymbol{e}\left(tq_{p^a}(\delta) - r(\delta,\lambda)_{p^a} + (\mu,\delta)_{p^a}\right). \tag{3.126}$$

Let us rewrite the sum over $\nu \in A$ as a sum over $\gamma \in A(p^a)$, with $p^a\gamma = \nu$. This map is not one-to-one; its kernel is given by $\frac{1}{p^a}L/L \subseteq A(p^a)$. Thus we need to divide by $\left|\frac{1}{p^a}L/L\right| = p^{a\dim L}$. We get:

$$\begin{aligned}\rho_L^{-1}(\beta_{b,a})e_\lambda = &\frac{p^{-\frac{3a}{2}\dim L}}{|A|^2} \\ &\times \sum_{\rho,\mu\in A}\sum_{\gamma\in A(p^a)} \boldsymbol{e}\left(brq(\lambda) - p^{2l}tq_{p^a}(\gamma) - (p^a\gamma,\rho) - b(\mu,\lambda) + p^l(\mu,p^a\gamma)\right) e_\rho \\ &\times \sum_{\delta\in A(p^a)} \boldsymbol{e}\left(tq_{p^a}(\delta) - (p^a\delta, r\lambda - \mu)\right). \end{aligned} \tag{3.127}$$

We then do a shift $\delta \mapsto \delta - p^l\gamma$. Since $p^l\gamma \in A(p^{a-l}) \subseteq A(p^a)$, the shift does not change the sum. We get

$$\begin{aligned}\rho_L^{-1}(\beta_{b,a})e_\lambda = &\frac{p^{-\frac{3a}{2}\dim L}}{|A|^2} \\ &\times \sum_{\rho,\mu\in A}\sum_{\gamma\in A(p^a)} \boldsymbol{e}\left(brq(\lambda) - (p^a\gamma,\rho) - b(\mu,\lambda) + rp^l(p^a\gamma,\lambda)\right) e_\rho \\ &\times \sum_{\delta\in A(p^a)} \boldsymbol{e}\left(tq_{p^a}(\delta) - (p^a\delta, r\lambda - \mu) - tp^l(\delta,\gamma)_{p^a}\right). \end{aligned} \tag{3.128}$$

We now do a shift $\mu \mapsto \mu + r\lambda$ to get

$$\begin{aligned}\rho_L^{-1}(\beta_{b,a})e_\lambda = &\frac{p^{-\frac{3a}{2}\dim L}}{|A|^2} \\ &\times \sum_{\rho,\mu\in A}\sum_{\gamma\in A(p^a)} \boldsymbol{e}\left(-brq(\lambda) - (\gamma,\rho)_{p^a} - b(\mu,\lambda) + rp^l(\gamma,\lambda)_{p^a}\right) e_\rho \\ &\times \sum_{\delta\in A(p^a)} \boldsymbol{e}\left(tq_{p^a}(\delta) + (p^a\delta, \mu) - tp^l(\delta,\gamma)_{p^a}\right). \end{aligned} \tag{3.129}$$

The sum over $\mu \in A$ is non-zero and equal to $|A|$ if and only if $p^a\delta = b\lambda \bmod L$. Moreover, as we have seen in the calculation for the $l \geq a$ case, we can substitute $\delta = b\delta'$ and replace the sum over $\delta \in A(p^a)$ such that $p^a\delta = b\lambda$ by a sum over $\delta' \in A(p^a)$ such that $p^a\delta' = \lambda$. Thus we can write

$$\rho_L^{-1}(\beta_{b,a})e_\lambda = \frac{p^{-\frac{3a}{2}\dim L}}{|A|}\sum_{\rho\in A}\sum_{\gamma\in A(p^a)} \boldsymbol{e}\left(-brq(\lambda) - (\gamma,\rho)_{p^a} + rp^l(\gamma,\lambda)_{p^a}\right) e_\rho \times \sum_{\substack{\delta'\in A(p^a)\\ p^a\delta'=\lambda}} \boldsymbol{e}\left(tb^2 q_{p^a}(\delta') - btp^l(\delta',\gamma)_{p^a}\right). \quad (3.130)$$

Using $bt = rp^a - 1$,

$$\rho_L^{-1}(\beta_{b,a})e_\lambda = \frac{p^{-\frac{3a}{2}\dim L}}{|A|} \times \sum_{\rho\in A}\sum_{\gamma\in A(p^a)} \boldsymbol{e}\left(-(\gamma,\rho)_{p^a}\right) e_\rho \sum_{\substack{\delta'\in A(p^a)\\ p^a\delta'=\lambda}} \boldsymbol{e}\left(-bq_{p^a}(\delta') + p^l(\delta',\gamma)_{p^a}\right). \quad (3.131)$$

The sum over $\gamma \in A(p^a)$ is non-zero and equal to $|A(p^a)|$ if and only if $\rho = p^l\delta'$ mod L. In particular, since $\rho \in A$, this implies that the sum can be non-zero only for $\delta' \in A(p^l) \subseteq A(p^a)$. Thus we get:

$$\begin{aligned}\rho_L^{-1}(\beta_{b,a})e_\lambda =& \frac{p^{-\frac{3a}{2}\dim L}|A(p^a)|}{|A|}\sum_{\substack{\delta'\in A(p^l)\\ p^a\delta'=\lambda}} \boldsymbol{e}\left(-bp^{a-l}q_{p^l}(\delta')\right) e_{p^l\delta'}\\ =& p^{-\frac{a}{2}\dim L}\sum_{\substack{\mu\in A\\ p^{a-l}\mu=\lambda}}\sum_{\substack{\delta'\in A(p^l)\\ p^l\delta'=\mu}} \boldsymbol{e}\left(-bp^{a-l}q_{p^l}(\delta')\right) e_\mu. \qquad (3.132)\end{aligned}$$

□

In the next section we will compare the Hecke operators of Bruinier-Stein with the Hecke operators constructed in section 3.2. The formula for the case $l \geq a$ that we obtain is equivalent to the one presented in Theorem 5.2 of [St], but our formula for the case $l < a$ is not. However, we get a precise match after incorporating the fix of Proposition 3.3.3.

3.4 Comparison of new Hecke operators to Bruinier-Stein

Let us now compare the Bruinier-Stein construction with our Hecke operators $\mathcal{H}_{p^{2l}}$. Since the Bruinier-Stein operators are defined for holomorphic vector-valued modular forms, we also restrict our construction to holomorphic vector-valued modular forms. In particular, we only consider weights of the form $(v, \bar{v}) = (k, 0)$. Recall that the Hecke operators $T_{p^{2l}}$ of Bruinier-Stein are given by,

$$T_{p^{2l}}^{(BS)}[\psi](\tau) = p^{2l(k-1)} \sum_{i} \sum_{\lambda \in A} \phi_{\delta_i}(\tau)^{-2k} \psi_\lambda(\delta_i \tau) \rho_L^{-1}(\delta_i) e_\lambda, \tag{3.133}$$

where the first sum is over the left coset representatives δ_i in (3.106) and we use the subscript BS to distinguish them from the Hecke operators of [BCJ].

Substituting the formulae in Propositions (3.3.2) and (3.3.3) in (3.105) and simplifying, we can write

$$T_{p^{2l}}^{(BS)}[\psi](\tau) = C_\alpha^{(BS)}(\tau) + \sum_{a=1}^{2l-1} \sum_{b \in (\mathbb{Z}/p^a\mathbb{Z})^*} C_{\beta_{b,a}}^{(BS)}(\tau) + \sum_{b=0}^{p^{2l}-1} C_{\gamma_b}^{(BS)}(\tau), \tag{3.134}$$

with

$$C_{\alpha}^{(BS)}(\tau) = p^{2l(k-1)} \sum_{\lambda \in A} \psi_\lambda(p^{2l}\tau) e_{p^l \lambda}, \tag{3.135}$$

$$C_{\beta_{b,a}}^{(BS)}(\tau) = \begin{cases} p^{(2l-a)k-2l-\frac{a}{2}\dim L} \sum_{\lambda \in A} \sum_{\substack{\delta \in A(p^a) \\ p^a\delta = \lambda}} \boldsymbol{e}\left(-bq_{p^a}(\delta)\right) \psi_\lambda \left(\frac{p^{2l-a}\tau + b}{p^a}\right) e_{p^{l-a}\lambda} & \text{if } l \geq a, \\ p^{(2l-a)k-2l-\frac{a}{2}\dim L} \sum_{\rho \in A} \sum_{\substack{\delta \in A(p^l) \\ p^l\delta = \rho}} \boldsymbol{e}\left(-bp^{a-l}q_{p^l}(\delta)\right) \psi_{p^{a-l}\rho} \left(\frac{p^{2l-a}\tau + b}{p^a}\right) e_\rho & \text{if } l < a, \end{cases} \tag{3.136}$$

$$C_{\gamma_b}^{(BS)}(\tau) = p^{-2l} \sum_{\lambda \in A} \boldsymbol{e}(-bq(\lambda)) \psi_{p^l\lambda} \left(\frac{\tau + b}{p^{2l}}\right) e_\lambda. \tag{3.137}$$

3.4.1 Comparison

Let us now compare the Bruinier-Stein construction with our Hecke operators $\mathcal{H}_{p^{2l}}$.

From Lemma 3.2.13, we can write:

$$\mathcal{H}_{p^{2l}}[\psi](\tau) = p^{2l(k-1)} \sum_{\lambda \in A} \left(\sum_{\substack{\gamma \in A(p^{2l}) \\ p^l\gamma = \lambda}} \sum_{a=0}^{2l} \frac{1}{p^{a\left(k+\frac{1}{2}\dim L\right)}} \right.$$
$$\left. \times \sum_{b=0}^{p^a-1} \Delta_{p^{2l}}(\gamma, p^l) \Delta_{p^{2l}}(\gamma, p^{2l-a}) \boldsymbol{e}\left(-\frac{b}{p^{2l-a}} q_{p^{2l}}(\gamma)\right) \psi_{p^a\gamma} \left(\frac{p^{2l-a}\tau + b}{p^a}\right)\right) e_\lambda. \tag{3.138}$$

We will consider a slightly different definition of the Hecke transform in this section. We use a double coset definition as in Bruinier and Stein, such that sum over left coset representatives is restricted to the sum over b to those that are in $(\mathbb{Z}/p^a\mathbb{Z})^*$.

To compare with the Hecke operator of Bruinier and Stein, we define:

$$\mathcal{H}_{p^{2l}}[\psi](\tau) = C_\alpha(\tau) + \sum_{a=1}^{2l-1} \sum_{b \in (\mathbb{Z}/p^a\mathbb{Z})^*} C_{\beta_{b,a}}(\tau) + \sum_{b=0}^{p^{2l}-1} C_{\gamma_b}(\tau), \tag{3.139}$$

with

$$C_\alpha(\tau) = p^{2l(k-1)} \sum_{\lambda \in A} \sum_{\substack{\gamma \in A(p^{2l}) \\ p^l\gamma = \lambda}} \Delta_{p^{2l}}(\gamma, p^l) \Delta_{p^{2l}}(\gamma, p^{2l}) \psi_\gamma \left(p^{2l}\tau\right) e_\lambda, \tag{3.140}$$

$$C_{\gamma_b}(\tau) = p^{-2l-l\dim L} \sum_{\lambda \in A} \sum_{\substack{\gamma \in A(p^{2l}) \\ p^l\gamma = \lambda}} \Delta_{p^{2l}}(\gamma, p^l) \Delta_{p^{2l}}(\gamma, 1) \boldsymbol{e}\left(-bq_{p^{2l}}\left(\gamma\right)\right) \psi_{p^{2l}\gamma} \left(\frac{\tau + b}{p^{2l}}\right) e_\lambda, \tag{3.141}$$

and

$$C_{\beta_{b,a}}(\tau) = p^{(2l-a)k-2l-\frac{a}{2}\dim L} \sum_{\lambda \in A} C_{\beta_{b,a}}(\lambda, \tau) e_\lambda, \tag{3.142}$$

where

$$C_{\beta_{b,a}}(\lambda, \tau) = \sum_{\lambda \in A} \sum_{\substack{\gamma \in A(p^{2l}) \\ p^l\gamma = \lambda}} \Delta_{p^{2l}}(\gamma, p^l) \Delta_{p^{2l}}(\gamma, p^{2l-a}) \boldsymbol{e}\left(-\frac{b}{p^{2l-a}} q_{p^{2l}}\left(\gamma\right)\right) \psi_{p^a\gamma} \left(\frac{p^{2l-a}\tau + b}{p^a}\right). \tag{3.143}$$

We need to compare these three expressions with the corresponding expressions obtained by Bruinier and Stein in equations (3.135), (3.136) and (3.137).

The α coset

We have:

$$C_\alpha(\tau) = p^{2l(k-1)} \sum_{\lambda \in A} \sum_{\substack{\gamma \in A(p^{2l}) \\ p^l \gamma = \lambda}} \Delta_{p^{2l}}(\gamma, p^l) \Delta_{p^{2l}}(\gamma, p^{2l}) \psi_\gamma \left(p^{2l}\tau\right) e_\lambda. \tag{3.144}$$

The $\Delta_{p^{2l}}(\gamma, p^{2l})$ implies that $\gamma \in A$, in which case the condition $\Delta_{p^{2l}}(\gamma, p^l)$ becomes trivial. So we can rewrite this expression as

$$C_\alpha(\tau) = p^{2l(k-1)} \sum_{\lambda \in A} \psi_\lambda \left(p^{2l}\tau\right) e_{p^l\lambda}, \tag{3.145}$$

which is precisely $C_\alpha^{(BS)}(\tau)$ in (3.135).

The γ_b cosets

Let us move on to the γ_b cosets. We have:

$$C_{\gamma_b}(\tau) = p^{-2l - l \dim L} \times \sum_{\lambda \in A} \sum_{\substack{\gamma \in A(p^{2l}) \\ p^l \gamma = \lambda}} \Delta_{p^{2l}}(\gamma, p^l) \Delta_{p^{2l}}(\gamma, 1) \boldsymbol{e}\left(-b q_{p^{2l}}(\gamma)\right) \psi_{p^{2l}\gamma} \left(\frac{\tau + b}{p^{2l}}\right) e_\lambda. \tag{3.146}$$

The $\Delta_{p^{2l}}(\gamma,1)$ condition is trivial, while the $\Delta_{p^{2l}}(\gamma,p^l)$ condition imposes that $\gamma \in A(p^l) \subseteq A(p^{2l})$. So we can write

$$\begin{aligned} C_{\gamma_b}(\tau) =& p^{-2l-l\dim L} \sum_{\lambda\in A} \sum_{\substack{\gamma\in A(p^l)\\ p^l\gamma=\lambda}} \boldsymbol{e}\left(-bq\left(p^l\gamma\right)\right) \psi_{p^{2l}\gamma}\left(\frac{\tau+b}{p^{2l}}\right) e_{p^l\gamma} \\ =& p^{-2l-l\dim L} \sum_{\lambda\in A} \boldsymbol{e}\left(-bq\left(\lambda\right)\right) \psi_{p^l\lambda}\left(\frac{\tau+b}{p^{2l}}\right) e_\lambda \left(\sum_{\substack{\gamma\in A(p^l)\\ p^l\gamma=\lambda}} 1\right) \\ =& p^{-2l} \sum_{\lambda\in A} \boldsymbol{e}\left(-bq\left(\lambda\right)\right) \psi_{p^l\lambda}\left(\frac{\tau+b}{p^{2l}}\right) e_\lambda, \end{aligned} \tag{3.147}$$

where we evaluated the summation as in (3.41). This is precisely $C^{(BS)}_{\gamma_b}(\tau)$ in (3.137).

The $\beta_{b,a}$ cosets

The remaining cases correspond to the $\beta_{b,a}$ cosets. We have:

$$\begin{aligned} &C_{\beta_{b,a}}(\tau) = p^{(2l-a)k-2l-\frac{a}{2}\dim L)} \\ \times&\sum_{\lambda\in A} \sum_{\substack{\gamma\in A(p^{2l})\\ p^l\gamma=\lambda}} \Delta_{p^{2l}}(\gamma,p^l)\Delta_{p^{2l}}(\gamma,p^{2l-a})\boldsymbol{e}\left(-\frac{b}{p^{2l-a}}q_{p^{2l}}\left(\gamma\right)\right) \psi_{p^a\gamma}\left(\frac{p^{2l-a}\tau+b}{p^a}\right) e_\lambda. \end{aligned} \tag{3.148}$$

The $\Delta_{p^{2l}}(\gamma, p^l)$ condition imposes that $\gamma \in A(p^l) \subseteq A(p^{2l})$. We rewrite

$$
\begin{aligned}
&C_{\beta_{b,a}}(\tau) = p^{(2l-a)k-2l-\frac{a}{2}\dim L} \\
&\quad \times \sum_{\lambda \in A} \sum_{\substack{\gamma \in A(p^l) \\ p^l \gamma = \lambda}} \Delta_{p^{2l}}(\gamma, p^{2l-a}) \boldsymbol{e}\left(-\frac{b}{p^{2l-a}} q_{p^{2l}}(\gamma)\right) \psi_{p^a \gamma}\left(\frac{p^{2l-a}\tau + b}{p^a}\right) e_\lambda \\
&= p^{(2l-a)k-2l-\frac{a}{2}\dim L} \sum_{\gamma \in A(p^l)} \Delta_{p^{2l}}(\gamma, p^{2l-a}) \boldsymbol{e}\left(-\frac{b}{p^{2l-a}} q_{p^{2l}}(\gamma)\right) \psi_{p^a \gamma}\left(\frac{p^{2l-a}\tau + b}{p^a}\right) e_{p^l \gamma}.
\end{aligned}
\tag{3.149}
$$

$\boxed{l \geq a}$

Let us consider first the case when $l \geq a$. The $\Delta_{p^{2l}}(\gamma, p^{2l-a})$ condition then imposes that $\gamma \in A(p^a)$. Since $l \geq a$, $A(p^a) \subseteq A(p^l)$, hence we can write

$$
\begin{aligned}
C_{\beta_{b,a}}(\tau) &= p^{(2l-a)k-2l-\frac{a}{2}\dim L} \sum_{\gamma \in A(p^a)} \boldsymbol{e}\left(-b q_{p^a}(\gamma)\right) \psi_{p^a \gamma}\left(\frac{p^{2l-a}\tau + b}{p^a}\right) e_{p^l \gamma} \\
&= p^{(2l-a)k-2l-\frac{a}{2}\dim L} \sum_{\lambda \in A} \left(\sum_{\substack{\gamma \in A(p^a) \\ p^a \gamma = \lambda}} \boldsymbol{e}\left(-b q_{p^a}(\gamma)\right) \right) \psi_\lambda \left(\frac{p^{2l-a}\tau + b}{p^a}\right) e_{p^{l-a}\lambda}.
\end{aligned}
\tag{3.150}
$$

This is precisely $C^{(BS)}_{\beta_{b,a}}(\tau)$ with $l \geq a$ in (3.136).

$\boxed{l < a}$

We start with

$$C_{\beta_{b,a}}(\tau) = p^{(2l-a)k-2l-\frac{a}{2}\dim L}$$
$$\times \sum_{\lambda\in A} \sum_{\substack{\gamma\in A(p^l)\\ p^l\gamma=\lambda}} \Delta_{p^{2l}}(\gamma, p^{2l-a}) \boldsymbol{e}\left(-\frac{b}{p^{2l-a}} q_{p^{2l}}(\gamma)\right) \psi_{p^a\gamma}\left(\frac{p^{2l-a}\tau+b}{p^a}\right) e_\lambda. \quad (3.151)$$

The $\Delta_{p^{2l}}(\gamma, p^{2l-a})$ condition imposes that $\gamma \in A(p^a)$, but since $l < a$, $A(p^l) \subseteq A(p^a)$, hence this condition is vacuous. Thus we get:

$$C_{\beta_{b,a}}(\tau) = p^{(2l-a)k-2l-\frac{a}{2}\dim L}$$
$$\times \sum_{\lambda\in A} \left(\sum_{\substack{\gamma\in A(p^l)\\ p^l\gamma=\lambda}} \boldsymbol{e}\left(-bp^{a-l} q_{p^l}(\gamma)\right) \right) \psi_{p^{a-l}\lambda}\left(\frac{p^{2l-a}\tau+b}{p^a}\right) e_\lambda. \quad (3.152)$$

This is precisely $C^{(BS)}_{\beta_{b,a}}(\tau)$ with $l < a$ in (3.136).

Chapter 4

Discussion and applications

In this thesis, Hecke operators acting on vector-valued modular forms of the Weil representation were constructed and some of their algebraic relations were computed. As discussed in Chapter 1, vector-valued modular forms offer a deeper insight into the theory of modular forms and Jacobi forms. They have also found applications in diverse areas such as enumerative geometry and VOAs. In this chapter we give a brief survey of some applications of the theory of vector-valued modular forms. We also list some possible directions for further research.

4.1 Borcherds products

Product formulae are relations between formal power series and infinite product expansions of functions and offer insights into the algebra and geometry of automorphic forms. In physics, product formulae for automorphic forms arise from generating functions of elliptic genera of symmetric product orbifolds (see [DMVV]). Borcherds found a set of product formulae for modular forms for orthogonal groups $O(2,n)$ through a correspondence with classical modular forms by using vector-valued modular forms of the Weil representation as a tool for computation. The product formula for $O(2,2)$ for example gives the product formula of j-invariant discovered by Koike-Norton-Zagier as the denominator identity of the Monster Lie algebra and used by Borcherds ([BOR92]) in his proof of Monstrous moonshine,

$$j(\tau) - j(\tau^{'}) = (e^{-2\pi i\tau'} - e^{-2\pi i\tau}) \prod_{m,n=1}^{\infty} (1 - e^{2\pi i(m\tau + n\tau')})^{c(mn)}, \tag{4.1}$$

where $j(q) = 744 + q^{-1} + \sum_{n=1}^{\infty} c(n)q^n$ is the q-expansion of $j(\tau)$ and τ and τ' are in the upper half plane. The Borcherds lift or the singular theta correspondence gives a map from the space of (almost holomorphic) vector-valued modular forms of the Weil representation to automorphic forms for orthogonal subgroups. This construction is beyond the scope of this thesis but we urge the interested reader to refer to [Br], [Bo1] and [Ker] for further details. We give a brief sketch of the correspondence below where we have made extensive use of the references mentioned above. Let $(L, (.,.))$ be an even lattice of type $(n,2)$ and $(.,.)$ its bilinear form. The associated quadratic form is denoted by $q(.) = \frac{(.,.)}{2}$. The quadratic and bilinear forms can be linearly extended to $V = L \otimes_{\mathbb{Z}} \mathbb{C}$ and the corresponding projective space $\mathbb{V}(\mathbb{C})$. We now define automorphic/modular forms for orthogonal groups in the spirit of [Br]. We are interested in the complex manifold,

$$\mathcal{K} = \{[Z] \in \mathbb{P}(V \otimes_{\mathbb{R}} \mathbb{C}) : q(Z) = 0, (Z, \bar{Z}) > 0\}. \tag{4.2}$$

This space has two connected components related to each other by complex conjugation. We denote the pre-image of one of these components under projection by $\widetilde{\mathcal{K}}^{+}$. This acts as a model of the upper half plane for the complex manifold $\mathcal{K}$. We denote the orthogonal group on L by $O(L)$ and the component preserving subgroup by

$$O(L)^+ = \{\sigma \in O(L) : \sigma(\widetilde{\mathcal{K}}^+) = \widetilde{\mathcal{K}}^+\}. \tag{4.3}$$

If Γ is a subgroup of $O(L)^+$, an automorphic form of weight k with character χ is a meromorphic function $\phi : \widetilde{\mathcal{K}}^+ \to \mathbb{C}$ if,

$$\Psi(tZ) = t^{-k}\psi(Z) \tag{4.4}$$

$$\Psi(\sigma(Z)) = \chi(\sigma)\Psi(\sigma) \quad \forall Z \in \widetilde{\mathcal{K}}^+, t \in \mathbb{C}^\times, \sigma \in \Gamma. \tag{4.5}$$

Other models exist for the Grassmannian submanifold $\widetilde{\mathcal{K}}^+$. For example, Borcherds in [Bo1] shows that the space of maximal positive definite subspaces of V denoted by $\mathrm{Gr}(L)$ can be given the complex structure of $\widetilde{\mathcal{K}}^+$.

Let $F(\tau)$ be a weakly holomorphic vector-valued modular form of type ρ_L and weight $1 - \frac{n}{2}$ with Fourier expansion $F(\tau) = \sum_{\gamma \in L'/L} \sum_{m=-q(\gamma)+\mathbb{Z}} c_\gamma(m) q^m e_\gamma$ and $c_\gamma(m) \in \mathbb{Z}$ and $c_0(0) \in 2\mathbb{Z}$. The Borcherds or theta lift of the vector-valued modular form $F(\tau)$ is obtained by integrating $F(\tau)$ alongside the Siegel theta function $\Theta_L(\tau, \alpha, \beta, v, p)$ that was studied in Section 2.4 where $\tau \in \mathbb{H}, \alpha, \beta \in L \otimes R, v \in \widetilde{\mathcal{K}}^+$ and p is a certain homogenous polynomial. In the current context we set $\alpha, \beta = 0, p = 0$. The theta lift is given by an integral over a fundamental domain $\mathrm{SL}_2(\mathbb{Z})/\mathbb{H}$,

$$\Phi(v, F) = \int_{\mathcal{F}} \langle F(\tau), \Theta_L(\tau, v)\rangle y \frac{dxdy}{y^2} \tag{4.6}$$

where $v \in \widetilde{\mathcal{K}}^+$ and $\tau = x + iy$. In the above equation $\langle F, \Theta\rangle$ is the standard pairing on $\mathbb{C}[L'/L]$ as defined in (2.32). The integral above diverges but it can be "regularized" through a procedure first introduced by Harvey and Moore in [HM98] and formalized by Borcherds. We denote the regularized integral by Φ_r and let $\Psi(F) = \exp(\Phi_r(., F))$. The subgroup stabilizing F

$$O(L, F)^+ = \{\sigma \in O(L)^+ : F^\sigma = F\}, \tag{4.7}$$

is a finite index subgroup of $O(L)^+$. We state below the following important theorem of Borcherds (See [Bo1][Theorem 13.3] for the complete statement).

Theorem 4.1.1. *Let L be an even, non-degenerate lattice of type $(n, 2)$ and $F(\tau), \Psi(F)$ be as above. Then $\Psi(F)$ is an automorphic form of weight $c_0(0)/2$ for $O(L, F)^+$ with some unitary character of $O(L, F)^+$.*

Theorem 13.3 of Borcherds also gives a convergent product expansion for $\Phi(F)$ and such product expansions are thus called Borcherds products. Automorphic forms in several areas of mathematics arise as theta lifts with convergent product expansions.

Example 4.1.2. The Igusa cusp form Φ_{10} of bosonic string theory can be obtained as a lift of the elliptic genus of $K3$ surfaces. See [Ka] for further details.

This theta lift can be used to obtain new examples of generalized Kac-Moody algebras with denominator functions that are automorphic forms (See [Sch], [Sch1], [Sch2]).

Example 4.1.3. The Borcherds function Φ_{12} (See [Bo2]) is an automophic form for $O(II_{26,2})^+$ where $II_{26,2}$ is an even unimodular lattice of signature $(26, 2)$. This is the theta lift of the inverse of the discriminant form $1/\Delta$ and the singular theta correspondence recovers the denominator identity of the fake monster Lie algebra.

The techniques of this thesis could be applied in the study of the singular theta correspondence. One could ask the following question:

Problem 4.1.4. How does the singular theta correspondence of Theorem 4.1.1 behave under the action of Hecke operators? Are the Hecke operators of this thesis compatible with the singular theta correspondence? (See [Bo, Problem 16.5] for further details and related questions.)

4.2 Donaldson-Thomas invariants

Vector-valued modular forms and Jacobi forms are ubiquitous in quantum field theory, string theory and algebraic geometry. For example, the elliptic genus of a Calabi-Yau manifold of dimension d is a Jacobi form with integral coefficients of weight 0 and index $d/2$. Counting problems in string theory and gauge theory (such as counting of degeneracies of BPS states) correspond to counting of topological invariants such as the Euler characteristic of moduli spaces, and partition functions in string theory and gauge theory thus gives us a way to write down generating functions of these topological or enumerative geometric invariants. In addition, these generating functions satisfy modularity properties thanks to dualities such as the S and T duality in gauge theory and string theory. Two physical theories are said to be dual if the correlation functions or observables of the two theories are mapped to each other by a transformation of a parameter. The topic of this thesis is inspired from the counting of vertical $D4-D2-D0$ states in type IIA string theory or generalized Donaldson-Thomas invariants on a smooth projective Calabi-Yau threefold X satisfying certain technical conditions (See [B–S, Section 2]). In mathematical terms a supersymmetric $D4-D2-D0$ configuration is a "Gieseker semistable torsion coherent sheaf" with respect to a certain Kahler class ω on X. Such a sheaf has numerical invariants $\gamma = (r, d, n) \in \mathbb{Z}_{n\geq 1} \times L' \times \mathbb{Z}$ where L' is a dual lattice of an even lattice L with quadratic form $q(.)$ that we will describe below. In [B–S] a formula for the partition function of these generalized Donaldson-

Thomas invariants $\{Z_{DT}(X, r, \delta; \tau)\}_{\delta \in A(r)}$ was obtained using the Gromov-Witten/stable-pair correspondence where $A(r)$ is a rescaled discriminant form. The construction of [B–S] starts with a vector-valued modular form $\psi(\tau)$ of type ρ_L and weight $(v, \bar{v}) = \left(-1 - \frac{\dim L}{2}, 0\right)$ and then the partition functions $(Z_{DT}(X, r, \delta; \tau))$ are obtained in terms of a Hecke-like transform. This is precisely the Hecke operator that we constructed in section 3.2. Then it is proved that $\{Z_{DT}(X, r, \delta; \tau)\}_{\delta \in A(r)}$ is a vector-valued modular form of weight $(-1 - \dim L/2)$ of the Weil representation $\rho_{L(r)}$ through direct calculations based on Shintani's formula of [Shin] for the matrix elements of the Weil representation. With the construction proposed in this thesis and the paper [BCJ], such a modularity statement follows directly from Corollary 3.2.3.

We briefly state below the setup and results of [B–S]. The interested reader is urged to refer to [B–S] and the references within for further details.

4.2.1 Generalized Donaldson-Thomas invariants

We begin by giving a non-technical overview of the definition of the integral Donaldson-Thomas (DT) invariants and the generalization by Joyce and Song to rational DT invariants.

Definition 4.2.1. Let X be a Calabi-Yau(CY) 3-fold. A holomorphic vector bundle $\pi : E \to X$ of rank r is a complex manifold E with a holomorphic map $\pi : E \to X$ whose fibres are complex vector spaces $\mathbb{C}^r$. These have topological invariants given by the Chern character $\mathrm{ch}_*(E) \in H^{even}(X, \mathbb{Q})$ with $\mathrm{ch}_0(E) = r$.

However, we are more interested in the larger category of coherent sheaves as their moduli space is better behaved. We do not give here a formal definition of coherent sheaves and Geiseker semistable coherent sheaves as it is outside the scope of this thesis. However, we will attempt to give a more intuitive understanding of these terms to paint a rough picture. Roughly speaking, coherent sheaves can be understood as possibly singular vector bundles $E \to Y$ on complex submanifolds Y in X.

In particular we are interested in Geiseker (semi)stable coherent sheaves. A coherent sheaf E is Geiseker semi(stable) if all subsheaves $F \subset E$ satisfy some numerical conditions (See [B–S, Section 2.3]) depending on the Kahler class $[\omega] \in H^2(X, \mathbb{R})$. In the setup of [B–S] X is assumed to be projective and E to be set theoretically supported on a finite union of $K3$ fibers. Then, such a sheaf has numerical invariants $\gamma = (r, d, n) \in \mathbb{Z}_{n\geq 1} \times L' \times \mathbb{Z}$ where L' is a dual lattice of an even lattice $L \subseteq H_2(X, \mathbb{Z})$. The bilinear form on L' is given by the restriction of the intersection form on $H^2(X_p, \mathbb{Z})$ for a smooth $K3$ fibre X_p to L (See [B–S, Section 3] for more details). The lattice L depends on the even integral homology of X. For example we have the relation $H_4(X, \mathbb{Z}) \cong \mathbb{Z} \oplus L'$. The set of all isomorphism classes of the Geiseker-semistable sheaves defined above can be encoded in a *coarse moduli scheme* denoted by $\mathcal{M}_\omega(\gamma)$. In the absense of semistable objects, the DT invariants were defined by Richard Thomas. In this case there is a $\mathbb{Z}-$valued constructible function on $\mathcal{M}_\omega(\gamma)$ called the "Behrend function" that counts the the number of points with multiplicity. In terms of this function we have,

$$DT_\omega(\gamma) = \int_{\mathcal{M}_\omega(\gamma)} \nu \, d\chi. \tag{4.8}$$

Thomas showed that the $DT_\omega(\gamma)$s are unchanged under deformations of the complex structure ω and thus this label can be omitted. These were generalized to the case of semistable objects by Joyce and Song in [JS] through a very technical procedure and are called generalized DT invariants $DT_\omega(\gamma) \in \mathbb{Q}$ that are independent of ω as well. They also conjectured the existence of integral invariants $\Omega_\omega(\gamma) \in \mathbb{Z}$ related to rational DT invariants via the multicover formula,

$$DT_\omega(\gamma) = \sum_{\substack{k\in\mathbb{Z} \\ \gamma = k\gamma'}} \frac{1}{k^2}\Omega_\omega(\gamma'). \tag{4.9}$$

The $\Omega(\gamma) := \Omega_\omega(\gamma)$ count the BPS degeneracies of the $D4 - D2 - D0$ bound states in the string theory picture.

4.2.2 Partition function and modularity

Now we define the relevant partition function and discuss its modularity. In this section L is the lattice described above with a bilinear form of type $(1, \dim L - 1)$ that comes from the restriction of an intersection form. In addition we also have the discriminant group $A = L'/L$ and the rescaled discriminant group $A(r) = L(r)'/L$ defined in Section 2.2. Let $\Omega(\gamma)$ be the conjectured integral invariants in (4.9). For a fixed r of the triples $\gamma = (r, d, n)$ the partition function for the integral DT invariants is given by

$$Z_{BPS}(X, r, \delta; \tau) = \sum_{n\in\mathbb{Z}} \Omega(\gamma) e^{2\pi i\tau(n + q_r(\frac{d}{r}) - 1)}, \quad \frac{d}{r} = \delta + L \in L' \tag{4.10}$$

where δ is a coset representative of $\frac{d}{r}$ in $A(r)$. We pair these with the complex conjugate of the Siegel theta functions

$$\bar{\theta}_{L(r)+\frac{d}{r}}(\tau, \bar{\tau}) = \sum_{\alpha\in L} e^{-2\pi i\tau q_r(\frac{d}{r}+\alpha)_- - 2\pi i\bar{\tau} q_r(\frac{d}{r}+\alpha)_+} \tag{4.11}$$

and studied in section 2.4 (See (2.64)). We are more interested in the partition function for rational invariants defined through the multicover formula. For $r = kr'$ and $r, r' \in \mathbb{N}$ we note that we have the following injective homomorphism between lattices that acts by multiplication

$$f_{r',k} : A(r') \to A(r), \qquad f_{r',k}(\alpha) = k\alpha. \tag{4.12}$$

The rank r partition function for the rational invariants is defined as,

$$Z_{DT}(X, r; \tau, \bar{\tau}) = \sum_{\delta \in A(r)} Z_{DT}(X, r, \delta; \tau) \bar{\theta}_{L(r)+\delta}(\tau, \bar{\tau}) \tag{4.13}$$

where

$$Z_{DT}(X, r, \delta; \tau) = \sum_{\substack{k \in \mathbb{Z}, k \geq 1 \\ (r,\delta) = (kr', f_{r',k}(\delta'))}} \frac{1}{k^2} Z_{BPS}(X, r', \delta'; k\tau). \tag{4.14}$$

Remark 4.2.2. The formulae in [B–S] look slightly different from the formulae above and in the rest of the section. We have rephrased the formulas in [B–S] in terms of rescaled lattices through the isomorphism,

$$\phi : (L'/rL, \frac{q(x)}{2r}) \to (A(r), rq(x)), \qquad \phi : \alpha + rL \to \frac{\alpha}{r} + L. \tag{4.15}$$

In [B–S] the authors provide an explicit formula for the coefficients of the rational partition function $Z_{DT}(X, r; \tau, \bar{\tau})$ in terms of a holomorphic vector-valued modular form $\tilde{\Phi}$ for the Weil representation of A and weight $11 - \dim L/2$. The modular form $\tilde{\Phi}$ has a geometrical interpretation in terms of Noether-Lefschetz numbers but we will not discuss that here. This formula for the "theta expansion coefficients" (4.14) of the rational partition function is given by,

$$Z_{DT}(X, r, \delta; \tau) = \frac{1}{r^2} \left(\sum_{\substack{k,l>0 \\ kl=r}} l \sum_{s=0}^{l-1} \Delta_r(\delta, k) \boldsymbol{e}\left(-\frac{s}{k} q_r(\delta)\right) (\Delta^{-1}\tilde{\Phi})_{l\mu} \left(\frac{k\tau + s}{l}\right) \right) \tag{4.16}$$

where $\Delta_r(\delta, k)$ is a non-vanishing condition defined as in (2.72) and $\Delta(\tau) = \eta(\tau)^{24}$ is the discriminant cusp form. As mentioned before these partition functions enjoy a modularity property and this was proved in [B–S].

Theorem 4.2.3. *[B–S, Section 6] For any $r \in \mathbb{Z}, r \geq 1$*

$$Z_r(\tau) = \sum_{\delta \in A(r)} Z_{DT}(X, r, \delta; \tau)\, e_\delta \tag{4.17}$$

is a holomorphic vector-valued modular form for $\rho_{L(r)}$ and weight $(-1 - \dim L/2)$.

As mentioned before this statement was proved in [B–S] through rather long and tedious calculations. However, this is now a direct consequence of the construction of Hecke operators in Section 3.2 and Corollary 3.2.3 of this thesis.

Physical constraints such as those coming from S-duality often hint that generating functions of moduli space invariants are related to modular forms. The results surveyed in this section then prompt the following questions:

Problem 4.2.4. Are there other examples of partition functions of Donaldson-Thomas invariants or other moduli space invariants that can be obtained as Hecke transforms of vector-valued modular forms?

4.3 Vertex Operator Algebras

As discussed in the previous sections, the j-invariant appears as a trace function of the Moonshine module $V^\natural$ which is a vertex operator algebra (VOA). This is one of the many inter-connections between the subjects of VOAs and modular forms. VOAs have proved to be a rich source of modular forms due to Zhu's modular invariance theorem and we will now discuss this here. Modular invariance of partition functions on a torus is a natural consequence of the partition function being independent of the choice of period of the torus in the worldsheet interpretation of bosonic string theory. Similarly, properties of rational conformal field theories (RCFTs) from physics have inspired various results on modular invariance of characters of C_2-cofinite rational VOAs of CFT-type such as the theorem of Zhu proved in [Zhu]. We just give a sketch of the definitions and theorems and the interested reader should refer to [Zhu] for further details.

4.3.1 Rational VOAs

Definition 4.3.1 (Definition 2.24, [MT]). A vertex operator algebra (VOA) is a quadruple $(V, Y, \mathbf{1}, \omega)$ where $V = \bigoplus_{n\in -N}^{\infty} V_n$ is a $\mathbb{Z}$-graded linear space and we have two distinguished elements $\mathbf{1}, \omega \in V, \mathbf{1} \neq 0$. The map Y is called the state-operator correspondence map and gives operator-valued formal Laurent series,

$$Y : V \to \operatorname{End} V[[z, z^{-1}]], v \to Y(v, z) = \sum v_n z^{-n-1}. \tag{4.18}$$

The fields $Y(v, z)$ are mutually local that is for all $v_1, v_2 \in V$, there exists a $k \in \mathbb{Z}_{\geq 0}$ such that

$$(z_1 - z_2)^k [Y(v_1, z_1), Y(v_2, z_2)] = 0. \tag{4.19}$$

The fields are creative so that all non-negative modes kill $\mathbf{1}$,

$$v_n \mathbf{1} = 0, n \geq 0, \qquad v_{-1}\mathbf{1} = v. \tag{4.20}$$

In addition the following axioms hold:

1. Let $Y(\omega, z) = \sum L_n z^{-n-2}$ and c a constant called the central charge. Then
$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m+n} c Id_V. \tag{4.21}$$

2. The graded components
$$V_n = \{v \in V_n | L_0 v = nv\} \tag{4.22}$$
are finite-dimensional i.e. $\dim V_n < \infty$.

3. $$Y(L_{-1}u, z) = \partial Y(u, z). \tag{4.23}$$

Remark 4.3.2. In the rest of this chapter, we will restrict attention to CFT's that are CFT-type which means that V has L_0-grading $V = \bigoplus_{n=0}^{\infty} V_n$ and $V_0 = \mathbb{C}\mathbf{1}$.

The representation theory of V can be described by the so-called weak modules of a VOA. These are defined as follows:

Definition 4.3.3 (Definition 6.1, [MT]). A weak V-module is a pair (V, Y_M) where,

$$Y_M : V \to \operatorname{End} M[[z, z^{-1}]], v \to Y_M(v, z) = \sum_n v_n^M z^{-n-1} \tag{4.24}$$

is a linear map satisfying the following conditions for $u, v \in V$:

$$Y_M(\mathbf{1}, z) = Id_M \tag{4.25}$$
$$(z_1 - z_2)^k [Y(u, z), Y(v, z)] = 0 \tag{4.26}$$

for some $k \in \mathbb{Z}_{\geq 0}$. In addition we have the associativity axiom which states that there exists a large enough k

$$(z_2 + z_2)^k Y_M(u, z_1 + z_2) Y_M(v, z_2) w = (z_1 + z_2)^k Y_M(Y_M(u, z_1)v, z_2) w \quad (4.27)$$

where $w \in M$.

In this section we are interested in special types of modules called admissible modules.

Definition 4.3.4. An admissible V-module is a weak V-module M which carries a $\mathbb{N}$-grading $M = \bigoplus_{n \geq 0} M_n$ such that

$$v \in V_k \implies v_n^M : M_m \to M_{m+k-n-1}. \quad (4.28)$$

We note here that the graded subspaces M_n are not required to be finite but this condition will need to be composed when we consider characters in the next section. This allows us to finally define rational and C_2-cofinite VOAs.

Definition 4.3.5.

1. V is said to be rational if every admissible V-module is completely reducible, i.e. a direct sum of irreducible, admissible V-modules.

2. V is C_2-cofinite if the graded subspace $C_2(V) = \langle u_{-2}v | u, v \in v \rangle$ is of finite codimension in V.

We remark here on the relation between C_2-cofinite and rational VOAs. There are lots of examples of C_2-cofinite VOAs that are not rational VOAs, but it has been conjectured that every rational VOA is C_2-cofinite.

4.3.2 Characters of VOAs

We now introduce characters of V-modules and discuss their modular properties. Let M be an admissible V-module such that each of the graded subspaces are finite-dimensional. If $v \in V_k$ be a homogenous state of degree k then from (4.28),

$$v \in V_k \implies v_n^M : M_m \to M_{m+k-n-1}. \tag{4.29}$$

The zero mode $o(v)$ of the state v is defined to be v_{k-1}^M and this is a zero weight operator so that,

$$o(v) : M_m \to M_m. \tag{4.30}$$

Definition 4.3.6. The character of M denoted by $Z(.,q) : V \to q^{-\frac{c}{24}}\mathbb{C}[[q,q^{-1}]]$ is a graded trace and defined as,

$$Z_M(v,q) = \mathrm{Tr}_M o(v) q^{L_0 - \frac{c}{24}} = q^{-\frac{c}{24}} \sum_n \mathrm{Tr}_{M_n} o(v) q^{L_0} \tag{4.31}$$

where c is the central charge.

In particular, the character associated to the vacuum $\mathbf{1}$ is simply,

$$Z_V(\mathbf{1}) = q^{-c/24} \sum_n \dim V_n q^n \tag{4.32}$$

and is called the graded dimension of V. Strictly speaking characters are formal Laurent series in q. But in most cases we can interpret these as convergent Fourier expansions for holomorphic functions on the upper half plane $\mathbb{H}$ through the substitution $q = e^{2\pi i \tau}$ and $\tau \in \mathbb{H}$.

All characters as defined in (4.31) can be shown to satisfy the properties of the one-point function on the torus as described in [Mil, Definition

5]. Let M be a rational C_2-cofinite VOA with we denote its irreducible V-modules by $\{M_i\}_{i\in I}$. In this case, it can be shown that the characters $a \to \mathrm{tr}_{M_i} o(a) q^{L(0)-\frac{c}{24}}$ are a basis of the space of one-point functions. However, we are most interested in the modularity invariance theorem of the characters of a VOA. This can be stated as follows.

Theorem 4.3.7 ([Zhu]). *Let V be a rational C_2-cofinite vertex operator algebra of CFT-type. Then for every homogeneous $a \in V$ with respect to $L[0]$, the expressions $\{\mathrm{tr}_{M_i} o(a) q^{L(0)-c/24}\}_{i\in I}$ span a* holomorphic *vector-valued modular form of weight* $\deg(a)$.

An important example of rational C_2-cofinite VOAs are lattice VOAs. Let L be an even non-degenerate lattice of rank $\dim L$ with quadratic form $q(.)$. Then one can construct a vertex operator algebra V_L associated to the lattice L with $|L'/L|$ number of irreducible V_L-modules denoted by $V_{L+\lambda}$ for every $\lambda \in L'/L$. The characters of $V_{L+\lambda}$ are given by,

$$Z_{V_{L+\lambda}}(1) = \frac{\sum_{\gamma\in\lambda+L} \boldsymbol{e}(q(\gamma))}{\eta(\tau)^{\dim L}} = \frac{\theta_{L+\lambda}(\tau)}{\eta(\tau)^{\dim L}}, \tag{4.33}$$

where η is the Dedekind eta function of (2.13). By Zhu's theorem this is a vector-valued modular form for some representation of $\mathrm{SL}_2(\mathbb{Z})$. In particular if L is self-dual so that $L' = L$ due to modularity of the theta function and $\eta(\tau)$ the character is a modular form of weight $\dim L$ but with a multiplier. This multiplier disappears if $\dim L$ is a multiple of 24 due to (2.14) and the character is a modular function in this case. Similarly it is easy to see that in the non self-dual case when $\dim L$ is a multiple of 24 the character is a vector-valued modular form of the Weil representation. Mason and Krauel obtained a similiar result in [KrM] for vector-valued weak Jacobi forms that arise as graded trace functions of VOAs with a Jacobi variable. These are closely related to the ellptic genera appearing in Mathieu moonshine.

4.3.3 Harvey and Wu's construction

In this section V denotes a simple, rational, C_2-cofinite, self-contragredient vertex operator algebra of CFT-type. A simple, rational, C_2-cofinite, self-contragradient VOA of CFT-type is also called strongly rational and we will use this terminology for brevity. In [HW], Hecke operators on characters of rational conformal field theory (RCFTs) are discussed. Let V, V' be strongly rational VOAs. A rational conformal field theory in mathematical terms is a representation $\mathcal{H}$ of $V \otimes V'$ satisfying certain additional constraints. Thus V and V' have a finite number of non-isomorphic irreducible modules which we denote by $\{A_i\}_{i\in I}$ and $\{B_j\}_{j\in J}$ for some finite index sets I and J respectively. The "Hilbert space" $\mathcal{H}$ of an RCFT has a decomposition,

$$\mathcal{H} = \bigoplus_{i\in I, j\in J} \mathcal{N}_{ij} A_i \otimes B_j \tag{4.34}$$

where $\mathcal{N}_{ij} \in \mathbb{Z}_{\geq 0}$.

As discussed in section 4.3 characters of a rational C_2-cofinite VOA V with irreducible modules A_i defined by

$$\chi_i(\tau) = q^{-\frac{c}{24}} \mathrm{Tr}_{A_i} q^{L_0} \tag{4.35}$$

are vector-valued modular forms for some representation ρ of $\mathrm{SL}_2(\mathbb{Z})$. In addition, each of the components $\chi_i(\tau)$ are weight 0 modular functions for a congruence subgroup $\Gamma(N)$ with positive integer Fourier coefficients (See [BG]). In other words, the principal congruence subgroup $\Gamma(N)$ lies in the kernel of ρ. Conversely, a vector-valued modular form with individual components modular functions of $\Gamma(N)$ can be characters of a rational VOA only if it satisfies the Verlinde formula for the fusion coefficients. (See [HW, Section 2.1]) . A rich source of vector-valued modular forms that are characters of strongly rational VOAs are modular linear differential equations (MLDEs). Characters of VOAs such as the Yang-Lee model and various models can be obtained as solutions of MLDEs. Another important notion is that of Galois symmetry of a VOA. Adjoining matrix elements of the representation ρ to $\mathbb{Q}$ gives a finite abelian extension $K \subseteq \mathbb{Q}[\zeta_N]$ where ζ_N is the primitive N-th root of unity. The integer N is called the conductor and is the same as the order of $\rho(T)$ and hence the level of the principal congruence subgroup $\Gamma(N)$. The Galois symmetry acts on the matrix ρ by entry-wise Frobenius automorphisms,

$$f_{N,l} : \zeta_N \to \zeta_N^l, \quad l \in (\mathbb{Z}/N\mathbb{Z})^* \tag{4.36}$$

Action of the Galois symmetries on the representation ρ is given by,

$$f_{N,l} : \rho(T) \to \rho(T)^l \tag{4.37}$$

$$f_{N,l} : \rho(S) \to G_l \rho(S) \tag{4.38}$$

for a matrix G_l defined in [HW, Section 2.2]. In [HW], the authors construct Hecke operators T_p for prime p such that $\gcd(p, N) = 1$ that map a vector-valued modular form of type ρ to a vector-valued modular form of type $\rho^{(p)}$. The representation $\rho^{(p)}$ is defined by its action on the generators,

$$\rho^{(p)}(T) = \rho(T^{\bar{p}}) \tag{4.39}$$

$$\rho^{(p)}(S) = \rho(\sigma_p S) \tag{4.40}$$

where $\bar{p}$ is the multiplicative inverse of p in $(\mathbb{Z}/N\mathbb{Z})^*$ and σ_p is a choice of an element in the preimage of $\left(\begin{smallmatrix} \bar{p} & 0 \\ 0 & p \end{smallmatrix}\right)$ under the reduction mod N map between $\mathrm{SL}_2(\mathbb{Z})$ to $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$. The action of Hecke operators T_p of Harvey and Wu on a vector-valued modular form $f(\tau) = \{f_i(\tau)\}$ is computed by applying the standard Hecke operators for $\Gamma(N)$ on each of the scalar-valued modular functions $f_i(\tau)$. The important result is that these Hecke operators $T_{\bar{p}}$ extend the Galois symmetries $f_{N,l}$ to characters of RCFT. In other words,

$$f_{N,l}(\rho(T)) = \rho^{(\bar{p})}(T) \tag{4.41}$$

$$f_{N,l}(\rho(S)) = \rho^{(\bar{p})}(S). \tag{4.42}$$

Finally, in [HW] the authors apply the Hecke operator thus defined to some example of RCFTs with two and three independent characters. In particular, for the examples considered they find that some of the Hecke operators considered transform characters of known RCFTs into characters of other known RCFTs or vector-valued modular forms that are solutions to MLDEs.

Harvey and Wu's work on Hecke transforms of RCFT characters opens up the possibility of using Hecke operators to probe Galois symmetry or other VOA symmetries.

Problem 4.3.8. Do the Hecke operators of section 3.2 match with the Hecke operators considered in Harvery and Wu's paper? Can Hecke operators provide information about VOAs that are Galois associates of each other or more broadly can it be used to connect VOAs related by other types of morphisms?

4.4 Jacobi forms and mock modular forms

As outlined in Theorem 2.3.3 there is a bijective correspondence between the space of Jacobi forms and vector-valued modular forms of the Weil representation. In [EZ] and [DMZ], Eichler and Zagier give "Hecke-like" operators that act on rank 1 Jacobi forms. In particular they define operators T_l that sends $J_{k,m}$ to $J_{k,m}$ and operators $V_{k,t}(t \geq 1)$ that send $J_{k,m}$ to $J_{k,tm}$. One can ask the following question:

Problem 4.4.1. How do the Hecke operators $\mathcal{T}_n$ and $\mathcal{H}_{n^2}$ compare with the operators T_n and $V_{k,n}$ of Eichler-Zagier under the isomorphism of Theorem 2.3.3?

Secondly, as the techniques developed in this thesis also work for more general automorphic objects such as Harmonic Mass forms, one can ask if the results of this thesis can be extended to vector-valued mock modular forms.

Problem 4.4.2. Do the results of this thesis extend to vector-valued mock modular forms and can they be used to obtain interesting mock modular form relations and identities?

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