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INVESTIGATIONS ON THE QUANTIZATION OF SPIN 3/2 FIELD

by

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled

INVESTIGATIONS ON THE QUANTIZATION OF SPIN 3/2 FIELD

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ABSTRACT

In this thesis the problem of quantization of spin 3/2 field interacting with an electromagnetic field is critically examined. An attempt is made to trace the origin of the difficulties occuring in this problem.

The first three chapters review briefly the nature of the problem, presenting the stand point from which we analyze the quantization procedure of spin 3/2 field, as well as establishing the notations to be used.

The Rarita-Schwinger field with mass m and spin 3/2 minimally coupled to electromagnetic field is quantized in Chapter 4 first by extending the method used by Takahashi and Umezawa, an extension of Kimel and Nath's work is then presented. It was found that our quantization procedure suffers the same inconsistency as Johnson and Sudarshan.

The existence of the S-Matrix is proved in chapter 5 under the weak field assumption.

Our conclusion is presented in Chapter 6 with discussion on the possibility for further investigation.

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TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

The problem of quantization for system with higher spin has been studied extensively ever since the earlier development of quantum field theory. Nevertheless, a consistent set of rules for the quantization of interacting higher spin field is still absent.

In 1936, Dirac (1) published a paper on relativistic wave equations for spins greater than a half. However, as was pointed out by Fierz and Pauli (2), if Dirac's wave equations were coupled to an external electromagnetic field by substituting $\partial_\mu - ieA_\mu$ for ∂_μ, they become inconsistent. To avoid the immediate algebraic inconsistency that arises in the presence of interactions, when the constraints are postulated independently of the equations of motion, Firez and Pauli suggested the method of higher-spin Lagrangians. It has since become a popular technique to construct Lagrangians for free higher spin particles which yield both the equations of motion and the constraints. However, Velo and Zwanzinger (3) have shown that the Lagrangian device by itself does not automatically provide satisfactory wave equations. The wave equations for the Rarita-Schwinger (4) Spin 3/2 field minimally coupled to electromagnetic field prossess noncausal modes of propagation. This problem is not resolved by the addition of (non-minimal) magnetic moment terms to the interaction Lagrangian (5).

It has been shown by Johnson and Sudarshan (6) that the basic problem in the quantization of Fermi-Dirac fields of spin greater than a half is that the existence of secondary constraints on the fields necessarily brings the dynamics into consideration. They have furthermore, shown that for the case of spin 3/2 field, a coupling to an external electromagnetic field renders the anticommutators indefinite and thereby inconsistent with a positive definite metric when it is quantized in terms of the Schwinger's action principle. The indefiniteness arises from the presence of the factor $R = 1 - (2e/3m^2)\vec{\sigma}\cdot\vec{B}$ in the anticommutation relation, $\vec{B}$ is the magnetic field strength. The anticommutator is positive definite only if $|(2e/3m^2)\vec{B}| < 1$ everywhere. The quantization by Johnson and Sudarshan is consistent if the inequality $(2e/3m^2)^2\vec{B}^2 < 1$ is satisfied. This situation shall be refered to as the "weak field case".

Gupta and Repko (7), demonstrated that the choice of canonical variables corresponding to the commutation relations found by Johnson and Sudarshan (6) suffers the additional inconsistency of not being compatible with the Heisenberg's equations of motion. By making a judicious transformation of the canonical variables for the spin 3/2 and the electromagnetic fields, they have explicitly constructed the Interaction Hamiltonian and the commutation relations for the field variables which are consistent with the Heisenberg's equations of motion. Hence they conclude that the quantization of charged spin 3/2 field, although creating enormous mathematical complications, does not appear to involve any fundamental difficulty or inconsistency. Recently Kimel and Nath (8) reexamined the problem using the Yang-Feldman (9) formalism. Their work seems to confirm and complement the results and conclusions of Gupta and Repko. However, they reserve the possibility of internal inconsistency within the Yang-Feldman formalism when certain conditions between the Heisenberg Field operators and the asymptotic fields break down.

It was not clear whether the Johnson-Sudarshan's inconsistency is inherent in the Schwinger's action principle (10) or it stems from the peculiar propagation character of the fundamental field equations. In this thesis, we critically examine the problem of quantization of spin 3/2 field interacting with an electromagnetic field and attempt to trace the origin of the difficulties. We present in Chapter 2 a summary of the fundamental structure of Quantum Field Theory which provides us with a stand point from which we analyze the quantization procedure of spin 3/2 field. In Chapter 3, we review briefly the general properties of the Rarita-

Schwinger wave equations. The Rarita-Schwinger field with mass m and spin 3/2 minimally coupled to electromagnetic field is quantized in Chapter 4 first by extending the method used by Takahashi and Umezawa (11) (in particular the second paper of Ref. 11, see also Katayama (12)) for the case of interactions with higher derivative coupling. This method is essentially an extension of the Yang-Feldman Technique (9) where the Interaction Hamiltonian is defined through the wave equation in integral form, and from the Interaction Hamiltonian, the S-matrix is constructed. The Interaction Hamiltonian for our case is found to be an infinite power series of the coupling constant (the charge e), and apart from the lowest order term, all terms of higher order depend on the normal to a hypersurface. The explicit form of the Interaction Hamiltonian is determined up to the fourth order in e. In contrast to the work of Kimel and Nath (8) where the secondary constraints has to be used explicitly to eliminate one of the d[illegible]ndent field in the source term, our method does not make any reference to the constraints. The S-matrix formally constructed from the Interaction Hamiltonian is explicitly demonstrated to satisfy the "generalized Matthews'rules" (13) in the perturbation calculation to fourth order in e. In the context of this formalism we can generally argue that the anticommutation relation of the Heisenberg field is always positive. Thus we seems to have obtained a consistent quantization scheme for the interacting spin 3/2 field in terms of perturbation calculation. However, after going through all the tedious and lengthly calculations, the anticommutators thus obtained coincides with that of Johnson and Sudarshan at least to fourth order in e if we expand their result in the form of power series. It seems highly unlikely that our result will disagree with those of Johnson and Sudarshan's in higher order. We are oblidged to conclude that inconsistency does occur in this formalism. We also extended the work by Kimel and Nath to fourth order in the charge in perturbation calculation and found that the results agrees with ours. We then employ a smearing technique (14) to our field to study the possibility of existence of the Scattering matrix. We found that we have to restrict ourselves to the weak field case (which is also the case where the previous method is consistent) in order to solve the Cauchy problem. We conclude our study in Chapter 6 and discuss the possibility for further investigation.

CHAPTER 2

FUNDAMENTAL STRUCTURE OF QUANTUM FIELD THEORY

To provide us with the foundation for analyzing the quantization procedure of the spin 3/2 field, we summarize here the fundamental structure of quantum field theory.

We begin with the field equations for the Heisenberg fields (say $\underset{\sim}{\psi}(x)$);

$$\underset{\sim}{\Lambda}(\partial)\underset{\sim}{\psi}(x) = \underset{\sim}{J}[\underset{\sim}{\psi}(x)] \,. \qquad 2.1$$

Here $\Lambda(\partial)$ is a differential operator and $\underset{\sim}{J}$, the source, is a functional of the field operators and their derivatives. We shall be interested in the case where the solutions of 2.1 can be expressed in terms of a certain set of free fields (say $\psi(x)$) which satisfy the linear, homogeneous wave equations;

$$\tilde{\lambda}(\partial)\psi(x) = 0 \,, \qquad 2.2$$

with proper propagation characteristic (i.e., the wave equations 2.2 should be hyperbolic). When there exist such solutions, we have;

$$\psi = \psi[\underset{\sim}{\psi}] , \qquad 2.3$$

and the particles associated with the field $\underset{\sim}{\psi}$ is referred to as quasiparticle.

We also require that any functional of $\underset{\sim}{\psi}$ should be expressible in terms of $\underset{\sim}{\psi}$. This means that the quasiparticle field $\underset{\sim}{\psi}$ forms a complete set (i.e., an irreducible operator ring). We must emphasize here the completeness of $\underset{\sim}{\psi}$ such that the above requirement is satisfied, since in the quantization procedure to be discussed in Chapter 4 (The Takahashi-Umezawa Method), we obtain a relation of the form;

$$\underset{\sim}{\psi} = \underset{\sim}{\psi}[\psi] , \qquad 2.4$$

(which is always valid as $\underset{\sim}{\psi}$ form a complete set), and by inverting 2.4 we find the structure of $\psi[\underset{\sim}{\psi}]$ in 2.3. This procedure may lead to inconsistent results if $\underset{\sim}{\psi}$ do not form a complete set*.

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*This may be the reason why the quantization of the spin 3/2 fields as presented later leads to inconsistent results, since we do assume the completeness of the free fields.

The particle picture is obtained when one chooses the Fock space of these ψ as the Hilbert space. To introduce the Fock space, let a_k stands for annihilation operators for the quasiparticles and $|0\rangle$ denotes the vacuum state;

$$a_k|0\rangle = 0 . \qquad 2.5$$

The Fock space is the Hilbert space which has as its basis the set*;

$$\{|a_{k_1}\cdots\cdots a_{k_n}\rangle = c a^{+}_{k_1}\cdots\cdots a^{+}_{k_n}|0\rangle\}$$

which contains all the states with finite n. c is the normalization constant.

To consider the problem of finding the structure of $\underset{\sim}{\psi}[\psi]$ in 2.3, let us begin with the total Hamiltonian $H(\phi)$ of the system in the "Schrodinger picture", where ϕ denotes both the canonical variables and their conjugates. H(ϕ) must be a hermitian operator in order to be identified as the energy operator of the system. Hence, we introduce a unitary transformation u which diagonalizes $H(\phi)$ as follows;

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*Strictly speaking we should use wave packet by smearing the a_k's with a complete set of square-integrable functions.

$$u^{-1}H(\phi)u = H_0(\phi) \, , \tag{2.7}$$

where $H_0(\phi)$ is a diagonalized matrix operator. If we could find such a u which determines the $H_0(\phi)$, the structure of 2.3 is established, since, as to be shown 2.14 that the Heisenberg fields are related to the asymptotic fields by $U(t)$ which is connected to u by 2.13 . We have then solved our problem. However, in quantum field theory, we make the basic assumption that $H_0(\phi)$ is of bilinear form in ϕ (plus c-number) which is the same as the noninteracting part of the total Hamiltonian $H(\phi)$ (apart from the possible mass renomalization term), and then we determine the $U(t)$. The validility of this procedure is guaranteed by the self-consistency of the theory. If inconsistency does arise in our result, we may try to modify our choice of $H_0(\phi)$ and repeat the whole procedure.

To see how the knowledge of $H_0(\phi)$ enables us to determine the u, we introduce the operators ϕ_0 related to ϕ through u as;

$$\phi_0 \equiv u\phi u^{-1} \, . \tag{2.8}$$

It follows from 2.7 that,

$$H(\phi) = H_0(\phi_0). \tag{2.9}$$

We introduce the Heisenberg operators $\underset{\sim}{\phi}(t)$ by the operator $e^{iH(\phi)t}$, as follow;

$$\underset{\sim}{\phi}(t) \equiv e^{iH(\phi)t}\phi\, e^{-iH(\phi)t}. \tag{2.10}$$

Similiarily we introduce the "asymptotic operators" as;

$$\phi_0(t) \equiv e^{iH_0(\phi_0)t}\phi\, e^{-iH_0(\phi_0)t}. \tag{2.11}$$

The reason for calling $\phi_0(t)$ as the asymptotic operators will become apparent later.

The relation;

$$H(\underset{\sim}{\phi}(t)) = H_0(\phi_0(t)), \tag{2.12}$$

is established from 2.9 together with the transformations 2.10 and 2.11 .

Defining

$$\begin{aligned} U(t) &\equiv e^{iH(\phi)t}u e^{-iH(\phi)t} \\ &= e^{iH_0(\phi_0)t}u e^{-iH_0(\phi_0)t}, \end{aligned} \tag{2.13}$$

because of 2.12 , the Heisenberg operators can be related to the asymptotic operators as;

$$\phi(t) = U^{-1}(t)\phi_0(t)U(t) . \qquad 2.14$$

We rewrite $U(t)$ as

$$\begin{aligned} U(t) &= e^{iH(\phi)t}ue^{-iH(\phi)t} \\ &= ue^{iH_0(\phi)t}e^{-iH(\phi)t} \\ &= uV(t), \end{aligned} \qquad 2.15$$

where

$$V(t) \equiv e^{iH_0(\phi)t}e^{-iH(\phi)t} . \qquad 2.16$$

The second step in 2.15 is due to 2.7 .

The initial condition for $U(t)$ is

$$U(0) = u . \qquad 2.17$$

We can explicitly determine $U(t)$ by defining

$$u \equiv 1 - T , \qquad 2.18$$

and

$$H(\phi) - H_0(\phi) \equiv H'(\phi) , \qquad 2.19$$

we have from 2.7 that;

$$H(\phi)T - TH_0(\phi) = H'(\phi) . \qquad 2.20$$

A solution to 2.20 can be constructed from;

$$T_\varepsilon = -i\int_{-\infty}^{0} dt' \, e^{iH(\phi)t'} H'(\phi) e^{-iH_0(\phi)t'} e^{\varepsilon t'} , \quad 2.21$$

by

$$T = \lim_{\varepsilon \to 0} T_\varepsilon , \quad 2.22$$

provided,

$$\lim_{\varepsilon \to 0} \varepsilon T_\varepsilon = 0 .$$

We can rewrite T_ε as;

$$T_\varepsilon = -i\int_{-\infty}^{0} dt' \, V^{-1}(t') H_I(t') e^{\varepsilon t'} \quad 2.23$$

where

$$H_I(t) \equiv e^{iH_0(\phi)t} H'(\phi) e^{-iH_0(\phi)t} . \quad 2.24$$

From the definition of $V(t)$, 2.16 , we have

$$i\frac{d}{dt}V(t) = H_I(t)V(t) , \quad 2.25$$

with initial condition,

$$V(0) = 1 .$$

Multiplying 2.25 from the right by $V^{-1}(t_0)$, we have;

$$i\frac{d}{dt}V(t)V^{-1}(t_0) = H_I(t)V(t)V^{-1}(t_0) . \quad 2.26$$

A solution to 2.26 is;

$$V(t)V^{-}(t_0) = 1 - i\int_{t_0}^{t} dt' \, H_I(t')V(t')V^{-1}(t_0)$$

or,

$$V(t) = V(t_0) - i\int_{t_0}^{t} dt'\, H_I(t')V(t') .$$

Taking the limits $t = 0$ and $t_0 = -\infty$, we have;

$$V(-\infty) = 1 + i\int_{-\infty}^{0} dt'\, H_I(t')\tilde{V}(t'),$$

hence,

$$V^{-1}(-\infty) = 1 - i\int_{-\infty}^{0} dt'\, V^{-1}(t')H_I(t') . \qquad 2.27$$

Comparing 2.27 with 2.23 and from 2.18, we obtain;

$$u = V^{-1}(-\infty) , \qquad 2.28$$

and hence from 2.15 we have,

$$U(t) = V^{-1}(-\infty)V(t) . \qquad 2.29$$

From 2.29, the initial condition for $U(t)$ is

$$U(-\infty) = 1 . \qquad 2.30$$

Differentiating 2.13 with respect to t we have;

$$\begin{aligned} i\frac{d}{dt}U(t) &= e^{iH(\phi)t}[\, u,H(\phi)\,]e^{-iH(\phi)t} \\ &= e^{iH_0(\phi_0)t}H'(\phi_0)ue^{-iH_0(\phi_0)t} \\ &= H'(\phi_0(t))U(t) . \end{aligned} \qquad 2.31$$

The solution to 2.31 with initial condition 2.30 is

$$U(t) = 1 - i\int_{-\infty}^{t} dt' \; H'(\phi_0(t'))U(t') \; . \qquad 2.32$$

From the initial condition 2.30 and the relation 2.14 we observe that as $t \to -\infty$, the Heisenberg operators $\underset{\sim}{\phi}(t)$ and the asymptotic operators $\phi_0(t)$ coincides. This is the reason why $\phi_0(t)$ is called the asymptotic operators. We also note from 2.12 that the Total Hamiltonian expressed in terms of the Heisenberg operators is equal to the diagonalized Hamiltonian H_0 expressed in terms of the asymptotic fields.

We usually introduce the operators in the interaction representation by;

$$\phi_I(t) = e^{iH_0(\phi)t}\phi e^{-iH_0(\phi)t} , \qquad 2.33$$

which are related to the asymptotic fields as;

$$\phi_I(t) = u^{-1}\phi_0(t)u = V(-\infty)\phi_0(t)V^{-1}(-\infty) . \qquad 2.34$$

Again, we note here that the $H_0(\phi)$ may differ from the one obtained by just taking the noninteracting part of the total Hamiltonian.

We should not conclude from,

$$H(\phi(t)) = H_0(\phi_0(t)) , \qquad 2.12$$

that there are no reactions among the particles. Let us return to the Schrodinger representation and denote the time dependent state vectors as $\chi(t)$. Since,

$$H(\phi) = H_0(\phi_0),$$

we have

$$H(\phi) = u^{-1}\{H_0(\phi_0) + H'(\phi_0)\}u . \qquad 2.35$$

The Schrodinger equation for the state vector $\chi(t)$ is

$$i\frac{\partial}{\partial t}u\chi(t) = \{H_0(\phi_0) + H'(\phi_0)\}u\chi(t) . \qquad 2.36$$

Introducing the state;

$$\Psi(t) \equiv e^{iH_0(\phi_0)t}u\chi(t) , \qquad 2.37$$

we have from 2.36 that;

$$i\frac{\partial}{\partial t}\Psi(t) = H'(\phi_0(t))\Psi(t) . \qquad 2.38$$

Using the definition of $U(t)$, 2.13 or 2.15 , we can rewrite 2.37 as;

$$\Psi(t) = U(t)e^{iH(\phi)t}\chi(t) . \qquad 2.39$$

Due to the initial condition 2.30 of $U(t)$ $U(-\infty) = 1$,

$e^{iH(\phi)t}\chi(t)$ is in fact time independent, we write;

$$\Psi(-\infty) = e^{iH(\phi)t}\chi(t),$$

or

$$\Psi(t) = U(t)\Psi(-\infty) . \qquad 2.40$$

$\Psi(-\infty)$ is then the state vector in the Heisenberg representation while $\Psi(t)$ is the state vector in the asymptotic representation.

The relations between the Heisenberg representation, the asymptotic representation and the Schrodinger representation are as follows;

$$\begin{aligned}&(\Psi(-\infty), F(\phi(t))\Psi(-\infty))\\ &= (\Psi(t), F(\phi_0(t))\Psi(t)),\end{aligned} \qquad 2.42$$

and

$$\begin{aligned}&(\Psi(-\infty), F(\phi(t))\Psi(-\infty))\\ &= (\chi(t), e^{-iH(\phi)t}F(\phi(t))e^{iH(\phi)t}\chi(t))\\ &= (\chi(t), F(\phi)\chi(t)).\end{aligned} \qquad 2.43$$

where F is a functional of the fields.

It is now apparent from 2.40 that the operator $U(t)$ describes the evolution of the states in the asymptotic representation. The transition probability from the state a to the state b is given by;

$$W_{ba}(t) = |(\Psi_b(-\infty), U(t)\Psi_a(-\infty))|^2 .$$

In the case when the transition from a to b occurs in an infinite time interval, the $U(t)$ becomes the S - matrix, i.e.,

$$S \equiv U(\infty) .$$

This concluded our summary for the fundamental structure of quantum field theory within which frame work our quantization proceeds.

CHAPTER 3

GENERAL PROPERTIES OF THE RARITA-SCHWINGER SPIN 3/2 WAVE EQUATIONS

Before we proceed to quantize the fields within the frame work of Quantum Field Theory summarized in the previous chapter, we review briefly the general properties of the wave equations we are concerned with. This also serves to establish the notations we use in the later chapters.

We begin with the free field. To describe particle of spin 3/2 and unique mass m, Rarita and Schwinger (4) used a sixteen component vector-spinor field ψ_μ with each component satisfying the Dirac equations;

$$(\gamma_\lambda \partial_\lambda + m)\psi_\mu = 0 . \qquad 3.1$$

It is well known that the vector-spinor field transforms according to the $(1,\frac{1}{2}) \oplus (\frac{1}{2},1) \oplus (\frac{1}{2},0) \oplus (0,\frac{1}{2})$ representation of the Lorentz group and hence describes a spin 3/2 field as well as two spin ½ fields. The subsidiary conditions;

$$\gamma_\mu \psi_\mu = 0 \quad ; \quad \partial_\mu \psi_\mu = 0 , \qquad 3.2$$

are imposed to eliminate the two unwanted spin ½ components from the vector-spinor field in order that the system describes pure spin 3/2 field.

Starting with a general wave equation containing three arbitrary real parameters a, b and c ;

$$\Lambda_{\mu\nu}(a,b,c;\partial)\psi_{\nu}(x)$$
$$= -[\ \Gamma_{\lambda}(a,b)\partial_{\lambda} + m\beta(c)\ ]_{\mu\nu}\psi_{\nu}(x) = 0, \qquad 3.3$$

where

$$\Gamma_{\lambda,\mu\nu}(a,b) \equiv \gamma_{\lambda}\delta_{\mu\nu} + a(\gamma_{\mu}\delta_{\lambda\nu}+\gamma_{\nu}\delta_{\lambda\mu}) + b\gamma_{\mu}\gamma_{\lambda}\gamma_{\nu},$$

and

$$\beta_{\mu\nu}(c) \equiv \delta_{\mu\nu} - c\gamma_{\mu}\gamma_{\nu},$$

we can derive both the equations of motion 3.1 and the subsidiary conditions 3.2 from 3.3 by imposing the relations;

$$a \neq -\tfrac{1}{2},$$
$$b = \tfrac{1}{2}(\ 1+2a+3a^2\ ),$$
$$c = (\ 1+3a+3a^2\ ). \qquad 3.4$$

The parameter a remains free except that it cannot be $-\frac{1}{2}$. The particular value of a is without any physical significance since the Lagrangian density*;

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*$\Lambda_{\mu\nu}(a;\partial)$ denotes the $\Lambda_{\mu\nu}(a,b,c;\partial)$ when b and c are related to a by 3.4.

$$\mathcal{L}_{R.S.} = \bar{\psi}_\mu(x)\Lambda_{\mu\nu}(a;\partial)\psi_\nu(x) \quad , \qquad 3.5$$

where

$$\bar{\psi}_\mu(x) \equiv \psi^+_\nu(x)\eta_{\nu\mu} \quad , \qquad 3.6$$

with

$$\eta_{\mu\nu} \doteq \gamma_4 g_{\mu\nu} \; ; \quad \eta \doteq \eta^{-1} = \eta^\dagger ,$$

is form invariant under the point transformation (15);

$$\psi_\mu \to \psi'_\mu = \psi_\mu + \frac{a' - a}{2(2a+1)}\gamma_\mu\gamma_\nu\psi_\nu \; .$$

This transformation merely mixes the two spin ½ components leaving the spin 3/2 components unchanged. In most of the following discussions, we choose for convience a = -1, then;

$$\Gamma_{\lambda,\mu\nu} = \gamma_\lambda\delta_{\mu\nu} - (\gamma_\mu\delta_{\lambda\nu}+\gamma_\nu\delta_{\lambda\mu}) + \gamma_\mu\gamma_\lambda\gamma_\nu$$

and

$$B_{\mu\nu} = \delta_{\mu\nu} - \gamma_\mu\gamma_\nu \; .$$

The differential operator with this value of a will be denoted simply by $\Lambda_{\mu\nu}(\partial)$.

In this case the projection matrices for the two sets of spin ½ components are;

$$(P_{\frac{1}{2}})_{\mu\nu} = \tfrac{1}{2}(\delta_{\mu\sigma}+g_{\mu\sigma})(\tfrac{1}{3}\gamma_\sigma\gamma_\rho)\tfrac{1}{2}(\delta_{\rho\nu}+g_{\rho\nu}).$$

and

$$(P_{\frac{1}{2}}')_{\mu\nu} = \tfrac{1}{2}(\delta_{\mu\nu} - g_{\mu\nu}) .$$

The corresponding sets of components characterizing spin ½ under space rotation are;

$$\gamma_i \psi_i \quad \text{and} \quad \psi_4 .$$

The spin 3/2 components, which are independent of the choice of a , are;

$$\phi_i = P_{ij}\psi_j = (\delta_{ij} - \tfrac{1}{3}\gamma_i\gamma_j)\psi_j . \qquad 3.7$$

One readily verifies that;

$$\gamma_i P_{ij} = P_{ij}\gamma_j = 0 \quad ; \quad P_{in}P_{nj} = P_{ij} \quad ; \quad P_{ij}^{\dagger} = P_{ij} .$$

The "Klein-Gordon divisor" , $d_{\mu\nu}(\partial)$, which reduces our wave equations to the Klein-Gordon Equations, i.e.,

$$d_{\mu\lambda}(\partial)\Lambda_{\lambda\nu}(\partial) = \Lambda_{\mu\lambda}(\partial)d_{\lambda\nu}(\partial) = (\Box - m^2)\delta_{\mu\nu} , \qquad 3.8$$

is given by*;

$$d_{\mu\nu}(\partial) = (\gamma_\lambda \partial_\lambda - m)d^{l}_{\mu\nu}(\partial) = d^{r}_{\mu\nu}(\partial)(\gamma_\lambda\partial_\lambda - m) \qquad 3.9$$

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*For the general case with $\Lambda_{\mu\nu}(a;\partial)$, a contact term

$$-\frac{1}{3m^2}\cdot\frac{a+1}{(2a+1)^2}(\Box - m^2)\left|\{\tfrac{1}{2}(a+1)\gamma_\lambda\partial_\lambda + am\}\gamma_\mu\gamma_\nu + (2a+1)\gamma_\mu\partial_\nu + a\gamma_\nu\partial_\mu\right|$$

should be added to the right hand side of 3.9

where,

$$d^{l}_{\mu\nu}(\partial) = d^{r}_{\mu\nu}(-\partial) = -\{\delta_{\mu\nu} - \frac{1}{3}\gamma_\mu\gamma_\nu + \frac{1}{3m}(\gamma_\mu\partial_\nu - \gamma_\nu\partial_\mu) - \frac{2}{3m^2}\partial_\mu\partial_\nu\} .$$

We observe that;

$$\{d(\partial)\eta\}^\dagger = d(-\partial)\eta .$$

By virtue of the existence of the Klein-Gordon divisor, the solutions to the wave equations can be decomposed in a Lorentz invariant fashion into positive and negative frequencies, satisfying the time independent orthonormality conditions;

$$i\int d\sigma_\lambda(x)\ \bar{u}^{r}_{\mu\vec{p}}(x)\Gamma_{\lambda,\mu\nu}u^{r'}_{\nu\vec{p}'}(x) = \delta_{rr'}\delta(\vec{p}-\vec{p}') ,$$

$$i\int d\sigma_\lambda(x)\ \bar{v}^{s}_{\mu\vec{q}}(x)\Gamma_{\lambda,\mu\nu}v^{s'}_{\nu\vec{q}'}(x) = \delta_{ss'}\delta(\vec{q}-\vec{q}') , \qquad 3.10$$

and the closure conditions;

$$\sum_{r=1}^{4}\int d^3p\ u^{r}_{\mu\vec{p}}(x)\bar{u}^{r}_{\nu\vec{p}}(x') = id_{\mu\nu}(\partial)\Delta^+(x-x') ,$$

$$\sum_{s=1}^{4}\int d^3q\ v^{s}_{\mu\vec{q}}(x)\bar{v}^{s}_{\nu\vec{q}}(x') = id_{\mu\nu}(\partial)\Delta^-(x-x') . \qquad 3.11$$

$u^{r}_{\mu\vec{p}}(x)$ ($v^{s}_{\mu\vec{q}}(x)$) denotes the positive (negative) frequency wave function with three momenta $\vec{p}$ ($\vec{q}$) and helicity r (s) = 1,2,3,4. Δ^+ and Δ^- are the solutions of the Klein-Gordon Equation.

The explicit form of the wave functions can be constructed from the wave functions for field with spin ½ and spin 1 using the Clebsch-Gordon Coefficients (16).

To implement minimal electromagnetic interaction to the spin 3/2 field, we make the relativistic and gauge invariant substitution;

$$\partial_\mu \rightarrow \pi_\mu = \partial_\mu - ieA_\mu$$

into the wave equations, with $A_\mu(x)$ being the vector potential of the electromagnetic field. We consider simultaneously the following set of equations;

$$\Lambda_{\mu\nu}(\partial)\psi_\nu = J_\mu = -ie(\Gamma_\lambda A_\lambda)_{\mu\nu}\psi_\nu , \qquad 3.12$$

and

$$\Box A_\lambda = I_\lambda = -ie\bar{\psi}_\mu \Gamma_{\lambda,\mu\nu}\psi_\nu . \qquad 3.13$$

For $\mu = 4$ in 3.12 we have the "primary constraints" (6)*;

$$\pi_i\psi_i = -(\gamma_i\pi_i - m)\gamma_j\psi_j . \qquad 3.14$$

Note that 3.14 does not contain any time derivatives, hence, is a pure constraint equation.

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*For $\Lambda(a;\partial)$, instead of taking $\mu=4$, we multiply from the left by $\gamma_\lambda\{(a+1)(\delta_{\lambda\mu}+g_{\lambda\mu})-(3a+1)(\delta_{\lambda\mu}-g_{\lambda\mu})\}$, an additional term $-(a+1)\{\gamma_i\pi_i-\frac{3}{2}m\}\gamma_\mu\psi_\mu$ appears on the right of 3.14 .

The "secondary constraints" whose structure depends on the dynamics of the system is obtained by contracting 3.12 by γ_μ and π_μ respectively as;*

$$\gamma_\mu\psi_\mu - i\varepsilon(\gamma_\lambda F_{\lambda\nu} - \tfrac{1}{2}\gamma_\lambda F_{\lambda\mu}\gamma_\mu\gamma_\nu)\psi_\nu = 0 , \qquad 3.15$$

where,

$$\varepsilon \equiv (2e/3m^2).$$

The commutation relation

$$[\pi_\mu, \pi_\nu] = -ie(\partial_\mu A_\nu - \partial_\nu A_\mu) \equiv -ieF_{\mu\nu}$$

has been used.

3.15 can also be written in a compact form;

$$\gamma_\mu\psi_\mu = \tfrac{1}{2}\varepsilon\gamma_5\gamma_\lambda F^d_{\lambda\mu}\psi_\mu , \qquad 3.16$$

where

$$F^d_{\mu\nu} \equiv \tfrac{1}{2}\varepsilon_{\mu\nu\sigma\rho}F_{\sigma\rho}$$

by using the relation $\{\gamma_\mu, \sigma_{\nu\lambda}\} = 2i\varepsilon_{\mu\nu\lambda\rho}\gamma_\rho\gamma_5$, or in a noncovariant form;

$$\gamma_\mu\psi_\mu = i\varepsilon R^{-1}\{\xi_j + (\gamma_4\partial_4 A_n)\beta_{nj}\}\psi_j , \qquad 3.17$$

where

$$R \equiv 1 - \varepsilon\vec{\sigma}\cdot\vec{B} ,$$

$$i\vec{\sigma}\cdot\vec{B} \equiv \tfrac{1}{2}\gamma_i F_{ij}\gamma_j = -\partial_i\beta_{ij}A_j ,$$

$$\zeta_j \equiv \gamma_i F_{ij} - (\partial_i\gamma_4 A_4)\beta_{ij} .$$

---

*For $\Lambda(a;\partial)$ the term $-(a+1)\{2\gamma_\mu\psi_\mu + \frac{i}{2}\varepsilon\gamma_\lambda F_{\lambda\nu}\gamma_\nu\gamma_\mu\psi_\mu\}$ appears on right of 3.15 .

The primary constraints 3.14 and the secondary constraints 3.15 together reduces the number of independent components of the interacting field to eight as in the free case.

Let us analyze the equations of motion 3.12 more closely. As we have observed that 3.12 implies both the primary and secondary constraints, also the time derivative of the components ψ_4 never appear in these equations at all. Hence mathematically speaking, 3.12 cannot be regarded as the true equations of motion for the field ψ_μ.

We review briefly how wave fields may be described mathematically: Wave propagation is usually associated with hyperbolic systems of partial differential equations. Such equations allow an initial value problem to be posed on a class of surfaces, called "spacelike" with respect to the equations, and they possess solutions with wave fronts that travel along rays at finite velocities. The rays through any point form a ray cone that is entirely determined by the coefficients of the highest derivatives. Thus, for hyperbolic systems, when coupling occurs only in lower derivatives, the ray cone is the same in the interacting and free case. The free Klein-Gordon and Dirac equations are familiar examples of hyperbolic systems, and so, when they are coupled through lower-order derivatives, the ray cone remains the light cone. On the other hand, for spin greater than one-half, the free equations are not hyperbolic, but constitute instead a degenerate system because they imply constraints. However, it may be shown that they are equivalent to a system of hyperbolic equations, which describe the wave propagation, supplemented by constraints that are conserved in time. But it is not true that, if any low or nonderivative coupling term is added to the free higher-spin Lagrangian, the resulting equations remain equivalent to a hyperbolic system with the light cone as ray cone, supplemented by the same number of constraints. Even when the system remain hyperbolic, the ray cone may be extended making the propagation acausal. In some cases the system may lose hyperbolicity making it unsuitable for the description of wave propagation. Some constraints may become equation of motion, thus increasing the degree of freedom of the field. These difficulties were observed by Velo and Zwanziger (3) when they investigated spin 1, 2 and 3/2.

To determine the velocity of propagation of the wave fronts, the concept of characteristic surfaces is introduced. The characteristic surfaces play the role of wave fronts whose normals n_μ for a linear system of the form;

$$(L_\lambda \partial_\lambda + B)\psi = 0,$$

are determined by the solutions to the equation

$$D(n) = \det L_\lambda n_\lambda = 0. \qquad 3.18$$

Also, a "spacelike" surface is the surface on which initial conditions can be posed and is necessarily non-characteristic.

When we apply the criterion 3.18 directly to the Rarita-Schwinger equation 3.12, we find that every surface is a characteristic surface, corresponding to the fact that there are constraints. Hence, we cannot pose the Cauchy problem associated with 3.12 with prescribed data on a "spacelike" surface. To circumvent this difficulty, Velo and Zwanziger (3) proposed to study the new equations of motion by substituting the constraints back into the original equations. The new equations of motion are;

$$\begin{aligned} M_{\mu\nu}(\pi)\psi_\nu &= d^1_{\mu\lambda}(\pi)\Lambda_{\lambda\nu}(\pi)\psi_\nu \\ &= \{(\gamma_\lambda\pi_\lambda+m)\delta_{\mu\nu} - i\varepsilon(\pi_\mu-\tfrac{1}{2}m\gamma_\mu)\gamma_5\gamma_\lambda F^d_{\lambda\nu}\}\psi_\nu \\ &= 0 \end{aligned} \qquad 3.19a$$

or in hermitian form;

$$\begin{aligned} &M'_{\mu\nu}(\pi)\psi_\nu \\ &= \{M_{\mu\nu} - i\varepsilon F^d_{\mu\lambda}\gamma_\lambda\gamma_5(\pi_\nu-\tfrac{1}{2}m\gamma_\nu) - \varepsilon^2 F^d_{\mu\lambda}\gamma_\lambda\gamma_5(\gamma_\sigma\pi_\sigma-2m)\gamma_5\gamma_\rho F^d_{\rho\nu}\}\psi_\nu \\ &= 0. \end{aligned} \qquad 3.19b$$

It can be shown that 3.19 preserves the constraints 3.14 and 3.15, (i.e., every solution of 3.19 which satisfies the constraints at a given time satisfies them for all time), and every solution of 3.19 which satisfies the constraints at a given time is a solution of the original equation 3.12 . In addition, 3.19 specifies the time derivative of ψ_μ for any given component. 3.19 is hence called the true equation of motion.

Bellissard and Seiler (17) have also proved that the knowledge of a fundamental solution E of $\Lambda(\pi)d^r(\pi)$ allows the construction of a fundamental soltuion E' (which in fact is just $d^r E$) of $\Lambda(\pi)$ by a purely algebraic procedure.

Applying 3.18 to 3.19a and 3.19b we obtain respectively;

$$(n^2)^6\{n^2+(i\varepsilon n\cdot F^d)^2\}^2 = 0 ;$$

and

$$(n^2)^4\{n^2+(i\varepsilon n\cdot F^d)^2\}^4 = 0 .$$

or, in a noncovariant form by taking n = $(\vec{0},n)$, as;

$$n^{16}\{1 - (\varepsilon\vec{B})^2\}^2 = 0 , \qquad 3.20a$$

and

$$n^{16}\{1 - (\varepsilon\vec{B})^2\}^4 = 0 . \qquad 3.20b$$

We observed that only in the "weak-field case" the system 3.19 is equivalent to a hyperbolic system of partial differential equations, allowing the definition of "spacelike" surface and "future and past cones" with respect to the system be made. These differ, however, from the familiar spacelike surfaces and light cones of special relatively, since, one infer from the second factor in 3.20 that the ray cone is extended, in other word, the propagation of the wave front is acausal.

If the magnetic field strength is strong, the system ceases to be hyperbolic and is not suitable for the description of wave phenomena.

With this review, we conclude our discussion of the classical wave equations. We will now turn our attention to the quantization of the Rarita-Schwinger field.

CHAPTER 4

QUANTIZATION

We divide the discussion in this chapter into five sections as follows; we review briefly in 4-1, the Takahashi and Umezawa formalism for the quantization of free fields as well as fields with general interaction. The identities for the determination of the interaction Hamiltonian is derived. The general relation between the Heisenberg fields and the asymptotic fields is established.

In 4-2, the interaction Hamiltonian expressed in term of the asymptotic fields is explicitly determined up to fourth order in e, using the formulas stipulated in 4-1. We also explicitly express the Heisenberg operators in terms of the asymptotic operators to this order in e.

From the interaction Hamiltonian determined in 4-2, we construct in 4-3 formally the S-matrix and demonstrate that the "generalized Matthew's rule" [13] is satisfied.

An argument concerning the positive definiteness of the equal time anti-commutation relation for the Heisenberg operator is presented in 4-4. The explicit form of this anti-commutator is calculated.

Section 4-5, is devoted to the extension of an alternate computational techniques within the present formalism proposed by Kimel and Nath to fourth order calculation in e. In contrast to the work in section 4-2, the constraints have to be used explicitly here.

4-1 The Takahashi and Umezawa Formalism

A method to quantize fields without the canonical formalism was proposed by Takahashi and Umezawa [11]. In this method, the Heisenberg equation of motion;

$$-i\partial_\mu F(x) = [F(x), P_\mu] \quad , \qquad 4.1$$

where F(x) is any dynamical variable, is regarded as the most fundamental equation. The interpretation of 4.1 may be stated as follows: from a knowledge of the evolution of F(x), the quantity P_μ as well as the commutator can be determined. According to this interpretation there are too many unknowns in the equation. In order to arrive at a physically meaningful theory, the following restrictions are imposed on the quantity P_μ:

1) P_i and $H = -iP_4$ are Hermitian.
2) H is non-negative (for it to be the energy) and represents the total Hamiltonian of the system.
3) Only bosons and fermions exist in nature.
4) P_μ is a four-vector.
5) All physical quantities at finite distance exterior to the light-cone are commutative.

The method of quantization may be summarized as follows: The starting point is the field equation which contains all the subsidiary conditions. For free fields the equation is linear and of the general form

$$\Lambda_{\alpha\beta}(\partial)\phi_\beta(x) = 0 \quad . \tag{4.2}$$

We demand that the operator property of the field ϕ_α, i.e. its commutation relation, must be determined in such a way that 4.2 is consistent with the Heisenberg equations

$$-i\partial_\mu\phi_\alpha = [\phi_\alpha, P_\mu] \quad . \tag{4.3}$$

It is important to note that in this approach the operator P_μ itself is an unknown quantity which has to be determined. In performing the quantization, we first solve the c-number linear wave equation 4.2

under certain boundary conditions. When the solutions are a complete set of orthogonal functions, we impose a normalization condition which is independent of time. The real advantage of the normalization, lies in the fact that when the field ϕ_α is expanded in terms of the c-number solutions so normalized, the expansion coefficients are nothing but the creation and annihilation operators where simply commutation relations are imposed. The operator property of ϕ_α is then determined. With these creation and annihilation operators, the Fock space is constructed.

The quantization of interacting fields is essentially the same. A source term is added to the right hand side of 4.26 making it a nonlinear equation. To obtain a consistent solution of the field equation, in such a way that 4.3 is satisfied, is the whole story of field quantization.

Let us consider the general field $\underset{\sim}{\phi}_\alpha$, (the label α denotes different types of field as well as the components of each field) whose field equations are of the form,

$$\Lambda_{\alpha\beta}(\partial)\underset{\sim}{\phi}_\beta(x) = \underset{\sim}{J}_\alpha(x) \qquad 4.4$$

where the source $\underset{\sim}{J}_\alpha(x)$ takes on the general form,

$$J_{\alpha}(x) = j_{\alpha}(x) + \partial_{\mu} j_{\alpha\mu}(x) + \ldots \equiv D^{a} j_{\alpha a}(x) .$$

On account of the second identity, we rewrite the field equation into the integral form,

$$\phi_{\alpha}(x) = \phi_{\alpha}(x) + \int_{-\infty}^{\infty} dx' D^{a} d_{\alpha\beta}(\partial) \Delta^{ret}(x-x') j_{\beta a}(x') \qquad 4.5$$

where ϕ_{α} is the asymptotic solution of the general field equation 4.4, satisfying the free field equation (i.e. with $J_{\alpha} = 0$), and so we regard it as the quantity in the asymptotic representation. Our objective is to introduce a unitary transformation $U(\sigma)$ connecting the Heisenberg representation with the asymptotic representation. We may not be able to define $U(\sigma)$ as the transformation combining ϕ_{α} and ϕ_{α} directly since in general ϕ_{α} contains dependent variables. We introduce the auxiliary field operation $\varphi_{\alpha}(x,\sigma)$ which is a functional of a space-like surface σ in addition to its dependence on the point x which may not lie on σ. We demand that $\varphi_{\alpha}(x,\sigma)$ satisfy the following requirements:

i) $\phi_{\alpha}(x,\sigma)$ is connected with $\phi_{\alpha}(x)$ by the unitary transformation $U(\sigma)$ which depends only on σ:

$$\phi_{\alpha}(x,\sigma) = U^{+}(\sigma)\phi_{\alpha}(x)U(\sigma) .$$

ii) $\phi_{\alpha}(x)$ is a functional of $\phi_{\alpha}(x,\sigma)$ at the same space-time point x,

$$\underset{\sim}{\phi}_\alpha(x) = F[\phi_\alpha(x,\sigma)]_{x/\sigma}$$

where x/σ means the point x lies on the surface σ.

From requirement (i) we deduce immediately that $\phi_\alpha(x,\sigma)$ satisfies the field equation and commutation relation of the free field for a fixed σ;

$$[\phi_\alpha(x,\sigma),\phi_\beta(x',\sigma)]_\pm = id_{\alpha\beta}(\partial)\Delta(x-x')$$

$$\Lambda_{\alpha\beta}(\partial)\phi_\beta(x,\sigma) = 0 \qquad 4.6$$

where $d_{\alpha\beta}(\partial)$ is the Klein-Gordon divisor.

Since 4.6 is required for every surface σ, it follows that there exists a unitary transformation $U(\sigma,\sigma')$ such that;

$$\phi_c(x,\sigma) = U^+(\sigma,\sigma')\phi_\alpha(x,\sigma')U(\sigma,\sigma') = U^+(\sigma)\underset{\sim}{\phi}_\alpha(x)U(\sigma),$$

with $U(\sigma) = U(\sigma,-\infty)$.

For an arbitrary functional $F[x/\sigma]$, we have in general;

$$F[x/\sigma] = U^+(\sigma)F[x]U(\sigma),$$

where $F[x/\sigma]$ and $F[x]$ are identical functions of $\phi_\alpha(x/\sigma)$ and $\underset{\sim}{\phi}_\alpha(x)$ respectively.

We assume that;

$$U(-\infty) = U^{+}(-\infty) = 1$$

and

$$U(\sigma)U^{+}(\sigma) = U^{+}(\sigma)U(\sigma) = 1 \quad . \qquad 4.7$$

These assumptions can be justified a posteriori.

The assumption 4.40 implies

$$i \frac{\delta U(\sigma)}{\delta\sigma(x)} U^{+}(\sigma) = -iU^{+}(\sigma) \frac{\delta U(\sigma)}{\delta\sigma(x)} \qquad 4.8$$

therefore, we may put

$$i \frac{\delta U(\sigma)}{\delta\sigma(x)} = \mathcal{H}(n,x)U(\sigma) \qquad 4.9$$

where $\mathcal{H}(n,x)$ is a functional of n_μ, the normal to the surface σ, and its derivatives. To be consistent with 4.8, $\mathcal{H}(n,x)$ must be a hermitian operator. We can later identify $\mathcal{H}(n,x)$ as the interaction Hamiltonian.

We obtain from 4.6 and 4.9 the relation;

$$i \frac{\delta\phi_\alpha(x,\sigma)}{\delta\sigma(x')} = U^{+}(\sigma)[\phi_\alpha(x), \mathcal{H}(n,x')]U(\sigma)$$

$$= [\phi_\alpha(x,\sigma), \mathcal{H}(n,x',\sigma)] \quad . \qquad 4.10$$

From 4.5 and the requirements (i) and (ii) we may write $\phi_\alpha(x,\sigma)$ in the following form:

$$\phi_\alpha(x,\sigma) = \phi_\alpha(x) + \int_{-\infty}^{\sigma} dx' D^a d_{\alpha\beta}(\partial)\Delta(x-')\underset{\sim}{j}_{\beta a}(x')$$

$$+ \int_{-\infty}^{\sigma} dx' \underset{\sim}{G}_\alpha(x,x') \qquad 4.11$$

where

$$\Lambda_{\alpha\beta}(\partial)\underset{\sim}{G}_\beta(x,x') = 0 \; . \qquad 4.12$$

The term G is as yet arbitrary except that for the case x/σ, it must be function which is independent of the fields in the past and hence must be a four divergence term. The condition 4.12 ensures that $\phi(x,\sigma)$ satisfies the free field equation.

For the surface σ which passes through the point x, we can establish a relation between $\phi(x,\sigma)$ and $\underset{\sim}{\phi}(x)$;*

$$\phi(x/\sigma) = \phi(x) + \int_{-\infty}^{\infty} dx' \theta(x_o - x_o') D^a d(\partial)\Delta(x-x')\underset{\sim}{j}_a(x')$$

$$+ \int_{-\infty}^{\sigma(x)} dx' \; \underset{\sim}{G}(x,x')$$

---

* We have suppressed the indices α and β's.

$$= \underset{\sim}{\phi}(x) + \int_{-\infty}^{\infty} dx' [\theta(x_o - x_o'), D^a d(\partial)] \Delta(x-x') \underset{\sim}{j}_a(x')$$

$$+ \int_{-\infty}^{\sigma(x)} dx' \underset{\sim}{G}(x,x')$$

$$= \underset{\sim}{\phi}(x) + \widehat{D^a d}(\partial) \underset{\sim}{j}_a(x) + \int_{-\infty}^{\sigma(x)} dx' \underset{\sim}{G}(x,x'). \qquad 4.13$$

Since $[\theta(x_o - x_o'), D^a d(\partial)] \Delta(x-x')$ is proportional to $\delta(x-x')$, we write

$$[\theta(x_o - x_o'), D^a d(\partial)] \Delta(x-x') = \widehat{D^a d}\ \delta(x-x') .$$

Using the notation

$$M(\partial)\phi(x/\sigma) \equiv [M(\partial)\phi(x,\sigma)]_{x/\sigma}$$

where $M(\partial)$ is a general differential operator, we have in general

$$M(\partial)\phi(x/\sigma) = M(\partial)\underset{\sim}{\phi}(x) + \widehat{MD^a d}\ \underset{\sim}{j}_a(x) + \int_{-\infty}^{\sigma(x)} dx' M(\partial)\underset{\sim}{G}(x,x') \qquad 4.14$$

If we substitute 4.14 into the right hand side of 4.13, we can successively express $\underset{\sim}{\phi}(x)$ in terms of $\phi(x/\sigma)$, $\partial_\mu\phi(x/\sigma)$, ... Let us write such a relation as

$$\underset{\sim}{\phi}_\alpha(x) = \phi_\alpha(x/\sigma) + g_\alpha(n,x/\sigma) \qquad 4.15$$

where $g(n,x/\sigma)$ is a functional of n_μ, $\phi(x/\sigma)$, $\partial_\mu\phi(x/\sigma)$ .... and $\partial_\mu \cdots \partial_\nu\phi(x/\sigma)$. The functional form of $g(n,x/\sigma)$ is dictated by the type of field and its source and is in general an infinite series. Relation 4.15 is equivalent to the assumption that the condition (ii) can be inverted.

Differentiating 4.11 with respect to σ, we have,

$$i\,\frac{\delta\phi(x,\sigma)}{\delta\sigma(x')} = iD^a d(\partial)\Delta(x-x')\underset{\sim}{j}_a(x') + i\underset{\sim}{G}(x,x') \qquad 4.16$$

In conjunction with 4.10, we arrive at the identity

$$[\phi(x,\sigma), \mathcal{H}(n,x'/\sigma)] = iD^a d(\partial)\Delta(x-x')\underset{\sim}{j}_a(x') + i\underset{\sim}{G}(x,x'). \qquad 4.17$$

This identity serves to determine both the $\mathcal{H}(n,x'/\sigma)$ and $\underset{\sim}{G}(x,x')$. To obtain the Hamiltonian practically, the first thing we do is to express the Heisenberg operators on the right hand side of 4.17 by the quantities of x/σ using 4.15, we then use the assumed commutation relations for $\phi(x/\sigma)$ to determine $\mathcal{H}(n,x/\sigma)$ by first setting $G = 0$. If the $\mathcal{H}$ so determined turn out to be nonhermitian, we then choose G in such a way that the right hand side of 4.17 enables us to obtain a hermitian $\mathcal{H}$. This is due to the fact as stated right after relation 4.9 that $\mathcal{H}$ must be hermitian.

The $\mathcal{H}$ thus determined is in general a power series of the coupling constant whose region of convergence is very difficult to determine. Assuming the convergence for $\mathcal{H}$, it can be shown that $\mathcal{H}(n,x)$ (obtained by replacing $\phi(x/\sigma)$ by $\phi(x)$ in $\mathcal{H}(n,x/\sigma)$ satisfies the integrability condition, also the Heisenberg equation of motion is satisfied by defining

$$\underset{\sim}{P}_{\mu} = U^{+}(\sigma)\{P_{\mu} - \int d\sigma_{\mu}(x)\mathcal{H}(n,x)\}U(\sigma) \qquad 4.18$$

with P_{μ}, the space time displacement operator for the free field in interaction representation.

The commutation relations for the Heisenberg operator can be determined from the assumed commutation relations for $\phi(x/\sigma)$ 4.6 through 4.15.

We remark that for the case in which the current $\underset{\sim}{J}$ contains at most first order derivative coupling and the spin of the field is at most one;

$$\underset{\sim}{G}(x,x') = 0 \ .$$

When $\underset{\sim}{J}$ contains derivative coupling of higher degree or the field operator with higher spin, $\mathcal{H}(n,x)$ and $\underset{\sim}{\phi}(x)$ expressed in terms of $\phi(x/\sigma)$ are the infinite series of terms containing the higher derivatives.

To relate this formalism with the fundamental structure of quantum field theory discussed in Chapter 2, we observe here that the fields $\phi_\alpha(x)$ in 4.5 corresponds to the asymptotic fields $\phi_o(t)$ in 2.12, while the auxiliary fields $\phi_\alpha(x/\sigma)$ take the role of canonical independent variables in the Heisenberg operator. And the $U(\sigma)$ in the requirements i) corresponds to the operator $U(t)$ in 2.13. The requirement ii), implies the invertibility between the Heisenberg operators and the asymptotic operators since $\phi_\alpha(x/\sigma)$ and $\phi_\alpha(x)$ are related by a unitary transformation. Finally 4.18 is the covariant version of 2.19 in terms of the Heisenberg operators.

4-2 Determination of the Interaction Hamiltonian

The interaction Hamiltonian for the Rarita-Schwinger spin 3/2 field $\underset{\sim}{\psi}_\mu$ minimally coupled to the electromagnetic field $\underset{\sim}{A}_\lambda$ will now be determined using the formulas stipulated in the previous section.

We start with the field equations,

$$\Lambda_{\mu\nu}(\partial)\ \underset{\sim}{\psi}_\nu(x) = \underset{\sim}{J}_\mu(x) \qquad 4.19a$$

$$\Box\ \underset{\sim}{A}_\lambda(x) = \underset{\sim}{I}_\lambda(x) \qquad 4.19b$$

and define the auxiliary operators as follows,*

$$\psi_\mu(x,\sigma) = \psi_\mu(x) + \int_{-\infty}^{\sigma} dx' d_\mu \Delta(x-x') \underset{\sim}{J}(x') + \int_{-\infty}^{\sigma} dx' \underset{\sim}{G}_\mu(x,x') \quad , \tag{4.20a}$$

$$A_\lambda(x,\sigma) = A_\lambda(x) + \int_{-\infty}^{\sigma} dx' D_\lambda(x-x') \underset{\sim}{I}(x') + \int_{-\infty}^{\sigma} dx' \underset{\sim}{g}_\lambda(x,x') \quad . \tag{4.20b}$$

Since

$$[\theta(x_o - x_o'), d_{\mu\nu}(\partial)]\Delta(x-x') = \hat{d}_{\mu\nu}(n,\partial)\delta(x-x')$$

$$= \frac{2}{3m^2}\,(\xi + \xi_{:\sigma}\partial_\sigma^s + \xi' n\partial)_{\mu\nu}\ \delta(x-x')^{**}$$

and

$$[\theta(x_o - x_o'), D_{\lambda\lambda'}(x-x')] = 0$$

we have from 4.20 when x/σ as follows;

$$\psi_\mu(x/\sigma) = \underset{\sim}{\psi}_\mu(x) + \hat{d}_\mu \underset{\sim}{J}(x) + \int_{-\infty}^{\sigma(x)} dx' \underset{\sim}{G}_\mu(x,x') \tag{4.21a}$$

and

$$A_\lambda(x/\sigma) = \underset{\sim}{A}_\lambda(x) + \int_{-\infty}^{\sigma(x)} dx' \underset{\sim}{g}_\lambda(x,x') \quad . \tag{4.21b}$$

---

* Four-vector indices which are contracted have been suppressed.

** $\partial_\sigma^s \equiv (\delta_{\sigma\rho} + n_\sigma n_\rho)\partial_\rho$, the explicit form of ξ's are given in Appendix.

According to 4.17, the Hamiltonian must satisfy the following identities simultaneously,

$$[\psi_\mu(x,\sigma),\, \mathcal{H}(n,x'/\sigma)] = id_\mu(\partial)\Delta(x-x')\underset{\sim}{J}(x') + i\underset{\sim}{G}_\mu(x,x')\ ,$$ 4.22a

$$[A_\lambda(x,\sigma),\, \mathcal{H}(n,x'/\sigma)] = iD_\lambda(x-x')\underset{\sim}{I}_\lambda(x') + i\underset{\sim}{g}_\lambda(x,x')\ .$$ 4.22b

For the case e = 0 in 4.21, we have;*

$$\underset{\sim}{\psi}^{o}_\mu(x) = \psi_\mu(x/\sigma) \quad ; \quad \underset{\sim}{A}^{o}_\lambda(x) = A_\lambda(x/\sigma) \qquad 4.23a,b$$

substituting this to the right hand side of 4.54, using the commutation relations;

$$\{\psi_\mu(x,\sigma),\bar{\psi}_\nu(x'/\sigma)\} = id_{\mu\nu}(\partial)\Delta(x-x')$$

and

$$[A_\lambda(x,\sigma),A_{\lambda'}(x'/\sigma)] = iD_{\lambda\lambda'}(x-x')$$

we obtain

$$\mathcal{H}^1(n,x/\sigma) = -ie\bar{\psi}\Gamma A\psi \Big|^{**}_{:x/\sigma} \qquad 4.24$$

---

* The superscript indicates the power in e.

** The symbol :x/σ at the right hand corner indicates that all field operators on this side of the equation are the x/σ fields.

which is just the interaction Lagrangian with opposite sign. Hence $\mathcal{H}^1$ is hermitian, so we set $\underset{\sim}{G}^1$ and $\underset{\sim}{g}^1$ equals to zero. We have then;

$$\underset{\sim}{\psi}_\mu^1 = -\hat{d}_\mu J \qquad :x/\sigma \quad . \qquad 4.25$$

Substituting 4.25 into 4.22, the right hand side of 4.22 becomes;

$$id_\mu(\partial)\Delta(x-x')\{ie\Gamma A'\hat{d}(n,\partial')J'\} + i\underset{\sim}{G}_\mu^2(x,x')^* \qquad :x'/\sigma \qquad 4.26a$$

and

$$iD_{\lambda\lambda'}(x-x')\{ie\bar{\psi}'\Gamma_{\lambda'}\hat{d}(n,\partial')J' + ie[\bar{J}'\hat{d}(n,-\overleftarrow{\partial}')]\Gamma_{\lambda'}\psi'\} + ig_\lambda^2(x,x') \quad :x'/\sigma \quad . \qquad 4.26b$$

We are no longer able to find a hermitian $\mathcal{H}^2$ by assuming $\underset{\sim}{G}$ and $\underset{\sim}{g}$ vanishes. This is because the differentiation in d does not enter symmetrically. Hence we set:

$$\underset{\sim}{G}_\mu^2(x,x') = -\frac{ie}{2}\left(\frac{2}{3m^2}\right)d_\mu(\partial)\Delta(x-x')\Gamma\underset{\sim}{A}'\hat{\partial}_\lambda'(\xi_{:\sigma}\delta_{\sigma\lambda}^s + \xi' n_\lambda)\underset{\sim}{J}' \qquad 4.27a$$

---

* $F' \equiv F(x'/\sigma)$.

and

$$g_\lambda^2(x,x') = -\frac{ie}{2}\left(\frac{2}{3m^2}\right)D_{\lambda\lambda'}(x-x')\{\bar{\psi}'\Gamma_{\lambda'}\hat{\partial}'_\rho(\xi_{:\sigma}\delta^S_{\sigma\rho}+\xi' n_\rho)J'$$

$$-\bar{J}'\hat{\partial}'_\rho(\xi_{:\sigma}\delta^S_{\sigma\rho}+\xi' n_\rho)\Gamma_{\lambda'}\psi'\} \quad . \qquad 4.27b$$

We observe that both G and g are four dimensional divergence terms in accordance with the requirement discussed in section 4-1.

With 4.26 and 4.27, the hermitian second order Hamiltonian $\mathcal{H}^2$ is determined to be

$$\mathcal{H}^2(n, x/\sigma) = -\bar{J}\hat{d}(n,\tfrac{1}{2}\overleftrightarrow{\partial})J_{:x/\sigma} \quad . \qquad 4.28$$

To determine $\mathcal{H}^3$, we must first find the expression for ψ^2_μ in terms of the auxiliary fields. We have from 4.20 that

$$n\partial\psi_\mu(x/\sigma) = n\partial\psi_\mu(x) + \hat{d}_\mu n\partial J(x) + \frac{2}{3m^2}b_\mu J$$
$$+\int_{-\infty}^{\sigma(x)} dx' n\partial G_\mu(x,x') \qquad 4.29a$$

and

$$n\partial A_\lambda(x/\sigma) = n\partial A_\lambda(x) + \int_{-\infty}^{\sigma(x)} dx' n\partial g_\lambda(x,x') \quad , \qquad 4.29b$$

since

$$[\theta(x-x'), n\partial d_{\mu\nu}]\Delta(x-x') = \{\hat{d}_{\mu\nu} n\partial + \frac{2}{3m^2} b_{\mu\nu}\}\delta(x-x')^*$$

and

$$[\theta(x-x'), n\partial]D_{\lambda\lambda'}(x-x') = 0.$$

The expression for $\underset{\sim}{\psi}^2_\mu$ is then found to be

$$\underset{\sim}{\psi}^2_\mu(x) = -ie\hat{d}_\mu \Gamma A\hat{d}J - ie\left(\frac{2}{3m^2}\right)^2 \xi'_\mu \Gamma AbJ - \frac{ie}{2}\left(\frac{2}{3m^2}\right)^2 b_\mu \Gamma A\xi' J \qquad :x/\sigma \tag{4.30}$$

The third term on the right hand side of 4.30 is due to the contribution from the G term as follows;

$$\int^{\sigma(x)} dx' \underset{\sim}{G}^2_\mu(x,x') = -\frac{ie}{2}\left(\frac{2}{3m^2}\right)\int dx' d_\mu(\partial)\Delta(x-x')\Gamma \underset{\sim}{A}' \times$$

$$\frac{1}{2}[\varepsilon(x_o-x'_o), \partial'_\lambda][\xi_{:\sigma}\delta^s_{\sigma\lambda} + \xi' n_\lambda)\underset{\sim}{J}'$$

$$= \frac{ie}{2}\left(\frac{2}{3m^2}\right)^2 b_\mu \Gamma \underset{\sim}{A}\xi' \underset{\sim}{J} \quad ,$$

since

$$\frac{1}{2}[\varepsilon(x_o-x'_o), n\partial']d_{\mu\nu}(\partial)\Delta(x-x') = -\frac{2}{3m^2} b_{\mu\nu}\ \delta(x-x')^* .$$

By repeating the same procedure, and after a very lengthy and tedious calculation, we arrive at the following results;

---

* See Appendix.

$$\mathcal{H}(n,x/\sigma) \mathrel{\hat{=}} \sum_{k=1}^{\infty} \mathcal{H}^k(n,x/\sigma)$$

$$= -ie\bar{\psi}\Gamma A\psi - \bar{J}\hat{d}(n,\tfrac{1}{2}\overleftrightarrow{\partial})J$$

$$-ie\bar{J}\hat{d}(n,\tfrac{1}{2}\overleftrightarrow{\partial})\Gamma A\hat{d}(n,\tfrac{1}{2}\overleftrightarrow{\partial})J$$

$$-\frac{i\epsilon^2}{2e}\bar{J}\{b(n,-\overleftarrow{\partial})\Gamma A\xi' + \xi'\Gamma A b(n,\partial)\}J$$

$$-(ie)^2\bar{J}\hat{d}(n,\tfrac{1}{2}\overleftrightarrow{\partial})\Gamma A\hat{d}(n,\tfrac{1}{2}\overleftrightarrow{\partial})\Gamma A\hat{d}(n,\tfrac{1}{2}\overleftrightarrow{\partial})J$$

$$+\frac{1}{2}\epsilon^2\bar{J}\{b(n,-\overleftarrow{\partial})\Gamma A\xi'\Gamma A\hat{d}(n,\partial)$$

$$+\hat{d}(n,-\overleftarrow{\partial})\Gamma A\xi'\Gamma A b(n,\partial)$$

$$+\xi'\Gamma A\hat{d}(n,-\overleftarrow{\partial})\Gamma A b(n,\partial)$$

$$+b(n,-\overleftarrow{\partial})\Gamma A\hat{d}(n,\partial)\Gamma A\xi'$$

$$+\xi'\Gamma A b(n,-\overleftarrow{\partial})\Gamma A\hat{d}(n,\partial)$$

$$+\hat{d}(n,-\overleftarrow{\partial})\Gamma A b(n,\partial)\Gamma A\xi'$$

$$-\frac{1}{4}\left(\frac{2}{3m^2}\right)\xi'\Gamma A(n\overleftarrow{\partial}+[\gamma\overleftarrow{\partial}^S+m]n\gamma)b(n,\partial)\Gamma A\xi'$$

$$+\frac{1}{4}\left(\frac{2}{3m^2}\right)\xi'\Gamma A b(n,-\overleftarrow{\partial})(n\partial+[\gamma\overleftarrow{\partial}^S+m]n\gamma)\Gamma A\xi'\}J$$

$$+(\epsilon/2)^2\{(\bar{\psi}\Gamma_\lambda\xi' J)^2 + (\bar{J}\xi'\Gamma_\lambda\psi)^2$$

$$-2\bar{\psi}\Gamma_\lambda\xi' J\bar{J}\xi'\Gamma_\lambda\psi\} \quad :x/\sigma \qquad 4.31$$

and

$$G_\mu(x,x') = -d_\mu(\partial)\Delta(x-x')\Gamma \underset{\sim}{A}'\{\frac{1}{2}\, i\varepsilon\hat{\partial}'_\lambda(\xi_{:\sigma}\delta^S_{\sigma\lambda}+\xi' n_\lambda)\underset{\sim}{J}'$$

$$+(ie)^2\hat{d}(n,\tfrac{1}{2}\overleftrightarrow{\partial}')\Gamma\underset{\sim}{A}'(\hat{d}(n,\partial')-\hat{d}(n,-\overleftarrow{\partial}'))\underset{\sim}{J}'$$

$$+(i\varepsilon)^2(b(n,\partial')-b(n,-\overleftarrow{\partial}'))\Gamma\underset{\sim}{A}'\xi'\underset{\sim}{J}' \qquad 4.32$$

$$\{(ie)^2 i\varepsilon[\hat{d}(n,\tfrac{1}{2}\overleftrightarrow{\partial})\Gamma\underset{\sim}{A}\hat{d}(n,\tfrac{1}{2}\overleftrightarrow{\partial})\Gamma\underset{\sim}{A}\hat{\partial}_\sigma(\xi_{:\sigma} + \xi' n_\sigma)$$

$$+ \frac{1}{2}\,(i\varepsilon)^3+\; \hat{\partial}_\sigma(\xi_{:\sigma} + \xi' n_\sigma)\Gamma\underset{\sim}{A}\xi'\Gamma\underset{\sim}{A}b(n,\partial)$$

$$+ ie(i\varepsilon)^2+\; (b(n,\partial) - b(n,-\overleftarrow{\partial}))\Gamma\underset{\sim}{A}\hat{d}(n,\partial)\Gamma\underset{\sim}{A}\xi'$$

$$+ ie(i\varepsilon)^2+\; \xi'\Gamma\underset{\sim}{A}(b(n,\partial) - b(n,-\overleftarrow{\partial}))\Gamma\underset{\sim}{A}\hat{d}(n,\partial)$$

$$+ (i\varepsilon)^3+\; \hat{\partial}_\sigma(\xi_{:\sigma} + \xi' n_\sigma)\Gamma\underset{\sim}{A}b(n,\partial)\Gamma\underset{\sim}{A}\xi'$$

$$- \frac{1}{4}(i\varepsilon)^3-\; \xi'\Gamma\underset{\sim}{A}(n\hat{\partial} - \gamma\hat{\partial}^S n\gamma)b(n,\partial)\Gamma\underset{\sim}{A}\xi']\underset{\sim}{J}\}$$

while

$$\underset{\sim}{\psi}_\mu(x) = \psi_\mu - \hat{d}_\mu J - ie\hat{d}_\mu \Gamma A\hat{d}J - \frac{2i\varepsilon}{3m^2}\{\xi'\Gamma Ab + \frac{1}{2} b_\mu \Gamma A\xi'\}J$$

$$- (ie)^2 \hat{d}_\mu \Gamma A\hat{d}\Gamma A\, \hat{d}J$$

$$- (i\varepsilon)^2 (\xi_\mu' \Gamma Ab\Gamma A\hat{d} + \xi_\mu' \Gamma A\hat{d}\Gamma Ab + \hat{d}\Gamma A\xi'\Gamma Ab)J$$

$$- \frac{1}{2}(i\varepsilon)^2 (\hat{d}_\mu \Gamma Ab\Gamma A\xi' + b_\mu \Gamma A\hat{d}\Gamma A\xi' + b_\mu \Gamma A\xi'\Gamma A\hat{d})J$$

$$+ \frac{1}{3m^2}(i\varepsilon)^2 \xi_\mu' \Gamma A\{n\partial - (\gamma\partial^s - m)n\gamma\}b\Gamma A\xi'\ J$$

$$- \frac{1}{2}\ (i\varepsilon)^2 \xi_\mu' \Gamma_\lambda \psi\{\bar{\psi}\Gamma_\lambda \xi' J - \bar{J}\xi'\Gamma_\lambda \psi\}$$

$$- (ie)^3 \hat{d}_\mu \Gamma A\hat{d}\Gamma Ad\Gamma AJ + ie\varepsilon^2\{\hat{d}_\mu \Gamma A\hat{d}\Gamma A(\xi'\Gamma Ab + \frac{1}{2} b\Gamma A\xi')$$

$$+ (\xi_\mu' \Gamma Ab + \frac{1}{2}\ b_\mu \Gamma A\xi')\Gamma A\hat{d}\Gamma A\hat{d}$$

$$+ \hat{d}_\mu \Gamma A\xi'\Gamma A(\hat{d}\Gamma Ab + b\Gamma A\hat{d})$$

$$+ \frac{1}{2}\ (\hat{d}_\mu \Gamma Ab + b_\mu \Gamma A\hat{d})\Gamma A\xi'\Gamma A\hat{d}$$

$$+ \xi_\mu' \Gamma A\hat{d}\Gamma A(\hat{d}\Gamma Ab + b\Gamma A\hat{d})$$

$$+ \frac{1}{2}\ (b_\mu \Gamma A\hat{d} + \hat{d}_\mu \Gamma Ab)\Gamma A\hat{d}\Gamma A\xi'\}J$$

$$+ \frac{i\varepsilon^4}{e}\{\xi_\mu' \Gamma A \quad \xi'\Gamma Ab + \frac{1}{2}\ b\Gamma A\xi') + \frac{3}{8}\ b_\mu \Gamma A\xi'\Gamma Ab\Gamma A\xi'\}J$$

$$- \frac{1}{2}\ i\varepsilon^3\{(\hat{d}_\mu \Gamma A\xi' + \xi'\Gamma A\hat{d})\Gamma A(n\partial - [\gamma\partial^s - m]n\gamma)b\Gamma A\xi'$$

$$+ \xi_\mu' \Gamma A(n\partial - [\gamma\partial^s - m]n\gamma)b\Gamma A(\xi'\Gamma A\hat{d} + \hat{d}\Gamma A\xi')\}J$$

:x/σ

+ ξ_μ' times terms containing three ψ_μ

4.33

We have explicitly determined $\mathcal{H}^k(n,x/\sigma)$ and have expressed the Heisenberg operators ψ_μ in terms of the auxiliary operators and their derivatives to fourth order in the coupling constant e. In principle, it is possible to evaluate the higher order terms by repeating the same procecure, though the calculations are enormously lengthy and complicated. Let us now examine some features of the procedure and the results obtained.

We observe that to obtain the n-th order expansion for ψ_μ^n in terms of the auxiliary operators, we have to use the (n-1)th order expression to construct the n-th order Hamiltonian $\mathcal{H}^n(n,x/\sigma)$ whereby determining the n-th order G and g.

The highest order of derivatives in the n-th order Hamiltonian as well as the expansion of $\psi^n(x)$ in terms of x/σ fields is $n-1$. It seems that the time derivative $(n\partial)$ in $\mathcal{H}^n$ can take the order of $n-1$, however closer examination reveals that the highest time derivative in the Hamiltonian is at most of order one.

In the expansion o ψ_μ in terms of x/σ fields, we observe that $\psi_\mu(x)$ ar $\psi_\mu(x/\sigma)$ are not related linearly, yet, all the nonlinear terms are proportional to n_μ, since $\xi'_{\mu\nu} = n_\mu n\gamma n_\nu$. Thus for the space component of $\underset{\sim}{\psi}_{\mu}^{s}$, we can write the expansion in a compact form as follows;

$$\underset{\sim}{\psi}_{\mu}^{s}(x) = \delta_{\mu\lambda} Q_{\lambda\nu}^{-1} \psi_{\nu} \quad :x/\sigma$$

where $Q_{\mu\nu}^{-1}$ is a functional of $A_{\lambda}(x/\sigma)$ and its derivatives as well as derivative operators acting on ψ_{ν}.

4-3 The S-Matrix

According to the general theory, the required interaction Hamiltonian $\mathcal{H}(n,x)$ in the interaction picture is found merely by replacing the auxiliary fields in $\mathcal{H}(n,x/\sigma)$ by the asymptotic fields (i.e. fields in asymptotic representation).

With the $\mathcal{H}(n,x)$ given by 4.31, we can formally write down the S-matrix of our system as follows;

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int dx_1 \ldots dx_n \; T(\mathcal{H}(n,x_1), \ldots, \mathcal{H}(n,x_n)) \qquad 4.34$$

where T stands for the chronological ordering

$$T(A(x),B(x')) = \theta(x_o - x_o')A(x)B(x') \pm \theta(x_o' - x_o)B(x')A(x) .$$

Our S-matrix can be written in perturbation expansion explicitly in terms of the $\mathcal{H}^n$'s given by 4.31

to fourth order in e as;

$$S = 1 + (-i)\int dx\ \{\mathcal{H}^1(n,x) + \mathcal{H}^2(n,x) + \mathcal{H}^3(n,x) + \mathcal{H}^4(n,x)\}$$

$$+ \frac{(-i)^2}{2!}\int dx dx_1 \{T(\mathcal{H}^1(n,x),\mathcal{H}^1(n,x_1)) + 2T(\mathcal{H}^1(n,x),\mathcal{H}^2(n,x_1))$$

$$+ 2T(\mathcal{H}^1(n,x),\mathcal{H}^3(n,x_1)) + T(\mathcal{H}^2(n,x),\mathcal{H}^2(n,x_1))\}$$

$$+ \frac{(-i)^3}{3!}\int dx dx_1 dx_2 \{T(\mathcal{H}^1(n,x),\mathcal{H}^1(n,x_1),\mathcal{H}^1(n,x_2))$$

$$+ 3T(\mathcal{H}^1(n,x),\mathcal{H}^1(n,x_1),\mathcal{H}^2(n,x_2))\}$$

$$+ \frac{(-i)^4}{4!}\int dx dx_1 dx_2 dx_3 T(\mathcal{H}^1(n,x),\mathcal{H}^1(n,x_1),\mathcal{H}^1(n,x_2),$$

$$\mathcal{H}^1(n,x_3)) \quad . \qquad 4.35$$

the coefficients directly in front of the T product arises from the symmetry of the interchange of the space-time coordinates.

We define the T*-product which is related to the T-product for the field operators as follows;

$$T(\psi_\mu(x),\bar{\psi}_\nu(x')) = T^*(\psi_\mu(x),\bar{\psi}_\nu(x')) + i\hat{d}_{\mu\nu}\delta(x-x')$$

$$T(n\partial\psi_\mu(x),\bar{\psi}_\nu(x')) = T^*(n\partial\psi_\mu(x),\bar{\psi}_\nu(x')) + i\hat{d}_{\mu\nu}n\partial\delta(x-x')$$

$$+ \frac{2i}{3m^2}\, b_{\mu\nu}\ \delta(x-x')$$

$$T(n\partial\psi_\mu(x), n\partial'\bar{\psi}_\nu(x')) = T^*(n\partial\psi_\mu(x), n\partial'\bar{\psi}_\nu(x'))$$

$$- i[\hat{d}_{\mu\nu}(n\partial)^2 + \frac{2}{3m^2}(n\partial+[\gamma\partial^s-m]n\gamma)b_{\mu\nu}]\delta(x-x') \qquad 4.36$$

and

$$T(A_\mu(x,)A_\nu(x')) = T^*(A_\mu(x), A_\nu(x'))$$

$$T(n\partial A_\mu(x), A_\nu(x')) = T^*(n\partial A_\mu(x), A_\nu(x'))$$

$$T(n\partial A_\mu(x), n\partial' A_\nu(x')) = T^*(n\partial A_\mu(x), n\partial A_\nu(x'))$$

$$- i\,\delta_{\mu\nu}\delta(x-x') . \qquad 4.36$$

With 4.36 we can replace the T-products in the S-matrix 4.35 by T*-products. After a lengthy calculation, all the normal dependent terms vanish, and the S-matrix is indeed given by

$$S = \sum_{n=0} \frac{(-i)^n}{n!}\int dx_1 - dx_n \; T^*(\mathcal{H}^1(n,x_1)\ldots\mathcal{H}^1(n,x_n)) . \qquad 4.37$$

Hence, we have explicitly demonstrated that in the S-matrix calculation, at least to fourth order in e, we can replace $\mathcal{H}(n,x)$ by

$$(\mathcal{H})_{effective} = \mathcal{H}^1 = -\mathcal{L}_I$$

so long as we also replace the T-product by the T*-product. This property of the S-matrix, referred to as the 'generalized Matthew's rule', is satisfied in our case. This is indeed the result one expects in a consistent theory.

4-4 Commutation Relations

To complete our quantization, we now derive the commutation relations for the Heisenberg field operators. Since we have expressed the Heisenberg operators in terms of the auxiliary fields whose commutation relations have been assumed, we are now able to determine the commutators of the Heisenberg operators.

From 4.33, we write for $\mu = i$, the relation,

$$\underset{\sim}{\psi}_i(x) = Q^{-1}_{i\nu}(n,x/\sigma)\psi_\nu(x/\sigma) \qquad 4.38$$

with

$$Q^{-1}_{i\nu} = \delta_{i\nu} - ie\{(\partial_i + \frac{m}{2}\gamma_i)M + ieA_i N\}A_k\beta_{k\ell}\delta_{\ell\nu} \qquad 4.38a$$

where

$$M = -\frac{1}{2}\left(\frac{2}{3m^2}\right)\{2 + i\varepsilon\ (\vec{A}.\vec{V} - i\vec{\sigma}.\vec{B}) - \varepsilon^2([\vec{A}.\vec{V}]^2 + [i\vec{\sigma}.\vec{B}]^2)$$

$$+ i\varepsilon^3(\frac{3}{4}\vec{V}.\vec{A}\ \vec{A}.\vec{V}\ \vec{V}.\vec{A} - [\vec{V}.\vec{A}]^2 i\vec{\sigma}.\vec{B} - i\vec{\sigma}.\vec{B}[\vec{A}.\vec{V}]^2 + \vec{A}.\vec{V}[i\vec{\sigma}.\vec{B}]^2$$

$$+ \text{higher order terms}\} \qquad :x/\sigma \qquad 4.38b$$

$$N = \frac{1}{2}\left(\frac{2}{3m^2}\right)\{1 + \varepsilon i\vec{\sigma}.\vec{B} - \varepsilon^2(\frac{1}{4}\,\vec{A}.\vec{\nabla}\;\vec{\nabla}.\vec{A} + [i\vec{\sigma}.\vec{B}]^2)\}$$

$$+ \text{ higher order terms} \qquad 4.38c$$

$:x/\sigma$

and

$$V_k \equiv \beta_{k\ell}\partial_\ell - m\gamma_k = \beta_{k\ell}(\partial_\ell + \frac{1}{2}\,m\gamma_\ell) \;.$$

The following equalities, which can easily be verified, have been used to arrive at the above result:

$$\hat{d}_{i\lambda}(\Gamma A)_{\lambda\nu} = \left(\frac{2}{3m^2}\right)(\partial_i + \frac{m}{2}\,\gamma_i)A_\kappa\beta_{k\ell}\delta_{\ell\nu}$$

$$b_{i\lambda}(\Gamma A)_{\lambda\nu}n_\nu = \left(\frac{3m^2}{2}\right)A_i + (\partial_i + \frac{m}{2}\,\gamma_i)\vec{V}.\vec{A}$$

$$\hat{d}_\mu \cdot \Gamma A \cdot n = -\left(\frac{2}{3m^2}\right)n_\mu\vec{V}.\vec{A}$$

$$n \cdot \Gamma A \cdot \hat{d}_\nu = -\left(\frac{2}{3m^2}\right)n\gamma\vec{A}.\vec{\nabla}n\gamma n_\nu$$

$$n \cdot \Gamma A \cdot b \cdot \Gamma A \cdot n = -n\gamma\vec{A}.\vec{\nabla}\;\vec{\nabla}.\vec{A}$$

Since $Q^{-1}_{i\nu} \neq 0$ only for $\nu = j$, making use of the primary constraint,

$$\gamma_i\psi_i = \frac{3}{2}\left(\vec{\gamma}.\vec{\nabla} - \frac{3m}{2}\right)^{-1}\partial_i\phi_i$$

enables us to obtain from 4.38 the relation

$$\underset{\sim}{\phi}_i(x) = S^{-1}_{ij}\, \phi_j \Big|_{x/\sigma} \qquad 4.39$$

where

$$S^{-1}_{ij} = P_{ir} Q^{-1}_{rs} [\delta_{sj} + \tfrac{1}{2}(\vec{\gamma}.\vec{\nabla} - \tfrac{3m}{2})^{-1}\, \partial_j] .$$

The equal time commutation relation for $\underset{\sim}{\phi}_i$ becomes

$$\{\underset{\sim}{\phi}_i(x),\ \underset{\sim}{\phi}^+_j(x')\}_{E.T} = S^{-1}_{ir}\{\phi_r, \phi^+_s\} S^{-1+}_{sj} \Big|_{x,x'/\sigma}$$

$$= S^{-1}_{ir}(\delta_{rs} - \frac{2}{3m^2}\, \partial_r \partial_s)\, \delta(x-x')\, S^{-1+}_{sj} \qquad 4.40$$

since Q^{-1}, consequently S^{-1}, contains only space-derivatives hence the electromagnetic field A_μ's commute at equal time. The relation

$$\{\phi_i(x/\sigma)\phi^+_j(x'/\sigma)\} = P_{ir}(\delta_{rs} - \frac{2}{3m^2}\, \partial_r \partial_s) P_{sj}\ \delta(x-x')$$

has been used.

If we now introduce an arbitrary complex spinor function U(x), we can write 4.40 as

$$\int dx dx' U(x) \{\underset{\sim}{\phi}_i(x,)\underset{\sim}{\phi}^+_j(x')\}_{E.T}\ U^+(x')$$

$$= \int dx\ U(x) S^{-1}_{ir}(-\overleftarrow{\partial})(\delta_{rs} + \frac{2}{3m^2}\, \overleftarrow{\partial}_r \partial_s) S^{-1+}_{sj}(+\partial) U^+(x) \qquad 4.41$$

When i equals to j, the right hand side of 4.41 is obviously positive definite. The problem with the indefinite nature of the anti-commutator for $\underset{\sim}{\phi}_i$ obtained through the canonical quantization does not seem to be present here. We may then conclude that our method of quantization for the charged spin-3/2 is consistent.

However, let us determine the commutator for $\underset{\sim}{\phi}_i$ explicitly to fourth order in e. From 4.38 and its hermitian conjugate, we have

$$\{\underset{\sim}{\phi}_i(x), \underset{\sim}{\phi}_j^+(x')\}_{E.T} = P_{in}\{\underset{\sim}{\psi}_n(x), \underset{\sim}{\psi}_r^+(x')\}_{E.T}\, P_{rj}$$

$$= P_{in}[(\delta_{nr} - \frac{2}{3m^2}\partial_n\partial_r) - ie(\partial_n M(\partial) + ieA_n N(\partial)) \times$$

$$\times\ (A_r - \frac{2}{3m^2}\vec{A}.\vec{\nabla}\partial_r)\delta(x-x')$$

$$+\ ie(A_n + \frac{2}{3m^2}\partial_n\vec{\nabla}.\vec{A})\delta(x-x')(M^+(-\overleftarrow{\partial}')\overleftarrow{\partial}_r' - ieN^+(-\overleftarrow{\partial}')A_r')$$

$$-(ie)^2(\partial_n M(\partial) + ieA_n N(\partial))\frac{2}{3m^2}\vec{A}.\vec{\nabla}\ \vec{\nabla}.\vec{A}\,\delta(x-x')\times$$

$$\times\ (M^+(-\overleftarrow{\partial}')\overleftarrow{\partial}_r' - ieN^+(-\overleftarrow{\partial}')A_r')]P_{rj}$$

$$+ \text{ higher order terms} \quad .. \qquad 4.42$$

Since,

$$\{\psi_i, \psi_j^+\}_{E.T} = [\delta_{ij} - \frac{1}{3m}(\gamma_i\partial_j - \gamma_j\partial_i) - \frac{2}{3m^2}\partial_i\partial_j]\,\delta(x-x')$$

$$A_k\beta_{ki}\{\psi_i, \psi_j'^+\}_{E.T} = [A_j - \frac{2}{3m^2}A.\vec{\nabla}(\partial_j - \frac{m}{2}\gamma_j)]\,\delta(x-x')$$

$$\{\psi_i, \psi_k'^+\}_{E.T}\,\beta_{k\ell}A_\ell' = [A_i + \frac{2}{3m^2}(\partial_i + \frac{m}{2}\gamma_i)\vec{\nabla}.\vec{A}]\,\delta(x-x')$$

$$A_k\beta_{ki}\{\psi_i, \psi_j'^+\}\beta_{j\ell}A_\ell' = \frac{2}{3m^2}\vec{A}.\vec{\nabla}\;\vec{\nabla}.\vec{A}\;\delta(x-x')$$

the relation $\delta(x-x')\overleftarrow{\partial}'_{\mu_1}..\overleftarrow{\partial}'_{\mu_n}g(x') = (-1)^n\partial_{\mu_1}..\partial_{\mu_n}g(x) \times \delta(x-x')$ for arbitrary function g has been used. Expressing all the A_μ's and ∂_μ' on the right hand side of 4.42 in terms of the unprimed ones, we finally arrive at:

$$\{\underset{\sim}{\phi}_i(x), \underset{\sim}{\phi}_j^+(x')\}_{E.T} = P_{in}[\delta_{nr} - \frac{2}{3m^2}\partial_n\partial_r$$

$$- \frac{2}{3m^2}\{\partial_n(-ieA_r) + (-ieA_n)\partial_r + \partial_n(\varepsilon\vec{\sigma}.\vec{B})\partial_r\}$$

$$- \frac{2}{3m^2}\{(-ieA_n)(-ieA_r) + \partial_n(\varepsilon\vec{\sigma}.\vec{B})(-ieA_r)$$

$$+ (-ieA_n)(\varepsilon\vec{\sigma}.\vec{B})\partial_r + \partial_n(\varepsilon\vec{\sigma}.\vec{B})\partial_r\}$$

$$- \frac{2}{3m^2}\{(-ieA_n)(\varepsilon\vec{\sigma}.\vec{B})(-ieA_r) + \partial_n(\varepsilon\vec{\sigma}.\vec{B})^2(-ieA_r)$$

$$+ (-ieA_n)(\varepsilon\vec{\sigma}.\vec{B})^2\partial_r + \partial_n(\varepsilon\vec{\sigma}.\vec{B})^3\partial_r\}$$

$$- \frac{2}{3m^2} \{(ieA_n)(\epsilon\vec{\sigma}.\vec{B})^2(-ieA_r)+\partial_n(\epsilon\vec{\sigma}.\vec{B})^3(-ieA_r)$$

$$+(-ieA_n)(\epsilon\vec{\sigma}.\vec{B})^3\partial_r+\partial_n(\epsilon\vec{\sigma}.\vec{B})^4\partial_r\}]P_{rj}\delta(x-x)$$

+ higher order terms .

The right hand side of 4.43 coincides to fourth order in e with the power series expansion of the expression

$$P_{in}\{\delta_{nr} - \frac{2}{3m^2}(\partial_n-ieA_n)(1-\epsilon\vec{\sigma}.\vec{B})^{-1}(\partial_r-ieA_r)\}P_{rj}\delta(x-x')$$

with

$$(1 - \epsilon\vec{\sigma}.\vec{B})^{-1} = \sum_{n\geq 0} (\epsilon\vec{\sigma}.\vec{B})^n$$

after collecting terms of the same power in e. This in fact is the well known result of Johnson and Sudarshan. We believe that if we had determined the higher order terms, the agreement will still hold. As long as we use perturbation expansion, the anti-commutator for ϕ_i is positive since to every order in e the above expression is positive definite. It is only when we have summed the infinite series that the indefiniteness becomes apparent. Our formalism hence suffers the same inconsistency as Johnson and Sudershan.

Before we discuss further the inconsistency of our quantization, let us look at an alternate way of deriving the results.

4-5 Extension of Kimel and Nath's Work

As we have seen in Section 4-2 that if we had set $\underset{\sim}{G} = 0$ in 4.22, the $\mathcal{U}(n,x/\sigma)$ that satisfies this identity when w express the right hand side in terms of the auxiliary fields is not hermitian apart from the first order term in e. In order to obtain a hermitian Hamiltonian to second order in e that satisfies the identity without introducing the G term, Kimel and Nath [8] proposed to modify the source $\underset{\sim}{J}_\mu$. The dependent components $\underset{\sim}{\psi}_4$ is first eliminated from the sources $\underset{\sim}{J}_\mu$ and $\underset{\sim}{I}_\mu$ using the secondary constraint explicitly. The currents $\underset{\sim}{J}_i$ and $\underset{\sim}{I}_i$ then contains terms of the form $\partial_\mu \underset{\sim}{A}_\lambda(x)$. With this derivative on the $\underset{\sim}{A}_\mu$'s, $\underset{\sim}{J}_i$ and $\underset{\sim}{I}_i$ are then rewritten into two parts, one of which has the form of deriative coupling. The general procedure for quantization as discussed in Section 4-1 with $\underset{\sim}{G} = 0$, is then applied.

According to the above discussion, they write $\underset{\sim}{J}_\mu$ and $\underset{\sim}{I}_\mu$ as

$$\underset{\sim}{J}_4(x) = \underset{\sim}{J}_4(x)$$

$$\underset{\sim}{J}_i(x) = \underset{\sim}{j}_i(x) + \partial_\mu \underset{\sim}{j}_{i:\mu}(x)$$

$$\underset{\sim}{I}_4(x) = \underset{\sim}{I}_4(x)$$

$$\underset{\sim}{I}_i(x) = \iota_i(x) + \partial_\mu \iota_{i:\mu}(x) \quad . \qquad 4.44$$

Then from 4.13 one arrives at

$$\psi_i(x/\sigma) = \underset{\sim}{\psi}_i(x) + \hat{d}_{i4}\underset{\sim}{J}_4(x) + \hat{d}_{ik}(\underset{\sim}{j}_k(x) + \partial_\mu \underset{\sim}{j}_{k:\mu}(x))$$

$$+ \frac{2}{3m^2}\,\delta_{\mu 4} b_{ik} \underset{\sim}{j}_{k;\mu}(x)$$

and

$$A_\lambda(x/\sigma) = \underset{\sim}{A}_\lambda(x) \quad . \qquad 4.45$$

The identities to be satisfied by $\mathcal{U}(n,x/\sigma)$ becomes

$$[\psi_\mu(x,\sigma), \mathcal{U}(n,x/\sigma)] = id_{\mu 4}\Delta(x-x')\underset{\sim}{J}_4(x') + id_{\mu\ell}\Delta(x-x')\underset{\sim}{j}_\ell(x')$$

$$+ id_{\mu\ell}\partial_\lambda \Delta(x-x')\underset{\sim}{j}_{\ell:\lambda}(x')$$

and

$$[A_\mu(x,\sigma), \mathcal{U}(n,x/\sigma)] + iD_{\mu 4}(x-x')\underset{\sim}{I}_4(x') + iD_{\mu\ell}(x-x')\underset{\sim}{\iota}_\ell(x')$$

$$+ i\partial_\lambda D_{\mu\ell}(x-x')\underset{\sim}{\iota}_{\ell:\lambda}(x') \quad . \qquad 4.46$$

With these modifications, they have determined the Hamiltonian and the relation between $\underset{\sim}{\psi}_\mu$ and the auxiliary fields to second order in e. Their results coincide with ours given in Section 4-2. However, to calculate higher order terms in this method, as we can easily deduced from our calculations in 4-2, the relations 4.44 should be modified into the form

$$\underset{\sim}{J}_i(x) = \sum_{k=0}^{n} \partial_{\mu_1} \cdots \partial_{\mu_k} \underset{\sim}{j}_{i:\mu_1 \cdots \mu_k}(x)$$

and

$$\underset{\sim}{I}_\ell(x) = \sum_{k=0}^{n} \partial_{\mu_1} \cdots \partial_{\mu_k} \iota_{\ell:\mu_1 \cdots \mu_k}(x) \qquad 4.47$$

where n depends on the order in e to which we want to calculate, for instance, the fourth order terms, $n = 3$. The difficulty involved in writing the J_i in the form 4.47 will prohibit one to extend the calculations to higher order term. However, if we observe that the additional contribution to 4.45 from writing the currents in the form of 4.41 is due only to the time derivative, by singling out the role of the time derivative and allowing the space derivatives to be symmetrized wherever needed, and further uses the equation of motion to keep the time derivative in $\underset{\sim}{J}_i$ and $\underset{\sim}{I}_i$ to at most of first order, we are able to compute higher order terms by this method. This is justified since three-divergence terms may be added to or subtracted from the Hamiltonian density without altering the physical content of the theory.

Writing the equation of motion in the form

$$\gamma_4 \partial_4 \beta_{ij} \underset{\sim}{\psi}_j = \{\eta_{ij} - ie\chi_{ij}\}\underset{\sim}{\psi}_j + \gamma_4 (V_i - ie\beta_{ik} \underset{\sim}{\mathring{A}}_k) \gamma_4 \gamma_\mu \underset{\sim}{\psi}_\mu \qquad 4.48$$

where

$$\eta_{ij} = \gamma_i \partial_j - \delta_{ij} (\vec{\gamma} \cdot \vec{\nabla} + m)$$

$$\chi_{ij} = \gamma_i A_j - \vec{\gamma} \cdot \vec{A} \delta_{ij} - \gamma_4 A_4 \beta_{ij} \quad .$$

With 4.48 and 3.17, we rewrite $\underset{\sim}{J}_\mu$ in the form

$$\underset{\sim}{J}_4 (x) = ie\gamma_4 A_\ell \beta_{\ell k} \psi_k \qquad 4.49a$$

$$\underset{\sim}{J}_i (x) = ie\underset{\sim}{\chi}_{ij} \underset{\sim}{\psi}_j + ie\beta_{ik} A_k \gamma_\mu \underset{\sim}{\psi}_\mu$$

$$= ie\underset{\sim}{\chi}_{ij} \underset{\sim}{\psi}_j + (ie)^2 \frac{2}{3m^2} \underset{\sim}{R}^{-1} \{\zeta_j + [\gamma_4 \partial_4 \underset{\sim}{A}_\ell] \beta_{\ell j}\} \psi_j$$

$$= \underset{\sim}{j}_i (x) + \gamma_4 \partial_4 \underset{\sim}{j}_{i;4} (x) \qquad 4.49b$$

where

$$j_{i,4} = \frac{(ie)^2}{3m^2}\beta_{ik}A_k\{1 - \frac{2ie}{3m^2}\vec{\nabla}.\vec{A} + (\frac{2ie}{3m^2}) \times$$

$$\times (\frac{3}{4}\vec{\nabla}.\vec{A}\ \vec{A}.\vec{\nabla} + \frac{1}{2}\vec{\nabla}.\vec{A}i\vec{\sigma}.\vec{B} + \frac{1}{4}[i\vec{\sigma}.\vec{B}]^2\} \times$$

$$\times A_\ell \beta_{\ell j}\psi_j$$

$$+ O(e^5) \qquad 4.50a$$

and

$$j_i(x) = ie\chi_{ij}\psi_j + \frac{(ie)^2}{3m^2}\beta_{ik}\{A_k(2\zeta_j - A_\ell\eta_{\ell j}) + [A_k\gamma_4\overleftrightarrow{\partial}_4 A_\ell]\psi_{\ell j}\}\psi_j$$

$$+ \frac{1}{4}(\frac{2ie}{3m^2})^3\beta_{ik}A_k\{\frac{3m^2}{2}A_\ell\chi_{\ell j} - i\vec{\sigma}.\vec{B}\zeta_j - \gamma_4\vec{\nabla}.\vec{A}\gamma_4(\zeta_j - A_\ell\eta_{\ell j})$$

$$- i\vec{\sigma}.\vec{B}[\gamma_4\partial_4 A_\ell]\beta_{\ell j}\}\psi_j + \frac{1}{4}(\frac{2ie}{3m^2})^3\beta_{ik}[\gamma_4\partial_4 A_k\vec{\nabla}.\vec{A}]A_\ell\beta_{\ell j}\psi_j$$

$$+ \frac{1}{4}(\frac{2ie}{3m^2})^4\beta_{ik}A_k\gamma_4\{-\frac{3m^2}{2}\vec{\nabla}.\vec{A}\gamma_4 A_\ell\chi_{ij}$$

$$+ (\vec{\nabla}.\vec{A}\ \vec{\nabla}.\vec{A} + i\vec{\sigma}.\vec{B}\ i\vec{\sigma}.\vec{B})\gamma_4(\zeta_j - A_\ell\eta_{\ell j})\}\psi_j$$

$$+ \frac{1}{4}(\frac{2ie}{3m^2})^4\beta_{ik}A_k\gamma_4\{\frac{1}{4}\vec{\nabla}.\vec{A}\ \vec{A}.\vec{\nabla} + \frac{1}{2}\vec{\nabla}.\vec{A}\ i\vec{\sigma}.\vec{B}$$

$$+ \frac{3}{4}(i\vec{\sigma}.\vec{B})^2\}\partial_4 A_\ell\beta_{\ell j}\psi_j$$

$$- \frac{1}{4}(\frac{2ie}{3m^2})^4\beta_{ik}[\gamma_4\partial_4 A_k\{\frac{3}{4}\vec{\nabla}.\vec{A}\ \vec{A}.\vec{\nabla} + \frac{1}{2}\vec{\nabla}.\vec{A}\ i\vec{\sigma}.\vec{B}$$

$$+ \frac{1}{4}(i\vec{\sigma}.\vec{B})^2\}]A_\ell\beta_{\ell j}\psi_j$$

$$+ O(e^5) \qquad 4.50b$$

Note: all field operators in 4.50 are Heisenberg operators.

Substituting 4.49 into 4.45 we obtain

$$\psi_i(x/\sigma) = \underset{\sim}{Q}_{ij}\underset{\sim}{\psi}_j(x) \qquad 4.51$$

where

$$\begin{aligned}\underset{\sim}{Q}_{ij} &= \delta_{ij} - i\varepsilon(\partial_i + \frac{m}{2}\gamma_i)A_\ell\beta_{\ell j} \\ &+ \frac{1}{2}\varepsilon^2\{\frac{3m^2}{2}A_i + (\partial_i + \frac{m}{2}\gamma_i)\vec{V}.\vec{A}\} \\ &\times\{-1 + \frac{2ie}{3m^2}\vec{V}.\vec{A} - \frac{1}{4}(\frac{2ie}{3m^2})(3\,\vec{V}.\vec{A}\,\vec{A}.\vec{V} + 2\,\vec{V}.\vec{A}i\vec{\sigma}.\vec{B} \\ &\qquad + (i\vec{\sigma}.\vec{B})^2)\}A_\ell\beta_{\ell j} \\ &+ O(e^5) .\end{aligned}$$

Again all operators in 4.51 are Heisenberg operators. We have taken $n_\mu = (0,i)$. The component $\psi_4(x/\sigma)$ can be obtained through 4.51 by the relation $\gamma_\mu\psi_\mu(x/\sigma) = 0$.

We formally construct the inverse operator Q^{-1} such that we can express the Heisenberg field $\underset{\sim}{\psi}_i$ in term of the auxiliary fields as

$$\underset{\sim}{\psi}_i(x) = Q^{-1}_{ij}\psi_j \quad :x/\sigma . \qquad 4.52$$

The Q^{-1}_{ij} in 4.52 is identical to that given in 4.38 to third order in e. The fourth order term differs slightly as follows:

The e^3 term in the expression for M (4.38b) is replaced by

$$-i\varepsilon^3\{\frac{3}{4}\ (\vec{\nabla}.\vec{A})^2\vec{A}.\vec{\nabla} - \frac{1}{2}\ \vec{\nabla}.\vec{A}i\sigma.B\vec{A}.\vec{\nabla} + \frac{1}{4}\ \vec{\nabla}.\vec{A}(i\vec{\sigma}.\vec{B})^2 - (i\vec{\sigma}.\vec{B})^3 - i\sigma.B(\vec{A}.\vec{\nabla})^2\}$$

while the e^2 term in N (4.38c) is replaced by

$$-\varepsilon^2\ \{\frac{1}{2}\ \vec{\nabla}.\vec{A}\ \vec{A}.\vec{\nabla} - i\vec{\sigma}.\vec{B}\vec{A}.\vec{\nabla} + \frac{3}{2}\ (i\vec{\sigma}.\vec{B})^2\}\ .$$

These differences can be attributed to the fact that the separation of the currents $\underset{\sim}{J}_i$ into two parts is not unique. We have indeed rewritten the fourth order terms in $\underset{\sim}{j}_i$ and $\underset{\sim}{j}_{i,4}$ in such a way that no differences occur. We retain the present form to illustrate that even though the relation between $\underset{\sim}{\psi}_i$ and the x/σ fields may appear to be different, the Hamiltonian as well as the commutation relations remain the same.

With 4.52, we express the currents 4.49 and 4.50 in terms of the auxiliary fields with the help of the following relations which can easily be verified;

$$\chi_{in}(\partial_n + \frac{1}{2} m\gamma_n) = \gamma_i\vec{A}.\vec{\nabla} - \vec{\gamma}.\vec{A}\partial_i - \beta_{ik}\gamma_4 A_4 \partial_k - m\beta_{ik}A_k - m\gamma_i\gamma_4 A_4$$

$$ie(\zeta_j - A_\ell \eta_{\ell j})\psi_j = ie\{\nabla.A\gamma_j - \partial_j \gamma.A - \partial_\ell \gamma_4 A_4 \beta_{\ell j} - mA_\ell\beta_{\ell j} - m\gamma_4 A_4 \gamma_j\}\psi_j$$

$$- ie\gamma_4 V.A\psi_4 + ie(\gamma.\nabla + 2m)A_\ell \beta_{\ell j}\psi_j$$

$$= (\partial_k + \frac{1}{2} m\gamma_k)J_k + \frac{m}{2}\gamma_4 J_4$$

$$\chi_{in}A_n = - \beta_{in}A_n\gamma_\mu A_\mu$$

$$A_n\chi_{nj} = - \gamma_4 A_4 \beta_{nj}$$

$$\zeta_n(\partial_n + \frac{1}{2} m\gamma_n) = \frac{1}{2} i\sigma.B\gamma.\nabla - \frac{1}{2}\gamma.\nabla i\sigma.B + \frac{1}{2}[\vec{\gamma}.\vec{\nabla}i\vec{\sigma}.\vec{B}]$$

$$- \partial_\ell \gamma_4 A_4 \beta_{\ell k}\partial_k + mi\sigma.B - m[\gamma.\vec{\nabla}\gamma_4 A_4]$$

$$A_\ell \eta_{\ell k}(\partial_k + \frac{1}{2} m\gamma_k) = -\gamma_4 \vec{A}.\vec{\nabla}\gamma_4(\gamma.\nabla + 2m) + \frac{3}{2} m^2\vec{\gamma}.\vec{A} \quad ,$$

we have determined $\mathcal{H}(x/\sigma)$ from 4.46 to fourth order in e. This $\mathcal{H}(x/\sigma)$ is indeed identical to the $\mathcal{H}(n',x/\sigma)$ given by 4.31' when we write them later in a noncovariant form with $n_\mu = (0,i)$. The commutation relations for the Heisenberg fields derived through 4.52 is also identical to those obtained in Section 4-2.

The present technique is in fact equivalent to that used in Section 4-2, in the sense that the separation of currents into two parts, one of which has the form of derivative coupling, takes the place of introducing the G's in Section 4-2, which has the effect of symmetrizing the derivatives in the current $\underset{\sim}{J}$. The technique in 4-2 has the advantage that all field components are treated on the same footing while in the present case one has to eliminate the component $\underset{\sim}{\psi}_4$ from the current using the constraints explicitly. The mathematical manipulations in the present method is far more tedious than that encountered in the previous sections.

CHAPTER 5

EXISTENCE OF THE S-MATRIX

We have explicitly constructed the scattering matrix for our system in Chapter 4. The validity of our procedure in this particular situation and hence its consequences is somewhat limited by perturbations. Further as shown, the interaction Hamiltonian determined is a power series of the coupling constant e, whose region of convergence is very difficult to determine. However, in order to prove just the existence of the S-matrix and not to determine it explicitly, we need not resort to perturbation.

Let us start from the integral equation 4.5 which can be written in the form

$$(1 + K^r)\underset{\sim}{\phi}(x) = \phi(x)$$

where

$$K^r\underset{\sim}{\phi}(x) \equiv \int dx' D^a d(\partial)\Delta^{ret}(x-x')\underset{\sim}{j}_a(x') .$$

Then

$$\underset{\sim}{\phi}(x) = L\phi(x) + \sum_i c_i\phi_i(x)$$

where L satisfies

$$L = 1 + K^r L$$

and

$$(1 + K^r)\phi_i = 0 ,$$

c_i's are arbitrary coefficients. The term ϕ_i is completely ignored in most cases. However, when the fields $\phi(x)$ do not form a complete set, the ϕ_i may be the complementary set such that ϕ and ϕ_i together form an irreducible operator ring.

The operator L is closely related to the S-matrix whose existence we want to prove. We shall first ignore these ϕ_i and later discuss its possible role.

We prove the existence of the S-matrix by employing the technique used by Capri (14) where the Q-number problem is first reduced to a closely related c-number problem. The formalism will be presented in the next section.

5-1 Formalism

We shall sketch here, as an illustration of the technique, for the case of a spin 1/2 field (say $\psi(x)$) interacting with external electromagnetic.

We start with the basic equation

$$\Lambda(\partial)\underset{\sim}{\psi}(x) = -(\gamma_\lambda \partial_\lambda + m)\underset{\sim}{\psi}(x) = -ie\gamma_\lambda \underset{\sim}{A}_\lambda(x)\underset{\sim}{\psi}(x) \qquad 5.1$$

where $\underset{\sim}{A}_\mu(x)$ is a suitable smooth function that decreases in a prescribed manner as any one of the components of x approaches $\pm\infty$. Rewriting 5.1 in integral form, we have

$$\underset{\sim}{\psi}(x) = \psi(x)^{in(out)} - (+) ie \int dx' \; G^{r(a)}(x-x') \gamma_\lambda \underset{\sim}{A}_\lambda (x') \underset{\sim}{\psi}(x') \tag{5.2}$$

where $G^{r(a)}(x-x') = d(\partial)\Delta^{ret(adv)}(x-x')$,

$d(\partial)$ being the Klein-Gordon divisor for the differential operator $\Lambda(\partial)$ in 5.1. Smearing 5.2 with four-component test functions $f(x)$, we get;

$$\underset{\sim}{\psi}(T^{r(a)} f) = \psi^{in(out)}(f) \; , \tag{5.3}$$

where

$$\psi(f) \equiv \int dx \; f(x)\psi(x) \; ,$$

$$T^{r(a)} f \equiv f + (-) ie (f * G^{r(a)}) \gamma_\lambda A_\lambda \tag{5.4}$$

and $(f*G)(y) \equiv \int dx \; f(x) G(x-y)$ is the convolution product of f and G.

So if $T^{r^{-1}}$ and $T^{a^{-1}}$ exist, and describe a continuous mapping of test function space onto itself, then

$$\underset{\sim}{\psi}(f) = \psi^{in}(T^{r^{-1}} f) = \psi^{out}(T^{a^{-1}} f) \qquad 5.5$$

and the whole problem is essentially solved.

To find the inverse mapping $T^{r^{-1}} f$ requires finding a function h such that

$$T^r h = h + ie(h{*}G^r)\gamma_\lambda A_\lambda = f \quad . \qquad 5.6$$

We introduce the auxiliary function

$$g = h{*}G^r \quad . \qquad 5.7$$

Then g along with all its derivatives vanish as $x_o \to -\infty$. Also g satisfies

$$g(x)\Lambda(-\overleftarrow{\partial}) = h(x) \qquad 5.8$$

and replacing h in terms of g in 5.6 we get

$$g\Lambda(-\overleftarrow{\partial}) + ieg\gamma_\lambda A_\lambda = f \qquad 5.9$$

with zero initial conditions at $x_o = -\infty$. If we can then solve this initial value c-number, we can use 5.8 to compute h and hence we will have computed $T^{r^{-1}} f$. In a similar manner we compute $T^{a^{-1}} f$ by solving 5.9 with zero final conditions at $x_o \to +\infty$. This means that if we have proved the existence of solutions g for the inhomogeneous wave equation 5.7

with prescribed initial conditions, the existence of the inverse mapping $T^{r^{-1}}$ and $T^{a^{-1}}$ is established. this allows us to write

$$\psi^{out}(f) = \psi^{in}(T^{r^{-1}} T^{a} f) \,, \qquad 5.10$$

which enables us to show that the in and out fields satisfy the same commutation relations. We need only to show that, corresponding to the out-field, there exists a vacuum state in the Hilbert space of the in-states, which was first constructed by assuming that the in fields are the same as free fields, to conclude that the two fields are unitarily equivalent [19]. The unitary operator connecting them is the S-matrix.

In the following sections, we shall apply this formalism to the spin 3/2 field, the electromagnetic field will be assumed as external.

5-2 Reduction to c-number Problem

Since the Rarita-Schwinger field ψ_μ contains sixteen components, we shall use the sixteen component test function $f_\mu(x)$ with each component taken from the test function space $\mathcal{S}$ of infinitely differentiable functions that are rapidly decreasing at infinity.

The analogues of the spin 1/2 equat ons 5.3 and 5.4 become;

$$\underset{\sim}{\psi}_{\mu}(T^{r(a)}_{\mu\nu} f_{\nu}) = \psi^{in(out)}_{\mu}(f_{\mu}) \qquad 5.11$$

and

$$T^{r(a)}_{\mu\nu} f_{\nu} = f_{\mu} + (-)ie(f_{\nu} * G^{r(a)}_{\nu\lambda})(\Gamma \underset{\sim}{A})_{\lambda\mu} \qquad 5.12$$

where

$$G^{r(a)}_{\mu\nu}(x-x') \equiv d_{\mu\nu}(\partial)\Delta^{ret(adv)}(x-x') \quad .$$

By restricting the external electromagnetic potential A_{μ} to be a function in $\mathcal{S}$, the mappings $T^{r}_{\mu\nu}$ and $T^{a}_{\mu\nu}$ define continuous mappings from $\mathcal{S}$ into $\mathcal{S}$, since $f_{\mu} \in \mathcal{S}$ and $G^{r(a)}_{\mu\nu}$ is a tempered distribution, hence $f_{\mu} * G^{r(a)}_{\mu\nu} \in O_{M}$, the space of infinitely differentiable functions of slow growth which are multipliers for $\mathcal{S}$. Hence we can obtain the analogues to 5.9 as

$$g_{\mu}\Lambda_{\mu\nu}(-\overleftarrow{\pi}) = f_{\nu} \qquad 5.13$$

with either zero initial conditions or zero final conditions, where

$$g_{\mu}\Lambda_{\mu\nu}(-\overleftarrow{\partial}) = h_{\nu}$$

and

$$g_\mu(x) = [h_\nu * G^*_{\nu\mu}](x) \quad .$$

The next section is devoted to solving 5.13.

5-2 The Cauchy Problem

We have established that "if $T^r_{\mu\nu} h_\nu = f_\mu$ then $h_\mu = T^{r^{-1}}_{\mu\nu} f_\nu$ is given by $g_\nu \Lambda_{\nu\mu}(-\overleftarrow{\partial})$ where $g_\nu \Lambda_{\nu\mu}(-\overleftarrow{\pi}) = f_\mu$ with zero initial data.

Conversely, given a $g_\mu \epsilon\, O_M$ such that $g_\mu \Lambda_{\mu\nu}(-\overleftarrow{\pi}) = f_\nu$ with zero initial data, then $T^{r^{-1}}_{\mu\nu} f_\nu$ exists and is given by $g_\nu \Lambda_{\nu\mu}(-\overleftarrow{\partial})$."

We now want to study the solutions to the system of homogeneous partial differential equations:

$$g_\mu(x) \Lambda_{\mu\nu}(-\overleftarrow{\pi}) = f_\nu(x) \qquad 5.13$$

with vanishing initial data.

However, since 5.13 contains constraints, this Cauchy problem is not well posed, we are thus unable to study the solutions to this problem directly. From the discussion in Chapter 3, we know that every solution of the system

$$g_\mu \bar{M}_{\mu\nu}(-\overleftarrow{\pi}) = g_\mu \Lambda_{\mu\lambda}(-\overleftarrow{\pi}) d^r_{\lambda\nu}(-\overleftarrow{\pi}) = 0 \ , \qquad 5.14$$

which satisfies the constraints initially, is a solution of the homogeneous equations,

$$g_\mu \Lambda_{\mu\nu}(-\overleftarrow{\pi}) = 0 \ .$$

Furthermore, in the 'weak field case' 5.14 is equivalent to a hyperbolic system of partial differential equations where we can prescribe initial data on the surface x_o= constant. And only in this case that we can expect reasonable solutions to the system exist.

Assuming that our system satisfies the 'weak field condition', we then have a well posed Cauchy problem;

$$g_\mu(x)\bar{M}_{\mu\nu} = \not{f}_\nu(x) \qquad 5.15$$

where

$$\not{f}_\nu(x) = f_\mu(x) d^r_{\mu\nu}(-\overleftarrow{\pi})$$

with vanishing initial data.

If we can solve this system for g_μ, we then have proved the existence of solutions to our original system 5.13. To solve 5.15, considered as a matrix equation with sixteen components, we first diagonalized the principal part of the matrix operator $\bar{M}$ by multiplying from the right by the operator

$$\bar{\lambda}_{\mu\nu}(-\overleftarrow{\pi}) = -\gamma_\eta \overleftarrow{\pi}_\lambda [\overleftarrow{\pi}^2 - (\frac{2e}{3m^2}\ \overleftarrow{\pi}.F^d)^2]\delta_{\mu\nu}$$

$$+ \frac{2ie}{3m^2}\gamma_\lambda \overleftarrow{\pi}_\lambda [2F^d_\mu . \overleftarrow{\pi}\gamma_5 - \gamma_\alpha \overleftarrow{\pi}_\alpha F^d_\mu . \gamma\gamma_5 + \frac{2ie}{3m^2}\ \bar{F}^d_\mu . \gamma\overleftarrow{\pi}.F^d . \gamma]\overleftarrow{\pi}_\nu .$$

Taking the transpose of the resulting equation, we obtain

$$\mathbb{1}\, L\, g = N g + \omega$$

with

$$L = [\Box - (\frac{2e}{3m^2} F^d . \partial)^2]\Box$$

$$\omega = [f\, \bar{\lambda}(-\vec{\pi})]^t$$

g and ω are sixteen component column matrices, N is a sixteen by sixteen square matrix operator with at most third order in derivatives, $\mathbb{1}$ is the sixteen by sixteen unit matrix. Also $\omega \in \mathcal{J}$, since $f \in \mathcal{J}$, d^r and $\bar{\lambda}$ are both multipliers of $\mathcal{J}$.

Working in the Lorentz frame in which only the magnetic field surviue, the operator L is expressible as

$$L = (a^2 \partial_o^2 - \vec{\nabla}^2)(\partial_o^2 - \vec{\nabla}^2)$$

with

$$a^2 = 1 - (\frac{2e}{3m^2} \vec{B})^2 > 0 .$$

We shall now sketch the proof for the existence and uniqueness of solutions to the Cauchy problem

$$Lg = Ng + \omega \qquad 5.16$$

with initial values of g's prescribed by

$$\partial_o^k g(x)\Big|_{x_o=T} = \mathcal{G}_k(T,\vec{x}) \qquad k=0,1,2, \text{ and } 3$$

with

$$\lim_{T\to-\infty} \mathcal{G}_k(T,\vec{x}) = 0 \quad , \text{ for all } k \quad .$$

The form of $\mathcal{G}_k$ is not specified since our results depend only on the limiting values.

The proof uses the technique of F. John (19) for system of hyperbolic equations, and is broken up into three parts.

We first use the method of 'Energy Integrals' to obtain a priori estimates for the solutions to the single scalar equation of fourth order of the form:

$$Lg = (a^2\partial_o^2 - \vec{\nabla}^2)(\partial_o^2 - \vec{\nabla}^2)g = \omega$$

with vanishing initial conditions at $x_o = 0$, i.e.

$$\partial_o^n g\Big|_{x_o=0} = 0 \quad , \qquad n=0,1,2, \text{ and } 3 \ .$$

The a priori estimates are given in the form of an inequality that bounds the L^2-norm of the solution g in terms of that for ω.

Secondly, from the a priori estimates for the solutions of the principal part of 5.16, an iterative procedure is used to obtain the required solutions for the whole equations. The sequence of solutions used in the iterative process is shown to converge in the prescribed norm. Pointwise convergence is achieved by means of Sobolev's lemma.

Finally, the solutions are proved to be unique by showing that the solution g to the system

$$\Pi L g = Ng$$

with vanishing initial data, vanishes identically.

The detailed proof is very involved and will not be given here. In the existence proof, the support of the solution g is determined to be contained in the forward ray cone with normal n_μ defined by the equation

$$a^2 n_o^2 - \vec{n}^2 = 0$$

subtended by the support of h. This ray cone is extended from the originary light cone for nonvanishing mangetic field, hence the effect is acausal.

We thus conclude that for all $f \in \mathcal{S}$ we can find $T^r f$, $T^a f$, $T^{r^{-1}} f$ and $T^{a^{-1}} f$ in $\mathcal{S}$. Furthermore, all of these mappings are continuous. The support of $T^{r^{-1}} f$ and $T^{a^{-1}} f$ is contained in the forward and backward ray cone subtended by the support of f.

5-4 Commutation Relations and the Out Vacuum

Since we have established the existence of the inverse mapping $T^{r^{-1}}$ and $T^{a^{-1}}$, the fields ψ_μ and ψ_μ^{out} can then be related directly to the in-fields as follows;

$$\psi_\mu(f_\mu) = \psi_\mu^{in}(T^{r^{-1}}_{\mu\nu} f_\nu)$$

$$\psi_\mu^{out}(f_\mu) = \psi_\mu^{in}(T^{r^{-1}}_{\mu\lambda} T^a_{\lambda\nu} f_\nu) .$$

From the commutation relation for the $\psi_\mu^{in}(x)$, which when smeared is of the form,

$$\{\psi_\mu^{in}(f_\mu), \bar{\psi}_\nu^{in}(\bar{g}_\nu) = iG^a_{\mu\nu}(f_\mu, g_\nu) + iG^r_{\mu\nu}(f_\mu, g_\nu)$$

where

$$\bar{\psi}_\mu^{in}(\bar{f}_\mu) = [\psi_\mu^{in}(f_\mu)]^* = \int dx\ \psi_\mu^+ \eta_{\mu\lambda}\eta_{\lambda\nu} g_\nu^+ = \int dx [g_\mu \psi_\mu]^*$$

$$G^{a(r)}_{\mu\nu}(f_\mu, g_\nu) = \int dx\, dy\ f_\mu(x) G^{a(r)}_{\mu\nu}(x-y)\eta_{\nu\lambda} g_\lambda^+(y) ,$$

we are able to derive the commutation relations for the ψ_μ and ψ_μ^{out}.

The following identities which can easily be proved by writing out the expression in full are needed for our derivation,

$$G^{r}_{\mu\nu}(T^{r(a)}_{\mu\lambda} f_{\lambda}, g_{\nu}) = G^{r}_{\mu\nu}(f_{\mu}, T^{a(r)}_{\nu\lambda} g_{\lambda})$$

$$G^{a}_{\mu\nu}(T^{r(a)}_{\mu\lambda} f_{\lambda}, g_{\nu}) = G^{a}_{\mu\nu}(f_{\mu}, T^{a(r)}_{\nu\lambda} g_{\lambda}) \qquad 5.17$$

$$G^{r}_{\mu\nu}(T^{a}_{\mu\lambda} f_{\lambda}, g_{\nu}) + G^{a}_{\mu\nu}(f_{\mu}, T^{a}_{\nu\lambda} g_{\lambda}) = G^{r}_{\mu\nu}(f_{\mu}, g_{\nu}) + G^{a}_{\mu\nu}(f_{\mu}, g_{\nu})$$

the corresponding identities in terms of the inverse mapping can also be obtained easily.

We now show that the in-field and the out-field satisfy the same commutation relations, that is,

$$\{\psi^{out}_{\mu}(f_{\mu}), \bar{\psi}^{out}_{\nu}(\bar{g}_{\nu})\} = \{\psi^{in}_{\mu}(f_{\mu}) \bar{\psi}^{in}_{\nu}(\bar{g}_{\nu})\} .$$

We have,

$$\{\psi^{out}_{\mu}(f_{\mu}), \bar{\psi}^{out}_{\nu}(\bar{g}_{\nu})\} = \{\psi^{in}_{\mu}(T^{r^{-1}}_{\mu} . T^{a} . f), \bar{\psi}^{out}_{\nu}(\overline{T^{r1}_{\nu} . T^{a} . g})\}$$

$$= iG^{a}_{\mu\nu}(T^{r1}_{\mu} . T^{a} . f, T^{r^{-1}}_{\nu} . T^{a} . g) + iG^{r}_{\mu\nu}(T^{r^{-1}}_{\mu} . T^{a} . f, T^{r^{-1}}_{\nu} . T^{a} . g)$$

$$= iG^{a}_{\mu\nu}(T^{a^{-1}}_{\mu} . T^{a} . f, T^{a^{-1}}_{\nu} . T^{a} . g) + iG^{r}_{\mu\nu}(T^{a^{-1}}_{\mu} . T^{a} . f, T^{a^{-1}}_{\nu} . T^{a} . g)$$

$$= iG^{a}_{\mu\nu}(f_{\mu}, g_{\nu}) + iG^{r}_{\mu\nu}(f_{\mu}, g_{\nu})$$

$$= \{\psi^{in}_{\mu}(f_{\mu}), \psi^{in}_{\nu}(g_{\nu})\} . \qquad 5.18$$

Relations 5.17 have been used.

This result 5.18 implies that the out-field can be expanded in terms of creation and annihilation operators which satisfy the same commutation relations as the in-operators. Our task now is to show that corresponding to the out-operators there exists a unique vacuum state in the Hilbert space to conclude that the in and out fields are unitarily equivalent (18). The unitary operator connecting them is the S-matrix.

Recall that

$$\psi_{\mu}^{out}(f_{\mu}) = \psi_{\mu}^{in}(T^{r^{-1}}_{\mu\lambda} T^{a}_{\lambda\nu} f_{\nu})$$

We first show that

$$T^{r^{-1}}_{\mu\lambda} T^{a}_{\lambda\nu} f_{\nu} = f_{\mu} + f_{\mu}^{o} \qquad 5.19$$

where support of f_{μ}^{o} is contained in the support of the A_{μ}. From 5.11 if we replace f_{μ} by $f'_{\lambda}\Lambda_{\lambda\mu}(-\overleftarrow{\partial})$, we have

$$T^{r}_{\mu\lambda}(f'_{\nu}\Lambda_{\nu\lambda}(-\overleftarrow{\partial})) = f'_{\nu}\Lambda_{\nu\mu}(-\overleftarrow{\pi}) .$$

We arrived at

$$T^{r^{-1}}_{\mu\lambda} T^{a}_{\lambda\nu} f_{\nu} = g_{\nu}\Lambda_{\nu\mu}(-\overleftarrow{\partial})$$

where g_{μ} is a solution of

$$g_{\nu}\Lambda_{\nu\mu}(-\overleftarrow{\pi}) = T^{a}_{\mu\nu} f_{\nu} = f_{\mu} - ie[f_{\nu} * G^{a}_{\nu\lambda}](\Gamma A)_{\lambda\mu} \qquad 5.20$$

corresponding to zero initial conditions.

A particular solution g_μ of 5.22 is

$$- ie[f_\nu * G^a_{\nu\mu}]$$

hence

$$g_\mu = -ie[f_\nu * G^a_{\nu\mu}] + g^o_\mu$$

where

$$g^o_\mu \Lambda_{\mu\nu}(-\overleftarrow{\pi}) = 0$$

with initial conditions

$$g^o_\mu = ie[f_\nu * (G^a_{\nu\mu} + G^r_{\nu\mu})] .$$

Thus,

$$f^o_\mu = g^o_\nu \Lambda_{\nu\mu}(-\overleftarrow{\partial}) = ieg^o_\nu (\Gamma A)_{\nu\mu}$$

and

$$\text{Support of } f^o_\mu \subset \text{Support of } A_\mu \qquad 5.21$$

as asserted.

We pick the test function f_μ such that

$$\int dx \; f_\mu(x) u^r_{\vec{p}\mu}(x) = 0$$

then by construction we have

$$\psi^{in}_\mu(f_\mu)|0>^{in} = \sum_{r=1}^{4} \int dx \; d^3p \; f_\mu(x) u^r_{\vec{p}\mu}(x) a^r(\vec{p})|0>^{in} = 0 .$$

We now show that the equation

$$\psi_\mu^{out}(f_\mu)|0>^{out} = 0$$

has a solution that $|0>^{out}$ is in the in-Hilbert space. Let us expand the state $|0>^{out}$ in terms of the complete set of orthonormal state-vectors in the in-Hilbert space as follows:

$$|0>^{out} = \sum_{n,m} B^{n,m}\psi^{n,m}$$

where $B^{n,m} = B(\vec{p}_1,r_1;\ldots;\ \vec{p}_n,r_n:\ \vec{q}_1,s_1;\ldots;\ \vec{q}_m,s_m)$ are the expansion coefficients. If the norms of B's are finite, the above expansion implies that the out-vacuum state lies in the in-Hilbert space.

Setting

$$\psi_\mu^{out}(f_\mu)|0>^{out} = 0$$

we get

$$\psi_\mu^{in}(f_\mu+f_\mu^o)|0>^{out} = \int dx\ d^3\vec{p}\{(f_\mu+f_\mu^o)u^r_{\vec{p}\mu}\, a^r(p) + f_\mu^o v^r_{\vec{p}\mu}\, b^{r^+}(p)\} \sum_{n,m} B^{n,m}\psi^{n,m}$$

$$= 0 \qquad 5.22$$

Equating the state with the same number of particles we obtain

$$\int dx\, d^3\vec{p}\; B^{n+1,m}\sqrt{m}\sqrt{n+1}\;(f_\mu + f_\mu^o)u^r_{\vec{p}\mu}\; a^r(p)\,\psi^{n+1,m}$$

$$= \int dx\; B^{n,m+1}\; f^o_\mu \upsilon^s_{\vec{q}_i\mu}\; b^{s^+}(q_i)\,\psi^{n,m-1} \quad .$$

Taking the inner product of the above expression with itself we get

$$|B^{n+1,m}|^2 I_{n+1,m} = \frac{1}{m(n+1)}|B^{n,m-1}|^2 K_{n,m-1} \qquad 5.23$$

where

$$I_{n+1,m} = \int dx\, dx' d\vec{p}\, d\vec{p}\,'[(f_\nu(x')+f^o_\nu(x'))u^{r'}_{p'_\nu}(x')]^*$$

$$\times[f_\mu(x)+f^o_\mu(x)u^r_{p\mu}(x)](a^{r'}(p')\psi^{n+1,m}, a^r(p)\psi^{n+1,m})$$

$$5.24$$

and

$$K_{n,m-1} = \int dx\, dx'[f^o_\nu(x')\upsilon^{s'}_{q_i}(x')]^*[f^o_\mu(x)\upsilon_{q_i\mu}(x)]$$

$$\times\;(b^{s'^+}(q_i)\psi^{n,m-1},\; b^{s^+}(q_i)\psi^{n,m-1}) \qquad 5.25$$

From 5.23 we can express $|B^{n+m,m}|^2$ in terms of $|B^{n,o}|^2$ and similarly $|B^{n,m+n}|^2$ in terms of $|B^{o,m}|^2$, however since

$$\psi^{in}_\mu(f_\mu+f^o_\mu)\psi^{n-1,o} = \int dx\, d\vec{p}\sqrt{n}(f_\mu+f^o_\mu)u^r_{\vec{p}\mu}\; a^r(p)\,\psi^{n,o}$$

hence

$$B^{n,o} = 0 \quad \text{for} \quad n \geq 1 \ .$$

Similarly we have

$$B^{o,m^*} = 0 \quad \text{for} \quad m \geq 1 \ .$$

Thus the only nonvanishing terms in I and K are $I_{n,n}$ and $K_{n,m}$. Using the commutation relations of a's and b's and integrating out the variables and since f and f^o are square integrable, we arrive at

$$I_{n+1,n+1} = |\theta_2|^2 > 0$$

and

$$K_{n,n} = (n+1)|\theta_1|^2 < \infty.$$

Then from 5.25 we have

$$|B^{n+1,n+1}|^2 \leq |B^{n,n}|^2 \ \frac{1}{n+1} \left|\frac{\theta_1}{\theta_2}\right|^2$$

and hence

$$|B^{n,n}|^2 \leq \frac{1}{n!} \ |B^{o,o}|^2 \ \left|\frac{\theta_1}{\theta_2}\right|^{2n}$$

or

$$\sum_n |B^{n,n}|^2 \leq |B^{o,o}|^2 \ e^{|\theta_1/\theta_2|^2} < \infty \ .$$

Thus we have shown that the out-field processes a vacuum state in the in-Hilbert space. However, this method does not allow us to show the uniqueness of the out-vacuum. Assuming the proof of uniqueness of the out-vacuum by Labonté and Capri (20) for the case of spin 1/2 field is valid when generalized to the present situation, we can then conclude that a unitary S-matrix exists for the Rarita-Schwinger spin 3/2 field interacting with external electromagnetic when we restrict ourselves to the weak field case. If the field is strong, we are simply unable to solve the Cauchy problem, and hence this technique is not applicable. It may worth investigating how to include the ϕ_i discussed in the beginning of this chapter and what are the consequences of this inclusion to the Cauchy problem.

CHAPTER 6

CONCLUSION AND DISCUSSION

We have discussed at length the quantization of the Rarita-Schwinger's spin 3/2 field in the presence of minimal electromagnetic interaction. Within the framework of quantum field theory summarized in Chapter 2, and under the assumption that the diagonalized Hamiltonian H_o in (2.7) is the same as the noninteracting part of the total Hamiltonian, we proceeded via the Takahashi and Umezawa method to determine the interaction Hamiltonian $\mathcal{H}(n,x/\sigma)$ from which the time evolution operator $U(\sigma)$ for the state vectors in the asymptotic representation is determined. It follows from the structure of theory that if such a $U(\sigma)$ exists, and the assumption concerning the H_o is valid, the relation

$$\phi(x/\sigma) = F(\phi(x)) \qquad 6.1$$

is invertible. This invertibility is presupposed before we perform our calculation. By formally inverting this relation and the requirement that $\mathcal{H}(n,x/\sigma)$ be a Hermitian operator (since the total Hamiltonian, and the free

Hamiltonian must be Hermitian), we used the identity (4.22) to determine the $\mathcal{H}(n,x/\sigma)$ as well as fixing the four-divergence term G. The complete cancellation of the normal dependent terms in the S-matrix formally constructed from the interaction Hamiltonian when we replace the T-product by the T*-product was also demonstrated. This cancellation called the "Generalized Matthew's rule" is in fact a consequence of a consistent theory, and as long as we stay in the perturbation expansion of the various quantities of interest as a power series of the coupling constant e (i.e., in the weak field case), the anticommutation relations for the Heisenberg fields remain positive definite. The commutation relations and the Hamiltonian derived are compatible with the Heisenberg equations of motion since it was by demanding this compatibility that the operator properties of the fields were established and the Hamiltonian was determined. Hence we can conclude that in the weak field case the charged spin 3/2 field can be quantized consistently. This conclusion is also complemented by the work in Chapter 5 where the existence of the S-matrix was proved by the technique of mapping in the test function space under the weak field assumption.

However, the equal time anti-commutator $\{\phi_i(x), \phi_j^+(x')\}_{x_o=x_o'}$ we determined, coincides with that given by Johnson and Sudarshan evaluated up to the fourth order in the coupling constant, and that we believe this agreement will hold to whatever order in e our tedious calculation allows us to go. This compels us to admit that our quantization procedure suffers the same inconsistency as pointed out by Johnson and Sudarshan when the electromagnetic field is strong. We also note that without the weak field restriction, the mapping technique simply cannot be applied since the Cauchy problem cannot be posed.

Thus we conclude that the quantization procedure we used (also the work of Kimel and Nath, since their results agree with ours even up to fourth order in e), cannot consistently quantize the Rarita-Schwinger spin 3/2 field minimally coupled to the electromagnetic field when the weak field condition is not satisfied.

It has been speculated by Kimel and Nath that the invertibility assumption of relation 5.1 may become invalid rendering the quantization procedure inconsistent. We have indirectly shown that when the field is strong invertibility does not hold. This means that our set of asymptotic fields is incomplete and hence does not form an irreducible operator ring. Recalling that the invertibility between the Heisenberg operators and the asymptotic operators is a basic structure of quantum field theory if the H_o in (2.7) is the correct diagonalized matrix operator. Thus the basic assumption that H_o is the same as the noninteracting part of the total Hamiltonian may not be justified in our case.

The noninvertibility may also be due to the acausal propagation characteristic of our fundamental field equation. It has been known [21] that if we do not constraint the parameters b and c in our general wave equations (3.3) as we did (3.4), (i.e. we include into our consideration the two spin 1/2 fields) the general wave equations process proper propagation behaviour. This may also be the case for a consistent quantization. Further investigation is necessary to see if irreducibility can be satisfied by including the spin 1/2 fields into our consideration.

APPENDIX

EXPRESSION FOR $[\theta(x_o-x_o'), d_{\mu\nu}(\partial)]\Delta(x-x')$

We define

$$[\theta(x_o-x_o'), d_{\mu\nu}(\partial)]\Delta(x-x') = \hat{d}_{\mu\nu}(\partial)\delta(x-x'),$$

$$[\theta(x_o-x_o'), n\partial d_{\mu\nu}(\partial)]\Delta(x-x') = n\partial d_{\mu\nu}(\partial)\delta(x-x'),$$

where,

$$d_{\mu\nu}(\partial) = -(\gamma_\lambda\partial_\lambda-m)\{\delta_{\mu\nu}-\frac{1}{3}\gamma_\mu\gamma_\nu+\frac{1}{3m}(\gamma_\mu\partial_\nu-\gamma_\nu\partial_\mu)-\frac{2}{3m^2}\partial_\mu\partial_\nu\}.$$

We have;

$$\hat{d}_{\mu\nu} = \frac{2}{3m^2}(\xi_{\mu\nu}+\xi_{\mu\nu:\sigma}\partial_\sigma^s+\xi'_{\mu\nu}n\partial),$$

and

$$n\partial d_{\mu\nu} = \hat{d}_{\mu\nu}n\partial+\frac{2}{3m^2}b_{\mu\nu}$$

where,

$$\xi_{\mu\nu} = -\frac{m}{2}(2n_\mu n_\nu-\gamma_\mu^s n\gamma n_\nu-n_\mu n\gamma\gamma_\nu^s),$$

$$\xi_{\mu\nu:\sigma} = n_\mu\gamma_\sigma^s n_\nu+\delta_{\mu\sigma}^s n\gamma n_\nu+n_\mu n\gamma\delta_{\nu\sigma}^s$$

$$\xi'_{\mu\nu} = -n_\mu n\gamma n_\nu,$$

$$b_{\mu\nu} = \zeta_{\mu\nu}+\zeta_{\mu\nu:\sigma}\partial_\sigma^s+\zeta_{\mu\nu:\sigma\rho}\partial_\sigma^s\partial_\rho^s,$$

$$\zeta_{\mu\nu} = \frac{m^2}{2}\{3\delta_{\mu\nu}^s n\gamma+\gamma_\mu^s n\gamma\gamma_\nu^s\},$$

$$\zeta_{\mu\nu:\sigma} = \frac{m}{2}\{\delta_{\mu\sigma}^s(3n_\nu-n\gamma\gamma_\nu^s)+(3n_\mu-\gamma_\mu^s n\gamma)\delta_{\sigma\nu}^s-\gamma_\mu^s\gamma_\sigma^s n_\nu-n_\mu\gamma_\sigma^s\gamma_\nu^s\},$$

$$\zeta_{\mu\nu:\sigma\rho} = -\gamma_\sigma^s(n_\mu\delta_{\rho\nu}^s+\delta_{\rho\mu}^s n_\nu)-\delta_{\mu\sigma}^s n\gamma\delta_{\rho\nu}^s-n_\mu n\gamma n_\nu\delta_{\sigma\rho}^s.$$

We also have;

$$\delta(x-x')d_{\mu\nu}(\partial)\Delta(x-x') = \frac{2}{3m^2}b_{\mu\nu}\delta(x-x').$$

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