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A Framework for Analysis of Problem Solving

University — Université
University of Alberta

Degree for which thesis was presented — Grade pour lequel cette thèse fut présentée
Master of Education

Year this degree conferred — Année d'obtention de ce grade	Name of Supervisor — Nom du directeur de thèse
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A FRAMEWORK FOR ANALYSIS OF PROBLEM SOLVING

by

© GWENDOLYN ELAINE JANET MCDONALD HALABISKY

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE

OF MASTER OF EDUCATION

DEPARTMENT OF SECONDARY EDUCATION

EDMONTON, ALBERTA

FALL, 1981

THE UNIVERSITY OF ALBERTA

RELEASE FORM

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DEGREE FOR WHICH THESIS WAS PRESENTED Master of Education

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DEDICATION

To my Family and Friends

who have sustained me throughout

ABSTRACT

An artificial intelligence conceptualization of learning has been posed by Robert Davis, Elizabeth Jockusch, and Curtis McKnight. They have used it to analyze students' mathematical thinking. The purpose of the present study was to investigate the feasibility of this conceptualization for use by teachers to analyze problem solving and to create a remedial program for students based on this analysis.

The study employed twelve grade eight students from two classes taught by the same teacher. The students were volunteers. The topic area covered was rate and percent.

The students were interviewed while they solved problems and protocols were made up from the students' verbalizations and written work. This work was analyzed using the artificial intelligence conceptualization.

It was found that the conceptualization could be used to analyze the students' work and to make recommendations for remedial work for the students based on this analysis. Recommendations were also made for teaching strategies.

ACKNOWLEDGEMENTS

Acknowledgement and gratitude is given for the assistance, co-operation and contributions made by others in order that this project could be completed.

Dr. A.T. Olson deserves special credit for his patience, guidance, suggestions and assistance. Thanks also go to Dr. T. Kieren and Dr. D. Sawada for serving on my examining committee and to Medhat Rahim for his encouragement.

Appreciation is expressed to the Mathematics Division of the Edmonton Separate School Board and to the principal and teachers of St. Gabriel School and, in particular, Gloria Holoiday for their courtesy and co-operation.

Several friends and relatives helped with family duties and offered much encouragement throughout this program. They are: Heddie Halabisky, Velma Mant, Linda Spero, Bertha Ayers and Sharon Nicholson.

A special word of thanks to my husband, Wayne, for his support and to my sons, Lorne and Brian, for their co-operation.

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CHAPTER I

THE PROBLEM

Introduction

"Use your head", was the advice given by Professor Polya as the final statement of a discussion about problem solving at the 1974 annual meeting of the American Mathematical Society (Lester, 1978). Professor Polya's advice should not be taken lightly. But, in what way can people learn to better "use their heads"?

The constant problem of mathematics educators is the improvement of problem solving skills. However, they can not be improved without first being understood. An assessment or evaluation of problem solving skills will help one to better understand them. Much evaluation has an orientation toward the product, or final answer, rather than the process involved in getting the final answer. Both the product and the process are essential components of the problem solving experience.

Davis and McKnight (1979a), Kantowski (1977), and Kilpatrick (1969), have all noted the product orientation of much research on problem solving. They comment that little research has been undertaken in the analysis of processes involved in problem solving. Krutetskii (1976) says that "failing to reveal the mental processes leading to a given result...[does not] provide a complete picture of the phenomenon under scrutiny" (p. 13). Because of this incomplete picture, teachers are at a loss in offering remediation or changing their presentation to enhance learning of problem solving skills.

When looking at processes one must often use subjective rather than objective means. Kilpatrick (1978) suggests that a valid way of studying the dynamics of a process is to have subjects report what they are doing as they solve problems and "it is no less valuable for being subjective. [Their] report provides data that can be obtained in no other way" (p. 13).

Interview situations are a viable way to have subjects report their cognitions as they see them. David and McKnight (1979a) have found that "the road which children travel in learning mathematics looks very different when assessed by interview procedures from the way it looks when assessed by commonly-used standardized tests" (p. 15). They also state that on standardized tests, "not only are the 'measurements' wrong, but they are derived from incorrect basic conceptualizations of the nature of learning" (op. cit., p. 15).

The following paragraphs discuss what conceptualization Davis and McKnight would consider correct.

> One often seeks to measure student achievement (or "knowledge, or skill") in mathematics: but in order to measure something, one needs to know quite a bit about it. In the case of mathematics learning and mathematical performance, we do not know very much about "what the thing really is" (Davis and McKnight, 1979, p. 1).

What must we find out in order to know "what the thing really is"? Hatfield (1978a) states that what is missing in studies he reviewed on heuristics, "are discussions of the mental operations and cognitive structures" (p. 13) which are used to assimilate and recall heuristics and are "'triggered' by the recall or suggestion of an heuristical statement" (p. 13).

How can we study and discuss these two phenomena, mental operations and cognitive structures? Krutetskii (1976) says that Pavlov indicated "an analytic-synthetic approach to the study of a complex psychological phenomenon requires first of all an analysis of its structure, an isolation of its components" (p. 77).

However, before an analysis of the components of a phenomenon can be made, a suitable metaphor or model must be discovered which can be used to represent the components.

Davis et al. (1978), after discussing the inadequacies of the metaphors, the "switchboard", and Tolman's "map", which have previously been used to represent thinking, offer an alternative.

> Today, of course, we are all at least somewhat conversant with computers, and they provide an even more sophisticated, more powerful, more flexible metaphor to represent thinking.
>
> Computers have led to what is virtually a new discipline, so-called artificial intelligence (or "A.I.") - the study of information-handling processes as they are carried out by sophisticated computer programs, and the search for parallels between computer information handling and human information handling (p. 14).

Radatz (1979) agrees that the mechanisms of information processing can be useful, particularly in the analysis of the causes of errors.

Davis and McKnight (1979) summarize many years work at the task of description and conceptualization, by saying: "We have in fact assembled a sizeable collection of instances of mathematical behavior, and have developed a conceptualization of the relevant cognitive processes that does allow effective analysis of these instances" (p. 19). Davis and his colleagues have used these analyses not only to describe learning but also to offer remediation to students.

Need for the Study

> The needs of the teacher with regard to the diagnostic teaching of mathematics make it necessary to complement research on error patterns...with cognitive models of the causes of errors. The mechanisms of information processing seem to offer one possible basis for classifying such causes (Radatz, 1979, p. 164).

There is a need for teachers to be more analytic in their teaching. However, in order to be analytic, one needs a model or conceptualization to represent what one is analyzing. Davis and his colleagues have provided such a conceptualization and they have found it of much use in describing learning.

Of interest to this study will be the question: Can a classroom teacher make use of this conceptualization and methodology in order to enhance instruction and to provide remediation?

Purpose of the Study and Research Questions

This study constitutes an application of the conceptualization and methods depicted in Davis and McKnight (1979a, 1979b), and Davis Jockusch and McKnight (1978). These will be further described in Review of the Related Research and Literature. This conceptualization and these methods will be used to analyze the problem solving processes used by grade eight students during task oriented interview situations.

The purpose of this study has been formulated into four research questions:

1. Can other researchers use the conceptualization posed by Davis et al. in order to analyze problem solving done by children?
2. If the answer to the question is yes or partly yes, what does this analysis reveal?

3. If the answer to question one is no or partly no, what changes or refinements in the conceptualization would facilitate better use?
4. Using this analysis, can suggestions be made which may enhance instruction or offer guidelines for remedial work?

Significance of the Study

If, on analysis, inappropriate cognitive structures and processes are found, suggestions could be made to correct these via remediation or, better yet, prevent them via changes in instruction. This study could add to a data base which provides specific information for classroom situations:

Definition of Terms

For the purpose of this study, the following terms will be used as defined:

Problem: "a situation in which an individual or group is called upon to perform a task for which there is no readily accessible algorithm which determines the method of solution" (Lester, 1978, p. 54).

Problem Solving: "a situation in which previous experiences, knowledge, and intuition must be coordinated in an effort to determine the outcome of a situation for which a procedure for determining the outcome is not known" (op. cit., p. 54).

Artificial Intelligence: (A.I.) - "the study of information-handling processes as they are carried out by sophisticated computer programs, and the search for parallels between computer information handling and human information handling" (Davis et al., 1978).

Heuristics: any resource or strategy which provides aid or direction in the solution of a problem.

Analysis of Problem Solving: Problem solving investigated using analogies with sophisticated information processing by computers.

Assumptions

1. It is assumed that the protocols will be partial reflections of the cognitive processes students use while solving rate, ratio and percent problems.
2. It is assumed that the requirement of thinking aloud will not significantly distort the problem solving procedure.
3. It is assumed that the subjects involved were typical grade eight students.
4. It is assumed that consistent patterns of observed behaviour are caused by stable thought processes.

Delimitations

1. The study was confined to grade eight students because the investigator was best prepared to assess performance at that level.

2. The study was confined to one curriculum topic (rate, ratio, and percent) since patterns were more likely to be seen using only one topic.

3. The study was restricted to classes from one junior high school.

Limitations

1. In traditional research, limitations are often related to generalizations from the sample to a larger population. Generalizations made in this study, however, pertain only to individuals. Some generalizations made regarding individual students would be affected by the lack of depth to which each individual's problem solving was analyzed.

CHAPTER II

REVIEW OF THE RELATED RESEARCH AND LITERATURE

Introduction

This review covers three major topics. Attention will be first given to the methodology involved in data collection. The second topic will be an artificial intelligence (A.I.) conceptualization used to represent learning. Finally, a review of research on analysis of problem solving will be given and compared to the A.I. conceptualization.

Methodology

The methodology used in the present study was formulated after consideration was given to the methods discussed in this section.

Although Piaget encountered success using clinical methodology in which the subjects being studied were encouraged to "think aloud", this methodology has not had extensive use by either psychologists or mathematics educator researchers studying problem solving.

> One of the few psychologists to study how complex mathematical problems are solved was Karl Dunker...who used the "thinking aloud" technique. Thinking aloud, although out of favor for years, has recently reappeared with the advent of information-processing approaches to the study of problem solving (Kilpatrick, 1969, p. 526).

Kilpatrick (1969) suggests that a researcher who chooses "to investigate problem solving in mathematics is probably best advised to undertake clinical studies of individual subjects...because our ignorance in this area demands clinical studies as precursors to larger efforts" (p. 532). He says, nine years later, that "too much developmental work is needed before experimentation could be effective" (Kilpatrick, 1978, p. 18). Hatfield (1978) agrees that exploratory rather than experimental studies might prove more valuable given the present state of our knowledge.

Kantowski (1977) conducted longitudinal observations using a clinical methodology. Verbal and written protocols of eight ninth grade subjects were collected and looked at in order to search for regularities and variation of processes used while solving geometry problems. The study was exploratory and indicated that further clinical exploratory studies were needed and suggested areas in which experimental studies might prove fruitful.

She outlined limitations of her study. The small sample limits generalizability. The presence of an observer and tape recorder may have put constraints on the subjects. Having to verbalize the process they were using may have hampered (or helped) the subjects in their problem solving. Finally, although objectivity was attempted, the coding was subjective and was done by only the one investigator, thus making the reliability of the study problematic.

Krutetskii (1977) confronted the following difficulties in using a think aloud methodology:

1. There could be a distortion of the real picture because of the unnaturalness of thinking aloud.
2. The subject may explain the solution so it can be understood by others rather than to think aloud.
3. The subject is dealing with self-observation, and "the very purpose of observing, as is known, can completely distort the picture of thought" (Krutetskii, 1977, pp. 92-93).

He claims to have circumvented the above difficulties by:

1. Telling the subject "that he not tell about how he was thinking but that he simply think out loud" (op. cit., p. 93) as if he were doing the problem alone at home.
2. Telling the subject that the process and not the product was of prime interest.
3. The subject was shown during interviews what was expected of him or her until he or she "gradually became used to thinking aloud" (op. cit., 93).
4. If verbalizing seemed to be presenting obstacles to the subjects' train of thought, the experimenter did not persist. Instead, discussion after solution and a record of the solution was used.
5. If the subject appeared indisposed, fatigued, or lacking in interest the experiment was postponed.

Clement (1979) encouraged a third grader to think aloud during and after solving a problem. Analysis proceeded "on the assumption that these comments [were] partial reflections of the cognitive process used to find the solution" (p. 59).

Erlwanger (1975) employed a procedure which "involved flexible and exploratory observations, discussions and interviews with each child extending over a semester's time in order to explore each child's ideas, beliefs and views about mathematics and the process of learning mathematics" (p. 157). The interviews were audio-taped and were conducted in such a way so as not to draw attention to the pupils in the study. He attempted to help the subjects verbalize by initially discussing "current and previous work with them, as well as their opinions about various aspects of school... It is not clear how effective these attempts were. [Mathematics] was still something one did -- silently" (op. cit. p. 188).

Erlwanger found himself in a role conflict in that he wanted to discover misconceptions but also felt an ethical responsibility to provide remediation. He provided remedial work when it was deemed absolutely necessary. "The view adopted otherwise was that remedial work would be more effective when the child's misconception was better understood" (Erlwanger, 1975, p. 188). He realized, of course, that when he did not offer remediation when a child committed an error, he was running the risk of the student inferring adult agreement.

Davis and McKnight (1979a) claim that interview situations in which subjects think aloud and not tests "provide us appropriate 'peep-holes' to look through surface phenomena and discern a bedrock foundation of the kind of cognitive growth that must occur if one is to 'learn mathematics'" (p. 16).

They observed students via task based interviews. The students had pens or pencils and other equipment if necessary. A pen proved more effective because it was permanent. The interviews were usually audio-taped and sometimes video-taped. They found that most students could verbalize with little apparent difficulty.

The researcher tried to be as unobtrusive as possible unless it was deemed necessary to determine exactly how a student was thinking. At these times the researcher might pursue a fairly insistent line of questionning.

> This may take one of several forms:
>
> (a) Asking "can you explain that to me?";
> (b) Asking "how did you (or "why did you...) decide that...?";

(c) .Commenting: "Suppose I told you that I think there is one mistake on paper...Do you suppose you could find it?";
(d) Posing a <u>new question</u> that may reveal something about how the student is thinking - e.g.: "How would you solve this problem if that were an x^2 term instead of and x^3 term?"
(Davis and McKnight, 1979a, p. 23).

Typed transcripts were later made from the audio tapes and these were used in the subsequent analysis.

Davis and his colleagues (Davis, R., 1975a, 1975b), (Davis et al., 1978), (Davis and McKnight, 1979a, 1979b), and (Davis, et al., 1979) did their analysis and developed a conceptualization at the same time. "It should be noted that the theoretical structure did NOT precede the observations; both were built up simultaneously, and the present state of each influenced the further development of the other." (Davis and McKnight, 1979a, p. 25).

The present study uses this conceptualization in order to examine problem solving. A description of the theoretical structure is the topic of the next section.

Conceptualization

The work of Davis and his colleagues has resulted in a conceptualization of mathematics learning modelled on artificial intelligence (A.I.). They have used the concepts involved in sophisticated computer programming to analyze information processing done by students in various contexts. A detailed summary of the concepts is contained in Appendix C: Summary of Theoretical Postulates, Observed Behaviors and Conjectures. The following is a review and explanation of what Davis and his colleagues consider to be the key concepts in their formalization.

Hypothetical Mechanisms in Mathematical Thought

1. Procedure: a definite sequence of steps, aimed at achieving some quite specific result. Procedure B is a sub-procedure of procedure A and A is a super-procedure if A can call upon B in order to accomplish part of the assignment of procedure A.

Example: a) Addition and subtraction of positive integers are sub-procedures of addition and subtraction of signed integers. b) Buttering the bread is a sub-procedure of making a sandwich.

2. Visually-Moderated Sequences (VMS): a sequence beginning with a visual (and/or auditory) input which is a stimulus to which a person responds by activating some procedure; followed by this procedure producing a new visual input and so on until some final outcome is reached or the sequence is terminated in some way.

Example: a) One may not be able to exactly describe the route to a new home but could drive home because once the sequence is set in motion visual inputs remind one of the next step. b) A grade four student may have to execute a long division problem in order to describe it.

3. Integrated Sequences: holistic entities which can be contemplated without executing the entire sequence, step by step.

Example: a) If the route home is simple or one has lived there for several years the route could be contemplated and described. b) An adult can visualize and describe a long division problem without actually going through the process.

4. Integrative Capabilities: the mechanism which makes possible the transformation of a VMS sequence, by practice, into an integrated sequence. This involves the capability to identify similarities between two portions of procedures, and a "look ahead" capability to recognize, before hand, what is coming next. This must be inferred from behavior that exhibits an integrated sequence.

5. Record Keeping Capabilities: the mechanism which makes possible the recording of procedures actually used. This must also be inferred from behavior that exhibits the subject has "remembered" some procedure.

6. Assimilation Paradigms:

> This concept is drawn primarily from the work of Piaget...[It is] an idea in the subjects's mind that is retrieved in response to some information-processing 'trigger', such as a cue in some visual input, and which then plays a central role in subsequent information processing... [Example:] In the Pigetian 'rotating jar' task, a young child (about age 3 years) may draw a fluid in a jar as if it were a frozen block of ice, when in fact it is not; this is sometimes analyzed by assuming that the child retrieved an 'assimilation paradigm' (AP) based on experience with solids rather than fluids. Since the AP then influences subsequent processing, the child's drawings are essentially those of a solid, not a liquid...'Frames' [defined below], drawn from A.I., are relatively precise and detailed; AP's are vague and general. (David and McKnight, 1979a, pp. 27-28).

7. Frames, (Scripts, or Schemata: No distinction is made in this conceptualization of learning between "frame", "script", or "schemata" as used by Minsky, Schank, and Rumelhart and Ortony respectively. Minsky (1975) defines frame as:

> a data-structure for representing a stereotyped situation, like being in a certain kind of living room, or going to a child's birthday party. Attached to each frame are several kinds of information. Some is about what one can expect to happen next. Some is about what to do if these expectations are not confirmed (p. 212).

Example: If birthday party is mentioned to children, they assume they will have to buy presents, get dressed up, be fed cake, the "party person" will blow out candles, etc.

8. Frame Retrieval Mechanisms: a mechanism for retrieving an appropriate frame, if one can be found in memory.

9. Frame Appropriateness Judgements: judgements made after a frame is retrieved as to whether it should be accepted or rejected. "This corresponds, at least roughly, to a part of the 'assimilation' vs. 'accommodation' alternatives of Piaget" (Davis and McKnight, 1979a, p. 29). Example: When one is invited to a party a frame is retrieved regarding dress. If the party is an informal house party a frame with a tuxedo in it would be rejected.

10. Assigning Values to "Variable Slots" in a Frame:

> We can think of a frame as a network of nodes and relations. The "top levels" of a frame are fixed, and

> represent things that are always true about the supposed situation. The lower levels have many terminals-"slots" that must be filled by specific instances or data. Each terminal can specify conditions its assignments must meet...Simple conditions are specified by markers that might require a terminal assignment to be a person, an object of sufficient value, or a pointer to a sub-frame of a certain type. More complex conditions can specify relations among things assigned to several terminals...
>
> A frame's terminals are normally already filled with "default" assignments...
>
> The default assignments are loosely attached to their terminals, so that they can be easily displaced by items that better fit the current situation. (Minsky, 1975, p. 213)

Thus, on hearing about a dog, readers will probably call to mind some dog in their past, say a black Cocker Spaniel. The fixed part of the frame would include, among other things, four legs, two ears, two eyes and hair. The lower level or variable slot for hair would contain the condition that the color of the hair be one of white, some shade of brown, or black, or a combination of these. One complex condition would specify that the eyes be on the head. The default assignment "black" (assigned because of the dog in the past) would be replaced by reddish-brown on hearing that the dog is an Irish Setter. Thus, as more information comes in about the size, sex, age, etc. of the Irish Setter, this would replace the default assignments which describe the Cocker Spaniel.

Davis and McKnight (1979a) describe the process by which frames are formed:

> Where Do Frames Come From?
>
> Our observations suggest a sequence generally like this: For novel problems something resembling VMS sequences

> will be learned (from books or teachers), which - given sufficient practice - will become schemata [frames]. Given more experience (and perhaps additional explicit instruction), a meta-language will be created. This is the general pattern, though individual students may not complete the entire process, and simpler problems may not require it....
>
> As the VMS procedure is acquiring the wholeness of an integrated sequence, as it is acquiring appropriate meta-language descriptors, it is in fact becoming a frame (pp. 34-35).

Davis and McKnight (1979b) comment on the difference between Frames and VMS sequences:

> 1. Frames allow for decisions to be made early in the process...
>
> 2. Because of the preceding property, frames make possible "top down" vs. "bottom up" processing. When a frame is retrieved early, it can guide us in looking for that input data which is most essential...[In "bottom up" processing used by VMS sequences one uses more minute pieces] of data not yet related to any specific "larger" schemata...[Followed by other data, it can lead us to the selection of an appropriate frame.]
>
> 3. Frames allow for "random access" variable binding [see Appendix C, Th. 3.1B]...whereas procedures are presumably more sequential...
>
> 4. Of course of the greatest importance is the question of default values for variables. If I say "Leslie is a teacher," you can immediately retrieve one (or more) probably relevant frames and make default evaluations...and modify them as incoming data may require.
>
> 5. People probably display more confidence when their data-processing is based on frames than when it is based on VMS sequences or other purely sequential processes (p. 100).

11. Meta Language: a set of appropriate descriptors for frames. Example: given a problem, a subject may have to retrieve one or more frames and sequence them properly. Possession of an appropriate meta-language of these frames can help. Given a problem that starts like: A grocer has two kinds of tea...one automatically thinks it is going to be a "mixture" problem and starts retrieving the mixture frame.

Given the above key concepts in the conceptualization for learning, the reader is advised to turn to Appendix C, for a summary of the other concepts. The following section applies the above conceptualization to research on problem solving.

Analysis

Davis and his colleagues, as mentioned earlier, did not have a predetermined conceptualization. They built their conceptualization as they did the analysis. The results are reported in Appendix C and the key concepts were discussed in the preceding section.

The conceptualization can be used to analyze both correct and incorrect procedures and the results of research of others who may not have based their studies on an A.I. conceptualization. Examples of analyses follow.

Erlwanger (1975) found that most visually-moderated-sequence errors made by fourth, fifth, and sixth grade arithmetic students he studied were made in super-procedures which were the subject of recent learning, and not in the sub-procedures. "A typical example: .3 + .4 = .07. The sub-procedure that inputs 3 and 4 and outputs 7, performed correctly; the subprocedure that counted decimal places and added them worked correctly;

but, of course, this last sub-procedure ought not to have been activated for this problem" (Davis and McKnight, 1979a, p. 103)

Other errors found by Erlwanger were the result of "Binary switches" in which procedure B is used when procedure A should be used. These resulted when the notation for the two procedures is similar, for example + and X. This error is a result of confusion between two visually-similar initial inputs.

Radatz (1979) also makes reference to this type of error in which there is interference from a previously learned skill or algorithm.

Another error found by Erlwanger (1975) which often occurs during a VMS sequence is the subject misreading his own written notation. For example, a carelessly written x may go through a metamorphis and become a y.

Davis and McKnight (1978) refer to the potential ambiguity of a stimulus and say that "various procedures must be activated at the outset to interpret the input signal" (p. 27). Radatz (1979) refers to several investigations and says that "the iconic representation of mathematical situations can involve great difficulties in information processing and that perceptual analysis and synthesis often make greater demands on the pupil than does the mathematical problem itself" (p. 165).

Parallel to CL 7.0 A (Natural language statements often lead students into error, see Appendix C), Radatz (1979) found in the research that errors evolved in the "translation from semantic network in natural language to a more formal network in mathematical language" (p. 165).

Radatz also discusses errors in assimilation in which there are mistakes in seeing, or reading or hearing. He also makes reference to studies that showed insufficient knowledge of necessary concepts. Several other studies, mentioned below, refer to this and would relate to the concept in A.I. of inadequate or inappropriate frame retrieval.

Kantowski (1977), addressing the problem of prerequisite knowledge, found that "the theorems committed to memory (as shown by the prerequisite-knowledge test) were easily activated for use in solutions, whereas those not remembered were most often not used even though reference sheets were available" (p. 177). She concludes that "minimal 'grounding' in content... may be necessary before the use of heuristics can be of any value to a problem solver" (p. 177).

Knifong and Holtan (1976) found that because students did not know the correct procedure, they committed 5% of their errors on average and area questions even though these questions constituted only 10% of a problem solving test. This would indicate, it seems, they did not fully understand the concepts of average and area; that is, they had incomplete frames.

Hatfield (1978) on reviewing problem-solving research found that several studies indicated lack of background knowledge hampered students problem solving. One study he reviewed (by Webb) showed that "conceptual knowledge was the variable with the highest relation to problem solving ability in mathematics" (Hatfield, 1978, p. 29). This shows the need for a careful building up of mathematical ideas.

Krutetskii (1976) describes some studies which do not use artificial intelligence as a model, but can be explained through the use of frames and frame retrieval mechanisms. Menchiskaya has shown that both the features of a problem and the pupil's understanding of the concept involved in the problem will influence whether the pupil can discriminate and abstract the relevant features and thus recognize the problem's type.

Kalmykova found that solving problems of a single type helped students to gradually discriminate and generalize the relevant features and guided them to solution. V.L. Yaroshcuk stated that a pupil was a more successful problem solver if he could classify the problem according to type.

Nesher and Teubal (1975) found there is an association between the choice of addition or subtraction and the verbal cue more or less given in the formulation of a problem. They suggest that the usual way of training to solve problems misses the target because the training encourages direct translation from the verbal formulation to a mathematical expression. Because the same word will sometimes appear as a cue and sometimes as a distractor, the main goal in a "genuine arithmetic word problem should be to reveal the underlying quantitative relations" (Nesher and Teubal, 1975, p. 50). This revealing of the underlying quantitative relations shows the student has an internal representation of the physical situation, as mentioned by Paige and Simon.

> Paige and Simon (1966) compared the protocols of subjects asked to think aloud as they solved algebra word problems with the processes used in a computer program for translating English sentences into equations and then solving them. Analysis of the protocols showed that subjects used some kind of internal representation of the physical situation described by the problem in framing their equations (Kilpatrick, 1969, p. 527).

This internal representation could be thought of as a frame. Kinsley et al. (1977) showed that appropriate frames were retrieved by adult subjects doing several word problems.

> 1. Subjects do recognize problem categories. There are at least 16 problem categories which our subjects agree about. Presumably, more could be identified if we were to search more thoroughly.
>
> 2. Subjects can in many cases recognize a problem's category early in reading the problem. Sometimes reading as little as the initial noun phrase is sufficient.
>
> 3. Subjects have information about the problem categories which is useful for formulating problems for solution. This information includes knowledge about useful equations and diagrams and appropriate procedures for making relevance judgements
>
> 4. Subjects can and often do use this information in solving algebra word problems when their instructions are simply to solve the problems and do not in any way call special attention to problem classification (p. 104).

Krutetskii's (1976) analysis of his own results lead him to state that there seems to be

> three basic stages in the process of solving mathematical problems:
>
> 1. Receiving information about the problem (related to an initial orientation toward its terms, an attempt to understand it).
>
> 2. Processing (transforming) the obtained information for the purpose of solving the problem, and obtaining the desired result.
>
> 3. Retaining information about the problem (p. 184).

Parallels can be drawn to the work of Davis and colleagues, and the three stages could be described as:

1. Receiving information would be the beginning of a VMS sequence. Orientation toward its terms would be retrieval of frames using a meta-

language and frame appropriateness judgements.

2. Processing the information would involve top-down processing driven by the frame or bottom-up processing, related to a VMS sequence.

3. Retaining information about the problem would use record keeping capabilities and the frame associated with the problem would be refined.

Depending on the success the student had in completing the above three steps while solving one problem, there would be, according to Davis et al. (1978), three possibilities in relation to solving a similar problem.

1. The student is unable to solve the problem.
2. The student uses a VMS sequence to solve the problem but has no clear over-all picture of the process.
3. The student has in mind an overall plan of action. Thus a frame for the problem has been activated. To assess the stage at which the student is, clinical judgement is required (see Appendix C: CL 3.2, p. 137).

Knifong and Holtan (1976), using a standardized word problem test with sixth-grade children, found that 49% of errors were found to be solely attributed to faulty computation. They suggested that "improved computational skills could have eliminated nearly half of the word problem errors and can be strongly recommended as a teaching strategy" (p. 111).

Davis et al. (1978) would refer to these computation errors as Behav. 1.5 (What should be a reliable unit sub-procedure may sometimes malfunction).

In Hatfield's (1978) review of some research, it was reported that Webb found that use of a pictorial representation helped students solve problems. This use of an alternative representation is CL 9.5 in Appendix C. Webb, as reported by Hatfield, also found that those students who used a wide range of heuristic strategies, on the average, were better problem solvers.

Hatfield (1978) also reports that McClintock found that students who verbalized what they recalled having done in solving problems were better able to transfer their methods to new problems on the same topic than those students who did not verbalize. In A.I. terminology this could be thought of as forming a meta-language of the procedures.

Lester (1978) gives an account of some research done at Indiana University concerning problem solving processes used by fifth-graders. It was found that:

> 1. Students often misread or misinterpreted problems.
>
> 2. Students had difficulty retaining and coordinating multiple conditions in a problem.
>
> 3. Students do not appear to use any strategies during problem solving (Lester, 1978, p. 83).

These would be paralleled, respectively by:

1. Problems in assimilation (Behav. 3.0).
2. Difficulties in creating and maintaining a long strategic planning sequence (CL 6.3, 6.4, 6.5).
3. Problems involving heuristics.

Kantowski (1977) found (in studies of grade-nine students) that the more successful problem solvers (those above the median) made use of goal-oriented heuristics more often than did those below the median. She states that "the introduction of a heuristic related to the goal seemed to correspond to 'insight'. From that point, the path to the goal was, in most cases, clear and characterized by regular analysis-synthesis patterns" (p. 166).

This would be explained using artificial intelligence terminology by saying a frame was called up (the insight, and goal oriented heuristic) which direct top-down processing. This top-down processing would stop at times and bottom-up processing (conducted by a VMS) would be used.

Kantowski (1977) also found that:

> Persistence did seem to be affected by prerequisite knowledge and by personality factors. When a problem was repeated, some subjects tried new techniques if they had been unsuccessful previously, whereas others abandoned the search for a solution immediately if they recognized a problem as one they had tried before without success (p. 169).

The former would be related to the heuristical method of trying something new. Assuming they should be able to do the problem, the latter could be related to a wrong evaluation of knowing the boundaries of what they should know (CL 8.9).

Looking at the above research using an A.I. conceptualization, not only gives one a new perspective on the research but also increases ones sensitivity to what might be "going on" in pupil's minds.

CHAPTER III

RESEARCH PROCEDURES

Introduction

The purpose of this chapter is to describe the pilot work, research design, the sample used, data collection, and the analysis methodology.

Pilot Work

Thirty problems involving rate, ratio or percent were piloted using grade nines in order to choose questions which seemed most fruitful for analysis purposes. Three heterogeneous classes of grade nines were asked to do ten problems during a class period and to put all their work on paper and to write down what they were thinking as they solved the problems.

Using these protocols, nineteen questions were chosen for the study and the wording of one question was changed for the study. Two questions were added in order to make three sets of seven parallel questions. Two questions, which were not rate, ratio or percent questions, were later added to each of sets B and C. This was done in order to see if the students would try to set up a proportion to solve them. The questions were typical of what one might find in any grade eight text book on the topic.

Each question was typed at the top of a page with the remainder of the page being left for the students' work. The questions used in the study can be found in Appendix A: Problem Sets.

Four adult women were interviewed to practice the interview technique and iron out problems that might occur using a tape recorder and requiring thinking to be done out loud.

Research Design

This project was a descriptive study using clinical methodology for data collection. Analysis of the data was done using a conceptualization of learning based on an Artificial Intelligence Metaphor. Observations were made of twelve subjects and neither a control nor a comparison group was used.

Sample

The sample of subjects came from two heterogeneous grade eight classes in a city junior high school. Both classes were taught by the same teacher. Students were asked to volunteer. From the volunteer list and a rank ordering of the subjects by IQ; two low, two average and two high IQ subjects were randomly chosen from each class. This resulted in twelve subjects taking part in the study. All twelve completed the study. It was decided that problems pertaining to a small unit of study would be better than using a large variety of problems. Thus, problems involving rate, ratio and percent were chosen. Students study this topic briefly in grade seven and return to it in the spring of grade eight.

Data Collection

Observations were made during April, May and June in order to examine subjects' problem solving before and after instruction in rate, ratio and percent.

The data was collected during tape recorded interviews in a small room where disturbances were unlikely to occur. The subjects were interviewed one at a time during school hours and each interview lasted approximately the length of a class period, thirty-six minutes. The students were given a pen and the problem sheets for the session. Using a tape recorder and the pen, permanent records of their work and verbalizations were made.

An explanation of the purpose of the interview was given to each subject prior to the interviews.

The subjects were told at the beginning of the interviews that none of the results would count toward report card marks, and that the interviewer was interested only in what they were thinking while they solved the problems. They were encouraged to "think out loud". They were told that questionning by the interviewer did not imply that they were doing something wrong, it merely meant that the interviewer wanted a further explanation of what they were doing.

During the interview, the interviewer was as unobtrusive as possible. If the subject was reluctant to verbalize his or her thoughts, only the work done on the problem sheet was used for analysis purposes. The interviews were flexible, however; and if it seemed prudent, the interviewer asked questions of the subjects in order to further reveal their thoughts if these weren't made evident by their verbalizations or their written work. Typical questions would be:

1. Does that seem reasonable? (Asked after unreasonable or reasonable answer were given if it seemed of interest to know if the subject had considered the reasonableness of an answer.

2. Why did you add there?

3. Can you tell me why you have that proportion?

If it seemed necessary, a new question was posed in order to reveal more of the thinking involved or to try to trace where the thinking may have originated.

If subjects seemed frustrated with a question at any point they were asked if they would like to go on to the next question.

The role conflict of feeling guilty for not offering remediation was resolved by offering pointers at the end of the interview if it was thought such pointers would not interfere with future interviews. Otherwise, remediation was not offered. Very apparent difficulties were exposed to the teacher at the end of the interview sessions and the responsibility for remediation was left to her.

During the first interview session, it became apparent that the subjects were having difficulty doing division. Therefore, a separate interview with each subject was set up in order to, hopefully, divulge the cause of these difficulties. The questions asked in this session are in Appendix B: Division Questions. Thus, at the end of the interview sessions there were four sets of data collected on each individual:

1. Problem set A, collected prior to the students being taught the rate, ratio, percent unit.
2. Division Questions.
3. Problem set B, collected within a week of the students finishing the unit.
4. Problem set C, collected three weeks after they finished the unit.

These sets were transcribed by the interviewer and three complete transcripts of protocols are contained in Appendix D: Sample Protocols. The names of subjects have been changed in order that they remain anonymous.

Analysis Methodology

Analysis involved repeated examination of all the protocols, transcripts from the interviews, and the students' written work. Of particular interest was the "fit" of the data to the conceptualization posed by Davis and his colleagues and what recommendations could be made regarding teaching.

To test the fit, each protocol was examined in relation to the following three topics.

I. Eleven hypothetical mechanisms discussed in Chapter II.

II. Three possible problem solving stages.

III. Other mechanisms of Appendix C not mentioned above.

These three topics are discussed below.

I. The general framework on which the conceptualization is based as outlined by the eleven hypothetical mechanisms discussed in chapter II.

1. Procedures
2. Visually-Moderated Sequences
3. Integrated Sequences
4. Integrative Capabilities
5. Record Keeping Capabilities
6. Assimilation Paradigms
7. Frames
8. Frame Retrieval Mechanisms
9. Frame Appropriateness Judgements
10. Assigning Values to Variable Solts in a Frame

Meta-Language

II. The stage at which the student was on approaching the problem.

1. Stage CL 3.2A Student was unable to solve the problem.

The solution was classified in this category if the student didn't try to solve the problem or tried but called up an inappropriate frame or used an incorrect process.

2. Stage CL 3.2B Algorithmic stage (using VMS sequence which may have occasional errors).

The solution was classified in this category if the student used an appropriate method of solution but seemed to use each step to guide decisions as to what to do next.

3. Stage CL 3.2C The stage of comprehension, defined in terms of the student's ability to state an a priori "plan of action" for solving the problem. The student can discuss how and why the pieces fit together.

The solution was classified in this category if the processing seemed to be top-down. That is, if the subject read the problem and without hesitation seemed to retrieve a frame for the solution of this type of problem, then looked for data in the problem to fit into this frame. There may have existed minor sub-procedure errors after this but it was still classified here if the subject had the correct over-all plan.

4. Aspects of the problem which could be classified as supportive of any of the theoretical postulates, observed behaviors or

conjectures listed in Appendix C and not mentioned above.

Clinical judgement was necessary but all decisions were made after careful consideration and repeated assessment of all protocols for each subject.

While the analysis was being done, particular attention was paid to areas where recommendations could be made regarding teaching strategies and remedial work.

CHAPTER IV

ANALYSIS AND IMPLICATIONS

Introduction

The purpose of this chapter is to discuss the analyses of the protocols and the implications of these analyses for teaching. It should be noted that the inclusion of implications in this chapter, rather than in the concluding chapter, is a deviation from the traditional format. This has been done for ease in referencing the implications with the data and analysis.

Analysis of Division Protocols and Implications for Teaching

During the first interview session; when the students were solving problems prior to being taught rate, ratio and percent; it became evident that many of the students had some difficulty with division. They did not have a method of approximating the partial quotients and encountered difficulty when decimals were involved.

Since division, by grade eight, should be a reliable procedure, it was decided to further analyze the students' division procedures. A separate interview was set up with each student in order to do this. The questions asked are in Appendix B and samples of protocols of interviews are in Appendix D.

The following paragraphs contain an analysis and discussion of some of the division protocols and implications that can be drawn from these. In the discussion, S_1, S_2, S_3....designate statements 1, 2, 3.... in a protocol. For example, S_4 in Kurt's protocol for problem 1a of session 1 is:

4. K: Yeah, I hope I'm not buying it 'cause that's an awful lot of money.

References will be made to the listing in Appendix C which summarizes the conceptionalization. In most cases the listed topic will be paraphrased. If it is not paraphrased, the page number on which it is contained will be given.

Of the twelve students discussed here, the first four had little difficulty with the division questions.

Trina:

As can be inferred from the statements below, Trina has a "correct" frame for the relationship between multiplication and division.

1 Div.

1. I: What is the relationship between multiplication and division?
2. T: Well like you can find out how many of one number can go into another number.
3. I: Can you give me some examples?
4. T: Well you've got 8 times 7 is 56, right? Then you know that 7 goes into 56, 8 times. Like if you know the answer to one you can almost do the other one.
5. I: Almost, but not quite?
6. T: Well yeah, if you know times you can do divide.

Trina completed all of the division questions except for 3DQ with no problems. She arrived at the correct answer for 3DQ but was definitely at a VMS stage for solving this kind of division question and, as can be seen by S4 and S10 below, she did not have an appropriate meta-language with which to talk about the procedure used.

One explanation for S4 is that Trina made an incorrect variable retrieval, Behav. 2.0. She retrieved "negatives" instead of "decimal numerals". Another explanation could be that Trina thinks that any digit after the decimal point represents a negative number. Her frame for the decimal numeration system would have to be exposed further in other interview situations. Questions like: "What fractional number is represented by .62?" or "Which is greater 0 or .003?" would expose where her frame for decimal numerals would have to go under further accommodation.

3 DQ: (32 ÷ 429)

1. T: (Writes the question in algorithmic form.)
2. T: Is that right, 32 divided by.....?
3. I: Yes.
4. T: Well I couldn't do it unless I bring down the zeros and then you'd have to go into negative numbers.
5. I: Well put whatever you think the answer might be.
6. T: We've never had anything like this before.
7. T: (Adds two zeros after the decimal in the 32, puts the decimal in the answer above the decimal in the 32.)
8. T: You can't use 32, and you can't use 320 so that's zero (puts the 0 in the tenths place in the answer).
9. I: Can you tell me why you have to put that zero there?
10. T: Well that decimal 9 lets say means 9 tenths of a hundred but it might be less than that like 1 ninth of a thousand or something of a hundred so you have to get it over.
11. T: (Tries 3 times 429 out at the side.)

12. I: Can you tell me why you picked 3?

13. T: Well I wasn't sure like how far it would go, and once I get a low number there (after multiplying) I can see about how much another 429 would be.

14. T: (Gets 2145 for 3 x 429) Let's see it could go up another 2. (Multiplies by 5.) (Multiplies 429 by 7.)

15. T: (Puts 7 in the hundredths place in the answer and gets a remainder of 197.)

Sue:

Sue was unable to give the names for divisor, quotient and dividend but did answer the other preliminary questions in a manner that indicated she had a "well developed" frame regarding division.

When asked to estimate in 6DQ, Sue rounded the divisor down rather than up. This indicates that an incorrect sub-procedure has been called up by the division super-procedure.

6 DQ a: (8245 ÷ 85)

6. S: Because I just rounded it off and 80 would go into 800, 10 times.

6 DQ c: (24156 ÷ 732)

2. S: Now I'm just rounding it off to 700 so the first digit would be 3.

Don:

As can be seen in Appendix D, where Don's complete protocols are given, (pp. 182-191) Don encountered little difficulty with division.

In the second division question, 2DQ, he was asked to check his answer using multiplication. He did so easily (S8) and corrected his error, an example of CL 24.5C, ability to criticize work and make changes where necessary. In 2DQ, also, he gave an informal meta-analysis of movement of decimal points (S1, S3).

Jane:

Jane, also, had little difficulty except for 3DQ. In S1 she exhibits a negative instance of CL 8.9, knowing the boundaries of what you are supposed to know. She should know how to handle division questions in which the divisor is less than the dividend. S1 also is an instance of CL 24.5A, retention of earlier restrictions even after one has learned they are no longer necessary.

Following the hint offered in S6 and S8 Jane illustrates CL 9.7C to solve the problem; she uses an illustrative example to solve an assigned problem.

3 DQ: (32 ÷ 429)

1. J: You can't divide 429 into 32.
2. I: Why?
3. J: Because 32 is too small.
4. I: Are there any sixes in 3?
5. J: No.
6. I: Is there half a 6 in 3?
7. J: Yes.
8. I: .5, does that give you a hint what to do?
9. J: (Puts .0 in the divisor.)
10. I: Why did you put the 0 there?

11. J: Because 429 won't go into 320 so I put the 0 there and add another 0. (Finishes and gets .07 remainder 197.)

Liz:

Liz answered the preliminary questions with no trouble. The only division questions she had trouble with involved decisions about whether to put zeros in as place holders in the quotient. This problem also occured in question 1c of the problems. Thus, what should be a reliable unit sub-procedure is not (Behav. 1.5). This problem could probably be clarified by discussing with her the structure of the decimal numeration system and the purpose of zero as a place-holder. This discussion should take the form of directed questioning. This could then be followed by her doing some questions of this type under supervision.

The above five students, particularly Don, encountered no real difficulties with division. However, each of the four girls would profit from some help in "ironing out the bugs" in their procedures. This would probably be best done on a one to one basis and would likely take about a half-hour per student. Occasional drill in division would help them keep their skills at an appropriate level. This drill should involve decimals, divisors greater than dividends, and quotients with zeros in them.

The students should be expected to approximate the answer prior to actually doing the question.

The other seven students did not do as well on the division questions. Since many of the seven encountered much difficulty the format for the discussion of the analysis of these seven protocols is different than the previous five. The data has been collapsed to give a general impression of the difficulties encountered. Samples of some protocols are included in order to give examples of explicit difficulties.

1 Div. : What is the relationship of multiplication and division?

Of the seven, only Kurt and Nan were unable to answer this question adequately, indicating they had failed to "cross reference" (CL 9.4) the information they new about multiplication and division.

S12, below indicates that Nan sees multiplication as a sub-procedure of division, which it is in this case, but she is unable to express the relationship of the two.

Nan:

1 Div. (Relationship of multiplication and division)

6. N: Oh, umh they're both multiplying like you know...

7. I: Division is multiplying? (M: Yeah) Can you tell me why?

8. N: Well umh one number has to go into another you know?

9. I: Can you give me an example with numbers Nan?

10. N: Like 2 into 20. (I: 2 into 20...Yeah) Goes 10 times. (I: Umhm.) You put 20 underneath the 20 and you need a 0 under there.

11. I: Now how is that question, that division question related to multiplication?

12. N: You're multiplying that number times (I: The 2?) Yeah, the 2 times the 10. To get that answer, 20.

13. I: I see.

Both Kurt and Nan should be given examples like $3 \times 2 = 6$ then $6 \div 3 = 2$ and $10 \times 3 = 30$ then $30 \div 10 = 3$ and slowly work up to greater numbers like $18 \times 24 = 432$ then $432 \div 18 = 24$, in order that they could generalize that if $a \times b = c$ then $c \div a = b$.

2 Div.: Students were asked to name the a, b and c when shown $c \div a = b$, written in algorithmic form.

Mat was the only one of the seven who got these all correct, indicating his "short list" (Th. 5.1) did not yet have too many entries for him to remember. For the others, their short lists had become ambiguous and they either confused the names for divisor, quotient, and dividend with other names like numerator and sum or were unable to recall more than one of them (the divisor) correctly.

If the short list has become ambiguous then Behav. 2.0 is exhibited; that is, incorrect or no variables are retrieved.

3 Div. : Students are shown $\begin{array}{r} b \\ a\overline{)c} \end{array}$ and are asked to write it using "$\div$".

For four of the seven, the "divided by" sign was not ambiguous. To the other three, the visual input was ambiguous, as can be seen from the excerpts of protocols below.

It could be that the visual input was ambiguous because they did not know how to process the symbol string $c \div a = b$, negative instances of CL 6.10. They would, thus, not produce it when given $\begin{array}{r} b \\ a\overline{)c} \end{array}$.

They have also not "cross-referenced" $c \div a = b$ with $a\overset{b}{\overline{)c}}$, negative instances of CL 9.4, cross referencing of material.

As stated in As. 7.0, natural language statements are not in proper "command" or "procedure" form, additional processing is required to extract the command from any natural language statement. Thus, *c* divided by *a* is not in "command" form, particularly if the numbers are large. The processing required would be to put it into algorithmic form.

It could be that these students, when confronted with $a\overset{b}{\overline{)c}}$ have stored in memory a record of what procedures have been used (Conj. 3.7), rather than the relationship of the *a*, the *b* and the *c*.

If the two representations for division are going to be used, teachers must make sure that both are understood. for the students to whom this is not clear, practice should be given in switching from one representation to the other until they can do so with ease.

Nan:

3 Div.

6. N: (Writes: $a \div b = c$)

7. I: Now can you read that to me?

8. N: Uh, *a* divided by *b* equals *c*.

9. I: And that means the same as this ($a\overset{b}{\overline{)c}}$)?

10. N: Yeah.

Kurt:

3 Div.

2. K: (Writes: $a \div b = c$)

3. I: Can you read that?

4. K: Well a has to go into c to equal b.

5. I: Is that what this sign ($\div$) means to you, has to go into?

6. K: Well, divide yeah.

7. I: Can you give me an example with small numbers to show that?

8. K: Oh I did that wrong. The a and the c have to be switched around so that it is c divided by a.

Brett:

3 Div.

2. B: (Writes $a \div c = b$)

3. I: Could you read that to me please?

4. B: a divided by c equals b.

5. I: And that means the same (as $a\,\overset{b}{\overline{)c}}$)?

6. B: Yeah.

4 Div. : In (a$\overset{b}{\overline{)c}}$, written out) if c is constant and a increases, what does b do?

5 Div. : In (a$\overset{b}{\overline{)c}}$, written out) if a is constant and c increases, what does b do?

6 Div. : In (a$\overset{b}{\overline{)c}}$, written out) if b is constant and a increases, what does c do?

For questions 4 Div., 5 Div. and 6 Div.; Kurt, Hank and Cathy were able to give answers which showed they had adequate frames for the relationship of the divisor, dividend and quotient.

Kurt:

Kurt's complete protocols are in Appendix D; of interest are statements S12 - S22 of question 6 Div., below. In these he realizes that a and c must vary at the same rate but does not possess a meta-language with which to discuss it. This would be an example of regression under cognitive stress, CL 6.8, he has regressed to differences, rather than using quotients.

6 Div.

1. I: If b is constant and a increases what would happen to c?
2. K: The c would increase the same as a.
3. I: Why?
4. K: Because if you get the same answer it has to go in the same as before.
5. I: So if a went from 2 to 6, c would have to go from...
6. K: 6 to 10.
7. I: Can you show me that?
8. K: (Writes in algorithmic form, $6 \div 2 = 3$.)
9. I: So 2 goes into 6, 3 times, and a goes from 2 to 6 and b is still the same so c would have to go to 10?
10. K: Yeah, (laughs) I doubt that would work out though.
11. I: 6 doesn't go into 10 three times eh? What would c have to be?

12. K: I wouldn't know.

13. I: To get 3 for an answer what would c have to be?

14. K: It would have to be the same number apart like it was before. Like it would increase but it would have to be the same number apart, like before it was 2 and 6.

15. I: That's 4 apart.

16. K: (laughs) Oh that's what I had before. Just say it would have to divide the same. 2 into 6 would go 3, it would still have to divide the same.

17. I: What would it have to be, then?

18. K: 18.

19. I: So what is the word for that, you said it had to increase the same amount but that didn't work out for us. So you should say it has to increase...

20. K: Uh...

21. I: Increase at the same...

22. K: Uh...ratio...rate.

Hank:

Hank seems to understand the relationship but he does not possess adequate descriptors with which to express himself. He says "changes the same amount", regression under cognitive stress (CL 6.8) to subtraction, rather than changes at the same rate.

4 Div.

6. H: Well if a gets bigger it won't go in as many times.

5 Div.

2. H: Increases because a would go in more times as c gets bigger.

6 Div.

2. H: It would stay the same, no it wouldn't do that, it would have to move the same amount as a.

3. I: What do you mean the same amount, can you give me an example?

4. H: Well if a is 10 and c is uh 10 and a went to 100 then c would have to go to 100 for b to be the same.

Cathy:

Cathy answers the questions adequately but it is difficult to tell if given numerical examples she would regress to using subtraction rather than ratio. She did not use numerical examples in her explanation; and, unfortunately, was not asked to.

4 Div.

1. I: If c is constant and a increases what does b do?

2. C: Decreases.

3. I: Why?

4. C: Because you're dividing a bigger number into another so less of that number can go into it.

5 Div.

1. I: If a is constant and c increases, what does b do?

2. C: It increases, because as c gets larger this number (a) is going to go into it more times.

6 Div.

1. I: If b is constant and a increases what does c do?

2. C: It increases because if b is constant and a increases then c has to increase to make this (b) the same if this (a) is going to go into it (c) the same amount of times.

The other four students had some difficulty expressing the relationship between the divisor, dividend and the quotient.

Rick:

Rick answered 4 Div. and 5 Div. adequately, using multiplication in his explanation.

4 Div.

6. R: To get c you have to multiply a by something; a times something will get you c.

12. R: Well a [illegible] a bigger number so b will have to be smaller. If b [illegible] larger then a would have to be smaller.

5 Div.

2. R: The b would have to get bigger.

3. I: Why? a stays the same...

4. R: So to get c you have to multiply higher numbers.

In 6 Div. the interviewer gave numerical examples until Rick said "It is increasing. Okay I get it."

Brett:

Brett gives an adequate explanation for 4 Div. and 5 Div. but does not answer 6 Div. correctly. He seems to think that something is actually happening to c when it is divided by a, (see S4, S10-15). He confuses the original dividend with the partial dividends or remainder after each step.

6 Div.

1. I: b is constant and a increases, what does c do?
2. B: It would decrease.
3. I: Can you tell me why?
4. B: Well, you'd take the a and divide it into c, then c would become smaller.
5. I: b stays the same.
6. B: So you divide, say this (a) is 2 and this (c) was 10 so you'd have an answer 5.
7. I: Do you want to write that down?
8. B: (Writes 10÷2 = 5, in algorithmic form.)
9. I: So 2 into 10 goes 5, and b stays the same and a gets bigger, what would c have to do?
10. B: 2 times 5 is 10. (Writes the product under the quotient and writes in the zero remainder.)
11. I: But a gets bigger than 2.
12. B: It would become 10 'cause you take two "5's" and get 10 down here. After that it would decrease because it would come to zero.
13. I: What would decrease?
14. B: The c because it would come down to zero.
15. I: Oh, you're subtracting there.

Mat:

Mat answered 4 Div. correctly and gave examples. He answered 5 Div. correctly but was unable to say why.

5 Div.

2. M: It would be larger.

3. I: Why?

4. M: Okay, a stays the same, c increases, b...uh...

5. I: Could you show me with numbers?

6. M: Umh,...no.

Nan:

Nan, also, gets the correct answer but is unable to say why. Of special interest is S4 which is an example of Behav. 2.0, incorrect variable retrieval. She retrieves "denominator" rather than "divisor".

In S6 she says that c has to get bigger or have zeros added on to it. She may think that adding zeros onto a number does not change the value of the number. This, unfortunately, was not discussed further with her. However, in 3DQ she adds two zeros onto 32 in order to divide it by 429 but makes no compensating changes in the placement of the decimal in the quotient. She makes a similar mistake in 2DQ, adding two zeros onto 60 and gets 139 for an answer rather than 13.9.

It is possible that at some time a teacher said something like, adding zero's on does not increase the number because zero is nothing. The teacher, of course, would have meant adding zeros on after the decimal point but Nan took this out of context and applied it whenever zeros are added on. This is an example of CL 7.0A, natural language statements often lead students into error. Her frame for zero as a place-holder before and after the decimal point has to undergo some accommodation.

6 Div.

1. I: If b is constant, now b is staying the same, and a increases what does c do? Now a is staying the same b is getting bigger, what does c do?

2. N: c is getting bigger.

3. I: c is getting bigger, can you tell me why?

4. N: Because if, if uh the denominator is bigger then this has to be....(I: If a is bigger...) Yeah then this (I: Then c...) has to be bigger.

5. I: Why?

6. N: Because it won't divide into it if it's not bigger. Unless you add zeros on.

7. I: Oh if a....a won't divide into....(N: c) unless it's bigger?

8. N: Yeah, c has to be bigger.

9. I: So c increases.

In order that these four students better understand this relationship they could be given examples like 18÷1 = _____, 18÷2 = _____, 18÷3 = _____, 18÷6 = _____, where one of the three is held constant, one increases and the students are asked to say what is happening to the other. Similar examples could be given holding a constant and varying c, and holding b constant and varying a and asking the students what is happening to the third one.

Kurt:

(Complete protocols for Kurt are in Appendix D, pp. 156 - 170). Kurt handled the division questions quite well. His sub-procedures for division; movement of decimals in divisor and quotient, and multiplication seem to be working efficiently.

In 4DQ (54.2 ÷ 26) he, at first, did not put a zero in as a place holder and got an answer of 2.8 remainder 12 instead of 2.08 remainder 12.

His confusion about how to check his answer, indicated by S3 - S11, is partially cleared up later, as indicated by S12 - S28; but further teaching would be required to ensure that the frames for divisor, quotient, dividend and remainder are embedded properly in his frame for division in order that the relationship between them is clear to Kurt. This should be done starting with small numbers like 7 ÷ 3 and slowly working up to larger numbers.

Kurt should also be given questions with zeros in the quotients and do them under the guidance of a teacher. Thus, the sub-procedure of putting in zeros as place-holders could become firmly established.

Cathy:

(Complete protocols for Cathy are in Appendix D, pp 171-181). In 1DQ, Cathy arrived at the correct answer despite the "sloppy" placement of the digits in the quotient relative to those in the dividend. However, incorrect placement of digits for the quotient in 4DQ caused her to arrive at an answer of 20 rather than 2.

To correct this error Cathy should do some questions under teacher guidance with the teacher reminding her of correct placement after she has been shown why it is necessary.

Cathy made a minor error in 2DQ. She copied 387 over as 380.

In 3DQ (32÷429), Cathy originally says "I can't do this" an instance of CL 23.2, since she is accurate in judging whether she has an adequate schema (frame) to deal with this type of problem. She is, in fact, unable to solve the problem.

The subsequent dialogue and that in 4DQ indicates she does not have an adequate frame for decimal numerals and their relationship to fractional numerals.

Remediation of this problem should involve changing examples like .6 to 6/10, then .65 to 65/100 and also to 6/10 + 5/100. Following this, she should be required to change 65/100 to .65. When she can do these easily, Cathy could be shown how to change numerals like 1/5, 2/25 and 3/50 to decimal numerals. Thus, a slow advancement to changing any fractional numeral to a decimal numeral could be achieved.

3 DQ:

23. C: Then it would be 1/14.

24. I: And what would that be as a decimal?

25. C: Wouldn't 1/14 be point one four?

4 DQ:

3. C: Umh 22 over 26, that's about three fourths so it's about point 75? I don't know if that's right.

5. C: Oh, wouldn't it be point two two?

Hank:

Hank solved 1DQ, 2DQ and 5DQ with only minor problems caused by incorrect variable retrieval when he multiplied 56 by 8 in 1DQ and 43 by 3 in 2DQ.

In 3DQ (32÷429), Hank decided he couldn't do it unless he went to "negatives". This is the same mistake Trina made and could be analyzed in the same way. See page 35 for this analysis and suggestions for remediation.

When asked, he decided that four cookies divided among eight people would result in them each getting 1/2 or .5 of a cookie. However, when asked to solve 4÷8, written in algorithmic form, he added .5 to the 4 and put .5 above as the quotient.

He, clearly, has no sub-procedure to call upon to deal with division questions in which the divisor is greater than the dividend.

To answer 4DQ, Hank did not use zero as a place holder and got 2.8 rather than 2.08 for an answer. This sub-procedure must be clarified for him also.

The last two problems could probably be solved by working with Hank on an individual basis, giving him easy examples to follow and do, and slowly working up to more difficult ones.

Brett:

Brett arrived at the correct answers for 1DQ and 2DQ.

3DQ: (32÷429)

Brett at first writes the question as if it were 429÷32 and is corrected. However he has inadequate sub-procedures to call upon to finish the question correctly.

S7 - S13 indicate he does not fully realize he has changed the value of 32 by adding two zeros, although he says that the number is larger. It is unfortunate the interviewer did not pursue this further by getting him to check his answer using multiplication.

Brett should be shown how to solve something like 4÷8; have practice solving examples like this and then procede on to more difficult examples under teacher guidance.

3DQ: (32÷429)

3. B: (Re-writes it properly.) I don't know how to do this. Umh, add a zero. It still won't work out.

4. I: It still won't work out, what if you add another zero?

5. B: (Adds two zeros to the 32, has not put a decimal point in.)

6. I: When you add those zeros, does it change the number?

7. B: Yeah, it makes it larger.

8. I: It makes 32 larger so you can divide eh? (M: Yeah.) Now do you think you'll get the right answer?

9. B: Yeah, I'll take it and divide it into whatever I get and probably it will be the right answer.

10. B: (Trys 7 at the side, then 8 x 429 = 3422.) So I'll have to go back to this (7) to get the right answer.

11. B: (Gets 7 remainder 397.)

12. I: So what's your answer?

13. B: 7 point, no, 7 remainder 397.

4DQ (54.2÷26)

Brett places the decimal point for the quotient above the decimal point in the dividend and gives his reason as, "Well probably to get the right answer."

This could be an instance of CL 8.12, the expectation that learning math consists of building VMS sequences and nothing more.

Brett should be shown why the placement of the decimal in the quotient above the decimal in the dividend gives the correct answer. This could be done by incorrect placement and checking using multiplication.

He has no adequate sub-procedure for estimating partial quotients.

4DQ: (54.2 26)

12. B: (Multiplies 26 by 9 out at the side and gets 234.)

13. I: Why are you multiplying by 9?

14. B: To see how close I come to 542. See it isn't close so I go back and divide 26 into 54.

Mat:

Mat decides on values for partial quotients by multiplying the divisor by two digit numbers out at the side until he comes close to the dividend. In 5DQ, where he is asked to estimate the first digit of the quotient, he can not understand what he is being asked.

He places the digits of the quotient inappropriately in relation to those of the dividend.

Mat, clearly needs major remedial work in division to learn the sub-procedures involved.

Rick:

The only question Rick answered correctly was 1DQ. His procedures for answering the other questions were very inadequate.

Both Mat and Rick need drill in the division facts. They would probably benefit from using manipulative materials like dividing plastic bread ties into egg cartons. They should, under teacher monitoring to correct faulty procedures and frames, slowly advance from small numbers to larger numbers and decimal numerals. This would probably be best done for about 15 minutes per day until they have achieved success.

Nan:

Nan's sub-procedure involving decimal placement in the quotient indicate she needs remedial work in this area. All of her mistakes involved this one problem. Working on several questions under teacher supervision and direction would probably remedy this problem.

Summary and Implications of Division Analysis

As can be seen by the preceding discussion, the artificial intelligence conceptualization can be used to help teachers disclose specific problem areas in division. There are many sub-procedures called upon by the division super-procedure that often require remedial work. Assessment of each sub-procedure could be done by giving several questions of one type which would require that sub-procedure to be used. For example, several questions could be given in which the quotient has zeros as place holders. Any students getting these questions incorrect could then be given remedial teaching. Of course, further interviews would be required to determine if it is just a faulty unit sub-procedure that is causing the problem or an inaccurate frame for the functions of zero in the decimal numeration system. Questions like the following could be asked:

a) Which is greater 2.08 or 2.8?

b) Convert each digit of 306.502 to a fractional number to show its value.

c) What has the greater value, the zero in 206 or the zero in 3.02?

If the questions were incorrectly answered then the teacher would have to explain to the student what the various places mean in the decimal numeration system and what function zero plays. Explanation at first should involve only whole numbers then numbers with digits after the decimal point could be included. Once the student understood this, division questions with zero in the quotient could be given and worked through slowly. Similar tests could be given for the other sub-procedures involved, estimation of partial quotients, movement of decimals in divisor and quotient and checking the answer using multiplication.

Time, of course, would be involved in construction, administration and correction of the tests and in provision for remediation. It would be wrong, though, to teach the curriculum at hand assuming the students have an adequate knowledge base with which to handle it.

Generalized division drill, with questions involving movement of decimals, quotients with zeros, and divisors greater than dividends should be given and corrected by the teacher in order that all of the sub-procedures are maintained. Class correction to find answers only would allow important mistakes to be bypassed unless students were required to fully analyze their mistakes.

ANALYSIS OF PROBLEM PROTOCOLS

Introduction

This section discusses the analyses of the problem protocols and implications of these analyses for teaching. Each student's protocols have been analyzed separately and recommendations have been made for remedial work for each student. A summary and general recommendations follow the individual analyses.

Decisions have been based on clinical judgement following repeated examination of all of the protocols for each subject. A sketch of each subject's cognitive structure for rate and percent has been inferred from his or her explicit behavior as recorded on the work sheets and the transcripts of the protocols. A few samples of protocols have been included and references have been made to others in order to illustrate procedures used by each subject. This sketch not only illustrates the viability of the artificial intelligence conceptualization for analysis but is also used to make recommendations regarding remedial work for each student.

To give the reader an estimate of the students' abilities, the analysis for each student begins with the number of problems falling into each stage of CL 3.2: Clinical judgement as to the stage of a student's approach to solving specific problems. These were discussed in Chapter III and are outlined below.

Stage CL 3.2A   Student was unable to solve the problem.

Stage CL 3.2B   Algorithmic stage using VMS which may have occasional errors.

Stage CL 3.2C   The stage of comprehension. The student can discuss how and why the pieces fit together.

The problem sets are in Appendix A pages 122 - 126.

In the following discussion several references will be made to "guidance" and "directed questioning". Both of these would exist in a one to one situation and would be determined by the specific situation. Guidance refers to watching the student as he or she solves a problem and pointing out where mistakes have been made and making suggestions for corrections. Directed questioning refers to asking questions of the student in order that he or she focuses on the pertinent information, retrieves appropriate frames and uses procedures and sub-procedures correctly.

Analysis

Don:

Complete protocols for Don are in Appendix D (pp. 182 - 189). Don was at stage CL 3.2A for 4 problems, stage CL 3.2B for 5 problems and stage CL 3.2C for 17 problems.

His typical response to a problem after reading it (visual input) was to pause for a moment (while he searched for and retrieved a frame) and then go into a description (meta-analysis) of his solution. The comments above, in parentheses are artificial intelligence terms inferred from the behavior.

The solution process sometimes involved a visually moderated sequence.

Below is an example of a typical response.

\# 6c: Mary read 50 pages of a novel in 80 minutes. Lorne read 30 pages of the same novel in 40 minutes. Who is the faster reader?

1. D: If 40 minutes is half of 80 so if she went 50 pages in 80 minutes and then to make it equal, Lorne to equal it you times 30 by 2 and get 60 so Lorne read more pages.

Don has clearly retrieved a "comparison" frame, used it, and described what he was doing. The descriptors he has chosen for comparing (equal) and multiplying (times), however, are examples of CL 7.1, poor use of language.

Don solved 5b by setting up a proportion and solving it and then said, "An easier way of doing it would be, 300 is just three times as much so you could just times the 35 by 3. "In 6m he set up the correct proportion but instead of solving it said, "Then to take a short cut, 50 is half of 100 so then half of 51 is 25.5, instead of going through all the timsing and stuff so 25 and a half people have brown eyes." He later decided that either 25 or 26 would be correct. Both of these are instances of CL 9.11, solving problems using more than one method.

He changed methods in 3c also, but probably because he was monitoring his output. The first frame he called up was inappropriate, resulting in an incorrect answer. Calling up the wrong frame is an example of Behav. 3.0. He seems to have monitored his work (CL 9.6), and decided to use another method. He used a VMS (CL 15.3) in S2 and solved the problem correctly. Each number he generates in S2 seems to determine his next step.

3c: The population of a town increased from 7500 to 9000. What was the percent increase in the population?

1. D: (Writes 7500/9000 = x/100, cross multiplies) so 9000x = 7500 and you just cross off the three zeros from the 9000 and three from the 750000 so 9x equals 750. It's 83.3 and the 3 will keep on going.

2. D: Oh I could do it another way, if 7500 is increased 1500 then 7500 is 100% and...and 1500 goes into 7500 five times so 5 into 100 goes 20 and it's increased once more so it's increased by 20%, the population.

In 3b, also a percent increase problem, he monitored his output S1 but did not monitor his second output S7, getting an incorrect answer. There is a possibility he was monitoring his output but no evidence for it. It seems he does not have an adequate frame for percent increase with which to monitor.

There is a possibility he was not using 3.6 as the "base" because of CL 24.5B, critical reliance upon an incorrect or inappropriate assumption. He seems to have been assuming that the base has to be greater than the first component. He, therefore, didn't consider using 3.6 as the base.

Even though he got the wrong answer and did not realize it, it seems in S7, that he was trying to show, an instance of CL 8.6A, that his answer was correct and unique.

3b: A baby weighed 3.6 kg at birth and 10.8 kg at the end of one year. What was the percent of increase in weight.

1. D: It would be 3.6 over 10.8 equals n over 100 and then you cross multiply and get 10.8x equals 36. So it's 10.8 into 36 and you have to move the decimal so it's 108 into 360, divide (and gets 3) and it would go about 2 times but then to average there is still just point 1 here so n would be about 3.3. Oh, percent uh I started this out wrong.

2. I: You are going to start again?

3. D: Yeah, 3.6 over 10.8 and n over 100.

4. I: That's the same as you had before.

5. D: Yeah, let's see 10.8n equals 36 uh...

6. I: Let's just leave that as it is. How much did the baby increase in weight?

7. D: Oh, now I know what to do. You'd minus 10.8 and 3.6 so that would be 7.2. So that would be 7.2 over 10.8 equals n over 100, so that would be 10.8 into 72. Move the decimal over, so it's 108 into 720. It goes 6 times

(multiplies and subtracts and gets 820 remainder) so it's about 67% increase. But then it was 3 uh 33 before so then you could just minus that from 100 and then you would have got 67 and the increase is 67%.

In problem 1c, Don at first retrieved the wrong frame for conversion of kilometres to miles. He then monitored his output and retrieved a different frame for solving the problem, one that partially matched the situation (CL 15.5). He failed to monitor the output this time resulting in a random error going unnoticed (Behav. 17A). He said he would double 288 "then take off a bit" but he forgets to double it.

The use of .6 rather than .625 would be an example of incorrect variable storage and retrieval. Of course, for this problem 1.6 should have been used, based on the information given in the problem.

1c: The distance from Calgary to Edmonton is 288 kilometres or 180 miles. The distance from Edmonton to Saskatoon is 350 miles. What is the distance in kilometers from Calgary to Saskatoon via Edmonton?

1. D: To get miles to kilometres you are supposed to times it by point 6 so times 350 miles times point 6 and that's 210 so 210 kilometres from Edmonton to Saskatoon. Then you add 288 kilometres that it takes to get from Edmonton to Calgary and then the 210 kilometres that it uh... (long pause) oh it can't work out because there are more kilometres than miles up here (288 and 180) so it would have to be uh try a rate pair...180 miles and you just double it and it's 360 so you could just double the 288

then take off a bit so it would be...270 (adds 270 and 288) about so 558 kilometres from Calgary to Saskatoon vis Edmonton.

In 2c Don retrieved the frame for rate problems an instance of CL 15.6, recognizing a problem as a specific instance of some general frame. During division, however, (CL 15.3, use of a VMS sequence for part of the problem) he made a random error, Behav. 1.7A. He was monitoring his output, however, and switched to the use of a VMS sequence to solve the problem, again CL 15.3.

In his second solution, S3, Don exhibits CL 14.2, representing a quantity as some sort of "block" amount that can be treated cognitively as a single entity. He has represented 60 minutes as one hour and used the one hour as an entity.

2c: David delivered 60 newspapers in 75 minutes. How many can he deliver in an hour and a half?

1. D: If he delivers 60 in 75 minutes and so you put 60 over an hour and a half is 90 minutes, no, over 75 and it equals how ever many paper in 90 minutes (has 60/75 = x/90) cross multiply so 75x equals 5400. (divides and gets 61.2, multiplies 75 by 61.2 and gets 4590.0.)

2. I: So you divided and got 61.2 and multiplied to check and got 4590 so you think you did something wrong?

3. D: Yeah, so I'll divide again. If I take 75 minutes, that's 5 quarters of an hour so you divide 5 into 60 and get 12 so then you need one more quarter of an hour so then he'd be able to do 72 papers.

In 4b, also, he made a random error; but, because he was monitoring his work, he corrected it and got the right answer.

In 4a and 6a, Don found something he could do and did it (CL 13.3 iv). The "something" was not correct and he was successful in neither case.

Don seems to need little remedial work. Practice solving more problems is recommended. He should be encouraged to approximate the answers before starting a problem. This would help him monitor his work.

He should be encouraged to continue finding other methods of solution but should also be more critical of his mistakes and find logical reasons for the ambiguities arising from his mistakes. For example, in 2c, he was dividing 75 into 5400. His first digit in the quotient was 6 rather than 7 resulting in the remainder for that step being larger than the divisor. He checked his division using multiplication and found the quotient to be wrong. He did the division again, repeating his earlier mistake. Rather than finding the reason for this ambiguity he switched to an entirely different method of solution. This was practical in this case; but if he could not have thought of another method of solution, he would have failed to solve the problem.

Don should also be encouraged to take just a little more care since he made some trivial errors that easily could have been avoided.

Rick:

Rick was at stage CL 3.2A for 22 problems and at stage CL 3.2C for the remaining 4 problems.

Rick did not try most of the questions, saying something like, "I don't get this" or, "This is too complicated".

In 3c(i) he arrived at the correct answer but used an incorrect procedure. An example like this shows that interviewing is superior to merely requiring written answers.

It could be that Rick, using a VMS noticed that 31 is about half of 60 so he found something he could do and did it, CL 13.3(iv); he found half of 60. Again, using a VMS, he added 1/2 to 30 1/2 because the 30 1/2 didn't seem reasonable. The numbers he got seem to determine what he did next.

3c(i): 15% of the students in grade 8 are honor students. If 30 of the 61 students are boys, how many are girls.

1. R: 31 would be girls.

2. I: How did you get that?

3. R: Well it says 15%, and if there is 61 people and half of that is 30 1/2 and it says only 15% there and well half of 30, I don't know. I just took half of 61 and added 1 'cause it ain't even.

S1 of 3b indicates he had some frame for rate but certainly an inadequate one. This is an example of CL 24.2, vagueness of new material. This could also be an instance of CL 8.12, the expectation that learning math consists of building VMS sequences and nothing more.

3b: A baby weighed 3.6 kg at birth and 10.8 kg at the end of one year. What was the percent of increase in weight?

1. R: I don't understand these because our teacher told us to do these we should set up a rate pair to compare two things and this question doesn't have two things. It just has the baby's weight.

Rick solved 3b(i) correctly but only after using a VMS to search for data which might fit a "rate" assimilation paradigm (assimilation paradigm, rather than frame because of the vagueness).

His descriptor "dividing" used in S1 is inappropriate to describe what he actually did in S3.

3b(i): In the student union election for president, Betty, Jim and Lorne were running. Of a total of 400 ballots cast, Betty got 110 votes and Jim got 220 votes. How many votes did Lorne get?

1. R: This is too complicated, it seems like a dividing question instead of rate pair.
2. I: Well I didn't guarantee that they would be rate pair questions. Can you solve it some other way?
3. R: Well if you subtract 220 from 400 and 110 from that answer then that is how many votes Lorne got.
4. I: What is the answer then.
5. R: He got 70 votes.

Rick solved 5b(i) by retrieving frames for 50% and "about" and used them to solve the problem but did not have much confidence in his solution. Poor use of language (CL 7.1) is exhibited in S11 where he said "on an average" rather than approximate.

5b(i): Approximately 50% of the students in grade 8 have brown eyes. If there are 51 students in grade 8, about how many have brown eyes?

1. R: This isn't a rate pair question, I could do it without doing a rate pair.

2. I: Okay.

3. R: Well just divide it by half. 50% is half so if there are 51 students then divide it by 2 and get 25.5. So about 25 students have brown eyes. It's easy. That can't be right though.

4. I: Why, because it is so easy?

5. R: Yeah.

6. I: Does that answer make sense?

7. R: Yeah.

8. I: You think it is right then?

9. R: Yes.

10. I: Would 26 be just as good?

11. R: Yeah, because it is on an average.

Rick solved 6c with no trouble.

6c: Mary read 50 pages of a novel in 80 minutes. Lorne read 30 pages of the same novel in 40 minutes. Who is the faster reader?

1. I: You've written down that Lorne would be the better reader, can you tell me why, Rick?

2. R: Well, if it took Mary 80 minutes to read 50 pages and half of that is 40 and if Lorne read 30 pages if he read 40 minutes longer which would be just as long as Mary read he would have doubled his pages which would have made him 60 and she would have had 50.

3. I: Okay.

4. R: Good?

5. I: Yes.

Rick needs much remedial work, certainly more than can be given in a regular classroom. A remedial teacher would have to work with him closely monitoring his work and correcting misconceptions as they are discovered in further remedial interview situations. Reteaching would not be sufficient since it would not disclose the misconceptions he may have.

Jane:

Jane was at stage CL 3.2A for 7 problems, stage CL 3.2B for 4 problems and stage CL 3.2C for 15 problems.

Jane profited from instruction in this unit, she used addition rather than setting up proportions for three questions in set A. She did not make this mistake after instruction.

Monitoring her output helped Jane correct mistakes in two problems, 3c and 5b.

Jane made a mistake in her division to get 106 instead of 16.6. This could be classified as a random error in light of analysis of her other division questions. Had she got 16.6, however, the frame she had retrieved originally may have passed the adequacy test (see S3) and she may not have gone back and retrieved the proper frame. She failed to put the equals sign in her proportions.

3c: The population of a town increased from 7500 to 9000. What was the percent of increase in the population?

1. J: (Writes: 1500/9000 n/100 solves for n and gets 106.) That's wrong, it can't be 106 out of 100.

2. I: Why not?

3. J: 1500 increase wouldn't be 106% increase because it's only a small amount out of 9000.

4. J: (Writes 1500/7500 n/100)

5. I: Where did the 1500 come from?

6. J: Well that's 7500 to 9000 is 1500.

7. I: Are you putting it over 7500 just because you didn't like the answer you got?

8. J: Yeah.

9. I: Well which one makes more sense?

10. J: The 7500 because 1500 out of the 7500 because there isn't 9000 yet. (Solves and gets 20%).

Had her first solution in 5b been a whole number the frame may, again, have passed the adequacy test.

Both 3c and 5b are instances of CL 13.3(v), try something even if it's wrong, then try to learn from what happens.

5b: Mary's mother received 35 cents an hour as a babysitter when she was a teenager. Mary received 300% of what her mother received. How much does Mary earn per hour when babysitting?

1. J: (Writes: 300/100 35/n, solves and gets 11.6) I'll start again.

2. I: Why?

3. J: Because I've got the n wrong. (Writes 300/100 n/35, solves and gets 105.) (Writes: Mary gets $1.05 per hour.)

Lack of monitoring her output prevented Jane from arriving at the correct solution for 1b. She retrieved an inappropriate frame Behav. 3.0. She retrieved a frame for rate rather than for area.

1b: A room is 6 metres by 4 metres. If the owner wants to make the area of the room twice as large as it now is, and adds 2 metres to the length, how much will he have to add to the width?

1. J: (Writes: 4/6 n/8, solves for n and gets 5.3.) n equals 5.3.
2. I: So does that answer the question?
3. J: He will have to add 5.3 metres.

Jane's frame for the meaning of the "equals" sign and when it should be used seemed to be incomplete. In only one of 11 instances did she use the equals sign in a proportion; in 10 it was ommitted. In two instances, she used the equals sign when it should not have been used. Jane's work should be monitored by the teacher and incorrect use or lack of use of the equals sign should be corrected and explained each time it occurs until it becomes habitual for her to use it properly.

Jane should be encouraged to approximate her answers prior to solution to facilitate monitoring output.

Practice with ratio and percent problems rather than remedial work seems to be what is required for Jane.

Brett:

Brett had 16 solutions at stage CL 3.2A, and 10 at stage CL 3.2C. Brett's frame for area was not adequate in 1a. He said that the area is the width times the length times the width times the length. In 1b he failed to call the frame for area up when it was required. He probably needs to review area concepts and do some associated problems.

In 4a Brett picked a possible answer and decided it was reasonable, although it was not. The correct answer was 31.66 and he picked 43. In 5a, also, he picked a possible answer and tested it to see if it would fit the problem. He is exhibiting CL 14.2, representing a quantity as some sort of "block" amount that can be treated as a single entity. The "40" is being treated as a single entity in S13.

Brett needs more exposure to ratio problems and guidance as to how to set up proportions for these. Practice in simply setting up proportions for problems would probably be helpful.

5a: If John got 80% on a test and had 30 questions correct, how many questions were on the test?

1. B: If John had 80% on a test and had 30 questions right... if he had 20 more percent right then he would have 100% then he'd have to have more questions correct so I'd bring it up 20%.

2. I: Uh hu.

3. B: That would mean I'd have to add another 10.

4. I: Oh, and...

5. B: That would add up to 40 questions on....

6. I: Oh, I see, uh hu.

7. B: But then if I took. If I had 40....

8. I: Uh hu.

9. B: If I took, if I had 40 and then I had to get 100 out of uh percent to get 80%...

10. I: Out of 40.

11. B: Get that and see if I get 30.

12. I: Oh, I see, you're sort of picking 40 and seeing if uh 30 would be 80%.

13. B: Yeah, 40 would be 100%.

14. I: Uh hu.

15. B: And what would equal 80%?

16. I: Of 40?

17. B: Then every 10 questions would equal 20%.

18. I: Um hm.

19. B: That wouldn't be right either.

20. I: So you think the answer is about 40 eh?

21. B: Um hm. (divides 80 by 30).

22. I: (after a long pause) Now what are you figuring?

23. B: Well I'm just seeing what number I could times this with 80 and get 30.

24. I: Um hm.

25. B: Get twice remainder 2.

26. I: So you're wondering how many times 30 goes into 80?

27. B: Yeah.

28. I: Uh hu.

29. B: So get 30 times 2 equal 6, 20 left over be 6 times and 2 left over the 6 would just keep on going on.

30. I: The 6 would keep on going.

31. B: Oh 26 remainder 2 (writes: 26.2).

32. I: Oh so 26 with remainder 2 is just 26.2.

33. B: Um hm. (looks at pile of problems left to do).

34. I: Are you wondering how many more you have to do?

35. B: Yeah.

36. I: (laughs) Just three.

37. B: If you took 26 over 80. If I added 26.2 to 30 I'd get 56.2 and times it by....divide it into 80 whatever I'd get thats how many questions I got.

38. I: 56.2 into 80 and that's how many questions.

39. B: Yeah.

40. I: About how many times do you think that would go?

41. I: (after a long pause) Would you like to just leave that one and go on to another one?

42. B: Yes.

Brett's sub-procedures of putting the percent over 100 and for solving proportions were both correct, as indicated in 3b. However, he does not have a frame for percent increase problems. He should be exposed to illustrations of solutions for this type of problem so he could use them as examples to follow.

3b: A baby weighed 3.6 kg at birth and 10.8 kg at the end of one year. What was the percent of increase in weight?

1. B: (Writes: n/100 = 3.6/10.8, solves and gets 33.3). 33 repeating itself would be the percent.

2. I: Why did you in your original proportion put n over 100.

3. B: Well when you have percent it's a number over percent so it would be 100% and then you'd have 3.6 over 10.8 and then you multiply these two and then divide.

4. I: So you cross-multiplied and divided?

5. B: Yeah.

Brett solved 4c with no problem and used it as an example to help him solve 5c. This is an illustration of CL 9.7B, using an illustrative example and using parts to match the probl[illegible] at hand. The interviewer suggested he do this. Since Brett could use an illustrative example here, he could probably do the same for percent increase problems.

5c: John earns $4.00 an hour. John's father earns 400% of what John earns. How much does John's father earn?

1. B: This is the kind I don't know whether to put the n on the top or the bottom. I guess I put it on the top.

2. I: Why the top?

3. B: Because I want to find out what John earns, not what his father earns so it's on the bottom. This 400% would be compared to the 4 dollars. Because that's how much money his father earns.

4. I: Look at 4c and see if it will help you. You were finding 6.5% of 2000, and now you're finding...?

5. B: 400%

6. I: Of....?

7. B: Of $4.00.

8. I: Can you use 4c to help you?

9. B: (Writes 400/100 and n/400.)

10. I: Can you tell me why the n goes on the top?

11. B: Because this is percent comparing to 2,000 bushels and this is percent comparing to $4.00.

12. I: So you're just putting them in the same positions?

13. B: Yeah, (Solves for <u>n</u> and gets 1600.) So John's father would earn $1,600.00.

14. I: $1,600.00?

15. B: No that's not right. $16.00 an hour.

16. I: Why did you change it to $16.00?

17. B: Because those two zeros wouldn't matter.

18. I: Why not?

19. B: Well the two zeros in the $4.00 to make it 400¢.

20. I: Oh, so you just moved the decimal to make it dollars.

21. B: Yeah.

In all Brett seems to need some more experience setting up proportions from rate and percent problem. Probably encouraging him to use illustrative examples and seeing parallels would be best. Of course he should be encouraged to analyze the illustrative example in order that his behavior is at least at level CL 9.7C (Attempting to understand the illustrative example, and to use this understanding to solve the assigned problem). He could then be encouraged to analyze problems given before referring to the illustrative example (CL 9.7D).

Brett's work should be monitored in order that correct use of the equals sign is encouraged.

Mat:

Mat was at stage CL 3.2A for 16 problems, stage CL 3.2B for 4 problems and stage CL 3.2C for 6 problems.

Mat, also, fails to use an equals sign in his proportions.

Mat exhibited CL 24.2, vagueness of new material, in most of his work. He failed to "put the original" across from the "100" in percent problems. In 2b he put "100" in his proportion, although it wasn't a percent question.

The teacher would have to go through some illustrative examples slowly and analytically with him in order to clear up any misconceptions he may have. This should be followed by his setting up proportions for problems under teacher guidance and directed questioning. Once Mat had a proportion set up, his sub-procedure for solving the proportion was correct. This sub-procedure even called up a reducing sub-procedure; he reduced 15/100 to 3/20. Therefore, no remedial work is indicated here.

In 3c(i), below he failed to focus on the relevant information, a negative instance of CL 14.2. This type of behavior, again, could probably be corrected by his reading and interpreting the problem under teacher guidance in order that the ambiguities in the "natural language" in the problem are sorted out by directed questioning by the teacher. His sub-procedure of putting the percent over 100, however, was correct, as shown in S5.

3c(i): 15% of the students in grade 8 are honor students. If 30 of the 61 students are boys, how many are girls?

1. M: (Sets up several rate pairs).

2. I: Okay, that's far enough Mat. You've set up a rate pair here 15 over 100 equals 30 over 61 and you reduced 15 over 100 to 3 over 20 and you cross multiplied and you got 183 and 600 and now you're dividing 183 into 600 and you get 3.27. Now does that 3.27 answer the question?

3. M: No.

4. I: It doesn't make much sense eh? Why did you put 15 over 100 and 30 over 61?

5. M: I put percent over 100 percent.

6. I: Okay.

7. M: And 30 over 61 because it would be uh out of uh 30 of the students are boys this other part I'm trying to figure out.

8. I: Trying to find out how many were girls. Uh usually when you set up a proportion, one of the rate pairs has an unknown in it. Neither of these do.

9. M: I should put n over 60.

10. I: And what would that n stand for.

11. M: How many girls.

12. I: Would that relate back to this first sentence. If you had 15 over 100 equals n over 61, this 15% of the students in grade eight are honor students...does the number of girls have something to do with the number of honor students?

13. M: (Shakes head.)

14. I: You're shaking your head no?

15. M: I don't know.

16. I: Umh, read the question again carefully, Mat.

17. M: (Reads the question again. Writes down 15/61 and n/30.)

18. I: Okay, now you've got 15 over 61 and n over 30. What does the 15 stand for?

19. M: Percent.

20. I: Percent, and what does percent mean?

21. M: Over 100.

22. I: So does that 15 over 61, does that take care of that percent then?

23. M: No.

24. I: No, so something seems not quite right.

25. I: Could you solve this problem Mat by just reading the last sentence? Read the last sentence.

26. M: 31.

27. I: Does the answer change using this part of it? Will the number of girls change because of this 15% are honor students? Does this first sentence have anything to do with the problem?

28. M: No.

29. I: So what is the answer to the problem?

31. M: 31.

32. I: 31, why did you use rate pairs to solve the problem?

33. M: 'Cuz I was trying to figure out what percent was girls.

In 4b, below, Mat called up a frame to find 8.5% of 42,000 but it seems he lost sight of why he was finding it. This is an illustration of CL 6.12B, losing sight of the purpose of some sub-goal. Another possibility is that he did not have a frame for percent increase problems.

Mat first divided 8.5 into 42,000. He called up a frame which partially matched the situation, CL 15.5. He knew he wanted a number smaller than 42,000 for the amount of increase (S1 - S4). Statements S6 - S9 exhibit his frame for the relationship between division and subtraction.

After the discussion, S10 - S18, he set up a proportion to find 8.5% of 42,000 but didn't seem to have a clear goal in mind, a negative instance of CL 6.12A. His procedures for setting up the proportion and solving it were correct, but a sub-procedure fails. He cross multiplied and correctly got 357,000.0; but on dividing by 100, he crossed off the last two zeros, the one to the right of the decimal point being one of them. This is an instance to illustrate Conjecture 3.7, when a task has been performed, it may be that what is stored in memory is the record of what procedures have been used. It seems he remembered merely that one crosses off two zeros when dividing by 100.

With some coaching by the interviewer he monitored his output, S28 - S44, and decided his answer was not correct S44 - S49 and gave a good approximation of what the answer should be.

4b: Mr. Brown's salary increased this year by 8.5%. His salary last year was $42,000. What is his salary this year?

1. M: (Divides 85 into 42000 and gets 495.1.)

2. I: Can you tell me why you are dividing 85 into 42,000?

3. M: To try to get to figure out the salary uh what is his salary.

4. I: And will that give it to you Mat, dividing 85? Why didn't you multiply or add, why are you dividing, why are you dividing rather than...

5. M: Because, it divides is like it takes it brings the salary down to it would increase less.

6. I: Oh, dividing would bring the salary down, I see. Subtracting would also, so why didn't you subtract?

7. M: 'Cuz I'd have to keep subtracting.

8. I: Oh, you'd have to keep subtracting, how many times would you have to subtract?...Quite a few or...?

9. M: Yeah about 495 times.

10. I: Oh, 495 times okay. On what basis are you dividing, like

you didn't set up any equation there like your teacher has been teaching you. Okay what type of question is this Mat?

11. M: It's a rate pair.

12. I: It's a rate pair but what else uh what special type of rate pair is it?

13. M: Percent.

14. I: Okay, and your teacher asked you to set up what when you are doing percent questions?

15. M: An equation.

16. I: Okay and you didn't set that up. What about setting that up, or did you set that up in your head and then decide to divide by 85.

17. M: (Does not answer.)

18. I: Could you set up a proportion for that problem?

19. M: (Writes 8.5/100 and n/42,000.)

20. I: Okay you've got 8.5 over 100, why did you put 8.5 over 100 Mat?

21. M: Because (can't be understood).

22. I: Why is it 8.5 over 100 rather than _n_ over 100 or...

23. M: Because that's your percentage.

24. I: I see, 8.5 is your percentage. Okay why do you have _n_ over 42,000.

25. M: Because I'm trying to find his salary.

26. I: Uh hu. Why don't you have this the other way around? 42 hundred over _n_?

27. M: Because I'm going to multiply that and then subtract that and divide by 100.

28. I: Um hm, okay so you've got 35,700 for an answer Mat?

29. M: I'm going to move the decimals.

30. I: Oh you haven't done the decimals yet?

31. M: (Writes 35.7.)

32. I: Okay, now what does that...35.7 what?

33. M: Percent, like salary increase.

34. I: Okay, does that answer the question, Mat?

35. M: Shakes head no.

36. I: You don't know or you don't....I'm not sure if you're shaking your head because you don't know or because...

37. M: No, I don't know.

38. I: Okay, read the last sentence in the question and tell me what kind of an answer your answer should be. Should it be in percent, should it be in feet, should it be in pounds, what kind of an answer should you get?

39. M: (Points to the word salary.) [illegible] this.

40. I: What is his salary this year?

41. M: How much he made this year.

42. I: Okay, would he make it in percent?

43. M: No, uh...

44. I: So if you were doing that question on a test you would just put down 35.7% and go on to the next question?

45. M: Yeah, probably. It would be 35,700.

46. I: 35,700, okay. Now does that make sense. (N: No.) Why not?

47. M: Because it increased by 8.5.

48. I: Why isn't that okay then?

49. M: Because it should be higher than 42,000.

50. I: About how much higher then Mat, can you give me an approximate answer?

51. M: About 50,000.

In 1c, Mat seemed to have a partially correct frame for rates, CL 15.5, but it was far from adequate. To convert 350 miles to kilometres, he approximated by using some information given in the problem.

1c: The distance from Calgary to Edmonton is 288 kilometres or 180 miles. The distance from Edmonton to Saskatoon is 350 miles. What is the distance in kilometres from Calgary to Saskatoon via Edmonton?

23. M: Well the 288 kilometres is 180 miles, right? So I took 350 and rounded it off to 500 kilometres.

Mat seems to need some more developmental work setting up proportions after reading the problem. He should work under teacher guidance just setting up proportions; the teacher could guide his thought processes through directed questioning.

Once he had the proportions set up he could solve them; therefore, no remedial work is required here. He would probably profit from practice though.

He also needs work with division, discussed earlier.

Sue:

Sue was at stage CL 3.2A for 15 problems, stage CL 3.2B for 2 problems and stage CL 3.2C for 9 problems.

In some problems Sue monitored her output and made corrections; in others she failed to monitor her output and got wrong answers. Problem 2c is an example of the former, 4c an example of the latter. She should be encouraged to consistently monitor her output.

In 2c, below, S1 indicates CL 24.2, vagueness of recently acquired information. In S5 she is using CL 9.6, monitoring own output.

2c: David delivered 60 newspapers in 75 minutes. How many can he deliver in an hour and a half?

1. S: I'm not sure how to do this.
2. I: Can you set up a rate pair to help.
3. S: (Writes: 60/75 = x/90, cross multiplies and gets 75x = 5400, divides by 75 and gets 74 remainder 50.)
4. I: Is that the answer?
5. S: No because you wouldn't deliver half a paper.
6. I: So that answer doesn't make any sense? Is it close to a reasonable answer?
7. S: Yeah, if it were 74.
8. S: (Subtracts 75 from 90 and gets 15.) There is 15 minutes difference and there are 14 papers so something is wrong but it is kind of reasonable..
9. I: Do you think you might have made a mistake in your arithmetic somewhere?
10. S: Yeah, probably in the division.

11. I: Is there a possibility you might have made a mistake in your multiplication?

12. S: Oh, yeah. (Writes down 75x = 5400, and solves for x and gets 72.) He delivers 72 papers.

13. I: So does that seem reasonable?

14. S: Yes.

The dialogue in 4c indicates CL 24.2, vagueness of new material. Even after prompting by the interviewer, she did not immediately change the proportion and when she did, she still did not get the correct answer. She did not question the reasonableness of her answer. Her sub-procedure for multiplying by 6.5 was faulty. She got 130000 and should have gotten 13000.0. This could be classified as Behav. 1.7A, a random error.

Statements S8 - S13 illustrate the potential ambiguity of visual inputs, Th. 1.3A. The percent sign (%) in mathematics is a precise symbol meaning a rate pair which has a second component of 100. Unfortunately it is often used with little precision in natural language.

This exemplifies the need for precise language useage at all times in order to avoid ambiguity. When one uses the word percent, one should mean exactly "out of 100".

4c: A farmer increased his crop yield by 6.5% by adding fertilizer. If his yield without fertilizer was 2,000 bushels per acre, what will his yield be with the fertilizer?

1. S: (Writes 1/2000 = x/6.5, then 6.5x = 2000.)

2. I: Okay, from that proportion, you get this eh?

3. S: (Writes 2000x = 6.5.)

4. I: Oh, it should be 2000x equals 6.5?

5. S: Yeah I did it backwards.

6. I: Can you tell me about this proportion?

7. S: It's 2000 bushels per one acre and then you want to find out what the yield is with fertilizer if he increased his crop yield by 6.5%.

8. I: I see. What does percent mean? What does this (%) sign mean?

9. S: Like part of.

10. I: Part of what?

11. S: Like how much he increased his crop yield.

12. I: Without refering to the question at all, what does 6.5% mean? What does that number mean?

13. S: I'm not sure.

14. I: Does it mean out of 100?

15. S: Yeah.

16. I: So 6.5% could say 6.5 per 100?

17. S: Over 100.

18. S: (Writes 6.5/100 = x/2000.)

19. I: So with that little hint you changed your proportion. What does this new proportion mean?

20. S: I want to find out how much his new yield will be like the 2000 bushels.

21. I: This x refers to what?

22. S: The increased amount.

23. S: (Cross multiplies and gets 100x = 130000, divides by 100 and gets 1300.) The yield with the fertilizer would be 1300.

In 1c Sue monitors her output; in S1 she says "Oh, in kilometres." She used the correct procedure to find how many kilometres in a mile, S3 and explains why, S7. She again monitors in S4 and switches to a different sub-procedure to find the answer. She seems to know what she is doing but analysis of similar problems would have to be done to be sure. She is at stage CL 3.2B for this problem since she is relying on VMS outputs before she decides what to do next.

1c: The distance from Calgary to Edmonton is 288 kilometres or 180 miles. The distance from Edmonton to Saskatoon is 350 miles. What is the distance in kilometres from Calgary to Saskatoon via Edmonton?

1. S: To get the distance from Calgary to Saskatoon, you'd have to add the two distances together and it comes out to 530 miles. Oh in kilometres.
2. I: Is there something in the question that will help you?
3. S: Well I could divide. (Divides 180 into 288 and gets 1.6).
4. S: (Divides 1.6 into 530, goes as far as the first digit in the solution (3) and stops.)
5. S: Multiplies 530 by 1.6 and gets 318. Adds 530 to the 318 and gets 848.
6. I: Why did you divide 180 into 288.
7. S: To find how many kilometres there are in a mile.
8. I: And then you divided 1.6 into 530.
9. S: Oh, that's wrong but I tried it anyway.
10. I: So does this answer make sense, 848?
11. S: Yes.

Sue's sub-procedures of cross-multiplying and dividing to solve proportions seems to work well. She should be encouraged to reduce before cross-multiplying.

Practice in setting up proportions under teacher guidance and directed questioning would help her.

Trina:

Trina was at stage CL 3.2A for 14 problems, stage CL [illegible] for 4 problems and stage CL 3.2C for 8 problems.

Trina had trouble in problem 5b and 6a because she failed to limit the input and focus on what was relevant, a negative instance of CL 14.1A. In 5b she used hour and should only have used the 35 cents and the 300%. In 6a she based her answer on the price per cookie rather than the price per gram.

In 4a, Trina realized percent has something to do with 100, but she is not sure what. In 4c and 5c, also, Trina exhibits CL 24.2, vagueness of new material. The new material being percent. The dialogue in 5c indicates more guidance and experience doing problems would probably clear up this vagueness.

4c: A farmer increased his crop yield by 6.5% by adding fertilizer. If his yield without fertilizer was 2,000 bushels per acre, what will his yield be with the fertilizer?

7. I: Would that be about right, if it is an increase of 6.5% would 13,000 be reasonable? How much is 6.5% of something, is it very much, or quite a bit or....

8. T: I think that would be about right because it would be 100% would be about 20,000 or 20,000,000 so 6.5% wouldn't be that much of an increase.

5c: John earns $4.00 an hour. John's father earns 400% of what John earns. How much does John's father earn?

5. T: If John earns $4.00 an hour, right? John's father earns 400% of what John earns, so out of $4.00 he earns 400% of it so that would be about that would be $4.00.

6. I: 400% of $4.00 is $4.00?

7. T: Yeah.

8. I: So they earn the same amount than?

9. T: Yeah.

10. I: Does that make sense in the question?

11. T: No, but I don't no if this means 4 times as much or just....

12. I: John's father earns 400% of what John earns. Can you replace this "of what John earns" (underlines it in problem) with some number?

13. T: n. It would be 400 over 100.

14. I: This 400 over 100, the 400 is the?

15. T: 400%. (Writes and solves 400/100 n/400, and gets 16.) His dad earns $16.00.

16. I: Does that make sense?

17. T: Yeah.

18. I: Is that 16, 400% of what John earns?

19. T: Well that is 4 times as much.

20. I: What is?

21. T: That's 400% of $4.00 because $4.00 would be 100% so you have to do that by 4.

In 2c Trina exhibited Behav. 1.3B, misreading own notations. She wrote 72 but said 75.

In 1c, Trina does not call up a rate frame to convert kilometres to miles. Instead, she uses addition. This could be regression under cognitive stress, CL 6.8. She realizes, in S3, that she does not have an adequate schema with which to convert miles to kilometres. This realization would be classified as 23.2, reasonable accuracy in judging whether one has an adequate schema for some problem type. She thus turns to heuristics and does something she can do; she adds. Doing something she can do would be classified as 13.3(iv).

1c: The distance from Calgary to Edmonton is 288 kilometres or 180 miles. The distance from Edmonton to Saskatoon is 350 miles. What is the distance in kilometres from Calgary to Saskatoon via Edmonton?

1. T: You have 180 miles and 288 kilometres so you probably just add 108 onto that (350). So you take the 350 miles and add 108 and get 458. 458 and 288 is 746. 746 km.
2. I: Can you tell me why you added those?
3. T: Well we didn't learn how to convert miles to kilometres so and up here it says 288 km equals 180 miles so that's 108 more than that so I just added 108 onto that (350).
4. I: Okay, and why did you add this 288 on?
5. T: When I converted the 350 miles to kilometres and then I have to add the two kilometres to see how far it was.
6. I: Oh the total distance?
7. T: Yeah.

Trina, also, did not use an equals sign in her proportions. This should be watched and corrected each time it occurs.

Remedial work for Trina should involve guidance and directed questioning by a teacher while Trina reads and interprets many percent and ratio problems. Practice with problems containing extra "noise" would probably be beneficial.

Cathy:

Cathy's complete protocols are in Appendix D.

Cathy was at stage CL 3.2A for 9 problems, stage 3.2B for 3 problems and stage CL 3.2C for 14 problems.

Cathy exhibited the same faulty sub-procedure twice. In 1c and 3c she multiplied by 100 by adding only one zero. Another faulty sub-procedure that resulted in incorrect answers was her "sloppy" placement of digits in the quotient relative to the dividend digits when dividing. Both of these problems could probably be remediated easily by the teacher's pointing out the correct procedure and monitoring Cathy's work until the correct procedure becomes habitual.

For 1c she converted the miles to kilometers using the ratio of 60 to 100 rather than 62.5 to 100. This is an example of CL 15.5, attacking a problem using a frame which partially matches the problem.

In 2a below, Cathy has trouble interpreting the 2/3. Her frame for rational numbers has to undergo some accommodation. She should have some explanation and drill converting fractions to decimal numerals.

In S6 there is evidence of some regression under cognitive stress, CL 6.8. She seems to be tempted to use subtraction but doesn't; she approximates instead.

In S8 she says 2/3 is around 66% which is correct, but she doesn't convert it to a decimal numeral.

2a(i): Mr. Big is 6 buttons tall while Mr. Small is only 4 buttons tall. If Mr. Big is 4 paperclips tall, what is Mr. Small's height measured in paperclips.

1. C: Okay, 2/3 and Mr. Big is 4 something. Mr. Small is 4/6 the height of Mr. Big so umh the common denominator is 28, I mean 24.

2. I: Of 6 and 4?

3. C: Yeah 24 and 6 goes into that 4 times so 4 does into it 6 times. Oh it's not working here...umh 2/3 (reduces 4/6).

4. C: (Trys to find 2/3 of 40) Okay if this is 2/3 then what is the percentage of the 4?

5. C: Okay, so this is 2 bigger than this so this is 2 smaller than would be...uh that wouldn't work...

6. C: This would be about 2 1/2 or....

7. I: Oh 2 and a little bit but you're not sure how to get the number?

8. C: No. Well I'll just say 3/4 or something, well it's around 66% or something like that.

9. I: Uhhu. Now what is your answer to the question?

10. C: Okay, Mr. Small's height is 3 paperclips, no 2 paperclips (laughs) 2 1/2 paperclips.

11. I: Somewhere between 2 and 3?

12. C: Yeah (Writes: 2 - 3)

In 5c(i) Cathy focused attention at first on only part of the input, the table and cloth and their measurements. She then focused attention on the fact that the cloth was centered. This is an example of CL 14.1, input limitation or focus.

She also exhibited CL 9.5, student habit of referring data to an alternative representation, by drawing the diagram of the table and cloth. She drew diagrams for 1a and 2a also.

5c(i): A table is 3 metres by 3 metres. How much of a square tablecloth, which has a side of 4 metres, will hang over the edge of the table? The cloth is centered on the table.

1. C: This 4 metres here, does it mean that it is 4 by 4 by 4 by 4?

2. I: Read the second sentence again. (C: Reads it aloud.) Does something in that sentence help you?

3. C: It just says it has a side of 4.

4. I: A side, and what kind of a tablecloth is it?

5. C: Oh, yeah...umh...(Draws a table with the cloth on it but not centered, the cloth covers the table and just hangs over two sides, she adds a bit on to the other two sides.) It would hang over a half a metres.

6. I: How did you get that.

7. C: Because if you have a table and you add on here (one side) that makes that four and here (the other side) and here that makes that four and here, that makes that four. And if the tablecloth is centered.

8. I: Oh, you don't have it centered?

9. C: No, so if you have it centered then half of this would be over here so half a metre.

She had a definite frame for percentages that are multiples of 100. In 5b (300%) and 5c (400%) she multiplied by three and four respectively.

In both 4b and 4c Cathy was aware she had no adequate schema for solving percent increase problems, an instance of CL 23.2. Some illustrative examples would probably remedy this situation.

Cathy needs some work done with her in division, as discussed earlier. Practice under teacher guidance of percent and rate problems would probably cause her frames for these two types of problems to undergo the required accommodation.

Liz:

Liz was at stage CL 3.2A for 5 problems, at stage 3.2B for 9 problems and at stage CL 3.2C for 12 problems. When Liz did not have a frame she could call up immediately she used CL 15.2, heuristic analysis. She usually used CL 13.2(iv), try something you <u>can</u> do and do it or CL 13.3(v), try something even if it's wrong and learn from it. She monitored her work and made adjustments where necessary. 3b is a typical example of this.

Although Liz incorrectly answered the question asked in S13, she used it as an example to help her solve the problem, CL 9.7. It is difficult to tell at what level of CL 9.7 this would be classified.

Liz exhibits CL 14.1 in S30, she has fixed certain aspects of the problem firmly in mind.

3b: A baby weighed 3.6 kg at birth and 10.8 kg at the end of one year. What was the percent of increase in weight?

1. L: Okay I'm going to just do this equation that our teacher taught us. Like n over 100, n means the percentage. Then I'm going to put 3.6 on the bottom and 10.8 on the top.

2. I: Why did you put the 3.6 on the bottom?

3. L: 'Cause that is what he weighed originally and you are going to have to find the increase.

4. I: And why is the 10.8 on the top?

5. L: Because that is what he weighed after. (Solves for n and gets 300, writes 300%)

6. L: This must be wrong.

7. I: Why?

8. L: Because if the baby weighed 3.6 at first and then weighed 10.8 at the end of a year, well that is only about 3 times as much. Like 30%.

9. I: 30%?

10. L: Well like 3 out of 4, something like that.

11. I: 75%?

12. L: Around there, yeah.

13. I: Could something increase 300%? If you started out with 2, what would an increase of 300% take the 2 to?

14. L: Okay...it would be 6.

15. I: Does that seem reasonable?

16. L: That 300% of 2 is 6? Yeah.

17. I: But you said this (the answer to the question) isn't reasonable , you said it should be 30 and then 75%.

18. L: I've just got to figure something out here. (Subtracts 3.6 from 10.8).

19. I: Why did you subtract?

20. L: Okay, that is the amount of increase.

21. I: So does that answer the question, what is that 7.2?

22. L: That is kilograms.

23. I: So now can you answer the question?

24. L: I've just got to think about it a minute.

25. L: (Writes n/100 = 7.2/3.6) 3.6 is the original and 100 is kind of the original and 7.2 is how much it increased and n is the percentage it is going to increase.

26. L: (Solves for n and gets 200% for an answer.)

27. I: So is that the answer?

28. L: Yeah either that or the 300%.

29. I: Which one would you vote for.

30. L: Well the 200 I guess. Because of the way that equation is. The 100 is kind of the original and the 3.6 is the original. The <u>n</u> is the percentage of increase and the 7.2 <u>is</u> the increase.

Liz's assimilation paradigm for percent was still undergoing some accommodation as can be seen in 3b above and in 5c, below.

5c: John earns $4.00 an hour. John's father earns 400% of what John earns. How much does John's father earn?

1. L: (Writes 400/n = n/4, solves for n and gets 16, puts a dollar sign in front of the 16.)

2. I: So $16 is you're answer? (L: Yeah.) Is that reasonable.

3. L: Well, yeah, I don't have any idea, I guess it is kind of reasonable.

4. I: Why do you say it is <u>kind</u> of reasonable?

5. L: Well I thought he would get more than that.

6. I: Why did you think he should get more than that?

7. L: Well because 400% of 4 seems like it should be more than 16.

8. I: What is 100% of 4?

9. L: 4.

10. I: What's 200% of 4?

11. L: It's 8 I guess.

12. I: You're not sure?

13. L: Yeah it's 8.

14. I: And what's 300% of 4?

15: L: 12.

16. I: And what's 400% of 4?

17L: L: 16, I guess that's (her answer) is right.

Liz got all but one of the rate problems correct. The one she failed to get correct was 2a(i) in the first set. Although she had the correct answer in 2a(i) she did not realize how to deal with the repeating decimal in 2.6. She then switched to using subtraction with "sort of" a ratio, an example of regression under cognitive stress, CL 6.8.

2a(i): Mr. Big is 6 buttons tall while Mr. Small is only 4 buttons tall. If Mr. Big is 4 paperclips tall, what is Mr. Small's height measured in paperclips?

1. L: Okay, I'm going to divide 4 into 6. I get 1 point 5.
2. L: Then I'm going to times 1.5 by 4. And I get...6 umh (laughs).
3. I: Lo and behold (laughs).
4. L: Yeah. I don't think that was right. Okay.
5. L: Oh yeah, okay, now I'm going to divide 1.5 into 4. (gets $2.\overline{66}$) Oh goody (sarcastically) this is going to be fun.
6. I: You don't like the 2,6,6?
7. L: I don't like that thing over there. (the bar over the 6).
8. I: Oh I see the repeating decimal. You think there is something wrong when you've got that.
9. L: Well no but I don't know how to figure out when I've got that. I don't know how to figure out the answer.
10. I: Oh, I see, What else do you have to figure out now?
11. L: I was going to find the difference between 4 and 6 in this one (points to 6 divided by 4).
12. L: Then, with that difference between them...I was going to times it by 4....
13. L: Now I'm going to subtract them (gets 2.5). Okay 2.5 so that means that he was 2.5 (refers to the question again) paperclips.
14. I: And that's your answer, 2.5?
15. L: (reads the question again) Okay.

Liz requires little remedial work. The accommodation required of her frames for rate and percent would probably be facilitated by just doing more problems.

Her work, of course should be monitored by the teacher in order that any misconceptions are detected and corrected early.

Hank:

Hank was at stage CL 3.2A for 16 problems, stage CL 3.2B for 2 problems and stage 3.2C for 8 problems.

In 6a Hank did not exhibit CL 14.1, input focus, he tried to solve the problem using the cost of the cookies and the number of cookies rather than the weight of the cookies. Possibly directed questioning would alert Hank to be more aware of what is relevant in problems.

His frame for percent was still undergoing accommodation. In 5b and 5b(i), respectively, he correctly said 300% is three times as much and 50% is half. However in 5c he said 400% of $4.00 is $4.00 and in 3C he said 1,500 equals 15%. In 2a(i) he said four is 75% of 6.

Hank's sub-procedure for setting up proportions for percent problems was faulty. He did not put the "100" across from the "original" in 4c and 5b. He should have practice doing this under direct guidance.

In 5b, below, he monitored his work, changed his strategy and got the correct answer.

5b: Mary's mother received 35 cents an hour as a babysitter when she was a teenager. Mary received 300% of what her mother received. How much does Mary earn per hour when babysitting?

1. H: (Writes 300/100 = 35/x, solves for x and gets 11.6)
(Multiplies 35 by 3 and gets 105.)

2. I: Why did you multiply by 3?

3. H: Well to get 300% more like 100% more would be another 35 so 300% more would be 3 times 35.

4. I: So what is your answer then?

5. H: $1.05.

In 1b, below, he did not call up the correct frame for area Behav. 3.0. Unfortunately the interviewer did not ask him what the original area was. It is, therefore, difficult to tell if he had a frame for finding the area of a rectangle or not. Assuming he had a frame for finding the area of a rectangle, S7 - S11 could be interpreted as regression under cognitive stress CL 6.8. He regressed to finding the perimeter. It could also be that he just had difficulty processing the natural language statement, the problem, and interpreting it into a mathematical context (As. 7.0, additional processing is required to extract the command information from any natural language statement).

1b: A room is 6 meters by 4 meters. If the owner wants to make the area of the room twice as large as it now is, and adds 2 meters to the length, how much will he have to add to the width?

1. H: I don't understand what they're trying to say.

2. I: Try drawing a picture it might help.

3. H: (Draws a rectangle and indicates it is 6 by 4...) He'd keep the width the same.

4. I: Would he?

5. H: Well he wants to make it longer not wider.

6. I: He wants to make the area twice as big.

7. H: Oh...(re-reads question). Yeah. He'd have to add more than two then if he wants to make it twice as big.

8. I: What would he have to add?

9. H: Well he'd have to add 6 and 4 on, 6 to the length and 4 to the width.

10. I: Oh, and he only added two on to the length. Well can you do that question then?

11. H: Well the only way he can make it twice as big is by adding 6 and 4 on. If it's only 8 how can it be twice as big?

12. I: So what would you say for that problem then?

13. H: I don't know, I guess I can't do it.

In S7 of 1c below, Hank displays regression under cognitive stress. He wants to take the difference rather than the quotient.

However, by S11 he is calling up a partially correct frame, CL 15.5. He is using a "sort of" ratio.

He has the aspects of the problem fixed firmly in mind, CL 14.1B. He realizes that he must triple the 288 to get an approximate answer. He also realizes that 288 was a little high because 180 is more than half of 350. Use could be made here of his partially correct frame by asking him to solve the problem his way and then suggesting that he set up a proportion to solve it. If he failed to set up a proportion by following directed questions like, "What ratio could you write in the first sentence?" and "What ratio could you write for the second sentence if the distance in kilometers is $\underline{x}$?" then the teacher could set up the proportion for him. This could then be used as an example for him to follow.

1c? The distance from Calgary to Edmonton is 288 kilometres or 180 miles. The distance from Edmonton to Saskatoon is 350 miles. What is the distance in kilometres from Calgary to Saskatoon via Edmonton?

1. H: All I can figure out is that it is 480 miles, I can't figure it out in kilometres.

2. I: Where did you get the 480 miles from?

3. H: The distance from Calgary to Edmonton is 180 miles and the distance from Edmonton to Saskatoon is 350 miles and I just added up the two.

4. I: And got 480.

5. H: Yeah, no (adds again.) Its 530 miles.

6. I: It's 530 miles, and you don't know how to switch it to kilometres. Is there anything in the question that would help you switch it to kilometres?

7. H: Yeah you could kind of take the difference between uh... it would be 108.

8. I: Oh, the difference between miles and kilometres.

9. H: Yeah, 108.

10. I: But you're not sure what to do with it?

11. H: It would be pretty close to double that because 180 plus 180 is 360 so it would be pretty close to triple that so it would be 288 and 288 and 288 would be pretty close to triple that but it would be a little less though because 180 is just about half of 350.

12. I: Why would it be triple then?

13. H: Because there are two of them in here (in the distance

from Edmonton to Saskatoon) plus 288 for Edmonton to Calgary.

14. I: So it would be about triple 288?

15. H: Yeah.

Hank's remedial work should make use of what partially correct frames he brings to situations. Guidance and directed questioning would facilitate accommodation of his frames.

Nan:

Nan was at stage CL 3.2A for 19 problems, stage CL 3.2B for 2 problems and stage CL 3.2C for 3 problems.

As can be seen, Nan was not very successful. It was not, however, for lack of trying. Her typical response to a problem was to find something she could do and do it, CL 13.3(iv). Unlike Liz, however, she was seldom successful.

She did not seem to question the reasonableness of her answers. In 4a she said "The customer had to pay 40%". In 4b she said, "His salary is 432.5%". In 5b Nan said that a babysitter would get 11.3 dollars per hour even though she only gets one dollar per hour. In 5b(i) she said that 102 students have brown eyes even though there were only 51 students under consideration. Nan should be encouraged to re-read the problem to make sure her answer makes "sense" and is reasonable.

In 3b she called up a frame for a rate problem rather than for percent and did not have enough numbers to complete the proportion. Her reaction to question 3b is typical of the frustration she seemed to experience in most of the problems.

3b: A baby weighed 3.6 kg at birth and 10.8 kg at the end of one year. What was the percent of increase in weight?

1. N: Okay, I put let x equal the percent of increase in weight.
2. I: Umhm.
3. N: Okay, now I compare time and kilogram. (Writes Time/kg)
4. I: Umhm.
5. I: Okay, now you said you'd compare time to kilograms and now you've got n over 3.6, can you tell me about that?
6. N: Well I don't understand I uh like it says that a baby weighed 3.6 at birth and I don't know how much at birth stands for.
7. I: Oh so it is the n is it?
8. N: Yeah....(She has written down so far: n/3.6 /10.8) I'm not too sure how to put it over. (I: Umhm.).... I know this (the blank over the 10.8) has to be something.
9. I: You mean something other than the n or the n is okay?
10. N: No it should be something other than the n.
11. I: Oh. Well maybe read the question again and see if you can get some more information.
12. N: (Reads the question again) I think it's not time it's weight I think...weight over kilograms. (Crosses out Time /see number 3/ and writes down weight.)
13. I: What is kilograms what does that measure?....Like what does seconds measure?
14. N: Time.
15. I: What does pounds measure?
16. N: Weight.

17. I: What does kilograms measure?

18. N: Weight, so maybe it should be weight over kilograms then.

19. I: Umh.

20. I: (After a pause) You don't know how to do it?

21. N: (Shakes her head no)

22. I: Okay do you just want to go on to another one? You gave it a good try.

Nan commented to her teacher after the sessions that the problems were very difficult.

Nan experienced many difficulties. She should be given extensive remedial work, starting at a very low level. More interviews would be necessary in order to assess her present abilities. Certainly, it seems unfair that a student should experience such little success in problem solving. She should be given manipulative experiences and simple problems that she could solve in order to build up her confidence.

Kurt:

Complete protocols for Kurt are in Appendix D (pp. 156-170).

Kurt was at stage CL 3.2A for 18 problems, stage CL 3.2B for 3 problems and at stage CL 3.2C for 5 problems.

At times Kurt monitored his work, CL 9.6, and questioned the reasonableness of answers and changed them; at other times he didn't. Questions 3a, 3c(i) and 4a are examples of the former. Questions 3c and 4c are examples of the latter.

In 6b he called up a frame for percent, set up and solved the proportion getting the correct answer. The sub-procedure for solving proportions functioned properly on other occasions also. He, however, should not have called up a frame for percent. This is an example of how interviewing discloses more than does just examining final answers.

6b: Mrs. Kane can reach a shopping centre from her home by driving 8 km east and 15 km north. Would she be speeding if the speed limit is 50 km/hr and she gets to the shopping centre in half an hour?

1. K: Umh, this is a tough one. They're not going to fool me on this one, I've got it. (Writes: 23/50 n/100, solves for n and gets 46). So I get 46 for an answer.

2. K: So would she be speeding if she were going 46 and the speed limit is 50? No she would not be speeding.

3. I: Why did you pick 100.

4. K: Well it's percent, I don't know it's just always done like that.

7b: A family had an increase of 10% in the cost of rent, but a 10% decrease in the cost of food. Explain why (or why not) the family expenses are the same as before.

1. K: Well they got first off 10% more and then second off they got 10% less so that equals off, 10% minus 10% is zero.

Kurt enjoyed little more success than did Nan. He did not seem to take it quite so seriously, though. He should however receive further assessment ~~and~~ extensive remedial work.

SUMMARY AND GENERAL IMPLICATIONS

Summary:

Examples of many of the theoretical entities, behaviors, observations and conjectures listed in Appendix C could be inferred from the behavior exhibited in the protocols.

The use of the artificial intelligence conceptualization was used to discover specific problem areas quickly and the use of descriptors for concepts in the conceptualization facilitated finding parallel problems in other contexts.

The stages, CL 3.2A - C were found useful in giving approximate descriptions of the students' abilities. They can not be considered precise, though, because there were often qualitative differences in solutions that were classified the same.

There were some instances where classifications could not be found to describe certain observations. These instances are listed below with suggestions for classifications.

1. Poor use of mathematical symbols could possibly be classified as Behav. 7.1B(i) while changing "poor use of language" to 7.1A from 7.1.
2. No use made of mathematical symbols at appropriate times could be classified as 7.1B(ii).
3. Poor written work, "sloppiness", could be classified as 7.1C.
4. Assessing the "reasonableness" of answers could be classified under CL 9.6, student habit of monitoring work, but the description of CL 9.6 would have to be expanded to include this. This could possibly also be classified under CL 8.4, the ability to justify a mathematical statement by reference to the real world.

There were several observations made which were common to many of the students. These are listed below and references made to the listing in Appendix C where appropriate.

1. Equals signs weren't used in proportions.
2. The words "timsing" was used instead of multiplying, CL 7.1, poor use of language.
3. Fractions weren't reduced before cross multiplying.
4. The reasonableness of the final answer was not assessed.
5. Written work was often "sloppy" and unorganized.
6. Proportions, once set up were usually solved correctly.
7. Diagrams were used as an alternative representation, CL 9.5 use of an alternative representation. They often, though, did not seem to help.
8. Many students, when unable to call up an appropriate frame, exhibited CL 13.3(iv), find something you can do and do it. This ocassionally proved successful, depending on how they monitored their results.
9. Many had problems with dividing, Behav. 1.5, what should be a reliable unit sub-procedure malfunctions.
10. Many students had inadequate frames for percent and rate problems, CL 24.2, vagueness in new information.
11. Unit sub-procedures involving multiplication, addition and subtraction functioned well.
12. The level of generality at which many of the students seemed to be operating indicates that their processing was based on assimilation paradigms rather than frames.

Implications:

> The gathering of qualitative data is but one step in progress toward understanding the processes of mathematical thinking. Methods of analyzing the data are needed, and more importantly methods for communicating the results to other researchers, and to classroom teachers must be explored. Finally, the ultimate goal is to find ways to use the data to improve classroom instruction and to positively affect mathematics learning and problem solving in the classroom. (Kantowski, 1978, p. 50)

The purpose of this section is to draw from the analysis general implications for teaching.

I. Perhaps teachers should break down a procedure to be taught into sub-procedures and make sure the sub-procedures are adequate before attempting to teach the super-procedure. "It is possible the basic skills have not been learned sufficiently automatically; it may be that they still require too much conscious thought, thus taking attention away from the decision-making of the super-procedure, which results in super-procedure errors." (Davis, et al. 1978, p. 25). The adequacy of the sub-procedures, of course, will depend on the adequacy of the frames for concepts involved in them.

The sub-procedures involved in rate and percent problems are:

A. Interpreting the problem into mathematical context

B. Setting up proportions

1. Getting the "original" across from the "100"

2. Having the same units across from each other; for example, hours across from hours

C. Solving the proportion

1. Reducing before cross multiplying
2. Cross multiplying
    a. Cross multiplying the proper terms with each other
    b. Multiplication of whole numbers, fractions and decimal fractions
3. Division
    a. Estimation partial quotients
    b. Moving decimal points
    c. Multiplying
    d. Subtracting
    e. Dealing with repeating decimals
    f. Dealing with remainders

D. Interpreting the answer in the context of what is asked

Certainly none of the above should be assumed to be known. Pretests should be given for B and C. A and D would be specific to the new super-procedure.

Of particular importance is the frame each student has for ratios and percent. This unit is the last unit taught in the late spring of Grade 5 and Grade 6. It is possible, then that it receives only superficial treatment.

Because of limited resources: materials, preparation time, money and instructional time; subject matter is often not developed in a way that is most beneficial to learning. Students are often given exercises that merely require a VMS sequence to be followed and nothing more. It is little wonder, then, that their perception of what mathematics involves is not correct.

The first introduction to ratio in grade five should probably not involve number manipulation at all but simple "hands on" experience drawing patterns, dividing construction paper off and making comparisons, dividing colored counters up and writing ratios for the different situations, measuring body parts and finding ratios between the various parts.

The students should be made aware of ratios in real life. For example: the faster the speed the longer the distance travelled, the more men working the greater the amount of work done, and the taller the object the longer the shadow. They should be encouraged to think of examples themselves.

In grade six the students could make scale drawings and models involving simple ratios. They could conduct experiments with colored water like: which mixture is "more green", the one with 3 parts clear water and 2 parts green water, or the one with 5 parts clear water and 3 parts green water? Why?

Using this developmental process, the students should come to realize that comparisons can be made between two things using ratios rather than just subtraction or addition. They should also come to realize when each type of comparison is appropriate.

By grade seven, the students should be able to deal with simple problems involving small whole numbers. They should be able to do group projects like planning a cross country bike trip based on how far they can bicycle each day.

The setting up and solving of proportions should be developed slowly with the students getting much experience in setting up proportions for problem situations. This could be done as a class with them justifying their proportions which requires analytical thought. The students should give reasons for cross multiplying and should be encouraged to reduce before they cross multiply.

Percents should be introduced as special kinds of ratios in which the second components are 100. Care must be taken introducing the percent sign in order that this visual stimulus not become ambiguous. The students should realize that 37 percent, 37%, 37/100, .37 and even 74/200 are all alternative representations of the same thing. They should be given examples of percents greater than 100.

In order to emphasize the usefulness of percent, a simple scientific or social studies report could be written using ratios with different bases and a parallel report written using percent.

Care must be taken in teaching the setting up of proportions for percent problems in order that the sub-procedure of putting the "original" across from the "100" is learned.

The students should be able to convert fractions to percent and should know some without having to convert, like 1/2, 1/4, 3/4, 1/3 and 2/3.

By grade 8 the students should be able to handle more complex ratio and percent problems and be introduced to percent increase and decrease.

None of the preceding should be assumed though and the teacher must continually monitor the students' work in order to spot faulty sub-procedures and evidence of inadequate frames.

Any remedial work done must take advantage of what the student knows but must also provide for accommodation of frames that are incorrect or inadequate.

II. Perhaps teachers should "think aloud" while solving problems rather than telling students how to do them. It would be more likely, then, that the students would file away procedures for solving the problem.

III. Perhaps students should be encouraged to approximate the solution before they actually solve a problem in order to facilitate monitoring the outcome.

IV. Perhaps <u>teachers</u> should correct students' written work rather than just read out the final answer for the students. This would facilitate early detection of where remedial work is required.

V. Perhaps teachers should supply lots of experience to the students in different contexts in order that the students' frames for concepts can go through the necessary accommodation.

VI. "Perhaps far more effort should be expended in teaching beginning students a useful set of descriptors capable of describing mathematical formula, concepts and other entities" (Davis and McKnight, 1979a, p. 81).

VII. Perhaps teachers should be more careful to make precise use of language and symbols and be ever alert for careless use of language and symbols by their students.

CHAPTER V

CONCLUSION

The artificial intelligence conceptualization discussed in this study is a valuable tool for analysis of problem solving. Such analysis can be used to make suggestions which may enhance instruction or offer guidelines for remedial work. "We can claim no direct access to another person's thoughts through language. The best we can do in talking about someone else's cognitive processes is to try to model those processes. This involves making an educated guess, but it is possible to become more and more 'educated' as one becomes more familiar with the particular person's behavior" (Clement, 1979, p. 65).

It was found in this study that repeated analysis of several protocols of students' problem solving facilitated the making of recommendations for remedial work for these students. It was also found that recommendations based on the analyses could be made for teaching strategies.

It was evident that students could not solve problems on a topic unless they had adequate frames (schemas) for the concepts covered by the topic. Skemp, 1971, comments on this:

> The central importance of the schema as a tool of learning means that inappropriate early schemas make the assimilation of later ideas much more difficult, perhaps impossible....
>
> An appropriate schema means one which takes into account the long-term learning task and not just the immediate one (Skemp, 1971, p. 51).

It was found that those students who had a meta-language for the topic were more successful. Skemp (1971) comments on this, saying:

> Once we have become able to reflect, to some degree, on our own schemas, important further steps can be taken. We can communicate them...We can set up new schemas...We can replace old schemas by new ones. We can correct errors in existing schemas (p. 57).

Recommendations for Future Research

The present study investigated the possibility of using an artificial intelligence conceptualization for analysis of problem solving. Suggestions were made for remedial work following the analysis.

The following recommendations have been made for future research:

1. It is recommended that studies similar to this one be conducted using various mathematics topics to determine if changing the topic will produce differing results regarding the viability of the conceptualization for analysis and remedial work recommendations.
2. It is recommended that a study be done to determine what thought processes are going on in the minds of successful problem solvers during the initial stages of interpreting the "natural language" of the problem statement to a mathematical context.
3. It is recommended that a study be done to see if "monitoring one's work" and "having a plan" can be taught or if they are related to maturation.
4. It is recommended that a study be done with mathematics education students to see if awareness of this conceptualization enhances teaching strategies.

5. It is recommended that a longitudinal study be made in which analysis based on the artificial intelligence conceptualization is followed by recommendation. The results of following up on these recommendations could then be assessed.

6. It is recommended that a study be done in which students are given "guidance" and "directed questioning" in order to assess if this improves problem solving ability.

The above recommended studies would further advance the field of artificial intelligence as applied to the learning and teaching of mathematics.

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APPENDIX A

PROBLEM SETS

In the following problem sets there are parallel questions. By "parallel" it is meant they require approximately the same solution process.

Other than questions 3b(i), 3c(i), 5b(i) and 5c(i), questions with the same numbers are parallel. Thus, questions 1a, 1b and 1c are parallel to each other, 2a, 2a(i), 2b and 2c are parallel to each other and so on.

Questions 3b(i), 3c(i) and 5c(i) were included because they were not rate problems but could be confused with rate problems because there were 3 numbers given and one unknown.

Question 5b(i) was put in to assess the subjects idea of "approximation".

PROBLEM SET A

1a. If a carpet costs $5.95 a square metre. How much will it cost to put it in a room which measures 4 metres by 5 metres?

2a. If the length and width of a rectangular photograph are 12mm by 5mm respectively, what is the width of the enlarged photograph if its length is 18mm?

2a(i). Mr. Big is 6 buttons tall while Mr. Small is only 4 buttons tall. If Mr. Big is 4 paperclips tall, what is Mr. Small's height measured in paper clips?

3a. Bill had $50 on Friday and earned $25 over the weekend. The amount he had on Monday is what percent of the amount he had on Friday?

4a. A store bought an item for $28 and marked the price up by 12% before selling it. What does the customer have to pay for the item?

5a. If John got 80% on a test and had 30 questions correct, how many questions were on the test?

6a. Which is the better buy, a 450 gm package of 24 cookies that costs $1.50 or a 300 gm package of 24 cookies that costs $1.10?

7a. Mary got 75 questions correct on a Social Studies test and only 30 questions correct on a Science test. Explain why and if she should get a higher percent mark in Social than in Science.

PROBLEM SET B

1b. A room is 6 metres by 4 metres. If the owner wants to make the area of the room twice as large as it now is, and adds 2 metres to the length, how much will he have to add to the width?

2b. A calculating machine gives the answer to a multiplication example in .02 second. At that rate, how many multiplications could the machine do in 2 seconds.

3b. A baby weighed 3.6 kg at birth and 10.8 kg at the end of one year. What was the percent of increase in weight?

3b(i).
In the student union elections for president, Betty, Jim and Lorne were running. Of a total of 400 ballots cast, Betty got 110 votes, Jim got 220 votes. How many votes did Lorne get?

4b. Mr. Brown's salary increased this year by 8.5%. His salary last year was $42,000. What is his salary this year?

5b. Mary's mother received 35 cents an hour as a babysitter when she was a teenager. Mary receives 300% of what her mother received. How much does Mary earn per hour when babysitting?

5b(i).
Approximately 50% of the students in grade 8 have brown eyes. If there are 51 students in grade 8, about how many have brown eyes?

6b. Mrs. Kane can reach a shopping centre from her home by driving 8 km east and 15 km north. Would she be speeding if the speed limit is 50 km /hr and she gets to the shopping centre in half an hour?

7b. A family had an increase of 10% in the cost of rent, but a 10% decrease in the cost of food. Explain why (or why not) the family expenses are the same as before.

PROBLEM SET C

1c. The distance from Calgary to Edmonton is 288 kilometres or 180 miles. The distance from Edmonton to Saskatoon is 350 miles. What is the distance in kilometres from Calgary to Saskatoon via Edmonton?

2c. David delivered 60 newspapers in 75 minutes. How many can he deliver in an hour and a half?

3c. The population of a town increased from 7500 to 9000. What was the percent of increase in the population?

3c(i). 15% of the students in grade 8 are honor students. If 30 of the 61 students are boys, how many are girls?

4c. A farmer increased his crop yield by 6.5% by adding fertilizer. If his yield without fertilizer was 2,000 bushels per acre, what will his yield be with the fertilizer?

5c. John earns $4.00 an hour. John's father earns 400% of what John earns. How much does John's father earn?

5c(i). A table is 3 metres by 3 metres. How much of a square tablecloth, which has a side of 4 meters, will hang over the edge of the table? The cloth is centered on the table.

6c. Mary read 50 pages of a novel in 80 minutes. Lorne read 30 pages of the same novel in 40 minutes. Who is the faster reader?

7c. Each of two employees of a company received a monthly increase in salary of $200. Was the percent increase in salary the same?

APPENDIX B

DIVISION QUESTIONS

The preliminary questions were asked to assess the degree to which the student possessed a meta-language with which to discuss division. The "completeness" of the students' frames for division was also determined by these questions along with the actual division questions they were asked to solve.

The division questions that required solution also tested the accuracy of various super- and sub-procedures.

DIVISION QUESTIONS

PRELIMINARY QUESTIONS

The following questions were asked orally. Depending on the degree to which the answer was supported by explanation, the interviewer went on to the next question or asked for an explanation.

1. Div. Can you tell me the relationship between multiplication and division?

2. Div. Can you tell me what the a, the b and the c are called in this? (Interviewer writes $c \div a = b$ in algorithmic form.)

3. Div. Can you write that (from question 2) using this ($\div$) sign?

4. Div. If (in $c \div a = b$) a increases and c stays constant, what does b do?

5. Div. If (in $c \div a = b$) a stays constant and c increases, what does b do?

6. Div. If (in $c \div a = b$) b stays constant and a increases, what does c do?

The questions on the following page were given as shown on the page. A further explanation was given for number 6 if the student did not seem to understand the question. The students were asked to try to do number 7 in their heads.

1. 1008 ÷ 56

2. 60 ÷ 4.3

3. 32 ÷ 429

4. 54.2 ÷ 26

5. 8.71 ÷ .37

6. Estimate the number of digits to the left of the decimal point there will be in the quotient and estimate what the first digit will be.

a) 8245 ÷ 85 b) 9356 ÷ 54 c) 24156 ÷ 732

d) 143115 ÷ 145

7. Divide 4328 by a) 10 b) 100 c) 1000.

APPENDIX C

SUMMARY OF THEORETICAL POSTULATES

OBSERVED BEHAVIORS AND CONJECTURES

This appendix is taken from Davis, Jockusch and McKnight (1978), pages 295-310. The pages in parentheses after each item are provided by Davis et al. (1978) for ease of reference to the body of their paper.

In order to make the summary more comprehensible to the reader. of this study, some comments, examples or quotations (from the text of Davis et al. (1978) have been added.

Davis et al. (1978) numbered the various entries in this summary after the section in which that entry was discussed. Thus, those entries with numbers close to each other are closely connected conceptually.

SUMMARY

Explanation of abbreviations used in the summary:

Th. : "indicates a <u>theoretical entity</u> (akin to 'electron' or 'ion') as opposed to, for example, an <u>observed behavior</u> (akin, say, to producing a black sediment at the bottom of a test tube). (Davis et al, 1978, P. 24)

Behav.: An observed behavior.

CL. : Clinical observations.

Conj. : Explicit conjectures.

As. : Assumption.

I. Commands and Procedures

Th. 1.0 A Commands

A command is a basic step in information processing, such as "Add 3 + 5". When we speak of a "command" we are treating the process as a single unitary entity. (Of course, earlier in the person's life this "command" was, presumably, built up from even more elementary processes; but at the present stage, it functions mainly as a single unit, and is not thought of in terms of its smaller component parts.) (page 24)

Th. 1.0 B Procedures

A procedure is a definite sequence of commands, intended to accomplish some definite goal. Procedures are often given names; a procedure may thus come to function as a single unified entity, in which case it has metamorphosed into a command. Thus, the list of commands in one person's reportoire is continually being extended. (page 24)

Th. 1.1 Sub-procedures and Super-procedures

Sometimes one procedure (call it P_1) activates another procedure (call it P_2) to accomplish some definite task. We describe this relationship between P_1 and P_2 by saying that P_1 is a <u>super-procedure</u>, and that P_2 is a <u>sub-procedure</u> of P_1. (page 25). "When a super-procedure calls upon a sub-procedure, it is often necessary for the super-procedure to give the sub-procedure certain specific instructions. For example: Find only positive values for the solution." (Davis et al, 1978, p. 25)

Th. 1.2 "Refinement" Procedures

The procedure P_2 is called a "refinement" of procedure P_1 if it was created as a result of trials of P_1, and is intended to improve the performance of P_1. (page 26) "Refinement procedures come into play during complex procedures. For example a boy beginning piano studies reported a conscious correction procedure for some specific relocation of his hand which he usually underestimated. He would, as he come to this point in playing, say to himself 'Remember, now, go a little bit further!'" (Davis et al, 1978, p. 26)

Th. 1.3 A Visual or Auditory Inputs

Visual or auditory inputs may be the stimulus to which a person responds by activating some procedure. (page 28)

Behav. 1.3 A

If two visual inputs are visually similar the more recently learned input will sometimes elicit the procedure which was learned first. (Thus, "4 x 4 = " will sometimes elicit the response "8", because addition is, usually, learned before multiplication.) (page 28)

Th. 1.3 B

Visual inputs that feed into an information-processing activity during the course of the activity (and not at the beginning). (page 28) [The partial products in 78 times 26 would be inputs of this kind.]

Behav. 1.3 B

Misreading your own written notations. (page 29)

Th. 1.4 Basic Visually-Moderated Sequence ("VMS")

This is a sequence initiated by some triggering mechanism, that has something of a periodic character: after a visual input I_1 some internal processing P_1 yields some written output I_2; this written output then becomes the visual input for the initiation (or shaping) of the next internal processing episode P_2, which then yields some new written output. The usual algorithm for long division would be a familiar example. (page 29)

Th. 1.5 "Unit" Sub-Procedures

This refers to some procedure that a student learned earlier in life, and now uses reliably. The importance of "unit sub-procedures" is that they allow us to terminate the dissection of a complex procedure at the level of reasonably complex, but reliable, components, without needing to trace our way back to the very primitive, very basic unitary procedures of (say) an infant. Thus we may accept "Add 3 + 4" as a unit, needing no analytical decomposition into its component parts. (page 30)

Behav. 1.5

What should be a reliable unit sub-procedure may, for some particular student, on some particular occasion, malfunction. (This has important diagnostic implications when it is observed to occur.) (page 31)

Th. 1.6

Let P be a procedure whose functioning has been judged unsatisfactory. To modify the performance of P, cues can be inserted at appropriate points, in order to call up sub-procedures S_1, $S_2, \ldots S_k$, and thus to modify the functioning of procedure P. Thus, a new procedure, P′, has been created: P′ results from inserting these correction cues into procedure P. (page 31)

Behav. 1.7 A

An observed behavior on non-systematic ("intermittent" or "random") errors. (page 32)

Th. 1.7 B

Newly-inserted cues may not function correctly until after a period of de-bugging. (page 32)

Behav. 1.7 B

Observed student errors where the timing, in relation to instruction, suggests that the student is going through a period of de-bugging newly-inserted cues that are intended to transform some defective procedure P into an improved procedure P′. (page 28)

II. A. I. Variables

Th. 2.0

Any symbol string may include variables - that is, spaces that will be filled in by the contents of some memory register. [The 'top' of a fraction is called the ________.] (page 34)

Behav. 2.0

Observable outputs of symbol strings (usually spoken or written) that contain errors apparently related to an incorrect variable entry, or to an incorrect variable retrieval. [The 'top' of a fraction is called the denominator.] (page 34)

III. The Representation of Knowledge in the Mind

Th. 3.0

It is often the case that some entity, stored in memory, can be retrieved and in some way made to correspond to some visual (or auditory) input. We call such an entity an assimilation paradigm. [See definition of assimilation paradigm in this paper.] (page 40)

Behav. 3.0

Selection of an incorrect or inappropriate assimilation paradigm [or schema (or frame).] [Example: 3 x 4 matched to assimilation paradigm for 3 + 4.] (page 40, 44)

Th. 3.1 A

At least some of the knowledge stored in a person's mind may be thought of as organized in the form of schemata in the sense of Rumelhart and Ortony. [Or of frames in the sense of Minsky (Minsky, 1975), (Davis and McKnight, 1979 b, 96.] (page 43)

Th. 3.1 B

Visual or other informational inputs may be thought of as mapping into schemata by "variable binding" in the sense of Rumelhart and Ortony. (page 51)

"The process of 'mapping reality into schemata' consists of three parts:

First, a memory search must lead to the identification and retrieval of some schema that is at least a plausible candidate to be the 'image' or 'range' into which the data might be mapped.
Second, the variables in this schema must be replaced by appropriate items from the input data or from elsewhere in memory (this is called 'binding the variables')....'Variable' always means something essentially similar to our idea of a memory-space that has a name....and contents....Basically, in its technical sense, a variable is a place where you store information.
Third, some process of evaluation must confirm the (at least reasonable) appropriateness of the resulting mapping."
(Davis et al, 1978, p. 43-44)

Th. 3.2

Schemata may be related by embedding as described by Rumelhart and Ortony. (page 55) [The schema for pupil is embedded in the schema for eye, the schema for eye is embedded in the schema for face, etc. Thus, the feet of a person are connected to that person rather than to the feet of a person standing beside him when the person schema is used]

CL. 3.2

Clinical judgement as to the stage of a student's approach to solving some specific problem:

Stage CL 3.2 A - Student unable to solve problem
Stage CL 3.2 B - "Algorithmic" stage (using VMS sequence) [may have occasional errors]
Stage CL 3.2 C - The stage of "comprehension," defined in terms of the student's ability to state an *a priori* "plan of action" for solving the problem. [The student can discuss how and why the pieces fit together] (page 58)

Conj. 3.7

When a task has been performed, it may be that what is stored in memory is the record of what procedures have been used. (page 69)

Th. 3.9 A

There exist "observer" procedures in one's repertoire that make it possible, as it were, to stand aside and observe the execution of a sequence of operations at the same time that the execution is taking place. (page 78)

"This can lead to correct predictions as to what is coming next, to "meta" descriptions, to planned variations in the procedure, to various forms of "chunking", and to names for the various chunks." (Davis et al, 1978, p. 78)

Th. 3.9 B

A particular sub-set of Th. 3.9 A "observer" procedures play a "predictor" or "look ahead" role. Given an input of the n^{th} stage of a sequence, they come - after repeated uses of the sequence - to be able to predict the $(n + 1)^{st}$ stage. (page 78) [For example, a person knows the way home and can describe it]

CL 5.1 <u>Binary confusions.</u>

Answering "4 x 4 =" with the answer "8", or "2^3" with the answer "6". In general, suppose that stimulus S_1 should activate procedure P_1, and that stimulus S_2 should activate procedure P_2. Then the Cl 5.1 phenomenon occurs when a student, faced with stimulus S_2, responds with procedure P_1. In every case that we have observed,

i) S_1 is visually similar to S_2.

ii) S_1 and P_1 were apparently learned earlier in the student's life than S_2 and P_2 were.

(page 85)

Assumption AS 5.1

Humans do not ordinarily "recognize" perceptual information, or match up items stored in memory, by an exhaustive study of every last attribute. On the contrary, relatively little information is ordinarily used; in reading, as soon as we have enough clues to make possible a plausible recognition of a word, we jump to the conclusion that we have identified the word. Clues may include overall word profile, initial letter, presence of "striking" sequences such as "ing" or "ph", presence of certain letters, word length, etc. (page 85)

Th. 5.1 <u>The "Short List"</u>

This refers to an assumed use of a short "look-up" list that becomes ambiguous as the number of items listed is increased. (page 86) [For example, one can remember only so many phone numbers]

CL 5.2

[Failing to make necessary distinctions (as between 0 vs. θ, for example), unless some additional mechanism (such as binary confusion appears to be involved.] (page 87)

SIMILARITY OF FORM: This is one of the fundamental information-processing questions in "thinking mathematically." It has several aspects, which we code separately:

CL 6.1 A

Obtaining Form A from Form B by using U.V. (replacement of variables). (page 88)
Category CL 6.1 A is further sub-divided

CL 6.1 A-1

Obtaining Form A from Form B (via formal or informal use of U.V.), when visual similarity serves as a helpful guide. (page 90) [For example: Form B is $a \cdot b = 0 \Rightarrow a = 0$ or $b = 0$, Form A is $(x - 3)(x - 2) = 0 \Rightarrow x - 3 = 0$ or $x - 2 = 0$]
"The negative instances of 'Similarity of form' contribute to some of the most common and persistent student errors. By any count, 'one of the most common and most persistent' errors had to include incorrect cancellation of fractions." (Davis et al, 1978, p. 92)

CL 6.1 A-2

Obtaining Form A from Form B (via formal or informal use of U.V.), when there is no obvious visual similarity to help guide the selection of a suitable correspondence. (page 90)

CL 6.1 B

Recognizing that Form A cannot be obtained from Form B by using U.V. (page 88) [For example: Form B is ($a \cdot b = 0 \Rightarrow a = 0$ or $b = 0$), Form A is $(x - 3)(x - 2) = 8$ and it does not follow that $x - 3 = 8$ or $x - 2 = 8$]

CL 6.2 A

This refers to observing a pattern such as

$$(a + 3)^2 = a^2 + 6a + 9$$
$$(a + 4)^2 = a^2 + 8a + 16$$

and "generalize" so as to be able to complete

$(a + 5)^2 =$.

(page 93)

CL 6.2 B

This refers to observing a pattern, such as

$(a + 3)^2 = a^2 + 6a + 9$

$(a + 4)^2 = a^2 + 8a + 16,$

and writing it in standard "variable" notation, perhaps as

$(a + n)^2 = a^2 + (2n)a + n^2.$

(page 93)

CL 6.2 C

Similar to CL 6.2 B, except that the "variation" has not been explicitly indicated in the actual stimulus material. Hence, it is necessary for the student to recognize the "general". and to separate it from the "specific", by imagining variation where no actual variation has been presented. (page 95) "In such a case, it becomes necessary to generate variation yourself. One must look, say, at □ + 3 = 3 + □ and ask is that '3' necessarily a '3'?" (Davis et al, 1978, p. 92)

(A special aspect of this task we code separately:

CL 6.2 C-1 Recognition of the appropriate attribute for some classification task.) (page 96)

CL 6.2 D

Given two mathematical instances A and B, devising something more general, C, such that both A and B are special cases of C. (Thus, if A were "addition", and B were "multiplication", C might be "binary operation.") (page 95)

CL 6.2 E "Seeing the General in the Particular"

Given one specific example, being able to interpret it correctly as a pattern for dealing with the general case (or "a more general case"). (page 97) [Specifically: 25 x 3 = 20 x 3 + 5 x 3 = 60 + 15 = 75. Generally: When multiplying a two digit number, one must multiply the digit in the one's place and the digit in the ten's place, then add the products]

CL 6.3

Creating a long strategic planning sequence. (page 98)

CL 6.4

Executing a long sequence. (page 99)

CL 6.5

Maintaining correct "sense of direction" in a long execution sequence. (page 99)

CL 6.6 A

Recognizing ambiquities. (page 100) [When two students are asked to add the integers from 3 to 6, one may get 18 and the other may get 6. They must realize they did not make an arithmetic mistake but that they interpreted the question differently.]

CL 6.6 B

Resolving ambiquities. (page 100) [The two students above, should realize there is a need to be more specific as to which numbers are to be added.]

CL 6.7

Coping with the dual nature of mathematical notations: "x + 2" is both an instruction to do something, and also a name for the answer. (page 100)

CL 6.8

Regression under cognitive stress. (page 101) [A student may make an incorrect cancellation when given a fraction involving many terms.]

CL 6.9

The ability to match up vertices of similar triangles. (page 101)

CL 6.10

Correct sequential order in processing a symbol string - "the ability to know which thing to look at next." (page 102)

CL 6.11 A

Decision trees that are incomplete. There may be possibilities the student has not thought of. (page 103)

CL 6.11 B

Decision trees with vaguely-defined alternatives. (page 103)

CL 6.11 C

Premature pruning of the "possibilities" tree. (page 103) "It is as if one said: Why, if I move my queen there she will be captured! Hence, cross out that idea immediately. But with such reasoning one could never recognize a queen sacrifice leading to checkmate." (Davis et al, 1978, p. 104)

CL 6.12 A

Decomposing an initial task into a heirarchy of goals and sub-goals. (page 105)

CL 6.12 B

Losing sight of the purpose of some sub-goal, in a CL 6.12 A-type hierarchy. (page 105)

Assumption AS 7.0

Humans possess a rather sizeable collection of procedures designed specifically for the task of processing natural-language statements. (Perhaps the point here is to emphasize that a natural-language statement will presumably NOT be in proper "command" or "procedure" form, and hence will not be directly executable in the way that ordinary commands and procedures are. Hence, additional processing is required to extract the command information from any natural language statement.) (page 107)

CL 7.0 A

Natural language statements often lead students into error. (page 107) [For example: 2, and 20 are the same because zero means nothing.]

CL 7.0 B

Nonetheless, experienced teachers do in fact make extensive use of natural-language statements in attempts to clarify mathematical ideas. (Perhaps the main point here is to find out just how such statements can be used successfully, given the known dangers of error.) (page 107)

CL 7.1

Poor use of language (as in reading x^3 as "x three"). (page 113)

CL 8.1 A

The realization that the abstract idealizations of mathematics are "perfect" objects, and are quite distinct from anything that can ever be found in "reality". (pa-e 114)

CL 8.1 B

The realization that, despite CL 8.1 A, there can be great simplicity and efficiency in studying "perfect" abstractions. (page 114)

CL 8.1 C

The realization that one does not deal with these "perfect" abstractions by attempting to make imperfect measurements on imperfect instances of them, but that, on the contrary, one deals with them by careful logical reasoning [as, for example, in proving (via the Theorem of Pythagoras) that $\sin 45^\circ = \frac{1}{\sqrt{2}}$.] (page 114)

CL 8.1 D

The realization, that, despite CL 8.1 A, these "perfect" abstractions are entirely appropriate tools to use in solving real problems. (page 114)

CL 8.2 A

The realization that every step in a proof (or in the solution of an equation) requires an explicit logical justification. (page 115)

CL 8.2 B

The realization that there exists an explicit list of citable logical justifications. (page 115)

CL 8.2 C

The realization that the CL 8.2 B list is based upon agreements between people. (page 115)

CL 8.3

The ability to identify explicit logical errors (as, for example, in recognizing the precise rule violation made by Speedy Sam [who, on solving $(x - 4)(x - 3) = 2$ concluded $x - 4 = 2$ or $x - 3 = 2$.] (page 116)

CL 8.4

Ability to identify errors that are NOT logical errors. (page 116)

CL 8.5

The ability to justify a mathematical statement by reference to the real world (together with a conscious awareness that that is what you are doing). (page 117)

CL 8.6 A

Recognition of the need to prove that answers are correct and complete (or unique). (page 118)

CL 8.6 B

Skill in making CL 8.6 A-type proofs. (page 118)

CL 8.6 C

Recognition of the need for methods <u>that are guaranteed to work</u>. (page 118)

CL 8.6 D

Skill in devising CL 8.6 C-type methods. (page 118)

CL 8.7 A

Recognition of boundaries of defined entities - e.g., the realization that 0! has NOT been defined by defining n! for positive integers n. (page 118)

CL 8.7 B

Recognition that, despite CL 8.7 A, there may be "value" or "strategic" or "purposeful" reasons for preferring some specific extension of a definition - even though no direct logical implication exists. (page 118)

CL 8.8

Analytical evaluative judgement in extending systems beyond defined boundaries [as in selecting definitions for 2°, 0!, etc.]. (page 119)

CL 8.9

Knowing the boundaries of what you know (i.e., of "what you are supposed to know"). (page 120)

CL 8.10

Use of plausible, but incorrect, re-writing rules.

Eg., Assuming $\sqrt{(x - a)^2 + y^2}$ can be written as $\sqrt{(x - a)^2} + \sqrt{y^2}$ and thus, ultimately, as $x - a + y$. (page 120)

CL 8.11

Inappropriate disbelief in the meaningful truth of various mathematical statements. (In simpler language, this refers to students' suspicions that "this is all nonsense," "none of it makes sense.") (page 121)

CL 8.12

The expectation that learning math consists of building VMS sequences (or S-algorithms) in the mind of the learner - and nothing more! (page 121)

CL 8.13

Understanding that "the equation of a plane (or other geometric configuration)" refers to an algebraic statement about the coordinates (x, y, z) or a "moving" of "generic" point P (x, y, z), such that the truth of the algebraic statement is necessary and sufficient for the point P (x, y, z) to lie on the plane (or other configuration).

(This has several component parts: recognition of the role of the point P (x, y, z); recognition that the "x,y,z" in the final answer should refer to P (x, y, z) (and not to some of the other points that may be dealing with during the course of the problem); and so on.) (page 125)

CL 8.14 A

Recognition of the correct role of variables to describe some "moving" point, or "the general case." (page 125)

CL 8.14 B

Recognition of the specificity of "general points" in relation to specific geometric figures. (E.g., the "r" of a circle need not equal the "r" of a sphere, if the circle is less than a great circle on the sphere.) (page 125)

CL 8.14 C

Understanding the correct rules for replacement of variables (especially in troublesome questions, as in using "n + 1" as a replacement for the variable "n"). (page 126)

CL 9.1

The student strategy of "checking with numbers" - checking by making one or more numerical replacements for the variables. (page 127)

CL 9.2

The student strategy of thinking in terms of the meanings of the various symbols. (page 127) [For example: x^2 should be thought of as x times x.]

LEVELS OF PERFORMANCE IN "LOOKING FOR PATTERNS"

"Note that, in Section 9.3, we are really concerned with habits of thinking, and only indirectly with the information-handling capabilities which may be required." (Davis et al, 1978, p. 129)

CL 9.3 A (Level 1)

No search for patterns. (page 128)

CL 8.3 B (Level 2)

Looking for patterns within a single problem, but NOT looking for patterns by making comparisons across problems (not comparing or contrasting different problems). (page 129)

CL 9.3 C (Level 3)

"Comparing and contrasting across problems, and hence finding patterns at this level. This depends upon the cognitive abilities we have coded as CL 6.2 A. (page 129)

CL 9.3 D (Level 4)

Going beyond CL 9.3 C, in the sense of recognizing the patterns and writing it in correct "variable" notation. (page 129)

CL 9.3 E (Level 5)

Writing the general case, and dealing with it analytically. (page 130)

CL 9.3 F (Level 6)

Subitizing; subsuming CL 9.3 D and CL 9.3 E in an "instantaneous" recognition. (This seems to match Krutetskii's concept of "curtailment".) (page 131)

CL 9.4

The student habit of "cross-referencing" current information with ideas (even remote ones) that are already known. (page 134) "Like the rich getting richer, the student with the best developed associations could do a better job of cross-referencing each new piece of information he encounters, thus increasing even more his advantage in the future." (Davis et al., 1978, p. 134)

CL 9.5

The student habit of referring input data to an alternative representation, as in drawing a graph, or rewriting small numbers in scientific notation, etc. (page 134)

CL 9.6

The student habit of monitoring one's own output. (page 134)

CL 9.7

Deals with student behavior when confronted by an illustrative example, and a problem to be solved:

CL 9.7 A

Attempting the assigned problem with no evidence of serious analytical thought, and with no reference to the illustrative example. (page 135)

CL 9.7 B

Attempting to solve the assigned problem by step-by-step imitation of the illustrative example; matching parts of the problem with seemingly similar parts of the illustrative example; to be a CL 9.7 B response, there must be no evidence of any serious attempt at problem analysis beyond this level of "rote" matching by apparent similarities. (page 135)

CL 9.7 C

Attempting to understand the illustrative example, and to use this understanding to solve the assigned problem. (page 135)

CL 9.7 D

Attempting to analyze the assigned problem before referring to the illustrative example. (CL 9.7 D can be distinguished from CL 9.7 A by assessing the amount and relevance of analytical thinking that is applied to the assigned problem. In simple language, CL 9.7 A is the level of "careless indifference", whereas CL 9.7 D is the level of testing yourself as a serious autonomous mathematics student.) (page 135)

CL 9.8

The student behavior of posing and solving the general case. (page 140)

CL 9.9

The student habit of responding to something novel or surprising by thinking about it afterwards, often outside of class. (page 141)

CL 9.10

The student habit of making frequent use of mathematical ideas in seemingly "non-mathematical" settings - e.g., while watching a "private eye" show on TV, computing how much money the private detective seems to earn per week, per month or per year. (page 143)

CL 9.11

The student habit of asking: "Is there another way to solve this problem?" Trying to solve problems several times, by several different methods. (page 145)

Conj. 11.1

Assuming that humans find "meaning" in perceptions by matching them up with assimilation paradigms, then the phenomenon described by Minsky and Papert - where the key chess pieces, say, stand out, set apart from their background, "as if they had suddenly changed color" - may mark the point at which the appropriate assimilation paradigm has been retrieved from memory and the key pieces on the board have been matched up correctly with the "variables" in the assimilation paradigm (that is to say, "variable binding" has taken place, in Elcock's sense). At this point the key pieces should be perceived differently; they now have a dual existence: they are the specifics of this actual situation, but they also are the general actors in a known general script (i.e., the schema or assimilation paradigm), and they consequently enjoy both the advantages of tangible, concrete specificity, and also all the known general properties for the characters in the script. (page 156)

CL 12.2

Student mastery of N ways of looking at a problem, when M ways are desired [N<M] (page 161)

CL 13.1

The use of explicit, ad hoc heuristics in algebra. (page 163) [We have to be aware we are using it.]

CL 13.2

The use of explicit, ad hoc heuristics in Euclidean synthetic geometry. (page 163)

CL 13.3

The use of general heuristics, such as:

"i) Try to break the problem up into pieces sub-goals, deal separately with the pieces, and then assemble the results into an appropriate whole....

ii) Try to make up (or to find) a similar, but easier example. See if you can solve the easier problem. Then see if you can learn enough from the way that you solved the easier problem so that you will be able to solve the original problem.

iii) Try to find exactly where the main obstacles are. Then see if you can eliminate them, circumvent them, or otherwise deal with them.

iv) Find something you CAN do, and do it. (This is the most dangerous heuristic, since after all it could lead you in ANY direction, including away from your goal; but it also seems to be the one that helps students the most. It is NOT as obvious as it seems: most students do not automatically use it.)

v) Try something even if it's wrong. Then try to learn from what happens." (Davis et al, 1978, pp. 164-165)

CL 13.4

The use of non-explicit "intuitive" heuristics. An explicit heuristic will become implicit if we use it so often it becomes an unconcious habit. (page 165)

CL 13.5

The use of "backward reaching" heuristics: work backward from the desired result. (page 165)

CL 13.6

To prove that A $\neq$ B, find some attribute of A which is NOT an attribute of B. (page 166)

CL 14.1 A

Input limitation or "focus" - rejection of part of the available informational input, in order to focus attention of what is, hopefully, most relevant at the moment. (page 169)

CL 14.1 B

Fixing certain aspects of a problem firmly in mind. (page 169)

CL 14.1 C

Separating "identified" aspects of a problem from the rest of the possible input data, and treating the rest of this data, for a moment, as "background noise". (page 169)

CL 14.1 D

The strategy, in geometry, of carrying out CL 14.1 C by looking "locally" at some restricted portion of a total diagram, rather as in a "tight close-up" of this part of the figure. (page 169)

CL 14.2

Representing a quantity as some sort of "block" amount, that can be treated cognitively as a single entity. (We do this, for example, when we take the abstract quantity "36 inches" and think of it as "1 wooden yardstick". We can then use information-processing procedures appropriate to the multiplication of the more tangible yardstick.) (page 171)

CL 14.3

Use by a student of highly original, and unorthodox, methods. (CL 14.3 implies some rather insistent reluctance to adopt more conventional methods.) (page 178)

CL 14.4

The heuristic strategy of picking a "possible" solution, and then studying it. In most cases, the "possible" solution will NOT turn out to be a solution - but in the process of studying it, you may learn how to modify it so that it is more nearly a solution. (page 179)

SIX POSSIBLE GENERAL MECHANISMS FOR PROBLEM SOLVING

CL 15.1 "Forward-moving" steps. (page 180)

The use of some obvious tactical step that transforms the problem, and may bring you nearer to your goal.

CL 15.2

Heuristic analysis of the problem situation. (page 181)

CL 15.3

Use of VMS sequences to solve the problem, or some part of it. (page 181)

CL 15.4

Attacking a problem by means of some "meta" level analysis. (page 181) "The defining characteristic of 'meta' analysis is that we talk about procedures (or schemata), without necessarily using them." (Davis et al, 1978, p. 181)

CL 15.5

Partially matching schema. "When we use a schema, we are on known, familiar territory. We can use a partially-matching schema in either of two ways:

a) The schema may accomplish part, but not all, of what we have in mind;
b) The schema may not match our problem situation correctly, and may require some modification before we can use it. (This second use may represent part of Piaget's concept of accommodation.)

(Davis et al, 1978, pp. 181-182)

CL 15.6

Recognizing the problem as a specific instance of some general schema, and then solving the problem. (page 182)

CL 22.1

Goal distinctions: the extent to which a student is attempting to complete a problem or assignment (so as to satisfy immediate external requirements, such as "pleasing the teacher" or "getting it in on time"), vs. the extent to which the student is "trying to learn mathematics" (as indicated by building up his cognitive structure, testing its adequacy, examining implications or apparent inconsistencies, preparing for problems that may be met in the more distant future, etc.). (page 276)

CL 23.1

Recognition of the value of "understanding". (page 281)

CL 23.2

Reasonable accuracy in judging whether you have an adequate schema for some situation, phenomenon, or problem type. (page 283)

CL 23.3

Reasonable accuracy in judging whether you have filed away some information so that you are likely to be able to retrieve it later on when it is needed. (page 283)

CL 23.4

Reasonable accuracy in judging whether you have related new information to enough knowledge that you already possess (for example: have you visualized the graph of a function; do you see how to <u>prove</u> some new statement; and so on). (page 283)

CL 23.5

Reasonable accuracy in judging whether you have been alert to recognize important <u>patterns</u> in the new information, or in its relation to things already known. (page 283)

CL 24.2

Vagueness in recently-acquired information. (page 284)

CL 24.5 A

Retention of earlier restrictions even after one has learned (in some intellectual sense) that they are no longer necessary. (page 286) [For example: Maintaining that one can't subtract a greater number from a lesser number even after learning about negative numbers.]

CL 24.5 B

Critical reliance upon an incorrect or inappropriate assumption. (Presumably the theoretical formulation of this would be Th. 24.5 B: use of an assimilation paradigm which has a critical failure in the sense of not matching the present instance faithfully enough.) (page 286)

CL 24.5 C

The ability to criticize your own work when you go over it again (and to see with reasonable accuracy just where changes need to be made). (page 286)

CL 24.8

Repetition of an earlier CL 24.5 B type error, even after the error had been recognized and corrected earlier. (page 293)

CL 24.9

Believing in the value of sharp delineation of [the key] ideas, [in order to make sense of what is being done and so as not to be overwhelmed by "hazy" ideas.] (page 293)

APPENDIX D

SAMPLE PROTOCOLS

The following transcripts are of three complete audio-taped protocols. They are representative of low, average, and high success rates on the problems.

PROTOCOL I   LOW SUCCESS RATE

KURT SESSION 1

1a
1. K: (Draws two sides of a rectangle) (Multiplies 5.95 by 4 and gets 23.80, and 5.95 by 5 and gets 27.75) Across 5 at 5.95 a square metre times 5 that times the square metres.
2. K: (Adds $27.75 and $23.80 and gets $51.55) So all together.... I hope it costs $51.55.
3. I: You hope it costs?
4. K: Yeah, I hope I'm not buying it cause that's an awful lot of money.
5. I: So that's your answer?
6. K: I guess so.

2a
1. K: Okay if you've got 12 metres....length....width 5....5 metres. So then you'd have 18 metres across in length. 12...umh.... (reads the question again)
2. K: Oh an enlarged photograph. Oh. This is the same photograph except it's enlarged?
3. I: Yeah.
4. K: So the length would be 18mm. Enlarged about 5 times cause this was 12 and if it's enlarged about 5 times it would be....if it was enlarged 6 times it would be 18 mm so you've got to enlarge this 6 times. It would be about 11mm.
5. I: Okay
6. K: So it would be 11mm width.

2a'
1. K: Mr. Big (sighs and reads remainder of the question aloud) Another trick question?
2. K: That would be 2 right? How simple. I've got 6 buttons for Mr. Big and had 4 for Mr. Small, that's a difference of 2.
3. K: Oh, this is too simple....something wrong....I think it would be 2 but it's too simple. I've got to read this again. (reads the question again) Yeah, I'll say 2. It's another trick question.

3a
1. K: $50 on Friday, $25 over the weekend...So 50...so that would be $75 altogether. He had 50 and got an extra 25, so actually, he had 75 altogether right? And he didn't spend any?
2. I: I don't think so, he was working to hard.
3. K: (Laughs) Okay he had 75. I'm trying to find out the percent.
4. I: Yeah.
5. K: On Friday he had 50....Okay. Try uh...try 25 (divides 25 into 75 and gets 3, divides 50 into 75 and gets 15, looks at the 3 and the 15). It's about 5% 5%.

3a (con't)
6. I: Okay.
7. K: I don't know, it's a tough one, so I'll say about 5%.
8. I: Okay

4a
1. K: 12%...(multiplies 28 by 12 and gets 336, divides 336 by 12 and gets 28) (laughs) I got $28. (knocks on table) I got stuck.
2. I: Got stuck on 28 eh?
3. K: Yeah I did everything and then I got 28 again. Maybe I multiplied wrong (checks his multiplication). No.
4. K: Okay. (divides 28 into 336 and gets 12) 12 oh great! I got $28 and 12%.
5. I: (laughs) Won't let you get out of that one eh?
6. K: It's the only way I can figure out how to do it...Maybe 12 into 28 ah ha! (divides 12 into 28 and gets 2.3) Oh great!
7. K: Oh well. (multiplies 28 by 2) I'm stumped. I'll just make a guess here....40.
8. I: That's good enough, there is no point in suffering. At least you've got a sense of humor.
9. K: (laughs) I just can't take this no more.

5a
1. K: Okay 30 questions correct, 80%
Great! (sarcastically)
2. K: (Divides 5 into 80 and gets 12) I don't know why I did that, useless completely. All I got was 12 I don't know what I'm going to do with 12.
3. K: Maybe there's 12 questions and he got 30 right. No that doesn't sound too good.
4. I: (laughs)
5. K: (Divides 30 into 80 and gets 11) (knocks on table)
6. I: This isn't a test remember, don't worry about it.
7. K: Oh yeah I'm going to have a nervous breakdown pretty soon.
8. K: Oh, here? (laughs)
9. K: Yes...30 ahh...hold it, just a minute. If there's 30 questions I got it, yeah. You get a mark. It must go up every number up. Like one question it would go up 4 then 8, then 12, then 16, something like that.
10. I: Each question would be worth so much?
11. K: So try 3. 5, 10, 15, 20, 25...I lost count, great! 5, 10, 15, 20, 25, 30, 35, 40, 45, 50. There you'd have 50 questions, easy. I went by 2's. So each question is worth 2.
12. I: Uh hu.
13. K: Hold it. 30 questions and he had 30 right (knocks on the table). That means he had to get...Okay 50 questions.
14. I: Makes you sweat eh?
15. K: These are tough, I can't handle these percents.

6a

1. K: 450 well I don't even know what "gm" means.
2. I: Grams.
3. K: Grams, I knew that. A package of 24 cookies that cost $1.50 or ah...300 gm ...well its an easy one. Cause it costs, it costs a little extra. Ah. This is a trick question. They both have 24 cookies. I mean what difference does it make if they both have 24 cookies you go for the lower price, right? Right.
4. I: I don't know. What would you do?
5. K: I'll take 450 gm , then I'll take 300. (writes: 300 under 450) No I won't do that. I'll take 450 and 24. (writes: 24 under 450) No, I won't do that, I take 450 and 150 (multiplies 450 by 150) that's a lot. Forget about that. I'll try 450 and 24 (multiplies 450 by 24) Okay that's 108. Okay, then I'll take 300 and 24 (writes down 300 and 24) well it has to be less of course it has to be less....
6. K: The 300 gm one. I made my decision it's a trick question.
7. I: (laughs) So are you just guessing 300 then?
8. K: Nope. If they both have 24 cookies, what difference does it make? Cause this costs less. It's grams, I mean who cares if they both have 24 cookies.
9. I: Okay. Then you would go out and buy the 300 gm. one?
10. K: Yes.

7a

1. K: (reads the question aloud) I got it. You don't get a higher percent in Social because...There are more questions in the social test than are in the Science test. Yeah. (taps desk)
2. K: So you had 75 questions correct only 30 questions on the science test. So that means if there were more questions on the Social Studies test...that means...and less on the Science test it would make a difference really of what you got correct. It's of how much.
3. K: If you got 200 questions on the Social test and only 60 on the Science test then you could fail the Social test with 75 and pass the science test with 30.
4. K: Another trick question. So I'll say...(taps desk) (looks at question again) Explain why, oh I've just said that, it's hard to put it all in writing though.
5. I: Well try one sentence, two words or less (laughs)
6. K: (laughs)
7. I: If you were writing it on a test what would you write down for a sentence answer?
8. K: It would make a difference on how many questions. That would be good. (writes, while saying) If the Social test, this is a long sentence, if the Social test had more questions than the Science test then it would not matter. (laughs) Oh, I don't know if I'm right or not. I said it better than what I'm writing it here.
9. I: That's okay.

KURT SESSION 2

1 Div.

1. I: What is the relationship between multiplication and division?
2. K: When you multiply you times two numbers. Like 6 times 4 you have 4, 6 times or 6, 4 times. Well division, you try to put a number into another number. Like 2 into 6 would be 3.
3. I: But you haven't told me how they are related.
4. K: Well they both have two [illegible]. Not necessarily but that is the least you could have.
5. I: For instance?
6. K: Well like 4 and 2, or you could have the same number. In some cases you could have the same number.
7. I: For division and multiplication?
8. K: Yeah, they won't be the same answer but you could still have the same numbers.
9. I: Okay, give me a division question.
10. K: Okay, 4 into 8....so to get 4 into 8 you'd have to times like 4 times 2 is 8.

2 Div.

1. I: In this: a into c goes b (writes it down in algorithmic form.) what is the a, the b and the c called.
2. K: Uh, I can't remember. The divisor and the dividend but I don't know which is which.
3. I: (Tells him.)

3 Div.

1. I: Can you write that (number 2) with that sign (÷) so that the a, the b and the c mean the same?
2. K: (Writes: a ÷ c = b.)
3. I: Can you read that?
4. K: Well **a** has to go into c to equal b.
5. I: Is that what this sign (÷) means to you, has to go into?
6. K: Well, divide yeah.
7. I: Can you give me an example with small numbers to show that.
8. K: Oh I did that wrong. The a and the c have to be switched around so that it is c divided by a.

4 Div.

1. I: In this question a goes into c, b times or c divided by a equals b, if c stays the same and a increases what happens to b?
2. K: The b would get smaller.
3. I: Why?
4. K: Because it means it wouldn't take as big a number to get into c.

5 Div.

1. I: If a is constant and c is increasing, what is b doing?

5 Div. (con't)
2. K: Getting bigger.
3. I: Why?
4. K: Because it would take a lot more to get into c, so that means b would have to get bigger.

6 Div.
1. I: If b is constant and a increases what would happen to c?
2. K: The c would increase the same as a.
3. I: Why?
4. K: Because if you get the same answer it has to go in the same as before.
5. I: So if a went from 2 to 6, c would have to go from...
6. K: 60 to 10.
7. I: Can you show me that?
8. K: (Writes in algorithmic form, 6÷2 = 3.)
9. I: So 2 goes into 6, 3 times, and a goes from 2 to 6 and b is still the same so c would have to go to 10?
10. K: Yeah, (laughs) I doubt that would work out though.
11. I: 6 doesn't go into 10 three times eh? What would c have to be?
12. K: I wouldn't know.
13. I: To get 3 for an answer what would c have to be?
14. K: It would have to be the same number apart like it was before. Like it would increase but it would have to be the same number apart, like before it was 2 and 6.
15. I: That's 4 apart.
16. K: (laughs) Oh that's what I had before. Just say it would have to divide the same. 2 into 6 would go 3, it would still have to divide the same.
17. I: What would it have to be, then?
18. K: 18.
19. I: So what is the word for that, you said it had to increase the same amount but that didn't work out for us. So you should say it has to increase...
20. K: Uh...
21. I: Increase at the same...
22. K: Uh...ratio...rate.

Division Questions

1 DQ 1008÷56
1. K: (Writes 1008÷ 56 in algorithmic form and puts a 1 above the second 0.)
2. I: Why did you put the 1 there?
3. K: Because I'm dividing into 100 so I have to show I'm using the whole number.
4. I: Is there some other place you could put it or do you have to put it there?
5. K: Well it works better, you're better able to understand what you're doing. It doesn't have to though.

1 DQ (con't)

6. K: Let's see (He has multiplied, subtracted and is now deciding how many times 56 goes into 448. About 8.
7. I: Why 8?
8. K: Well I'm thinking of a round number, 50 so you go 50, 100, 150... (Multiplies 56 by 8 at the side and is pleased it is 448.) (Finishes the question.)

2 DQ 60÷4.3

1. K: It would be 600 because you can't divide by a decimal so you have to move the decimal here (in 4.3) so you have to move the decimal here one (change 60 to 600.)
2. I: Why?
3. K: Well if this (divisor) increases this (dividend) has to increase, something like that.
4. K: (Finishes the question by multiplying out at the side before putting the answer down and gets 13.9) 10 no that's 9.
5. I: What if it would have gone 10 times. Where would have you put that 10?
6. K: Here (where the 9 is)...no I would need another zero here (in dividend) for it to go 10 times.

3 DQ 32÷429

1. K: Oh (Writes the question correctly in algorithmic form.) (Puts the decimal point in the correct place in the quotient.
2. K: Lets see it will go (429 into 3200) about 17 no, I'll try 27 times.
3. I: Okay, if it goes 27 times where will you put the 27?
4. K: Right here (after the decimal point).
5. I: Wouldn't it be better to put the 2 first and multiply?
6. K: No it depends on what the person does the best.
7. K: (Writes 429 times 27) No I'll try 7. Because 400 times 7 (Is close to 3200).
8. K: (Puts the 7 above the second zero.) We need a zero here. (As a place holder after the decimal point.)
9. K: So the answer is point zero seven.

4 DQ 54.2÷26

1. K: (Writes the question down in algorithmic form, writes the multiplications out at the side and gets 2.8 remainder 12).
2. I: Now if I asked you to correct that using multiplication could you?
3. K: I'll try this, 26 times 2.8. (Multiplies and gets 72.8) That won't work.
4. K: I'll multiply by 12 (the remainder) but that won't work.
5. K: I'd have to do this. (Multiply 54.2 by 26 writes it down.)
6. I: Will you get 2.8 when you multiply 54.2 by 26?
7. K: No. (Multiplies 26 by 2.8 again.)
8. I: Since you're not getting anywhere, do you think you made a mistake in you're division somewhere?

4 DO (con't)

9. K: No I don't think so. It's hard to make a mistake in division.
10. I: You made a mistake in the division.
11. K: I made a mistake!
12. K: Okay, 26 into 54, that goes 2 remainder 2 so that is 22 (after bring down the other 2) Oh! I used 2 of these (he had also brought down a zero) so that means I have to have a double answer then.
13. I: So what should go up here then?
14. K: A zero (as a place holder).
15. I: Okay could you check it using multiplication now?
16. K: (Writes 54.2 times 26)
17. I: What do you want to get for an answer?
18. K: 2.08.
19. I: Will you get 2.08?
20. K: No. There's got to be some catch.
21. I: At the begining of this interview we said how miltiplication and division are related and we said 4 goes into 8 twice because 2 times 4 is 8. So how would you check this?
22. K: (Multiplies 2.08 by 26 and gets 54.08)
23. I: You get 54.08, it doesn't work out quite right. It's close isn't it, why didn't it work out?
24. K: Maybe because of the remainder.
25. I: Do 3 into 7 and check it using multiplication.
26. K: 3 times 2 is 6 plus the one remainder is 7.
27. I: So how can that example help you?
28. K: 2.08 times 26 plus the 12 is....54.20 but you don't need the zero at the end.

5 DQ 8.71÷.37
Not done.

6a DQ 8245÷85

1. K: The first digit will be 9 and that's it there is just one digit before the decimal.
2. I: Can you tell me why it is 9?
3. K: I was thinking 8 and then 16 then 24 and all that.
4. I: So you are dividing 85 into 824 and you get 9 and it isn't going to be 90 or 900 or something like that?
5. K: Yeah it's 9.

6b DQ 9356÷54

1. K: The first digit will be 1.
2. I: And where will that 1 be, how many places to the left of the decimal will it be?
3. K: 3.
4. I: Why?
5. K: Well I was thinking 54 into 93 and there are two extra numbers left.

6c DQ 24156 ÷ 732
1. K: The first number will be 3 because 700 as a round number into 24000.
2. K: There will be two digits.
3. I: Why?
4. K: Because there will be one left after the first.

6d DQ 143115 ÷ 145
1. K: 10, no it can't be, yeah it can, ten.
2. I: So ten is the first digit.
3. K: (laughs) 10 is two digits so it's 9.
4. I: Just about 10, but it's 9 eh, where will that 9 go?
5. K: Right here so there will be 3 digits.

7a DQ 4328 ÷ 10
1. I: Can you divide 4328 by 10 in your head?
2. K: 430, something like that.
3. I: Okay, write it down and do it.
4. K: (Writes it in algorithmic form and gets 432.8)

7b DQ 4328 ÷ 100
1. I: Using the answer for 10, divide 4328 by 100 in your head.
2. K: Four, three, two eight with a decimal.
3. I: Where is the decimal, write it down.
4. K: (Writes 4328.)
5. I: How do you read that number?
6. K: Four thousand two hundred and twenty-eight.
7. I: So 4328 divided by 100 is 4328?
8. K: No it isn't. Forty three point two eight.

7c DQ 4328 ÷ 1000
1. K: I got it, 4.328.
2. I: What about by 10,000?
3. K: .4328.
4. I: And 100,000?
5. K: .04328.

KURT SESSION 3

1b
1. K: (Writes 6/4 n/2, solves for n and gets 3.) The answer for n is 3.
2. I: Okay does that make sense?
3. K: I hope so, that is the answer I got, I hope it's right.
4. I: Why did you put 6 over 4?
5. K: Because it's being compared, the 6 and the 4, at least I hope it is.

2b
1. K: (Writes: n/2) Oh no....
2. I: What's the problem?
3. K: Because I'm stuck.
4. I: Why did you put n over 2?
5. K: Because you're trying to find the answer for the 2 seconds.
6. I: I see, and you don't know what else you've got?
7. K: Yeah I should have another number.
8. I: Read the question out loud.
9. K: (Reads it out loud.)
10. I: What does "at that rate mean".
11. K: Well the same rate as it does it.
12. I: What is that rate.
13. K: Umh, ...well every .02 seconds just well you know .02 then .04 and .06 just keep adding on.
14. I: Instead of adding on can you multiply or divide or something; to find the answer?
15. K: That's what I'm confused about.
16. I: Is that n going to be seconds or what.
17. K: Well I don't know because that's what I'm trying to find out.
18. I: What would you do if that were on a test.
19. K: I'd faint. Well I'd divide .02 into 2.
20. I: Okay, do that.
21. K: (Divides and it goes 100 times.) Well 100 seconds could be the answer.
22. I: Does that make sense for an answer to that question, 100 seconds?
23. K: No, I don't think seconds, I don't know what. I hope the answer is 100 but I don't know what.
24. I: But you're not sure what. 100 bananas?
25. K: No something to do with time.
26. I: What does the question ask you to find?
27. K: How many multiplications can the machine do in 2 sec. Oh yeah, 100 multiplications.
28. I: Does that seem reasonable?
29. K: Yeah.

3b
1. K: (Writes 3.6/10.8 n/100, cross multiplies and solves for n and gets 305 with the 5 repeating.)
2. I: So what is the answer?
3. K: 305%, ohhh.
4. I: Why are you groaning?
5. K: Well that's kind of like a lot, 305%. I'm sure I did this right, I'm almost certain.
6. I: What would seem reasonable. Are you familiar with kilograms or are you more familiar with pounds?
7. K: I'm not familiar with either.
8. I: How much should a baby weigh when it is born, in pounds?
9. K: About 6 or 7.
10. I: And about how much does a year old weigh?

3b (con't)
11. K: About 25.
12. I: Okay, just to make it easier, let's say it weighed 7 pounds at birth and 21 pounds at one year, what would the per cent increase in weight be?
13. K: About 3%. Because 7 into 21 goes 3. Maybe my decimal (in this division) is wrong. Yeah, my decimal is wrong.
14. I: Is 21, 3% of 7?
15. K: Uh, yeah.
16. I: What is 300% of 7?
17. K: You'd have to divide 7 into 300 (divides and gets 47.8).

3b (i)
1. K: (Writes 110/400 220/400) That 400 is the total, and I think this (220) goes over 400. I'm not sure.
2. K: (Multiplies 400 by 220 and gets 88000 and 400 by 110 and gets 44,000.) I'm definitely not sure. I did something wrong there. I'm really confused. So you get 44,000 into 88,000 and you'd get 2.
3. K: (Reads the question again.) Oh all I'd have to do is plus 220 and 110. (Adds 220 and 110 and gets 330) Oh so 70 is the answer. The answer when you add 220 and 110 is 330 so you have to subtract it from 400 because there are 400 altogether.

4b
1. K: (Writes 8.5%/100) This is the difficult part (deciding where the n and the 42,000 go).
2. K: (Writes n/42000) (Cross multiplies the 8.5 and the 42,000 and gets 3,570,000.) Oh, 3,570,000, that's a lot! Oh divide 100 into 3,570,000 and it goes, oh this cancels so it goes 35,700.
3. I: So what is the answer then?
4. K: $35,700, that's his salary for this year.
5. I: Is that reasonable?
6. K: Yeah, I hope so.

5b
1. K: (Writes n/35 300%/100, cross multiplies 35 by 300 and gets 10,500.) Oh, 10,500 (Divides by 100 and gets 105.)
2. K: That's the answer, 105.
3. I: 105 what?
4. K: Umh, 105 uh a dollar five.
5. I: Is that reasonable, do you babysit?
6. K: Yeah for my rich brother.
7. I: What do you get.
8. K: $10.00
9. I: How long do you babysit?
10. K: About 6 hours so that's about a dollar and a half an hour.

5b (i)

1. K: (Writes n/51 and 50%/100, cross multiplies and divides and gets 25.5.) (As he is solving it says:) This will be an easy one.
2. K: I think 25.5 is the answer.
3. I: What would your sentence answer be?
4. K: Uhm...yeah it's 25.5, so 25.5 students have brown eyes.
5. I: Does that make sense? What does that point 5 mean?
6. K: A half, a half a person, I don't know. That's just what I got. I could go 100 into (without cancelling the zero in the 100 and the 2550) but I'd probably get the same answer. Yeah, I'd get the same answer.
7. I: Well what does that 25.5 mean for an answer to this question?
8. K: 25.5 have brown eyes, that's 50% of the class. That's right you know because that point 5 and another point 5 would make it 51, you can't fool me.

6b

1. K: Umh, this is a tough one. They're not going to fool me on this one, I've got it. (Writes: 23/50 n/100, solves for n and gets 46) So I get 46 for an answer.
2. K: So would she be speeding if she were going 46 and the speed limit is 50? No, she would not be speeding.
3. I: Why did you pick 100.
4. K: Well it's per cent, I don't know it's just always done like that.

7b

1. K: Well they got first off 10% more and then second off they got 10% less so that equals off. 10% minus 10% is zero.

KURT SESSION 4

1c

1. K: Umh...Calgary to Saskatoon...through Edmonton?
2. I: Yes, that's what that via means.
3. K: Okay, Calgary 288 and Saskatoon to Edmonton is 350. So 350 plus 288 is 638. 638 miles.
4. I: Miles?
5. K: Oh, 538 miles.
6. I: Okay, but the question asks in kilometres.
7. K: Oh, so I have to find out what that miles would be in kilometres. I wouldn't know.
8. K: (Subtracts 180 from 288 and gets 108, and subtracts the 108 from 350 and gets 242, then adds 350 and 180 and gets 530.)
9. K: I guess that's it 530. I don't know.
10. I: You subtracted 180 from 288, can you tell me why?
11. K: So I would know what to subtract 350 from.

1c (con't)

12. I: What to subtract from 350? (K: Yeah.) And then you subtracted that 108 from 350 and you got 242, what does that 242 mean?
13. K: I think that's the kilometres for Edmonton to Saskatoon. Then that plus the 288 is 530.

2c

1. K: An hour and a half is 90 minutes so 60 into 75 goes (works it out) 1.25. I don't know what that has to do with the answer.
2. K: (Adds 90 and 75 and gets 165 and gets 2.75) 2.75, umhm ahh 2.75 has to mean something. So how many can he deliver.
3. K: (Divides 3 into 75 and gets 25.)
4. I: Why are you dividing 3 into 75?
5. K: So I can see how many newspapers he can deliver every minute. So if every 3 minutes he delivered...oh....
6. I: Where did the 3 come from?
7. K: Well, that's just a number out of my head. I tried to think of a number that would go.
8. I: If he delivers 60 newspapers in 75 minutes....
9. K: So it would be 25 newspapers in 3 minutes, no that isn't right. 3 newspapers in 25 minutes, no that wouldn't be right either.
10. I: What does that 3 have to do with the problem?
11. K: It would make a difference between the 90 minutes and how many minutes it would go into 60. Like it would find out how many minutes it would take for each newspaper.
12. I: Oh.
13. K: I'll divide 5 into 75 (gets 15).
14. I: Where did the 5 come from?
15. K: Just a number I can work with I guess. And like every...I'll try 15 into 60. (Works it out and gets 4.) Oh yeah! 15 into 60 that's 4 and like 4 papers, 4 minutes every 15. Now I think I got something. 10 into 60 goes 6, so that's 6 papers every 10 minutes. Umh, I'm not doing too good. 5 into 60 uh 12...
16. K: I'm going to work with 20, so 20 every 3 minutes. Every 20 seconds 3 papers. Uh, it looks so easy. Maybe I'll try a rate pair.
17. I: Okay, just draw a line across the paper to separate it.
18. K: 60. Uh I don't know (what goes on the bottom)
19. I: 60 papers is it?
20. K: Yeah. An hour and a half is 90 minutes so I guess it would be 75 over 90. (Puts 100 under the 60.)
21. I: Where did the 100 come from Kurt?
22. K: It's kind of uh, you're just supposed to use 100 I guess.
23. K: (Cross multiplies and writes down 5400 - 7500, then decides to subtract 5400 from 7500 and gets 2100.) Ah! That looks better. (Than the 2.75) It's hard to deliver 2.75, but it's hard to deliver 2100 papers.
24. I: This 75 is minutes and the 90 is minutes and the 60 is papers and the 100 is what?
25. K: Well, it's like uh...I could have used <u>x</u>.
26. I: You could have used <u>x</u>.

2c

27. K: I should have used x.
28. K: (Writes: 75/60 90/x) 75, 90 let's see where would the x go... the 60 would go here. So it's 75x and 5400.
29. K: (Divides 75 into 5400 and gets 72.) Ha! Ha! 72 papers.
30. I: Is that your answer, then, 72 papers?
31. K: Yeah 72.

3c

1. K: (Writes 9000/7500 x/100, cross multiplies, cancels the final two zeros in 900000 and 7500 and divides 75 into 9000 and gets 120.) That's it 120%...
2. I: 120%?
3. K: ....Yeah.

3c (i)

1. K: 31.
2. I: You're sure?
3. K: Yeah I'm sure.
4. I: What does the 15% have to do with it.
5. K: Well...(Reads the question aloud.) I don't know what this 15% is.
6. I: But you're sure there are 31 girls?
7. K: 31, yeah.

4c

1. K: (Writes 6.5/100 x/2000, cross multiplies to get 100x and 13000, divides the 13000 by 100 and gets 130) 130 I don't know. "What will his yield be?" 130. Well that's the answer I got, 130 I don't know.
2. I: 130 what, bananas or what?
3. K: I don't know exactly what this means, what the yield will be?
4. I: Yield is what he gets off the land, like his yield is 2000 bushels....
5. K: Oh, then 130 bushels.
6. I: And does that make sense in the question?
7. K: (Considers for a moment.) I don't know, if I had more zeros... like 100 into this 13000...
8. I: Well 100 goes into 13000, 130 times.
9. K: So I'd get 130.

5c

1. K: (Divides 400 into 4 and gets .01.)
2. K: I guess he would have to have .01 of everything that John has. I don't know if that makes sense or not...
3. K: He would have to have 1% of everything that John has, like 1 cent of everything... .01.
4. I: Can you tell me why you divided 400 into $4.00?

5c (con't)

5. K: 400%, that will give me how much I think John's father would make. Like every time John would get a cent he would get .01 of that.
6. I: So your answer is .01, then?
7. K: Not really, cause that's what John's father makes so if he earns $4.00 an hour, he would have to earn 400% more...could be 400% times.
8. K: (Multiplies 400 by 400 and gets 1600.) So 1600.
9. I: What if you set up a rate pair for that.
10. K: (Writes 4.00/400 x/100, cross multiplies, cancels the two zeros and divides 400 by 4 and gets 100.) 100.
11. I: 100 what?
12. K: $100.00
13. I: Every week or...?
14. K: An hour, but that's got to be a lot, though, $100.00 an hour. This would be...hold it...it could be $1.00 too. It's either $100.00 or $1.00 but if John's father earns 400% of what John earns...it doesn't tell you if he earns more or if he earns less.
15. I: Oh, I see, that question doesn't tell you if John's father earns more or less?
16. K: No. It says John's father earns 400% more.
17. I: It says John's father earns 400% <u>of</u> what John earns.
18. K: I guess $100.00 is a lot so it is probably $1.00. But the father would earn a lot more than the son would, so I'll say $100.00.

5c (i)

1. K: (Writes: 3/3 x/4, solves for x and gets 4) 4 metres will hang over.
2. I: Could you draw a picture of that to show what it would look like?
3. K: (Draws a table 3 x 3) Let's see, this would be 4 metres cause 4 metres hang over. No this would be 1 cm because this is 3 metres and this is 4 metres so 1 metre would hang over.

6c

1. K: (Adds 50 and 30 and gets 80.)
2. I: Why did you add 50 and 30?
3. K: Well because that's the pages and that's the minutes.
4. I: Oh, I see.
5. K: (Divides 80 into 120, cancels the zeros and gets 1.5.) 1.5, that doesn't answer my question.
6. K: Okay, 50 over 80 and 30 over 40. (Cross multiplies and gets 2000 and 2400. Divides 2000 into 2400 and gets 12.)
7. K: Who is the faster reader? Ohhh, it's the differences. I think Lorne would be the faster reader because of this difference here is 10 and this is 30.
8. I: A difference between 30 and 40 of 10 and a difference between 50 and 80 of 30?
9. K: Yeah, Lorne would be the faster reader.

7c

1. K: It would depend on what they got before. Well if one got more money before the increase, it would depend on what they got before.

PROTOCOL II AVERAGE SUCCESS RATE

CATHY SESSION 1

1a

1. C: (Draws a rectangle with width 4 and length 5) Okay, if the room measures 4 metres by 5 metres then you times 4 by 5 to get 20.
2. I: Umhm.
3. C: Then you times 20 by the amount it cost of a metre for the carpet.
4. I: Umhm.
5. C: (Talks while multiplying 5.95 by 20 and gets 119.00) So...get 119 dollars to carpet the room.
6. I: Could you write that down.
7. C: (Writes: 119.00 to carpet the room.)

2a

1. C: (Draws a rectangle 5 by 12 and draws a larger one with the small one in the corner of the large one) Do I have to enlarge this (the length) to 18 mm?
2. I: Umhm.
3. C: Okay, to enlarge that it would be 6 more there so then you would have to enlarge this (the width) to 6 more so that would be 6 more there.
4. I: Okay, so that's your answer then?
5. C: You mean what's the answer to the question?
6. I: Yeah.
7. C: 11.

2a'

1. C: Okay, 2/3 and Mr. Big is 4 something. Mr. Small is 4/6 the height of Mr. Big so umh the common denominator is 28, I mean 24.
2. I: Of 6 and 4?
3. C: Yeah 24 and 6 goes into that 4 times so 4 goes into it 6 times. Oh it's not working here...umh 2/3 (reduces 4/6)
4. C: (Trys to find 2/3 of 40) Okay if this is 2/3 then what is the percentage of the 4?
5. C: Okay, so this is 2 bigger than this so this is 2 smaller than would be...uh that wouldn't work...
6. C: This would be about 2 1/2 or...
7. I: Oh 2 and a little bit but you're not sure how to get the number?
8. C: No. Well I'll just say 3/4 or something, well it's around 66% or something like that.
9. I: Uhhu. Now what is your answer to the question?
10. C: Okay, Mr. Small's height is 3 paper clips, no 2 paper clips (laughs) 2 1/2 paper clips.
11. I: Somewhere between 2 and 3?
12. C: Yeah (Writes: 2 - 3).

3a
1. C: If he had $50 then he had 75 on Monday.
2. I: Umhm.
3. C: Oh...???....25% (Writes: 25%).
4. I: 25%, can you tell me why 25%?
5. C: Well he had 50 and made 25 more so thats 25% umh he earned $75 and so had 50 on Friday and 75 on Monday, so he had 25% more than he had earned.

4a
1. C: (Talking while dividing 100 into 280 and gets 28). Oh it comes out to $28. (laughs)
2. C: Oh I know. (Writes down 50-14, 25-7, 12 1/2 - 3 1/2)
3. I: Can you Tell me what you're doing there?
4. C: I'm trying to figure out how much each percent would be.
5. I: Oh I see, so...
6. C: 14 is 50%, 7 is 25%, and 3 1/2 is 12 1/2%....Oh so, let's see... that would be about...I don't know how to do it...
7. C: Oh now I know. (Writes 12 x 3 and gets 36)
8. C: (Writes: 28.35¢) It's probably wrong though, $28.36.
9. I: I see, because every percent is...?
10. C: 3 or someting.

5a
1. C: (Divides 30 into 80 and gets 2 2/3).
2. C: (Writes: 40 questions on the test)
3. I: Can you tell me where that 40 came from?
4. C: Okay if he had 80% and 30 into 80 goes 2 2/3 times and 80 questions is 2 2/3, no 30 questions is 2 2/3 of the test so if there is 3/3. Oh hang on, wait a minute, that would mean 30 questions...2/3... oh that would be 45 (changes the 40 to 45).
5. I: Oh I see.
6. C: 30 into well if 2/3 of the questions were 30 then you would have to divide that into half 'cause there is a 2 there (in the 2/3) and that would be 15 and we are missing 1/3 so that would be 15 more.
7. I: Oh.

6a
1. C: (Divides 150 into 450 and gets 3)
2. C: (Divides 110 into 300 and gets 2 remainder 80)
3. C: (Reduces 8/110 to 40/55)
4. C: The 300 gm package that cost $1.10 cause for every, hold it, hang on, for every....it's 2 and 40/55 and here it's 3 so it's the 300 gm package that cost $1.10.
5. I: Is the cheaper one?
6. C: Yeah.
7. I: Okay
8. I: Now I'm not sure why you decided that.
9. C: Neither am I.

6a (con't)
10. I: Oh you're not. You just did this...
11. C: Yeah that's what I think.
12. I: Do you think that's (the answer) from what you did here?
13. C: Yeah (laughs)...Well we know how many of these we got right and that?
14. I: When we are finished all of the interviewing I could tell you.

7a
1. C: In this question right here how much was it out of like...cause there would be...Like she got 75 questions correct on social and 30 on science but the social could be out of 100?
2. I: Okay. If that question were on a test what would you write down. Would you write down what you just told me?
3. C: Yeah. It depends on what the two tests were out of.

CATHY SESSION 2 DIVISION

1
1. I: What is the relationship between multiplication and division; can you tell me something about multiplication and division that ties them together?
2. C: Well when you times a number by another number and if you take whatever you get, your sum or whatever you call it. (I: Product.) Yeah, product, if you take that number and you divided it by one of the numbers you'd get the other number.
3. I: Could you give me an example using numbers?
4. C: Say 7 times 9 is 63 and 63 into (means divided by) equals 7. (Writes 7 x 9 = 63, and 63 divided by 9 equals 7 in algorithmic form.)

2
1. I: What are the a, the b and the c called in a into c equals b? (Writes it in algorithmic form.)
2. C: The a is the divisor. I don't know what c or b are.
3. I: The c is the dividend, and b is the quotient.

3
1. I: Can you write this (as in 2) so it means the same, using this sign (÷)?
2. C: (Writes $c \div a = b$). That's c divided by a equals b.

4
1. I: If c is constant and a increases what does b do?
2. C: Decreases.
3. I: Why?
4. C: Because you're dividing a bigger number into another so less of that number can go into it.

5
1. I: If a is constant and c increases, what does b do?
2. C: It increases, because as c gets larger this number (a) is going to go into it more times.

6
1. I: If b is constant and a increases what does c do?
2. C: It increases because if b is constant and a increases then c has to increase to make this (b) the same if this (a) is going to go into it (c) the same amount of times.

Division Questions

1 DQ 1008 ÷ 56
1. C: (Divides while talking out loud, and tests 8 x 56 out at the side, gets 18 remainder 0.)

2 DQ 60 ÷ 4.3
1. C: You have to move the decimal here (in 4.3) so you have to move it here (in 60) so you add a zero.
2. I: Why?
3. C: Because what you do to one you have to do to the other. I don't know, I learned it somewhere.
4. I: Why did you have to move it in the 4.3?
5. C: Because you can't divide by decimals, it makes it easier, I guess.
6. C: (Divides 43 into 60, gets 1 remainder 17, brings down the zero.)
7. C: (Tests 4 x 43 (because 4 x 4 = 16), finds it too big, and uses 3 and gets 41 remainder.) So it's 13 remainder 41 over 30.
8. I: Could you take it to one decimal place?
9. C: (Tests 9 x 43 at the side and gets 387, puts 380 rather than 387 under the 410, subtracts, and gets 30 remainder.) So it's 13.9 remainder 30.

3 DQ 32 ÷ 429
1. C: (Writes question as if it were 429 divided by 32.)
2. I: You've written that down wrong.
3. C: (Rewrites it properly). I can't do this.
4. I: You don't know how to do it?
5. C: Well I could add some zeros.
6. I: Would it change that number if you added zeros?
7. C: Yeah, there's no decimal points.
8. I: Can you divide 6 into 3?
9. C: Yeah, it's 1/2.
10. I: What's that as a decimal?
11. C: .5.
12. I: Does that help you?
13. C: Nope, I have an idea, but I wouldn't do it right.
14. I: Try it.
15. C: I'll divide 32 into 429 to see if it's a fraction.

3 DQ (con't)
16. I: And then what will you do?
17. C: I don't know.
18. I: Well let's just say that 32 goes into 429 uh 14.2 times. Then what would you say?
19 C: I'd say that 429 goes into 32, 1/14 times.
20. I: And what would you do with that decimal 2?
21. C: I don't know.
22. I: And if it went exactly 14 times,
23. C: Then it would be 1/14.
24. I: And what would that be as a decimal?
25. C: About point 1,...
26. I: Does that help you?
27. C: Wouldn't 1/14 be point one four?
28. I: So can you do the question now?
29. C: That's the only way I know how to do it, (divide 32 into 429). (Multiplies 32 by 14 and gets 448, and decides 14 is too big. Multiplies 13 by 32 and gets 416.) That would be about right.
30. C: (Divides 13 into 429 and gets 33) So I'm one off.
31. I: So what is the answer to the original question?
32. C: Point 13. Point one three.
33. I: How would you check that?
34. C: By timsing dividing 429 by 13 and see.
35. I: Could you do that?
36. C: I already did that and it came out to 33.
37. I: Oh, so .13 isn't quite right.
38. C: No.

4 DQ 54.2[illegible]
1. C: (Divides 26 into 54 and places the 2 of the quotient above and slightly to the left of the 5, rather than above the 4, gets 22 remainder and decides the answer is 20 remainder 22.)
2. I: Could you take that to a couple decimal places?
3. C: Umh 22 over 26, that's about three fourths, so it's about point 75? I don't know if that's right.
4. I: Could you find the answer without using that remainder like that?
5. C: Oh, wouldn't it be point two two?
6. I: And how would you check it?
7. C: Umh, times 20.22 times 26. (Gets 525.72)
8. I: Is it the right answer?
9. C: No, oh maybe this (the decimal point) would be there. (Moves the decimal so it is 52.572). Yeah it's close to the right answer. I moved the decimal the wrong way, I moved it this way (from the left) and should have moved it this way (from the right).
10. I: I see, you moved it over from the 2 and you should have moved it from the 5. Can you tell me a reason for moving it over two places either way?
11. C: Because that's the way you multiply decimals, if the 26 were 2.6 then you'd have to move it over three places because of the one in the 2.6 and the two in the 20.22.

4 DQ (con't)
12. I: If you didn't have a pen to do this multiplication, could you approximate what 20.22 times 26 would be?
13. C: Umh...20 times 26 would be about 400 and uh...oh yeah.
14. I: So it would be in the hundreds then?
15. C: Yeah.
16. I: So is your answer to your division question right?
17. C: No!
18. I: Do you think you could fix up your division answer without actually dividing again, using the information you got from your multiplication?
19. C: No.
20. I: Could you do something to that 20.22 so that when you multiplied by 26 you got closer to 54.2?
21. C: I don't think so.
22. I: Okay let's just go on to number 5.

5 DQ 8.71÷37
1. C: (Moves the decimal in .37 and in 8.71 so they become 37 and 871, puts decimal for quotient above decimal of dividend.) 37 goes into 87 twice so that's 2 (puts the 2 above and between the 8 and the 7, continues and gets 23 remainder 20.)
2. I: Could you divide [illegible] into 35 and check it?
3. C: (Divides and gets 6 remainder 5, multiplies 6 by 5 and adds the 5 remainder to get 35.

6 DQ 8245÷85
1. C: There would be two.
2. I: Why?
3. C: Because the 85 doesn't go into 82 so you'd have to go to the 824. It would be 8 times because 10 is too big and 9 would probably be a bit too big because 9 times 5 is 45 and you'd have to carry the 4.
4. I: What if it were 81 instead of 85?
5. C: 9 would probably be okay then.

6b DQ 9356÷54
1. C: There would be 3 digits because 54 goes into 93, and the first digit would be 1.

6c DQ 24156÷732
1. C: There would be two digits because 732 won't go into 241[illegible]t would have to go into 2415 and the first number would be 3.

6d DQ 143115÷145
1. C: There would be 3 digits because it would have to go into 1431 and the first digit would probably be 9.

7a DQ 4328÷10
1. I: If you can, do them in your head.
2. C: Umh, 10 won't go into 4 so 10 into 43 goes 4.

7a DQ (con't)
3. I: So you're imagining that it is written down.
4. C: Can you estimate it?
5. I: Yeah.
6. C: Four, three, two....432 and 1/8.
7. I: 1/8 for that 8 there eh?
8. C: Or 8/10.
9. I: Which is it, 1/8 or 8/10?
10. C: 8/10.
11. I: What would that be as a decimal?
12. C: Point 8, I think or point zero eight, one or the other.

7b DQ 4328 ÷ 100
1. C: That's 43.28.

7c DQ 4328 ÷ 1000
1. C: That's 4.328.

CATHY SESSION 3

1b
1. C: (Draws room and indicates [illegible] 6 x 4.) If he adds 2 to the lengt[illegible]to add 2 to the width.
2. I: Why?
3. C: Becau[illegible]e does here (to the length) he has to do here (to [illegible]).
4. I: Would [illegible]satisfy the question?
5. C: Yes I [illegible] so, yes you have to add 2 to the width.

2b
1. C: (Divides .02 into 2 and gets 100.) So in 2 seconds the machine could do 100, yowee, that's wrong.
2. I: Why?
3. C: Because if it can [illegible] oh yeah, that's right. In 2 seconds it could do 100.
4. I: What made you decide to divide .02 into 2?
5. C: Because you have to find out how many times .02 goes into 2 to find out how many times it could do it. If it does this once in .02 seconds then you have to find how many .02 seconds there are in 2 seconds to find how many it could do in 2 sec.

3b
1. C: You have to minus this (10.8 - 3.6) and you get 7.2 then add this (7.2 and 7.2) to find out what the total is and you get 14.4 and 7.2 is half of 14.4 so it's 50%.
2. I: It's 50% of what?
3. C: Of his weight.
4. I: Of which weight?

3b (con't)
5. C: Uh of the weight he was when he was born, he gained 50 more per cent of that weight.
6. I: Of 3.6, what would 50% of 3.6 be?
7. C: (Divides 3.6 by 2.) It's 1.8.
8. I: Did he gain that much, 1.8?
9. C: No he gained 7.2.
10. C: (Divides 1.8 into 7.2 and gets 4.) Umh?
11. I: You divided but you're not sure why?
12. C: No.
13. I: If 1.8 is 50% what is 7.2?
14. C: 200%.
15. I: Why?
16. C: Because 1.8 is 50% and 1.8 goes into 7.2 four times and 4 times 50 is 200.
17. I: Does that seem like a reasonable answer.
18. C: Yeah, I guess so. (Writes 7.2/3.6 2/0.) Yeah it's 200, because that's (7.2) is twice as much as that (3.6) so it's 200.

3b (i)
1. C: There were 400 altogether so you add these two (110 and 220) and get 330 and minus it from 400, so he got 70.

4b
1. C: I don't know how to do this so I'll just guess at it. That's about 10 more per cent so 10 goes into this (42000) 42 times. So 10% more of his salary, okay, I'll just say 10% more of his salary would be about 47,000 this year, I don't know.
2. I: Why 47,000?
3. C: Because 42 is almost half of one hundred so uh there's, oh I don't even know.
4. I: Why did you say 10%?
5. C: Because 8.5 is almost 10.
6. I: So now you're finding 10% of 42000? (C: Yeah.) Can you set up a rate pair up to solve that?
7. C: (Writes n/100.) I don't know if I'm doing this right.
8. I: Why did you put n over 100?
9. C: Because I don't know what the number is so I have to put n.
10. I: What did your teacher say to put over the 100?
11. C: She said the n.
12. I: You put n over if you don't know the what?
13. C: The number.
14. I: I see, let's go on to the next question.

5b
1. C: This is 300% and this (35) is 100% so that would be 3 more of what she's earning so it's 3 times 35 so it's $1.05.
2. I: Is that reasonable?
3. C: Yeah, I get $1.00, $1.25.

5b (i)

1. C: 25.5 per cent have brown eyes, uh 25.5 students have brown eyes so it would be 25 or 26.
2. I: Would 24 be just as good?
3. C: Well, it's approximate (it would be pretty close) but 25 and 26 would be the two best answers.

6b

1. C: (Adds 15 and 8) Add 8 km and 15 km to find out how far she has to go and if it's 50 km an hour then, uh let's see, in an hour she should go 50 km so in half an hour she should go 25 km so she was not speeding.
2. I: What would she have to go to be speeding?
3. C: 26 km or anything above that.

7b

1. C: I don't understand this.
2. I: Well, read it out loud.
3. C: (Reads it out loud.) Well, I don't think the expenses would be the same because the food might cost more or less than the cost of the rent.
4. I: Would they ever be the same?
5. C: If they paid 250 dollars for food and 250 dollars for rent.
6. I: When would the expenses be more?
7. C: If the food or the rent cost more than the money, oh, hang on here, if the rent cost more than the food.

CATHY SESSION 4

1c

1. C: Okay, it's 180 miles from Calgary to Edmonton and 350 miles from Edmonton to Saskatoon, so add those two and get 530 miles from Calgary to Saskatoon.
2. I: Does that answer the question?
3. C: Oh, it's in kilometres. So I'll divide 60 into 530.
4. I: How come?
5. C: Because there are 60 km to oh no, there are 60 miles to every 100 kilometres.
6. I: Could you set up a rate pair for that, 60 miles for every 100 kilometres or is it 60 kilometres for every 100 miles.
7. C: Yeah, it's 60 miles for every 100 kilometres so it would be 530 over <u>x</u> and 60 over 100. (Solves for <u>x</u> and gets 88 1/3.) (She had 530 X 100 = 5300.)
8. I: So what is the answer?
9. C: Eighty uh no that's not right, it doesn't look right.
10. I: What should it be about?
11. C: Eight hundred. Oh the decimal should be here (moves the decimal so it is 883 1/3.) 883 1/3 miles.

1c (con't)
12. I: How did you make a mistake?
13. C: I guess I din't need to bring down that extra zero (after the decimal point in 5300). I don't know. (Realizes she did do the division correctly but doesn't realize she made the mistake in the multiplication.) It just doesn't seem right.
14. I: So what's your answer?
15. C: 883 1/3 miles.

2c
1. C: (Writes 75/60 = 90/x, solves and gets 72.) No that isn't right either.
2. I: 72 isn't right, why?
3. C: Because if he delivered 60 newspapers in, oh yes that's right, I thought he delivered 75 but he delivered 60.

3c
1. C: (Writes 7500/100 = 1500/x, cross multiplies and gets 7500 and 15000, divides and gets 2.) I got 2 but I'm not sure why. I don't know where to put all these.
2. I: You don't know where to put all these numbers? Why have you got 7500 over 100?
3. C: I don't know, because that's the population over 100.
4. I: Why over 100?
5. C: I'm not sure.
6. I: Okay let's go on to the next question.

3c (i)
1. C: How come they have that 15% in the question?
2. I: I don't know, what does it have to do with the question?
3. C: Nothing.
4. I: Can you answer the question without it then?
5. C: Umhm, 31.

4c
1. C: (Writes 2000/100 x/6.5, multiplies 2000 by 6.5 and gets 130000 and puts the decimal between the 1 and the 3, writes 13.000 over at the side then scratches it out.) I don't know how to do these per cent questions.
2. I: Okay, let's go on.

5c
1. C: John's father makes $16.00 an hour.
2. I: Where did the 16 come from?
3. C: Because he makes four times as much as him.
4. I: How do you know he makes four times as much?
5. C: Because he makes 400% of what John earns.

5c (i)
1. C: This 4 metres here, does it mean that it is 4 by 4 by 4 by 4?

5c (i) (con't)

2. I: Read the second sentence again. (C: Reads it aloud.) Does something in that sentence help you?
3. C: It just says it has a side of 4.
4. I: A side, and what kind of a tablecloth is it?
5. C: Oh, yeah...umh...(Draws a table with the cloth on it but not centred, the cloth covers the table and just hangs over two sides, she adds a bit on to the other two sides.) It would hang over a half a metre.
6. I: How did you get that.
7. C: Because if you have a table and you add on here (one side) that makes that four and here (the other side) and here that makes that four and here, that makes that four. And if the tablecloth is centred.
8. I: Oh, you don't have it centred?
9. C: No, so if you have it centred then half of this would be over here, so half a metre.

6c

1. C: It doesn't have to be an exact answer does it?
2. I: Well I want to know why you picked your answer.
3. C: Okay, Lorne is the faster reader because in 40 minutes he read 30 pages and she read 50 pages in 80 minutes, so if you double this he would have read 60 pages, so Lorne is the faster reader.

7C

1. C: It depends on how much each of them make.
2. I: I see.
3. C: If one made $1,000 and the other made $900 a month then uh...I guess uh...if he made more it would be less of a percent increase and if he made less it would be more of a percent increase.

PROTOCOL III HIGH SUCCESS RATE

DON SESSION 1

1a
1. D: Uh, okay if it's $5.95 a square metre and it's 4 metres by 5 metres so then that makes it...you times 4 by 5 and get 20 by $5.95. (Multiplies 5.95 by 20 and gets 119.00.)
2. D: Then you get $119.
3. I: Are you finished, that's your answer?
4. D: Yeah.
5. I: Okay.

2a
1. D: Okay you put 12 if you put...let's see, 12 over 5 (Writes: 12/5) if the 12 would become 18 then like the same number that goes into each so this is 8 to 6 so you add another 6 (to the 12) and that gives 18, so half of 5, so it would be 7.5 mm.
2. I: Umhm, and that's your answer?
3. D: (Hands the paper to I.)

2a'
1. D: So you'd get 6 over 4 and then this group is 4 then you'd have to get another fraction that equals it so that means that must have been divided by...I don't know...probably be...oh...so he would be 2.5 paper clips tall cause you'd cross multiply and put them equal. (Has written down 6/4 = 4/2.5.)
2. I: Cross multiply?
3. D: (Indicates to multiply 6 by 2.5 and 4 by 4.)
4. I: Oh, I see and then they're equal when you cross multiply.
5. D: Yeah, so Mr. Small would be 2.5 paper clips tall.
6. I: Okay.

3a
1. D: Okay you've got to uh divide $25 into $50 and which gives you 2 so $25 is half of 50, so it would be...it would be 150% because if you add the 50, so you double it and then double the 25, so it's...I think...(To this point all he has written down is 150%)
2. I: Oh you double the 50.
3. D: Yeah, you double the 50, then double the 25.
4. I: Oh, I see.

4a
1. D: Okay, uh (a long pause then he divides 12 into 2800).
2. D: I think you add 2 zeros to the 28 and divide by 12...and the 3 keeps on repeating so it's 2 hundred and 33 with the 3 repeating, so then you would put decimal places so it's $2.33, so you add $28.00 with the $2.33 and get $30.33.

4a (con't)

3. I: For your answer?
4. D: Yeah.

5a

1. D: (Knocks on desk, long pause.)
2. I: You're not telling me what you're thinking.
3. D: I'm trying to figure out how you do it.
4. D: I think this time you add two zeros to the 30 and then divide by 80 (talks while dividing 80 into 3000 and gets 375). I think there would be...I don't know how it worked out but...37 1/2 questions.
5. I: Is that your answer then?
6. D: Yeah, it's wrong though cause they wouldn't have a question for half a mark.
7. I: Oh, half a question?
8. D: Yeah.
9. I: Okay.

6a

1. D: Okay, you'd divide the $1.50 into 450, so can't have two zeros there, so move the zeros and add two zeros to the end of the other one so that would be 3, 3000 and then you'd take off from the end so that in the first one each cookie would be 30 grams.
2. D: And the second one you'd divide $1.10 into the 300 grams then you'd switch the zeros over and add them so that would be twice and (remainder) 8000, so that would be 7000...700....that would leave you with 300 and 110 that would keep on going, it would be 272.7 and keep on repeating the 27.
3. I: Umhm.
4. D: So the answer for that would be 27.27 repeating and gee...then you'd divide the 30 into the $1.50 which would give you 5 cents and the 27 into $1.10 so...and that would be about (approximates it in his head) 3.7 cents or something like that.
5. I: Umhm.
6. D: Hm...So let's say...I'm not sure but probably the first one would be better.
7. I: A better buy?
8. D: Yeah.
9. I: Okay.

7a

1. D: (Writes: It all depends on how many questions.)

DON SESSION 2 DIVISION

1

1. I: Can you tell me the relationship between multiplication and division?

1 (con't)
2. D: Well, if you have two numbers say 7 and 6, 7 times 6 is 42 and then if you had 42 and just one of the numbers say 7 then it's 6, they're all the same three numbers that you use.

2
1. I: If you have a into c equals b (writes it in algorithmic form) what are the a, b, and the c called?
2. D: Uh, a is the divisor, b is the quotient and c is the dividend.

3
1. I: Could you write that using this sign (÷)?
2. D: It would be c divided by a equals b (Writes: c ÷ a = b.)

4
1. I: If c is constant and a increases, what happens to b?
2. D: Uh...it decreases.
3. I: Why?
4. D: If a gets bigger b has to get smaller to make c the same.

5
1. I: If a is constant and c increases, what does b do?
2. D: Uh...b would get bigger too because if c gets bigger then if a stays the same then you have to times it by a bigger number. So it would increase.

6
1. I: If b is constant and a increases what does c do?
2. D: Uh..c would increase too because if a gets bigger then that would mean a bigger number if you times the two (a and b).

Division Questions

1 DQ 1008 ÷ 56
1. D: It's 56 into 1008 so 56 doesn't go into 10, so 56 goes into 100 once, so it's 44 remainder and bring down the 8 and 56 into 44 and 56 is closer to 6 so 6 into 44 about 8 times (tests 8 out at the side). So it is 8 and the answer is 18.

2 DQ 60 ÷ 4.3
1. D: You can't have the decimal here in the divisor so I move it (4.3 becomes 43) and I'll just add another to the dividend.
2. I: Why do you have to add the zero in the dividend.
3. D: What you do to the divisor you have to do to the dividend so it will equal out.
4. D: 43 goes into 60 once (puts the 1 in the tens place multiplies and subtracts and gets 17 remainder). So 43 into 170 would go about 4 times and I'll just multiply it to check (multiplies

2 DQ (con't)

out at the side and gets 172). It's 2 over so it will go 3 times and 43 times 3 is uh 129 and 41 remainder.

5. I: Could you work that question out to one decimal place?
6. D: You have 41 remainder and I put the decimal down so it's 41.0 and 43 goes into it (checks 9 out at the side) uh 9 times and (multiplies, subtracts and gets 3.3 remainder). So it's 13.9 and 3.3 remainder.
7. I: Could you check that without dividing again?
8. D: It will be 43 times 13.9 (multiplies and gets 597.70, he describes what he is doing as he is doing it) and then you add the remainder 3.3 and oh I made a mistake, the remainder is 2.3 so I add 2.3 and get what I divided into. (Changes the remainder to 2.3.)

3 DQ 32÷429

1. D: 429 into 32 and that doesn't go into it so you have to use decimals so 429 goes into you can kind of like say 3200.
2. I: Oh, you're just ignoring the decimals.
3. D: Yeah, uh 429 into 3200 goes (tests 7 and gets 3003) 7 times (puts a decimal zero, then the 7.)
4. I: Why did you put that zero there?
5. D: Well, 429 doesn't go into 320 so you have to put the zero there to show it is two places over. (Finishes and gets .07 remainder 1.97).

4 DQ 54.2÷26

1. D: That's 26 into 54.2, so 26 into 54 goes twice (multiplies, subtract and gets 2.2 remainder) so that's 2.2 left over if you go to decimal places and 26 won't go into 22, so I put a zero up at the top and bring down the zero and now it's 26 into 220, so it's 8 times, I'll times it at the side.
2. I: What made you try 8?
3. D: Well, I rounded 26 off to 30 and I thought 3 into 22 and 8 times 3 is 24 but it is a little bit less. (Multiplies 26 by 8 and gets 208, finishes and gets 2.08 and .12 remainder.)

5 DQ 8.71÷37

1. D: Okay it's .37 into 8.71 and you can't have the decimal so...
2. I: Do you think you could do it and then place the decimal?
3. D: Yes, but it would take too long. You move it two to the right and move it in the dividend over two so it's 37 into 871. 37 goes into 87 twice, so it's 74 and subtract and get 13 remainder. So it's 37 into 131 and it goes 3 times and that's 111 and subtract and get 20 remainder and you have to go another decimal point over, so it's 37 into 200 and it would go 6 times (starts multiplying the 6 and realizes it is too big and changes it to 5.)
4. I: Where did you get the 6 from?
5. D: Well, I was thinking 37 but 37 is closer to 40 so it's 4 into 20, so it should be 5. Decimal 5 times 37 is (multiplies and

5 DQ

gets 175 rather than 185, subtracts and gets 25 remainder). So it's 23.5 and 25 remainder.

6a DQ 8245÷85

1. D: There would be 2 digits because 85 can't go into 82, so you'd have to use the next one and the first number would probably be about uh...I'd say 8 or 9.
2. I: Which would you try first?
3. D: The 9 but 9 times 9 would be 81 and 9 times 5 it would be over then.

6b DQ 9356÷54

1. D: There would be 3 because right away 54 goes into 93 and the first one would be 1.

6c DQ 24156÷732

1. D: There would be two digits in front because 732 wouldn't go into 241, so you'd have to go to the 5. The first digit would be 3.
2. I: Why 3?
3. D: 7 times 3 is 21 and that would be close to 24 but if it was 4 it would go into 28 and be too big.

6d DQ 143115÷145

1. D: It would be 3 because 145 almost goes into 143 but not quite, so you'd have to go to the next one and the first number would be 9 because 145 is so close to 143.

7a DQ 4328÷10

1. D: I think it would be 432.8.
2. I: You think?
3. D: Well 10 you just take off the zero from there and take away one of the values from there (from 4328).

7b DQ 4328÷100

1. D: It would be two decimal points over for the 100 so it would be 43.28.

7c DQ 4328÷1000

1. D: It would be three over so it would be 4.328.
2. I: Could you explain that?
3. D: Well, if your dividing with a number with zeros like 10 or 100 then the answer will be the same number as the dividend but with the decimal point moved over unless the divisor is 1.
4. I: Could you justify this one using mulitiplication?
5. D: (Multiplies 4.328 times 1000 and gets 4328.)

DON SESSION 3

1b

1. D: 6 metres by 4 metres so that is 6 over 4 and if he adds 2 metres to the length, so that would be 8 over x. 6x equals 32 uh... I think it's 6 over 4 and x over 8 so that would be about 5.4 so you'd have to add about 1.4 metres to the width.
2. I: Would that double the area?
3. D: No, uh...it would be 6 over 4 equals x over 8. It would be 4x equals 48, so then x equals 12. So then it would double the area.
4. I: What would?
5. D: It would be wider by 6 metres.

Note: We returned to question 1b after doing 7b, the protocal follows:

1b

1. I: Let's go back to number 1 and you draw a picture for it. You got the wrong answer before, so try again.
2. D: (Draws a rectangle 4 by 6.) It's 4 metres here and here and 6 metres here and here. And he wants to make the area twice as large and he adds 2 metres to the length so that would mean it's 8 on the sides, so to make it, it would be let's see....8 over 6 and n over 4...oh that's the same way as I had before. Oh, he added one third of 6 to give it 8 so he added 1/3, so 1/3 of 4 would be about 1 1/3 then across he would have to add more, so it would be 4 and 1 and about point 3 so it would be about 5 1/3 across to double it.
3. I: Does that double the area?
4. D: Let's see....(writes down 24 below the first rectangle and 48 below the second rectangle) so this (width of the second rectangle) would have to be 6 then.
5. I: So the area of the first rectangle is...?
6. D: 24 and you have to add 2 to this so it's 8 and then 24 twice cause doubled is 48 and 8 into 48 is 6, so this side would have to be 6.
7. I: So what is the answer to the question?
8. D: He would have to add 2 metres.
9. I: Does this proportion you set up have something to do with the question?
10. D: Uh, yeah but I'm not sure what.

2b

1. D: In .02 seconds there would be (divides .02 into 1 and gets 50) so then if that's for 1 then for 2 seconds you'd times it by 2 and get 100 so it could do 100 in 2 seconds.

3b

1. D: It would be 3.6 over 10.8 equals n over 100 and then you cross multiply and get 10.8x equals 36. So it's 10.8 into 36 and you

3b (con't)
have to move the decimal so it's 108 into 360, divide (and gets 3) and it would go about 2 times but then to average there is still just point 1 here so n would be about 3.3. Oh, per cent, uh I started this out wrong.

2. I: You are going to start again?
3. D: Yeah, 3.6 over 10.8 and n over 100.
4. I: That's the same as you had before.
5. D: Yeah, let's see 10.8n equals 36 uh...
6. I: Let's just leave that as it is. How much did the baby increase in weight?
7. D: Oh, now I know what to do. You'd minus 10.8 and 3.6 so that would be 7.2. So that would be 7.2 over 10.8 equals n over 100, so that would be 10,8 into 72. Move the decimal over, so it's 108 into 720. It goes 6 times (multiplies and subtracts and gets 820 remainder) so it's about 67% increase. But then it was 3 uh 33 before, so then you could just minus that from 100 and then you would have got 67 and the increase is 67%.

3b (i)
1. D: Add 110 and 220 and get 330, then minus 330 from 400, so Lorne got 70 votes.

4b
1. D: You have the per cent for both because 8.5 goes over 100, because use 100 for the base, and then you want to know what the increase is so it would be over 42,000, n over 42,000. So that would be (cross multiplies and gets 100n = 357,000) and then if you divide 100 into it you could just knock off the two zeros off the 100 and the 357,000, so it's 3570. So if that was his increase you would add 42000 and 3570 (writes them down and subtracts). So that would be, oh ugh, this is adding, I was minusing. (Adds 42,000 and 3570 and gets 45,570.) So his pay this year is 45,570.
2. I: Is that a reasonable answer?
3. D: Yes, because it has gone up but not that much.

5b
1. D: Uh, let's see, he got 300%, so that goes over 100 and if her mom got 35 cents then that would go with the 100 (on the bottom) and then n would be over 35 (writes it down and solves it and gets 105) so n equals 105 so then she got or Mary got $1.05. An easier way of doing it would be, 300 is just 3 times as much so you could just times the 35 by 3.

5b (i)
1. D: You know that 50%, so 50 over 100 and then n over 51 and then to take a short cut 50 is half of 100, so then half of 51 is 25.5 instead of going through all the timsing and stuff, so 25 and a half people have brown eyes.

5b (i) (con't)
2. I: Is that a reasonable answer, 25 and a half people have brown eyes?
3. D: Well, it could be like and a half, so then if you rounded it off so it could be 26 then.
4. I: Would 25 be just as good for an answer?
5. D: Yes.
6. I: What about 24?
7. D: It would be getting off more but then if half of them have it then it would be good because 24 and 24 is 48 and there's 51 students, so that would be about half.

6b
1. D: Okay, so you'd add 8 and 15 so that would give you 23 kilometres and the speed limit is 50 and she gets to the shopping centre in half an hour, so that means she would have to be travelling half of...like if it's 50 kilometres per hour then in half an hour it would be 25 so if she was travelling 23 and then if half is 25 then she wouldn't be speeding, she would be going 2 km under.

7b
1. D: Umh, they could either be more or...let's see...it could either be the same or less but then if the rent was 100 dollars it would be 110 dollars but then if the food cost 50 dollars it would go up to 55, so it would be the same then because they would both equal out to the same amount. They'd have to spend more then, oh decrease, so if the rent is 100 dollars and it went up 10% it would be 110 dollars and if food was 50 dollars, then 10% of 50 would equal out to 45 dollars so...they would have before 105 dollars total for both rent and food and now they pay 155 dollars for both rent and food, so now it would be more.
2. I: Would it necessarily have to be more?
3. D: Well, if it was 100 dollars each, then it would be the same. And if the rent was less than what the food cost then it would be less.

DON SESSION 4

1c
1. D: To get miles to kilometres you are supposed to times it by point 6, so times 350 miles times point 6 and that's 210, so 210 kilometres from Edmonton to Saskatoon. Then you add 288 kilometres that it takes to get from Edmonton to Calgary and then the 210 kilometres that it uh...(long pause) oh, it can't work out because there are more kilometres than miles up here (288 and 180) so it would have to be uh, try a rate pair...180 miles and you just double it and it's 360, so you could just double the 288 then take off a bit so it would be...270 about, so 558

1c (con't)
kilometres from Calgary to Saskatoon via Edmonton.

2c
1. D: If he delivers 60 in 75 minutes and so you put 60 over an hour and a half is 90 minutes, no, over 75 and it equals how ever many paper in 90 minutes (has 60/75 = x/90) cross multiply so 75x equals 5400. (Divides and gets 61.2, multiplies 75 by 61.2 and gets 4590.0.)
2. I: So you divided and got 61.2 and multiplied to check and got 4590, so you think you did something wrong?
3. D: Yeah, so I'll divide again. If I take 75 minutes, that's 5 quarters of an hour, so you divide 5 into 60 and get 12, so then you need one more quarter of an hour, so then he'd be able to do 72 papers.

3c
1. D: (Writes 7500/9000 = x/100, cross multiplies) So 9000x = 75000 and you just cross off the three zeros from the 9000 and three from the 750000, so 9x equals 750. It's 83.3 and the 3 will keep on going.
2. D: Oh, I could do it another way, if 75000 is increased 1500 then 75000 is 100% and...and 1500 goes into 75000 five times so 5 into 100 goes 20 and it's increased once more so it's increased by 20%, the population.
3. I: It's one more 1500, you mean?
4. D: Yeah, it's 20%.

3c (i)
1. D: Does this question ask how many girls there are or how many honor students?
2. I: Just how many girls.
3. D: If there were 61 students you take 30 from 61, so there were 31 girls.

4c
1. D: If it increased by 6.5%, then that goes over 100 and uh try the 2000 on the bottom (writes x/2000 = 6.5/100) (cross multiplies but multiplies 2000 by 6.2 rather than 6.5 and gets 12400.0, divides by 100 and gets 124). So x equals 124 and that would be the 6.5% and you add that to the 2000 bushels, so it's 2124 altogether.

5C
1. D: The easiest way to do it would be to just times 400% over 4 just times by 400, no because if $4.00 is 100% then 4 times 4 is 16, so his dad earns $16.00 an hour.
2. I: Why did you multiply the 4 by 4?
3. D: Because if $4.00 is 100% and he gets 400% more that's 4 times as much.

5c (i)

1. D: (Draws a square and indicates that it is 3 metres and draws the tablecloth hanging over one side.) It's 3 metres by 3 metres and the tablecloth is 4 metres on either side then the tablecloth would be 1 metre farther out but if it's even on the table it will be half a metre so half a metre will hang down.

6c

1. D: If 40 minutes is half of 80, so if she went 50 pages in 80 minutes and then to make it equal, Lorne to equal it you times 40 by 2 to give you 80 and the 30 by 2 and get 60, so Lorne read more pages.

7c

1. D: I don't know what this question is asking.
2. I: Read it out loud.
3. D: (Reads it out loud.)
4. I: They each had <u>a</u> salary and they each received $200 more and I'm asking if the percent increase is the same for each of them?
5. D: It depends if who has the more money, like if one has $100.00 and one has $200.00, then if the guy with $100.00 has $200.00 added on to his, then his increase will be twice as much but the guy that has $200.00 gets $200.00, his will only be doubled and the other guy's will be tripled.