Positive definite functions and spherical h-harmonic expansions with negative indices on spheres

by

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Abstract

This thesis consists of the following two parts:

Part I: Positive definite functions on unit spheres.

Part II: Spherical h-harmonic expansions with negative indices.

In the first part, we study positive definite functions on the unit sphere $\mathbb{S}^d$ of the Euclidean space $\mathbb{R}^{d+1}$ equipped with the usual geodesic distance ρ. A continuous function $g : [-1,1] \to \mathbb{R}$ is called positive definite on $\mathbb{S}^d$ if for any $N \in \mathbb{N}$ and any set of N distinct points $\mathbf{X}_N := \{x_1, \cdots, x_N\}$ on $\mathbb{S}^d$, the $N \times N$ matrix $g[\mathbf{X}_N] := [g(\rho(x_i, x_j))]_{i,j=1}^N$ is positive definite, and it is said to be positive semi-definite if the matrix $g[\mathbf{X}_N]$ is nonnegative positive definite. Our main interest in this part is the following longstanding conjecture on positive definite functions on spheres:

Conjecture. *Let $\delta \geq \frac{d+1}{2}$. Then for any $\theta \in (0, \pi)$, the function*

$$f_{\theta,\delta}(t) = (\theta - t)_+^\delta$$

is isotropic positive definite on $\mathbb{S}^d$.

We first confirm this conjecture in the case when the dimension d is odd.

Theorem 0.0.1. *Let d be an odd integer ≥ 3. Suppose that g is a continuous function on $[0, \infty)$ with compact support in $[0, \pi]$. If g is isotropic positive definite on $\mathbb{R}^d$, then so it is on $\mathbb{S}^d$.*

Our next result reveals a close connection between the positive definite functions on $\mathbb{R}^d$ and $\mathbb{S}^d$. In this situation, the restriction on dimensions can be removed. We state it as below.

Theorem 0.0.2. *Suppose that g is a continuous function on $[0, \infty)$ with compact support on $[0, \pi]$. Let $d \in \mathbb{N}$, if g is isotropic positive semi-definite on $\mathbb{S}^d$, then it is also positive*

semi-definite on $\mathbb{R}^d$.

We also partially confirm the conjecture for small parameters θ when the dimension d is even. The method used there works for all higher even dimensions. To give numerical estimate on error terms, we will present proof when $d = 2$.

Theorem 0.0.3. *Let $d = 2$ and $\delta \geq \frac{3}{2}$. The function $f_{\theta,\delta}(t) = (\theta - t)_+^\delta$ is isotropic positive definite on $\mathbb{S}^2$ when $0 < \theta < C_{A,B}$, where $C_{A,B}$ is an absolute constant.*

In the second part, we study the spherical h-harmonic expansions with negative indices. We extend some of the classical results from Chanillo and Muckenhoupt [10] to the Cesàro means of the weighted orthogonal polynomial expansions (WOPEs) in several variables with respect to the weight function

$$h_\kappa(x) := \prod_{i=1}^{d} |x_i|^{\kappa_i}$$

on the unit sphere $\mathbb{S}^{d-1}$ for all parameters $\kappa_1, \cdots, \kappa_d > -\frac{1}{2}$. It is worth to point out that when the index is nonnegative, that is $\kappa_{\min} := \min_{1 \leq i \leq d} \kappa_i \geq 0$, the problem has been studied in a series of papers [15–18, 30], where the nonnegative assumption is essential in these works. However, many arguments do not work if one of the parameters κ_i is negative, in which case that the above mentioned WOPEs on the sphere $\mathbb{S}^{d-1}$ is still well defined if $\kappa_{\min} > -\frac{1}{2}$.

Our aim is to deal with the negative indices. More precisely, we develop a new technique to establish sharp pointwise estimates for the corresponding Cesàro kernels, which works for the full range of $\kappa_{\min} > -\frac{1}{2}$. This means we fully settle the problem for the case when $\min_{1 \leq j \leq d} \kappa_j < 0$. We believe that this new technique will, in particular, lead to a simpler proof of the estimates of Chanillo and Muckenhoupt [10, Theorem 14.1] on the Cesàro kernels of the Jacobi polynomial expansions with parameters $\alpha, \beta > -1$. We also establish similar results for the corresponding WOPEs on the ball $\mathbb{B}^d = \{x \in \mathbb{R}^d : \|x\| \leq 1\}$ and the simplex $\mathbb{T}^d = \{x \in \mathbb{R}^d : x_1 \geq 0, \ldots, x_d \geq 0, x_1 + \cdots + x_d \leq 1\}$, as was observed by Xu [39].

Preface

Part I of this thesis is contained in a joint paper [23] as H. Feng and Y. Ge, Isotropic positive definite functions on spheres. (will appear soon)

Part II of this thesis has been published in [14] as F. Dai and Y. Ge, Sharp estimates of the Cesàro kernels for weighted orthogonal polynomial expansions in several variables, Journal of Functional Analysis, Volume 280, Issue 4, 15, February 2021.

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Contents

Part I

Positive definite functions on the unit sphere $\mathbb{S}^{d-1}$

Chapter 1

Introduction

Positive definite functions are very important in both theory and applications of approximation theory, probability, and statistics. One major application of positive definite functions theory is to analyze radial basis functions for interpolating scattered data; see, for example [22],[29],[9] and [36]. In particular, identifying positive definite functions, or, more generally, positive definite kernels, is of great interest as interpolation problems corresponding to these kernels are guaranteed to be well-posed.

We begin with the general definition of (strictly) positive definite function.

Definition 1.0.4. *Given a metric space (Ω, ρ), a radial real-valued function K is called (strictly) positive definite on Ω if for any integer $N \in \mathbb{N}$ and $\mathbf{X}_N = \{x_1, \ldots, x_N\} \subset \Omega$, the corresponding $N \times N$ symmetric matrix*

$$K[\mathbf{X}_N] = \Big\{ K\Big(\rho(x_i, x_j)\Big) \Big\}_{i,j=1}^{N}$$

is positive definite, which means

$$\mathbf{c}^T K[\mathbf{X}_N]\mathbf{c} > 0, \quad \textit{for all nonzero } \mathbf{c} \in \mathbb{R}^N.$$

Motivations. We consider the problem of interpolating data measured at scattered locations in general metric space. Let Ω be a metric space. Given a finite set $\mathbf{X}_N := \{x_1, \cdots, x_N\}$ of distinct points in Ω and a target function $f : \Omega \to \mathbb{R}$, find a continuous function S_f (depending on f): $\Omega \to \mathbb{R}$ that satisfies the interpolation conditions

$$S_f(x_i) = f(x_i), \quad 1 \leq i \leq N.$$

Now, assume S_f is a linear combination of certain basis functions $K(\rho(x, x_j))$, $j = 1, 2 \ldots N$, that is

$$S_f(x) = \sum_{j=1}^{N} c_j K(\rho(x, x_j)),$$

for $c_j \in \mathbb{R}, 1 \leq j \leq N$. Solving the interpolation problem under this assumption leads to a system of linear equations of the form

$$K[\mathbf{X}_N] \cdot C = f.$$

The scattered data fitting problem will be feasible, that is a solution to the above problem will exist and be unique, if and only if the matrix $K[\mathbf{X}_N]$ is non-singular. Therefore, the situation is favourable if we know in advance that the matrix is positive definite. This is a motivation to investigate whether a function is positive definite in the given metric space.

Here are some important remarks of positive definite functions:

- Radial (or isotropic, spherically symmetric) function is a function whose value depends only on the distance between the input and some fixed point.

- If K is a real-valued positive definite function on Ω, then $K(\rho(x, y)) = K(\rho(y, x))$ for all $x, y \in \Omega$.

- Positive definite functions can also be defined for complexed valued functions. We can also similarly define positive semi-definite functions in the Definition 1.0.4 if the matrix $K[\mathbf{X}_N]$ is nonnegative definite.

- By Schur's theorem for Hadamard products of two positive definite matrices, it follows that if $K_1, K_2 : \Omega \times \Omega \to \mathbb{R}$ are two positive definite functions, so is $K_1 K_2$.

Since many problems are often generated by real-world applications, we then describe some results of positive definite functions on both Euclidean space and the unit sphere in the following chapters. For an abstract theory of positive definite functions, the readers are referred to the book [28]. In the next chapter, we will study the characterizations of positive definite functions on $\mathbb{R}^d$ and $\mathbb{S}^{d-1}$, including two remarkable results: Bochner's theorem and Schoenberg's theorem.

Chapter 2

Preliminaries

2.1 Positive definite functions on $\mathbb{R}^d$

From the previous definition and the discussion done on introduction, if we denote ρ be the Euclidean metric, we have the following definition.

Definition 2.1.1. *A function $f : \mathbb{R}^d \to \mathbb{C}$ is called positive definite on $\mathbb{R}^d$, if for each finite subset $\mathbf{X}_N = \{x_1, \ldots, x_N\} \subset \mathbb{R}^d$, the matrix $f[\mathbf{X}_N] = [f(x_i - x_j)]_{i,j=1}^N$ is a positive definite matrix. It is said to be positive semi-definite if the matrix $f[\mathbf{X}_N]$ is positive semi-definite.*

Comments.

- If $f : \mathbb{R}^d \to \mathbb{C}$ is positive definite on $\mathbb{R}^d$, then $f(-x) = \overline{f(x)}$ for each $x \in \mathbb{R}^d$, and $\max_{x\in\mathbb{R}^d} |f(x)| = f(0) > 0$.
- If $f : \mathbb{R}^d \to \mathbb{C}$ is positive definite on $\mathbb{R}^d$, and is continuous at 0, then f is uniformly continuous on $\mathbb{R}^d$.

Examples.

- If μ is a probability measure on $\mathbb{R}^d$, then its Fourier transform

$$\widehat{\mu}(x) = \int_{\mathbb{R}^d} e^{-2\pi i x \cdot \xi}\, d\mu(\xi) \tag{2.1.1}$$

  is positive semi-definite on $\mathbb{R}^d$.

- If $g \in L^1(\mathbb{R}^d)$ and $g(x) > 0$ for a.e. $x \in \mathbb{R}^d$, then $\widehat{g}(x)$ is a positive definite function on $\mathbb{R}^d$.

2.2 Characterizations of the positive definite functions on $\mathbb{R}^d$

One of the most celebrated results on positive semi-definite functions is their characterization in terms of Fourier transforms, which was established by Bochner in 1932 for $d = 1$ in [5], and 1933 for general d in [6]. If μ is a positive Borel measure on $\mathbb{R}^d$, then the computation in the example (2.1.1) can be extended to see that $\widehat{\mu}$ is a positive semi-definite function on $\mathbb{R}^d$. Bochner established the converse:

Theorem 2.2.1 (Bochner)**.** *In order that a function $f : \mathbb{R}^d \to \mathbb{C}$ be positive semi-definite and continuous, it is necessary and sufficient that it be the Fourier transform of a nonnegative finite-valued Borel measure on $\mathbb{R}^d$.*

This gives an important tool to characterize the positive semi-definite functions on $\mathbb{R}^d$. Some examples are worthwhile to list.

- The Gaussian $f(x) = e^{-\pi|x|^2}$ is a positive definite function as $\widehat{f}(\xi) = f(\xi) > 0$.

- $f(x) = e^{-\pi|x|}$ is a positive definite function since $\widehat{f}(x) = \frac{c_d}{(1+|x|^2)^{\frac{d+1}{2}}} > 0$, where c_d is a positive constant.

- The truncated power function $f_\delta(x) = (1-|x|)_+^\delta$ is a positive definite function on $\mathbb{R}^d$ when $\delta \geq \frac{d+1}{2}$. Indeed,

$$\begin{aligned}\widehat{f_\delta}(\xi) &:= c_d \int_0^1 r^{d-1}(1-r)^\delta j_{\frac{d-2}{2}}(2\pi r|\xi|)\,dr \\ &= c_d'|\xi|^{-d-\delta}\int_0^{2\pi|\xi|} t^{d-1}(2\pi|\xi|-t)^\delta j_{\frac{d-2}{2}}(t)\,dt,\end{aligned}$$

where $j_\alpha(z) = c_\alpha z^{-\alpha}J_\alpha(z)$ and $J_\alpha(z) := (\frac{z}{2})^\alpha \sum_{k=0}^\infty \frac{(-1)^k}{k!\Gamma(k+\alpha+1)}(\frac{z}{2})^{2k}$ is the Bessel function of the first kind for $\alpha > 0$. The following result was proved by Gasper [26], which can be stated as follows: for $\alpha > -\frac{1}{2}$ and $x > 0$,

$$\int_0^x (x-t)^{\alpha+\frac{3}{2}} t^{2\alpha+1} j_\alpha(t)\,dt > 0.$$

Setting $\alpha = \frac{d-2}{2}$, we conclude that if $\delta \geq \frac{d+1}{2}$, then $\widehat{f_\delta}(\xi) > 0$ for all $\xi \in \mathbb{R}^d$, and hence, $f_\delta(x) = (1-|x|)_+^\delta$ is a positive definite function on $\mathbb{R}^d$.

2.3 Positive definite functions on $\mathbb{S}^{d-1}$

For interpolation of some data, such as geodetic, geological, and meteorological information gathered over the Earth's surface, the data locations are known to lie on the sphere's surface. This leads us to consider a specialization of the basis function defined on unit spheres. In this section, we study positive definite functions on the unit sphere $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : \|x\| = 1\}$ equipped with the usual geodesic distance $\rho(x,y) := \arccos\langle x, y\rangle$, much of which originated with I. J. Schoenberg [34].

Definition 2.3.1. *A continuous function $g : [-1,1] \to \mathbb{R}$ is called positive definite on $\mathbb{S}^{d-1}$ if for any $N \in \mathbb{N}$ and any set of N distinct points $\mathbf{X}_N := \{x_1, \cdots, x_N\}$ on $\mathbb{S}^{d-1}$, the $N \times N$ matrix*

$$g[\mathbf{X}_N] := [g(\cos\rho(x_i,x_j))]_{i,j=1}^N = [g(\langle x_i, x_j\rangle)]_{i,j=1}^N$$

is positive definite.

Examples.

- An important result is that zonal spherical harmonics

$$Z_n(t) := \frac{n+\lambda}{n} C_n^{\frac{d-2}{2}}(t), \quad t \in [-1,1], \quad n \in \mathbb{N},$$

  is positive definite function on $\mathbb{S}^{d-1}$, as can be seen from the fact that the Gegenbauer polynomial $C_n^{\frac{d-2}{2}}(t)$ is positive definite on $\mathbb{S}^{d-1}$. Thus, positive definite functions are closely related to spherical harmonics.

- Positive definite functions on $\mathbb{R}^d$ can be restricted to $\mathbb{S}^{d-1}$. This can be seen from the norm relation: $\|x-y\|^2 = 2 - 2\langle x, y\rangle$.

2.4 Characterizations of the positive definite functions on $\mathbb{S}^{d-1}$

The importance of (strictly) positive definiteness is its connection with the posedness of interpolation. Therefore, some easy means of identifying (strictly) positive definite functions are of great interest as it will enable the assembly of a toolkit of different kernel-based interpolation methods. Positive semi-definite functions on the sphere have been studied by Schoenberg [34, Theorem 1], who proved the following theorem of Bochner type.

Theorem 2.4.1. *Let g be a continuous function on $[0,\pi]$. The function g is isotropic positive semi-definite on $\mathbb{S}^{d-1}$ if and only if g has the Gegenbauer expansion*

$$g(\theta) = \sum_{n=0}^{\infty} a_n C_n^{\frac{d-2}{2}}(\cos\theta), \quad \theta \in [0,\pi], \tag{2.4.1}$$

in which all of the coefficients

$$a_n := \int_0^\pi g(\theta) C_n^{\frac{d-2}{2}}(\cos\theta)(\sin\theta)^{d-2} d\theta \geq 0, \quad \forall\, n \in \mathbb{N}_0 \tag{2.4.2}$$

and $\sum_{n=0}^\infty a_n C_n^{\frac{d-2}{2}}(1) < \infty$.

The characterization of (strictly) positive definite functions on $\mathbb{S}^{d-1}$ came somewhat later. A simple sufficient condition [37] states that g is (strictly) positive definite if, in addition to the conditions of Theorem 2.4.1, all the Gegenbauer coefficients a_n are positive. Chen, Menegatto and Sun [11] showed that a necessary and sufficient condition for g to be (strictly) positive definite on $\mathbb{S}^{d-1}$, $d \geq 3$, is that, infinitely many of the Gegenbauer coefficients with odd indices, and infinitely many of those with even indices, are positive. This was later established for the case $\mathbb{S}^1$ in [32]. Dai and Xu also gave a self-contained proof in [16, Section 14.3].

Schoenberg's result to characterize positive semi-definite functions is classical. Unfortunately, given a function g, checking the signs of all the Gegenbauer coefficients can be an impossible task. Therefore, it is natural to seek simpler sufficient conditions that guarantee (strictly) positive definiteness. Confronting this difficulty, a Pólya type of criterion was established in [4] for $d \leq 8$.

Theorem 2.4.2. ***(Pólya type criterion)*** *Let* $d \in \{3, 4, ..., 8\}$ *and* $k = \lceil \frac{d-2}{2} \rceil$. *Let the real-valued function* g *on* $[0, \pi]$ *satisfy the following conditions:*

(i) $g \in C^k[0, \pi]$,

(ii) $supp(g) \subset [0, \pi)$,

(iii) the derivative, from the right, $g^{(k+1)}(0)$ *exists, and is finite,*

(iv) $(-1)^k g^{(k)}$ *is convex.*

Then g is a positive semi-definite function on $\mathbb{S}^{d-1}$. If, in addition to the above properties, $g^{(k)}$, restricted to $(0,\pi)$, does not reduce to a linear polynomial, then g is a (strictly) positive definite function on $\mathbb{S}^{d-1}$.

In the same reference [4], it showed that the Pólya type of criterion continues to hold for all $d > 8$, if we can prove the function

$$f_{\theta,\delta}(t) := (\theta - t)_+^{\delta} = \begin{cases} (\theta - t)^{\delta}, & t \leq \theta, \\ 0, & t > \theta, \end{cases}$$

where $\delta > 0$, $\theta \in (0,\pi]$, is (strictly) positive definite on $\mathbb{S}^{d-1}$ when $\delta \geq \lceil \frac{d}{2} \rceil$.

Our main interest is the following more general longstanding conjecture on positive definite functions on spheres, stated in [4]. The importance of this conjecture lies in the fact that it leads to a sharp Pólya type criterion for any dimensions.

Conjecture 1. *Let $\delta \geq \frac{d+1}{2}$. Then for any $\theta \in (0,\pi]$, the function*

$$f_{\theta,\delta}(t) = (\theta - t)_+^{\delta}$$

is isotropic positive definite on $\mathbb{S}^d$.

2.5 Some useful formulas

To better describe our results in the following chapters, in this section, we shall introduce the Jacobi polynomials, some needed preliminaries and standard notions, which will be used throughout the rest of this part.

2.5.1 Jacobi and related orthogonal polynomials

For parameters $\alpha, \beta \in \mathbb{R}$ and $n \in \mathbb{N}$, the Jacobi polynomials are defined by

$$P_n^{(\alpha,\beta)}(x) = \frac{1}{n!}\sum_{k=0}^{n} \frac{\Gamma(n+\alpha+\beta+k+1)}{\Gamma(n+\alpha+\beta+1)} \frac{\Gamma(\alpha+1+n)}{\Gamma(\alpha+k+1)} \Big(\frac{x-1}{2}\Big)^k,$$

where $x \in [-1,1]$. The Gegenbauer polynomials are defined by

$$C_n^\lambda(x) = \frac{\Gamma(\lambda+\frac{1}{2})\Gamma(n+2\lambda)}{\Gamma(2\lambda)\Gamma(n+\lambda+\frac{1}{2})} P_n^{(\lambda-\frac{1}{2},\lambda-\frac{1}{2})}(x).$$

We will use $R_n^\lambda(x) := \frac{C_n^\lambda(x)}{C_n^\lambda(1)}$ as the normalization of Gegenbauer polynomials.

Below, we collect several formulas and properties of the Jacobi polynomials needed in the proofs. Our main reference is the classical treatise by [35].

(i) [35, (7.32.2)]: For $\alpha > -1, \beta > -1$, and $x \in [-1,1]$,

$$|P_n^{(\alpha,\beta)}(x)| \le P_n^{(\alpha\,\beta)}(1) = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)}.$$

(ii) [35, (4.22.2)] For a positive integer $1 \le \ell \le n$ and $\beta \in \mathbb{R}$,

$$P_n^{(-\ell,\beta)}(x) = \frac{\binom{n+\beta}{\ell}}{\binom{n}{\ell}} \Big(\frac{x-1}{2}\Big)^\ell P_{n-\ell}^{(\ell,\beta)}(x), \quad n \ge \ell. \tag{2.5.1}$$

(iii) [35, (4.21.7)] For $\alpha, \beta \in \mathbb{R}$,

$$\frac{d}{dx} P_n^{(\alpha,\beta)}(x) = \frac{1}{2}(n+\alpha+\beta+1) P_{n-1}^{(\alpha+1,\beta+1)}(x). \tag{2.5.2}$$

(iv) [35, (4.1.3)] For $\alpha, \beta \in \mathbb{R}$,

$$P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x). \tag{2.5.3}$$

Using (2.5.1) and (2.5.3), we obtain that for any positive integers n, ℓ, m with $n > \ell + m$,

$$P_n^{(-\ell,-m)}(x) = (-1)^\ell \Big(\frac{1-x}{2}\Big)^\ell \Big(\frac{x+1}{2}\Big)^m P_{n-\ell-m}^{(\ell,m)}(x). \tag{2.5.4}$$

In particular,

$$P_n^{(-1,-1)}(x) = -\frac{1-x^2}{4} P_{n-2}^{(1,1)}(x), \tag{2.5.5}$$

which, by (2.5.2), also implies

$$\int P_n(x)\,dx = \frac{2}{n} P_{n+1}^{(-1,-1)}(x) = -\frac{1-x^2}{2n} P_{n-1}^{(1,1)}(x). \tag{2.5.6}$$

(v) [35, (8.21.18)] For $n^{-1} \le \theta \le \pi - n^{-1}$ and $\alpha, \beta > -\frac{1}{2}$,

$$P_n^{(\alpha,\beta)}(\cos\theta) = \pi^{-\frac{1}{2}} n^{-\frac{1}{2}} \Big(\sin\frac{\theta}{2}\Big)^{-\alpha-\frac{1}{2}} \Big(\cos\frac{\theta}{2}\Big)^{-\beta-\frac{1}{2}} \Big[\cos(N\theta+\tau) + O(1)(n\sin\theta)^{-1}\Big],$$

where $N = n + \frac{\alpha+\beta+1}{2}$ and $\tau = -\frac{\pi}{2}(\alpha+\frac{1}{2})$. In particular, in [27, Page 295], we have for $\theta \in (0, \frac{\pi}{2})$,

$$P_n(\cos\theta) = \frac{2}{\pi^{\frac{1}{2}}} \frac{\Pi(n)}{\Pi(n+\frac{1}{2})} \frac{\cos(N\theta - \frac{\pi}{4})}{(2\sin\theta)^{\frac{1}{2}}} + p_{n,1}(\cos\theta)$$

where

$$|p_{n,1}(\cos\theta)| \le \frac{4}{\pi^{\frac{1}{2}}} \frac{\Pi(n)}{\Pi(n+\frac{1}{2})} \frac{1}{2(2n+3)} \frac{1}{(2\sin\theta)^{\frac{3}{2}}},$$

and $\Pi(n) = \Gamma(n+1)$.

Applying the Gautschi's inequality: $x^{1-s} < \frac{\Gamma(x+1)}{\Gamma(x+s)} < (x+1)^{1-s}$ for $s \in (0,1)$, we have

$$\frac{\Pi(n)}{\Pi(n+\frac{1}{2})} = \frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})} < (n+\frac{1}{2})^{-\frac{1}{2}},$$

and thus

$$P_n(\cos\theta) \le \frac{2}{\pi^{\frac{1}{2}}} n^{-\frac{1}{2}} \frac{1}{(2\sin\theta)^{\frac{1}{2}}} \cos\left(N\theta - \frac{\pi}{4}\right) + \frac{1}{(2\pi)^{\frac{1}{2}}(\frac{2}{\pi})^{\frac{3}{2}}} \frac{1}{(n\theta)^{\frac{3}{2}}}. \tag{2.5.7}$$

(vi) [35, (7.32.5)] For α,$\beta \in \mathbb{R}$,

$$|P_n^{(\alpha,\beta)}(\cos\theta)| \le \frac{C}{n^{\frac{1}{2}}(n^{-1}+\theta)^{\alpha+\frac{1}{2}}(n^{-1}+\pi-\theta)^{\beta+\frac{1}{2}}}.$$

We will need some special cases in the proof:

In [35, (7.3.8)], for $\alpha = \beta = 0$ and $\theta \in (0, \frac{\pi}{2}]$,

$$|P_n(\cos\theta)| \le \frac{1}{n^{\frac{1}{2}}\theta^{\frac{1}{2}}}, \tag{2.5.8}$$

and for $\alpha = \beta = 1$,

$$|P_{n-1}^{(1,1)}(\cos\theta)| \le \frac{C}{(n-1)^{\frac{1}{2}}\theta^{\frac{3}{2}}}. \tag{2.5.9}$$

We can confirm the constant C of the upper bound in (2.5.9) by using the following result which can be found in [1, Page 213] and [3]:

Proposition 2.5.1. *For every $\theta \in (0, \frac{\pi}{2}]$, $N = n + \frac{\alpha+\beta+1}{2}$,*

$$\left|\sin(\frac{\theta}{2})^{\alpha+\frac{1}{2}} \cos(\frac{\theta}{2})^{\beta+\frac{1}{2}} P_n^{(\alpha,\beta)}(\cos\theta)\right| \le \binom{n+\alpha}{n} N^{-\alpha-\frac{1}{2}} \times 2.821. \tag{2.5.10}$$

Thus using (2.5.10) with $\alpha = \beta = 1$ and replace n with $n-1$, we have

$$\left|P_{n-1}^{(1,1)}(\cos\theta)\right| \le \frac{1}{(\frac{1}{2})^{\frac{3}{2}}(\sin\theta)^{\frac{3}{2}}} \frac{n}{(n+\frac{1}{2})^{\frac{3}{2}}} \times 2.821 \le \frac{2^{\frac{3}{2}} \times 2.821}{(\frac{2}{\pi}\theta)^{\frac{3}{2}} n^{\frac{1}{2}}} \le \frac{\pi^{\frac{3}{2}} \times 2.821}{(n-1)^{\frac{1}{2}}\theta^{\frac{3}{2}}}. \tag{2.5.11}$$

(vii) [2, p. 302] For $\lambda \neq 0$,

$$C_n^\lambda(\cos\theta) = \sum_{k=0}^{n} \frac{\Gamma(\lambda+k)\Gamma(\lambda+n-k)}{k!(n-k)![\Gamma(\lambda)]^2} \cos[(n-2k)\theta]. \tag{2.5.12}$$

(viii) [35, (4.5.4)]

$$P_n^{(\alpha,\beta+1)}(x) = \frac{2}{2n+\alpha+\beta+2} \frac{(n+\beta+1)P_n^{(\alpha,\beta)}(x) + (n+1)P_{n+1}^{(\alpha,\beta)}(x)}{1+x}. \tag{2.5.13}$$

(ix) [35, (4.1.5)]

$$P_{2n}^{(\alpha,\alpha)}(x) = \frac{\Gamma(2n+\alpha+1)\Gamma(n+1)}{\Gamma(n+\alpha+1)\Gamma(2n+1)} P_n^{(\alpha,-\frac{1}{2})}(2x^2-1). \tag{2.5.14}$$

2.5.2 An extensional relation about Jacobi polynomials

In this section, we prove an extensional relation Lemma 2.5.1 on the Jacobi(Gegenbauer) polynomials that will be useful in the proof of Theorem 3.1.1. Precisely, we can write $R_n^\lambda(\cos t)(\cos\frac{t}{2})^{2\lambda}$ in terms of the linear combination of the terms $R_{2j}^\lambda(\cos\frac{t}{2})$, with $j = n, ..., n+\lambda$, and more importantly, all the coefficients are nonnegative. This relation plays an important role in our proof.

Lemma 2.5.1. *If λ is a positive integer, then there exists a sequence $\{a_{n,j}^\lambda\}_{j=n}^{2n+2\lambda}$ of positive numbers such that for any $t \in [0,\pi]$,*

$$R_n^\lambda(\cos t)(\cos\frac{t}{2})^{2\lambda} = \sum_{j=n}^{n+\lambda} a_{n,j}^\lambda R_{2j}^\lambda(\cos\frac{t}{2}). \tag{2.5.15}$$

Proof. First, applying the formula (2.5.13), substituting x by $\cos(\frac{t}{2})$, α by $\alpha-\frac{1}{2}$, and β

by $\beta - \frac{1}{2}$, we get for $t \in [0, \pi]$

$$(\cos\frac{t}{2})^2 P_n^{(\alpha-\frac{1}{2},\beta+\frac{1}{2})}(\cos t) = A_n^{\alpha,\beta} P_n^{(\alpha-\frac{1}{2},\beta-\frac{1}{2})}(\cos t) + B_n^{\alpha,\beta} P_{n+1}^{(\alpha-\frac{1}{2},\beta-\frac{1}{2})}(\cos t), \tag{2.5.16}$$

where

$$A_n^{\alpha,\beta} = \frac{n+\beta+\frac{1}{2}}{2n+\alpha+\beta+1}, \quad B_n^{\alpha,\beta} = \frac{n+1}{2n+\alpha+\beta+1}.$$

Thus, using (2.5.16) λ times, we obtain

$$C_n^{\lambda}(\cos t)(\cos\frac{t}{2})^{2\lambda} = \sum_{j=n}^{n+\lambda} \gamma_j P_j^{(\lambda-\frac{1}{2},-\frac{1}{2})}(\cos t),$$

where $\gamma_j, n \leq j \leq n+\lambda$ are some nonnegative coefficients.

Next, to complete the proof, we will use the Lemma 2.5.14 by substituting x by $\cos(\frac{t}{2})$, α by $\alpha - \frac{1}{2}$, and get for $t \in [0, \pi]$ and $\alpha > -\frac{1}{2}$,

$$P_n^{(\alpha-\frac{1}{2},-\frac{1}{2})}(\cos t) = \frac{\Gamma(n+\alpha+\frac{1}{2})\Gamma(2n+1)}{\Gamma(2n+\alpha+\frac{1}{2})\Gamma(n+1)} P_{2n}^{(\alpha-\frac{1}{2},\alpha-\frac{1}{2})}(\cos\frac{t}{2}).$$

□

2.5.3 A useful bridge for the connection between Bessel functions and Jacobi polynomials

This section provides the following well-known relation [25] [35, (8.21.17)] on Jacobi polynomial and Bessel function that plays a crucial role in our proofs.

Proposition 2.5.2. *For $\alpha > -\frac{1}{2}$, we have for $t \in (0, \pi)$,*

$$\frac{P_n^{(\alpha,\alpha)}(\cos t)}{P_n^{(\alpha,\alpha)}(1)} = 2^{\alpha}\Gamma(\alpha+1)\Big(\frac{t}{\sin t}\Big)^{\alpha+\frac{1}{2}}\Big[j_{\alpha}(Nt) + O(n^{-1})\Big], \tag{2.5.17}$$

where $N = n + \alpha + \frac{1}{2}$, $j_{\alpha}(z) = z^{-\alpha}J_{\alpha}(z)$, and here and in what follows $J_{\alpha}(z)$ is the Bessel *function of the first kind,*

$$J_\alpha(z) = \sum_{v=0}^{\infty} \frac{(-1)^v (\frac{z}{2})^{\alpha+2v}}{v!\Gamma(v+\alpha+1)}.$$

The O-term is uniform with respect to $t \in [0, \pi - \varepsilon]$, ε being an arbitrary positive number.

In particular, when $\alpha = \beta = 0$, [25, (4.37), (4.38)] provides a useful consequence in Legendre polynomial case: for $t \in [0, \frac{\pi}{2}]$,

$$\left| P_n(\cos t) - \left(\frac{t}{\sin t}\right)^{\frac{1}{2}} J_0(Nt) \right| \leq \frac{0.1711}{n}. \tag{2.5.18}$$

2.5.4 Two asymptotic expansion formulas

The following useful asymptotic formula can be found in the book [12, p. 24, (11.5),(11.6)]:

Proposition 2.5.3. *Let $\phi(t)$ be v times continuously differentiable in $\alpha \leq t \leq \beta$. Let $\phi(t)$ and its first $v-1$ derivatives vanish when $t = \beta$. Then, if $0 < \lambda < 1$, as $N \to \infty$,*

$$\int_\alpha^\beta e^{iNt}(x-\alpha)^{\lambda-1}\phi(t)dt = \sum_{n=0}^{v-1} \frac{\Gamma(n+\lambda)}{n!N^{n+\lambda}} e^{iN\alpha + \frac{1}{2}\lambda\pi i + \frac{1}{2}n\pi i}\phi^{(n)}(\alpha) + O(N^{-v}),$$

where

$$O(N^{-v}) = \frac{1}{N^v}\int_\alpha^\beta |\phi^{(v)}(t)|(t-\alpha)^{\lambda-1}dt.$$

By a change of variable, we get

Proposition 2.5.4. *Let $\phi(t)$ be v times continuously differentiable in $\alpha \leq t \leq \beta$. Let $\phi(t)$ and its first $v-1$ derivatives vanish when $t = \alpha$. Then, if $0 < \mu < 1$, as $N \to \infty$,*

$$\int_\alpha^\beta e^{iNt}(\beta-t)^{\mu-1}\phi(t)dt = \sum_{n=0}^{v-1} \frac{\Gamma(n+\mu)}{n!N^{n+\mu}} e^{iN\beta - \frac{1}{2}\mu\pi i + \frac{1}{2}n\pi i}\phi^{(n)}(\beta) + O(N^{-v}).$$

Chapter 3

The relationship between positive definite functions on the unit sphere $\mathbb{S}^d$ and the Euclidean space $\mathbb{R}^d$

3.1 Main results

In this chapter, we study the relationship between positive definite functions on the unit sphere and the Euclidean space. We first consider odd dimensional cases and shall provide an approach to show that the positive definite property can be inherited from $\mathbb{R}^d$ to $\mathbb{S}^d$, which will immediately verify Conjecture 1 when d is odd. Our first main theorem states as follows.

Theorem 3.1.1. *Let d be an odd integer ≥ 3. Suppose that g is a continuous function on $[0, \infty)$ with compact support in $[0, \pi]$. If g is positive definite on $\mathbb{R}^d$, then so it is on $\mathbb{S}^d$.*

Some remarks are worthwhile to list.

Remark 3.1.1. *The condition of compact support is necessary. A counter example can* *be $g(t) = \exp(-t^2)$, which is positive define on all $\mathbb{R}$ but not on $\mathbb{S}^1$.*

Remark 3.1.2. *The result was claimed as well by Nie and Ma [33, Theorem 2.1]. Besides, another related and similar result was obtained due to Ma and Lu [31], in which the positive definite was proved to be preserved from Euclidean spaces to unit spheres when the dimensions are odd.*

Remark 3.1.3. *Our work is self-contained. In contrast with the proof of Theorem 3.1.1 in [33], we shall first proceed our proof to functions with high regularity, which allows us to simplify our problem by applying integration by parts. Then through inducing a proper identity approximation operator, additional regular assumptions will be avoided.*

Corollary 3.1.4. *Let d be odd and g be an isotropic continuous positive semi-definite function on $\mathbb{S}^d$ with compact support on $[0, \pi]$, then the function*

$$g(t)\Big(\frac{\sin t}{t}\Big)^{d-1}$$

is an isotropic positive semi-definite function on both $\mathbb{R}^d$ and $\mathbb{S}^d$.

The converse of Theorem 3.1.1 will be considered for the positive semi-definite functions. In this situation, the restriction on dimensions can be removed. Precisely, we state the following theorem.

Theorem 3.1.2. *Suppose that g is a continuous function on $[0, \infty)$ with compact support on $[0, \pi]$. For any dimension $d \geq 1$, if g is isotropic positive semi-definite on $\mathbb{S}^d$, then it is also positive semi-definite on $\mathbb{R}^d$.*

We also partially confirm the conjecture for small parameters θ when the dimension d is even. Actually, the method used there works for all higher even dimensions. To give a numerical upper estimate of θ, we will focus mainly on the case $d = 2$. Our main result can be stated as follow:

Theorem 3.1.3. *Let $d = 2$ and $\delta \geq \frac{3}{2}$. For $0 < \theta < C_{A,B}$, the function*

$$f_{\theta,\delta}(t) = (\theta - t)_+^\delta$$

is isotropic positive definite on $\mathbb{S}^2$, where $C_{A,B}$ is an absolute constant.

3.2 Positive definite functions on $\mathbb{S}^d$ generated from those on $\mathbb{R}^d$

3.2.1 The case of odd dimensions

We are now in a position to prove Theorem 3.1.1, that is, for an odd integer d and a continuous function g on $[0, \infty)$ with compact support in $[0, \pi]$, if g is isotropic positive definite on $\mathbb{R}^d$, so it is on $\mathbb{S}^d$. According to Schoenberg's theorem, we need to prove that the Gegenbauer coefficients of g

$$a_n = \int_0^\pi g(t) C_n^\lambda(\cos t) \sin^{2\lambda} t dt > 0, \quad \forall\, n \in \mathbb{N}_0, \tag{3.2.1}$$

and

$$\sum_{n=0}^\infty a_n C_n^\lambda(1) < \infty, \tag{3.2.2}$$

where $\lambda = \frac{d-1}{2}$.

We first verify the validity of equation (3.2.2). The proof follows a standard argument in [34]. The series $\sum_{n=0}^\infty a_n C_n^\lambda(\cos t)$ is Abel-summable for every $t \in [0, \pi]$. Hence, for $t = 0$,

$$\sum_{n=0}^N a_n C_n^\lambda(1) \leq \lim_{r \to 1^-} \sum_{n=0}^\infty a_n C_n^\lambda(1) r^n < \infty.$$

Thus we have $\sum_{n=0}^{\infty} a_n C_n^\lambda(1) < \infty$.

We now embark on proving equation (3.2.1), which will be carried out in three steps.

3.2.1.1 Step 1: Reduction

Let us begin with a series of reductions. In order to prove equation (3.2.1), we claim that it is enough to prove a slightly stronger statement: for any $n \in \mathbb{N}_0$, and $\theta \in (0, \pi]$,

$$\int_0^\theta g\Big(\frac{t\pi}{\theta}\Big) C_n^\lambda(\cos t)(\sin t)^{2\lambda}\, dt > 0, \tag{3.2.3}$$

and it is obvious to see equation (3.2.1) is exactly a particular case when $\theta = \pi$. We now continue to deal with the claim (3.2.3). Without loss of generality, in (3.2.3), we may replace $f(\frac{t}{\theta})$ with $g(\frac{t\pi}{\theta})$, i.e. $f(t) = g(\pi t)$, where f is isotropic continuous positive definite on $\mathbb{R}^d$ with compact support in $[0, 1]$. Hence, showing the claim (3.2.3) is equivalent to show for any $n \in \mathbb{N}_0$, $\theta \in (0, \pi]$,

$$\int_0^\theta f\Big(\frac{t}{\theta}\Big) C_n^\lambda(\cos t)(\sin t)^{2\lambda}\, dt > 0. \tag{3.2.4}$$

For simplicity in the proof, we will deal the following integral and verify its positivity:

$$I_n(\theta) := \theta^{-2\lambda-1} \int_0^\theta f\Big(\frac{t}{\theta}\Big) R_n^\lambda(\cos t)(\sin t)^{2\lambda}\, dt \tag{3.2.5}$$

where $\theta \in (0, \pi]$, and $n \in \mathbb{N}_0$.

Thus, by the arguments above, the previous claim equation (3.2.1) reduces to show $I_n(\theta) > 0$ for all $n \in \mathbb{N}_0$ and every $\theta \in (0, \pi]$. Furthermore, with the help of Lemma 2.5.1, we would restrict our analysis on the integral (3.2.5) on a small range of θ. More precisely, we have if $I_n(\theta) > 0$, where $\theta \in (0, \varepsilon_0)$, and ε_0 is a sufficiently small positive parameter, then $I_n(\theta) > 0$ for every $\theta \in (0, \pi]$. Indeed, this claim follows from the fact that for every $\theta \in (0, \pi]$, there exists a $\theta_0 \in (0, \varepsilon_0)$, where ε_0 is a sufficiently small positive parameter, such that $I_n(\theta)$ can be written as a sum of $I_n(\theta_0)$ with positive coefficients, which establishes the desired claim.

Therefore, in view of the whole reduction arguments above, the proof of the claim equation (3.2.1) is finally reduced to show the following Lemma:

Lemma 3.2.1. *Let d be an odd integer ≥ 3, $f \in C[0, \infty)$ with $\operatorname{supp}(f) \subset [0, 1]$. If f is an isotropic positive definite function on $\mathbb{R}^d$, then*

$$I_n(\theta) = \theta^{-2\lambda-1} \int_0^\theta f\Big(\frac{t}{\theta}\Big) R_n^\lambda(\cos t)(\sin t)^{2\lambda}\, dt > 0 \tag{3.2.6}$$

for all $n \in \mathbb{N}_0$ and every $\theta \in (0, \varepsilon_0)$, where $\varepsilon_0 \in (0, 1)$ is a sufficiently small positive parameter.

Now, we turn to the proof of the Lemma 3.2.1. We will give the proof in the following step 2 and 3.

3.2.1.2 Step 2: The proof under additional assumptions

In this step, we shall prove Lemma 3.2.1 for the functions with high regularity, that is, under the additional assumptions that $f \in C^{\lambda+3}[0, \infty)$, and $f'(0) < 0$. We divide the proof of (3.2.6) into the following three cases: (i) $n\theta \geq A$; (ii) $n\theta \leq B$; (iii) $B < n\theta < A$, where $A > 1$ and $B \in (0, 1)$ are certain parameters depending only on f.

Case (i). In this case, we shall prove that there exists a constant $A > 1$ depending only on f such that (3.2.6) holds whenever $n\theta \geq A$. First, we need a definition.

Definition 3.2.2. *Let λ be a positive integer. For $j = 1, \cdots, \lambda$, we define*

$$F_0(t) = f(\theta^{-1}t)(\sin t)^{2\lambda} \quad \text{and} \quad F_j(t) = \Big(\frac{F_{j-1}(t)}{\sin t}\Big)'.$$

Clearly, for $0 \le j \le \lambda$, $F_j \in C^{\lambda-j+3}[0,\infty)$, $F_j(t) = 0$ for $t \ge \theta$ and

$$F_j(t) = O(t^{2(\lambda-j)}), \quad \text{as } t \to 0^+.$$

The functions $F_j(t), j = 1, 2, ..., \lambda$ we defined in the above Definition 3.2.2 have the following decomposition.

Lemma 3.2.3. *For $0 \le t \le \theta$ and $\ell = 1, 2, \cdots, \lambda$,*

$$F_\ell(t) = \sum_{j=0}^{\ell} \theta^{-j} f^{(j)}(\theta^{-1}t) \sum_{k=0}^{[\frac{\ell-j}{2}]} \alpha_{\ell,k}^{(j)} (\sin t)^{2\lambda-2\ell+j+2k} (\cos t)^{\ell-j-2k}, \tag{3.2.7}$$

where the $\alpha_{\ell,k}^{(j)}$ are constants,

$$\alpha_{\ell,0}^{(0)} = (2\lambda-1)(2\lambda-3)\cdots(2\lambda-2\ell+1), \tag{3.2.8}$$

$$\alpha_{\ell+1,0}^{(1)} = \alpha_{\ell,0}^{(0)} + 2(\lambda-\ell)\alpha_{\ell,0}^{(1)}, \quad \alpha_{1,0}^{(1)} = 1. \tag{3.2.9}$$

In particular,

$$\begin{aligned} F_\lambda(t) = &\Big[(2\lambda-1)!! \cos^\lambda t + \sum_{k=1}^{[\frac{\lambda}{2}]} \alpha_{\lambda,k}^{(0)} (\sin t)^{2k} (\cos\ t)^{\lambda-2k}\Big] f(\theta^{-1}t) \\ &+ \sum_{j=1}^{\lambda} \theta^{-j} f^{(j)}(\theta^{-1}t) \sum_{k=0}^{[\frac{\lambda-j}{2}]} \alpha_{\lambda,k}^{(j)} (\sin t)^{j+2k} (\cos t)^{\lambda-j-2k}. \end{aligned} \tag{3.2.10}$$

Proof. The proof uses induction by ℓ.

When $\ell = 1$, we have

$$\begin{aligned} F_1(t) = \Big(\frac{F_0(t)}{\sin t}\Big)' &= \theta^{-1} f'(\frac{t}{\theta})(\sin t)^{2\lambda-1} + (2\lambda-1) f(\frac{t}{\theta})(\sin t)^{2\lambda-2} \cos t \\ &= \text{the right-hand side of } (3.2.7) \end{aligned}$$

and $\alpha_{1,0}^{(0)} = (2\lambda - 1)$, $\alpha_{1,0}^{(1)} = 1$.

Suppose that the statement holds for ℓ. Applying the Definition 3.2.2, we have

$$
\begin{aligned}
F_{\ell+1}(t) = &\Big(\frac{F_\ell(t)}{\sin t}\Big)' = \sum_{j=0}^{\ell} \sum_{k=0}^{[\frac{\ell-j}{2}]} \Big(\theta^{-j-1} f^{(j+1)}(\frac{t}{\theta}) \alpha_{\ell,k}^{(j)} (\sin t)^{2\lambda-2\ell+j+2k-1} (\cos t)^{\ell-j-2k} \\
&+ \theta^{-j} f^{(j)}(\frac{t}{\theta}) \alpha_{\ell,k}^{(j)} (2\lambda - 2\ell + j + 2k - 1)(\sin t)^{2\lambda-2\ell+j+2k-2} (\cos t)^{\ell-j-2k+1} \\
&+ \theta^{-j} f^{(j)}(\frac{t}{\theta}) \alpha_{\ell,k}^{(j)} (j + 2k - \ell)(\sin t)^{2\lambda-2\ell+j+2k} (\cos t)^{\ell-j-2k-1}\Big)
\end{aligned}
$$

Furthermore, to achieve our decomposition, we can rewrite the

$$
\begin{aligned}
F_{\ell+1}(t) = &\sum_{j=0}^{\ell-1} \sum_{k=0}^{[\frac{\ell-j}{2}]} \Big(\theta^{-j-1} f^{(j+1)}(\frac{t}{\theta}) \alpha_{\ell,k}^{(j)} (\sin t)^{2\lambda-2\ell+j+2k-1} (\cos t)^{\ell-j-2k}\Big) \\
&+ \theta^{-\ell-1} f^{(\ell+1)}(\frac{t}{\theta}) \alpha_{\ell,0}^{(\ell)} (\sin t)^{2\lambda-\ell-1} \\
&+ \sum_{j=0}^{\ell} \sum_{k=0}^{[\frac{\ell-j}{2}]} \Big(\theta^{-j} f^{(j)}(\frac{t}{\theta}) \alpha_{\ell,k}^{(j)} (2\lambda - 2\ell + j + 2k - 1)(\sin t)^{2\lambda-2\ell+j+2k-2} (\cos t)^{\ell-j-2k+1}\Big) \\
&+ \sum_{j=0}^{\ell} \sum_{k=0}^{[\frac{\ell-j}{2}]} \Big(\theta^{-j} f^{(j)}(\frac{t}{\theta}) \alpha_{\ell,k}^{(j)} (j + 2k - \ell)(\sin t)^{2\lambda-2\ell+j+2k} (\cos t)^{\ell-j-2k-1}\Big)
\end{aligned}
\tag{3.2.11}
$$

Observing the power of each terms in (3.2.11), we can exactly rewrite $F_{\ell+1}(t)$ by a double sum in terms of $\theta^{-j} f^{(j)}(\frac{t}{\theta})$, $(\sin t)^{2\lambda-2\ell+j+2k-2}$, and $(\cos t)^{\ell-j-2k+1}$ as j runs from 0 to $\ell+1$ and k runs from 0 to $[\frac{\ell+1-j}{2}]$. It remains to prove the coefficient satisfies (3.2.8) and (3.2.9), that is, we need to verify that

$$
\alpha_{\ell+1,0}^{(0)} = (2\lambda - 1)(2\lambda - 3) \cdots (2\lambda - 2\ell + 1)(2\lambda - 2\ell - 1), \tag{3.2.12}
$$

$$
\alpha_{\ell+1,0}^{(1)} = \alpha_{\ell,0}^{(0)} + 2(\lambda - \ell)\alpha_{\ell,0}^{(1)}, \quad \alpha_{1,0}^{(1)} = 1. \tag{3.2.13}
$$

To show (3.2.12), we need to consider the coefficient of the term $f(\frac{t}{\theta})(\sin t)^{2\lambda-2\ell-2}(\cos t)^{\ell}$

in (3.2.11), from which we have

$$\alpha^{(0)}_{\ell+1,0} = \alpha^{(0)}_{\ell,0}(2\lambda - 2\ell - 1) = (2\lambda - 1)(2\lambda - 3)\cdots(2\lambda - 2\ell + 1)(2\lambda - 2\ell - 1).$$

To show (3.2.13), we need to consider the coefficient of the term $\theta^{-1} f'(\frac{t}{\theta})(\sin t)^{2\lambda-2\ell-1}(\cos t)^{\ell}$ in (3.2.11), from which we have

$$\alpha^{(1)}_{\ell+1,0} = \alpha^{(0)}_{\ell,0} + \alpha^{(1)}_{\ell,0}(2\lambda - 2\ell).$$

This proves the identity (3.2.7) holds in the case of $\ell+1$ and completes the induction. $\square$

Recall the following well-known properties on Gegenbauer polynomials:

$$\Big(R^{\mu-1}_{n+1}(x)\Big)' = \frac{(n+1)(n+2\mu-1)}{2\mu-1} R^{\mu}_{n}(x), \tag{3.2.14}$$

$$\lim_{\mu\to 0+} R^{\mu}_{n}(\cos t) = \cos(nt). \tag{3.2.15}$$

Using (3.2.14), (3.2.15) and integration by parts λ times on (3.2.6), we obtain

$$\theta^{2\lambda+1} I_n(\theta) = c_n(\lambda) \int_0^{\theta} F_{\lambda}(t)\cos((n+\lambda)t)\, dt \tag{3.2.16}$$

for some constant $c_n(\lambda) > 0$.

Note that (3.2.8) and (3.2.9) imply that

$$\alpha^{(1)}_{\ell+1,0} + \alpha^{(0)}_{\ell+1,0} = 2(\lambda - \ell)\Big(\alpha^{(1)}_{\ell,0} + \alpha^{(0)}_{\ell,0}\Big),$$

which in turn implies that

$$\alpha_{\lambda,0}^{(1)} + \alpha_{\lambda,0}^{(0)} = 2^{\lambda}\lambda! = (d-1)!!.$$

Thus, by our assumptions on the function f, it is easily seen that F_λ has the following properties:

(i) $F_\lambda \in C^3[0,\infty)$ and F_λ is supported in $[0,\theta]$;

(ii)

$$F_\lambda'(0) = \left(\alpha_{\lambda,0}^{(0)} + \alpha_{\lambda,0}^{(1)}\right)\theta^{-1}f'(0) = (d-1)!!\theta^{-1}f'(0) < 0,$$

and $|F_\lambda'''(t)| \le C_f\theta^{-3}$ for any $t \in [0,\theta]$.

To this end, setting $N = n+\lambda$, and using integration by parts three times on the right-hand side of (3.2.16), we obtain

$$\begin{aligned}\int_0^\theta F_\lambda(t)\cos(Nt)\,dt &= -\frac{1}{N^2}F_\lambda'(0) + \frac{1}{N^3}\int_0^\theta F_\lambda'''(t)\sin(Nt)\,dt \\ &\ge N^{-2}\theta^{-1}\Big[-(d-1)!!f'(0) - C\frac{1}{N\theta}\Big],\end{aligned}$$

which is positive provided that $n\theta \ge A$ and $A = A_f$ is sufficiently large. This proves (3.2.6) in the case of $n\theta \ge A$.

Case (ii). In this case, we shall prove that there exists a constant $B \in (0,1)$ depending only on f such that (3.2.6) is true whenever $n\theta \le B$.

To see this, we first note that

$$
\begin{aligned}
I_n(\theta) &= \theta^{-2\lambda-1}\Big[\int_0^\infty f(\theta^{-1}t)t^{2\lambda}\,dt + \int_0^\theta f(\theta^{-1}t)\Big(R_n^\lambda(\cos t)\frac{\sin^{2\lambda} t}{t^{2\lambda}} - 1\Big)t^{2\lambda}\,dt\Big] \\
&\geq \int_0^\infty f(t)t^{d-1}\,dt - Cn\theta^{-2\lambda-1}\int_0^\theta t^{2\lambda+1}\,dt \\
&\geq c_d \int_{\mathbb{R}^d} f(|x|)\,dx - Cn\theta \\
&\geq c_d \widehat{f}_d(0) - CB > 0.
\end{aligned}
$$

For the first inequality, we used Bernstein's inequality for trigonometric polynomials. Indeed, for $n \geq 0$, we have

$$
\begin{aligned}
|R_n^\lambda(\cos t)(\frac{\sin t}{t})^{2\lambda} - 1| &\leq |R_n^\lambda(\cos t)(\frac{\sin t}{t})^{2\lambda} - (\frac{\sin t}{t})^{2\lambda}| + |(\frac{\sin t}{t})^{2\lambda} - 1| \\
&\leq |(\frac{\sin t}{t})^{2\lambda}| \cdot |R_n^\lambda(\cos t) - 1| + 2 \\
&\leq |R_n^\lambda(\cos t) - R_n^\lambda(\cos 0)| + 2 \\
&\leq nt\|R_n^\lambda(\cos t')\|_\infty + 2nt \leq Cn\theta,
\end{aligned}
$$

where $0 < t' < t$. The last inequality is positive under the condition $n\theta \leq B$ and B is sufficiently small.

Case (iii). In this case, we shall prove (3.2.6) for the remaining case $B \leq n\theta \leq A$. Let $N = n + \lambda$, and by substitution $t = \frac{t'}{N}$, we can write

$$
I_n(\theta) = N^{-1}\theta^{-2\lambda-1}\int_0^{N\theta} f\Big(\frac{t'}{N\theta}\Big)R_n^\lambda(\cos\frac{t'}{N})\Big(\sin\frac{t'}{N}\Big)^{2\lambda}\,dt'.
$$

Using the asymptotic formula (2.5.17), we have that for $0 \leq t' \leq N\theta \leq A$,

$$
R_n^\lambda(\cos\frac{t'}{N}) = c_\lambda\Big(\frac{t'/N}{\sin(t'/N)}\Big)^\lambda\Big[j_{\lambda-\frac{1}{2}}(t') + O(n^{-1})\Big],
$$

where $c_\lambda = 2^{\lambda-\frac{1}{2}}\Gamma(\lambda+\frac{1}{2})$. Set $O_{A,B}$ term is uniform in n, t', θ but may depend on the constants A and B. By using substitution $x = \frac{t'}{N\theta}$ and (2.5.17), we obtain that

$$\begin{aligned}
I_n(\theta) &= c_\lambda \int_0^1 f(x) j_{\lambda-\frac{1}{2}}(N\theta x) x^{2\lambda} \Big(\frac{\sin(\theta x)}{\theta x}\Big)^\lambda \, dx + O_{A,B}(n^{-1}) \\
&\geq c'_\lambda \int_0^1 f(x) j_{\lambda-\frac{1}{2}}(N\theta x) x^{2\lambda} \, dx + O_{A,B}(n^{-1}) \\
&\geq c'_\lambda \min_{B \leq u := N\theta \leq A} \int_0^\infty f(x) j_{\lambda-\frac{1}{2}}(ux) x^{2\lambda} \, dx + B^{-1}\theta O_{A,B}(1) \\
&= c''_\lambda \min_{B \leq |\xi| \leq A} \widehat{f}(|\xi|) - C_{A,B} B^{-1} \theta,
\end{aligned}$$

which is positive if θ is small enough. This shows (3.2.6) in this case.

3.2.1.3 Step 3: Additional assumptions avoided

In this step, we will show Lemma 3.2.1 holds for a general continuous positive definite function f defined on $\mathbb{R}^d$ without high regularity. The proof is based on the previous step 2. The main task in this proof is using the function f to construct a function f^ε with high regularity, such that the Gegenbauer coefficients of f^ε can approximate to the Gegenbauer coefficients of f. The construction requires two ingredients.

The first is a finite Borel measure defined on $\mathbb{R}^d$:

$$\mu_f := f_d \cdot \chi_{B_{1-\varepsilon}} + M_{f_\varepsilon} m_\varepsilon \delta_0,$$

where f_d to be the radial function on $\mathbb{R}^d$ given by $f_d(x) = f(|x|)$ for $x \in \mathbb{R}^d$; $B_r := \{x \in \mathbb{R}^d : |x| \leq r\}$ is the ball of radius r in $\mathbb{R}^d$; For $\varepsilon \in (0,1)$, $M_{f_\varepsilon} := \max_{t\in[1-\varepsilon,1]} |f(t)|$, δ_0 denotes the Dirac measure supported at the origin, and

$$m_\varepsilon := B_1 - B_{1-\varepsilon} = \Big|\{x \in \mathbb{R}^d : \ 1-\varepsilon \leq |x| \leq 1\}\Big| = d^{-1}|\mathbb{S}^{d-1}|\Big(1-(1-\varepsilon)^d\Big) \leq C\varepsilon.$$

Note that

$$\begin{aligned}
\widehat{\mu_f}(\xi) &= \int_{|x|\le 1-\varepsilon} f(|x|)e^{-2\pi i x\cdot\xi}\,dx + m_\varepsilon M_{f_\varepsilon} \\
&= \widehat{f}_d(\xi) + m_\varepsilon M_{f_\varepsilon} - \int_{1-\varepsilon<|x|\le 1} f(|x|)\cos(2\pi i x\cdot\xi)dx \\
&\ge \widehat{f}_d(\xi) + \int_{1-\varepsilon<|x|\le 1} (M_{f_\varepsilon} - f(|x|))dx > 0.
\end{aligned}$$

The second ingredient is an approximate identity:

$$\phi(x) := a_\lambda(1-|x|)_+^{\lambda+6}, \quad x\in\mathbb{R}^d,$$

where the constant $a_\lambda > 0$ is chosen so that $\int_{\mathbb{R}^d}\phi(x)\,dx = 1$. It has been known that $\widehat{\phi}(\xi) > 0$ for all $\xi\in\mathbb{R}^d$ (See [26]).

Let $x = \frac{t}{\theta}$. Note that when $|t| < \theta$, we have $0 < |x| = |\frac{t}{\theta}| < 1$. Define $f_d^\varepsilon(\frac{t}{\theta}) := f_d^\varepsilon(x)$ and

$$f_d^\varepsilon(x) := \phi_\varepsilon * \mu_f(x) = \int_{\mathbb{R}^d}\phi_\varepsilon(x-y)\,d\mu_f(y) = f_d\cdot\chi_{B_{1-\varepsilon}} * \phi_\varepsilon(x) + m_\varepsilon M_{f_\varepsilon}\phi_\varepsilon(x),$$

where $\phi_\varepsilon(x) = \varepsilon^{-d}\phi(x/\varepsilon)$.

Clearly, $f_d^\varepsilon(x) = f^\varepsilon(|x|)$ is a radial function on $\mathbb{R}^d$, and

$$\widehat{f_d^\varepsilon}(\xi) = \widehat{\phi}(\varepsilon\xi)\widehat{\mu_f}(\xi) > 0, \quad \xi\in\mathbb{R}^d.$$

Since $\phi_\varepsilon \in C_c^{\lambda+5}(\mathbb{R}^d)$ is supported in $\{x\in\mathbb{R}^d:\ |x|\le\varepsilon\}$, and since $f_d\cdot\chi_{B_{1-\varepsilon}}\in L^1(\mathbb{R}^d)$ is supported in $\{x\in\mathbb{R}^d:\ |x|\le 1-\varepsilon\}$, it follows that f^ε is a $(\lambda+5)$-times continuously differentiable function on $[0,\infty)$ that is supported in $[0,1]$, In addition,

$$(f^\varepsilon)'(0) = m_\varepsilon M_{f_\varepsilon}\phi_\varepsilon'(0) = -\varepsilon^{-d-1}m_\varepsilon(f_d\cdot\chi_{B_{1-\varepsilon}})a_\lambda(\lambda+6) < 0.$$

Applying the conclusion that has already been proven in step 2, we obtain that for any $\varepsilon \in (0,1)$ and $n \in \mathbb{N}_0$,

$$\int_0^\theta f^\varepsilon(\frac{t}{\theta}) R_n^\lambda(\cos t)(\sin t)^{2\lambda}\, dt > 0. \tag{3.2.17}$$

On the other hand, by $|R_n^\lambda(\cos t)| \le 1$, we have,

$$\begin{aligned} \Big|\int_0^\theta \Big(f^\varepsilon(\frac{t}{\theta}) - f(\frac{t}{\theta})\Big) R_n^\lambda(\cos t)(\sin t)^{2\lambda}\, dt\Big| &\le \int_0^\theta |f^\varepsilon(\frac{t}{\theta}) - f(\frac{t}{\theta})| t^{d-1}\, dt \\ &\le \int_0^\infty |f^\varepsilon(\frac{t}{\theta}) - f(\frac{t}{\theta})|\, t^{d-1}\, dt \\ &= \frac{1}{|\mathbb{S}^{d-1}|} \|f_d^\varepsilon - f_d\|_{L^1}. \end{aligned}$$

Furthermore, by the triangle inequality, we have

$$\begin{aligned} \|f_d^\varepsilon - f_d\|_{L^1} &\le \|f_d * \phi_\varepsilon - f_d\|_{L^1} + \|(f_d - f_d \cdot \chi_{B_{1-\varepsilon}}) * \phi_\varepsilon\|_{L^1} + \|(f_d \cdot \chi_{B_{1-\varepsilon}}) * \phi_\varepsilon - f_d^\varepsilon\|_{L^1} \\ &\le \|f_d * \phi_\varepsilon - f_d\|_{L^1} + \|f_d(1 - \chi_{B_{1-\varepsilon}})\|_{L^1} + C\varepsilon^{\lambda+1}. \end{aligned}$$

Now, let $\varepsilon \to 0+$, we then have $\|f_d * \phi_\varepsilon - f_d\|_{L^1} \to 0$ by the norm convergence of approximation to the identity, and the second and third terms go to 0 evidently.

Hence, we deduce that

$$\Big|\int_0^\theta \Big(f^\varepsilon(\frac{t}{\theta}) - f(\frac{t}{\theta})\Big) R_n^\lambda(\cos t)(\sin t)^{2\lambda}\, dt\Big| \le \varepsilon. \tag{3.2.18}$$

Thus, letting $\varepsilon \to 0+$ in (3.2.18), we obtain for all $n \in \mathbb{N}_0$,

$$\int_0^\theta f(\frac{t}{\theta}) R_n^\lambda(\cos t)(\sin t)^{2\lambda}\, dt \ge 0. \tag{3.2.19}$$

Finally, we show that if $\widehat{f_d}(\xi) > 0$ for all $\xi \in \mathbb{R}^d$, then (3.2.19) with strict inequality holds. To see this, we need to use the (2.5.15) to obtain that for every $\theta \in (0, \pi)$,

$$\begin{aligned}
I_n(\theta) &= \theta^{-2\lambda-1} \int_0^\theta f(\frac{t}{\theta}) R_n^\lambda(\cos t)(\sin t)^{2\lambda}\, dt \\
&= \theta^{-2\lambda-1} 2^{2\lambda} \int_0^\theta f(\frac{t}{\theta}) R_n^\lambda(\cos t)(\cos\frac{t}{2})^{2\lambda}(\sin\frac{t}{2})^{2\lambda}\, dt \\
&= \theta^{-2\lambda-1} 2^{2\lambda} \int_0^\theta f(\frac{t}{\theta}) \sum_{j=n}^{n+\lambda} a_{n,j}^\lambda R_{2j}^\lambda(\cos\frac{t}{2})(\sin\frac{t}{2})^{2\lambda}\, dt \\
&= \theta^{-2\lambda-1} 2^{2\lambda+1} \sum_{j=n}^{n+\lambda} a_{n,j}^\lambda I_{2j}(\frac{\theta}{2}).
\end{aligned}$$

Thus, for a fixed n, if we repeat the same process m-times, we have

$$I_n(\theta) = (\theta^{-2\lambda-1} 2^{2\lambda+1})^m \sum_{j_1=n}^{n+\lambda} \sum_{j_2=2j_1}^{2j_1+\lambda} \sum_{j_3=2j_2}^{2j_2+\lambda} \cdots \sum_{j_m=2j_{m-1}}^{2j_{m-1}+\lambda} a_{n,j_1}^\lambda a_{2j_1,j_2}^\lambda \cdots a_{2j_{m-1},j_m}^\lambda I_{2j_m}(\frac{\theta}{2^m}). \tag{3.2.20}$$

It follows from (3.2.19) that each term in (3.2.20) is nonnegative. We can conclude that for any $m \in \mathbb{N}$,

$$I_n(\theta) \geq c_{n,m} I_{2^m n}(2^{-m}\theta),$$

where $c_{n,m} > 0$ for any $m \in \mathbb{N}$. Noticing that $2^m n \cdot 2^{-m}\theta = n\theta$, which satisfies the case in section 3.2.3.3, letting $m \to \infty$ and following the same argument in Case (iii), we can conclude that

$$\begin{aligned}
I_{2^m n}(2^{-m}\theta) &= N^{-1} 2m(2\lambda+1) \int_0^{n+\lambda 2^{-m}} f(\frac{t}{N+2^{-m}\lambda}) R_{2^m n}^\lambda(\cos\frac{t}{N})(\sin\frac{t}{N})^{2\lambda} dt + O((2^m n)^{-1}) \\
&\geq c_\lambda \int_0^1 f(x) j_{\lambda-\frac{1}{2}}((n+2^{-m}\lambda)x) x^{2\lambda} dx - C_n 2^{-m} \\
&\geq c_\lambda \min_{n \leq u \leq n+\lambda} \int_0^1 f(x) j_{\lambda-\frac{1}{2}}(ux) x^{2\lambda} dx - C_n 2^{-m},
\end{aligned}$$

which is positive for a sufficiently large integer m. This in turn implies the strict inequality in (3.2.19).

3.2.2 Proof of a corollary

In this section, we will show the Corollary 3.1.4.

Proof of Corollary 3.1.4. Since g is an isotropic continuous positive semi-definite function on $\mathbb{S}^d$ with compact support on $[0, \pi]$, by the Schoenberg's theorem and Lemma 2.5.1, it can be easily seen that for each positive integer ℓ, and every nonnegative integer n, the function

$$R_n^\lambda(\cos 2^\ell t)\Big(\frac{\sin(2^\ell t)}{2^\ell \sin t}\Big)^{2\lambda}$$

can be expressed as a convex combination of the polynomials $R_j^\lambda(\cos t)$, $j \geq 0$. It follows that for any $\ell \in \mathbb{N}$,

$$g(2^\ell t)\Big(\frac{\sin(2^\ell t)}{\sin t}\Big)^{2\lambda} \tag{3.2.21}$$

is an isotropic continuous positive semi-definite function on $\mathbb{S}^d$.

Let $x > 0$ and $\ell \in \mathbb{N}$. Set $n = n_{x,\ell} \in \mathbb{N}$ be such that $\frac{n-1}{2^\ell} \leq x < \frac{n}{2^\ell}$. Then by (3.2.21)

$$\begin{aligned}
0 &\leq n^{2\lambda+1} 2^{-2\ell\lambda} \int_0^{2^{-\ell}\pi} g(2^\ell t)\Big(\sin(2^\ell t)\Big)^{2\lambda} R_n^\lambda(\cos t)\, dt \\
&= \int_0^{2^{-\ell} n\pi} g(2^\ell t n^{-1})\Big(\frac{\sin(2^\ell t n^{-1})}{2^\ell n^{-1} t}\Big)^{2\lambda} R_n^\lambda\Big(\cos(\frac{t}{n})\Big) t^{2\lambda}\, dt \\
&= \int_0^{x\pi} g(x^{-1} t)\Big(\frac{\sin(x^{-1} t)}{x^{-1} t}\Big)^{2\lambda} R_n^\lambda\Big(\cos(\frac{t}{n})\Big) t^{2\lambda}\, dt + o(1),
\end{aligned}$$

where the last step uses the fact that $|x^{-1} - 2^\ell n^{-1}| \leq \frac{1}{nx} \to 0$ as $n \to \infty$. Thus, letting

$n \to \infty$ and applying the asymptotic formula (2.5.17), we get

$$\begin{aligned}
0 &\le x^{-2\lambda-1} \int_0^{\pi x} g(x^{-1}t) \Big(\frac{\sin(x^{-1}t)}{x^{-1}t} \Big)^{2\lambda} j_{\lambda-\frac{1}{2}}(t) t^{2\lambda}\, dt \\
&= \int_0^{\infty} g(t) \Big(\frac{\sin(t)}{t} \Big)^{2\lambda} j_{\lambda-\frac{1}{2}}(xt) t^{2\lambda}\, dt \\
&= c_\lambda \widehat{G_d}(\xi),
\end{aligned}$$

where $\xi \in \mathbb{R}^d$, $|\xi| = x$ and

$$G(t) = g(t) \Big(\frac{\sin t}{t} \Big)^{2\lambda}, \quad t \ge 0.$$

By Bochner's theorem, G is an isotropic continuous positive semi-definite function on $\mathbb{R}^d$. Furthermore, when d is odd and by Theorem 3.1.1, G is also a positive semi-definite function on $\mathbb{S}^d$. $\square$

3.2.3 The case of even dimensions $(d = 2)$

In this section, we deal with the Conjecture 1 when $d = 2$. Since the convergence of series $\sum_{n=0}^{\infty} a_n C_n^\lambda(1) < \infty$ can be verified by using the same argument in Theorem 3.1.1, we only need to consider the following conjecture:

Conjecture 2. *Let $\delta \ge \lambda + 1$ and $\lambda = \frac{d-1}{2}$. Then for any $\theta \in (0, \pi)$ and $n \in \mathbb{N}_0$,*

$$\int_0^\theta (\theta - t)^\delta C_n^\lambda(\cos t) \sin^{2\lambda} t dt > 0. \tag{3.2.22}$$

We first deduce that to prove Conjecture 2, it suffices to consider the boundary case $\delta = \lambda + 1$. In fact, notice that for $a > -1, b > 0$, by using the substitution $t = u + s(\theta - u)$, we have

$$\begin{aligned}\int_0^\theta (\theta-t)^a \int_0^t (t-u)^b g(u)dudt &= \int_0^\theta g(u) \int_u^\theta (t-u)^b(\theta-t)^a dtdu \\ &= \int_0^\theta g(u) \int_0^1 (\theta-u)^{a+b+1}(1-s)^a s^b dsdu \\ &= B(b+1,a+1) \int_0^\theta g(u)(\theta-u)^{a+b+1}du.\end{aligned}$$

where $B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt$ is the Beta function. Then we have

$$\int_0^\theta g(u)(\theta-u)^{a+b+1}du = \frac{1}{B(b+1,a+1)} \int_0^\theta (\theta-t)^a \int_0^t (t-u)^b g(u)dudt.$$

Now for $\delta_1>\delta_2$, let $a=\delta_1-\delta_2-1$, $b=\delta_2$, and $g(u)=C_n^\lambda(\cos u)(\sin u)^{2\lambda}$. It follows immediately that

$$\begin{aligned}&\int_0^\theta (\theta-u)^{\delta_1} C_n^\lambda(\cos u)(\sin u)^{2\lambda}du \\ &\qquad = \frac{1}{B(\delta_2+1,\delta_1-\delta_2)} \int_0^\theta (\theta-t)^{\delta_1-\delta_2-1} \int_0^t (t-u)^{\delta_2} C_n^\lambda(\cos u)(\sin u)^{2\lambda}dudt.\end{aligned}$$

Hence, the boundary case $\delta=\lambda+1$ can yield 2.

Our main result is to show that Conjecture 2 is true when the dimension $d=2$ under the restriction when θ small. By the argument above, it suffices to consider the boundary case $\delta=\frac{3}{2}$. More precisely, we state the main theorem below.

Theorem 3.2.4. *Let $d=2$. The function*

$$f_{\theta,\delta}(t)=(\theta-t)_+^{\frac{3}{2}}$$

is isotropic positive definite on $\mathbb{S}^2$ when $0<\theta<C_{A,B}$, where $C_{A,B}$ is an absolute constant.

Remark 3.2.1. *In the proof, we give an upper estimate of $C_{A,B}$ and it can be taken at* *most* 1.2644×10^{-21}. *We believe this value could be improved.*

For the proof of Theorem 3.2.4, by the above analysis of the Schoenberg's theorem, it suffices to show that

$$\int_0^\theta (\theta - t)^{\frac{3}{2}} P_n(\cos t) \sin t dt > 0, \quad \forall\, n \in \mathbb{N}_0. \tag{3.2.23}$$

The proof of the positivity of (3.2.23) consists of three cases: (i) $n\theta \geq A$ (ii) $n\theta \leq B$ (iii) $B \leq n\theta \leq A$. Next, we will give detailed proofs of the three cases.

3.2.3.1 Case (i): $n\theta \geq A$

In this case, we shall prove that there exists a constant $A > 1$ such that (3.2.23) holds whenever $n\theta \geq A$. The proof is long and will be divided into several steps.

Step 1: Decomposition of the integral.

We first give the following decomposition on the integral (3.2.23):

Lemma 3.2.5. *For* $\theta \in (0, \frac{\pi}{2}]$ *and* $n \geq 1$,

$$\frac{4n(n+1)}{3} \int_0^\theta (\theta - t)^{\frac{3}{2}} P_n(\cos t) \sin t \, dt = I_{n,1}(\theta) + R_{n,1}(\theta) + R_{n,2}(\theta) - R_{n,3}(\theta),$$

where

$$\begin{aligned}
I_{n,1}(\theta) &= \frac{2}{n(n+1)} \int_0^\theta \Big[1 - P_n(\cos t)\Big] \frac{\sqrt{\theta - t}\cos t}{\sin^2 t}\, dt,\\
R_{n,1}(\theta) &= \frac{1}{2n} \int_0^\theta P_{n-1}^{(1,1)}(\cos t) \frac{\sin t \cos t}{\sqrt{\theta - t}}\, dt,\\
R_{n,2}(\theta) &= \frac{1}{n(n+1)} \int_0^\theta \Big[1 - P_n(\cos t)\Big] \frac{1}{(\sin t)\sqrt{\theta - t}}\, dt \geq 0.\\
R_{n,3}(\theta) &= \int_0^\theta P_n(\cos t) \frac{\sin t}{\sqrt{\theta - t}}\, dt,
\end{aligned}$$

Proof. Using (2.5.1) and integration by parts, we obtain that

$$\begin{aligned}
A_n &:= \int_0^\theta (\theta-t)^{\frac{3}{2}} P_n(\cos t)\sin t\, dt = -\frac{2}{n}\int_0^\theta (\theta-t)^{\frac{3}{2}} \Big(P_{n+1}^{(-1,-1)}(\cos t)\Big)' dt \\
&= -\frac{3}{n}\int_0^\theta P_{n+1}^{(-1,-1)}(\cos t)(\theta-t)^{\frac{1}{2}}\, dt = \frac{3}{4n}\int_0^\theta P_{n-1}^{(1,1)}(\cos t)\sin^2 t(\theta-t)^{\frac{1}{2}}\, dt,
\end{aligned}$$

where we used (2.5.2) in the first step, and (2.5.5) in the third step. Applying integration by parts to this last integral once again yields

$$\begin{aligned}
A_n &= -\frac{3}{2}\frac{1}{n(n+1)}\int_0^\theta (\theta-t)^{\frac{1}{2}}\Big(P_n(\cos t)\Big)'\sin t\, dt \\
&= -\frac{3}{4}\frac{1}{n(n+1)}\int_0^\theta P_n(\cos t)\frac{\sin t}{\sqrt{\theta-t}}\, dt + \frac{3}{2}\frac{1}{n(n+1)}\int_0^\theta P_n(\cos t)(\theta-t)^{\frac{1}{2}}\cos t\, dt \\
&=: A_{n,1} + A_{n,2}.
\end{aligned}$$

For the term $A_{n,2}$, we use integration by parts to obtain

$$\begin{aligned}
A_{n,2} &= -\frac{3}{n^2(n+1)}\int_0^\theta \Big(P_{n+1}^{(-1,-1)}(\cos t)\Big)'(\theta-t)^{\frac{1}{2}}\cot t\, dt \\
&= -\frac{3}{n^2(n+1)}\int_0^\theta P_{n+1}^{(-1,-1)}(\cos t)\left[\frac{1}{2}(\theta-t)^{-\frac{1}{2}}\cot t + \frac{(\theta-t)^{\frac{1}{2}}}{\sin^2 t}\right] dt \\
&= \frac{3}{8n^2(n+1)}\int_0^\theta P_{n-1}^{(1,1)}(\cos t)\frac{\sin t\cos t}{\sqrt{\theta-t}}\, dt + \frac{3}{4n^2(n+1)}\int_0^\theta P_{n-1}^{(1,1)}(\cos t)(\theta-t)^{\frac{1}{2}}\, dt,
\end{aligned}$$

where the third step uses (2.5.5). Putting the above together, we then obtian

$$\begin{aligned}
&\frac{4n(n+1)}{3}A_n = -\int_0^\theta P_n(\cos t)\frac{\sin t}{\sqrt{\theta-t}}dt \\
&+\frac{1}{2n}\int_0^\theta P_{n-1}^{(1,1)}(\cos t)\frac{\sin t\cos t}{\sqrt{\theta-t}}\, dt + \frac{1}{n}\int_0^\theta P_{n-1}^{(1,1)}(\cos t)(\theta-t)^{\frac{1}{2}}\, dt \\
&= -R_{n,3}(\theta) + R_{n,1}(\theta) + \frac{1}{n}\int_0^\theta P_{n-1}^{(1,1)}(\cos t)(\theta-t)^{\frac{1}{2}}\, dt.
\end{aligned}$$

For the last integral, we apply integration by parts again to obtain

$$\begin{aligned}
&\frac{1}{n}\int_0^\theta P_{n-1}^{(1,1)}(\cos t)(\theta-t)^{\frac{1}{2}}\,dt = \frac{2}{n(n+1)}\int_0^\theta \frac{(\theta-t)^{\frac{1}{2}}}{\sin t}\Big(P_n(1)-P_n(\cos t)\Big)'\,dt\\
&= \frac{2}{n(n+1)}\int_0^\theta \Big[1-P_n(\cos t)\Big]\frac{\sqrt{\theta-t}\cos t}{\sin^2 t}\,dt + \frac{1}{n(n+1)}\int_0^\theta \Big[1-P_n(\cos t)\Big]\frac{1}{\sin t\sqrt{\theta-t}}\,dt\\
&= I_{n,1}(\theta)+R_{n,2}(\theta).
\end{aligned}$$

This completes the proof of the lemma. □

Step 2: Estimates of $I_{n,1}(\theta)$, $R_{n,3}(\theta)$, and $R_{n,1}(\theta)$

In this step, we will give the upper bounds for $R_{n,1}(\theta)$ and $R_{n,3}(\theta)$ (see Lemma 3.2.6 and 3.2.7), and the lower bounds for $I_{n,1}(\theta)$(see Lemma 3.2.8).

Lemma 3.2.6. *If $n\theta \ge 5$ and $\theta \in (0, \frac{\pi}{2}]$, then*

$$|R_{n,1}(\theta)| = \frac{1}{2n}\Big|\int_0^\theta P_{n-1}^{(1,1)}(\cos t)\frac{\sin t\cos t}{\sqrt{\theta-t}}\,dt\Big| \le C(n\theta)^{-2}\theta^{\frac{3}{2}}.$$

where the constant C can be taken to be 42.6909.

Proof. We break the integral $\int_0^\theta$ into two parts $\int_{\theta-n^{-1}}^\theta + \int_0^{\theta-n^{-1}}$. For the first part, we use the estimate (2.5.11) to get

$$\begin{aligned}
\Big|\frac{1}{2n}\int_{\theta-\frac{1}{n}}^\theta P_{n-1}^{(1,1)}(\cos t)\frac{\sin t\cos t}{\sqrt{\theta-t}}\,dt\Big| &\le \frac{2.821\times\pi^{\frac{3}{2}}}{2n(n-1)^{\frac{1}{2}}}\int_{\theta-\frac{1}{n}}^\theta (\theta-t)^{-\frac{1}{2}}t^{-\frac{1}{2}}\,dt\\
&\le \frac{2.821\times\pi^{\frac{3}{2}}}{2n(n-1)^{\frac{1}{2}}}2\arctan\Big(\frac{\frac{1}{n}}{\theta-\frac{1}{n}}\Big)^{\frac{1}{2}}\\
&\le \frac{2.821\times\pi^{\frac{3}{2}}}{n(n-1)^{\frac{1}{2}}}\cdot\Big(\frac{\frac{1}{n}}{\theta-\frac{1}{n}}\Big)^{\frac{1}{2}}\\
&\le 2.821\times\pi^{\frac{3}{2}}\times\Big(\frac{5}{4-\pi}\Big)^{\frac{1}{2}}n^{-2}\theta^{-\frac{1}{2}}\\
&\le 37.9275\times n^{-2}\theta^{-\frac{1}{2}},
\end{aligned}$$

where we use $\arctan x \le x$ in the third inequality and use the following fact in the fourth inequality: when $n\theta \ge 5$,

$$n^2\theta^{\frac{1}{2}} \cdot \frac{1}{n(n-1)^{\frac{1}{2}}}\left(\frac{\frac{1}{n}}{\theta-\frac{1}{n}}\right)^{\frac{1}{2}} \le \left(\frac{5}{4-\pi}\right)^{\frac{1}{2}}.$$

For the second part, by (2.5.2), we use integration by parts and the fact (2.5.8) to obtain

$$\begin{aligned}
&\Big|\frac{1}{2n}\int_0^{\theta-\frac{1}{n}} P_{n-1}^{(1,1)}(\cos t)\frac{\sin t\cos t}{\sqrt{\theta-t}}\,dt\Big| = \Big|\frac{-1}{n(n+1)}\int_0^{\theta-\frac{1}{n}}(\theta-t)^{-\frac{1}{2}}(\cos t)\,d\Big(P_n(\cos t)\Big)\Big| \\
&\quad\le \frac{1}{n^2}\Big|P_n(\cos(\theta-\frac{1}{n}))\frac{\cos(\theta-\frac{1}{n})}{\sqrt{\frac{1}{n}}}\Big| + \frac{1}{n^2}\frac{1}{\theta^{\frac{1}{2}}} + \frac{1}{2n^2}\Big|\int_0^{\theta-\frac{1}{n}} P_n(\cos t)(\theta-t)^{-\frac{3}{2}}(\cos t)dt\Big| \\
&\quad+\frac{1}{n^2}\Big|\int_0^{\theta-\frac{1}{n}} P_n(\cos t)(\theta-t)^{-\frac{1}{2}}(\sin t)dt\Big| \\
&\quad\le \frac{\sqrt{5}}{2}\frac{1}{n^2\theta^{\frac{1}{2}}} + \frac{1}{n^2\theta^{\frac{1}{2}}} + \frac{1}{n^2\theta^{\frac{1}{2}}} + \frac{2(\frac{\pi}{2})^2}{3n^2\theta^{\frac{1}{2}}} \le \frac{4.7634}{n^2\theta^{\frac{1}{2}}},
\end{aligned}$$

where we use the triangle inequality in the first inequality. The second inequality, by (2.5.8) and $n\theta \ge 5$, is followed by the following estimates:

$$\frac{1}{n^2}\Big|P_n(\cos(\theta-\frac{1}{n}))\frac{\cos(\theta-\frac{1}{n})}{\sqrt{\frac{1}{n}}}\Big| \le \frac{1}{n^2}\frac{1}{n^{\frac{1}{2}}(\theta-\frac{1}{n})^{\frac{1}{2}}}\frac{1}{\sqrt{\frac{1}{n}}} \le \frac{1}{n^2(\frac{4}{5}\theta)^{\frac{1}{2}}} = \frac{\sqrt{5}}{2n^2\theta^{\frac{1}{2}}}$$

$$\frac{1}{2n^2}\Big|\int_0^{\theta-\frac{1}{n}} P_n(\cos t)(\theta-t)^{-\frac{3}{2}}(\cos t)dt\Big| \le \frac{1}{2n^2}\int_0^{\theta-\frac{1}{n}}\frac{1}{n^{\frac{1}{2}}t^{\frac{1}{2}}}(\theta-t)^{-\frac{3}{2}}dt = \frac{1}{2n^{\frac{5}{2}}}\frac{2\sqrt{\theta-\frac{1}{n}}}{\theta\sqrt{\frac{1}{n}}} \le \frac{1}{n^2\theta^{\frac{1}{2}}}$$

$$\frac{1}{n^2}\Big|\int_0^{\theta-\frac{1}{n}} P_n(\cos t)(\theta-t)^{-\frac{1}{2}}(\sin t)dt\Big| \le \frac{1}{n^2}\int_0^{\theta-\frac{1}{n}}\frac{1}{n^{\frac{1}{2}}t^{\frac{1}{2}}}(\theta-t)^{-\frac{1}{2}}tdt \le \frac{1}{n^{\frac{5}{2}}}n^{\frac{1}{2}}\int_0^{\theta-\frac{1}{n}}\sqrt{t}dt \le \frac{2(\frac{\pi}{2})^2}{3n^2\theta^{\frac{1}{2}}}.$$

Therefore, combining the above two parts, we get

$$|R_{n,1}(\theta)| = \frac{1}{2n}\Big|\int_0^\theta P_{n-1}^{(1,1)}(\cos t)\frac{\sin t\cos t}{\sqrt{\theta - t}}\, dt\Big| \le (37.9275 + 4.7634) \times n^{-2}\theta^{-\frac{1}{2}}$$
$$= 42.6909 \times n^{-2}\theta^{-\frac{1}{2}}.$$

$\square$

Lemma 3.2.7. *If* $n\theta \ge 5$ *and* $\theta \in (0, \frac{\pi}{2}]$, *then*

$$|R_{n,3}(\theta)| = \left|\int_0^\theta P_n(\cos t)\frac{\sin t}{\sqrt{\theta - t}}\, dt\ \right| \le \frac{\sqrt{2}}{n\theta}\sin(N\theta)\Big(\frac{\sin\theta}{\theta}\Big)^{\frac{1}{2}}\theta^{\frac{3}{2}} + C \times (n\theta)^{-\frac{3}{2}}\theta^{\frac{3}{2}},$$

where $N = n + \frac{1}{2}$, *and the constant* C *can be taken to be* 92.1237.

Proof. We first choose a function $\eta \in C^1(\mathbb{R})$ where

$$\eta(x) = \begin{cases} 1, & \frac{1}{2} \le x \\ \frac{1}{2}\cos(4\pi x) + \frac{1}{2}, & \frac{1}{4} \le x < \frac{1}{2} \\ 0, & x < \frac{1}{4} \end{cases}. \tag{3.2.24}$$

Then,

$$\int_0^\theta P_n(\cos t)\frac{\sin t}{\sqrt{\theta - t}}\, dt = \int_0^{\frac{\theta}{4}}(1 - \eta(\theta^{-1}t))P_n(\cos t)\frac{\sin t}{\sqrt{\theta - t}}\, dt + \int_{\frac{\theta}{4}}^{\frac{\theta}{2}}(1 - \eta(\theta^{-1}t))P_n(\cos t)\frac{\sin t}{\sqrt{\theta - t}}\, dt$$
$$+ \int_{\frac{\theta}{4}}^{\frac{\theta}{2}}\eta(\theta^{-1}t)P_n(\cos t)\frac{\sin t}{\sqrt{\theta - t}}\, dt + \int_{\frac{\theta}{2}}^{\theta}\eta(\theta^{-1}t)P_n(\cos t)\frac{\sin t}{\sqrt{\theta - t}}\, dt$$
$$= \int_0^{\frac{\theta}{2}}(1 - \eta(\theta^{-1}t))P_n(\cos t)\frac{\sin t}{\sqrt{\theta - t}}\, dt + \int_{\frac{\theta}{4}}^{\theta}\eta(\theta^{-1}t)P_n(\cos t)\frac{\sin t}{\sqrt{\theta - t}}\, dt.$$

For the first integral, by (2.5.6) and integration by parts, we have

$$\int_0^{\frac{\theta}{2}}(1-\eta(\theta^{-1}t))P_n(\cos t)\frac{\sin t}{\sqrt{\theta-t}}\,dt = -\int_0^{\frac{\theta}{2}}(1-\eta(\theta^{-1}t))\frac{1}{\sqrt{\theta-t}}d\Big(\frac{\sin^2 t}{2n}P_{n-1}^{(1,1)}(\cos t)\Big)$$
$$= -\frac{1}{4n}\int_0^{\frac{\theta}{2}} P_{n-1}^{(1,1)}(\cos t)(\sin t)^2(\theta-t)^{-\frac{3}{2}}g(t)dt,$$

where $g(t) := 1-\eta(\theta^{-1}t)-2\theta^{-1}(\theta-t)\eta'(\theta^{-1}t)$. Using (2.5.11) and noting that $|g(t)| \le 4\pi+2$, this implies that

$$\begin{aligned}
&\Big|\int_0^{\frac{\theta}{2}}(1-\eta(\theta^{-1}t))P_n(\cos t)\frac{\sin t}{\sqrt{\theta-t}}dt\Big| \\
&= \Big|\frac{1}{4n}\int_0^{\frac{\theta}{2}} P_{n-1}^{(1,1)}(\cos t)(\sin t)^2(\theta-t)^{-\frac{3}{2}}g(t)\,dt\Big| \\
&\le \frac{2.821\times\pi^{\frac{3}{2}}}{4n}\int_0^{\frac{\theta}{2}}\frac{1}{(n-1)^{1/2}t^{3/2}}(\sin t)^2(\theta-t)^{-\frac{3}{2}}|g(t)|dt \\
&\le \frac{2.821\times\pi^{\frac{3}{2}}\times(4\pi+2)}{4n(n-1)^{1/2}}\int_0^{\frac{\theta}{2}}\Big(\frac{\sin t}{t}\Big)^{\frac{3}{2}}(\sin t)^{\frac{1}{2}}(\theta-t)^{-\frac{3}{2}}dt \\
&\le \frac{2.821\times\pi^{\frac{3}{2}}\times(4\pi+2)}{4n(n-1)^{1/2}}\Big(\frac{\theta}{2}\Big)^{\frac{1}{2}}\int_0^{\frac{\theta}{2}}(\theta-t)^{-\frac{3}{2}}dt \\
&\le \frac{2.821\times\pi^{\frac{3}{2}}\times(4\pi+2)\times\sqrt{2}}{4n^{3/2}},
\end{aligned}$$

where the last inequality follows by $n\ge 2$ when θ small. Next, we estimate the second integral. We claim that

$$\frac{1}{\theta^{\frac{3}{2}}}\int_{\frac{\theta}{4}}^{\theta}P_n(\cos t)\frac{\eta(\theta^{-1}t)\sin t}{\sqrt{\theta-t}}\,dt \le \frac{\sqrt{2}}{n\theta}\sin(N\theta)\Big(\frac{\sin\theta}{\theta}\Big)^{\frac{1}{2}} + C\times(n\theta)^{-3/2}, \qquad (3.2.25)$$

where $N = n+\frac{1}{2}$ and C will be determined in equation (3.2.28) below. To see this, we use (2.5.7) to obtain

$$\int_{\frac{\theta}{4}}^{\theta} P_n(\cos t)\frac{\eta(\theta^{-1}t)\sin t}{\sqrt{\theta - t}}\,dt$$
$$\leq 2^{1/2}\pi^{-\frac{1}{2}}n^{-\frac{1}{2}}\int_{\theta/4}^{\theta}\eta(\theta^{-1}t)\cos(Nt-\frac{\pi}{4})(\sin t)^{\frac{1}{2}}(\theta-t)^{-\frac{1}{2}}\,dt + \frac{1}{(2\pi)^{\frac{1}{2}}(\frac{2}{\pi})^{\frac{3}{2}}}(n\theta)^{-\frac{3}{2}}\theta^{\frac{3}{2}}.$$

We then write

$$\begin{aligned}
&2^{1/2}\pi^{-\frac{1}{2}}n^{-\frac{1}{2}}\int_{\theta/4}^{\theta}\eta(\theta^{-1}t)\cos(Nt-\frac{\pi}{4})(\sin t)^{\frac{1}{2}}(\theta-t)^{-\frac{1}{2}}\,dt\\
&\quad= 2^{-1/2}\pi^{-\frac{1}{2}}n^{-\frac{1}{2}}\int_{\theta/4}^{\theta}\eta(\theta^{-1}t)\left[e^{i(Nt-\frac{\pi}{4})}+e^{i(-Nt+\frac{\pi}{4})}\right](\sin t)^{\frac{1}{2}}(\theta-t)^{-\frac{1}{2}}\,dt\\
&\quad=: I_n^+ + I_n^-.
\end{aligned}$$

However, using Proposition 2.5.3 and 2.5.4 with $\phi(t) = \eta(\frac{t}{\theta})(\sin t)^{\frac{1}{2}}$, $\theta \in (0, \frac{\pi}{2})$, $\alpha = \frac{\theta}{4}$, $\beta = \theta$, $\lambda = \frac{1}{2}$, $\mu = \frac{1}{2}$, and $v = 1$, we have

$$\int_{\frac{\theta}{4}}^{\theta}\eta(\theta^{-1}t)(\sin t)^{\frac{1}{2}}e^{iNt}(\theta-t)^{-\frac{1}{2}}\,dt \leq \frac{\Gamma(1/2)}{N^{\frac{1}{2}}}e^{iN\theta-\frac{1}{2}\cdot\frac{1}{2}\pi i}\eta(1)(\sin\theta)^{\frac{1}{2}} + \frac{13.0552}{N}, \tag{3.2.26}$$

and

$$\int_{\frac{\theta}{4}}^{\theta}\eta(\theta^{-1}t)(\sin t)^{\frac{1}{2}}e^{-iNt}(\theta-t)^{-\frac{1}{2}}\,dt \leq \frac{\Gamma(1/2)}{N^{\frac{1}{2}}}e^{-iN\theta+\frac{1}{2}\cdot\frac{1}{2}\pi i}\eta(1)(\sin\theta)^{\frac{1}{2}} + \frac{13.0552}{N}. \tag{3.2.27}$$

Therefore, we obtain that

$$
\begin{aligned}
I_n^+ &= 2^{-\frac{1}{2}}\pi^{-\frac{1}{2}}n^{-\frac{1}{2}}e^{-\frac{\pi i}{4}}\int_{\theta/4}^{\theta}\eta(\theta^{-1}t)e^{iNt}(\sin t)^{\frac{1}{2}}(\theta-t)^{-\frac{1}{2}}\,dt\\
&\leq 2^{-\frac{1}{2}}\pi^{-\frac{1}{2}}n^{-\frac{1}{2}}e^{-\frac{\pi i}{4}}\Big[\frac{\Gamma(1/2)}{N^{\frac{1}{2}}}e^{iN\theta-\frac{1}{2}\cdot\frac{1}{2}\pi i}\eta(\frac{\theta}{\theta})(\sin\theta)^{\frac{1}{2}}+13.0552\times N^{-1}\Big]\\
&\leq 2^{-\frac{1}{2}}n^{-1}e^{iN\theta}e^{-\frac{\pi}{2}i}(\sin\theta)^{\frac{1}{2}}+e^{-\frac{\pi}{2}i}\sqrt{\frac{1}{2\pi}}\cdot 13.0552\times(n\theta)^{-\frac{3}{2}}\theta^{\frac{3}{2}}\\
&\leq 2^{-\frac{1}{2}}n^{-1}e^{iN\theta}e^{-\frac{\pi}{2}i}(\sin\theta)^{\frac{1}{2}}+e^{-\frac{\pi}{2}i}5.2080\times(n\theta)^{-\frac{3}{2}}\theta^{\frac{3}{2}},
\end{aligned}
$$

and

$$
\begin{aligned}
I_n^- &= 2^{-\frac{1}{2}}\pi^{-\frac{1}{2}}n^{-\frac{1}{2}}e^{\frac{\pi i}{4}}\int_{\theta/4}^{\theta}\eta(\theta^{-1}t)e^{-iNt}(\sin t)^{\frac{1}{2}}(\theta-t)^{-\frac{1}{2}}\,dt\\
&\leq 2^{-\frac{1}{2}}n^{-1}e^{\frac{\pi i}{2}}e^{-iN\theta}(\sin\theta)^{\frac{1}{2}}+e^{-\frac{\pi}{2}i}5.2080\times(n\theta)^{-\frac{3}{2}}\theta^{\frac{3}{2}}.
\end{aligned}
$$

It follows that

$$
|I_n^+ + I_n^-| \leq 2^{\frac{1}{2}}n^{-1}\sin(N\theta)(\sin\theta)^{\frac{1}{2}}+10.4160\times(n\theta)^{-\frac{3}{2}}\theta^{\frac{3}{2}}.
$$

This proves

$$
\frac{1}{\theta^{\frac{3}{2}}}\int_{\theta/4}^{\theta}P_n(\cos t)\frac{\eta(\theta^{-1}t)\sin t}{\sqrt{\theta-t}}\,dt \leq \frac{\sqrt{2}}{n\theta}\sin(N\theta)\Big(\frac{\sin\theta}{\theta}\Big)^{\frac{1}{2}}+\Big(\frac{1}{(2\pi)^{\frac{1}{2}}(\frac{2}{\pi})^{\frac{3}{2}}}+10.4160\Big)(n\theta)^{-3/2}. \tag{3.2.28}
$$

This completes the proof of the claim (3.2.25). By the above two arguments, we thus conclude the desired estimate for $R_{n,3}(\theta)$:

$$
\begin{aligned}
|R_{n,3}(\theta)| =& \left| \int_0^\theta P_n(\cos t) \frac{\sin t}{\sqrt{\theta - t}}\, dt \right| \\
\leq& \frac{\sqrt{2}}{n\theta} \sin(N\theta) \Big(\frac{\sin\theta}{\theta} \Big)^{\frac{1}{2}} \theta^{\frac{3}{2}} \\
&+ \Big(\frac{1}{(2\pi)^{\frac{1}{2}} (\frac{2}{\pi})^{\frac{3}{2}}} + 10.4160 + \frac{2.821 \times \pi^{\frac{3}{2}} \times (4\pi + 2) \times \sqrt{2}}{4} \Big) (n\theta)^{-3/2} \theta^{3/2} \\
\leq& \frac{\sqrt{2}}{n\theta} \sin(N\theta) \Big(\frac{\sin\theta}{\theta} \Big)^{\frac{1}{2}} \theta^{\frac{3}{2}} + 92.1237 \times (n\theta)^{-3/2} \theta^{3/2}.
\end{aligned}
$$

□

Lemma 3.2.8. *For $n\theta \geq 5$ and $\theta \in (0, \frac{\pi}{2}]$, we have that*

$$
I_{n,1}(\theta) = \frac{2}{n(n+1)} \int_0^\theta \Big[1 - P_n(\cos t) \Big] \frac{\sqrt{\theta - t}\cos t}{\sin^2 t} dt \geq \frac{3}{4} \frac{2\sqrt{\theta}}{n} - C(n\theta)^{-\frac{3}{2}} \theta^{\frac{3}{2}}, \tag{3.2.29}
$$

where

$$
C = \frac{3}{4} \Big(10\sqrt{\frac{\pi}{2}} + 2\sqrt{\pi}\sqrt{\frac{\pi}{2}} + \frac{4}{\sqrt{5}} + 2\big(\frac{4}{\pi}(4 + \sqrt{\frac{\pi}{2}}) + 4 \big) \Big) < 30.1066.
$$

Proof. Firstly, we claim the following lower estimate.

$$
I_{n,1}(\theta) \geq \frac{3}{4} \frac{2\sqrt{\theta}}{n^2} \int_0^\theta \frac{1 - P_n(\cos t)}{t^2}\, dt - \frac{3}{2n^2} \theta^{-\frac{1}{2}} \log(n\theta) \Big[\Big(\frac{4}{\pi}(2 + \sqrt{\frac{\pi}{2}}) + 2 \Big) + 2 \Big]. \tag{3.2.30}
$$

Indeed, since that

$$
\frac{\cos t}{\sin^2 t} = \frac{1 - 2\sin^2 \frac{t}{2}}{\sin^2 t} = \frac{-2\sin^2 \frac{t}{2}}{\sin^2 t} + \frac{1}{t^2}\big(\frac{t^2}{\sin^2 t} - 1 \big) + \frac{1}{t^2},
$$

we may rewrite the integral $I_{n,1}(\theta)$ to

$$\begin{aligned}
\frac{n(n+1)}{2} I_{n,1}(\theta) =& - \int_0^\theta \Big[1-P_n(\cos t)\Big] \frac{2\sqrt{\theta-t}\sin^2\frac{t}{2}}{\sin^2 t}\,dt \\
&+ \int_0^\theta \frac{1-P_n(\cos t)}{t^2}\sqrt{\theta-t}\Big[\frac{t^2}{\sin^2 t}-1\Big]\,dt + \int_0^\theta \sqrt{\theta-t}\frac{1-P_n(\cos t)}{t^2}\,dt \\
\ge& \sqrt{\theta}\int_0^\theta \frac{1-P_n(\cos t)}{t^2}\,dt - \int_0^\theta \frac{1}{\sqrt{\theta}+\sqrt{\theta-t}}\frac{1-P_n(\cos t)}{t}\,dt - 2(\frac{2}{\pi})^2\theta^{\frac{3}{2}} \\
\ge& \sqrt{\theta}\int_0^\theta \frac{1-P_n(\cos t)}{t^2}\,dt - \left(\frac{4}{\pi}(4+\sqrt{\frac{\pi}{2}})+2\right)\theta^{-\frac{1}{2}}\log(n\theta) - 2(\frac{2}{\pi})^2(\frac{\pi}{2})^2\theta^{-\frac{1}{2}}\log(n\theta) \\
\ge& \sqrt{\theta}\int_0^\theta \frac{1-P_n(\cos t)}{t^2}\,dt - \theta^{-\frac{1}{2}}\log(n\theta)\left(\frac{4}{\pi}(4+\sqrt{\frac{\pi}{2}})+4\right),
\end{aligned}$$

where the first inequality follows that

$$\begin{aligned}
&\Big| - \int_0^\theta \Big[1-P_n(\cos t)\Big]\frac{2\sqrt{\theta-t}\sin^2\frac{t}{2}}{\sin^2 t}\,dt + \int_0^\theta \frac{1-P_n(\cos t)}{t^2}\sqrt{\theta-t}\Big(\frac{t^2}{\sin^2 t}-1\Big)\,dt + \\
&\int_0^\theta \sqrt{\theta-t}\frac{1-P_n(\cos t)}{t^2}\,dt - \sqrt{\theta}\int_0^\theta \frac{1-P_n(\cos t)}{t^2}\,dt + \int_0^\theta \frac{1}{\sqrt{\theta}+\sqrt{\theta-t}}\frac{1-P_n(\cos t)}{t}\,dt\Big| \\
=&\Big| - \int_0^\theta \Big[1-P_n(\cos t)\Big]\frac{2\sqrt{\theta-t}\sin^2\frac{t}{2}}{\sin^2 t}\,dt + \int_0^\theta \frac{1-P_n(\cos t)}{t^2}\sqrt{\theta-t}\Big(\frac{t^2}{\sin^2 t}-1\Big)\,dt\Big| \\
=&\Big|\int_0^\theta \Big[1-P_n(\cos t)\Big]\sqrt{\theta-t}\Big(\frac{-2\sin^2\frac{t}{2}}{\sin^2 t}+\frac{1}{\sin^2 t}-\frac{1}{t^2}\Big)\,dt\Big| \\
\le& 2\theta^{\frac{1}{2}}\int_0^\theta \Big|\frac{-2\sin^2\frac{t}{2}}{\sin^2 t}+\frac{1}{\sin^2 t}-\frac{1}{t^2}\Big|\,dt \le 2\theta^{\frac{3}{2}}(\frac{2}{\pi})^2.
\end{aligned}$$

And the second inequality follows that

$$\Big| - \int_0^\theta \frac{1}{\sqrt{\theta}+\sqrt{\theta-t}}\frac{1-P_n(\cos t)}{t}\,dt\Big| \le \left(\frac{4}{\pi}(4+\sqrt{\frac{\pi}{2}})+2\right)\theta^{-\frac{1}{2}}\log(n\theta). \tag{3.2.31}$$

Indeed, to show (3.2.31), we need to split the integral into two parts: $\int_0^{1/n} + \int_{1/n}^\theta$. For the first part, we use representation (2.5.12). Without loss of generality, we may assume that $n = 2m$ (The case when $n = 2m-1$ can be treated similarly). Then we can obtain that

$$P_{2m}(\cos t) = \frac{1}{\pi}\Big(\frac{\Gamma(m+\frac{1}{2})}{\Gamma(m+1)}\Big)^2 + \frac{2}{\pi}\sum_{j=1}^{m}\frac{\Gamma(m-j+\frac{1}{2})\Gamma(m+j+\frac{1}{2})}{\Gamma(m-j+1)\Gamma(m+j+1)}\cos(2jt). \tag{3.2.32}$$

By the identity

$$1 - \frac{1}{\pi}\Big(\frac{\Gamma(m+\frac{1}{2})}{\Gamma(m+1)}\Big)^2 = \frac{2}{\pi}\sum_{j=1}^{m}\frac{\Gamma(m-j+\frac{1}{2})\Gamma(m+j+\frac{1}{2})}{\Gamma(m-j+1)\Gamma(m+j+1)}, \tag{3.2.33}$$

we have

$$\frac{1-P_n(\cos t)}{t} = \frac{4}{\pi}\sum_{j=1}^{m}\frac{\Gamma(m-j+\frac{1}{2})\Gamma(m+j+\frac{1}{2})}{\Gamma(m-j+1)\Gamma(m+j+1)}\frac{\sin^2(jt)}{t}.$$

So,

$$\begin{aligned}
&\Big|\int_0^{1/n}\frac{1}{\sqrt{\theta}+\sqrt{\theta-t}}\frac{1-P_n(\cos t)}{t}\,dt\Big| \\
&\quad= \frac{4}{\pi}\sum_{j=1}^{m}\frac{\Gamma(m-j+\frac{1}{2})\Gamma(m+j+\frac{1}{2})}{\Gamma(m-j+1)\Gamma(m+j+1)}\int_0^{1/n}\frac{1}{\sqrt{\theta}+\sqrt{\theta-t}}\frac{\sin^2(jt)}{t}dt \\
&\quad\le \theta^{-\frac{1}{2}}\frac{4}{\pi}\Big[\sum_{j=1}^{m-1}\frac{1}{\sqrt{m-j}\sqrt{m+j}}\int_0^{1/n}\frac{\sin^2(jt)}{t}dt + \sqrt{\frac{\pi}{2m}}\int_0^{1/n}\frac{\sin^2(mt)}{t}dt\Big] \\
&\quad\le \theta^{-\frac{1}{2}}\frac{4}{\pi}\Big[\frac{1}{n}\sum_{j=1}^{m-1}\frac{j}{\sqrt{m^2-j^2}} + \sqrt{\frac{\pi}{2}}\Big] \le \theta^{-\frac{1}{2}}\frac{4}{\pi}\Big[\frac{1}{n}\int_0^{m}\frac{x}{\sqrt{m^2-x^2}}dx + \sqrt{\frac{\pi}{2}}\Big] \\
&\quad\le \theta^{-\frac{1}{2}}\frac{4}{\pi}\Big[2+\sqrt{\frac{\pi}{2}}\Big].
\end{aligned}$$

This means

$$\Big|\int_0^{1/n}\frac{1}{\sqrt{\theta}+\sqrt{\theta-t}}\frac{1-P_n(\cos t)}{t}\,dt\Big| \le \theta^{-\frac{1}{2}}\log(n\theta)\cdot\frac{4}{\pi}\Big(4+\sqrt{\frac{\pi}{2}}\Big).$$

While for the second part, we have

$$\Big|\int_{1/n}^{\theta} \frac{1}{\sqrt{\theta}+\sqrt{\theta-t}} \frac{1-P_n(\cos t)}{t}\, dt\Big| \le \frac{2}{\sqrt{\theta}} \int_{1/n}^{\theta} \frac{1}{t} dt = \frac{2}{\sqrt{\theta}} \log(n\theta).$$

Combining the two parts, we prove (3.2.31).

To show (3.2.30), since for a large number A and $\theta \in (0, \frac{\pi}{2})$, $n\theta \ge A \ge 5$ implies $n \ge \frac{A}{\theta} > \frac{10}{\pi} > 3$, we then have

$$\begin{aligned}
I_{n,1}(\theta) \ge& \frac{2}{n(n+1)} \Big[\sqrt{\theta} \int_0^{\theta} \frac{1-P_n(\cos t)}{t^2}\, dt - \theta^{-\frac{1}{2}} \log(n\theta) \left(\frac{4}{\pi}(4+\sqrt{\frac{\pi}{2}})+4\right)\Big] \\
\ge& \frac{2}{\frac{4}{3}n^2} \Big[\sqrt{\theta} \int_0^{\theta} \frac{1-P_n(\cos t)}{t^2}\, dt - \theta^{-\frac{1}{2}} \log(n\theta) \left(\frac{4}{\pi}(4+\sqrt{\frac{\pi}{2}})+4\right)\Big] \\
=& \frac{3}{4} \frac{2\sqrt{\theta}}{n^2} \int_0^{\theta} \frac{1-P_n(\cos t)}{t^2}\, dt - \frac{3}{2n^2} \theta^{-\frac{1}{2}} \log(n\theta) \left(\frac{4}{\pi}(4+\sqrt{\frac{\pi}{2}})+4\right).
\end{aligned}$$

This proves (3.2.30).

Next, we will deal with the integral

$$\int_0^{\theta} \frac{1-P_n(\cos t)}{t^2} dt.$$

We will apply the representation (2.5.12) again and assume that $n = 2m$ (The case when $n = 2m-1$ can be treated similarly). We then have the identities (3.2.32) and (3.2.33). Thus,

$$\frac{2\sqrt{\theta}}{n^2} \int_0^{\theta} \frac{1-P_n(\cos t)}{t^2}\, dt = \frac{8}{\pi} \frac{\sqrt{\theta}}{n^2} \sum_{j=1}^{m} \frac{\Gamma(m-j+\frac{1}{2})\Gamma(m+j+\frac{1}{2})}{\Gamma(m-j+1)\Gamma(m+j+1)} \int_0^{\theta} \frac{\sin^2(jt)}{t^2}\, dt. \tag{3.2.34}$$

Note that for $j \geq 1$,

$$\int_0^{\theta} \frac{\sin^2(jt)}{t^2}\,dt = j\int_0^{j\theta} \frac{\sin^2 t}{t^2}\,dt \geq j\int_0^{\infty} \frac{\sin^2 t}{t^2}\,dt - \frac{1}{\theta} = j\int_0^{\infty} \frac{\sin(t)}{t}\,dt - \frac{1}{\theta} = \frac{\pi}{2}j - \frac{1}{\theta}, \tag{3.2.35}$$

where we used the identity $\int_0^\infty \frac{\sin\theta}{\theta}\,d\theta = \frac{\pi}{2}$ in the last step (see [2, p. 50]). Thus, by (3.2.34) and (3.2.35), we obtain

$$\begin{aligned}
\frac{2\sqrt{\theta}}{n^2}\int_0^{\theta} \frac{1-P_n(\cos t)}{t^2}\,dt &\geq \frac{4\sqrt{\theta}}{n^2}\sum_{j=1}^{m} \frac{j\Gamma(m-j+\frac{1}{2})\Gamma(m+j+\frac{1}{2})}{\Gamma(m-j+1)\Gamma(m+j+1)} - 4n^{-2}\theta^{-\frac{1}{2}} \\
&\geq \frac{4\sqrt{\theta}}{n^2}\sum_{j=1}^{m-1} \frac{j}{\sqrt{m^2-j^2}} - 2\sqrt{\pi}\sqrt{\frac{\pi}{2}}n^{-\frac{3}{2}} - 4n^{-2}\theta^{-\frac{1}{2}} \\
&\geq \frac{4\sqrt{\theta}}{n^2}\int_0^{m-1} \frac{y}{(m^2-y^2)^{\frac{1}{2}}}\,dy - 2\sqrt{\pi}\sqrt{\frac{\pi}{2}}n^{-\frac{3}{2}} - \frac{4}{\sqrt{5}}\cdot n^{-\frac{3}{2}} \\
&= \frac{4\sqrt{\theta}}{n^2}(m-\sqrt{2m-1}) - \Big(2\sqrt{\pi}\sqrt{\frac{\pi}{2}} + \frac{4}{\sqrt{5}}\Big)(n\theta)^{-\frac{3}{2}}\theta^{3/2} \\
&\geq \frac{2\sqrt{\theta}}{n} - \Big(10\sqrt{\frac{\pi}{2}} + 2\sqrt{\pi}\sqrt{\frac{\pi}{2}} + \frac{4}{\sqrt{5}}\Big)(n\theta)^{-\frac{3}{2}}\theta^{3/2},
\end{aligned}$$

where the second inequality obtained by the Gautschi's inequality: $x^{1-s} < \frac{\Gamma(x+1)}{\Gamma(x+s)} < (x+1)^{1-s}$,

$$\begin{aligned}
&\Big|\frac{4\sqrt{\theta}}{n^2}\sum_{j=1}^{m} \frac{j\Gamma(m-j+\frac{1}{2})\Gamma(m+j+\frac{1}{2})}{\Gamma(m-j+1)\Gamma(m+j+1)} - \frac{4\sqrt{\theta}}{n^2}\sum_{j=1}^{m-1} \frac{j}{\sqrt{m^2-j^2}}\Big| \\
&\leq \frac{4\sqrt{\theta}}{n^2}\Big|\sum_{j=1}^{m-1} j\Big(\frac{1}{\sqrt{m-j}\sqrt{m+j}} - \frac{1}{\sqrt{m^2-j^2}}\Big) + \frac{m\Gamma(\frac{1}{2})}{\Gamma(1)}\frac{1}{(2m)^{\frac{1}{2}}}\Big| \\
&= \frac{4\sqrt{\theta}}{n^2}\frac{m\sqrt{\pi}}{\sqrt{2m}} \leq 2\sqrt{\pi}\sqrt{\frac{\pi}{2}}n^{-\frac{3}{2}},
\end{aligned}$$

and the third inequality follows by the fact $4n^{-2}\theta^{-\frac{1}{2}} = 4n^{-\frac{3}{2}}(n\theta)^{-\frac{1}{2}} \leq \frac{4}{\sqrt{5}}\cdot n^{-\frac{3}{2}}$. Hence, we have

$$
\begin{aligned}
I_{n,1}(\theta) \geq & \frac{3}{4}\frac{2\sqrt{\theta}}{n^2}\int_0^\theta \frac{1-P_n(\cos t)}{t^2}\,dt - \frac{3}{2n^2}\theta^{-\frac{1}{2}}\log(n\theta)\left(\frac{4}{\pi}(4+\sqrt{\frac{\pi}{2}})+4\right)\\
\geq & \frac{3}{4}\frac{2\sqrt{\theta}}{n} - \frac{3}{4}\Big(10\sqrt{\frac{\pi}{2}}+2\sqrt{\pi}\sqrt{\frac{\pi}{2}}+\frac{4}{\sqrt{5}}\Big)(n\theta)^{-\frac{3}{2}}\theta^{3/2} - \frac{3}{2n^2}\theta^{-\frac{1}{2}}\log(n\theta)\left(\frac{4}{\pi}(4+\sqrt{\frac{\pi}{2}})+4\right)\\
\geq & \frac{3}{4}\frac{2\sqrt{\theta}}{n} - \frac{3}{4}\Big(10\sqrt{\frac{\pi}{2}}+2\sqrt{\pi}\sqrt{\frac{\pi}{2}}+\frac{4}{\sqrt{5}}+2\big(\frac{4}{\pi}(4+\sqrt{\frac{\pi}{2}})+4\big)\Big)(n\theta)^{-\frac{3}{2}}\theta^{\frac{3}{2}}.
\end{aligned}
$$

$\square$

Step 3: Estimate of the integral

In this step, we will prove the following estimate:

Lemma 3.2.9. *For $n\theta \geq 5$ and $\theta \in (0, \frac{\pi}{2}]$,*

$$
\frac{4n(n+1)}{3\theta^{\frac{3}{2}}}\int_0^\theta (\theta-t)^{\frac{3}{2}}P_n(\cos t)\sin t\,dt \geq (\frac{3}{2}-\sqrt{2})(n\theta)^{-1} - 164.9212\times(n\theta)^{-\frac{3}{2}}.
$$

In particular, this implies that there exists a determined constant $A > 1$ such that

$$
\int_0^\theta (\theta-t)^{\frac{3}{2}}P_n(\cos t)\sin t\,dt > 0
$$

whenever $n\theta \geq A$.

Proof. By the Lemma 3.2.6 3.2.7 and 3.2.8, for $n\theta \geq 5$, we have

$$
\begin{aligned}
&\frac{4n(n+1)}{3}\int_0^\theta (\theta-t)^{\frac{3}{2}}P_n(\cos t)\sin t\,dt\\
&\geq \frac{3}{4}\cdot 2(n\theta)^{-1}\theta^{\frac{3}{2}} - 30.1066\times(n\theta)^{-\frac{3}{2}}\theta^{3/2} - \frac{\sqrt{2}}{n\theta}\sin(N\theta)\Big(\frac{\sin\theta}{\theta}\Big)^{\frac{1}{2}}\theta^{\frac{3}{2}}\\
&\quad - 92.1237\times(n\theta)^{-3/2}\theta^{3/2} - 42.6909\times(n\theta)^{-2}\theta^{\frac{3}{2}}.
\end{aligned}
$$

Thus,

$$\begin{aligned}
&\frac{4n(n+1)}{3\theta^{\frac{3}{2}}}\int_0^\theta (\theta-t)^{\frac{3}{2}} P_n(\cos t)\sin t\,dt \\
&\geq \frac{3}{2}(n\theta)^{-1} - 30.1066\times(n\theta)^{-\frac{3}{2}} - \frac{\sqrt{2}}{n\theta}\sin(N\theta)\Big(\frac{\sin\theta}{\theta}\Big)^{\frac{1}{2}} \\
&\quad - 92.1237\times(n\theta)^{-3/2} - 42.6909\times(n\theta)^{-2}. \\
&\geq (\frac{3}{2}-\sqrt{2})(n\theta)^{-1} - 164.9212\times(n\theta)^{-\frac{3}{2}} > 0
\end{aligned}$$

whenever

$$n\theta \geq \Big(\frac{164.9212}{\frac{3}{2}-\sqrt{2}}\Big)^2 > 3.6959\times 10^6. \tag{3.2.36}$$

This means there exists a determined positive constant A such that the desired integral (3.2.23) is positive when $n\theta \geq A$. □

3.2.3.2 Case (ii): $n\theta \leq B$

In this case, we shall prove that there exists a constant $B \in (0,1)$ such that (3.2.23) is true whenever $n\theta \leq B$.

To see this, we first note that

$$\int_0^\theta (\theta-t)^{\frac{3}{2}} P_n(\cos t)\sin t\,dt = \theta^{\frac{3}{2}}\int_0^\theta (1-\frac{t}{\theta})^{\frac{3}{2}} P_n(\cos t)\sin t\,dt.$$

Thus, what we need to show is

$$\int_0^\theta (1-\frac{t}{\theta})^{\frac{3}{2}} P_n(\cos t)\sin t\,dt > 0.$$

Indeed,

$$
\begin{aligned}
&\theta^{-2}\int_0^\theta (1-\frac{t}{\theta})^{\frac{3}{2}} P_n(\cos t)\sin t\,dt \\
&\quad= \theta^{-2}\Big[\int_0^\theta (1-\frac{t}{\theta})^{\frac{3}{2}} t\,dt + \int_0^\theta (1-\frac{t}{\theta})^{\frac{3}{2}}\Big(P_n(\cos t)\frac{\sin t}{t}-1\Big)t\,dt\Big] \\
&\quad\geq \theta^{-2}\int_0^\theta (1-\frac{t}{\theta})^{\frac{3}{2}} t\,dt - 2n\theta^{-2}\int_0^\theta t^2\,dt \\
&\quad= \int_0^1 (1-x)^{\frac{3}{2}} x\,dx - n\theta \\
&\quad= \frac{4}{35} - 2n\theta > 0.
\end{aligned}
$$

For the first inequality, we used Bernstein's inequality for trigonometric polynomials. Indeed, for $n \geq 0$, $0 < t < \theta < \frac{\pi}{2}$, we have

$$
\begin{aligned}
|P_n(\cos t)(\frac{\sin t}{t}) - 1| &\leq |P_n(\cos t)(\frac{\sin t}{t}) - \frac{\sin t}{t}| + |\frac{\sin t}{t} - 1| \\
&\leq |\frac{\sin t}{t}| \cdot |P_n(\cos t) - 1| + 1 \\
&\leq |P_n(\cos t) - P_n(\cos 0)| + 1 \\
&\leq nt||P_n(\cos t')||_\infty + nt < 2n\theta,
\end{aligned}
$$

where $0 < t' < t < \theta$. The last inequality is positive under the condition

$$
n\theta < \frac{2}{35}. \tag{3.2.37}
$$

This means there exists a determined positive constant B such that the desired integral (3.2.23) is positive when $n\theta \leq B$.

3.2.3.3 Case (iii): $B \leq n\theta \leq A$

In this case, we shall prove (3.2.23) for the remaining case $B \leq n\theta \leq A$.

We write

$$\int_0^\theta (\theta - t)^{\frac{3}{2}} P_n(\cos t) \sin t \, dt = \theta^{\frac{3}{2}} \int_0^\theta (1 - \frac{t}{\theta})^{\frac{3}{2}} P_n(\cos t) \sin t \, dt.$$

It suffices to prove

$$\int_0^\theta (1 - \frac{t}{\theta})^{\frac{3}{2}} P_n(\cos t) \sin t \, dt > 0.$$

To simplify our calculation, we will next show

$$\theta^{-2} \int_0^\theta (1 - \frac{t}{\theta})^{\frac{3}{2}} P_n(\cos t) \sin t \, dt > 0.$$

Let $N = n + \frac{1}{2}$, and by substitution $t = \frac{t'}{N}$, we have

$$\theta^{-2} \int_0^\theta (1 - \frac{t}{\theta})^{\frac{3}{2}} P_n(\cos t) \sin t \, dt = N^{-1}\theta^{-2} \int_0^{N\theta} (1 - \frac{t'}{N\theta})^{\frac{3}{2}} P_n(\cos \frac{t'}{N}) \Big(\sin \frac{t'}{N} \Big) \, dt'.$$

By (2.5.18) and the fact

$$\Big| N^{-1}\theta^{-2} \int_0^{N\theta} (1 - \frac{t'}{N\theta})^{\frac{3}{2}} \sin \frac{t'}{N} dt' \Big| \le N^{-1}\theta^{-2} \int_0^{N\theta} (1 - \frac{t'}{N\theta})^{\frac{3}{2}} |\sin \frac{t'}{N}| dt' \le N^{-1}\theta^{-2} \cdot N\theta \cdot \theta = 1,$$

and taking the substitution $x = \frac{t'}{N\theta}$, we have

$$\begin{aligned} &N^{-1}\theta^{-2} \int_0^{N\theta} (1 - \frac{t'}{N\theta})^{\frac{3}{2}} P_n(\cos \frac{t'}{N}) \Big(\sin \frac{t'}{N} \Big) \, dt' \\ &\qquad \ge N^{-1}\theta^{-2} \int_0^{N\theta} (1 - \frac{t'}{N\theta})^{\frac{3}{2}} \Big(\frac{t'}{N} \Big)^{\frac{1}{2}} \Big(\frac{\sin \frac{t'}{N}}{\frac{t'}{N}} \Big)^{\frac{1}{2}} j_0(t') \, dt' - 0.1711 \times n^{-1} \\ &\qquad = \int_0^1 (1 - x)^{\frac{3}{2}} j_0(N\theta x) x \Big(\frac{\sin(\theta x)}{\theta x} \Big)^{\frac{1}{2}} dx - 0.1711 \times n^{-1} \\ &\qquad \ge \sqrt{\frac{2}{\pi}} \int_0^1 (1 - x)^{\frac{3}{2}} j_0(N\theta x) x \, dx - 0.1711 \times n^{-1} \\ &\qquad \ge \sqrt{\frac{2}{\pi}} \int_0^1 (1 - x)^{\frac{3}{2}} j_0(ux) x \, dx - 0.1711 \times B^{-1}\theta, \end{aligned} \tag{3.2.38}$$

where $u := N\theta$. To make the last expression be positive, we will find a positive lower bound for the integral $\int_0^1 (1-x)^{\frac{3}{2}} j_0(ux)xdx$ when $B \le u \le A$. By the reference [24, Corollary 1.1, Page 551, 552, 556], we have

$$\int_0^1 (1-x)^{\frac{3}{2}} j_0(ux)xdx \ge \frac{\Gamma(2)\Gamma(\frac{5}{2})}{\Gamma(1)\Gamma(\frac{9}{2})} \begin{cases} 1 - \frac{2}{33}u^2, & 0 \le u \le \sqrt{\frac{33}{2}} \approx 4.0620 \\ \frac{2.0963}{u^3}, & u \ge \sqrt{12} \approx 3.4641 \end{cases} \tag{3.2.39}$$

Thus, in our case, for $B \le u \le A$, we have to separate two cases to find the positive lower bound:

(1) When $B \le u \le \sqrt{12}$, we have

$$\int_0^1 (1-x)^{\frac{3}{2}} j_0(ux)xdx = \frac{\Gamma(2)\Gamma(\frac{5}{2})}{\Gamma(1)\Gamma(\frac{9}{2})}(1 - \frac{24}{33}) = \frac{\Gamma(2)\Gamma(\frac{5}{2})}{\Gamma(1)\Gamma(\frac{9}{2})}\frac{3}{11}.$$

(2) When $\sqrt{12} \le u \le A$, we have

$$\int_0^1 (1-x)^{\frac{3}{2}} j_0(ux)xdx \ge \frac{\Gamma(2)\Gamma(\frac{5}{2})}{\Gamma(1)\Gamma(\frac{9}{2})}\frac{2.0963}{A^3}.$$

Notice that $\frac{3}{11} > \frac{2.0963}{A^3}$, thus we will use (2) as the the positive lower bound. Hence, the last expression (3.2.38) is positive, if we assume θ satisfying the following condition:

$$\theta < \frac{B}{0.1711} \cdot \sqrt{\frac{2}{\pi}} \cdot \frac{\Gamma(2)\Gamma(\frac{5}{2})}{\Gamma(1)\Gamma(\frac{9}{2})}\frac{1}{A^3} \times 2.0963 \le 1.2644 \times 10^{-21}.$$

This completes the proof of Theorem 3.2.4.

3.3 Positive semi-definite functions on $\mathbb{R}^d$ generated from those on $\mathbb{S}^d$

In this section, we will give the proof of Theorem 3.1.2.

Proof of Theorem 3.1.2. Fix $x > 0$. Let $n \geq 2x$ be an integer and let $\theta = \theta_n = \frac{x}{n}$. Since g is an isotropic positive semi-definite function on $\mathbb{S}^d$ and supported in $[0, \pi]$, we have for $\theta \in (0, 1)$,

$$0 \leq n^{2\lambda+1} \int_0^{\theta\pi} g(\theta^{-1}t) R_n^\lambda(\cos t)(\sin t)^{2\lambda} \, dt = \int_0^{x\pi} g\Big(\frac{t}{x}\Big) R_n^\lambda(\cos \frac{t}{n}) \Big(\frac{\sin(n^{-1}t)}{n^{-1}t}\Big)^{2\lambda} t^{2\lambda} \, dt.$$

By (2.5.17), letting $n \to \infty$ and $u = \frac{t}{x}$, we obtain

$$\begin{aligned}
\lim_{n\to\infty} \int_0^{x\pi} g\Big(\frac{t}{x}\Big) R_n^\lambda(\cos \frac{t}{n}) \Big(\frac{\sin(n^{-1}t)}{n^{-1}t}\Big)^{2\lambda} t^{2\lambda} \, dt &= \int_0^{x\pi} g\Big(\frac{t}{x}\Big) j_{\lambda-\frac{1}{2}}(t) t^{2\lambda} \, dt \\
&= x^{2\lambda+1} \int_0^{\infty} g(u) j_{\lambda-\frac{1}{2}}(ux) u^{2\lambda} \, du \\
&= c_\lambda x^{2\lambda+1} \widehat{g_d}(\xi) \geq 0,
\end{aligned}$$

where $\xi \in \mathbb{R}^d$, $|\xi| = x$. By Bochner's theorem, we prove that g is a positive semi-definite function on $\mathbb{R}^d$. $\square$

Part II

Spherical h-harmonic expansions with negative indices

Chapter 4

Introduction

It is well known the n-th Cesàro mean $\sigma_n^{(\alpha,\beta),\delta}f$ of order δ of the Jacobi polynomial expansion of $f \in C[-1,1]$ with parameters $\alpha > -1$ and $\beta > -1$ converges uniformly to f on $[-1,1]$ as $n \to \infty$ if $\delta > \max\{\max\{\alpha + \frac{1}{2}, \beta + \frac{1}{2}\}, 0\}$. This result was obtained in [7] for $\alpha, \beta \geq -\frac{1}{2}$, and in [10, p. 78] for $\alpha, \beta > -1$ from accurate pointwise estimates of the Cesàro kernel $K_n^{(\alpha,\beta),\delta}(s,t)$ given by

$$\sigma_n^{(\alpha,\beta),\delta}(f,s) := \int_{-1}^{1} f(t) K_n^{(\alpha,\beta),\delta}(s,t)(1-t)^{\alpha}(1+t)^{\beta}\, dt, \quad s \in [-1,1].$$

Indeed, accurate estimates of the Cesàro kernels have many other important applications, including the summability results of $\sigma_n^{(\alpha,\beta),\delta}f$ on L^p spaces if p lies between the critical values, the weak type estimates of the maximal Cesàro operators $\sup_n |\sigma_n^{(\alpha,\beta),\delta}f|$, and various multiplier theorems for the Jacobi polynomial expansions. (See [7] for $\alpha, \beta \geq -\frac{1}{2}$, and [10] for $\alpha, \beta > -1$). An accurate estimate of the kernel $K_n^{(\alpha,\beta),\delta}(s,t)$ was first obtained by Bonami and Clerc [7, Theorem 2.1] in 1973 for all $\delta > 0$ and $\alpha, \beta \geq -\frac{1}{2}$. However, problem for the remaining case of parameters $-1 < \min\{\alpha, \beta\} < -\frac{1}{2}$ looks much more difficult. This was finally solved by Chanillo and Muckenhoupt [10] in 1993, who established an accurate estimate of $K_n^{(\alpha,\beta),\delta}(s,t)$ for all parameters $\alpha, \beta > -1$.

Recently, some of these results have been extended to the Cesàro means of weighted orthogonal polynomial expansions (WOPEs) in several variables on the unit sphere $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : \|x\| = 1\}$, the unit ball $\mathbb{B}^d = \{x \in \mathbb{R}^d : \|x\| \le 1\}$, and the simplex $\mathbb{T}^d = \{x \in \mathbb{R}^d : x_1 \ge 0, \dots, x_d \ge 0, x_1 + \cdots + x_d \le 1\}$ in a series of papers ([15–18, 30]), with weights being given by

$$h_\kappa^2(x) := \prod_{i=1}^{d} |x_i|^{2\kappa_i}, \quad x \in \mathbb{S}^{d-1}, \tag{4.0.1}$$

$$W_\kappa^B(x) := \Big(\prod_{i=1}^{d} |x_i|^{2\kappa_i}\Big)(1 - \|x\|^2)^{\kappa_{d+1}-1/2}, \quad x \in \mathbb{B}^d \tag{4.0.2}$$

$$W_\kappa^T(x) := \Big(\prod_{i=1}^{d} x_i^{\kappa_i-1/2}\Big)(1 - x_1 - \cdots - x_d)^{\kappa_{d+1}-1/2}, \qquad x \in \mathbb{T}^d, \tag{4.0.3}$$

and with parameters $\kappa_1, \cdots, \kappa_{d+1} \in \mathbb{R}$ satisfying

$$\kappa_{\min} := \min_{1 \le i \le d+1} \kappa_i \ge 0. \tag{4.0.4}$$

Here and throughout this part, $\|\cdot\|$ denotes the Euclidean norm of $\mathbb{R}^d$. WOPEs on the sphere $\mathbb{S}^{d-1}$ turn out to be closely related to WOPEs on the ball $\mathbb{B}^d$ and the simplex $\mathbb{T}^d$, as was observed by Xu [39].

It should be pointed out that the condition (4.0.4) is essential in the works of [15–18, 30], where many arguments do not work if one of the parameters κ_i is negative. On the other hand, however, it is easily seen that the weights in (4.0.1)–(4.0.3) are integrable on the underlying domain if and only if $\kappa_{\min} > -\frac{1}{2}$, and as a result, the above mentioned WOPEs on the sphere $\mathbb{S}^{d-1}$, the ball $\mathbb{B}^d$ and the simplex $\mathbb{T}^d$ are well defined if $\kappa_{\min} > -\frac{1}{2}$.

One of the main purposes in this part is to establish accurate estimates of the Cesàro kernels of the above mentioned WOPEs with less restriction on the parameters κ_i (i.e., $\kappa_{\min} > -\frac{1}{2}$), from which we deduce the Cesàro summability results of the WOPEs for $\kappa_{\min} < 0$. We develop a new technique to establish accurate pointwise estimates of the Cesàro kernels, which works for the full range of $\kappa_{\min} > -\frac{1}{2}$. We believe that this new technique will, in particular, lead to a simper proof of the estimates of Chanillo and Muckenhoupt [10, Theorem 14.1] on the Cesàro kernels of the Jacobi polynomial expansions with parameters $\alpha, \beta > -1$.

Throughout this part we denote by c a generic constant that may depend on fixed parameters such as κ and d, whose value may change from line to line. Furthermore we write $A \sim B$ if there exists a constant $c > 0$ such that $A \geq cB$ and $B \geq cA$.

Chapter 5

Preliminaries

5.1 Weighted orthogonal polynomial expansions (WOPEs) on the sphere $\mathbb{S}^{d-1}$

Let $d\sigma(x)$ denote the usual surface Lebesgue measure on $\mathbb{S}^{d-1}$, and ρ the geodesic distance on $\mathbb{S}^{d-1}$; that is, $\rho(x, y) = \arccos x \cdot y$ for $x, y \in \mathbb{S}^{d-1}$. Denote by $B(x, \theta)$ the spherical cap with center $x \in \mathbb{S}^{d-1}$ and radius $\theta > 0$; that is, $B(x, \theta) := \{y \in \mathbb{S}^{d-1} : \ \rho(x, y) \le \theta\}$. For $\kappa := (\kappa_1, \cdots, \kappa_d) \in \mathbb{R}^d$, we define

$$h_\kappa(x) := \prod_{i=1}^{d} |x_i|^{\kappa_i}, \quad x \in \mathbb{S}^{d-1}. \tag{5.1.1}$$

Then $h_\kappa(x)$ is a homogeneous function of degree $\gamma_\kappa := \kappa_1 + \cdots + \kappa_d$, and h_κ^2 is integrable on $\mathbb{S}^{d-1}$ if and only if $\kappa_{\min} := \min_{1\le j\le d} \kappa_j > -\frac{1}{2}$. Unless otherwise stated, we will always assume that $\kappa_{\min} > -\frac{1}{2}$ throughout this part.

Next, we denote by μ_κ the probability measure on $\mathbb{S}^{d-1}$ given by $d\mu_\kappa(x) := \omega_d^\kappa h_\kappa^2(x)\, d\sigma(x)$, where

$$\omega_d^\kappa := \Big(\int_{\mathbb{S}^{d-1}} h_\kappa^2(x)\, d\sigma(x)\Big)^{-1} = \frac{2\Gamma(\kappa_1 + \frac{1}{2}) \cdots \Gamma(\kappa_d + \frac{1}{2})}{\Gamma(\gamma_\kappa + \frac{d}{2})}. \tag{5.1.2}$$

It is easily seen that for $0 < r \le \pi$, (see [13]),

$$\mu_\kappa(B(x,r)) \sim r^{d-1}\prod_{j=1}^d (|x_j|+r)^{2\kappa_j}, \quad x \in \mathbb{S}^{d-1}, \tag{5.1.3}$$

where the constants of equivalence depend only on d and κ. This in particular implies that μ_κ is a doubling measure on $\mathbb{S}^{d-1}$ satisfying that for any $x \in \mathbb{S}^{d-1}$ and $\theta \in (0, \frac{\pi}{4})$,

$$\mathrm{meas}_\kappa(B(x, 2^j\theta)) \le C2^{js_\kappa}\,\mathrm{meas}_\kappa(B(x,\theta)), \quad j = 1, 2, \cdots, \tag{5.1.4}$$

where $C > 0$ is a constant depending only on κ and d,

$$s_\kappa = d - 1 + 2\gamma_\kappa^+ - 2\max\{\kappa_{\min}, 0\} \quad \text{and} \quad \gamma_\kappa^+ := \sum_{j:\kappa_j>0} \kappa_j. \tag{5.1.5}$$

It is easily seen that s_κ is the optimal constant for which (5.1.4) holds.

Given $0 < p \le \infty$, we denote by $L^p(h_\kappa^2) \equiv L^p(h_\kappa^2; \mathbb{S}^{d-1})$ the Lebesgue L^p-space defined with respect to the measure $d\mu_\kappa$ on $\mathbb{S}^{d-1}$, and $\|\cdot\|_{\kappa,p}$ the L^p-norm of the space $L^p(d\mu_\kappa)$. A spherical polynomial of degree at most n on $\mathbb{S}^{d-1}$ is the restriction on $\mathbb{S}^{d-1}$ of an algebraic polynomial in d variables of total degree at most n. Denote by Π_n^d the space of all spherical polynomials of degree at most n on the sphere $\mathbb{S}^{d-1}$. Set $\Pi_{-1}^d = \{0\}$, and let $\mathcal{H}_n^d(h_\kappa^2)$ denote the orthogonal complement of the space Π_{n-1}^d in the Hilbert space $\Pi_n^d \subset L^2(h_\kappa^2)$ (relative to the norm of $L^2(h_\kappa^2)$). Then the $\mathcal{H}_n^d(h_\kappa^2)$, $n = 0, 1, \cdots$ are mutually orthogonal, finite-dimensional linear subspaces of $L^2(h_\kappa^2)$. Denote by $P_n(h_\kappa^2)$ the reproducing kernel of the space $\mathcal{H}_n^d(h_\kappa^2)$; that is,

$$P_n(h_\kappa^2; x, y) := \sum_{j=1}^{a_n^d} Y_{n,j}^\kappa(x)\overline{Y_{n,j}^\kappa(y)}, \quad x, y \in \mathbb{S}^{d-1}, \tag{5.1.6}$$

where $a_n^d = \dim \mathcal{H}_n^d(h_\kappa^2)$ and $\{Y_{n,j}^\kappa : \ j = 1, 2, \cdots, a_n^d\}$ is an orthonormal basis of the space

$\mathcal{H}_n^d(h_\kappa^2) \subset L^2(h_\kappa^2)$. Then the standard Hilbert space theory shows that each $f \in L^2(h_\kappa^2)$ can be represented as an orthogonal series converging in the norm of $L^2(h_\kappa^2)$:

$$f = \sum_{n=0}^{\infty} \operatorname{proj}_n(h_\kappa^2; f), \tag{5.1.7}$$

where $\operatorname{proj}_n(h_\kappa^2) : L^2(h_\kappa^2; \mathbb{S}^{d-1}) \mapsto \mathcal{H}_n^d(h_\kappa^2)$ is the orthogonal projection operator, which can be expressed as an integral operator

$$\operatorname{proj}_n(h_\kappa^2; f, x) = \omega_d^\kappa \int_{\mathbb{S}^{d-1}} f(y) P_n(h_\kappa^2; x, y) h_\kappa^2(y) d\sigma(y), \quad x \in \mathbb{S}^{d-1}. \tag{5.1.8}$$

Clearly, in the case of $h_\kappa(x) \equiv 1$, the orthogonal expansion in (5.1.7) coincides with the ordinary spherical harmonic expansion on $\mathbb{S}^{d-1}$.

We define the n-th Cesàro mean of order $\delta > 0$ of the WOPE (5.1.7) of f by

$$S_n^\delta(h_\kappa^2; f, x) := \frac{1}{A_n^\delta} \sum_{j=0}^{n} A_{n-j}^\delta \operatorname{proj}_j(h_\kappa^2; f, x), \quad x \in \mathbb{S}^{d-1},$$

where $A_j^\delta = \frac{\Gamma(j+\delta+1)}{\Gamma(j+1)\Gamma(\delta+1)}$ for $j = 0, 1, \cdots$. According to (5.1.8), the Cesàro (C, δ) operators $S_n^\delta(h_\kappa^2)$ can be represented as

$$S_n^\delta(h_\kappa^2; f, x) = \omega_d^\kappa \int_{\mathbb{S}^{d-1}} f(y) K_n^\delta(h_\kappa^2; x, y) h_\kappa^2(y)\, d\sigma(y), \quad x \in \mathbb{S}^{d-1}, \tag{5.1.9}$$

where

$$K_n^\delta(h_\kappa^2; x, y) = \sum_{j=0}^{n} \frac{A_{n-j}^\delta}{A_n^\delta} P_j(h_\kappa^2; x, y), \quad x, y \in \mathbb{S}^{d-1}. \tag{5.1.10}$$

In the case when $\kappa_{\min} := \min_{1 \le j \le d} \kappa_j \ge 0$, the Cesàro summability of the orthogonal expansions (5.1.7) has been well studied in a series of papers (see [17, 18, 30]). Indeed, in this case, each function in $\mathcal{H}_n^d(h_\kappa^2)$ is called a spherical h-harmonic of degree n, and the general theory of spherical h-harmonic analysis developed by Dunkl (see [19, 21]) is applicable. More importantly, Xu [41] proved that if $\kappa_{\min} \geq 0$, then the reproducing kernel $P_n(h_\kappa^2)$ of the space $\mathcal{H}_n^d(h_\kappa^2)$ can be expressed explicitly as

$$P_n(h_\kappa^2; x, y) = \frac{n + \lambda_\kappa}{\lambda_\kappa} \int_{[-1,1]^d} C_n^{\lambda_\kappa}\Big(\sum_{j=1}^d t_j x_j y_j\Big) \prod_{j=1}^d d\nu_j(t_j), \tag{5.1.11}$$

where $x = (x_1, \cdots, x_d) \in \mathbb{S}^{d-1}$, $y = (y_1, \cdots, y_d) \in \mathbb{S}^{d-1}$, $\lambda_\kappa := \frac{d-2}{2} + \gamma_\kappa$,

$$d\nu_j(t_j) = \frac{\Gamma(\kappa_j + \frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(\kappa_j)}(1 - t_j^2)^{\kappa_j - 1}(1 + t_j)dt_j \quad \text{if } \kappa_j > 0,$$

and $d\nu_j$ is Dirac measure supported at $x = 1$ if $\kappa_j = 0$; namely,

$$\int_{-1}^1 g(t)\, d\nu_j(t) = g(1) = \lim_{\lambda \to 0} \frac{\Gamma(\lambda + \frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(\lambda)} \int_{-1}^1 g(t)(1 + t)(1 - t^2)^{\lambda - 1} dt.$$

Here and throughout the paper, C_n^λ denotes the usual Gegenbauer polynomial of degree n with parameter $\lambda > -\frac{1}{2}$. It should be pointed out that this explicit integral representation of the reproducing kernel $P_n(h_\kappa^2)$ plays a crucial role in the previous works [17, 18, 30] on Cesàro summability of the spherical h-harmonic expansions. Unfortunately, this formula (5.1.11) is not applicable when $\kappa_{\min} < 0$.

5.2 The extended Jacobi polynomials

We denote by $P_k^{(\alpha,\beta)}$ the usual Jacobi polynomial of degree k with indices α and β. According to [35, (4.21.2)], we have

$$\begin{aligned} P_n^{(\alpha,\beta)}(x) = \frac{1}{n!} \sum_{v=0}^n \binom{n}{v} &(n + \alpha + \beta + 1) \cdots (n + \alpha + \beta + v) \times \\ &\times (\alpha + v + 1) \cdots (\alpha + n) \Big(\frac{x - 1}{2}\Big)^v, \end{aligned}$$

where the general coefficient

$$\binom{n}{v}(n+\alpha+\beta+1)\cdots(n+\alpha+\beta+v)(\alpha+v+1)\cdots(\alpha+n)$$

has to be replaced by $(\alpha+1)(\alpha+2)\cdots(\alpha+n)$ for $v=0$, and by $(n+\alpha+\beta+1)(n+\alpha+\beta+2)\cdots(2n+\alpha+\beta)$ for $v=n$. This formula furnishes the extension of the polynomial $P_n^{(\alpha,\beta)}(x)$ to arbitrary complex values of the parameters α and β. It is a polynomial in x, α, and β satisfying

$$\frac{d}{dx}P_n^{(\alpha,\beta)}(x) = \frac{n+\alpha+\beta+1}{2}P_{n-1}^{(\alpha+1,\beta+1)}(x). \tag{5.2.1}$$

In the case when α, β are both real, we have the following well known estimate on the Jacobi polynomials ([35, (7.32.5) and (4.1.3)]):

Lemma 5.2.1. *For an arbitrary real number α and $t \in [0,1]$,*

$$|P_n^{(\alpha,\beta)}(t)| \le cn^{-1/2}(1-t+n^{-2})^{-(\alpha+1/2)/2}. \tag{5.2.2}$$

The estimate on $[-1,0]$ follows from the fact that $P_n^{(\alpha,\beta)}(t) = (-1)^n P_n^{(\beta,\alpha)}(-t)$.

Next, we denote by C_n^λ the usual Gegenbauer polynomial of degree n with parameter $\lambda > -\frac{1}{2}$. As is well known, for $\alpha > -1$

$$C_n^{\alpha+\frac{1}{2}}(x) = \frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}\frac{\Gamma(n+2\alpha+1)}{\Gamma(n+\alpha+1)}P_n^{(\alpha,\alpha)}(x). \tag{5.2.3}$$

For later applications, we introduce the following normalized Jacobi polynomials:

$$E_n^\alpha(x) = \frac{\sqrt{\pi}(n+\alpha+\frac{1}{2})\Gamma(n+2\alpha+1)}{2^{2\alpha+1}\Gamma(n+\alpha+1)}P_n^{(\alpha,\alpha)}(x) \tag{5.2.4}$$

$$= \frac{(n+\alpha+\frac{1}{2})\Gamma(\alpha+\frac{1}{2})}{2}C_n^{\alpha+\frac{1}{2}}(x). \tag{5.2.5}$$

We also define $E_j^\alpha(x) = 0$ for $j < 0$. Since

$$E_0^\alpha(t) = \frac{\sqrt{\pi}}{2^{2\alpha+2}} \frac{\Gamma(2\alpha+2)}{\Gamma(\alpha+1)} P_0^{(\alpha,\alpha)}(t) = \frac{1}{2}\Gamma(\alpha + \frac{3}{2}),$$

it follows that E_k^α is an analytic function in the parameter α on the domain $\{\alpha \in \mathbb{C} : \operatorname{Re}\alpha > -\frac{3}{2}\}$ for each integer k. Moreover,

$$\frac{d}{dx} E_n^\alpha(x) = E_{n-1}^{\alpha+1}(x).$$

Chapter 6

Boundedness of projection operators and Cesàro means in weighted L^p space on the unit sphere

6.1 Main results on the sphere $\mathbb{S}^{d-1}$

In this section, we state our main results on the sphere. The proofs of these results will be given in later sections.

Our first result gives an explicit integral representation of the reproducing kernel $P_n(h_\kappa^2; x, y)$ of the space $\mathcal{H}_n^d(h_\kappa^2)$. Our main purpose is to extend the explicit integral representation (5.1.11) of Xu [41] to the case $\kappa_{\min} < 0$. To be precise, let $e_k(x_1, x_2, \cdots, x_d)$, $k = 0, 1, \cdots, d$ be the elementary symmetric polynomials in d variables given by

$$e_0(x_1, x_2, \cdots, x_d) = 1,$$
$$e_k(x_1, x_2, \cdots, x_d) = \sum_{1 \le j_1 < j_2 < \cdots < j_k \le d} x_{j_1} x_{j_2} \cdots x_{j_k}, \quad 1 \le k \le d.$$

In Section 6.2, we prove

Theorem 6.1.1. *Let $\kappa = (\kappa_1, \cdots, \kappa_d) \in \mathbb{R}^d$ be such that $\kappa_{\min} := \min_{1\leq j\leq d} \kappa_j > -\frac{1}{2}$. Let $\gamma_\kappa := \sum_{j=1}^d \kappa_j$ and $\lambda_\kappa = \frac{d-2}{2} + \gamma_\kappa$. Then for any $x, y \in \mathbb{S}^{d-1}$,*

$$P_n(h_\kappa^2; x, y) = \frac{1}{\pi^{d/2}} \Big(\prod_{j=1}^d \frac{\Gamma(\kappa_j + \frac{3}{2})}{\Gamma(\kappa_j + 1)} \Big) \sum_{\ell=0}^d P_{n,\ell}(h_\kappa^2; x, y), \tag{6.1.1}$$

where for $\ell = 0, 1, \cdots, d$,

$$\begin{aligned} P_{n,\ell}(h_\kappa^2; x, y) := \frac{(n+\lambda_\kappa)\Gamma(\lambda_\kappa+\ell)}{\Gamma(\lambda_\kappa+1)} \int_{[-1,1]^d} C_{n-\ell}^{\lambda_\kappa+\ell}\big(\sum_{j=1}^d x_j y_j t_j\big) \times \\ \times e_\ell\Big(\frac{x_1 y_1(1+t_1)}{2\kappa_1+1}, \cdots, \frac{x_d y_d (1+t_d)}{2\kappa_d+1}\Big) \Big(\prod_{j=1}^d (1-t_j^2)^{\kappa_j} dt_j\Big). \end{aligned} \tag{6.1.2}$$

Here and throughout the paper, it is agreed that $C_j^\lambda(t) = 0$ for $j < 0$.

After that, in Section 6.3, we prove an accurate estimate of a multiple integral that takes the form on the right hand side of (6.1.2) with $P_n^{(\alpha,\beta)}$ in place of $C_{n-\ell}^{\lambda_\kappa+\ell}$. Such an estimate together with Theorem 6.1.1 allows us to deduce the following sharp estimate of the reproducing kernel $P_n(h_\kappa^2; x, y)$:

Theorem 6.1.2. *Let $\kappa = (\kappa_1, \cdots, \kappa_d) \in \mathbb{R}^d$ be such that $\kappa_{\min} > -\frac{1}{2}$. Then for any $x, y \in \mathbb{S}^{d-1}$,*

$$|P_n(h_\kappa^2; x, y)| \leq C n^{d-2} \max_{\varepsilon\in\{\pm1\}^d} \frac{\prod_{j=1}^d (|x_j y_j| + n^{-1}\rho(x\varepsilon, y) + n^{-2})^{-\kappa_j}}{\Big(1 + n\rho(x\varepsilon, y)\Big)^{\frac{d-2}{2}}}.$$

Here and elsewhere, for $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$ and $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_d) \in \{\pm1\}^d$,

$$x\varepsilon := (x_1\varepsilon_1, x_2\varepsilon_2, \cdots, x_d\varepsilon_d).$$

In the case when $\kappa_{\min} \geq 0$, Theorem 6.1.2 was proved previously in [17, Theorem 2.1]. In Section 6.4, we develop a new technique to deduce the following sharp estimates of the Cesàro kernels, which were previously proved in [17, Theorem 2.1] in the case when $\kappa_{\min} \geq 0$:

Theorem 6.1.3. *If $\delta \geq 0$, then for any $x = (x_1, \cdots, x_d)$, $\quad y = (y_1, \cdots, y_d) \in \mathbb{S}^{d-1}$,*

$$|K_n^\delta(h_\kappa^2; x, y)| \leq \ cn^{d-1} \max_{\varepsilon \in \{\pm 1\}^d} \left[\frac{\prod_{j=1}^d (|x_j y_j| + n^{-1}\rho(x, y\varepsilon) + n^{-2})^{-\kappa_j}}{(1 + n\rho(x, \varepsilon y))^{\delta + \frac{d}{2}}} \right. \\ \left. + \frac{\prod_{j=1}^d (|x_j y_j| + (\rho(x, y\varepsilon))^2 + n^{-2})^{-\kappa_j}}{(n\rho(x, y\varepsilon) + 1)^d} \right]. \tag{6.1.3}$$

We point out here that it seems very hard to prove Theorem 6.1.3 following the method of [17] or [7]. Indeed, in the case when $\kappa_{\min} \geq 0$, the basic idea behind the proof of these estimates in [17] is to reduce the problem to estimating certain multiple integrals of Jacobi polynomials, using the following formula for the Cesàro kernels of the Jacobi polynomial expansions ([35, p. 261, (9.4.13)]):

$$K_n^{(\alpha,\alpha),\delta}(x, 1) = a_n^\delta(\alpha) P_n^{(\alpha+\delta+1,\alpha)}(x) + \sum_{v=1}^{\infty} b_{n,v}^\delta(\alpha) K_n^{(\alpha,\alpha),\delta+v}(x, 1), \tag{6.1.4}$$

where the explicit expression of the coefficients $a_n^\delta(\alpha)$ and $b_{n,v}^\delta(\alpha)$ can be found in [35, p. 261].

In the case when $\kappa_{\min} < 0$, it seems hard to deduce the desired estimates on Cesàro kernels from (6.1.4), as can be seen in the work of Chanillo and Muckenhoupt [10] on Cesàro kernels of the Jacobi polynomial expansions with parameter $\alpha, \beta > -1$.

One of the main difficulties comes from the fact that if $\kappa_{\min} < 0$, then the integral representation of the reproducing kernel $P_n(h_\kappa^2; x, y)$ stated in Theorem 6.1.1 involves derivatives of the Gegenbauer polynomials C_n^λ, and as a result, an application of (6.1.4) would require accurate estimates of certain multiple integrals involving the derivatives of the Cesàro kernels $K_n^{(\alpha,\alpha),\delta+v}(x, 1)$ with parameter $\alpha < -\frac{1}{2}$, which are very hard to prove. Our proof of Theorem 6.1.3 uses neither the formula 6.1.4 nor those estimates of Chanillo and Muckenhoupt [10].

Because of the doubling property (5.1.4) of the weight function $h_\kappa^2(x)$ on $\mathbb{S}^{d-1}$, in many applications it is more convenient to write the estimates stated in Theorems 6.1.2 and 6.1.3 in the following form:

Corollary 6.1.1. *Let $\kappa = (\kappa_1, \cdots, \kappa_d) \in \mathbb{R}^d$ be such that $\kappa_{\min} > -\frac{1}{2}$. Let*

$$\sigma_\kappa := \frac{s_\kappa - 1}{2} = \frac{d-2}{2} + \gamma_\kappa^+ - \max\{\kappa_{\min}, 0\}, \tag{6.1.5}$$

where s_κ is the optimal constant for which (5.1.4) holds. Then for $\delta \geq 0$ and $x, y \in \mathbb{S}^{d-1}$,

$$|P_n(h_\kappa^2; x, y)| \leq \frac{C}{n w_{n,\kappa}(x,y)} \max_{\varepsilon \in \{\pm 1\}^d} \Big(1 + n\rho(x\varepsilon, y)\Big)^{\sigma_\kappa + 1}, \tag{6.1.6}$$

$$|K_n^\delta(h_\kappa^2; x, y)| \leq \frac{C}{w_{n,\kappa}(x,y)} \max_{\varepsilon \in \{\pm 1\}^d} \Big[\frac{1}{(1 + n\rho(x\varepsilon, y))^{\delta - \sigma_\kappa}} + \frac{1}{1 + n\rho(x\varepsilon, y)}\Big], \tag{6.1.7}$$

where

$$w_{n,\kappa}(x,y) := \int_{B(x, \rho(x,y) + n^{-1})} h_\kappa^2(z)\, d\sigma(z), \quad x, y \in \mathbb{S}^{d-1}, \quad n \in \mathbb{N}.$$

The proof of Corollary 6.1.1 is given in Section 6.5. Corollary 6.1.1 together with the doubling property of the weight h_κ^2 implies the following result, which is also proved in Section 6.5:

Theorem 6.1.4. *Let $\kappa = (\kappa_1, \cdots, \kappa_d) \in \mathbb{R}^d$ be such that $\kappa_{\min} > -\frac{1}{2}$. Let $\sigma_\kappa := \frac{s_\kappa - 1}{2}$ be given in (6.1.5), and let $\delta \geq 0$. Then*

$$\sup_{x \in \mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} |P_n(h_\kappa^2; x, y)| h_\kappa^2(y)\, d\sigma(y) \leq C n^{\sigma_\kappa} \tag{6.1.8}$$

and

$$\sup_{x \in \mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} |K_n^\delta(h_\kappa^2; x, y)|\, h_\kappa^2(y)\, d\sigma(y) \leq C L_n^\delta, \tag{6.1.9}$$

where

$$L_n^\delta := \begin{cases} n^{\sigma_\kappa-\delta}, & \text{if } 0\le\delta<\sigma_\kappa;\\ \log n, & \text{if } \delta=\sigma_\kappa;\\ 1, & \text{if } \delta>\sigma_\kappa. \end{cases}$$

In particular, this implies that if $f\in L^p(h_\kappa^2;\mathbb{S}^{d-1})$ and $1\le p<\infty$ or $f\in C(\mathbb{S}^{d-1})$ and $p=\infty$, then for any $\delta>\sigma_\kappa$,

$$\lim_{n\to\infty}\|S_n^\delta(h_\kappa^2;f)-f\|_{\kappa,p}=0.$$

Since s_κ is the optimal constant for which (5.1.4) holds, which behalves like the dimension of the measure-metric space $(\mathbb{S}^{d-1},\rho,h_\kappa^2(x)d\sigma(x))$, it is very natural to expect that the stated estimates in Theorem 6.1.4 are sharp in the sense that the corresponding matching lower estimates of the integrals are also true. In the case when $\kappa_{\min}\ge 0$, this was proved in [17, Theorem 2.2]. In the case when $\kappa_{\min}<0$, we can prove the sharpness if there exists one and exactly one parameter κ_j that is negative. The following result is proved in Section 6.6:

Theorem 6.1.5. *If $\#\{j:\ 1\le j\le d,\kappa_j<0\}=1$, then there exists a constant $c>0$ depending only on κ and d such that*

$$\sup_{x\in\mathbb{S}^{d-1}}\int_{\mathbb{S}^{d-1}}|P_n(h_\kappa^2;x,y)|h_\kappa^2(y)\,d\sigma(y)\ge cn^{\sigma_\kappa} \tag{6.1.10}$$

$$\max_{1\le j\le n}\sup_{x\in\mathbb{S}^{d-1}}\int_{\mathbb{S}^{d-1}}|K_j^\delta(h_\kappa^2;x,y)|\,h_\kappa^2(y)\,d\sigma(y)\ge cL_n^\delta,\quad \delta\ge 0. \tag{6.1.11}$$

In particular, this implies that if $0\le\delta\le\sigma_\kappa$, then there exists $f\in C(\mathbb{S}^{d-1})$ such that

$$\lim_{n\to\infty}\|S_n^\delta(h_\kappa^2;f)\|_\infty=\infty.$$

In Sections 7.1 and 7.2, we shall also establish similar results for WOPEs on the unit ball $\mathbb{B}^d$ with respect to the weight function W_κ^B given in (4.0.2) as well as for WOPEs on the simplex $\mathbb{T}^d$ with weights given in (4.0.3), following the approaches developed previously by Xu [39, 40].

6.2 An explicit Integral representation of the reproducing kernels of the space $\mathcal{H}_n^d(h_\kappa^2)$

This section is devoted to the proof of Theorem 6.1.1, the integral representation of the reproducing kernel $P_n(h_\kappa^2; x, y)$ of the space $\mathcal{H}_n^d(h_\kappa^2)$. We start with the following lemma, which will play an important role in the proof of Theorem 6.1.1:

Lemma 6.2.1. *Let* $\mathbf{a} := (a_1, \cdots, a_d) \in \mathbb{R}^d$ *be such that* $\|\mathbf{a}\|_1 := |a_1| + \cdots + |a_d| \le 1$. *Assume that* $\tau = (\tau_1, \cdots, \tau_d) \in (0, \infty)^d$, $f \in C^d[-1, 1]$ *and* $-1 + \|\mathbf{a}\|_1 \le s \le 1 - \|\mathbf{a}\|_1$. *Then*

$$\Big(\prod_{j=1}^d \frac{2\tau_j}{2\tau_j + 1}\Big) \int_{[-1,1]^d} f(\mathbf{a} \cdot \mathbf{t} + s) d\mu_d(\mathbf{t}) \tag{6.2.1}$$
$$= \sum_{\ell=0}^d \int_{[-1,1]^d} f^{(\ell)}(\mathbf{a} \cdot \mathbf{t} + s) e_\ell \Big(\frac{a_1(1 + t_1)}{2\tau_1 + 1}, \cdots, \frac{a_d(1 + t_d)}{2\tau_d + 1}\Big) d\widehat{\mu}_d(\mathbf{t}),$$

where $\mathbf{t} = (t_1, \cdots, t_d) \in [-1, 1]^d$ *and*

$$d\mu_d(\mathbf{t}) := \prod_{j=1}^d (1 - t_j^2)^{\tau_j - 1}(1 + t_j) dt_j \quad \textit{and} \quad d\widehat{\mu}_d(\mathbf{t}) = \prod_{j=1}^d (1 - t_j^2)^{\tau_j} dt_j.$$

Proof. We use induction on the dimension d. For a positive integer d, set

$$I_d := \int_{[-1,1]^d} f(\mathbf{a} \cdot \mathbf{t} + s) d\mu_d(\mathbf{t}).$$

We start with the case of $d = 1$. Write

$$I_1 := \int_{-1}^{1} f(a_1 t + s)(1-t^2)^{\tau_1 - 1}(1+t)\,dt = \int_{-1}^{1} f(a_1 t + s)(1-t^2)^{\tau_1}\,dt + I_{1,1},$$

where

$$I_{1,1} := \int_{-1}^{1} f(a_1 t + s)(1-t^2)^{\tau_1 - 1} t(1+t)\,dt.$$

Integration by parts shows that

$$\begin{aligned} I_{1,1} &= -\frac{1}{2\tau_1}\int_{-1}^{1} f(a_1 t + s)(1+t)\Big((1-t^2)^{\tau_1}\Big)'\,dt \\ &= \frac{1}{2\tau_1}\int_{-1}^{1}\Big[f'(a_1 t + s)a_1(1+t) + f(a_1 t + s)\Big](1-t^2)^{\tau_1}\,dt. \end{aligned}$$

Thus,

$$I_1 = \frac{2\tau_1 + 1}{2\tau_1}\int_{-1}^{1} f(a_1 t + s)(1-t^2)^{\tau_1}\,dt + \frac{a_1}{2\tau_1}\int_{-1}^{1} f'(a_1 t + s)(1+t)(1-t^2)^{\tau_1}\,dt, \quad (6.2.2)$$

which implies (6.2.1) for $d = 1$.

Next, we assume (6.2.1) holds for some positive integer d. In the case of $d+1$, we write $\mathbf{t} \in [-1,1]^{d+1}$ in the form $\mathbf{t} = (\widetilde{\mathbf{t}}, t_{d+1})$, where $\widetilde{\mathbf{t}} := (t_1, \cdots, t_d) \in [-1,1]^d$. Then by Fubini's theorem, we have

$$I_{d+1} := \int_{-1}^{1}\Big[\int_{[-1,1]^d} f(\mathbf{a}\cdot\mathbf{t} + s)\,d\mu_d(\widetilde{\mathbf{t}})\Big](1-t_{d+1}^2)^{\tau_{d+1}-1}(1+t_{d+1})\,dt_{d+1}.$$

Applying the induction hypothesis to this last integral $\int_{[-1,1]^d}$ that is inside the brackets, we get

$$I_{d+1} = \Big(\prod_{j=1}^{d}\frac{2\tau_j + 1}{2\tau_j}\Big)\sum_{\ell=0}^{d} I_{d+1,\ell},$$

where

$$I_{d+1,\ell} := \int_{[-1,1]^d} \Big[\int_{-1}^{1} f^{(\ell)}(\mathbf{a}\cdot\mathbf{t}+s)(1-t_{d+1}^2)^{\tau_{d+1}-1}(1+t_{d+1})\,dt_{d+1}\Big]\times$$
$$\times\, e_\ell\Big(\frac{a_1(1+t_1)}{2\tau_1+1},\cdots,\frac{a_d(1+t_d)}{2\tau_d+1}\Big)d\widehat{\mu}_d(\widetilde{\mathbf{t}}).$$

Using (6.2.2) with $f^{(\ell)}$ and τ_{d+1} in place of f and τ_1 respectively, we have

$$\int_{-1}^{1} f^{(\ell)}(\mathbf{a}\cdot\mathbf{t}+s)(1-t_{d+1}^2)^{\tau_{d+1}-1}(1+t_{d+1})\,dt_{d+1} = \frac{2\tau_{d+1}+1}{2\tau_{d+1}}\int_{-1}^{1} f^{(\ell)}(\mathbf{a}\cdot\mathbf{t}+s)(1-t_{d+1}^2)^{\tau_{d+1}}\,dt_{d+1}$$
$$+\int_{-1}^{1} f^{(\ell+1)}(\mathbf{a}\cdot\mathbf{t}+s)\frac{a_{d+1}(1+t_{d+1})}{2\tau_{d+1}}(1-t_{d+1}^2)^{\tau_{d+1}}\,dt_{d+1}.$$

It follows that

$$I_{d+1,\ell} = \frac{2\tau_{d+1}+1}{2\tau_{d+1}}\int_{[-1,1]^{d+1}}\Big[f^{(\ell)}(\mathbf{a}\cdot\mathbf{t}+s)+f^{(\ell+1)}(\mathbf{a}\cdot\mathbf{t}+s)\frac{a_{d+1}(1+t_{d+1})}{2\tau_{d+1}+1}\Big]\times$$
$$\times\, e_\ell\Big(\frac{a_1(1+t_1)}{2\tau_1+1},\cdots,\frac{a_d(1+t_d)}{2\tau_d+1}\Big)\,d\widehat{\mu}_{d+1}(\mathbf{t}).$$

Thus,

$$I_{d+1} = \Big(\prod_{j=1}^{d+1}\frac{2\tau_j+1}{2\tau_j}\Big)\int_{[-1,1]^{d+1}} F_\ell(\mathbf{a},\mathbf{t},s)d\widehat{\mu}_{d+1}(\mathbf{t}),$$

where

$$F_\ell(\mathbf{a},\mathbf{t},s) := \sum_{\ell=0}^{d}\ \Big[f^{(\ell)}(\mathbf{a}\cdot\mathbf{t}+s)+f^{(\ell+1)}(\mathbf{a}\cdot\mathbf{t}+s)\frac{a_{d+1}(1+t_{d+1})}{2\tau_{d+1}+1}\Big]e_\ell\Big(\frac{a_1(1+t_1)}{2\tau_1+1},\cdots,\frac{a_d(1+t_d)}{2\tau_d+1}\Big).$$

Setting $u_j = \frac{a_j(1+t_j)}{2\tau_j+1}$ for $j = 1, \cdots, d+1$, we have

$$\begin{aligned}F_\ell(\mathbf{a}, \mathbf{t}, s) =& f(\mathbf{a}\cdot\mathbf{t}+s) + f^{(d+1)}(\mathbf{a}\cdot\mathbf{t}+s)e_{d+1}(u_1, \cdots, u_d, u_{d+1}) + \\ &+ \sum_{\ell=1}^{d} f^{(\ell)}(\mathbf{a}\cdot\mathbf{t}+s)\Big[e_\ell(u_1, \cdots, u_d) + u_{d+1}e_{\ell-1}(u_1, \cdots, u_d)\Big] \\ =& \sum_{\ell=0}^{d+1} f^{(\ell)}(\mathbf{a}\cdot\mathbf{t}+s)e_\ell(u_1, \cdots, u_d, u_{d+1}).\end{aligned}$$

Thus,

$$\begin{aligned}\Big(\prod_{j=1}^{d}\frac{2\tau_j}{2\tau_j+1}\Big)I_{d+1} &= \sum_{\ell=0}^{d} I_{d+1,\ell} \\ &= \sum_{\ell=0}^{d+1}\int_{[-1,1]^{d+1}} f^{(\ell)}(\mathbf{a}\cdot\mathbf{t}+s)e_\ell\Big(\frac{a_1(1+t_1)}{2\tau_1+1}, \cdots, \frac{a_{d+1}(1+t_{d+1})}{2\tau_{d+1}+1}\Big)d\widehat{\mu}_{d+1}(\mathbf{t}),\end{aligned}$$

proving (6.2.1) for the case of $d+1$. This completes the induction. $\square$

Now we are in a position to prove Theorem 6.1.1.

Proof of Theorem 6.1.1. Let

$$\Omega := \Big\{\mathbf{z} = (z_1, \cdots, z_d) \in \mathbb{C}^d : \quad \mathrm{Re} z_j > -\frac{1}{2}, \quad j = 1, 2, \cdots, d\Big\} \subset \mathbb{C}^d.$$

For $\mathbf{z} = (z_1, \cdots, z_d) \in \Omega$ and each nonnegative integer n, we define a function $H_n^{\mathbf{z}}$ on $\mathbb{S}^{d-1} \times \mathbb{S}^{d-1}$ by

$$\begin{aligned}&H_n^{\mathbf{z}}(x, y) = g(\mathbf{z})\int_{[-1,1]^d} E_n^{\alpha_{\mathbf{z}}}\Big(\sum_{j=1}^{d} x_j y_j t_j\Big)\Big(\prod_{j=1}^{d}(1-t_j^2)^{z_j}dt_j\Big) + \\ &+ g(\mathbf{z})\sum_{\ell=1}^{d}\sum_{1\le i_1<i_2<\cdots<i_\ell\le d}\int_{[-1,1]^d} E_{n-\ell}^{\alpha_{\mathbf{z}}+\ell}\Big(\sum_{j=1}^{d} x_j y_j t_j\Big)\Big(\prod_{j=1}^{\ell}\frac{x_{i_j}y_{i_j}(1+t_{i_j})}{2\kappa_{i_j}+1}\Big)\Big(\prod_{j=1}^{d}(1-t_j^2)^{z_j}dt_j\Big),\end{aligned}$$

where $x, y \in \mathbb{S}^{d-1}$, and

$$\alpha_{\mathbf{z}} := \frac{d-3}{2} + \sum_{j=1}^{d} z_j,$$

$$g(\mathbf{z}) := \frac{2}{\pi^{d/2}\Gamma(\alpha_{\mathbf{z}} + \frac{3}{2})} \prod_{j=1}^{d} \frac{\Gamma(z_j + \frac{3}{2})}{\Gamma(z_j + 1)}.$$

Clearly, for each fixed $x, y \in \mathbb{S}^{d-1}$, the function $H_n^{\mathbf{z}}(x, y)$ is analytic in each variable z_j on the domain Ω.

With the above notation, it is enough to show that for $\kappa = (\kappa_1, \cdots, \kappa_d) \in (-\frac{1}{2}, \infty)^d$, H_n^κ coincides with the reproducing kernel of $P_n(h_\kappa^2)$ of the space $\mathcal{H}_n^d(h_\kappa^2)$.

We first consider the case of $\kappa_{\min} > 0$. Indeed, by (5.2.5) and (5.1.11), if $\kappa_{\min} := \min_{1 \le j \le d} \kappa_j > 0$, then

$$P_n(h_\kappa^2; x, y) = c(\kappa) \int_{[-1,1]^d} E_n^{\alpha_\kappa}(\sum_{j=1}^{d} x_j y_j t_j) \prod_{j=1}^{d} (1 - t_j^2)^{\kappa_j - 1}(1 + t_j)\, dt_j, \tag{6.2.3}$$

where $\alpha_\kappa = \lambda_\kappa - \frac{1}{2}$ and

$$c(\kappa) := \frac{2\prod_{j=1}^{d} c_{\kappa_j}}{\Gamma(\lambda_\kappa + 1)} = \frac{2\Gamma(\kappa_1 + \frac{1}{2}) \cdots \Gamma(\kappa_d + \frac{1}{2})}{\pi^{d/2}\Gamma(\lambda_\kappa + 1)\Gamma(\kappa_1)\Gamma(\kappa_2) \cdots \Gamma(\kappa_d)}.$$

(6.2.3) together with Lemma 6.2.1 implies that for any $\kappa \in (0, \infty)^d$,

$$P_n(h_\kappa^2; x, y) = H_n^\kappa(x, y), \quad x, y \in \mathbb{S}^{d-1}. \tag{6.2.4}$$

Next, we prove (6.2.4) for the case of $-\frac{1}{2} < \kappa_{\min} \le 0$. By definition, clearly,

$$H_n^{\mathbf{z}}(x, y) = H_n^{\mathbf{z}}(y, x), \quad \forall x, y \in \mathbb{S}^{d-1}, \quad \forall \mathbf{z} \in \Omega.$$

Moreover, for each fixed $x \in \mathbb{S}^{d-1}$, $H_n^\kappa(x, y)$ is an algebraic polynomial in y of total degree at most n; that is,

$$H_n^\kappa(x,\cdot) \in \Pi_n^d, \quad \forall x \in \mathbb{S}^{d-1}.$$

Thus, to complete the proof, it suffices to verify that for each $\kappa \in (-\frac{1}{2},\infty)^d$,

$$H_n^\kappa(x,\cdot) \in \mathcal{H}_n^d(h_\kappa^2), \quad \forall x \in \mathbb{S}^{d-1}, \tag{6.2.5}$$

and

$$f(x) := \omega_d^\kappa \int_{\mathbb{S}^{d-1}} f(y) H_n^\kappa(x,y) h_\kappa^2(y) d\sigma(y), \quad \forall f \in \mathcal{H}_n^\kappa(h_\kappa^2), \quad \forall x \in \mathbb{S}^{d-1}, \tag{6.2.6}$$

where

$$\omega_d^\kappa := \Big(\int_{\mathbb{S}^{d-1}} h_\kappa^2(x)\, d\sigma(x)\Big)^{-1} = \frac{2\Gamma(\kappa_1+\frac{1}{2})\cdots\Gamma(\kappa_d+\frac{1}{2})}{\Gamma(\gamma_\kappa+\frac{d}{2})}. \tag{6.2.7}$$

Clearly, by (6.2.7), we can extend ω_d^κ to an analytic function $\mathbf{z} \to \omega_d^{\mathbf{z}}$ in each variable z_j, $j = 1, \cdots, d$ on the domain Ω.

We first prove (6.2.5). Since H_n^κ is the reproducing kernel of the space $\mathcal{H}_n^d(h_\kappa^2)$ of orthogonal polynomials for $\kappa \in (0,\infty)^d$, it follows that for $\mathbf{z} = \kappa \in (0,\infty)^d$ and any $f \in \Pi_{n-1}^d$,

$$0 = \omega_d^{\mathbf{z}} \int_{\mathbb{S}^{d-1}} f(y) H_n^{\mathbf{z}}(x,y) h_{\mathbf{z}}^2(y)\, d\sigma(y), \quad x \in \mathbb{S}^{d-1}. \tag{6.2.8}$$

Since for each fixed $f \in \Pi_n^d$ and $x \in \mathbb{S}^{d-1}$, the integral on the right hand side of (6.2.8) is an analytic function in each $z_j > -\frac{1}{2}$, it follows that (6.2.8) holds for all $z \in \Omega$. In particular, this implies that for all $\kappa \in (-\frac{1}{2},\infty)^d$,

$$\omega_d^\kappa \int_{\mathbb{S}^{d-1}} f(y) H_n^\kappa(x,y) h_\kappa^2(y)\, d\sigma(y) = 0, \quad \forall f \in \Pi_{n-1}^d, \quad \forall x \in \mathbb{S}^{d-1}. \tag{6.2.9}$$

Since $H_n^\kappa(x,\cdot) \in \Pi_n^d$ for each $x \in \mathbb{S}^{d-1}$, we prove (6.2.5).

Next, we prove (6.2.6). For each $\mathbf{z} \in \Omega$, we define

$$G_n^{\mathbf{z}}(x,y) := \sum_{j=0}^{n} H_j^{\mathbf{z}}(x,y), \quad x, y \in \mathbb{S}^{d-1}.$$

If $\kappa \in (0,\infty)^d$, then by (6.2.4), G_n^κ is the reproducing kernel of the space $\Pi_n^d \subset L^2(h_\kappa^2)$. Hence, for $\mathbf{z} = \kappa \in (0,\infty)^d$ and any $f \in \Pi_n^d$,

$$f(x) : = \omega_d^{\mathbf{z}} \int_{\mathbb{S}^{d-1}} f(y) G_n^{\mathbf{z}}(x,y) h_{\mathbf{z}}^2(y)\, d\sigma(y), \quad x \in \mathbb{S}^{d-1}. \tag{6.2.10}$$

Since the integral on the right hand side of (6.2.10) is analytic in each $z_j > -\frac{1}{2}$, (6.2.10) holds for all $z \in \Omega$.

Now combining (6.2.10) with (6.2.5), and taking into account the fact that $H_n^\kappa(x,y) = H_n^\kappa(y,x)$, we conclude that for any $\kappa \in (-\frac{1}{2},\infty)^d$, $f \in \mathcal{H}_n^d(h_\kappa^2)$, and $x \in \mathbb{S}^{d-1}$,

$$f(x) : = \omega_d^\kappa \int_{\mathbb{S}^{d-1}} f(y) G_n^\kappa(x,y) h_\kappa^2(y) d\sigma(y) = \omega_d^\kappa \int_{\mathbb{S}^{d-1}} f(y) H_n^\kappa(x,y) h_\kappa^2(y) d\sigma(y).$$

This proves (6.2.6). $\square$

6.3 Estimates of multiple integrals of Jacobi polynomials

For convenience, we shall write the integral representation in Theorem 6.1.1 in the form

$$P_n(h_\kappa^2; x, y) = \frac{n+\lambda_\kappa}{\lambda_\kappa} T_\kappa(C_n^{\lambda_\kappa})(x,y), \quad x, y \in \mathbb{S}^{d-1}, \tag{6.3.1}$$

where T_κ is an operator defined as follows.

Definition 6.3.1. *Given $\kappa = (\kappa_1, \cdots, \kappa_d) \in (-\frac{1}{2}, \infty)^d$ and $f \in C^d[-1,1]$, define*

$$(T_\kappa f)(x, y) : = c_2(\kappa) \sum_{\ell=0}^{d} \frac{1}{2^\ell} (T_{\kappa,\ell} f)(x, y), \quad x, y \in \mathbb{S}^{d-1},$$

where

$$c_2(\kappa) : = \frac{1}{\pi^{d/2}} \prod_{j=1}^{d} \frac{\Gamma(\kappa_j + \frac{3}{2})}{\Gamma(\kappa_j + 1)} \tag{6.3.2}$$

and

$$(T_{\kappa,\ell} f)(x, y) := \int_{[-1,1]^d} f^{(\ell)}(\sum_{j=1}^{d} x_j y_j t_j) e_\ell \Big(\frac{x_1 y_1 (1 + t_1)}{2\kappa_1 + 1}, \cdots, \frac{x_d y_d (1 + t_d)}{2\kappa_d + 1} \Big) \Big(\prod_{j=1}^{d} (1 - t_j^2)^{\kappa_j} dt_j \Big).$$

One of the main purposes in this section is to prove a sharp estimate of a multiple integral of Jacobi polynomials defined below.

Definition 6.3.2. *Let $\kappa = (\kappa_1, \cdots, \kappa_d) \in (-\frac{1}{2}, \infty)^d$. For $\alpha, \beta \in \mathbb{R}$, define*

$$\mathcal{P}_n^{(\alpha,\beta)}(x, y) := T_\kappa(P_n^{(\alpha,\beta)})(x, y), \quad x, y \in \mathbb{S}^{d-1}.$$

With the definition, we have

$$P_n(h_\kappa^2; x, y) = b_n(\kappa) \mathcal{P}_n^{(\alpha_\kappa, \alpha_\kappa)}(x, y),$$

where $\alpha_\kappa = \frac{d-3}{2} + \gamma_\kappa$, and

$$b_n(\kappa) = \frac{\Gamma(\alpha_\kappa + 1)}{(\alpha_\kappa + \frac{1}{2}) \Gamma(2\alpha_\kappa + 1)} \frac{(n + \frac{d-2}{2} + \gamma_\kappa) \Gamma(n + d - 2 + 2\gamma_\kappa)}{\Gamma(n + \frac{d-1}{2} + \gamma_\kappa)} \sim n^{\frac{d-1}{2} + \gamma_\kappa}.$$

Recall that for $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$ and $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_d) \in \{\pm 1\}^d$,

$$x\varepsilon := (x_1\varepsilon_1, x_2\varepsilon_2, \cdots, x_d\varepsilon_d),$$

and for $x, y \in \mathbb{S}^{d-1}$,

$$\rho(x, y) = \arccos(x \cdot y).$$

In this section, we will prove the following result, from which the stated estimate of the reproducing kernel $P_n(h_\kappa^2, x, y)$ in Theorem 6.1.2 will follow immediately.

Theorem 6.3.3. *Let $\kappa = (\kappa_1, \cdots, \kappa_d) \in (-\frac{1}{2}, \infty)^d$. If $\alpha \geq \beta$, then*

$$|\mathcal{P}_n^{(\alpha,\beta)}(x, y)| \leq C n^{\alpha - 2\gamma_\kappa} \max_{\varepsilon \in \{\pm 1\}^d} \frac{\prod_{j=1}^d (|x_j y_j| + n^{-1}\rho(x\varepsilon, y) + n^{-2})^{-\kappa_j}}{\Big(1 + n\rho(x\varepsilon, y)\Big)^{\alpha + \frac{1}{2} - \gamma_\kappa}}, \tag{6.3.3}$$

where $\gamma_\kappa = \kappa_1 + \cdots + \kappa_d$.

6.3.1 Technical lemmas

The following estimate of a multiple integral of Jacobi polynomials play a crucial role in the proof of Theorem 6.3.3:

Lemma 6.3.4. *Assume that $\tau = (\tau_1, \cdots, \tau_d) \in (0, \infty)^d$, and $\varphi_1, \cdots, \varphi_d \in C^\infty[-1, 1]$. If $\alpha \geq \beta$, then for any $x, y \in \mathbb{S}^{d-1}$,*

$$\begin{aligned} &\left| \int_{[-1,1]^d} P_n^{(\alpha,\beta)}\Big(\sum_{j=1}^d x_j y_j t_j\Big) \prod_{j=1}^d \varphi_j(t_j)(1 - t_j^2)^{\tau_j - 1}\, dt_j \right| \\ &\qquad \leq c n^{\alpha - 2|\tau|} \max_{\varepsilon \in \{\pm 1\}^d} \frac{\prod_{j=1}^d (|x_j y_j| + n^{-1}\rho(x, y\varepsilon) + n^{-2})^{-\tau_j}}{(1 + n\rho(x, y\varepsilon))^{\alpha + \frac{1}{2} - |\tau|}}, \end{aligned} \tag{6.3.4}$$

where $|\tau| = \sum_{j=1}^d \tau_j$.

Lemma 6.3.4 under the additional assumption $\alpha \geq \sum_{j=1}^{d} \tau_j - \frac{1}{2}$ was proved in [17, Theorem 3.1].

Here we have to remove this extra condition as it may not be satisfied in the case when $\kappa_j = \tau_j - 1$ and $\kappa_{\min} < 0$.

The proof of Lemma 6.3.4 relies on the following estimates proved in [17].

Lemma 6.3.5. *[17, Lemma 3.5] Let $a = (a_1, \cdots, a_m) \in \mathbb{R}^m$, $x \in \mathbb{R}$ be such that $a_j \neq 0$ for $1 \leq j \leq m$ and $\sum_{j=1}^{m} |a_j| + |x| \leq 1$. Let $\xi_1, \cdots, \xi_m \in C^\infty(\mathbb{R})$ be such that supp $\xi_j \subset \mathbb{R} \setminus [-1, -\frac{1}{2}]$ for $1 \leq j \leq m$. Then for $\tau = (\tau_1, \cdots, \tau_m) \in (0, \infty)^d$ and $\alpha \geq \beta$,*

$$\Big| \int_{[-1,1]^m} P_n^{(\alpha,\beta)} \Big(\sum_{j=1}^{m} a_j t_j + x \Big) \prod_{j=1}^{m} \xi_j(t_j)(1-t_j)^{\tau_j - 1} \, dt_j \Big|$$
$$\leq c n^{\alpha - 2|\tau|} \Big(\prod_{j=1}^{m} |a_j|^{-\tau_j} \Big) \Big(1 + n\sqrt{1 - |a_1 + a_2 + \cdots + a_m + x|} \Big)^{-\alpha - \frac{1}{2} + |\tau|},$$

where $|\tau| := \sum_{j=1}^{m} \tau_j$.

Now we are in a position to prove Lemma 6.3.4.

Proof of Lemma 6.3.4. For $1 \leq j \leq d$, let ξ_j denote a C^∞-function on $[-1, 1]$ such that $\xi_j(t) = 0$ for $-1 \leq t \leq -\frac{1}{4}$. By symmetry, it is enough to show that

$$\Big| \int_{[-1,1]^d} P_n^{(\alpha,\beta)} \Big(\sum_{j=1}^{d} x_j y_j t_j \Big) \prod_{j=1}^{d} \xi_j(t_j)(1-t_j^2)^{\tau_j - 1} \, dt_j \Big|$$
$$\leq c n^{\alpha - 2|\tau|} \frac{\prod_{j=1}^{d} (|x_j y_j| + n^{-1} \widetilde{\rho}(x,y) + n^{-2})^{-\tau_j}}{(1 + n\widetilde{\rho}(x,y))^{\alpha + \frac{1}{2} - |\tau|}}, \tag{6.3.5}$$

where

$$\widetilde{\rho}(x,y) = \sqrt{1 - \Big| \sum_{j=1}^{d} x_j y_j \Big|} \sim \max\Big\{ \rho(x,y), \rho(x,-y) \Big\}.$$

Without loss of generality, we may assume that

$$|x_jy_j| \geq c_0\Big(n^{-1}\rho(x,y)+n^{-2}\Big) \quad \text{for } j=1,2,\cdots,m, \tag{6.3.6}$$

$$|x_jy_j| < c_0\Big(n^{-1}\rho(x,y)+n^{-2}\Big) \quad \text{for } m<j\leq d, \tag{6.3.7}$$

where $c_0 \in (0,1)$ is a small constant to be specified later. For each fixed $\widetilde{\mathbf{t}} = (t_{m+1},\cdots,t_d) \in [-1,1]^{d-m}$, we define

$$f_n(x,y,\widetilde{\mathbf{t}}) := \Big|\int_{[-1,1]^m} P_n^{(\alpha,\beta)}(\sum_{j=1}^d x_jy_jt_j)\prod_{j=1}^m \xi_j(t_j)(1-t_j^2)^{\tau_j-1}\,dt_j\Big|.$$

Using Lemma 6.3.5, we obtain that

$$\begin{aligned} f_n(x,y,\widetilde{\mathbf{t}}) \leq{}& Cn^{\alpha-2\theta_m}\Big(\prod_{j=1}^m |x_jy_j|^{-\tau_j}\Big)\times \\ &\times\Big(1+n\sqrt{1-|\sum_{j=1}^d x_jy_j+\sum_{j=m+1}^d x_jy_j(t_j-1)|}\Big)^{-\alpha-\frac{1}{2}+\theta_m}, \end{aligned} \tag{6.3.8}$$

where $\theta_m = \sum_{j=1}^m \tau_j$.

We claim that that for any $\widetilde{\mathbf{t}} = (t_{m+1},\cdots,t_d) \in [-1,1]^{d-m}$,

$$1+n\sqrt{1-\Big|\sum_{j=1}^d x_jy_j+\sum_{j=m+1}^d x_jy_j(t_j-1)\Big|} \sim 1+n\widetilde{\rho}(x,y). \tag{6.3.9}$$

Indeed, by (6.3.7) ,

$$\Big|\sum_{j=m+1}^d x_jy_j(t_j-1)\Big| \leq 2\sum_{j=m+1}^d |x_jy_j| \leq 2(d-m)c_0(n^{-1}\widetilde{\rho}(x,y)+n^{-2}). \tag{6.3.10}$$

It follows that

$$
\begin{aligned}
&n\sqrt{1-\Big|\sum_{j=1}^{d} x_j y_j + \sum_{j=m+1}^{d} x_j y_j (t_j-1)\Big|} \\
&\le n\sqrt{\widetilde{\rho}(x,y)^2 + c' n^{-2}(1+n\widetilde{\rho}(x,y))} \le C(1+n\widetilde{\rho}(x,y)),
\end{aligned}
$$

which implies the upper estimate of (6.3.9). To show the lower estimate of (6.3.9), without loss of generality, we may assume that $\widetilde{\rho}(x,y) \ge n^{-1}$. (The lower estimate holds trivially if $\widetilde{\rho}(x,y) \le n^{-1}$.) Using (6.3.10), we then obtain

$$
\begin{aligned}
&1-\Big|\sum_{j=1}^{d} x_j y_j + \sum_{j=m+1}^{d} x_j y_j (t_j-1)\Big| \ge \widetilde{\rho}(x,y)^2 - Cc_0\Big(n^{-1}\widetilde{\rho}(x,y) + n^{-2}\Big) \\
&\ge c_1 \widetilde{\rho}(x,y)^2,
\end{aligned}
$$

provided that the constant c_0 is sufficiently small. This implies the lower estimate of (6.3.9).

Now using (6.3.9) and (6.3.8), we obtain

$$
\begin{aligned}
f_n(x,y,\widetilde{\mathbf{t}}) &\le Cn^{\alpha-2\theta_m}\Big(\prod_{j=1}^{m} |x_j y_j|^{-\tau_j}\Big)(1+n\widetilde{\rho}(x,y))^{-\alpha-\frac{1}{2}+\theta_m} \\
&\le Cn^{\alpha-2|\tau|}\frac{\prod_{j=1}^{d}\Big(|x_j y_j| + n^{-2} + n^{-1}\widetilde{\rho}(x,y)\Big)^{-\tau_j}}{(1+n\widetilde{\rho}(x,y))^{\alpha+\frac{1}{2}-|\tau|}},
\end{aligned}
$$

where the last step uses (6.3.6) and (6.3.7). The estimate (6.3.5) then follows by integrating both sides of this last inequality with respect to the measure $\prod_{j=m+1}^{d}(1-t_j^2)^{\tau_j-1}\xi_j(t_j)dt_j$ on $[-1,1]^{d-m}$.

$\square$

6.3.2 Proof of Theorem 6.3.3

By definition, we have

$$\left|\mathcal{P}_n^{(\alpha,\beta)}(x,y)\right| \le C_\kappa \sum_{\ell=0}^{d} \sum_{1\le i_1<i_2<\cdots<i_\ell\le d} B_{i_1,\cdots,i_\ell}^{\alpha,\beta}(x,y),$$

where

$$\begin{aligned}&B_{i_1,\cdots,i_\ell}^{\alpha,\beta}(x,y) := n^\ell |x_{i_1}y_{i_1}x_{i_2}y_{i_2}\cdots, x_{i_\ell}y_{i_\ell}|\times\\ &\times\Big|\int_{[-1,1]^d} P_{n-\ell}^{(\alpha+\ell,\beta+\ell)}(\sum_{j=1}^d x_jy_jt_j)\Big(\prod_{j=1}^{\ell}(1+t_{i_j})\Big)\Big(\prod_{j=1}^{d}(1-t_j^2)^{\kappa_j}dt_j\Big)\Big|.\end{aligned}$$

Invoking Lemma 6.3.4 with $\tau = (\kappa_1+1,\cdots,\kappa_d+1)$, we obtain that for $1\le i_1<\cdots< i_\ell \le d$,

$$\begin{aligned}B_{i_1,\cdots,i_\ell}^{\alpha,\beta}(x,y) &\le C\Big(\prod_{j=1}^{\ell}|x_{i_j}y_{i_j}|\Big)n^{\alpha+2\ell-2\gamma_\kappa-2d} \max_{\varepsilon\in\{\pm1\}^d} \frac{\prod_{j=1}^d\Big(|x_jy_j|+n^{-1}\rho(x,y\varepsilon)+n^{-2}\Big)^{-\kappa_j-1}}{(1+n\rho(x,y\varepsilon))^{\alpha+\ell+\frac12-\gamma_\kappa-d}}\\
&\le Cn^{\alpha+2\ell-2\gamma_\kappa-2d} \max_{\varepsilon\in\{\pm1\}^d} \frac{\prod_{j=1}^d\Big(|x_jy_j|+n^{-1}\rho(x,y\varepsilon)+n^{-2}\Big)^{-\kappa_j}}{(1+n\rho(x,y\varepsilon))^{\alpha+\ell+\frac12-\gamma_\kappa-d}}\Big(n^{-1}\rho(x,y\varepsilon)+n^{-2}\Big)^{-d+\ell}\\
&\le Cn^{\alpha-2\gamma_\kappa} \max_{\varepsilon\in\{\pm1\}^d} \frac{\prod_{j=1}^d\Big(|x_jy_j|+n^{-1}\rho(x,y\varepsilon)+n^{-2}\Big)^{-\kappa_j}}{(1+n\rho(x,y\varepsilon))^{\alpha+\frac12-\gamma_\kappa}}.\end{aligned}$$

6.4 Estimates of the Cesàro kernels

The main purpose in this section is to prove Theorem 6.1.3, which we restate as follows:

Theorem 6.4.1. *If $\delta \geq 0$, and $x = (x_1, \cdots, x_d)$, $\;y = (y_1, \cdots, y_d) \in \mathbb{S}^{d-1}$, then*

$$|K_n^\delta(h_\kappa^2; x, y)| \leq \; cn^{d-1} \max_{\varepsilon \in \{\pm 1\}^d} \left[\frac{\prod_{j=1}^d (|x_j y_j| + n^{-1}\rho(x, y\varepsilon) + n^{-2})^{-\kappa_j}}{(1 + n\rho(x, \varepsilon y))^{\delta + \frac{d}{2}}} \right. \\ \left. + \frac{\prod_{j=1}^d (|x_j y_j| + (\rho(x, y\varepsilon))^2 + n^{-2})^{-\kappa_j}}{(n\rho(x, y\varepsilon) + 1)^d} \right]. \tag{6.4.1}$$

For any $\delta > 0$, let

$$K_n^{(\alpha,\alpha),\delta}(t, 1) := \sum_{j=0}^n \frac{A_{n-j}^\delta}{A_n^\delta} \frac{(2j + 2\alpha + 1)\sqrt{\pi}}{2^{2\alpha+1}\Gamma(\alpha + \frac{3}{2})} \frac{\Gamma(j + 2\alpha + 1)}{\Gamma(j + \alpha + 1)} P_j^{(\alpha,\alpha)}(t), \quad t \in [-1, 1].$$

For each fixed $x \in [-1, 1]$, this last equation extends $K_n^{(\alpha,\alpha),\delta}(t, 1)$ to an analytic function of α on the domain $G := \{z \in \mathbb{C} : \; \mathrm{Re}\, z > -\frac{3}{2}\}$. For $x, y \in \mathbb{S}^{d-1}$, we have

$$\begin{aligned} K_n^\delta(h_\kappa^2; x, y) &= \sum_{j=0}^n \frac{A_{n-j}^\delta}{A_n^\delta} P_j(h_\kappa^2; x, y) = T_\kappa \Big[K_n^{(\alpha_\kappa,\alpha_\kappa),\delta}(\cdot, 1) \Big](x, y) \\ &= \sum_{j=0}^n \frac{A_{n-j}^\delta}{A_n^\delta} \frac{(2j + 2\alpha_\kappa + 1)\sqrt{\pi}}{2^{2\alpha_\kappa+1}\Gamma(\alpha_\kappa + \frac{3}{2})} \frac{\Gamma(j + 2\alpha_\kappa + 1)}{\Gamma(j + \alpha_\kappa + 1)} \mathcal{P}_j^{(\alpha_\kappa,\alpha_\kappa)}(x, y), \end{aligned}$$

where $\alpha_\kappa = \lambda_\kappa - \frac{1}{2}$.

We break the proof of Theorem 6.4.1 into several steps, which will be given in the next few subsections.

6.4.1 Decomposition

Let $\varphi_0 \in C^\infty[0, \infty)$ be such that $\chi_{[0,1]} \leq \varphi_0 \leq \chi_{[0,2]}$, and let $\varphi(t) := \varphi_0(t) - \varphi_0(2t)$. Clearly, φ is a C^∞-function supported in $(\frac{1}{2}, 2)$ and satisfying $\sum_{v=0}^\infty \varphi(2^v t) = \varphi_0(t)$ for all $t > 0$. We set

$$\widehat{S}_{n,v}^\delta(j) := \varphi\Big(\frac{2^v(n - j)}{n}\Big) \frac{A_{n-j}^\delta}{A_n^\delta}, \quad j = 0, 1, \cdots, n, \tag{6.4.2}$$

and define

$$S^\delta_{n,v} f := \sum_{j=0}^{n} \widehat{S}^\delta_{n,v}(j)\, \mathrm{proj}_j(h_\kappa^2; f), \qquad v = 0, 1, \cdots, \lfloor \log_2 n \rfloor + 2.$$

Since $\sum_{v=0}^{\lfloor \log_2 n\rfloor+2} \varphi\Big(\frac{2^v(n-j)}{n}\Big) = 1$ for $0 \le j \le n-1$, it follows that

$$S^\delta_n(h_\kappa^2; f) = \sum_{v=0}^{\lfloor \log_2 n\rfloor+2} S^\delta_{n,v} f + \frac{1}{A_n^\delta} \mathrm{proj}_n(h_\kappa^2; f). \tag{6.4.3}$$

Since $\widehat{S}^\delta_{n,v}(j) = 0$ whenever $n - j > \frac{n}{2^{v-1}}$ or $n - j < \frac{n}{2^{v+1}}$, it is easy to verify by the Leibniz rule that

$$\Big|\Delta^\ell(\widehat{S}^\delta_{n,v}(j))\Big| \le c2^{-v\delta}\Big(\frac{2^v}{n}\Big)^\ell, \qquad \forall \ell \in \mathbb{N},\ 0 \le j \le n. \tag{6.4.4}$$

Let

$$D^\delta_{n,v}(t) := \sum_{j=0}^{n} \widehat{S}^\delta_{n,v}(j) \frac{\lambda_\kappa + j}{\lambda_\kappa} C_j^{\lambda_\kappa}(t).$$

Then

$$S^\delta_{n,v} f(x) = \omega_\kappa^d \int_{\mathbb{S}^{d-1}} f(y) K^\delta_{n,v}(x, y) h_\kappa^2(y)\, d\sigma(y), \tag{6.4.5}$$

where

$$K^\delta_{n,v}(x, y) := T_\kappa\Big[D^\delta_{n,v}\Big](x, y). \tag{6.4.6}$$

Using (6.4.3), we obtain

Lemma 6.4.2. *For $\delta \ge 0$ and $x, y \in \mathbb{S}^{d-1}$,*

$$K^\delta_n(h_\kappa^2; x, y) = \sum_{v=0}^{[\log_2 n]+2} K^\delta_{n,v}(x, y) + \frac{1}{A_n^\delta} P_n(h_\kappa^2; x, y). \tag{6.4.7}$$

6.4.2 Estimates of the kernels $K^\delta_{n,v}(x,y)$

We start with the case of $v \geq 2$.

Lemma 6.4.3. *If $2 \leq v \leq \lfloor \log_2 n \rfloor + 2$, then for any given positive integer ℓ, and any $x, y \in \mathbb{S}^{d-1}$,*

$$|K^\delta_{n,v}(x,y)| \leq c n^{d-1} 2^{v(\ell-1-\delta)} \max_{\varepsilon \in \{\pm 1\}^d} \frac{\prod_{i=1}^d (|x_i y_i| + n^{-1}\rho(x\varepsilon, y) + n^{-2})^{-\kappa_i}}{(1 + n\rho(x\varepsilon, y))^{\frac{d-2}{2}+\ell}}.$$

Proof. We follow the proof of Lemma 3.3 of [8, p.413–414]. We shall use the following formula for Jacobi polynomials (see [Sz, (4.5.3) p.71]):

$$\begin{aligned}\sum_{n=0}^{k} \frac{(2n+\alpha+\beta+j+1)\Gamma(n+\alpha+\beta+j+1)}{\Gamma(n+\beta+1)} P_n^{(\alpha+j,\beta)}(t) \\ = \frac{\Gamma(k+\alpha+\beta+j+2)}{\Gamma(k+\beta+1)} P_k^{(\alpha+j+1,\beta)}(t),\end{aligned} \tag{6.4.8}$$

where $j = 0, 1, \cdots$.

Define a sequence of functions $\{a_{n,v,\ell}(\cdot)\}_{\ell=0}^\infty$ recursively by

$$\begin{aligned} a_{n,v,0}(j) &= 2(j+\lambda_\kappa)\widehat{S}^\delta_{n,v}(j), \\ a_{n,v,\ell+1}(j) &= \frac{a_{n,v,\ell}(j)}{2j+2\lambda_\kappa+\ell} - \frac{a_{n,v,\ell}(j+1)}{2j+2\lambda_\kappa+\ell+2}, \qquad \ell \geq 0. \end{aligned}$$

Using (6.4.8) and summation by parts finite times, we have for any integer $\ell \geq 0$,

$$D^\delta_{n,v}(t) = c_\kappa \sum_{j=0}^\infty a_{n,v,\ell}(j) \frac{\Gamma(j+2\lambda_\kappa+\ell)}{\Gamma(j+\lambda_\kappa+\frac{1}{2})} P_j^{(\lambda_\kappa+\ell-\frac{1}{2}, \lambda_\kappa-\frac{1}{2})}(t), \tag{6.4.9}$$

where $\lambda_\kappa = \frac{d-2}{2} + \sum_{j=1}^d \kappa_j$. It follows by (6.4.6) that

$$K^\delta_{n,v}(x,y) = c_\kappa \sum_{j=0}^\infty a_{n,v,\ell}(j) \frac{\Gamma(j+2\lambda_\kappa+\ell)}{\Gamma(j+\lambda_\kappa+\frac{1}{2})} \mathcal{P}_j^{(\lambda_\kappa+\ell-\frac{1}{2}, \lambda_\kappa-\frac{1}{2})}(x,y). \tag{6.4.10}$$

Note that $a_{n,v,\ell}(j) = 0$ if $j + \ell \le (1 - \frac{1}{2^{v-1}})n$ or $j \ge (1 - \frac{1}{2^{v+1}})n$, so that the sum is over $j \sim n$. Furthermore, it follows from the definition, (6.4.4) and Leibniz rule that

$$\left|\triangle^i a_{n,v,\ell}(j)\right| \le c2^{-v\delta} n^{-\ell+1} \Big(\frac{2^v}{n}\Big)^{i+\ell}, \qquad i, \ell = 0, 1, \cdots . \tag{6.4.11}$$

Consequently, using the pointwise estimates (6.3.3), and (6.4.11), we obtain

$$\begin{aligned}|K^\delta_{n,v}(x,y)| &\le cn^{d-3+2\ell} \max_{\varepsilon \in \{\pm 1\}^d} \frac{\prod_{i=1}^d (|x_i y_i| + n^{-1}\rho(x\varepsilon, y) + n^{-2})^{-\kappa_i}}{(1 + n\rho(x\varepsilon, y))^{\frac{d-2}{2}+\ell}} \sum_{\substack{j \sim n \\ n-j \sim \frac{n}{2^v}}} |a_{n,v,\ell}(j)| \\ &\le cn^{d-1} 2^{v(\ell-1-\delta)} \max_{\varepsilon \in \{\pm 1\}^d} \frac{\prod_{i=1}^d (|x_i y_i| + n^{-1}\rho(x\varepsilon, y) + n^{-2})^{-\kappa_i}}{(1 + n\rho(x\varepsilon, y))^{\frac{d-2}{2}+\ell}}.\end{aligned}$$

□

Next, we deal with the cases of $v = 0, 1$.

Lemma 6.4.4. *If $v = 0, 1$, then for any $x, y \in \mathbb{S}^{d-1}$,*

$$|K^\delta_{n,v}(x,y)| \le cn^{d-1} \max_{\varepsilon \in \{\pm 1\}^d} \frac{\prod_{i=1}^d (|x_i y_i| + \rho(x\varepsilon, y)^2 + n^{-2})^{-\kappa_i}}{(1 + n\rho(x\varepsilon, y))^d}.$$

Proof. The proof is very similar to that of Lemma 6.4.3. The difference here comes from the fact that the coefficients $\widehat{S}^\delta_{n,v}(j)$ for $v = 0, 1$ are supported in $0 \le j \le \frac{3}{4}n$ rather than $\frac{n}{2} \le j \le n$. Indeed, for the case of $v = 0, 1$, we have to replace the estimates (6.4.11) by

$$\left|\triangle^k a_{n,v,\ell}(j)\right| \le \begin{cases} cn^{-k-1}, & \text{if } \ell = 1, \\ cn^{-1}(j+1)^{-k-2\ell+2}, & \text{if } \ell \ge 2. \end{cases} \tag{6.4.12}$$

Using (6.4.10), and (6.4.12), we then obtain that for $v = 0, 1$ and any $\ell \ge 2$,

$$|K^\delta_{n,v}(x,y)| \le cn^{-1} \max_{\varepsilon \in \{\pm 1\}^d} \sum_{j=1}^n \frac{\prod_{i=1}^d (|x_i y_i| + j^{-1}\rho(x\varepsilon, y) + j^{-2})^{-\kappa_i}}{(1 + j\rho(x\varepsilon, y))^{\frac{d-2}{2}+\ell}} (j+1)^{d-1}.$$

Thus, to complete the proof, it is enough to show that if $\ell > \frac{d}{2} + 1 + \sum_{j=1}^{d} |\kappa_j|$, then for each $x, y \in \mathbb{S}^{d-1}$,

$$\begin{aligned} n^{-1} \sum_{j=1}^{n} & \frac{\prod_{i=1}^{d}(|x_i y_i| + j^{-1}\rho(x,y) + j^{-2})^{-\kappa_i}}{(1 + j\rho(x,y))^{\frac{d-2}{2}+\ell}} j^{d-1} \\ & \le c n^{d-1} \frac{\prod_{i=1}^{d}(|x_i y_i| + \rho(x,y)^2 + n^{-2})^{-\kappa_i}}{(1 + n\rho(x,y))^{d}}. \end{aligned} \tag{6.4.13}$$

To this end, we write

$$I := \{1, 2, \cdots, d\} = I_1 \cup I_2 \cup I_3,$$

where

$$\begin{aligned} I_1 &:= \{i \in I : \quad \kappa_i < 0\}, \\ I_2 &:= \Big\{i \in I : \quad \kappa_i \ge 0, \quad |x_i| \ge 4\rho(x,y) + n^{-1}\Big\}, \quad I_3 = I \setminus (I_1 \cup I_2). \end{aligned}$$

We also define

$$u_i(x,y,j) = \Big(|x_i y_i| + j^{-2} + j^{-1}\rho(x,y)\Big)^{-\kappa_i}, \quad i \in I, \quad j \ge 1.$$

If $i \in I_1$, then

$$u_i(x,y,j) \le C\Big[|x_i y_i|^{-\kappa_i} + j^{2\kappa_i}\big(1 + j\rho(x,y)\big)^{-\kappa_i}\Big].$$

If $i \in I_2$, then

$$u_i(x,y,j) \le |x_i y_i|^{-\kappa_i}.$$

If $i \in I_3$, then

$$u_i(x,y,j) \le j^{2\kappa_i}(1 + j\rho(x,y))^{-\kappa_i}.$$

Thus, setting

$$\kappa(J) := \sum_{j\in J} \kappa_j \quad \text{for } J \subset I,$$

we obtain

$$\prod_{i=1}^{d} u_i(x,y,j) \le Cj^{2\kappa(I_3)}(1+j\rho(x,y))^{-\kappa(I_3)}\Big(\prod_{i\in I_2}|x_iy_i|^{-\kappa_i}\Big)\prod_{i\in I_1}\Big(|x_iy_i|^{-\kappa_i}+j^{2\kappa_i}(1+j\rho(x,y))^{-\kappa_i}\Big)$$
$$\le \sum_{J\subset I_1}\Big(\prod_{i\in(I_1\cup I_2)\setminus J}|x_iy_i|^{-\kappa_i}\Big)j^{2\kappa(I_3\cup J)}(1+j\rho(x,y))^{-\kappa(I_3\cup J)},$$

where the sum is taken over all finite subsets J of I_1. Note that the index sets I_1, I_2, I_3 are independent of j.

Since $d+2\kappa(I_1) > 0$ and $\ell > \frac{d}{2}+1+\sum_{j=1}^{d}|\kappa_j|$, a straightforward calculation shows that for each fixed $J \subset I_1$,

$$\sum_{j=1}^{n}\frac{j^{d-1+2\kappa(I_3\cup J)}}{(1+j\rho(x,y))^{\frac{d-2}{2}+\ell+\kappa(I_3\cup J)}} \le C\Big(\rho(x,y)+n^{-1}\Big)^{-d-2\kappa(I_3\cup J)}. \tag{6.4.14}$$

This implies that

$$\begin{aligned}&\text{LHS of (6.4.13)}\\ &\le Cn^{-1}\max_{J\subset I_1}\left[\Big(\rho(x,y)+n^{-1}\Big)^{-d-2\kappa(I_3\cup J)}\Big(\prod_{i\in(I_1\cup I_2)\setminus J}|x_iy_i|^{-\kappa_i}\Big)\right]\end{aligned} \tag{6.4.15}$$

Set

$$v_i(x,y) = (|x_iy_i| + \rho(x,y)^2 + n^{-2})^{-\kappa_i}, \quad i = 1,\cdots,d.$$

If $i \in I_1$, then $-\kappa_i > 0$, and hence

$$|x_iy_i|^{-\kappa_i} \le v_i(x,y) \quad \text{and} \quad (\rho(x,y)^2+n^{-2})^{-\kappa_i} \le v_i(x,y).$$

If $i \in I_2$, then

$$|x_i y_i| \sim |x_i|^2 \geq (4\rho(x,y) + n^{-1})^2 \implies |x_i y_i|^{-\kappa_i} \sim v_i(x,y).$$

Finally, if $i \in I_3$, then

$$\begin{aligned} |y_i| \leq 5\rho + n^{-1} &\implies |x_i y_i| \leq C(\rho^2 + n^{-2}) \\ &\implies v_i(x,y) \sim (\rho(x,y)^2 + n^{-2})^{-\kappa_i}. \end{aligned}$$

Putting the above together, we obtain that for each $J \subset I_1$,

$$\begin{aligned} &\Big(\rho(x,y) + n^{-1}\Big)^{-2\kappa(I_3 \cup J)} \Big(\prod_{i \in (I_1 \cup I_2) \setminus J} |x_i y_i|^{-\kappa_i} \Big) \\ &= \Big(\prod_{i \in I_3 \cup J} (\rho(x,y) + n^{-1})^{-2\kappa_i} \Big) \Big(\prod_{i \in I_1 \setminus J} |x_i y_i|^{-\kappa_i} \Big) \Big(\prod_{i \in I_2} |x_i y_i|^{-\kappa_i} \Big) \\ &\leq C \Big(\prod_{i \in I_3 \cup J} v_i(x,y) \Big) \Big(\prod_{i \in I_1 \setminus J} v_i(x,y) \Big) \Big(\prod_{i \in I_2} v_i(x,y) \Big) = C \prod_{i \in I} v_i(x,y). \end{aligned}$$

Thus, using (6.4.15), we obtain

$$\text{LHS of (6.4.13)} \leq C \frac{n^{d-1}}{(1 + n\rho(x,y))^d} \prod_{i=1}^{d} v_i(x,y).$$

This proves (6.4.13), and hence completes the proof of Lemma 6.4.4. $\square$

6.4.3 Proof of Theorem 6.4.1

Let

$$A_n(x,y) := \prod_{i=1}^{d} (|x_i y_i| + n^{-1}\rho(x,y) + n^{-2})^{-\kappa_i}.$$

Let ℓ_0 be a smallest integer $\geq \delta + 4$. Using (6.4.7), Lemma 6.4.3, Lemma 6.4.4 and Theorem 6.1.2, we obtain

$$|K_n^\delta(h_\kappa^2;x,y)| \le Cn^{d-1} \max_{\varepsilon\in\{\pm1\}^d} \sum_{2\le v\le \log_2 n+2} \min_{1\le \ell\le \ell_0} 2^{v(\ell-1-\delta)} \frac{A_n(x\varepsilon,y)}{(1+n\rho(x\varepsilon,y))^{\frac{d-2}{2}+\ell}}$$

$$+ Cn^{-\delta+d-2} \max_{\varepsilon\in\{\pm1\}^d} \frac{A_n(x\varepsilon,y)}{\Big(1+n\rho(x\varepsilon,y)\Big)^{\frac{d-2}{2}}} + Cn^{d-1} \max_{\varepsilon\in\{\pm1\}^d} \frac{\prod_{i=1}^d (|x_iy_i|+\rho(x\varepsilon,y)^2+n^{-2})^{-\kappa_i}}{(1+n\rho(x\varepsilon,y))^d}.$$

However, for each fixed $\varepsilon\in\{\pm1\}^d$,

$$\begin{aligned} n^{-\delta+d-2}\frac{1}{\Big(1+n\rho(x\varepsilon,y)\Big)^{\frac{d-2}{2}}} &= \big(n^{-1}+\rho(x\varepsilon,y)\big)^{1+\delta} \frac{n^{d-1}}{\big(1+n\rho(x\varepsilon,y)\big)^{\delta+\frac{d}{2}}} \\ &\le C\frac{n^{d-1}}{\big(1+n\rho(x\varepsilon,y)\big)^{\delta+\frac{d}{2}}}. \end{aligned}$$

Thus, it suffices to prove that for each $x,y\in\mathbb{S}^{d-1}$,

$$\sum_{2\le v\le \log_2 n+2} \min_{1\le\ell\le\ell_0} \frac{2^{v(\ell-1-\delta)}}{(1+n\rho(x,y))^{\frac{d-2}{2}+\ell}} \le \frac{C}{\big(1+n\rho(x,y)\big)^{\delta+\frac{d}{2}}}. \tag{6.4.16}$$

If $0\le\rho(x,y)\le n^{-1}$, then setting $\ell=1$, we get

$$\text{LHS of (6.4.16)} \le \sum_{2\le v\le\log_2 n+2} 2^{-\delta v} \le C \sim \frac{C}{\big(1+n\rho(x,y)\big)^{\delta+\frac{d}{2}}}$$

If $\rho(x,y)>n^{-1}$, then we break the sum on the left hand side of (6.4.16) into two parts, $\sum_{2^v\le n\rho(x,y)} + \sum_{2^v>n\rho(x,y)}$, and set $\ell=\ell_0$ for the first part and $\ell=1$ for the second part, we then obtain

$$\begin{aligned} \text{LHS of (6.4.16)} &\le C(n\rho(x,y))^{-\frac{d}{2}-\ell_0+1} \sum_{2^v\le n\rho(x,y)} 2^{v(\ell_0-1-\delta)} + (n\rho(x,y))^{-\frac{d}{2}} \sum_{2^v>n\rho(x,y)} 2^{-\delta v} \\ &\le \frac{C}{\big(1+n\rho(x,y)\big)^{\delta+\frac{d}{2}}}. \end{aligned}$$

6.5 Cesàro summability of the WOPEs on $\mathbb{S}^{d-1}$

In this section, we prove Corollary 6.1.1 and Theorem 6.1.4. We start with the proof of Corollary 6.1.1.

Proof of Corollary 6.1.1. It is enough to consider the case $\kappa_{\min} < 0$ as the case $\kappa_{\min} \geq 0$ was treated in [18].

Using Theorem 6.1.3, we have

$$|K_n^\delta(h_\kappa^2; x, y)| \leq C \max_{\varepsilon \in \{\pm 1\}^d} \Big[E_n^\delta(h_\kappa^2; x\varepsilon, y) + R_n(h_\kappa^2; x\varepsilon, y) \Big], \tag{6.5.1}$$

where

$$\begin{aligned} E_n^\delta(h_\kappa^2; x, y) :=& n^{d-1} \frac{\prod_{j=1}^d (|x_j y_j| + n^{-1}\rho(x,y) + n^{-2})^{-\kappa_j}}{(1 + n\rho(x,y))^{\delta + \frac{d}{2}}}, \\ R_n(h_\kappa^2; x, y) :=& n^{d-1} \frac{\prod_{j=1}^d (|x_j y_j| + (\rho(x,y))^2 + n^{-2})^{-\kappa_j}}{(n\rho(x,y) + 1)^d}. \end{aligned}$$

Using (5.1.3), one can easily show that for $x, y \in \mathbb{S}^{d-1}$,

$$R_n(h_\kappa^2, x, y) \leq \frac{C}{(1 + n\rho(x,y)) w_{n,\kappa}(x,y)}. \tag{6.5.2}$$

Indeed, (6.5.2) for $\kappa_{\min} \geq 0$ was proved in [15, Lemma 3.9]. The proof there works equally well for the case $\kappa_{\min} < 0$.

To complete the proof of (6.1.7), it remains to show

$$E_n^\delta(h_\kappa^2; x, y) \leq \frac{C}{(1 + n\rho(x,y))^{\delta - \sigma_\kappa} w_{n,\kappa}(x,y)}. \tag{6.5.3}$$

To this end, let

$$u_j(x,y) := (|x_j y_j| + n^{-1}\rho(x,y) + n^{-2})^{-\kappa_j}, \quad j = 1, \cdots, d.$$

If $|x_j| \geq 4\rho(x,y)$, then

$$u_j(x,y) \sim \left(|x_j| + \rho + n^{-1}\right)^{-2\kappa_j}.$$

If $|x_j| < 4\rho(x,y)$ and $\kappa_j > 0$, then

$$u_j(x,y) \leq (n^{-1}\rho(x,y) + n^{-2})^{-\kappa_j} \sim (1 + n\rho(x,y))^{\kappa_j} \Big(|x_j| + \rho(x,y) + n^{-1}\Big)^{-2\kappa_j}.$$

If $|x_j| < 4\rho(x,y)$ and $\kappa_j < 0$, then

$$u_j(x,y) \leq (|x_j|^2 + \rho(x,y)^2 + n^{-1}\rho(x,y) + n^{-2})^{-\kappa_j} \sim \Big(|x_j| + \rho(x,y) + n^{-1}\Big)^{-2\kappa_j}.$$

Putting the above together, and using (5.1.3), we obtain

$$\prod_{j=1}^{d} u_j(x,y) \leq \frac{Cn^{-d+1}(1 + n\rho(x,y))^{\kappa(J)+d-1}}{w_{n,\kappa}(x,y)},$$

where $\kappa(J) = \sum_{j \in J} \kappa_j$, and

$$J = J(x,y) := \{1 \leq j \leq d : \quad \kappa_j \geq 0, \quad |x_j| \leq 4\rho(x,y)\}.$$

Since

$$\kappa(J) \leq \gamma_\kappa^+ = \sum_{j:\ \ \kappa_j > 0} \kappa_j,$$

we conclude that

$$\prod_{j=1}^{d} u_j(x,y) \le \frac{Cn^{-d+1}(1+n\rho(x,y))^{\gamma_\kappa^+ + d-1}}{w_{n,\kappa}(x,y)},$$

which implies the desired estimate (6.5.3). This shows the desired estimate (6.1.7) of the Cesàro kernel $K_n^\delta(h_\kappa^2; x, y)$.

Finally, the above proof with a slight modification also gives the desired estimate (6.2.2) of the reproducing kernel $P_n(h_\kappa^2; x, y)$.

□

Proof of Theorem 6.1.4. For simplicity, we write, for $E \subset \mathbb{S}^{d-1}$,

$$\mathrm{meas}_\kappa(E) = \int_E h_\kappa^2(x)\, d\sigma(x).$$

Then

$$w_{n,\kappa}(x,y) = \mathrm{meas}_\kappa\Big(B(x, \rho(x,y) + n^{-1})\Big), \quad x, y \in \mathbb{S}^{d-1}.$$

Let

$$a = a_\delta := \min\{\delta - \sigma_\kappa,\ 1\},$$

and define

$$A_n^\delta(x,y) := \frac{(1+n\rho(x,y))^{-a}}{w_{n,\kappa}(x,y)}, \quad x, y \in \mathbb{S}^{d-1}.$$

By Corollary 6.1.1, we then have

$$\begin{aligned}
&\max_{x\in\mathbb{S}^{d-1}}\int_{\mathbb{S}^{d-1}}|K_n^\delta(h_\kappa^2;x,y)|\,h_\kappa^2(y)\,d\sigma(y)\\
&\quad\le C\max_{x\in\mathbb{S}^{d-1}}\sum_{\varepsilon\in\{\pm1\}^d}\int_{\mathbb{S}^{d-1}}A_n^\delta(x\varepsilon,y)h_\kappa^2(y)\,d\sigma(y)\le C\max_{x\in\mathbb{S}^{d-1}}\int_{\mathbb{S}^{d-1}}A_n^\delta(x,y)h_\kappa^2(y)\,d\sigma(y)\\
&\quad\le C\max_{x\in\mathbb{S}^{d-1}}\Big[1+\sum_{1\le j\le\log(Cn)}\int_{2^{j-1}\le n\rho(x,y)\le 2^j}\frac{h_\kappa^2(y)d\sigma(y)}{2^{ja}\operatorname{meas}_\kappa(B(x,2^jn^{-1})}\Big]\\
&\quad\le C\sum_{0\le j\le\log(Cn)}2^{-ja}\le CL_n^\delta.
\end{aligned}$$

This proves (6.1.9).

Finally, the estimate (6.1.8) can be proved in a similar way.

□

6.6 Proof of Theorem 6.1.5

Given an operator T on spaces of functions on $\mathbb{S}^{d-1}$, we shall use the notation $\|T\|$ to denote the operator norm of T from $C(\mathbb{S}^{d-1})$ to $C(\mathbb{S}^{d-1})$; that is,

$$\|T\|:=\sup\{\|Tf\|_\infty:\quad f\in C(\mathbb{S}^{d-1}),\quad \|f\|_\infty\le 1\}.$$

Then

$$\begin{aligned}
\|\operatorname{proj}_n(h_\kappa^2)\|&=\omega_d^\kappa\sup_{x\in\mathbb{S}^{d-1}}\int_{\mathbb{S}^{d-1}}|P_n(h_\kappa^2;x,y)|h_\kappa^2(y)\,d\sigma(y),\\
\|S_n^\delta(h_\kappa^2)\|&=\omega_d^\kappa\sup_{x\in\mathbb{S}^{d-1}}\int_{\mathbb{S}^{d-1}}|K_n^\delta(h_\kappa^2;x,y)|h_\kappa^2(y)\,d\sigma(y).
\end{aligned}$$

As is well known (see, for instance, [42, Theorem 3.1.22, p. 78] and [42, Theorem 3.1.23, p. 78]), for any $\delta \geq 0$, there exists a constant $C_\delta > 0$,

$$\| \operatorname{proj}_n(h_\kappa^2)\| \leq C_\delta n^\delta \max_{0\leq j\leq n} \|S_j^\delta(h_\kappa^2)\|. \tag{6.6.1}$$

Without loss of generality, we may assume that $\kappa_1 < 0$ and $\kappa_j \geq 0$ for $1 < j \leq d$. Then

$$\sigma_\kappa = \frac{d-2}{2} + \sum_{j=2}^{d} \kappa_j.$$

First, we show that there exists a constant $C > 0$ independent of n such that

$$\| \operatorname{proj}_{2n}(h_\kappa^2)\| \geq C n^{\sigma_\kappa}, \tag{6.6.2}$$

which together with (6.6.1) will imply that for $0 \leq \delta < \sigma_\kappa$,

$$\max_{1\leq j\leq n} \|S_j^\delta(h_\kappa^2)\| \geq C n^{\sigma_\kappa - \delta}. \tag{6.6.3}$$

To show (6.6.2), let

$$\alpha_\kappa := \frac{d-3}{2} + \sum_{j=2}^{d} \kappa_j \quad \text{and} \quad \beta_\kappa := \kappa_1 - \frac{1}{2}.$$

Let

$$w_\kappa(t) := t^{\beta_\kappa}(1-t)^{\alpha_\kappa}, \quad t \in [0,1].$$

For $1 \leq p \leq \infty$, we denote by $\|\cdot\|_{p,\alpha_\kappa,\beta_\kappa}$ the Lebesgue L^p-norm defined with respect to the measure $w_\kappa(t)dt$ on $[0,1]$. Clearly, $\{P_j^{(\alpha_\kappa,\beta_\kappa)}(2t-1)\}_{j=0}^\infty$ is an orthogonal basis of the space $L^2([0,1], w_\kappa(t)\,dt)$, and hence, each $f \in L^2([0,1], w_\kappa(t)dt)$ has an orthogonal polynomial expansion converging to f in the norm of $\|\cdot\|_{2,\alpha_\kappa,\beta_\kappa}$:

$$f(t) = \sum_{j=0}^{\infty} \mathrm{proj}_j(w_\kappa; f, t),$$

where

$$\mathrm{proj}_j(w_k; f, t) := \widehat{f}(j) P_j^{(\alpha_\kappa,\beta_\kappa)}(2t-1), \quad j = 0, 1, \cdots,$$

and

$$\widehat{f}(j) = \|P_j^{(\alpha_\kappa,\beta_\kappa)}\|_{2,\alpha_\kappa,\beta_\kappa}^{-2} \int_0^1 f(t) P_j^{(\alpha_\kappa,\beta_\kappa)}(2t-1) t^{\beta_\kappa}(1-t)^{\alpha_\kappa}\, dt.$$

For each nonnegative integer n, we define a spherical polynomial $\varphi_n^\kappa : \mathbb{S}^{d-1} \to \mathbb{R}$ of degree $2n$ by

$$\varphi_n^\kappa(x) := P_n^{(\alpha_\kappa,\beta_\kappa)}(2x_1^2 - 1), \quad x = (x_1, x_2, \cdots, x_d) \in \mathbb{S}^{d-1}.$$

We claim that

$$\varphi_n^\kappa \in \mathcal{H}_{2n}^d(h_\kappa^2); \tag{6.6.4}$$

Since $\varphi_n^\kappa \in \Pi_{2n}^d$, for the proof of (6.6.4), it suffices to show that for each $\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{Z}_{\geq 0}^d$ with $|\alpha| = \alpha_1 + \cdots + \alpha_d < 2n$,

$$\int_{\mathbb{S}^{d-1}} \varphi_n^\kappa(x) x^\alpha h_\kappa^2(x)\, d\sigma(x) = 0. \tag{6.6.5}$$

By symmetry, (6.6.5) holds trivially if one of the α_j is odd. Thus, without loss of generality, we may assume that $\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{Z}_{\geq 0}^d$, $|\alpha| \leq 2n-2$ and each α_j is even. A straightforward calculation shows that for any $g \in L^1([0,1], w_\kappa(t)dt)$,

$$\int_{\mathbb{S}^{d-1}} g\Big(x_1^2\Big) h_\kappa^2(x)\, d\sigma(x) = c_\kappa \int_0^1 g(t) t^{\beta_\kappa} (1-t)^{\alpha_\kappa}\, dt, \tag{6.6.6}$$

where $c_\kappa > 0$. Since each α_j is even, $x^\alpha = \prod_{j=1}^d |x_j|^{\alpha_j}$ and $x^\alpha h_\kappa^2(x) = h_{\kappa+\alpha/2}^2(x)$. It follows that

$$\int_{\mathbb{S}^{d-1}} \varphi_n^\kappa(x) x^\alpha h_\kappa^2(x)\, d\sigma(x) = \int_{\mathbb{S}^{d-1}} \varphi_n^\kappa(x) h_{\kappa+\alpha/2}^2(x)\, d\sigma(x),$$

which, using (6.6.6), equals a constant multiple of

$$\int_0^1 P_n^{(\alpha_\kappa,\beta_\kappa)}(2t-1) t^{\alpha_1/2} \Big(\prod_{j=2}^d (1-t)^{\alpha_j/2}\Big) t^{\beta_\kappa}(1-t)^{\alpha_\kappa}\, dt = 0.$$

This proves (6.6.5) and hence the claim (6.6.4).

Now we define an operator $E : C[0,1] \to C(\mathbb{S}^{d-1})$ by

$$Ef(x) = f(x_1^2), \quad x \in \mathbb{S}^{d-1}, \quad f \in C[0,1].$$

From the claim (6.6.4), it is easily seen that for each $f \in C[0,1]$,

$$\mathrm{proj}_{2j+1}(h_\kappa^2; Ef) = 0, \quad j = 0, 1, \cdots \tag{6.6.7}$$

and

$$\mathrm{proj}_{2j}(h_\kappa^2; Ef)(x) = \mathrm{proj}_j(w_\kappa; f, x_1^2), \quad j = 0, 1, \cdots. \tag{6.6.8}$$

Thus,

$$\begin{aligned}\|\operatorname{proj}_{2n}(h_\kappa^2)\| &\ge \sup\Big\{\|\operatorname{proj}_{2n}(h_\kappa^2; Ef)\|_{L^\infty(\mathbb{S}^{d-1})}: \quad f\in C[0,1], \quad \|f\|_{C[0,1]}=1\Big\}\\ &= \|P_n^{(\alpha_\kappa,\beta_\kappa)}\|_\infty \sup\Big\{|\widehat{f}(n)| : f\in C[0,1], \quad \|f\|_{C[0,1]}=1\Big\}\\ &= \|P_n^{(\alpha_\kappa,\beta_\kappa)}\|_\infty \|P_n^{(\alpha_\kappa,\beta_\kappa)}\|_{2,\alpha_\kappa,\beta_\kappa}^{-2}\int_0^1 |P_n^{(\alpha_\kappa,\beta_\kappa)}(2t-1)|t^{\beta_\kappa}(1-t)^{\alpha_\kappa}\,dt.\end{aligned}$$

We need the following facts on Jacobi polynomials for $\alpha>-1$ and $\beta>-1$, which can be found in [35, (7.32.2)], [35, p.391], [35, (4.3.3)] respectively:

$$\|P_n^{(\alpha,\beta)}\|_\infty \sim n^{\max\{\alpha,\beta,-\frac12\}}, \tag{6.6.9}$$

$$\int_{-1}^1 (1+x)^\beta(1-x)^\alpha |P_n^{(\alpha,\beta)}(x)|\,dx \sim n^{-\frac12}, \tag{6.6.10}$$

$$\int_{-1}^1 (1-x)^\alpha(1+x)^\beta\Big(P_n^{(\alpha,\beta)}(x)\Big)^2\,dx \sim n^{-1}. \tag{6.6.11}$$

Since $-1<\beta_\kappa<-\frac12\le\alpha_\kappa$, it follows that

$$\|\operatorname{proj}_{2n}(h_\kappa^2)\| \ge Cn^{\alpha_\kappa+\frac12} = Cn^{\sigma_\kappa}.$$

This proves (6.6.2), and (6.6.3) as well.

Finally, we prove (6.1.11) for $\delta\ge\sigma_\kappa$. Since $S_n^\delta(h_\kappa^2,1)=1$, (6.1.11) holds trivially if $\delta>\sigma_\kappa$. Thus, it remains to prove (6.1.11) for $\delta=\sigma_\kappa$, or equivalently,

$$\max_{1\le j\le n}\|S_j^{\sigma_\kappa}(h_\kappa^2)\| \ge C\log n, \tag{6.6.12}$$

where $C>0$ is independent of n.

To prove (6.6.12), we denote by $S_n^\delta(w_\kappa; f)$ the n-th Cesàro mean of order δ of the WOPE of $f\in C[0,1]$ with respect to the weight w_κ on $[0,1]$. Let m be the integer such that $2m \le n < 2m+2$. Using (6.6.7) and (6.6.8), we have that for $x = (x_1, \cdots, x_d) \in \mathbb{S}^{d-1}$,

$$\begin{aligned}
S_m^\delta(w_\kappa; f, x_1^2) &= \sum_{j=0}^{m} \frac{A_{m-j}^\delta}{A_m^\delta} \operatorname{proj}_j(w_\kappa; f, x_1^2) = \sum_{j=0}^{m} \frac{A_{m-j}^\delta}{A_m^\delta} \operatorname{proj}_{2j}(h_\kappa^2; Ef, x) \\
&= \sum_{j=0}^{2m} \mu_j \operatorname{proj}_j(h_\kappa^2; Ef, x) + S_n^\delta(h_\kappa^2; Ef, x),
\end{aligned}$$

where

$$\mu_j = \begin{cases} -\frac{A_{n-j}^\delta}{A_n^\delta} + \frac{A_{m-j/2}^\delta}{A_m^\delta}, & 0 \le j \le 2m \\ 0, & j > 2m, \end{cases}$$

and

$$A_x^\delta := \frac{\Gamma(x+\delta+1)}{\Gamma(\delta+1)\Gamma(x+1)}, \quad \forall x \ge 0.$$

According to [15, (5.10)], we have that for $0 \le j \le 2m$,

$$|\Delta^i \mu_j| \le C m^{-\delta}(m - j/2 + 1)^{\delta - i - 1}, \quad i = 0, 1, \cdots . \tag{6.6.13}$$

Let ℓ be an integer such that $\delta - 1 < \ell \le \delta$. Summation by parts ℓ times shows that for any $f \in C[0,1]$ with $\|f\|_\infty = 1$,

$$\begin{aligned}
&\Big\|\sum_{j=0}^{2m} \mu_j \operatorname{proj}_j(h_\kappa^2; Ef)\Big\|_\infty \le C \sum_{j=0}^{2m-\ell} |\Delta^{\ell+1}\mu_j|(j+1)^\ell \|S_j^\ell(h_\kappa^2; Ef)\|_\infty \\
&+ C m^\ell \max_{0 \le i \le \ell} |\Delta^i \mu_{m-i}| \|S_{2m-i}^\ell(h_\kappa^2; Ef)\|_\infty,
\end{aligned}$$

which, using (6.6.13), is bounded by a constant multiple of $\max_{1\le j\le n} \|S_j^\delta(h_\kappa^2)\|$. It follows that

$$\max_{1\le j\le n} \|S_j^{\sigma_\kappa}(h_\kappa^2)\| \ge C \sup\Big\{\int_{\mathbb{S}^{d-1}} |S_m^{\sigma_\kappa}(w_\kappa; f, x_1^2)| h_\kappa^2(x)\, d\sigma(x) : \quad f \in C[0,1], \quad \|f\|_\infty = 1\Big\},$$

which, using (6.6.6), equals

$$C \sup\Big\{ \int_0^1 |S_m^{\sigma_\kappa}(w_\kappa; f, t)| w_\kappa(t)\, dt : \quad f \in C[0,1], \quad \|f\|_\infty = 1 \Big\} \geq C \log n.$$

Chapter 7

Boundedness of projection operators and Cesàro means in weighted L^p space on the unit ball and simplex

7.1 Weighted orthogonal polynomial expansions (WOPEs) on the ball and the simplex

In this section, we shall describe briefly some necessary notations and results for WOPEs on the unit ball $\mathbb{B}^d$ and the simplex $\mathbb{T}^d$. Unless otherwise stated, most of the results described in this section can be found in the paper [39] and the books [16, 21].

7.1.1 WOPEs in several variables

Let Ω denote a compact domain in $\mathbb{R}^d$ endowed with the usual Lebesgue measure dx. Given a weight function W on Ω, we denote by $L^p(W;\Omega)$ the usual L^p-space defined with respect to the measure Wdx on Ω, and $\mathcal{V}_n^d(W)$ the space of orthogonal polynomials of degree n with respect to the weight function W on Ω. Thus, if we denote by Π_n^d the space of all algebraic polynomials in d variables of total degree at most n, then $\mathcal{V}_n^d(W)$ is the orthogonal complement of Π_{n-1}^d in the space Π_n^d with respect to the inner product of $L^2(W;\Omega)$, where it is agreed that $\Pi_{-1}^d = \{0\}$.

Since Ω is compact, each function $f \in L^2(W;\Omega)$ has a weighted orthogonal polynomial expansion on Ω, $f = \sum_{n=0}^{\infty} \mathrm{proj}_n(W;f)$, converging in the norm of $L^2(W;\Omega)$, where $\mathrm{proj}_n(W;f)$ denotes the orthogonal projection of f onto the space $\mathcal{V}_n^d(W)$. Let $P_n(W;\cdot,\cdot)$ denote the reproducing kernel of the space $\mathcal{V}_n^d(W)$; that is,

$$P_n(W;x,y) := \sum_{j=1}^{a_n^d} \varphi_{n,j}(x)\overline{\varphi_{n,j}(y)}, \quad x,y \in \Omega$$

for an orthonormal basis $\{\varphi_{n,j} : \quad 1 \le j \le a_n^d := \dim \mathcal{V}_n^d(W)\}$ of the space $\mathcal{V}_n^d(W)$.

The orthogonal projection operator $\mathrm{proj}_n(W) : L^2(W;\Omega) \mapsto \mathcal{V}_n^d(W)$ can be expressed as an integral operator

$$\mathrm{proj}_n(W;f,x) = \int_\Omega f(y)P_n(W;x,y)W(y)dy, \quad x \in \Omega, \tag{7.1.1}$$

which also extends the definition of $\mathrm{proj}_n(W;f)$ to all $f \in L^1(W;\Omega)$ since the kernel $P_n(W;x,y)$ is a polynomial in both x and y.

Let $S_n^\delta(W;f)$, $n = 0,1,\cdots$, denote the Cesàro (C,δ)-means of the WOPEs of $f \in L^1(W;\Omega)$. Each $S_n^\delta(W;f)$ can be expressed as an integral against a kernel, $K_n^\delta(W;x,y)$, called the Cesàro (C,δ)- kernel,

$$S_n^\delta(W;f,x) := \int_\Omega f(y)K_n^\delta(W;x,y)W(y)dy, \quad x \in \Omega,$$

where

$$K_n^\delta(W;x,y) := (A_n^\delta)^{-1} \sum_{j=0}^{n} A_{n-j}^\delta P_j(W;x,y), \quad x,y \in \Omega.$$

7.1.2 WOPEs on the unit ball $\mathbb{B}^d$

The weight function W_κ^B we consider on the unit ball $\mathbb{B}^d$ is given in (4.0.2) with $\kappa := (\kappa_1, \cdots, \kappa_{d+1}) \in (-\frac{1}{2}, \infty)^d$. It is related to the h_κ on the sphere $\mathbb{S}^d$ of $\mathbb{R}^{d+1}$ by

$$h_\kappa^2(x, \sqrt{1 - \|x\|^2}) = W_\kappa^B(x)\sqrt{1 - \|x\|^2}, \quad x \in \mathbb{B}^d, \tag{7.1.2}$$

in which h_κ is defined in (5.1.1) with $\mathbb{S}^d$ in place of $\mathbb{S}^{d-1}$. Furthermore, under the change of variables $y = \phi(x)$ with

$$\phi : x \in \mathbb{B}^d \mapsto (x, \sqrt{1 - \|x\|^2}) \in \mathbb{S}_+^d := \{y \in \mathbb{S}^d : \; y_{d+1} \geq 0\}, \tag{7.1.3}$$

we have

$$\int_{\mathbb{S}^d} g(y)d\sigma(y) = \int_{\mathbb{B}^d} \left[g(x, \sqrt{1 - \|x\|^2}\,) + g(x, -\sqrt{1 - \|x\|^2}\,)\right] \frac{dx}{\sqrt{1 - \|x\|^2}}. \tag{7.1.4}$$

The orthogonal structure is preserved under the mapping (7.1.3) and the study of orthogonal expansions for W_κ^B on $\mathbb{B}^d$ can be essentially reduced to that of h_κ^2 on $\mathbb{S}^d$. More precisely, we have

$$\begin{aligned} P_n(W_\kappa^B; x, y) =& \frac{1}{2}\left[P_n(h_\kappa^2; (x, x_{d+1}), (y, y_{d+1}))\right. \\ &\left. + P_n(h_\kappa^2; (x, x_{d+1}), (y, -y_{d+1}))\right] \end{aligned} \tag{7.1.5}$$

where $x, y \in \mathbb{B}^d$, and $x_{d+1} = \sqrt{1 - \|x\|^2}$, $y_{d+1} = \sqrt{1 - \|y\|^2}$. As a consequence, the orthogonal projection, $\mathrm{proj}_n(W_\kappa^B; f)$, of $f \in L^2(W_\kappa^B; \mathbb{B}^d)$ onto $\mathcal{V}_n^d(W_\kappa^B)$ can be expressed in terms of the orthogonal projection of $F(x, x_{d+1}) := f(x)$ onto $\mathcal{H}_n^{d+1}(h_\kappa^2)$:

$$\mathrm{proj}_n(W_\kappa^B; f, x) = \mathrm{proj}_n(h_\kappa^2; F, X), \qquad \text{with } \; X := (x, \sqrt{1 - \|x\|^2}). \tag{7.1.6}$$

This relation allows us to deduce results on the convergence of orthogonal expansions with respect to W_κ^B on $\mathbb{B}^d$ from those of h-harmonic expansions on $\mathbb{S}^d$.

7.1.3 WOPEs on the simplex

The weight functions we consider on the simplex $\mathbb{T}^d$ are defined by (4.0.3), which are related to W_κ^B, hence to h_κ^2. In fact, W_κ^T is exactly the product of the weight function W_κ^B under the mapping

$$\psi : (x_1, \ldots, x_d) \in \mathbb{B}^d \mapsto (x_1^2, \ldots, x_d^2) \in \mathbb{T}^d \tag{7.1.7}$$

and the Jacobian of this change of variables. Furthermore, the change of variables shows

$$\int_{\mathbb{B}^d} g(x_1^2, \ldots, x_d^2) dx = \int_{\mathbb{T}^d} g(x_1, \ldots, x_d) \frac{dx}{\sqrt{x_1 \cdots x_d}}. \tag{7.1.8}$$

The orthogonal structure is preserved under the mapping (7.1.7). In fact, $R \in \mathcal{V}_n^d(W_\kappa^T)$ if and only if $R \circ \psi \in \mathcal{V}_{2n}^d(W_\kappa^B)$. The orthogonal projection, $\mathrm{proj}_n(W_\kappa^T; f)$, of $f \in L^2(W_\kappa^T; \mathbb{T}^d)$ onto $\mathcal{V}_n^d(W_\kappa^T)$ can be expressed in terms of the orthogonal projection of $f \circ \psi$ onto $\mathcal{V}_{2n}^d(W_\kappa^B)$:

$$\mathrm{proj}_n(W_\kappa^T; f, \psi(x)) = \frac{1}{2^d} \sum_{\varepsilon \in \mathbb{Z}_2^d} \mathrm{proj}_{2n}(W_\kappa^B; f \circ \psi, x\varepsilon), \quad x \in \mathbb{B}^d. \tag{7.1.9}$$

7.2 Results for Cesàro means of WOPEs on the unit ball and the simplex

It was observed by Xu [39, 40] that WOPEs on $\mathbb{B}^d$ and $\mathbb{T}^d$ are closely related to WOPEs on the sphere $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ and results on WOPEs on $\mathbb{B}^d$ and $\mathbb{T}^d$ can often be deduced from the corresponding results on the unit sphere $\mathbb{S}^d$.

The main purpose in this section is to establish similar results for WOPEs on $\mathbb{B}^d$ and $\mathbb{T}^d$. Throughout this section, we will use a slight abuse of notations. The letter κ denotes a fixed, nonzero vector $\kappa := (\kappa_1, \cdots, \kappa_{d+1}) \in (-\frac{1}{2}, \infty)^{d+1}$. Accordingly, we define

$$\kappa_{\min} := \min_{1\leq j\leq d+1} \kappa_j, \quad \gamma_\kappa = \sum_{j=1}^{d+1} \kappa_j, \quad \gamma_\kappa^+ := \sum_{j:\kappa_j>0} \kappa_j, \tag{7.2.1}$$

$$\tag{7.2.2}$$

7.2.1 Results on the ball

For $x \in \mathbb{B}^d$, we set $x_{d+1} := \sqrt{1-\|x\|^2}$. Let $\rho_B : \mathbb{B}^d \times \mathbb{B}^d \to [0, \pi]$ denote the metric on $\mathbb{B}^d$ given by

$$\rho_B(x, y) = \arccos\big(x \cdot y + x_{d+1}y_{d+1}\big), \quad x, y \in \mathbb{B}^d.$$

For $x \in \mathbb{B}^d$ and $\theta > 0$, define

$$B^B(x, \theta) := \{y \in \mathbb{B}^d : \quad \rho_B(x, y) \leq \theta\}.$$

We write

$$\mathrm{meas}_\kappa^B(E) := \int_E W_\kappa^B(x)dx, \qquad E \subset \mathbb{B}^d,$$

where W_κ^B is the weight function on $\mathbb{B}^d$ given in (4.0.2) with $\kappa_{\min} > -\frac{1}{2}$. It is easily seen that for $x \in \mathbb{B}^d$ and $\theta \in (0, \pi]$,

$$\mathrm{meas}_\kappa^B(B^B(x, \theta)) \sim \theta^d \prod_{j=1}^{d+1}(|x_j| + \theta)^{2\kappa_j}.$$

This in particular implies that meas_κ^B is a doubling measure on $\mathbb{B}^d$ satisfying that for any $x \in \mathbb{B}^d$ and $\theta \in (0, \pi]$,

$$\mathrm{meas}_\kappa^B(B^B(x, 2^j\theta)) \leq C 2^{js_\kappa} \mathrm{meas}_\kappa^B(B^B(x, \theta)), \quad j = 1, 2, \cdots, \tag{7.2.3}$$

where $C > 0$ is a constant depending only on κ and d, and

$$s_\kappa = d + 2\gamma_\kappa^+ - 2\max\{\kappa_{\min}, 0\}. \tag{7.2.4}$$

It is easily seen that s_κ is the optimal constant for which (7.2.3) holds.

Recall that $P(W_\kappa^B, x, y)$ denotes the reproducing kernel of the space $\mathcal{V}_n^d(W_\kappa^B)$ of orthogonal polynomials of degree n with respect to the weight W_κ^B on $\mathbb{B}^d$, $S_n^\delta(W_\kappa^B; f)$ denotes the n-th Cesàro mean of order $\delta \geq 0$ of the WOPE of f with respect to the weight function W_κ^B on $\mathbb{B}^d$, and $K_n^\delta(W_\kappa^B; x, y)$ is the Cesàro kernel of the operator $S_n^\delta(W_\kappa^B)$.

Theorem 7.2.1. *Let $\kappa = (\kappa_1, \cdots, \kappa_{d+1}) \in \mathbb{R}^{d+1}$ be such that $\kappa_{\min} > -\frac{1}{2}$. Let*

$$\sigma_\kappa := \frac{s_\kappa - 1}{2} = \frac{d-1}{2} + \gamma_\kappa^+ - \max\{\kappa_{\min}, 0\}. \tag{7.2.5}$$

Then for $\delta > 0$ and $x, y \in \mathbb{B}^d$,

$$\begin{aligned}
|P_n(W_\kappa^B; x, y)| \leq & \frac{C}{n w_{n,\kappa}^B(x, y)} \max_{\varepsilon \in \{\pm 1\}^d} \Big(1 + n\rho_B(x\varepsilon, y)\Big)^{\sigma_\kappa + 1}, \\
|K_n^\delta(W_\kappa^B; x, y)| \leq & \frac{C}{w_{n,\kappa}^B(x, y)} \max_{\varepsilon \in \{\pm 1\}^d} \Big[\frac{1}{(1 + n\rho_B(x\varepsilon, y))^{\delta - \sigma_\kappa}} + \frac{1}{1 + n\rho_B(x\varepsilon, y)}\Big],
\end{aligned}$$

where

$$w_{n,\kappa}^B(x, y) := \int_{B^B(x, \rho_B(x,y) + n^{-1})} W_\kappa^B(z)\, dz, \quad x, y \in \mathbb{B}^d, \quad n \in \mathbb{N}$$

and $x\varepsilon = (x_1\varepsilon_1, \cdots, x_d\varepsilon_d)$ for $x = (x_1, \cdots, x_d) \in \mathbb{B}^d$ and $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_d) \in \{\pm 1\}^d$.

Given an operator T on spaces of functions on $\mathbb{B}^d$, we set

$$\|T\| := \sup\{\|Tf\|_\infty : \quad f \in C(\mathbb{B}^d), \quad \|f\|_\infty = 1\}.$$

Theorem 7.2.2. *Let $\kappa = (\kappa_1, \cdots, \kappa_{d+1}) \in \mathbb{R}^{d+1}$ be such that $\kappa_{\min} > -\frac{1}{2}$. Let σ_κ be given in (7.2.5). Then there exists a constant $C > 0$ independent of n such that*

$$\|S_n^\delta(W_\kappa^B)\| \le C \begin{cases} 1, & \delta > \sigma_\kappa \\ \log n, & \delta = \sigma_\kappa \\ n^{-\delta+\sigma_\kappa}, & 0 \le \delta < \sigma_\kappa, \end{cases}$$

and

$$\|\operatorname{proj}_n(W_\kappa^B)\| \le C n^{\sigma_\kappa}.$$

In particular, if $\delta > \sigma_\kappa$ and $f \in C(\mathbb{B}^d)$, then $S_n^\delta(W_\kappa^B; f)$ converges uniformly to f on $\mathbb{B}^d$.

In the case when $\kappa_{\min} \ge 0$, Theorem 7.2.1 and Theorem 7.2.2 were previously proved in [17]. These results can be deduced directly from the corresponding results on the sphere $\mathbb{S}^d$. Since the proofs are almost identical to those in [17], we skip the details here.

7.2.2 Results on the simplex

For $x = (x_1, \cdots, x_d) \in \mathbb{T}^d$, let $|x| = x_1 + x_2 + \cdots + x_d$ and $x_{d+1} := 1 - |x|$. Let $\rho_T : \mathbb{T}^d \times \mathbb{T}^d \to [0, \pi]$ be the metric on $\mathbb{T}^d$ given by

$$\rho_T(x, y) = \arccos\Big(\sum_{j=1}^{d+1} \sqrt{x_j y_j}\Big), \quad x, y \in \mathbb{T}^d.$$

For $x \in \mathbb{B}^d$ and $\theta > 0$, define

$$B^T(x,\theta) := \{y \in \mathbb{T}^d : \quad \rho_T(x,y) \le \theta\}.$$

We write

$$\mathrm{meas}_\kappa^T(E) := \int_E W_\kappa^T(x)dx, \qquad E \subset \mathbb{T}^d,$$

where W_κ^T is the weight function on $\mathbb{T}^d$ given in (4.0.3) with $\kappa_{\min} > -\frac{1}{2}$. It is easily seen that for $x \in \mathbb{T}^d$ and $\theta \in (0,\pi]$,

$$\mathrm{meas}_\kappa^T(B^T(x,\theta)) \sim \theta^d \prod_{j=1}^{d+1} (\sqrt{x_j} + \theta)^{2\kappa_j}.$$

This in particular implies that meas_κ^T is a doubling measure on $\mathbb{T}^d$ satisfying that for any $x \in \mathbb{T}^d$ and $\theta \in (0,\pi]$,

$$\mathrm{meas}_\kappa^T(B^T(x,2^j\theta)) \le C2^{js_\kappa}\,\mathrm{meas}_\kappa^T(B^T(x,\theta)), \quad j = 1,2,\cdots, \tag{7.2.6}$$

where $C > 0$ is a constant depending only on κ and d,

$$s_\kappa = d + 2\gamma_\kappa^+ - 2\max\{\kappa_{\min}, 0\}. \tag{7.2.7}$$

It is easily seen that s_κ is the optimal constant for which (7.2.6) holds.

Recall that $P(W_\kappa^T, x, y)$ denotes the reproducing kernel of the space $\mathcal{V}_n^d(W_\kappa^T)$ of orthogonal polynomials of degree n with respect to the weight W_κ^T on $\mathbb{T}^d$, $S_n^\delta(W_\kappa^T; f)$ denotes the n-th Cesàro mean of the WOPE of f with respect to the weight function W_κ^T on $\mathbb{T}^d$, and $K_n^\delta(W_\kappa^T; x, y)$ is the Cesàro kernel of the operator $S_n^\delta(W_\kappa^T)$.

Theorem 7.2.3. *Let $\kappa = (\kappa_1, \cdots, \kappa_{d+1}) \in \mathbb{R}^{d+1}$ be such that $\kappa_{\min} > -\frac{1}{2}$. Let*

$$\sigma_\kappa := \frac{s_\kappa - 1}{2} = \frac{d-1}{2} + \gamma_\kappa^+ - \max\{\kappa_{\min}, 0\}. \tag{7.2.8}$$

Then for $\delta \geq 0$ and $x, y \in \mathbb{T}^d$,

$$\begin{aligned} |P_n(W_\kappa^T; x, y)| \leq & \frac{C}{n w_{n,\kappa}^T(x,y)} \Big(1 + n\rho_T(x,y)\Big)^{\sigma_\kappa + 1}, \\ |K_n^\delta(W_\kappa^T; x, y)| \leq & \frac{C}{w_{n,\kappa}^T(x,y)} \Big[\frac{1}{(1+n\rho_T(x,y))^{\delta - \sigma_\kappa}} + \frac{1}{1 + n\rho_T(x,y)}\Big], \end{aligned}$$

where

$$w_{n,\kappa}^T(x,y) := \int_{B^T(x, \rho_T(x,y) + n^{-1})} W_\kappa^T(z)\, dz, \quad x, y \in \mathbb{T}^d, \quad n \in \mathbb{N}$$

Given an operator T on spaces of functions on $\mathbb{T}^d$, we set

$$\|T\| := \sup\{\|Tf\|_\infty : \quad f \in C(\mathbb{T}^d), \quad \|f\|_\infty = 1\}.$$

Theorem 7.2.4. *Let $\kappa = (\kappa_1, \cdots, \kappa_{d+1}) \in \mathbb{R}^{d+1}$ be such that $\kappa_{\min} > -\frac{1}{2}$. Let σ_κ be given in (7.2.8). Then there exists a constant $C > 0$ independent of n such that*

$$\|S_n^\delta(W_\kappa^T)\| \leq C \begin{cases} 1, & \delta > \sigma_\kappa \\ \log n, & \delta = \sigma_\kappa \\ n^{-\delta + \sigma_\kappa}, & 0 \leq \delta < \sigma_\kappa, \end{cases}$$

and

$$\|\operatorname{proj}_n(W_\kappa^T)\| \leq C n^{\sigma_\kappa}.$$

In particular, if $\delta > \sigma_\kappa$ and $f \in C(\mathbb{T}^d)$, then $S_n^\delta(W_\kappa^T; f)$ converges uniformly to f on $\mathbb{T}^d$.

In the case when $\kappa_{\min} \geq 0$, Theorem 7.2.3 and Theorem 7.2.4 were previously proved in [17]. These results can be deduced largely from the corresponding results on the ball $\mathbb{B}^d$. Since the proofs are similar to those in [17], we skip the details here.

Bibliography

[1] M. Alfaro, J. S. Dehesa, F. J. Marcellan, J. L. Rubio de Francia, and J. Vinuesa, eds., Orthogonal Polynomials and Their Applications, Lecture Notes in Mathematics, Vol. 1329 (SpringerVerlag, Berlin, 1988) 329-330. MR 89f:00027.

[2] G. E. Andrews, R. Askey and R. Roy, Special functions, *Encyclopedia of Mathematics and its Applications* 71, Cambridge University Press, Cambridge, (1999).

[3] P. Baratella, Bounds for the error term in Hilb formula for Jacobi polynomials, Atti Acc. Scienze Torino, Cl. Sci. Fis. Mat. Natur., 120 (1986).

[4] R. Beatson, W. zu Castell and Y. Xu, A Pólya criterion for (strictly) positive-definiteness on the sphere, IMA J. Numer. Anal. **34** (2014), 550-568.

[5] S. Bochner, Vorlesungen über Fouriersche Integrale. Leipzig, Akademischer Verlagsgesellschaft, (1932).

[6] S. Bochner, Monotone Funktionen, Stieltjes Integrale und harmonische Analyse. Math. Ann., 108: 378-410, (1933).

[7] A. Bonami and J-L. Clerc, Sommes de Cesàro et multiplicateurs des développements en harmoniques sphériques, Trans. Amer. Math. Soc. **183** (1973), 223-263.

[8] G. Brown and F. Dai, Approximation of smooth functions on compact two-point homogeneous spaces, J. Funct. Anal. **220** (2005), no. 2, 401-423.

[9] M. D. Buhmann, Radial Basis Functions: Theory and Implementations, Cambridge Monographs on Applied and Computational Mathematics, vol. 12, (2003).

[10] S. Chanillo and B. Muckenhoupt, Weak type estimates of Jacobi polynomial series, Memoirs of the American Mathematical Society. **102**(487), (1993).

[11] D. Chen, V. A. Menegatto and X. Sun, A necessary and sufficient condition for strictly positive definite functions on spheres, Proc. Amer. Math. Soc., 131 (2003), 2733-2740.

[12] E. T. Copson, Asymptotic expansions, Cambridge Tracts in Mathematics and Mathematical Physics, No. 55. Cambridge University Press, New York, (1965).

[13] F. Dai, Multivariate polynomial inequalities with respect to doubling weights and A_∞ weights, J. Funct. Anal., volume 235, (2006), 137–170.

[14] F. Dai and Y. Ge, Sharp estimates of the Cesàro kernels for weighted orthogonal polynomial expansions in several variables, Journal of Functional Analysis, Volume 280, Issue 4, 15, February 2021.

[15] F. Dai, S. Wang, and W. R. Ye , Maximal estimates for the Cesàro means of weighted orthogonal polynomial expansions on the unit sphere, J. Funct. Anal., volume 265, 2013, 2357–2387.

[16] F. Dai, Y. Xu, Approximation theory and harmonic analysis on spheres and balls, Springer Monographs in Mathematics, Springer, New York, (2013).

[17] F. Dai, Y. Xu, Cesàro means of orthogonal expansions in several variables, Const. Approx. **29** (2009), 129–155.

[18] F. Dai, Y. Xu, Boundedness of projection operators and Cesàro means in weighted L^p space on the unit sphere. Trans. Amer. Math. Soc. **361** (2009), 3189–3221.

[19] C. F. Dunkl, Differential-difference operators associated to reflection groups, Trans. Amer. Math. Soc. **311** (1989), 167–183.

[20] C. F. Dunkl, Integral kernels with reflection group invariance, Canad. J. Math. **43** (1991), 1213-1227.

[21] C. F. Dunkl, Y. Xu, Orthogonal polynomials of several variables, Encyclopedia of Mathematics and its Applications, volume 155, Cambridge University Press, Cambridge, 2014.

[22] G.E. Fasshauer and L.L. Schumaker, Scattered data fitting on the sphere. In Mathematical Methods for Curves and Surfaces, II (Lillehammer, 1997) (M. Daehlen, T. Lyche and L.L. Schumaker, eds.). Innov. Appl. Math. 117166. Nashville, TN: Vanderbilt Univ.Press.

[23] H. Feng and Y. Ge, Isotropic positive definite functions on spheres. (will appear soon)

[24] J. L. Fields and M. Ismail, On the positivity of some ${}_1F_2$'s, SIAM J. Math. Anal. **6** (1975), 551–559.

[25] C.L. Frenzen and R. A. Wong, A uniform asymptotic expansion of the Jacobi polynomials with error bounds, Canad. J. Math. **37** (1985), no. 5, 979–1007.

[26] G. Gasper, Positive integrals of Bessel functions, SIAM J. Math. Anal. 5 (1975), 868-881.

[27] E. W. Hobson, The Theory of Spherical and Ellipsoidal Harmonics. Chelsea Publishing Company, (1955).

[28] E. Kaniuth and A. T.-M. Lau, Fourier and Fourier-Stieltjes Algebras on Locally Compact Groups, American Math Society Mathematical Surveys and Monographs, Volume 231 (2018).

[29] Q. T. Le Gia, I. H. Sloan, and H. Wendland, Multiscale analysis in Sobolev spaces on the sphere. SIAM J. Numer. Anal. 48 (2010) 2065-2090.

[30] Z. K. Li and Y. Xu, Summability of orthogonal expansions of several variables, J. Approx. Theory, **122** (2003), 267-333.

[31] T. S. Lu, C. S. Ma, Isotropic covariance matrix functions on compact two-point homogeneous spaces. J Theor Probab 33(3) (2020): 1630-1656.

[32] V. A. Menegatto, C. P. Oliveira, and A. P Peron, Strictly positive definite kernels on subsets of the complex plane, Comput. Math. Appl. 51 (2006), 1233-1250.

[33] Z. H. Nie and C. S. Ma, Isotropic positive definite functions on spheres generated from those in Euclidean spaces, Proc. Amer. Math. Soc. 147 (2019), 3047-3056.

[34] I. J. Schoenberg, Positive definite functions on spheres, Duke Math. J. 9 (1942), 96-108.

[35] G. Szegö, Orthogonal polynomials, edition 4, American Mathematical Society, Colloquium Publications, Vol. XXIII, (1975).

[36] H. Wendland, Scattered Data Approximation, in: Cambridge Monogr. Appl. Comput. Math., vol. 17, Cambridge Univ. Press, Cambridge, (2005).

[37] Y. Xu and W. Cheney, Strictly positive definite functions on spheres, Proc. Amer. Math. Soc. 116 (1992), 977-981.

[38] Y. Xu, Positive definite functions on the unit sphere and integrals of Jacobi polynomials, *Proc. Amer. Math. Soc.* 146 (2018), 2039-2048.

[39] Y. Xu, Orthogonal polynomials and summability in Fourier orthogonal series on spheres and on balls, Math. Proc. Cambridge Philos. Soc. **31** (2001), 139-155.

[40] Y. Xu, Orthogonal polynomials and cubature formulae on balls, simplices, and spheres. Numerical analysis 2000, Vol. V, Quadrature and orthogonal polynomials. J. Comput. Appl. Math. **127** (2001), no. 1-2, 349-368.

[41] Y. Xu, Orthogonal polynomials for a family of product weight functions on the spheres, Canadian J. Math., **49** (1997), 175-192.

[42] A. Zygmund, Trigonometric Series: Vols. I, II, Cambridge University Press, London, (1968).