JOURNAL OF DIFFERENTIAL EQUATIONS **136**, 35–75 (1997)
ARTICLE NO. DE963244

Averaging for Fundamental Solutions of Parabolic Equations

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Received December 13, 1995; revised August 27, 1996

Herein, an averaging theory for the solutions to Cauchy initial value problems of arbitrary order, ε-dependent parabolic partial differential equations is developed. Indeed, by directly developing bounds between the derivatives of the fundamental solution to such an equation and derivatives of the fundamental solution of an "averaged" parabolic equation, we bring forth a novel approach to comparing x-derivatives of

$$\partial_t u^\varepsilon(x, t) = \sum_{|k| \leqslant 2p} A_k(x, t/\varepsilon)\, \partial_x^k u^\varepsilon(x, t) + f^\varepsilon(x, t), \qquad u^\varepsilon(x, 0) = \varphi^\varepsilon(x)$$

on $\Re^d \times [0, T]$ to like derivatives of

$$\partial_t u(x, t) = \sum_{|k| \leqslant 2p} A_k^0(x)\, \partial_x^k u^\varepsilon(x, t) + f(x, t), \qquad u(x) = \varphi(x)$$

(as $\varepsilon \to 0$) under general regularity conditions and our basic hypothesis that

$$\left\| \int_0^t A_k(x, s/\varepsilon) - A_k^0(x)\, ds \right\| \xrightarrow{\varepsilon \to 0} 0$$

for each x, t (i.e., pointwise). The flexibility afforded by studing fundamental vis-à-vis specific solutions of these equations not only permits ε-dependent Cauchy data and provides a unified method of treating all x-derivatives of u^ε up to order $2p-1$ but also proves an invaluable tool when considering related problems of stochastic averaging. Our development was motivated by and retains a strong resemblance to the classical theory of parabolic partial differential equations. However, it will turn out that the classical conditions under which fundamental solutions are known to exist are somewhat unsuitable for our purposes and a modified set of conditions must be used.

* The author gratefully acknowledges support from NSF, Lockheed Martin Air Traffic Management-Eagan MN, the Canadian Laboratory for Research in Statistics and Probability, and NSERC (in the form of a postdoctoral fellowship and funding from a collaborative grant in stochastic partial differential equations). This work would not have been possible without this support. E-mail:kouritzi@ima.umn.edu.

0022-0396/97 $25.00

1. INTRODUCTION

Apparently, the method of averaging to compare a non-linear ordinary differential equation

$$\frac{d}{d\tau} z^{\varepsilon}(\tau) = \varepsilon F(z^{\varepsilon}(\tau), \tau) \qquad \text{subject to} \quad z^{\varepsilon}(0) = x_0 \tag{1}$$

for small $\varepsilon > 0$ over intervals like $[0, T/\varepsilon]$ or, equivalently, the time-changed equation

$$\frac{d}{dt} x^{\varepsilon}(t) = F(x^{\varepsilon}(t), t/\varepsilon) \qquad \text{subject to} \quad x^{\varepsilon}(0) = x_0 \tag{2}$$

over compact intervals $[0, T]$, with a time-homogeneous differential equation

$$\frac{d}{dt} x^0(t) = \bar{F}(x^0(t)) \qquad \text{subject to} \quad x^0(0) = x_0 . \tag{3}$$

was first used in celestial mechanics centuries ago. The primary additional regularity justifying such a comparison as $\varepsilon \to 0$ is the ability to "average out" the t-dependence on the right hand side of (1) and the definition

$$\bar{F}(x) \doteq \lim_{t \to \infty} \frac{1}{t} \int_0^t F(x, s)\, ds \qquad \forall x \in \Re^d. \tag{4}$$

However, precise conditions under which x^{ε} converges uniformly over $[0, T]$ to x^0 were not established until the works of Bogoliubov (see [4]), Gikhman [7], and Besjes [3]. Subsequently, averaging principles were extended (using probabilistic methods) by Khas'minskii [10] to second order parabolic partial differential equations (pdes) with the form

$$\partial_t u^{\varepsilon}(x, t) = \sum_{i, j=1}^{d} a_{ij}\left(x, \frac{t}{\varepsilon}\right) \partial_{x_i x_j} u^{\varepsilon}(x, t) + \sum_{i=1}^{d} b_i\left(x, \frac{t}{\varepsilon}\right) \partial_{x_i} u^{\varepsilon}(x, t)$$
$$+ c\left(x, \frac{t}{\varepsilon}\right) u^{\varepsilon}(x, t) + d\left(x, \frac{t}{\varepsilon}\right). \tag{5}$$

More recently, Bensoussan *et al.* [2], Henry [8], and Zhikov *et al.* [13] developed other averaging principles for parabolic pdes and Watanabe [12] initiated investigations of stochastic averaging principles for second order parabolic equations with random coefficients.

In the present note, we extend the theory of (deterministic) averaging for parabolic pdes by developing a theory directly for the derivatives of the fundamental solution of arbitrary-order parabolic equations. For concreteness, we will use the multi-index notation of L. Schwartz (see Section 2 to follow) and consider the $\mathscr{C}^N$-valued system of parabolic equations (for each $\varepsilon > 0$)

$$\partial_t u^\varepsilon(x,t) = \sum_{|k| \leqslant 2p} A_k(x, t/\varepsilon)\, \partial_x^k u^\varepsilon(x,t) + f^\varepsilon(x,t)$$
$$\text{subject to} \quad u^\varepsilon(x,0) = \varphi^\varepsilon(x) \tag{6}$$

with the limit equation

$$\partial_t u(x,t) = \sum_{|k| \leqslant 2p} A_k^0(x)\, \partial_x^k u(x,t) + f(x,t)$$
$$\text{subject to} \quad u(x,0) = \varphi(x). \tag{7}$$

Then, under general regularity conditions on the coefficients (see Theorem A of Section 2) fundamental solutions Γ^ε and Γ exist for (6) respectively (7), and, furthermore,

$$u^\varepsilon(x,t) \doteq \int_{\mathfrak{R}^d} \Gamma^\varepsilon(x,t;\xi,0)\, \varphi^\varepsilon(\xi)\, d\xi - \int_0^t \int_{\mathfrak{R}^d} \Gamma^\varepsilon(x,t;\xi,\tau)\, f^\varepsilon(\xi,\tau)\, d\xi\, d\tau \tag{8}$$

and

$$u(x,t) \doteq \int_{\mathfrak{R}^d} \Gamma(x,t;\xi,0)\, \varphi(\xi)\, d\xi - \int_0^t \int_{\mathfrak{R}^d} \Gamma(x,t;\xi,\tau)\, f(\xi,\tau)\, d\xi\, d\tau \tag{9}$$

are continuous solutions to respectively (6) and (7). By bounding the difference

$$\partial_x^k [\Gamma^\varepsilon(x,t;\xi,\tau) - \Gamma(x,t;\xi,\tau)] \ \forall x, \xi \in \mathfrak{R}^d,$$
$$0 \leqslant \tau \leqslant t \leqslant 1,\ \varepsilon > 0,\ |k| < 2p; \tag{10}$$

and making use of classical bounds for $\partial_x^k \Gamma^\varepsilon$, $|k| < 2p$, $\varepsilon > 0$ (see Theorem A of Section 2); our approach allows one to compare readily

$$\begin{aligned}
&\partial_x^k u^\varepsilon(x,t) - \partial_x^k u(x,t) \\
&\quad = \int_{\mathfrak{R}^d} \partial_x^k \Gamma^\varepsilon(x,t;\xi,0)[\varphi^\varepsilon(\xi) - \varphi(\xi)]\, d\xi \\
&\qquad + \int_{\mathfrak{R}^d} \partial_x^k [\Gamma^\varepsilon(x,t;\xi,0) - \Gamma(x,t;\xi,0)]\, \varphi(\xi)\, d\xi \\
&\qquad - \int_0^t \int_{\mathfrak{R}^d} \partial_x^k \Gamma^\varepsilon(x,t;\xi,\tau)[f^\varepsilon(\xi,\tau) - f(\xi,\tau)]\, d\xi\, d\tau \\
&\qquad - \int_0^t \int_{\mathfrak{R}^d} \partial_x^k [\Gamma^\varepsilon(x,t;\xi,\tau) - \Gamma(x,t;\xi,\tau)]\, f(\xi,\tau)\, d\xi\, d\tau
\end{aligned} \tag{11}$$

simultaneously for all $|k|<2p$ without any apriori constraints on φ, φ^ε, f, f^ε. Moreover, in the stochastic setting $A_k(x, t/\varepsilon)$ whence $\Gamma^\varepsilon(x, t; \xi, \tau)$ will not only be ε-dependent but also random. However, $A_k^0(x)$ and $\Gamma(x, t; \xi, \tau)$ will remain non-random and the bounds on (10) applied almost surely will permit replacing the random ε-dependent kernel $\Gamma^\varepsilon(x, t; \xi, \tau)$ with the non-random, ε-homogeneous, averaged kernel $\Gamma(x, t; \xi, \tau) = \Gamma(x, t-\tau, \xi)$ in problems of stochastic averaging. This approach has been employed in Dawson and Kouritzin [5].

In many applications the pdes of interest will not immediately have the desired form but rather will satisfy equations like

$$\partial_\tau v^\varepsilon(x, \tau) = \varepsilon \sum_{|k| \leqslant 2p} A_k(x, \tau)\, \partial_x^k v^\varepsilon(x, \tau) + g^\varepsilon(x, \tau)$$

$$\text{subject to} \quad v^\varepsilon(x, 0) = \varphi^\varepsilon(x). \tag{12}$$

However, (6) can easily be recovered via the substitutions $t = \tau\varepsilon$, $u^\varepsilon(x, t) \doteq v^\varepsilon(x, t/\varepsilon)$ and $f^\varepsilon(x, t) \doteq (1/\varepsilon)\, g^\varepsilon(x, t/\varepsilon)$. Alternatively, in other applications the original pdes may have higher order derivatives in t. However, by introducing new variables (see e.g. pp. 238–9 of Friedman [6]) these equations can often be reduced to the case considered here.

Our proof will utilize several well-known bounds for fundamental solutions, introduce supplementary equations where the x-dependence of the coefficients in (6) and (7) is replaced with an auxiliary parameter, and adhere to the long-established parametrix method. Therefore, our development will retain many similarities to the classical literature for parabolic pdes. On the other hand, our proof is not short of novelties. For instance, through modest use of analysis and pde theory we reduce our problem to that of establishing convergence for certain objects (defined in (38) and (47) of Subsection 3.1) as $\varepsilon \to 0$ on spaces of continuous functions with unbounded domain. Relative compactness for these objects is then established by imposing only a slightly strengthened version of the regularity conditions required for existence of our fundamental solutions to (6) and (7). It is only then that we will require our basic hypothesis that

$$\left\| \int_0^t A_k\left(y, \frac{\varepsilon}{s}\right) - A_k^0(y)\, ds \right\| \to 0 \qquad \text{as} \quad \varepsilon \to 0, \tag{13}$$

for each $y \in \Re^d$ and $t \in [0, T]$ (i.e., pointwise) to show that the only possible limit for either object in (38) or (47) is 0. Moreover, the classical conditions for existence of fundamental solutions to (6) (for each ε) which are uniformly bounded in ε and thereby useful for our problem would require an assumption like:

(A) For each $|k| = 2p$, (the principle coefficient) $A_k(x, t/\varepsilon)$ is continuous in t uniformly with respect to $(x, t, \varepsilon) \in \mathfrak{R}^d \times [0, \infty) \times (0, 1]$.

Obviously, this condition would not allow our principle coefficients $\{A_k\}_{|k|=2p}$ to depend on t or ε and our work would result in a rather uninteresting averaging theory. Therefore, we eschew this condition entirely and instead show (in Lemma 5 of Subsection 3.4) that the classical theory still holds without Assumption (A) if one imposes a slightly stronger uniform-parabolic-type condition than is customary.

Our note is organized as follows: Section 2 contains the notation and conditions required to state and prove our result as well as the result itself and some motivation for its use. The proof of this result is first sketched in Subsection 3.1 and then proved in Subsections 3.2 and 3.3. To avoid complicating the proof unnecessarily several subsidiary lemmas have been placed in Subsection 3.4. *The reader may find it beneficial to keep a separate copy of Subsection* 3.1 *handy while reading Subsections* 3.2, 3.3, *and* 3.4.

2. NOTATION, CONDITIONS, AND RESULT

Throughout this note; p, N, and d are fixed positive integers; and $|\cdot|$ denotes absolute value as well as modulus. For technical reasons it will be most convenient to define our norms on $\mathscr{C}^N$ and $\mathscr{C}^d$ via

$$|\zeta| \doteq \left[\sum_{j=1}^{N} |\zeta_j|^r\right]^{1/r} \quad \text{and} \quad |x| \doteq \left[\sum_{j=1}^{d} |x_j|^r\right]^{1/r}, \quad r = \frac{2p}{2p-1} \tag{14}$$

for all $\zeta \in \mathscr{C}^N$ and $x \in \mathscr{C}^d$. Then, $\|\cdot\|$ will be used for the $|\cdot|$ −induced norm for $\mathscr{C}^{N\times N}$ matrices. Moreover, for vectors $k = (k_1, k_2, ..., k_d)$ of non-negative integers, we define

$$|k| \doteq k_1 + k_2 + \cdots + k_d \tag{15}$$

and let "$\sum_{|k|\leqslant 2p}$" denote the summation over all possible d-tuples k of non-negative integers such that $|k| \leqslant 2p$. (It will always be clear from the context whether $|\cdot|$ is being used as absolute value, modulus, norm in $\mathscr{C}^N$, norm in $\mathscr{C}^d$, or the sum of non-negative integers). Next, letting $e_i \doteq (0, ..., 0, 1, 0, ..., 0)^T \in \mathfrak{R}^d$ with the 1 in the ith row, and k be as above, we define

$$\partial_x^k \doteq \partial_{x_1}^{k_1} \partial_{x_2}^{k_2} \cdots \partial_{x_d}^{k_d} \ \forall \ x \in \mathfrak{R}^d \tag{16}$$

$$d_x \doteq \partial_{x_1} e_1 + \partial_{x_2} e_2 + \cdots + \partial_{x_d} e_d \ \forall \ x \in \mathfrak{R}^d. \tag{17}$$

Likewise, for any vector $\zeta \in \mathscr{C}^d$ and d-tuple of non-negative integers k, we define

$$\zeta^k \doteq (\zeta_1)^{k_1} (\zeta_2)^{k_2} \cdots (\zeta_N)^{k_d}. \tag{18}$$

Finally, Re A and Im A will denote the real and imaginary parts of a complex matrix A, $a \vee b$ and $a \wedge b$ will be used to denote the maximum respectively minimum of two real numbers a, b, and $a_{m,n} \ll^{n,m} b_{m,n}$ will imply that there is a constant $c > 0$ such that $|a_{m,n}| \leqslant c\,|b_{m,n}|$ for all n, m. The latest notation is a natural extension to the Vinogradov symbol.

The following Conditions will be assumed throughout this note:

(C1) The system (6) is uniformly parabolic in the sense that

$$-\sup_{t \geqslant 0} \sup_{x \in \Re^d} \max_j \sup_{|\xi| = 1} \lambda_j(\xi; x, t) > 0, \tag{19}$$

where $\{\lambda_j(\xi; x, t)\}_{j=1}^{2N}$ are the (real) roots of the polynomial

$$\det\left(\sum_{|k|=2p} \begin{bmatrix} \mathrm{Re}[A_k(x,t) + A_k^T(x,t)] & -\mathrm{Im}[A_k(x,t) - A_k^T(x,t)] \\ \mathrm{Im}[A_k(x,t) - A_k^T(x,t)] & \mathrm{Re}[A_k(x,t) + A_k^T(x,t)] \end{bmatrix} \times (i\xi)^k - \lambda I_{2N} \right) \tag{20}$$

for all ξ, $x \in \Re^d$, and $t \geqslant 0$, I_{2N} being the identity matrix in $\Re^{2N \times 2N}$.

(C2) (7) is uniformly parabolic in the sense that

$$-\sup_{x \in \Re^d} \max_l \sup_{|\xi|=1} \mathrm{Re}\{\lambda_l^0(\xi; x)\} > 0, \tag{21}$$

where $\{\lambda_l^0(\xi; x)\}_{l=1}^{N}$ are the roots of the polynomial

$$\det\left(\sum_{|k|=2p} A_k^0(x)(i\xi)^k - \lambda I_N \right) \tag{22}$$

for all ξ, $x \in \Re^d$, I_N being the identity matrix in $\mathscr{C}^{N \times N}$.

(C3) For each $|k| \leqslant 2p$: (i) A_k is continuous in t over $[0, \infty)$, and (ii) A_k and A_k^0 are uniformly bounded in $\Re^d \times [0, \infty)$ respectively $\Re^d$.

(C4) For each $|k| \leqslant 2p$, $\partial_{x_i} A_k$ and $\partial_{x_i} A_k^0$ exist and are uniformly bounded in $\Re^d \times [0, \infty)$ respectively $\Re^d$ for $i = 1, ..., d$.

(C5) When $|k| = 2p$, $\partial_{x_i} A_k$ and $\partial_{x_i} A_k^0$ are Hölder continuous in x with exponent $0 < \varsigma \leqslant 1$ uniformly in $\Re^d \times [0, \infty)$ respectively $\Re^d$ for $i = 1, ..., d$.

The following theorem is a variation on Theorems 2 and 3 in Chapter 9 of Friedman [6]. It can be proved by combining Lemma 5 of Subsection 3.4 herein with the proofs of said theorems on pp. 251–257 of [6]. Actually, this theorem would still hold under a weaker version of (C4).

THEOREM A. *Suppose Regularity Conditions* (C1–C4) *hold. Then, there exist* (*forward*) *fundamental solutions* Γ^ε *and* Γ *to the equations*

$$\partial_t z^\varepsilon(x,t) = \sum_{|k| \leqslant 2p} A_k\left(x, \frac{t}{\varepsilon}\right) \partial_x^k z^\varepsilon(x,t)$$

and

$$\partial_t z(x,t) = \sum_{|k| \leqslant 2p} A_k^0(x)\, \partial_x^k z(x,t). \tag{23}$$

Moreover, these fundamental solutions satisfy

$$\|\partial_x^b \Gamma^\varepsilon(x,t;\xi,\tau)\| \vee \|\partial_x^b \Gamma(x,t;\xi,\tau)\| \leqslant \frac{C}{|t-\tau|^{(d+|b|)/2p}} \exp\left[-c\left|\frac{|x-\xi|^{2p}}{t-\tau}\right|^{1/(2p-1)}\right] \tag{24}$$

with constants $C, c > 0$ (*depending only on the constants in* (C1–C4) *and, in particular, independent of* $\varepsilon > 0$) *for all* $|b| < 2p$, $\varepsilon \in (0,1]$, $0 \leqslant \tau \leqslant t \leqslant T$ *and* $x, \xi \in \Re^d$. *Finally, suppose* f^ε, f *are continuous, bounded functions on* $\Re^d \times [0,T]$ *and* φ^ε, φ *are continuous, bounded functions on* $\Re^d$. *Then, there exist continuous, bounded solutions to* (6) *and* (7) *on* $\Re^d \times [0,T]$ *which are given by*

$$u^\varepsilon(x,t) \doteq \int_{\Re^d} \Gamma^\varepsilon(x,t;\xi,0)\, \varphi^\varepsilon(\xi)\, d\xi - \int_0^t \int_{\Re^d} \Gamma^\varepsilon(x,t;\xi,\tau)\, f^\varepsilon(\xi,\tau)\, d\xi\, d\tau \tag{25}$$

$$u(x,t) \doteq \int_{\Re^d} \Gamma(x,t;\xi,0)\, \varphi(\xi)\, d\xi - \int_0^t \int_{\Re^d} \Gamma(x,t;\xi,\tau)\, f(\xi,\tau)\, d\xi\, d\tau. \tag{26}$$

We now state the main result in this note which compliments Theorem A and is an averaging principle for derivatives of such fundamental solutions. The phrase "slightly strengthened version of the regularity conditions" in the second last paragraph of our introduction refers to the fact that whereas Theorem A holds without (C5) and with (C4) replaced by Hölder continuity, Theorem 1 below requires (C4) and (C5) as stated above.

THEOREM 1. *Suppose the Regularity Conditions* (C1–C5) *hold and for each* $y \in \Re^d$ *and each* $t \in (0, T]$ *we have that*

$$\left\| \int_0^t A_k\left(y, \frac{s}{\varepsilon}\right) - A_k^0(y)\, ds \right\| \to 0 \qquad \text{as} \quad \varepsilon \to 0. \tag{27}$$

Then, for any $0 < \chi,\ \nu < 1$ *it follows that there exists a positive constant* $\tilde{c} = \tilde{c}_{\chi, \nu}$ *and a* $\Re$*-valued function* $\gamma(\cdot) = \gamma_{\chi, \nu}(\cdot)$ *satisfying* $\lim_{\varepsilon \to 0} \gamma(\varepsilon) = 0$ *such that*

$$\|\partial_x^b [\Gamma^\varepsilon(x, t; \xi, \tau) - \Gamma(x, t; \xi, \tau)]\| \leqslant \frac{\gamma(\varepsilon)\, |1 + |\xi|^2|^\nu}{(t-\tau)^{(d + |b| + \chi)/2p}} \exp\left[-\tilde{c} \left| \frac{|x-\xi|^{2p}}{t-\tau} \right|^{1/(2p-1)} \right] \tag{28}$$

for all $0 \leqslant |b| < 2p$, $\varepsilon \in (0, 1]$, $0 \leqslant \tau \leqslant t \leqslant T$ *and* $x, \xi \in \Re^d$, *where* Γ^ε *and* Γ *are the fundamental solutions introduced in Theorem A above.*

Remark 1. By comparing the right hand side of (28) to that of (24), one can see that they are of the same form with the exception that (28) has the extra multiplicative term

$$\gamma(\varepsilon)\, |1 + |\xi|^2|^\nu\, (t-\tau)^{-\chi/2p}. \tag{29}$$

Whereas the first factor in (29) establishes that $\partial_x^b \Gamma^\varepsilon$ will approach $\partial_x^b \Gamma$ as $\varepsilon \to 0$ the remaining two factors are required to allow a uniform result over all $0 \leqslant \tau \leqslant t \leqslant T$ and $x, \xi \in \Re^d$. Indeed, it can be seen from the proof in the sequel that these factors can be replaced by other functions that grow even slower as $|\xi| \to \infty$ and $t - \tau \to 0$. The only motivation for the present factors was to simplify the right hand side of (28).

Remark 2. Of course, it is sufficient by continuity for (27) to hold in a dense subset of $\Re^d \times [0, T]$.

As we mentioned in the introduction, Theorem 1 can immediately be used to compare $\partial_x^k u^\varepsilon$ to $\partial_x^k u$ whereas several additional assumptions would be required to use, for example, Exercise 1 and Theorem 7.5.2 pp. 218–221 of Henry [8]. Most critically; k, N, d, and p would have to be respectively 0, 1, 1, and 1; f^ε and f would both have to be 0; the principle coefficients could not depend on ε; and the limits in (27) would have to be replaced by more restrictive uniform limits. In fact, there are several other differences between Henry's work and ours. However, the four points mentioned previously are enough to distinguish our work from all other work on averaging for parabolic equations.

Both equivalent formulations (6) and (12) for our original system of parabolic equations suggest that our results provide a means to find an appropriate solution to a more complicated partial differential equation via a simpler one. For example, our condition (27) is equivalent to

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T A_k(y,\tau)\,d\tau = A_k^0(y) \qquad \forall y \in \mathfrak{R}^d \tag{30}$$

and our coefficients form Krylov–Bogoliubov–Mitropolsky vector fields (see Sanders and Verhulst [11]). Hence, $\partial_x^k v^\varepsilon(x,\tau)$ of our system (12) with almost-periodic coefficients (see p. 55 of [11]) can be approximated uniformly by $\partial_x^k u(x,\varepsilon\tau)$ over $\mathfrak{R}^d \times [0, T\varepsilon^{-1}]$. However, probing this application further and investigating a simple $\mathfrak{R}$-valued Itô stochastic differential equation

$$dX_t^\varepsilon = m(X_t^\varepsilon, \xi_{t/\varepsilon})\,dt + \sigma(\xi_{t/\varepsilon})\,dW_t, \qquad X_0^\varepsilon = \rho, \tag{31}$$

where $\{W_t,\ t \geqslant 0\}$ is a standard Brownian motion, $m\colon \mathfrak{R}^2 \to \mathfrak{R}$, $\sigma\colon \mathfrak{R} \to \mathfrak{R}$ are non-linear functions and $\{\xi_t,\ t \geqslant 0\}$ is a (non-anticipative) stochastic process, we find under certain conditions that the law of X_t given $\sigma\{\xi_{s/\varepsilon}, 0 \leqslant s \leqslant t\}$ has a density

$$\begin{aligned}\partial_t p^\varepsilon_{X\mid\xi}(x,t) &= \tfrac{1}{2}\sigma^2(\xi_{t/\varepsilon})\,\partial_x^2 p^\varepsilon_{X\mid\xi}(x,t) - m(x,\xi_{t/\varepsilon})\,\partial_x p^\varepsilon_{X\mid\xi}(x,t)\\ &\quad - \partial_x m(x,\xi_{t/\varepsilon})\,p^\varepsilon_{X\mid\xi}(x,t), \\ p^\varepsilon_{X\mid\xi}(x,0) &= p_\rho(x).\end{aligned} \tag{32}$$

Thus, our results also provide a means of approximating the law of a controlled diffusion (31) under the mild ergodic conditions that there exist a^0, m^0, and b^0 such that

$$\begin{aligned}&\int_0^t \sigma^2(\xi_{\tau/\varepsilon}) - a^0\,d\tau, \qquad \int_0^t m(x,\xi_{\tau/\varepsilon}) - m^0(x)\,d\tau,\\ &\int_0^t \partial_x m(x,\xi_{\tau/\varepsilon}) - b^0(x)\,d\tau \xrightarrow{\varepsilon\to 0} 0\end{aligned} \tag{33}$$

pointwise for all t, x in a dense set of $[0, T] \times \mathfrak{R}$. This idea can, of course, be extended to more complicated, multi-dimensional diffusions and recent work (see e.g. Hochberg and Orsingher [9]) connecting densities of certain stochastic processes with higher order linear parabolic equations suggests that similar applications of our results with higher order parabolic equations may be in the offing. Still, our personal motivation for our results was applications in stochastic partial differential equations and the stochastic averaging of parabolic equations with random coefficients for which we refer the reader to Dawson and Kouritzin [5].

3. PROOF OF THEOREM 1

Inasmuch as the value of T does not change the following proof in any significant way, we will take $T=1$ in the sequel. Moreover, to ease the notation in the following proof we define

$$\tilde{A}_k(y,s) \doteq A_k(y,s) - A_k^0(y), \qquad q \doteq 2p, \tag{34}$$

and (cf. Conditions (C1–C2))

$$\delta \doteq -\tfrac{1}{2}\{\sup_{t\geqslant 0}\ \sup_{x\in\mathfrak{R}^d}\ \max_{j,l}\ \sup_{|\xi|=1}\ [\lambda_j(\xi;x,t) \vee \mathrm{Re}\{\lambda_l^0(\xi;x)\}\,]\} > 0. \tag{35}$$

3.1. *Sketch of Proof.*

Our proof relies heavily on the classical theory summarized in Chapters 1 and 9 of Friedman [6]. Indeed, we will follow the general plan outlined there by first, in Subsection 3.2, using Fourier transform techniques to establish bounds for the fundamental solutions of

$$\partial_t v^\varepsilon(x,t,y) = \sum_{|k|=q} A_k\left(y, \frac{s}{\varepsilon}\right) \partial_x^k v^\varepsilon(x,t,y) \tag{36}$$

for all $y \in \mathfrak{R}^d$. To make our presentation manifest, suppose Z^ε and Z denote the (forward) fundamental solutions to (36) and

$$\partial_t v(x,t,y) = \sum_{|k|=q} A_k^0(y)\, \partial_x^k v(x,t,y) \tag{37}$$

and V^ε respectively V denote the Fourier transforms of Z^ε and Z. Furthermore, suppose $\lambda_1 > 0$ is a constant whose value will be fixed later and $\delta_1 \in (0,\delta)$ with δ as in (35). Then, Subsection 3.2 follows the following outline:

(i) Using only the Regularity Conditions (C1–5), show that $(t,\tau;y,\zeta) \to \phi^\varepsilon(t,\tau;y,\zeta)$, $\varepsilon>0$ are appropriately bounded and equicontinuous, where

$$\phi^\varepsilon(t,\tau;y,\zeta) \doteq \psi^\varepsilon(t,\tau;y,\zeta)(1+|y|^2)^{-\nu} \tag{38}$$

$$\psi^\varepsilon(t,\tau;y,\zeta) \doteq \eta^\varepsilon(t,\tau;y,\zeta)\exp\{[\delta_1|\alpha|^q - \lambda_1|\beta|^q](t-\tau)\}(t-\tau)^{\chi-1} \tag{39}$$

and

$$\eta^{\varepsilon}(t, \tau; y, \zeta) \doteq \int_{\tau}^{t} V^{\varepsilon}(t, s; y, \zeta) \sum_{|k|=q} \tilde{A}_k\left(y, \frac{s}{\varepsilon}\right) V(s, \tau; y, \zeta)\, ds \tag{40}$$

for all $\varepsilon > 0$, $0 \leqslant \tau \leqslant t \leqslant 1$, $\zeta \doteq \alpha + i\beta \in \mathscr{C}^d$ and $y \in \mathfrak{R}^d$.

(ii) Use (i) to show that $\{\phi^{\varepsilon}\}_{\varepsilon>0}$ is relatively compact in a space of continuous, bounded $\mathscr{C}^{N\times N}$-valued functions with unbounded domain. Since this space will be complete the argument reduces to showing $\{\phi^{\varepsilon}\}_{\varepsilon>0}$ is totally bounded which is done by showing restrictions onto compact sets are totally bounded and then convolving a finite $(\eta/4)$-net for such restrictions with "nice" kernels to produce an appropriate net for the unrestricted functions.

(iii) Next, our main hypothesis (27) is used to show that the set of limit points (as $\varepsilon \to 0$) for ϕ^{ε} is the single point 0. The convergence of ϕ^{ε} to 0 implied by (ii) and (iii) will yield a desirable bound on $\eta^{\varepsilon}(t, \tau; y, \zeta)$.

(iv) Then, a variation-of-constants-based argument is used to show that $V^{\varepsilon} - V$ is bounded in terms of η^{ε} from which it follows by taking inverse Fourier transforms and applying Cauchy's integral theorem that for any d-vector of non-negative integers b there exists a constant $c = c_{b,\chi,\nu} > 0$ and a function $\gamma_1(\cdot) = \gamma_{1,b,\chi,\nu}(\cdot)$ independent of (x, t, ξ, τ, y) and satisfying $\lim_{\varepsilon\to 0} \gamma_1(\varepsilon) = 0$ such that

$$\begin{aligned} &\|\partial_x^b [Z^{\varepsilon}(x-\xi, t, y, \tau) - Z(x-\xi, t, y, \tau)]\| \\ &\quad \leqslant \frac{\gamma_1(\varepsilon)}{(t-\tau)^{(d+|b|+\chi)/q}} \exp\left[-c\left|\frac{|x-\xi|^q}{t-\tau}\right|^{1/(q-1)}\right] (1+|y|^2)^{\nu} \end{aligned} \tag{41}$$

for all $\varepsilon \in (0, 1]$, $x, y, \xi \in \mathfrak{R}^d$, and $0 \leqslant \tau \leqslant t \leqslant 1$.

In Subsection 3.3, we also follow the classical theory somewhat by using the parametrix method to develop our desired bounds between the fundamental solutions of (6) and (7).

(v) Initially, we follow the program outlined in (i–iii) above. However, we must redefine ϕ^{ε} which requires some notation. First, we fix constants $c, C > 0$ such that (41) with $0 \leqslant |b| \leqslant q$ as well as the classical-type bounds in Lemma 5(i–iv) of Subsection 3.4 hold. Secondly, we fix vectors of non-negative integers m, k such that $0 \leqslant |m| \leqslant q$, $0 \leqslant |k| \leqslant q$, make the simplifying definitions

$$A_m^0(x, y) \doteq A_m^0(x) - A_m^0(y), \qquad \tilde{A}_k\left(y, \xi, \frac{s}{\varepsilon}\right) \doteq \tilde{A}_k\left(y, \frac{s}{\varepsilon}\right) - \tilde{A}_k\left(\xi, \frac{s}{\varepsilon}\right), \tag{42}$$

$$\hat{Z}(x, t; y, s) \doteq Z(x-y, t, y, s), \qquad \hat{Z}^{\varepsilon}(x, t; y, s) \doteq Z^{\varepsilon}(x-y, t, y, s) \tag{43}$$

(with Z and Z^ε as in the previous paragraph and also defined in (83) and (82) of Subsection 3.2), and define $\eta^\varepsilon = \eta^\varepsilon_{m,k}(x, t; y, s; \xi, \tau)$ by

$$\eta^\varepsilon \doteq \begin{cases} \partial_x^m \hat{Z}(x, t; y, s)\, \tilde{A}_k\left(y, \frac{s}{\varepsilon}\right) \partial_y^k \hat{Z}(y, s; \xi, \tau) \\ \qquad |m|, |k| < q \\ \partial_x^m \hat{Z}(x, t; y, s)\, \tilde{A}_k\left(y, \xi, \frac{s}{\varepsilon}\right) \partial_y^k \hat{Z}(y, s; \xi, \tau) \\ \qquad |m| < |k| = q \\ A_m^0(x, y)\, \partial_x^m \hat{Z}(x, t; y, s)\, \tilde{A}_k\left(y, \frac{s}{\varepsilon}\right) \partial_y^k \hat{Z}(y, s; \xi, \tau) \\ \qquad |k| < |m| = q \\ A_m^0(x, y)\, \partial_x^m \hat{Z}(x, t; y, s)\, \tilde{A}_k\left(y, \xi, \frac{s}{\varepsilon}\right) \partial_y^k \hat{Z}(y, s; \xi, \tau) \\ \qquad |m|, |k| = q. \end{cases} \tag{44}$$

Furthermore, for all $x,\ y,\ \xi \in \Re^d$ and $0 \leqslant \tau \leqslant s \leqslant t$ we define $Y^\varepsilon = Y^\varepsilon_{m,k}(x, t; y, s; \xi, \tau)$ by

$$Y^\varepsilon \doteq \eta^\varepsilon_{m,k}(x, t; y, s; \xi, \tau) \exp\left\{c_1 \left| \frac{|x-\xi|^q}{t-\tau} \right|^{1/(q-1)}\right\} (t-\tau)^{(d+|m| \wedge (q-\varsigma))/q}, \tag{45}$$

where ς is as in Condition (C5) and $0 < c_1 < c$ is a constant, let

$$\begin{aligned} \psi^\varepsilon(x, t; \xi, \tau) &= \int_\tau^{(\tau+t)/2} \int_{\Re^d} Y^\varepsilon(x, t; y, s; \xi, \tau)\, dy\, ds \\ &\quad + \int_{(\tau+t)/2}^t \int_{\Re^d} Y^\varepsilon(x, t; y, s; \xi, \tau)\, dy\, ds \\ &\doteq \psi_1^\varepsilon(x, t; \xi, \tau) + \psi_2^\varepsilon(x, t; \xi, \tau), \end{aligned} \tag{46}$$

and define

$$\phi^\varepsilon_{m,k}(x, t; \xi, \tau) \doteq \psi^\varepsilon_{m,k}(x, t; \xi, \tau)(1+|\xi|^2)^{-\nu}. \tag{47}$$

Then, using only basic vector calculus, we show $\phi^\varepsilon_{m,k}$ converges to 0 in a space of continuous, bounded functions with unbounded domain according to the program in (i–iii) above and conclude that there exists a function $\gamma_2(\cdot)$ independent of $(x, t; \xi, \tau, m, k)$ and satisfying $\lim_{\varepsilon \to 0} \gamma_2(\varepsilon) = 0$ such that

$$\left\| \int_r^t \int_{\mathfrak{R}^d} \eta^{\varepsilon}_{m,k}(x, t; y, s; \xi, \tau)\, dy\, ds \right\| \leqslant \frac{\gamma_2(\varepsilon)\, |1 + |\xi|^2|^{\nu}}{|t-\tau|^{(d+|m| \wedge (q-\varsigma))/q}} \exp\left\{ -c_1 \left| \frac{|x-\xi|^q}{t-\tau} \right|^{1/(q-1)} \right\}. \tag{48}$$

(vi) Next, on the basis of (41), (44), (48), standard bounds (Lemma 5 of Subsection 3.4), and a calculus-based bound (Lemma 4(ii) of Subsection 3.4), one concludes that there exists a function $\gamma_3(\cdot)$ independent of $(x, t; \xi, \tau)$ satisfying $\lim_{\varepsilon \to 0} \gamma_3(\varepsilon) = 0$ such that

$$\left\| \int_\tau^t \int_{\mathfrak{R}^d} \partial_x^m \hat{Z}(x, t; y, s)[K^{\varepsilon}(y, s; \xi, \tau) - K(y, s; \xi, \tau)]\, dy\, ds \right\| \leqslant \frac{\gamma_3(\varepsilon)\, |1 + |\xi|^2|^{\nu}}{(t-\tau)^{(d+|m|)/q}} \exp\left\{ -c_1 \left| \frac{|x-\xi|^q}{t-\tau} \right|^{1/(q-1)} \right\} \tag{49}$$

when $|m| < q$ and

$$\left\| \int_\tau^t \int_{\mathfrak{R}^d} A_m^0(x, y)\, \partial_x^m \hat{Z}(x, t; y, s)[K^{\varepsilon}(y, s; \xi, \tau) - K(y, s; \xi, \tau)]\, dy\, ds \right\| \leqslant \frac{\gamma_3(\varepsilon)\, |1 + |\xi|^2|^{\nu}}{(t-\tau)^{1+(d-\varsigma)/q}} \exp\left\{ -c_1 \left| \frac{|x-\xi|^q}{t-\tau} \right|^{1/(q-1)} \right\} \tag{50}$$

when $|m| = q$, where (as in Friedman [6], p. 252)

$$K^{\varepsilon}(y, s; \xi, \tau) \doteq \sum_{|k|=q} A_k\left(y, \xi, \frac{s}{\varepsilon}\right) \partial_y^k \hat{Z}^{\varepsilon}(y, s; \xi, \tau) + \sum_{|k|<q} A_k\left(y, \frac{s}{\varepsilon}\right) \partial_y^k \hat{Z}^{\varepsilon}(y, s; \xi, \tau) \tag{51}$$

$$K(y, s; \xi, \tau) \doteq \sum_{|k|=q} A_k^0(y, \xi)\, \partial_y^k \hat{Z}(y, s; \xi, \tau) + \sum_{|k|<q} A_k^0(y)\, \partial_y^k \hat{Z}(y, s; \xi, \tau). \tag{52}$$

(vii) Next, defining (see Friedman [6] p. 252 for motivation) Φ, Φ^{ε} via the integral equations

$$\Phi(x, t; \xi, \tau) = K(x, t; \xi, \tau) + \int_{\tau}^{t} \int_{\Re^d} K(x, t; y, s)\, \Phi(y, s; \xi, \tau)\, dy\, ds \tag{53}$$

$$\Phi^{\varepsilon}(x, t; \xi, \tau) = K^{\varepsilon}(x, t; \xi, \tau) + \int_{\tau}^{t} \int_{\Re^d} K^{\varepsilon}(x, t; y, s)\, \Phi^{\varepsilon}(y, s; \xi, \tau)\, dy\, ds, \tag{54}$$

fixing (ξ, τ), and letting

$$u^{\varepsilon}(x, t) \doteq \left\| \int_{\tau}^{t} \int_{\Re^d} K(x, t; y, s)[\Phi^{\varepsilon}(y, s; \xi, \tau) - \Phi(y, s; \xi, \tau)]\, dy\, ds \right\|, \tag{55}$$

we can use (49) and (50) to bound u^{ε} recursively (in terms of $u^{\varepsilon}(x, s)$ for $s \leqslant t$) and then expand this using a little operator theory to get a non-recursive bound. It then follows immediately from (49), classical bounds for Φ^{ε}, $\partial_x^b \hat{Z}$ (Lemma 5), this bound for u^{ε}, and the triangle inequality that there are constants $\tilde{c}$, $\tilde{C} > 0$ with $\tilde{c} < c_1$ such that

$$\begin{aligned} &\left\| \int_{\tau}^{t} \int_{\Re^d} \partial_x^b \hat{Z}(x, t; y, s)[\Phi^{\varepsilon}(y, s; \xi, \tau) - \Phi(y, s; \xi, \tau)]\, dy\, ds \right\| \\ &\quad \leqslant \frac{\tilde{C} \gamma_3(\varepsilon)\, |1 + |\xi|^2|^{\nu}}{(t-\tau)^{(d+|b|)/q}} \exp\left\{ -\tilde{c} \left| \frac{|x-\xi|^q}{t-\tau} \right|^{1/(q-1)} \right\} \qquad \forall 0 \leqslant |b| \leqslant q. \end{aligned} \tag{56}$$

(viii) Finally, noting (again see Friedman [6] p.252 for motivation) that

$$\Gamma(x, t; \xi, \tau) \doteq \hat{Z}(x, t; \xi, \tau) + \int_{\tau}^{t} \int_{\Re^d} \hat{Z}(x, t; y, s)\, \Phi(y, s; \xi, \tau)\, dy\, ds \tag{57}$$

respectively

$$\Gamma^{\varepsilon}(x, t; \xi, \tau) \doteq \hat{Z}^{\varepsilon}(x, t; \xi, \tau) + \int_{\tau}^{t} \int_{\Re^d} \hat{Z}^{\varepsilon}(x, t; y, s)\, \Phi^{\varepsilon}(y, s; \xi, \tau)\, dy\, ds \tag{58}$$

form our fundamental solutions for (7) and (6), one finds from the triangle inequality, (41), a classical bound for Φ^{ε} (Lemma 5), Lemma 4(ii), and (56) that

$$\begin{aligned} &\|\partial_x^b [\Gamma^{\varepsilon}(x, t; \xi, \tau) - \Gamma(x, t; \xi, \tau)]\| \\ &\quad \leqslant \frac{\gamma(\varepsilon)\, |1 + |\xi|^2|^{\nu}}{(t-\tau)^{(d+|b|+\chi)/q}} \exp\left\{ -\tilde{c} \left| \frac{|x-\xi|^q}{t-\tau} \right|^{1/(q-1)} \right\} \end{aligned} \tag{59}$$

for some γ as in the statement of Theorem 1 and all $|b|<q$, $\varepsilon \in (0, 1]$, $0 \leqslant \tau \leqslant t \leqslant 1$, and x, $\xi \in \mathfrak{R}^d$ as desired.

For expository reasons we have relegated much of the details in Subsections 3.2 and 3.3 to subsidiary lemmas in Subsection 3.4.

3.2. *Bounds for the Fundamental Solutions of* (36)

We start by using Fourier transform methods and defining V^ε and V by

$$V^\varepsilon(t, \tau; y, \zeta) = I + \int_\tau^t \sum_{|k|=q} A_k\left(y, \frac{s}{\varepsilon}\right) (i\zeta)^k \, V^\varepsilon(s, \tau; y, \zeta) \, ds \tag{60}$$

and

$$V(t, \tau; y, \zeta) = \exp\left\{ \sum_{|k|=q} A_k^0(y)(i\zeta)^k \, (t-\tau) \right\} = V(t-\tau; y, \zeta)$$

so

$$\begin{aligned} V(t, \tau; y, \zeta) &= I + \int_\tau^t \sum_{|k|=q} A_k^0(y)(i\zeta)^k \, V(s-\tau; y, \zeta) \, ds \\ &= I + \int_\tau^t \sum_{|k|=q} A_k^0(y)(i\zeta)^k \, V(t-s; y, \zeta) \, ds \end{aligned} \tag{61}$$

for all $0 \leqslant \tau \leqslant t \leqslant 1$, $\zeta \in \mathscr{C}^d$ and $y \in \mathfrak{R}^d$. Now, letting $\zeta = \alpha + i\beta$, and recalling the definition of δ from (35), one finds by Lemma 5 of Subsection 3.4 that

$$\| V^\varepsilon(t, s; y, \zeta) \| \overset{\varepsilon, t, s, y, \zeta}{\ll} \exp\{ [\lambda \, |\beta|^q - \delta \, |\alpha|^q](t-s) \} \tag{62}$$

and

$$\| V(t, s; y, \zeta) \| \overset{s, \tau, y, \zeta}{\ll} \exp\{ [\lambda \, |\beta|^q - \delta \, |\alpha|^q](s-\tau) \} \tag{63}$$

for some $\lambda > 0$ and all $0 \leqslant \tau \leqslant s \leqslant t \leqslant 1$, $\zeta \in \mathscr{C}^d$ and $y \in \mathfrak{R}^d$. Next, fixing a $\lambda_1 > \lambda$, recalling the definition of ψ^ε from (39), and defining

$$\bar{\delta} \doteq \delta - \delta_1, \qquad \bar{\lambda} \doteq \lambda_1 - \lambda, \tag{64}$$

we find from Lemma 6 of Subsection 3.4 that $(t, \tau; y, \zeta) \to \psi^\varepsilon(t, \tau; y, \zeta)$, $\varepsilon \in (0, 1]$ are equicontinuous and

$$\|\psi^\varepsilon(t, \tau; y, \zeta)\| \overset{\varepsilon, t, \tau, y, \zeta}{\ll} \exp\{ -[\bar{\lambda} \, |\beta|^q + \bar{\delta} \, |\alpha|^q](t-\tau) \} (t-\tau)^\chi \tag{65}$$

for all $\varepsilon > 0$, $0 \leqslant \tau \leqslant t \leqslant 1$, $\zeta \doteq \alpha + i\beta \in \mathscr{C}^d$ and $y \in \mathfrak{R}^d$.

Now, we recall definition (38) and show that $\{\phi^{\varepsilon},\ 0<\varepsilon\leqslant 1\}$ is totally bounded in the Banach space of continuous, bounded $\mathscr{C}^{N\times N}$-valued functions

$$(C_B(\Delta), \sup_{\Delta} \|\cdot\|), \qquad \Delta \doteq \{(t, \tau, y, \zeta)\in[0,1]^2\times\Re^d\times\mathscr{C}^d : \tau\leqslant t\}. \tag{66}$$

Indeed, letting $\eta>0$ be an arbitrary positive constant and availing ourselves of (65) and (38), we can find a $C=C_\eta>1$ such that $\|\phi^{\varepsilon}(t,\tau,y,\zeta)\|\leqslant\eta/4$ for all $\varepsilon>0$, $0\leqslant\tau\leqslant t\leqslant 1$, $\zeta=\alpha+i\beta\in\mathscr{C}^d$ and $y\in\Re^d$ such that $\max_{j=1,2,...,d}|y^j|\vee|\alpha^j|\vee|\beta^j|>C-1$. Now, for each $r=0, \frac{1}{2}, 1$, we define

$$\Lambda^r \doteq \{(y,\alpha,\beta)\in\Re^{3d} : |y^j|\vee|\alpha^j|\vee|\beta^j|\leqslant C-r\ \forall j=1,2,...,d\} \tag{67}$$

$$\Omega^r \doteq \{(t,\tau,y,\alpha+i\beta)\in\Delta : (y,\alpha,\beta)\in\Lambda^r\} \tag{68}$$

and consider the restrictions, $\phi^{\varepsilon}|_{\Omega^0}$, of ϕ^{ε} to Ω^0. Clearly, Ω^0 is a compact subset and $\{\phi^{\varepsilon}|_{\Omega^0},\ 0<\varepsilon\leqslant 1\}$ is relatively compact whence totally bounded in $C_B(\Omega^0)$. Let $\{\phi_1^0, ..., \phi_n^0\}$ be a finite collection of functions on all of Δ such that support $(\phi_l^0)=\Omega^0$ for each $l=1,2,...,n$ and $\{\phi_1^0|_{\Omega^0}, ..., \phi_n^0|_{\Omega^0}\}$ forms a $(\eta/4)$-net for $\{\phi^{\varepsilon}|_{\Omega^0},\ 0<\varepsilon\leqslant 1\}$. Now, define

$$\phi_l^a(t,\tau,y,\alpha+i\beta)\doteq\int_{\Re^{3d}}\phi_l^0(t,\tau,\theta,v+i\vartheta) \times\rho^a((y,\alpha,\beta)-\Theta)\,d\Theta, \qquad \Theta=(\theta,v,\vartheta) \tag{69}$$

for each $a>0$, $l=1,2,...,n$, $0\leqslant\tau\leqslant t\leqslant 1$, and $y,\alpha,\beta\in\Re^d$ where $\rho^a(\cdot)$, $a>0$ are the Laplace distributions

$$\rho^a(\Theta)\doteq\left(\frac{a}{2}\right)^{3d}\exp\left\{-a\sum_{j=1}^{3d}|\Theta^j|\right\}, \qquad \Theta\in\Re^{3d},\quad a>0. \tag{70}$$

(The functions $\{\phi_1^0, ..., \phi_n^0\}$ will be continuous on the interior of Ω^0 and 0 on $(\Omega^0)^c$. However, they will in general be discontinuous on the boundary $\partial\Omega^0$ and hence will not be in $C_B(\Delta)$. Therefore, we concolve them with nice kernels ρ^a which approach the Dirac delta distribution as $a\to\infty$. Actually, there are a variety of other kernels that could have been used instead of the Laplace distributions; the appeal of the Laplace distributions is that they form a simple single parameter class.) Clearly, $\{\phi_1^0, ..., \phi_n^0\}$ are uniformly bounded by $D>0$ say and one can fix an open ball of $\Re^{3d}$ $B=B(0,\kappa)$ with $0<\kappa\leqslant\frac{1}{2}$ such that

$$\{(t,\tau,y-\theta,\zeta-v-i\vartheta):(t,\tau,y,\zeta)\in\Omega^{1/2},\ (\theta,v,\vartheta)\in B\}\subset\Omega^0 \tag{71}$$

and

$$\|\phi_l^0(t,\tau,y-\theta,\zeta-v-i\vartheta)-\phi_l^0(t,\tau,y,\zeta)\|\leqslant\frac{\eta}{4} \tag{72}$$

for all $l=1,2,...,n$, $(t,\tau,y,\zeta)\in\Omega^{1/2}$, and $(\theta,v,\vartheta)\in B$. Finally,

$$\int_{\{\Theta\in\Re^{3d}:\,|\Theta^j|\geqslant\gamma\}}\rho^a(\Theta)\,d\Theta=\exp\{-a\gamma\} \tag{73}$$

for all a, $\gamma>0$ and $j=1,2,...,d$ so by (69), (72), and (73) it follows that there is some $a_\kappa>0$ such that

$$\begin{aligned}
&\|\phi_l^0(t,\tau,y,\zeta)-\phi_l^a(t,\tau,y,\zeta)\|\\
&\quad\leqslant\int_B\|\phi_l^0(t,\tau;y,\zeta)-\phi_l^0(t,\tau;y-\theta,\zeta-v-i\vartheta)\|\,\rho^a(\Theta)\,d\Theta\\
&\qquad+\int_{B^c}[\,\|\phi_l^0(t,\tau;y,\zeta)\|+\|\phi_l^0(t,\tau;y-\theta,\zeta-v-i\vartheta)\|\,]\,\rho^a(\Theta)\,d\Theta\\
&\quad\leqslant\frac{\eta}{4}+2D\exp\left\{-\frac{a\kappa}{\sqrt[r]{3d}}\right\}<\frac{\eta}{2}\quad\forall(t,\tau;y,\zeta)\in\Omega^{1/2},\ l=1,...,n
\end{aligned}$$

provided $a>a_\kappa$, r being defined in (14). Hence, fixing an arbitrary $\varepsilon>0$ and finding an l such that

$$\|\phi^\varepsilon(t,\tau,y,\zeta)-\phi_l^0(t,\tau,y,\zeta)\|<\frac{\eta}{4}\ \forall(t,\tau,y,\zeta)\in\Omega^0, \tag{75}$$

we find by (74) and (75) that

$$\sup_{\Omega^{1/2}}\|\phi^\varepsilon(t,\tau,y,\zeta)-\phi_l^a(t,\tau,y,\zeta)\|<\eta \tag{76}$$

for any $a>a_\kappa$. On the other hand; using (69), (75), and (73); and fixing a large enough a (which is independent of ε); we find that

$$\sup_{(\Omega^{1/2})^C} \|\phi^\varepsilon(t,\tau,y,\zeta)-\phi_l^a(t,\tau,y,\zeta)\|$$

$$\leqslant \frac{\eta}{4}+\sup_{(\Omega^{1/2})^c}\int_{(\Lambda^1)^c}\|\phi_l^0(t,\tau,\theta,v+i\vartheta)\|\,\rho^a((y,\alpha,\beta)-\Theta)\,d\Theta$$

$$+\sup_{(\Omega^{1/2})^c}\int_{\Lambda^1}\|\phi_l^0(t,\tau,\theta,v+i\vartheta)\|\,\rho^a((y,\alpha,\beta)-\Theta)\,d\Theta$$

$$\leqslant \frac{\eta}{4}+\frac{\eta}{2}+D\exp\left\{-\frac{a}{2}\right\}<\eta. \tag{77}$$

It follows easily by (76) and (77) that $\{\phi_1^a, ..., \phi_n^a\}$ forms a finite η-net for $\{\phi^\varepsilon, 0<\varepsilon\leqslant 1\}$ and $\{\phi^\varepsilon, 0<\varepsilon\leqslant 1\}$ is relatively compact in $C_B(\Delta)$.

Now, we show that the only possible limit point for $\{\phi^\varepsilon, 0<\varepsilon\leqslant 1\}$ as $\varepsilon\to 0$ is 0. In fact, it follows from integration by parts, (60–63), Condition (C3), our hypothesis (27) and the dominated convergence theorem that

$$\left\|\int_\tau^t V^\varepsilon(t,s;y,\zeta)\sum_{|k|=q}\tilde{A}_k\left(y,\frac{s}{\varepsilon}\right)V(s,\tau;y,\zeta)\,ds\right\|$$

$$\leqslant \left\|\int_\tau^t \sum_{|k|=q}\tilde{A}_k\left(y,\frac{s}{\varepsilon}\right)ds\,V(t,\tau;y,\zeta)\right\|$$

$$+\left\|\int_\tau^t \partial_s V^\varepsilon(t,s;y,\zeta)\int_\tau^s\sum_{|k|=q}\tilde{A}_k\left(y,\frac{\sigma}{\varepsilon}\right)d\sigma\,V(s,\tau;y,\zeta)\,ds\right\|$$

$$+\left\|\int_\tau^t V^\varepsilon(t,s;y,\zeta)\int_\tau^s\sum_{|k|=q}\tilde{A}_k\left(y,\frac{\sigma}{\varepsilon}\right)d\sigma\,\partial_s V(s,\tau;y,\zeta)\,ds\right\|\to 0 \tag{78}$$

for each fixed (t,τ,y,ζ) and it follows by (38–40) that $\phi^\varepsilon\to 0$ in $C_B(\Delta)$.

Next, we use this convergence to establish our bound between the fundamental solutions of (36) and (37). Letting $W^\varepsilon(t,\tau;y,\zeta)\doteq V^\varepsilon(t,\tau;y,\zeta)-V(t,\tau;y,\zeta)$, we find by (60), (61), and (34) that

$$W^\varepsilon(t,\tau;y,\zeta)=\int_\tau^t\sum_{|k|=q}A_k\left(y,\frac{s}{\varepsilon}\right)(i\zeta)^k\,W^\varepsilon(s,\tau;y,\zeta)\,ds$$

$$+\int_\tau^t\sum_{|k|=q}\tilde{A}_k\left(y,\frac{s}{\varepsilon}\right)(i\zeta)^k\,V(s,\tau;y,\zeta)\,ds \tag{79}$$

or by variation of constants and (40) that

$$\|W^\varepsilon(t,\tau;y,\zeta)\|\leqslant\|\eta^\varepsilon(t,\tau;y,\zeta)\|\cdot|\zeta|^q\ \forall\varepsilon>0,(t,\tau,y,\zeta)\in\Delta. \tag{80}$$

Consequently, using (80), (38–40), and the fact $\phi^\varepsilon \to 0$, one finds that there exist constants $C, \delta_2, \lambda_2 > 0$ and a function $\gamma_1(\cdot)$ satisfying $\lim_{\varepsilon \to 0} \gamma_1(\varepsilon) = 0$ such that

$$\begin{aligned} &\| V^\varepsilon(t, \tau; y, \zeta) - V(t, \tau; y, \zeta) \| \\ &\qquad \leqslant \frac{\gamma_1(\varepsilon)\, |\zeta|^q}{(t-\tau)^{\chi-1}} \exp\{[\lambda_1 |\beta|^q - \delta_1 |\alpha|^q](t-\tau)\} (1+|y|^2)^\nu \\ &\qquad \leqslant \frac{C\gamma_1(\varepsilon)}{(t-\tau)^{\chi}} \exp\{[\lambda_2 |\beta|^q - \delta_2 |\alpha|^q](t-\tau)\} (1+|y|^2)^\nu \end{aligned} \tag{81}$$

for all $\varepsilon > 0$, $(t, \tau, y, \zeta) \in \Delta$. Now, one finds (see Friedman [6] Section 2, Chapter 9) that with any $\beta \in \Re^d$

$$\begin{aligned} Z^\varepsilon(x-\xi, t, y, \tau) \doteq \frac{1}{(2\pi)^d} \int_{\Re^d} &\exp\{(i\alpha - \beta)(x-\xi)\} \\ &\times V^\varepsilon(t, \tau; y, \alpha + i\beta)\, d\alpha \end{aligned} \tag{82}$$

$$\begin{aligned} Z(x-\xi, t, y, \tau) \doteq \frac{1}{(2\pi)^d} \int_{\Re^d} &\exp\{(i\alpha - \beta)(x-\xi)\} \\ &\times V(t, \tau; y, \alpha + i\beta)\, d\alpha \end{aligned} \tag{83}$$

form fundamental solutions for (36) and (37) respectively and, using (81) as well as the argument on pp. 245–6 of [6], that for any vector of non-negative integers b there exist constants $c, C = c_{b,\chi,\nu}, C_{b,\chi,\nu} > 0$ such that

$$\begin{aligned} &\|\partial_x^b [Z^\varepsilon(x-\xi, t, y, \tau) - Z(x-\xi, t, y, \tau)]\| \\ &\qquad \leqslant \frac{C\gamma_1(\varepsilon)}{(t-\tau)^{(d+|b|+\chi)/q}} \exp\left[-c \left| \frac{|x-\xi|^q}{t-\tau} \right|^{1/(q-1)} \right] (1+|y|^2)^\nu \end{aligned} \tag{84}$$

for all $\varepsilon \in (0, 1]$, $x, y, \xi \in \Re^d$, and $0 \leqslant \tau \leqslant t \leqslant 1$.

3.3. *Bounds in the Parametrix Method*

For simplicity, we define

$$A_k\left(y, \xi, \frac{s}{\varepsilon}\right) \doteq A_k\left(y, \frac{s}{\varepsilon}\right) - A_k\left(\xi, \frac{s}{\varepsilon}\right) \quad \forall y, \xi \in \Re^d, \quad 0 \leqslant s \leqslant 1, \quad \varepsilon > 0. \tag{85}$$

Next, we recall from Subsection 3.1 that:

(i) $c, C > 0$ are constants such that (84) holds for all $0 \leqslant |b| \leqslant q$ and Lemma 5 (i–iv) of Subsection 3.4 also holds; and

(ii) $\tilde{c}$, c_1 are constants satisfying $0<\tilde{c}<c_1<c$. Then, we show that (48–50) and (59) of Subsection 3.1 hold as follows: From (84) of Subsection 3.2, (43) of Subsection 3.1, (85), Condition (C4), Lemma 5(i), and Lemma 4(ii) (both to follow), we find that for small enough $a>0$

$$\begin{aligned}
&\left\| \int_\tau^t \int_{\mathfrak{R}^d} \partial_x^m \hat{Z}(x,t;y,s)\, A_k\left(y,\xi,\frac{s}{\varepsilon}\right) \partial_y^k [\hat{Z}^\varepsilon(y,s;\xi,\tau) - \hat{Z}(y,s;\xi,\tau)]\, dy\, ds \right\| \\
&\quad \ll \int_\tau^t \int_{\mathfrak{R}^d} \frac{1}{(t-s)^{(d+|m|)/q}} \left| \frac{|y-\xi|^q}{s-\tau} \right|^{1/(q-1)} \exp\left\{ -a \left| \frac{|y-\xi|^q}{s-\tau} \right|^{1/(q-1)} \right\} \\
&\qquad \times \frac{\gamma_1(\varepsilon)\, |1+|\xi|^2|^\nu}{(s-\tau)^{(d+q-1+\chi)/q}} \\
&\qquad \times \exp\left\{ -c \left| \frac{|x-y|^q}{t-s} \right|^{1/(q-1)} - (c-a) \left| \frac{|y-\xi|^q}{s-\tau} \right|^{1/(q-1)} \right\} dy\, ds \\
&\quad \overset{\varepsilon,x,\xi,t,\tau}{\ll} \frac{\gamma_1(\varepsilon)\, |1+|\xi|^2|^\nu}{(t-\tau)^{(d+|m|+\chi-1)/q}} \exp\left\{ -c_1 \left| \frac{|x-\xi|^q}{t-\tau} \right|^{1/(q-1)} \right\} \qquad (86)
\end{aligned}$$

for all $\varepsilon \in (0,1]$, $x, \xi \in \mathfrak{R}^d$, and $0 \leqslant \tau \leqslant t \leqslant 1$ when $|m|<q$ and $|k|=q$. In a similar manner, we discover that

$$\begin{aligned}
&\left\| \int_\tau^t \int_{\mathfrak{R}^d} \partial_x^m \hat{Z}(x,t;y,s)\, A_k\left(y,\frac{s}{\varepsilon}\right) \partial_y^k [\hat{Z}^\varepsilon(y,s;\xi,\tau) - \hat{Z}(y,s;\xi,\tau)]\, dy\, ds \right\| \\
&\quad \ll \frac{\gamma_1(\varepsilon)\, |1+|\xi|^2|^\nu}{(t-\tau)^{(d+|m|+\chi-1)/q}} \\
&\qquad \times \exp\left\{ -c_1 \left| \frac{|x-\xi|^q}{t-\tau} \right|^{1/(q-1)} \right\} \qquad \text{when} \quad |m|,\ |k|<q, \qquad (87)
\end{aligned}$$

$$\begin{aligned}
&\left\| \int_\tau^t \int_{\mathfrak{R}^d} A_k^0(x,y)\, \partial_x^m \hat{Z}(x,t;y,s)\, A_k\left(y,\frac{s}{\varepsilon}\right) \right. \\
&\qquad \left. \times \partial_y^k [\hat{Z}^\varepsilon(y,s;\xi,\tau) - \hat{Z}(y,s;\xi,\tau)]\, dy\, ds \right\| \\
&\quad \ll \frac{\gamma_1(\varepsilon)\, |1+|\xi|^2|^\nu}{(t-\tau)^{1+(d+\chi-2)/q}} \\
&\qquad \times \exp\left\{ -c_1 \left| \frac{|x-\xi|^q}{t-\tau} \right|^{1/(q-1)} \right\} \qquad \text{when} \quad |m|=q,\ |k|<q, \qquad (88)
\end{aligned}$$

and

$$\left\| \int_\tau^t \int_{\Re^d} A_k^0(x, y)\, \partial_x^m \hat{Z}(x, t; y, s)\, A_k\left(y, \xi, \frac{s}{\varepsilon}\right) \times \partial_y^k [\hat{Z}^\varepsilon(y, s; \xi, \tau) - \hat{Z}(y, s; \xi, \tau)]\, dy\, ds \right\| \leqslant \frac{\gamma_1(\varepsilon)\, |1 + |\xi|^2|^\nu}{(t-\tau)^{1 + (d + \chi - 2)/q}} \exp\left\{ -c_1 \left| \frac{|x - \xi|^q}{t - \tau} \right|^{1/(q-1)} \right\} \quad \text{when} \quad |m|, |k| = q. \tag{89}$$

Moreover, recalling definitions (45–47) and availing one's self of Lemma 2 of Subsection 3.4, one establishes (48) of Subsection 3.1. Hence; combining (44) and (48) with (86–89); recalling definitions (51) and (52); and making use of Minkowski's inequality; we can now establish (49) and (50) of Subsection 3.1. Consequently, an additional application of (52) together with Condition (C3) yields a $C > 0$ such that

$$w^\varepsilon(x, t; \xi, \tau) \doteq \left\| \int_\tau^t \int_{\Re^d} K(x, t; y, s)[K^\varepsilon(y, s; \xi, \tau) - K(y, s; \xi, \tau)]\, dy\, ds \right\| \leqslant \frac{C\gamma_3(\varepsilon)\, |1 + |\xi|^2|^\nu}{(t-\tau)^{1 + (d - \varsigma)/q}} \exp\left\{ -c_1 \left| \frac{|x - \xi|^q}{t - \tau} \right|^{1/(q-1)} \right\} \tag{90}$$

for all $\varepsilon \in (0, 1]$, $x, \xi \in \Re^d$, $0 \leqslant \tau \leqslant t \leqslant 1$. Moreover, fixing (ε, ξ, τ); recalling definitions (53–55); interchanging the order of integration; utilizing the bound $(1 + |z|^2)^\nu \ll^{z, \xi} (1 + |\xi|^2)^\nu + |z - \xi|^{2\nu}$ for all $z, \xi \in \Re^d$; and using (90), Lemma 5(iii), (ii) and Lemma 4(ii); one finds that there exist constants $C > 0$, $c_2 \in (\tilde{c}, c_1)$ independent of $(\varepsilon, \xi, \tau, x, t)$ such that

$$\begin{aligned} u^\varepsilon(x, t) &\leqslant w^\varepsilon(x, t; \xi, \tau) + \int_\tau^t \int_{\Re^d} w^\varepsilon(x, t; z, \sigma)\, \|\Phi^\varepsilon(z, \sigma; \xi, \tau)\|\, dz\, d\sigma \\ &\quad + \int_\tau^t \int_{\Re^d} \|K(x, t; y, s)\|\, u^\varepsilon(y, s)\, dy\, ds \\ &\leqslant \frac{C\gamma_3(\varepsilon)\, |1 + |\xi|^2|^\nu}{(t-\tau)^{1 + (d - \varsigma)/q}} \exp\left\{ -c_2 \left| \frac{|x - \xi|^q}{t - \tau} \right|^{1/(q-1)} \right\} \\ &\quad + \int_\tau^t \int_{\Re^d} \frac{C}{(t-s)^{1 + (d-1)/q}} \\ &\quad \times \exp\left\{ -c_2 \left| \frac{|x - y|^q}{t - s} \right|^{1/(q-1)} \right\} u^\varepsilon(y, s)\, dy\, ds \end{aligned} \tag{91}$$

for all $x \in \Re^d$, $t \in [\tau, 1]$.

Next, letting (ξ, τ) remain fixed, we use our recursive bound in (91) to establish an absolute bound for u^ε. First, we let a, $\{a_i\}_{i=0}^{q+d}$ be constants such that $\tilde{c} < a < a_{i+1} < a_i < c_2$, let $\{C_{W,i}\}_{i=0}^{q+d}$ denote the Banach spaces of continuous $\Re$-valued functions φ on $\Re^d \times (\tau, 1]$ such that

$$|\varphi|_i \doteq \sup_{x,t} \left\{ |\varphi(x,t)| \, (t-\tau)^{1+(d-\varsigma-i)/q} \exp\left\{ a_i \left| \frac{|x-\xi|^q}{t-\tau} \right|^{1/(q-1)} \right\} \right\} < \infty, \tag{92}$$

and define the operators $T_i: C_{W,i} \to C_{W,i+1}$ (c.f. Lemmas 3 and 4(ii) of Subsection 3.4) by

$$T_i \varphi(x,t) \doteq \int_\tau^t \int_{\Re^d} \frac{C}{(t-s)^{1+(d-1)/q}} \times \exp\left\{ -c_2 \left| \frac{|x-y|^q}{t-s} \right|^{1/(q-1)} \right\} \varphi(y,s) \, dy \, ds. \tag{93}$$

(The completeness of each $C_{W,i}$ follows from the completeness of the space of continuous, bounded functions on $\Re^d \times (\tau, 1]$ and isometry.) Now, it follows by (55), Lemma 5(ii), (iii), and Lemma 4(ii) that

$$u^\varepsilon(x,t) \overset{\varepsilon, x, t; \xi, \tau}{\ll} \frac{1}{|t-\tau|^{(q+d-2)/q}} \exp\left\{ -c_1 \left| \frac{|x-\xi|^q}{t-\tau} \right|^{1/(q-1)} \right\} \tag{94}$$

for all $\varepsilon \in (0, 1]$, $x \in \Re^d$, $\tau \leqslant t \leqslant 1$. Hence, it follows by Lemmas 3 and 4(ii) that

$$v^\varepsilon(x,t) \doteq T_{d+q-1} \cdots T_0 u^\varepsilon(x,t) \quad \text{and} \quad g^\varepsilon(x,t) \doteq T_{d+q-1} \cdots T_0 f^\varepsilon(x,t) \tag{95}$$

are well defined, where

$$f^\varepsilon(x,t) \doteq \frac{C\gamma_3(\varepsilon) \, |1+|\xi|^2|^v}{(t-\tau)^{1+(d-\varsigma)/q}} \exp\left\{ -c_2 \left| \frac{|x-\xi|^q}{t-\tau} \right|^{1/(q-1)} \right\}. \tag{96}$$

Moreover, letting $C_B(\Re^d \times [\tau, 1])$ be the Banach space of bounded, continuous $\Re$-valued functions with supremum norm and $|||\cdot|||$ denote the operator norm on $C_B(\Re^d \times [\tau, 1])$, we find from (91), (93), and (95) that

$$u^\varepsilon(x,t) \leqslant f^\varepsilon(x,t) + \sum_{j=0}^{d+q-2} T_j T_{j-1} \cdots T_0 f^\varepsilon(x,t) + v^\varepsilon(x,t) \tag{97}$$

and

$$v^{\varepsilon}(x,t) \leqslant g^{\varepsilon}(x,t) + \sum_{i=1}^{n-1} S^i g^{\varepsilon}(x,t) + |||S^n||| \cdot |v^{\varepsilon}|_{C_B} \qquad \forall n = 1, 2, ..., \tag{98}$$

where $S: C_B(\mathfrak{R}^d \times [\tau, 1]) \to C_B(\mathfrak{R}^d \times [\tau, 1])$ is also defined by

$$S\varphi(x,t) \doteq \int_{\tau}^{t} \int_{\mathfrak{R}^d} \frac{C}{(t-s)^{1+(d-1)/q}} \exp\left\{-c_2 \left|\frac{|x-y|^q}{t-s}\right|^{1/(q-1)}\right\} \varphi(y,s)\, dy\, ds. \tag{99}$$

Now, it follows from (93), (95), (96), and Lemma 3 as well as Lemma 4(ii) that there is a $c_3 \in (\tilde{c}, a)$ such that

$$\sum_{j=0}^{d+q-2} T_j \cdots T_0 f^{\varepsilon}(x,t) + g^{\varepsilon}(x,t) \overset{x,\,t;\,\varepsilon,\,\xi,\,\tau}{\ll} \frac{\gamma_3(\varepsilon)\, |1+|\xi|^2|^{\nu}}{(t-\tau)^{1+(d-1-\varsigma)/q}} \exp\left\{-c_3 \left|\frac{|x-\xi|^q}{t-\tau}\right|^{1/(q-1)}\right\} \tag{100}$$

and for some $C' > 0$

$$S^n\varphi(x,t) \leqslant \begin{cases} \displaystyle\int_{\tau}^{t} \int_{\mathfrak{R}^d} \frac{C'}{(t-s)^{1+(d-n)/q}} \exp\left\{-a_n \left|\frac{|x-y|^q}{t-s}\right|^{1/(q-1)}\right\} \\ \qquad \times |\varphi(y,s)|\, dy\, ds \qquad n < q+d \\ \displaystyle\int_{\tau}^{t} \int_{\mathfrak{R}^d} \frac{C' A^n}{\Gamma(1+(n-d)/q)} \exp\left\{-a \left|\frac{|x-y|^q}{t-s}\right|^{1/(q-1)}\right\} \\ \qquad \times |\varphi(y,s)|\, dy\, ds \qquad n \geqslant q+d \end{cases} \tag{101}$$

for all $\varphi \in C_B(\mathfrak{R}^d \times [\tau, 1])$ so

$$|||S^n||| \overset{n}{\ll} \frac{A^n}{\Gamma\left(1 + \dfrac{n-d}{2}\right)} \to 0 \qquad \text{as} \quad n \to \infty. \tag{102}$$

Therefore, letting $n \to \infty$ in (98), substituting the resultant into (97), exploiting (101) and estimates for the gamma function, applying (96), (100) and (95), and availing ourselves of Lemma 3 as well as Lemma 4(i), one finds that

$$u^{\varepsilon}(x,t) \leqslant f^{\varepsilon}(x,t) + \sum_{j=0}^{d+q-1} [T_j \cdots T_0 f^{\varepsilon}(x,t) + S^j g^{\varepsilon}(x,t)]$$

$$+ \int_{\tau}^{t} \int_{\Re^d} \sum_{j=d+q}^{\infty} \frac{C' A^j}{\Gamma\left(1 + \frac{j-d}{q}\right)}$$

$$\times \exp\left\{-a \left|\frac{|x-y|^q}{t-s}\right|^{1/(q-1)}\right\} g^{\varepsilon}(y,s)\, dy\, ds$$

$$\overset{x,\, t;\, \varepsilon,\, \xi,\, \tau}{\ll} \frac{\gamma_3(\varepsilon)\, |1+|\xi|^2|^{\nu}}{(t-\tau)^{1+(d-\varsigma)/q}}$$

$$\times \exp\left\{-c_3 \left|\frac{|x-\xi|^q}{t-\tau}\right|^{1/(q-1)}\right\} \qquad \forall x \in \Re^d, \quad t \in [\tau, 1]. \quad (103)$$

Piecing everything together, we find by (53–55), the fact $|1+|z|^2|^{\nu} \overset{z,\,\xi}{\ll} |1+|\xi|^2|^{\nu} + |z-\xi|^{2\nu}$ for all $z,\ \xi \in \Re^d$, (49), and (103) as well as Lemmas 5(i), (iii) and 4(ii) that

$$\left\| \int_{\tau}^{t} \int_{\Re^d} \partial_x^b \hat{Z}(x,t;y,s)[\Phi^{\varepsilon}(y,s;\xi,\tau) - \Phi(y,s;\xi,\tau)]\, dy\, ds \right\|$$

$$\leqslant \left\| \int_{\tau}^{t} \int_{\Re^d} \partial_x^b \hat{Z}(x,t;y,s)[K^{\varepsilon}(y,s;\xi,\tau) - K(y,s;\xi,\tau)]\, dy\, ds \right\|$$

$$+ \int_{\tau}^{t} \int_{\Re^d} \left\| \int_{\sigma}^{t} \int_{\Re^d} \partial_x^b \hat{Z}(x,t;y,s)[K^{\varepsilon}(y,s;z,\sigma) - K(y,s;z,\sigma)]\, dy\, ds \right\|$$

$$\times \|\Phi^{\varepsilon}(z,\sigma;\xi,\tau)\|\, dz\, d\sigma$$

$$+ \int_{\tau}^{t} \int_{\Re^d} \|\partial_x^b \hat{Z}(x,t;y,s)\|\, u^{\varepsilon}(y,s)\, dy\, ds$$

$$\overset{\varepsilon,\, x,\, t,\, \xi,\, \tau}{\ll} \frac{\gamma_3(\varepsilon)\, |1+|\xi|^2|^{\nu}}{(t-\tau)^{(d+|b|)/q}} \exp\left\{-\tilde{c} \left|\frac{|x-\xi|^q}{t-\tau}\right|^{1/(q-1)}\right\} \qquad (104)$$

for all $\varepsilon \in (0, 1]$, $0 \leqslant |b| < q$, $0 \leqslant \tau \leqslant t \leqslant 1$, and $x,\ \xi \in \Re^d$. Finally, from (58), (57), and (43) of Subsection 3.1, (84) of Subsection 3.2, Lemma 5(iii), the fact that $(1+|y|^2)^{\nu} \ll (1+|\xi|^2)^{\nu} + |y-\xi|^{2\nu}$, Lemma 4(ii), and (104) it follows that

$$\begin{aligned}
&\|\partial_x^b[\Gamma^\varepsilon(x,t;\xi,\tau)-\Gamma(x,t;\xi,\tau)]\| \\
&\quad\leqslant \|\partial_x^b[\hat{Z}^\varepsilon(x,t;\xi,\tau)-\hat{Z}(x,t;\xi,\tau)]\| \\
&\qquad+\int_\tau^t\int_{\mathfrak{R}^d}\|\partial_x^b[\hat{Z}^\varepsilon(x,t;y,s)-\hat{Z}(x,t;y,s)]\|\,\|\Phi^\varepsilon(y,s;\xi,\tau)\|\,dy\,ds \\
&\qquad+\left\|\int_\tau^t\int_{\mathfrak{R}^d}\partial_x^b\hat{Z}(x,t;y,s)[\Phi^\varepsilon(y,s;\xi,\tau)-\Phi(y,s;\xi,\tau)]\,dy\,ds\right\| \\
&\quad\leqslant\frac{\gamma(\varepsilon)\,|1+|\xi|^2|^\nu}{(t-\tau)^{(d+|b|+\chi)/q}}\exp\left\{-\tilde{c}\left|\frac{|x-\xi|^q}{t-\tau}\right|^{1/(q-1)}\right\}
\end{aligned}\tag{105}$$

for all $\varepsilon\in(0,1]$, $0\leqslant|b|<q$, $0\leqslant\tau\leqslant t\leqslant 1$, and $x,\ \xi\in\mathfrak{R}^d$. ∎

3.4. *Subsidiary Results*

Our first lemma is used in Subsection 3.3 to establish ϕ^ε, as defined in (47), converges to 0 in the space of bounded, continuous $\prod_{|m|,\,|k|\leqslant q}\mathscr{C}^{N\times N}$-valued functions

$$\begin{gathered}
(C_B(\Delta^1),\ \sup_{\Delta^1}\|\cdot\|), \\
\Delta^1\doteq\{(x,t,\xi,\tau)\in\mathfrak{R}^d\times[0,1]\times\mathfrak{R}^d\times[0,1]:\tau\leqslant t\}.
\end{gathered}\tag{106}$$

LEMMA 2. *Suppose ϕ^ε is defined as in* (47) *of Subsection* 3.1 *and Conditions* (C1–5) *are satisfied. Then, $\phi^\varepsilon\to 0$ in $C_B(\Delta^1)$.*

To ease the notation in the following proof we define

$$\alpha=\alpha_m\doteq\frac{|m|\wedge(q-\varsigma)}{q}\qquad\text{and}\qquad\beta=\beta_k\doteq\frac{|k|\wedge(q-1)}{q},\tag{107}$$

where ς is the constant of Condition (C5).

Proof. Suppose $C,\ c>0$ are constants such that Lemma 5(i–iv) hold and, as in Subsection 3.1, $c_1\in(0,c)$. Then, it follows from (45), (46), Lemma 5(i), Conditions (C3) and (C4), and both parts of Lemma 4 that there exist constants $C_2,\ c_2>0$ such that

$$\begin{aligned}
&\|\psi^\varepsilon_{m,k}(x,t;\xi,\tau)\| \\
&\quad\leqslant\exp\left\{-c_2\left|\frac{|x-\xi|^q}{t-\tau}\right|^{1/(q-1)}\right\}\int_\tau^t\int_{\mathfrak{R}^d}\frac{(t-\tau)^{\alpha+(d/q)}\,dy\,ds}{(t-s)^{\alpha+(d/q)}\,(s-\tau)^{\beta+(d/q)}} \\
&\qquad\times\exp\left\{-c_3\left|\frac{|x-y|^q}{t-s}\right|^{1/(q-1)}-c_3\left|\frac{|y-\xi|^q}{s-\tau}\right|^{1/(q-1)}\right\} \\
&\quad\leqslant C_2(t-\tau)^{1/q}\exp\left\{-c_2\left(\frac{|x-\xi|^q}{t-\tau}\right)^{1/(q-1)}\right\},
\end{aligned}\tag{108}$$

for all $(\varepsilon; x, t; \xi, \tau; m, k)$, where $c_3 \doteq c - c_1 - c_2 > 0$. Now, we show that the functions $(x, t; \xi, \tau) \to \psi^\varepsilon(x, t; \xi, \tau)$, $0 < \varepsilon \leqslant 1$ are equicontinuous in $C_B(\Delta^1)$. First, we fix arbitrary $0 \leqslant \tau \leqslant t \leqslant 1$ and $0 \leqslant \tau' \leqslant t' \leqslant 1$ such that $t \geqslant t'$ and claim that (108) implies that

$$\begin{aligned} &\|\psi^\varepsilon(x, t; \xi, \tau)\| + \|\psi^\varepsilon(x, t'; \xi, \tau')\| \\ &\qquad \leqslant 2C_2 \cdot 2^{1/q}[|t - t'|^{1/q} + |\tau - \tau'|^{1/q}] \end{aligned} \tag{109}$$

when $t' \leqslant (t + \tau)/2$ or $\tau' \geqslant (t + \tau)/2$ or $\tau \geqslant (t' + \tau')/2$. (To show for example the case $t' \leqslant (t + \tau)/2$ one could consider the subcases $\tau' \geqslant \tau$ and $\tau \geqslant (t' + \tau')/2$ separately and note that $(t' + \tau')/2 \geqslant \tau'$.) On the other hand; assuming for the moment that there exist $\hat{c}, \hat{C} > 0$ such that

$$\begin{aligned} &\|d_\sigma Y^\varepsilon_{m, k}(x, \sigma; y, s; \xi, \tau)\| \\ &\leqslant \begin{cases} \dfrac{\hat{C}}{(\sigma - \tau)(s - \tau)^{\beta + (d/q)}} \exp\left\{-\hat{c}\left|\dfrac{|y - \xi|^q}{s - \tau}\right|^{1/(q-1)}\right\} & s \leqslant \dfrac{\sigma + \tau}{2} \\ \dfrac{\hat{C}\,(\sigma - \tau)^{\alpha - \beta}}{(\sigma - s)^{1 + \alpha + (d/q)}} \exp\left\{-\hat{c}\left|\dfrac{|x - y|^q}{\sigma - s}\right|^{1/(q-1)}\right\} & s \geqslant \dfrac{\sigma + \tau}{2} \end{cases} \end{aligned} \tag{110}$$

for all $x, y, \xi \in \mathfrak{R}^d$, $0 \leqslant \tau \leqslant s \leqslant \sigma \leqslant 1$, $\varepsilon \in (0, 1]$, $|m|, |k| \leqslant q$; we would find that

$$\begin{aligned} &\int_{t'}^{t} \int_{\tau}^{(\sigma + \tau)/2} \int_{\mathfrak{R}^d} \|d_\sigma Y^\varepsilon(x, \sigma; y, s; \xi, \tau)\|\, dy\, ds\, d\sigma \\ &\qquad \overset{\varepsilon, x, \xi, \tau, t, t'}{\ll} \int_{t'}^{t} \frac{1}{\sigma - \tau} \int_0^{(\sigma - \tau)/2} s^{-\beta}\, ds\, d\sigma \\ &\qquad \ll (t - \tau)^{1/q} - (t' - \tau)^{1/q} \leqslant (t - t')^{1/q} \end{aligned} \tag{111}$$

when $t' \geqslant (t + \tau)/2$ and

$$\begin{aligned} &\int_{t'}^{t} \int_{(\sigma + \tau)/2}^{t'} \int_{\mathfrak{R}^d} \|d_\sigma Y^\varepsilon(x, \sigma; y, s; \xi, \tau)\|\, dy\, ds\, d\sigma \\ &\qquad \overset{\varepsilon, x, \xi, \tau, t, t'}{\ll} \int_{t'}^{t} \int_{(\sigma + \tau)/2}^{t'} (\sigma - s)^{-\alpha - 1}\, ds (\sigma - \tau)^{\alpha - \beta}\, d\sigma \\ &\qquad \overset{\varepsilon, x, \xi, \tau, t, t'}{\ll} \int_{t'}^{t} \left[(\sigma - t')^{-\alpha} - \left(\frac{\sigma - \tau}{2}\right)^{-\alpha}\right] (\sigma - \tau)^{\alpha - \beta}\, d\sigma \\ &\qquad \overset{\varepsilon, x, \xi, \tau, t, t'}{\ll} \int_{t'}^{t} (\sigma - t')^{-(\alpha \vee \beta)}\, d\sigma \ll (t - t')^{\varsigma/q} \end{aligned} \tag{112}$$

when $t' \geqslant (t+\tau)/2$. Furthermore, availing ourselves once again of (95), Conditions (C3–C4), and Lemmas 5(i) and 4(i), we would find that

$$\left\| \int_{t'}^{t} \int_{\mathfrak{R}^d} Y^\varepsilon(x, t; y, s; \xi, \tau)\, dy\, ds \right\| \overset{\varepsilon, x, \xi, \tau, t, t'}{\ll} \int_0^{t-t'} \frac{(t-\tau)^{\alpha-\beta}}{s^\alpha}\, ds$$

$$\ll \int_0^{t-t'} s^{-(\alpha \vee \beta)}\, ds \ll (t-t')^{\varsigma/q} \tag{113}$$

when $t' \geqslant (t+\tau)/2$. Therefore, it would follow from (109) when $t' \leqslant (t+\tau)/2$ or otherwise from (46), and (111–113) that

$$\|\psi^\varepsilon(x, t; \xi, \tau) - \psi^\varepsilon(x, t'; \xi, \tau)\| \overset{\varepsilon, x, \xi, \tau, t, t'}{\ll} |t-t'|^{\varsigma/q} \tag{114}$$

for all $\varepsilon \in (0, 1]$ and $x, \xi \in \mathfrak{R}^d$, if $0 \leqslant \tau \leqslant t' \leqslant t \leqslant 1$. Moreover, postulating existence of constants $\hat{c}, \hat{C} > 0$ such that

$$\|d_\sigma Y^\varepsilon_{m,k}(x, t'; y, s; \xi, \sigma)\|$$

$$\leqslant \begin{cases} \dfrac{\hat{C}}{(s-\sigma)^{1+\beta+d/q}} \exp\left\{ -\hat{c} \left| \dfrac{|x-\xi|^q}{s-\sigma} \right|^{1/(q-1)} \right\} & s \leqslant \dfrac{t'+\sigma}{2} \\ \dfrac{\hat{C}(t'-\sigma)^{\alpha-\beta-1}}{(t'-s)^{\alpha+d/q}} \exp\left\{ -\hat{c} \left| \dfrac{|x-y|^q}{t'-s} \right|^{1/(q-1)} \right\} & s \geqslant \dfrac{t'+\sigma}{2} \end{cases} \tag{115}$$

for all $(\varepsilon; x, t; y, s; \xi, \sigma; m, k)$ and repeating the above arguments, one would find that

$$\|\psi^\varepsilon(x, t'; \xi, \tau) - \psi^\varepsilon(x, t'; \xi, \tau')\| \overset{\varepsilon, x, \xi, \tau, \tau', t'}{\ll} |\tau-\tau'|^{\varsigma/q} \tag{116}$$

for all $\varepsilon \in (0, 1]$ and $x, \xi \in \mathfrak{R}^d$ if $\tau \leqslant t'$. Combining (109), (114), and (116), we would find that

$$\|\psi^\varepsilon(x, t; \xi, \tau) - \psi^\varepsilon(x, t'; \xi, \tau')\| \overset{\varepsilon, x, \xi, \tau, \tau', t, t'}{\ll} |t-t'|^{\varsigma/q} + |\tau-\tau'|^{\varsigma/q} \tag{117}$$

for all $\varepsilon \in (0, 1]$, $0 \leqslant \tau \leqslant t \leqslant 1$, $0 \leqslant \tau' \leqslant t' \leqslant 1$, and $x, \xi \in \mathfrak{R}^d$. Next; suppose there exist constants $\hat{c}, \hat{C} > 0$ such that

$$\|\partial_{\xi_i} Y^\varepsilon_{m,k}(x, t; y+\xi, s; \xi, \tau)\|$$

$$\leqslant \hat{C}\, \frac{\exp\left\{ -\hat{c} \left| \dfrac{|y|^q}{s-\tau} \right|^{1/(q-1)} - \hat{c} \left| \dfrac{|x-\xi|^q}{t-\tau} \right|^{1/(q-1)} \right\}}{(t-\tau)^{1/q}\,(s-\tau)^{1+(d-\varsigma)/q}} \tag{118}$$

for any $i=1,2,\ldots,d$ if $s\leqslant(t+\tau)/2$ and $|k|\geqslant q-1$, with ς being the constant of Condition (C5);

$$\|\partial_{\xi_i} Y^{\varepsilon}_{m,k}(x,t;y,s;\xi,\tau)\| \leqslant \hat{C}\,\frac{\exp\left\{-\hat{c}\left|\frac{|y-\xi|^q}{s-\tau}\right|^{1/(q-1)}-\hat{c}\left|\frac{|x-\xi|^q}{t-\tau}\right|^{1/(q-1)}\right\}}{(s-\tau)^{1+(d-1)/q}} \tag{119}$$

for any $i=1,2,\ldots,d$ if $s\leqslant(t+\tau)/2$ and $|k|<q-1$; or

$$\|\partial_{\xi_i} Y^{\varepsilon}_{m,k}(x,t;y,s;\xi,\tau)\| \leqslant \frac{\hat{C}(t-\tau)^{\alpha-\beta-(1/q)}}{(t-s)^{\alpha+(d/q)}}\exp\left\{-\hat{c}\left|\frac{|x-\xi|^q}{t-s}\right|^{1/(q-1)}\right\} \tag{120}$$

for any $i=1,2,\ldots,d$ if $s\geqslant(t+\tau)/2$. Then, we would fix $\xi',\xi\in\Re^d$, define

$$\xi^i \doteq (\xi_1,\ldots,\xi_{i-1},\xi'_i,\ldots,\xi'_d)^T, \qquad i=0,1,\ldots,d+1, \tag{121}$$

and note that

$$\|\psi^{\varepsilon}_1(x,t;\xi,\tau)-\psi^{\varepsilon}_1(x,t;\xi',\tau)\| \leqslant \sum_{i=1}^{d}\|\psi^{\varepsilon}_1(x,t;\xi^{i+1},\tau)-\psi^{\varepsilon}_1(x,t;\xi^i,\tau)\|. \tag{122}$$

However, in the case $|k|\geqslant q-1$ and $\xi_i\geqslant\xi'_i$, it would then follow by (46), (121), (118), and Hölder's inequality that

$$\begin{aligned}
&\|\psi^{\varepsilon}_{1,m,k}(x,t;\xi^{i+1},\tau)-\psi^{\varepsilon}_{1,m,k}(x,t;\xi^i,\tau)\|\\
&\quad\leqslant\int_0^{\xi_i-\xi'_i}\int_\tau^{(t+\tau)/2}\int_{\Re^d}\|\partial_{\xi_i}Y^{\varepsilon}_{m,k}(x,t;y+\xi^i+ue_i,s;\xi^i+ue_i,\tau)\|\,dy\,ds\,du\\
&\quad\ll\int_0^{|\xi_i-\xi'_i|}\int_\tau^{(t+\tau)/2}\frac{ds}{(t-\tau)^{1/q}(s-\tau)^{1-(\varsigma/q)}}\\
&\qquad\times\exp\left\{-c\left|\frac{|x-\xi^i-ue_i|^q}{t-\tau}\right|^{1/(q-1)}\right\}du\\
&\quad\ll\int_0^{|\xi_i-\xi'_i|}(t-\tau)^{(\varsigma-1)/q}\left|\frac{|x-\xi^i-ue_i|^q}{t-\tau}\right|^{(\varsigma-1)/q}du\\
&\quad\ll\int_0^{|\xi_i-\xi'_i|}|x_i-\xi'_i-u|^{\varsigma-1}\,du\leqslant|\xi_i-\xi'_i|^{\varsigma^2}\left[\int_0^1|x_i-\xi'_i-u|^{-1/(1+\varsigma)}\,du\right]^{1-\varsigma^2}\\
&\quad\overset{\varepsilon,x,t,\tau,\xi,\xi'}{\ll}|\xi-\xi'|^{\varsigma^2}
\end{aligned} \tag{123}$$

for all $i=1, ..., d$, $\varepsilon \in (0, 1]$, $0 \leqslant \tau \leqslant t \leqslant 1$, and x, ξ', $\xi \in \Re^d$ with $|\xi - \xi'| \leqslant 1$. The cases $|k| < q-1$, and $|k| \geqslant q-1$, $\xi_i < \xi_i'$ can be handled similarly if one substitutes (119) for (118) when $|k| < q-1$. Moreover, one would find by (46), (120) as well as substitutions of variables that

$$\|\partial_{\xi_i} \psi_2^\varepsilon(x, t; \xi, \tau)\| \ll \int_0^{(t-\tau)/2} \frac{(t-\tau)^{\alpha-\beta-1/q}}{s^\alpha} \times \int_{\Re^d} \frac{1}{s^{d/q}} \exp\left\{-\hat{c}\left|\frac{|y|^q}{s}\right|^{1/(q-1)}\right\} dy\, ds \ll 1 \quad (124)$$

for all $i=1, 2, ..., d$, $0 \leqslant \tau \leqslant t \leqslant 1$, and x, $\xi \in \Re^d$. Hence, it follows easily from (124), the mean value theorem, (122) and (123) that

$$\|\psi^\varepsilon(x, t; \xi, \tau) - \psi^\varepsilon(x, t; \xi', \tau)\| \overset{\varepsilon, x, \xi, \xi', \tau, t}{\ll} |\xi - \xi'|^{\varsigma^2} \quad (125)$$

for all $\varepsilon \in (0, 1]$, $0 \leqslant \tau \leqslant t \leqslant 1$, and x, ξ', $\xi \in \Re^d$ with $|\xi - \xi'| \leqslant 1$. Finally; assuming existence of constants $\hat{C}$, $\hat{c} > 0$ such that

$$\|\partial_{x_i} Y_{m,k}^\varepsilon(x, t; y, s; \xi, \tau)\| \leqslant \frac{\hat{C}}{(t-\tau)^{1/q}(s-\tau)^{1+(d-1)/q}} \times \exp\left\{-\hat{c}\left|\frac{|y-\xi|^q}{s-\tau}\right|^{1/(q-1)}\right\} \quad (126)$$

for any $i=1, 2, ..., d$ if $s \leqslant (t+\tau)/2$;

$$\|\partial_{x_i} Y_{m,k}^\varepsilon(x, t; y+x, s; \xi, \tau)\| \leqslant \hat{C}\, \frac{\exp\left\{-\hat{c}\left|\frac{|y|^q}{t-s}\right|^{1/(q-1)} - \hat{c}\left|\frac{|x-\xi|^q}{t-\tau}\right|^{1/(q-1)}\right\}}{(t-\tau)^{\beta+(1/q)-\alpha}(t-s)^{\alpha+(d/q)}} \quad (127)$$

for any $i=1, 2, ..., d$ if $s \geqslant (t+\tau)/2$ and $|m| \geqslant q-1$; or

$$\|\partial_{x_i} Y_{m,k}^\varepsilon(x, t; y\, s; \xi, \tau)\| \leqslant \hat{C}\, \frac{\exp\left\{-\hat{c}\left|\frac{|x-y|^q}{t-s}\right|^{1/(q-1)} - \hat{c}\left|\frac{|x-\xi|^q}{t-\tau}\right|^{1/(q-1)}\right\}}{(t-\tau)^{\beta-\alpha}(t-s)^{\alpha+(d+1)/q}} \quad (128)$$

for any $i=1,2,\dots,d$ if $s\geqslant(t+\tau)/2$ and $|m|<q-1$; one would find by (126–128), and an argument similar to the previous one that

$$\|\psi^{\varepsilon}(x,t;\xi,\tau)-\psi^{\varepsilon}(x',t;\xi,\tau)\| \overset{\varepsilon,x,x',\xi,\tau,t}{\ll} |x-x'|^{\varsigma^2} \tag{129}$$

for all $\varepsilon\in(0,1]$, $0\leqslant\tau\leqslant t\leqslant 1$, and $x, x', \xi\in\mathfrak{R}^d$ with $|x-x'|\leqslant 1$. Equicontinuity on Δ^1 follows from (129), (125), and (117) and one can easily adapt the arguments in (66–77) of Subsection 3.2 (with aid of (108), (47), and this equicontinuity) to discover that $\{\phi^{\varepsilon},\ 0<\varepsilon\leqslant 1\}$, defined by (47), is realtively compact in $C_B(\Delta^1)$. To show $\phi^{\varepsilon}\to 0$ as $\varepsilon\to 0$ we only have to show $\phi^{\varepsilon}_{m,k}(x,t;\xi,\tau)\to 0$ for a dense set of $(x,t;\xi,\tau;m,k)$. To do this we note by (47), (46), and the argument used in (108) as well as the dominated convergence theorem that we only have to show

$$\int_{\tau}^{t} Y^{\varepsilon}_{m,k}(x,t;y,s;\xi,\tau)\,ds\to 0 \qquad \text{as} \quad \varepsilon\to 0 \tag{130}$$

for almost all $(x,t;\xi,\tau;y;m,k)$. Indeed, suppose $|m|<q$, $|k|=q$, $x\neq y$ and $y\neq\xi$. Then, noting by Lemma 5(i) that $\lim_{s\to t}\partial^m_x\hat{Z}(x,t;y,s)=\lim_{s\to\tau}\partial^k_y\hat{Z}(y,s;\xi,\tau)=0$ (pointwise), one finds from integration by parts that

$$\begin{aligned}
&\int_{\tau}^{t}\partial^m_x\hat{Z}(x,t;y,s)\,\tilde{A}_k\left(y,\xi,\frac{s}{\varepsilon}\right)\partial^k_y\hat{Z}(y,s;\xi,\tau)\,ds\\
&\quad=-\int_{\tau}^{t}\partial^m_x\hat{Z}(x,t;y,s)\int_{\tau}^{s}\tilde{A}_k\left(y,\xi,\frac{\sigma}{\varepsilon}\right)d\sigma\,\partial_s\partial^k_y\hat{Z}(y,s;\xi,\tau)\,ds\\
&\qquad-\int_{\tau}^{t}\partial_s\partial^m_x\hat{Z}(x,t;y,s)\int_{\tau}^{s}\tilde{A}_k\left(y,\xi,\frac{\sigma}{\varepsilon}\right)d\sigma\,\partial^k_y\hat{Z}(y,s;\xi,\tau)\,ds
\end{aligned} \tag{131}$$

for all $0\leqslant\tau\leqslant t\leqslant 1$. Now, using (43), (83), and (61), we discover that

$$\partial_s\partial^m_x\hat{Z}(x,t;y,s)=-\sum_{|b|=q}A^0_k(y)\,\partial^{b+m}_x\hat{Z}(x,t;y,s) \tag{132}$$

and, adding Condition (C3) and Lemma 5(i), that

$$\begin{aligned}
\|\partial_s\partial^m_x\hat{Z}(x,t;y,s)\| &\overset{s}{\ll}\frac{1}{(t-s)^{2+(d-1)/q}}\\
&\qquad\times\exp\left\{-c\left|\frac{|x-\xi|^q}{t-s}\right|^{1/(q-1)}\right\}\overset{s}{\ll}1
\end{aligned} \tag{133}$$

(since $x \neq y$). Hence, after creating similar bounds for the other $\hat{Z}$ terms, we find by such bounds, (44), (45), (42), (34), our hypothesis (27), and the dominated convergence theorem that

$$\int_\tau^t Y^\varepsilon_{m,k}(x, t; y, s; \xi, \tau)\, ds \to 0, \qquad |m| < q, \quad |k| = q \tag{134}$$

as $\varepsilon \to 0$ provided $x \neq y$ and $y \neq \xi$. The cases $|m| < q$, $|k| < q$; $|m| = q$, $|k| < q$; and $|m| = q$, $|k| = q$ are handled similarly and it only remains to establish (110), (115), (118), (119), (120), (126), (127), and (128).

Inasmuch as the proofs of all eight bounds under all combinations of the conditions: $|m| < q$ or $|m| = q$, and $|k| < q$ or $|k| = q$ are very similar, we will only prove (110) under the conditions $|m| = q$, $|k| = q$ and (118) under the conditions $|m| < q$, $|k| = q$ here. The remaining bounds follow through similar arguments. For (110) with $|m| = q$, $|k| = q$, it follows from (45), (44), an entirely similar argument to (132–133), Lemma 5(iii), Condition (C4), and Lemma 4(i) that there are $0 < c_1 < c_2 < c$ and $\hat{c} > 0$ such that

$$\begin{aligned}
&\|d_t Y^\varepsilon_{m,k}(x, t; y, s; \xi, \tau)\| \\
&\quad \ll \|A^0_m(x, y)\| \left\{ \|d_t \partial^m_x \hat{Z}(x, t; y, s)\| + \frac{\|\partial^m_x \hat{Z}(x, t; y, s)\|}{t-\tau} \right. \\
&\quad\quad \times \left. \left[1 + \left| \frac{|x-\xi|^q}{t-\tau} \right|^{1/(q-1)} \right] \right\} \exp\left\{ c_1 \left| \frac{|x-\xi|^q}{t-\tau} \right|^{1/(q-1)} \right\} \\
&\quad\quad \times \left\| \tilde{A}_k\left(y, \xi, \frac{s}{\varepsilon} \right) \partial^k_y \hat{Z}(y, s; \xi, \tau) \right\| \cdot |t-\tau|^{\alpha + (d/q)} \\
&\quad \ll \frac{|x-y|}{(t-s)^{2+(d/q)}} \exp\left\{ -c \left| \frac{|x-y|^q}{t-s} \right|^{1/(q-1)} \right\} \\
&\quad\quad \times \left\{ 1 + \left| \frac{t-s}{t-\tau} \right| \right\} \exp\left\{ c_2 \left| \frac{|x-\xi|^q}{t-\tau} \right|^{1/(q-1)} \right\} \\
&\quad\quad \times \frac{|y-\xi|}{(s-\tau)^{1+(d/q)}} \exp\left\{ -c \left| \frac{|y-\xi|^q}{s-\tau} \right|^{1/(q-1)} \right\} (t-\tau)^{\alpha+(d/q)} \\
&\quad \overset{\varepsilon;\, x,\, t;\, y,\, s;\, \xi,\, \tau}{\ll} \frac{(t-\tau)^{\alpha+(d/q)}}{(t-s)^{2+(d-1)/q}\,(s-\tau)^{1+(d-1)/q}} \\
&\quad\quad \times \exp\left\{ -\hat{c} \left| \frac{|x-y|^q}{t-s} \right|^{1/(q-1)} - \hat{c} \left| \frac{|y-\xi|^q}{s-\tau} \right|^{1/(q-1)} \right\}
\end{aligned} \tag{135}$$

for all $(\varepsilon; x, t; y, s; \xi, \tau)$ and (110) follows by considering $s \leqslant (t+\tau)/2$ and $s \geqslant (t+\tau)/2$ separately.

Now for (118) with $|m|<q$, $|k|=q$, one finds by (42), Conditions (C4) and (C5) as well as Lemma 5(i), (iv) that there are $0<c_2<c$ and $C, C'>0$ such that

$$\left\|\partial_{\xi_i}\left[\tilde{A}_k\left(y+\xi,\xi,\frac{s}{\varepsilon}\right)\partial_y^k\hat{Z}(y+\xi,s;\xi,\tau)\right]\right\|$$

$$\leqslant\left\|\tilde{A}_k\left(y+\xi,\frac{s}{\varepsilon}\right)-\tilde{A}_k\left(\xi,\frac{s}{\varepsilon}\right)\right\|\,\|\partial_{\xi_i}\partial_y^k\hat{Z}(y+\xi,s;\xi,\tau)\|$$

$$\times\left\|\partial_{\xi_i}\left[\tilde{A}_k\left(y+\xi,\frac{s}{\varepsilon}\right)-\tilde{A}_k\left(\xi,\frac{s}{\varepsilon}\right)\right]\right\|\,\|\partial_y^k\hat{Z}(y+\xi,s;\xi,\tau)\|$$

$$\leqslant\frac{C\,|y|^{\varsigma}}{(s-\tau)^{1+(d/q)}}\exp\left\{-c\left|\frac{|y|^q}{s-\tau}\right|^{1/(q-1)}\right\}$$

$$\leqslant\frac{C'}{(s-\tau)^{1+(d-\varsigma)/q}}\exp\left\{-c_2\left|\frac{|y|^q}{s-\tau}\right|^{1/(q-1)}\right\}\tag{136}$$

for all $x, y, \xi\in\Re^d$, $0\leqslant\tau\leqslant s\leqslant 1$ and, defining $z=x-y-\xi$, $\zeta=y+\xi$, one finds by Lemma 5(i), (iv) that

$$\|\partial_{\xi_i}\partial_x^m\hat{Z}(x,t;y+\xi,s)\|$$

$$=\|\partial_{\zeta_i}\partial_z^m\hat{Z}(z+\zeta,t;\zeta,s)-\partial_{z_i}\partial_z^m\hat{Z}(z+\zeta,t;\zeta,s)\|$$

$$\ll\frac{1}{(t-s)^{(d+|m|+1)/q}}\exp\left\{-c\left|\frac{|x-y-\xi|^q}{t-s}\right|^{1/(q-1)}\right\}\tag{137}$$

for all $x, y, \xi\in\Re^d$, $0\leqslant s\leqslant t\leqslant 1$. Hence, since $s\leqslant(t+\tau)/2$ we find by (45), (44), (137), (136), Lemma 5(i), and Lemma 4(i) that there are $0<c_1<c_{3/2}<c_2<c$ and $\hat{c}>0$ such that

$$\|\partial_{\xi_i}Y^{\varepsilon}_{m,k}(x,t;y+\xi,s;\xi,\tau)\|$$

$$=(t-\tau)^{\alpha+(d/q)}$$

$$\left\|\partial_{\xi_i}\left\{\partial_x^m\hat{Z}(x,t;y+\xi,s)\,\tilde{A}_k\left(y+\xi,\xi,\frac{s}{\varepsilon}\right)\partial_y^k\hat{Z}(y+\xi,s;\xi,\tau)\right.\right.$$

$$\left.\left.\times\exp\left\{c_1\left|\frac{|x-\xi|^q}{t-\tau}\right|^{1/(q-1)}\right\}\right\}\right\|$$

$$\overset{\varepsilon;\,x,\,t;\,y,\,s;\,\xi,\,\tau}{\ll}\frac{(t-\tau)^{\alpha+(d/q)}}{(t-s)^{(d+|m|)/q}\,(s-\tau)^{1+(d-\varsigma)/q}}\left[\frac{1}{(t-s)^{1/q}}+1+\frac{1}{(t-\tau)^{1/q}}\right]$$

$$\times \exp\left\{-c\left|\frac{|x-\xi-y|^q}{t-s}\right|^{1/(q-1)} - c_2\left|\frac{|y|^q}{s-\tau}\right|^{1/(q-1)} + c_{3/2}\left|\frac{|x-\xi|^q}{t-\tau}\right|^{1/(q-1)}\right\}$$

$$\overset{\varepsilon;\, x,\, t;\, y,\, s;\, \xi,\, \tau}{\ll} \frac{1}{(t-\tau)^{1/q}\,(s-\tau)^{1+(d-\varsigma)/q}} \times \exp\left\{-\hat{c}\left|\frac{|y|^q}{s-\tau}\right|^{1/(q-1)} - \hat{c}\left|\frac{|x-\xi|^q}{t-\tau}\right|^{1/(q-1)}\right\} \tag{138}$$

for all $\varepsilon \in (0, 1]$, $x, y, \xi \in \mathfrak{R}^d$, and $0 \leqslant \tau \leqslant s \leqslant t \leqslant 1$ such that $s \leqslant (t+\tau)/2$. ∎

LEMMA 3. *Suppose n is a positive integer; τ, C, c_2, a, and $\{a_i\}_{i=1}^{q+d}$ are non-negative constants satisfying $0 < a < a_{i+1} < a_i < c_2$ for all $i = 1, ..., q+d-1$; and $(U, \mathscr{D}(U))$ is the operator on the vector space of continuous functions on $\mathfrak{R}^d \times [\tau, 1]$ defined by*

$$U\varphi(x, t) \doteq \int_\tau^t \int_{\mathfrak{R}^d} \frac{C}{(t-\sigma)^{1+(d-1)/q}} \times \exp\left\{-c_2\left|\frac{|x-z|^q}{t-\sigma}\right|^{1/(q-1)}\right\} \varphi(z, \sigma)\, dz\, d\sigma \tag{139}$$

for all $\varphi \in \mathscr{D}(U)$, $t \in [\tau, 1]$. $x \in \mathfrak{R}^d$, $\mathscr{D}(U)$ being an appropriately defined domain. Then; for any $\varphi \in \mathscr{D}(U)$ such that $U^m\varphi \in \mathscr{D}(U)$ for $m = 1, 2, ..., n-1$; we find that

$$U^n\varphi(x, t) \leqslant \begin{cases} \displaystyle\int_\tau^t \int_{\mathfrak{R}^d} \frac{B^n}{(t-s)^{1+(d-n)/q}} \exp\left\{-a_n\left|\frac{|x-y|^q}{t-s}\right|^{1/(q-1)}\right\} \times |\varphi(y, s)|\, dy\, ds & n < q+d \\ \displaystyle\int_\tau^t \int_{\mathfrak{R}^d} \frac{B' A^n (t-s)^{(n-d)/q-1}}{\Gamma(1+(n-d)/q)} \exp\left\{-a\left|\frac{|x-y|^q}{t-s}\right|^{1/(q-1)}\right\} \times |\varphi(y, s)|\, dy\, ds & n \geqslant q+d \end{cases} \tag{140}$$

for all $t \in [\tau, 1]$, $x \in \mathfrak{R}^d$; where $B > 0$ does not depend on x, t or φ; $B' \doteq B^{q+d}$; and $A \doteq [C \cdot \int_{\mathfrak{R}^d} \exp\{-(c_2 - a)\, |z|^{q/(q-1)}\}\, dz \cdot \Gamma(1/2p)] \vee 1$.

Proof. (140) holds for $n = 1$ so we assume that it holds for all $1 \leqslant n \leqslant n_0 < q+d$. Then, we find by (139), (140), and Lemma 4(ii) that

$$U^{n_0+1}\varphi(x,t)\leqslant U\left(\int_\tau^\sigma\int_{\mathfrak{R}^d}\frac{B^{n_0}\,|\varphi(y,s)|}{(\sigma-s)^{1+(d-n_0)/q}}\right.$$
$$\left.\times\exp\left\{-a_{n_0}\left|\frac{|z-y|^q}{\sigma-s}\right|^{1/(q-1)}\right\}dy\,ds\right)(x,t)$$
$$\leqslant\int_\tau^t\int_{\mathfrak{R}^d}\int_s^t\int_{\mathfrak{R}^d}\frac{dz\,d\sigma\,dy\,ds\,CB^{n_0}\,|\varphi(y,s)|}{(t-\sigma)^{1+(d-1)/q}\,(\sigma-s)^{1+(d-n_0)/q}}$$
$$\times\exp\left\{-c_2\left|\frac{|x-z|^q}{t-\sigma}\right|^{1/(q-1)}-a_{n_0}\left|\frac{|z-y|^q}{\sigma-s}\right|^{1/(q-1)}\right\}$$
$$\leqslant\int_\tau^t\int_{\mathfrak{R}^d}\frac{B^{n_0+1}}{(t-s)^{1+(d-n_0-1)/q}}$$
$$\times\exp\left\{-a_{n_0+1}\left|\frac{|x-y|^q}{t-s}\right|^{1/(q-1)}\right\}|\varphi(y,s)|\,dy\,ds \qquad (141)$$

for all $t\in[\tau,1]$, $x\in\mathfrak{R}^d$ provided $B>C$ is large enough. Moreover, if (140) holds for all $1\leqslant n\leqslant n_0$ with $n_0\geqslant q+d$ then by (139), (140), and Lemma 4(i), it follows that

$$U^{n_0+1}\varphi(x,t)\leqslant U\int_\tau^\sigma\int_{\mathfrak{R}^d}\frac{B'A^{n_0}\,|\sigma-s|^{(n_0-d-q)/q}}{\Gamma\left(1+\dfrac{n_0-d}{q}\right)}$$
$$\times\exp\left\{-a\left|\frac{|z-y|^q}{\sigma-s}\right|^{1/(q-1)}\right\}|\varphi(y,s)|\,dy\,ds\,(x,t)$$
$$\leqslant\int_\tau^t\int_{\mathfrak{R}^d}\int_s^t\int_{\mathfrak{R}^d}\exp\left\{-(c_2-a)\left|\frac{|z|^q}{t-\sigma}\right|^{1/(q-1)}\right\}\frac{CB'A^{n_0}\,dz}{(t-\sigma)^{d/q}}$$
$$\times\frac{(\sigma-s)^{(n_0-d-q)/q}\,d\sigma}{(t-\sigma)^{1-(1/q)}\,\Gamma\left(1+\dfrac{n_0-d}{q}\right)}$$
$$\times\exp\left\{-a\left|\frac{|x-y|^q}{t-s}\right|^{1/(q-1)}\right\}|\varphi(y,s)|\,dy\,ds$$
$$\leqslant\int_\tau^t\int_{\mathfrak{R}^d}\frac{B'A^{n_0+1}(t-s)^{(n_0+1-d-q)/q}}{\Gamma\left(1+\dfrac{n_0+1-d}{q}\right)}$$
$$\times\exp\left\{-a\left|\frac{|x-y|^q}{t-s}\right|^{1/(q-1)}\right\}|\varphi(y,s)|\,dy\,ds. \quad\blacksquare \qquad (142)$$

The following lemma is used throughout Subsections 3.3 and 3.4. Part (i) follows inter alia from the proof of Theorem 2 (see the equation following Equation (4.15) on p. 254) in Friedman [6]. Part (ii) is a simple consequence of Lemma 7 p. 253 of Friedman [6].

LEMMA 4. *Let $0 \leqslant \alpha, \beta < 1$, q be an even positive integer,*

$$f(y, s; z, \sigma) \doteq \left|\frac{|y|^q}{s}\right|^{1/(q-1)} + \left|\frac{|z|^q}{\sigma}\right|^{1/(q-1)} \qquad \forall y, z \in \mathfrak{R}^d,\ s, \sigma > 0 \tag{143}$$

and $J_a \doteq J_a(x, t; \xi, \tau)$ be defined by

$$J_a \doteq \int_\tau^t \int_{\mathfrak{R}^d} \frac{dy\, ds}{(t-s)^{(d/q)+\alpha}\,(s-\tau)^{(d/q)+\beta}} \exp\{-a\, f(x-y, t-s; y-\xi, s-\tau)\}$$

for all $a > 0$, $x, \xi \in \mathfrak{R}^d$ and $0 \leqslant \tau \leqslant t \leqslant 1$. Then,

$$\text{(i)}\quad f(y, s; z, \sigma) \geqslant \left|\frac{|y+z|^q}{s+\sigma}\right|^{1/(q-1)} \qquad \forall y, z \in \mathfrak{R}^d,\ s, \sigma > 0 \tag{144}$$

and for any $0 < a' < a$ there exists a $M_{a'} = M_{a', \alpha, \beta} > 0$ such that

$$\text{(ii)}\quad J_a \leqslant \frac{M_{a'}}{(t-\tau)^{(d/q)+\alpha+\beta-1}} \exp\left\{-a' \left|\frac{|x-\xi|^q}{t-\tau}\right|^{1/(q-1)}\right\} \tag{145}$$

for all $0 \leqslant \tau \leqslant t \leqslant 1$, and $x, \xi \in \mathfrak{R}^d$.

LEMMA 5. *Suppose Conditions* (C1–C4) *of Section* 2 *are satisfied and V^ε, V are as defined in* (60) *and* (61). *Then, there exist $\lambda, K > 0$ independent of $\varepsilon \in (0, 1]$, $0 \leqslant \tau \leqslant t \leqslant 1$, $\zeta = \alpha + i\beta \in \mathscr{C}^d$, and $y \in \mathfrak{R}^d$ such that*

$$\|V^\varepsilon(t, \tau; y, \zeta)\| \vee \|V(t, \tau; y, \zeta)\| \leqslant K \exp\{[\lambda\, |\beta|^q - \delta\, |\alpha|^q](t-\tau)\}, \tag{146}$$

where $\delta > 0$ is the constant of (35). *Moreover, $\hat{Z}$, $\hat{Z}^\varepsilon$, K, K^ε, Φ, and Φ^ε, as defined in* (43) *and* (51–54), *exist as continuous function on*

$$\Delta^1 \doteq \{(x, t, \xi, \tau) \in \mathfrak{R}^d \times [0, 1] \times \mathfrak{R}^d \times [0, 1] : \tau < t\}, \tag{147}$$

$\hat{Z}$ and $\hat{Z}^\varepsilon$ are continuously differentiable to any order with respect x, and there exist $c, C > 0$ independent of $(x, t; \xi, \tau; j, a)$ and ε such that

$$\text{(i)}\quad \|\partial_x^a \hat{Z}(x, t; \xi, \tau)\| \vee \|\partial_x^a \hat{Z}^\varepsilon(x, t; \xi, \tau)\| \leqslant \frac{C}{(t-\tau)^{(d+|a|)/q}} \exp\left\{-c \left|\frac{|x-\xi|^q}{t-\tau}\right|^{1/(q-1)}\right\} \tag{148}$$

(ii) $\|K(x, t; \xi, \tau)\| \vee \|K^{\varepsilon}(x, t; \xi, \tau)\|$

$$\leqslant \frac{C}{(t-\tau)^{1+(d-1)/q}} \exp\left\{-c\left|\frac{|x-\xi|^q}{t-\tau}\right|^{1/(q-1)}\right\} \tag{149}$$

(iii) $\|\Phi(x, t; \xi, \tau)\| \vee \|\Phi^{\varepsilon}(x, t; \xi, \tau)\|$

$$\leqslant \frac{C}{(t-\tau)^{1+(d-1)/q}} \exp\left\{-c\left|\frac{|x-\xi|^q}{t-\tau}\right|^{1/(q-1)}\right\} \tag{150}$$

(iv) $\|\partial_{\xi_j}\partial_x^a \hat{Z}(x+\xi, t; \xi, \tau)\|$

$$\leqslant \frac{C}{(t-\tau)^{(d+|a|)/q}} \exp\left\{-c\left|\frac{|x|^q}{t-\tau}\right|^{1/(q-1)}\right\} \tag{151}$$

for all $(x, t, \xi, \tau) \in \Delta^1$, $\varepsilon \in (0, 1]$, $j = 1, ..., d$ *and* $0 \leqslant |a| \leqslant 2q$ *(say).*

Proof. Suppose we showed that there is a constant $\lambda' > 0$ such that

$$\|V^{\varepsilon}(t, \tau; y, \zeta)\|^2 \leqslant N \exp\{[\lambda' |\beta|^q - \delta |\alpha|^q](t-\tau)\} \tag{152}$$

for all $(x, t, \xi, \tau) \in \Delta^1$, $0 \leqslant \tau \leqslant t \leqslant 1$, $y \in \Re^d$, $\zeta = \alpha + i\beta$, and $\varepsilon \in (0, 1]$. Then, a similar bound for V could be established by elementary methods and (146) would follow. Furthermore, since the theory on pp. 245–246 of [6] does not require Assumption (A) of the introduction once (146) has been established, (i) above would follow. (ii) would follow from (i), (51–52), and Conditions (C3–C4) and (iv) would follow from Condition (C4) and (3.11) of Friedman [6]. Finally, (iii) would result from (ii) and the development on pp. 252–255 (top) of Friedman [6]. Hence, it only remains to show (152).

To establish (152) we adapt the method of Agarwal and Gupta [1] pp. 174–5. Thus, we define the norm

$$|||B||| = \sqrt{\sum_{m, n=1}^{N} |B_{m,n}|^2}, \tag{153}$$

$B_{m,n}$ representing the (m, n)th element of B, for all $\mathscr{C}^{N\times N}$-matrices B and let

$$\begin{aligned} R^{\varepsilon}(t, \tau, y, \zeta) &= \mathrm{Re}[V^{\varepsilon}(t, \tau; y, \zeta)], \\ I^{\varepsilon}(t, \tau, y, \zeta) &= \mathrm{Im}[V^{\varepsilon}(t, \tau; y, \zeta)] \end{aligned} \tag{154}$$

so $V^\varepsilon = R^\varepsilon + iI^\varepsilon$. Then; suppressing the dependence on (y, ζ) for V^ε, R^ε, I^ε; recalling definition (60); letting

$$\mathscr{A}(s) = \sum_{|k|=q} A_k(y, s)(i\zeta)^k; \tag{155}$$

and utilizing properties of symmetric and skew-symmetric matrices; one finds that

$$\begin{aligned}
&|||V^\varepsilon(t, \tau)|||^2 - |||V^\varepsilon(t', \tau)|||^2 \\
&\quad = 2 \sum_{m,n} \int_{t'}^{t} [R^\varepsilon_{m,n}(s, \tau)\{\mathrm{Re}[\mathscr{A}(s/\varepsilon)\, V^\varepsilon(s, \tau)]\}_{m,n} \\
&\qquad + I^\varepsilon_{m,n}(s, \tau)\{\mathrm{Im}[\mathscr{A}(s/\varepsilon)\, V^\varepsilon(s, \tau)]\}_{m,n}]\, ds \\
&\quad = \sum_n \int_{t'}^{t} [R^\varepsilon_n(s, \tau)^T\, I^\varepsilon_n(s, \tau)^T]\, \mathscr{S}(s/\varepsilon) \begin{bmatrix} R^\varepsilon_n(s, \tau) \\ I^\varepsilon_n(s, \tau) \end{bmatrix} ds \\
&\quad \leqslant \int_{t'}^{t} \lambda_{\max}(\mathscr{S}(s/\varepsilon)) \cdot |||V^\varepsilon(s, \tau)|||^2\, ds \qquad \forall 0 \leqslant t' \leqslant t \leqslant 1, \quad \varepsilon \in (0, 1],
\end{aligned} \tag{156}$$

where $R^\varepsilon_n(s, \tau)$ denotes the nth column of $R^\varepsilon(s, \tau)$ and $\mathscr{S}(\sigma) = \mathscr{S}(\sigma, \zeta, y)$ is defined by

$$\begin{aligned}
\mathscr{S}(\sigma) \doteq \sum_{|k|=q} (i\zeta)^k \\
\times \begin{bmatrix} \mathrm{Re}[A_k(y, \sigma) + A_k^T(y, \sigma))] & -\mathrm{Im}[A_k(y, \sigma) - A_k^T(y, \sigma))] \\ \mathrm{Im}[A_k(y, \sigma) - A_k^T(y, \sigma))] & \mathrm{Re}[A_k(y, \sigma) + A_k^T(y, \sigma))] \end{bmatrix}
\end{aligned} \tag{157}$$

for all $0 \leqslant \sigma < \infty$, $y \in \mathfrak{R}^d$, and $\zeta \in \mathscr{C}^d$. Now, it follows from Condition (C1), (35), and Friedman [6] Lemma 1, p. 242 that the maximum eigenvalue of $\mathscr{S}(\sigma, \zeta, y)$ satisfies

$$\lambda_{\max}(\mathscr{S}(\sigma, \zeta, y)) \leqslant -\delta\, |\alpha|^q + \lambda'\, |\beta|^q \tag{158}$$

for all $0 \leqslant \sigma < \infty$, $\zeta = \alpha + i\beta \in \mathscr{C}^d$, and $y \in \mathfrak{R}^d$, where $\delta > 0$ is the constant of (35) and $\lambda' > 0$ does not depend on σ, ζ, or y. Hence, it follows by (156) and (158) that

$$d_t[\, |||V^\varepsilon(t, \tau; y, \zeta)|||^2 \exp\{[\delta\, |\alpha|^q - \lambda'\, |\beta|^q](t - \tau)\}\,] \leqslant 0 \tag{159}$$

for all $0 \leqslant \tau \leqslant t \leqslant 1$, $\varepsilon \in (0, 1]$, $y \in \mathfrak{R}^d$, $\zeta \in \mathscr{C}^d$, and (152) follows since $|||V^\varepsilon(\tau, \tau; y, \zeta)|||^2 = N$. ∎

LEMMA 6. *Suppose ψ^{ε} is as defined in* (39) *of Subsection* 3.1; *and* $\bar{\lambda}$, $\bar{\delta}$ *are as in* (64) *of Subsection* 3.2. *Then, under Conditions* (C1–5) *of Section* 2 *it follows that* $(t, \tau; y, \zeta) \to \psi^{\varepsilon}(t, \tau; y, \zeta)$, $\varepsilon > 0$ *are equicontinuous and*

$$\|\psi^{\varepsilon}(t, \tau; y, \zeta)\| \overset{\varepsilon, t, \tau, y, \zeta}{\ll} \exp\{-[\bar{\lambda}\, |\beta|^q + \bar{\delta}\, |\alpha|^q](t-\tau)\}(t-\tau)^{\chi}. \tag{160}$$

Proof. (160) follows immediately via (40), (39), Condition (C2) and (62–64). Moreover, as a result of this argument one finds that

$$\|\eta^{\varepsilon}(t, \tau; y, \zeta)\| \overset{\varepsilon, t, \tau, y, \zeta}{\ll} \exp\{[\lambda\, |\beta|^q - \delta\, |\alpha|^q](t-\tau)\}(t-\tau). \tag{161}$$

In preparation for the equicontinuity argument, we note that it follows from the argument in (79–80), Equations (60–4), and Condition (C4) that

$$\begin{aligned} &\| V^{\varepsilon}(t, s; y, \zeta) - V^{\varepsilon}(t, s; y', \zeta)\| \\ &\leqslant \sum_{|k|=q} \left\| \int_s^t V^{\varepsilon}(t, \sigma; y, \zeta) \left[A_k\left(y, \frac{\sigma}{\varepsilon}\right) - A_k\left(y', \frac{\sigma}{\varepsilon}\right)\right] V^{\varepsilon}(\sigma, s; y', \zeta)\, d\sigma \right\| \cdot |\zeta|^q \\ &\overset{\varepsilon, t, \tau, y, \zeta}{\ll} |y - y'| \exp\{[\lambda_1\, |\beta|^q - \delta_1\, |\alpha|^q](t-s)\}, \end{aligned} \tag{162}$$

and

$$\begin{aligned} &\| V(s, \tau; y, \zeta) - V(s, \tau; y', \zeta)\| \\ &\qquad \overset{s, \tau, y, \zeta}{\ll} |y - y'| \exp\{[\lambda_1\, |\beta|^q - \delta_1\, |\alpha|^q](s-\tau)\} \end{aligned} \tag{163}$$

for all $\varepsilon \in (0, 1]$, y, y', α, $\beta \in \Re^d$, $\zeta = \alpha + i\beta$ and all $0 \leqslant \tau \leqslant s \leqslant t \leqslant 1$. Hence, it follows from (40), Minkowski's inequality, (62), (63), (162), (163), (34), and Conditions (C3) and (C4) that

$$\begin{aligned} &\|\eta^{\varepsilon}(t, \tau; y, \zeta) - \eta^{\varepsilon}(t, \tau; y', \zeta)\| \\ &\qquad \overset{\varepsilon, t, \tau, y, y', \zeta}{\ll} |y - y'|\, (t-\tau) \exp\{[\lambda_1\, |\beta|^q - \delta_1\, |\alpha|^q](t-\tau)\} \end{aligned} \tag{164}$$

from which it follows immediately from (39) that

$$\|\psi^{\varepsilon}(t, \tau; y, \zeta) - \psi^{\varepsilon}(t, \tau; y', \zeta)\| \overset{\varepsilon, t, \tau, y, y', \zeta}{\ll} |y - y'| \tag{165}$$

for all $\varepsilon \in (0, 1]$, y, $y' \in \Re^d$, $\zeta \in \mathscr{C}^d$, and $0 \leqslant \tau \leqslant t \leqslant 1$. As for $\zeta = \alpha + i\beta$, we note by (60), variation of constants, Condition (C3) and (62) that

$$\|\partial_{\alpha_j} V^\varepsilon(t, s; y, \zeta)\|$$

$$= \left\| \int_s^t V^\varepsilon(t, \sigma; y, \zeta) \sum_{|k|=q} A_k\left(y, \frac{\sigma}{\varepsilon}\right) [\partial_{\alpha_j}(i\alpha - \beta)^k]\, V^\varepsilon(\sigma, s; y, \zeta)\, d\sigma \right\|$$

$$\overset{\varepsilon, t, \tau, y, \zeta}{\ll} (t-s)^{1/q + (q-1)/q}\, |\zeta|^{q-1} \exp\{[\lambda\, |\beta|^q - \delta\, |\alpha|^q](t-s)\}$$

$$\overset{\varepsilon, t, \tau, y, \zeta}{\ll} \exp\{[\lambda_1\, |\beta|^q - \delta_1\, |\alpha|^q](t-s)\}, \tag{166}$$

and a similar bound holds for $\partial_{\alpha_j} V(s, \tau; y, \zeta)$ so by (40), (64), and Condition (C3)

$$\|\partial_{\alpha_j} \eta^\varepsilon(t, \tau; y, \zeta)\| \overset{\varepsilon, t, \tau, y, \zeta}{\ll} (t-\tau) \exp\{[\lambda_1\, |\beta|^q - \delta_1\, |\alpha|^q](t-\tau)\} \tag{167}$$

for all $j = 1, 2, ..., d$, $\varepsilon \in (0, 1]$, $y, \alpha, \beta \in \Re^d$, and $0 \leqslant \tau \leqslant t \leqslant 1$. Furthermore, since $d_\alpha\, |\alpha|^q = q\, |\alpha|^{q(q-2)/(q-1)} \sum_{j=1}^d |\alpha_j|^{1/(q-1)} \operatorname{sgn}(\alpha_j) e_j$ by (14), we find

$$|d_\alpha \exp\{[\delta_1\, |\alpha|^q - \lambda_1\, |\beta|^q](t-\tau)\}|$$

$$\leqslant (t-\tau)\, |\zeta|^{q-1} \exp\{[\delta_1\, |\alpha|^q - \lambda_1\, |\beta|^q](t-\tau)\} \tag{168}$$

so we easily discover through (39), the mean value theorem, (167), (161), and (168) that

$$\|\psi^\varepsilon(t, \tau; y, \alpha + i\beta) - \psi^\varepsilon(t, \tau; y, \alpha' + i\beta)\|$$

$$\leqslant (t-\tau)^{\chi - 1}\, \|d_\alpha[\eta^\varepsilon(t, \tau; y, \alpha^* + i\beta)$$

$$\times \exp\{[\delta_1\, |\alpha^*|^q - \lambda_1\, |\beta|^q](t-\tau)\}](\alpha - \alpha')\|$$

$$\overset{\varepsilon, t, \tau, y, \zeta, \zeta'}{\ll} |\alpha - \alpha'|\, (t-\tau)^\chi \overset{\varepsilon, t, \tau, y, \zeta, \zeta'}{\ll} |\alpha - \alpha'| \tag{169}$$

for all $\varepsilon \in (0, 1]$, $y, \alpha, \alpha', \beta \in \Re^d$, and $0 \leqslant \tau \leqslant t \leqslant 1$, where α^* in some point on the line connecting α and α'. A similar bound can be established in terms of β. Next, we consider the uniform continuity in the pair (τ, t). Indeed, one finds by Condition (C3), (62), and (63) that

$$\left\| \int_{t'}^t V^\varepsilon(t, s; y, \zeta) \sum_{|k|=q} \tilde{A}_k\left(y, \frac{s}{\varepsilon}\right) V(s, \tau; y, \zeta)\, ds \right\| (t-\tau)^{\chi - 1} \tag{170}$$

$$\overset{\varepsilon, t, t', \tau, y, \zeta}{\ll} (t-t')^\chi \exp\{[\lambda\, |\beta|^q - \delta\, |\alpha|^q](t-\tau)\}$$

for all $0 \leqslant \tau \leqslant t' \leqslant t \leqslant 1$ and by (60), (62–64), and Condition (C3), that

$$\left\| \int_{t'}^{t} \int_{\tau}^{t'} d_\sigma \{ V^\varepsilon(\sigma, s; y, \zeta) \exp\{[\delta_1 |\alpha|^q - \lambda_1 |\beta|^q](\sigma - \tau)\} (\sigma - \tau)^{\chi - 1} \} \right.$$

$$\left. \times \sum_{|k| = q} \tilde{A}_k \left(y, \frac{s}{\varepsilon} \right) V(s, \tau; y, \zeta)\, ds\, d\sigma \right\|$$

$$\overset{\varepsilon, t, t', \tau, y, \zeta}{\ll} (t' - \tau) \int_{t'}^{t} \left\{ |\zeta|^q + \frac{1}{\sigma - \tau} \right\}$$

$$\times \exp\{ -[\bar{\lambda} |\beta|^q + \bar{\delta} |\alpha|^q](\sigma - \tau)\} (\sigma - \tau)^{\chi - 1}\, d\sigma$$

$$\overset{\varepsilon, t, t', \tau, y, \zeta}{\ll} (t' - \tau) \int_{t'}^{t} (\sigma - \tau)^{\chi - 2}\, d\sigma$$

$$\leqslant \frac{(t - \tau)^\chi - (t' - \tau)^\chi}{\chi} \overset{\varepsilon, t, t', \tau, y, \zeta}{\ll} (t - t')^\chi \tag{171}$$

for all $0 \leqslant \tau \leqslant t' \leqslant t \leqslant 1$. Therefore, it follows by (39), (40), (64), (170) and (171) that

$$\| \psi^\varepsilon(t, \tau; y, \zeta) - \psi^\varepsilon(t', \tau; y, \zeta) \| \overset{\varepsilon, t, t', \tau, y, \zeta}{\ll} (t - t')^\chi \tag{172}$$

for all $\varepsilon \in (0, 1]$, $y \in \Re^d$, $\zeta \in \mathscr{C}^d$, and $0 \leqslant \tau \leqslant t' \leqslant t \leqslant 1$. In exactly the same manner, we find that

$$\| \psi^\varepsilon(t', \tau; y, \zeta) - \psi^\varepsilon(t', \tau'; y, \zeta) \| \overset{\varepsilon, t, \tau', \tau, y, \zeta}{\ll} |\tau - \tau'|^\chi \tag{173}$$

for all $\varepsilon \in (0, 1]$, $y \in \Re^d$, $\zeta \in \mathscr{C}^d$, and $0 \leqslant \tau \leqslant \tau' \leqslant t \leqslant 1$ or $0 \leqslant \tau' \leqslant \tau \leqslant t' \leqslant 1$. Moreover, if $t' < \tau$ then $t - \tau \leqslant t - t'$ and $t' - \tau' \leqslant \tau - \tau'$ and it follows from (160), (172), and (173) that

$$\| \psi^\varepsilon(t, \tau; y, \zeta) - \psi^\varepsilon(t', \tau'; y, \zeta) \| \overset{\varepsilon, t, t', \tau, \tau', y, \zeta}{\ll} |t - t'|^\chi + |\tau - \tau'|^\chi \tag{174}$$

for all $\varepsilon \in (0, 1]$, $y \in \Re^d$, $\zeta \in \mathscr{C}^d$, $0 \leqslant \tau \leqslant t \leqslant 1$ and $0 \leqslant \tau' \leqslant t' \leqslant 1$. Hence, it follows easily from (165), (169), and (174) that $(t, \tau; y, \zeta) \to \psi^\varepsilon(t, \tau; y, \zeta)$ are equicontinuous. ∎

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