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EXTENDED OBJECTS IN QUANTUM SYSTEMS

by

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A THESIS

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled "Extended Objects in Quantum Systems" submitted by Pasquale Sodano in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Theoretical Physics.

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ABSTRACT

Extended solutions of field theories with spontaneously broken symmetries have been widely investigated in recent years. The properties of these solutions have been analysed in several models by means of the soliton solutions of the Euler equations for the classical fields. An interesting feature of models exhibiting such solutions is the appearance of the so called topological charge. The thesis is intended to present, in the framework of the boson theory, a quantum field theoretical study of the properties of the extended objects appearing in the quantum ordered state; by quantum ordered state we mean the state with spontaneous breakdown of symmetries.

As a first step of our analysis we study the process of creation of the ordered state. We analyse models with global and gauge symmetries. We see that the spontaneous creation of an ordered state is always caused by a symmetry rearrangement when the symmetry of the Heisenberg fields is global. Then, we analyse peculiar features of gauge theories: we show that, in ordinary quantum electrodynamics, the dynamical rearrangement of symmetry takes place even when no ordered state is created.

As a second step of our analysis we study the creation of extended objects in the quantum ordered state. In our approach the spacetime properties of extended objects are described by means of a c-number field $\phi^f(x)$ constructed

from the quantum theory by the boson transformation. We show that the c-number field $\phi^f(x)$ constructed by the boson method becomes the soliton solution of the Euler equations when the Planck constant, h, is ignored, implying that the soliton solution can be regarded as an extended object with quantum origin. Starting from the Heisenberg equations of the quantum theory we obtain the equations for $\phi^f(x)$ which can be regarded as classical Euler equations. When the tree approximation is used, the Euler equations for $\phi^f(x)$ have the same form as the original Heisenberg equations. The general argument is supplemented by a concrete example which shows how the boson transformation applied to a quantum system leads to the static soliton solution in the (1 + 1) dimensional $\lambda\phi^4$ model. Using the tree approximation, we prove also the finiteness of the energy of the soliton constructed by the boson method. After this pioneering work our method has been successfully used in the analysis of soliton solutions of more complicated models.

Finally, we analyse the relation between the basic symmetry of the theory and the topological charge. We reach the conclusion that, although the basic symmetry does not restrict the shape of the extended objects appearing in the ordered state, it strongly influences the answer to the question asking which extended object can be classified by topological quantum number. Requiring the single valuedness of the c-number field $\phi^f(x)$, we express the condition for the topological quantization of an extended object in terms of the asymptotic behavior of the boson function. The general argument is supplemented by concrete examples of models with U(1)-symmetry.

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CHAPTER I

INTRODUCTION: EXTENDED OBJECTS IN Q.F.T.

1.1 The problem and its motivations

In recent years the structure of extended objects, appearing in a quantum field theory with spontaneous breakdown of symmetries, have been widely investigated both in high energy and solid state physics.

In high energy physics the interest in the study of extended solutions of field theories has been stimulated in connection with the problem of quark confinement[(1)] and with the fact that dual models seem to require an extended structure of the hadrons[(2)]. In fact, many high energy physicists feel that the experimental work in particle physics during the past years has given an increasing support to the idea of an extended model for the structure of the particles which participate in strong interactions. The present view, held by most, is that any model which has a reasonable chance of accounting for all the diverse phenomena must include quarks as the basic ingredients in the composition of the physical hadrons. Although the quarks thus follow in the historic traditions of the explanation of the structure of the matter as the latest element in the sequence of elementary parts, nature has made a qualitative change in this case: in fact, it is a common belief among physicists that quarks cannot exist in isolation and that they are permanently confined constituents of the hadrons. Several models[1] have been proposed in order to construct a scheme in which quark confinement could be understood in the framework o[illegible]ntum field theory; a common feature of all these [illegible]s is the introduction of an extended domain in which quarks could be confined. Furthermore, it has been proposed that the spectrum of the dual resonance models[3] could be regarded as the quantum states of a relativistic string. The Veneziano model[4], which incorporated the feature of duality and which, when generalized to include multi-particle processes, took shape as the dual resonance model, had a rather rich spectrum of states in quite good agreement with the observed hadron states. If this duality would indeed originate from the string structure, as many physicists feel, then the study of extended objects would become a central problem in analysis of hadron physics. The elementary structure, instead of being a geometrical point, is a sequence of geometrical points connected in a linear chain in the case of the string or a closed surface in the case of bags. Thus, the indication coming from the phenomenology of the hadrons suggest looking for extended solutions of interacting quantum field theories with spontaneously broken symmetries. This problem is far from being a trivial one. In fact, the usual perturbative approach of Q.F.T. cannot be applied in this case and new methods to perform accurate but approximate calculations in quantum field theory are needed. In the course of the last few years, there has been a revival in the use of semi-classical methods toward this end, leading to many interesting results within this period. The central idea is to look for classical solutions to non-linear field equations, and then evaluate "quantum corrections" to the classical solution.

We find it useful to divide these methods into two categories: In the first method, quantum effects are regained by quantizing the classical solution through semi-classical Jordan-Wentzel-Kramers-Brillouin (J.W.K.B.) methods (Dashen-Hasslacher-Neveu[5], Korepin-Fadeev[6]). In the second approach, the full quantum theory is expanded in the Born-Oppenheimer fashion, for which the first term is computed classically, and quantum corrections are found in a series expansion (Goldstone-Jackiw[7]). Both the above mentioned methods have been illustrated in the case of 1+1 dimensional scalar theories[5,7]. Although the practical significance of all this for describing the present experimental data is obscure, we have learned from these studies that a quantum field theory might give rise to phenomena of a much richer variety than had been believed heretofore.

In principle, with some generalizations, the above outlined methods could be applied in exactly the same way to more realistic theories; a big difficulty in these cases lies at the classical level of solving exactly coupled non-linear differential equations. A host of interesting papers have emerged on approximate non-perturbative classical solutions to four dimensional theories. An important set, whose ancestry can be traced to the Nambu string[8], includes the work of Nielsen and Olesen[9], 't Hooft[10], Faddeev[11], Wu and Wu[12], Mandelstam[13], Polyakov[14] and D.H.N.[5]. These people obtain approximate extended solutions to gauge theories. The extended solutions appear to have a separate conserved quantum number related to their topology (topological charge). This is a generalization of the "kink" number of the $\lambda\phi^4$ theory introduced by Finkelstein[15]. The 't Hooft solution is in addition a spherically symmetric magnetic monopole.

The interest in the study of extended solutions of field theories with spontaneous breakdown of symmetries is not motivated only by problems in high energy physics. In solid state physics, one can find many examples of extended objects in the quantum ordered state (by quantum ordered state we mean the state with spontaneous breakdown of symmetries). The vortex in superconductivity, the dislocations in crystals, the Weiss domains in ferromagnetic materials are well-known examples. The study of the properties of the extended objects is performed in this case by means of the Ginzburg-Landau equations[16], which are regarded as a kind of classical Euler equation for the order parameter. Also in this case the solutions exhibit a quantized topological charge: quantization of the e.m. flux[17] in superconductors, and quantization of the Burger's vectors[18] in the theory of crystal dislocations are the well-known examples.

In the following we call solitons the extended solutions of the classical Euler equations. Although all of the above mentioned extended systems have usually been described by means of classical equations, a complete understanding of their properties must be found in the framework of the quantum theory. For this purpose, it is important to observe that many of the systems in which extended objects appear, when they are studied with the methods of quantum field theory, are in certain ordered states whose structure is more or less complicated according to the complexity of the system.

It is a remarkable fact that most classical objects manifest a certain order. To make this statement clearer, we consider the case of crystals. The equation for interacting molecules is translationally and rotationally invariant, as the equation for a molecular gas should be. Under certain conditions, the molecular gas system manifests the crystal lattice order (creation of order) and the translational and rotational symmetry disappear from the observations (spontaneous breakdown of symmetry). In this perfect crystal state, we can excite phonons and many other quantum levels. In other words, the perfect crystal state without boundaries is a quantum system of phonons and other excitations. However, we can modify the situation in such a way that there appear, in this quantum system, many kinds of extended objects (dislocation, point defects, ... etc.), thus creating a situation in which classical and quantum objects coexist.

In the course of the last years, a method which provides us with a very systematic approach to the study of extended systems in quantum ordered states, has been formulated by H. Umezawa and coworkers. It has been called the boson theory[19].

The thesis is intended to present, in the framework of the boson theory, a detailed study of the properties of the extended objects appearing in the quantum ordered state. In general terminology, we will be concerned with the following processes:

(a) creation of an ordered state(equivalent to the spontaneous breakdown of symmetry).

(b) creation of extended objects in quantum systems.

A detailed analysis of step (a) will be presented in Chapter II. There the phenomenon of the dynamical rearrangement of symmetry[20] will be carefully studied. It will be shown that the phenomenon of spontaneous breakdown of symmetry (equivalent to the creation of an ordered state) is always caused by these symmetry rearrangements when the symmetry of the Heisenberg fields is global. Then, we analyse gauge theories: we show[21] that in Q.E.D. the dynamical rearrangement of symmetry takes place even when there is no creation of an ordered state.

Analysis of step (b) leads to a systematic formulation of a theory for extended objects in a quantum system (the boson theory)[22]. The method was first applied to the analysis of vortices in type II superconductors[23] and led to results in good agreement with experiment. Then, it was applied to the Nielsen-Olesen vortex solutions[24] of a relativistic theory. More recently, the boson theory has been successfully applied to the study of extended structures appearing in crystals such as dislocations[25], point defects, surface phenomena[25] etc. A merit of the boson theory is that it allows us to avoid any use of classical arguments in the discussion of extended objects, thus implying the quantum origin of these structures.

Chapter II will be devoted to a study of the boson theory. There, we will be mainly concerned with the observable effects of the massless bosons. The problem of the relation between the basic symmetry of the theory and the topological charge will be analysed. It will be shown that, although the basic symmetry does not restrict the "shape" of the extended objects[26], it profoundly influences the answer to the question asking which extended object should be quantized. In fact, the basic group symmetry strongly controls the structure of the dynamical map[27] through which the topological quantum number is identified by the requirement of single valuedness of the observables of the theory.

To shed further light on these results, I briefly summarize how the boson theory is constructed. The construction of the boson theory can be summarized in two essential steps:

i) use of the in-field (quasi-particle in solid state physics) picture in the study of the microscopic properties of the system.

ii) use of the boson-transformation.

In the first step, one usually starts from a given set of Heisenberg field operators which satisfy known field equations:

$$\Lambda(\partial)\psi(x) = F[\psi] \quad . \tag{1.1}$$

One looks for solutions of eqs. (1.1) which can be expressed in terms of normal products of a set of certain free field operators $\{\phi\}$:

$$\psi(x) = \psi[x;\phi(x)] \quad . \tag{1.2}$$

This equation, which has been called "dynamical map" has to be read as a weak relation, in the sense that equality holds only among matrix elements. The existence and the properties of these free fields are determined in the course of a self-consistent calculation, in which the

Hilbert space is chosen as the Fock space of the same free fields. The Heisenberg fields operate in this space through the expression of the dynamical map (1.2). At this stage, all the calculations are performed by assuming that the order parameter is homogeneous (i.e. space time independent).

The second step[19] is considered when one wants to describe phenomena related to the presence of a space - and/or time-dependent order parameter. The spatial and temporal dependence is introduced by means of the boson transformation[19]. Let ϕ be one of the boson fields appearing in the dynamical map satisfying the free field equation

$$\lambda(\partial)\phi(x) = 0 . \qquad (1.3)$$

Then, we perform the substitution called the boson transformation

$$\phi(x) \rightarrow \phi(x) + f(x) \qquad (1.4)$$

where f(x) is a c-number function satisfying the equation:

$$\lambda(\partial)f(x) = 0 . \qquad (1.5)$$

As a result of the substitution (1.4) the Heisenberg operators transform according to

$$\psi(x) \rightarrow \psi^{f}(x) = \psi(x;\phi(x)+f(x),\ldots) . \qquad (1.6)$$

It can be proved that $\psi^{f}(x)$ satisfies the same Heisenberg equation of motion as $\psi(x)$ does. Such a statement is the content of the boson transformation theorem[19]. The intuitive content of the substitution (1.4) can be understood as follows: eqs. (1.4) and (1.5) show that the functions $f(x)$ are created by the condensation of the Bose quanta $\phi(x)$ in the vacuum of the physical Fock space. The result of this condensation is the appearance of extended objects. Due to the fact that $f(x)$ is a c-number function, the extended objects created by $f(x)$ behave classically in the sense that the quantum fluctuations are much smaller than the macroscopic effects of the condensed bosons

$$\text{i.e.} \quad \frac{h\Delta N}{hN} \longrightarrow 0 \quad \text{as} \quad N \longrightarrow \infty \; .$$

The c-number field $\phi^f(x)$ defined by

$$\phi^f(x) \equiv \langle 0|\psi^f(x)|0\rangle$$

is, at least when ψ is a boson field, the space-time dependent order parameter describing the properties of the extended object created by a particular choice of $f(x)$. How is $\phi^f(x)$, constructed by the quantum field, related to the Higgs field $\phi(x)$ describing the space time properties of the extended solutions of the classical Euler equations? The answer to this question is provided in Chapter III: we show[28] that $\phi^f(x)$ coincides with the soliton solution of the Euler equation when the Planck constant, h, is ignored, thus implying the quantum origin of solitons. The argument presented there guarantees also the finiteness of the energy of the soliton. One sees then, that use of boson transformation provides us with a technique to derive classical equations from the quantum theory. Furthermore, use of boson transformation simplifies the study of many classical equations. The different classical solitons correspond to different solutions of the classical Euler equations, solved under different boundary conditions. The advantage presented by the use of the boson transformation is that one does not have to apply the boundary conditions directly to the classical equations of motion, but only to the choice of that particular f(x) which will create the particular soliton solution in which one is interested.

It is clear that this offers a remarkable simplicity when we consider the fact that f(x) satisfies a simple homogeneous differential equation (the free field equation). When we try to apply the theory of extended objects to the particle physics, we meet a serious difficulty which arises from the fact that the extended objects behave classically, while we want to have quantum particles. To overcome this difficulty we might need an enlarged Hilbert space which contains all the states with topological objects. A very important result in this direction has been recently obtained[29]. It has been shown that it is possible to associate with the soliton a set of canonical variables (q,p): the space of the states of a system containing classical solitons is then the product of the Fock space $\mathcal{H}$ of the in-fields and the Hilbert space of the canonical variables (q,p).

Once the canonical coordinates are assigned it is possible to associate[29] a quantum coordinate Q with the soliton. In Chapter III we review some of these results. Here, it is important to point out that, depending on the observability or unobservability of the quantum fluctuation associated with Q, the soliton behaves as a quantum or a classical object. The relevance of this result for hadron physics is clear: in ref.[30] it was shown that a small object with an enclosed surface singularity behaves as the MIT bag. There, the quantum coordinate was neglected, and therefore, the system behaved as a classical object. The considerations of ref.[29] show that a careful treatment of the quantum coordinate could show the quantum behavior of the bag.

1.2 Extended solutions of classical Euler equations

This section is to provide a review of the results obtained in the study of the soliton solutions of the classical Euler equations. The general framework will be the following: starting from a classical Lagrangian field theory, invariant under an internal symmetry group G, one obtains the classical Euler equations by minimizing the action integral. Among the solutions of these equations one can distinguish, in principle, three types:

i) constant solutions (time and space independent)

ii) static solutions (time independent but space dependent)

iii) time and space dependent solutions.

In the following we will be mainly interested in solutions of class ii. Among these solutions, the solitons will be identified with solutions of the classical Euler equations which satisfy "peculiar" boundary conditions. The class iii of solutions is relevant for the study of multisolitons and Euclidean instantons[31].

The set of "peculiar" boundary conditions is usually assigned by taking into account the fact that, due to the spontaneous breakdown of the internal symmetry, the constant solutions of the classical Euler equations describe a manifold[32] parametrized by the parameters of a set M, which is related to the original symmetry group G by the relation $M = G/H$ (H an invariant subgroup). We will call this manifold the manifold of the constant solutions. Once the manifold of the constant solutions is specified, it is required[32] that the asymptotic values of the classical fields can have a non-trivial mapping onto this manifold. A non-trivial mapping is one that cannot, by continuous small changes, be deformed into the trivial mapping in which all the points of one manifold are mapped into a single point of the other.

The mathematical study of such maps is called homotopy theory (an introduction to the homotopy theory is provided in ref. [33]). Each map is characterized by an integer which, in all the cases known, is related to the topological charge. Thus, in this approach, the question of existence of solitons is seen as a boundary value problem, to wit: to construct solutions of the classical Euler equations with the requirement that the asymptotic values of the fields have a non-trivial mapping onto the manifold of the constant solutions of the Euler equations. The following will be dedicated to an analysis of some specific soliton solutions of the classical Euler equation and a study of their topological properties.

a) Static solitons in 1+1 dimensional scalar models

In order to encounter first in a simple setting the ideas that I wish to review, let us consider a field theory of a spinless field $\phi(x,t)$ in one spatial dimension.

The Lagrange density is assumed to be of the form

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - U(\phi) \quad . \tag{2.1}$$

For the energy to be positive definite, we take the field potential $U(\phi)$ to be non-negative

$$U(\phi) \geq 0 \quad . \tag{2.2}$$

$U(\phi)$ will in general depend on various numerical parameters (coupling constants). In the models we will be considering in the following $U(\phi)$ depends on the coupling constant g in a scaled fashion

$$U(\phi) = U(\phi;g) = \frac{1}{g^2} U(g\phi;1) . \qquad (2.3)$$

The equation satisfied by ϕ is:

$$\Box\phi + U'(\phi) = (\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2})\phi + U'(\phi) = 0 . \qquad (2.4)$$

I shall discuss first static c-number solutions to this field equation. Time-dependent solutions will be analyzed only later. In the static case, eq. (2.4) reads:

$$\frac{d^2}{dx^2} \phi = U'(\phi) \qquad (2.5)$$

and the static energy functional has the form

$$E_c(\phi) = \int dx[\frac{1}{2} (\frac{d}{dx} \phi)^2 + U(\phi)] . \qquad (2.6)$$

A first integral of the one dimensional equation (2.5) is given, for arbitrary U, by

$$\frac{1}{2} (\frac{d}{dx} \phi)^2 = U(\phi) + \text{const.} \qquad (2.7)$$

Use of (2.7) implies the following form for the static energy functional

$$E_c(\phi) = \int dx[2U(\phi) + \text{const.}] . \qquad (2.8)$$

In order to eliminate an unwanted infrared divergence of the classical energy we choose the value of the integration constant in (2.7) equal to 0.

To integrate (2.7) we need an explicit expression for $U(\phi)$. Two examples will be discussed explicitly; namely

a) the ϕ^4-theory;

$$U(\phi) = \frac{m^4}{2g^2}\left[1 - g^2\frac{\phi^2}{m^2}\right]^2 \qquad (2.9a)$$

b) the sine-Gordon theory;

$$U(\phi) = \frac{m^4}{g^2}\left[1 - \cos\left(\frac{g}{m}\phi\right)\right] \quad . \qquad (2.9b)$$

Both theories possess discrete symmetries

$$\phi^4\text{-theory;} \qquad \phi \to -\phi \qquad (2.10a)$$

$$\text{sine-Gordon theory;} \quad \phi \to \pm\phi + 2\pi n\frac{m}{g} \qquad (2.10b)$$
$$n = 0, \pm 1, \ldots$$

The constant solutions to (2.7) are

$$\phi^4\text{-theory;} \qquad \phi = \pm\frac{m}{g} \qquad (2.11a)$$

$$\text{sine-Gordon theory;} \quad \phi = 2\pi n\left(\frac{m}{g}\right) \quad . \qquad (2.11b)$$

The boundary conditions of eq. (2.7) are assigned by requiring that the asymptotic values of the fields are not the same in different directions (in this case it means that $\phi(+\infty) \neq \phi(-\infty)$).

Position-dependent solutions of (2.7), under the above specified boundary conditions, are the following:

$$\phi^4\text{-theory;} \qquad \phi_c(x) = \frac{m}{g}\tanh m(x-x_o) \qquad (2.12a)$$

$$\text{sine-Gordon theory;} \quad \phi_c(x) = 4\frac{m}{g}\tan^{-1}\exp \pm m(x-x_o) \quad . \qquad (2.12b)$$

The occurrence of the parameter x_o specifies the center of the soliton.

The classical energy of the solution is finite and it is given by:

ϕ^4-theory; $$E_c(\phi) = \frac{4}{3}\frac{m^3}{g^2} \tag{2.13a}$$

sine-Gordon theory; $$E_c(\phi) = 8\,\frac{m^3}{g^2} \tag{2.13b}$$

Let us observe that, in both theories the static solutions (2.12) are of order $0(g^{-1})$, just as are the constant solutions (2.11). They interpolate between the constant solutions as x ranges from $-\infty$ to $+\infty$. Furthermore, the energy of the soliton solutions (2.12) is localized around the point x_o and depends upon the reciprocal of the coupling constant. As we will see in the following, the dependence of the energy upon the reciprocal of the coupling constant is a general and important feature of all the soliton solutions of the classical Euler equations. Because the field equations are Lorentz-invariant, once we have the solution $\phi_c(x)$, we also have the boosted solution $\phi_c(\frac{x - vt}{\sqrt{1 - v^2}})$ for arbitrary v, $|v| < 1$.

We turn next to the definition of the topological charge in the 1 + 1 dimensional models under consideration. For this purpose, we observe that each scalar field in two dimensions provides a conserved current, and hence a time-independent charge. This was noticed long ago in the analysis of the Schwinger[34] and Thirring models[35].

The proof is simple: let ϕ be a scalar field in (1+1) spacetime; then

$$J_\mu = \varepsilon_{\mu\nu} \partial^\nu \phi \qquad (2.14)$$

is naturally conserved because it is the divergence of an antisymmetric tensor. The charge associated with it is

$$Q = \int dx\, J_o(x) = \phi(x=+\infty) - \phi(x=-\infty) \; . \qquad (2.15)$$

Q is the topological charge. It is clear from (2.15) that the topological charge Q is different from zero if and only if the asymptotic values of the scalar field $\phi(x)$ are not the same in different directions (i.e. $\phi(+\infty) \neq \phi(-\infty)$).

For the static solutions (2.12a,b) $Q = \pm 1$. In order to classify the possible values of the topological charge (2.15) in the two 1+1 dimensional models we are considering, we notice that the charge Q, defined by (2.15), could be interpreted in the framework of the algebraic topology as the homotopic index of the mapping, induced by ϕ, between the boundary of R and the manifold of constant solutions. The possible values of the topological charge are classified by the homotopy group $\Pi_o(M) = \Pi_o(G/H)$. We have for

ϕ^4-theory; $$\Pi_o(Z_2) = Z_2 \Rightarrow Q = \begin{cases} +1 \\ 0 \\ -1 \end{cases} \qquad (2.16a)$$

sine-Gordon theory; $$\Pi_o\left(\frac{D_\infty}{Z_2}\right) = \Pi_o(Z) = Z \Rightarrow Q = n \; . \qquad (2.16b)$$

For the sine-Gordon theory the values of the topological charge such that $|Q| > 1$ cannot be realized as static solitons, the physical reason being that solitons "interact" (the defining equations are non-linear) and the two-soliton configuration, for example, cannot be time independent. We are thus motivated to study the n-soliton time dependent solutions of the (2.11) S.G. equation. These solutions have the following properties. The n-soliton solution depends on 2n parameters. As $t \to -\infty$ the solution becomes a superposition of n one-soliton solutions and the 2n parameters correspond to asymptotic velocities $v^{(i)}$ and $x_0^{(i)}$ $(i = 1, \ldots, n)$. As $t \to +\infty$ the solution again decomposes into a superposition of n one-soliton solutions. The asymptotic final velocities are the same as the initial ones, the asymptotic final positions differ from the initial ones by an amount that can be ascribed to a time delay in the multisoliton-collision.

The explicit form of the two-soliton solution is:

$$\phi_{ss} = 4\frac{m}{g}\tan^{-1}\frac{u \sinh m\gamma x}{\cosh m\gamma ut} \qquad (2.17)$$

$$\gamma = (1 - u^2)^{\frac{1}{2}} \qquad u^2 < 1$$

soliton-antisoliton;

$$\phi_{ss} = 4\frac{m}{g}\tan^{-1}\frac{1}{u}\frac{\sinh m\gamma ut}{\cosh m\gamma x} \qquad (2.18)$$

$$\gamma = (1 - u^2)^{-\frac{1}{2}} \qquad u^2 < 1 .$$

Here u is the relative velocity of the two solitons.

The total momentum of each solution is zero; the energy is $2M_o\gamma$, $M_o = 8m^3/g^2$. The analysis of the asymptotic form of the two solutions shows that in both cases the time delay is given by

$$\Delta t(u) = \frac{2}{mu\gamma} \log u . \tag{2.19}$$

There is another class of exact solutions of the time dependent sine-Gordon equation: the so-called "doublet" solutions. Such doublets are easily obtained by making the relative velocity parameter u in (2.18) imaginary. When we set $u = iv$ in ϕ_{ss}, (2.18) becomes

$$\phi_v = 4\,\frac{m}{g}\tan^{-1}\frac{1}{v}\,\frac{\sinh m\gamma vt}{\cosh m\gamma x} . \tag{2.20}$$

This function is still real, and a classical solution for all finite v. Its shape as a function of time continues, roughly, to resemble ϕ_{ss}, but with an important difference. Instead of separating into a soliton-antisoliton pair infinitely far apart as $t' \to \pm\infty$ the relative separation here oscillates in time with period $\tau \simeq 2\pi(1+v^2)^{\frac{1}{2}}/mv$. This doublet solution is a "breathing" solution and can be thought of as a bound soliton-antisoliton pair.

Note that the same procedure does not work for a soliton-soliton pair. If we set $u = iv$ in ϕ_{ss} the field ϕ_{ss} is not real any more. There seem to be no soliton-soliton bound pairs.

It is also possible to construct exact solutions involving an arbitrary number of solitons. We will not write them down here, but they are discussed in ref.[36]. A systematic way to obtain them is provided by the Backlund transformation[37]. In fact, the Backlund transformation acts as a sort of creation operator, creating the soliton out of a constant solution, the two soliton out of the soliton, building soliton-antisoliton pairs (including the doublets), etc.

b) The vortex in superconductivity-like models

The possibility of vortices in superconductivity was first demonstrated by Abrikosov[38]. He showed that they naturally occurred as solutions to the Ginzburg-Landau theory of superconductivity in the presence of an external magnetic field. Following this pioneering work, the existence of these objects was verified experimentally and many of their properties were investigated. More recently, Nielsen and Olesen[9] pointed out that relativistic field theories like the Abelian Higgs model also possessed static vortex solutions. In these theories, the scalar Higgs field plays the role of the order parameter. This development subsequently provided the inspiration for 't Hooft's treatment of the magnetic monopole[10], which will be discussed later. The original idea behind the theory proposed by Nielsen and Olesen was to provide a bridge between field theoretic and dual-string descriptions of the hadronic world; in fact there is a limit where the Nielsen-Olesen vortex behaves like a Nambu string[8]. In the following discussion of vortices we will review some of the findings of Nielsen and Olesen and in addition outline a variety of properties exhibited by vortices.

The Lagrangian density for the Abelian Higgs model[9] is given by

$$\mathcal{L} = \frac{1}{2} (\partial_\mu - ieA_\mu)\phi^*(\partial^\mu + ieA^\mu)\phi - \frac{1}{4g^2} (\mu^2 - g^2\phi^*\phi)^2 - \frac{1}{4} F_{\mu\nu}F^{\mu\nu} \tag{2.21}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ gauge group U(1).

The classical Euler equations are:

$$(\partial_\mu + ieA_\mu)(\partial^\mu + ieA^\mu)\phi - \mu^2\phi + g^2(\phi^*\phi)\phi = 0 \tag{2.22a}$$

$$\partial_\mu F_{\mu\nu} = \Box A_\nu = J_\nu = \frac{ie}{2} (\phi^*\partial_\nu\phi - \partial_\nu\phi^*\phi) - e^2\phi^*\phi A^\nu . \tag{2.22b}$$

Parametrizing the Higgs field by $\phi = |\mu/g| \exp i\chi$ eqs. (2.22) become

$$[\partial_\mu\partial^\mu + i(\partial_\mu\partial^\mu\chi) + 2i(\partial_\mu\chi + eA_\mu)\partial^\mu - (\partial_\mu\chi + eA_\mu)^2]u = \mu^2 u - g^2 u^3 \tag{2.23a}$$

$$\partial_\mu F^{\mu\nu} = \Box A^\nu = J^\nu = -eu^2(\partial^\nu\chi + eA^\nu) . \tag{2.23b}$$

Therefore

$$A_\mu = -\frac{1}{e^2u^2} J_\mu - \frac{1}{e}\partial_\mu\chi . \tag{2.24}$$

Before actually showing that eqs. (2.23) possess static vortex solutions, let us describe the topological properties such vortices must exhibit.

Let us evaluate the flux of the e.m. field through the area bounded by an infinitely large closed path C

$$\Phi = \int F_{\mu\nu} dS^{\mu\nu} = \oint_C dx^\mu A_\mu = -\frac{1}{e} \oint_C \partial_\mu \chi dx^\mu \; . \tag{2.25}$$

In obtaining (2.25) we assumed $J_\mu = 0$ along C. The assumption is needed because in this approach we make no use of the Maxwell equations. The requirement that $\phi(x)$ be single valued then implies

$$\Phi = \frac{1}{e} [\chi(2\pi) - \chi(0)] = \frac{2\pi}{e} n = n \Phi_o \tag{2.26}$$

$$n = 0, \pm 1, \pm 2, \ldots \; .$$

This quantized magnetic flux is the total topological charge of the vortices encircled by the path C. To illustrate the homotopic nature of the flux quantization, let us recall that the manifold of the constant solutions of the equation for the Higgs field (2.22a) is specified by means of the conditions

$$|\phi| = \frac{\mu}{g} \qquad (\partial_\mu + ieA_\mu)\phi = 0 \; . \tag{2.27}$$

However (2.27) specifies ϕ up to the phase χ. There exists a circle of constant solutions in the complex plane parametrized by χ. Now suppose C is a circle in the (x,y) plane, then as one moves around this circle the phase $\chi(x,y) = \chi(\theta)$ can change from 0 to $2\pi n$. Therefore, $\chi(\theta)$ provides a mapping from a real circle in the (x,y) plane onto the manifold of the constant solutions of the equation for the Higgs field. This is the map $U(1) \to S_1$ characterized by:

$$\Pi_1(U(1)) = Z \quad \text{(the set of integers)}. \tag{2.28}$$

The integer labelling each homotopy class is called the winding number. It indicates the number of complete revolutions in the ϕ plane that correspond to a single 2π-revolution in the (x,y) plane. The net vortex flux is proportional to this winding number.

It is a very difficult task to show that vortex solutions to the field equations (2.23) with non-vanishing flux actually exist. The equations of motion are too difficult to solve and we need to make an ansatz for the fields which decreases the degree of difficulty of the coupled non-linear differential equations which must be solved. In the following we will restrict ourselves to the study of the static case.

Choosing the gauge $A_o = 0$, eqs. (2.23a,b) become

$$(\vec{\nabla} + ie\vec{A})(\vec{\nabla} + ie\vec{A})\, ue^{i\chi} + \mu^2 ue^{i\chi} - g^2u^3e^{i\chi} = 0 \tag{2.29a}$$

$$\Delta\vec{A} = eu^2(\vec{\nabla}\chi + e\vec{A}) \; . \tag{2.29b}$$

Eqs. (2.29a,b) are the relativistic analogue of the Landau-Ginzburg equations for type II superconductors. A cylindrically symmetric ansatz which corresponds to a vortex with n units of magnetic flux is given by

$$\chi = n\varphi \qquad n\vec{\nabla}\varphi = \vec{\nabla}\chi \qquad \vec{\nabla}\chi = \frac{n}{r} \tag{2.30a}$$

$$\vec{A} = a(r)r\nabla\varphi \qquad |\vec{B}| = \frac{1}{r}\frac{d}{dr}(ra) \ . \tag{2.30b}$$

If we introduce the quantity

$$Q = a + \frac{n}{er}$$

eqs. (2.29) become

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) - e^2Q^2u + \mu^2u - g^2u^3 = 0 \tag{2.31a}$$

$$\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(rQ)\right) - e^2u^2Q = 0 \tag{2.31b}$$

$$|\vec{B}| = \frac{1}{r}\frac{d}{dr}(rQ) \ . \tag{2.31c}$$

The vortex solution is obtained(9) when we require

$$u \xrightarrow[r\to\infty]{} \frac{\mu}{g} \qquad Q \xrightarrow[r\to\infty]{} 0 \qquad |\vec{B}| \rightarrow 0 \tag{2.32}$$

(i.e. when we require that the asymptotic value of the fields is mapped onto the manifold of the constant solutions of (2.22)).

There are two distinct mass scales in eqs. (2.31) $m_s = \sqrt{2}\,\mu$ and $m_v = \mu e/g$ which correspond respectively to the masses of the displaced Higgs field $\phi' = \phi - \frac{\mu}{g}$ and the vector gauge field A_μ. In analogy with the study of superconductors, the coherence length $\xi \equiv \sqrt{2}/m_s$ provides the scale for spatial variations in the Higgs field, while the penetration depth $\delta \equiv 1/m_v$ describes the spatial variations in the e.m. effects. A well defined vortex line is obtained when $\delta \approx \xi$. This is the "soliton" of the Nielsen and Olesen theory; nothing is known about multisoliton solutions, nor of soliton-soliton interaction.

In the strong coupling limit ($\delta \to 0$) we have just the constant solution throughout, except for a line with a strong magnetic field. In this limit the vortex solution of the Nielsen-Olesen model is identified [9] with the Nambu string [8].

The energy of the vortex solution is calculated by $E = \int d_3 x \mathcal{H}$ with

$$\mathcal{H}(x) = \frac{1}{2} (\vec{\nabla} u)^2 + \frac{1}{2} u^2 (\vec{\nabla}\chi + e\vec{A})^2 + \frac{1}{4g^2} (\mu^2 - g^2 u^2)^2 + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} . \tag{2.33}$$

The energy is divergent due to the infinite length of the vortex solution. However, if one evaluates the energy per unit length under the assumption $g > e$ (i.e. $\delta \approx \xi > 0$ smeared vortex) one finds [39]

$$\varepsilon \approx n^2 \frac{\mu^2}{g^2} \log \frac{g}{e} . \tag{2.34}$$

As in the 1+1 dimensional models examined before, also in this case the energy density depends upon the reciprocal of the coupling constant.

c) The monopole in non Abelian gauge theories

The possibility of spherically symmetric solitons was first discussed by 't Hooft [10] and Polyakov [14]. They considered an SO(3) gauge invariant Lagrangian which describes the interaction of a gauge field and scalar Higgs isovectors:

$$\mathcal{L} = \frac{1}{2} D_\mu \phi^i D^\mu \phi^i - \frac{1}{4g^2} (\mu^2 - g^2 \phi^i \phi^i)^2 - \frac{1}{4} F^i_{\mu\nu} F^{\mu\nu i} \qquad (2.35)$$

where

$$D_\mu \phi^i = \partial_\mu \phi^i + e\varepsilon^{ijk} A^j_\mu \phi^k = (\delta^{ik} \partial_\mu + \varepsilon^{ijk} A^j_\mu) \phi^k = D^{ik}_\mu \phi^k$$

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + e\varepsilon^{ijk} A^j_\mu A^k_\nu \quad .$$

This model (when fermions are added) is the Georgi-Glashow model of weak and electromagnetic interactions[40]. It describes one massless photon and two massive charged intermediate vector bosons which obtain their masses by the Higgs mechanism. We look for static and spherically symmetric solutions of the classical Euler equations of this model. For this purpose, we assume (Wu and Yang) that:

$$A_o = 0 \qquad A^i_\mu = \varepsilon_{\mu ik} \frac{x^k}{r} a(r) \qquad (2.36a)$$

$$\phi^i = \frac{x^i}{r} u(r) \qquad \phi^i \phi^i = u^2 \quad . \qquad (2.36b)$$

Eqs. (2.36) lead to

$$D_\mu \phi^i = \frac{x_\mu}{r} \frac{x^i}{r} \frac{du}{dr} + (ea + \frac{1}{r}) (\delta^i_\mu - \frac{x_\mu x^i}{r^2}) u \qquad (2.37a)$$

$$D_\mu \phi^i D^\mu \phi^i = (\frac{du}{dr})^2 + 2(\frac{1}{r} + ea)^2 u^2 \quad . \qquad (2.37b)$$

Here the sum is over the space indices. Therefore,

$$F^i_{\mu\nu} = 2\varepsilon_{\mu\nu i}\frac{a}{r} + (\varepsilon_{\nu ik}\frac{x_\mu}{r} - \varepsilon_{\mu ik}\frac{x_\nu}{r})\frac{x^k}{r}(\frac{da}{dr} - \frac{a}{r}) + e\varepsilon_{\mu\nu k}\frac{x^i x^k}{r^2}a^2 \tag{2.38a}$$

$$F^i_{\mu\nu}F^{\mu\nu i} = 8a^2(\frac{ea}{2} + \frac{1}{r}) + 4(\frac{da}{dr} + \frac{a}{r})^2 \quad . \tag{2.38b}$$

Then (2.35) becomes

$$\mathcal{L} = -\frac{1}{2}(\frac{du}{dr})^2 - \mu^2(ea + \frac{1}{r})^2 - \frac{1}{4g^2}(\mu^2 - g^2u^2) - (\frac{da}{dr} + \frac{a}{r}) + 2a^2(\frac{ea}{2} + \frac{1}{r})^2 \tag{2.39}$$

and the energy is given by:

$$E = 4\pi\int_0^\infty r^2 dr[-\mathcal{L}] \quad . \tag{2.40}$$

The equations of motion obtained from (2.39) are

$$\frac{d^2u}{dr^2} + \frac{2}{r}\frac{du}{dr} - 2(ea + \frac{1}{r})^2 u + (\mu^2 - g^2u^2)u = 0 \tag{2.41a}$$

$$\frac{d^2a}{dr^2} + \frac{2}{r}\frac{da}{dr} - (a + \frac{1}{er})[(ea + \frac{1}{r})^2 - \frac{1}{r^2}] - e^2u^2(ea + \frac{1}{r}) \quad . \tag{2.41b}$$

We again require that the asymptotic value of the Higgs field is not the same in different directions. Then we assume:

$$u(0) = a(0) = 0 \tag{2.42}$$

$$u(\infty) = \frac{\mu}{g} \qquad a(r) \xrightarrow[r\to\infty]{} a_\infty r^{-n} \quad . \tag{2.43}$$

Then use of (2.41b), together with (2.42) and (2.43), leads to

$$a(r) \xrightarrow[r\to\infty]{} -\frac{1}{er} \qquad A^i_\mu \xrightarrow[r\to\infty]{} -\varepsilon_{\mu ik}\frac{x^k}{er^2}$$

$$\phi^i \xrightarrow[r\to\infty]{} \frac{x^i}{r}\frac{\mu}{g} \tag{2.44}$$

The solution we have found is called a monopole or hedgehog(10,14). The name hedgehog is derived from the behavior of the Higgs field at infinity ($\phi^i \sim x^i$); it points radially outwards.

The complete functions, known only numerically, smoothly interpolate, without nodes, between their asymptotic values. The energy or mass of this monopole is computable by numerical methods and found to be

$$E = \frac{4\pi\mu}{e^2}\left(\frac{e}{g}\right)C(\beta) \; ; \qquad \beta = \frac{g}{e} \; . \tag{2.45}$$

The topological invariant in the model is the magnetic charge. In order to illustrate the non-trivial topology of his solution, 't Hooft constructed a gauge invariant electromagnetic field tensor

$$F_{\mu\nu} = \frac{1}{|\phi|}\left[\phi^i F^i_{\mu\nu} - \frac{1}{e|\phi|^2}\varepsilon^{ijk}\phi^i (D_\mu\phi^j)(D_\nu\phi^k)\right] . \tag{2.46}$$

Merely inserting the asymptotic values of the fields (2.44) into (2.46) and taking into account that

$$\phi^i F^i_{\mu\nu} = -\varepsilon_{\mu\nu i}\frac{x^i}{r}\frac{u}{egr^2} \qquad D_\mu\phi^i = 0 \tag{2.47}$$

we obtain:

$$F_{\mu\nu} = -\varepsilon_{\mu\nu i}\frac{x^i}{r}\frac{1}{er^2} \tag{2.48}$$

which corresponds to a radial magnetic field:

$$B^i = \frac{x^i}{er^3} \tag{2.49}$$

with magnetic flux

$$\phi = \frac{4\pi}{e} . \tag{2.50}$$

Hence, this solution is a magnetic monopole. It satisfies Schwinger's condition

$$eg = 1 \tag{2.51}$$

(in units where h = 1). The anti-monopole is obtained by changing the sign of the Higgs field in (2.44).

To illustrate the homotopic nature of the magnetic charge we follow the discussion of Arafune-Freund-Goebel[41]. We notice that 't Hooft's electromagnetic tensor can be rewritten as

$$F_{\mu\nu} = M_{\mu\nu} + H_{\mu\nu} \tag{2.52}$$

where

$$\begin{aligned} M_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu \\ B_\mu &= \hat{\phi}^i A^i_\mu \\ H_{\mu\nu} &= \frac{1}{e} \varepsilon_{ijk} \hat{\phi}^i \partial_\mu \hat{\phi}^j \partial_\nu \hat{\phi}^k \quad ; \quad \hat{\phi}^i \equiv \frac{\phi^i}{|\phi|} . \end{aligned} \tag{2.53}$$

We now define the magnetic current as

$$J_\mu = \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} \partial^\nu F^{\rho\sigma} . \tag{2.54}$$

If there are no string singularities in B_μ, then the magnetic current is given by

$$J_\mu = \frac{1}{2e}\, \varepsilon_{\mu\nu\rho\sigma}\varepsilon_{ijk}\partial^\nu \hat{\phi}^i \partial^\rho \hat{\phi}^j \partial^\sigma \hat{\phi}^k \quad . \tag{2.55}$$

The remarkable feature is that the magnetic current is completely specified in terms of the scalar triplet of Higgs fields. It is independent of the Yang-Mills fields A_μ^a. Such a current is trivially conserved because it is the divergence of an antisymmetric tensor. The magnetic charge is given by:

$$M = \frac{1}{4\pi}\int J_o d^3x = \frac{1}{8\pi e}\oint \varepsilon_{ijk}\varepsilon_{abc}\partial_i(\hat{\phi}_a\partial_j\hat{\phi}_b\partial_k\hat{\phi}_c)d^3x$$

$$= \lim_{R\to\infty}\frac{1}{8\pi e}\int_{S_R^2} \varepsilon_{ijk}\varepsilon_{abc}\hat{\phi}^a\partial_j\hat{\phi}^b\partial_k\hat{\phi}^c (d^2\sigma)_i \tag{2.56}$$

where S is a sphere of radius R (in the limit $R\to\infty$). Since a sphere can be parametrized by two coordinates ξ_α ($\alpha = 1,2$) (2.56) becomes

$$M = \frac{1}{8\pi e}\lim_{R\to\infty}\int_{S_R^2} \varepsilon_{\alpha\beta}\varepsilon_{abc}\hat{\phi}^a\partial_\alpha\hat{\phi}^b\partial_\beta\hat{\phi}^c\, d^2\xi = \frac{1}{4\pi e}\int d^2\xi\sqrt{g} \tag{2.57}$$

where $g = \det(\partial_\alpha\hat{\phi}^a\partial_\beta\hat{\phi}^a)$. The integral is known to be four times the Kronecker index of the map $S_R^2 \to S_\phi^2$ which must be an integer; therefore $Q_m = n/e$.

The homotopic nature of the magnetic charge is thus clarified.

CHAPTER II

SPONTANEOUS CREATION OF THE ORDERED STATE: OBSERVABLE EFFECTS OF MASSLESS BOSONS

2.1 The structure of the dynamical map: in field expansion of the observables

Our point of departure is the formal structure of the field equations for a given set $\{\psi\}$ of Heisenberg field operators

$$\Lambda(\partial)\psi(x) = F[\psi] \quad . \tag{1.1}$$

Here F is a functional of $\psi(x)$ and $\Lambda(\partial)$ is the differential operator appropriate to the spin of the field ψ. Because of the operator nature of the field, the solution of the field equations consists in realizing $\psi(x)$ by matrices in an appropriate Hilbert space. This is a central problem of the theory. It is important to realize, however, that, in quantum field theory, the choice of an appropriate Hilbert space on which to evaluate the matrix elements of the Heisenberg field operators is a far from trivial problem because of the existence of infinitely many unitarily inequivalent irreducible realizations of the canonical variables[42].

This is a statement about the kind of situation that one should expect once the infinite number of degrees of freedom characteristic of a quantum field theoretical system is taken into account, or, equivalently, once the non-separability of the space of the states is considered[42]. It is obvious that, a priori, nobody can tell which, among the many realizations of the canonical variables, is the physically relevant one. It turns out that this choice is a self-consistent one in that it will be the specific feature of the detailed dynamics involved in the basic field equations that selects "self-consistently" among all possible inequivalent realizations which will be the one related to the "physical particles".

At this point, however, we must specify what we mean by "physical particles": consider a scattering process between two or more particles. We can distinguish in such a process, a first stage in which we can identify by convenient measurements the kind, the number, the energy, etc. of the particles before they interact (incoming particles); a second stage, i.e. the one of interaction; a third stage in which again we can measure the kind, the number, the energy, etc. of the particles after the interaction (outgoing particles). What is observed is that in such a process the sum of the energies of the incoming particles is equal to the sum of the energies of the outgoing particles; we will refer to the incoming and to the outgoing particles as "physical particles" or else as "observed" or "free" particles, where the word "free" does not exclude the possibility of interaction among them.

It only means that the total energy of the system is given by the sum of the energies of the observed particles. Furthermore, in analogy to the Quantum Mechanics, we require that the energy of the physical particles is determined as a certain function of their momenta. This requirement needs some care as we will see later. In solid state physics the physical particles are usually called quasi-particles.

The self-consistency of the problem of finding the physical realization appears in the mathematical formalism as soon as we consider the equations for the basic fields as operator equations in the physical Fock space. This is because the physical particles can be characterized only by exploiting the detailed features of the dynamics involved, that is, only by solving the field equations themselves while, in order to solve the equations, we need a specification of the physical Fock space. We can recognize a similar dilemma in the Lehman-Symanzik-Zimmermann formalism[43] too: there the incoming fields are established as an asymptotic (weak) limit of the Heisenberg fields, while to perform the weak limit requires a knowledge of the Hilbert space associated with the incoming fields.

In order to begin the self-consistent computation we prepare as a candidate for the set of physical fields, a set of free fields by appealing to various physical considerations and then expand the Heisenberg field operators in terms of these free fields. By using the Heisenberg field equations, we obtain a set of coupled equations for the expansion coefficients. If one can solve these equations, one determines the expansion coefficients together with the energy spectrum of the physical particles. If these equations do not admit any solution one modifies the initial set of free fields and repeats the computations. Such a modification of the initial set of free fields is frequently made by introducing more free fields. This is the way in which many composite particles are successively brought into the theory[44].

As we will analyse in more detail in Chapter III, soliton-meson bound states are introduced in the fully quantum theory, through this self-consistent procedure[29]. Since the dynamics plays a crucial role in fixing the expansion of the Heisenberg fields in terms of the physical fields, such an expansion has been called a "dynamical map"[45,46]. From a physical point of view, the dynamical map self-consistently interpolates the Heisenberg field operators, carriers of the fundamental properties of a given physical system, and the physical field operators, carriers of all the properties which are directly detectable. In this context, a decisive step has been marked in exploiting the connection among basic and physical fields, by answering the question as to how observable symmetries of the physical fields are consistently related to the invariance of the Heisenberg field operators. In investigating this problem, the guiding principle is that the properties of invariance of the Heisenberg equations cannot disappear but should remain at every stage if we want the theory to be internally consistent.

However, due to the non-linearity of the field equations, it can sometimes happen that the transformation takes an entirely different shape when it is rewritten in terms of physical fields. When this happens, we do not recognize the original symmetry in observations. Thus, we may say that the invariance of the basic equations cannot be broken, but the shape of the symmetry can change. This point of view has been cast in the concise expression: dynamical rearrangement of symmetries[20,47]. The spontaneous breakdown of a symmetry of the Heisenberg equations is caused by these symmetry rearrangements. A detailed analysis of this phenomenon is presented in Section 2 where we analyse in detail the problem of the creation of the ordered state. Here we only point out that in all known cases in which the dynamical rearrangement of symmetries takes place massless boson fields are present among the physical fields and that such symmetry rearrangements are caused by infrared effects of these massless bosons.

Let us now study in more detail the structure of the dynamical maps. For this purpose, we must construct the Fock space, $\mathcal{H}$, associated with the physical particles. To do this, we introduce the annihilation and creation operators of physical particles (and their antiparticles):

these operators will be denoted by α_k^{τ} and $\alpha_k^{\tau\dagger}$ $(\beta_k^{\tau}, \beta_k^{\tau\dagger})$, where $\vec{k}$ signifies the particle and antiparticle momentum and τ the helicity states. For the sake of simplicity, we denote these operators by α_k and $\alpha_k^{\dagger}$ $(\beta_k, \beta_k^{\dagger})$, ignoring the helicity superscript. They satisfy

$$[\alpha_{\vec{k}}, \alpha_{\vec{\ell}}^{\dagger}] = \delta(\vec{k} - \vec{\ell}) \quad . \tag{1.2}$$

Strictly speaking, one cannot use plane waves as wave functions of physical particles, because the plane waves are not normalizable: one should use wave packets. For this purpose, one introduces a set $\{f_i(\vec{k})\}$ of functions which form an orthonormal set in the L_2-space and then defines the annihilation (creation) operators for physical particles in wave packet states:

$$\alpha_i = \int d_3k \ f_i(\vec{k}) \ \alpha_{\vec{k}} \tag{1.3}$$

$$\beta_i = \int d_3k \ f_i(k) \ \alpha_{\vec{k}} \quad . \tag{1.4}$$

These operators satisfy

$$[\alpha_i, \alpha_j^{\dagger}] = \delta_{ij} \quad . \tag{1.5}$$

Using α_i and β_i and following the well-known steps[48], one builds the Fock space $\mathcal{F}$. The vacuum state in $\mathcal{F}$ will be denoted by $|0\rangle$:

$$\begin{aligned} \alpha_i|0\rangle &= 0 \\ \beta_i|0\rangle &= 0 \quad . \end{aligned} \tag{1.6}$$

The physical field is then defined by:

$$\phi(x) = \int \frac{d_3k}{(2\pi)^{3/2}} [u(\vec{k})\, \alpha_{\vec{k}}\, e^{i\vec{k}x - iE_k t} + v(\vec{k})\, \beta_{\vec{k}}\, e^{-i\vec{k}x + iE_k t}] \tag{1.7}$$

The field ϕ satisfies the free field equation:

$$\lambda(\partial)\phi(x) = 0 \tag{1.8}$$

and this requirement determines the wave functions $u(\vec{k})$ and $v(\vec{k})$. These functions are then orthonormalized according to:

$$\int d_3x\; u_{\vec{k}}(x)\Gamma(\partial,-\overleftarrow{\partial})u_{\vec{\ell}}(x) = \delta(\vec{k}-\vec{\ell}) ,$$

$$\int d_3x\; v_{\vec{k}}(x)\Gamma(\partial,-\overleftarrow{\partial})v_{\vec{\ell}}(x) = \pm\delta(\vec{k}-\vec{\ell}) \tag{1.9}$$

$$\int d_3x\; u_{\vec{k}}(x)\Gamma(\partial,-\overleftarrow{\partial})v_{\vec{\ell}}(x) = 0 . \tag{1.10}$$

In the second equation the $\pm$ signs on the right correspond to Fermions and Bosons respectively[48]. The differential operator $\Gamma(\partial,-\overleftarrow{\partial})$ is uniquely determined by $\lambda(\partial)$.[48]

Having defined the concept of physical field, we state the problem of the dynamical maps as the one of expressing the Heisenberg field operator obeying eq. (1.1) in terms of the physical fields. It has been shown[27] that the general form of the dynamical map is:

$$\psi(x) = \chi + Z^{\frac{1}{2}}\phi + \int d_4\xi_1 \int d_4\xi_2 f(x-\xi_1; x-\xi_2) : \phi(\xi_1)\phi(\xi_2) : + \ldots\ldots \tag{1.11}$$

Here ϕ stands for both ϕ and $\phi^\dagger$. The dots denote those terms which contain higher order normal products. χ is zero unless ψ is a boson field. Z is the wave function renormalization constant and the symbol $Z^{\frac{1}{2}}\phi$ means $\sum_i Z_i^{\frac{1}{2}}\phi_i$, when there are many kinds of physical fields.

As it was pointed out before, (1.11) must be interpreted as a weak equality: any matrix element of the right hand side of (1.11) is equal to the corresponding matrix element of the left hand side. Therefore, the normal product expansion is convenient for our purposes because each expansion coefficient corresponds to a matrix element of ψ: for example the matrix element of $\langle 0|\psi|\alpha_k\alpha_\ell\rangle$ is given by the Fourier transform of $f(x-\xi_1;-\xi_2+x)$. The expansion coefficients and therefore the matrix elements of the Heisenberg field operator are determined by inserting (1.11) into the Heisenberg field equation (1.1) and then starting the self-consistent computation. However, in (1.11) there is still an indeterminacy in the choice of the physical fields; in fact, $\hat{\phi}(x) = u^{-1}\phi u$ can also be used as a physical field when u is a unitary operator. Among an infinite number of choices for the physical fields, one usually chooses the incoming fields as the physical fields. The choice is established by requiring that the expansion coefficients in (2.11) be retarded functions[48]:

$$f(x-\xi_1;x-\xi_2) = \theta(t_x-t_{\xi_1})\theta(t_x-t_{\xi_2})\bar{f}(x-\xi_1;x-\xi_2) . \qquad (1.12)$$

As a result of this requirement the time-integrations involved in (1.11) are well defined only when contributions from the infinite past vanish. Such vanishing occurs when we take the matrix elements of the integrals appearing in (1.11) between wave packet states; contributions from the infinite past do not vanish, but make the integrals undeterminate when the plane wave states are used.

This is due to the fact that in evaluating matrix elements of the Heisenberg field $\psi(x)$ the time integration $\int_{-\infty}^{t_x} dt$ is defined only if

$$\lim_{t\to-\infty} \int dE\ f(E)\, e^{iEt} = 0 \tag{1.13}$$

and that (1.13) holds due to the Riemann Lebesgue theorem, only when f(E) is square integrable.

Once the dynamical map (1.11) has been determined we are able to construct the in-field expansion for all the observables of physical interest of the theory under study. To calculate the dynamical map of a generic observable $O_H(\psi)$, we need to define the products of ψ. The products are defined by the computational rule which states that one first evaluates the products of normal products of the in-fields and then rearrange them into a linear combination of normal products. The space-time integrations should be performed only after the products of normal products are well taken care of. We assume that, when certain divergences appear through the course of the calculation, they can be eliminated by some regularization procedure. Once the products of ψ are determined, we can evaluate all the matrix elements of $O_H(\psi)$ in the physical Fock space.

Let us now make some remarks on the structure of the Hamiltonian in the physical representation. Since the physical fields satisfy certain free field equations (1.8), their Hamiltonian H_o is given by

$$H_o = \int d_3k \; E_k (\alpha_k^\dagger \alpha_k + \beta_k^\dagger \beta_k) \; . \qquad (1.14)$$

Here E_k stands for the energy of the physical particle. We then have[45,46,20,27]

$$\frac{1}{i} \frac{\partial}{\partial t} \phi(x) = [H_o, \phi(x)] \qquad (1.15)$$

which is a weak equality, i.e.

$$<a| \frac{1}{i} \frac{\partial}{\partial t} \psi(x) |b> = <a| [H_o, \psi(x)] |b> \qquad (1.16)$$

where $|a>$ and $|b>$ stand for wave packet states in the physical Fock space. On the other hand, if one introduces the Heisenberg Hamiltonian operator H such that

$$<a| [H(\psi), \psi(x)] |a> = <a| \frac{1}{i} \frac{\partial}{\partial t} \psi(x) |b> \qquad (1.17)$$

one finds, by comparing (1.16) and (1.17), that

$$<a| [H(\psi), \psi(x)] |b> = <a| [H_o, \psi(x)] |b> \qquad (1.18)$$

$$\forall |a>, |b> \in \mathcal{H}$$

(1.18) implies that when $\mathcal{H}$ is an irreducible representation for the Heisenberg operators $\psi(x)$, the Heisenberg Hamiltonian is weakly equal to the Hamiltonian of the physical fields

$$H[\psi] = H_o(\phi) + c \ . \tag{1.19}$$

where c is a c-number constant. Indeed, in deriving (1.19), we considered the fact that any quantity which commutes with an irreducible set of in-field operators must be a c-number constant. We observe also that eq. (1.19), where H is written in terms of the Heisenberg fields, while H_o is in terms of the physical fields, is a direct consequence of the dynamical map (1.11), and thus is a weak equality. Furthermore, eq. (1.19) relies heavily upon the fact that the physical fields $\phi(x)$ are Fourier-transformable solutions of linear homogeneous equations (the free field equations). Only in this case, in fact, the energy of the physical particles is determined as a certain function of their momenta. As we will see later, some care is needed in evaluating the vacuum expectation value of the Heisenberg Hamiltonian in the presence of an extended object, created by a boson function f(x), which is not Fourier-transformable.

We will see that eq. (1.19) cannot be used in this case. From an analysis of (1.19), we should not conclude that there are no reactions among the physical particles. In fact, the outgoing particles are obtained by the usual L.S.Z. procedure

$$c\phi^{out} = \lim_{t\to+\infty} (\psi(x)-x) \tag{1.20}$$

where the limit is understood to be a weak limit. Since the expansion coefficients $f(x-\xi_1;x-\xi_2)$ in (1.11) are retarded functions, they contribute to (1.20) although they do not contribute to

$$\lim_{t\to-\infty} \psi(x) = c\phi^{in}(x) \quad . \tag{1.21}$$

This shows that $\phi^{out}(x)$ is different from the incoming field $\phi^{in}(x)$. Therefore, there exist reactions among the physical particles. The calculation of the S-matrix in the physical representation is the same as that in the usual L.S.Z. formalism.

2.2 Original symmetry and observable symmetry: the spontaneous creation of the ordered state

As we mentioned earlier, many symmetry patterns, which we can recognize in the physical investigation, appear to be violated to a certain degree in actual observations; usually one says that, in such circumstances, the symmetry is broken. The term "spontaneous breakdown" tends to emphasize the disappearance of the original dynamical symmetry and is popular in high energy physics; in fact, much of the recent progress in this branch of science is based upon the assumption that the underlying dynamical symmetry is modified at the phenomenological level by some kind of dynamical mechanism. The same kind of phenomena is very frequent also in the theory of solids; in fact, the crystals do not possess translational and rotational invariance though the Hamiltonian of molecular gas is translationally and rotationally invariant. In ferromagnetism, we observe a similar situation concerned with the rotational invariance, and in superconductivity and in superfluidity we are concerned with the phenomenological disappearance of phase invariance, and so on. What characterizes this set of phenomena is that the loss of the fundamental invariance does not lead to chaos at the observational level, but instead, to structures of high degree of order: lattice structure in crystals(18), spin-polarized state in ferromagnetism(49) etc. From this point of view the designation "ordered state" which is given to such systems by solid state physicists is very appropriate.

It is clear, however, that the invariant properties of the fundamental equations cannot simply disappear, since the theory must be internally consistent(45). Then, it is evident that the analysis of the problem of "broken symmetry" or "creation of the ordered state" requires the study of the relation between original symmetry and phenomenological symmetry; we are thus still facing the problem of the mapping between the two "languages" used in Quantum Field Theory: the basic or Heisenberg field "language" and the physical field "language". The designation of the above considered phenomena with the term "symmetry rearrangement" is then more apt, since it bridges the two aspects of the phenomenon by focusing on the change of the dynamical symmetry into the observable symmetry of the ordered state[47]. In the self-consistent method, the connection between original and phenomenological symmetry is quite obviously displayed through the dynamical map. Due to the non-linear character of the dynamical map, which reflects the non-linear dynamical effects, one naturally expects that the original symmetries can manifest themselves, through the mechanism of "dynamical symmetry rearrangement", at the level of physical fields. In fact, to the comments on the dynamical map, given in 2.1, we should add that the left and right hand sides of (1.11) must have the same symmetry properties, although not necessarily term-by-term. To be more specific, let us consider a theory which is invariant under an internal transformation of the Heisenberg fields:

$$\psi(x) \longrightarrow \psi'(x) = G[\psi(x)] \quad . \tag{2.1}$$

We, then, consider the set of physical fields $\{\phi_\alpha(x)\}$, related to the Heisenberg fields by means of the dynamical map

$$\psi(x) = \psi[x;\phi_\alpha(x)] \quad . \tag{2.2}$$

We assume that there exists an invariant transformation of the physical fields

$$\phi_\beta \longrightarrow \phi'_\beta = q(\phi_\alpha) \tag{2.3}$$

such that, through the dynamical map (2.2), we have

$$\psi'(x) = \psi(x;\phi'_\beta) \quad . \tag{2.4}$$

If the q-transformation and the G-transformation are different we speak of dynamical rearrangement of symmetry and we say that the G-symmetry is dynamically rearranged into the q-symmetry[20,27,47].

Note that (2.2-4) are weak equalities. Let us note also that from the above point of view the breakdown of symmetry (or creation of ordered state) is interpreted as a dynamical effect in the sense that the basic equations are fully invariant, while because of the dynamics, the symmetry can appear in a different form at the physical level. Eq. (2.4) is one of the conditions which determine the q-transformation. There are two further important requirements[27]. Namely, (2.3) must leave the S-matrix and the in-field equations of motion invariant

$$S[\phi'] = S[\phi] \tag{2.5}$$

$$\Lambda(\partial)\phi'(x) = 0 \ . \tag{2.6}$$

An important question to ask in the study of the symmetry rearrangement is how the G-symmetry and the q-symmetry are related to each other. There is, in fact, a rich variety of forms that the dynamical rearrangement of symmetries can take. It is possible[47] to classify the various possibilities into three classes as follows:

(i) R$_1$-class: The G and q mappings correspond to the same algebra, but the Heisenberg field ψ and the physical field ϕ provide different realizations of the algebra. This is the case for Abelian symmetries[19,20,27].

(ii) R$_2$-class: The G and q mappings correspond to different algebras[49,50,51].

(iii) R$_3$-class: The G mapping generators form an algebra, but the q mapping does not[52].

The study of models in which the dynamical rearrangement of symmetries is of R_2-type suggests[53] the rule that the rearrangement of a simple Lie group is a group contraction[54]. It has been proved, in fact, that if $G = SU(2)$ with an invariant subgroup U(1), then $q = E(2)$ (the two dimensional Euclidean group) and that[51], when $G = SU(2) \times SU(2)$, with an SU(2) invariant subgroup, we have E(3).

However, the analysis of models of class R_3 does not lead to the same conclusion: if $G = SU(2)$ with no invariant subgroup the observable symmetry q does not form a group[52]. It is very difficult to determine the set of conditions to be verified in order to guarantee that the dynamical rearrangement of a simple Lie group follows the mechanism of the group contraction. A very recent result[55] shows that a necessary condition for the rearrangement to become a group contraction is that the invariant subgroup is a maximal subgroup of G. The origin of the symmetry rearrangement lies in the fact that the dynamical map is a weak relation. When G_i stands for the generators of the G symmetry, the matrix elements have the form:

$$<a|G_i|B> = \int d_3x <a|\rho_i(x)|b> \quad .$$

When $<a|\rho_i|b>$ contains a term of order $1/v$ (v: volume) such a term does not contribute to $<a|G_i|b>$, because the limit $v \to \infty$ should be taken before the sptial integration is made. The limiting process $1/v \to 0$ induces the symmetry rearrangement[19].

The dynamical rearrangement of symmetries has been studied in several models using either approximate methods in the course of the self-consistent calculation[20,27] or a formal derivation without any approximation by means of the path-integral formalism[56,57]. As a result of these studies, it has been shown that the creation of any ordered state (i.e. spontaneous breakdown of any symmetry) is caused by a certain symmetry rearrangement. The symmetry q is the one which is manifested in observations on the ordered state. We will analyse the implications of the previous statement in the following.

A basic characteristic of theories in which an ordered state is created is the appearance of massless Goldstone bosons[19] in the set of the in-fields; the massless bosons provide the long range force necessary for maintaining the order: phonons in crystals[18], magnons in ferromagnets[49] are the well known examples. In the case of the crystals, the phonon is the one which correlates the motion of a large number of molecules in such a way that they are prevented from going out of their stationary positions (i.e. the lattice points). The effects of the Goldstone bosons are essential for the understanding of the properties of the ordered state: first, the observable symmetry q is regulated by the condensation of the Goldstone field, and second, the creation of the ordered state is caused by the infrared effect of the Goldstone boson. When these effects become too strong they destroy order[58]. We analyse these aspects in Section 3, where we discuss the crucial role of the infrared boson in creating the ordered state; there we also discuss a case in which the symmetry rearrangement takes place without creating any ordered state. In fact, we have shown[21] that in gauge theories such as quantum electrodynamics, there always appear among the set of in-fields certain massless bosons and a symmetry rearrangement takes place even when no ordered state is created.

A general rule is that, whenever an ordered state is created the observable symmetry q contains the translation of the Goldstone bosons $\chi_\alpha^{in} \rightarrow \chi_\alpha^{in} + \theta_\alpha$. This leads to the possibility of observing in spacetime measurements the structure of the internal q symmetry of the ordered state.

The physical mechanism, through which the internal symmetry q has a spacetime manifestation, is the condensation of bosons in the ground state of the physical Fock space. The boson condensation acts as a printing process of these symmetry properties on the ground state. The result of this "printing" is manifested as the creation of an extended structure in the spacetime which, in a way to be specified later, manifests in spacetime the effects of the translations of the observable symmetry q. Since the number of condensed bosons is infinite, the extended object, created through the condensation process, behaves classically: intuitively speaking, the quantum fluctuation becomes much smaller than the macroscopic effect of the condensed bosons

$$\frac{h\Delta N}{hN} = \frac{\Delta N}{N} \to 0 \quad \text{as} \quad N \to \infty .$$

To study the situation in which a classically behaving extended object is created in the quantum ordered state, we use the boson theory[19,25]. We will see that the vacuum expectation value of the boson-transformed observables constructed from the dynamical map of the Heisenberg field ψ, provides a spacetime manifestation of the translations $\chi_\alpha^{in} \to \chi^{in} + \theta_\alpha$ of the observable internal symmetry q. We will analyse this problem in more detail in Section 4. There we study also the problem of the topological quantum number and see how the question asking which extended object should be quantized is controlled by the translations of the observable internal symmetry q.

The possibility of creating a classically behaving extended structure in the quantum ordered state is a very beautiful feature of the boson method. It leads us immediately to the question: how are the extended structures, created through the boson condensation process, related to the ones described by means of the Higgs field in a Ginzburg-Landau-type[16,9,10] approach to the study of the properties of the ordered state? Chapter III is devoted to the analysis of this problem.

Let us now analyse the role played by the infrared bosons in creating the quantum ordered state.

2.3 Infrared effects of the Goldstone bosons: the observable symmetry of the ordered state

a) A useful method of analysis

To study the process of creation of the ordered state, we need the relation between the Heisenberg fields (in terms of which the basic equations are expressed) and the physical fields (which are revealed in observations). For this purpose, the path-integral formalism[59] is an excellent tool of analysis. A reason is that this formalism provides very compact expressions for the S-matrix and the Heisenberg operators in terms of the physical fields. According to the L.S.Z.[43] formulation of the field theory, the S-matrix and the Heisenberg fields can be written respectively as

$$S = \sum_{n=0}^{\infty} \frac{1}{n!} \int d_4x_1 \ldots d_4x_n \mathcal{S}_n(x_1 \ldots x_n) :\phi^{in}(x_1) \ldots \phi^{in}(x_n): \quad (3.1)$$

$$S\psi_H(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \int d_4x_1 \ldots d_4x_n \mathcal{S}_n^{\psi}(x_1 \ldots x_n) :\phi^{in}(x_1) \ldots \phi^{in}(x_n): \quad (3.2)$$

where

$$\mathcal{S}_n(x_1 \ldots x_n) = (-iZ^{-\frac{1}{2}})^n K(x_1) \ldots K(x_n) \langle 0|T\{\psi_H(x_1) \ldots \psi_H(x_n)\}|0\rangle \quad (3.3)$$

and

$$\mathcal{S}_n^{\psi}(x_1 \ldots x_n) = (-iZ^{-\frac{1}{2}})^n K(x_1) \ldots K(x_n) \langle 0|T\{\psi_H(x) \ldots \psi_H(x_n)\}|0\rangle \quad (3.4)$$

Here Z stands for the wave function renormalization constant of the ψ field, T is the chronological product and

$$K(x) = -\partial_\mu \partial^\mu - m^2 . \quad (3.5)$$

For simplicity, we assumed here that the theory describes a single spinless massive particle. Eqs. (3.1-2) are weak relations; the matrix elements refer to wave packet states of incoming particles. When we use the path-integral formalism[59], the vacuum expectation values in (3.3) and (3.4) are given by certain functional average. Suppose that our model is characterized by a Lagrangian $\mathcal{L}(x) = \mathcal{L}(\psi, \partial_\mu \psi)$ and define the generating functional

$$W[J] = \frac{1}{N}\int [d\psi]\, \exp i\int d_4x[\mathcal{L}(x) + J(x)\psi(x)] \tag{3.6}$$

with $J(x)$ an external source and N given by

$$N \equiv \int [d\psi]\, \exp i\int d_4x\, \mathcal{L}(x) \ . \tag{3.7}$$

As it is well known(59), W[J] generates the Green's functions of the model by repeated differentiations with respect to the source $J(x)$. For any functional $g[\psi(x)]$, we define

$$\langle g[\psi(x)]\rangle_J \equiv \frac{1}{N}\int [d\psi] g[\psi(x)]\, \exp i\int d_4x[\mathcal{L}(x) + J(x)\psi(x)] \tag{3.8}$$

and

$$\langle g[\psi(x)]\rangle \equiv \langle g[\psi(x)]\rangle_{J=0} \ . \tag{3.9}$$

With these definitions, $\langle g[\psi(x)]\rangle$ coincides with the vacuum expectation value $\langle 0|Tg[\psi_H(x)]|0\rangle$. Here ψ_H denotes the Heisenberg field and ψ stands for the integration variable in a functional integral.

Using these notations eqs. (3.1-2) can be rewritten as(60)

$$S = :\langle e^{-\mathcal{A}(\phi^{in})}\rangle: \tag{3.10}$$

$$S\psi_H(x) = :\langle\psi(x)\, e^{-\mathcal{A}(\phi^{in})}\rangle: \tag{3.11}$$

where

$$\mathcal{A}(\phi^{in}) = i\int d_4x\, Z^{-\frac{1}{2}}\phi^{in}(x)\vec{K}(x)\psi(x) \ . \tag{3.12}$$

Here, the arrow on $\vec{K}(x)$ signifies that $K(x)$ should be always acting on $\psi(x)$ and the brackets on the right hand side of

(3.10-11) denote functional average, not expectation value in the Hilbert space. The symbol : : means that the annihilation and creation operators of ϕ^{in} stand in normal order.

The equations (3.10) and (3.11) are the functional equivalents of the familiar L.S.Z. reduction formulae. Eq. (3.11) expresses the exact form of the dynamical map; it can be generalized as

$$Sg[\psi_H(x)] = :\langle g[\psi(x)\, e^{-\mathcal{A}(\phi^{in})}]\rangle: \quad . \tag{3.13}$$

What makes the path-integral formalism very useful in the analysis of problems of spontaneous breakdown of symmetries is that symmetry transformations can be reduced by changes of the functional integration variables.

It has been realized, however, that special care is needed to be able to distinguish between symmetric and spontaneously broken solutions[56]. The reason for this lies in the fact that the field equations (and therefore the Lagrangian) are not sufficient to determine the solutions uniquely. One also needs to specify the boundary conditions.

If spontaneous breakdown of a continuous symmetry is possible in the model then there exists a continuous infinity of spontaneously broken solutions each one corresponding to a different choice of the boundary conditions. On the other hand, once W[J] is specified, all the Green's functions (and therefore a solution of the theory) can be obtained from it by appropriate functional differentiation. Thus, the information as to which particular solution is picked must be contained in the definition of the generating functional itself. In order to obtain the asymmetric solutions we add to the Lagrangian an "ε-term"(56) of the form $i\varepsilon f[\phi]$, which violates the invariance of the Heisenberg equations. The limit $\varepsilon \to 0$ has to be taken when all calculations are performed(56). In general, the generating functional for spontaneously broken solutions should be

$$W[J] = \frac{1}{N} \int \Pi [d\phi_i] \exp i \int d_4x \left(\mathcal{L}(\phi) + \sum_i J_i \phi_i + \text{[illegible]} f[\phi] \right) \qquad (3.14)$$

where N is equal to the numerator of W[J] when $J_i(x)$ are all zero, and $f[\phi_i]$ is a functional not invariant under the original invariant transformations of the theory. In the case of the complex scalar field model W[J] becomes:

$$W[J] = \frac{1}{N} \int [d\phi][d\phi^*] \exp i \int d_4x [\mathcal{L}(\phi) + J^*\phi + J\phi^* + i\varepsilon |\phi - v|^2] \qquad (3.15)$$

with

$$N = \int [d\phi][d\phi^*] \exp i \int d_4x [\mathcal{L}(\phi) + i\varepsilon |\phi - v|^2] \quad . \qquad (3.16)$$

Since each functional derivative $\delta/\delta J(x)$ ($\delta/\delta J^*(x)$) acting on W[J] generates the factor $i\phi^*(x)$ ($i\phi(x)$), the vacuum expectation value of any chronological product of $\phi^*(x)$ and $\phi(x)$; i.e. the Green's function, can be obtained by repeated operations of $\delta/\delta J$ and $\delta/\delta J^*$ followed by the limit $\varepsilon \to 0$. To investigate the symmetry properties of the theory, we can thus derive the Ward-Takahashi identities[61] which express such properties. To this end, we make the change of variables $\phi \to e^{i\theta}\phi$ in the numerator of (3.15). Since the integral is unaltered by a change of variables, we must have

$$\frac{\partial W[J]}{\partial \theta} = 0 \tag{3.17}$$

which, evaluated at $\theta = 0$, leads to the basic identity:

$$i\int d_4x \langle J_2\psi - J_1\chi\rangle_{\varepsilon,J} = \sqrt{2}\,\varepsilon v\, d_4x \langle\chi\rangle_{\varepsilon,J} \tag{3.18}$$

Here, the notation is as follows:

$$\langle F(\phi)\rangle_{\varepsilon,J} \equiv \frac{1}{N}\int [d\phi][d\phi^*]F(\phi)\exp i\int d_4x[\mathcal{L}(\phi)+J^*\phi+J\phi^*+i\varepsilon|\phi-v^2|]$$

$$\phi(x) = \frac{1}{\sqrt{2}}[\psi(x) + i\chi(x)] \tag{3.19}$$

$$J(x) = \frac{1}{\sqrt{2}}[J_1(x) + iJ_2(x)] .$$

Further shorthand notations which will be extensively used, are

$$\langle F(\phi)\rangle_\varepsilon \equiv \langle F(\phi)\rangle_{\varepsilon,J=0} \tag{3.20}$$

$$\langle F(\phi)\rangle \equiv \langle F(\phi)\rangle_{\varepsilon=J=0} . \tag{3.21}$$

Before we proceed with the investigation of (3.18), let us point out that $\langle\chi(x)\rangle_\varepsilon = 0$ because both the Lagrangian and the ε-term are invariant under the transformation

$\chi(x) \to -\chi(x)$. Then, the vacuum expectation value of the $\phi(x)$ is due entirely to that of $\psi(x)$:

$$<\phi(x)> = \frac{1}{\sqrt{2}} <\psi(x)> \equiv \frac{\tilde{v}}{\sqrt{2}} \quad . \tag{3.22}$$

The second equality defines the quantity $\tilde{v}$. $\tilde{v}$ is the homogeneous order parameter. It is related to the original v in the following way: the phase of $\tilde{v}$ is completely controlled by the phase of v, the magnitude of $\tilde{v}$ does not depend upon the magnitude of v, as long as $v \neq 0$.(56) By successive functional differentiations of the (3.18) with respect to $J_1(x)$ and/or $J_2(x)$ we can obtain all the identities relating the Green's functions (i.e.: W.T. identities)(56).

The ones we use in the following are:

$$<\psi(x)>_\varepsilon = \sqrt{2}\,\varepsilon v \int d_4 y <\chi(x)\chi(y)>_\varepsilon \tag{3.23}$$

$$<\rho(x)\rho(y)>_\varepsilon - <\chi(x)\chi(y)>_\varepsilon = \sqrt{2}\,\varepsilon v \int d_4 z <\chi(z)\chi(x)\rho(y)> \tag{3.24}$$

where

$$\rho(x) = \psi(x) - <\psi(x)>_\varepsilon . \tag{3.25}$$

To rewrite these identities in momentum space, let us introduce the Fourier transforms:

$$<\chi(x)\chi(y)> = \frac{i}{(2\pi)^4} \int d_4 p\, e^{-ip(x-y)}\, \Delta_\chi(p) \tag{3.26a}$$

$$<\rho(x)\rho(y)> = \frac{1}{(2\pi)^4} \int d_4 p\, e^{-ip(x-y)}\, \Delta_\rho(p) \tag{3.26b}$$

and

$$\langle\chi(x)\chi(y)\rho(z)\rangle = -\frac{1}{(2\pi)^8}\int d_4p\, d_4q\, d_4\tau\, e^{-i(px+qy+\tau z)}\,\delta(p+q+\tau)\,\times$$

$$\Delta_\chi(p)\Delta_\chi(q)\Delta_\rho(\tau)\Gamma_{\chi\chi\rho}(p,q,\tau) \quad . \qquad (3.27)$$

Here we used $g_{oo} = -g_{ii} = 1$. The propagation function $\Delta_\chi(p)$ has the form:

$$\Delta_\chi(p) = \lim_{\varepsilon\to 0}\left[\frac{Z_\chi}{p^2 - m_\chi^2 + i\varepsilon a_\chi} + \begin{matrix}\text{continuum}\\ \text{contributions}\end{matrix}\right] \qquad (3.28)$$

Here Z_χ is the wave function renormalization constant of the field χ. The continuum contribution comes from states with more than one particle.

Eq. (3.26a) implies

$$\tilde{v} = \sqrt{2}\,\frac{Z_\chi}{a_\chi}\,v \qquad \text{with} \qquad m_\chi^2 = 0 \qquad (3.29)$$

$$\tilde{v} = 0 \qquad \text{with} \qquad m_\chi^2 \neq 0 \qquad (3.30)$$

whereas (3.26b) yields

$$\Delta_\chi^{-1}(p) - \Delta_\rho^{-1}(p) = \tilde{v}\,\Gamma_{\chi\chi\rho}(0,p,-p) \quad . \qquad (3.31)$$

Eq. (3.29) is a statement of the Goldstone theorem[62]: if $\tilde{v} \neq 0$, $\chi(x)$ must be a zero mass field.

Eq. (3.31) gives the relation among the two propagators and the vertex function, and leads to restrictions on the renormalization constants. The presence of the ε-term generates the asymmetric solutions. If we had not introduced such term, the relation corresponding to (3.23) would have been $\langle\phi(x)\rangle = 0$ and the possibility of spontaneous breakdown would not have arisen. The situation is quite different if the effects of the ε-term are taken into account: if the integral (3.23) remains finite when $\varepsilon \to 0$ we get the symmetric solution; if it behaves as $1/\varepsilon$ in the limit, the ordered state is created. The Goldstone theorem comes from such $1/\varepsilon$-dependence. We also note that the ε-term prescription is equivalent in this model to the replacement $J \to J - i\varepsilon v$ in $W[J]$. One can then regard $J - i\varepsilon v$ as a new source J' and assume that J' does not vanish as $t \to \pm\infty$. However, such a prescription is not a general one, since, for example, it cannot be applied to models involving fermion fields or composite Goldstone bosons.

As already mentioned, the invariance of a theory cannot disappear when an ordered state is created. This feature of an invariant theory is expressed by the conservation of the local current corresponding to the symmetry transformation. In the path-integral formalism, one can derive W.T. identities for the divergence of the current by subjecting the integrand to local gauge transformations $\phi(x) \to e^{i\alpha(x)}\phi(x)$, which are not invariant transformations of this model. Through a procedure based essentially on functional differentiations of $W[J]$, which we omit for brevity, it is possible[56] to verify the conservation of the local current even when the ordered state is created.

The path integral analysis of a particular model in which an ordered state is created showed us two general features of such models: the appearance of massless (Goldstone) particles and the conservation of the local current corresponding to the symmetry transformations of the theory.

We are now ready to study the key role played by the massless bosons in rearranging the original symmetry G into the observable symmetry q of the in fields. For our discussion we use the complex scalar field model.

b) Dynamical rearrangement and boson transformations of global symmetries

Our starting point is the dynamical map and the possibility of expressing it in a compact form through the path-integral formulation. As we mentioned, this is due to the fact that the matrix elements of local operators such as $\phi_H(x)$ can be obtained by suitable limiting operations (e.g. those of the L.S.Z. formalism) from the Green's functions of the theory, while the latter are compactly summarized in the generating functional.

In the case of the complex scalar field model there is only one asymptotic field, which we shall denote by $\chi_{in}(x)$. It corresponds to $\chi(x)$, the imaginary part of the complex field

$$\phi(x) = \frac{1}{\sqrt{2}}\{\tilde{v} + \rho(x) + i\chi(x)\} . \tag{3.32}$$

There is no asymptotic field corresponding to $\rho(x)$, since ρ becomes unstable in presence of spontaneous breakdown[56]. In the Fock space of the in-fields, we define the S operator by[60]:

$$S = \langle :\exp\{iZ_{\chi}^{\frac{1}{2}}\int d_4x\ \chi_{in}(x)\ \vec{K}(x)\cdot\chi(x)\}:\rangle \quad . \tag{3.33}$$

The symbol K(x) denotes the d'Alambertian

$$K(x) = -\partial_{\mu}\partial^{\mu} \quad .$$

The arrow on K(x) signifies that it should be always acting on $\chi(x)$. Let us also remember that the bracket on the right hand side of (3.33) denotes functional average, not expectation value in the Fock space.

In a similar way, we can define the Heisenberg field operator $\phi_H(x)$ by means of the formula:

$$S\phi_H = \langle\phi(x):\exp\{-iZ_{\chi}^{-\frac{1}{2}}\int d_4x\,\chi_{in}(x)\vec{K}(x)\chi(x)\}:\rangle \quad . \tag{3.34}$$

This expression, which must be understood in the weak sense, is the functional equivalent of the familiar L.S.Z. reduction formula.

Let us introduce the in-field transformation

$$\chi_{in}(x) \longrightarrow \chi_{in}(x) + \alpha(x) \tag{3.35}$$

where the c-number function $\alpha(x)$ satisfies the same field equation as $\chi_{in}(x)$:

$$K(x)\alpha(x) = 0 \quad . \tag{3.36}$$

The transformation (3.35) is called the <u>boson transformation</u>. We want to prove that the Heisenberg field ϕ_H^f which is obtained through the dynamical mapping (3.34) from the boson-transformed $\chi_{in}(x)$ is also a solution of the field equation.

Let us start with the identity

$$<\chi(x)Q> = \int d_4x' <\chi(x)\chi(x')>q(x') \qquad (3.37)$$

which is valid for any operator Q and suitable choice of q(x'). Eq. (3.37), together with (3.23) and (3.29), implies the basic relation:

$$-i\frac{\tilde{v}}{Z_\chi} \int d^4x\, \alpha(x) K(x) <\chi(x)Q> = \lim_{\varepsilon\to 0} \sqrt{2}\, \varepsilon v \int d^4x\, \alpha(x) <\chi(x)Q> \qquad (3.38)$$

since the limit $\varepsilon \to 0$ serves to pick up the zero-mass pole and $\alpha(x)$ is a solution of the equation (3.36). Next, define a new functional average with space-dependent ε-term:

$$<F(\phi(x))>_{v(x)} \equiv N'^{-1} \int [d\phi][d\phi^*] F[\phi(x)] \times$$

$$\exp\left(i\int d_4z [\mathcal{L}(\phi) + i\varepsilon|\phi(z) - v(z)|^2\right] \qquad (3.39a)$$

with

$$N' = \int [d\phi][d\phi^*] \exp i \int d^4z [\mathcal{L}(\phi) + i\varepsilon|\phi(z) - v(z)|^2] \qquad (3.39b)$$

and[60]

$$V(z) = v[1 + i\frac{Z_\chi^{1/2}}{\tilde{v}}\alpha(z)] \qquad (3.40)$$

Then, if we define the boson-transformed S and $S\phi_H$:

$$S[\chi^{in}(x) + \alpha(x)] \equiv \langle :\exp\{-iZ_\chi^{-\frac{1}{2}}\int d_4x[\chi_{in}(x) + \alpha(x)]\vec{K}(x)\chi(x): \rangle \tag{3.41}$$

$$S\phi_H[x;\chi^{in}(x) + \alpha(x)] \equiv \langle\ \phi(x):\exp\{-iZ_\chi^{-\frac{1}{2}}\int d^4x[\chi_{in}(x) + \alpha(x)] \times \vec{K}(x)\chi(x):\ \rangle \tag{3.42}$$

we find with the aid of (3.38) and the definitions (3.39), (3.40):

$$S[\chi^{in}(x) + \alpha(x)] = C\langle:\exp\{-iZ_\chi^{-\frac{1}{2}}\int d_4x\,\chi_{in}(x)\vec{K}(x)\chi(x)\}:\rangle_{v(x)} \tag{3.43}$$

and

$$S\phi_H[x;\chi_{in}(x) + \alpha(x)] = C\langle\phi(x):\exp\{-iZ_\chi^{-\frac{1}{2}}\int d_4x\,\chi^{in}(x)\vec{K}(x)\chi(x)\}:\rangle_{v(x)} \tag{3.44}$$

The factor

$$C \equiv \lim_{\varepsilon\to 0} \langle \exp\sqrt{2}\,\varepsilon v\,\frac{Z_\chi^{\frac{1}{2}}}{\tilde{v}}\int d^4x\ \alpha(x)\chi(x)\}\rangle_\varepsilon \tag{3.45}$$

comes from the denominator N' defined in (3.39b). Eqs. (3.43), (3.44) prove the assertion made earlier: <u>the two field operators $\phi_H[x;\chi_{in}(x)]$ and $\phi_H[x;\chi_{in}(x) + \alpha(x)]$ differ by ε-terms, and therefore are solutions of the same field equations</u>[60].

Eqs. (3.41) and (3.42) yield an explicit expression[60] for the boson-transformed order parameter:

$$\tilde{v}(x) \equiv \langle 0|\phi_H[x;\chi_{in}(x) + \alpha(x)|0\rangle$$

$$= \lim_{\varepsilon\to 0} N'^{-1} \int [d\phi][d\phi^*]\phi(x)\exp i\int d_4z[\mathcal{L}(\phi) + i\varepsilon|\phi(z) - v(z)|^2]. \tag{3.46}$$

When $\alpha(x)$ is a normalizable function, we can rewrite eq. (3.46) as:

$$\tilde{v}(x) = \lim_{\varepsilon\to 0} \tilde{N}^{-1}\int [d\phi][d\phi^*]\phi(x) \exp i\int d_4z[\mathcal{L}(\phi) + i\varepsilon|\phi - ve^{i\lambda(z)}|^2] \tag{3.47a}$$

with

$$\lambda(x) = \frac{Z_\chi^{1/2}}{\tilde{v}}\alpha(x)$$

and

$$\tilde{N} = \int [d\phi][d\phi^*] \exp i\int d_4z[\mathcal{L}(\phi) + i\varepsilon|\phi(z) - ve^{i\lambda(z)}|^2] \quad . \tag{3.47b}$$

This follows from the fact that $\alpha^n(x)$ with $n \geq 2$ gives no contribution to (3.47), since their momentum space supports are not limited to the hypersurface $K^2 = 0$. As a consequence, the boson-transformed order parameter has the following structure[60]:

$$\tilde{v}(x) = \exp\{i\frac{Z_\chi^{1/2}}{\tilde{v}}\alpha(x)\}[\tilde{v} + V(i\frac{Z_\chi^{1/2}}{\tilde{v}}\partial_\mu\alpha(x)] \tag{3.48}$$

where the functional V has the property that

$$V(i\frac{Z_\chi^{1/2}}{\tilde{v}}\partial_\mu\alpha(x)) \longrightarrow 0 \qquad \text{for} \quad \partial_\mu\alpha(x) \longrightarrow 0 \quad . \tag{3.49}$$

Eq. (3.48) defines the boson-transformed order parameter as a functional of the boson function.

The normalizability of $\alpha(x)$ is essential in deriving eq. (3.47) from (3.46). If $\alpha(x)$ is a not normalizable solution of (3.36), the singular c-number translation of the in-field must be viewed as the limit of a boson transformation by a square-integrable c-number function. The limit must be taken at the end of all calculations (e.g. after all matrix elements have been calculated). An example of such situation is the translation of $\chi_{in}(x)$ by a constant c-number:

$$\alpha(x) \longrightarrow \alpha \quad .$$

Then (3.48) yields

$$\tilde{v}(x) = \exp\{i \frac{Z_\chi^{1/2}}{\tilde{v}} \alpha\} \tilde{v} \quad . \tag{3.50a}$$

More generally, we might consider a boson function satisfying

$$\lim_{|\vec{x}| \to \infty} \alpha(x) = \alpha(\vec{n}) \tag{3.50b}$$

where $\vec{n}$ is a direction of approaching the space-infinity. The same considerations, which yielded (3.50a), lead to

$$\lim_{|\vec{x}| \to \infty} \tilde{v}(x) = \exp\{i \frac{Z_\chi^{1/2}}{\tilde{v}} \alpha(\vec{n})\} \tilde{v} \quad . \tag{3.50c}$$

We may say that the boson-transformed order parameter induced by a boson function satisfying (3.50b) asymptotically manifests in space-time the structure of the original internal group.

We will return to the analysis of (3.50b) in section 4. We are now in a position to prove the basic theorem about the dynamical rearrangement of phase symmetry in our model. The only in-field transformation

$$\chi_{in}(x) \longrightarrow \chi_{in}(x;\alpha) \tag{3.51}$$

which, through the dynamical mapping, induces a constant phase transformation of the Heisenberg field $\phi_H(x)$, is the boson transformation (3.36) with $\alpha(x) \rightarrow$ const.[60]. Precisely speaking, the problem of dynamical rearrangement in this model involves finding an operator $\tilde{\chi}^{in}(x;\alpha)$, which is a function of $\chi^{in}(x)$ and a phase parameter α, such that the following two conditions are satisfied:

$$S[\tilde{\chi}^{in}(x;\alpha)] = S[\chi^{in}(x)] \tag{3.52}$$

$$S\phi_H[\tilde{\chi}^{in}(x,\alpha)] = e^{i\alpha}S\phi_H[\chi_{in}(x)] \quad . \tag{3.53}$$

The operators on the left hand sides of the above equations are defined through the dynamical mapping (3.33), (3.34):

$$S[\tilde{\chi}^{in}(x;\alpha)] = \langle :\exp\{-iA[\tilde{\chi}^{in}(x;\alpha)]\}:\rangle \tag{3.54}$$

$$S\phi_H[\tilde{\chi}^{in}(x,\alpha)] = \langle \phi(x):\exp\{-iA[\tilde{\chi}_{in}(x;\alpha)]\}:\rangle \tag{3.55}$$

with

$$A[\tilde{\chi}^{in}(x,\alpha)] \equiv Z_\chi^{-1/2}\int d_4x\ \tilde{\chi}^{in}(x;\alpha)K(x)\chi(x) \quad . \tag{3.56}$$

In addition, since the transformation (3.51) must be a symmetry transformation at the in-field level, we must impose the condition:

$$K(x)\tilde{\chi}^{in}(x;\alpha) = 0 . \tag{3.57}$$

The constraints (3.52), (3.53) lead to the following conditions by differentiation with respect to α:

$$<:(-i)\int d_4 z\ Z_\chi^{-1/2}\ \frac{\partial\tilde{\chi}^{in}(z;\alpha)}{\partial\alpha}\ \vec{K}(z)\chi(z)\exp\{-iA[\tilde{\chi}^{in}(x,\alpha)]\}:> = 0 \tag{3.58}$$

and

$$<\phi(x):(-i)\int d_4 z\ Z_\chi^{-1/2}\ \frac{\partial\tilde{\chi}^{in}(z,\alpha)}{\partial\alpha}\ \vec{K}(z)\chi(z)\exp\{-iA[\tilde{\chi}^{in}(x;\alpha)]\}:>$$

$$= i<\phi(x):\exp\{-iA[\tilde{\chi}^{in}(x;\alpha)]\}:> . \tag{3.59}$$

On the other hand, the basic identity (3.23) of the model, when evaluated with $J_1(x) = 0$ and $J_2(x) = -Z_\chi^{-1/2}\tilde{\chi}^{in}(x;\alpha)K(x)$ yields

$$<:(-i)Z_\chi^{-1/2}\int d_4 z\,\tilde{\chi}^{in}(z;\alpha)\ \vec{K}(z)(\tilde{v}+\rho(z))\exp\{-iA[\tilde{\chi}^{in}(x;\alpha)]\}:>$$

$$= <:(-i)\int d^4 z\ \frac{\tilde{v}}{Z_\chi}\ K(x)\chi(x)\exp\{-iA[\tilde{\chi}^{in}(x;\alpha)]\}:>. \tag{3.60}$$

Eq. (3.38) was used on the right hand side of this relation. Since $\rho(z)$ has no zero-mass pole and $\chi^{in}(z;\alpha)$ has a momentum space support confined on the hypersurface $K^2 = 0$ by virtue of (3.57), the left hand side of (3.60) is zero. Therefore

$$<:(-i)\int d^4 z\,\frac{\tilde{v}}{Z_\chi}\ K(x)\chi(x)\exp\{-iA[\tilde{\chi}^{in}(x;\alpha)]\}:> = 0 . \tag{3.61}$$

In a similar fashion, we can use the identity obtained by applying $-i(\frac{\delta}{\delta J_1}+i\frac{\delta}{\delta J_2})$ on both sides of (3.34) to prove that

$$<\phi(x):(-i)\int d_4z\ \frac{\tilde{v}}{Z_\chi}\ K(z)\chi(z)\exp\{-iA[\chi^{in}(x;\alpha)]\}:>$$

$$= i<\phi(x):\exp\{-iA[\tilde{\chi}^{in}(x;\alpha)]\}:> \quad . \qquad (3.62)$$

Comparison of (3.58), (3.59) with (3.61), (3.62) leads to the following necessary and sufficient condition on $\tilde{\chi}^{in}$:

$$\frac{\partial}{\partial\alpha}\tilde{\chi}^{in}(x;\alpha) = \frac{\tilde{v}}{Z_\chi^{\frac{1}{2}}} \quad .$$

This differential equation, with the initial condition

$$\tilde{\chi}^{in}(x;\alpha=0) = \chi^{in}(x) \qquad (3.63)$$

implies that:

$$\tilde{\chi}^{in}(x;\alpha) = \chi^{in}(x) + \frac{\tilde{v}}{Z_\chi^{\frac{1}{2}}}\alpha \quad . \qquad (3.64)$$

Clearly, this solution satisfies the requirement (3.57). The in-field transformation (3.64) must be understood as the limit of the boson transformation

$$\chi^{in}(x) \longrightarrow \chi^{in}(x) + \frac{\tilde{v}}{Z_\chi^{\frac{1}{2}}}\alpha g(x) \qquad (3.65)$$

with g(x) a normalizable c-number function that tends to 1, and satisfies

$$K(x)g(x) = 0 \quad .$$

The limit $g(x) \to 1$ must be taken after all the matrix elements have been calculated. The reason is the same as in (3.50): due to infrared singularities caused by Feynman diagrams with many momentumless and energyless external lines(63), $S[\chi_{in}+\alpha]$ and $S\phi_H[\chi_{in}+\alpha]$ are ill-defined. On the other hand, $S[\chi_{in}+\alpha g(x)]$ and $S\phi_H[\chi_{in}+\alpha g(x)]$, with g(x) a normalizable c-number function, are well defined.

Note that g(x) must satisfy the free field massless equations through the limiting process $g(x) \to 1$. The fact that the invariant transformation has different forms at the level of the Heisenberg fields and at the level of the in-fields expresses the dynamical rearrangement of the phase symmetry. Eq. (3.65) shows the crucial role played by the condensation of the Goldstone field, induced by a normalizable boson function, in determining the difference between the original and phenomenological symmetry. The need for a normalizable g(x) comes from the requirement that the generator of the in-field translation must be a well defined operator in the in-field Fock space. It is well known, in fact, that the generator of (3.32)

$$D = \int d_3x \; \dot{\chi}_{in} \frac{\tilde{v}}{Z_\chi^{1/2}} \qquad (3.66)$$

is ill defined(64), whereas

$$D_g = \int d_3x \; \frac{\tilde{v}}{Z_\chi^{1/2}} \left[g(x) \overleftrightarrow{\frac{\partial}{\partial t}} \chi^{in}(x) \right] \qquad (3.67)$$

is well defined. In fact, the Heisenberg field transformation is implemented through:

$$\lim_{g\to 1}\left\{e^{i\alpha D_g}\,\phi_H\,e^{-i\alpha D_g}\right\} = e^{i\alpha}\,\phi_H \quad . \tag{3.68}$$

Note that D_g is time-independent when $g(x)$ satisfies (3.36). In summary, <u>the phase transformation of the Heisenberg field</u>:

$$\phi_H \longrightarrow e^{i\alpha}\,\phi_H \tag{3.69}$$

<u>is induced by the in-field transformation</u>

$$\chi_{in}(x) \longrightarrow \chi_{in}(x) + \frac{\tilde{v}}{Z_\chi^{1/2}}\,\alpha g(x)$$

<u>with $g(x)$ a normalizable solution of $K(x)g(x) = 0$, when the limit $g(x) \to 1$ is performed after all matrix elements of ϕ_H</u> have been evaluated.

The necessity of introducing $g(x)$ in (3.65) leads to an intuitive understanding of the rearrangement of symmetry. Consider the generator of the G-transformation $\phi_H(x) \to e^{i\theta}\phi_H(x)$:

$$D(\phi_H) = \int d_3x\;\rho(\phi_H(x)) \quad .$$

Here

$$\rho(\phi_H) = \phi^\dagger \partial_o \phi \quad .$$

According to eq. (3.68), we treat $D(\phi_H)$ as the limit for $g \to 1$ of

$$D_g(\phi_H) = \int d_3x\; g(x)\;\rho[\phi_H(x)] \quad . \tag{3.70}$$

The weak equality in the dynamical map of ϕ_H leads to the following relation between $D_g(\phi_H)$ and the corresponding in-field generator $D^{in}(g)$:

$$\langle a|D_g^{in}|b\rangle = \int d_3x\, g(x)\, \langle a|\rho(\phi_H(x))|b\rangle \quad . \qquad (3.71)$$

Here, the states $|a\rangle$ and $|b\rangle$ are in particle wave packet states and D_g^{in} generates the q-transformation, in the limit $g \to 1$. Since $g(x)$ is square integrable, the integration involved in (3.71) is insensitive to locally infinitesimal terms which may be present in $\langle a|\rho(\phi_H(x))|b\rangle$ (i.e.: terms of order $1/V$, where V is the volume of the space). Such terms do not contribute to D_g^{in} with $g \to 1$ when it is evaluated by means of (3.71); on the other hand, since they are uniformly distributed over the space, they contribute to $D(\phi_H)$ by a finite amount, thus leading to a finite difference between $D(\phi_H)$ and D^{in}. The condensation of infrared Goldstone bosons produces such locally infinitesimal contributions, and when they are summed up, they account for the difference between the G and q transformations[47].

The example of the complex scalar field model illustrates two important features of the dynamical rearrangement, first, the observable q-symmetry is regulated by the condensation of the Goldstone field, and second, the rearrangement is caused by the infrared effects of the Goldstone bosons. The analysis of many models with non Abelian symmetries [50,51,52,49] confirms the generality of these results. Furthermore, we showed the remarkable way in which the original symmetry of the Heisenberg fields is recovered, asymptotically, when g(x) is a not normalizable solution of the free massless equation satisfying the boundary conditions (3.50b). This result is expected to be true also in non Abelian theories - at least the ones in which the dynamical rearrangement of symmetry is of R_2 type - since the original symmetry is rearranged into a phenomenological symmetry which contains, besides the invariant generators, only the generators of the massless in-field translations.

c) Dynamical rearrangement and boson transformations of gauge theories

We have seen that the creation of ordered states, in the case of a global symmetry, implies the existence of massless bosons, the Goldstone particles, which play a key role in the rearrangement of the original symmetry of the Heisenberg fields into the observable symmetry of the in-fields. Intuitively speaking, the phenomenological systematic structure of the ordered state is due to the presence of these massless bosons which act as long range correlation modes.

In particle physics, it seems hard, however, to explain the observed symmetry violations in terms of Goldstone bosons since there is no experimental observation of such massless particles. In non-relativistic phenomena it is known[65] that the presence of the Coulomb interaction affects the Goldstone theorem in the sense that the excitation modes have finite mass. Intuitively, this fact suggests that the role of the Goldstone mode in the creation of the systematic structure of the ground state is played by the long range Coulomb force. Following these ideas, it has been shown(66) that in particle physics the presence of a gauge field affects a spontaneously broken symmetry theory by eliminating the massless Goldstone particles (Higgs phenomenon). By taking advantage of the Higgs phenomenon, a unified theory of weak and electromagnetic interactions has been proposed(67) and many investigations of spontaneously broken gauge theories have been done, especially in connection with the renormalizability of such theories(68).

It has been observed(69) that if long range forces are present in the theory, the conservation of the total charge associated with the symmetry transformation is not valid, since surface contributions of the current do not vanish. This fact invalidates the Goldstone theorem and Goldstone particles completely disappear from the theory. This situation is, however, unsatisfactory since the Goldstone bosons play the crucial role of preserving the local conservation of currents associated with the invariance of the theory(19,45) (cf. section 2.2), and the absence of such Goldstone particles would make the theory internally inconsistent from the point of view of the invariance properties.

Furthermore, since the gauge fields participate in local gauge transformations only, they cannot have an effect in the rearrangement of the global invariance, which normally is mediated by the Goldstone bosons. The question arises then: if Goldstone bosons disappear from the theory what is the agent of the global symmetry transformations? A clue to the answer can be found in the Gupta Bleuler formulation of electrodynamics[70]; in this scheme, the physical Hilbert space is defined as a subspace of the Fock space of the theory and unphysical states do not contribute to processes because of compensation between longitudinal and scalar photons.

It has been shown[71] that a similar situation occurs in the Anderson-Higgs-Kibble mechanism: the dynamics naturally creates a boson of negative norm (ghost), and the contribution of this ghost cancels the contribution of the Goldstone bosons in any physical process. The Goldstone boson therefore does not disappear from the theory, and it can still participate in the rearrangement of a global symmetry. It should be mentioned that this cancellation between ghost and Goldstone quanta appears when the ground state is translationally invariant. Very interesting effects occur when a local condensation of Goldstone bosons induces a space-time dependent order parameter. These effects are a consequence of the physical (Gupta-Bleuler type) condition to eliminate unwanted components. Such a condition has to be preserved even after the boson transformation and this necessitates a redefinition of the field operators and induces the Meissner current in addition to the boson current. The situation is analogous to the one found in superconductivity[19,23], where a boson condensation is allowed also in the presence of Coulomb force; in this sense the Goldstone theorem is still valid and the gapless energy modes recover the symmetry properties of the theory.

Let us summarize briefly the model of ref.[71] and the way the boson transformation appears in the presence of gauge fields. The Heisenberg fields consist of a massless fermion field ψ and a gauge field A_μ and the Lagrangian is assumed invariant under global and local chiral transformations:

$$\psi(x) \to e^{i\theta\gamma_5}\psi(x) \qquad A_\mu(x) \to A_\mu(x)$$

$$\psi(x) \to e^{ie_o\lambda(x)\gamma_5}\psi(x) \qquad A_\mu(x) \to A_\mu + \partial_\mu\lambda(x) \quad . \tag{3.72}$$

The symmetry (3.72) is broken spontaneously by means of the condition:

$$\tilde{v} = \langle 0|\frac{1}{\sqrt{2}}[\bar{\psi}(x)\psi(x) + \bar{\psi}(x)\gamma_5\psi(x)]|0\rangle \neq 0 \quad .$$

We work in the Lorentz gauge, and this is ensured by adding to the Lagrangian a term

$$B(x)\,\partial^\mu A_\mu(x) \quad .$$

$B(x)$ is put into the theory in order to introduce the canonical conjugate of A_o.

The set of in-fields consists of:

(i) a fermion massive field $\psi^{in}(x)$.

(ii) a massive vector field $U_\mu^{in}(x)$ satisfying the Proca equation: $(-\partial^2-m^2)U_\mu^{in}=0,\ \partial^\mu U_\mu^{in}=0$

(iii) a massless Goldstone field $\chi_{in}(x)$, which is fermion-antifermion bound state, and

(iv) a massless field $b^{in}(x)$ with negative norm (ghost).

The relation between the S-matrix and the in-field is of the form:

$$S = S[J_\mu^{in},\psi^{in}] \qquad (3.73)$$

with[71]

$$J_\mu^{in} = [m^2 - (Z_3^{-1} - 1)(-\partial^2 - m^2)]U_\mu^{in} + m\, Z_3^{-1}\partial_\mu(b^{in} - \chi^{in})$$

and the dynamical map of the various Heisenberg operators is given by:

$$\psi(x) = :\exp\{i\,\frac{Z_\chi^{1/2}}{2\tilde{v}}\,\chi^{in}(x)\gamma_5\}[Z_\psi\psi^{in}(x) + F_\psi(x;\psi^{in},J_\mu^{in})]: \qquad (3.74a)$$

$$A_\mu(x) = \frac{Z_\chi^{1/2}}{2e_o\tilde{v}}\,\partial_\mu b^{in}(x) + A_\mu^o(x) \qquad (3.74b)$$

$$A_\mu^o = Z_3^{-1/2}U_\mu^{in}(x) + :F_\mu[x;J_\nu^{in},\psi^{in}]: \qquad (3.74c)$$

$$B(x) = \frac{2e_o\tilde{v}}{Z_\chi^{1/2}}\,(b^{in}(x) - \chi^{in}(x)) \qquad (3.74d)$$

$$\phi(x) \equiv \frac{1}{\sqrt{2}}\,(\bar{\psi}\psi + \psi\gamma_5\psi) = :\exp[i\frac{Z_\chi^{1/2}}{\tilde{v}}\,\chi^{in}(x)]\{\tilde{v} + F[\psi^{in},J_\mu^{in}]\}: \qquad (3.74e)$$

In these relations, the Z's are renormalization constants satisfying the constraints:

$$m^2 = \frac{Z_3}{Z_\chi} (2e_o\tilde{v})^2 \quad . \tag{3.75}$$

The various F's appearing in eqs. (3.74) are at least bilinear in their indicated arguments. $A^o_\mu(x)$ plays the role of the vector potential in the Lorentz gauge. The current $J_\mu(x)$ can be expressed in terms of the in-fields by means of the equation of motion:

$$-\partial^2 A_\mu(x) = -\partial^2 A^o_\mu(x) = J_\mu(x) - \partial_\mu B(x) \quad .$$

Correspondence with the classical Maxwell equation requires that for physical states $|a>$, $|b>$:

$$-\partial^2 <a|A^o_\mu(x)|b> = <a|J_\mu(x)|b> \quad .$$

This imposes on the physical states the Gupta-Bleuler kind of condition:

$$(\chi^{in(-)} - b^{in(-)})|a> = 0 \quad . \tag{3.76}$$

As can be seen from eqs. (3.74), the local gauge transformation of the Heisenberg fields is induced by:

$$\begin{aligned} \chi^{in}(x) &\longrightarrow \chi^{in}(x) + \frac{2\tilde{v}e_o}{Z_\chi^{1/2}} \lambda(x) \\ b^{in}(x) &\longrightarrow b^{in}(x) + \frac{2\tilde{v}e_o}{Z_\chi^{1/2}} \lambda(x) \quad . \end{aligned} \tag{3.77}$$

The S-matrix is clearly gauge invariant. Notice the special dependence on U^{in}_μ, χ^{in} and b^{in}; in fact, any gauge-invariant operator G has a dynamical map of the form:

$$G = G[J_\mu^{in}, \psi^{in}] \quad . \tag{3.78}$$

Because of the physical state condition (3.76), the boson transformation in this case is more complicated than in the case of a global symmetry. It was shown in ref.[19] that the boson transformation:

$$\chi_{in}(x) \longrightarrow \chi_{in}(x) + \frac{\tilde{v}}{Z_\chi^{\frac{1}{2}}} f(x) \tag{3.79}$$

must be accompanied by the transformation:

$$U_\mu^{in} \longrightarrow U_\mu^{in} + u_\mu(x) \tag{3.80}$$

where $f(x)$ is a solution of the d'Alembert equation:

$$\partial^2 f(x) = 0 \tag{3.81}$$

and $u_\mu(x)$ satisfies:

$$(-\partial^2 - m^2) u_\mu(x) = -\frac{m^2}{2e} \partial_\mu f(x) \; ; \qquad \partial^\mu u_\mu = 0 \quad . \tag{3.82}$$

In this last equation, e. is the renormalized charge:

$$e = Z_3^{\frac{1}{2}} e_o \quad .$$

Under the transformation (3.79-80), the gauge-invariant operator G (eq. (3.78)) transforms into

$$G^f = G[J_\mu^{in} + \frac{1}{m^2} J_\mu, \psi^{in}] \tag{3.83}$$

where

$$J_\mu(x) = m^2 [u_\mu(x) - \frac{1}{2e} \partial_\mu f(x)] \quad . \tag{3.84}$$

The vector potential becomes

$$A^{o}_{\mu f}(x) = Z_3^{\frac{1}{2}}[U^{in}_{\mu} + u_{\mu}(x)] + \ldots . \qquad (3.85)$$

The renormalized c-number vector potential induced by the boson transformation is then

$$a_{\mu}(x) \equiv \langle 0| Z_3^{-\frac{1}{2}} A^{o}_{\mu f}(x) |0\rangle = u_{\mu}(x) + \ldots . \qquad (3.86)$$

The corresponding c-number current is obtained through:

$$-\partial^2 a_{\mu} = J_{\mu} + \ldots . \qquad (3.87)$$

Here, the dots stand for contributions coming from the functional F^{o}_{μ} appearing in the expression of the dynamical map of the gauge potential $A^{o}_{\mu H}$ (see eq. (3.74c)).
The field strength is defined as

$$u_{\mu\nu} = \partial_{\mu} u_{\nu} - \partial_{\nu} u_{\mu} \quad . \qquad (3.88)$$

Solving (3.82) and substituting in (3.85) and (3.88) we can express the c-number vector potential and field strength as functionals of the boson function f(x):

$$u_{\mu}(x) = \frac{m^2}{2e} \int dx' \Delta_c(x-x') \partial'_{\mu} f(x') = a_{\mu}(x) \qquad (3.89)$$

$$J_{\mu}(x) = -\frac{m^2}{2e} \int dx' \Delta_c(x-x') \partial'^{\lambda} (\partial'_{\mu}\partial'_{\nu} - \partial'_{\nu}\partial'_{\mu}) f(x) \qquad (3.90)$$

$$u_{\mu\nu}(x) \equiv F_{\mu\nu}(x) = \frac{m^2}{2e} \int dx' \Delta_c(x-x') (\partial'_{\mu}\partial'_{\nu} - \partial'_{\nu}\partial'_{\mu}) f(x) \qquad (3.91)$$

where

$$(\partial^2 + m^2) \Delta_c(x) = \delta(x) \quad .$$

From the structure of the e.m. observables (3.89-91) we see that, if f(x) is Fourier transformable everywhere (i.e. $[\partial_\mu, \partial_\nu] f(x) = 0$) $J_\mu = 0$: the gauge invariant observables are unaffected by the boson transformation. This is due to the fact that the boson transformation (3.79) in this case is a gauge transformation, and therefore does not lead to any observable effect. Thus, the boson function f(x) should not be Fourier transformable in order to produce macroscopically observed effects.

Our analysis showed that, also in a gauge model, the Goldstone bosons appear in the dynamical map of the Heisenberg operators and that their infrared effects cause the dynamical rearrangement of the global symmetry. Furthermore it showed a peculiarity of gauge theories: the only macroscopic effects of the local condensation of the massless bosons are obtained by a boson function which is not Fourier transformable. When there is no gauge field, persistent flows without singularities (such as linear flows) and with singularities (vortices) are both possible.

The symmetry considered in this model is Abelian; in this case the arguments are simpler and the main results appear more clearly. It will be interesting to consider also non-Abelian gauge theories and to analyse the connection, if any, between the unphysical Goldstone mode and the fictitious scalar field introduced by Faddeev and Popov[72] in the perturbative treatment of a Yang-Mills[73] field theory. However, the requirement that the boson function must be singular (not Fourier transformable) in order to produce observable macroscopic e.m. effects should hold also in a non-Abelian theory, due to the fact that this is only a consequence of a Gupta-Bleuler type condition.

There is another peculiar feature of gauge theories; in fact, we have shown[21] that, when a theory has a gauge symmetry the dynamical rearrangement of symmetry takes place even without creating any ordered state. To illustrate this point, we discuss the gauge transformation of the conventional (3+1) dimensional quantum electrodynamics from the point of view of the in-field transformation.

We start with the Lagrangian

$$\mathcal{L}(x) = \mathcal{L}_o(\psi(x), A_\mu(x)) + B(x)\partial^\mu A_\mu(x) + \frac{1}{2}\alpha B^2(x) \qquad (3.92)$$

where $\psi(x)$ is the electron field and $A_\mu(x)$ is the gauge field, and $\mathcal{L}_o(\psi(x), A_\mu(x))$ is a gauge invariant Lagrangian.

$$\mathcal{L}_o(e^{ie\lambda(x)}\psi(x), A_\mu(x) + \partial_\mu\lambda(x)) = \mathcal{L}_o(\psi(x), A_\mu(x)). \qquad (3.93)$$

The supplementary field B(x) introduces the canonical conjugate of A_o and fixes the gauge condition[74]. The choice of the metric is $g_{oo} = -g_{ii} = 1$.

The W.T. relations obtained from the gauge invariance are

$$\langle \partial^2 B_{x_1} \ldots \partial^2 B_{x_n},\ \partial^2 B_x - \partial^\mu J_\mu(x) + ie[\tilde{\eta}\psi - \tilde{\psi}\eta]\rangle_J = 0 \qquad (3.94)$$

and

$$<\partial^{\mu}A_{\mu} + \alpha B(x)>_{J,\eta} = 0 \qquad (3.95)$$

where, as usual, $<F(A_{\mu},\psi)>$ means the functional average with the source terms. The derivative operators (3.94) and (3.95) commute with the bracket symbols for the functional average[75]. The W.T. relations for the two point functions are

$$\partial^2 <B(x), A_{\mu}(y)> + i\partial^x_{\mu}\delta(x-y) = 0 \qquad (3.96a)$$

$$\partial^2_x \partial^2_y <B(x)B(y)> = 0 \qquad (3.96b)$$

$$\partial^{\mu}<A_{\mu}(x)A_{\nu}(y)> + \alpha<B(x)A_{\nu}(y)> = 0 \quad . \qquad (3.96c)$$

It follows from (3.96a) and (3.96c) that the propagator of the vector boson satisfies

$$p^{\mu}\Delta_{\mu\nu}(p) = -\alpha\frac{p_{\nu}}{p^2} \quad , \qquad (3.97)$$

where

$$<A_{\mu}(x)A_{\nu}(y)> = \frac{i}{(2\pi)^4}\int d_4p\, e^{-ip(x-y)}\Delta_{\mu\nu}(p) \quad . \qquad (3.98)$$

When we restrict the choice of the gauge by the requirement that the singularity in $\Delta_{\mu\nu}(p)$ be a pole or cut only (<u>i.e. the theory admits the particle representation</u>), we find, by using (3.97), that pole terms in $\Delta_{\mu\nu}(p)$ are determined as

$$\Delta_{\mu\nu}(p) = Z_3\frac{-g_{\mu\nu}}{p^2} + Z_3^{(m)}\frac{-g_{\mu\nu}+p_{\mu}p_{\nu}/m^2}{p^2-m^2} - \frac{p_{\mu}p_{\nu}}{m^2p^2} + \cdots \qquad (3.99)$$

with $\alpha = Z_3$. In (3.99) the dots stand for terms with certain cut singularities. We have $Z_3 \neq 0$ and $Z_3^m = 0$ in the case of conventional Q.E.D., while $Z_3 = 0$ and $Z_3^m \neq 0$ in the model analysed before. When the latter case was studied, it was assumed that the system appeared in an ordered state; i.e. there was a phase order parameter which did not vanish. It is still an open question to ask whether or not the massiveness of the gauge boson always requires the presence of a certain non vanishing order parameter when the space dimension is more than one. The answer to such a question requires detailed dynamical calculations.

We analyse the conventional Q.E.D. i.e., the case $Z_3 \neq 0$ and $Z_3^{(m)} = 0$. It is straightforward to see that the relations (3.96a-c) and $\Delta_{\mu\nu}(p) = Z_3 \frac{-g_{\mu\nu}}{p^2} + \ldots$ are satisfied by the following in-field expressions:

$$A_o(x) = Z_3^{\frac{1}{2}} \frac{1}{(-\nabla^2)^{\frac{1}{2}}} b^{in}(x) + \ldots. \tag{3.100a}$$

$$\vec{A}(x) = Z_3^{\frac{1}{2}}[\vec{A}_T^{in}(x) + \frac{\vec{\nabla}}{(-\nabla^2)^{\frac{1}{2}}} \chi_{in}(x)] + \ldots. \tag{3.100b}$$

$$B(x) \stackrel{\perp}{=} Z_3^{-\frac{1}{2}}(-\nabla^2)^{\frac{1}{2}}[b^{in} - \chi^{in}] + \ldots. \tag{3.100c}$$

Here A_T^{in} is the transverse photon in-field, which satisfies

$$<0|T\{A_{Ti}^{in}(x), A_{Tj}^{in}(y)\}|0> = \frac{i}{(2\pi)^4} \int d_4p e^{-ip(x-y)} \frac{\delta_{ij} - P_iP_j/|P|^2}{P^2} \tag{3.101}$$

and $\chi_{in}(x)$ is a massless in-field with positive norm and $b^{in}(x)$ is a massless in-field with negative norm (the Gupta-Bleuler ghost).

The dots in (3.100a)-(3.100c) stand for the higher order normal products. The relations (3.100a)-(3.100c) are weak ones. Once the asymptotic fields are identified, one can derive more information about the structure of the in-field expression of the Heisenberg fields by means of the Lehman-Symanzik-Zimmermann formula together with successive uses of the W.T. relations.

The results are:

$$\psi(x) = \exp[ie \frac{Z_3^{\frac{1}{2}}}{(-\nabla^2)^{\frac{1}{2}}} b^{in}(x)]\{Z^{\frac{1}{2}}\psi^{in}(x) + F[A_T^{in},\psi^{in},\chi^{in} - b^{in}]\} \tag{3.102a}$$

$$\vec{A}(x) = Z_3^{\frac{1}{2}}[A_T^{in}(x) + \frac{\vec{\nabla}}{(-\nabla^2)^{\frac{1}{2}}} \dot{\chi}^{in}(x)] + \vec{F}[A_T^{in},\psi^{in},\chi^{in} - b^{in}] \tag{3.102b}$$

$$A_o(x) = Z_3^{\frac{1}{2}} \frac{1}{(-\nabla^2)^{\frac{1}{2}}} b^{in}(x) + F_o[A_T^{in},\psi^{in},\chi^{in} - b^{in}] \tag{3.102c}$$

$$B(x) = Z_3^{\frac{1}{2}} (-\nabla^2)^{\frac{1}{2}}(b^{in} - \chi^{in}) \quad . \tag{3.102d}$$

Here ψ^{in} is the electron in-field, and $\vec{F}$, F_o, and F are certain linear combinations of higher order normal products of in-fields. The Gupta-Bleuler condition for any observable state $|a\rangle$ reads as follows:

$$(-\nabla^2)^{\frac{1}{2}}(b^{in} - \chi^{in})^{(-)}|a\rangle = 0 \quad . \tag{3.103}$$

This ensures that neither the longitudinal photon χ^{in} or the ghost b^{in} appear in gauge invariant observations. The relations (3.102a)-(3.102d) imply that the gauge transformation ($\psi \to \psi e^{ie\lambda}$, $A_\mu \to A_\mu + \partial_\mu \lambda$) in conventional Q.E.D. is induced by

$$\chi_{in}(x) \longrightarrow \chi_{in}(x) + \frac{1}{Z_3^{1/2}} (-\nabla^2)^{1/2} \lambda(x) \tag{3.104a}$$

$$b^{in}(x) \longrightarrow b^{in}(x) + \frac{1}{Z_3^{1/2}} (-\nabla^2)^{1/2} \lambda(x) \quad . \tag{3.104b}$$

The free field equations for χ_{in} and b_{in} are invariant under (3.104a)-(3.104b) when and only when $\partial^2\lambda = 0$. This is consistent with the fact that the gauge condition $\partial_\mu A^\mu + Z_3 B = 0$ is preserved when and only when $\partial^2\lambda = 0$. In other words, when $\lambda(x)$ does not satisfy $\partial^2\lambda = 0$, the gauge condition is modified. Equations (3.104a) and (3.104b) show how the gauge symmetry is dynamically rearranged in Q.E.D. . Although, due to (3.103), the ghost field and the longitudinal photon χ_{in} are not observable, they induce through (3.104a)-(3.104b) the dynamical rearrangement of the gauge symmetry in conventional Q.E.D.

So far, we have been concerned only with the local gauge transformation. Now, we consider the global phase transformation of the Q.E.D. . Knowing the canonical variables, we can construct the generator $Q[\lambda]$ for the gauge transformation:

$$Q[\lambda] = \int d_3x [J_o(x)\lambda(x) + F_{oi}\partial^i\lambda(x) - B(x)\partial_o\lambda(x)] \quad . \tag{3.105}$$

When the Heisenberg equation

$$-\partial^\nu F_{\nu\mu} = J_\mu - \partial_\mu B(x) \tag{3.106}$$

is considered, (3.105) can be rewritten as

$$Q[\lambda] = \int d_3x[\dot{B}(x)\lambda(x) - B(x)\dot{\lambda}(x)] \qquad (3.107)$$

where an integration by parts has been made.

On the other hand, the global phase transformation $\psi \to e^{ie\theta}\psi$, $A_\mu(x) \to A_\mu(x)$ is generated by

$$Q = \int d_3x[-\partial^i F_{oi} + \dot{B}(x)] \qquad (3.108)$$

If the first term in (3.108) were to vanish, Q would be identical to $\lim_{\lambda\to 1} Q[\lambda]$. However, this is not true in the case of Q.E.D. because F_{oi} contains a massless component.

To calculate the matrix elements of Q in the case of Q.E.D., let us recall that the current $J_\mu(x)$ is written as:

$$J_\mu(x) = -\partial^2 A_\mu(x) + (1-\alpha)\partial_\mu B(x) \qquad (3.109)$$

where use was made of

$$\partial^\nu A_\nu(x) = -\alpha B(x) \qquad \alpha = Z_3 .$$

We are interested in the matrix elements $\langle p|(-\partial^2)A_\mu|p'\rangle$, where p and p' are certain one electron states. Since the d'Alembertian of the renormalized photon field gives a renormalized electric source, one has

$$\langle p|-\partial^2 A_\mu(x)|p'\rangle = e_r Z_3^{\frac{1}{2}}\langle p|\tilde{\psi}^{in}\gamma_\mu\psi^{in}|p'\rangle . \qquad (3.110)$$

Using the fact that $e_r = eZ_3^{\frac{1}{2}}$, we have

$$J_\mu(x) = eZ_3\tilde{\psi}^{in}\gamma_\mu\psi^{in} + (1-Z_3)\partial_\mu B(x) + \ldots . \qquad (3.111)$$

where the dots stand for the higher order normal products of in-fields. Therefore the generator Q is written as

$$Q = \int d_3x[eZ_3\tilde{\psi}^{in}\gamma_o\psi^{in} + (1-Z_3)\dot{B}(x)] \quad . \tag{3.112}$$

Note that in the calculation of any matrix element of operators which involve Q, the spatial integration should be performed only after the matrix element of the integrand is calculated. Note also that every particle state is a wave packet state.

When the in-field expression of B is considered, we see that the first term in (3.112) generates the "renormalized" phase transformation of ψ^{in} ($\psi^{in} \rightarrow e^{ieZ_3\theta}\psi^{in}$), while the second term generates the field translation of $Z_3^{\frac{1}{2}}b^{in}/(-\nabla^2)^{\frac{1}{2}}$ and $Z_3^{\frac{1}{2}}\chi^{in}/(-\nabla^2)^{\frac{1}{2}}$ according to

$$\frac{Z_3^{\frac{1}{2}}b^{in}}{(-\nabla^2)^{\frac{1}{2}}} \longrightarrow \frac{Z_3^{\frac{1}{2}}b^{in}}{(-\nabla^2)^{\frac{1}{2}}} + (1-Z_3)\,\theta \tag{3.113}$$

and

$$\frac{Z_3^{\frac{1}{2}}\chi_{in}}{(-\nabla^2)^{\frac{1}{2}}} \longrightarrow \frac{Z_3^{\frac{1}{2}}\chi_{in}}{(-\nabla^2)^{\frac{1}{2}}} + (1-Z_3)\,\theta \ . \tag{3.114}$$

Then the dynamical map of ψ shows the remarkable way that Q as a whole generates the phase transformation $\psi \rightarrow \exp(ie\theta)\psi$

$$e^{-i\theta Q}\,\psi\, e^{i\theta Q} = e^{ieZ_3\theta}\, e^{ie(1-Z_3)\theta}\,\psi = e^{ie\theta}\,\psi(x) \quad . \tag{3.115}$$

Then we see that the global phase transformation of $\psi(x)$ is induced by the transformation

$$\psi^{in} \longrightarrow e^{ieZ_3\theta}\psi^{in}$$

$$\frac{Z_3^{\frac{1}{2}}b^{in}}{(-\nabla^2)^{\frac{1}{2}}} \longrightarrow \frac{Z_3^{\frac{1}{2}}b^{in}}{(-\nabla^2)^{\frac{1}{2}}} + (1 - Z_3)\,\theta \qquad (3.116)$$

$$\frac{Z_3^{\frac{1}{2}}\chi_{in}}{(-\nabla^2)^{\frac{1}{2}}} \longrightarrow \frac{Z_3^{\frac{1}{2}}\chi_{in}}{(-\nabla^2)^{\frac{1}{2}}} + (1 - Z_3)\,\theta$$

which is the combination of the field translations of b^{in} and χ^{in} and the phase transformation of the electron field with the renormalized charge $e_r = eZ_3$. In this way, the B term in (3.112) plays a significant role in creating the difference between "bare charge" phase and "renormalized charge" phase while it does not contribute to the observable total charge because of the Gupta-Bleuler condition. Thus, we say that the charge difference $(e - e_r)$ is in the unobservable sector of the Hilbert space[76].

We showed that the change of the total charge through renormalization is explained by the fact that the missing charge is in the unobservable sector of the Hilbert space. Our choice of the gauge condition is:

$$\partial^\mu A_\mu = \alpha B \quad , \text{ with } \quad \alpha = Z_3 . \qquad (3.117)$$

Any gauge transformation with $\partial^2\lambda = 0$ does not change this gauge condition. Furthermore, this choice of gauge does not introduce any dipole ghost. To move to another choice of gauge without introducing any c-number space-time function in the in-field expression of the Heisenberg operators, we should perform the gauge transformation with a q-number λ. In order to move to the gauge with $\alpha \neq Z_3$, we need λ with the property

$$\partial^2\lambda = (\alpha - Z_3)\, B \; . \tag{3.118}$$

which shows that λ contains a dipole ghost because $\partial^2 B = 0$. This shows how the dipole ghost appears when $\alpha \neq Z_3$.

In the case of a Yang-Mills theory, $\partial^\mu A_\mu$ changes under the gauge transformation even when $\partial^2\lambda=0$. It is therefore an open question to ask if there is any choice of gauge in which the Yang-Mills-Heisenberg field contains no dipole ghost. In the case of Q.E.D. we saw that the boson transformation

$$\chi_{in} \longrightarrow \chi_{in} + f$$

$$b_{in} \rightarrow b_{in} + f$$

induces the local gauge transformation, without modifying the Gupta-Bleuler condition.

The boson-transformed vector potential $a_\mu(x)$ and the c-number field strength $F_{\mu\nu}$ are given by:

$$a_\mu \equiv \langle 0|A^f_\mu(x)|0\rangle = \frac{1}{e}\,\partial_\mu f \tag{3.119a}$$

$$F_{\mu\nu} = \frac{1}{e}\,[\partial_\mu,\partial_\nu]\, f \quad . \tag{3.119b}$$

Note that a non-vanishing e.m. field is obtained only if $f(x)$ is multivalued.

2.4 Spacetime manifestation of internal symmetries: creation of topologically non-trivial extended objects in the ordered state

In the previous section we analysed the role of the Goldstone bosons in the dynamical rearrangement of the symmetry of the Heisenberg fields into the observable symmetry of the physical fields: we saw that, whenever a symmetry rearrangement occurs, the observable symmetry of the in-fields contains the translations of the Goldstone bosons

$$\chi_\alpha^{in} \to \chi_\alpha^{in} + \theta_\alpha f_\alpha(x) \qquad (4.1)$$

$\alpha = 1,\ldots,n$ $\qquad$ n = number of Goldstone bosons .

Here θ_α are translation parameters and the f_α are normalizable c-number solutions of

$$\partial^2 f_\alpha = 0 \qquad (4.2)$$

introduced in order to properly define[64] the generators of the in-field translations

$$D^{in}(f_\alpha) \simeq \int d_3x\, f_\alpha(x) \overleftrightarrow{\frac{\partial}{\partial t}} \chi_\alpha^{in}(x) \qquad (4.3)$$

in the Fock space of the physical fields.

Eq. (4.3) led us to the conclusion that the infrared effects of the Goldstone bosons are responsible for the dynamical rearrangement of symmetries.

Here we analyse an observable effect of the translations (4.1): namely, how the condensation of the massless bosons in the ground state of the physical Fock space leads to observable spacetime manifestation of the internal symmetry group. To understand intuitively how this can happen we denote by hN a quantum number carried by these bosons, and by $h\Delta N$ the quantum fluctuation. When the condensation of the bosons makes N very large, then $h\Delta N/hN = \Delta N/N$ becomes very small and the system created by the condensation process behaves classically. In this way from a system of quanta, a system, in which a classically behaving extended object coexists with the quantum fields, is created. The classical system manifests in spacetime the effects of the translations since the transformations (4.1) regulate the condensation of the massless bosons.

The above intuitive considerations have been put in a mathematical form: the boson theory[19,22,18]. The boson theory can be summarized in two steps:

(i) use of the dynamical map

(ii) use of the boson transformation.

In the first step (see 2.1) one usually starts from a given set of Heisenberg field operators (say ψ) which satisfy known field equations

$$\Lambda(\partial)\psi(x) = F(\psi(x)) \quad . \qquad (4.4)$$

We look then for a solution of eq. (4.4) which can be weakly expressed in terms of normal products of a set of certain free field operators

$$\psi(x) = \psi[x;\phi_\alpha(x)] \quad .$$

The dynamical map of any Heisenberg operator is then obtained from (4.5) (see 2.1)

$$O_H[\psi] = O_H[x;\phi_\alpha(x)] \quad . \tag{4.6}$$

Eqs. (4.2) and (4.3) determine the structure of the Heisenberg operators when no extended objects are created in the quantum system.

The second step is considered when one wants to describe phenomena related to the appearance of a classically behaving extended object. Extended objects are introduced in the theory by means of boson transformations. Let ϕ_α be a boson field appearing in (4.5) satisfying the free field equation

$$\lambda_\alpha(\partial)\phi_\alpha(x) = 0 \quad . \tag{4.7}$$

Let us perform the boson transformation

$$\phi_\alpha(x) \longrightarrow \phi_\alpha(x) + f_\alpha(x) \tag{4.8}$$

where $f_\alpha(x)$ is a c-number function satisfying the equation

$$\lambda_\alpha(\partial)f_\alpha(x) = 0 \quad . \tag{4.9}$$

As an effect of (4.8) the Heisenberg operator is modified as

$$\psi(x) \longrightarrow \psi^f(x) = \psi[x;\phi_\alpha(x)+f_\alpha(x),\ldots.] \tag{4.10}$$

and the boson-transformed order parameter (here we assume ψ to be a boson field)

$$\phi^f(x; f_\alpha, \partial_\mu f_\alpha) \equiv \langle 0|\psi^f|0\rangle = \langle 0|\psi(x; \phi_\alpha + f_\alpha, \ldots)|0\rangle \qquad (4.11)$$

describes the spacetime properties of the system in which a classical extended object coexists with the quantum fields. It can be proved that

$$\Lambda(\partial)\,\psi^f = F(\psi^f)\,. \qquad (4.12)$$

This is the content of the boson transformation theorem[19,28]. The transformation (4.8), together with the condition (4.9), shows that f_α are created by the condensation of the bosons $\phi_\alpha(x)$.

It has been widely known that to perform the transformation $\phi_\alpha(x) \to \phi_\alpha$+c-number is one way to cover many unitarily inequivalent representation of canonical commutation relations[42]. However, when we speak of boson transformations, we are concerned not only with the canonical commutation relations, but also with the Heisenberg field equations; the boson transformation is conditioned with the requirement that the Heisenberg equation should not change. This condition is the one which leads us to (4.9). It should be noted that, when $f_\alpha(x)$ is singular (see appendix), the original Heisenberg equation should hold even at the singular points. Note that the boson transformation (4.8) can be performed on any boson field, massless or massive. However, when the theory allows massless bosons to appear in the dynamical map of the Heisenberg operators, owing to energy considerations, we expect that condensations of massless bosons will be favored. We consider in the following only the condensation of massless bosons. A case in which an extended object is created by the condensation of a massive boson will be considered in Chapter III.

Let us consider then a quantum field theoretical model in which an ordered state is created. As we mentioned, the condensation of the massless bosons is induced by the translations (4.1); the result of the condensation is the creation of an extended object in the ordered state. There exists a wide variety of possible extended structures, each one characterized by different choices of $f_\alpha(x)$ satisfying eq. (4.2)(see appendix). However, it is interesting to note that, depending upon the model under consideration, there exists <u>some</u> extended structures with a conserved quantity, the so-called topological quantum number, associated with them. We analyse the question asking which restrictions must be imposed on the boson transformation parameters f_α in order to create topologically non-trivial extended objects. We will see that an answer to such a question is strongly controlled by the translations of the rearranged group q.

Our starting point is the requirement that the boson-transformed order parameter (4.11) should be singled valued; namely

$$\phi(x;f_\alpha+\delta f_\alpha^c,\ldots) = \phi(x;f_\alpha) \tag{4.13}$$

where δf_α^C denotes the change of the boson function when x moves along a closed path C in the physical space:

$$\delta f_\alpha^C = \int d\vec{s}\ \vec{\nabla} f_\alpha \ . \tag{4.14}$$

When the condition (4.13) leads to $\delta f_\alpha^C \sim \nu$, with ν integer, then we say that the extended object created by f_α is topologically non-trivial and we define ν as the topological quantum number.

When an extended object is such that

$$\delta f_\alpha^C = 0$$

we say that the extended object is topologically trivial. How does the rearranged group control the choice of the f_α such that $\delta f_\alpha^C \neq 0$? For the sake of simplicity we assume the circle c to be infinitely large.

We recall that the creation of the extended objects in the ordered state is induced by the translations of the Goldstone bosons (4.1). Further, we assume that $f_\alpha(x)$ depends only upon the space coordinates (static case) and is such that

$$\lim_{|\vec{x}|\to\infty} f_\alpha(\vec{x}) = g_\alpha(\vec{n}) \tag{4.15}$$

$$\lim_{|\vec{x}|\to\infty} f_\alpha(x) = 0 \ .$$

Here n denotes a direction of approaching infinity. When we consider the asymptotic form of the boson transformed order parameter induced by boson functions satisfying (4.15), we have:

$$\lim_{|\vec{x}|\to\infty} \phi^f(x;f_\alpha,\partial_\mu f_\alpha) = \phi(x;g_\alpha(\vec{n})) \quad . \tag{4.16}$$

Upon identifying the translation parameters θ_α appearing in (4.1) with the asymptotic value of the boson function $g_\alpha(\vec{n})$ we obtain

$$\phi(x;g_\alpha(\vec{n}),0) = \phi(x;\theta_\alpha)\Big|_{\theta_\alpha = g_\alpha(\vec{n})} \quad . \tag{4.17}$$

Eq. (4.17) implies that, when (4.15) is satisfied, the asymptotic form of the extended object with different $\vec{n}$ corresponds to different values of θ_α. In fact, upon changing the direction n continuously, the functions $g_\alpha(\vec{n})$ are mapped on a continuous path in the manifold of the translation parameters θ_α of the observable symmetry group. In this sense we say that the asymptotic form of the extended object is a spacetime manifestation of the internal symmetry.

The topological constant δf_α^C can be quantized only if, changing the direction of $\vec{n}$ continuously by a full turn in the physical space, the functions $g_\alpha(\vec{n})$ are not mapped into a closed path of the space of the θ_α's. In fact,

$$\delta f_\alpha^C \equiv \int_{C\to\infty} d\vec{s}\; \vec{\nabla} f_\alpha = \delta g_\alpha = \delta\theta_\alpha \quad . \tag{4.18}$$

Let us illustrate this situation for the case of the U(1)-symmetry. In this case, the boson-transformed order parameter is given by (see 2.3)

$$\phi(x;f;\partial_\mu f) = e^{if(x)} F(\partial_\mu f) \quad . \tag{4.19}$$

The single-valuedness condition requires

$$\delta f^C = \int_C d\vec{s}\ \vec{\nabla} f = 2\pi\nu. \tag{4.20}$$

The eq. (4.15) implies

$$\lim_{|\vec{x}|\to\infty} \phi(x;\partial_\mu f,f) = e^{ig(\vec{n})}F(0) \tag{4.21}$$

and (4.20) leads to

$$\delta g = 2\pi\nu \tag{4.22}$$

where δg is the change of $g(\vec{n})$ by a full turn of $\vec{n}$. Eq. (4.22) shows that δg is the topological quantum number for the U(1)-symmetry. $\delta g = 0$ if and only if the change of $g(\vec{n})$ under a full turn of $\vec{n}$ can be mapped on a closed path in the manifold of the translation parameter θ.

In the above specified way, the asymptotic form of $\phi(x;f;\partial_\mu f)$ manifests the internal symmetry and the integer ν classifies the homotopy classes of the mapping between $\{g(\vec{n})\}$ and θ. The same argument can be repeated even when we have a gauge field. The only modification in this case is due to the fact that the vacuum expectation value of the boson-transformed Heisenberg field has the form (see 2.3)

$$\phi(x;f;a_\mu-\partial_\mu f) = e^{if(x)}F(a_\mu-\partial_\mu f) \tag{4.23}$$

Eq. (4.23) completely clarifies the topological nature of the flux quantization in vortices. In fact, when one considers the vortex line created by the boson transformation [19,23], one has that

$$r = \sqrt{x_i^2 + x_j^2} \longrightarrow \infty \qquad a_\mu \simeq \frac{1}{2e}\,\partial_\mu f \tag{4.24}$$

and

$$\lim_{|\vec{x}|\to\infty} f(x) = \theta \equiv \text{polar angle in cylindrical coordinates.} \tag{4.25}$$

Use of (4.24) and (4.25) leads to

$$\lim_{|\vec{x}|\to\infty} \phi^f(x;f;\partial_\mu f) = e^{i\theta} F(0) \tag{4.26}$$

implying that

$$\delta f^C \equiv \int_C d\vec{s}\,\vec{\nabla} f = 2\pi\nu \sim \oint_C ds_\mu a^\mu \tag{4.27}$$

which shows that the quantization of the e.m. flux is only a consequence of the single-valuedness of the order parameter.

CHAPTER III

CONSTRUCTION OF SOLITON SOLUTION OF THE CLASSICAL EULER EQUATIONS

The aim of this chapter is to present a description of the relation between the boson theory of extended objects and the approach based on the soliton solutions of the Euler equations for the classical fields[77,78] (see also 1.2). We will see that the extended objects constructed by the boson method become the soliton solutions of the Euler equations when the Planck constant, h, is ignored, implying that the soliton solutions can be regarded as the extended objects with a quantum origin.

To make the essence of our approach more transparent, we consider a simple case, in which no composite particles appear and the perturbative expension is usable.

Consider a Heisenberg equation

$$\Lambda(\partial)\psi = F(\psi) \quad . \tag{1.1}$$

Eq. (1.1) leads to a Yang-Feldman equation[79]:

$$\psi = \varphi^{in} + [\Lambda(\partial)]^{-1} F(\psi) \tag{1.2}$$

where φ^{in} is a free boson field satisfying

$$\Lambda(\partial)\, \varphi^{in} = 0 \quad . \tag{1.3}$$

It is important to observe that $F(\psi)$ in (1.1) is chosen in such a way that the mass in $\Lambda(\partial)$ is the physical mass, and therefore, that φ^{in} is the renormalized (or physical) free field. To understand this, we rewrite (1.2) in the form

$$<a|\psi(x)|b> = <a|\varphi^{in}|b> + (-i)\int d_4x\,\Delta(x-y) <a|F_y(\psi)|b>. \quad (1.4)$$

Here $<a|$ and $|b>$ are vectors in the Fock space of φ^{in} and $F_y[\psi]$ means $F[\psi]$ at the space-time position y. The function $\Delta(x)$ in (1.4) is the Green's function

$$\Lambda(\partial)\Delta(x) = \delta(x) . \quad (1.5)$$

The second term on the right hand side of (1.4) diverges unless $<a|F_y(\psi)|b>$ vanishes in a reasonable manner in the limit $t_y \to \pm\infty$. Such a divergence is avoided when φ^{in} is renormalized; it is eliminated by the mass counter term introduced in the renormalization procedure.

The above argument implies that we regard the Heisenberg equation as an equation among matrix elements (i.e. the weak relation). All the equations for Heisenberg operators in the following should be understood as weak relations. When we solve (1.2) by successive iteration, we are led to the usual perturbative expansion. The result is an expression for ψ in terms of φ^{in}:

$$\psi(x) = \psi(x; \varphi^{in}) \quad (1.6)$$

which is called the dynamical map. It is the expression of the field ψ in terms of a linear combination of normal products of the physical fields; it should be understood as a weak relation.

Let us now introduce a c-number function f(x) which satisfies the equation for φ^{in}:

$$\Lambda(\partial)\, f = 0 \quad . \tag{1.7}$$

Then, we can generalize the Yang-Feldman equation as

$$\psi^{f} = \varphi^{in} + f + [\Lambda(\partial)]^{-1}\, F(\psi^{f}) \quad . \tag{1.8}$$

Solving this by successive iteration, we obtain a new solution of the Heisenberg equation (1.1):

$$\psi^{f}(x) = \psi(x;\varphi^{in} + f) \quad . \tag{1.9}$$

ψ^{f} is obtained from ψ by means of the boson transformation.

The fact that both ψ and ψ^{f} satisfy the same Heisenberg equation is the content of the boson transformation theorem:

$$\Lambda(\partial)\, \psi^{f} = F(\psi^{f}) \quad . \tag{1.10}$$

We introduce the c-number field ϕ^{f} as

$$\phi^{f}(x) = \langle 0|\psi^{f}(x)|0\rangle \quad . \tag{1.11}$$

When we ignore h (the Planck constant), ϕ^{f} will be denoted by ϕ^{f}_{o}:

$$\phi^{f}_{o} = \lim_{h\to 0} \phi^{f} \quad . \tag{1.12}$$

Note that now the difference between the vacuum expectation value of products of ψ^f and the product of vacuum expectation values of ψ^f is due to the contraction of the in-fields, which create the loop diagrams in the course of successive iteration applied to (1.8). Since the contraction of the in-fields creates terms which vanish at $h = 0$, we have:

$$<0|F(\psi^f)|0> = F(\phi^f) + O(h) . \qquad (1.13)$$

Here $O(h)$ stands for terms which vanish in the limit $h \to 0$. Thus, the boson theorem (1.10) leads to the classical Euler equation[28]

$$\Lambda(\partial)\phi_0^f(x) = F(\phi_0^f) . \qquad (1.14)$$

The above argument shows that ϕ_0^f is given by tree diagrams only. We can formulate[28] a general method of construction of soliton solutions ϕ_0^f of the classical Euler equation as follows. First, construct the dynamical map by means of the tree approximation and then perform the boson transformation. We obtain ψ^f in the limit $h \to 0$. The vacuum expectation value of ψ^f thus obtained is the ϕ_0^f which satisfies the classical Euler equations.

The situation becomes very involved when ψ is a fermion field, because $<0|\psi^f|0>$ vanishes. In this case one can still construct various boson-like operators by means of products of an even number of ψ. When one finds a set of boson-like operators (say $\varphi_\alpha(\psi)$; $\alpha = 1, \ldots, n$) which satisfy a closed set of Heisenberg equations, we can write down the classical equations of the same form. The latter classical equations are regarded as the Euler equations, and our consideration can be applied to these Euler equations. However, it usually happens that the Heisenberg equations are not closed by a finite number of boson-like operators; to obtain a closed set of Heisenberg equations for boson-like operators, we usually need certain approximations. The Gor'kov equation for the order parameter $\Delta(x)$ (80) in the theory of superconductivity is a well-known example of this kind of Euler equation. When $\Delta(x)$ is very small, the Gor'kov equation becomes the Ginzburg-Landau equation. When $\Delta(x)$ is not small, its Euler equation has a very complicated structure.

Furthermore, when we denote the boson-transformed electron Heisenberg field by ψ^f, the order parameter is equal to $\langle 0|\psi^f_\uparrow \psi^f_\downarrow|0\rangle$ and the Goldstone boson is a bound state of two electrons. Thus, the tree approximation should be formulated in such a way that the internal lines include the propagation function of composite particles. Such a tree approximation has been formulated by means of the W.T. relations and has been called the generalized tree approximation[81]. Using this tree approximation, we can calculate the dynamical maps of the electron field $\psi_{\uparrow\downarrow}$ and of its boson-transformed $\psi^f_{\uparrow\downarrow}$. Then the vacuum expectation value of $\psi^f_\uparrow \psi^f_\downarrow$ gives $\Delta(x)$, which satisfies the Euler equation for $\Delta(x)$. However, in the boson method for superconductivity, we do not need to calculate $\Delta(x)$, because we can calculate the classical physical quantities such as the electromagnetic field and current by applying the boson transformation to the dynamical maps of the electromagnetic Heisenberg field and electron Heisenberg field[19,82].

It is an interesting question whether or not ϕ_o^f, with all the choices of $f_\alpha(x)$ satisfying $\Lambda_\alpha(\partial) f_\alpha(x) = 0$, covers all the soliton solutions of the classical Euler equation. The fact that the boson transformation covers all the states in which certain extended objects are created in the vacuum of the in-field Fock space suggests that the soliton solution, ϕ_o^f, constructed by the boson transformation method may cover the cases in which the asymptotic limit of the soliton solutions at infinite distance is well defined (see 2.4).

Let us illustrate the construction of the soliton solution ϕ_o^f from the boson transformation by using the following model in (1+1) dimensions:

$$(-\partial^2 - \mu^2)\psi(x) = \lambda\psi^3(x) \quad . \tag{1.15}$$

Here $x = (x_o x_1)$ and $\psi(x)$ is a scalar Heisenberg field. Using the notation,

$$v = \langle 0|\psi(x)|0\rangle \tag{1.16}$$

we define the Heisenberg operator $\rho(x)$ by the relation

$$\psi(x) = v + \rho(x) \quad . \tag{1.17}$$

Eq. (1.15) then leads to

$$(-\partial^2 - m^2)\,\rho(x) = \frac{3}{2}\, mg\rho^2(x) + \frac{1}{2}\, g^2\rho^3(x) \tag{1.18}$$

and also to $\lambda v^2 = -\mu^2$. Here $g = \sqrt{2\lambda}$ and $m^2 = 2\lambda v^2$. Let $\rho^{in}(x)$ denote the in-field which is the asymptotic limit of ρ. We have

$$(-\partial^2 - m^2)\rho^{in}(x) = 0 . \tag{1.19}$$

In the following we consider the extended objects created by the condensation of $\rho_{in}(x)$. We can therefore ignore all the in-fields except ρ_{in}. The tree approximation leads to the following dynamical map for ρ:

$$\begin{aligned}\rho(x) = \rho_{in}(x) &+ \frac{3}{2}\, mg(-i)\int d^2y\, \Delta(x-y) :(\rho_{in}(y))^2: + \\ &+ [\frac{1}{2}\, g^2\, (-i)\int d^2y\, \Delta(x-y) :\rho^3_{in}(y): + \\ &+ \frac{9}{2}\, m^2 g^2 (-i)\int d^2y\, \Delta(x-y)\Delta(y-z) :\rho_{in}(y)\rho^2_{in}(z):] .\end{aligned} \tag{1.20}$$

Here $\Delta(x)$ is the Green's function satisfying

$$(-\partial^2 - m^2)\,\Delta(x) = i\delta^{(2)}(x) . \tag{1.21}$$

We put (1.20) in the form

$$\rho(x) = \sum_{n=1}^{\infty} \rho^{(n)}(x) \tag{1.22}$$

where (n) denotes the order of the normal products. Then the following recursive relation holds in the tree approximation:

$$\rho^{(n)}(x) = \frac{3}{2}\, mg(-i)\int d^2y\,\Delta(x-y) : \sum_{i+j=n} \rho^{(i)}(y)\rho^{(j)}(y) : + \tag{1.23}$$
$$+ \frac{1}{2}\, g^2(-i)\int d^2y\,\Delta(x-y) : \sum_{i+j+k=n} \rho^{(i)}(y)\rho^{(j)}(y)\rho^{(k)}(y) :$$

We now perform the boson transformation

$$\rho_{in}(x) \longrightarrow \rho_{in}(x) + f(x) \tag{1.24}$$

where the c-number function f satisfies

$$(-\partial^2 - m^2)\, f = 0 \,. \tag{1.25}$$

Denoting the boson-transformed ρ-field operator by ρ^f, we have

$$\phi_o^f = v + \langle 0|\psi^f|0\rangle \,. \tag{1.26}$$

We consider the static case. The space coordinates x, y, will be simply denoted by x, y, ... Then (1.25) and (1.26) read as

$$\frac{d^2}{dx^2} f(x) = m^2 f(x) \tag{1.27}$$

and

$$\phi_o^f(x) = v + f(x) + \frac{3}{2}\, mg\int dy K(x-y) f^2(y) + [\frac{1}{2} g^2\int dy K(x-y) f^3(y) +$$
$$+ \frac{9}{2}\, m^2 g^2\int dy K(x-y) f(y)\int dz K(y-z) f^2(z)] + \ldots \tag{1.28}$$

where the Green's function K(x-y) is defined by

$$[\frac{d^2}{dx^2} - m^2] K(x-y) = \delta(x-y) \,. \tag{1.29}$$

The recursive relation (1.23) becomes

$$\rho_f^{(n)}(x) = \frac{3}{2}\, mg \int dy K(x-y) : \sum_{i+j=n} \rho_f^{(i)} \rho_f^{(j)} : +$$

$$+ \frac{1}{2} g^2 \int dy K(x-y) : \sum_{i+j+k=n} \rho_f^{(i)} \rho_f^{(j)} \rho_f^{(k)} : . \qquad (1.30)$$

As a solution of (1.27), we choose $f(x)$ which diverges at $x=-\infty$ and regular at $x=+\infty$; $f(x) = A \exp[-mx]$. Since $f(x)$ diverges at $x=-\infty$, we choose the Green's function K in such a way that $K(x-y) = 0$ for $x > y$:

$$K(x-y) = -\theta(y-x) \frac{1}{m} \sinh m(x-y) . \qquad (1.31)$$

Noticing that $\rho_f^{(1)}(x) = f(x)$, we can easily see from (1.30) that

$$\rho_f^{(n)}(x) = C_n (e^{-mx})^n \qquad (1.32)$$

with C_n satisfying the recurrence relation

$$C_n = \frac{1}{m^2(n^2-1)} \left[\frac{3}{2} mg \sum_{i+j=n} C_i C_j + \frac{1}{2} g^2 \sum_{i+j+k=n} C_i C_j C_k\right]$$

$$C_1 = A . \qquad (1.33)$$

Solving (1.33), together with the relation $v = m/g$, we have

$$C_n = 2v\left(\frac{A}{2v}\right)^n \qquad (1.34)$$

which leads to

$$\phi_o^f = v + 2v \sum_{n=1}^{\infty} \left(\frac{A}{2v} e^{-mx}\right)^n = v \frac{1 + \frac{A}{2v} e^{-mx}}{1 - \frac{A}{2v} e^{-mx}} \qquad (1.35)$$

When A is chosen as

$$A = -2ve^{ma} \tag{1.36}$$

then $\phi_o^f(x)$ is obtained as

$$\phi_o^f(x) = v \tanh[\frac{m}{2}(x-a)] \tag{1.37}$$

which is the well-known static solution of the Euler equation

$$(-\partial^2 - \mu^2)\,\phi_o^f(x) = \lambda[\phi_o^f(x)]^3 . \tag{1.38}$$

When $A = 2ve^{ma}$, $\phi_o^f(x)$ is given by

$$\phi_o^f(x) = v \coth[\frac{m}{2}(x-a)] \tag{1.39}$$

For the static case (1.37) and (1.39) are the only solutions which satisfy (1.38) with the condition $\phi_o^f(x) \to v(x \to +\infty)$. The same approach yields the static soliton of the sine-Gordon model.

The N-soliton solutions of the time-dependent sine-Gordon equation have been carefully studied by a group of people at Strasbourg[83]. They showed that the choice

$$f^N(t,x) = \sum_{j=1}^{N} \exp[\alpha_i x + \beta_i t - \delta_i] \tag{1.40}$$

with the condition

$$\alpha_i^2 - \beta_i^2 = m^2$$

leads to the N-soliton solutions[83,36] of the classical sine-Gordon equation. The construction of the two soliton solutions by means of f^N with $N = 2$ was explicitly made in ref.[83]. In their approach the boson transformation parameter for the N-soliton is related by a linear law to the single soliton parameters (see eq. (1.40)). So far we have been concerned with the explicit construction of the soliton solutions of the Euler equations for the classical fields ϕ. We have seen how, starting from a quantum theory in which the Heisenberg equations are treated as weak relations defined in the Fock space of the physical fields, one can construct a quantity $\phi_o^f(x)$ which is a soliton solution of the Euler equations for the classical field ϕ.

We want now to consider the energy of our solution. Let H denote the Hamiltonian of the Heisenberg field ϕ.

$$H = \int d_3x \, \mathcal{H}[x;\psi] \tag{1.41}$$

where $\mathcal{H}[x;\psi]$ is the Hamiltonian density. The boson-transformed Hamiltonian H^f is given by

$$H^f = \int d_3x \, \mathcal{H}[x;\psi^f] \, . \tag{1.42}$$

When the tree approximation is used, one can prove, by means of the same argument used in the derivation of the classical Euler equations from the quantum Heisenberg equations, that

$$\lim_{h\to 0} \langle 0|H^f|0\rangle = \int d_3x \, \mathcal{H}[x;\phi_o^f] \, . \tag{1.43}$$

Eq. (1.43), when used in the calculation of the energy of the soliton solutions of the 1+1 dimensional models analysed before, leads to

$$E_c = \begin{cases} \frac{2}{3}\frac{m^3}{g^2} & \phi_2^4 \\ 8\frac{m^3}{g^2} & \text{S.G.} \end{cases} \quad (1.44)$$

When the tree approximation is not used, the soliton energy is given by

$$\langle 0|H^f|0\rangle = \int d_3x \langle 0|\mathcal{H}[x;\psi^f]|0\rangle . \quad (1.45)$$

It is important to note that the space integration should be made only after the vacuum expectation value of the boson-transformed Hamiltonian density is calculated. This is true only when f has a certain singularity which prohibits its Fourier transform. In fact, it is well known that the Heisenberg Hamiltonian $H(\psi)$ is weakly equal to the Hamiltonian of the physical field $H_o(\phi^{in})$

$$\langle a|H(\psi)|b\rangle = \langle a|H_o(\phi^{in})|b\rangle \quad (1.46)$$

and that, as we saw in (2.1), the proof of (1.46) heavily relies upon the fact that ϕ^{in} is a Fourier transformable solution of the free field equation. Thus, when f(x) is Fourier transformable, we can generalize (1.46) as

$$\langle 0|H^f|0\rangle = \langle 0|H_o(\phi^{in} + f)|0\rangle \quad (1.47)$$

which gives the energy of the classically behaving extended object created by a regular boson function.

When f has a certain singularity which prohibits its

Fourier transform, eq. (1.47) is not true and we use (1.45).

Singular boson functions are very frequent in static models. In fact we have shown[28] that, except for the case in which the energy of the ψ^{in}-quantum vanishes for a certain non-vanishing value of the momentum, f(x) for a static soliton always has a singularity which prohibits its Fourier transform. The boson theory simplifies many of the problems which we usually meet in a quantum theory of extended objects.

First of all, the problem of evaluating quantum corrections to the classical soliton becomes clearer. We recall that the c-number field ϕ^f contains h, it has quantum effects in it, although it describes a classical object: the quantum correction is given by the loop diagrams appearing in the complete ($\hbar \neq 0$) expression of the dynamical map. Furthermore, the interaction between the classical system and quanta is consistently studied in the boson theory, since for this purpose we need only to calculate other matrix elements of ϕ^f and find the expression of the boson-transformed S-matrix.

The interaction between a classically behaving extended system and the quanta has been studied very recently[29]. The starting point is the Yang-Feldman equation (1.2) for the real boson field ψ and the expression of the dynamical map of ψ^f in (1.9) by means of successive iteration together with the tree approximation

$$\psi^{f}(x) = \phi_{o}^{f}(x) + \int d_4 y K(x,y) \varphi^{in}(y) + \ldots \quad . \qquad (1.48)$$

Here φ^{in} is the free field which satisfies (1.3). The second term of the dynamical map is called the linear boson term. Note that from (1.48)

$$<0|\psi^{f}|1> = \int d_4 y K(x,y) <0|\varphi^{in}|1> \equiv u(x) \qquad (1.49)$$

is the one boson wavefunction in presence of solitons. Then, use of the boson theorem leads [29] to the following equation for u(x)

$$[\Lambda(\partial) - F'(\phi_{o}^{f})] u(x) = 0 \qquad (1.50)$$

where $F'(\phi_{o}^{f}) = \delta F / \delta \phi_{o}^{f}$. This equation shows <u>that the φ^{in} quantum feels the self-consistent potential, which is induced by the soliton</u>.

A detailed study of eq. (1.50) has been presented in ref. [29] and will not be repeated here. An important feature of eq. (1.50) is the fact that it admits as solutions zero energy bound states: the so-called translation modes [77,78].

It has been shown [24] that, in order to include "self-consistently" the translation modes into the quantum theory, we need to enlarge the physical Fock space. The important result contained in ref. [29] is that this can be done if we associate a quantum coordinate $\vec{Q}$ to the soliton in such a way that

$$\psi^{f}(x) = \phi_{o}(\vec{x} + \vec{Q}) + F(\vec{x} + \vec{Q}) \quad . \tag{1.51}$$

An important conclusion which can be reached from (1.51) is the following: when the size of the extended object is much larger than the quantum fluctuation of $\vec{Q}$, the extended object behaves as a classical object. When the quantum fluctuation of $\vec{Q}$ becomes as large as the size of the object, the extended object behaves quantum-mechanically.

Various small domains which appear in the quantum ordered states both in solid state and high energy physics could be these quantum extended objects.

REFERENCES

1. A. Chodos, R. Jaffe, K. Johnson, C. Thorn, V. Weisskopf: Phys. Rev. D9, 3471 (1974); D10, 2599 (1974).
   W.A. Bardeen, M.S. Chanowitz, S.D. Drell, M. Weinstein, T.M. Yau: Phys. Rev. D11, 1094 (1975).
2. Y. Nambu: Phys. Rep. 23C, 250 (1976).
3. G. Veneziano: Phys. Rep. 9C, 201 (1974).
4. G. Veneziano: Nuovo Cimento 57A, 190 (1968).
5. R.F. Dashen, B. Hasslacher, A. Neveu: Phys. Rev. D10, 4114 (1974); Phys. Rev. D10, 4130 (1974); Phys. Rev. D10, 4138 (1974).
6. V. Korepin, L. Faddeev: Theor. Mat. Fiz. 25, 147 (1975).
7. J. Goldstone, R. Jackiw: Phys. Rev. D11, 1486 (1975).
8. Y. Nambu: Proc. Int. Conf. on Symmetries and Quarks (Wayne State University, 1969).
9. H.B. Nielsen, P. Olesen: Nucl. Phys. B61, 45 (1973).
10. G. 't Hooft: Nucl. Phys. B79, 276 (1974).
11. L. Faddeev: JETP Lett. 21, 64 (1975).
12. A.C.T. Wu, T.T. Wu: J. Math. Phys. 15, 53 (1974).
13. S. Mandelstam: Phys. Rev. D11, 3026 (1975).
14. A.M. Polyakov: JETP Lett. 20, 194 (1974).
15. D. Finkelstein: J. Math. Phys. 7, 1218 (1966).
16. V.L. Ginzburg, L.D. Landau: Zh. Eksp. Teor. Fiz. 20, 1064 (1950).
17. C.G. Kuper: An Introduction to the Theory of Superconductivity, Clarendon Press-Oxford (1968) pp. 120-125.

18. H. Matsumoto, M. Wadati, Y. Takahashi, H. Umezawa: Phys. Lett. 62A, 255 (1977) and Fortschritte der Physik (to be published).

19. L. Leplae, H. Umezawa and F. Mancini: Phys. Rep. 10C, 151 (1974); H. Matsumoto, N.J. Papastamatiou, H. Umezawa: Nucl. Phys. B82, 45 (1974); B97, 90 (1975).

20. H. Umezawa: Nuovo Cimento 40, 450 (1975); L. Leplae, R.N. Sen, H. Umezawa: Suppl. Prog. Theor. Phys. (Kioto) (Commemoration issue for the 30th Anniversary of the Meson Theory by Dr. H. Yukawa), 637 (1965).

21. H. Matsumoto, P. Sodano, H. Umezawa: Phys. Rev. D15, 2192 (1977).

22. M. Wadati, H. Matsumoto, H. Umezawa: Phys. Lett. 73B, 448 (1978); Phys. Rev. D18, 520 (1978).

23. L. Leplae, F. Mancini, H. Umezawa: Phys. Rev. B2, 3594 (1970).

24. H. Matsumoto, N.J. Papastamatiou, H. Umezawa: Nucl. Phys. B97, 90 (1975).

25. F. Mancini, H. Matsumoto, H. Umezawa, M. Wadati: Surface Phenomena in Superconductivity, Preprint, University of Alberta (1978).

26. The shape of the singularities depends on boundary condition: see ref. (22).

27. H. Umezawa: "Self-consistent Q.F.T. and Symmetry Breaking" from Renormalization and Invariance in Q.F.T., edited by E.R. Caianiello (Plenum Publishing, 1973).

28. H. Matsumoto, P. Sodano, H. Umezawa: Phys. Rev. D19, 511 (1979).

29. H. Matsumoto, G. Oberlechner, H. Umezawa, M. Umezawa: preprint, University of Alberta.

30. M. Wadati, H. Matsumoto, H. Umezawa: Phys. Rev. D18, 1192 (1978).

31. A. Belavin, A. Polyakov, A. Schwartz, Y. Tyupkin: Phys. Lett. B59, 85 (1975).

32. Yu. S. Tyupkin, V.A. Fateev, A.S. Shvarts: JETP Lett. 21, 42 (1975); M.I. Monastyrskii, A.M. Perelomov: JETP Lett. 21, 43 (1975).

33. W. Drechsler, M.E. Mayer: Fiber Bundle Techniques in Gauge Theories; Springer-Verlag (1977).

34. S. Schwinger: Phys. Rev. 128, 2425 (1962).

35. W. Thirring: Ann. Phys. 3, 91 (1969).

36. R. Hirota: J. Phys. Soc. Japan 33, 1459 (1972); A.C. Scott, F. Chiu, D. McLaughlin, Proc. IEEE 61, 1443 (1973).

37. D. McLaughlin, J. Math. Phys. 1, 96 (1975): see also 36 and references quoted there.

38. A.A. Abrikosov: Zh. Eksp. Teor. Fiz. 32, 1442 (1957).

39. A.M. Polyakov, JETP 41, 988 (1975).

40. H. Georgi, S. Glashow: Phys. Rev. Lett. 28, 1494 (1972).

41. J. Arafune, P.G.O. Freund, J. Goebel: J. Math. Phys. 16, 433 (1975).

42. L. Van Hove: Physica 18, 145 (1952); S.S. Schweber, A.S. Wightman: Phys. Rev. 98, 812 (1955); R. Haag: Mat. Fys. Medd. Dan. Vid. Selsk. 29, 12 (1955); L. Garding, A.S. Wightman: Proc. Nat. Acad. Sci. USA 40, 617 (1956);

K.O. Friedrichs: Mathematical Aspects of Quantum Theory of Fields (Interscience, N.Y., 1953); H. Umezawa: Lectures delivered at the UWM, 1974.

43. H. Lehmann, K. Symanzik, W. Zimmermann: Nuovo Cimento 11, 205 (1955).

44. H. Ezawa, T. Muta, H. Umezawa: Prog. Theor. Phys.(Kyoto) 29, 877 (1963).

45. H. Umezawa: Acta Phys. 19, 9 (1965).

46. H. Umezawa: Nuovo Cimento 40, 450 (1965); see also 27.

47. H. Matsumoto, N.J. Papastamatiou, H. Umezawa: Lectures delivered at the XIIth Winter School for Theoretical Physics, Karpacz, 1975.

48. H. Umezawa: Lectures delivered at UWM (1974).

49. M.N. Shah, H. Umezawa, G. Vitiello: Phys. Rev. B10, 4724 (1974).

50. H. Matsumoto, H. Umezawa, G. Vitiello, J.K. Wyly: Phys. Rev. D9, 2806 (1974).

51. Y. Fujimoto, N.J. Papastamatiou: Nuovo Cimento 40, 468 (1977).

52. H. Matsumoto, N.J. Papastamatiou, H. Umezawa: Phys. Rev. D13, 1054 (1976).

53. C. De Concini, G. Vitiello: Nucl. Phys. B116, 141 (1976); Phys. Lett. 70B, 355 (1977).

54. E. Inönü, E.P. Wigner: Proc. Natl. Acad. Sci. U.S. 39, 510 (1953).

55. M. Hongoh, H. Matsumoto, H. Umezawa: "The Dynamical Re-arrangement and Group Contraction" Preprint, U. of A. (1978).

56. H. Matsumoto, N.J. Papastamatiou, H. Umezawa: Phys. Lett. 46B, 73 (1973); Nucl. Phys. B68, 92 (1974).

57. H. Matsumoto, N.J. Papastamatiou, H. Umezawa, G. Vitiello: Nucl. Phys. B97, 61 (1975).

58. H. Matsumoto, N.J. Papastamatiou, H. Umezawa: Phys.Rev. D12, 1836 (1975).

59. R.P. Feynman: Rev. Mod. Phys. 20, 367 (1947); Phys. Rev. 74, 1430 (1948); J. Schwinger: Proc. Nat. Acad. of Sci. 37, 452 (1951); H. Umezawa, A. Visconti: Nuovo Cimento 1, 1079 (1955); K. Symanzik: Z. Naturf. 9, 809 (1954); D. Lurié: Particles and Fields (Interscience, New York, 1968).

60. H. Matsumoto, N.J. Papastamatiou, H. Umezawa: Nucl.Phys. B82, 45 (1974).

61. J.C. Ward: Phys. Rev. 78, 1824 (1950); Y. Takahashi: Nuovo Cimento 6, 370 (1957).

62. S. Goldstone: Nuovo Cimento 19, 154 (1961); J. Goldstone, A. Salam, S. Weinberg: Phys. Rev. 127, 965 (1962).

63. The existence of $f(x)$ in $A(\phi^{in})$ helps us in deriving relations in the momentum representation.

If $f(x) = 1$, $S\phi_H[\chi_{in} + \alpha]$ contains terms such as

$$\delta(p^2)\, p^2 \left(\frac{\alpha}{p^2 + i\epsilon}\right)^n$$

which are not well defined.

64. S. Coleman: Phys. Lett. 19, 144 (1965); E. Fabri, L.E. Picasso: Phys. Rev. Lett. 16, 408 (1966); R.F. Streater: Proc. Roy. Soc. 287A, 510 (1965).

65. P.W. Anderson: Phys. Rev. 130, 433 (1963).

66. M.P. Anderson: Phys. Rev. 110, 827 (1958); P. Higgs: Phys. Rev. 45, 1156 (1960); T.W.B. Kibble: Phys. Rev. 155, 1554 (1967).

67. S. Weinberg: Phys. Rev. Lett. 19, 1264 (1967); A. Salam, J. Strathdee: Nuovo Cimento 11A, 397 (1972).

68. G. 't Hooft: Nucl. Phys. B33, 173 (1971); B35, 167 (1971); B.W. Lee: Phys. Rev. D5, 823 (1972); B.W. Lee, J. Zinn-Justin: Phys. Rev. D5, 3121 (1972).

69. G.S. Guralnik, C.R. Hagen, T.W.B. Kibble: Phys. Rev. Lett. 13, 585 (1964).

70. S. Gupta: Proc. Soc. A63, 681 (1950); K. Bleuler: Helv. Phys. Acta 23, 567 (1950).

71. H. Matsumoto, N.J. Papastamatiou, H. Umezawa, G. Vitiello: Nucl. Phys. B97, 61 (1975).

72. L. Faddeev, V.N. Popov: Phys. Lett. 25B, 29 (1967).

73. C.N. Yang, R.L. Mills: Phys. Rev. 96, 191 (1954).

74. N. Nakanishi: Progr. Theor. Phys. 49, 640 (1973); 50, 1388 (1973).

75. For the sake of readers who do not like the functional method, Takahashi recently derived certain operator relations. All the W.T. relations are vacuum expectation values of these Takahashi operator relations; Y. Takahashi: Phys. Rev. D15, 1589 (1977); Y. Takahashi, T. Goto: University of Alberta Report (1976).

76. J.S. Schwinger: Phys. Rev. 76, 790 (1949).

77. R. Jackiw: Rev. Mod. Phys. 49, 681 (1977).

78. R. Rajaramaran: Phys. Rep. 21C, 227 (1975); J.L. Gervais, A. Neveu: Phys. Rep. 23C, 237 (1976).

79. C.N. Yang, D. Feldman: Phys. Rev. 79, 772 (1950).

80. L.P. Gor'kov: Zh. Eksp. Teor. Fiz. 36, 1918 (1959); 37, 1407 (1959).

81. H. Matsumoto, S. Seki, M. Tachiki, H. Umezawa: Phys.Rev. B17, 2296 (1978).

82. H. Matsumoto, H. Umezawa: Fortschritte der Physik 24, 357 (1976); H. Matsumoto, M. Tachiki, H. Umezawa: Fortschritte der Physik 25, 273 (1977).

83. G. Oberlechner, H. Umezawa, Ch. Zenses: "The Boson Transformation and the Soliton Solutions of the Sine-Gordon Equaton, University of Strasbourg Preprint CRN/HE 78-18 (to be published in Lett. Nuovo Cimento).

APPENDIX

Singular solutions of $\lambda_\alpha(\partial) f_\alpha(x) = 0$

We showed how soliton solutions of the classical Euler equations are constructed from a fully quantum theory by means of translations of boson fields (boson transformations)

$$\varphi_\alpha^{in} \longrightarrow \varphi_\alpha^{in} + f_\alpha \quad .$$

The only requirement imposed on $f_\alpha(x)$ is that it must satisfy

$$\lambda_\alpha(\partial) f_\alpha(x) = 0 \qquad (1.1)$$

where $\lambda_\alpha(\partial)$ is a free field boson operator.
We want to examine now the singular solutions of eq.(1.1).

We consider first the static case (i.e. $f_\alpha = f_\alpha(\vec{x})$, $\vec{x} \equiv$ space coordinate). In a relativistic theory, eq.(1.1) becomes

$$(\nabla^2 - m_\alpha^2) f_\alpha(x) = 0 \ . \qquad (1.2)$$

When $m_\alpha = 0$, this equation has a trivial solution, i.e. $f_\alpha(x) = \text{const}$. When we disregard the trivial solution, eq. (1.2) with $m_\alpha^2 \geq 0$ does not admit any solution which is Fourier transformable. Thus, the static extended objects created by the condensation of bosons carry certain singularities which prohibit the Fourier transformability of $f_\alpha(\vec{x})$.

The situation is different in non-relativistic cases, since the energy momentum relations allow for a situation in which

$$\omega_\alpha(\vec{p}) = 0 \qquad \text{for} \qquad \vec{p} \neq \vec{p}_a \neq 0 \qquad (a = 1, \ldots, n)$$

Here $\omega_\alpha(\vec{p})$ denotes the energy of the boson f_α^{in} with momentum $\vec{p}$. Assuming that $f_\alpha(\vec{x})$ is Fourier transformable, eq. (1.1) for static cases leads to

$$\omega_\alpha(p) f_\alpha(p) = 0$$

implying that $f_\alpha(\vec{x})$ is a multiperiodic function of the form

$$f_\alpha(\vec{x}) = \sum_a c_a e^{i\vec{p}_a \vec{x}} .$$

When $f_\alpha = f_\alpha(\vec{x}, t)$, eq. (1.1) admits both regular and singular (non-Fourier transformable) solutions.

For a relativistic theory the analysis of the static case of eq. (1.2) implies that $f_\alpha(\vec{x})$ has either a divergent singularity or a topological singularity. Here, divergent singularity means that $f_\alpha(\vec{x})$ diverges at $|\vec{x}| = \infty$ at least in certain direction of x. By topological singularity we mean that $f_\alpha(\vec{x})$ is not single valued.

We want to show that topological singularities are associated with the condensation of massless bosons[28]. For this purpose let us recall that a topologically singular boson function is defined by the relation

$$G^{(\alpha)\dagger}_{\mu\nu}(x) \neq 0 \quad \text{for certain} \quad x,\mu,\nu,\alpha \tag{1.3}$$

with $G^{\dagger}_{\mu\nu}(x)$ given by

$$G^{(\alpha)\dagger}_{\mu\nu}(x) \equiv [\partial_\mu, \partial_\nu] f_\alpha(x) \ . \tag{1.4}$$

The definitions (1.3-4) express the path-dependence of $f_\alpha(\vec{x})$. Furthermore, existence of the path-dependent $f_\alpha(\vec{x})$ requires that $\partial_\rho f_\alpha(\vec{x})$ should be single valued:

$$[\partial_\mu, \partial_\nu] \partial_\rho f_\alpha(x) = 0 \ . \tag{1.5}$$

Eq. (1.1) together with (1.4) and (1.5) leads to

$$\partial_\nu f_\alpha(x) = \frac{1}{\partial^2 + m_\alpha^2} \partial^\mu G^{(\alpha)\dagger}_{\mu\nu}(x) \tag{1.6}$$

where the derivative operator $1/\partial^2 + m_\alpha^2$ is defined in terms of the Fourier representation as follows:

$$\frac{1}{\partial^2 + m_\alpha^2} e^{ipx} = - \frac{1}{p^2 + m_\alpha^2} e^{ipx} \ .$$

Thus, $1/\partial^2 + m_\alpha^2$ is the Green's function of the Klein-Gordon equation with mass m_α. Note that, due to eq. (1.5), $G^{(\alpha)\dagger}_{\mu\nu}$ is Fourier transformable. Eq. (1.6) then leads to $\partial^2 f_\alpha(x) = 0$ implying that $m_\alpha = 0$.

Thus, <u>the extended objects created by a topologically singular boson function come from the condensation of massless bosons</u>. Let us recall here that the definition of topological singularity of the boson method is in sharp contrast with the commonly used definition: the topologically singular domains are usually defined as the domains where the Higgs field vanishes. The method of construction of solutions of the classical Euler equations, presented in Chapter III, provides a bridge between the two approaches.

A systematic study of extended objects with topological singularities has been presented in ref.[22]. Here, we only sketch the results of this construction. Use of eq. (1.6) leads to:

$$\partial_\mu f_\alpha(x) = \int d_4 y D(x-y) \partial^\nu G^{(\alpha)\dagger}_{\mu\nu}(y) \quad . \tag{1.7}$$

Eq. (1.7) expresses $\partial_\mu f_\alpha$ in terms of the tensor $G^{(\alpha)\dagger}_{\mu\nu}$ characterizing the topologically singular domain. In order to construct $G^{(\alpha)\dagger}_{\mu\nu}(x)$, it is useful to introduce the tensor

$$G^{(\alpha)}_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu}{}^{\lambda\rho} G^{(\alpha)\dagger}_{\lambda\rho} \quad . \tag{1.8}$$

Here, $\varepsilon_{o123} = 1$; $g_{oo} = g_{ii} = 1 \quad (i = 1,2,3)$.

The single valuedness of $\partial_\mu f_\alpha(x)$ leads to

$$\partial^\mu G^{(\alpha)}_{\mu\nu}(x) = 0 \quad . \tag{1.9}$$

It can be easily shown that the condition (1.9) is sufficient for the relation (1.7) to reproduce (1.4).

Thus, the construction of boson functions $f_\alpha(x)$ with topological singularities proceeds as follows: First, look for $G^{\alpha}_{\mu\nu}$ which satisfies the divergenceless condition (1.9) and then construct $G^{(\alpha)\dagger}_{\mu\nu}$ according to (1.8). Then, $\partial_\mu f_\alpha(x)$ can be calculated by means of (1.7). The multivalued function $f_\alpha(x)$ is obtained through a path-integral of $\partial_\mu f_\alpha(x)$, the existence of which is guaranteed by (1.4) as long as the path does not cross the singularities. In fact, (1.5) is the integrability condition for $f_\alpha(\vec{x})$ outside the singular domain. Although $f_\alpha(x)$ is path-dependent, an explicit expression of the function $f_\alpha(x)$ for x outside of the topologically singular domain can be obtained.

It is important to observe that for the construction of a topologically singular boson function the only input is the assignment of a tensor $G^\alpha_{\mu\nu}$ satisfying the divergence condition.

The convenience of using such a tensor as a starting point for our construction relies on the fact that $G^\alpha_{\mu\nu}$ is directly related to the parametrization of the topologically singular domain[22]. Furthermore, the condition (1.9) imposes some restrictions on the domain of topological singularity: it has been shown[22] that, in order for $G_{\mu\nu}(x)$ to satisfy (1.9), the domain of singularity must be without end points.

As an example, we consider a set of topologically singular domains $y^a_\mu(\tau,\sigma)$ $(a = 1,\ldots,n)$ which depends on two parameters τ and σ. Making use of the notation

$$\frac{\partial[y_\mu, y_\nu]}{\partial[\tau,\sigma]} = \frac{\partial y_\mu}{\partial\tau}\frac{\partial y_\nu}{\partial\sigma} - \frac{\partial y_\nu}{\partial\sigma}\frac{\partial y_\mu}{\partial\tau}. \tag{1.10}$$

we construct

$$G^\alpha_{\mu\nu}(x) = \sum_{a=1}^{n} \mathcal{M}^{\alpha a}\int d\tau\int d\sigma\, \frac{\partial[y^a_\mu, y^a_\nu]}{\partial[\tau,\sigma]}\,\delta^{(4)}(x-y^a(\tau,\sigma)) \tag{1.11}$$

such that:

$$\partial^{\mu} G^{(\alpha)}_{\mu\nu}(x) = \sum_{a=1}^{n} \mathcal{M}^{\alpha a} \int d\tau\, d\sigma \{ -\frac{\partial y^a_\nu}{\partial \sigma}\frac{\partial}{\partial \tau} + \frac{\partial y^a_\nu}{\partial \tau}\frac{\partial}{\partial \sigma} \} \delta(x - y^{(a)}(\tau,\sigma)) . \tag{1.12}$$

As mentioned before, the condition (1.12) restricts the choice of the surfaces $y^a(\tau,\sigma)$. For example, when we have only one surface (say $y(\tau,\sigma)$), (1.12) requires that the surface $y(\tau,\sigma)$ should not have any boundary.

In the following we analyse the consequences of (1.12) for a system of strings.

Systems of strings[22]:

We assume that τ is the time-like parameter and we choose y^a_o as

$$y^a_o(\tau,\sigma) = \tau \qquad \text{for all} \quad a \ . \tag{1.13}$$

Then $y^a_\mu(\tau,\sigma)$ appear to be lines at each instant τ. These lines are parametrized by the spatial parameter σ. In this case the extended object is called the string. The vortices in superconductors[23] and the dislocations[18] in crystals are well known examples.

Use of (1.12) and (1.13) leads to

$$\begin{aligned} \partial^{\mu} G^{(\alpha)}_{\mu o}(x) &= \sum_a \mathcal{M}^{\alpha a} \int d\tau\, d\sigma \{ -\frac{\partial y^a_o}{\partial \sigma}\frac{\partial}{\partial \tau} + \frac{\partial y^a_o}{\partial \tau}\frac{\partial}{\partial \sigma} \} \delta^{(4)}(x - y^a(\tau,\sigma)) \\ &= \sum_a \mathcal{M}^{\alpha a} \int d\tau\, d\sigma \frac{\partial}{\partial \sigma} \delta^{(4)}(x - y^{(a)}(\tau,\sigma)) \\ &= \sum_a \mathcal{M}^{\alpha a} \int d\sigma \frac{\partial}{\partial \sigma} \delta^{(3)}(\vec{x} - \vec{y}^{(a)}(t,\sigma)) \\ &= \sum_a \mathcal{M}^{\alpha a} [\delta^{(3)}(\vec{x} - \vec{y}^{(a)}(t,\sigma^a_1)) - \delta^{(3)}(\vec{x} - \vec{y}^{a}(t,\sigma^a_2))] \end{aligned} \tag{1.14}$$

where σ_1^a and σ_2^a are the end points of the line $y^{(a)}(\tau,\sigma)$ at the time t.

On the other hand, the divergence condition with $\nu = 0$ reads

$$\partial_j G_{jo}^{(\alpha)}(x) = 0 \tag{1.15}$$

which gives

$$\int_V d_3x \; \partial_j G_{jo}^{(\alpha)}(x) = 0 \; . \tag{1.16}$$

where V is a spatial domain enclosed by the surface S. Suppose now that a line (say $y^a(\tau,\sigma)$) has an end point (say $y^a(\tau,\sigma^a)$) which is not shared by any other lines. We can choose V in such a way that it contains no end points other than $y^a(\tau,\sigma^a)$. In this case, (1.14) gives:

$$\int_V d_3x \; \partial_j G_{jo}^{(\alpha)}(x) = \mathcal{M}^{\alpha a}$$

which contradicts with (1.16). Thus, the system of all the lines should form a network which does not have any end point. A joint point of more than two lines is called a vertex (or node). Consider a vertex denoted by $y(\tau)$. The lines which are joint with each other at $y(\tau)$ will be denoted by $y^b(\tau,\sigma)$ $(b = 1, 2, \ldots)$. Then choosing V to contain no other vertices than $y(\tau)$, we obtain from (1.16) and (1.14) that:

$$\sum_b (\pm \mathcal{M}^{ab}) = 0 \tag{1.17}$$

where + (−) sign corresponds to the first (second) term in right hand side of (1.14). This relation, which means that the strength of the string is conserved at each vertex, will be referred as the continuity relation. Conversely, it is possible to show that $\partial^{\mu} G^{\alpha}_{\mu\nu} = 0$ for all ν when (1.17) holds at each vertex, implying that (1.17) is the complete condition for the divergence condition to be satisfied. Therefore, the lines $y^{a}_{\mu}(\tau,\sigma)$ $(a = 1,2,\ldots)$ at time τ should form a network without any end point and the continuity relation (1.17) should hold at each vertex of the network.

The significance of the study of a system of strings lies in the fact that, by combining many strings and deforming them, one can construct a full variety of extended objects with topological singularities. Furthermore, it sheds light on the consequences of the divergence condition (1.9); in fact due to (1.9) the system of strings should form a network without any end points. This implies that only the singular domains, which can be formed by assembling either open lines or rings, are acceptable in this construction.

We will call the topological extended objects obtained by the assembly of open lines an open system, while the extended objects obtained by the assembly of rings a closed system. It is important to note that closed and open systems are extremely different in the asymptotic properties of the boson function. In fact, it is possible to show that open systems are characterized by

$$\lim_{|\vec{x}|\to\infty} f_\alpha(\vec{x}) \sim g_\alpha(\vec{n}) \qquad (1.18a)$$

$$\lim_{|\vec{x}|\to\infty} \partial_\mu f_\alpha(\vec{x}) \sim \frac{1}{|\vec{x}|} \qquad (1.18b)$$

while for closed systems we have

$$\lim_{|\vec{x}|\to\infty} f_\alpha(\vec{x}) = 0 \qquad (1.19a)$$

$$\lim_{|\vec{x}|\to\infty} \partial_\mu f_\alpha(x) = \frac{1}{|x|^\gamma} \qquad (1.19b)$$

In (1.18a-b) $\vec{n}$ is the direction along which the limit $|\vec{x}| \to \infty$ is performed. Eqs. (1.18a-b) lead us to the conclusion that the only extended objects which could give rise to an extended objects with a quantized topological charge are the open systems (see 2.4).

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1) S. De Martino, S. De Siena, F. Guerra, P. Sodano: Lett. Nuovo Cimento <u>16</u>, 569 (1976).

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4) M. Martellini, P. Sodano, G. Vitiello: Nuovo Cimento <u>48A</u>, 341 (1978).

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