University of Alberta

AN ENHANCED SOLVER FOR THE GAME OF AMAZONS

by

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Abstract

The game of Amazons is a young board game with simple rules, nice mathematical properties yet a high complexity between chess and Go. The state of the art Amazons solver was presented by Martin Müller in 2001 with which he solved the Amazons 5×5 starting position as a first player win.

This thesis presents our work on building Amazons endgame databases, improving the bounds heuristics and using ideas from combinatorial game theory to enhance the solver. With the improvements, we solve the 5×6 Amazons starting position as a first player win.

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Chapter 1

Introduction

1.1 Games and Artificial Intelligence

Since the advent of Artificial Intelligence (AI) in the 1950s, the relationship between games and AI research has been a reciprocal one. Games, as abstractions of real life problems, have finite state spaces, well defined rules and quantifiable goals, therefore offering ideal testbeds for AI research. AI research, on the other hand, leads to competitive opponents for human game players and also often offers interesting insights of the games.

As of today, the achievements of AI research on games have been prominent. Computer programs that can play at the same level or beyond world-class human players exist for many popular games:

- The **chess** program *Deep Blue* defeated the World Chess Champion *Garry Kasparov* with a score of $2-1-3$ (two wins, one loss, and three draws) in 1997 [29].
- The **Othello** program *Logistello* defeated the World Othello Champion *Takeshi Murakami* with a score of $6-0-0$ in 1997 [9].
- The **Scrabble** program *MAVEN* defeated the World Scrabble Champion *Joel Sherman* and runner-up *Matt Graham* with a composite score of $6-3-0$ in 1998 [30].

Moreover, numerous non-trivial games have been solved in the past thirty years:

- **Connect Four** was solved to be a first player win in 1989 [33].
- **Qubic** was solved to be a first player win in 1992 [4].
- **Go-Moku** was solved to be a first player win in 1996 [3].
- **Nine Men's Morris** was solved to be a draw in 1996 [17].
- 5×5 **Amazons** was solved to be a first player win in 2001 [23].
- **Awari** was solved to be a draw in 2002 [26].

- **Checkers** was solved to be a draw in 2007 [28].

The game of Amazons is a young board game with simple rules yet a high complexity between chess and Go [19]. With a high branching factor (average branching factor around 500 in 10×10 [18]), the game of Amazons is an ideal testbed for selective search algorithms and Monte Carlo tree searches. Since its board naturally decomposes into independent subgames as a game proceeds, the game of Amazons also offers a nice domain for combinatorial game theory studies.

1.2 Thesis Outline

In this thesis, we present various techniques we used to solve 5×6 Amazons. This thesis is organized as follows: Chapter 1 motivates AI research in games and introduces the game of Amazons. Chapter 2 covers the essential background on game solving algorithms, combinatorial game theory and our methodology for solving Amazons games. Chapter 3 discusses different types of endgame databases we built for solving 5×6 Amazons and various techniques for reducing their sizes. Chapter 4 gives details on how bounds of each area are computed and combined for solving an Amazons position. Chapter 5 shows some interesting statistics of the game itself and the experimental results of the techniques we developed. Chapter 6 discusses potential research topics for further improving the Amazons solver.

1.3 The Game of Amazons

The game of Amazons (Amazons for short) was invented by Walter Zamkauskas of Argentina in 1988. Amazons is a two-player board game played on a rectangular board with each player controlling 4 *amazons* (or *queens*) which are placed on the board before a game starts. The two players, *Black* and *White*, always move alternately until the game terminates when the player to move has no legal move. *White* usually plays first, and the winner is the player who made the last move. For example, the starting positions of 6×6 Amazons and 5×6 Amazons are shown in Figure 1.1a and Figure 1.1b, correspondingly.

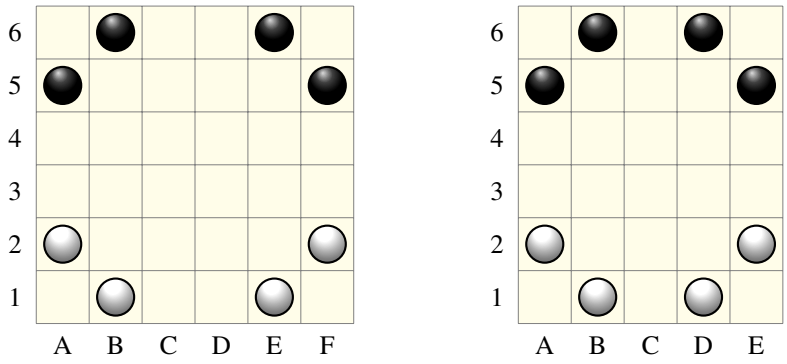
A move in Amazons is comprised of two compulsory phases.

1. **queen move**

   One player picks one of his queens q and moves it from its *origin square* to a different *destination square* in a straight line either horizontally, vertically or diagonally with the constraint that it may not cross or enter a square occupied by an amazon of either color or a burnt-off square.

2. **arrow shot**

   After q has moved, it has to shoot an arrow, which travels in the exact same way as a queen, from the square q moved to in the previous phase. The destination square of the arrow how-

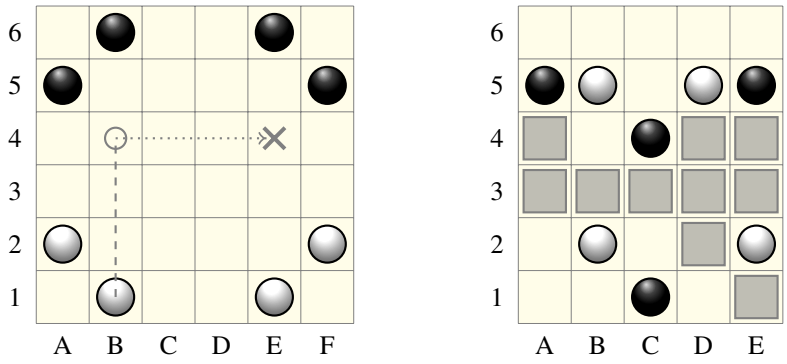


(a) 6×6 starting position (b) 5×6 starting position

Figure 1.1: Amazons starting positions

ever, is *burnt-off* permanently from the board such that no further queen moves or arrow shots can enter or cross this square.

Since exactly one empty square is burnt off in each move and at least one empty square is needed to make a move, an Amazons game is guaranteed to terminate with its total number of moves upper bounded by the number of empty squares. For example, a typical opening move for *White* in 6×6 Amazons is to move queen $B1$ to $B4$ and shoot to $E4$ (abbreviated as $B1 - B4 \times E4$), which is shown in Figure 1.2a with the square marked with the grey circle being the destination square for the queen move phase and the square marked with the grey cross being the square to be burnt off. A sample 5×6 position after 10 moves is shown in Figure 1.2b.



(a) Typical starting move (b) 10 moves played, *White* to play

Figure 1.2: Amazons moves

1.4 Related Work

Amazons has simple rules, but the computational complexity of an Amazons game is usually high due to its high branching factor (e.g., the first player has 544 moves in a 6×6 starting position). Buro has shown that deciding whether a queen can make certain number of moves in simple one-player

Amazons puzzles is NP-complete [10]. Furtak et al. have shown that determining the winner of a generalized Amazons game is PSPACE-complete [16].

Since an Amazons board naturally decomposes into independent subgames as the game proceeds, Amazons has attracted the attention of many researchers in combinatorial game theory. Berlekamp looked at positions with one queen per player on boards of size $2 \times N$ and calculated the thermographs for these positions [7]. Snatzke has computed the canonical forms of all Amazons positions up to size 2×11 by exhaustive searches and found some interesting positions [31].

Besides building combinatorial-game-theoretical databases, Tegos has used Amazons endgame databases for building a strong Amazons player [32]. Also, 5×5 Amazons was solved to be a first player win by Müller in 2001 [23].

Chapter 2

Background

2.1 Solving Games

Games discussed in this chapter are two player, zero sum, perfect information games with no chance. Such a game has no chance element (e.g., dice rolling) or hidden information (e.g., card dealing to a certain player), so both players know everything about the game. For example, Go, checkers and Amazons are all such games while Poker, which may have more than two players and cards hidden from the opponents, is not. Since there is no element of chance and the playing order is fixed once a game starts, such games are completely deterministic.

Minimax

The theory of *minimax* was introduced to simplify reasoning about games by representing them as *minimax trees*. A minimax tree T is a tree such that [13]:

- each node of the tree is either of type *max* (a.k.a. *OR*) or of type *min* (a.k.a *AND*),
- each node has zero or more children,
- there exists a unique special node called *root* which has no parent,
- the children of a node of type *max* are of type *min*,
- the children of a node of type *min* are of type *max*,
- a node which has no children is called a *leaf*.

Conventionally, max nodes are drawn as squares, min nodes are drawn as circles and the root player is the max player. For example, Figure 2.1 shows a typical minimax tree.

The **minimax value** of a node n with a set of children c in a minimax tree is defined recursively as follows [13]:

$$value(n) = \begin{cases} eval_m(n) & \text{if } n \text{ is a leaf} \\ \max\limits_{x \in c} value(x) & \text{if } n \text{ is a non-leaf max node} \\ \min\limits_{x \in c} value(x) & \text{if } n \text{ is a non-leaf min node} \end{cases}$$

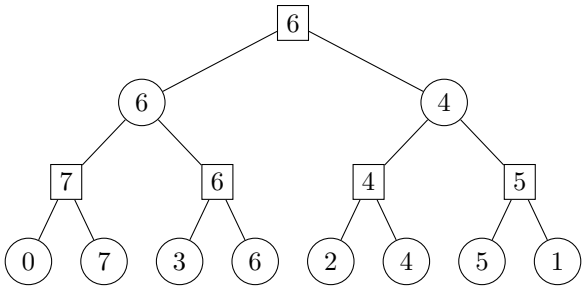


Figure 2.1: A typical minimax tree

where $eval_m(n)$ is the *evaluation function* assigning scores to the leaves from the max player's point of view. In other words, in a minimax tree, each max node maximizes amongst its children's values and each min node minimizes amongst its children's values. The minimax value of the root is also called the *value of the game*. For example, the leaf values in Figure 2.1 are assigned by the evaluation function and non-leaf values are propagated in a bottom-up fashion from the leaves. The value of the game is 6 from the max player's point of view.

A minimax tree contains all possible game positions reachable from the root as its nodes. An edge from a node p to its child n in a minimax tree corresponds to the move that changes p to n. The **strategy** for max in a minimax tree T is defined as a tree S such that:

- the root of S is the same as the root of T,
- the children of a non-leaf max node in S is a single node with the maximum minimax value in T (ties are broken arbitrarily),
- the children of a non-leaf min node in S are the same as that in T.

In other words, the strategy of T is built by selecting a child with the maximum minimax value at a max node and all children at a min node in a top-down fashion starting from the root of T. The strategy of a game gives a playing strategy that guarantees an outcome that is no inferior to the value of this game. For example, the strategy of Figure 2.1 is shown in Figure 2.2.

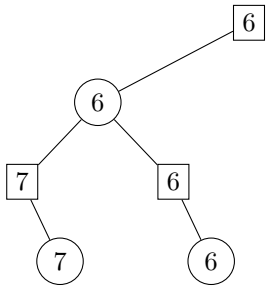


Figure 2.2: The strategy of Figure 2.1

Based on the fact that $\max(-a, -b) = -\min(a, b)$, *negamax* formulates the value of a node n with a set of children c from the player-to-move's point of view as follows [20]:

$$value(n) = \begin{cases} eval_n(n) & \text{if } n \text{ is a leaf} \\ \max\limits_{x \in c} -value(x) & \text{otherwise} \end{cases}$$

where $eval_n(n)$ assigns scores to the leaves from the player-to-move's point of view. For example, the negamax formulation of the tree in Figure 2.1 is shown in Figure 2.3. All the nodes are drawn as squares in a negamax tree since they are all maximizing among the negation of their children's values. The negamax formulation is used in the following discussions.

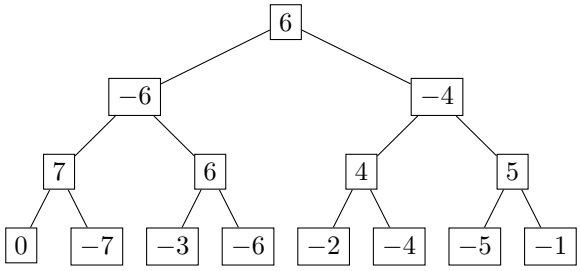


Figure 2.3: Negamax formulation of Figure 2.1

Solving Levels

To solve a game is to determine its *game-theoretic value* (win, loss or draw). Based on how much information is retained after a game is solved, there are three levels of solving a game [2]:

1. For an *ultraweakly solved* game, the game-theoretic value of the starting position, but not a corresponding strategy, is known. For example, Hex is a first player win but the winning strategy is unknown for large boards [34].

2. For a *weakly solved* game, both the game-theoretic value of the starting position and a corresponding strategy is known. For example, 5×5 Amazons is a first player win [23].

3. For a *strongly solved* game, the game-theoretic values for all possible positions that can arise in the game are known. For example, Awari is strongly solved and the starting position is a draw [26].

2.1.1 Game Solving Algorithms

Minimax Based

The minimax theory can be used to solve a game by assigning *exact scores* instead of heuristic scores to solved leaves in the evaluation function. In the negamax formulation, such an evaluation fuction can be defined as

$$eval_s(n) = \begin{cases} \infty & \text{if } n \text{ is a leaf and the player-to-move wins} \\ -\infty & \text{if } n \text{ is a leaf and the player-to-move loses} \\ 0 & \text{if } n \text{ is a leaf that draws} \\ eval_n(n) & \text{otherwise} \end{cases}$$

where $eval_n(n)$ assigns heuristic scores to leaves from the player-to-move's point of view. When ∞ or $-\infty$ is propagated back to a non-leaf node, this node is solved as win or loss for the player-to-move. Specifically, a game tree is solved when its root is solved.

With an evaluation function defined as such, the $\alpha\beta$ search algorithm [20] can be used to solve a game by traversing its minimax tree efficiently. For example, a solved minimax tree is shown in Figure 2.4 with its strategy highlighted in red bold lines. The player to move at the root could win the game by following any path from the root to a leaf in the strategy.

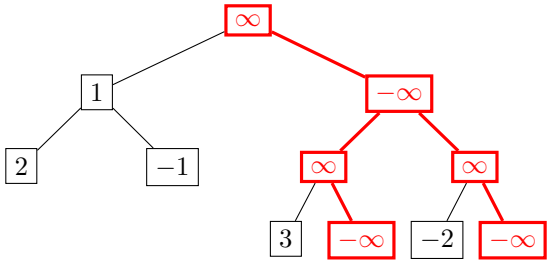


Figure 2.4: A solved minimax tree with highlighted strategy

Proof Number Search Based

If only the winner of a game is wanted, the binary question "Is this node a win for the player at the root?" can be asked. Proof and disproof numbers, first introduced by Allis in 1994 [5], are specifically designed to quickly find an answer to such a question. For any node n, the proof number of n (denoted as $n.pn$) is the minimum number of leaf nodes in its subtree that must be proven for n to be proven (i.e., n is a win for the player at the root). Similarly, the disproof number of n (denoted as $n.dn$) is the minimum number of leaf nodes in its subtree that must be disproven for n to be disproven (i.e., n is a loss for the player at the root). The proof and disproof number of a solved node s are set according to Equation 2.1 and Equation 2.2.

$$s.pn = 0, s.dn = \infty \qquad \text{if } s \text{ is proven} \tag{2.1}$$

$$s.pn = \infty, s.dn = 0 \qquad \text{if } s \text{ is disproven} \tag{2.2}$$

Proof-number search (PNS), introduced in the same paper [5], is a best-first search algorithm which uses proof and disproof numbers to guide the search. PNS is an iterative search algorithm which is terminated when the root position is solved. There are three phases in a single PNS iteration:

1. In the *descending* phase, a *most proving node (MPN)* (a leaf that is guaranteed to decrease either the proof or disproof number of the root if solved) is selected. An MPN is found by selecting a child with the minimum proof number at an OR node and a child with the minimum disproof number at an AND node until a leaf is reached.

2. In the *expanding* phase, the selected MPN is evaluated. If the MPN is solved, its proof and disproof number are set as in Equation 2.1 and Equation 2.2. Otherwise, the MPN is expanded

by generating all its children and assigning to each child both a proof number and a disproof number of 1.

3. In the *updating* phase, the ancestor nodes of the selected MPN are updated as in Equation 2.3 and Equation 2.4.

$$n.pn = \min_{x \in c} x.pn, n.dn = \sum_{x \in c} x.dn \qquad \text{if } n \text{ is an OR node} \tag{2.3}$$

$$n.pn = \sum_{x \in c} x.pn, n.dn = \min_{x \in c} x.dn \qquad \text{if } n \text{ is an AND node} \tag{2.4}$$

For example, Figure 2.5 shows a PNS tree with each node's proof and disproof number separated by a horizontal line with the proof numbers on top and the disproof numbers at bottom. Leaf n_8 is proven and leaves n_5 and n_{10} are disproven. The single winning child n_8 at the OR node n_3 is sufficient to prove n_3 and the single losing child n_5 at the AND node n_2 is sufficient to disprove n_2. The highlighted path from n_0 to n_9 illustrates that n_9 is found as an MPN (another MPN in the tree is n_{11}) in the descending phase. The proof number of n_0 indicates that only one leaf (either n_9 or n_{11}) needs to be proven for n_0 to be proven and the disproof number of n_0 indicates that two leaves (both n_9 and n_{11}) need to be disproven for n_0 to be disproven. If n_9 is disproved (i.e., $n_9.pn = \infty$ and $n_9.dn = 0$), then in the updating phase, all its ancestors (n_4, n_1 and n_0) will be updated as to have both the proof and disproof number to be 1 and the next descent will select n_{11} as the MPN.

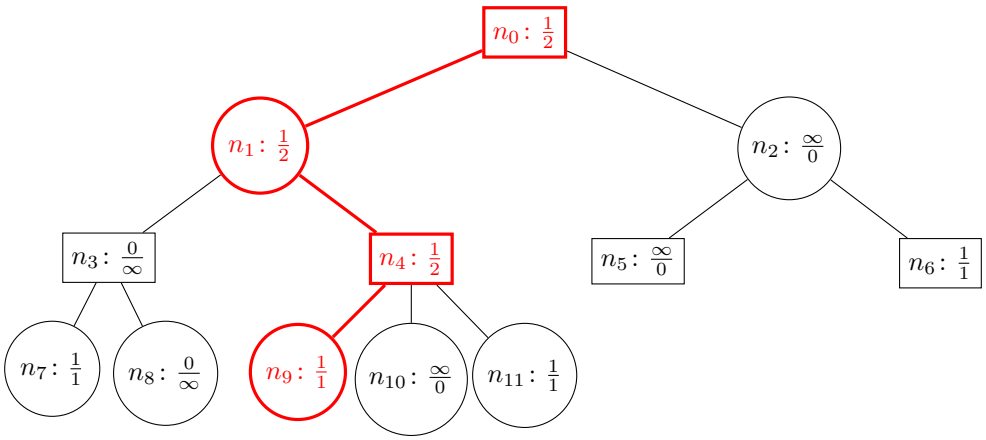


Figure 2.5: A PNS tree, proof numbers on top, disproof numbers at bottom

Since PNS is a best-first search algorithm, all nodes generated have to be stored in memory to select the next MPN. To avoid such a huge memory usage, various proof-number search variants have been proposed. Depth first proof-number search (DFPN) [25] turns PNS into a depth first search algorithm by staying within a subtree for as long as possible.

Let p be a node and c be the set of its children. Let c_1 and c_2 be p's most and second most proving child respectively. p does not need to be updated as long as the next MPN will stay within c_1's subtree, which means that c_1's proof number is smaller than or equal to all its siblings' proof

numbers if p is an OR node, and c_1's disproof number is smaller than or equal to all its siblings' disproof numbers if p is an AND node. This is achieved by setting up two thresholds before expanding an MPN. If p is an OR node, the proof threshold of c_1 ($c_1.th_{pn}$) and disproof threshold of c_1 ($c_1.th_{dn}$) are set as in Equation 2.5 and Equation 2.6. If p is an AND node, the proof and disproof threshold of c_1 are set as in Equation 2.7 and Equation 2.8. After setting up the thresholds, the search stays within c_1's subtree (i.e., no updating for c_1 and its ancestors) until $c_1.pn \geq c_1.th_{pn}$ or $c_1.dn \geq c_1.th_{dn}$.

$$c_1.th_{pn} = \min(p.th_{dn}, c_2.pn + 1) \qquad \text{if } p \text{ is an OR node} \tag{2.5}$$

$$c_1.th_{dn} = p.th_{pn} + c_1.pn - \sum_{x \in c, x \neq c_1} x.pn \qquad \text{if } p \text{ is an OR node} \tag{2.6}$$

$$c_1.th_{pn} = p.th_{dn} + c_1.dn - \sum_{x \in c, x \neq c_1} x.dn \qquad \text{if } p \text{ is an AND node} \tag{2.7}$$

$$c_1.th_{dn} = \min(p.th_{pn}, c_2.dn + 1) \qquad \text{if } p \text{ is an AND node} \tag{2.8}$$

2.2 Combinatorial Game Theory

2.2.1 Combinatorial Games

A **combinatorial game** is a two-player game with perfect information. Two players, often called *Left* and *Right*, play alternately until the player whose turn it is to move has no legal moves available.

The winner of a combinatorial game can be determined in various ways (e.g., counting stones) and it is possible to *draw* in some games (e.g., chess). In this paper, we assume there is a unique winner when the game terminates determined on the basis of who made the last move. A game where the player who made the last move wins is called *normal play* [1]. A game where the player who made the last move loses is called *misère play* [1].

A game is said to be *loopy* if it is possible to return to a previously seen position during gameplay (e.g., checkers) and therefore infinitely long move sequences are possible [1]. In this paper, we assume non-loopy games.

Under such assumptions, the outcome of a combinatorial game can be categorized into 4 classes as shown in Table 2.1. Game positions where the first or second player to move can force a win fall in the $\mathcal{N}$ or $\mathcal{P}$ class respectively. If *Left* or *Right* can always force a win regardless of who moves first in a game position, this position falls into the $\mathcal{L}$ or $\mathcal{R}$ class respectively.

Outcome Classes		**When Right moves first**	
		Right wins	Left wins
When Left moves first	Left wins	$\mathcal{N}$	$\mathcal{L}$
	Right wins	$\mathcal{R}$	$\mathcal{P}$

Table 2.1: Outcome classes [1]

Combinatorial game theory is a branch of mathematics developed to represent and analyze two-player combinatorial games [8][11]. The basic idea of the theory is to represent a two-player

game G as an ordered pair. The first element of the pair, called G's *Left Options*, is the set of games reached by playing each of *Left*'s legal moves on G. The second element of the pair, called G's *Right Options*, is the set of games reached by playing each of *Right*'s legal moves on G. Formally a game is defined as:

$$G = \{G^L \mid G^R\}$$

where G^L and G^R are *Left Options* and *Right Options* respectively. It is easy to see how this recursive definition of G leads to a tree structure when games in G's *Left Options* and *Right Options* are expanded in the same way [12]. With G being the root, each node in this tree represents a reachable game from G and each branch from a parent node to its child represents the corresponding move.

For example, according to its rules, Amazons is a non-loopy game subject to normal play. Following the convention of [23], let *Black* be the *Left* player and *White* be the *Right* player. A typical Amazons position can be represented as a rooted tree as shown in Figure 2.6. The root position G has 3 left branches and 3 right branches since both *Black* and *White* have 3 moves to choose from at G. The *Left* and *Right* options are

$$G^L = \{G_1^L, G_2^L, G_3^L\}$$
$$G^R = \{G_1^R, G_2^R, G_3^R\}$$

respectively. Each game in G^L and G^R can be analyzed in the same way and the tree can be grown recursively until all leaves are terminated games where no player has moves available.

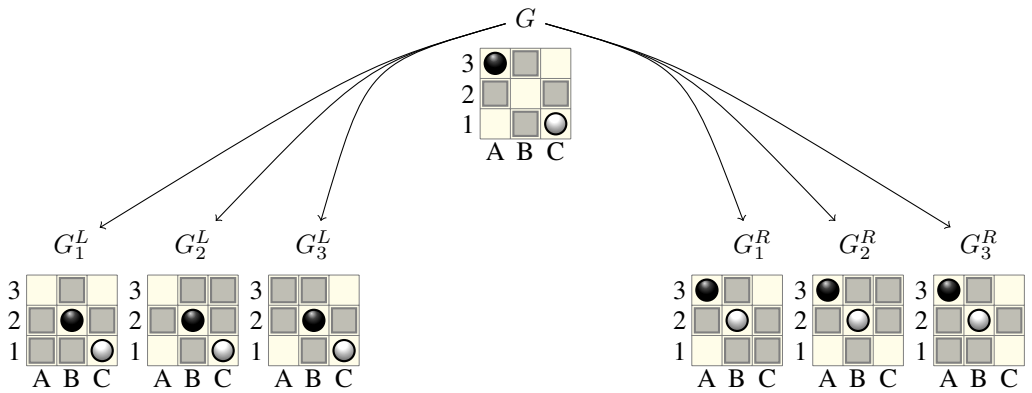


Figure 2.6: An Amazons position as a tree

A full game tree constructed in this way is usually not the most simplified form. For example, the game tree in Figure 2.6 has redundant branches since symmetrical therefore equivalent game positions exist (e.g., G_1^L and G_2^L). Luckily, every game G has a unique smallest equivalent game that is called G's *canonical form* [1]. A game can be simplified to its canonical form by following a methodical procedure [1].

2.2.2 Naming Games

Adding and Comparing Games

Besides viewing a game as a rooted tree, combinatorial game theory also enables games as defined in the previous section to be *summed* and *compared*. In particular, games form a *partially ordered abelian group* [1]. Given two games $G = \{G^L \mid G^R\}$ and $H = \{H^L \mid H^R\}$, the *summation*, *negation* and *subtraction* of games are defined in Equation 2.9 to Equation 2.11 and the *comparison* of games are defined in Equation 2.12 to Equation 2.17. An operation applied on a set of games (e.g., G^L) means the operation is applied to each element in this set.

$$G + H \equiv \quad \{G^L + H, G + H^L \mid G^R + H, G + H^R\} \tag{2.9}$$

$$-G \equiv \quad \{-G^R \mid -G^L\} \tag{2.10}$$

$$G - H \equiv \quad G + (-H) \tag{2.11}$$

$$G \geq H \equiv \quad \forall X \textit{ Left} \text{ wins } G + X \text{ whenever } \textit{Left} \text{ wins } H + X. \tag{2.12}$$

$$G \leq H \equiv \quad \forall X \textit{ Right} \text{ wins } G + X \text{ whenever } \textit{Right} \text{ wins } H + X. \tag{2.13}$$

$$G = H \equiv \quad G \geq H \text{ and } G \leq H. \tag{2.14}$$

$$G > H \equiv \quad G \geq H \text{ and } G \neq H. \tag{2.15}$$

$$G < H \equiv \quad G \leq H \text{ and } G \neq H. \tag{2.16}$$

$$G \mid\mid H \equiv \quad G \ngeq H \text{ and } G \nleq H. \tag{2.17}$$

The above definitions have intuitive interpretations. $G + H$ means that the two games G and H are placed side by side and the player to move can pick either game that he still has legal moves and move in it, leaving the other one intact. $-G$ means that the players in G have switched color (i.e., *Left* becomes *Right* and *Right* becomes *Left*). $G \geq H$ means that *Left* has a bigger advantage over *Right* in G than in H. Similarly, *Right* can win over *Left* for more in G than in H if $G \leq H$. $G = H$ simply means the two players are indifferent between G and H. The relationship of $G \mid\mid H$ (G is *confused* with H), means neither game offers a sure advantage over the other for the players.

Numbers

Assuming $G = \{G^L \mid G^R\}$ in canonical form, if each game in G^L is less than each game in G^R, then G is a **number**. A number is an **integer** if at least one of G^L and G^R is an empty set. If G^L and G^R are both empty, this integer is defined as 0. Formally, integers are defined as [1]:

$$0 = \{ \mid \} \qquad n = \{n - 1 \mid \}$$

$$-n = \{ \mid -n + 1\}$$

A positive integer n indicates n free moves for *Left* while *Right* has no moves available. Similarly, a negative integer $-n$ means n free moves for *Right*. 0 means no player can make a move, thus the first player to move in a 0 loses. For example, the position shown in Figure 2.7a is a -1 position

since there are no *Black* queens and *White* can only make 1 move. The position in Figure 2.7b is a 0 position since neither player can make a move.

(a) A -1 position (b) A 0 position

Figure 2.7: Amazons positions – integers

The only numbers occurring in finite non-loopy games are integers and *dyadic rationals* (**fractions** for short), rationals whose denominator is a power of 2. Formally, a fraction is defined as [1]:

$$\frac{m}{2^j} = \{\frac{m-1}{2^j} \mid \frac{m+1}{2^j}\} \quad \text{(for m odd)}$$

For example, the position in Figure 2.8 is $\frac{1}{2} = \{0 \mid 1\}$ since *Black*'s best move is $B2 - B1 \times B2$, which leads to a 0 position and the only move *White* has leads to a 1 position.

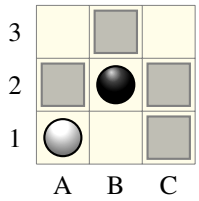


Figure 2.8: A $\frac{1}{2}$ Amazons position

Games with A Confusion Interval

Given $G = \{G^L \mid G^R\}$, $LS(G)$ (the left stop of G) and $RS(G)$ (the right stop of G) are defined in a mutually recursive fashion as [1]:

$$LS(G) = \begin{cases} x & \text{if } G \text{ is the number x} \\ max(RS(G^L)) & \text{if } G \text{ is not a number} \end{cases}$$

$$RS(G) = \begin{cases} x & \text{if } G \text{ is the number x} \\ min(LS(G^R)) & \text{if } G \text{ is not a number} \end{cases}$$

For example, if $G = \{\{3 \mid 2\} \mid -2\}$, then $LS(G) = 2$ and $RS(G) = -2$. The left/right stop of a game G can be viewed as the minimax value of G when G is played with *Left*/*Right* being the first player respectively, assuming the players stop playing as soon as the game becomes a number.

If G is not a number, then G has a *confusion interval* from $LS(G)$ to $RS(G)$, and G compares with all other games besides $LS(G)$ and $RS(G)$ according to Table 2.2. How G compares with $LS(G)$ and $RS(G)$ is game dependent. G could either be greater than or confused with $LS(G)$ and G could either be less than or confused with $RS(G)$ [1].

$G \mid\mid v$	$\forall v \in (RS(G), LS(G))$
$G > v$	$\forall v < RS(G)$
$G < v$	$\forall v > LS(G)$

Table 2.2: Games with a confusion interval

Infinitesimals

If G satisfies the condition that $-v < g < v$ for all positive numbers v, then g is called an **infinitesimal** [1]. The left stop and right stop of any infinitesimals therefore are both 0. In other words, for either player, an infinitesimal provides a utility that is strictly less than any number in his favor. For example, the position shown in Figure 2.9 is the infinitesimal $\{0 \mid 0\}$ and it is confused with 0. The winner of this game will be whoever makes the first move because it will lead to a 0 position for his opponent.

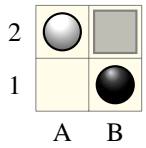


Figure 2.9: A $*$ Amazons position

The infinitesimal $\{0 \mid 0\}$ is called a "$*$" (read *star*). Two other common infinitesimals are *up* ($\uparrow = \{0 \mid *\}$) and *down* ($\downarrow = \{* \mid 0\}$). An infinitesimal doesn't have to be confused with 0. There are positive infinitesimals (e.g., $\uparrow > 0$) and negative infinitesimals (e.g., $\downarrow < 0$). Moreover, 0 itself is an infinitesimal. The sum of any two infinitesimals is still an infinitesimal. Specifically, the sum of any two positive/negative infinitesimals is still a positive/negative infinitesimal [1]. Figure 2.10 summarizes the relative ordering of numbers, $\uparrow$, $\downarrow$, and $*$. By convention, the number line is reversed to put the *Left* player on the left-hand side.

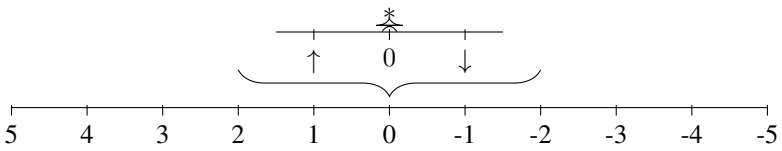


Figure 2.10: Relative ordering of numbers, $\uparrow$, $\downarrow$, and $*$ [1]

Hot Games

If $LS(G) > RS(G)$, then G is called a **hot game**. For example, in Figure 2.11, the position is $\{2 \mid -2\}$ because the first player to move in this position could completely block his opponent and turn this position into a number in his favor.

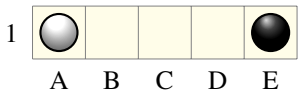


Figure 2.11: A $\{2 \mid -2\}$ Amazons position

Outcome Classes Revisited

How a game G compares to 0 completely determines its outcome class according to Table 2.3 [1].

G **Compares to** 0	**Outcome Class**
$G > 0$	$\mathcal{L}$
$G = 0$	$\mathcal{P}$
$G < 0$	$\mathcal{R}$
$g \mid\mid 0$	$\mathcal{N}$

Table 2.3: Outcome class of game G

2.2.3 Thermography

Although we can easily locate a number on the reversed number line (see Figure 2.10), it is impossible to find a single point that suits a hot game. Therefore, we need to "cool down" such a game before we can plot it. A game G cooled by t is defined as [1]:

$$G_t = \begin{cases} \mu & \text{if } \exists t' < t\text{: } LS(G_{t'}) = RS(G_{t'}) = \mu \\ \{G_t^L - t \mid G_t^R + t\} & \text{otherwise} \end{cases}$$

An insightful view of G_t is that a third party is levying a tax t on each move in G. Players are eager to move in a hot game, therefore they are interested in paying t for the privilege. When the tax gets high enough to just offset the gain from moving first, players will be indifferent of moving or not.

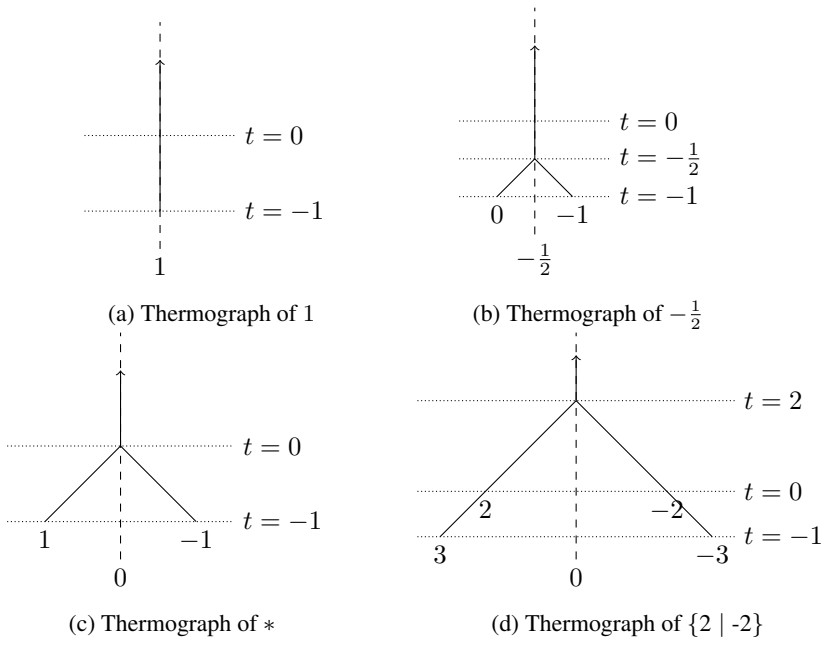
The cooled game is designed to find out this minimum t that makes G_t infinitesimally close to a number. Such a t always exists for a non-integer game since t is going be deducted from G's left options and will be added to G's right options to make the difference between G^L and G^R smaller. This minimum t, called the **temperature** of G ($TEMP(G)$), measures the urgency for either player to move in G. The number G_t approaches is called the **mean value** of G ($MEAN(G)$).

The higher G's temperature, the more interested are the players in playing in it. Temperatures and mean values of four different types of games are shown in Table 2.4. Any integer has a temperature of -1 and its mean value is the integer itself. The temperature of a fraction $\frac{m}{2^j}$ (m odd) is $-\frac{1}{2^j}$ and its mean value is the fraction itself. The fact that numbers have negative temperatures indicates that moves in such games are not desirable since moves on a number change it to a worse number for the player that makes this move. Both the temperature and the mean value of an infinitesimal are 0. A hot game has a positive temperature and a mean value of this game cooled by its temperature plus any positive ϵ.

G	$TEMP(G)$	$MEAN(G)$
integer	-1	G
fraction $\frac{m}{2^j}$	$-\frac{1}{2^j}$	G
infinitesimal	0	0
hot game	> 0	$G_{TEMP(G)+\epsilon}(\forall \epsilon > 0)$

Table 2.4: $TEMP(G)$ and $MEAN(G)$ characteristics of a game G

Since hot games behave differently at different temperatures, to plot it, we need to add a vertical axis for the temperature. The resulting graph is called the **thermograph** of G. At each temperature t from -1 to positive infinity, we plot the corresponding cooled game G_t's left and right stop. The collection of all the left(right) stops is called G's **left(right) wall**. Beyond G's temperature, G_t is equal to the number $MEAN(G)$, which is represented by a vertical *mast* with an up-pointing arrow [8]. Typical thermographs for four different types of games are shown in Figure 2.12.



(a) Thermograph of 1

(b) Thermograph of $-\frac{1}{2}$

(c) Thermograph of $*$

(d) Thermograph of {2 | -2}

Figure 2.12: Typical thermographs

All integers and fractions form a dense set at temperature 0 [6]. However, the possible values of thermographs become sparse at negative temperatures. At temperature -1, all thermographs rest on a discrete set of pilings located at the integer values; At t = $-\frac{1}{2}$, there are twice as many pilings located at half-integers, etc [6]. This effect is shown in Figure 2.13.

2.3 Solving Amazons

2.3.1 Methodology

Because Amazons is a normal play game, we only need to know which player can make more moves than his opponent to determine the winner. In other words, we are not interested in the

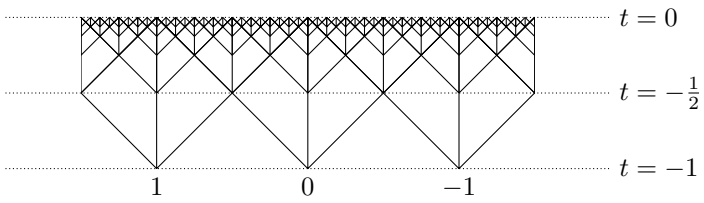


Figure 2.13: The foundation of thermographs [6]

absolute number of moves either player can make but rather the difference in the number of moves the players can make. Without loss of generality, we define the **bounds** of a game G as the range of the number of moves *Black* can make more than *White* in G. The bounds of a game are written in the form $[lower, upper]$ where $lower$ is the minimum number of moves *Black* can make more than *White*, and $upper$ is the maximum number of moves *Black* can make more than *White*. For example, if the bounds of a game are $[-3, -1]$, it is interpreted as *Black* can make at least -3 and at most -1 more moves than *White* (i.e., *White* has at least 1 and at most 3 more moves than *Black*). Therefore *White* wins.

More on Bounds

The bounds of a game G are abbreviated as $bounds(G)$. The lower and upper bound of bounds b are abbreviated as $l(b)$ and $u(b)$ respectively. For any game G, Equations 2.18 and 2.19 hold between its bounds and stop values due to their definitions.

$$l(bounds(G)) \leq RS(G) \tag{2.18}$$

$$u(bounds(G)) \geq LS(G) \tag{2.19}$$

The *summation*, *negation* and *subtraction* operators of the two bounds b_1 and b_2 are defined from Equation 2.20 to Equation 2.22.

$$b_1 + b_2 \equiv [l(b_1) + l(b_2), u(b_1) + u(b_2)] \tag{2.20}$$

$$-b_1 \equiv [-u(b_1), -l(b_1)] \tag{2.21}$$

$$b_1 - b_2 \equiv b_1 + (-b_2) \tag{2.22}$$

Given two games G and H, the bounds of $G+H$, $-G$ and $G-H$ are shown from Equation 2.23 to Equation 2.25. In Chapter 4, we will introduce techniques for improving the bounds of a game based on its next player. The improved bounds cannot be summed directly. Instead, we will discuss the combination of such bounds in Section 4.3.

$$bounds(G + H) = bounds(G) + bounds(H) \tag{2.23}$$

$$bounds(-G) = -bounds(G) \tag{2.24}$$

$$bounds(G - H) = bounds(G) - bounds(H) \tag{2.25}$$

Judging the Winner

The winner of a game G can sometimes be determined based on its bounds according to the following rules [23].

- *Black* wins if $l(bounds(G)) > 0$ or $l(bounds(G)) = 0$ and it is *White* to move;
- *White* wins if $u(bounds(G)) < 0$ or $u(bounds(G)) = 0$ and it is *Black* to move;
- In other cases, $bounds(G)$ does not provide enough information for judging the winner of G.

2.3.2 Amazons Solver Architecture

Since an arrow is shot on the board for each move and neither the following arrows nor queens can pass through a burnt-off square, the burnt-off squares practically establish a "barrier" between different parts of the board. As the game progresses and squares are burnt off, the board is naturally split into several independent subgames (or areas). For example, in Figure 2.14, these areas are delimited by the blue solid lines.

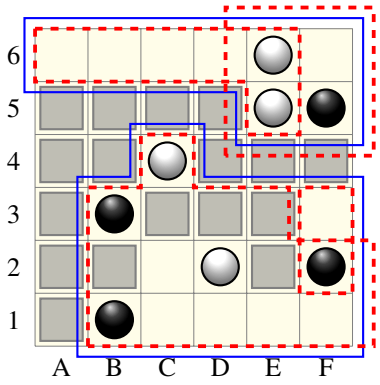


Figure 2.14: Board partition

Bounds of these independent areas can be computed individually and summed up into a single global bound, which can then be used to determine the winner of the whole board according to the rules in Section 2.3.1.

Board Partition

Intuitively, areas can be found by finding all the 8-connected components of empty squares and queens of both players on the board. This *basic partition* yields the two areas delimited by the blue solid lines in Figure 2.14.

An *improved partition* can be achieved by using queens of the same color to block his opponents [23]. In the basic partition, if part of an area is inaccessible to one player given that his opponent wouldn't move, then this part of the board along with the blocking queens (called **blockers**) constitutes a new area for his opponent. The non-blocker queens are also called *normal* queens. For

example, in Figure 2.14, the improved partitions are delimited by the red dashed lines and the blockers are *White* queens $E5$ and $E6$ and the *Black* queen $F2$. An improved partition splits an area from the basic partition into multiple smaller areas with blockers. A blocker therefore must be part of all the areas it separates to illustrate the fact that they can make moves in, potentially, all of them.

Solver Overview

Algorithm 1 illustrates the abstract main loop of the solver. Function `NextPosition` gets the next position to be evaluated depending on the search algorithm (e.g., DFPN will compute a most proving node). This chosen position is then partitioned and the bounds of each resulting area are computed. Finally, the computed bounds are summed up into `globalBounds` which is passed to function `UpdateSolver` to determine the winner of this position according to the rules in Section 2.3.1 and update the solver's internal state (e.g., DFPN will update the proof or disproof numbers as necessary). This process is repeated until the starting position is solved.

Algorithm 1 Abstract Solver Main Loop

```
function SolverMain(position)
    while position is not solved do
        position ← Nextposition()
        areas ← BoardPartition(position)
        ComputeBounds(areas)
        globalBounds ← SumBounds(areas)
        UpdateSolver(globalBounds)
    end while
end function
```

2.3.3 Areas in Amazons

As shown in Section 2.3.2, areas are the major building blocks of the solver. They need to be carefully analyzed before they can be used as a summand of the entire game. An **area** in Amazons is a set of 8-connected squares which can contain empty squares and queens of either color. The presence of at least one queen (of either color) and at least one reachable empty square is essential for an area to be of interest for the player of that queen. An area containing only queens or only empty squares is called a *dead area* [23]. Neither player can make moves in a dead area, therefore dead areas can be safely discarded when analyzing a position and any of its descendants. Areas don't contain any burnt-off squares. However in the figures, burnt-off squares are often included for simplicity.

In a basic partition, the resulting areas are all independent since they are found as 8-connected components to begin with. In an improved partition however, areas can overlap on blockers. Based on whether both players have queens in an area and whether blockers exist, areas can be classified as follows:

	has-blockers	no-blockers
queens-of-both-colors	active	
queens-of-one-color	simple territory	blocker territory

Table 2.5: Classification of areas

Furthermore, we define the **size** of an area to be the $(width \times height)$ pair of its minimum bounding rectangle. For example, Figure 2.15 shows different types of areas in a 6×6 position. Area A is a *White blocker territory* of size 1×2 while area B, though having exactly the same elements, is a *White simple territory* because *White* queen $A2$ is a blocker but $F6$ is not; Area C is a 2×2 *active area* because it contains queens of both colors; Both Area D and Area E are dead areas since Area D contains only queens and Area E contains only empty squares. *White* has an obvious advantage in this position since its territories are large enough to guarantee him a win.

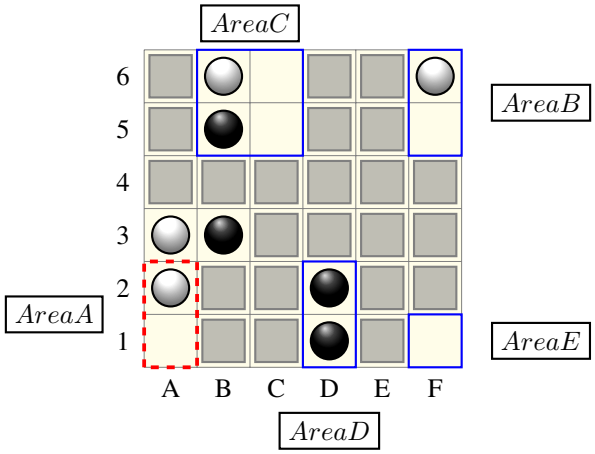


Figure 2.15: Area classification example

Simple Territories

A **simple territory** contains queens of only one color and does not overlap with other areas. For example, area B in Figure 2.15 is a *White* simple territory. Since only one player has moves in a simple territory, one of the game's *Left Options* and *Right Options* has to be empty. Therefore a simple territory is an integer game whose absolute value is the maximum number of moves the owner of this simple territory can make.

Determining the exact value of a simple territory is not trivial. First, simple territories are not always completely fillable. A simple territory that provides fewer moves than the number of empty squares it contains is said to be **defective** [24]. For example, even though there are three empty squares in Figure 2.16, Black can make at most two moves.

Buro showed that determining whether a queen can make a certain number of moves in an simple territory is an NP-complete problem [10], which means that very likely there is no polynomial time algorithm to determine whether a simple territory is defective or not. Tegos showed that having more

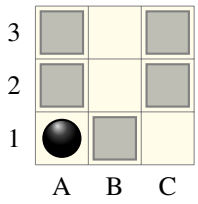
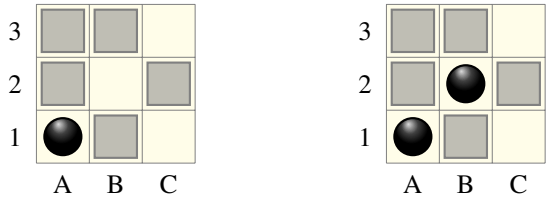


Figure 2.16: A defective simple territory

queens in a territory could possibly make an originally non-defective territory defective [32]. For example, Figure 2.17 shows a simple territory which is non-defective with one queen (Figure 2.17a) but defective with two queens (Figure 2.17b). Even though defective territories are relatively rare (see Table B.1), they need to be identified properly to ensure the correctness of the solver.



(a) Non-Defective (b) Defective

Figure 2.17: More queens can make a simple territory defective [32]

Active Areas

An **active area** contains at least one queen of each color. For example, the initial 6×6 board position shown in Figure 1.1a is a 6×6 active area itself. An active area may have any valid *Left Options* and *Right Options*, therefore it can be an integer, a fraction, an infinitesimal or a hot game.

Playing in an active area is usually good as it gives a player the chance to mark off more squares from his opponent. However, moving in an active area is not always desirable because **zugzwang** positions (where playing first is a disadvantage) exist in Amazons [24]. Zugzwang positions are always integers.

For example, Müller and Tegos showed the zugzwang position in Figure 2.18. In this position, if *White* is to move, the best move is moving to $C2$ or $E2$ and shooting back to $D3$ to prevent *Black* from getting the other side. Doing this leaves *White* with only one more move in this position. However, if *Black* is forced to move first because of running out of moves in other parts of the board, his best move is $D4 - C5 \times D4$. Then *White* is left with 4 more moves. This position is $\{3 \mid 6\} = 4$, therefore both players prefer to move second in this position.

Blocker Territories

A **blocker territory** contains queens only of one color and overlaps with one or more active areas on its blockers. Like simple territories, blocker territories are also one-player games, therefore they

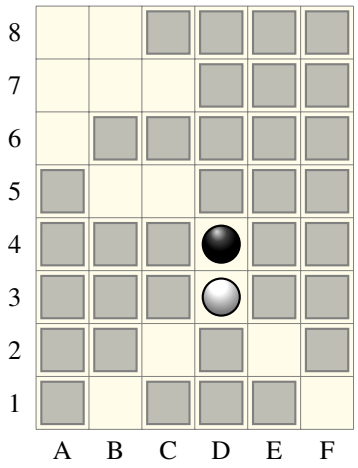


Figure 2.18: A Zugzwang Position [24]

are integer games as well.

Having blocker territories separated provides advantages for both playing and solving the game. From a strategic point of view, having blockers provides a player the advantage of securing his territory-to-be as well as threatening other parts of the board. For the solver, evaluating the blocker territory plus the remaining active area(s) yields a total bound in favor of the owner of the blocker territory, which could potentially lead to an earlier proof. Also, separating blocker territories shrinks the size of the original active area, thus making the active area databases potentially more useful.

The number of blockers in a blocker territory is only upper-bounded by the total number of queens available. For example, assuming we have 8 *White* queens, we could set up a *White* blocker territory with all available *White* queens as blockers as shown in Figure 2.19.

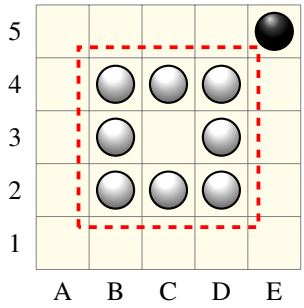


Figure 2.19: Multiple blockers in one blocker territory

2.3.4 From Thermographs to Bounds

The bounds of an Amazons game G can be computed based on its thermograph. If G is a number or a hot game, the best possible bounds of G is $[RS(G), LS(G)]$, i.e., the stop values of G. For example, from Figure 2.12a and Figure 2.12b, it is easy to see the bounds of 1 and $-\frac{1}{2}$ are [1,1] and $[-\frac{1}{2}, -\frac{1}{2}]$ respectively. The left stop is always greater than the right stop of a hot game. For

example, the bounds of $\{2 \mid -2\}$ are [-2,2] as shown in Figure 2.12d.

However if G is an infinitesimal, using its stop values overlooks its subtlety. An infinitesimal has a temperature of 0 (i.e., it is already "cooled down" at temperature 0) and a mean value of 0 (i.e., $RS(G_0) = LS(G_0) = 0$). Therefore using stop values at 0 as bounds will confuse infinitesimals with a true integer 0, which is always a second player win. To differentiate infinitesimals from a true integer 0, the stop values at temperature -1 are used because the stop values of an infinitesimal at -1 are fixed based on how an infinitesimal compares with 0 [6]. Bounds of infinitesimals are summarized in Table 2.6. For example, the bounds of $*$, $\uparrow$ and $\downarrow$ are [-1,1] (see Figure 2.12c for the illustration), [0,1] and [-1,0], respectively.

Infinitesimal i	**Bounds of** i **at temperature** -1
$i > 0$	[0,1]
$i < 0$	[-1,0]
$i \mid\mid 0$	[-1,1]

Table 2.6: Bounds of infinitesimals at temperature -1

Chapter 3

Endgame Databases

Based on whether the full board breaks down into subgames in an endgame, endgame databases can be categorized into *full-board* or *partial-board* endgame databases.

A full-board endgame database contains pre-computed thorough analysis of endgame positions (whose definition is application specific) for a certain game. For a *converging game*, where the game complexity reduces as the game progresses, full-board endgame databases are a performance asset for the game-playing program in reducing its search space by replacing heuristic evaluations with perfect knowledge [27]. For example, 10-piece endgame databases databases have been successfully used to solve checkers [28].

For non-converging games such as Havannah, it makes no sense to build full-board endgame databases [15]. However, if the game board can decompose into independent subgames as in Amazons, partial-board endgame databases can provide the perfect information for the subgames and they can also be used regardless of the original board size. For example, Tegos has built and used partial-board endgame database to enhance his Amazons playing program *Antiope* [32].

Our Amazons endgame databases are built separately for the three different area types as in Section 2.3.3. In this chapter, we mainly discuss the building of the blocker territory databases used in our solver. Since databases of the other two types have been built and used before, we only give the differences in our implementation.

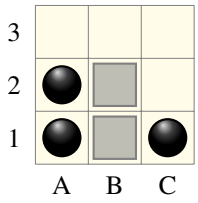
3.1 Building Blocker Territory Databases

3.1.1 Database Format

There are three parameters for a single blocker territory database: the number of normal queens, the number of blockers and the size of the blocker territory. The three parameters are stored in the beginning of a database file. The number of entries is also stored to avoid dynamic memory allocation when loading the database.

Each entry in a database file stores the **position** of the blocker territory, its **value** and the corresponding **best move** for achieving its value. The position of a blocker territory is computed by

representing each square with a single character from top-left to bottom-right in a row-major fashion, using "E" for empty squares, "B" for *Black* queens, "A" for arrows and "O" for blockers. The value of a blocker territory is the maximum number of moves achievable in this position stored as an integer (see Appendix A). For example, Figure 3.1 shows a 3×3 *Black* blocker territory with 2 blockers $A2$ and $C1$ and 1 normal queen $A1$ and its corresponding entry in the database.



Position	Value	Best Move
EEEOAEBAO	4	10758697

Figure 3.1: Blocker Territory Database Entry Format

3.1.2 Building Process

The computational technique used to generate the blocker territory databases is *retrograde analysis*. Also known as backward induction, retrograde analysis starts by generating terminal positions whose values are known statically. It then computes positions that lead to a terminal position directly by doing a 1-ply lookup. After this, positions that are 2-ply away from the terminal positions can also be generated by a 1-ply lookup, and so on. This process is repeated until either some stopping criteria (such as the memory limit) or the beginning of the game is reached. Retrograde analysis has been successfully applied for generating endgame databases [28] and even for completely solving non-trivial games [26].

Retrograde analysis cannot be applied directly in generating a blocker territory database due to the presence of blockers. A blocker can also move in a blocker territory, but in order to prevent the opponent from getting into this territory, it must always shoot back to its origin square. This property is called the *blocker constraint*. After its move, a blocker turns into a normal queen and can move freely thereafter. The possibility of a blocker turning into a normal queen makes the generation of a blocker territory database dependent on positions that are not part of the database. Therefore, we need to resolve the dependencies by building potentially needed positions prior to building the blocker territory database itself.

To be more specific, the positions in a blocker territory database of size $w \times h$ with B blockers, Q normal queens and $E = w \times h - B - Q$ empty squares depend on positions of the same size with b ($0 \leq b \leq B$) blockers, correspondingly $Q + B - b$ normal queens and e ($0 \leq e \leq E - B + b$) empty squares. For example, Figure 3.2 shows that the 3×3 blocker territory database with 2 blockers and 1 normal queen depends on 3×3 positions with 1 blocker, 2 normal queens and up to 5 empty

squares and 3×3 positions with 0 blockers, 3 normal queens and up to 4 empty squares.

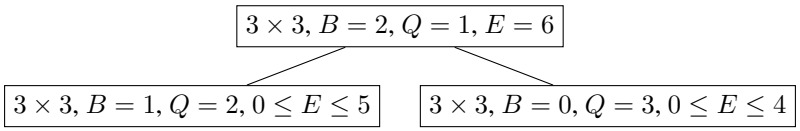


Figure 3.2: Blocker territory database dependencies example

After the dependencies are built, a regular retrograde analysis process can be applied directly by generating all positions of size $w \times h$ with B blockers and Q normal queens with e ($0 \leq e \leq E$) empty squares. The building process is given in function `RetroBT` in Algorithm 2. Function `NewDatabase` returns a newly constructed database whose entries have the format as stated in Section 3.1.1. Function `BlockerTerritoryGenerator` iterates through all valid blocker territories given the width, height, number of normal queens, number of blockers and number of empty squares of these positions.

Algorithm 2 Blocker Territory Retrograde Analysis

```
function RetroBT(w, h, B, Q)
    dependencies ← NewDatabase()
    toBuilt ← NewDatabase()
    E ← w × h − B − Q
                                                    ▷ Dependencies Handling
    for b = 0 to B − 1 do
        MaxDepE ← E − B + b
        q ← Q + B − b
        for e = 0 to MaxDepE do
            for all position ← BlockerTerritoryGenerator(w, h, q, b, e) do
                dependencies.Add(position)
            end for
        end for
    end for
                                                    ▷ Retrograde Analysis
    for e = 0 to E do
        for all position ← BlockerTerritoryGenerator(w, h, Q, B, e) do
            toBuilt.Add(position)
        end for
    end for
    return toBuilt
end function
```

3.2 Reducing Databases

The databases built right after the retrograde analysis process are too big to be used effectively. To reduce the size of the databases generated, we prune out the positions that are redundant (Section 3.2.1) and trade time for memory by storing only the normal form of a position (Section 3.2.2). By pruning out the redundant positions and storing only the normal form of a position, the reduction

rates of the blocker territory databases are shown in the last column of Table B.3 in Appendix B.

3.2.1 Redundant Positions

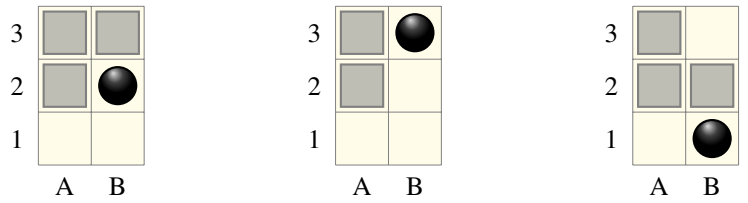
During retrograde analysis, some generated positions, while necessary for the building process, are redundant once the computation is done. These positions are simply pruned out before the database is stored on disk. Redundant positions can be classified into two categories:

1. **Positions that do not span the full area**

   For example, when building a 2×3 blocker territory database with only 1 blocker, we will generate the position in Figure 3.3a during retrograde analysis. This position is necessary during the building process since it is reachable from other positions (e.g., Figure 3.3b). However, it is redundant for the final database since all the squares in the top row in this position are burnt-off and this position will be looked up in the 2×2 databases instead.

2. **Positions that are not 8-connected**

   For example, the position in Figure 3.3c is also a reachable position from position Figure 3.3b in the building process. After building, it is removed from the 2×3 blocker territory database and it will be looked up in a 1×2 database.



(a) Does not span the full area. Redundant.

(b) Potential parent

(c) Not 8-connected. Redundant

Figure 3.3: Redundant positions

3.2.2 Transformation Detection and Normal Form

A group of different-looking positions might be structurally equivalent due to geometric transformations. An arbitrary number of *rigid* transformations (*reflection*, *rotation* and *translation*) can be applied to a position without changing its structure. For a position in the Amazons databases, we need to consider 3 transformations: reflection against the x-axis (*ref-x*), reflection against the y-axis (*ref-x*) and $90°$ counterclockwise rotation (*rot-c90*). The 8 unique combination sequences (named t_0 to t_7) of these 3 transformations are shown in Table 3.1. t_0 means no transformations are applied. For example in Figure 3.4, assuming p_0 is the original position, p_0 through p_7 are positions after applying corresponding transformation sequences t_0 to t_7.

t_0 : none	t_4 : rot-c90
t_1 : ref-x	t_5 : rot-c90 $\rightarrow$ ref-x
t_2 : ref-y	t_6 : rot-c90 $\rightarrow$ ref-y
t_3 : ref-x $\rightarrow$ ref-x	t_7 : rot-c90 $\rightarrow$ ref-x $\rightarrow$ ref-y

Table 3.1: Transformation Sequences

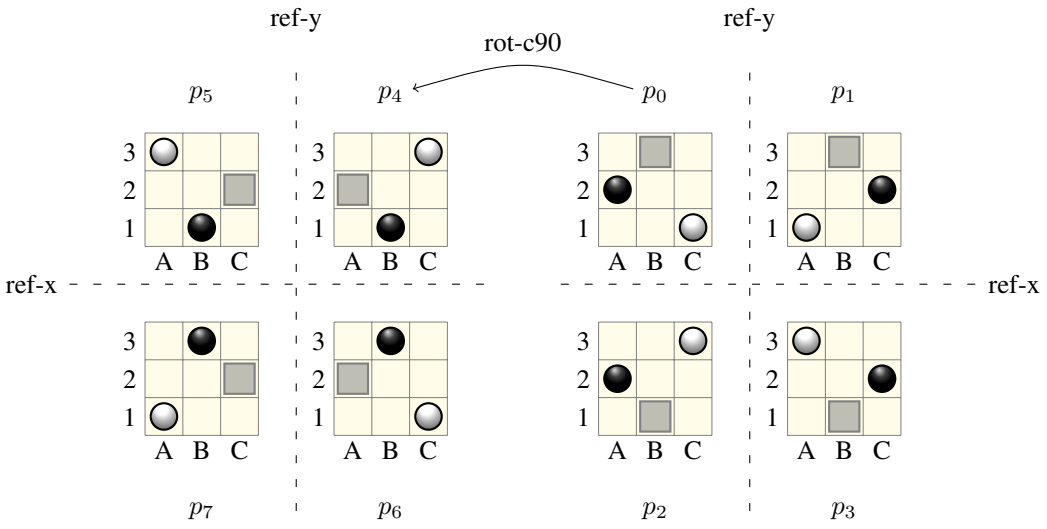


Figure 3.4: Transformations

For a group of structurally equivalent positions, keeping one of them in the database is sufficient. Define the *code* of a square on an Amazons position to be "B" for *Black* queens, "W" for *White* queens, "A" for arrows, "E" for empty squares and "O" for blockers. Define the *signature* of an Amazons position to be the concatenation of the codes of all its squares from its lower-left corner to its upper-right corner in a row-major fashion (i.e., first row from left to right then second row from left to right, etc.). We define the **normal form** of a $m \times n$ position, amongst all its 8 possible transformations, as:

- the one with the lexicographically smallest signature if $m = n$, and
- the "slimmer" one (i.e., its width is smaller than its height) with the lexicographically smallest signature if $m \neq n$.

In the databases, only these normal forms are stored.

For example, the signatures of positions p_0 to p_7 in Figure 3.4 are shown in Table 3.2. Therefore, the normal form of positions p_0 to p_7 is p_2.

p_0 : EEWBEEEAE	p_4 : EBEAEEEEW
p_1 : WEEEEBEAE	p_5 : EBEEEAWEE
p_2 : **EAEBEEEEW**	p_6 : EEWAEEEBE
p_3 : EXEEEBWEE	p_7 : WEEEEAEBE

Table 3.2: Signatures

3.3 Databases Used

Simple territory databases

The simple territory databases we built are similar to the ones Tegos used [32]. However, we used the blocker territory database entry format for the simple territory databases rather than *line segment graphs* as in Tegos' implementation. Using line segment graphs can potentially further reduce the size of the database but it also takes longer to query a position because a position has to be converted into such a graph for query, which is more complex than computing its normal form. We computed 48 simple territory databases and their statistics are summarized in Table B.1 in Appendix B.

Active area databases

For active area databases, we used the implementation by Enzenberger which is part of the Arrow code base. [1] For each entry, we store the position as in the territory databases and its corresponding thermograph. The active area databases are not built with the blocker constraint. Therefore they cannot be used to query a remaining active area when blocker territories are partitioned out in an improved partition. Table B.2 in Appendix B shows the statistics of the active area databases. We computed 87 active areas in total but only 86 of them are used in solving. The active area database of size 3×6 with 2 queens for each color is not used because it is too large (43433015 entries) to be loaded quickly enough. The loading time of all the active area databases with this database is 12 hours and it is 45 minutes without. Such 3×6 areas does not occur too often in solving the 5×6 board (see Table C.4 in Appendix C).

Blocker territory databases

Statistics on the 106 blocker territory databases we built are summarized in Table B.3 in Appendix B.

[1] The Arrow code base contains C++ source code for building Amazons playing and solving programs. It was built and maintained by Martin Müller since 2002. Enzenberger was a major contributor. The Arrow code base may go open source in the future [22].

Chapter 4

Improving Area and Global Bounds

In this chapter, we discuss how the bounds of different types of areas are computed. In Section 4.1, we show how bounds each type of area are initialized and computed with the heuristics or local search. In Section 4.2, we discuss how the bounds of an area of each type are computed with the databases. Finally in Section 4.3, we show how to combine bounds computed in different ways into global bounds for the entire board.

4.1 Computing Bounds by Heuristics

4.1.1 Simple Territory

The bounds of a *Black* simple territory with v empty squares are initialized to $[0, v]$. The bounds of a *White* simple territory with v empty squares are initialized to $[-v, 0]$. These initial bounds simply indicate that the opponent cannot move in this area, but are not claiming any moves for the owner. Since there is no easy way to determine whether a simple territory can be completely filled or not (see Section 2.3.3), the simple *plodding* heuristic in solving 5×5 is used to quickly compute an basic lower bound for a simple territory [23]. The plodding heuristic counts the number of moves a queen in a simple territory can make by greedly making as many *plodding moves* (moving into a neighboring square and shooting back) as possible. The lower bound of a simple territory is set to the sum of the number of plodding moves over all queens in this simple territory where each empty square is used by at most one queen. With the plodding heuristic, the upper bound always equals to the number of empty squares in a simple territory.

For example, the bounds of the position shown in Figure 4.1 are $[1, 2]$ since the heuristic can only make 1 move with this *Black* queen. Even though this is a defective simple territory and the maximum number achievable moves is 1, the heuristic can not determine whether the remaining square is reachable or not.

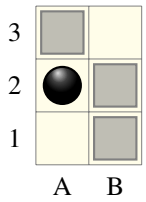


Figure 4.1: Simple territory bounds heuristics, bounds $[1, 2]$

4.1.2 Blocker Territory

The bounds of a territory are initialized the same way as a simple territory. In filling a blocker territory, the plodding heuristic has one additional constraint which is to avoid using blockers if possible.

For example, the position shown in Figure 4.2 is a *Black* blocker territory with the *Black* queen $B2$ as its blocker. The bounds computed by the heuristic are $[1, 3]$ since the blocker has to shoot back when it moves, thus blocking its access to other empty squares.

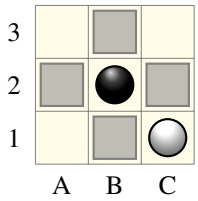


Figure 4.2: Blocker territory bounds heuristics, bounds $[1, 3]$

4.1.3 Active Area

The bounds of an active area with v empty squares are initialized to $[-v, v]$. These crude bounds can be improved either by static evaluation or by local minimax searches. The static evaluation improves bounds by quickly identifying safe moves for both players, which is which is rather weak for large areas but fast enough to be applied to arbitrary positions. The local search, on the other hand, provides the bounds for an active area by exhaustive minimax search. It is only conducted on small positions as a complement to the databases, so as not to slow down the solving process.

Static Evaluation

A queen needs at least one *adjacent empty square* (AES) to make a move. The purpose of static evaluation is to find out if there are safe (i.e., guaranteed) moves for both players and improve the bounds according to the following rules:

1. For every *Black* safe move, the lower bound is increased by 2;

2. For every *White* safe move, the upper bound is decreased by 2.

Each safe move changes the bounds by 2 because one safe move for a player also means one less move for his opponent. For example, if *Black* has 2 safe moves and *White* has 1 in an active area with v empty points, the improved bounds will be $[-v+4, v-2]$.

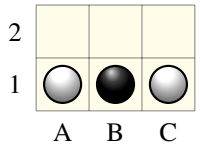
How many safe moves a queen can make depends on its AES status, the opponent's queen distribution and whose turn it is to move next. Müller has shown the following rules for finding safe moves [23]:

1. A queen with 3 or more AES has a safe move. For example, in Figure 4.3a, *Black* queen $B2$ has a safe move because it has 3 AES.

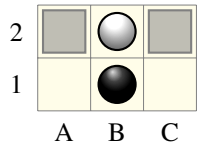
2. A queen with 2 AES such that the opponent cannot block both in one move. For example, the *Black* queen $B1$ in Figure 4.3b has a safe move because it has two AES that the *White* queen $B2$ cannot block in a single move.

3. A queen with 2 AES which the opponent can block in one move has a safe move if and only if in blocking these two empty squares, the opponent has to open up another AES for the same queen. For example, in Figure 4.3b, *Black* queen $B1$ can take both empty squares in one move. However in doing so, $B1$ will be a new AES for the *White* queen $B2$, therefore $B2$ also has a safe move.

4. A queen with 1 AES which the opponent cannot block has a safe move.



(a) *Black* queen $B2$ has 3 AES [23]



(b) *Black* queen $B1$ has 2 safe AES

Figure 4.3: Static evaluation Rules

The fourth rule can be improved by taking the original square of the opponent queen which blocks the single AES into account. For example, in Figure 4.4 the *White* queen $A2$ has 1 AES $B2$ which the *Black* queen $D2$ can block easily by $D2 - C3 \times B2$ as shown in the figure. But in blocking this AES, *Black* has to free the *White* queen $D1$ it blocked. Therefore, a safe move can be claimed for the two *White* queens combined.

The second rule above did take the opponent square into consideration but only for the same queen with the 2 AES and thus can be improved in a similar fashion. For example, in Figure 4.5, the *White* queen $D1$ has 2 AES and the *Black* queen $A3$ can block both by moving to $C1$ and shooting to $C2$. However, in doing this, *Black* has to open up his original square $A3$ which has an adjacent *White* queen $B4$. Therefore, *White* can claim a safe move for $D1$ and $B4$ combined.

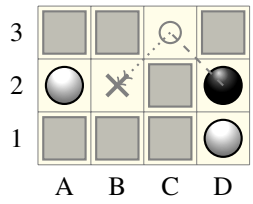


Figure 4.4: New static evaluation rule for queens with 1 AES

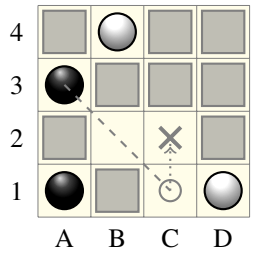


Figure 4.5: New static evaluation rule for queens with 2 AES

Müller also pointed out that there are potential dependencies amongst queens [23]. Therefore, the static evaluation is only performed once at most for each player in an active area . For example, in Figure 4.6, there are 4 empty squares in the active area (original bounds $[-4, 4]$) and the *Black* queen $C1$ and both *White* queens $A2$ and $E2$ can find a safe move. However, *White* cannot claim two safe moves because it can make only one move if *Black* moves first by $C1 - B1 \times D1$. Bounds of this area are improved only to $[-2, 2]$ due to the dependencies of the *White* queens.

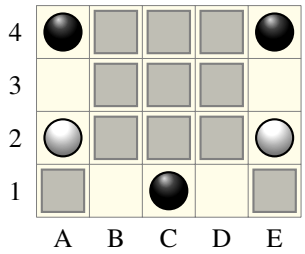


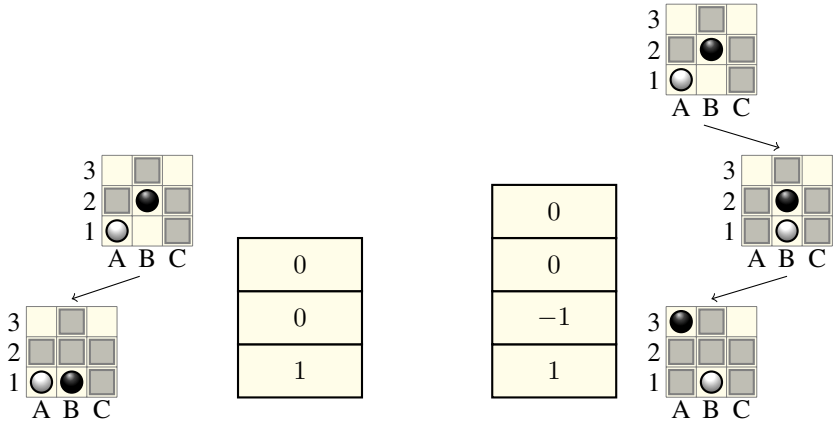
Figure 4.6: Static evaluation dependencies example

Local Search

Bounds of an active area can also be computed by local minimax searches on the fly. To compute the bounds of an active area A, two local minimax searches are started on A with *Black* and *White* being the first player respectively. In order to make the search results consistent with the databases, we search for the number of moves the first player can make more than his opponent. This is achieved by maintaining a single stack *s* for a local search. Name the first player to move in the local search *fp*, each time *fp* plays a legal move, $+1$ is pushed onto *s*. Each time *fp*'s opponent plays a legal move, -1 is pushed onto *s*. 0 is pushed onto s if the player to move passes. Passes are allowed for

both players even if they have other legal moves so as to handle zugzwang positions correctly. The game is terminated after two consecutive passes.

For example, Figure 4.7a and Figure 4.7b shows the principal variation in the local search of the position in Figure 2.8 with *Black* and *White* being the first player respectively. The stack s when the search terminates is also shown along with the principal variation.



(a) *Black* plays first, search score 1 (b) *White* plays first, search score 0

Figure 4.7: Local search of Figure 2.8

Let the local search value be v_w and v_b for *White* and *Black* moving first in an active area A respectively. The bounds of A are set to $[-v_w, v_b]$. For example, in Figure 4.7, the bounds are set to $[0, 1]$ after the local search. For efficiency reasons, local searches are only performed on positions whose total number of queens and empty squares is less than 9 in the solver.

4.2 Computing Bounds from Databases

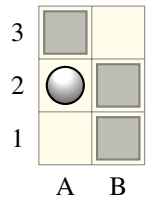
4.2.1 Simple Territory Databases

When a *Black* simple territory is encountered during the search, it is changed to its normal form and queried against the simple territory database of the corresponding size and queen number. If the query is successful, its exact value v is returned and the bounds for this territory are set to $[v, v]$.

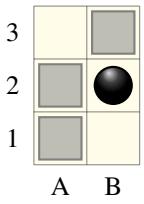
If it was a *White* simple territory, the color of the queens is changed to *Black* (i.e., negate the game) and the position is queried as a *Black* simple territory as described above. The return value is negated after the query succeeded. For example, Figure 4.8 shows a *White* simple territory of size 2×3 with only one queen and its corresponding querying position. Because this is a defective territory, the query returns a value of 1. Thus the bounds for this *White* simple territory are $[-1, -1]$.

4.2.2 Active Area Databases

During the search, if an active area is encountered and the number of its *Black* queens is no less than the number of its *White* queens, this active area is first converted to its normal form and then



(a) *White* simple territory

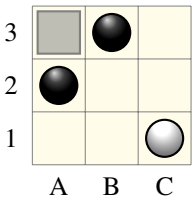


(b) Querying position

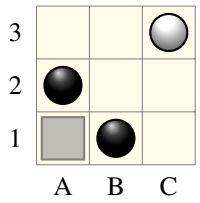
Figure 4.8: Simple territory query, bounds $[-1, -1]$

queried against the active area database with the corresponding size and number of queens of both colors. If the query succeeds, the thermograph of this area is retrieved and bounds are computed as in Section 2.3.4.

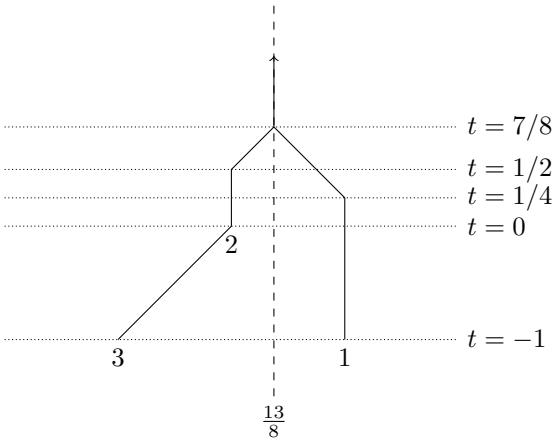
For example, Figure 4.9 shows the query process for the active area in Figure 4.9a. Figure 4.9b shows the normal form which is queried against the active area database of size 3×3 with 2 *Black* queens and 1 *White* queen. The thermograph retrieved is shown in Figure 4.9c with the left stop and right stop at temperature 0 being 2 and 1 respectively. Therefore, the bounds for this active area are $[1, 2]$.



(a) The active area



(b) The normal form



(c) The thermograph

Figure 4.9: Active area query, bounds $[1, 2]$

Since we built our active area databases in an asymmetrical way (see Section 3.3), we need to switch the color of the queens in an active area (i.e., negate the game) before querying if the number of *Black* queens in this area is less than the number of *White* queens. If the query succeeds, the

bounds computed from the thermograph will be negated to account for the color switching.

For example, the position in Figure 4.10 has 2 *White* queens and 1 *Black* queen. Therefore the color of the queens will be switched before the query. The position after the color switching is identical to Figure 4.9a, thus the query process is the same as in Figure 4.9 except that the resulting bounds will be negated to [-2,-1].

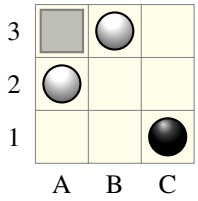


Figure 4.10: Active area query with more *White* than *Black* queens, bounds [-2,-1]

4.2.3 Blocker Territory Databases

Blocker territories are the result of an improved partition of an active area. Therefore if the original active area itself is queryable as a whole, there is no need to identify blocker territories within. The original active area will be queried and processed as described in Section 4.2.2. If an active area is too large to be queried, an improved partition is performed on it. If no blocker territories can be partitioned out, then the bounds of this active area are computed as shown in Section 4.1.3.

If an improved partition succeeds, bounds of the resulting blocker territories and active areas are computed separately. A blocker can be used to improve the bounds of at most one area it is part of. For example, Figure 4.11 shows a blocker territory (delimited by the red dashed line) with 1 normal queen $A1$ and 1 blocker $B2$. The blocker $B2$ can either be used to fill the blocker territory or make moves in the remaining active area.

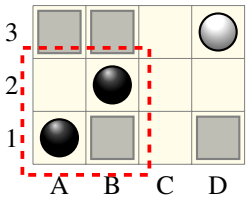


Figure 4.11: Blocker territory with 1 normal queen and 1 blocker

Because neither our active area databases nor local searches consider the blocker constraint, bounds of the remaining active areas currently are only computed with static evaluations as stated in Section 4.1.3, with an additional constraint that blockers that are used in other areas must not be used to find safe moves (either by itself or combined with other queens). Therefore the more blockers we can spare from using in blocker territories, the better the chances are there for tighter active area bounds. For example, in Figure 4.11, if the blocker $B2$ is used in filling the blocker territory, then it cannot be used in improving the bounds of the remaining active area. The bounds of the remaining

active area are therefore [-4,2] since *White* can find a safe move with 3 AES. However, if the blocker $B2$ is used to improve the bounds of the remaining active area, its bounds will be [-2,2] because now *Black* can also find a safe move.

Unfortunately, in order to reduce the size of blocker territory databases, we did not store the information as to which queens are used in filling a blocker territory. Therefore, if we query a blocker territory directly, we need to assume all its blockers and normal queens are used for filling, even if it can be filled with a small subset of them. For example, the blocker territory in Figure 4.11 is clearly completely fillable by either the normal queen $A1$ or the blocker $B2$. However, if we queried the blocker territory as is, we have to assume both queens are used in filling it.

Changing the Querying Position

To resolve this problem, we built 3 heuristics for selecting a subset of all the queens in a blocker territory B to be in its querying position B' to minimize the number of blockers used. However, there is one drawback for eliminating certain queens in the querying position B' of a blocker territory B: if B' is defective, we are not sure whether B is defective or not. In terms of bounds, $bounds(B) = [u(bounds(B'), e]$, where e is the number of empty squares in B. Notice that if B' is non-defective, B has to be non-defective since this value can be achieved by moving only the queens in B' while leaving the other queens untouched. For example, if the blocker $C2$ is taken out from the blocker territory in Figure 4.12, then this territory is defective and its bounds are $[1, 2]$.

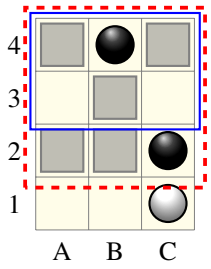


Figure 4.12: *Black* territory is defective if the blocker $C2$ is taken out

The selection process is based on the queens' types. Algorithm 3 summarizes the queen selection process with the details explained in the rest of the section.

1. **Normal Queens Or Blockers Blocking Dead Active Areas**

   In a blocker territory, two types of queens are always included in the querying position:

   (a) *normal queens.*

   Normal queens are always included because they have no constraints in choosing moves. They are the best candidates for filling a blocker territory.

   (b) *blockers that block a dead active area.*

   Due to the process of building the blocker territory databases, a blocker will always shoot

Algorithm 3 Blocker territory queen selection

```
function BTQueenSelect(bt)
                                                    ▷ bt is a blocker territory
    normal ← 0
    blockDead ← 0
    blockOne ← 0
    blockMore ← 0
                                    ▷ separate the queens based on their types
    for all q ← bt.queens do
        if q is a normal queen then
            normal.Add(q)
        else if q blocks a dead active area then
            blockDead.Add(q)
        else if q blocks a single territory then
            blockOne.Add(q)
        else
            blockMore.Add(q)
        end if
    end for
                                                          ▷ queens selection
    if normal.size > 0 or blockDead.size > 0 then
        return normal + blockDead
    else if blockOne.size > 0 then
        return SelectBlockOne(blockOne)
    else
        return SelectBlockMore(blockMore)
    end if
end function
```

back when it is used to fill a blocker territory. Therefore blockers that block dead active areas can be used freely since the active areas they belong to can not be improved and they will always block their opponents when they move.

For example, in the blocker territory delimited by the red dashed line in Figure 4.13, the *Black* queen $A4$ is a normal queen and $B2$ is a blocker blocking a dead active area. Therefore the querying position (delimited by the blue solid line) of this blocker territory does not include the blocker $C3$, which blocks a non-dead active area. Based on informal tests, if queens of these two types exist, they are the only queens included in the querying position.

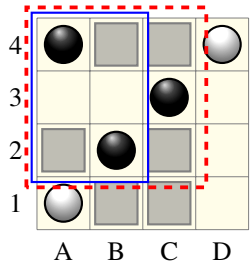


Figure 4.13: Normal queens and blockers blocking dead active area exist

2. **Blockers Blocking A Single Territory**

If normal queens and blockers that block dead active areas are not available in a blocker territory, then blockers that block only this territory are identified. Among the identified blockers, the one that is the nearest to the center of the position is selected to be present in the querying position. For example, in Figure 4.14 the *Black* blocker territory delimited by the red dashed line has 3 blockers $E5$, $C3$ and $D3$. $E5$ also belongs to the active area in the upper right corner while $C3$ and $D3$ also belong to the active area in the lower left. Since all three blockers block only this territory, we choose $D3$ to be in the querying position since it is closest to the center of the blocker territory. The querying position is delimited by the blue solid line.

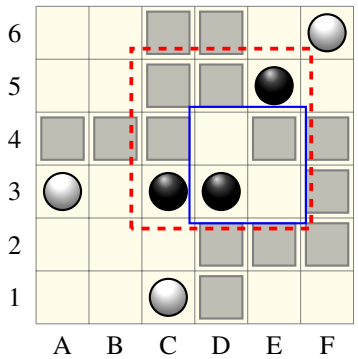


Figure 4.14: Blockers blocking a single territory, assigned to $D3$

3. **Blockers Blocking Multiple Territories**

Blockers blocking multiple territories are only considered when queens of other types are not available. If the only queen available in a blocker territory blocks multiple territories, it can only guarantee the biggest one by walking in it and shooting back to block the opponent. Instead of computing bounds for each of the blocker territories it blocks, a single bound is computed for all involved blocker territories combined [23]. Let $\mathcal{B}$ be the set of bounds for all the blocker territories blocked by a common blocker, then the combined bounds are:

$$[\max_{b \in \mathcal{B}} l(b), \sum_{b \in \mathcal{B}} u(b)]$$

In other words, the combined lower bound is the maximum lower bound in $\mathcal{B}$ and the combined upper bound is the sum of all upper bounds in $\mathcal{B}$.

For example, in Figure 4.15, the *Black* blocker $B2$ is blocking two territories (delimited by the red dashed line) with bounds of $[1, 1]$ and $[2, 2]$ respectively. The combined bounds are therefore $[\max(1, 2), (1 + 2)] = [2, 3]$, which means *Black* can make at least two moves in these two blocker territories and could make at most three.

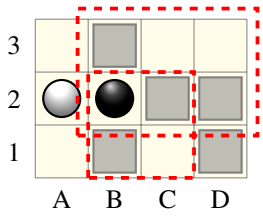


Figure 4.15: $B2$ blocks two territories, combined bounds $[2, 3]$

If all queens in a blocker territory block multiple territories, we "assign" this territory to the blocker whose maximum blocker territory (excluding this one) is the smallest. The reason for such a choice is that this blocker territory should be used to assist the "weakest" blocker to achieve a better overall bound. For example, in Figure 4.16 the shared *Black* blocker territory (delimited by the red dashed line) will be assigned to $B3$ since it blocks another territory of size 1 and $D3$ blocks another territory of size 2. The querying position is delimited by the blue solid line.

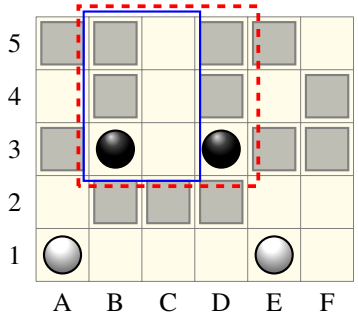


Figure 4.16: Both *Black* blockers block multiple territories, assigned to $B3$

4.3 Combining Bounds

After a game position is partitioned into areas, and bounds of each area are computed, we need to combine all the area bounds into a single global bound and try to determine the winner of this position based on the rules in Section 2.3.1.

The overall structure of the bounds combination process is summarized in Algorithm 4. For *simple territories* (computed by heuristics or from the databases), *blocker territories* (computed by heuristics or from the databases, assuming blockers blocking multiple blocker territories are handled properly as stated in Section 4.2.3) and *active areas* computed by *static evaluation and local search*, the combining process is a simple bounds summation. For *active areas computed from the databases* however, the combining process is more complex because we have better knowledge of these games and can achieve better bounds. The handling of these games is explained in the next two subsections.

Algorithm 4 Area bounds combination

```
function AreasCombine(areas)
    infAreas ← EmptySet
    hotAreas ← EmptySet
    globalBounds ← [0, 0]
                                  ▷ sum up everything besides infinitesimals and hot games
    for all a ← areas do
        if a is an infinitesimal then
            infAreas.Add(a)
        else if a is a hot game from the database then
            hotAreas.Add(a)
        else
            globalBounds ← globalBounds + bounds(a)
        end if
    end for
                                                  ▷ hot games from the databases handling
    globalBounds ← HotCombine(globalBounds, hotAreas)
                                                          ▷ infinitesimals handling
    globalBounds ← InfCombine(globalBounds, infAreas)

    return globalBounds
end function
```

Handling Hot Games From the Databases

In theory, all hot games should be summed to a single game and the bounds of the resulting game will be used. However, we cannot do this in our current framework with thermographs. Instead, we improve the global bounds by assuming the next player will move in the hot game with the widest bounds. The *widest bounds* is the bounds whose upper and lower bound have the biggest absolute difference. Let the widest bounds be b_w. If the next player is *Black*, we add $[u(b_w), u(b_w)]$ instead of b_w; If the next player is *White*, we add $[l(b_w), l(b_w)]$ instead of b_w. For example, if the widest bounds are $[-2, 2]$, then $[-2, -2]$ are added if the next player is *White*, and $[2, 2]$ are added if the

next player is *Black*.

We can add the widest bounds from the next player's point of view because this value is guaranteed by its thermograph and the next player can achieve such a value by moving in the area that provides the widest bounds. Adding the widest bounds uses the next player's privilege of moving first, which can be used at most once during the entire bounds combining process. All other bounds besides the widest are simply added to the global bounds.

The entire process for handling hot games from the databases is shown in Algorithm 5. Function `NextPlayer` will return the next player and function `RevokeToMovePrivilege` marks the next player's privilege of moving first as used.

Algorithm 5 Hot games from the databases bounds combination

```
function HotCombine(globalBounds, hotAreas)
    maxRange ← 0
    widest ← NULL
                                          ▷ find the area with the widest bounds
    for all a ← hotAreas do
        range ← u(bounds(a)) − l(bounds(a))
        if range > maxRange then
            maxRange ← range
            widest ← a
        end if
    end for
                                  ▷ add on the widest in favor of the next player
    if NextPlayer() is Black then
        globalBounds ← globalBounds + [u(bounds(widest)), u(bounds(widest))]
    else
        globalBounds ← globalBounds + [l(bounds(widest)), l(bounds(widest))]
    end if
    RevokeToMovePrivilege()
                                          ▷ add on the rest of the bounds
    for all a ← hotAreas do
        if a is not widest then
            globalBounds ← globalBounds + bounds(a)
        end if
    end for

    return globalBounds
end function
```

Handling Infinitesimals

Infinitesimals are strictly less than any positive number and greater than any negative number. Therefore if the bounds of all other areas combined (called $oBounds$) are in one player's favor (i.e., $l(oBounds) > 0$ or $u(oBounds) < 0$), the winner is determined and there is no need to add the bounds of any infinitesimals. If $oBounds$ does not favor any player, bounds of infinitesimals have to be added on.

Bounds of infinitesimals are handled in two parts according to whether an infinitesimal is a $*$.

1. $*$s are counted and are combined into a single variable $starBounds$ as follows:

$$starBounds = \begin{cases} [-1, 1] & \text{if there is an odd number of } *\text{s} \\ [0, 0] & \text{if there is an even number of } *\text{s} \end{cases}$$

2. All other infinitesimals are summed together to produce a single variable $iBounds$ as follows:

$$iBounds = \begin{cases} [0, 1] & \text{all other infinitesimals are positive} \\ [-1, 0] & \text{all other infinitesimals are negative} \\ [-1, 1] & \text{otherwise} \end{cases}$$

If the next player's privilege of moving first was not used in the hot game handling phase, it can also be combined with the infinitesimals to provide one free move for the next player. The criteria for this free move is shown below and the process is summarized in Algorithm 6.

- *Black* as the next player has a free move if
  - no other infinitesimals exist and there is an odd number of $*$s;
  - all other infinitesimals are positive and there is an even number of $*$s.
- *White* as the next player has a free move if
  - no other infinitesimals exist and there is an odd number of $*$s;
  - all other infinitesimals are negative and there is an even number of $*$s.

Algorithm 6 Decide whether infinitesimals can provide a free move for the next player

```
function HasInfMove(starBounds, iBounds)
                                          ▷ odd number of stars and no other infinitesimals
    if iBounds = [0, 0] and starBounds = [−1, 1] then
        return true
    end if
                                          ▷ odd number of stars and no other infinitesimals
    if iBounds ≠ [0, 0] and starBounds = [0, 0] then
        if NextPlayer() is Black and l(iBounds) ≥ 0 then
            return true
        else if NextPlayer() is White and u(iBounds) ≤ 0 then
            return true
        end if
    end if
                                                                          ▷ otherwise
    return false
end function
```

If the next player's privilege of moving first is used in the hot game handling phase or the infinitesimals are not strong enough to provide a move, $starBounds$ and $iBounds$ are then combined and added to $oBounds$. This entire infinitesimals handling process is shown in Algorithm 7. Function `HasToMovePrivilege` returns true if the next player's privilege of moving first is not used yet and false otherwise.

Algorithm 7 Infinitesimal bounds combination

function `InfCombine`(`oBounds`, `infAreas`)

▷ inifinitesimal bounds are not necessary

  if $\texttt{l}(\texttt{oBounds}) > 0$ **or** $\texttt{u}(\texttt{oBounds}) < 0$ **then**
    return `oBounds`
  end if

▷ count the stars and sum the rest

  $\texttt{starCount} \leftarrow 0$
  $\texttt{iBounds} \leftarrow [0, 0]$
  for all $\texttt{a} \leftarrow \texttt{infAreas}$ **do**
    if `a` **is** $*$ **then**
      $\texttt{starCount} \leftarrow \texttt{starCount} + 1$
    else
      $\texttt{iBounds} \leftarrow [\min(\texttt{l}(\texttt{iBounds}), \texttt{l}(\texttt{bounds}(\texttt{a}))), \max(\texttt{u}(\texttt{iBounds}), \texttt{u}(\texttt{bounds}(\texttt{a})))]$
    end if
  end for

▷ sum the stars

  $\texttt{starBounds} \leftarrow \texttt{NULL}$
  if `starCount` **is** odd **then**
    $\texttt{starBounds} \leftarrow [-1, +1]$
  else
    $\texttt{starBounds} \leftarrow [0, 0]$
  end if

▷ check if the next player has a free move

  if `HasToMovePrivilege`() **and** $\texttt{HasInfMove}(\texttt{starBounds}, \texttt{iBounds})$ **then**
    if `NextPlayer`() **is** *Black* **then**
      $\texttt{oBounds} \leftarrow \texttt{oBounds} + [1, 1]$
    else
      $\texttt{oBounds} \leftarrow \texttt{oBounds} + [-1, -1]$
    end if
    `RevokeToMovePrivilege`()
  else
    $\texttt{iBounds} \leftarrow [\min(\texttt{l}(\texttt{starBounds}), \texttt{l}(\texttt{iBounds})), \max(\texttt{u}(\texttt{starBounds}), \texttt{u}(\texttt{iBounds}))]$
    $\texttt{oBounds} \leftarrow \texttt{oBounds} + \texttt{iBounds}$
  end if

  return oBounds
end function

4.3.1 Board Evaluation Examples

Examples shown in this section are endgame positions from self-play games by the program Arrow 2.1 (a *UCT* player) given 500 seconds per game.

A $*$ Win

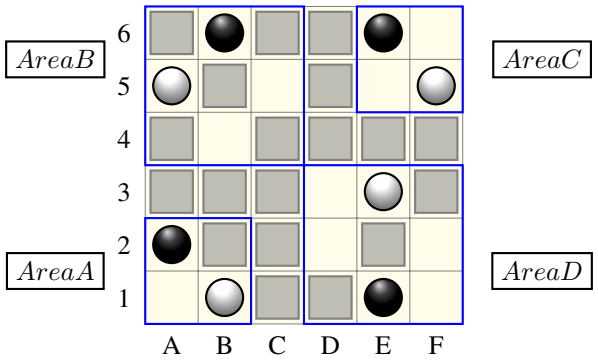


Figure 4.17: 19 moves played, *Black* to move, global bounds $[1, 1]$

Figure 4.17 shows a 6×6 Amazons endgame position after 19 moves have been played and there are 9 empty squares left. The board is partitioned into 4 active areas and all the areas can be looked up in the databases. Area A is the infinitesimal $*$ and Areas B, C and D are all the integer 0. Therefore, the global bounds after summing up everything besides hot games and infinitesimals are still $[0, 0]$. The infinitesimal handling phase will recognize the $*$ and use the to-move privilege to claim a free move for *Black* from the $*$. The overall global bounds therefore are $[1, 1]$ and *Black* can win the game by moving in Area A, leaving the rest of the board (a 0 game) to *White*.

Without the databases, all areas would be locally searched with the bounds of Area A being $[-1, 1]$ and the bounds of Areas B, C and D being $[0, 0]$. The global bounds therefore are $[-1, -1]+$ $[0, 0] + [0, 0] + [0, 0] = [-1, 1]$. Because all the results comes from the local search, we cannot use *Black*'s privilege of moving next to claim a move for him. Therefore the winner of this position cannot be determined yet.

A New Static Evaluation & Infinitesimal Win

Figure 4.18 shows a 6×6 Amazons endgame position after 18 moves have been played and there are 10 empty squares left. Area A is a 2×6 active area with 2 *White* queens and 1 *Black* queen which can be queried from the database. The query result of Area A is $-4 + i$ where i is some infinitesimal that is confused with 0 (i is actually a $*$, but the database information is not strong enough to prove it). Area B is an active area with 4 empty squares wherein a safe move can be found for *White* with the two *White* queens $C4$ and $E1$ combined with the new static evaluation rules. *Black* has a safe move in Area B using the old static evaluation rule, therefore the bounds of Area B are $[-2, 2]$. Area C is a *Black* blocker territory with bounds of $[1, 1]$. Without the infinitesimal i, the global bounds

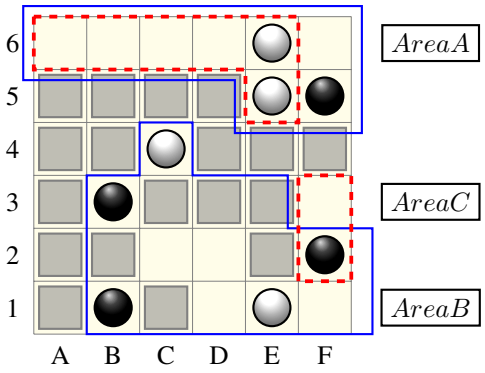


Figure 4.18: 18 moves played, *White* to move, global bounds $[-5, -1]$

are $[-4, -4] + [-2, 2] + [1, 1] = [-5, -1]$. Because the upper bound already indicates a win for *White*, there is no need to add on the infinitesimal part. This position therefore is a *White* win.

Without the databases, Area A would be partitioned into a *White* blocker territory (delimited by red dashed line in Figure 4.18) with bounds of $[-4, -4]$. The remaining active area has 1 empty square and neither player can find a safe move (bounds $[-1, 1]$). Therefore, the global bounds are $[-4, -4] + [-1, 1] + [-2, 2] + [1, 1] = [-6, 0]$. Since it is *White* to move, the winner cannot be determined yet.

A Local Search Win

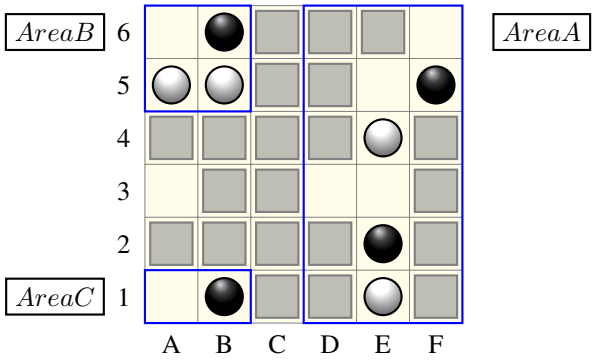


Figure 4.19: 21 moves played, *White* to move, global bounds $[1, 3]$

Figure 4.19 shows a 6×6 Amazons endgame position after 21 moves have been played and there are 6 empty squares left. Area B is looked up in the active databases as a $*$. Area C is looked up in the simple territory databases with its bounds being $[1, 1]$. Area A has a size of 3×6 so it is too large to be in the database. However, Area A can be locally searched with its bounds being $[0, 2]$. The global bounds without the infinitesimal therefore are $[0, 2] + [1, 1] = [1, 3]$. Since the lower bound is greater than 0, *Black* wins this position and there is no need to add the infinitesimal bounds.

Without the databases, Areas B and C will be evaluated as $[-1, 1]$ and $[1, 1]$ respectively. The

global bounds then are $[0, 2] + [1, 1] + [-1, 1] = [0, 4]$. Because it is *White* to move and the lower bound is 0, this position is still evaluated as a *Black* win.

However, without the local search, Area A will be evaluated as $[-2, 2]$ because both players can find a safe move in Area A. The global bounds then will be $[-2, 2] + [1, 1] + [-1, 1] = [-2, 4]$. The winner of the position cannot be determined based on these bounds.

Chapter 5

Experiments and Results

5.1 Area Evolution

Figure 5.1, 5.2, 5.3 and 5.4 illustrates how different types of areas evolve during game play on 5×5, 5×6, 6×5 and 6×6 boards respectively. For each figure, the statistics are averaged over 100 self-play games by program Arrow 2.1 (a *UCT* player) given 500 seconds per game. In each figure, subfigure (a) shows a summary of all the areas and subfigure (b) shows statistics of only the active areas in more detail. The legends in these figures are explained in Table 5.1.

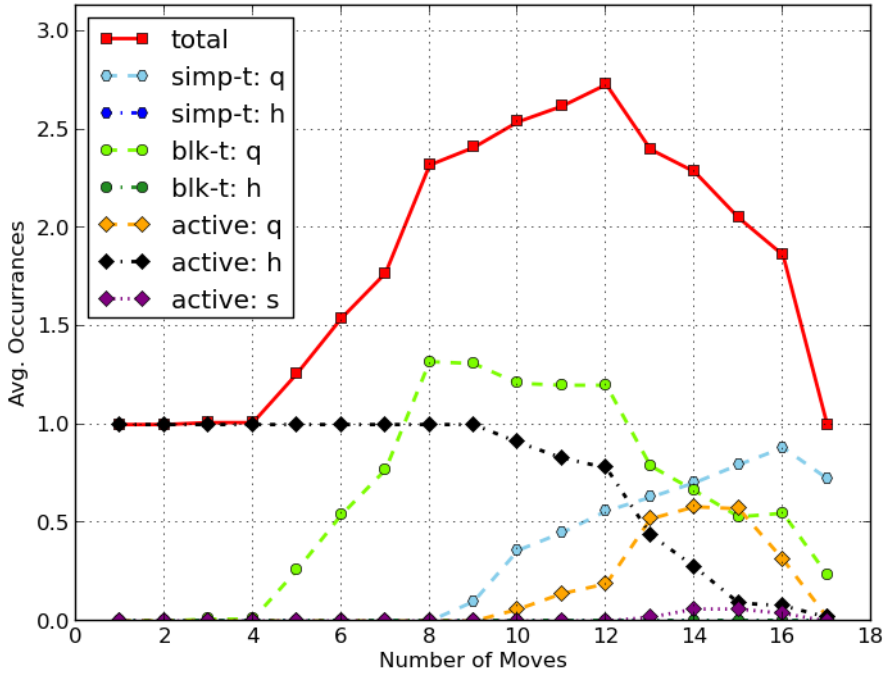
Legend	Meaning
total	the sum of all types of areas
simp-t: q	simple territories queried from the databases
simp-t: h	simple territories computed by heuristics
blk-t: q	blocker territories queried from the databases
blk-t: h	blocker territories computed by heuristics
active: q	active areas queried from the databases
active: h	active areas computed by heuristics
active: s	active areas computed by local searches
infi	infinitesimals in active: q
number	numbers in active: q
hot	hot games in active: q

Table 5.1: Legends of Figure 5.1 to 5.4

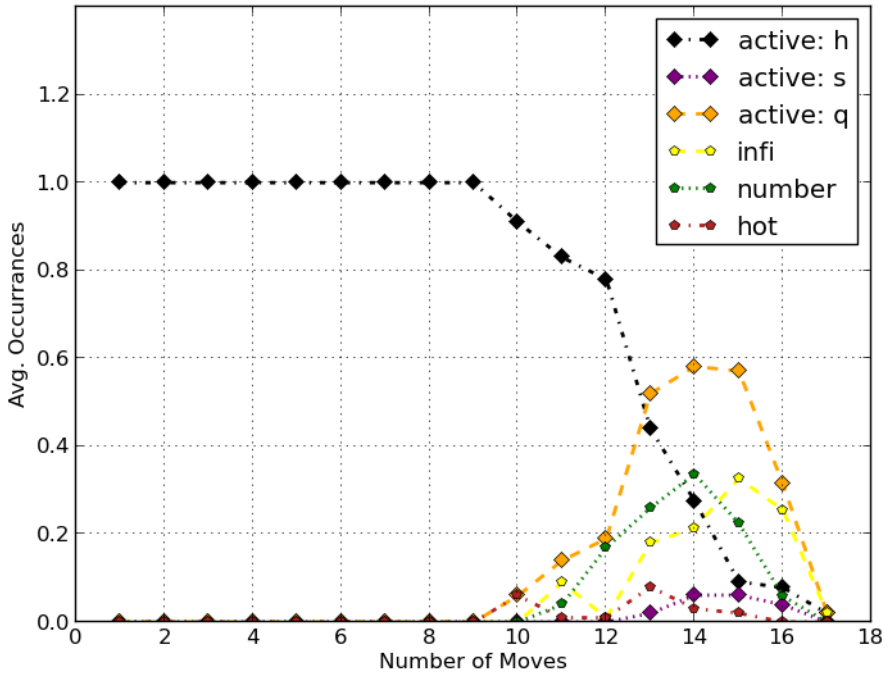
These figures clearly illustrate how an Amazons board decomposes into separate areas as the game progresses. For all board sizes, the number of simple and blocker territories computed by heuristics is either 0 or very close to 0. This means that almost all territories occurring during game play fall in the databases. The number of active areas that are locally searched is also close to 0 and it happens only towards the end of the game. Among the queried active areas, numbers and infinitesimals are predominant towards the end of the game.

Several interesting data points in the figures are compiled in Table 5.2. The name of the points are:

- avl**:** the average length of the games.

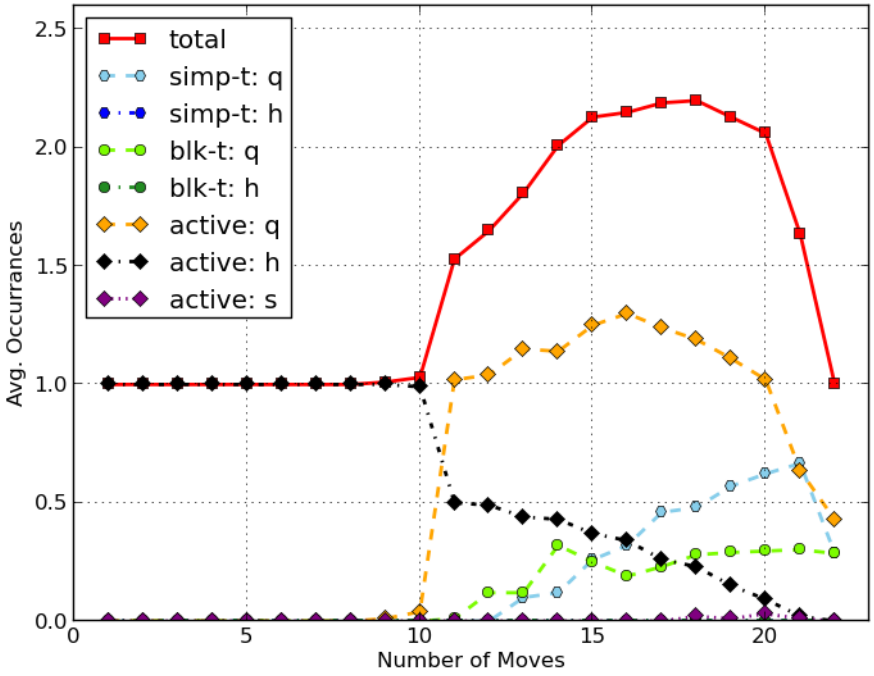


(a) Summary

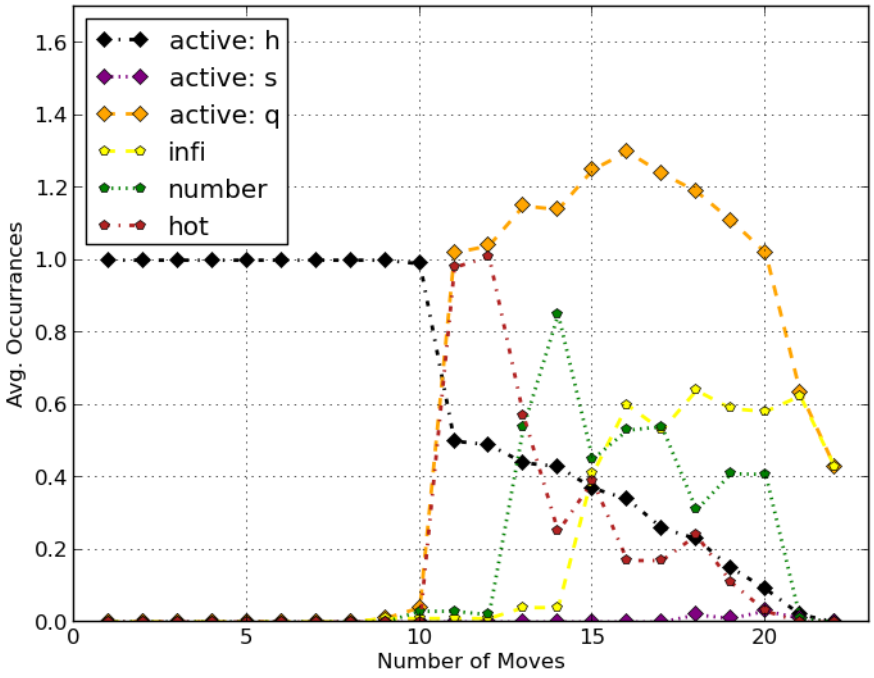


(b) Active areas only

Figure 5.1: Area evolution on 5×5 board

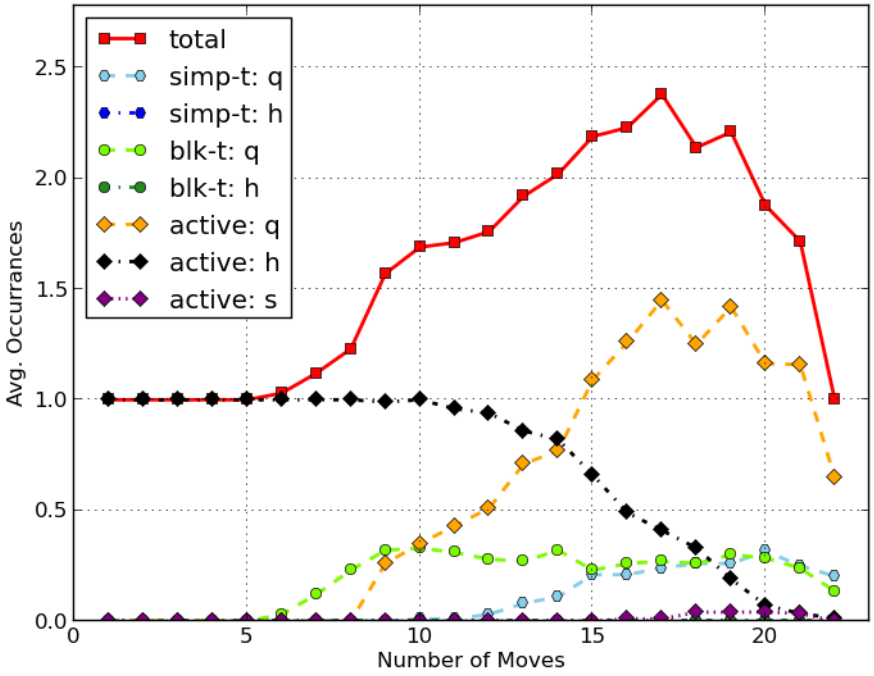


(a) Summary

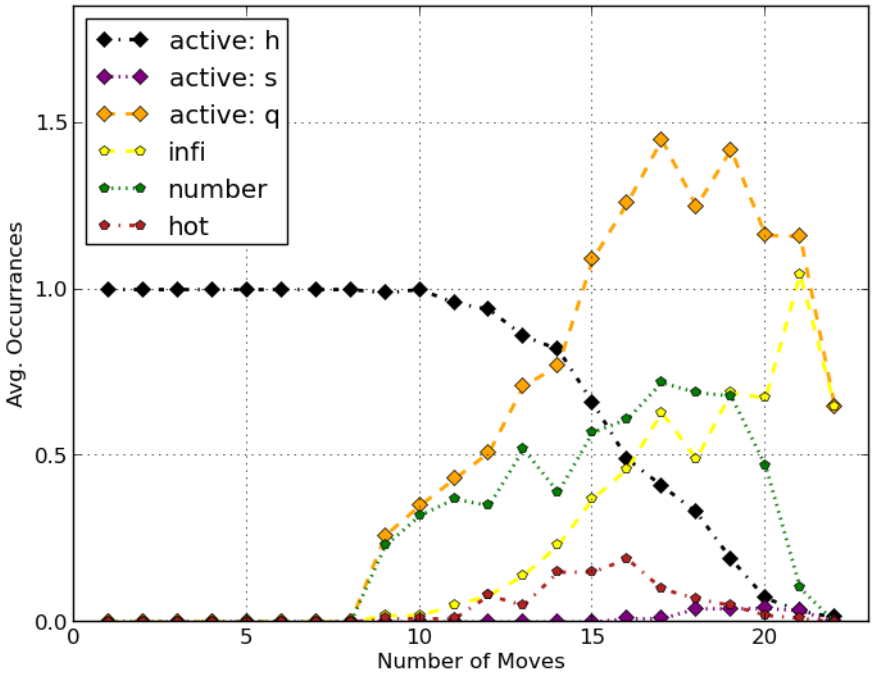


(b) Active areas only

Figure 5.2: Area evolution on 5×6 board

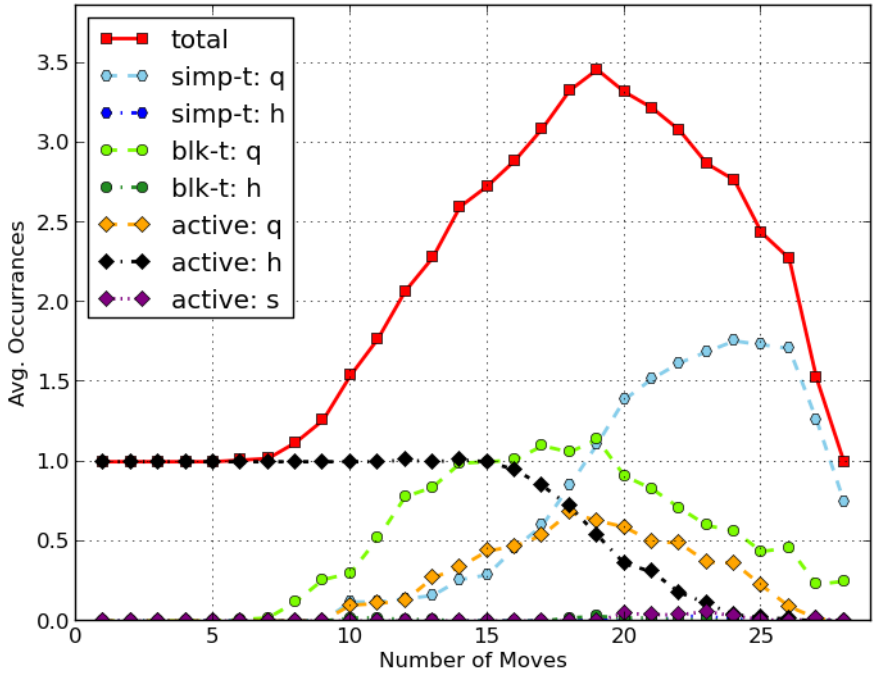


(a) Summary

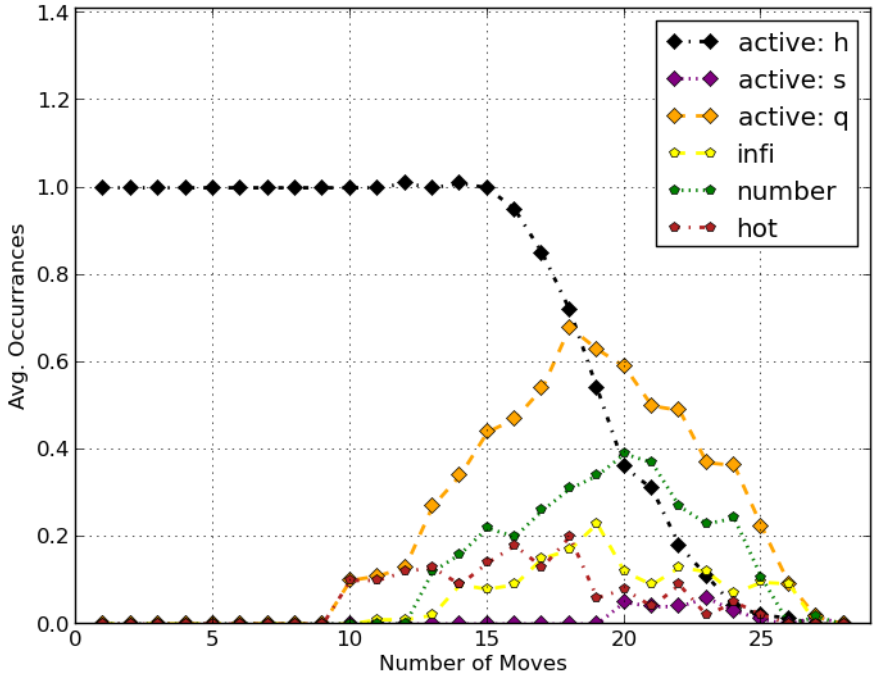


(b) Active areas only

Figure 5.3: Area evolution on 6×5 board



(a) Summary



(b) Active areas only

Figure 5.4: Area evolution on 6×6 board

- $a1l$**:** the average length of the opening of the games where the number of active areas computed by heuristics is within 2% of 1. This data point measures the average length where the entire board is the predominant active area.

- tip**:** the point where the total number of areas starts increasing. Smaller areas start emerging at this point. This point is usually close to bip, since the players try to block off parts of the board as their blocker territories at this stage.

- tdp**:** the point where the total number of areas starts decreasing. This point usually indicates that the game is entering the endgame phase when the areas are gradually filled up and become dead areas.

- tpv**:** the peak value of the total number of areas.

- sip**:** the point where the number of simple territories starts increasing. sip is always later than bip. This indicates strong play from both players because usually no queens should be completely isolated from the rest of the board at this stage.

- sdp**:** the point where the number of simple territories starts decreasing. sdp always comes after bdp. This point usually indicates when the game enters the territory-filling stage.

- bip**:** the point where the number of blocker territories starts increasing.

- bdp**:** the point where the number of blocker territories starts decreasing.

- sbp**:** the point where the number of simple territories exceeds the number of blocker territories.

- aip**:** the point where the number of queried active areas starts increasing. This is the earliest point where the active area databases start being helpful.

- adp**:** the point where the number of queried active areas starts decreasing.

- ahp**:** the point where the number of queried active areas exceeds the number of active areas computed by the heuristics. After this point, most of the active areas on board can be queried from the database. Solving positions hereafter is usually easy.

Data Point	5×5	5×6	6×5	6×6
avl	15.98	21.08	21.60	26.54
$a1l$	9	10	10	15
tip	4	9	5	7
tdp	12	18	17	19
tpv	2.73	2.2	2.38	3.46
sip	8	12	11	9
sdp	16	21	20	24
bip	4	11	5	6
bdp	8	14	14	19
sbp	14	15	20	20
aip	9	9	8	9
adp	15	16	17	18
ahp	13	11	15	19

Table 5.2: Interesting data points in area evolution

5.2 Solving Test Cases

We have two solvers for solving Amazons: *ab* (a minimax based solver) and *dfpn* (a DFPN solver). To gauge the performance of the solvers and the databases, the test set from [23] is used. This test set contains three test games named $f1$, $f2$ and $f3$, which were played by an early version of our Amazons playing program Arrow against a human expert [23]. $f1$ and $f2$ were played on the 5×5 board and $f3$ was played on the 6×6 board. All three game files can be found at [21]. Test cases in the test set are positions selected from these games. Configurations of the solvers are shown in Table 5.3. All endgame databases as described in Section 3.3 are used in these tests. The time limit is 10000 seconds per test case for all solvers.

Solver Name	Hash Table	Others
ab	2^{25}	
ab_db	2^{25}	endgame databases
dfpn	2^{22}	
dfpn_db	2^{22}	endgame databases

Table 5.3: Solver configurations – test set

The search results for $f1$, $f2$ and $f3$ are shown in Figure 5.5, 5.6 and 5.7 respectively. For each picture, the horizontal axis is labelled with the test case number and the difficulty of the test cases increases from left to right. The vertical axis shows the number of nodes searched to solve a test case in the logarithm scale. Even with a hash table of $\frac{1}{8}$ of the size of that used by *ab*, *dfpn* searches fewer nodes than *ab*, sometimes by orders of magnitude, in all the test cases except for position 2 in $f3$. The statistics for *ab* on position 10 in $f3$ is missing because it did not finish the search within time limit. In these test cases, *dfpn_db* never searches more nodes than *dfpn*. Sometimes *dfpn_db* can reduce the number of nodes needed by *dfpn* by an order of magnitude.

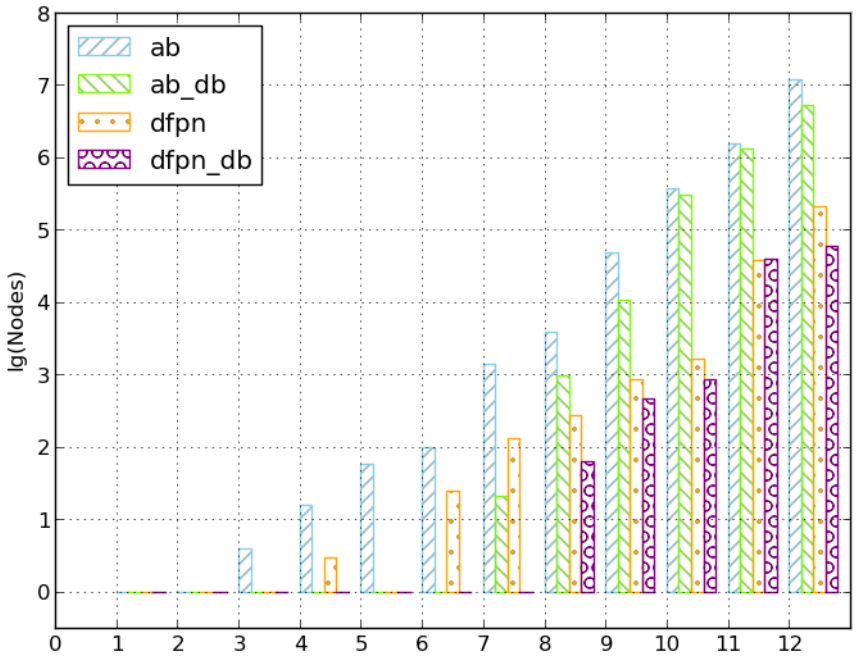


Figure 5.5: Search result of $f1$

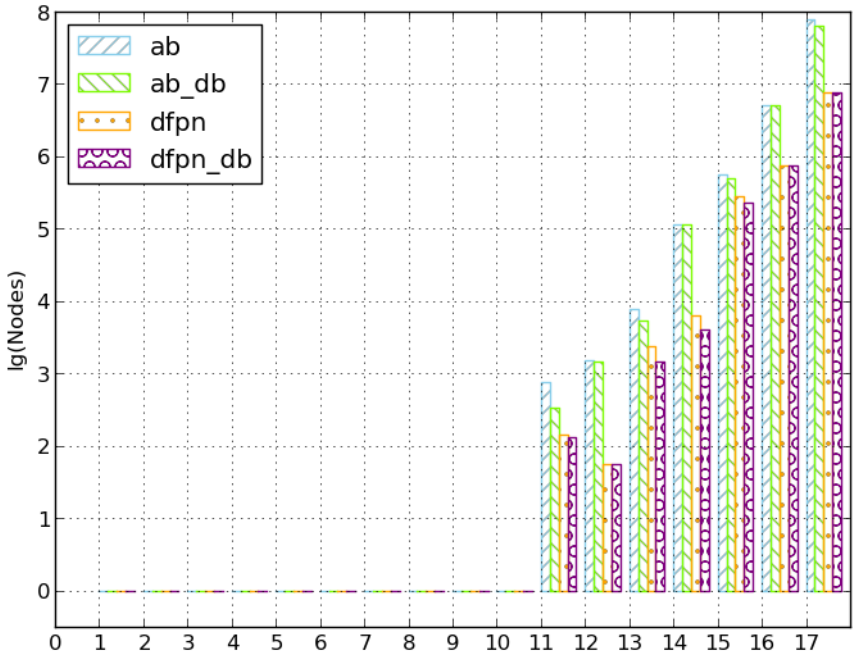


Figure 5.6: Search result of $f2$

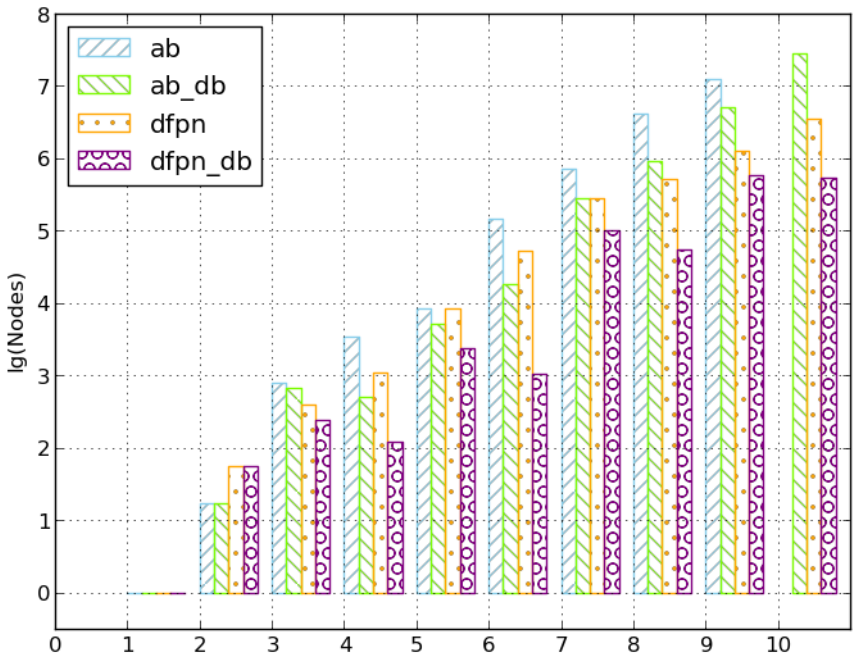
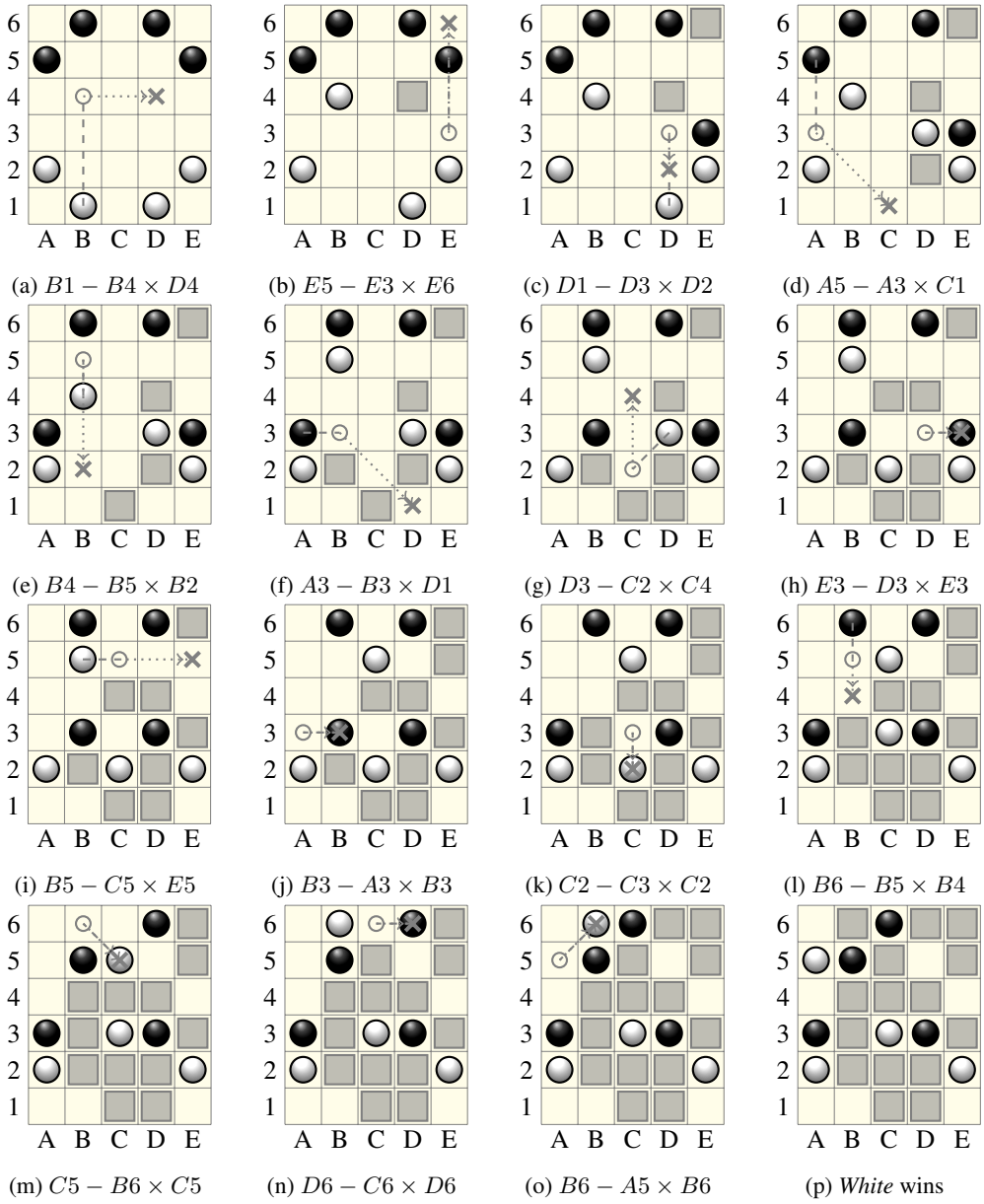


Figure 5.7: Search result of $f3$

5.3 Solving 5×6 Amazons: A First Player Win

Using the *dfpn* solver with a hash table of 2^{30} entries (32 bytes per entry) plus all the endgame databases as described in Section 3.3, we weakly solved the 5×6 starting position (shown in Figure 1.1b) as a first player win. The search was run in a single thread and took 31.4 hours. The search traversed 333310325 nodes in total. There were 331565697 territories evaluated in total and 99.82% of them were evaluated as their exact values. There were 502346758 active areas evaluated in total while only 1.35% of these were evaluated as exact. 83.55% of the active areas had a size of 5×6. The statistics on the occurrences of territories and active areas are shown in Table C.1 and C.2 in Appendix C. Detailed database querying statistics for simple territories, active areas and blocker territories are shown in Table C.3, C.4 and C.5 respectively in Appendix C.

A complete strategy for winning 5×6 Amazons was stored, and the principal variation of the search is shown in Figure 5.8. During the principal variation, the entire board is the only active area of size 5×6. In the final position of the principal variation (Figure 5.8p), *White* has two blocker territories (bounds $[-1, -1]$ and $[-2, -2]$ respectively) and *Black* has one blocker territory (bounds $[2, 2]$). The remaining active area in the top left corner is evaluated as $[0, 0]$ since both players can find a safe move. The global bounds are $[-1, -1]$ and *White* can win regardless of who moves first.



(a) $B1 - B4 \times D4$ (b) $E5 - E3 \times E6$ (c) $D1 - D3 \times D2$ (d) $A5 - A3 \times C1$

(e) $B4 - B5 \times B2$ (f) $A3 - B3 \times D1$ (g) $D3 - C2 \times C4$ (h) $E3 - D3 \times E3$

(i) $B5 - C5 \times E5$ (j) $B3 - A3 \times B3$ (k) $C2 - C3 \times C2$ (l) $B6 - B5 \times B4$

(m) $C5 - B6 \times C5$ (n) $D6 - C6 \times D6$ (o) $B6 - A5 \times B6$ (p) *White* wins

Figure 5.8: 5×6 board principal variation, *White* moves first and wins

5.4 Solving Early 6×5 and 6×6 Positions

In order to evaluate the solver's performance on bigger boards, we also solved some early 6×5 and 6×6 positions. For each board size, we generated 100 self-play games with program Arrow 2.1 (a *UCT* player). Then we started solving each game from the end of the game with 500 seconds per position until we ran into the first position that exceeds the time limit. The solver we used was *dfpn* with all the databases and a hash table of 2^{22} entries.

# Moves	# Solved	# Avg Node
22	74	1.00
21	88	1.00
20	98	1.01
19	100	1.11
18	100	4.28
17	100	5.37
16	100	38.38
15	100	21.25
14	100	446.39
13	100	566.03
12	100	5386.52
11	100	6978.62
10	100	65212.43
9	100	83433.01
8	99	1339502.87
7	59	623908.36
6	25	700953.92
5	16	548137.81

Table 5.4: Solving early 6×5 positions

Table 5.4 and 5.5 show the results on 6×5 and 6×6 boards respectively. Within 500 seconds, 99 out of the 100 6×5 games are solved after 8 moves, and 95 out of the 100 6×6 games are solved after 10 moves. These results suggest that to solve 6×5 or 6×6 Amazons, a forward search of at most 8 or 10 plies is needed before the current solver can solve a position reasonably fast. The number of games solved beyond these points decreases rapidly. For example, 16 games are solved after 5 moves on the 6×5 board, and only 2 games are solved after 7 moves on the 6×6 board.

# Moves	# Solved	# Avg Node
28	24	1.00
27	61	1.00
26	76	1.00
25	94	1.00
24	99	1.00
23	100	1.00
22	100	1.03
21	100	3.21
20	100	4.90
19	100	16.08
18	100	32.72
17	100	155.01
16	100	318.39
15	100	1432.08
14	100	2778.09
13	100	17043.12
12	100	41942.31
11	97	193850.19
10	95	501987.98
9	77	685108.74
8	53	634355.26
7	2	2548.00

Table 5.5: Solving early 6×6 positions

Chapter 6

Future Work

6.1 Parallel Computing

All the solvers we are using now are single-threaded. However, as the problems we are trying to solve get larger, it makes more sense to have a multi-threaded/distributed solver with a master process/machine generating unsolved nodes and a farm of slave processes/machines actually solving them.

6.2 Summing Games

Storing the thermographs of games is cheaper than storing their full combinatorial-game-theoretical values. However, it also restricts us from employing the full power of the theory and the databases since thermographs do not support game addition [6]. For example, Figure 6.1 shows the position that is one move before the position shown in Figure 4.17. Areas A, C and D are the same as in Figure 4.17 and they are $*$, 0 and 0 respectively. Area B is actually a $*$, however from the query result, we can only tell that Area B is an infinitesimal that is confused with 0. Therefore we cannot use the next player's moving privilege to provide a free move and the global bounds are $[-1, 1]$. If we could sum up the games, this position is 0 and *White* loses.

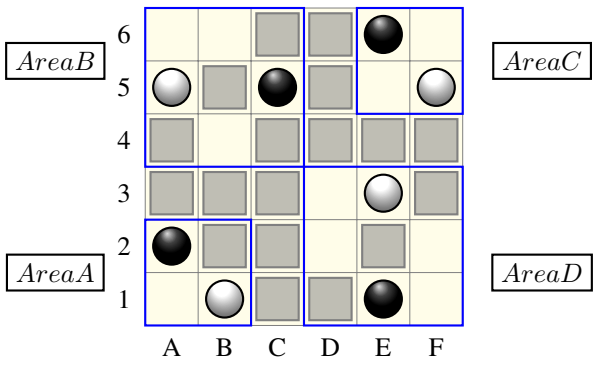


Figure 6.1: *White* to move, all areas queried, cannot sum

6.3 Database Improvements

Currently the territory databases store both defective and non-defective positions because because the search engines used require a best move in each non-terminal position. However, this is not necessary because once the exact value v of a position is known, playing the best move in it will lead to a position of $v - 1$. Therefore, we can prune out the non-defective positions from the databases and the best move field for the defective entries. This will reduce the size of the territory database drastically as is evident from Table B.1 in Appendix B.

Active area databases can also be built with the blocker constraint such that the remaining active areas in an improved partition can also be queried.

6.4 Other Improvements

- **Local search improvements.** Currently, the local search module is not using the databases and it does not consider the blocker constraint. Both of these can be added on to achieve better local search results.
- **Proof number initialization.** The proof and disproof numbers of a newly created node can be initialized to indicate how easy or hard it is expected to solve a node [25]. Combined with a decent evaluation function, this could achieve better search results.
- **Area caching.** Caching search results of areas that are not in the databases but arise frequently in the search can potentially give the solver a performance boost.
- **Search unknown areas only.** If the perfect information on most of the areas is known, it might make sense to restrict the global search to only the unknown areas.

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Appendix A

Database Move Encoding

A move in Amazons is associated with three squares on board: the origin square of the moving queen (s_0), the destination square this queen moves to (s_1), and the square where the arrow is shot (s_2). Assuming that the function `index()` extract the index of a square on an Amazons board, an Amazons move can be encoded by concatenating the indexes of the three squares as shown in Figure A.1.

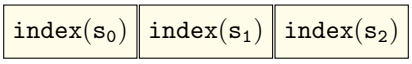

Figure A.1: Move encoding

In our implementation, we used the *smartgame* library [14] which uses 9 bits for indexing a single square. Therefore an Amazons move can be encoded in a 32 bit integer.

Appendix B

Databases Used

Table B.1: Simple territory database statistics

Size	Queens	Defective	ReducedTotal
1×2	1	0	1
1×3	1	0	2
1×3	2	0	2
1×4	1	0	2
1×4	2	0	4
1×4	3	0	2
1×5	1	0	3
1×5	2	0	6
1×5	3	0	6
1×5	4	0	3
1×6	1	0	3
1×6	2	0	9
1×6	3	0	10
1×6	4	0	9
2×2	1	0	4
2×2	2	0	5
2×2	3	0	2
2×3	1	2	28
2×3	2	0	47
2×3	3	0	37
2×3	4	0	18
2×4	1	0	106
2×4	2	0	252
2×4	3	0	298
2×4	4	0	233
2×5	1	12	411
2×5	2	6	1205
2×5	3	0	1998
2×5	4	0	2141
2×6	1	50	1455
2×6	2	0	5271
2×6	3	0	11060
2×6	4	0	15504
3×3	1	4	194
3×3	2	6	446

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Table B.1 – continued from previous page

Size	Queens	Defective	ReducedTotal
3×3	3	3	591
3×3	4	1	500
3×4	1	40	3216
3×4	2	38	10414
3×4	3	54	19798
3×4	4	46	25025
3×5	1	380	27670
3×5	2	353	113015
3×5	3	523	280668
3×5	4	715	474610
3×6	1	3423	222988
3×6	2	3309	1108291
3×6	3	4956	3403023

Table B.2: Active area database statistics

Size	*Black*	*White*	ReducedTotal
1×3	1	1	3
1×4	1	1	6
1×4	2	1	6
1×5	1	1	10
1×5	2	1	16
1×5	2	2	16
1×5	3	1	10
1×6	1	1	15
1×6	2	1	30
1×6	2	2	48
1×6	3	1	30
1×6	3	2	30
1×6	4	1	15
2×2	1	1	6
2×2	2	1	4
2×3	1	1	84
2×3	2	1	104
2×3	2	2	93
2×3	3	1	59
2×3	3	2	32
2×3	4	1	16
2×4	1	1	480
2×4	2	1	894
2×4	2	2	1350
2×4	3	1	884
2×4	3	2	1000
2×4	3	3	540
2×4	4	1	500
2×4	4	2	420
2×4	4	3	140
2×5	1	1	2357

Continued on next page

Table B.2 – continued from previous page

Size	*Black*	*White*	ReducedTotal
2×5	2	1	5948
2×5	2	2	12702
2×5	3	1	8432
2×5	3	2	14898
2×5	3	3	14072
2×5	4	1	7449
2×5	4	2	10602
2×5	4	3	7724
2×5	4	4	3100
2×6	1	1	10434
2×6	2	1	33180
2×6	2	2	92673
2×6	3	1	61665
2×6	3	2	148650
2×6	3	3	203170
2×6	4	1	74325
2×6	4	2	152595
2×6	4	3	173460
2×6	4	4	120156
3×3	1	1	820
3×3	2	1	1659
3×3	2	2	2771
3×3	3	1	1836
3×3	3	2	2478
3×3	3	3	1722
3×3	4	1	1246
3×3	4	2	1308
3×3	4	3	672
3×3	4	4	178
3×4	1	1	20612
3×4	2	1	59124
3×4	2	2	149092
3×4	3	1	99294
3×4	3	2	216970
3×4	3	3	270332
3×4	4	1	108537
3×4	4	2	203030
3×4	4	3	212200
3×4	4	4	136592
3×5	1	1	224933
3×5	2	1	839751
3×5	2	2	2840274
3×5	3	1	1893105
3×5	3	2	5762803
3×5	3	3	10442026
3×5	4	1	2881951
3×5	4	2	7833601
3×5	4	3	12560595
3×5	4	4	13221370
3×6	1	1	2213888
3×6	2	1	10203480

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Table B.2 – continued from previous page

Size	*Black*	*White*	ReducedTotal
3×6	2	2	43433015
4×4	1	1	284708
4×4	2	1	1121379
4×4	2	2	4033994
4×4	3	3	17260100

Table B.3: Blocker territory database statistics

Size	Normal Queens	Blockers	ReducedTotal	UnreducedTotal	ReductionRate
1×2	0	1	1	4	75.00%
1×3	0	1	2	12	83.33%
1×3	0	2	2	6	66.67%
1×3	1	1	3	12	75.00%
1×4	0	1	2	32	93.75%
1×4	0	2	4	24	83.33%
1×4	0	3	2	8	75.00%
1×4	1	1	6	48	87.50%
1×4	1	2	6	24	75.00%
1×4	2	1	6	24	75.00%
1×5	0	1	3	80	96.25%
1×5	0	2	6	80	92.50%
1×5	0	3	6	40	85.00%
1×5	0	4	3	10	70.00%
1×5	1	1	10	160	93.75%
1×5	1	2	16	120	86.67%
1×5	1	3	10	40	75.00%
1×5	2	1	16	120	86.67%
1×5	2	2	16	60	73.33%
1×5	3	1	10	40	75.00%
1×6	0	1	3	192	98.44%
1×6	0	2	9	240	96.25%
1×6	0	3	10	160	93.75%
1×6	0	4	9	60	85.00%
1×6	1	1	15	480	96.88%
1×6	1	2	30	480	93.75%
1×6	1	3	30	240	87.50%
1×6	2	1	30	480	93.75%
1×6	2	2	48	360	86.67%
1×6	3	1	30	240	87.50%
2×2	0	1	4	32	87.50%
2×2	0	2	5	24	79.17%
2×2	0	3	2	8	75.00%
2×2	1	1	6	48	87.50%
2×2	1	2	4	24	83.33%
2×2	2	1	4	24	83.33%
2×3	0	1	28	192	85.42%
2×3	0	2	47	240	80.42%
2×3	0	3	37	160	76.88%

Continued on next page

Table B.3 – continued from previous page

Size	Normal Queens	Blockers	ReducedTotal	UnreducedTotal	ReductionRate
2×3	0	4	18	60	70.00%
2×3	1	1	84	480	82.50%
2×3	1	2	104	480	78.33%
2×3	1	3	59	240	75.42%
2×3	2	1	104	480	78.33%
2×3	2	2	93	360	74.17%
2×3	3	1	59	240	75.42%
2×4	0	1	106	1024	89.65%
2×4	0	2	252	1792	85.94%
2×4	0	3	298	1792	83.37%
2×4	0	4	233	1120	79.20%
2×4	1	1	480	3584	86.61%
2×4	1	2	894	5376	83.37%
2×4	1	3	884	4480	80.27%
2×4	2	1	894	5376	83.37%
2×4	2	2	1350	6720	79.91%
2×4	3	1	884	4480	80.27%
2×5	0	1	411	5120	91.97%
2×5	0	2	1205	11520	89.54%
2×5	0	3	1998	15360	86.99%
2×5	0	4	2141	13440	84.07%
2×5	1	1	2357	23040	89.77%
2×5	1	2	5948	46080	87.09%
2×5	1	3	8432	53760	84.32%
2×5	2	1	5948	46080	87.09%
2×5	2	2	12702	80640	84.25%
2×5	3	1	8432	53760	84.32%
2×6	0	1	1455	24576	94.08%
2×6	0	2	5271	67584	92.20%
2×6	0	3	11060	112640	90.18%
2×6	0	4	15504	126720	87.77%
2×6	1	1	10434	135168	92.28%
2×6	1	2	33180	337920	90.18%
2×6	1	3	61665	506880	87.83%
2×6	2	1	33180	337920	90.18%
2×6	2	2	92673	760320	87.81%
2×6	3	1	61665	506880	87.83%
3×3	0	1	194	2304	91.58%
3×3	0	2	446	4608	90.32%
3×3	0	3	591	5376	89.01%
3×3	0	4	500	4032	87.60%
3×3	1	1	820	9216	91.10%
3×3	1	2	1659	16128	89.71%
3×3	1	3	1836	16128	88.62%
3×3	2	1	1659	16128	89.71%
3×3	2	2	2771	24192	88.55%
3×3	3	1	1836	16128	88.62%
3×4	0	1	3216	24576	86.91%
3×4	0	2	10414	67584	84.59%
3×4	0	3	19798	112640	82.42%
3×4	0	4	25025	126720	80.25%

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Table B.3 – continued from previous page

Size	Normal Queens	Blockers	ReducedTotal	UnreducedTotal	ReductionRate
3×4	1	1	20612	135168	84.75%
3×4	1	2	59124	337920	82.50%
3×4	1	3	99294	506880	80.41%
3×4	2	1	59124	337920	82.50%
3×4	2	2	149092	760320	80.39%
3×4	3	1	99294	506880	80.41%
3×5	0	1	27670	245760	88.74%
3×5	0	2	113015	860160	86.86%
3×5	0	3	280668	1863680	84.94%
3×5	1	1	224933	1720320	86.92%
3×5	1	2	839751	5591040	84.98%
3×5	2	1	839751	5591040	84.98%
3×5	2	2	2840274	16773120	83.07%
3×5	3	1	1893105	11182080	83.07%
4×4	1	1	284708	3932160	92.76%
4×4	2	1	1121379	13762560	91.85%

Appendix C

Solving 5×6 Amazons Statistics

Table C.1: Territory occurrence statistics in solving 5×6

Size	Simple Territory	Blocker Territory	Total
1×2	3996342	66003317	69999659(21.11%)
1×3	431227	3911984	4343211(1.31%)
1×4	56579	652646	709225(0.21%)
1×5	22540	14898	37438(0.01%)
1×6	1123	0	1123(0.00%)
2×2	2410303	128333623	130743926(39.43%)
2×3	1306360	71870716	73177076(22.07%)
2×4	125752	17142314	17268066(5.21%)
2×5	155471	7218604	7374075(2.22%)
2×6	5158	28858	34016(0.01%)
3×3	67103	9147174	9214277(2.78%)
3×4	7775	9034087	9041862(2.73%)
3×5	46172	3914078	3960250(1.19%)
3×6	197	46694	46891(0.01%)
4×4	116	1515350	1515466(0.46%)
4×5	3103	3761521	3764624(1.14%)
4×6	1	11430	11431(0.00%)
5×5	46	304022	304068(0.09%)
5×6	169	18844	19013(0.01%)
		Grand Total	331565697

Table C.2: Active area occurrence statistics in solving 5×6

Size	Total
1×3	290960(0.06%)
1×4	394287(0.08%)
1×5	104117(0.02%)
1×6	73840(0.01%)
2×2	3078181(0.61%)
2×3	4441047(0.88%)
2×4	2237604(0.45%)
	Continued on next page

Table C.2 – continued from previous page

Size	Total
2×5	$4146069(0.83\%)$
2×6	$1556823(0.31\%)$
3×3	$2081977(0.41\%)$
3×4	$1900830(0.38\%)$
3×5	$9297435(1.85\%)$
3×6	$1968195(0.39\%)$
4×4	$693490(0.14\%)$
4×5	$7454827(1.48\%)$
4×6	$4277333(0.85\%)$
5×5	$38659102(7.70\%)$
5×6	$419690641(83.55\%)$
Grand Total	502346758

Table C.3: Simple territory query statistics in solving 5×6

Size	Queens	Defective	Total
1×2	1	0	$6352946(46.33\%)$
1×3	1	0	$653973(4.77\%)$
1×4	1	0	$87340(0.64\%)$
1×5	1	0	$23009(0.17\%)$
1×6	1	0	$877(0.01\%)$
2×2	1	0	$3007564(21.93\%)$
2×3	1	62256	$1746394(12.74\%)$
2×4	1	0	$89596(0.65\%)$
2×5	1	674	$19904(0.15\%)$
2×6	1	33	$1758(0.01\%)$
3×3	1	1185	$52895(0.39\%)$
3×4	1	107	$3113(0.02\%)$
3×5	1	60	$3244(0.02\%)$
3×6	1	2	$44(0.00\%)$
1×3	2	0	$54542(0.40\%)$
1×4	2	0	$17011(0.12\%)$
1×5	2	0	$15475(0.11\%)$
1×6	2	0	$988(0.01\%)$
2×2	2	0	$633126(4.62\%)$
2×3	2	0	$475949(3.47\%)$
2×4	2	0	$71671(0.52\%)$
2×5	2	576	$95508(0.70\%)$
2×6	2	0	$4980(0.04\%)$
3×3	2	518	$56980(0.42\%)$
3×4	2	132	$7559(0.06\%)$
3×5	2	174	$24849(0.18\%)$
3×6	2	0	$258(0.00\%)$
1×4	3	0	$574(0.00\%)$
1×5	3	0	$1255(0.01\%)$
2×2	3	0	$5945(0.04\%)$
2×3	3	0	$13267(0.10\%)$
2×4	3	0	$15543(0.11\%)$

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Table C.3 – continued from previous page

Size	Queens	Defective	Total
2×5	3	0	131324(0.96%)
2×6	3	0	61(0.00%)
3×3	3	3	3440(0.03%)
3×4	3	6	1411(0.01%)
3×5	3	16	36755(0.27%)
3×6	3	0	33(0.00%)
1×5	4	0	18(0.00%)
2×3	4	0	12(0.00%)
2×4	4	0	29(0.00%)
2×5	4	0	271(0.00%)
3×3	4	0	16(0.00%)
3×4	4	0	6(0.00%)
3×5	4	0	272(0.00%)
		Grand Total	13711785

Table C.4: Active area query statistics in solving 5×6

Size	*Black*	*White*	Integer	Fraction	Infini	Hot	Total
1×3	1	1	0	0	57796	0	57796(0.19%)
1×4	1	1	652363	0	103546	3693	759602(2.50%)
1×5	1	1	52944	0	18356	4640	75940(0.25%)
1×6	1	1	19644	0	11681	2599	33924(0.11%)
2×2	1	1	1476705	0	2276449	0	3753154(12.33%)
2×3	1	1	945423	690296	49736	1451068	3136523(10.31%)
2×4	1	1	494522	37613	115026	170818	817979(2.69%)
2×5	1	1	129163	14389	37978	119854	301384(0.99%)
2×6	1	1	75148	8789	16582	46569	147088(0.48%)
3×3	1	1	221722	116982	139296	392293	870293(2.86%)
3×4	1	1	30080	9757	14076	43355	97268(0.32%)
3×5	1	1	26312	3335	12370	55199	97216(0.32%)
3×6	1	1	1911	275	610	3098	5894(0.02%)
4×4	1	1	197	18	40	206	461(0.00%)
1×4	2	1	0	0	25609	0	25609(0.08%)
1×5	2	1	21253	0	32519	2322	56094(0.18%)
1×6	2	1	11562	0	16707	3368	31637(0.10%)
2×2	2	1	0	0	343084	0	343084(1.13%)
2×3	2	1	375999	0	481975	320385	1178359(3.87%)
2×4	2	1	176766	4827	470618	305454	957665(3.15%)
2×5	2	1	447634	122083	444569	1340850	2355136(7.74%)
2×6	2	1	219634	17914	76094	206036	519678(1.71%)
3×3	2	1	229538	66753	245963	395133	937387(3.08%)
3×4	2	1	95312	49962	69402	198467	413143(1.36%)
3×5	2	1	226674	108671	174430	1167457	1677232(5.51%)
3×6	2	1	22904	4128	8773	41000	76805(0.25%)
4×4	2	1	1765	501	892	3027	6185(0.02%)
1×5	2	2	0	0	979	0	979(0.00%)
1×6	2	2	29477	0	18885	133	48495(0.16%)
2×3	2	2	16514	0	33638	1116	51268(0.17%)

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Table C.4 – continued from previous page

Size	*Black*	*White*	Integer	Fraction	Infini	Hot	Total
2×4	2	2	39359	0	63067	16004	118430(0.39%)
2×5	2	2	112077	14453	161228	332096	619854(2.04%)
2×6	2	2	341024	12010	299786	554827	1207647(3.97%)
3×3	2	2	29112	2996	57488	48800	138396(0.45%)
3×4	2	2	57350	12285	78403	106164	254202(0.84%)
3×5	2	2	259645	97032	265001	1072566	1694244(5.57%)
4×4	2	2	5934	2302	7029	13925	29190(0.10%)
1×5	3	1	0	0	636	0	636(0.00%)
1×6	3	1	125	0	72	20	217(0.00%)
2×3	3	1	11141	0	15592	8834	35567(0.12%)
2×4	3	1	15213	44	15449	19563	50269(0.17%)
2×5	3	1	284503	2558	474570	799040	1560671(5.13%)
2×6	3	1	8082	480	9367	15350	33279(0.11%)
3×3	3	1	8961	836	9620	7985	27402(0.09%)
3×4	3	1	13474	1763	17160	21407	53804(0.18%)
3×5	3	1	308278	48305	336749	1787377	2480709(8.15%)
1×6	3	2	0	0	45	0	45(0.00%)
2×3	3	2	0	0	1358	0	1358(0.00%)
2×4	3	2	4047	0	5316	755	10118(0.03%)
2×5	3	2	94765	386	99160	140710	335021(1.10%)
2×6	3	2	66144	657	71693	100049	238543(0.78%)
3×3	3	2	2986	92	4319	2722	10119(0.03%)
3×4	3	2	13224	1138	16842	15968	47172(0.16%)
3×5	3	2	343380	49639	420752	1514668	2328439(7.65%)
2×4	3	3	30	0	50	12	92(0.00%)
2×5	3	3	2971	0	4401	377	7749(0.03%)
2×6	3	3	9383	17	9298	13711	32409(0.11%)
3×3	3	3	84	0	132	43	259(0.00%)
3×4	3	3	820	24	1117	683	2644(0.01%)
3×5	3	3	26333	2054	31207	69956	129550(0.43%)
4×4	3	3	1666	104	1897	1666	5333(0.02%)
2×3	4	1	0	0	14	0	14(0.00%)
2×4	4	1	25	0	57	36	118(0.00%)
2×5	4	1	1175	1	1047	2023	4246(0.01%)
2×6	4	1	7	0	11	25	43(0.00%)
3×3	4	1	53	0	48	24	125(0.00%)
3×4	4	1	181	8	178	168	535(0.00%)
3×5	4	1	14347	1433	12002	61116	88898(0.29%)
2×4	4	2	19	0	10	0	29(0.00%)
2×5	4	2	366	0	578	297	1241(0.00%)
2×6	4	2	502	0	423	497	1422(0.00%)
3×3	4	2	10	0	12	7	29(0.00%)
3×4	4	2	149	9	165	102	425(0.00%)
3×5	4	2	6807	599	6683	29558	43647(0.14%)
2×5	4	3	11	0	9	4	24(0.00%)
2×6	4	3	129	0	143	109	381(0.00%)
3×3	4	3	0	0	1	0	1(0.00%)
3×4	4	3	14	0	15	4	33(0.00%)
3×5	4	3	807	55	858	2183	3903(0.01%)
						Grand Total	30427857

Table C.5: Blocker territory query statistics in solving 5×6

Size	Normal Queens	Blockers	Defective	Total
1×2	0	1	0	66003307(20.83%)
1×3	0	1	0	2568413(0.81%)
1×4	0	1	0	547037(0.17%)
1×5	0	1	0	13482(0.00%)
2×2	0	1	0	63073574(19.90%)
2×3	0	1	0	18806124(5.93%)
2×4	0	1	0	884067(0.28%)
2×5	0	1	0	163792(0.05%)
2×6	0	1	0	2863(0.00%)
3×3	0	1	7503	1335886(0.42%)
3×4	0	1	259	171016(0.05%)
3×5	0	1	11	10789(0.00%)
1×3	0	2	0	1200956(0.38%)
1×4	0	2	0	37605(0.01%)
2×2	0	2	0	63014400(19.88%)
2×3	0	2	0	45869647(14.47%)
2×4	0	2	0	10461224(3.30%)
2×5	0	2	0	2255803(0.71%)
2×6	0	2	0	17394(0.01%)
3×3	0	2	9381	5475967(1.73%)
3×4	0	2	11479	4629020(1.46%)
3×5	0	2	157	543420(0.17%)
2×2	0	3	0	2026607(0.64%)
2×3	0	3	0	3941361(1.24%)
2×4	0	3	0	3798085(1.20%)
2×5	0	3	0	3597907(1.14%)
2×6	0	3	0	2477(0.00%)
3×3	0	3	896	962657(0.30%)
3×4	0	3	4874	2966685(0.94%)
3×5	0	3	177	2113193(0.67%)
2×3	0	4	0	27715(0.01%)
2×4	0	4	0	97055(0.03%)
2×5	0	4	0	215822(0.07%)
2×6	0	4	0	21(0.00%)
3×3	0	4	17	54383(0.02%)
3×4	0	4	405	142225(0.04%)
1×3	1	1	0	142760(0.05%)
1×4	1	1	0	68001(0.02%)
1×5	1	1	0	1417(0.00%)
2×2	1	1	0	209968(0.07%)
2×3	1	1	0	3071317(0.97%)
2×4	1	1	0	1189582(0.38%)
2×5	1	1	0	253137(0.08%)
2×6	1	1	0	5075(0.00%)
3×3	1	1	6	1076274(0.34%)
3×4	1	1	416	371863(0.12%)
3×5	1	1	6	102742(0.03%)
2×2	1	2	0	9352(0.00%)

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Table C.5 – continued from previous page

Size	Normal Queens	Blockers	Defective	Total
2×3	1	2	0	122446(0.04%)
2×4	1	2	0	590538(0.19%)
2×5	1	2	0	505803(0.16%)
2×6	1	2	0	856(0.00%)
3×3	1	2	1	192848(0.06%)
3×4	1	2	41	667369(0.21%)
3×5	1	2	10	591251(0.19%)
2×3	1	3	0	402(0.00%)
2×4	1	3	0	4462(0.00%)
2×5	1	3	0	10795(0.00%)
2×6	1	3	0	1(0.00%)
3×3	1	3	0	1398(0.00%)
3×4	1	3	2	41748(0.01%)
2×2	2	1	0	483(0.00%)
2×3	2	1	0	32820(0.01%)
2×4	2	1	0	115491(0.04%)
2×5	2	1	0	210688(0.07%)
2×6	2	1	0	171(0.00%)
3×3	2	1	0	46670(0.01%)
3×4	2	1	0	33964(0.01%)
3×5	2	1	0	131997(0.04%)
2×3	2	2	0	280(0.00%)
2×4	2	2	0	1781(0.00%)
2×5	2	2	0	4515(0.00%)
3×3	2	2	0	703(0.00%)
3×4	2	2	0	9983(0.00%)
3×5	2	2	0	35391(0.01%)
2×3	3	1	0	62(0.00%)
2×4	3	1	0	127(0.00%)
2×5	3	1	0	415(0.00%)
3×3	3	1	0	239(0.00%)
3×4	3	1	0	275(0.00%)
3×5	3	1	0	6050(0.00%)
4×6	3	1	0	5(0.00%)
			Grand Total	316895494