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ON VIBRATION TRANSMISSION RECIPROCITY IN MODULATED MATERIALS

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***Abstract*—Modulated materials are artificially structured materials with effective properties that vary periodically in time in response to an external stimulus. The time-dependent elastic properties of a modulated material enable them to exhibit vibration transmission characteristics that are not typical of regular materials. Modulated materials have a demonstrated ability to restrict the transmission of waves to only one direction. Thus, the transmission properties become dependent on the direction of the propagating waves, in contrast with the reciprocity principle in linear time-invariant materials. In this work, we study the reciprocity of vibration transmission in a discrete model of a modulated material with two degrees of freedom. We highlight the role of the difference in modulation phases of the two units on controlling the reciprocity bias in the linear operating regime. We then investigate and discuss the influence of nonlinear elasticity on the transmission reciprocity, highlighting the significant role played by the type of nonlinearity. This work facilitates further parametric studies on the combined effects of modulation and nonlinearity on reciprocity in modulated materials.

Keywords-reciprocity; metamaterials; nonlinear dynamics; temporal-spatial modulation

I. INTRODUCTION

Propagation of mechanical waves in elastic materials has been studied for about three centuries, dating back to Sir Newton's study of sound propagation in air [1]. For a regular material with constant density and Young's modulus, wave propagation characteristics between two arbitrary points remain invariant after interchanging the locations of the vibration source and receiver. This symmetry property, known as the principle of reciprocity, remains valid for propagation of small-amplitude waves in materials with properties that do not change with time; i.e. linear time-invariant systems. Within this context, it is not possible for waves to have different transmission characteristics (e.g. changes in amplitude and phase) depending on the direction of travel between two points. Such asymmetric wave transmission can be utilized for development of novel vibration mitigation devices and energy harvesting mechanisms, for example. Accordingly, there the physics and engineering of nonreciprocal propagation of elastic waves has recently drawn the attention of many researchers [2].

Nonreciprocal wave propagation has recently been investigated in the context of periodic materials, both for discrete and continuous models. Periodic materials provide an amenable context for this study because their wave propagation characteristics are dictated by the properties of their repeating sub-structure, also known as the unit cell. Nonreciprocal and directional propagation was analyzed in a discrete infinite-long modulated metamaterial, in which a wave-like temporal-spatial modulation was added to stiffness coefficient of the resonant spring in every unit cell [3]. Within the one-dimensional structure, directional scattered waves are generated because of the modulation. The scattered waves are coupled to the incident wave at certain frequencies, resulting in nonreciprocal propagation. For uniform continuous media, researchers found the appearance of nonreciprocity due to temporal-spatial modulation in Young's modulus of the media [4-6], as well as both Young's modulus and density (two-phase modulation) [7]. A similar nonreciprocal wave propagation phenomenon can be realized in elastic metasurfaces by means of temporal-spatial modulation of resonant springs at the surface [8,9].

In experimental demonstration of nonreciprocity due to temporal-spatial modulations, periodic systems naturally comprise only a few units. For example, the temporal-spatial modulations have been realized by means of magnetic forces [10, 11]. The spatial modulation, in particular, corresponds to a constant phase shift between the modulated elasticity of adjacent units. This spatial phase shift is as essential in breaking reciprocity as the temporal modulation. To investigate this in more detail, we focus in this work on the special case of a system with two degrees of freedom (2DoF). Our goal is to systematically study the influence of system parameters on the reciprocity of vibration transmission in this system, highlighting the significance of modulation phase shift and nonlinear elasticity. This will be the first building block for investigating the combined effects of modulation and nonlinearity in modulated materials.

In Section 2, the problem formulation and methodology are introduced. In Section 3, the effects of system parameters on nonreciprocity are presented in the linear operating range. The influence of nonlinear elasticity on nonreciprocity is described

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in Section 4. We conclude in Section 5 by summarizing our findings and pointing out directions for future work.

II. Analysis of a 2DoF System with Modulation

We consider a 2DoF system composed of two identical masses, viscous dampers, coupling springs and weakly modulated grounding springs with nonlinearity. On each of the two masses, there is an external force applied. The two external forces have the same frequency. See Fig. 1.

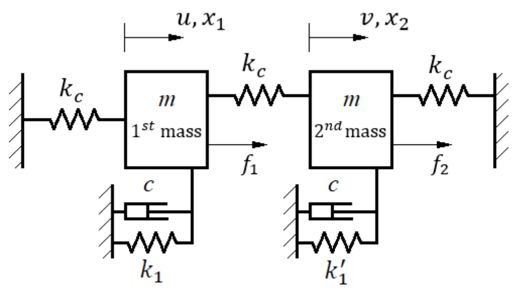


Figure 1. Scheme of the 2DoF system. Each grounding spring has three components: a constant term, a time-dependent term and a nonlinear (amplitude-dependent) term.

A. *Formulation of the Problem*

The equations of motion for the system in Fig. 1 are:

$$\begin{cases} m\ddot{u} + c\dot{u} + (2k_c + k_1)u - k_c v = F_1 \cos(\omega_f t) \\ m\ddot{v} + c\dot{v} + (2k_c + k_1')v - k_c u = F_2 \cos(\omega_f t) \end{cases}, \tag{1}$$

where $k_1 = k_L + k_m \cos(\omega_m t) + k_N u^2$ and $k_1' = k_L + k_m \cos(\omega_m t - \varphi) + k_N v^2$. The phase shift φ represents a spatial modulation in the grounding stiffness of each mass. We introduce the following parameters to non-dimensionalize the governing equations: $t = \tau/\omega_0$, $\omega_0^2 = (2k_c + k_L)/m$, $\omega_m = \Omega_m \omega_0$, $\omega_f = \Omega_f \omega_0$, $c = 2\zeta m \omega_0$, $k_c = K_c(2k_c + k_L)$, $k_m = K_m(2k_c + k_L)$, $k_N = K_N(2k_c + k_L)/a^2$, $F_1 = a(2k_c + k_L)P_1$, $F_2 = a(2k_c + k_L)P_2$, $u(t) = a x_1(\tau)$ and $v(t) = a x_2(\tau)$, where a is a representative length, The governing equations (1) are therefore rewritten as:

$$\begin{cases} \dfrac{d^2}{d\tau^2}x_1 + 2\zeta \dfrac{d}{d\tau}x_1 + [1 + K_m \cos(\Omega_m \tau)]x_1 \\ \qquad + K_N x_1^3 - K_c x_2 = P_1 \cos(\Omega_f \tau), \\ \dfrac{d^2}{d\tau^2}x_2 + 2\zeta \dfrac{d}{d\tau}x_2 + [1 + K_m \cos(\Omega_m \tau - \varphi)]x_1 \\ \qquad + K_N x_2^3 - K_c x_1 = P_2 \cos(\Omega_f \tau). \end{cases} \tag{2}$$

Our focus is on investigating the steady-state response of the system. In order to distinguish the two directions of wave propagation (left to right versus right to left), two configurations are defined: (i) the forward configuration with $P_1 = P, P_2 = 0$ where the output is the steady-state response of the second mass $x_2^F(\tau)$; (ii) the backward configuration with $P_1 = 0, P_2 = P$ where the output is the steady-state response of the first mass $x_1^B(\tau)$. If and only if $x_2^F(\tau) = x_1^B(\tau)$, vibration transmission through the system is reciprocal. The reciprocity bias R is introduced to quantify the degree of nonreciprocity between the outputs of forward and backward configurations:

$$R = \lim_{T\to\infty} \sqrt{\frac{1}{T}\int_0^T [x_2^F(\tau) - x_1^B(\tau)]^2 d\tau}. \tag{3}$$

If $R = 0$, the vibration transmission is reciprocal; otherwise, the transmission is nonreciprocal [12]. Output norms N^F and N^B are introduced to represent the response in the forward and backward configurations respectively:

$$\begin{aligned} N^F &= \lim_{T\to\infty} \sqrt{\frac{1}{T}\int_0^T [x_2^F(\tau)]^2 d\tau}, \\ N^B &= \lim_{T\to\infty} \sqrt{\frac{1}{T}\int_0^T [x_1^B(\tau)]^2 d\tau}. \end{aligned} \tag{4}$$

In direct numerical simulations, the norms in (3) & (4) are evaluated after the steady state is reached.

B. *Solution Methodology*

Approximating the solutions of (2) is the key strategy to estimate the displacement output in the steady-state. The steady-state displacement output of the 2DoF system in different configurations is expressed by the expansion in Fourier series:

$$\begin{aligned} x_2^F(\tau) &= \sum_{n=-\infty}^{\infty} \left[\left(\frac{\hat{\xi}_n}{2} e^{in\Omega_m \tau}\right) e^{i\Omega_f \tau} + cc.\right], \\ x_1^B(\tau) &= \sum_{n=-\infty}^{\infty} \left[\left(\frac{\hat{\eta}_n}{2} e^{in\Omega_m \tau}\right) e^{i\Omega_f \tau} + cc.\right], \end{aligned} \tag{5}$$

where $cc.$ represents the corresponding complex conjugate, $\hat{\xi}_n$ and $\hat{\eta}_n$ are complex-valued amplitudes for a given Ω_f. The representation of the displacement in (5) points out an important characteristic of the steady-state response of the systems subject to simultaneous external and parametric excitation: the response contains spectral components not only at the frequency of the external force, Ω_f, but also at $\Omega_f \pm \Omega_m$, $\Omega_f \pm 2\Omega_m$ and so on. To find the amplitudes, $\hat{\xi}_n$ and $\hat{\eta}_n$, we substitute (5) into (2) and integrate the result over one modulation period. This procedure yields a system of nonlinear algebraic equations for the amplitudes $\hat{\xi}_n$ and $\hat{\eta}_n$, which can be solved numerically. We refer to this procedure as the averaging method.

The transient response is computed by using the Runge-Kutta method [13]. We use the results from direct numerical integration of the governing equations to validate the predictions made by the averaging method.

Fig. 2 shows the output norms calculated for (2) for the following parameters: $\Omega_m = 0.15$, $K_m = 0.1$, $K_c = 0.44$, $\zeta =$

0.02 and $P = 0.1$. There is very good agreement between the analytical and numerical prediction of the steady-state response. Thus, we will use the analytical approach in the remainder of this work. Unless otherwise stated, these parameters are used in examples in other sections as well.

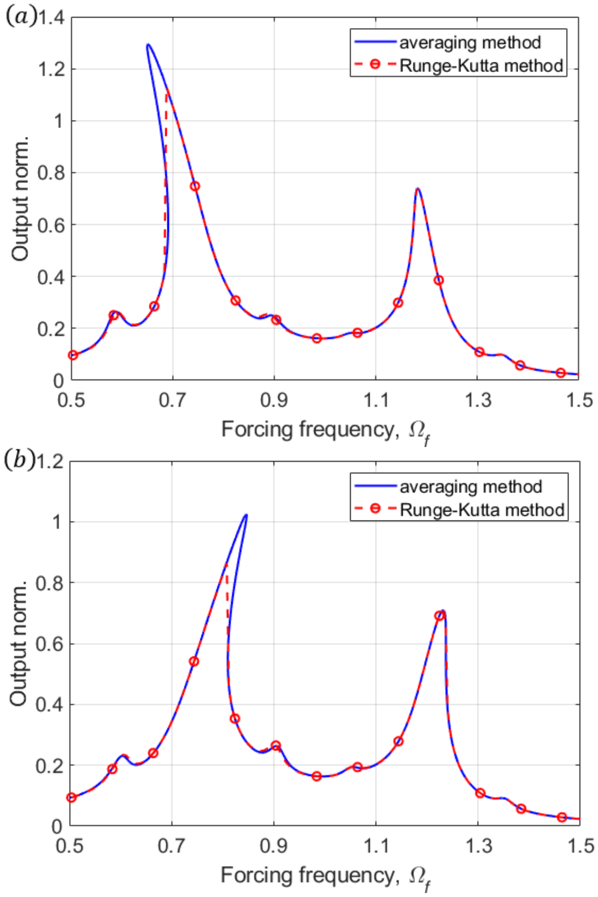


Figure 2. Comparison between the results of averaging method and numerical simulation. (a): forward configuration with $K_N = -0.05$, $\varphi = 0.5\pi$; (b): backward configuration with $K_N = 0.1$, $\varphi = 0.5\pi$.

III. Nonreciprocal Vibration Transmission in Linear Modulated Systems

We first investigate nonreciprocity in the linear modulated system; i.e. $K_n = 0$ in (2). This will establish the importance of linear system parameters on breaking reciprocity. In this work, we only discuss the case of weakly modulated systems ($K_m \leq 0.1$). The methodology presented in Section II.B, however, remains valid even for strong modulations.

A. *Effects of K_c and Ω_m in the Modulated Linear Systems*

We start by considering the system with temporal modulations only; i.e. $\varphi = 0$. In this case, the parameter K_c determines the two natural frequencies (dimensionless) of an unmodulated linear system: $\sqrt{1 \pm K_c}$, which are approximately the primary resonant frequencies of a weakly modulated linear system (indicated by the green solid arrows in Fig. 3). The secondary resonant frequencies of a weakly modulated linear system occur near frequencies $\sqrt{1 \pm K_c} \pm \Omega_m$ (indicated by the green hollow arrows in Fig. 3), with each pair corresponding to one of the natural frequencies of the unmodulated system. Due to the mirror-symmetry of the system, response is reciprocal.

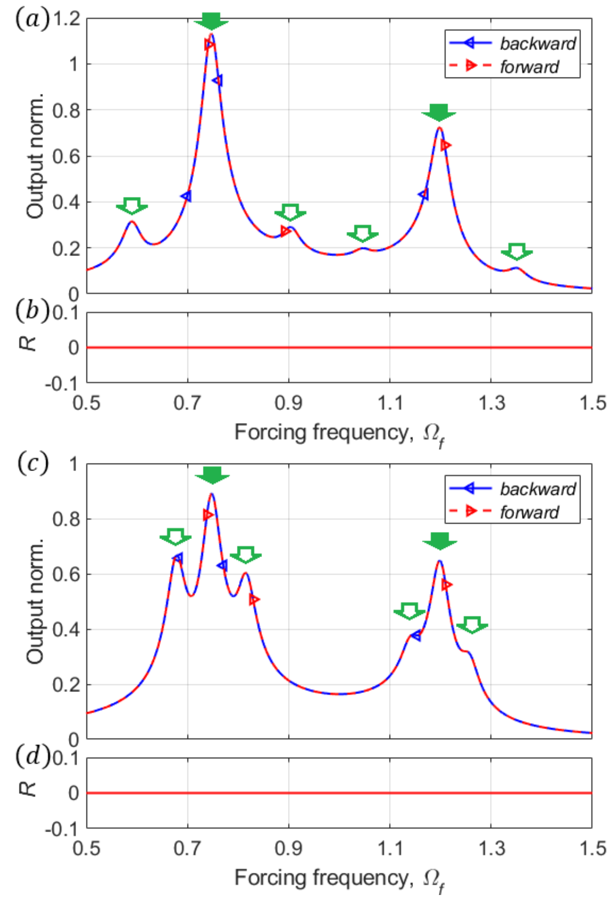


Figure 3. Effect of K_c and Ω_m on the response of the system with temporal modulation ($\varphi = 0$). (a) and (c) are plots of the output norms for forward and backward configurations with respect to forcing frequency. (b) and (d) are plots of reciprocity bias of the systems represent by (a) and (c) respectively. The green solid arrows indicate primary resonances, the green hollow arrows indicate secondary resonances. In both 2DoF systems: $K_N = 0$, (a) and (b): $\Omega_m = 0.15$, (c) and (d): $\Omega_m = 0.05$. In (b) and (d), reciprocity bias is equal to zero over the frequency range, responses of both systems are reciprocal.

B. *Effect of φ in the Modulated Linear System*

When $\varphi \neq 0$, the modulated system is no longer mirror-symmetric. Thus, the transmission is no longer reciprocal. Fig. 4 shows the output norms and reciprocity bias of the system at two different values of φ. As expected, the degree of nonreciprocity can be controlled by φ, the difference in the modulation phases of the two degrees of freedom.

Notice that the output norms in panels (a) and (e) are very similar, but the corresponding reciprocity bias in panels (b) and (f) remain non-zero. This implies that a significant contribution to non-reciprocity is possibly due to the phase difference between the response in the forward and backward configurations. It can be verified in panels (d) and (h). A similar phenomenon may occur when $K_m = 0$ and $K_n \neq 0$ [12].

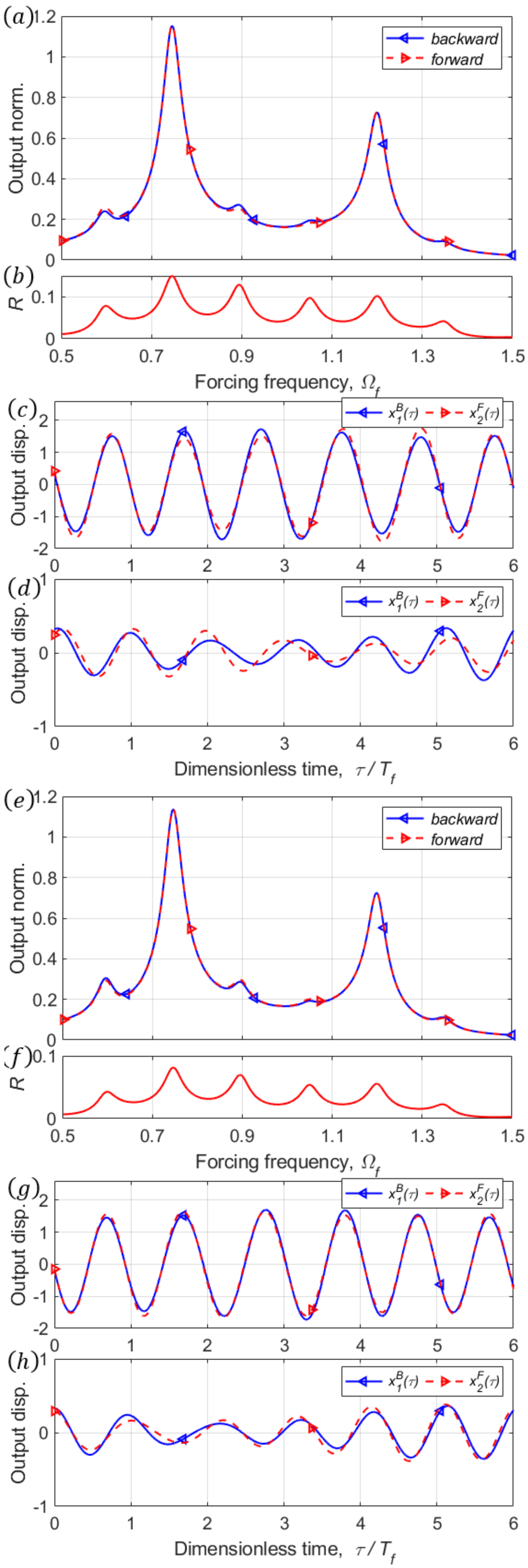
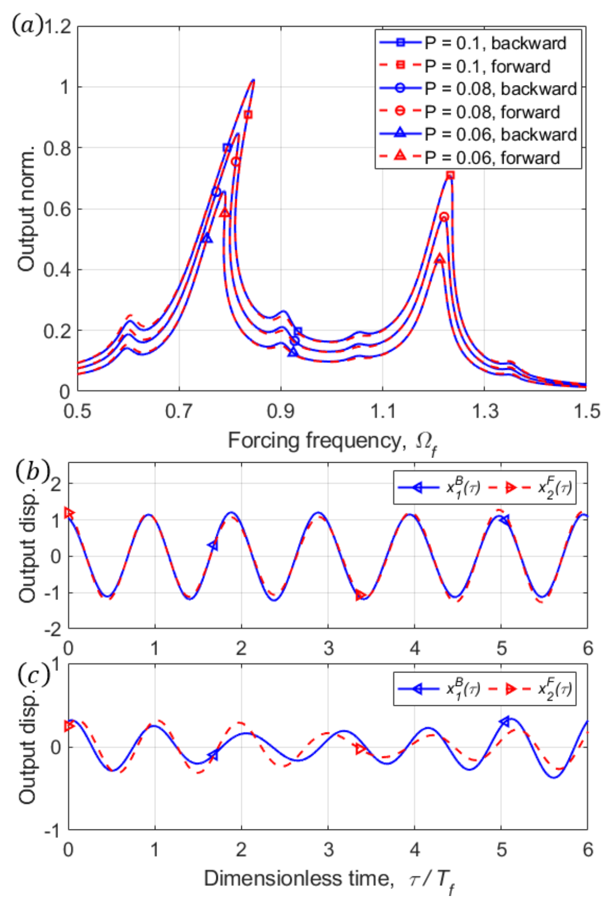


Figure 4. Effect of φ on reciprocity for the linear system, $K_N = 0$. Panels (a)-(d): $\varphi = 0.5\pi$, (e)-(h): $\varphi = -0.25\pi$. Panels (c), (d), (g) and (h): output displacement at different values of forcing frequency, Ω_f, (c) and (g) $\Omega_f = 0.748$, (d) and (h) $\Omega_f = 1.05$. $T_f = 2\pi/\Omega_f$.

IV. Nonreciprocal Vibration Transmission in Nonlinear Modulated Systems

Although the operation of mechanical systems is traditionally based on their linear response, it is sometimes beneficial or necessary to consider the influence of nonlinear forces, particularly in experiments [10]. Therefore, we investigate the influence of nonlinear elasticity on the nonreciprocity of vibration transmission in our 2DoF system.

We consider both the hardening ($K_N = 0.1 > 0$) and softening ($K_N = -0.05 < 0$) types of nonlinearity. Fig. 5 shows the output norms of the nonlinear system as a function of forcing amplitude, P. Given that the coupling force is linear, the influence of nonlinearity is larger on the response near the in-phase modes (near 0.75). The steady-state response of (2) with $K_N \in \{-0.05, 0.1\}$ and $P \in \{0.06, 0.08, 0.1\}$ is calculated using the averaging method, and the output norms for forward and backward configurations are shown in Fig. 5. As expected, the primary and secondary resonant frequencies are all amplitude-dependent.



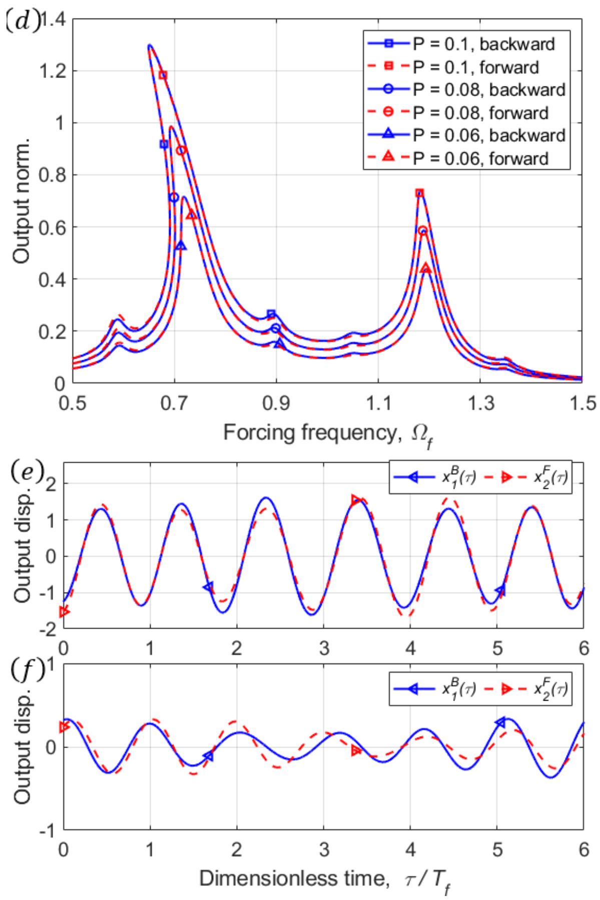


Figure 5. Steady-state responses of 2DoF nonlinear systems in forward and backward configurations with different excitations, (a)-(c): $K_N = 0.1$, $\varphi = 0.5\pi$;(d)-(f): $K_N = -0.05$, $\varphi = 0.5\pi$. Panels (b), (c), (e) and (f): output displacement at different values of forcing frequency, Ω_f, (b) $\Omega_f = 0.8$, (c) and (f) $\Omega_f = 1.05$, (e) $\Omega_f = 0.7$.

We use the normalized reciprocity bias, R/P, to study the effect of the forcing amplitude on the degree of nonreciprocity. For the linear system, the degree of nonreciprocity does not depend on the amplitude of motion; this can be inferred by the three overlapping curves in Fig. 6(a). For the nonlinear system, panels (b) and (c) in Fig. 6 show that the degree of nonreciprocity depends on the forcing amplitude, as expected. Interestingly, increasing the forcing amplitude increases the normalized reciprocity bias for the system with hardening nonlinearity, while it has the opposite effect in the system with softening nonlinearity. Note that the most significant effect of nonlinearity is observed near the primary in-phase resonance peak in both cases.

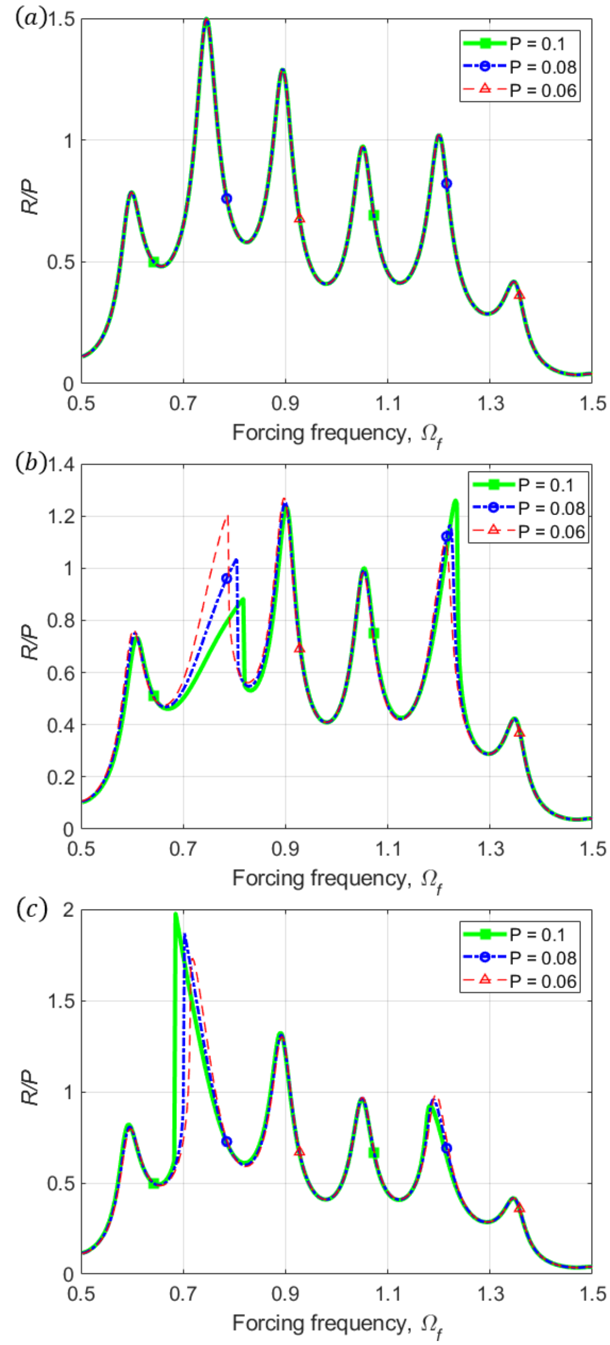


Figure 6. Plots of normalized reciprocity bias, R/P. (a): a 2DoF linear modulated system, $K_N = 0$ and $\varphi = 0.5\pi$. (b): a 2DoF nonlinear modulated system, $K_N = 0.1$ and $\varphi = 0.5\pi$. (c): a 2DoF nonlinear modulated system, $K_N = -0.05$ and $\varphi = 0.5\pi$.

V. Conclusion

We studied the reciprocity of vibration transmission in a discrete model of modulated materials. Temporal-spatial modulation in the stiffness coefficient is the key factor in breaking the reciprocity invariance in coupled systems, within which the phase shift between two adjacent modulations presents the modulation in space. Nonreciprocity in coupled modulated systems can be quantified using the reciprocity bias. We observed that having equal output norms for the forward and backward configurations is not a sufficient test for reciprocity. We presented scenarios in which it is the phase difference that plays a major role in increasing the reciprocity bias. In weakly modulated linear systems, the locus of the primary and secondary resonance can be adjusted by manipulating the coupling stiffness and frequency of modulation. We emphasized the role of the difference in modulation phases of the two units on controlling the reciprocity bias in the linear operating regime. We then reported the influence of cubic (on-site) nonlinearity on the reciprocity bias and highlighted the significance of the type of nonlinearity on how reciprocity depends on the forcing amplitude.

The analysis of coupled modulated systems provides a new perspective on nonreciprocal wave propagation in nonlinear materials. The methodology described in this work facilitates further parametric studies in this context.

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