University of Alberta

DYNAMICS OF DIFFERENTIAL EQUATIONS ON INVARIANT MANIFOLDS

by

Benjamin D. Wards ©

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To my parents, for their support and encouragement.

ABSTRACT

The simplification resulting from reduction of dimension involved in the study of invariant manifolds of differential equations is often difficult to achieve in practice. Appropriate coordinate systems are difficult to find or are essentially local in nature thus complicating analysis of global dynamics. Li and Muldowney [8] developed an approach that avoids the selection of coordinate systems on the manifold. Conditions were given for the stability of equilibria and periodic orbits in terms of stability of compound equations of the linearized systems at the equilibrium or periodic orbit. When the manifold is a finite dimensional Euclidean space, results in [11] and [7] show that if these conditions are satisfied by the linearized system at any bounded orbit, then the omega limit set is respectively an equilibrium or a periodic orbit. The thesis provides a survey of these topics and develops a new approach that extends the results on the existence of equilibria and periodic orbits to systems with invariant manifolds.

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Chapter 1

Introduction

The existence and stability of periodic motions for dynamical systems is an area of much interest in the sciences. From an applied perspective, nonlinear ordinary differential equations are routinely used to model physical phenomena. Periodic motions are common for these models especially for electrical and biological systems. Understanding when stable periodic motions arise provides insight into these systems. From the perspective of pure mathematics, generalizing well-known results in the plane such as the Poincaré-Bendixson theorem is a fruitful path for research.

In Chapter 2, concepts and terminology necessary for the study of dynamical systems are discussed. To understand the long-term behavior of a dynamical system, properties of limit sets are considered. It is seen that stability of an orbit has strong implications for the stability of its limit sets, as well as restricting what the limit set can be. Sufficient conditions are then reviewed for the existence of an asymptotic equilibrium and a phase asymptotic periodic orbit.

In Chapter 3, dynamical systems generated by non-linear autonomous systems of differential equations in $\mathbb{R}^n$ are considered. When $n = 2$, the dynamics are well understood. Here, the Poincaré-Bendixson theorem characterizes limit sets of a bounded orbit. This theorem can be used to deduce the existence of a periodic orbit. However, this theorem is not valid in higher dimensional systems. The variational equation at a solution of a differential equation can be used to study the behavior of this solution. Muldowney showed in [11] that when the linearized system about a bounded solution is uniformly asymptotically stable, the flow is asymptotic at this solution and limits to an asymptotic equilibrium. Li and Muldowney in [7] showed that when the second compound of the variation equation at a bounded solution is uniformly asymptotically stable, the flow is phase asymptotic at this solution and it limits to a phase asymptotic periodic orbit if the orbit does not get close to any equilibrium. At the end of Chapter 3, we present an alternative proof of this result. It will be used in Chapter 4 when we extend this result to flows on invariant manifolds.

In Chapter 4, we consider flows on invariant manifolds generated by autonomous differential equations. Invariant manifolds arise, for example, in physical systems from the existence of conserved quantities. The traditional approach to dynamical systems with an invariant manifold is to use coordinate systems on the manifold to reduce the dimension of the problem. This approach is difficult to implement. Li and Muldowney in [8] study the stability of periodic orbits and equilibria with respect to the flow on an invariant manifold. They developed stability criteria for equilibria and periodic orbits without resorting to special coordinates. Some properties of flows on an invariant manifold are reviewed. The chapter concludes with Theorem 4.9, where we provide sufficient conditions for the existence of a phase asymptotic periodic orbit on an invariant manifold in terms of a linear equation associated with the variational equation.

In Chapter 5, Theorem 4.9 is applied to a series of examples to demonstrate the existence of a phase asymptotically stable periodic orbit.

In Appendix A, we collect facts on compound matrices and compound differential equations necessary for the development in this thesis. Appendix B, contains a technical result used in the proofs of Theorems 3.11 and 4.9.

Chapter 2

Dynamical Systems

The focus of this thesis is on the existence and stability of periodic orbits for dynamical systems generated by an autonomous differential equation. A parallel discussion of similar questions for equilibria is included throughout as motivation.

Dynamical systems were developed to model physical phenomena. In many physical applications, an understanding of the long-term behavior of the systems is of interest. This can be facilitated by finding the invariant sets and limit sets, and understanding the behavior of neighboring orbits. The simplest of these invariant sets are periodic orbits and equilibria. These problems can be explored by studying the stability of an equilibrium or a periodic orbit.

In this chapter, dynamical systems and notions of stability for equilibria and periodic orbits are defined. A review of known results that can be used to establish the existence of an equilibrium or a periodic orbit from the attraction of its neighbors by a bounded orbit is given.

2.1 Definitions

Some terminology for dynamical systems is now reviewed. Let $\mathbb{R}$ denote the real numbers, $\mathbb{R}^+$ the nonnegative real numbers, and $M_{n\times m}$ the $n \times m$ real matrices. A metric space (X, d) is a set X with a map $d : X \times X \mapsto \mathbb{R}^+$

called the metric such that $\mathbf{x}, \mathbf{y}, \mathbf{z} \in X$ implies $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x})$ and $d(\mathbf{x}, \mathbf{y}) \leq d(\mathbf{x}, \mathbf{z}) + d(\mathbf{z}, \mathbf{y})$.

Definition 2.1. Let (X, d) be a metric space. Suppose the function $(t, \mathbf{x}) \mapsto \varphi(t, \mathbf{x})$ is continuous for $t \in \mathbb{R}$ and $\mathbf{x} \in X$. Then $\varphi(t, \mathbf{x})$ is a *flow* or a *(continuous) dynamical system* if the following occur:

(a) $\varphi(t + s, \mathbf{x}) = \varphi(s, \varphi(t, \mathbf{x}))$ for all $\mathbf{x} \in X$ and $s, t \in \mathbb{R}$.

(b) $\varphi(0, \mathbf{x}) = \mathbf{x}$, for all $\mathbf{x} \in X$.

Let $B_\delta(\mathbf{x}) := \{\mathbf{y} \in X : |\mathbf{x} - \mathbf{y}| < \delta\}$ and $B_\delta(A) := \{\mathbf{y} \in X : |\mathbf{a} - \mathbf{y}| < \delta, \mathbf{a} \in A\}$ for $\delta > 0$ where $\mathbf{x} \in X$ and $A \subset X$. A set $B \subset X$ is *positively invariant* with respect to the flow on X if $\varphi(t, B) \subset B$ for $t \geq 0$. It is *invariant* if $\varphi(t, B) \subset B$ for $t \in \mathbb{R}$, where $\varphi(t, B) = \{\varphi(t, \mathbf{b}) : \mathbf{b} \in B\}$. If $B \subset X$ and $\mathbf{x} \in X$, the distance from B to $\mathbf{x}$ is defined as

$$d(\mathbf{x}, B) := \inf_{\mathbf{y} \in B} d(\mathbf{x}, \mathbf{y}).$$

Important classes of invariant sets are the ω-limit and α-limit sets. A point $\mathbf{y} \in X$ is defined to be in the *ω-limit set* at $\mathbf{x}$, $\Omega(\mathbf{x})$, if there exists a sequence $\{t_n\}$ such that $\lim_{n\to\infty} t_n = \infty$ and $\lim_{n\to\infty} \varphi(t_n, \mathbf{x}) = \mathbf{y}$. A point $\mathbf{y} \in X$ is defined to be in the *α-limit set* at $\mathbf{x}$, $A(\mathbf{x})$, if there exists a sequence $\{t_n\}$ such that $\lim_{n\to\infty} t_n = -\infty$ and $\lim_{n\to\infty} \varphi(t_n, \mathbf{x}) = \mathbf{y}$.

Definition 2.2. Suppose $\mathbf{x} \in X$.

(a) The set $\Gamma_+(\mathbf{x}) := \{\varphi(t, \mathbf{x}) : t \geq 0\}$ is the *positive semiorbit* at $\mathbf{x}$.

(b) A point $\mathbf{x}$ is an *equilibrium* if $\varphi(t, \mathbf{x}) = \mathbf{x}$ for all $t \in \mathbb{R}$.

(c) The positive orbit $\Gamma_+(\mathbf{x})$ is *periodic* with period $\omega > 0$ if $\varphi(t + \omega, \mathbf{x}) = \varphi(t, \mathbf{x})$, for $t \in \mathbb{R}$ and $\mathbf{x}$ is not an equilibrium.

(d) The flow $\varphi(t, \mathbf{x})$ is *Lagrange Stable* at $\mathbf{x}$ if $\overline{\Gamma_+(\mathbf{x})}$ is compact, where the bar indicates topological closure.

(e) The flow $\varphi(t, \mathbf{x})$ is *Lyapunov stable* at S if for each $\epsilon > 0$ there exists a $\delta > 0$ such that $d(\mathbf{x}, S) < \delta$ implies $d(\varphi(t, \mathbf{x}), S) < \epsilon$ for $t \geq 0$.

(f) If $S \subset X$ is not Lyapunov stable then it is *unstable*.

(g) The flow is *asymptotic* at $S \subset X$ if there exists a $\rho > 0$ such that $\mathbf{x}_0 \in S$ and $d(\mathbf{x}, \mathbf{x}_0) < \rho$ implies $\lim_{t\to\infty} d(\varphi(t, \mathbf{x}), \varphi(t, \mathbf{x}_0)) = 0$.

(h) The flow $\varphi(t, \mathbf{x})$ is *phase asymptotic* at the set $S \subset X$ if there exist $\rho, \eta > 0$ such that, for each $\mathbf{x}_0 \in S$, there is a real valued phase function $\mathbf{x} \mapsto h(\mathbf{x})$ with $|h(\mathbf{x})| < \eta$ and such that $d(\mathbf{x}, \mathbf{x}_0) < \rho$ implies

$$\lim_{t\to\infty} d(\varphi(t + h(\mathbf{x}), \mathbf{x}), \varphi(t, \mathbf{x}_0)) = 0.$$

The concepts of phase asymptoticity and asymptoticity for a flow will be important throughout this thesis. A flow may be asymptotic at a set but not be phase asymptotic. In the definitions of phase asymptotic and asymptotic, often the set S will be taken to be an orbit or a point.

Note: to apply these notions of stability to a periodic orbit with path Γ one would let $S = \Gamma$. In this case, if the flow is Lyapunov stable, asymptotic, or phase asymptotic at Γ, then it is said that the periodic orbit is *orbitally stable*, *orbitally asymptotically stable*, or *orbitally phase asymptotically stable*, respectively.

2.2 Limit Sets and the Existence of Periodic Orbits

In this section, sufficient conditions are stated for a Lagrange stable orbit to limit to a phase asymptotically stable closed orbit.

Proposition 2.3. *Suppose the flow is Lagrange stable at* $\mathbf{x} \in X$. *Then* $\Omega(\mathbf{x})$ *is nonempty, compact, connected, and invariant.*

The following proposition states that the ω-limit set of an orbit attracts this orbit. However, it is not necessarily the case that an ω-limit set attracts all neighboring orbits.

Proposition 2.4. *Suppose that the flow is Lagrange stable at* $\mathbf{x}_* \in X$. *Then for every open neighborhood* V *of* $\Omega(\mathbf{x}_*)$ *there exists a* τ *such that* $\varphi(t, \mathbf{x}_*) \in V$ *for* $t \geq \tau$.

A proof of Propositions 2.3 and 2.4 can be found in [13] chapter II.

Proposition 2.5. *Suppose the flow,* $\varphi(t, \mathbf{x})$, *is Lagrange stable at* $\mathbf{x}_*$. *Then* $\varphi(t, \mathbf{x})$ *is Lyapunov stable, asymptotic, or phase asymptotic at* $\Gamma_+(\mathbf{x}_*)$ *if and only if it has the same property at* $\Omega(\mathbf{x}_*)$.

Proof. The proposition is proved only for the "phase asymptotic" statement. The others are proved similarly.

Suppose φ is phase asymptotic at $\Gamma_+(\mathbf{x}_*)$ and $\mathbf{x}_0 \in \Omega(\mathbf{x}_*)$. Let ρ and η be as in the definition of phase asymptotic. There exists $\mathbf{x}_1 \in \Gamma_+(\mathbf{x}_*)$ such that $|\mathbf{x}_1 - \mathbf{x}_0| < \rho/2$. If $|\mathbf{x}_0 - \mathbf{x}| < \rho/2$, then by the triangle inequality $|\mathbf{x}_1 - \mathbf{x}| < \rho$. Since the flow is phase asymptotic at $\Gamma_+(\mathbf{x}_*)$, the previous inequalities imply that

$$\lim_{t\to\infty} d(\varphi(t, \mathbf{x}_1), \varphi(t + h(\mathbf{x}_1, \mathbf{x}_0), \mathbf{x}_0))) = 0, \tag{2.1}$$

$$\lim_{t\to\infty} d(\varphi(t, \mathbf{x}_1), \varphi(t + h(\mathbf{x}_1, \mathbf{x}), \mathbf{x}))) = 0. \tag{2.2}$$

From the triangle inequality, (2.1), (2.2), and the properties of a flow,

$$\lim_{t\to\infty} d\left(\varphi(t, \mathbf{x}_0), \varphi\left(t + h(\mathbf{x}_1, \mathbf{x}) - h(\mathbf{x}_1, \mathbf{x}_0), \mathbf{x}\right)\right) = 0.$$

Let $h := h(\mathbf{x}_1, \mathbf{x}) - h(\mathbf{x}_1, \mathbf{x}_0)$. Then

$$\lim_{t\to\infty} d(\varphi(t, \mathbf{x}_0), \varphi(t + h, \mathbf{x}))) = 0,$$

where $|h| < 2\eta$. We conclude that the flow is phase asymptotic at $\Omega(\mathbf{x}_*)$.

Conversely, suppose the flow is phase asymptotic at $\Omega(\mathbf{x}_*)$. Let ρ and η be as in the definition of phase asymptotic. Then there exists a t_1 such that for $t \geq t_1$, $d(\varphi(t,\mathbf{x}_*),\Omega(\mathbf{x}_*)) < \rho/2$. Let $\mathbf{y}_1 := \varphi(t_1,\mathbf{x}_*)$. If $\mathbf{x}_1 \in \Gamma_+(\mathbf{y}_1)$, then there exists $\mathbf{x}_0$ in $\Omega(\mathbf{x}_*)$ so that $d(\mathbf{x}_0,\mathbf{x}_1) < \rho/2$. By the triangle inequality, $d(\mathbf{x},\mathbf{x}_1) < \rho/2$ implies that $d(\mathbf{x},\mathbf{x}_0) < \rho$. Then

$$\lim_{t\to\infty} d(\varphi(t,\mathbf{x}_0),\varphi(t+h(\mathbf{x}_0,\mathbf{x}_1),\mathbf{x}_1))) = 0, \tag{2.3}$$

$$\lim_{t\to\infty} d(\varphi(t,\mathbf{x}_0),\varphi(t+h(\mathbf{x}_0,\mathbf{x}),\mathbf{x}))) = 0. \tag{2.4}$$

From the triangle inequality, (2.3), (2.4), and the properties of a flow,

$$\lim_{t\to\infty} d(\varphi(t,\mathbf{x}_1),\varphi(t+h,\mathbf{x}))) = 0$$

where $h := h(\mathbf{x}_0,\mathbf{x}) - h(\mathbf{x}_0,\mathbf{x}_1)$ with $|h| < 2\eta$. It has been shown that $\Gamma_+(\mathbf{y}_1) \subset \Gamma_+(\mathbf{x}_*)$ is phase asymptotic.

We will now show that φ is phase asymptotic at $\Gamma_+(\mathbf{x}_*)$. Since φ is uniformly continuous on $[0,t_1] \times \Gamma_+(\mathbf{x}_*)$, there exists ρ_1, $0 \leq \rho_1 \leq \rho$ such that if $\varphi(s_1,\mathbf{z}_1) = \mathbf{y}_1$ with $s_1 \in [0,t_1]$, then $|\mathbf{z}_1 - \mathbf{x}| < \rho_1$ implies $|\mathbf{y}_1 - \varphi(s_1,\mathbf{x})| < \rho$. Since the flow is phase asymptotic at $\mathbf{y}_1$,

$$\begin{aligned}
&\lim_{t\to\infty} |\varphi(t+h(\varphi(s_1,\mathbf{x}),\mathbf{y}_1),\mathbf{x}) - \varphi(t,\mathbf{z}_1)| \\
&\quad= \lim_{t\to\infty} |\varphi(t+s_1+h(\varphi(s_1,\mathbf{x}),\mathbf{y}_1),\mathbf{x}) - \varphi(t+s_1,\mathbf{z}_1)| \\
&\quad= \lim_{t\to\infty} |\varphi(t+h(\varphi(s_1,\mathbf{x}),\mathbf{y}_1),\varphi(s_1,\mathbf{x})) - \varphi(t,\mathbf{y}_1)| \\
&\quad= 0
\end{aligned}$$

Hence, the flow is phase asymptotic at $\Gamma_+(\mathbf{x}_*)$. □

The following theorem was first proved in [11]. It can be used to detect an asymptotic equilibrium.

Theorem 2.6. *Suppose $(t,\mathbf{x}) \mapsto \varphi(t,\mathbf{x})$ is a flow which is Lagrange stable at $\mathbf{x}_*$. Then the following are equivalent:*

(i) *The flow* φ *is asymptotic [and Lyapunov stable] at* $\Gamma_+(\mathbf{x}_*)$, *the positive image of* $\mathbf{x}_*$ *under the flow.*

(ii) *The* ω*-limit set* $\Omega(\mathbf{x}_*)$ *is an equilibrium at which* φ *is asymptotic [and Lyapunov stable].*

The phrase in square brackets may be included or omitted throughout.

Proof. Suppose the flow is asymptotic at $\Gamma_+(\mathbf{x}_*)$. Let ρ be as in the definition of asymptotic. Choose $\mathbf{x}_1 \in \Gamma_+(\mathbf{x}_*)$ and $w > 0$ so that $|\mathbf{x}_1 - \varphi(\epsilon, \mathbf{x}_1)| < \rho$ for $|\epsilon| \leq w$. Then

$$\lim_{t\to\infty} d(\varphi(t,\mathbf{x}_1), \varphi(t, \varphi(\epsilon, \mathbf{x}_1))) = \lim_{t\to\infty} d(\varphi(t,\mathbf{x}_1), \varphi(\epsilon, \varphi(t, \mathbf{x}_1))) = 0 \tag{2.5}$$

since φ is asymptotic at $\Gamma_+(\mathbf{x}_*)$. Let $\mathbf{x}_0 \in \Omega(\mathbf{x}_*)$. Then there exists a sequence $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$ such that $\varphi(t_n, \mathbf{x}_1) \to \mathbf{x}_0$ as $n \to \infty$. From (2.5),

$$\lim_{n\to\infty} d(\varphi(t_n,\mathbf{x}_1), \varphi(\epsilon, \varphi(t_n, \mathbf{x}_1))) = 0, \quad |\epsilon| \leq w. \tag{2.6}$$

Therefore,

$$\lim_{n\to\infty} \varphi(t_n, \mathbf{x}_1) = \lim_{n\to\infty} \varphi(\epsilon, \varphi(t_n, \mathbf{x}_1)) \tag{2.7}$$

or equivalently $\mathbf{x}_0 = \varphi(\epsilon, \mathbf{x}_0)$ for $|\epsilon| \leq w$. This implies that $\mathbf{x}_0$ is an equilibrium. From Proposition 2.5, the flow is asymptotic at $\mathbf{x}_0$ since the flow is asymptotic at $\Gamma_+(\mathbf{x}_*)$. Further, there exists an open neighborhood, N, of $\mathbf{x}_0$ such that $\mathbf{x} \in N$ implies that $\varphi(t, \mathbf{x})$ limits to $\mathbf{x}_0$. This implies that $\mathbf{x}_0$ is the only equilibrium in N. Since $\Omega(\mathbf{x}_*)$ is connected, $\mathbf{x}_0 = \Omega(\mathbf{x}_*)$. If the the flow is Lyapunov stable at $\Gamma_+(\mathbf{x}_*)$, then from Proposition 2.5 the flow is Lyapunov stable at $\Omega(\mathbf{x}_*)$. This proves that (i) implies (ii).

Conversely, suppose the flow is asymptotic at $\mathbf{x}_0 = \Omega(\mathbf{x}_*)$. Then, from Proposition 2.5 the flow is asymptotic at $\Gamma_+(\mathbf{x}_*)$. If the flow is Lyapunov stable at $\Omega(\mathbf{x}_*)$, then from Proposition 2.5 the flow is Lyapunov stable at $\Gamma_+(\mathbf{x}_*)$. This proves that (ii) implies (i). □

The following theorem was proved by Li and Muldowney in [7]. It is a generalization of a theorem in [14] due to Sell. Sell's result does not assume boundedness of the phase function and has Lyapunov stability as a requirement rather then as an option. The boundedness of the phase function is in fact a consequence of this requirement.

Theorem 2.7. *Suppose $(t, \mathbf{x}) \mapsto \varphi(t, \mathbf{x})$ is a flow which is Lagrange stable at $\mathbf{x}_*$. Then the following are equivalent:*

(i) *The flow φ is phase asymptotic [and Lyapunov stable] at $\Gamma_+(\mathbf{x}_*)$, the positive semiorbit of $\mathbf{x}_*$.*

(ii) *The ω-limit set $\Omega(\mathbf{x}_*)$ is a periodic orbit at which φ is phase asymptotic [and Lyapunov stable].*

The phrase in square brackets may be included or omitted throughout.

Proof. Suppose the flow is phase asymptotic at $\Gamma_+(\mathbf{x}_*)$. Let ρ,η be as in the definition of phase asymptotic.

Choose $\mathbf{x}_1, \mathbf{x}_2 \in \Gamma_+(\mathbf{x}_*)$ such that $|\mathbf{x}_1 - \mathbf{x}_2| < \rho$ and $\mathbf{x}_2 = \varphi(t_1, \mathbf{x}_1)$ with $t_1 > \eta$. Since the flow is Lagrange stable at $\mathbf{x}_*$, we take $t_1 > \eta$. Since the flow is phase asymptotic at $\Gamma_+(\mathbf{x}_*)$, there exists a phase $h = h(\mathbf{x}_1, \mathbf{x}_2)$ with $|h| < \eta$ so that

$$\lim_{t\to\infty} d(\varphi(t, \mathbf{x}_1), \varphi(h, \varphi(t, \mathbf{x}_2)))$$
$$= \lim_{t\to\infty} d(\varphi(t, \mathbf{x}_1), \varphi(h + t_1 + t, \varphi(t, \mathbf{x}_1))) = 0. \quad (2.8)$$

Let $\omega := h + t_1 > 0$ and $\mathbf{x}_0 \in \Omega(\mathbf{x}_*)$. Then, there exists a sequence t_n, $t_n \to \infty$ as $n \to \infty$, such that $\lim_{n\to\infty} \varphi(t_n, \mathbf{x}_1) = \mathbf{x}_0$. With (2.8) this implies that

$$\lim_{n\to\infty} d(\varphi(t_n, \mathbf{x}_1), \varphi(\omega, \varphi(t_n, \mathbf{x}_1))) = 0$$

or equivalently $\mathbf{x}_0 = \varphi(\omega, \mathbf{x}_0)$. Hence, $\Gamma_+(\mathbf{x}_0)$ is a periodic orbit.

From Proposition 2.5 the flow is phase asymptotic at the orbit $\Gamma_+(\mathbf{x}_0) \subset \Omega(\mathbf{x}_*)$. Hence,

$$\lim_{t\to\infty} d(\varphi(t, \mathbf{x}_*), \Gamma_+(\mathbf{x}_0)) = 0. \quad (2.9)$$

To obtain a contradiction, suppose that $\mathbf{y}_1 \in \Omega(\mathbf{x}_*)\backslash\Gamma_+(\mathbf{x}_0)$. Then,

$$d(\mathbf{y}_1, \Gamma_+(\mathbf{x}_0)) > 0,$$

and there exists a sequence t_n, $t_n \to \infty$ as $n \to \infty$, such that

$$\lim_{n\to\infty} d(\varphi(t_n, \mathbf{x}_*), \mathbf{y}_1) = 0. \tag{2.10}$$

This is a contradiction, since an orbit cannot come arbitrarily close to $\Gamma_+(\mathbf{x}_0)$ as $t \to \infty$ and be bounded away from it. Hence, $\Omega(\mathbf{x}_*) = \Gamma_+(\mathbf{x}_0)$ is a periodic orbit, and the flow is phase asymptotic at $\Omega(\mathbf{x}_*)$. If the flow is Lyapunov stable at $\Gamma(\mathbf{x}_*)$, then from Proposition 2.5 the flow is Lyapunov stable at $\Omega_+(\mathbf{x}_*)$. This proves that (i) implies (ii).

Conversely, suppose the flow is asymptotic at $\Gamma_+(\mathbf{x}_0) = \Omega(\mathbf{x}_*)$. Then, from Proposition 2.5 the flow is asymptotic at $\Gamma_+(\mathbf{x}_*)$. If the flow is Lyapunov stable at $\Omega(\mathbf{x}_*)$, then from Proposition 2.5 the flow is Lyapunov stable at $\Gamma_+(\mathbf{x}_*)$. This proves that (ii) implies (i).

□

Chapter 3

Dynamics of Differential Equations in $\mathbb{R}^n$

In the previous chapter, we reviewed some results that can be used to establish the existence of an equilibrium or a periodic orbit from the attraction of its neighbors by a bounded orbit. This chapter presents a similar review of known results for dynamical systems associated with autonomous differential equations.

Results in this chapter are all well-known. However, the approach of Section 3.6 gives a new proof of the known Theorem 3.11 on the existence of a phase asymptotically stable periodic orbit. This approach plays an essential role in extending Theorem 3.11 to flows on invariant manifolds in Chapter 4.

Let $C^k(D \mapsto \mathbb{R}^m)$ denote the class of k-differentiable functions from D to $\mathbb{R}^m$, where D is an open subset of $\mathbb{R}^n$.

Suppose $\mathbf{f} \in C^1(D \mapsto \mathbb{R}^n)$, and let $\varphi(t, \mathbf{x}) = \mathbf{x}(t)$ be a solution of

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{f}(\mathbf{x}(t)) \tag{3.1}$$

where $\mathbf{x} \in D$ and $t \in \mathbb{R}$, such that $\varphi(t, \mathbf{x})$ exists for all $t \in \mathbb{R}$ and is uniquely determined by the given initial condition $\varphi(0, \mathbf{x}) = \mathbf{x}(0)$. Then, $\varphi(t, \mathbf{x})$ is a flow or dynamical system.

3.1 Planar Autonomous Systems

In this section, dynamical systems generated by a system of autonomous differential equations in the plane $\mathbb{R}^2$ are considered. In the plane, a periodic orbit can be detected using the Poincaré-Bendixson theorem, which states that any bounded solution which does not get close to any equilibria limits to a periodic orbit.

Suppose $P(x, y)$ and $Q(x, y)$ are continuously differentiable functions from $\mathbb{R}^2$ into $\mathbb{R}$. Consider the following system of equations

$$\begin{cases} \frac{dx}{dt} &= P(x, y), \\ \frac{dy}{dt} &= Q(x, y). \end{cases} \tag{3.2}$$

Theorem 3.1 (Poincaré-Bendixson Theorem). *Suppose $(x(t), y(t))$ is a bounded solution to (3.2) with the initial condition $(x(0), y(0)) = (x_0, y_0)$. If $\Omega(x_0, y_0)$ contains no equilibria, then $(x(t), y(t))$ limits to a periodic orbit. In particular, $\Omega(x_0, y_0)$ is a periodic orbit.*

Theorem 3.1 does not provide any information about the stability characteristics of the periodic orbit.

We give a brief outline of the proof. A detailed proof can be found in [12].

- A closed line segment l is a transversal for (3.2) if it contains no equilibria and no points where l is tangential to the corresponding vector field.
- Every non-equilibrium point is an interior point of a transversal.
- A transversal has a neighborhood such that every trajectory that intersects this neighborhood must cross the transversal. All crossing are in the same sense (see Figure 3.1).
- Choose a transversal l through a point $(x_1, y_1) \in \Omega(x_0, y_0)$. The trajectory enters a neighborhood of l infinitely many times, and so the trajectory crosses l infinitely many times.

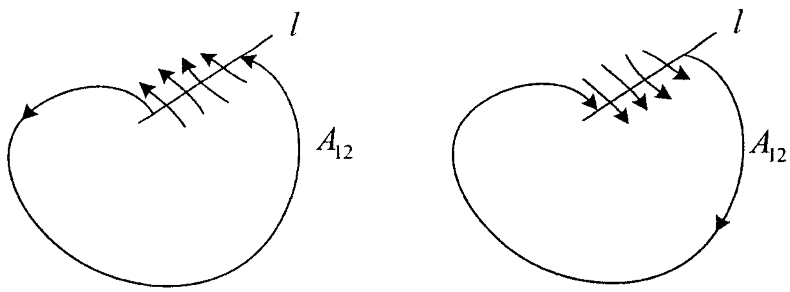

Figure 3.1: Solutions cross l in the same sense.

- The Jordan Curve Theorem implies that successive crossings are monotone on l, so it follows that the trajectory spirals to a periodic orbit.

3.2 Definitions of Stability

Definitions of stability are now introduced for solutions of the following nonautonomous differential equation

$$\frac{d\mathbf{x}}{dt} = \mathbf{h}(t, \mathbf{x}). \tag{3.3}$$

Definition 3.2. Suppose $\mathbf{x}(t)$ and $\mathbf{z}(t)$ are solutions of (3.3), where $\mathbf{x}(t)$ is defined for all $t \geq t_0$, where t_0 is some fixed number.

(a) The solution $\mathbf{x}(t)$ is *stable* with respect to the interval $[t_0, \infty]$, if for each $\epsilon > 0$ there is a $\delta > 0$ such that $|\mathbf{x}(0) - \mathbf{z}(0)| < \delta$ implies $\mathbf{z}(t)$ exists and satisfies $|\mathbf{x}(t) - \mathbf{z}(t)| < \epsilon$ for $t \geq t_0$.

(b) The solution $\mathbf{x}(t)$ is *asymptotically stable* if it is stable and

$$\lim_{t \to \infty} |\mathbf{x}(t) - \mathbf{z}(t)| = 0$$

whenever $|\mathbf{x}(0) - \mathbf{z}(0)|$ is sufficiently small.

(c) The solution $\mathbf{x}(t)$ is *uniformly stable* if for each $\epsilon > 0$ there exists a $\delta > 0$ such that if $|\mathbf{x}(t_1) - \mathbf{z}(t_1)| < \delta$ for some $t_1 \geq t_0$, then $|\mathbf{x}(t) - \mathbf{z}(t)| < \epsilon$ for

$t \geq t_1$.

(d) The solution $\mathbf{x}(t)$ is *uniformly asymptotically stable* if it is uniformly stable and there is a $\delta_0 > 0$ such that for every $\epsilon > 0$ there exists a $T > 0$ such that if $|\mathbf{x}(t_1) - \mathbf{z}(t_1)| < \delta_0$ for some $t_1 \geq t_0$ then $|\mathbf{x}(t) - \mathbf{z}(t)| < \epsilon$ for $t \geq t_1 + T$.

Suppose $\mathbf{x}(t)$ and $\mathbf{y}(t) + \mathbf{x}(t)$ are solutions of (3.1). Then $\mathbf{y}(t)$ satisfies

$$\frac{d\mathbf{y}(t)}{dt} = \mathbf{f}(\mathbf{y}(t) + \mathbf{x}(t)) - \mathbf{f}(\mathbf{x}(t)). \tag{3.4}$$

If $|\mathbf{y}(t)|$ is sufficiently small and $\mathbf{f} \in C^1$, then

$$\mathbf{f}(\mathbf{y} + \mathbf{x}) - \mathbf{f}(\mathbf{x}) \approx \frac{\partial \mathbf{f}}{\partial x}(\mathbf{x})\mathbf{y},$$

and so (3.4) is approximately

$$\frac{d\mathbf{y}}{dt} = \frac{\partial \mathbf{f}}{\partial x}(\mathbf{x}(t))\mathbf{y} \tag{3.5}$$

where $\frac{\partial \mathbf{f}}{\partial x}(\mathbf{x})$ is the Jacobian matrix of $\mathbf{f}$. We are thus led to believe that under certain circumstances the stability of $\mathbf{x}(t)$ can be reduced to the stability of the zero solution of the non-autonomous linear equation (3.5).

Let $A \in C(\mathbb{R} \mapsto M_{n \times n})$ where $M_{n \times n}$ is the set of $n \times n$ real matrices. We know that each solution of the linear system

$$\frac{d\mathbf{y}}{dt} = A(t)\mathbf{y} \tag{3.6}$$

is stable, uniformly stable, asymptotically stable, or uniformly asymptotically stable if and only if the zero solution of (3.6) is stable, uniformly stable, asymptotically stable, or uniformly asymptotically stable, respectively. Thus, the equation (3.6) is said to be stable, uniformly stable, asymptotically stable, or uniformly asymptotically stable if the zero solution to the equation has the same property, respectively.

Also, we have the following equivalent conditions for stability.

Proposition 3.3. *Let $Y(t)$ be a fundamental matrix solution of* (3.6). *Then the equation* (3.6) *is*

(i) *stable if and only if there exists a $K > 0$ such that*

$$|Y(t)| \leq K \quad \text{for all } t \geq t_0.$$

(ii) *uniformly stable if and only if there exists a $K > 0$ such that*

$$|Y(t)Y^{-1}(s)| \leq K \quad \text{for all } t_0 \leq s \leq t < \infty.$$

(iii) *asymptotically stable if and only if*

$$\lim_{t\to\infty} |Y(t)| = 0.$$

(iv) *uniformly asymptotically stable if and only if there exist positive constants K and α such that*

$$|Y(t)Y^{-1}(s)| \leq Ke^{-\alpha(t-s)} \quad \text{for all } t_0 \leq s \leq t < \infty.$$

A proof of the above proposition can be found in Chapter 3, [1].

Consider the autonomous time-independent linear system,

$$\frac{d\mathbf{y}}{dt} = A\mathbf{y} \tag{3.7}$$

where $A \in M_{n \times n}$.

Proposition 3.4. *Let $Re(\lambda)$ denote the real part of the eigenvalue λ of A. The equation* (3.7) *is uniformly asymptotically stable if and only if $Re(\lambda) < 0$ for all λ.*

For a proof, see [3].

3.3 Stability of Equilibria

The linearization of (3.1) with respect to a solution $\mathbf{x}(t)$ is

$$\frac{d\mathbf{y}}{dt} = \frac{\partial \mathbf{f}}{\partial x}(\mathbf{x}(t))\mathbf{y}. \tag{3.8}$$

An equilibrium $\mathbf{x}_0$ of (3.1) is said to be *hyperbolic* if none of the eigenvalues of $\frac{\partial \mathbf{f}}{\partial x}(\mathbf{x}_0)$ have zero real parts. It is said to be *stable hyperbolic* if all the eigenvalues of $\frac{\partial \mathbf{f}}{\partial x}(\mathbf{x}_0)$ have negative real parts. As a consequence, we have the following proposition.

Proposition 3.5. *An equilibrium* $\mathbf{x}_0$ *of* (3.1) *is stable hyperbolic if and only if the linearization* (3.8)*, with respect to* $\mathbf{x}(t) = \mathbf{x}_0$*, is uniformly asymptotically stable.*

In principle, the problem of finding all stable equilibria can be solved by the following procedure. First, solve $\mathbf{f}(\mathbf{x}) = 0$ for all the equilibria, and then calculate all the eigenvalues of the linearized system. However, in practice, this can be quite difficult to achieve, since the equations may not be algebraic or the domain may be unbounded.

3.4 The Existence of Stable Equilibria

By Proposition 3.3(iv), the equation (3.8) is uniformly asymptotically stable if and only if there exist positive constants K and α such that

$$\left|\frac{\partial \varphi}{\partial x}(t, \mathbf{x})\right| \leq Ke^{-\alpha t} \tag{3.9}$$

for $\mathbf{x} \in \Gamma_+(\mathbf{x}(0))$ and $t \geq 0$, since $Y(t)Y^{-1}(s) = \frac{\partial \varphi}{\partial x}(t-s, \mathbf{x})$ when $\mathbf{x} = \varphi(s, \mathbf{x}_0)$ and $Y(t)$ is a fundamental matrix solution of (3.8). The matrix valued function $\frac{\partial \varphi}{\partial x}(t, \mathbf{x}(0))$ is also fundamental solution of (3.8) with $\frac{\partial \varphi}{\partial x}(0, \mathbf{x}(0)) = I$.

Proposition 3.6. *Suppose the flow is Lagrange stable at* $\mathbf{x}_*$ *and* (3.9) *holds for* $\mathbf{x} \in \Gamma_+(\mathbf{x}_*)$ *and* $t \geq 0$*. For any* $L > K$ *and* $0 < \gamma < \alpha$*, there exist an* *open neighborhood U of $\overline{\Gamma_+(\mathbf{x}_*)}$ and a $\delta > 0$ such that $\mathbf{y} \in U$ and $|\mathbf{y} - \mathbf{z}| < \delta$ implies that $|\varphi(t, \mathbf{y}) - \varphi(t, \mathbf{z})| \leq Le^{-\gamma t}|\mathbf{y} - \mathbf{z}|$ for all $t \geq 0$. In particular, the flow is asymptotic and Lyapunov stable at U.*

Proof. Since the flow $\varphi(t, x)$ is a C^1 function with respect to $\mathbf{x}$, (3.9) is satisfied if $t \geq 0$ and $\mathbf{x} \in \overline{\Gamma_+(\mathbf{x}_*)}$. Choose a constant $T > 0$ so that $Le^{-(\alpha-\gamma)T} < 1$. Since $|\frac{\partial \varphi}{\partial x}(t, \mathbf{y})e^{\alpha t}|$ is uniformly continuous with respect to $t \in [0, T]$, $\mathbf{y} \in \overline{B_\eta(\Gamma_+(\mathbf{x}_0))}$ if $\eta > 0$, it follows that there exists $\delta > 0$ such that $|\mathbf{y} - \mathbf{x}| < \delta$ implies

$$\left|\frac{\partial \varphi}{\partial x}(t, \mathbf{y})\right| \leq Le^{-\alpha t}. \tag{3.10}$$

Let $t \geq 0$, $\mathbf{x} \in \Gamma_+(\mathbf{x}_*)$, and $\mathbf{x}_k := \varphi(kT, \mathbf{x})$ for $k = 0, 1, 2, \ldots$. There exist an $s \in [0, T]$ and a positive integer k so that $t = s + Tk$. By the chain rule

$$\begin{aligned}
\tfrac{\partial}{\partial x}\varphi(t, \mathbf{x}) &= \tfrac{\partial}{\partial x}\varphi(s + kT, \mathbf{x}) \\
&= \tfrac{\partial}{\partial x}\varphi(s + (k-1)T, \varphi(T, \mathbf{x})) \\
&= \tfrac{\partial \varphi}{\partial x}(s + (k-1)T, \mathbf{x}_1)\tfrac{\partial \varphi}{\partial x}(T, \mathbf{x}) \\
&\;\vdots \\
&= \tfrac{\partial \varphi}{\partial x}(T + s, \mathbf{x}_{k-1}) \ldots \tfrac{\partial \varphi}{\partial x}(T, \mathbf{x}_1)\tfrac{\partial \varphi}{\partial x}(T, \mathbf{x}) \\
&= \tfrac{\partial \varphi}{\partial x}(s, \mathbf{x}_k)\tfrac{\partial \varphi}{\partial x}(T, \mathbf{x}_{k-1}) \ldots \tfrac{\partial \varphi}{\partial x}(T, \mathbf{x}_1)\tfrac{\partial \varphi}{\partial x}(T, \mathbf{x}).
\end{aligned} \tag{3.11}$$

Recall that $Le^{-(\alpha-\gamma)T} < 1$, therefore from (3.10) and (3.11),

$$\begin{aligned}
\left|\tfrac{\partial}{\partial x}\varphi(t, \mathbf{x})\right| &\leq \left|\tfrac{\partial}{\partial x}\varphi(s, \mathbf{x}_k)\right|\left|\tfrac{\partial}{\partial x}\varphi(T, \mathbf{x}_{k-1})\right| \ldots \left|\tfrac{\partial}{\partial x}\varphi(T, \mathbf{x})\right| \\
&\leq Le^{-\alpha s}(Le^{-\alpha T})^k \\
&\leq Le^{-\gamma s}e^{-\gamma kT} \\
&= Le^{-\gamma t}.
\end{aligned} \tag{3.12}$$

Let $U := B_\delta(\Gamma_+(\mathbf{x}_*))$ and $\mathbf{x}_0 \in \Gamma_+(\mathbf{x}_*)$. Suppose $|\mathbf{y} - \mathbf{x}_0| < \delta$, $|\mathbf{y} - \mathbf{z}| < \delta$, and $\mathbf{x}(\lambda) := (1 - \lambda)\mathbf{z} + \lambda\mathbf{y}$ for $0 \leq \lambda \leq 1$. Immediately, we have that $\frac{d}{d\lambda}\mathbf{x}(\lambda) = \mathbf{y} - \mathbf{z}$. If the line segment $\mathbf{x}(\lambda)$, $0 \leq \lambda \leq 1$ is evolved under the flow φ, then the curve $\varphi(t, \mathbf{x}(\lambda))$, $0 \leq \lambda \leq 1$ will join the two points $\varphi(t, \mathbf{y})$ and $\varphi(t, \mathbf{z})$. Further, the length of the this curve, $\int_0^1 \left|\frac{d}{d\lambda}\varphi(t, \mathbf{x}(\lambda))\right| d\lambda$ will be no less than the distance from $\varphi(t, \mathbf{y})$ to $\varphi(t, \mathbf{z})$. Then

$$
\begin{aligned}
|\varphi(t, \mathbf{y}) - \varphi(t, \mathbf{z})| &\leq \int_0^1 \left| \frac{d}{d\lambda} \varphi(t, \mathbf{x}(\lambda)) \right| d\lambda \\
&\leq \int_0^1 \left| \frac{\partial \varphi}{\partial x}(t, \mathbf{x}(\lambda))(\mathbf{y} - \mathbf{z}) \right| d\lambda \\
&\leq \int_0^1 L e^{-\beta t} |\mathbf{y} - \mathbf{z}| d\lambda \\
&\leq L e^{-\beta t} |\mathbf{y} - \mathbf{z}| \quad \text{for } t \geq 0.
\end{aligned}
$$

where the second last inequality follows from (3.12). This implies that the flow is asymptotic and Lyapunov stable at U. □

The following theorem allows us to establish the existence of a stable hyperbolic equilibrium.

Theorem 3.7. *Suppose the flow is Lagrange stable at a solution of* (3.1), $\mathbf{x}(t)$. *Then* $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{x}_0$, *where* $\mathbf{x}_0$ *is a hyperbolic and stable equilibrium, if and only if the linearization* (3.8) *of* (3.1) *with respect to* $\mathbf{x}(t)$ *is uniformly asymptotically stable.*

Proof. (I) Suppose $\mathbf{x}_0$ is a stable hyperbolic equilibrium. From the proof of Proposition 3.6, there exist positive constants L and γ and an open neighborhood U of $\mathbf{x}_0$, such that

$$
\left| \frac{\partial}{\partial x} \varphi(t, \mathbf{x}) \right| \leq L e^{-\gamma t} \tag{3.13}
$$

for $t \geq 0$ and $\mathbf{x} \in U$. We now show that (3.13) holds for $\mathbf{x} \in \Gamma_+(\mathbf{x}(0))$ and $t \geq 0$. If $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{x}_0$, then $\mathbf{x}(t)$ in U for sufficiently large t, which implies constants γ and L can be chosen so that (3.13) is valid for $\mathbf{x} \in \Gamma_+(\mathbf{x}_0)$ and $t \geq 0$. Thus, the equation (3.8) is uniformly asymptotically stable.

Conversely, suppose the linearization (3.8) of the flow at $\mathbf{x}(t)$ is uniformly asymptotically stable. Then, from Proposition (3.6), there exist positive constants δ,L,γ and an open neighborhood U of $\overline{\Gamma_+(\mathbf{x}(0))}$ so that $\mathbf{z} \in U$ and

$|\mathbf{z} - \mathbf{y}| < \delta$ imply that

$$|\varphi(t, \mathbf{y}) - \varphi(t, \mathbf{z})| \leq Le^{-\gamma t}|\mathbf{y} - \mathbf{z}| \tag{3.14}$$

for $t \geq 0$. It follows from Theorem 2.6 that $\mathbf{x}_0$ is an asymptotic equilibrium. Since (3.8) is uniformly asymptotically stable, if we replace $\mathbf{x}(t)$ with $\mathbf{x}_0$, the equation (3.8) will still be uniformly asymptotically stable. Thus, by Proposition 3.5, $\mathbf{x}_0$ is stable hyperbolic. □

Remark 1. The alternative proof of Theorem 3.7, given below, is simpler than the preceding one. However, it does not have an apparent analogue for periodic orbits. The first proof has been given as motivation of the approach to periodic orbits based on the analysis developed in Chapter 2 on the consequences of an orbit attracting its neighbors asymptotically.

Proof. (II) Let $\mathbf{y}(t) := d\mathbf{x}(t)/dt = \mathbf{f}(\mathbf{x}(t))$. Then $\mathbf{y}(t)$ is a solution of (3.8). Suppose the linearization (3.8) of the flow is uniformly asymptotically stable at $\mathbf{x}(t)$. Then

$$\lim_{t\to\infty} \mathbf{f}(\mathbf{x}(t)) = \lim_{t\to\infty} \mathbf{y}(t) = 0. \tag{3.15}$$

Take $\mathbf{x}_0 \in \Omega(\mathbf{x}(0))$ (which is nonempty by Lagrange stability of the flow at $\mathbf{x}_0$). Then there exists a sequence $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$, such that $\lim_{n\to\infty} \varphi(t_n, \mathbf{x}(0)) = \mathbf{x}_0$, so by (3.15), $\lim_{n\to\infty} \mathbf{f}(\varphi(t_n, \mathbf{x}(0))) = 0$. Hence, by the continuity of $\mathbf{f}$, $\mathbf{x}_0$ is an equilibrium.

Since (3.8) is uniformly asymptotically stable, equation (3.9) is valid for $t \geq 0$ and $\mathbf{x} \in \Gamma_+(\mathbf{x}(0))$. Moreover, by continuity of $\partial\varphi/\partial x$, (3.9) is also valid for $\mathbf{x} = \mathbf{x}_0$ and $t \geq 0$. Hence, $\mathbf{x}_0$ is a stable hyperbolic equilibrium. In particular, $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{x}_0$. Further, $\mathbf{x}_0$ being a stable hyperbolic equilibrium implies that there exists an open neighborhood V of $\mathbf{x}_0$ such that any solution of (3.1) with initial condition in V limits to $\mathbf{x}_0$. Consequently, there are no equilibria different from $\mathbf{x}_0$ in V. Therefore, $\Omega(\mathbf{x}(0))$ being a connected set of only equilibria, is a single equilibrium. □

3.5 Stability of Periodic Orbits

In this section, we will review sufficient conditions for a periodic orbit to be stable.

Let $\mathbf{p}(t)$ be a non-trivial periodic solution of (3.1). The linearization about $\mathbf{p}(t)$ is given by

$$\frac{d\mathbf{y}}{dt} = \frac{\partial \mathbf{f}}{\partial x}(\mathbf{p}(t))\mathbf{y}. \tag{3.16}$$

By Floquet's Theorem, a fundamental matrix solution $Y(t)$ of the linear periodic system (3.16) can be expressed as $Y(t) = P(t)e^{Lt}$, where $P(t)$ has the same period as $\mathbf{p}(t)$ and $L \in M_{n\times n}$. The eigenvalues of L are called the *characteristic exponents* or *Floquet exponents* of (3.16).

Theorem 3.8. *Let* $\mathbf{p}(t)$ *be a non-trivial periodic solution of* (3.1). *If* 0 *is a simple characteristic exponent of the linearization* (3.16) *and the other* $n-1$ *characteristic exponents have real part strictly less than zero, then the periodic solution* $\mathbf{p}(t)$ *is orbitally phase asymptotically stable.*

For a proof of the above result see page 82, [1].

Theorem 3.9 (Poincaré's Stability Criterion). *Suppose* $\mathbf{p}(t)$ *is an* ω*-periodic solution of* (3.1$\int_0^{\omega}$ *When* $n = 2$, $\mathbf{p}(t)$ *is orbitally asymptotically stable if*

$$\int_0^{\omega} \operatorname{div} \mathbf{f}(\mathbf{p}(t))dt < 0.$$

Poincaré's Stability Criterion was extended to higher dimensions by Muldowney in [10] as follows:

Theorem 3.10. *A sufficient condition for* $\mathbf{p}(t)$ *to be orbitally asymptotically stable is the linear system*

$$\frac{d\mathbf{y}}{dt} = \frac{\partial \mathbf{f}}{\partial x}^{[2]}(\mathbf{p}(t))\mathbf{y}$$

being asymptotically stable.

For an $n \times n$ matrix A, the $\binom{n}{k} \times \binom{n}{k}$ matrix $A^{[2]}$ is the second additive compounded matrix which is defined in Appendix A.

3.6 The Existence of Stable Periodic Orbits

In this section, we provide an alternative geometric proof to Theorem 4.1(a) in [7], where it was shown that if the second compound of the variational equation is uniformly asymptotically stable, then any Lagrange stable orbit, which does not get close to any equilibria, limits to a phase asymptotically stable periodic orbit. The proof was carried out by studying an integral equation in a Banach space. Here, we take a different approach. It will be used extensively in Chapter 4, where we discuss flows on invariant manifolds.

The linear variational equation of (3.1) at $\Gamma_+(\mathbf{x}_0)$ is

$$\frac{d\mathbf{y}}{dt} = \frac{\partial \mathbf{f}}{\partial x}(\varphi(t, \mathbf{x}_0))\mathbf{y}, \tag{3.17}$$

and the second compound is

$$\frac{d\mathbf{z}}{dt} = \frac{\partial \mathbf{f}}{\partial x}^{[2]}(\varphi(t, \mathbf{x}_0))\mathbf{z}, \tag{3.18}$$

where $B^{[2]}$ denotes the second additive compound of a matrix B (see Appendix A for properties and definitions of compound matrices).

Theorem 3.11. *Suppose that the flow $\varphi(t, \mathbf{x})$ of (3.1) is Lagrange stable at $\mathbf{x}_0$, $\overline{\Gamma_+(\mathbf{x}_0)} \subset D$, the ω-limit set $\Omega(\mathbf{x}_0)$ contains no equilibria, and* (3.18) *is uniform asymptotically stable. Then,*

(i) *there exist positive constants K, ρ, γ and a bounded function $h : D \mapsto \mathbb{R}$ such that for all $\mathbf{y} \in D$, $t \geq 0$, $|\mathbf{y} - \mathbf{x}_0| < \rho$ implies*

$$|\varphi(t + h(\mathbf{y}), \mathbf{y}) - \varphi(t, \mathbf{x}_0)| \leq K|\mathbf{y} - \mathbf{x}_0|e^{-\gamma t}.$$

(ii) *$\Omega(\mathbf{x}_0)$ is a phase asymptotically stable periodic orbit.*

For convenience, we introduce the following notations. Define the *flow box* $F(\mathbf{z}, \eta, \delta) := \{\varphi(t, \mathbf{x}) : |t| \leq \eta,\ |\mathbf{x} - \mathbf{z}| \leq \delta,\ \mathbf{f}(\mathbf{z}) \cdot (\mathbf{x} - \mathbf{z}) = 0\}$ and put $\Pi(\mathbf{a}) := \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a} \cdot \mathbf{x} = 0\}$. The affine plane $\mathbf{z} + \Pi(\mathbf{a})$ is a *transverse section*

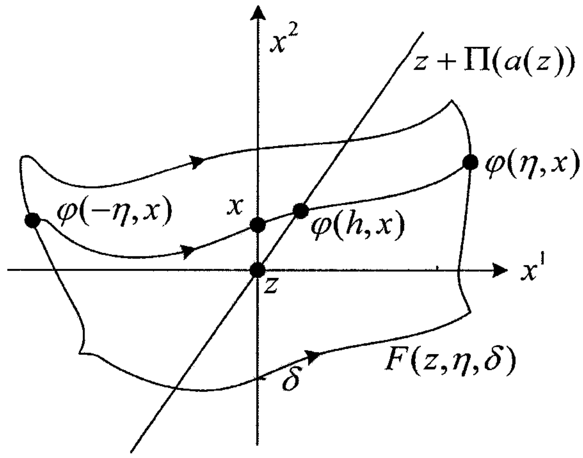

Figure 3.2: The flow box $F(\mathbf{z},\eta,\delta)$ and the affine plane $\mathbf{z}+\Pi(\mathbf{a}(\mathbf{z}))$.

of the flow box $F(\mathbf{z},\eta,\delta)$ if for each $\mathbf{x}\in\mathbf{z}+\Pi(\mathbf{f}(\mathbf{z}))\cap F(\mathbf{z},\eta,\delta)$, there exists an $h\in[-\eta,\eta]$ such that $\varphi(h,\mathbf{x})\in\mathbf{z}+\Pi(\mathbf{a})$.

To prove Theorem 3.11, we need the following lemma:

Lemma 3.12. *Suppose that* $\mathbf{a}:\Gamma_+(\mathbf{x}_0)\mapsto\mathbb{R}^n$ *and that*

(i) *the flow* $\varphi(t,\mathbf{x})$ *of (3.1) is Lagrange stable at* $\mathbf{x}_0$*;*

(ii) *the function* $\mathbf{f}(\mathbf{z})\neq 0$ *for* $\mathbf{z}\in\overline{\Gamma_+(\mathbf{x}_0)}$*. By compactness, we put* $l:=\inf_{\{\mathbf{z}\in\overline{\Gamma_+(\mathbf{x}_0)}\}}|\mathbf{f}(\mathbf{z})|>0$*;*

(iii) *there is a* $\zeta\in(0,1)$ *such that for* $\mathbf{z}\in\overline{\Gamma_+(\mathbf{x}_0)}$ *and* $0\neq\mathbf{a}(\mathbf{z})\in\mathbb{R}^n$*, we have*

$$\zeta|\mathbf{a}(\mathbf{z})|\,|\mathbf{f}(\mathbf{z})|\leq\mathbf{a}(\mathbf{z})\cdot\mathbf{f}(\mathbf{z}). \tag{3.19}$$

Let $\eta:\mathbb{R}^+\mapsto\mathbb{R}^+$ *be defined by* $\eta(\delta)=\frac{4\delta}{l\zeta}$*. Then,*

(a) *there exists a* $\delta_*>0$ *such that for* $0<\delta<\delta_*$*, the affine plane* $\mathbf{z}+\Pi(\mathbf{a}(\mathbf{z}))$ *is a transverse section of* $F(\mathbf{z},\eta(\delta),\delta)$ *for each* $\mathbf{z}\in\Gamma_+(\mathbf{x}_0)$*;*

(b) *there exists a constant* b *such that* $B_{\delta/2}(\mathbf{z})\subset F(\mathbf{z},\eta(\delta),\delta)\subset B_{\delta(1+4b)}(\mathbf{z})$*.*

Proof. (a) By condition (i) we have the uniform continuity of $\mathbf{f}$ on $\overline{\Gamma_+(\mathbf{x}_0)}$, which implies that there exists a constant $b > 0$ such that

$$\zeta bl/2 > |\mathbf{f}(\mathbf{z})| \tag{3.20}$$

for $\mathbf{z} \in \Gamma_+(\mathbf{x}_0)$. Without loss of generality, we assume that $\mathbf{z} = 0$. By condition (ii), $\mathbf{f}(\mathbf{z}) \neq 0$, so we can choose an orthogonal basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ of $\mathbb{R}^n$ with $\mathbf{e}_1 = \mathbf{f}(0)/|\mathbf{f}(0)|$. Thus, given an $\mathbf{x} \in \mathbb{R}^n$ we write $\mathbf{x} = (x_1, \mathbf{x}_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ where x_1 and $\mathbf{x}_2$ are the corresponding coordinates with respect to $\{\mathbf{e}_1\}$ and $\{\mathbf{e}_2, \ldots, \mathbf{e}_n\}$, and $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), \mathbf{f}_2(\mathbf{x}))$ similarly. Clearly, $(0, \mathbf{x}_2) \in \Pi(\mathbf{e}_1) = \Pi(\mathbf{f}(0))$. In addition,

$$\mathbf{f}(0) = (f_1(0), 0) \quad \text{and} \quad f_1(0) = |\mathbf{f}(0)| \geq l > 0. \tag{3.21}$$

By definition of the flow box and uniform continuity of $\mathbf{f}$, we can choose a sufficiently small δ_* such that $|\mathbf{f}(\mathbf{x}) - \mathbf{f}(0)|$ is arbitrary small for all $\mathbf{x} \in F(0, \eta(\delta_*), \delta_*)$. Also taking into account (3.20) and (3.21), we have a choice of δ_* such that

$$\zeta bl > f_1(\mathbf{x}) > l/2 \tag{3.22}$$

and

$$|\mathbf{f}_2(\mathbf{x})| < l\zeta/8. \tag{3.23}$$

Consequently, for any $\delta \in (0, \delta_*)$ and $\mathbf{x} \in \Pi(\mathbf{f}(0))$ with $|\mathbf{x}| < \delta$, we have

$$\begin{aligned} \mathbf{a}(0) \cdot \varphi(\eta(\delta), \mathbf{x}) &= a_1(0)\varphi_1(\eta(\delta), \mathbf{x}) + \mathbf{a}_2(0) \cdot \varphi_2(\eta(\delta), \mathbf{x}) \\ &\geq a_1(0)\tfrac{l}{2}\eta(\delta) - |\mathbf{a}_2(0)|(\delta + \tfrac{1}{8}\zeta l \eta(\delta)) \\ &\geq \tfrac{2a_1(0)\delta}{\zeta} - |\mathbf{a}_2(0)|\tfrac{3\delta}{2}. \end{aligned} \tag{3.24}$$

In addition, combining (3.19) and (3.21), we obtain $0 < a_1(0)$ and $|\mathbf{a}_2(0)| < a_1(0)/\zeta$, so (3.24) implies that

$$\mathbf{a}(0) \cdot \varphi(\eta(\delta), \mathbf{x}) > \frac{1}{2\zeta}\delta a^1(0) > 0. \tag{3.25}$$

Similarly,

$$\mathbf{a}(0) \cdot \varphi(-\eta(\delta, \mathbf{x}) < 0 \tag{3.26}$$

holds for $\mathbf{x} \in \Pi(\mathbf{f}(0))$ and $|\mathbf{x}| < \delta$. Therefore, by the intermediate value theorem, there exists $h \in (-\eta(\delta), \eta(\delta))$ such that $\mathbf{a}(0) \cdot \varphi(h, \mathbf{x}) = 0$, which in turn implies $\Pi(\mathbf{a}(0))$ is a transverse section of $F(0, \eta(\delta), \delta)$. □

Proof. (b) As in the proof of (a), without loss of generality we assume $\mathbf{z} = 0$. By (3.22), for $0 < \delta < \delta_*$, $\mathbf{x} \in \Pi(\mathbf{f}(0))$ with $|\mathbf{x}| < \delta$, and $|t| < \eta(\delta)$, we have

$$\begin{aligned} |\varphi(t, \mathbf{x})| &\leq \zeta b l \eta(\delta) + \delta \\ &\leq \delta(1 + 4b) \end{aligned}$$

i.e. $F(0, \eta(\delta), \delta) \subset B_{\delta(1+4b)}(0)$.

Put

$$\partial F(0, \eta(\delta)/4, \delta) = E_1 \cup E_2,$$

where

$$E_1 = \left\{ \varphi(\pm\frac{\eta(\delta)}{4}, \mathbf{x}) : \mathbf{x} \in \Pi(\mathbf{f}(0)),\ |\mathbf{x}| \leq \delta \right\},$$

and

$$E_2 = \left\{ \varphi(t, \mathbf{x}) : \mathbf{x} \in \Pi(\mathbf{f}(0)),\ |\mathbf{x}| = \delta,\ -\frac{\eta(\delta)}{4} \leq t \leq \frac{\eta(\delta)}{4} \right\}.$$

Take $\mathbf{y} \in \partial F(0, \eta(\delta)/4, \delta)$. If $\mathbf{y} \in E_1$, then by 3.22 $|y_1| \geq (l/2)\eta(\delta)/4 > \delta/2$.

Otherwise if $\mathbf{y} \in E_2$ with $\mathbf{y} = \varphi(t, \mathbf{x})$ and $\mathbf{x} \in \Pi(\mathbf{f}(0))$, then by (3.23) $|\mathbf{x}_2 - \mathbf{y}_2| < (\eta(\delta)/4)(l\zeta/8) = \delta/8$, and so

$$|\mathbf{y}_2 - 0| \geq |\mathbf{x}_2 - 0| - |\mathbf{x}_2 - \mathbf{y}_2| > \frac{7\delta}{8} > \frac{\delta}{2}.$$

Thus, $B_{\frac{\delta}{2}}(0) \subset F(0, \eta(\delta)/4, \delta)$. Therefore, $B_{\frac{\delta}{2}}(0) \subset F(0, \eta(\delta), \delta)$ and the result follows. □

We are now in a position to prove Theorem 3.11.

Proof. From Theorem B.3 with $m = 0$,

$$|Y(t)P_2Y^{-1}(s)| \leq Ce^{-\alpha(t-s)} \quad \text{for } t \geq s \geq 0, \tag{3.27}$$

where $Y(t)$ is the fundamental solution of (3.17) with $Y(0) = I$, P_2 is an $(n-1)$-dimensional projection matrix, and α, C are some positive constants. Notice, $Y(t) = \frac{\partial \varphi}{\partial x}(t, x_0)$. Since $Y(t)Y^{-1}(s)$ and $\frac{\partial \varphi}{\partial x}(t-s, \mathbf{x})$ coincide when $t = s$ and by uniqueness of solutions, we have

$$Y(t)Y^{-1}(s) = \frac{\partial \varphi}{\partial x}(t-s, \mathbf{x})$$

where $\mathbf{x} = \varphi(s, \mathbf{x}_0)$. Thus,

$$\begin{aligned} Y(t)P_2Y^{-1}(s) &= Y(t)Y^{-1}(s)Y(s)P_2Y^{-1}(s) \\ &= \frac{\partial \varphi}{\partial x}(t-s, \mathbf{x})Y(s)P_2Y^{-1}(s) \end{aligned}$$

where $\mathbf{x} = \varphi(s, \mathbf{x}_0)$ and $P_2(\mathbf{x}) = Y(s)P_2Y^{-1}(s)$ is a projection matrix since

$$\begin{aligned} (P_2(\mathbf{x}))^2 &:= Y(s)P_2Y^{-1}(s)Y(s)P_2Y^{-1}(s) \\ &= Y(s)P_2^2Y^{-1}(s) \\ &= Y(s)P_2Y^{-1}(s) \\ &= P_2(\mathbf{x}). \end{aligned}$$

Therefore, (3.27) is equivalent to

$$\left|\frac{\partial \varphi}{\partial x}(t, \mathbf{x})P_2(\mathbf{x})\right| \leq Ce^{-\alpha t} \quad \text{for } t \geq 0$$

with $t - s$ replaced by t.

By continuity of $P_2(\mathbf{x})$ and compactness of $[0, T]$, $\overline{B_\eta(\Gamma_+(\mathbf{x}_0))}$, and $\overline{\Gamma_+(\mathbf{x}_0)}$, $\frac{\partial \varphi}{\partial x}(t, \mathbf{y})P_2(\mathbf{x})e^{\alpha t}$ is uniformly continuous with respect to $t \in [0, T]$, $\mathbf{x} \in \overline{\Gamma_+(\mathbf{x}_0)}$, and $\mathbf{y} \in \overline{B_\eta(\Gamma_+(\mathbf{x}_0))}$. It follows that there exists $\kappa > 0$ such that $|\mathbf{y} - \mathbf{x}| < \kappa$

implies

$$\left|\frac{\partial \varphi}{\partial x}(t, \mathbf{y}) P_2(\mathbf{x})\right| \leq 2Ce^{-\alpha t}. \tag{3.28}$$

The range of the projection matrix $P_2(\mathbf{x})$ for each $\mathbf{x} \in \Gamma_+(\mathbf{x}_0)$ is a plane. Take a non-zero normal vector $\mathbf{a}(\mathbf{x})$ to this plane. From the proof of Theorem B.3, the angle between $\Pi(\mathbf{f}(\mathbf{x}))$ and $\Pi(\mathbf{a}(\mathbf{x}))$ is bounded away from $\pm\pi/2$, which implies that there exists a constant ζ, $0 < \zeta < 1$, such that $\zeta|\mathbf{a}(\mathbf{x})||\mathbf{f}(\mathbf{x})| < \mathbf{f}(\mathbf{x}) \cdot \mathbf{a}(\mathbf{x})$. By condition (ii), $\mathbf{f}(\mathbf{x}) \neq 0$ for $\mathbf{x} \in \overline{\Gamma_+(\mathbf{x}_0)}$. Also, the flow $\varphi(t, \mathbf{x})$ is Lagrange stable at $\mathbf{x}_0$. Thus, the conditions of Lemma 3.12 are satisfied. Let δ_* the sufficiently small number we chose as in the lemma and l, b some constants in the corresponding proof.

Let $\mathbf{x}_k := \varphi(Tk, \mathbf{x}_0)$ for $k = 0, 1, \ldots$. Suppose $|\mathbf{u} - \mathbf{x}_k| < \kappa$ and $(\mathbf{u} - \mathbf{x}_k) = P_2(\mathbf{x}_k)(\mathbf{u} - \mathbf{x}_k) \in \Pi(\mathbf{a}(\mathbf{x}_k))($ immediately, $(\mathbf{u} - \mathbf{x}_k) = P_2(\mathbf{x}_k)(\mathbf{u} - \mathbf{x}_k))$. Define $\mathbf{x}(\lambda) := \mathbf{x}_k + \lambda(\mathbf{u} - \mathbf{x}_k)$ for $0 \leq \lambda \leq 1$. Then, $d\mathbf{x}(\lambda)/d\lambda = \mathbf{u} - \mathbf{x}_k$, and

$$\begin{aligned} \tfrac{d}{d\lambda}\varphi(t, \mathbf{x}(\lambda)) &= \tfrac{\partial \varphi}{\partial x}(t, \mathbf{x}(\lambda))(\mathbf{u} - \mathbf{x}_k) \\ &= \tfrac{\partial \varphi}{\partial x}(t, \mathbf{x}(\lambda)) P(\mathbf{x}_k)(\mathbf{u} - \mathbf{x}_k). \end{aligned} \tag{3.29}$$

So (3.28) and (3.29) imply that

$$\begin{aligned} |\varphi(t, \mathbf{u}) - \varphi(t, \mathbf{x}_k)| &\leq \int_0^1 \left|\tfrac{d}{d\lambda}\varphi(t, \mathbf{x}(\lambda))\right| d\lambda \\ &= \int_0^1 \left|\tfrac{\partial \varphi}{\partial x}(t, \mathbf{x}(\lambda)) P(\mathbf{x}_k)(\mathbf{u} - \mathbf{x}_k)\right| d\lambda \\ &\leq \int_0^1 2Ce^{-\alpha t} |\mathbf{u} - \mathbf{x}_k| \, d\lambda \\ &\leq 2Ce^{-\alpha t}|\mathbf{u} - \mathbf{x}_k| \quad \text{for } 0 \leq t \leq T. \end{aligned} \tag{3.30}$$

We choose κ sufficiently small so that $\kappa < \delta_*$ and (3.28) holds. Take T large enough so that $4Ce^{-\alpha T}(1 + 4b) < 1/2$. We prove the following statement by induction:

Statement (S): If $|\mathbf{y} - \mathbf{x}_0| < \rho := \kappa/(2 + 8b)$, then $\varphi\left(kT + \sum_{i=0}^{k} h_i, \mathbf{y}\right) \in \Pi(\mathbf{a}(\mathbf{x}_k)) + \mathbf{x}_k$, $\left|\varphi\left(kT + \sum_{i=0}^{k} h_i, \mathbf{y}\right) - \mathbf{x}_k\right| < 2^{-k}|\mathbf{y} - \mathbf{x}_0|$, and $|h_k| \leq \frac{8}{\zeta l 2^k}|\mathbf{y} - \mathbf{x}_0|$ for all $k = 0, 1, \ldots$.

Base Step: By choice of κ and ρ, $|\mathbf{y} - \mathbf{x}_0| < \rho$ implies $\delta_0 := 2|\mathbf{y} - \mathbf{x}_k| < \delta_*$.

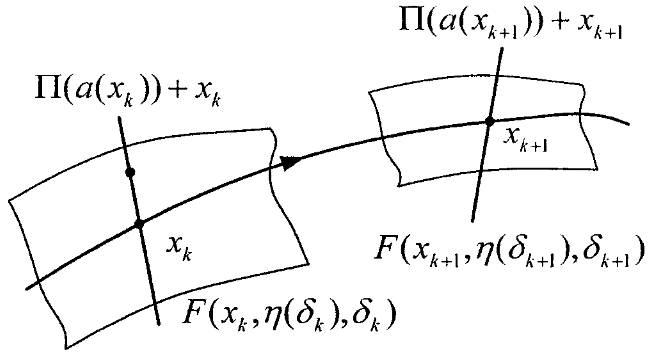

Figure 3.3: An induction step.

So, by Lemma 3.12(a) with $\delta = \delta_0$ there exists an m_0 with $|m_0| < \eta(\delta)$ such that $\varphi(m_0, \mathbf{y}) \in \Pi(\mathbf{f}(\mathbf{x}_0)) + \mathbf{x}_0$ and $\varphi(m_0, \mathbf{y}) \in F(\mathbf{x}_0, \eta(\delta_0), \delta_0)$. Further, Lemma 3.12(a) implies that there exists an ξ_0 satisfying $|m_0 + \xi_0| \leq 2\eta(\delta_0) = 16|\mathbf{y} - \mathbf{x}_0|/l\zeta$ so that $|\varphi(m_0 + \xi_0, \mathbf{y}) - \mathbf{x}_0| < 2(1 + 4b)|\mathbf{y} - \mathbf{x}_0| < \kappa$ and $\varphi(m_0 + \xi_0, \mathbf{y}) \in \Pi(\mathbf{a}(\mathbf{x}_0)) + \mathbf{x}_0$. Then, the Statement (S) holds for $k = 0$ with $h_0 := m_0 + \xi_0$.

Inductive Step: Suppose that Statement (S) is true for arbitrary positive integer k. Then, by (3.30) with $t = T$, $\mathbf{u} = \varphi\left(kT + \sum_{i=0}^{k} h_i, \mathbf{y}\right)$ combined with the fact that $\mathbf{x}_{k+1} = \varphi(T(k+1), \mathbf{x}_0) = \varphi(T, \mathbf{x}_k)$, we have the following

$$\begin{aligned}\left|\varphi\left(T, \varphi\left(kT + \sum_{i=0}^{k} h_i, \mathbf{y}\right)\right) - \varphi(T, \mathbf{x}_k)\right| &= \\ \left|\varphi\left((k+1)T + \sum_{i=0}^{k} h_i, \mathbf{y}\right) - \mathbf{x}_{k+1}\right| & \\ < 2Ce^{-\alpha T}\left|\varphi\left(kT + \sum_{i=0}^{k} h_i, \mathbf{y}\right) - \mathbf{x}_k\right|. & \qquad (3.31)\end{aligned}$$

We put $\delta_{k+1} = 4Ce^{-\alpha T}|\varphi(kT + \sum_{i=0}^{k} h_i, \mathbf{y}) - \mathbf{x}_k|$, they by Lemma 3.12 there exists an h_{k+1} (see Base Step for details) satisfying

$$\begin{aligned}|h_{k+1}| &\leq 2\eta(\delta_{k+1}) \\ &\leq \tfrac{16}{\zeta l 2^{k+1}}|\mathbf{y} - \mathbf{x}_0|.\end{aligned} \qquad (3.32)$$

so that

$$\begin{aligned}
&\left|\varphi\left((k+1)T+\sum_{i=0}^{k+1} h_i, \mathbf{y}\right)-\mathbf{x}_{k+1}\right| \\
&\leq 4Ce^{-\alpha T}(1+4b)\left|\varphi\left(kT+\textstyle\sum_{i=0}^{k} h_i, \mathbf{y}\right)-\mathbf{x}_k\right| \\
&\leq \tfrac{1}{2}\left|\varphi\left(kT+\textstyle\sum_{i=0}^{k} h_i, \mathbf{y}\right)-\mathbf{x}_k\right| \\
&\leq 2^{-(k+1)}|\mathbf{y}-\mathbf{x}_0|
\end{aligned} \tag{3.33}$$

and $\varphi\left((k+1)T+\sum_{i=0}^{k+1} h_i, \mathbf{y}\right) \in \Pi(\mathbf{a}(\mathbf{x}_{k+1}))+\mathbf{x}_{k+1}$. The Statement (S) is true for $k+1$, which concludes the induction proof.

From (3.30),

$$\left|\varphi\left(kT+\sum_{i=0}^{k} h_i+t, \mathbf{y}\right)-\varphi(t, \mathbf{x}_k)\right| \leq 2Ce^{-\alpha t}\left|\varphi\left(kT+\sum_{i=0}^{k} h_i, \mathbf{y}\right)-\mathbf{x}_k\right|$$

for $0 \leq t \leq T$, or equivalently,

$$|\varphi(\sum_{i=0}^{k} h_i+t, \mathbf{y})-\varphi(t, \mathbf{x}_0)| \leq 2Ce^{-\alpha(t-kT)}|\varphi(kT+\sum_{i=0}^{k} h_i, \mathbf{y})-\varphi(kT, \mathbf{x}_0)|$$

for $kT \leq t < (k+1)T$. Combining with (3.33), we have

$$|\varphi(\sum_{i=0}^{k} h_i+t, \mathbf{y})-\varphi(t, \mathbf{x}_0)| \leq 2Ce^{-\alpha(t-kT)}2^{-k}|\mathbf{y}-\mathbf{x}_0| \tag{3.34}$$

In addition, if $kT \leq t < (k+1)T$, then

$$\exp(-\alpha(t-kT)) \leq 1 \quad \text{for } \alpha > 0 \tag{3.35}$$

and

$$\begin{aligned}
2^{-k} &= \exp(\ln(2))\exp(-\tfrac{\ln(2)(k+1)T}{T}) \\
&\leq \exp(\ln(2))\exp(-\tfrac{\ln(2)t}{T}).
\end{aligned} \tag{3.36}$$

The above inequalities (3.34), (3.35), and (3.36) yield that

$$|\varphi(\sum_{i=0}^{k} h_i+t, \mathbf{y})-\varphi(t, \mathbf{x}_0)| \leq 2C\exp(\ln(2))\exp(-\frac{\ln(2)t}{T})|\mathbf{y}-\mathbf{x}_0|. \tag{3.37}$$

Let $h := \sum_{i=0}^{\infty} h_i$, so that by (3.32) $|h| \leq \frac{32}{\zeta l}|\mathbf{y} - \mathbf{x}_0|$. Combined with (3.33), we obtain

$$\begin{aligned} \left|\sum_{i=k+1}^{\infty} h_i\right| \zeta l b &\leq 16b2^{-k}|\mathbf{y} - \mathbf{x}_0| \\ &\leq 16b\exp(\ln(2))\exp(-\tfrac{\ln(2)t}{T})|\mathbf{y} - \mathbf{x}_0|. \end{aligned} \tag{3.38}$$

Hence, (3.37) and (3.38) imply that

$$\begin{aligned} &|\varphi(h+t,\mathbf{y}) - \varphi(t,\mathbf{x}_0)| \\ &\leq \left|\varphi(h+t,\mathbf{y}) - \varphi(t + \textstyle\sum_{i=0}^{k} h_i, \mathbf{y})\right| + \left|\varphi(t + \textstyle\sum_{i=0}^{k} h_i, \mathbf{y}) - \varphi(t,\mathbf{x}_0)\right| \\ &\leq \left|\textstyle\sum_{i=k+1}^{\infty} h_i\right| \zeta l b + 2C\exp(\ln(2))\exp(-\tfrac{\ln(2)t}{T})|\mathbf{y} - \mathbf{x}_0| \\ &\leq (16b + 2C)\exp(\ln(2))\exp(-\tfrac{\ln(2)t}{T})|\mathbf{y} - \mathbf{x}_0|. \end{aligned}$$

Let $\gamma := \ln(2)/T$ and $K := (16b + 2C)\exp(\ln(2))$. Then, Theorem 3.11(i) follows. In turn the statement(i) implies that the flow is phase asymptotic at $\Gamma_+(\mathbf{x}_0)$, so, from Theorem 2.7, the ω-limit set $\Omega(\mathbf{x}_0)$ is a phase asymptotically stable periodic orbit, which proves (ii). □

Chapter 4

Dynamics of Differential Equations on Invariant Manifolds

In this chapter, an autonomous differential equation with an invariant manifold is considered. An invariant manifold is a smooth surface that is invariant with respect to the dynamical system. The restriction of the dynamical system to the invariant manifold is also a dynamical system. The presence of an invariant manifold, in principle, simplifies the study of the dynamics. The traditional approach to these systems is to use the invariance to reduce the number of variables of the system. Selecting an appropriate coordinate system on the invariant manifold can be very difficult. Moreover, sometimes all that is known is the existence of the invariant manifold, so that a change of coordinates cannot be considered. Instead, we focus on the implications of the variational equation and its associated compounded differential equations. Criteria are developed for the existence of a phase asymptotically stable periodic orbit with respect to the dynamics on an invariant manifold.

An orbit being phase asymptotically stable with respect to the flow on the invariant manifold means that solutions on the invariant manifold are attracted in phase to the orbit. It is possible that solutions off the invariant manifold are repelled away from a phase asymptotically stable orbit with respect to the flow on the invariant manifold. This thesis does not investigate solutions off the invariant manifold.

4.1 Stability of Equilibria and Periodic Orbits

In this section, we consider the stability of equilibria and periodic orbits as well as some general facts concerning the flow on an invariant manifold. The treatment in this section is based on [8]. The proofs of all of the propositions and theorems in this section can be found there.

In what follows, let D be an open set in $\mathbb{R}^n$, $\mathbf{f} \in C^1(D \mapsto \mathbb{R}^n)$, and $\mathbf{g} \in C^2(D \mapsto \mathbb{R}^m)$ where $0 \leq m \leq n$. Let $\varphi(t, \mathbf{x})$ be the flow defined by the following autonomous differential equation

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}). \tag{4.1}$$

Let $\Sigma := \{\mathbf{x} \in D : \mathbf{g}(\mathbf{x}) = 0\}$. Then, Σ is a manifold of dimension $n - m$ if $\mathrm{rk}(\frac{\partial \mathbf{g}}{\partial x}(\mathbf{x})) = m$ for all $\mathbf{x} \in \Sigma$, where $\mathrm{rk}(\frac{\partial \mathbf{g}}{\partial x}(\mathbf{x}))$ is the rank of Jacobian matrix of $\mathbf{g}$ at $\mathbf{x}$. The case $m = 0$ will correspond to the case that $\Sigma = D$. The manifold Σ is an *invariant manifold* with respect to (4.1) if $\mathbf{x} \in \Sigma$ implies $\mathbf{g}(\varphi(t, \mathbf{x})) = 0$ for any $t \in \mathbb{R}$. The function $\tilde{\mathbf{g}}(\mathbf{x})$ is a *first integral* if $\tilde{\mathbf{g}}(\varphi(t, \mathbf{x})) = \tilde{\mathbf{g}}(\mathbf{x})$ for all $\mathbf{x} \in D$ and $t \in \mathbb{R}$. If $\tilde{\Sigma} := \{\tilde{\mathbf{g}}(\mathbf{x}) - \mathbf{c} = 0\}$ is a manifold, then it is also an invariant manifold for each constant $\mathbf{c} \in \mathbb{R}^m$.

Let $\mathbf{v} \in C^2(\mathbb{R}^n \mapsto \mathbb{R}^m)$. Recall, the derivative of $\mathbf{v}$ along solutions of (4.1) at $\mathbf{x}$ is the continuously differentiable $\mathbb{R}^m$-valued function

$$\mathbf{v}'_{(4.1)}(\mathbf{x}) = \frac{\partial \mathbf{v}}{\partial x}(\mathbf{x})\mathbf{f}(\mathbf{x}), \tag{4.2}$$

since $\frac{d}{dt}\mathbf{v}(\mathbf{x}(t)) = \frac{\partial \mathbf{v}}{\partial x}(\mathbf{x}(t))\mathbf{x}'(t) = \frac{\partial \mathbf{v}}{\partial x}(\mathbf{x}(t))\mathbf{f}(\mathbf{x}(t))$ if $\mathbf{x}(t)$ is a solution of (4.1).

Proposition 4.1. *Let $\Sigma := \{\mathbf{x} \in D : \mathbf{g}(\mathbf{x}) = 0\}$ be a manifold of dimension $n - m$. Then, Σ is an invariant manifold with respect to* (4.1) *if and only if* *there is a continuous $m \times m$ matrix valued function $N(\mathbf{x})$ such that*

$$\mathbf{g}'_{(4.1)}(\mathbf{x}) = N(\mathbf{x})\mathbf{g}(\mathbf{x}) \quad \textit{for } \mathbf{x} \in D. \tag{4.3}$$

Suppose, in what follows, that Σ is an invariant manifold with respect to (4.1). Then, by Proposition 4.1, such $N(\mathbf{x})$ always exists. We denote by $\nu(\mathbf{x})$ the trace of the matrix $N(\mathbf{x})$. Additionally, $\mathcal{T}_{\mathbf{x}}$ will be used to denote the tangent space to Σ at $\mathbf{x}$.

Let $\tilde{\mathbf{g}}(\mathbf{x})$ be a first integral. Choose $\mathbf{c} \in \mathbb{R}^m$ and suppose $\tilde{\Sigma} := \{\mathbf{x} \in \mathbb{R}^n : \tilde{\mathbf{g}}(\mathbf{x}) = c\}$ is a manifold. Then, for a solution $\mathbf{x}(t)$ of (4.1) with $\tilde{\mathbf{g}}(\mathbf{x}(0)) = \mathbf{c}$,

$$\begin{aligned} \tilde{\mathbf{g}}'_{(4.1)}(\mathbf{x}(t)) &= \frac{\partial \tilde{\mathbf{g}}}{\partial x}(\mathbf{x}(t))\mathbf{f}(\mathbf{x}(t)) \\ &= \frac{\partial \tilde{\mathbf{g}}}{\partial x}(\mathbf{x}(t))\frac{d}{dt}(\mathbf{x}(t)) \\ &= \frac{d}{dt}\tilde{\mathbf{g}}(\mathbf{x}(t)) \\ &= \frac{d}{dt}\mathbf{c} = 0. \end{aligned}$$

This implies $\tilde{\mathbf{g}}'_{(4.1)}(\mathbf{x}) = 0$ for $\mathbf{x} \in D$. Then, by Proposition 4.1 with $\mathbf{g} = \tilde{\mathbf{g}} - c$, $N(\mathbf{x}) = 0$. Consequently, $\nu(\mathbf{x}) = \mathrm{Tr}(N(\mathbf{x}) = 0$.

Take $\mathbf{x} \in \Sigma$. Let $Y(t) := \frac{\partial \varphi}{\partial x}(t, \mathbf{x})$. Then, $Y(t)$ is the fundamental matrix for the linearization of (4.1) with respect to the solution $\varphi(t, \mathbf{x})$:

$$\frac{d\mathbf{y}}{dt} = \frac{\partial \mathbf{f}}{\partial x}(\varphi(t, \mathbf{x}))\mathbf{y}, \tag{4.4}$$

such that $Y(0) = I$.

The m-dimensional equation,

$$\frac{d\mathbf{u}}{dt} = -N^*(\varphi(t, \mathbf{x}))\mathbf{u} \tag{4.5}$$

is the adjoint of the equation

$$\frac{d\mathbf{v}}{dt} = N(\varphi(t, \mathbf{x}))\mathbf{v}.$$

Let $U(t)$ be an $m \times m$ solution matrix of (4.5). Then, it can be shown (see page 301 in [8]) that $W(t) = (\frac{\partial \mathbf{g}}{\partial x})^*(\varphi(t,\mathbf{x}))U(t)$ is an $n \times m$ solution matrix to the adjoint of (4.4).

Proposition 4.2. *Let* $\mathbf{w}^i(t) = (\frac{\partial \mathbf{g}}{\partial x})^*(\varphi(t,\mathbf{x}))U(t)\mathbf{e}^i$, *where* $\mathbf{x} \in \Sigma$, $\{\mathbf{e}^i : i = 1,\ldots,m\}$ *is the canonical basis on* $\mathbb{R}^m$, *and* $U(t)$ *is a solution matrix of* (4.5) *with* $U(0) = I_{m\times m}$. *Then,*

(i) *each* $\mathbf{w}^i(t)$ *is orthogonal to* $\mathcal{T}_{\varphi(t,\mathbf{x})}$ *and* $\{\mathbf{w}^i(t) : i = 1,\ldots,m\}$ *spans a m-dimensional solution subspace of the adjoint equation of* (4.4), *which is orthogonal to* $\mathcal{T}_{\varphi(t,\mathbf{x})}$.

(ii) *If* $\mathbf{y}(t)$ *is a solution of* (4.4) *and* $\mathbf{y}(0) \in \mathcal{T}_{\mathbf{x}}$, *then* $\mathbf{y}(t) \in \mathcal{T}_{\varphi(t,\mathbf{x})}$ *for all* $t \in \mathbb{R}^n$.

Remark 2. Statements (i) and (ii) in Proposition 4.2 are equivalent. If $\mathbf{y}(t)$ and $\mathbf{w}(t)$ are solutions of (4.4) and the adjoint of (4.4), respectively, then $\mathbf{y}(t)\cdot\mathbf{w}(t) = \mathbf{y}(0)\cdot\mathbf{w}(0)$ for all $t \in \mathbb{R}$. Hence, if $\mathbf{y}(0) \in \mathcal{T}_x$ and $\mathbf{w}(0)$ is orthogonal to $\mathcal{T}_{\mathbf{x}}$, then $\mathbf{y}(t)\cdot\mathbf{w}(t) = 0$ for $t \geq 0$. This implies that $\mathbf{y}(t) \in \mathcal{T}_{\varphi(t,\mathbf{x})}$ for $t \geq 0$ if and only if $\mathbf{w}(t)$ is orthogonal to $\mathcal{T}_{\varphi(t,\mathbf{x})}$ for $t \geq 0$.

Remark 3. The differential $\frac{\partial\varphi}{\partial x}(t,\mathbf{x})$ of the map $\mathbf{x} \mapsto \varphi(t,\mathbf{x})$ satisfies the property $\frac{\partial\varphi}{\partial x}(t,\mathbf{x})\mathcal{T}_{\mathbf{x}} = \mathcal{T}_{\varphi(t,\mathbf{x})}$.

Proposition 4.3. *Let* $U(t)$, $\mathbf{w}^i(t)$, $i = 1,\ldots,m$, *be as defined in Proposition* 4.2. *Then,*

$$\left|\mathbf{w}^1(t)\wedge\ldots\wedge\mathbf{w}^m(t)\right| \leq C\exp\left(-\int_0^t \nu(\varphi(s,\mathbf{x}))ds\right)\left|\mathbf{w}^1(t)\wedge\ldots\wedge\mathbf{w}^m(t)\right|$$

where $C = \sup\left\{\left|\bigwedge^m(\frac{\partial \mathbf{g}}{\partial x})^*(\mathbf{y})\right| / \left|\bigwedge^m \frac{\partial \mathbf{g}}{\partial x}(\mathbf{x})\right| : \mathbf{y} \in \Gamma_+(\mathbf{x})\right\}$.

Let $A \in M_{n\times n}$ and $\mathcal{T}$ be a subspace of $\mathbb{R}^n$. The restriction of A to $\mathcal{T}$ is denoted by $A|_{\mathcal{T}}$. In addition,

$$|\,A|_{\mathcal{T}}\,| = \sup_{\mathbf{x}\in\mathcal{T},\ |\mathbf{x}|=1} |A\mathbf{x}|.$$

Proposition 4.4. *Under the assumptions of Proposition* 4.3. *We have*

$$\left| \left. \bigwedge^{k} \frac{\partial \varphi}{\partial x}(t, \mathbf{x}) \right|_{\mathcal{T}_x} \right| \leq \left| \bigwedge^{(m+k)} \frac{\partial \varphi}{\partial x}(t, \mathbf{x}) \right| e^{-\int_0^t \nu(\varphi(s))ds}.$$

The following proposition indicates a close relation between the function $\nu(\mathbf{x}) = \mathrm{Tr}(N(\mathbf{x}))$ and the spectrum of $\frac{\partial \mathbf{f}}{\partial x}(\mathbf{x})$.

Proposition 4.5. *Let* $\lambda_1(\mathbf{x}), \ldots, \lambda_n(\mathbf{x})$ *be the eigenvalues of* $\frac{\partial \mathbf{f}}{\partial x}(\mathbf{x})$. *If* $\mathbf{x} \in \Sigma$ *and* $\lambda_{m+1}(\mathbf{x}), \ldots, \lambda_n(\mathbf{x})$ *are the eigenvalues which correspond to the* $(n-m)$*-dimensional tangent space* $\mathcal{T}_{\mathbf{x}}$*, then*

$$\nu(\mathbf{x}) = \lambda_1(\mathbf{x}) \cdots \lambda_m(\mathbf{x}).$$

Suppose $\mathbf{x}_0 \in \Sigma$ is an equilibrium of (4.1). Recall, $\mathbf{x}_0$ is *stable hyperbolic* with respect to the dynamics on Σ, if the $(n-m)$ the eigenvalues λ_i of $\frac{\partial \mathbf{f}}{\partial x}(\mathbf{x}_0)$ corresponding to the invariant subspace $\mathcal{T}_{\mathbf{x}_0}$ satisfy $Re(\lambda_i) < 0$. Let $\varphi(t, \mathbf{x}_0)$ be an ω-periodic orbit for some $\omega > 0$. By Proposition 4.2, the tangent space to Σ at $\mathbf{x}_0$, $\mathcal{T}_{\mathbf{x}_0}$ is mapped to $\mathcal{T}_{\varphi(\omega, \mathbf{x}_0)}$ by $\frac{\partial \varphi}{\partial x}(\omega, \mathbf{x}_0)$. Further, by the periodicity of $\varphi(t, \mathbf{x}_0)$, $\frac{\partial \varphi}{\partial x}(\omega, \mathbf{x}_0)\mathcal{T}_{\mathbf{x}_0} = \mathcal{T}_{\varphi(\omega, \mathbf{x}_0)} = \mathcal{T}_{\mathbf{x}_0}$. That is, $\mathcal{T}_{\mathbf{x}_0}$ is invariant under $\frac{\partial \varphi}{\partial x}(\omega, \mathbf{x}_0)$. The eigenvalue, μ_n, of $\frac{\partial \varphi}{\partial x}(\omega, \mathbf{x}_0)\big|_{\mathcal{T}_{x_0}}$ associated with the eigenvector $\dot{\varphi}(0, x_0)$ satisfies $\mu_n = 1$. Recall, we call the periodic orbit $\varphi(t, \mathbf{x}_0)$ *stable hyperbolic* with respect to the dynamics on Σ, if the $(n-m-1)$ remaining eigenvalues μ_i of $\frac{\partial \varphi}{\partial x}(\omega, \mathbf{x}_0)\big|_{\mathcal{T}_{x_0}}$ satisfy $|\mu_i| < 1$.

Theorem 4.6. *Let* $\mathbf{x}_0 \in \Sigma$ *be an equilibrium of* (4.1).

(i) *A sufficient condition for* $\mathbf{x}_0$ *to be a stable hyperbolic equilibrium with respect to the dynamics of* (4.1) *on* Σ *is that*

$$\frac{d\mathbf{z}}{dt} = \left[\frac{\partial \mathbf{f}}{\partial x}^{[m+1]}(\mathbf{x}_0) - I\nu(\mathbf{x}_0) \right] \mathbf{z} \tag{4.6}$$

is asymptotically stable, where I *is the* $\binom{n}{m+1} \times \binom{n}{m+1}$ *identity matrix.*

(ii) *The sufficient condition in* (i) *is also necessary if the system*

$$\frac{d\mathbf{u}}{dt} = -N^*(\mathbf{x}_0)\mathbf{u}$$

is stable.

For a ω-periodic orbit, we have the following analog of Theorem 4.6. It is also an analog of Poincaré's Stability Criterion for stability of a periodic solution of a 2-dimensional autonomous differential equation.

Theorem 4.7. *Take* $\varphi(0) \in \Sigma$, *and suppose* $\varphi(t)$ *is a non-trivial* ω*-periodic solution of* (4.1) *where* $\omega > 0$.

(i) *A sufficient condition for* $\Gamma_+(\varphi(0))$ *to be a stable hyperbolic with respect to the dynamics on* Σ *is that*

$$\frac{d\mathbf{z}}{dt} = \left[\frac{\partial \mathbf{f}}{\partial x}^{[m+2]}(\varphi(t)) - I\nu(\varphi(t))\right]\mathbf{z} \tag{4.7}$$

is asymptotically stable, where I *is the* $\binom{n}{m+2} \times \binom{n}{m+2}$ *identity matrix.*

(ii) *The sufficient condition in* (i) *is also necessary if the system*

$$\frac{d\mathbf{u}}{dt} = -N^*(\varphi(t))\mathbf{u}$$

is stable.

4.2 The Existence of Equilibria

The following is an analog of Theorem 3.7 for a flow on an invariant manifold.

Theorem 4.8. *Suppose that the flow* $\varphi(t, \mathbf{x})$ *is Lagrange stable at* $\mathbf{x}(t)$ *a solution of* (4.1) *on* Σ. *Then* $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{x}_0$ *is a stable hyperbolic equilibrium with respect to the dynamics on* Σ *if*

$$\frac{d\mathbf{z}}{dt} = \left[\frac{\partial \mathbf{f}}{\partial x}^{[m+1]}(\mathbf{x}(t)) - I\nu(\mathbf{x}(t))\right]\mathbf{z} \tag{4.8}$$

is uniformly asymptotically stable.

Proof. Let $Y(t)$ be a fundamental solution of the following

$$\frac{d\mathbf{y}}{dt} = \frac{\partial \mathbf{f}}{\partial x}(\mathbf{x}(t))\mathbf{y}. \tag{4.9}$$

Suppose (4.8) is uniformly asymptotically stable. Then, there exist positive constants α, K so that $Y^{(m+1)}(t)e^{-\int_0^t \nu(\mathbf{x}(s))ds}$, a fundamental solution of (4.8), satisfies

$$|Y^{(m+1)}(t)|e^{-\int_0^t \nu(\mathbf{x}(s))ds} < Ke^{-\alpha t} \quad \text{for } t \geq 0.$$

As a result, from Proposition 4.4 with $k = 1$, we have

$$\begin{aligned} \left|Y(t)|_{\mathcal{T}_{x_0}}\right| = \left|Y^{(1)}(t)|_{\mathcal{T}_{x_0}}\right| &\leq \left|Y^{(m+1)}(t)|_{\mathcal{T}_{x_0}}\right| e^{-\int_0^t \nu(\mathbf{x}(s))ds} \\ &\leq Ke^{-\alpha t} \quad \text{for } t \geq 0. \end{aligned} \tag{4.10}$$

Let $\mathbf{y}_1(t) = \mathbf{f}(\mathbf{x}(t))$. Then $\mathbf{y}_1(t)$ is a solution of (4.9), and since $\mathbf{f}(\mathbf{x}(0)) \in \mathcal{T}_{\mathbf{x}(0)}$, we have $\mathbf{y}_1(t) = Y(t)|_{\mathcal{T}_{\mathbf{x}(0)}}\mathbf{y}(0)$. This implies with (4.10) that

$$\lim_{t\to\infty} \mathbf{y}(t) = 0. \tag{4.11}$$

Let $\mathbf{x}_* \in \Omega(\mathbf{x}(0))$. Then, there exists a sequence $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$, such that $\mathbf{x}(t_n) \to \mathbf{x}_*$ as $n \to \infty$. This implies with (4.11) that $\lim_{n\to\infty} \mathbf{y}_1(t_n) = \lim_{n\to\infty} \mathbf{f}(\mathbf{x}(t_n)) = \mathbf{f}(\mathbf{x}_*) = 0$. Consequently, $\mathbf{x}_*$ is an equilibrium. By continuity, (4.8) is uniformly asymptotically stable when $\mathbf{x}(t)$ is replaced by $\mathbf{x}_*$. Therefore, from Theorem 4.6(i), $\mathbf{x}_*$ is a stable hyperbolic equilibrium. □

4.3 The Existence of Periodic Orbits

In this section, a similar result to Theorem 3.11 is proved for a flow on an invariant manifold. That is, sufficient conditions are given for the existence of a periodic orbit which is phase asymptotically stable with respect to the flow on an invariant manifold.

For the system of differential equations (4.1), the variational equation at $\varphi(t, \mathbf{x}_0)$ is

$$\frac{d\mathbf{y}}{dt} = \frac{\partial \mathbf{f}}{\partial x}(\varphi(t, \mathbf{x}_0))\mathbf{y}. \tag{4.12}$$

Recall, we suppose that $\Sigma = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{g}(\mathbf{x}) = 0\}$ is an $(n-m)$-dimensional invariant manifold where $0 \leq m \leq n$ and $\mathbf{g} \in C^2(\mathbb{R}^n \mapsto \mathbb{R}^m)$. The linear equation

$$\frac{d\mathbf{z}}{dt} = \left[\frac{\partial \mathbf{f}}{\partial x}^{[m+2]}(\varphi(t, \mathbf{x}_0)) - I\nu(\varphi(t, \mathbf{x}_0))\right]\mathbf{z}, \tag{4.13}$$

plays an important role in the analysis that follows.

Theorem 4.9. *Suppose that* $\overline{\Gamma_+(\mathbf{x}_0)} \subset D$ *and that*

(i) *the flow* $\varphi(t, \mathbf{x})$ *generated by* (4.1) *is Lagrange stable at* $\mathbf{x}_0 \in \Sigma$*;*

(ii) *the* ω*-limit set* $\Omega(\mathbf{x}_0)$ *contains no equilibria;*

(iii) *equation* (4.13) *is uniformly asymptotically stable.*

Then, the following holds,

(a) *there exist positive constants* K, ρ, γ *and a bounded function* $h : \Sigma \mapsto \mathbb{R}$ *such that for all* $y \in \Sigma$, $|\mathbf{y} - \mathbf{x}_0| < \rho$ *implies*

$$|\varphi(t + h(\mathbf{y}), \mathbf{y}) - \varphi(t, \mathbf{x}_0)| \leq K|\mathbf{y} - \mathbf{x}_0|e^{-\gamma t} \quad \text{for } t \geq 0;$$

(b) $\Omega(\mathbf{x}_0)$ *is a non-trivial periodic orbit.*

Proof. Under the assumptions (i)-(iii), the conditions of Theorem $B.3$ are satisfied. Hence, there exist positive constants C and α so that

$$|Y(t)P_2Y^{-1}(s)| \leq \frac{C}{2}e^{-\alpha(t-s)} \quad \text{for } 0 \leq s \leq t, \tag{4.14}$$

where P_2 is a projection matrix of rank $(n-m-1)$ and $Y(t) := \frac{\partial \varphi}{\partial x}(t, \mathbf{x}_0)$ is the fundamental solution of (4.12), such that $Y(0) = I$. As shown in Theorem 3.11,

$$\left|\frac{\partial \varphi}{\partial x}(t, \mathbf{z})P_2(\mathbf{z})\right| \leq \frac{C}{2}e^{-\alpha t} \quad \text{for } t \geq 0,$$

where $P_2(\mathbf{z}) := Y(s)P_2Y^{-1}(s)$ for $\mathbf{z} = \varphi(s, \mathbf{x}_0)$ is a projection matrix.

Take $\mathbf{z} \in \Gamma_+(\mathbf{x}_0)$. Let the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ be a basis of the normal space to Σ at $\mathbf{z}$, and let $\{\mathbf{v}_m, \ldots, \mathbf{v}_{n-1}\}$ be a basis for the subspace $P_2(\mathbf{z})\mathbb{R}^n$, a subset of the tangent space to Σ at $\mathbf{z}$. Then, $\{\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}\}$ is a basis for an $(n-1)$-dimensional plane. Let $\mathbf{a}(\mathbf{z})$ be a normal vector to this plane at $\mathbf{z}$. We shall show that the angle between the two planes $\Pi(\mathbf{f}(\mathbf{z}))$ and $\Pi(\mathbf{a}(\mathbf{z}))$ is uniformly bounded away from $\pi/2$ for $\mathbf{z} \in \Gamma_+(\mathbf{x}_0)$ where $\Pi(\mathbf{a}(\mathbf{z}))$ is the plane with normal vector $\mathbf{a}(\mathbf{z})$. Recall that the conditions of Theorem B.3 are satisfied. Let $\mathcal{Z}_1$ and $\mathcal{Z}_2$ be as defined in Theorem B.3 i.e. $\mathcal{Z}_1 := \text{span}\{\mathbf{y}_1(\cdot)\}$, with $\mathbf{y}_1(t) := \mathbf{f}(\varphi(t, \mathbf{x}_0))$ and $\mathcal{Z}_2$ is the set of all solutions of (4.12) such that $\mathbf{y}(0) \in P_2\mathbb{R}^n$ which is the $(n-m-1)$-dimensional subspace of all solutions going to zero as $t \to \infty$. Also, $\mathcal{Z}_1$ and $\mathcal{Z}_2$ are uniformly bounded away from each other by (B.28). By definition of $P_2(\mathbf{z})$ and the fact that $Y(s)$ has full rank, we have

$$
\begin{aligned}
P_2(\mathbf{z})\mathbb{R}^n &= Y(s)P_2Y^{-1}(s, \mathbf{x}_0)\mathbb{R}^n \\
&= Y(s)P_2\mathbb{R}^n \\
&= \{\mathbf{y}(t) : \mathbf{y}(\cdot) \in \mathcal{Z}_2,\ \mathbf{z} = \varphi(s, \mathbf{x}_0)\}.
\end{aligned}
$$

Thus, the angle between the vector $\mathbf{f}(\mathbf{z})$ and any nonzero vector in $P_2(\mathbf{z})\mathbb{R}^n$ is bounded away from 0. Since the normal space to Σ at $\mathbf{z}$ is orthogonal to $\mathbf{f}(\mathbf{z})$, the angle between the vector $\mathbf{f}(\mathbf{z})$ and any nonzero vector in $\Pi(\mathbf{a}(\mathbf{z}))$ is bounded away from 0. Therefore, the angle between the planes $\Pi(\mathbf{a}(\mathbf{z}))$ and $\Pi(\mathbf{f}(\mathbf{z}))$ is uniformly bounded away from $\pi/2$, which implies that there exists a constant ζ, $0 < \zeta < 1$, such that

$$\zeta|\mathbf{a}(\mathbf{z})||\mathbf{f}(\mathbf{z})| < \mathbf{a}(\mathbf{z}) \cdot \mathbf{f}(\mathbf{z}).$$

Moreover, by the uniform boundedness of the angle, such a choice of ζ does not depend on $\mathbf{z} \in \Gamma_+(\mathbf{x}_0)$.

Since $\left|\frac{\partial \varphi}{\partial x}(t, \mathbf{y})P_2(\mathbf{x})e^{\alpha t}\right|$ is uniformly continuous on any compact subset of

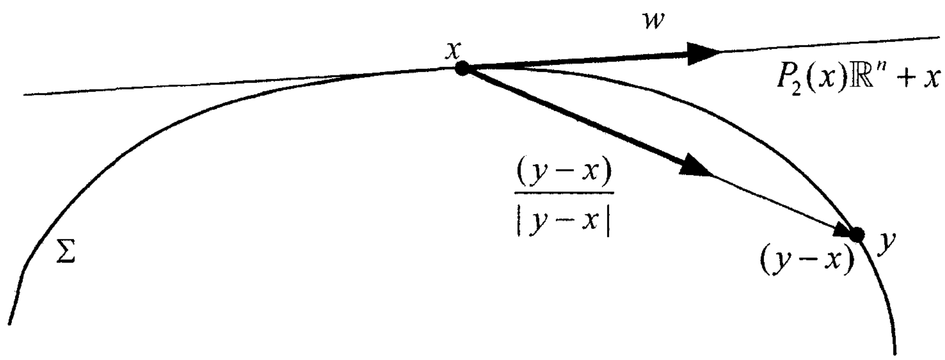

Figure 4.1: The existence of the vector $\mathbf{w}$

a finite-dimensional space, we can choose $T > 0$ and $\eta > 0$ such that for all $t \in [0,T]$, $\mathbf{y} \in \overline{B_\eta(\Gamma_+(\mathbf{x}_0))}$, and $\mathbf{x} \in \overline{\Gamma_+(\mathbf{x}_0)}$, there exists a $\kappa > 0$ such that $|\mathbf{x} - \mathbf{y}| < \kappa$ implies that

$$\left|\frac{\partial \varphi}{\partial x}(t, \mathbf{y}) P_2(\mathbf{x})\right| \leq Ce^{-\alpha t}. \tag{4.15}$$

Since $[0,T] \times \overline{B_\eta(\Gamma_+(\mathbf{x}_0))}$ is compact, there exists a constant M such that $|\frac{\partial \varphi}{\partial x}(t,\mathbf{y})| < M$ when $t \in [0,T]$ and $\mathbf{y} \in \overline{B_\eta(\Gamma_+(\mathbf{x}_0))}$. Further, (4.15) yields that for $t \in [0,T]$, $\mathbf{x} \in \Gamma_+(\mathbf{x}_0)$, $\mathbf{w} \in P_2(\mathbf{x})\mathbb{R}^n$ with $|\mathbf{w}| = 1$, $|\mathbf{x}-\mathbf{y}| < \kappa$ implies $|\frac{\partial \varphi}{\partial x}(t,\mathbf{y})\mathbf{w}| < Ce^{-\alpha t}$. In addition, for $\tilde{\mathbf{w}}$ such that $|\tilde{\mathbf{w}} - \mathbf{w}| < Ce^{-\alpha T}/M$, we have

$$\begin{aligned}
\left|\frac{\partial \varphi}{\partial x}(t, \mathbf{y})\tilde{\mathbf{w}}\right| &\leq \left|\frac{\partial \varphi}{\partial x}(t, \mathbf{y})\mathbf{w}\right| + |\tilde{\mathbf{w}} - \mathbf{w}| \left|\frac{\partial \varphi}{\partial x}(t, \mathbf{y})\right| \\
&\leq Ce^{-\alpha t} + M\frac{Ce^{-\alpha T}}{M} \\
&\leq 2Ce^{-\alpha t}.
\end{aligned} \tag{4.16}$$

The compactness of $\Sigma \cap \overline{B_\eta(\Gamma_+(\mathbf{x}_0))}$ and the continuity of Σ imply that we can chose $\kappa > 0$ small enough such that for all $\mathbf{x} \in \Gamma_+(\mathbf{x}_0)$, $\mathbf{y} \in \Sigma \cap \Pi(\mathbf{a}(\mathbf{x}))$ with $0 < |\mathbf{x} - \mathbf{y}| < \kappa$, there exists a $\mathbf{w} \in P_2(\mathbf{x})\mathbb{R}^n$ (see Figure 4.1) such that

$$\left|\frac{(\mathbf{y} - \mathbf{x})}{|\mathbf{y} - \mathbf{x}|} - \mathbf{w}\right| < \frac{Ce^{-\alpha T}}{M}. \tag{4.17}$$

For $k = 0, 1 \ldots$, we put $\mathbf{x}_k = \varphi(T \cdot k, \mathbf{x}_0)$. Take $y \in \Sigma \cap \Pi(\mathbf{a}(\mathbf{x}))$ such that $|\mathbf{y} - \mathbf{x}_k| < \kappa$. In particular, by (4.17), there exists a $\mathbf{w}$ such that $\left| \frac{(\mathbf{y}-\mathbf{x}_k)}{|\mathbf{y}-\mathbf{x}_k|} - \mathbf{w} \right| < \frac{Ce^{-\alpha T}}{M}$. Consequently, (4.16) applies with $\tilde{\mathbf{w}} = (\mathbf{y} - \mathbf{x}_k)/|\mathbf{y} - \mathbf{x}_k$. On the other hand, let $\mathbf{x}(\lambda) = \mathbf{x}_k + \lambda(\mathbf{y} - \mathbf{x}_k)$ for $0 \le \lambda \le 1$. Then, as showed in the proof of Theorem 3.11,

$$\begin{aligned} |\varphi(t, \mathbf{y}) - \varphi(t, \mathbf{x}_k)| &\le \int_0^1 \left| \tfrac{d}{d\lambda} \varphi(t, \mathbf{x}(\lambda)) \right| d\lambda \\ &\le \int_0^1 \left| \tfrac{\partial \varphi}{\partial x}(t, \mathbf{x}(\lambda))(\mathbf{y} - \mathbf{x}_k) \right| d\lambda \\ &\le |\mathbf{y} - \mathbf{x}_k| \int_0^1 \left| \tfrac{\partial \varphi}{\partial x}(t, \mathbf{x}(\lambda)) \tfrac{(\mathbf{y}-\mathbf{x}_k)}{|\mathbf{y}-\mathbf{x}_k|} \right| d\lambda \end{aligned} \tag{4.18}$$

for $0 \le t \le T$. Therefore, we have

$$\begin{aligned} |\varphi(t, \mathbf{y}) - \varphi(t, \mathbf{x}_k)| &\le |\mathbf{y} - \mathbf{x}_k| \int_0^1 2Ce^{-\alpha t} d\lambda \\ &\le 2Ce^{-\alpha t} |\mathbf{y} - \mathbf{x}_k| \end{aligned} \tag{4.19}$$

for $0 \le t \le T$.

Moreover, since (i)-(ii) and (4.3) holds, the conditions of Lemma 3.12 applies. Therefore, by using the induction argument as in the proof of Theorem 3.11 with $\mathbf{y} \in \Sigma$, the result follows.

□

Chapter 5

Examples

We consider three examples where Theorem 4.9 is used to find a phase asymptotically stable periodic orbit with respect to the flow on an invariant manifold. In the first example (cf. Section 5.2), a system with an invariant sphere corresponding to a first integral is considered. Then, we discuss a general system with an invariant manifold in Section 5.3. Two examples are given for the system in Sections 5.4 and 5.5. The first is a 3-dimensional system with a 2-dimensional invariant cylinder, where we show there exists a unique phase asymptotic periodic orbit. The other example considers a 4-dimensional system which has a 3-dimensional invariant cylinder.

5.1 A Review of First Integrals

For convenience we summarize properties of first integrals which were discussed in Section 4.1.

First integrals commonly arise from the existence of conservation principle in dynamical systems. They are functions along which solutions to a differential equation are constant.

Suppose $\mathbf{f} \in C^1(\mathbb{R}^n \mapsto \mathbb{R}^n)$ and $\tilde{\mathbf{g}} \in C^2(\mathbb{R}^n \mapsto \mathbb{R}^m)$ where $(m \leq n)$.

Consider the following differential equation

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}). \tag{5.1}$$

Let $\varphi(t, \mathbf{x})$ be the flow generated by (5.1). Recall, the function $\tilde{\mathbf{g}}(\mathbf{x})$ is called a first integral for (5.1) if $\tilde{\mathbf{g}}(\varphi(t, \mathbf{x})) = \tilde{\mathbf{g}}(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$ and $t \in \mathbb{R}$. The set $\Sigma := \{\mathbf{x} \in \mathbb{R}^n : \tilde{\mathbf{g}}(\mathbf{x}) = c\}$ is an $(n-m)$-dimensional invariant manifold for each given constant c, if the matrix $\frac{\partial \tilde{\mathbf{g}}}{\partial x}(\mathbf{x})$ is of rank m for $\mathbf{x} \in \Sigma$. Moreover, by Proposition 4.1 for an invariant manifold, there always exists a corresponding function $N : \mathbb{R}^n \mapsto M_{n \times n}$ defined by (4.3) with $\mathbf{g} = \tilde{\mathbf{g}} - c$. In the case of a first integral, $N = 0$, so we have $\nu(\mathbf{x}) := \mathrm{Tr}(N(\mathbf{x}) = 0$.

5.2 An Invariant Sphere

Consider a system in $\mathbb{R}^3$ as follows

$$\begin{aligned} \frac{dx_1}{dt} &= -x_2 + x_1 x_3^2 \\ \frac{dx_2}{dt} &= x_1 + x_2 x_3^2 \\ \frac{dx_3}{dt} &= -x_3(x_1^2 + x_2^2), \end{aligned}$$

where $x_i \in \mathbb{R}$, $i = 1, 2, 3$. We write it concisely as

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}). \tag{5.2}$$

If $\mathbf{x}(t)$ is a solution of (5.2), then

$$\begin{aligned} \frac{d}{dt}(x_1^2(t) + x_2^2(t) + x_3^2(t)) &= 2x_1(t)\frac{dx_1}{dt}(t) + 2x_2(t)\frac{dx_2}{dt}(t) + 2x_3(t)\frac{dx_3}{dt}(t) \\ &= 0. \end{aligned}$$

Hence, $(x_1^2 + x_2^2 + x_3^2)$ is a first integral. Let $g(\mathbf{x}) := x_1^2 + x_2^2 + x_3^2 - 1$ and let $\Sigma := \{\mathbf{x} \in \mathbb{R}^n : g(\mathbf{x}) = 0\}$. Then, Σ is an invariant manifold and $\nu(\mathbf{x}) = 0$.

Using the cylindrical coordinates, $x_1 = r\cos\theta$, $x_2 = r\sin\theta$, and $x_3 = x_3$ where $r = (x_1^2 + x_2^2)^{(1/2)}$, the system is

$$\begin{aligned}\frac{dr}{dt} &= rx_3^2,\\ \frac{d\theta}{dt} &= 1,\\ \frac{dx_3}{dt} &= -x_3r^2.\end{aligned} \tag{5.3}$$

It is evident from (5.3) that the only equilibria on Σ are $(0,0,1)$ and $(0,0,-1)$. Take $\mathbf{x}(t)$ to be a solution of (5.2) on Σ with $|x_3(0)| < 1/3$. If $0 < x_3(0) < 1/3$, then $dx_3/dt \leq -8x_3/9$, and hence $x_3(t) < 1/3$ for $t \geq 0$. Otherwise if $-1/3 < x_3(0) \leq 0$, then $dx_3/dt \geq -8x_3/9$, and hence $x_3(t) > -1/3$ for $t \geq 0$. Therefore, the ω-limit set of an orbit in Σ with $|x_3(0)| < 1/3$ does not contain any equilibria.

Since the sphere is bounded, to apply Theorem 4.9 to the solution $\mathbf{x}(t)$, it remains to show that the linear 1-dimensional equation

$$\frac{dz}{dt} = \frac{\partial \mathbf{f}}{\partial x}^{[3]}(\mathbf{x}(t))z \tag{5.4}$$

is uniformly asymptotically stable, where $\partial \mathbf{f}^{[3]}/\partial x$ is the third additive compound of the Jacobian matrix of $\mathbf{f}$. Take $z(t)$ to be a solution of (5.4). Then $|z(t)| \leq |z(s)| \exp(\int_s^t \frac{\partial \mathbf{f}}{\partial x}^{[3]}(\mathbf{x}(\tau))d\tau$ for $t \geq s \geq 0$. This implies that if there exists a constant $c > 0$ such that $\frac{\partial \mathbf{f}}{\partial x}^{[3]}(\mathbf{x}(t)) \leq -c < 0$, then by Proposition 3.3(iv) the equation (5.4) will be uniformly asymptotically stable. Since

$$\frac{\partial \mathbf{f}}{\partial x}^{[3]}(\mathbf{x}) = \operatorname{div}\mathbf{f}(\mathbf{x}) = 2x_3^2 - x_1^2 - x_2^2 < \frac{2}{9} - \frac{8}{9} \leq -\frac{2}{3} \quad \text{for } \mathbf{x} \in \Gamma_+(\mathbf{x}(0)),$$

we have that (5.4) is uniformly asymptotically stable.

The conditions of Theorem 4.9 are satisfied for the solution $\mathbf{x}(t)$. Hence $\mathbf{x}(t)$ limits to a phase asymptotic periodic orbit in the region $|x_3| < 1/3$ contained in Σ. From (5.3), the only nontrivial periodic orbit must be in the plane defined by $x_3 = 0$. This periodic orbit is given by

$$(x_1(t), x_2(t), x_3(t)) = (\cos(t - \phi_0), \sin(t - \phi_0), 0)$$

where ϕ_0, a constant, is the phase shift.

5.3 A System with an Invariant Manifold

In this section, a system of differential equations with an invariant manifold is considered. Suppose $\mathbf{F} \in C^1(\mathbb{R}^n \mapsto \mathbb{R}^n)$, and suppose H is a $n \times n$ constant nonzero real matrix. Consider the autonomous differential equation

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}) := \mathbf{F}(\mathbf{x}) - [\mathbf{x}^* H^* H \mathbf{F}(\mathbf{x})]\mathbf{x}, \tag{5.5}$$

where the asterisk denotes the transpose.

Proposition 5.1. *Let* $\Sigma := \{\mathbf{x} \in \mathbb{R}^n : |H\mathbf{x}|^2 = 1\}$, *where* $|H\mathbf{x}|^2 = \mathbf{x}^* H^* H \mathbf{x}$. *Then* Σ *is an invariant manifold for* (5.5).

Proof. For $g \in C^1(\mathbb{R}^n \mapsto \mathbb{R})$, ∇g will be used to denote the gradient of g. Let $g(\mathbf{x}) := |H\mathbf{x}|^2 - 1 = \mathbf{x}^* H^* H \mathbf{x} - 1$. The set Σ is a manifold of dimension $n - 1$ if $g \in C^1(\mathbb{R}^n \mapsto \mathbb{R})$ and $\text{rk}(\nabla g(\mathbf{x})) = 1$ for any $\mathbf{x} \in \Sigma$. This is equivalent to showing that $\nabla g(\mathbf{x}) \neq 0$ for $\mathbf{x} \in \Sigma$. Since $(\nabla g)^* = 2\mathbf{x}^* H^* H = 0$, we have $\mathbf{x}^* H^* H \mathbf{x} = 0$. This contradicts the fact that $\mathbf{x}^* H^* H \mathbf{x} = 1$ for $\mathbf{x} \in \Sigma$. Therefore, Σ is a manifold of dimension $n - 1$.

The manifold Σ is an invariant with respect to (5.5) if for all $\mathbf{x} \in \Sigma$, $\mathbf{f}(\mathbf{x})$ is tangent to Σ at $\mathbf{x}$, that is if $\nabla g(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}) = 0$ for $\mathbf{x} \in \Sigma$. In fact for all $\mathbf{x} \in \Sigma$,

$$\begin{aligned} \nabla g(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}) &= 2\mathbf{x}^* H^* H \mathbf{F}(\mathbf{x}) - 2[\mathbf{x}^* H^* H \mathbf{F}(\mathbf{x})][\mathbf{x}^* H^* H \mathbf{x}] \\ &= 2[\mathbf{x}^* H^* H \mathbf{F}(\mathbf{x})](1 - \mathbf{x}^* H^* H \mathbf{x}) \\ &= 0, \end{aligned}$$

and so Σ is an invariant manifold. $\square$

Let $\nu(\mathbf{x}) = \text{Tr}(N(\mathbf{x}))$ where $N(\mathbf{x})$ is the matrix valued function that satisfies

$$\nabla g(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}) = N(\mathbf{x})g(\mathbf{x}).$$

Such a function always exists (see Section 4.1).

In this case, $\nu(\mathbf{x}) = N(\mathbf{x})$. We obtain,

$$\begin{aligned} \nabla g(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}) &= \nabla g(\mathbf{x}) \cdot (\mathbf{F}(\mathbf{x}) - [\mathbf{x}^* H^* H \mathbf{F}(\mathbf{x})]\mathbf{x}) \\ &= -2\mathbf{x}^* H^* H \mathbf{F}(\mathbf{x}) g(\mathbf{x}). \end{aligned}$$

Therefore,

$$\nu(\mathbf{x}) = -2\mathbf{x}^* H^* H \mathbf{F}(\mathbf{x}). \tag{5.6}$$

5.4 An Invariant 2-Dimensional Cylinder

Proposition 5.1 will now be used as an example of Theorem 4.9 on an invariant cylinder.

Let $\mathbf{F}(\mathbf{x}) := (F_1(\mathbf{x}), F_2(\mathbf{x}), F_3(\mathbf{x}))$ where

$$\begin{aligned} F_1(\mathbf{x}) &:= x_2 + \frac{x_1^2}{10} \\ F_2(\mathbf{x}) &:= -x_1 + \frac{x_2^2}{10}, \\ F_3(\mathbf{x}) &:= -x_3 + q(x_1, x_2) + x_3(x_1 F_1(\mathbf{x}) + x_2 F_2(\mathbf{x})), \end{aligned}$$

and $q \in C^1(\mathbb{R}^2 \mapsto \mathbb{R})$. From Proposition 5.1 with $H :=$ Diagonal$[1, 1, 0]$, $\Sigma := \{\mathbf{x} : |H\mathbf{x}|^2 = x_1^2 + x_2^2 = 1\}$ is an invariant manifold for the following system

$$
\begin{aligned}
\frac{dx_1}{dt} &= F_1(\mathbf{x}) - x_1(x_1F_1(\mathbf{x}) + x_2F_2(\mathbf{x})) \\
&= x_2 - \frac{x_1}{10}(-x_1 + x_1^3 + x_2^3) := f_1(\mathbf{x}), \\
\frac{dx_2}{dt} &= F_2(\mathbf{x}) - x_2(x_1F_1(\mathbf{x}) + x_2F_2(\mathbf{x})) \\
&= -x_1 - \frac{x_2}{10}(-x_2 + x_1^3 + x_2^3) := f_2(\mathbf{x}), \\
\frac{dx_3}{dt} &= F_3(\mathbf{x}) - x_3(x_1F_1(\mathbf{x}) + x_2F_2(\mathbf{x})) \\
&= -x_3 + q(x_1, x_2) := f_3(\mathbf{x}).
\end{aligned}
$$

We Write concisely as

$$d\mathbf{x}/dt = \mathbf{f}(\mathbf{x}). \tag{5.7}$$

Suppose that $\mathbf{x} \in \Sigma$ and $f_1(\mathbf{x}) = f_2(\mathbf{x}) = 0$. Then,

$$
\begin{aligned}
0 = 10x_2f_1(\mathbf{x}) - 10x_1f_2(\mathbf{x}) &= 10(x_1^2 + x_2^2) + x_2x_1^2 - x_1x_2^2 \\
&> 10 - 2 = 8,
\end{aligned}
$$

which contradicts $f_1(\mathbf{x}) = f_2(\mathbf{x}) = 0$. Hence, there are no equilibria on Σ.

Next, we show that any solution on the cylinder is bounded. By continuity of q and compactness of the set $\{(x_1, x_2) : x_1^2 + x_2^2 = 1\}$ we let $M := \max_{\mathbf{x}\in\Sigma} q(x_1, x_2)$. If $x_3 > M$, then $dx_3/dt < 0$, and hence a solution is bounded from above. Similarly, if $x_3 < -M$, then $dx_3/dt > 0$, and hence any solution is bounded from below. Therefore, all solutions are bounded.

Finally, we show that the 1-dimensional linear equation

$$\frac{dz}{dt} = \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}^{[3]}(\varphi(t)) - \nu(\varphi(t))\right) z$$

is uniformly asymptotically stable, where $\varphi(t)$ is any solution of (5.7) on Σ. As in Section 5.2, it is sufficient to show that there exists a constant c so that $\operatorname{div}\mathbf{f}(\mathbf{x}) - \nu(\mathbf{x}) \le -c < 0$ for $\mathbf{x} \in \Sigma$. From (5.6), $\nu(\mathbf{x}) = -2\mathbf{x}^* H^* H\mathbf{F}(\mathbf{x}) =$

$-(x_1^3 + x_2^3)/5$. Thus,

$$
\begin{aligned}
\operatorname{div} \mathbf{f}(\mathbf{x}) - \nu(\mathbf{x}) &= \frac{-10 + 2x_1 + 3x_1^3 + 2x_2 + 3x_2^3}{10} \\
&< \frac{-10 + 4 + 3(x_1^2 + x_2^2)}{10} \\
&< -\frac{3}{10} \quad \text{for } \mathbf{x} \in \Sigma.
\end{aligned}
$$

As established in the previous three paragraphs, the conditions of Theorem 4.9 are satisfied for any solution of (5.7) on Σ. Hence, all these solutions are phase asymptotically stable with respect to the flow on Σ, and they limit to a phase asymptotically stable periodic orbit on Σ. Further, we show that this periodic orbit is unique on Σ. In this case, any solution on Σ will be attracted in phase with respect to the dynamics on Σ to a single periodic orbit.

Suppose that a periodic orbit on Σ is homotopic to a point on Σ. Then, it must contain an equilibrium in Σ, clearly this is impossible since there are no equilibria in Σ. As a consequence, any periodic solution goes around the cylinder exactly once.

By continuous differentiability of any solution of (5.7) and since $\frac{dx_3}{dt} \neq 0$ if $|x_3| \geq M$, all periodic orbits on Σ must be contained in the region $|x_3| < M$ Suppose there are infinitely many periodic orbits on Σ. Since they are contained in a finite region, there exists a periodic orbit $\Gamma(\mathbf{x}_*)$ where every open neighborhood of $\Gamma(\mathbf{x}_*)$ contains at least two periodic orbits. This contradicts $\Gamma(\mathbf{x}_*)$ being phase asymptotically stable with respect to the dynamics on Σ. We conclude that there are a finite number of periodic orbits on Σ.

Let $\mathbf{x}_* \in \Sigma$ be some point between the two periodic orbits. We can adapt Theorem 3.1, the Poincaré-Bendixson theorem, to Σ bounded between the periodic orbits. As a result the α-limit set, $A(\mathbf{x}_*)$, contains either a periodic orbit or an equilibrium. Since there are no equilibria on Σ, $A(\mathbf{x}_*)$ contains a periodic orbit. This periodic orbit cannot be phase asymptotically stable. Hence there is at most one periodic orbit on Σ.

5.5 An Invariant 3-Dimensional Cylinder

Proposition 5.1 is used as an example of Theorem 4.9 on a 3-dimensional invariant cylinder. Let

$$\begin{aligned}
F_1(\mathbf{x}) &:= x_2 + \frac{x_1^2}{50}, \\
F_2(\mathbf{x}) &:= -x_1 + \frac{x_2^2}{50}, \\
F_3(\mathbf{x}) &:= -x_3 + x_1/10, \\
F_4(\mathbf{x}) &:= -x_4 + x_2/10.
\end{aligned}$$

Let $H := \mathit{Diagonal}[1,1,0,0]$. Then, Proposition 5.1 implies that $\Sigma := \{\mathbf{x} : |H\mathbf{x}|^2 = x_1^2 + x_2^2 = 1\}$ is an invariant manifold for the differential equation

$$\begin{aligned}
\frac{dx_1}{dt} &= x_2 + \frac{x_1^2}{50} - x_1\frac{(x_1^3 + x_2^3)}{50}, \\
\frac{dx_2}{dt} &= -x_1 + \frac{x_2^2}{50} - x_2\frac{(x_1^3 + x_2^3)}{50}, \\
\frac{dx_3}{dt} &= -x_3 + \frac{x_1}{10} - x_3\frac{(x_1^3 + x_2^3)}{50}, \\
\frac{dx_4}{dt} &= -x_4 + \frac{x_2}{10} - x_4\frac{(x_1^3 + x_2^3)}{50},
\end{aligned}$$

written concisely as

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}). \tag{5.8}$$

Let $E := \{\mathbf{x} \in \Sigma : |x_1|, |x_2| < 1\}$. We show that all solutions to (5.8) on Σ will eventually enter and stay in E. Take $\mathbf{x}(t)$ to be a solution of (5.8) on Σ. Suppose $x_3(0) \geq 0$. Then,

$$\begin{aligned}
\frac{dx_3}{dt} &\leq -x_3 + \frac{x_1}{10} - x_3\frac{(x_1^3 + x_2^3)}{50} \\
&\leq -x_3 + \frac{1}{10} + \frac{2x_3}{50} \\
&\leq -24x_3/25 + \frac{1}{10}
\end{aligned}$$

and so $x_3(t) < 5/48 + C\exp(-24t/25)$. Hence, $x_3(t)$ is eventually less than 1. Similarly, if $x_3(0) < 0$, then $x_3(t)$ is eventually greater than -1. Therefore, all solutions are bounded in the x_3 direction. Likewise, all solutions are eventually enter and remain in the region $|x_4| < 1$. Therefore the region E is positively invariant under the flow generated by (5.8). In particular, all solutions are bounded.

Suppose for $\mathbf{x} \in \Sigma$ that $\mathbf{f}(\mathbf{x}) = 0$. Then,

$$\begin{aligned} 0 &= x_2 f_1(\mathbf{x}) - x_1 f_2(\mathbf{x}) \\ &= 50(x_1^2 + x_2^2) + x_1^2 x_2 - x_1 x_2^2 \\ &> 50 - 2 = 48. \end{aligned}$$

Clearly this is impossible, which implies that are no equilibria in Σ.

Take $\mathbf{x}(t)$ to be a solution of (5.8) in E. We show that

$$\frac{d\mathbf{z}}{dt} = \left(\frac{\partial \mathbf{f}}{\partial x}^{[3]}(\mathbf{x}(t)) - \nu(\mathbf{x}(t)) I_4\right)\mathbf{z} \tag{5.9}$$

is uniformly asymptotically stable. By the definition of compound matrices

$$\frac{\partial \mathbf{f}}{\partial x}^{[3]}(\mathbf{x}) - \nu(\mathbf{x}) I_4 =$$

$$\begin{pmatrix} \frac{x_1-2x_1^3+x_2-2x_2^3-25}{25} & 0 & 0 & 0 \\ 0 & \frac{x_1-2x_1^3+x_2-2x_2^3-25}{25} & 0 & 0 \\ \frac{-5+3x_4x_2^2}{50} & -\frac{3x_2^2x_3}{50} & \frac{2x_1-4x_1^3-x_2^3-100}{50} & \frac{50-3x_1x_2^2}{50} \\ -\frac{3x_4x_1^2}{50} & -\frac{5+3x_3x_1^2}{50} & \frac{-50-3x_2x_1^2}{50} & \frac{2x_1-x_1^3+2x_2-4x_2^3-100}{50} \end{pmatrix}.$$

Let $a_{ij}(\mathbf{x})$ be the components of this matrix. For $\mathbf{z} \in \mathbb{R}^4$, consider the Lyapunov function for the system (5.9), $V(\mathbf{z}) = ||\mathbf{z}||_\infty = \max\{|z_1|, |z_2|, |z_3|, |z_4|\}$. From [1] page 58,

$$\frac{d}{dt}V(\mathbf{z}) \leq \mu(\mathbf{x}(t))V(\mathbf{z}), \tag{5.10}$$

where

$$\mu(\mathbf{x}) = \max_{1\le i\le 4}\{\mu_i(\mathbf{x})\} \quad \text{and} \quad \mu_i(\mathbf{x}) = \{a_{ii}(\mathbf{x}) + \sum_{i=1,\, i\neq j}^{4} |a_{ij}(\mathbf{x})|\}.$$

By the theory of Lyapunov functions (see [5] page 305), if $V(\mathbf{z})$ is positive definite and $\frac{d}{dt}V(\mathbf{z})$ is negative definite, then (5.9) will be uniformly asymptotically stable. Since the norm is positive definite, $V(\mathbf{z})$ is positive definite. From (5.10), $\frac{d}{dt}V(\mathbf{z})$ will be negative definite if for some constant c, $\mu(\mathbf{x}(t)) \le -c < 0$ for $t \ge 0$.

For $\mathbf{x} \in E$, we have

$$\mu_1(\mathbf{x}) = \frac{-25 + x_1 - 2x_1^3 + x_2 - 2x_2^3}{25} < \frac{-25+1+2+1+2}{25} < -\frac{19}{25},$$

$$\mu_2(\mathbf{x}) = \frac{-25 + x_1 - 2x_1^3 + x_2 - 2x_2^3}{25} \le -\frac{19}{25},$$

$$\begin{aligned}\mu_3(\mathbf{x}) = \left|\frac{-5+3x_4x_2^2}{50}\right| + \left|-\frac{3x_2^2x_3}{50}\right| + \left|\frac{50-3x_1x_2^2}{50}\right| + \frac{-100+2x_1-4x_1^3-x_2^3}{50} \\ < \frac{5+3+3+50+3-100+2+4+1}{50} \le \frac{-29}{50},\end{aligned}$$

$$\begin{aligned}\mu_4(\mathbf{x}) = \left|\frac{3x_4x_1^2}{50}\right| + \left|\frac{5+3x_3x_1^2}{50}\right| + \left|\frac{50+3x_2x_1^2}{50}\right| + \frac{-100-x_1^3+2x_2-4x_2^3}{50} \\ < \frac{3+5+3+50+3-100+1+2+4}{50} \le \frac{-29}{50}.\end{aligned}$$

Consequently, Since E is positively invariant set for system (5.8), $\mu(\mathbf{x}(t)) \le -29/50$ for $t \ge 0$. We conclude that (5.9) is uniformly asymptotically stable. Thus, the conditions of Theorem 4.9 are satisfied. Hence, there exists a phase asymptotically stable periodic orbit with respect to the dynamics on Σ.

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Appendix A

Compound Equations

In this appendix, compound matrices and compound equations are considered.

A.1 Compound Matrices

Suppose X is an $n \times m$ real or complex matrix. Let $x^{j_1 \dots j_k}_{i_1 \dots i_k}$ denote the minor of X determined by the rows $(i_1, \dots, i_k)$ and the columns $(j_1, \dots, j_k)$ of X where $1 \leq i_1 < i_2 < \dots < i_k \leq n$ and $1 \leq j_1 < j_2 < \dots < j_k \leq m$.

Definition A.1. The *k-th* multiplicative compound, $X^{(k)}$, of X is the $\binom{n}{k} \times \binom{m}{k}$ matrix whose entries written in lexicographical order are $x^{j_1 \dots j_k}_{i_1 \dots i_k}$.

Theorem A.2 (Binet-Cauchy Theorem). *Let A and B be $n \times l$ and $l \times m$ real or complex matrices, respectively. Then*

$$(AB)^{(k)} = A^{(k)} B^{(k)}. \tag{A.1}$$

A proof can be found in [9] page 17.

Definition A.3. Suppose X is a real or complex $n \times n$-matrix, and let k be an integer such that $1 \leq k \leq n$. The *k-th* additive compound matrix of X is defined by $X^{[k]} = D(I + hX)^{(k)}|_{h=0}$.

Proposition A.4. *Suppose A and B are real or complex $n \times n$ matrices. Then*

(i) $(A^{-1})^{(k)} = (A^{(k)})^{-1}$ *if A is nonsingular.*

(ii) $I^{(k)}_{n\times n} = I_{N\times N}$ *where* $N = \binom{n}{k}$.

(iii) $(A+B)^{[k]} = A^{[k]} + B^{[k]}$.

The components of the k-*th* additive compound matrix can be computed as follows. For the integers $i = 1, \ldots, \binom{n}{k}$, let $(i) = (i_1, \ldots, i_k)$ be the i^{th} index in the lexicographic ordering of all k-tuples of integers such that $1 \le i_1 < i_2 < \ldots < i_k \le n$. If $Y = X^{[k]}$, then

$$y_i^j = \begin{cases} x_{i_1}^{i_1} + \ldots + x_{i_k}^{i_k} & \text{if } (i) = (j), \\ (-1)^{r+s} x_{i_s}^{i_r} & \text{if exactly one entry } i_s \text{ in } (i) \text{ does not occur} \\ & \text{in } (j) \text{ and } j_r \text{ does not occur in} (i), \\ 0 & \text{if } (i) \text{ differs from } (j) \text{ in two or more entries.} \end{cases}$$

When $n = 3$, the additive compound matrices are

$$A^{[1]} = A = \begin{bmatrix} a_1^1 & a_1^2 & a_1^3 \\ a_2^1 & a_2^2 & a_2^3 \\ a_3^1 & a_3^2 & a_3^3 \end{bmatrix},$$

$$A^{[2]} = \begin{bmatrix} a_1^1 + a_2^2 & a_2^3 & -a_1^3 \\ a_3^2 & a_1^1 + a_3^3 & a_1^2 \\ -a_3^1 & a_2^1 & a_2^2 + a_3^3 \end{bmatrix},$$

$$A^{[3]} = a_1^1 + a_2^2 + a_3^3 = \mathrm{Tr}[A].$$

When $n = 4$, the additive compound matrices are

$$A^{[1]} = A = \begin{bmatrix} a_1^1 & a_1^2 & a_1^3 & a_1^4 \\ a_2^1 & a_2^2 & a_2^3 & a_2^4 \\ a_3^1 & a_3^2 & a_3^3 & a_3^4 \end{bmatrix},$$

$$A^{[2]} = \begin{bmatrix} a_1^1 + a_2^2 & a_2^3 & a_2^4 & -a_1^3 & -a_1^4 & 0 \\ a_3^2 & a_1^1 + a_3^3 & a_3^4 & a_1^2 & 0 & -a_1^4 \\ a_4^2 & a_4^3 & a_1^1 + a_4^4 & 0 & a_1^2 & a_1^3 \\ -a_3^1 & a_2^1 & 0 & a_2^2 + a_3^3 & a_3^4 & -a_2^4 \\ -a_4^1 & 0 & a_2^1 & a_4^3 & a_2^2 + a_4^4 & a_2^3 \\ 0 & -a_4^1 & a_3^1 & -a_4^2 & a_3^2 & a_3^3 + a_4^4 \end{bmatrix},$$

$$A^{[3]} = \begin{bmatrix} a_1^1 + a_2^2 + a_3^3 & a_3^4 & -a_2^4 & a_1^4 \\ a_4^3 & a_1^1 + a_2^2 + a_4^4 & a_2^3 & -a_1^3 \\ -a_4^2 & a_3^2 & a_1^1 + a_3^3 + a_4^4 & a_1^2 \\ a_4^1 & -a_3^1 & a_2^1 & a_2^2 + a_3^3 + a_4^4 \end{bmatrix},$$

$$A^{[4]} = a_1^1 + a_2^2 + a_3^3 + a_4^4 = \mathrm{Tr}[A].$$

In addition, the additive and multiplicative matrices can be defined in a more general setting. Let $T : \mathbb{R}^n \mapsto \mathbb{R}^n$ be a bounded linear operator. Define two linear operators $T^{[k]}$ and $T^{(k)}$ from the wedge product space $\bigwedge^k \mathbb{R}^n$ to $\mathbb{R}$ for $\mathbf{u}_1, \ldots, \mathbf{u}_k \in \mathbb{R}^n$ as

$$T^{(k)}(\mathbf{u}_1 \wedge \ldots \wedge \mathbf{u}_k) = T\mathbf{u}_1 \wedge \ldots \wedge T\mathbf{u}_k \tag{A.2}$$

and

$$T^{[k]}(\mathbf{u}_1 \wedge \ldots \wedge \mathbf{u}_k) = \sum_i \mathbf{u}_1 \wedge \ldots \wedge T\ \mathbf{u}_i \wedge \ldots \wedge \mathbf{u}_k, \tag{A.3}$$

respectively. If we identify T, $T^{[k]}$, and $T^{(k)}$ with their matrix representation with respect to the canonical basis on $\mathbb{R}^n$, then $T^{[k]}$ and $T^{(k)}$ will be the *k-th* additive compound and *k-th* multiplicative compound matrices of T, respectively. In this definition, the space $\mathbb{R}$ may be replaced with $\mathbb{C}$, the complex numbers.

Proposition A.5. *Suppose that $\lambda_1, \ldots, \lambda_n$, are the eigenvalues of a $n \times n$ matrix A. Then*

(i) *the eigenvalues of $A^{(k)}$, counting multiplicities, are given the $\binom{n}{k}$ possible products of the form:*

$$\lambda_{i_1} \ldots \lambda_{i_k}, \quad 1 \leq i_1 < \ldots < i_k \leq n.$$

(ii) *the eigenvalues of $A^{[k]}$, counting multiplicities, are the $\binom{n}{k}$ possible sums of the form:*

$$\lambda_{i_1} + \ldots + \lambda_{i_k}, \quad 1 \leq i_1 < \ldots < i_k \leq n.$$

(iii) *suppose that $\mathbf{x}_1, \ldots, \mathbf{x}_k$ are independent eigenvectors of A corresponding to the eigenvalues $\lambda_{i_1}, \ldots, \lambda_{i_k}$. Then $\mathbf{x}_1 \wedge \ldots \wedge \mathbf{x}_k$ is an eigenvector of $A^{(k)}$ and $A^{[k]}$ corresponding to the eigenvalue $\lambda_{i_1} \ldots \lambda_{i_k}$ and $\lambda_{i_1} + \ldots + \lambda_{i_k}$, respectively.*

A.2 Linear Differential Equations and Compound Differential Equations

Suppose $A \in C(\mathbb{R} \mapsto M_{n \times n})$. Consider the linear non-autonomous differential equation

$$\frac{d\mathbf{x}}{dt} = A(t)\mathbf{x} \tag{A.4}$$

and its *k-th* compound

$$\frac{d\mathbf{y}}{dt} = A^{[k]}(t)\mathbf{y} \tag{A.5}$$

where $k \in \mathbb{N}$, $1 \leq k \leq n$.

Theorem A.6. *Suppose $X(t)$ is a fundamental solution of (A.4). Then $X^{(k)}(t)$ is a fundamental solution of (A.5).*

If $k = n$, then $X^{(n)}(t) = \det(X(t))$ and $A^{[n]}(t) = \mathrm{Tr}(A(t))$. In this case the above theorem reduces to the Abel-Jacobi-Liouville formula:

$$\frac{d}{dt}\det(X(t)) = \mathrm{Tr}(A(t))\det(X(t))$$

which implies

$$\det(X(t)) = \det(X(0))\exp\left(\int_0^t \mathrm{Tr}(A(s))ds\right).$$

Let χ be a subspace of $C([0,\infty] \mapsto \mathbb{R}^n)$ and let χ^k denote its k-th exterior power, $1 \leq k \leq n$:

$$\chi^k = span\{\mathbf{x}^1 \wedge \ldots \wedge \mathbf{x}^k : \mathbf{x}^i \in \chi\}.$$

Further, define

$$\chi_0 = \{\mathbf{x} \in \chi : \lim_{t\to\infty} \mathbf{x}(t) = 0\}$$

Theorem A.7. *Suppose for* $\mathbf{x}(t) \in \chi$ *that*

(i) $\limsup_{t\to\infty} |\mathbf{x}(t)| < \infty$.

(ii) $\liminf_{t\to\infty} |\mathbf{x}(t)| = 0$ *implies that* $\lim_{t\to\infty} \mathbf{x}(t) = 0$.

Then

$$codim(\chi_0) < k \Leftrightarrow \chi_o^{(k)} = \chi^{(k)}. \tag{A.6}$$

This theorem is proved in [10].

Corollary A.8. *Suppose the system (A.4) is uniformly stable. Then a necessary and sufficient condition that (A.4) have an $(n-k+1)$-dimensional set of solutions satisfying* $\lim_{t\to\infty} \mathbf{x}(t) = 0$ *is that the system (A.5) be uniformly asymptotically stable.*

Appendix B

A Linear Result

In this section, sufficient conditions are provided for the existence of a dichotomy of the solutions to the variational equation of a differential equation on the tangent space to an invariant manifold. Theorem B.3 is similar to Proposition 3.1 in [7]. In their result, the invariant manifold is all of $\mathbb{R}^n$. A few adjustments to the proof in [7] are required for the proof in the more general setting.

Let $D \subset \mathbb{R}^n$, $\mathbf{f} \in C^1(D \mapsto \mathbb{R}^n)$, and $\mathbf{g} \in C^2(D \mapsto \mathbb{R}^m)$ where m is an integer and $0 \leq m \leq n$. Consider the autonomous differential equation

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}). \tag{B.1}$$

Let $\varphi(t, \mathbf{x})$ be the flow generated by (B.1). The linear variational equation of (B.1) at a solution $\varphi(t, \mathbf{x}_0)$ is

$$\frac{d\mathbf{y}}{dt} = \frac{\partial \mathbf{f}}{\partial x}(\varphi(t, \mathbf{x}_0))\mathbf{y}, \tag{B.2}$$

where $\frac{\partial \mathbf{f}}{\partial x}(\mathbf{x})$ is the Jacobian matrix of $\mathbf{f}$ at $\mathbf{x}$. An equation that will be important in what follows is the second compound:

$$\frac{d\mathbf{w}}{dt} = \frac{\partial \mathbf{f}}{\partial x}^{[2]}(\varphi(t, \mathbf{x}_0))\mathbf{w}, \tag{B.3}$$

where $\frac{\partial \mathbf{f}}{\partial x}^{[2]}(\mathbf{x})$ is the second additive compound of the Jacobian matrix of $\mathbf{f}$ at $\mathbf{x}$. See Appendix A for the definitions of additive and multiplicative compound matrices. Let $\Sigma := \{\mathbf{x} \in \mathbb{R}^n : \mathbf{g}(\mathbf{x}) = 0\}$ be a $(n-m)$-dimensional invariant manifold and let $\nu(\mathbf{x})$ be the corresponding function defined in Section 4.1. A linear equation associated with the linear variational equation is

$$\frac{d\mathbf{z}}{dt} = \left[\frac{\partial \mathbf{f}}{\partial x}^{[m+2]}(\varphi(t, \mathbf{x}_0)) - \nu(\varphi(t, \mathbf{x}_0))\right] \mathbf{z} \tag{B.4}$$

where $B^{[m+2]}$ is the $(m+2)$-th additive compound matrix of the matrix B.

The matrix $B^{(2)}$ is the second multiplicative compound of the matrix B. If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of the $n \times n$ real matrix A. Then, $\lambda_i \lambda_j$ and $\lambda_i + \lambda_j$, $1 \leq i < j \leq n$, are the eigenvalues of $A^{(2)}$ and $A^{[2]}$, respectively. The numbers $\sigma_1^2 \geq \ldots \geq \sigma_n^2 \geq 0$ are the singular values of A if $\sigma_1^2, \ldots, \sigma_n^2$ are the eigenvalues of the symmetric matrix A^*A. For the l^2-norm on $\mathbb{R}^n$, $|\mathbf{x}| = (\mathbf{x}^*\mathbf{x})^{1/2}$ and the matrix norm it induces, $|A| = \sigma_1$ and $|A^{(2)}| = \sigma_1 \sigma_2$.

The following two propositions are used in the proof of Theorem B.3 below.

Proposition B.1. *Let V be a subspace of $\mathbb{R}^n$. Suppose that V decomposes into a direct sum $V = V_1 + V_2$ of subspaces of V, and P_1 and $P_2 = I|_V - P_1$ are the corresponding projections onto these subspaces. Then, the following estimate is valid:*

$$1/|P_k| \leq 2\sin(\theta/2) \leq 2/|P_k| \quad \text{for } k = 1, 2 \tag{B.5}$$

where θ is the angle between the two subspaces V_1 and V_2.

The above proposition is proved in [4] page 156. There, V is a Banach space and V_1, V_2 are closed subspaces of V.

Proposition B.2. *Suppose that there exist positive constants α, M, N, J and supplementary projection matrices P_1, P_2 on $\mathcal{T}_{\mathbf{x}_0}$ such that $P_1 + P_2 = I|_{\mathcal{F}_{x_0}}$,*

$$|Y(t)P_1\xi| \leq Me^{-\alpha(t-s)}|X(s)P_1\xi| \text{ for } 0 \leq s \leq t, \tag{B.6}$$

$$|Y(t)P_2\xi| \le N|X(s)P_2\xi| \; for\; 0 \le t \le s, \tag{B.7}$$

$$|Y(t)P_1Y^{-1}(t)| \le J \; for\; t \ge 0, \tag{B.8}$$

for any $\xi \in \mathbb{R}^n$. Then, there exist positive constants M' and N' such that

$$|Y(t)P_1Y^{-1}(s)| \le M'e^{-\alpha(t-s)}, \; for\; 0 \le s \le t, \tag{B.9}$$

$$|Y(t)P_2Y^{-1}(s)| \le N', \; for\; 0 \le t \le s. \tag{B.10}$$

Proof. Let the columns of $Y^{-1}(t)$ be $e_1(t), \dots, e_n(t)$. Then

$$\begin{aligned} |Y(t)P_1Y^{-1}(s)| &\le \sum_{i=1}^{n} |Y(t)P_1e_i(s)| \\ &\le \sum_{i=1}^{n} |Y(s)P_1e_i(s)|Me^{-\alpha(t-s)} \\ &\le nJMe^{-\alpha(t-s)} \end{aligned}$$

for $t \ge s \ge 0$. Thus, (B.9) holds with $M' = nJM$. Equation $(B.10)$ likewise holds. □

The result above can be found in [2].

Theorem B.3. *Suppose*

(i) *the flow is Lagrange stable at* $\mathbf{x}_0 \in \Sigma$;

(ii) *the ω-limit set, $\Omega(\mathbf{x}_0)$, contains no equilibria;*

(iii) *the equation* (B.4) *is uniformly asymptotically stable.*

Then, there exist positive constants N,M and supplementary matrix projections P_1,P_2 on $\mathbb{R}^n$, where $P_1 + P_2 = I|_{T_{x_0}}$, $\mathrm{rk}(P_1) = (n-m-1)$ and $\mathrm{rk}(P_2) = 1$ such that

$$|Y(t)P_1Y^{-1}(s)| \le Me^{-\alpha(t-s)} \quad for\; 0 \le s \le t,$$

$$|Y(t)P_2Y^{-1}(s)| \le N \quad for\; 0 \le t \le s.$$

Proof. Let $\mathbf{y}_1(t) := \frac{d}{dt}\varphi(t, \mathbf{x}_0)$. Then, $\mathbf{y}_1(t)$ is a solution of (B.2). We now show that there exists a $L > 1$ such that

$$|\mathbf{y}_1(t)| \leq L|\mathbf{y}_1(s)| \quad \text{for } s, t \geq 0. \tag{B.11}$$

Condition (ii) implies that there exists a b such that

$$0 < b < |\mathbf{y}_1(s)| \tag{B.12}$$

for $s > 0$. If not, then either $\Gamma_+(\mathbf{x}_0)$ contains an equilibrium or there exists a sequence $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$, such that $\mathbf{f}(\varphi(t_n)) = \mathbf{y}_1(t_n) \to 0$ as $n \to \infty$. This implies, from continuity and condition (i), that $\Omega(\mathbf{x}_0)$ contains an equilibrium. In either case, there is a contradiction to condition (ii). By the continuous differentiability of $\varphi(t, \mathbf{x}_0)$ in t and condition (i), there exists a constant a such that

$$|\mathbf{y}_1(t)| < a \quad \text{for } t \geq 0. \tag{B.13}$$

Then, (B.12) and (B.13) imply $|\mathbf{y}_1(t)|/|\mathbf{y}_1(s)| < a/b$ for $t, s \geq 0$. Suppose a had be chosen large enough so that $1 < a/b$. Then, the assertion is proved with $L := a/b$.

Suppose $\mathbf{y}(t)$ is a solution of (B.2). Condition (i) and the continuity of $\frac{\partial \mathbf{f}}{\partial x}(\mathbf{x})$ implies that there exists a constant β such that $|\frac{\partial \mathbf{f}}{\partial x}(\varphi(t, \mathbf{x}_0))| < \beta$ for $t \geq 0$. Thus,

$$\begin{aligned} |\mathbf{y}(t)| &\leq |\mathbf{y}(s)| + \int_s^t \left|\frac{\partial \mathbf{f}}{\partial x}(\varphi(\tau))\right| |\mathbf{y}(\tau)| d\tau \\ &\leq |\mathbf{y}(s)| + \int_s^t \beta |\mathbf{y}(\tau)| d\tau \quad \text{for } t \geq s \geq 0. \end{aligned}$$

Then, by Gronwall's inequality

$$|\mathbf{y}(t)| \leq |\mathbf{y}(s)| e^{\beta(t-s)} \quad \text{for } 0 \leq s \leq t. \tag{B.14}$$

Recall from Appendix A, if $A(t)$ is a continuous $n \times n$-matrix valued function and $X(t)$ is a fundamental solution of the linear differential equation $d\mathbf{x}/dt = A(t)\mathbf{x}$, then $X^{(k)}$ is fundamental solution of the linear equation $d\mathbf{w}/dt = A^{[k]}\mathbf{w}$.

Suppose $Y(t)$ is a fundamental solution of (B.2) such that $Y(0) = I$. This implies that $Y(t) = \frac{\partial \varphi}{\partial x}(t, \mathbf{x}_0)$. By direct substitution,

$$Z(t) := Y(t)^{(n-m+2)} e^{-\int_0^t \nu(\varphi(s))ds}$$

is a matrix solution of (B.4). Since $Z(0) = I$, $Z(t)$ is a fundamental solution of (B.4). From condition (iii), there exist positive constants α, C such that

$$\left| \bigwedge^{(m+2)} Y(t) \right| = \left| Y^{(m+2)}(t) \right| < Ce^{-\alpha t} e^{\int_0^t \nu(\varphi(s,\mathbf{x}_0))ds}.$$

By Proposition 4.4 with $k = 2$ and the previous equation,

$$\left| \bigwedge^{2} Y(t) \Big|_{\mathcal{T}_{\varphi(t,x_0)}} \right| \leq \left| Y^{(m+2)}(t) \right| e^{-\int_0^t \nu(\varphi(s,\mathbf{x}_0))ds} < CKe^{-\alpha t} \qquad \text{(B.15)}$$

for $t > 0$.

The tangent space to Σ at $\varphi(s, \mathbf{x}_0)$, $\mathcal{T}|_{\varphi(s,\mathbf{x}_0)}$, is mapped under the transformation $Y(t)Y^{-1}(s)$ to the tangent space $\mathcal{T}|_{\varphi(t-s,\mathbf{x})}$. The map

$$(Y^{-1}(s))^* Y^*(t) Y(t) Y^{-1}(s)$$

leaves the space $\mathcal{T}|_{\varphi(s,\mathbf{x}_0)}$ invariant. Let

$$\sigma_1(s,t) \geq \ldots \geq \sigma_{n-m}(s,t) \geq 0$$

be the singular values of the values of this map associated with the invariant subspace. Then, $\sigma_1(s,t) = \left| Y(t)Y^{-1}(s)|_{\mathcal{T}_{\varphi(s,x_0)}} \right|$, and hence from (B.14),

$\sigma_1(s,t) \le e^{\beta(t-s)}$ if $0 \le s \le t$. Similarly, from (B.11)

$$1/L \le \sigma_1(s,t) \quad \text{for } t, s \ge 0. \tag{B.16}$$

Further, (B.15) implies

$$\left|(Y(t)Y^{-1}(s)|_{T_{\varphi(s,x_0)}})^{(2)}\right| = \sigma_1\sigma_2(s,t) \le CKe^{-\alpha(t-s)} \quad \text{for } 0 \le s \le t. \tag{B.17}$$

It follows from (B.16) and (B.17) that

$$\sigma_2(s,t) \le LCKe^{-\alpha(t-s)} \quad \text{for } 0 \le s \le t. \tag{B.18}$$

Let $\delta > 0$. From (B.17) and (B.18), $T > 0$ can be chosen sufficiently large so that if $t = s + T$, then

$$\sigma_1\sigma_2(s,t) < \delta, \quad \text{and } \sigma_2(s,t) < \delta. \tag{B.19}$$

For a fixed s, let $\mathcal{Y}_1 := \text{span}\{\mathbf{y}_1(\cdot)\}$ and $\mathcal{Y}_2 := \text{span}\{\mathbf{y}_2(\cdot), \ldots, \mathbf{y}_{n-m}(\cdot)\}$, where $\mathbf{y}_i(s)$ is an eigenvector of $(Y^{-1}(s))^* Y^*(t) Y(t) Y^{-1}(s)$ corresponding to the eigenvalues $\sigma_i^2(t,s)$ for $i = 2, \ldots, n-m$ and $\mathbf{y}_i(\cdot)$ is a solution of (B.2) for $i = 1, \ldots, n-m$. Then, since $\mathbf{y}(t) = Y(t)Y^{-1}(s)\mathbf{y}(s)$, $\mathbf{y}(\cdot) \in \mathcal{Y}_2$ implies

$$|\mathbf{y}(t)| \le |\mathbf{y}(s)|\sigma_2(s,t). \tag{B.20}$$

The subspaces $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are supplementary, that is $\mathcal{Y}_1 \cap \mathcal{Y}_2 = \{0\}$ and $\mathcal{Y}_1 \oplus \mathcal{Y}_2 = \mathcal{T}|_{\varphi(s,x_0)}$, since a solution in $\mathcal{Y}_1$ is bounded away from zero and solutions in $\mathcal{Y}_2$ decay to zero. Let $t := s + T$. Suppose T had been chosen sufficiently large so that $0 < 1/L - \delta$. Let $y(\cdot) \in \mathcal{Y}_2$. From (B.16), (B.19), and (B.20),

$$0 < 1/L - \delta \le \left|\frac{\mathbf{y}_1(t)}{|\mathbf{y}_1(s)|} - \frac{\mathbf{y}(t)}{|\mathbf{y}(s)|}\right|. \tag{B.21}$$

Since $\frac{\mathbf{y}_1(t)}{|\mathbf{y}_1(s)|} - \frac{\mathbf{y}(t)}{|\mathbf{y}(s)|}$ is a solution of (B.2) tangent to the invariant manifold,

$$\left|\frac{\mathbf{y}_1(t)}{|\mathbf{y}_1(s)|} - \frac{\mathbf{y}(t)}{|\mathbf{y}(s)|}\right| \leq \left|\frac{\mathbf{y}_1(s)}{|\mathbf{y}_1(s)|} - \frac{\mathbf{y}(s)}{|\mathbf{y}(s)|}\right| \sigma_1(s,t).$$

This implies with (B.21) that the angular separation, $\inf |(\mathbf{y}_1(s)/|\mathbf{y}_1(s)|) - (\mathbf{y}(s)/|\mathbf{y}(s)|)|$, over $\mathbf{y}(\cdot) \in \mathcal{Y}_2$, between spaces of initial values $\{\mathbf{y}(s) : \mathbf{y}(\cdot) \in \mathcal{Y}_i\}$, $i = 1, 2$, is at least $(1-L\delta)/L\sigma_1(s,t)$. Therefore, projections $P_i(s)$, $i = 1, 2$ with $P_1(s) + P_2(s) = I|_{\mathcal{T}_{\varphi(s,\mathbf{x}_0)}}$ from $\mathcal{T}|_{\varphi(s,\mathbf{x}_0)}$ onto these initial value subspaces satisfy

$$|P_i(s)| \leq \gamma\sigma_1(s,t) \quad \text{for } i = 1, 2 \tag{B.22}$$

where $\gamma = 2L/(1 - L\delta)$, from Proposition B.1.

The space $\mathcal{Y}_1$ is independent of (s,t), while $\mathcal{Y}_2$ is not necessarily so, since the span of the eigenvectors $\mathbf{y}_i(s)$ is not necessarily so. Let $s_0 \geq 0$, $s_k := s_0 + kT$ for $k = 0, 1, \ldots$, and $\mathcal{Y}_{2,k}$ denote the space $\mathcal{Y}_2$ corresponding to $(s,t) = (s_k, s_{k+1})$. If $\mathbf{y}(\cdot)$ is a solution of (B.2) with $\mathbf{y}(0) \in \mathcal{T}_{\mathbf{x}_0}$, then for $k = 0, 1, 2, \ldots$ there exist $\mathbf{y}_{1,k}(\cdot) \in \mathcal{Y}_{1,k}$ and $\mathbf{y}_{2,k}(\cdot) \in \mathcal{Y}_{2,k}$ such that $\mathbf{y}(\cdot) = \mathbf{y}_{1,k}(\cdot) + \mathbf{y}_{2,k}(\cdot)$. Then,

$$\mathbf{y}_{i,k}(s_k) = P_i(s_k)\mathbf{y}(s_k) = P_i(s_k)\mathbf{y}_{1,k-1}(s_k) + P_i(s_k)\mathbf{y}_{2,k-1}(s_k)$$

for $i = 1, 2$. Simplifying,

$$\mathbf{y}_{1,k}(s_k) = \mathbf{y}_{1,k-1}(s_k) + P_1(s_k)\mathbf{y}_{2,k-1}(s_k), \tag{B.23}$$

for $k = 1, 2, \ldots$, since $\mathcal{Y}_{1,k}$ is independent of k, and

$$\mathbf{y}_{2,k}(s_k) = P_2(s_k)\mathbf{y}_{2,k-1}(s_k).$$

From (B.20) and (B.22),

$$\begin{aligned} |P_i(s_k)\mathbf{y}_{2,k-1}(s_k)| &\leq \gamma\sigma_1(s_k, s_{k+1})|\mathbf{y}_{2,k-1}(s_k)| \\ &\leq \gamma\sigma_1(s_k, s_{k+1})\sigma_2(s_{k-1}, s_k)|\mathbf{y}_{2,k-1}(s_{k-1})| \end{aligned}$$

for $i = 1, 2$ and $k = 1, 2 \ldots$. By the same reasoning $|\mathbf{y}_{2,0}(s_0)| = |P_2(s_0)\mathbf{y}(s_0)| \le \gamma\sigma_1(s_0, s_1)|\mathbf{y}(s_0)|$. Hence, by induction, (B.20), and (B.22)

$$\begin{aligned} |P_i(s_k)\mathbf{y}_{2,k-1}(s_k)| &\le \gamma^{k+1}\sigma_1(s_k, s_{k+1}) \prod_{j=1}^{k}[\sigma_1\sigma_2(s_{j-1}, s_j)]|\mathbf{y}(s_0)| \\ &\le \gamma^{k+1}\delta^k e^{\beta T}|\mathbf{y}(s_0)|. \end{aligned} \tag{B.24}$$

Let constants c_k be defined so that $\mathbf{y}_{1,k}(\cdot) = c_k\mathbf{y}_1(\cdot)$. Then, (B.23) is equivalent to $c_k = c_{k-1} + \Delta_k$ where $\Delta_k\mathbf{y}_1(s_k) = P_1(s_k)\mathbf{y}_{2,k-1}(s_k)$ for $k = 1, 2, \ldots$. Thus, by (B.24), $|\Delta_k| \le \gamma^{k+1}\delta^k e^{\beta T}|\mathbf{y}(s_0)|/|\mathbf{y}_1(s_k)|$. Additionally, $|c_0||y_1(s_0)| \le |P_1(s_0)||\mathbf{y}(s_0)|$. This implies that $|c_0| \le \gamma e^{\beta T}|\mathbf{y}(s_0)|/|\mathbf{y}_1(s_0)|$. Therefore, if δ had been chosen small enough so that $\gamma\delta < 1$, then

$$|c_k| \le |c_0| + \sum_{j=1}^{k}|\Delta_j| \le \frac{\gamma M_1 e^{\beta T}}{m_1(1-\gamma\delta)}|y(s_0)| \tag{B.25}$$

where $m_1 := \inf_{s\ge 0}|y_1(s)|$ and $M_1 := \sup_{s\ge 0}|y_1(s)|$. Further,

$$|\mathbf{y}_{2,k}(s_k)| = |P_2(s_k)\mathbf{y}_{2,k-1}(s_k)| \le \gamma^{k+1}\delta^k e^{\beta T}|\mathbf{y}(s_0)| \le \gamma e^{\beta T}|\mathbf{y}(s_0)|. \tag{B.26}$$

Together (B.25) and (B.26) imply that

$$|\mathbf{y}(s_k)| \le |\mathbf{y}_{1,k}(s_k)| + |\mathbf{y}_{2,k}(s_k)| \le \gamma e^{\beta T}\left[\frac{M_1}{m_1(1-\gamma\delta)} + 1\right]|\mathbf{y}(s_0)|.$$

From (B.14), $|\mathbf{y}(t)| \le e^{\beta T}|\mathbf{y}(s_k)|$ for $s_k \le t \le s_{k+1}$, and so

$$|\mathbf{y}(t)| \le H|\mathbf{y}(s_0)| \quad \text{for } 0 \le s_0 \le t \tag{B.27}$$

where $H := \delta e^{2\beta T}[M_1/m_1(1-\gamma\delta)+1]$. We conclude that (B.2) has a uniformly stable subspace.

Let χ be this uniformly stable subspace. Then, (B.27) implies that all solutions are bounded. Hence, condition (i) of Theorem (A.7 is satisfied.. Suppose $\mathbf{y}(t) \in \chi$ and $\liminf_{t\to\infty}|\mathbf{y}(t)| = 0$. Then, there exists a sequence $t_i \to \infty$ as $i \to \infty$, such that $\lim_{i\to\infty}|\mathbf{y}(t_i)| = 0$. Additionally, (B.27) implies that $|\mathbf{y}(t)| \leq H|\mathbf{y}(t_i)|$ if $0 \leq t_i < t$, and hence $\lim_{t\to\infty} \mathbf{y}(t) = 0$. Therefore, condition (ii) of Theorem A.7 is satisfied. From equation (B.15), $\chi^{(2)} = \chi_0^{(2)}$. This implies from Theorem A.7 that $\operatorname{codim}\chi_0 < 2$, and therefore $n - m - 1 \leq \dim(\chi_0) \leq n - m$. Since $\mathbf{y}_1(t)$ is bounded from below, $\dim(\chi_0) = n - m - 1$.

Let $\mathcal{Z}_1 = \operatorname{span}\{\mathbf{y}_1(\cdot)\}$ and $\mathcal{Z}_2 = \chi_0$. Then, to apply Proposition B.2 to prove the theorem we need to show that there exist positive constants α, M, N, and J and supplementary projection matrices P_1, P_2, $P_1 + P_2 = I|_{\mathcal{T}(x_0)}$, so that (B.6), (B.7), and (B.8) are satisfied. Equation (B.7) is equivalent to (B.11), for suppose $\mathbf{y}_1(t) = Y(t)\mathbf{v}$, $Y(0) = I$ for some vector $\mathbf{v} \in \mathbb{R}^n$. Then, $|Y(t)\mathbf{v}| \leq |Y(s)\mathbf{v}|L$. Let P_1 be a 1-dimensional projection matrix such that $P_1\mathbf{v} = \mathbf{v}$. Then $|Y(t)P_1\xi| \leq L|Y(s)P_1\xi|$ for $t, s \geq 0$ and $\xi \in \mathbb{R}^n$. Similarly, if we can show $\mathcal{Z}_2$ is uniformly asymptotically stable (B.6) will be satisfied, where P_2 a $(n-m-1)$-dimensional projection matrix defined such that $P_2\mathbf{y}(0) = \mathbf{y}(0)$ for $\mathbf{y}(\cdot) \in \mathcal{Z}_2$. Equation (B.8) is equivalent to subspaces being bounded away from each other from Proposition B.1. The theorem will, therefore, be established if we can show that,

(a) the subspace $\mathcal{Z}_2$ is uniformly asymptotically stable.

(b) if $\mathbf{y}(\cdot) \in \mathcal{Z}_2$ is nonzero, the angle $\theta(t)$ between $\mathbf{y}_1(t)$ and $\mathbf{y}(t)$ is bounded away from 0 uniformly with respect to $t \geq 0$.

We show that (b) implies (a). If $\mathbf{y}_1$ and $\mathbf{y}$ are vectors in $\mathbb{R}^n$ and if θ is the angle between them, then $|\mathbf{y}_1 \wedge \mathbf{y}| = \sin(\theta)|\mathbf{y}_1||\mathbf{y}|$. Since $\mathbf{w}(\cdot) = \mathbf{y}_1(\cdot) \wedge \mathbf{y}(\cdot)$ is a solution of $\frac{d\mathbf{w}}{dt} = \frac{\partial \mathbf{f}}{\partial x}^{[2]}(\varphi(t))\mathbf{w}$ with $\mathbf{w}(0) \in \mathcal{T}|_{x_0}$, (B.15) implies that

$$|\sin\theta(t)||\mathbf{y}_1(t)||\mathbf{y}(t)| \leq |\sin\theta(s)||\mathbf{y}_1(s)||\mathbf{y}(s)|CKe^{-\alpha(t-s)}$$

if $0 \leq s \leq t$. Further, (B.11) implies

$$|\mathbf{y}(t)| \leq \frac{CKL|\mathbf{y}(s)|e^{-\alpha(t-s)}}{|\sin\theta(t)|}.$$

Thus, proving condition (a).

Condition (b) is equivalent to showing $2|\sin(\theta(t)/2)| = |(\mathbf{y}_1(t)/|\mathbf{y}_1(s)|) - (\mathbf{y}(t)/|\mathbf{y}(s)|)|$ is bounded away from zero. Choose $t > s$ sufficiently large that $|\mathbf{y}(t)| < 1/2L|\mathbf{y}(s)|$. Then, (B.27) implies

$$0 < 1/2L < L - \frac{|\mathbf{y}_1(t)|}{|\mathbf{y}_1(s)|} \leq \left| \frac{\mathbf{y}_1(t)}{|\mathbf{y}_1(s)|} - \frac{\mathbf{y}(t)}{|\mathbf{y}(s)|} \right| \leq H \left| \frac{\mathbf{y}_1(s)}{|\mathbf{y}_1(s)|} - \frac{\mathbf{y}(s)}{|\mathbf{y}(s)|} \right|. \quad \text{(B.28)}$$

Therefore, $0 < L/(2H) < 2|\sin\theta(s)|$, and hence $\theta(s)$ is bounded away from zero for $s \geq 0$. □