ANALYSIS ON SEMIHYPERGROUPS

by

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Abstract

The concepts of semihypergroups and hypergroups were first introduced by C. Dunkl [8], I. Jewett [18] and R. Spector [29] independently around the year 1972. Till then, a variety of research has been carried out on different areas of hypergroups. However, no extensive study is found so far on the more general category of semihypergroups, which serves as building blocks of the hypergroup theory.

In this thesis, we initiate a systematic study of semihypergroups. We introduce and study several natural algebraic and analytic stuctures on semihypergroups, which are well-known in the case of topological semigroups and groups. In particular, we study semihypergroup actions, almost periodic and weakly almost periodic function spaces, ideals, homomorphisms and free structures in the category of semihypergroups, and finally investigate where the theory deviates from the classical theory of (topological) semigroups.

To Mathematics, a symphony of logic and paradox

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Chapter 1

Introduction

The theory of topological semigroups and groups have been a very important area of research in mathematics, especially in the study of abstract harmonic analysis, starting from early 1960's. But in practice, we often come across certain objects arising from groups (for example, coset and double-coset spaces, orbit spaces *etc.*), which although have a structure somewhat similar to groups, are not exactly groups. In particular, consider the following examples.

Example 1.0.1. Consider the topological group $GL_n(\mathbb{R})$ of all invertible $n \times n$ matrices for some $n \in \mathbb{N}$. We know that the space of orthogonal $n \times n$ matrices $O(n)$ is a non-normal subgroup of $GL_n(\mathbb{R})$.

Hence the coset space $GL_n(\mathbb{R})/O(n)$ is not even a semigroup.

Example 1.0.2. Consider the $n \times n$ orthogonal group $O(n)$ and the natural group action $\pi : O(n) \times \mathbb{R}^n \to \mathbb{R}^n$ of $O(n)$ on $\mathbb{R}^n$ given by $(A, x) \mapsto Ax$.

Recall that for any $x \in \mathbb{R}^n$ the orbit of x under the above action is given as

$$\mathcal{O}(x) = \{Ax : A \in O(n)\}$$

Now consider the orbit space $\{\mathcal{O}(x) : x \in \mathbb{R}^n\}$ for this action. Note that again, it is not even a topological semigroup.

Example 1.0.3. Consider the double coset space

$$GL_n(\mathbb{R})//O(n) := \{O(n)A\ O(n) : A \in GL_n\mathbb{R}\}$$

It can be seen immediately that this is not a topological group either.

Hence these kinds of structures that frequently appear while studying the classical theory of topological groups, fall out of the parent category and can not quite be studied or analysed with the existing general theory for topological groups and semigroups.

To mend this kind of situation, the concepts of semihypergroups and hypergroups were introduced around 1972 [8, 18, 29]. The theory of semihypergroups and hypergroups allows a detailed study of various important measure algebras. In turns out that these concepts are sufficiently general to cover a variety of interesting special cases including those described above, but yet have enough structure to allow an independent theory to develop. Often these structures can be expressed in terms of a convolution of measures on the underlying spaces.

A semihypergroup can be perceived in a number of ways. To start with, it can be seen simply as a generalization of locally compact semigroups where the product of two elements is a certain probability measure, rather than being a single element. Similarly a hypergroup, as we will see later, can be perceived simply as a generalization of locally compact groups. The structure and concept of a semihypergroup arises naturally from the quotient space of a locally compact group whereas the structure of a hypergroup arises naturally from the double coset space of a locally compact group.

Again, a semihypergroup is essentially a Hausdorff locally compact topological space where the measure space is equipped with a certain convolution product, turning it into an associative algebra, whereas in the case of a hypergroup, the measure algebra is also equipped with an identity and an involution. The concept of hypergroups and semihypergroups was first introduced early in 1972 by C. Dunkl [8], R. Spector [29] and I. Jewett [18] independently. Dunkl and Spector called their creations hypergroups (resp. semihypergroups) while Jewett preferred to call them convos (resp. semiconvos). The definitions given by these authors are not identical, although their core ideas are essentially the same (except that Dunkl's definition requires the convolution on hypergroups to be commutative). Hence most of the interesting examples of hypergroups and semihypergroups hold true according to all the definitions.

The theory of hypergroups has developed in several directions since then ([12], [26], [28], [32], [33] among others), including the area of commutative hypergroups, weighted hypergroups, amenability of hypergroups and several function spaces on it, most of which have been based on the definition introduced by Jewett [18]. In this text, we will also base our work on semihypergroups on Jewett's definition of semiconvos [18].

The lack of any algebraic structure on a semihypergroup poses a serious challenge in extending results from semigroups to semihypergroups. Also unlike hypergroups, the fact that a semihypergroup structure lacks the existence of a Haar measure or an involution in its measure algebra, creates a serious obstacle to generalize most group and semigroup theories and ideas naturally to semihypergroups.

In practice, although the double-coset spaces have a hypergroup structure, if the compact subgroup is not normal (which is of course, more often than not the case), then the left coset spaces and orbit spaces of a locally compact group as discussed in the examples before, are never a semigroup nor a hypergroup. However, these objects arising frequently in different areas of research (for example, matricial and classical abstract harmonic analysis on coset spaces, dynamical systems among others), fall in the more general category of semihypergroups.

Unlike hypergroups, no extensive systematic literature is found so far on semihypergroups. The main motivation behind this thesis is to develop a systematic theory on semihypergroups. The brief structure of this text will be as the following.

In the first chapter, we recall some important definitions and notations given by Jewett in [18], and introduce some new definitions required for further work. In the second chapter, we list some important useful examples of semihypergroups and hypergroups.

In the third chapter, we introduce the concepts of left, right and semitopological semihypergroups. We also introduce semihypergroup actions in terms of the space of non-negative measures and investigate the relation between joint and separate continuity near the centre of a hypergroup contained in a semihypergroup.

In the fourth chapter we discuss several properties of two of the most important function spaces on semihypergroups, namely the spaces of almost periodic and weakly almost periodic functions. Here we explore the relation of these function spaces to the compactness of the underlying space, as well as some other important identities related to them. We conclude the section with examining the behavior of Arens product on the duals of these function spaces.

In our fifth chapter, we introduce the concept of an ideal in (semitopological) semihypergroups and explore some of its basic properties as well as its relation with a more general form of homomorphism between semihypergroups. Furthermore, we investigate the structure of the kernel of a compact (semitopological) semihypergroup and finally explore the connection between minimal left ideals and the concept of amenability on a compact semihypergroup.

In the sixth chapter, we initiate the study of a free structure on semihypergroups. We introduce a free product structure and construct a specific topology and convolution for a family of semihypergroups such that the resulting semihypergroup abides by an universal property equivalent to the universal property for free products in the classical theory of topological groups, thus providing us with a whole new class of semihypergroups.

Finally, we conclude with some potential problems and areas which we intend to work on and explore further in near future, in addition to introducing and investigating similar structures as in [9], [10], [11], [13], [14], [15], [19], [25], [30], [31] for the case of semihypergroups and explore where and why the theory deviates from the classical theory of topological semigroups and groups .

Chapter 2

Preliminaries

In this chapter, we list some formal definitions with brief required background and introduce some new definitions, mostly analogous to the theory of semigroups, that we will need in the following chapters. We also list some important examples of semihypergroups and hypergroups, emphasising the fact that the category of semihypergroups indeed includes the examples outlined in the previous chapter, hence catering to the overall need for a more general theory as discussed before.

2.1 Notations and Definitions

Here we first list a set of basic notations that we will use throughout the text. Next we briefly recall the tools and concepts needed for the formal definition of a semihypergroup and a hypergroup [18]. Finally in addition to the formal definitions, we also recall some basic structures on semihypergroups and hypergroups, required for the following chapters.

All topologies in this text are assumed to be Hausdorff unless otherwise specified. Now for any locally compact Hausdorff topological space X, consider the following spaces:

$M(X)$	:=	Space of all regular complex Borel measures on X.
$M^+(X)$	:=	Subset of $M(X)$ consisting of all finite non-negative regular Borel measures on X.
$M_F(X)$	:=	Subset of $M(X)$ consisting of all Borel measures on X with finite support.
$M_F^+(X)$	:=	Subset of $M(X)$ consisting of all finitely supported non-negative Borel measures on X.
$P(X)$	:=	Set of all probability measures on X.
$P_c(X)$	:=	Set of all probability measures with compact support on X.
$\mathfrak{C}(X)$	:=	Space of all compact subsets of X.

$B(X)$:= Space of all bounded functions on X.

$C(X)$:= Space of all bounded continuous functions on X.

$C_c^+(X)$:= Space of all non-negative compactly supported continuous functions on X.

$\mathcal{B}(X)$:= Space of all Borel measurable functions on X.

Also note that for any $x \in X$, we denote by p_x the point-mass measure or Dirac measure on X at the point x.

First, let us introduce two very important topologies on the positive measure space and the space of compact subsets for any locally compact topological space X. Unless mentioned otherwise, we will always assume these two topologies on the respective spaces.

Definition 2.1.1 (Cone Topology)**.** *The* ***cone topology*** *on $M^+(X)$ is defined as the weakest topology on $M^+(X)$ for which the maps $\mu \mapsto \int_X f \, d\mu$ is continuous for any $f \in C_c^+(X) \cup \{1_X\}$ where 1_X denotes the characteristic function of X.*

Note that if X is compact then it follows immediately from the Riesz representation theorem that the cone topology coincides with the weak*-topology on $M^+(X)$ in this case.

Definition 2.1.2 (Michael Topology [22])**.** *The* ***Michael topology*** *on $\mathfrak{C}(X)$ is defined to be the topology generated by the sub-basis*

$$\{\mathcal{C}_U(V) : U, V \text{ are open sets in } X\},$$

where for any open sets $U, V \subset X$ we have

$$\mathcal{C}_U(V) = \{C \in \mathfrak{C}(X) : C \cap U \neq \emptyset, C \subset V\}.$$

Note that $\mathfrak{C}(X)$ actually becomes a locally compact Hausdorff space with respect to this natural topology.

Moreover if X is compact then $\mathfrak{C}(X)$ is also compact [22].

Now before we proceed to the formal definition of a semihypergroup, let us briefly recall the concepts of positive linear maps on the measure space of a locally compact topological space.

Definition 2.1.3. *Let X and Y be locally compact Hausdorff spaces. A linear map $\pi : M(X) \to M(Y)$ is called positive continuous if the following holds:*

1. *$\pi(\mu) \in M^+(Y)$ whenever $\mu \in M^+(X)$.*
2. *The map $\pi|_{M^+(X)} : M^+(X) \to M^+(Y)$ is continuous.*

For any locally compact Hausdorff space X and any element $x \in X$, we denote by p_x the point-mass measure or the Dirac measure at the point $\{x\}$.

Definition 2.1.4. *Let X and Y be locally compact Hausdorff spaces and $\pi : M(X) \to M(Y)$ is a positive continuous map. Then for any Borel function f*

on Y, the function f' on X is defined as

$$f'(x) := \int_Y f(y)\ d\pi(p_x)(y)$$

whenever the integral exists.

Proposition 2.1.5. *Let X and Y be locally compact Hausdorff spaces and $\pi : M(X) \to M(Y)$ is a positive continuous map. Then for any $\mu \in M(X)$ and $f \in C(Y)$ we have the following results:*

1. *f' is also a bounded continuous function on X.*
2. *$\int_X f'\ d\mu = \int_Y f\ d\pi(\mu)$.*

The proof of this Proposition can be found in [18].

Definition 2.1.6. *Let X, Y, Z be locally compact Hausdorff spaces. A bilinear map $\Psi : M(X) \times M(Y) \to M(Z)$ is called* ***positive continuous*** *if the following holds :*

1. *$\Psi(\mu, \nu) \in M^+(Z)$ whenever $\mu \in M^+(X), \nu \in M^+(Y)$.*
2. *The map $\Psi|_{M^+(X)\times M^+(Y)}$ is continuous.*

Now we are ready to state the formal definitions for a semihypergroup and a hypergroup. Note that we follow Jewett's notion [18] in terms of the definitions and notations, in most cases.

Definition 2.1.7. ***(Semihypergroup)*** *A pair $(K, *)$ is called a (topological) semihypergroup if they satisfy the following properties:*

(A1) *K is a locally compact Hausdorff space and $*$ defines a binary operation on $M(K)$ such that $(M(K), *)$ becomes an associative algebra.*

(A2) *The bilinear mapping $* : M(K) \times M(K) \to M(K)$ is positive continuous.*

(A3) *For any $x, y \in K$ the measure $p_x * p_y$ is a probability measure with compact support.*

(A4) *The map $(x, y) \mapsto$ supp$(p_x * p_y)$ from $K \times K$ into $\mathfrak{C}(K)$ is continuous.*

Note that for any $A, B \subset K$ the convolution of subsets is defined as the following

$$A * B := \cup_{x \in A, y \in B} \, supp(p_x * p_y) \ .$$

It is shown by Jewett in [18, Lemma 3.2] that the following basic properties hold for this convolution of sets.

Proposition 2.1.8. *Let $(K, *)$ be a semihypergroup. Then for any subsets $A, B, C \subset K$ the following statements hold true.*

1. *$\bar{A} * \bar{B} \subset \overline{(A * B)}$.*
2. *If A, B are compact, then so is $A * B$.*
3. *the convolution map $* : \mathfrak{C}(K) \times \mathfrak{C}(K) \to \mathfrak{C}(K)$ is jointly continuous in Michael topology.*

4. The convolution is associative, i.e, we have

$$A * (B * C) = (A * B) * C$$

Also recall that for any locally compact Hausdorff space X, a map $i : X \to X$ is called a topological involution if i is a homeomorphism and $i \circ i(x) = x$ for each $x \in X$. For semihypergroups, an involution is defined in the following way.

Definition 2.1.9 (Involution). *Let $(K, *)$ be a semihypergroup. Then a map $i : K \to K$ given by $i(x) := x^-$ is called an* ***involution*** *if it is a topological involution and if for any $\mu, \nu \in M(K)$ we have that*

$$(\mu * \nu)^- = \nu^- * \mu^-,$$

where for any measure $\omega \in M(K)$ we have that $\omega^-(B) := \omega(B^-) = \omega(i(B))$ for any Borel measurable subset B of K.

Definition 2.1.10. ***(Hypergroup)*** *A pair $(H, *)$ is called a (topological) hypergroup if it is a semihypergroup and satisfies the following conditions :*

(A5) *There exists an element $e \in H$ such that $p_x * p_e = p_e * p_x = p_x$ for any $x \in H$.*

(A6) *There exists an involution $x \mapsto x^-$ on H such that $e \in supp(p_x * p_y)$ if and only if $x = y^-$.*

The element e in the above definition is called the **identity** of H. Note that the identity and the involution are necessarily unique [18].

Remark 2.1.11. Given a Hausdorff topological space K, in order to define a continuous bilinear mapping $* : M(K) \times M(K) \to M(K)$, it suffices to only define the measures $(p_x * p_y)$ for each $x, y \in K$.

This is true since we can then extend '$*$' linearly to $M_F^+(K)$. As $M_F^+(K)$ is dense in $M^+(K)$ [18], we can further extend '$*$' to $M^+(K)$ and hence to the whole of $M(K)$ using linearity.

Definition 2.1.12 (Centre of a Semihypergroup)**.** *Let $(K, *)$ be a semihypergroup. The center of K, denoted as $Z(K)$ is defined as*

$$Z(K) := \{x \in K : \text{ } supp(p_x * p_y) \text{ is singleton for any } y \in K\}.$$

Definition 2.1.13 (Centre of a Hypergroup, [28])**.** *Let $(H, *)$ be a hypergroup. The center of H, denoted as $Z(H)$ is defined as*

$$Z(H) := \{x \in K : p_x * p_{x^-} = p_{x^-} * p_x = p_e\}.$$

Note that for any hypergroup, the center is always nonempty since it will always contain the identity. Also, it can easily be seen that the definition of center for a hypergroup coincides with that of the definition of center for a semihypergroup. The following is an example of a hypergroup with a non-trivial centre. Note that if G is a topological group, we immediately see that $Z(G) = G$.

Example 2.1.14. This example is due to Zeuner ([33]). Consider the hypergroup $(H, *)$ where $H = [0, 1]$ and the convolution is defined as

$$p_s * p_t = \frac{p_{|s-t|} + p_{1-|1-s-t|}}{2}.$$

Then it can be shown that $Z(H) = \{0, 1\}$.

Now we conclude this section with the definition of left and right translations for a (semitopological) semihypergroup.

Definition 2.1.15 (Translations)**.** *Let K be a semihypergroup and f be a continuous function on K. Then for $x \in K$ the left translation of f by x is the function L_x on K defined as the following.*

$$L_x(y) = f(x * y) := \int_K f(z)\ d(p_x * p_y)(z).$$

Similarly the right translation of f by x is defined as the function

$$R_x(y) := f(y * x) = \int_K f(z)\ d(p_y * p_x)(z).$$

2.2 Examples

In this section, we list some well known examples [18] of semihypergroups and hypergroups and thus explore how the shortcomings explained in the introduction are overcome by the category of semihypergroups and hypergroups.

Example 2.2.1. If $(S, \cdot)$ is a locally compact topological semigroup, then $(S, *)$ is a semihypergroup where $p_x * p_y = p_{x.y}$ for any $x, y \in S$.

Similarly, if $(G, \cdot)$ is a locally compact topological group, then $(G, *)$ is a hypergroup with the same bilinear operation $*$, identity element e where e is the identity of G and the involution on G defined as $x \mapsto x^{-1}$.

Example 2.2.2. Take any set with three elements $S = \{e, a, b\}$ and equip it with the discrete topology. Define

$$\begin{aligned}
p_e * p_a &= p_a * p_e = p_a \\
p_e * p_b &= p_b * p_e = p_b \\
p_a * p_b &= p_b * p_a = \frac{p_e + p_b}{2} \\
p_a * p_a &= \frac{p_a + p_b}{2} \\
p_b * p_b &= p_a
\end{aligned}$$

Then $(S, *)$ is a semihypergroup with identity e.

Example 2.2.3. Take $T = \{e, a, b\}$ and equip it with the discrete topology.

Define

$$\begin{aligned}
p_e * p_a &= p_a * p_e = p_a \\
p_e * p_b &= p_b * p_e = p_b \\
p_a * p_b &= p_b * p_a = z_1 p_a + z_2 p_b \\
p_a * p_a &= x_1 p_e + x_2 p_a + x_3 p_b \\
p_b * p_b &= y_1 p_e + y_2 p_a + y_3 p_b
\end{aligned}$$

where $x_i, y_i, z_i \in \mathbb{R}$ such that $x_1 + x_2 + x_3 = y_1 + y_2 + y_3 = z_1 + z_2 = 1$ and $y_1 x_3 = z_1 x_1$. Then $(T, *)$ is a commutative hypergroup with identity e and the identity function on T taken as involution.

Using the same technique, we can actually see that any finite set can be seen as a semihypergroup.

Example 2.2.4. Let G be a locally compact topological group and H be a compact subgroup of G. Also, let μ be the normalized Haar measure of H. Consider the left quotient space

$$S := G/H = \{xH : x \in G\}$$

and equip it with the quotient topology. For any $x, y \in G$, define

$$p_{xH} * p_{yH} = \int_H p_{(xty)H} \; d\mu(t).$$

Then $(S, *)$ is a semihypergroup.

For instance, take G to be the symmetric group S_4 and take H to be the dihedral group D_8. We know that D_8 is not a normal subgroup of S_4. Consider the left coset space

$$G/H = \{H, s_1H, s_2H\}$$

where $s_1 = (124)$ and $s_2 = (142)$.

Then the above formulation gives us that

$$p_{xH} * p_{yH} = \frac{1}{8}\sum_{h\in H} p_{(xty)H}$$

Hence the left coset space S_4/D_8 is a discrete semihypergroup where the convolution can explicitly be given by the following table:

$*$	p_H	p_{s_1H}	p_{s_2H}
p_H	p_H	$\frac{1}{2}(p_{s_1H} + p_{s_2H})$	$\frac{1}{2}(p_{s_1H} + p_{s_2H})$
p_{s_1H}	p_{s_1H}	$\frac{1}{2}(p_H + p_{s_2H})$	$\frac{1}{2}(p_H + p_{s_1H})$
p_{s_2H}	p_{s_2H}	$\frac{1}{2}(p_H + p_{s_1H})$	$\frac{1}{2}(p_H + p_{s_2H})$

Example 2.2.5. Let G be any locally compact topological group and H be a compact group. Recall that a map $\phi : G \to G$ is called affine if there exists a scaler α and an automorphism Ψ of G such that $\phi(x) = \alpha\Psi(x)$ for each $x \in G$.

An action π of H on G given by $\pi(h, g) = g^h$ for each $g \in G, h \in H$ is called a continuous affine action if $\pi : H \times G \to G$ is continuous and the map $g \mapsto g^h : G \to G$ is affine for each $h \in H$. For any continuous affine action π

of H on G, consider the orbit space

$$\mathcal{O} := \{x^H : x \in G\},$$

where $x^H = \mathcal{O}(x) = \{\pi(h, x) : h \in H\}$.

Let σ be the normalized Haar measure of H. Consider $\mathcal{O}$ with the quotient topology and the following convolution

$$p_{x^H} * p_{y^H} := \int_H \int_H p_{\pi(s,x)\pi(t,y)^H} \; d\sigma(s) d\sigma(t).$$

Then $(\mathcal{O}, *)$ becomes a semihypergroup.

Example 2.2.6. Similarly, consider the space of double cosets

$$K := G//H = \{HxH : x \in G\}$$

where G is a locally compact topological group and H is a compact subgroup of G.

Equip K with the usual quotient topology. Let μ be the normalized Haar measure of H and for any $x, y \in G$, define

$$p_{HxH} * p_{HyH} = \int_H p_{H(xty)H} \; d\mu(t).$$

Then $(K, *)$ is a hypergroup with the identity element $e = H$ and involution function $HxH \mapsto Hx^{-1}H$.

Chapter 3

Semitopological Semihypergroup and Actions

In this chapter we introduce the concept of semitopological semihypergroups analogous to the category of semitopological semigroups. Moreover, we introduce the concept of semihypergroup actions on a topological space with respect to the space of non-negative measures. Some of its properties and its behaviour near the centre of a hypergroup included in a semihypergroup is also investigated.

3.1 Semitopological Semihypergroups

First let us briefly recall some definitions regarding semigroups. Based on these concepts we introduce some more general kinds of semihypergroups that satisfy weaker continuity conditions than those of semihypergroups, and hence include a more varied family of objects than that of semihypergroups.

We know that a semigroup $(S, \cdot)$ is called a topological semigroup if the set S is equipped with a topology such that the multiplication map

$$(s, t) \mapsto s.t : S \times S \to S$$

is jointly continuous on $S \times S$.

Also for a semigroup $(S, \cdot)$, the left multiplication map l_s for some element $s \in S$ is defined as

$$\begin{aligned} l_s : \ S &\ \to \ S \\ x &\ \mapsto \ s.x \end{aligned}$$

Similarly the right multiplication map r_s for some $s \in S$ is defined as

$$\begin{aligned} r_s : \ S &\ \to \ S \\ x &\ \mapsto \ x.s \end{aligned}$$

A semigroup $(S, \cdot)$ is called a left topological semigroup if the set S is equipped with a topology such that the left multiplication map l_s is continuous on S for each $s \in S$.

Similarly, a semigroup $(S, \cdot)$ is called a right topological semigroup if the set S is equipped with a topology such that the right multiplication map r_s is continuous on S for each $s \in S$.

Finally, a semigroup $(S, \cdot)$ is called a semitopological semigroup if S is equipped with a topology such that the multiplication map

$$(s, t) \mapsto s.t : S \times S \to S$$

is separately continuous, *i.e,* if $(S, \cdot)$ is both left and right topological.

Now we introduce some analogous definitions to the class of semihypergroups.

Definition 3.1.1. *A pair $(K, *)$ is called a **left topological semihypergroup** if it satisfies all the conditions of Definition 2.1.7 with property **(A2)** replaced by the following:*

(A2′) *The bilinear map $(\mu, \nu) \mapsto \mu * \nu$ is positive and for each $\omega \in M^+(K)$ the left multiplication map*

$$\begin{aligned} L_\omega : M^+(K) &\to M^+(K) \\ \mu &\mapsto \omega * \mu \end{aligned}$$

is continuous on $M^+(K)$.

Similarly we can define the right multiplication map for semihypergroups and thus introduce the class of right topological semihypergroups.

Definition 3.1.2. *A pair* $(K, *)$ *is called a* ***right topological semihypergroup*** *if it satisfies all the conditions of Definition 2.1.7 with property* ***(A2)*** *replaced by the following:*

(A2$''$) *The bilinear map* $(\mu, \nu) \mapsto \mu * \nu$ *is positive and for each* $\omega \in M^+(K)$ *the right multiplication map given by*

$$\begin{aligned} R_\omega : M^+(K) &\rightarrow M^+(K) \\ \mu &\mapsto \mu * \omega \end{aligned}$$

is continuous on $M^+(K)$.

Definition 3.1.3. *A pair* $(K, *)$ *is called a* ***semitopological semihypergroup*** *if it is both left and right topological semihypergroup, i.e, if it satisfies all the conditions of Definition 2.1.7 with property* ***(A2)*** *replaced by the following:*

(A2$'''$) *The bilinear map* $(\mu, \nu) \mapsto \mu * \nu$ *is positive and the restricted map*

$$(\mu, \nu) \mapsto \mu * \nu : M^+(K) \times M^+(K) \rightarrow M^+(K)$$

is separately continuous.

Note that in particular whenever $(K, *)$ is a semigroup as in Example 2.2.1, all the definitions reduce to the classical definitions on semigroups as discussed in the beginning of this section.

3.2 Semihypergroup Actions

As in the previous section, here we first briefly recall the classical definition of an action of a topological semigroup on a topological space, and thereafter proceed to introduce an analogous concept of a semihypergroup action on an arbitrary topological space.

Let $(S, \cdot)$ be a topological semigroup and X be a topological space.

Definition 3.2.1. *A map $\sigma : S \times X \to X$ is called an action of S on the space X if it satisfies the following two conditions.*

1. *For each $s \in S$ the map*

$$\begin{aligned} \sigma_s : X &\to X \\ x &\mapsto \sigma(s, x) \end{aligned}$$

is continuous on X.

2. *For any $s, t \in S$ and $x \in X$ we have that*

$$\sigma(st, x) = \sigma(s, \sigma(t, x)).$$

Now we define actions on semihypergroups in such a way that whenever we take the semihypergroup to be a topological semigroup as outlined in Example 2.2.1 the definition coincides with that of topological semigroups.

Definition 3.2.2. *Let $(K, *)$ be a semihypergroup and X be any locally compact Hausdorff space. A map $\sigma : M^+(K) \times X \to X$ is called an action of K on X if the following two conditions hold :*

1. *For each $\omega \in M^+(K)$ the map $\sigma_\omega : X \to X$ given by $\sigma_\omega(x) = \sigma(\omega, x)$ is continuous.*

2. *For any $\mu, \nu \in M^+(K)$, $x \in X$ we have that $\sigma(\mu * \nu, x) = \sigma(\mu, \sigma(\nu, x))$*

Remark 3.2.3. Let $(K, *)$ be a semihypergroup and X be a locally compact Hausdorff space.

- We can define actions of left/ right/ semitopological semihypergroups on X in the same manner as in Definition 3.2.2.

- An action $\sigma : M^+(K) \times X \to X$ is called seperately continuous if for each $x \in X$ the map

$$\begin{aligned} \sigma^x : M^+(K) &\rightarrow X \\ \mu &\mapsto \sigma(\mu, x) \end{aligned}$$

is also continuous on $M^+(K)$.

- An action $\sigma : M^+(K) \times X \to X$ is called a continuous action if σ is continuous on $M^+(K) \times X$.
- For an action σ of K on X and any subset N of $M^+(K)$, V of X, we define

$$N.V := \{\sigma(\mu, x) : \mu \in N, x \in V\} \quad .$$

Now we are ready to state the main theorem of this section investigating when separate continuity forces joint continuity of the action of a left (or right) topological semihypergroup on a compact Hausdorff space.

Theorem 3.2.4. *Let $(K, *)$ be a compact left topological semihypergroup with identity e and σ is a separately continuous action of K on a compact Hausdorff space X. Let $H \subset K$ be a hypergroup containing e. Then σ is continuous at each point (p_g, x) where $g \in Z(H)$, $x \in X$.*

But before we proceed with the proof of the above theorem, we first prove a couple of lemmas imperative for proving the same. Note that the proof of this theorem follows similar ideas as outlined in [23], with the additional details required in order to generalize it for semihypergroups.

Lemma 3.2.5. *Assume that Theorem 3.2.4 holds for any separately continuous action σ for which the map $\sigma_{p_e} : X \to X$ is the identity mapping, i.e, $\sigma(p_e, x) = x$ for any $x \in X$.*

Then the theorem will hold for any separately continuous action σ on K.

Proof. Let σ be any separately continuous action on K and set

$$L := \{p_e\}.\ X \quad .$$

Note that for any $x \in X$ we have

$$\sigma(p_e, \sigma(p_e, x)) = \sigma(p_e * p_e, x) = \sigma(p_e, x) \quad .$$

Also, for any $\mu \in M^+(K)$, $x \in X$ we have that

$$\begin{aligned}(\mu, \sigma(p_e, x)) &= \sigma(\mu * p_e, x) \\ &= \sigma(p_e * \mu, x) \\ &= \sigma(p_e, \sigma(\mu, x)) \in L \ .\end{aligned}$$

Hence the restricted action $\tilde{\sigma} := \sigma|_{M^+(K)\times L} : M^+(K) \times L \to L$ is well defined and satisfies the initial assumption, *i.e,*

$$\tilde{\sigma}(p_e, \tilde{x}) = \tilde{x}$$

for any $\tilde{x} = \sigma(p_e, x) \in L$ where $x \in X$. Thus by the above hypothesis, $\tilde{\sigma}$ is continuous at each point $(p_g, \tilde{x})$, $g \in Z(H)$, $x \in X$.

Now pick any $g \in Z(H)$, $x \in X$ and let $\{(\mu_\alpha, x_\alpha)\}$ be any net in $M^+(K) \times X$ that converges to (p_g, x). Since σ is separately continuous, the net $\{\tilde{x}_\alpha\} := \{\sigma(p_e, x_\alpha)\}$ converges to $\tilde{x}$.

Hence $\{(\mu_\alpha, \tilde{x}_\alpha)\}$ converges to $(p_g, \tilde{x})$ in $M^+(K) \times L$. Then we have that

$$\begin{aligned}
\sigma(\mu_\alpha, x_\alpha) &= \sigma(\mu_\alpha * p_e, x_\alpha) \\
&= \sigma(\mu_\alpha, \sigma(p_e, x_\alpha)) \\
&= \tilde{\sigma}(\mu_\alpha, \tilde{x}_\alpha) \to \tilde{\sigma}(p_g, \tilde{x}) = \sigma(p_g * p_e, x) = \sigma(p_g, x).
\end{aligned}$$

□

Lemma 3.2.6. *Under the hypothesis of Theorem 3.2.4, if x, y are two distinct points in X, then we can find neighborhoods N of p_e (in the weak* topology of $M^+(K)$), U of x and V of y such that $N.U \cap V = \emptyset$.*

Proof. Using Urysohn's Lemma, let us first choose a continuous function $f : X \to [-1, 1]$ such that $f(x) \neq f(y) = 0$. Set $g := f \circ \sigma$.

Since K is compact, $M^+(K)$ is locally compact with respect to the weak* topology [5]. Also since σ is separately continuous, the function

$$g : M^+(K) \times X \to [-1, 1]$$

is separately continuous. Hence we can find[27] a dense G_δ subset M of $M^+(K)$ such that g is continuous at each point (μ, x) of $M^+(K) \times X$.

Now set

$$S := \{\mu \in M^+(K) : g(\mu, x) \neq g(\mu, y)\} \quad .$$

Note that since σ_{p_e} is identity, we have

$$\begin{aligned} g(p_e, x) &= f(\sigma(p_e, x)) \\ &= f(x) \\ &\neq f(y) \\ &= f(\sigma(p_e, y)) \\ &= g(p_e, y). \end{aligned}$$

Thus S is a non-empty open subset of $M^+(K)$ and hence $S \cap M \neq \emptyset$. Pick $\mu_0 \in S \cap M$ and set

$$\varepsilon := |g(\mu_0, x) - g(\mu_0, y)| > 0 \quad .$$

Since g is continuous at (μ_0, x) there exist neighborhoods N_0 of μ_0 and U of x

such that

$$|g(\nu, z) - g(\mu_0, x)| < \varepsilon/4 \quad \text{for any} \quad \nu \in N_0,\ z \in U. \tag{3.2.6.1}$$

Now since K is left topological, the map

$$\begin{aligned} L_{\mu_0} : M^+(K) &\to M^+(K) \\ \nu &\mapsto \mu_0 * \nu \end{aligned}$$

is continuous. In particular, $L_{\mu_0}(p_e) = \mu_0$ and hence we can find a neighborhood N of p_e in $M^+(K)$ such that $\{\mu_0\} * N \subseteq N_0$. Set

$$V := \{z \in X : |g(\mu_0, z) - g(\mu_0, y)| < \varepsilon/4\} \quad .$$

Clearly, V is an open neighborhood of y.

Now if possible, assume that $N.U \cap V \neq \emptyset$, *i.e,* there exists $\nu \in N$, $z \in U$ such that $\sigma(\nu, z) \in V$. Then we have

$$\begin{aligned} \varepsilon = |g(\mu_0, x) - g(\mu_0, y)| &\leq |g(\mu_0, x) - g(\mu_0 * \nu, z)| + |g(\mu_0 * \nu, z) - g(\mu_0, y)| \\ &< \varepsilon/4 + |g(\mu_0, \sigma(\nu, z)) - g(\mu_0, y)| \\ &< \varepsilon/4 + \varepsilon/4 \\ &< \varepsilon \end{aligned}$$

which can not be true. Note that the second inequality follows since

$$\mu_0 * \nu \in \{\mu_0\} * N \subseteq N_0$$

and the third inequality follows since $\sigma(\nu, z) \in V$ by Equation 3.2.6.1. $\square$

Now we return to the proof of our main theorem for this section.

Proof of Theorem 3.2.4: First using Lemma 3.2.5, without loss of generality we can assume that σ_{p_e} is the identity mapping on X. Pick and fix any $x \in X$ and let W be any open neighborhood of $\sigma(p_e, x) = x$ in X.

Pick $y \in X \setminus W$. By Lemma 3.2.6 we can find open neighborhoods N_y of p_e, U_y of x and V_y of y such that $N_y.U_y \cap V_y = \emptyset$. We will get such a set of neighborhoods corresponding to each $y \in X \setminus W$.

But $X \setminus W$ is compact and hence we can find $y_1, y_2, \ldots, y_n \in X \setminus W$ such that

$$X \setminus W \subseteq \cup_{i=1}^{n} V_{y_i}.$$

Set $N := \cap_{i=1}^{n} N_{y_i}$ and $U := \cap_{i=1}^{n} U_{y_i}$. Then clearly $N.U \cap V = \emptyset$. Thus we get neighborhoods N of p_e, U of x such that $N.U \subseteq W$, *i.e,* σ is continuous at (p_e, x). This is true for any $x \in X$.

Now pick any $g \in Z(H)$, $x \in X$. Let $\{(\mu_\alpha, x_\alpha)\}$ be a net in $M^+(K) \times X$ that converges to (p_g, x). Since K is left topological, $L_{p_{g^-}}$ is continuous and so

$$p_{g^-} * \mu_\alpha \longrightarrow p_{g^-} * p_g = p_e$$

since $g \in Z(H)$. But we know that σ is continuous at (p_e, x) and hence

$$\sigma(p_{g^-} * \mu_\alpha, x_\alpha) \longrightarrow \sigma(p_e, x) = x. \tag{3.2.6.2}$$

Finally we have

$$\begin{aligned}
\sigma(\mu_\alpha, x_\alpha) &= \sigma(p_e * \mu_\alpha, x_\alpha) \\
&= \sigma(p_g * p_{g^-} * \mu_\alpha, x_\alpha) \\
&= \sigma(p_g, \sigma(p_{g^-} * \mu_\alpha, x_\alpha)) \to \sigma(p_g, x)
\end{aligned}$$

where the convergence follows from Equation 3.2.6.2 and the fact that σ is separately continuous. □

Chapter 4

Almost Periodic and Weakly Almost Periodic Functions

Here we first recall some definitions of several important function spaces on a semihypergroup. We further explore some basic properties of the spaces of almost periodic and weakly almost periodic functions, and investigate the relation between these spaces and compactness of the underlying space. Finally, we conclude with examining Arens regularity [1] on dual of the space of almost periodic functions. The contents of this chapter is based on [2, Section 4].

4.1 Motivation and Definitions

Before we proceed to the main definitions regarding different function spaces on a (semitopological) semihypergroup, let us first recall the relevant definitions for the real line and semigroups in general.

Almost periodic functions on the real line $\mathbb{R}$ were first introduced by H. A. Bohr around 1925 as the closure of trigonometric polynomials with respect to the uniform norm $||\cdot||_\infty$ on $\mathbb{R}$. His definition is equivalent to the following definition.

Definition 4.1.1 (Bohr)**.** *A continuous function $f : \mathbb{R} \to \mathbb{R}$ is called almost periodic if for every $\varepsilon > 0$ there exists some $\delta_\varepsilon > 0$ such that for each interval $[t, t+\delta_\varepsilon] \subset \mathbb{R}$ there exists some $\tau(t) \in (t, t+\delta_\varepsilon)$ such that*

$$|f(t+\tau(t)) - f(t)| < \varepsilon.$$

Later around 1926, S. Bochner gave a simpler version of an equivalent definition using sequential compactness, which is closer to the definition we mostly use nowadays.

Definition 4.1.2 (Bochner)**.** *A continuous function $f : \mathbb{R} \to \mathbb{R}$ is called almost periodic if for any sequence $\{a_n\}$ in $\mathbb{R}$, the sequence $\{f_n\}$ of translations of f has a subsequence that converges uniformly on $(-\infty, \infty)$ where for each $t \in \mathbb{R}$*

$$f_n(t) := f(t + a_n).$$

Later the definition got generalized for semigroups and groups in terms of left and right almost periodic functions, using left and right translations of continuous functions on a semigroup.

Let $(S, \cdot)$ be a topological semigroup and f be a continuous function on S. Then for each $s \in S$, the left translation $l_s f$ is a continuous function on S defined as

$$l_s f(t) := f(st) \quad \text{for each} \;\; t \in S.$$

Similarly for each $s \in S$, the right translation $r_s f : S \to \mathbb{R}$ is the continuous function defined as

$$r_s f(t) := f(ts) \quad \text{for each} \;\; t \in S.$$

A function $f \in C(S)$ is called right almost periodic if the right orbit

$$\mathcal{O}_r(f) = \{r_s f : s \in S\}$$

is relatively compact in $C(S)$ with respect to the uniform norm $||\cdot||_\infty$ on $C(S)$.

Similarly, $f \in C(S)$ is called left almost periodic if the left orbit

$$\mathcal{O}_l(f) = \{l_s f : s \in S\}$$

is relatively compact in $C(S)$ with respect to the norm topology on $C(S)$.

On the other hand, a function $f \in C(S)$ is called right (resp. left) weakly almost periodic if the right (resp. left) orbit $\mathcal{O}_r(f)$ (resp. $\mathcal{O}_l(f)$) is relatively compact in $C(K)$ with respect to the weak topology on $C(K)$ (see [6], [7], [17] for further details regarding these functions spaces for a topological semigroup).

Now as a final step in the evolution, we get the definitions for left and right (weakly) almost periodic functions for semihypergroups as follows.

Recall that for any continuous function f on a (semitopological) semihypergroup K and each $x, y \in K$ we define the left and right translates of f (denoted as $L_x f$ and $R_x f$ respectively) as the following:

$$L_x f(y) = R_y f(x) = f(x * y) = \int_K f \; d(p_x * p_y)$$

Definition 4.1.3. *For any function $f \in C(K)$ we define the right orbit $\mathcal{O}_r(f)$ of f as*

$$\mathcal{O}_r(f) := \{R_x f : x \in K\}$$

Definition 4.1.4. *A function $f \in C(K)$ is called right almost periodic if we have that $\mathcal{O}_r(f)$ is relatively compact in $C(K)$ with respect to the norm topology.*

Similarly, a function $f \in C(K)$ is called right weakly almost periodic if we have that $\mathcal{O}_r(f)$ is relatively compact in $C(K)$ with respect to the weak topology on $C(K)$.

We denote these two classes of functions as:

$$AP_r(K) := \text{ Space of all right almost periodic functions on } K.$$
$$WAP_r(K) := \text{ Space of all right weakly almost periodic functions on } K.$$

Similarly we can define the left orbit $\mathcal{O}_l(f)$ of a function $f \in C(K)$ and hence define the set of all left almost periodic functions $AP_l(K)$ and left weakly almost periodic functions $WAP_l(K)$. In most instances, the results proved for the right case also hold true for the left case.

Now recall that for a topological group $(G, \cdot)$, the spaces $LUC(G)$ (resp. $RUC(G)$) denote the spaces of left (resp. right) uniformly continuous functions [17] and serves as two very important functions spaces. In the same light as for topological groups and semigroups, these function spaces can be defined for (semitopological) semihypergroups as well.

Definition 4.1.5. *A function $f \in C(K)$ is called left(right) uniformly continuous if the map $x \mapsto L_x f$ $(x \mapsto R_x f)$ from K to $C(K)$ is continuous.*

A function f is called uniformly continuous if it is both left and right uniformly continuous.

We denote these classes of functions as:

$$LUC(K) := \text{ Space of all left uniformly continuous functions on } K.$$
$$RUC(K) := \text{ Space of all right uniformly continuous functions on } K.$$

$$UC(K) \quad := \quad \text{Space of all uniformly continuous functions on } K.$$

Finally, we conclude this section with recalling the notion of amenability for semihypergroups.

Definition 4.1.6. *Let K be a (semitopological) semihypergroup with identity and $\mathcal{F}$ is a linear subspace of $C(K)$ containing constant functions. A function $m \in \mathcal{F}^*$ is called a **mean** of $\mathcal{F}$ if we have that*

$$||m|| = 1 = m(1).$$

Definition 4.1.7. *Let K be a (semitopological) semihypergroup with identity and $\mathcal{F}$ is a translation-invariant linear subspace of $C(K)$ containing constant functions. A mean m of $\mathcal{F}$ is called a **left invariant mean** (LIM) if we have that $m(L_x f) = m(f)$ for any $x \in K$, $f \in \mathcal{F}$.*

The space $\mathcal{F}$ is called left amenable if it admits a left invariant mean. Also, a semihypergroup K is called left amenable if $C(K)$ admits a left-invariant mean.

Similarly, we can define right-invariant means (RIM) on a translation-invariant linear subspace $\mathcal{F}$ of $C(K)$ containing constant functions, and K is called right amenable if $C(K)$ admits a right-invariant mean.

4.2 Basic Properties

Here we examine the relation between the spaces of left and right (weakly) almost periodic functions, and their relation to other function spaces as well. The translation-invariance properties of these spaces are also investigated.

Recall that for any subset A of a topological vector space X, the convex hull $co(A)$ is defined as

$$co(A) := \Big\{ \sum_{i=1}^{n} \alpha_i x_i : x_i \in A, \alpha_i > 0 \ \ \forall\ i, \sum_{i=1}^{n} \alpha_i = 1, n \in \mathbb{N} \Big\}.$$

The closed convex hull of X, *i.e,* the closure of $co(X)$ is denoted as $\overline{co}(X)$.

Also, the circled hull $\Gamma(A)$ of A is defined as

$$\Gamma(A) := \{\alpha x : x \in A, |\alpha| \leq 1\}.$$

The convex circled hull of A, denoted as $cco(A)$ is the convex hull of the circled hull $\Gamma(A)$ of A. Similarly, the closed convex circled hull $\overline{cco}(A)$ is the closure of $cco(A)$.

Now let us start with examining the relation between the spaces of left and right (weakly) almost periodic functions. But before we proceed to prove the first result on this aspect, first let us recall an important result on the existence of a vector integral on the function space of a general locally compact Hausdorff space, proved in [27].

Theorem 4.2.1. *Suppose that (X, τ) is a Hausdorff topological vector space where X^* separates points, and $\lambda \in P_c(Y)$ where Y is a locally compact Hausdorff space. If a function $F : Y \to X$ is continuous and $\overline{co}(F(Y))$ is compact in X then the vector integral $\omega := \int_Y F \, d\lambda$ exists and $\omega \in \overline{co}(F(Y))$.*

We divide the proof of our next theorem in a series of key steps for convenience, due to the length of the proof.

Theorem 4.2.2. *Let K be a semihypergroup. Then*

$$WAP_r(K) \cap UC(K) = WAP_l(K) \cap UC(K).$$

Proof. Pick any $f \in WAP_l(K) \cap UC(K)$. We need to show that $f \in WAP_r(K)$, *i.e,* we prove that $\mathcal{O}_r(f)$ is relatively weakly compact in $C(K)$. We will show this in five steps.

Step I: Embed K into $P_c(K)$ through the homomorphism $x \mapsto p_x$ [18]. Now define a map $\Phi : C(K) \to C(P_c(K))$ by $\Phi(f) = \tilde{f}$ for any $f \in C(K)$ where

$$\tilde{f}(\mu) := \int f \, d\mu \quad \forall \mu \in P_c(K).$$

We know that Φ is an isometry [18]. Thus Φ is continuous in the norm topologies of $C(K)$ and $C(P_c(K))$ and hence Φ is continuous in the weak topologies of $C(K)$ and $C(P_c(K))$.

Step II: For any $\mu \in P_c(K)$ define a function $\hat{L}_\mu f$ on K by

$$\hat{L}_\mu f(x) := \int_K f(y * x)\ d\mu(y)\ .$$

Since $f \in UC(K)$ the map $x \mapsto R_x f$ is continuous, and $\hat{L}_\mu f(x) = \int_K R_x f\ d\mu$ by the above construction. Hence $\hat{L}_\mu f \in C(K)$ for any $\mu \in P_c(K)$.

Now define a subset in $C(K)$ as the following.

$$\tilde{\mathcal{O}}_l(f) := \{\hat{L}_\mu f : \mu \in P_c(K)\}.$$

Consider the left-translation map ψ on K given as

$$\begin{aligned} \psi : K &\rightarrow C(K) \\ x &\mapsto L_x f. \end{aligned}$$

Since $f \in UC(K)$ ψ is continuous on K. Also since $f \in WAP_l(K)$ the left orbit of f namely $\psi(K)$ is relatively weakly compact in $C(K)$.

Hence by the Krein-Smulian Theorem we have that $\overline{co}(\psi(K))$ is weakly compact in $C(K)$, *i.e,* $\overline{co}(\mathcal{O}_l(f))$ is weakly compact in $C(K)$.

Step III: Now can use Theorem 4.2.1 by letting $(X, \tau) = (C(K),$ weak topology$)$, $Y = K$, $F = \psi$ and $\lambda = \mu$ for any $\mu \in P_c(K)$. Thus we see that

$$\omega_0 := \int_K \psi\ d\mu \in \overline{co}(\psi(K)).$$

Now for any $x \in K$ we have that

$$\begin{aligned}
\omega_0(x) &= \int_K (\psi(y))(x)\ d\mu(y) \\
&= \int_K L_y f(x)\ d\mu(y) \\
&= \int_K f(y * x)\ d\mu(y) = \hat{L}_\mu f(x).
\end{aligned}$$

Since $\psi(K) = \mathcal{O}_l(f)$ by construction, thus we see that $\hat{L}_\mu f \in \overline{co}(\mathcal{O}_l(f))$ for any $\mu \in P_c(K)$. Hence $\tilde{\mathcal{O}}_l(f) \subset \overline{co}(\mathcal{O}_l(f))$. But from Step II we know that $\overline{co}(\mathcal{O}_l(f))$ is weakly compact in $C(K)$. Hence $\tilde{\mathcal{O}}_l(f)$ is relatively weakly compact in $C(K)$.

Step IV: Note that $(\hat{L}_\mu f)\breve{} = L_\mu \tilde{f}$ for any $\mu \in P_c(K)$. This is true since for any $\nu \in P_c(K)$ we have that

$$\begin{aligned}
(\hat{L}_\mu f)\breve{}(\nu) &= \int_K \hat{L}_\mu f(x)\ d\nu(x) \\
&= \int_K \int_K f(y * x)\ d\mu(y)\ d\nu(x) \\
&= \int_K f\ d(\mu * \nu) \\
&= \tilde{f}(\mu * \nu) = L_\mu \tilde{f}(\nu).
\end{aligned}$$

where the thirds equality follows from [18, Theorem 3.1E]. Thus we see that

$$(\tilde{\mathcal{O}}_l(f))\breve{} = \{\tilde{f} : f \in \tilde{\mathcal{O}}_l(f) \subset C(K)\} = \mathcal{O}_l(\tilde{f}).$$

From Step III we know that $\tilde{\mathcal{O}}_l(f)$ is relatively weakly compact in $C(K)$, and from Step I we see that the map $f \mapsto \tilde{f}$ from $C(K)$ to $C(P_c(K))$ is continuous when both spaces are equipped with weak topology.

Hence the set $(\tilde{\mathcal{O}}_l(f))\breve{}$ is also relatively weakly compact in $C(P_c(K))$. Thus $\mathcal{O}_l(\tilde{f})$ is relatively weakly compact in $C(P_c(K))$,*i.e,* $\tilde{f} \in WAP_l(P_c(K))$.

Step V: We know that $(P_c(K), *)$ is a topological semigroup. Hence $WAP_l(P_c(K)) = WAP_r(P_c(K))$. Thus we have that $\tilde{f} \in WAP_r(P_c(K))$, *i.e,* $\mathcal{O}_r(\tilde{f})$ is relatively weakly compact in $C(P_c(K))$.

Hence the set $N := \{R_{p_x}\tilde{f} : x \in K\} \subset \mathcal{O}_r(\tilde{f})$ is also relatively weakly compact in $C(P_c(K))$.

Now consider the map $\phi : C(P_c(K)) \to C(K)$ given by $h \mapsto \check{h}$ where

$$\check{h}(x) := h(p_x)$$

for any $x \in K$. The fact that $\check{h}$ is continuous on K follows directly from the fact that the map $x \mapsto p_x$ is continuous. Also, ϕ is continuous since $||\check{h}|| = \sup_{x \in K} |h(p_x)| \leq ||h||$ for any $h \in C(P_c(K))$.

Note that for each $h \in C(K)$, $x \in K$, $\phi(R_{p_x}\tilde{h}) = R_x h$ since for any $y \in K$ we have that

$$\begin{aligned} \phi(R_{p_x}\tilde{h})(y) &= (R_{p_x}\tilde{h})\check{}(y) \\ &= R_{p_x}\tilde{h}(p_y) \end{aligned}$$

$$
\begin{aligned}
&= \tilde{h}(p_y * p_x) \\
&= \int_K h \; d(p_y * p_x) \\
&= h(y * x) = R_x h(y)
\end{aligned}
$$

Thus in particular, we have that $\phi(N) = \mathcal{O}_r(f)$. Since ϕ is continuous and N is relatively weakly compact in $C(P_c(K))$, we finally have that $\mathcal{O}_r(f)$ is relatively weakly compact in $C(K)$, *i.e,* $f \in WAP_r(K)$ as required.

Similarly for any $f \in WAP_r(K) \cap UC(K)$ we can show that $f \in WAP_l(K)$. Hence the proof is complete. □

Remark 4.2.3. In the proof of Theorem 4.2.2, the ideas involved in steps I-III are similar to [32, Theorem 2.4] as no specific involution properties are required. However, the above proof of those steps contains several important details, not found in the aforementioned text.

Theorem 4.2.4. *Let K be a semihypergroup. Then $AP_l(K) = AP_r(K)$.*

Proof. This can be proved in exactly similar manner as in Theorem 4.2.2. The only modification needed is, we need to consider the norm topologies on $C(K)$ and $C(P_c(K))$ opposed to the fact that we had to consider weak topologies on these two spaces in the proof of Theorem 4.2.2. □

Remark 4.2.5. For a (semitopological) semigroup $(S, \cdot)$ it is easy to see [23] that the spaces of left and right weakly almost periodic functions coincide, *i.e,*

$$WAP_r(S) = WAP_l(S).$$

But the question that whether it holds true in general for a semitopological semihypergroup $(K, *)$ is still open. For example, if we pick some $f \in WAP_l(K) \setminus UC(K)$, then as shown in the proof of Theorem 4.2.2, steps II and III may not hold true for f and hence we can not necessarily conclude if f will indeed be contained in $WAP_r(K)$ as well or not.

The following facts regarding the relation between the function spaces in question are proved for hypergroups in [32]. The same proof works for semihypergroups as well. We give a proof of the second statement later in the next section in Theorem 4.3.5.

Proposition 4.2.6. *Let K be a semihyergroup. Then the following statements hold true.*

1. *$WAP_l(K), WAP_r(K), AP_l(K), AP_r(K)$ are norm-closed, conjugate-closed subsets of $C(K)$ containing the constant functions.*

2. *$AP_l \subset UC(K)$ and $AP_r(K) \subset UC(K)$.*

3. *$AP_l(K) \subset WAP_l(K)$ and $AP_r(K) \subset WAP_r(K)$.*

Next we proceed to examine the translation-invariance properties of the spaces of (weakly) almost periodic functions on a (semitopological) semihypergroup.

Recall that a function space $\mathcal{F}$ on a (semitopological) semihypergroup K is called left (resp. right) translation-invariant if we have that $L_x f \in \mathcal{F}$ (resp. $R_x f \in \mathcal{F}$) for any $f \in \mathcal{F}$ and $x \in K$.

Also, $\mathcal{F}$ is simply said to be translation-invariant if it is both left and right translation-invariant.

Theorem 4.2.7. *Let K be any semitopological semihypergroup. Then $AP(K)$ is translation-invariant.*

Proof. Pick any $f \in AP(K)$, $x_0 \in K$. Now consider the following map on $C(K)$.

$$\begin{aligned} \Phi: C(K) &\to C(K) \\ g &\mapsto L_{x_0}g. \end{aligned}$$

First note that for any $x, y, t \in K$ we have

$$\begin{aligned} R_x L_y f(t) &= L_y f(t * x) \\ &= f(y * t * x) \\ &= R_x f(y * t) = L_y R_x f(t). \end{aligned}$$

Hence in turn we see that

$$\begin{aligned} \mathcal{O}_r(L_{x_0}f) &= \{R_x L_{x_0} f : x \in K\} \\ &= \{L_{x_0} R_x f : x \in K\} \\ &= \Phi(\mathcal{O}_r(f)). \end{aligned}$$

Now for any two functions $g, h \in C(K)$ we have that

$$\begin{aligned}
||\Phi(g) - \Phi(h)|| &= ||L_{x_0}g - L_{x_0}h|| \\
&= \sup_{y \in K} |(L_{x_0}g - L_{x_0}h)(y)| \\
&= \sup_{y \in K} |L_{x_0}(g - h)(y)| \\
&= \sup_{y \in K} |(g - h)(x_0 * y)| \\
&= \sup_{y \in K} \Big| \int_K (g - h)\ d(p_{x_0} * p_y) \Big| \\
&\leq ||g - h|| \sup_{y \in K} |(p_{x_0} * p_y)(K)| \\
&= ||g - h|| \ .
\end{aligned}$$

Thus we see that Φ is continuous in the norm topology on $C(K)$. Hence $\Phi(\overline{\mathcal{O}_r(f)})$ is compact since $f \in AP_r(K)$. Also as noted before, since

$$\mathcal{O}_r(L_{x_0}f) = \Phi(\mathcal{O}_r(f)) \subset \Phi(\overline{\mathcal{O}_r(f)})$$

we have that $\mathcal{O}_r(L_{x_0}f)$ is relatively compact in $C(K)$ with respect to the norm-topology and hence $L_{x_0}f \in AP_r(K) = AP(K)$.

Similarly, consider the map

$$\begin{aligned}
\Psi : \ C(K) &\rightarrow C(K) \\
g &\mapsto R_{x_0}g.
\end{aligned}$$

We can see similarly that Ψ is is acontinuous map on $C(K)$ with respect to the norm topology and hence the set $\Psi(\overline{\mathcal{O}_l(f)})$ is compact in $C(K)$. As before, we have that

$$\begin{aligned}
\mathcal{O}_l(R_{x_0}f) &= \{L_x R_{x_0} f : x \in K\} \\
&= \{R_{x_0} L_x f : x \in K\} \\
&= \Psi(\mathcal{O}_l(f)) \\
&\subseteq \Psi(\overline{\mathcal{O}_l(f)}).
\end{aligned}$$

Hence $R_{x_0}f \in AP_l(K) = AP(K)$ as required. $\square$

Theorem 4.2.8. *Let K be any semitopological semihypergroup. Then*

1. *$WAP_r(K)$ is left translation-invariant.*
2. *$WAP_l(K)$ is right translation-invariant.*

Proof. Pick any $f \in WAP_r(K)$, $x_0 \in K$. As in the proof of the above theorem, consider the map

$$\begin{aligned}
\Phi : C(K) &\to C(K) \\
g &\mapsto L_{x_0} g.
\end{aligned}$$

Note that $\overline{\mathcal{O}_r(f)}^w$ is compact in $C(K)$. But the weak topology on $C(K)$ is stronger than the topology of pointwise-convergence where later topology is Hausdorff. Hence these two topologies coincide on any compact subset on $C(K)$, particularly on $\overline{\mathcal{O}_r(f)}^w$.

Now let $g \in \overline{\mathcal{O}_r(f)}^w$ and $\{g_\alpha\}$ be a net in $\overline{\mathcal{O}_r(f)}^w$ such that $g_\alpha \xrightarrow{w} g$ on K.

Then in particular, we have that

$$
\begin{aligned}
& \int_K g_\alpha \, d(p_{x_0} * p_y) \to \int_K g \, d(p_{x_0} * p_y) \;\; \text{for each } y \in K. \\
\Rightarrow \;\; & g_\alpha(x_0 * y) \to g(x_0 * y) \;\; \text{for each } y \in K. \\
\Rightarrow \;\; & L_{x_0} g_\alpha \to L_{x_0} g \quad \text{pointwise on } K. \\
\Rightarrow \;\; & L_{x_0} g_\alpha \xrightarrow{w} L_{x_0} g \quad \text{on } C(K).
\end{aligned}
$$

Thus Φ is weak-weak continuous on $\overline{\mathcal{O}_r(f)}^w$ and hence $\Phi(\overline{\mathcal{O}_r(f)}^w)$ is weakly compact in $C(K)$. Now the fact that $L_{x_0} f \in WAP_r(K)$ follows from the fact that

$$
\begin{aligned}
\mathcal{O}_r(L_{x_0} f) &= \{R_x L_{x_0} f : x \in K\} \\
&= \{L_{x_0} R_x f : x \in K\} \\
&= \Phi(\mathcal{O}_r(f)) \\
&\subseteq \Phi(\overline{\mathcal{O}_r(f)}^w).
\end{aligned}
$$

In a similar manner we can also see that $WAP_l(K)$ is right translation-invariant. $\square$

Remark 4.2.9. Note that for a topological semigroup $(S, \cdot)$ it is trivial to see that both $AP(S)$ and $WAP(S)$ are translation-invariant since for any $s, t \in S$ we have that

$$l_s l_t = l_{ts} \quad , \quad r_s r_t = r_{st}.$$

But in general for a semitopological semihypergroup $(K, *)$, the left translation-invariance of $WAP_l(K)$ and similarly the right translation-invariance of $WAP_r(K)$ may not neccessarily hold.

4.3 AP(K) and WAP(K) on Compact Spaces

In this section, we examine the structure of the (weakly) almost periodic function spaces when the underlying (semitopological) semihypergroup is compact. In particular, we see that in this aspect, semihypergroups in general behave exactly like semigroups when the underlying space is compact.

The converse of the result, *i.e,* whether for an unknown semihypergroup, certain particular structure of these function spaces are sufficient to conclude that the underlying space is compact, is also investigated. We conclude the section with a sufficient condition for the existence of a left invariant mean on a semihypergroup K.

Before we proceed further, let us first quickly recall a widely-used result by A. Grothendieck [16].

Theorem 4.3.1 (Grothendieck)**.** *Let X be a compact Hausdorff space. Then a bounded set in $C(X)$ is weakly compact if and only if it is compact in the topology of pointwise convergence.*

The proof of the above theorem can be found in [16].

Proposition 4.3.2. *Let K be a semitopological semihypergroup and f is a continuous function on $C(K)$. Then the map $(x, y) \mapsto f(x * y)$ is separately* *continuous.*

Proof. Fix $x_0 \in K$. Since the map $(\mu, \nu) \mapsto \mu * \nu$ is separately continuous on $M(K) \times M(K)$, we have that the map $\nu \mapsto p_{x_0} * \nu$ is positive continuous on $M(K)$.

Thus by Proposition 2.1.5 we have that the map $y \mapsto f'(y)$ is continuous. But here $f'(y) = \int_K f \ d(p_{x_0} * p_y) = f(x_0 * y)$ and so the map $y \mapsto f(x_0 * y)$ is continuous.

Similarly, we can show that the map $x \mapsto f(x * y_0)$ is continuous for any fixed $y_0 \in K$. $\square$

Theorem 4.3.3. *If K is a compact semitopological semihypergroup, then $WAP_r(K) = C(K)$.*

Proof. Pick any $f \in C(K)$. We need to show that $\mathcal{O}_r(f)$ is weakly compact in $C(K)$.

Let $\{x_\alpha\}$ be a net in K converging to x. By Proposition 4.3.2 we see that for each $y \in K$ the map $x \mapsto f(x * y) = R_x f(y)$ is continuous and hence $R_{x_\alpha} f(y) \to R_x f(y)$. Thus the map $x \mapsto R_x f$ from K into $C(K)$ is continuous where $C(K)$ is equipped with the topology of pointwise convergence.

Since K is compact, by Theorem 4.3.1 we have that $\mathcal{O}_r(f)$ is relatively compact in $C(K)$ with respect to the weak topology. $\square$

It turns out that the result becomes much stronger if we have joint-continuity on the convolution product $*$ on $M(K)$, opposed to separate continuity as in the above case.

Theorem 4.3.4. *If K is a compact semihypergroup, then*

$$AP(K) = WAP_l(K) = WAP_r(K) = C(K).$$

Proof. Pick any $f \in C(K)$. We need to show that $\mathcal{O}_r(f)$ is relatively compact in $C(K)$ with respect to the strong (norm) topology.

Consider the map $\Phi : K \to C(K)$ given by $x \mapsto R_x f$. Since the map $(\mu, \nu) \mapsto \mu * \nu$ is continuous on $M^+(K) \times M^+(K)$ we have that the map $(x, y) \mapsto f(x * y)$ is continuous on $K \times K$.

Fix $y_0 \in K$ and pick any $\varepsilon > 0$. Then for each $x \in K$ we will get open neighborhoods V_x of x and W_x of y_0 such that

$$|f(x * y_0) - f(s * t)| < \varepsilon$$

where $(s, t) \in V_x \times W_x$.

Since $\{V_x\}_{x \in K}$ is an open cover of the compact space K, we will have a finite subcover $\{V_{x_i}\}_{i=1}^n$ that covers K. Set $W := \cap_{i=1}^n W_{x_i}$. Then for any $x \in K, t \in W$ we will have that

$$|R_{y_0} f(x) - R_t f(x)| = |f(x * y_0) - f(x * t)| < \varepsilon.$$

Thus we get an open neighborhood W of y such that $||R_{y_0}f - R_t f||_\infty < \varepsilon$ for any $t \in W$. Hence the map Φ is continuous and so $\mathcal{O}_r(f)$ is compact.

Since $AP(K) \subset WAP_l(K)$ and $AP(K) \subset WAP_r(K)$ the result follows. $\square$

The immediate question that naturally rises now is whether a converse to Theorem 4.3.3 and Theorem 4.3.4 also holds true. We will examine that in two parts in what follows.

Recall that a topological space X is called σ-compact if it is an union of countably many compact subspaces. Now before we proceed further, let us first note the following property of (weakly) almost periodic functions on a semitopological semihypergroup.

Theorem 4.3.5. *Let K be a semitopological semihypergroup and $f \in C(K)$. If $\mathcal{O}_r(f)$ is (weakly) relatively compact in $C(K)$ then the map $x \mapsto R_x f$ is (weakly) continuous on K.*

Proof. First assume that $\mathcal{O}_r(f)$ is relatively compact in $C(K)$ in norm topology. Let $\{x_\alpha\}$ be a net in K converging to $x \in K$. Then By Proposition 4.3.2 we have that $f(y * x_\alpha) \to f(y * x)$ for each $y \in K$ and hence $R_{x_\alpha} f \to R_x f$ pointwise in $C(K)$.

If the net $\{R_{x_\alpha} f\}$ has a limit point in $C(K)$, it has to be $R_x f$. But $\overline{\mathcal{O}_r(f)}$ is compact in $C(K)$ and hence $R_{x_\alpha} f \to R_x f$, as required.

The case where $\mathcal{O}_r(f)$ is relatively compact in $C(K)$ with respect to the weak topology, can be proved proceeding along the same lines. $\square$

Note that all the above results will also hold true for the spaces of left almost periodic functions. Now we are ready to investigate the converses as mentioned above.

Theorem 4.3.6. *Let K be a σ-compact semitopological semihypergroup such that $WAP_l(K) = C(K)$. Then K is compact.*

Proof. If possible, suppose that K is not compact.

Then we can get a strictly increasing sequence $\{K_i\}$ of compact sets in K such that $\cup_i K_i = K$. Set $U_0 := \emptyset$. For each $i \in \mathbb{N}$ repeat the following steps:

1. Pick $x_i \in K$ such that $(\{x_i\} * K_i) \cap U_{i-1} = \emptyset$.
2. Set $A_i := \{x_i\} * K_i$.
3. Pick $y_i \in K$ such that $(K_i * \{y_i\}) \cap (U_{i-1} \cup A_i) = \emptyset$.
4. Set $B_i := K_i * \{y_i\}$.
5. Set U_i to be a compact neighborhood of the compact set $(U_{i-1} \cup A_i \cup B_i \cup K_i)$.

Note that we can always get such x_i and y_i's since otherwise it will imply that K is compact. Now we have that each A_i and B_i are compact and $A_i \cap B_i = \emptyset$ for each i. Hence using Urysohn's Lemma we get $f \in C(K)$ such that $f \equiv 0$ on each A_i and $f \equiv 1$ on each B_i.

Now for any $y \in K_i$ we have that $supp(p_{x_i} * p_y) \subset A_i$ and hence

$$f(x_i * y) = \int_K f \, d(p_{x_i} * p_y) = \int_{supp(p_{x_i} * p_y)} f \, d(p_{x_i} * p_y) = 0 \ .$$

This is true for each i.

Pick any $y \in K$. From the contruction of K_i's there exists some $i_0 \in \mathbb{N}$ such that $y \in K_i$ for any $i \geq i_0$. Since the map $(x, y) \mapsto f(x * y)$ is separately continuous on $K \times K$ we have that

$$\lim_{i \to \infty} L_{x_i} f(y) = \lim_{i \to \infty, i \geq i_0} f(x_i * y) = 0 \ .$$

Hence the sequence $\{L_{x_i} f\}$ converges pointwise to 0 on K. Thus any weakly convergent subsequence of $\{L_{x_i} f\}$, if exists, will converge weakly to 0.

Following the same steps as above we see that $\lim_{i \to \infty} R_{y_i} f(x) = 1$ for any $x \in K$. Now for any $g \in C(K)$, we define $\phi(g) := \lim_{i \to \infty} g(y_i)$.

Set $N := \{g \in C(K) : \phi(g) \ \text{ exists } \}$. Then N is a linear subspace of $C(K)$, and ϕ is a well-defined linear functional on N. Therefore using Hahn-Banach Theorem we can extend ϕ to $C(K)$, and consider ϕ as a linear functional on $C(K)$.

In particular, for each $x_j \in K$ we have that

$$\phi(L_{x_j} f) = \lim_{i \to \infty} L_{x_j} f(y_i) = \lim_{i \to \infty} R_{y_i} f(x_j) = 1 \ .$$

But we know that any weakly convergent subsequence of $\{L_{x_j}f\}$, if exists, will converge weakly to the zero function. Hence the above equality implies that there does not exist any weakly convergent subsequence of $\{L_{x_j}f\}_{j\in\mathbb{N}}$, *i.e*, the set $\{L_x f : x \in K\}$ is not sequentially weakly compact.

This implies that $\mathcal{O}_l(f)$ is not weakly compact, contradicting the fact that $f \in C(K) = WAP_l(K)$. $\square$

Theorem 4.3.7. *Let K be a non σ-compact semitopological semihypergroup with identity such that $WAP_l(K) = C(K)$. Then K is compact.*

Proof. If possible, suppose that K is not compact.

Let $\{U_n\}$ be an increasing sequence of open sets in K such that $e \in U_1$, $\overline{U}_n \subset U_{n+1}$ for each n and $\overline{U}_n$ is compact for each n. Set $V_n := \overline{U}_n$ and for each $n, k \in \mathbb{N}$ define

$$V_n^k := \underbrace{V_n * V_n * \ldots * V_n}_{k \text{ times}} .$$

For any $n > 1$ since $V_{n-1} \subset V_n$ we have that

$$V_{n-1}^{n-1} \subset V_n^{n-1} \subset V_n^{n-1} * V_n = V_n^n$$

where the second inclusion holds since $e \in V_n$ and therefore for any $x \in V_n^{n-1}$ we have that $\{x\} = supp(p_x) = supp(p_x * p_e) \subset V_n^{n-1} * V_n$. Thus we see that $V_n^n \subset V_m^m$ whenever $n < m$.

Set $H := \cup_{k\in\mathbb{N}} V_k^k$. Note that for any $x, y \in H$ we can get $m, n \in \mathbb{N}$ such that $x \in V_m^m$ and $y \in V_n^n$. Pick any $l \in \mathbb{N}$ such that $l > m + n$. Then from the above result we have that $x, y \in V_l^l$ and hence

$$\{x\} * \{y\} \subset V_l^l * V_l^l = V_l^{2l} \subset V_{2l}^{2l}$$

where the last inclusion follows since $V_l \subset V_{2l}$. Thus we see that

$$H * H \subset H,$$

i.e, H is a subsemihypergroup of K, which is closed and σ-compact by construction.

If H is compact, then set L to be the subsemihypergroup of K generated by the union of countably many cosets of H in K. If H is non-compact, then set $L := H$. From Theorem 4.3.6 we see that $WAP_l(L) \neq C(L)$. Pick $f \in C(L) \setminus WAP_l(L)$ and extend f to $F \in C(K)$ by defining $F(x) = 0$ for any x outside L. Since $\mathcal{O}_l(f)$ is not weakly sequentially compact in $C(L)$ we have that $\mathcal{O}_l(F)$ is not weakly sequentially compact in $C(K)$.

Thus we see that F does not lie in $WAP_l(K)$ contradicting our given hypothesis. $\square$

Now the final result of this section gives us a sufficient condition for the left amenability of a semihypergroup K, *i.e,* the existence of a left-invariant mean on $C(K)$. Here we use the well-known fact that a commutative semigroup is always amenable.

Recall that a (semitopological) semihypergroup K is called *commutative* whenever we have that

$$\mu * \nu = \nu * \mu$$

for any $\mu, \nu \in M(K)$, or equivalently, whenever we have

$$p_x * p_y = p_y * p_x$$

for any $x, y \in K$.

Theorem 4.3.8. *Let K be a commutative semihypergroup. Then there exists a LIM on $C(K)$.*

Proof. Consider the function $\Phi : C(K) \to C(P_C(K))$ introduced in the proof of Theorem 4.2.2, defined as $f \mapsto \tilde{f}$ where

$$\tilde{f}(\mu) := \int_K f \, d\mu$$

for each $\mu \in P_C(K)$.

We know that $(P_C(K), *)$ is a commutative semigroup. Then there exists a LIM m on $P_C(K)$. Define

$$\tilde{m} := m \circ \Phi : C(K) \to \mathbb{C},$$

$i.e$, for each $f \in C(K)$ we have

$$\tilde{m}(f) = m(\tilde{f}).$$

Note that for any $f \in C(K)$ such that $f \geq 0$ we have that

$$\tilde{f}(\mu) = \int_K f \; d\mu \geq 0$$

for any $\mu \in P_C(K)$. Therefore

$$\tilde{m}(f) = m(\tilde{f}) \geq 0.$$

Moreover, $\tilde{m}(1) = m(\tilde{1}) = m(1) = 1$. Hence $\tilde{m}$ is a mean on $C(K)$.

Now pick any $x \in K$, $f \in C(K)$. For any $\mu \in P_C(K)$ we have that

$$\begin{aligned}
(L_x f)\tilde{}(\mu) &= \int_K L_x f \; d\mu \\
&= \int_K f \; d(p_x * \mu) \\
&= \tilde{f}(p_x * \mu) = L_{p_x} \tilde{f}(\mu).
\end{aligned}$$

Now the left invariance of $\tilde{m}$ follows as below

$$\begin{aligned}
\tilde{m}(L_x f) &= m((L_x f)\tilde{}) \\
&= m(L_{p_x} \tilde{f}) \\
&= m(\tilde{f}) = \tilde{m}(f).
\end{aligned}$$

Hence the result. $\square$

4.4 Introversion on AP(K)*

In the last section of this chapter, similar to the general theory of topological semigroups, we introduce the concept of introversion on a translation invariant function space on a (semitopological) semihypergroup K. For topological semigroups, the concept of introversion was introduced by M. M. Day in [4]. We conclude by exploring how the introversion operators help us in acquiring an algebraic structure on $AP(K)^*$.

Definition 4.4.1. *Let $\mathcal{F}$ be a translation-invariant linear subspace of $C(K)$. For each $\mu \in \mathcal{F}^*$ the left introversion operator T_μ determined by μ is the map $T_\mu : \mathcal{F} \to B(K)$ defined as*

$$T_\mu f(x) := \mu(L_x f)$$

for each $x \in K$.

Similarly, the right introversion operator U_μ determined by μ is the map $U_\mu : \mathcal{F} \to B(K)$ given by

$$U_\mu f(x) := \mu(R_x f).$$

Definition 4.4.2. *Let K be a (semitopological) semihypergroup and $\mathcal{F}$ be a translation-invariant linear subspace of $C(K)$. $\mathcal{F}$ is called left-introverted if*

$T_\mu f \in \mathcal{F}$ for each $\mu \in \mathcal{F}^*$, $f \in \mathcal{F}$.

Similarly, $\mathcal{F}$ is called right-introverted if $U_\mu f \in \mathcal{F}$ for each $\mu \in \mathcal{F}^*$, $f \in \mathcal{F}$.

We denote by $\mathcal{B}_1$ the closed unit ball of $AP(K)^*$. Now before we proceed further to define an algebraic structure on $AP(K)^*$, let us first explore some basic properties of introversion operators on $AP(K)$.

The next result gives us a necessary and sufficient condition for a function to be almost periodic, in terms of left and right introversion operators. Before we get on with the proof, let us first quickly recall the following version of Mazur's Theorem and another important result from [23].

Theorem 4.4.3 (Mazur)**.** *Let A be a compact subset of a Banach space X. Then the closed circled convex hull of A, denoted as $\overline{cco}(A)$ is also compact.*

Theorem 4.4.4. *Let K be a semihypergroup and $\mathcal{F}$ be a translation-invariant conjugation-closed linear subspace of $B(K)$ containing constant functions. For any $f \in \mathcal{F}$, the set $\{T_\mu f : ||\mu|| \leq 1\}$ is the closure of $cco(\mathcal{O}_r(f))$ in $B(K)$ with respect to the topology of pointwise convergence.*

Note that the above theorem is proved for topological semigroups in [23]. The proof for semihypergroups follows in exactly the same manner.

Theorem 4.4.5. *Let K be a semitopological semihypergroup. Then $f \in AP(K)$ if and only if the map*

$$\mu \mapsto T_\mu f : \mathcal{B}_1 \to B(K)$$

is $\sigma(\mathcal{B}_1, AP(K))$*-norm continuous.*

Proof. Let $f \in AP(K)$. Let $\Psi : \mathcal{B}_1 \to B(K)$ be the map given by

$$\Psi(\mu) := T_\mu f \text{ for each } \mu \in \mathcal{B}_1.$$

Let $\mu_\alpha \to \mu$ in $\mathcal{B}_1$ with respect to the topology $\sigma(\mathcal{B}_1, AP(K))$, *i.e,* we have that $\mu_\alpha(g) \to \mu(g)$ for each $g \in AP(K)$.

Also by Theorem 4.2.7 we know that $L_x f \in AP(K)$ for each $x \in K$. Hence in particular for each $x \in K$ we have that

$$\begin{aligned} \mu_\alpha(L_x f) &\to \mu(L_x f) \quad \text{for each } x \in K \\ \Rightarrow T_{\mu_\alpha} f(x) &\to T_\mu f(x) \quad \text{for each } x \in K \\ \Rightarrow \quad T_{\mu_\alpha} f &\to T_\mu f \quad \text{in topology of pointwise convergence.} \end{aligned}$$

Hence the map Ψ is continuous when $B(K)$ is equipped with the topology of pointwise convergence.

Also, $\Psi(\mathcal{B}_1)$ is the closure of $cco(\mathcal{O}_r(f))$ with respect to topology of pointwise convergence. Since $\mathcal{O}_r(f)$ is compact in $B(K)$, we have that $\overline{\Psi(\mathcal{B}_1)}$ is compact. Hence the topology of pointwise convergence coincides with the norm topology on $\Psi(\mathcal{B}_1)$.

Conversely, let the map $\Psi : \mathcal{B}_1 \to B(K)$ as defined above is $\sigma(\mathcal{B}_1, AP(K))$-norm continuous. Then it follows immediately that $f \in AP_r(K) = AP(K)$

since $\Psi(\mathcal{B}_1)$ is norm-compact and the following inclusion holds:

$$\mathcal{O}_r(f) \subset \overline{cco(\mathcal{O}_r(f))}^{pt.wise\ topology} = \Psi(\mathcal{B}_1).$$

$\square$

Note that of course, the right-counterpart of the above theorem holds true in a similar manner, *i.e,* we can also show that a function f is almost periodic if and only if the map

$$\mu \mapsto U_\mu f : \mathcal{B}_1 \to \mathcal{B}(K)$$

is $\sigma(\mathcal{B}_1, AP(K))$-norm continuous.

Now we see that $AP(K)$ is left and right introverted, enabling us to introduce left and right Arens product [1] on $AP(K)^*$.

Corollary 4.4.6. *Let K be any semitopological semihypergroup. Then $AP(K)$ is left and right introverted.*

Proof. We know that $AP(K)$ is a translation-invariant linear subspace of $C(K)$. Now pick any $f \in AP(K)$ and again consider the function $\Psi : \mathcal{B}_1 \to B(K)$ given by

$$\Psi(\mu) := T_\mu f \text{ for each } \mu \in \mathcal{B}_1.$$

As pointed out in the above proof, we have that the norm topology coincides with the topology of pointwise convergence on $\Psi(\mathcal{B}_1)$ and hence finally we have that

$$\Psi(\mathcal{B}_1) = \overline{cco}(\mathcal{O}_r(f)) \subset AP(K).$$

Hence after scaling by proper scalers we have that $T_\mu f \in AP(K)$ for each $\mu \in AP(K)^*$, as required.

The proof for the right-counterpart follows similarly. $\square$

Theorem 4.4.7. *Let K be a semitopological semihypergroup and $f \in AP(K)$. Then the map $\Phi : \mathcal{B}_1 \times \mathcal{B}_1 \to \mathbb{C}$ given by*

$$\Phi(\mu, \nu) := \mu(T_\nu f)$$

is continuous with respect to the topology $\sigma(\mathcal{B}_1, AP(K)) \times \sigma(\mathcal{B}_1, AP(K))$. Moreover, for any $\mu, \nu \in AP(K)^*$ we have that

$$\mu(T_\nu f) = \nu(U_\mu f).$$

Proof. Pick any $(\mu_0, \nu_0) \in \mathcal{B}_1 \times \mathcal{B}_1$ and consider the function Ψ as in Theorem 4.4.5. Then for any $\mu, \nu \in \mathcal{B}_1$ we have that

$$\begin{aligned}
|\phi(\mu,\nu) - \phi(\mu_0,\nu_0)| &= |\mu(T_\nu f) - \mu_0(T_{\nu_0} f)| \\
&\leq |\mu(T_\nu f) - \mu(T_{\nu_0} f)| + |\mu(T_{\nu_0} f) - \mu_0(T_{\nu_0} f)| \\
&\leq ||\Psi(\nu) - \Psi(\nu_0)|| + |(\mu - \mu_0)(T_{\nu_0} f)| \\
&= ||\Psi(\nu) - \Psi(\nu_0)|| + |(\mu - \mu_0)(\Psi(\nu_0))|
\end{aligned}$$

Since Ψ is $\sigma(\mathcal{B}_1, AP(K))$-norm continuous, we have that both the terms in the last inequality tends to zero whenever (μ, ν) tends to (μ_0, ν_0) in $\mathcal{B}_1 \times \mathcal{B}_1$ with respect to the topology $\sigma(\mathcal{B}_1, AP(K)) \times \sigma(\mathcal{B}_1, AP(K))$.

In a similar way, we can show that the map $\Phi' : \mathcal{B}_1 \times \mathcal{B}_1 \to \mathbb{C}$ given by

$$\Phi'(\mu, \nu) := \nu(U_\mu f)$$

is continuous with respect to the topology $\sigma(\mathcal{B}_1, AP(K)) \times \sigma(\mathcal{B}_1, AP(K))$.

Now for any $x \in K$ consider the evaluation map $E_x \in AP(K)^*$ given by

$$E_x(f) := f(x) \quad \text{for each } f \in AP(K).$$

We denote the set of all evaluation maps in $AP(K)^*$ as $\mathcal{E}(K)$, *i.e,* we have

$$\mathcal{E}(K) := \{E_x : x \in K\}.$$

Pick any $\mu = E_x, \nu = E_y$ for some $x, y \in K$. Then we have

$$\begin{aligned}
\mu(T_\nu f) &= E_x(T_\nu f) \\
&= T_\nu f(x) \\
&= E_y(L_x f) \\
&= L_x f(y) = f(x * y).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned} f(x * y) &= R_y f(x) \\ &= E_x(R_y f) \\ &= U_\mu f(y) \\ &= E_y(U_\mu f) = \nu(U_\mu f). \end{aligned}$$

Note that as $\mathcal{E}(K)$ is the set of extreme points, we have that the weak* closure of $cco(\mathcal{E}(K))$ equals $\mathcal{B}_1$ [23]. Hence it follows from the continuity of the maps Φ and Φ' that

$$\mu(T_\nu f) = \nu(U_\mu f)$$

for any $\mu, \nu \in \mathcal{B}_1$ and hence by proper scaling, for any $\mu, \nu \in AP(K)^*$. $\square$

The above theorem holds true even if we replace $AP(K)$ by any translation invariant conjugation-closed linear subspace $\mathcal{F}$ of $C(K)$ containing constant functions.

Now let us define a product $\star$ on $AP(K)^*$ given by

$$\mu \star \nu(f) := \mu(T_\nu f).$$

for any $(\mu, \nu) \in AP(K)^* \times AP(K)^*$.

Now recall the construction of the left Arens product $\Diamond$ for any Banach algebra X [1] and apply it on $AP(K)^*$. For any $\mu, \nu \in AP(K)^*$, $f \in AP(K)$ and

$x, y \in K$ we have that

$$\begin{aligned} f\Diamond x(y) &:= f(x * y). \\ \nu\Diamond f(x) &:= \nu(f\Diamond x). \\ \mu\Diamond\nu(f) &:= \mu(\nu\Diamond f). \end{aligned}$$

Hence in particular, we have that

$$\begin{aligned} f\Diamond x(y) &= L_x f(y). \\ \nu\Diamond f(x) &= \nu(L_x f) = T_\nu f(x). \\ \mu\Diamond\nu(f) &= \mu(T_\nu f) = \mu \star \nu(f). \end{aligned}$$

Note that here the second step is possible since $AP(K)$ is left translation-invariant and the last step is possible since $AP(K)$ is left introverted.

Thus the left Arens product coincides with $\star$ on $AP(K)^*$ and hence $(AP(K)^*, \star)$ becomes a Banach algebra.

Now in a similar manner, consider the right Arens product $\Box$ on $AP(K)^*$. Then we have that

$$\begin{aligned} x\Box f(y) &= f(y * x) = R_x f(y). \\ f\Box\mu(x) &= \mu(x\Box f) = \mu(R_x f) = U_\mu f(x). \\ \mu\Box\nu(f) &= \nu(f\Box\mu) = \nu(U_\mu f). \end{aligned}$$

Thus in view of Theorem 4.4.7 we see that the left and right Arens product coincide on $AP(K)^*$, *i.e*, $AP(K)^*$ is Arens regular.

Chapter 5

Ideals and Homomorphisms

In this chapter, we introduce the concept of an ideal in (semitopological) semihypergroups and explore some of its basic properties as well as its relation with a more general form of homomorphism between semihypergroups. Furthermore, we investigate the structure of the kernel of a compact (semitopological) semihypergroup and finally explore the connection between minimal left ideals and the concept of amenability on a compact semihypergroup. The contents of this chapter is based on [2, Section 5].

5.1 Motivation and Definitions

Here we introduce the basic definitions of ideals and homorphisms for semitopological semihypergroups. First we briefly recall the concepts of ideals and homomorphisms for a semigroup, in order to realise how the definitions for (semitopological) semihypergroups boil down to the classical definitions when restricted to topological semigroups.

Throughout this section unless otherwise mentioned, K and H will denote semitopological semihypergroups.

Definition 5.1.1. *Let $(S, \cdot)$ be a semigroup. A subset $I \subset S$ is called a left ideal if for any $a \in I, x \in S$ we have that*

$$a.x \in I.$$

Similarly, a subset $I \subset S$ is called a right ideal of S if for any $a \in I, x \in S$ we have that

$$x.a \in I.$$

Finally, a subset $I \subset S$ is called a two-sided ideal (or simply an ideal) in S if it is both a left and right ideal.

Now we define such structures for a semitopoloical semihypergroup.

Definition 5.1.2. *Let K be a (semitopological) semihypergroup. A Borel measurable set $I \subset K$ is called a left (resp. right) ideal of K if for any $a \in I$,*

$x \in K$ we have that

$$p_x * p_a(I) = 1 \quad \big(resp. \;\; p_a * p_x(I) = 1\big).$$

Equivalently, a Borel measurable set $I \subset K$ is called a left (resp. right) ideal of K if for any $a \in I$, $x \in K$ and any measurable set $E \subset K$ such that $E \cap I = \emptyset$, we have that

$$p_x * p_a(E) = 0 \quad \big(resp. \;\; p_a * p_x(E) = 0\big).$$

A subset $I \subset K$ is called a (two-sided) ideal if it is both a left and right ideal.

Remark 5.1.3. It follows immediately from the above definitions that a closed set $I \subset K$ is a left (resp. right) ideal in K if for any $x \in K$, $a \in I$ we have that

$$supp(p_x * p_a) \subset I \quad \big(resp. \;\; supp(p_a * p_x) \subset I\big).$$

Now let K be a topological semigroup as in Example 2.2.1. Then for any $a \in I, x \in K$ we have that

$$a.x = supp(p_{a.x}) = supp(p_a * p_x).$$

Hence the classical definition of a left ideal (and similarly for a right ideal) coincides with the above definition.

Throughout this chapter, we discuss the properties of left ideals and their relation to several algebraic and analytic objects associated to K. All these results hold for right ideals too, and can be proved in exactly similar manner as in the case of left ideals.

Now let us briefly recall the definition for a semigroup homomorphism.

Definition 5.1.4. *Let $(S, \cdot)$ and $(T, \cdot)$ be semigroups. A map $\phi : S \to T$ is called a homomorphism if for any $x, y \in S$ we have that*

$$\phi(x.y) = \phi(x).\phi(y)$$

We now define homomorphisms for (semitopological) semihypergroups in such a way, that a continuous homomorphism (in the classical definition) between two (semi)topological semigroups will also remain a homomorphism according to the new definition.

Definition 5.1.5. *Let K and H be two (semitopological) semihypergroups. A continuous map*

$$\phi : K \to H$$

is called a homomorphism if for any Borel measurable function f on H and for any $x, y \in K$ we have that

$$f \circ \phi(x * y) = f(\phi(x) * \phi(y)).$$

A homomorphism ϕ between K and H is called an isomorphism if the map ϕ is bijective.

Remark 5.1.6. Let K be a topological semigroup as in Example 2.2.1. Then for any Borel measurable function h on K and any $u, v \in K$ we have

$$\begin{aligned} h(u * v) &= \int_K h \; d(p_u * p_v) \\ &= \int_K h \; d(p_{uv}) \\ &= h(uv). \end{aligned}$$

Now let both K and H in the above definition be topological semigroups in the sense of Example 2.2.1. Then a continuous map

$$\phi : K \to H$$

will be called a homomorphism if for any measurable function f on H and for any $x, y \in K$ we have that

$$f \circ \phi(xy) = f(\phi(x)\phi(y)) \quad .$$

In particular, if we have that

$$\phi(xy) \neq \phi(x)\phi(y)$$

for some $x, y \in K$, then we can easily find a measurable function f on K such that

$$f\big(\phi(xy)\big) = 1, \quad f\big(\phi(x)\phi(y)\big) = 0$$

and hence ϕ will not be a homomorphism according to Definition 5.1.5.

Thus we see that the classical definition of a homomorphism between two semigroups coincide with Definition 5.1.5 when we restrict K and H to be (semi)topological semigroups.

Remark 5.1.7. In 1975 Jewett introduced the concept of orbital morphisms for hypergroups in [18]. If K and J are two hypergroups, then a map

$$\phi : k \to J$$

is called proper if $\phi^{-1}(C)$ is compact in K for every compact set C in J. A recomposition of ϕ was defined to be a continuous map from $J \to M^+(K)$ defined as $x \mapsto q_x$ such that

$$supp(q_x) = \phi^{-1}(\{x\}).$$

Roughly speaking, a proper open map $\phi : K \to J$ is called a orbital homomorphism if for any $x, y \in J$ we have that

$$p_x * p_y = \phi(q_x * q_y)$$

as measures, *i.e,* for nay Borel measurable function f on J we have that

$$\int_J f \; d(p_x * p_y) = \int_K f \circ \phi \; d(q_x * q_y).$$

Jewett defined homomorphism between hypergroups as a special case of orbital morphisms. According to the definition in [18] a continuous open map

$$\phi : K \to J$$

is called a homomorphism if

$$\phi(p_x * p_y) = p_{\phi(x)} * p_{\phi(y)}$$

for any $x, y \in K$.

Note that our definition of homomorphism is equivalent to that of Jewett's apart from the fact that in Definition 5.1.5 we do not need the map ϕ to be open and proper, and here the map is defined between two (semitopological) semihypergroups, instead of hypergroups.

Otherwise these two definitions coincide since similar to the definition of orbital morphisms, Jewett's definition of homomorphism implies that for any measurable function f on J, $x, y \in K$ we have that

$$\int_J f \; d(p_{\phi(x)} * p_{\phi(y)}) = \int_K f \circ \phi \; d(p_x * p_y).$$

Hence by definition we have

$$f(\phi(x) * \phi(y)) = f \circ \phi(x * y).$$

5.2 Basic Properties

Now that we have defined ideals and homomorphisms for semitopological semihypergroups, next we check if some of the basic properties of the classical case hold for these definitions as well. Since in the classical case, the multiplication of two points simply gives us another point, these properties hold trivially in that case.

Throughout this section, we see that all of these properties hold true for semitopological semihypergroups as well, with respect to Definition 5.1.2 and Definition 5.1.5. We first prove the following key lemma before we proceed to examine several specific properties sketching the interplay of ideals and homomorphisms in general for (semitopological) semihypergroups.

Lemma 5.2.1. *Let K and H be (semitopological) semihypergroups, and $\phi : K \to H$ is a homomorphism. Then for any $x, y \in K$, for almost all $z \in supp(p_x * p_y)$ with respect to the measure $(p_x * p_y)$ we have that*

$$\phi(z) \in supp(p_{\phi(x)} * p_{\phi(y)}).$$

Conversely for any $x, y, z \in K$, for almost all $\phi(z) \in supp(p_{\phi(x)} * p_{\phi(y)})$ with respect to the measure $(p_{\phi(x)} * p_{\phi(y)})$ we have that

$$z \in \phi^{-1}\phi(supp(p_x * p_y)).$$

Proof. Pick any $x, y \in K$ and let $z \in \mathrm{supp}(p_x * p_y)$.

Now set

$$A := \text{supp}(p_{\phi(x)} * p_{\phi(y)}) \subset H$$

and define a measurable function g on H such that

$$g \equiv \begin{cases} 0 & \text{on} \quad A. \\ 1 & \text{on} \quad A^c. \end{cases}$$

Then we have that

$$\begin{aligned} 0 &= \int_A g \; d(p_{\phi(x)} * p_{\phi(y)}) \\ &= \int_H g \; d(p_{\phi(x)} * p_{\phi(y)}) \\ &= g(\phi(x) * \phi(y)) \\ &= g \circ \phi(x * y) \\ &= \int_K g \circ \phi \; d(p_x * p_y). \end{aligned}$$

For any $z \in supp(p_x * p_y)$, for the relation

$$\int_K (g \circ \phi) \; d(p_x * p_y) = 0$$

to hold true we must have that $g \circ \phi(z) = 0$ almost everywhere on $supp(p_x * p_y)$ with respect to $(p_x * p_y)$. Thus by construction of g we have that $\phi(z) \in A$ for almost all $z \in supp(p_x * p_y)$ as required.

Now to prove the converse, pick any $x, y \in K$ and let $z \in K$ be such that

$\phi(z) \in \text{supp}(p_{\phi(x)} * p_{\phi(y)})$.

Set

$$A := \text{supp}(p_x * p_y)$$

and define a measurable function h on H such that

$$h \equiv \begin{cases} 0 & \text{on} \;\; \phi(A). \\ 1 & \text{on} \;\; \phi(A)^c. \end{cases}$$

Note that for any $u \in A$ we have that $\phi(u) \in \phi(A)$ and hence $h \circ \phi(u) = 0$. Thus we have

$$\begin{aligned} 0 &= \int_A h \circ \phi(u) \; d(p_x * p_y) \\ &= \int_K h \circ \phi \; d(p_x * p_y) \\ &= h \circ \phi(x * y) \\ &= h(\phi(x) * \phi(y)) \\ &= \int_H h \; d(p_{\phi(x)} * p_{\phi(y)}). \end{aligned}$$

where the first equality follows from the construction of h and the second equality follows from the construction of A.

Now if $z \in A$, then the result follows trivially. Let z does not lie in A. Since

$$\phi(z) \in supp(p_{\phi(x)} * p_{\phi(y)}) \subset H,$$

for the relation $\int_H h\ d(p_{\phi(x)} * p_{\phi(y)}) = 0$ to be true we must have that

$$h(\phi(z)) = 0$$

for almost all $\phi(z) \in supp(p_{\phi(x)} * p_{\phi(y)})$ with respect to $(p_{\phi(x)} * p_{\phi(y)})$.

Thus almost all $\phi(z) \in \phi(A)$ and so $z \in \phi^{-1}\phi(A)$ as required. $\square$

Remark 5.2.2. Roughly speaking, the above lemma implies that for two (semitpological) semihypergroups K and H and an isomorphism $\phi : K \to H$ between them, we have the following:

For any $x, y \in K$ we have that $z \in supp(p_x * p_y)$ if and only if

$$\phi(z) \in supp(p_{\phi(x)} * p_{\phi(y)}),$$

upto a set of measure zero.

Proposition 5.2.3. *Let $\phi : K \to H$ be a homomorphism. Then $\phi(K)$ is a semihypergroup.*

Proof. In order to verify if $\phi(K)$ is a semihypergroup we only need to check if $\phi(K)$ is closed under convolution, *i.e,*

$$\phi(K) * \phi(K) \subset \phi(K).$$

Pick $a, b \in K$ and set

$$A := \mathrm{supp}(p_{\phi(a)} * p_{\phi(b)}).$$

If possible, let $A \nsubseteq \phi(K)$. Hence $A \setminus \phi(K) \neq \emptyset$. Now define a measurable function f on H such that

$$f \equiv \begin{cases} 1 & \text{on } \ A \setminus \phi(K). \\ 0 & \text{on } \ A \cap \phi(K). \end{cases}$$

Then we have

$$\begin{aligned}
\int_{A\setminus\phi(K)} f \ d(p_{\phi(a)} * p_{\phi(b)}) &= \int_A f \ d(p_{\phi(a)} * p_{\phi(b)}) \\
&= \int_H f \ d(p_{\phi(a)} * p_{\phi(b)}) \\
&= f(\phi(a) * \phi(b)) \\
&= f \circ \phi(a * b) \\
&= \int_K f \circ \phi \ d(p_a * p_b) = 0.
\end{aligned}$$

where the first equality holds since $f \equiv 0$ on $A \cap \phi(K)$, the second equality follows from the construction of A and the last equality holds since for almost all $z \in supp(p_a * p_b)$ we have from Lemma 5.2.1 that $\phi(z) \in A$.

Thus $\phi(z) \in A \cap \phi(K)$ and hence $f \circ \phi(z) = 0$.

But $f \equiv 1$ on $A \setminus \phi(K)$ and $A \setminus \phi(K) \subset supp(p_{\phi(a)} * p_{\phi(b)})$. But $A \setminus \phi(K)$ can not be a set of $(p_{\phi(a)} * p_{\phi(b)})$-measure zero and hence

$$\int_{A\setminus\phi(K)} f \ d(p_{\phi(a)} * p_{\phi(b)}) \neq 0$$

and we arrive at a contradiction. □

Proposition 5.2.4. *Let K be a semitopological semihypergroup. Pick any $a \in K$. Then $I := K * \{a\}$ is a left ideal in K.*

Proof. Pick any $x \in K$ and $b \in I$. Since $I = \cup_{y\in K} supp(p_x * p_a)$ there exists some $y_0 \in K$ such that

$$b \in supp(p_{y_0} * p_a) = \{y_0\} * \{a\}.$$

Now the result follows:

$$\begin{aligned} supp(p_x * p_b) &= \{x\} * \{b\} \\ &\subset \{x\} * (\{y_0\} * \{a\}) \\ &= (\{x\} * \{y_0\}) * \{a\} \\ &\subset K * \{a\} = I. \end{aligned}$$

where the second equality follows from the associativity of convolution of sets (see Proposition 2.1.8). □

Proposition 5.2.5. *Let $\phi : K \to H$ is a homomorphism and $I \subset K$ is a left ideal. Then $\phi(I)$ is also a left ideal in $\phi(K)$.*

Proof. Pick any $a \in I$, $x \in K$. Set

$$B := supp(p_{\phi(x)} * p_{\phi(a)})$$

and define a function f on H such that

$$f \equiv \begin{cases} 1 & \text{on } \ \phi(I) \cap B. \\ 0 & \text{elsewhere .} \end{cases}$$

Then we have

$$\begin{aligned}
p_{\phi(x)} * p_{\phi(a)}(\phi(I)) &= \int_{\phi(I)\cap B} 1 \ d(p_{\phi(x)} * p_{\phi(a)}) \\
&= \int_H f \ d(p_{\phi(x)} * p_{\phi(a)}) \\
&= f(\phi(x) * \phi(a)) \\
&= f \circ \phi(x * a) \\
&= \int_K (f \circ \phi) \ d(p_x * p_a) \\
&= \int_{\phi^{-1}\phi(I)} 1 \ d(p_x * p_a) \\
&= (p_x * p_a)(\phi^{-1}\phi(I)) \\
&= (p_x * p_a)(I) = 1.
\end{aligned}$$

where the sixth equality follows from the fact that for almost all $z \in supp(p_x * p_a)$ Lemma 5.2.1 gives us that $\phi(z) \in B$ and hence $f(\phi(z)) = 0$ whenever $\phi(z)$ does not lie in $\phi(I)$, *i.e,* whenever z lies outside $\phi^{-1}\phi(I)$.

Also, the second last equality follows since $I \subset \phi^{-1}\phi(I)$ and $supp(p_x * p_a) \subset I$ as I is a left ideal in K.

$\square$

Proposition 5.2.6. *Let $\phi : K \to H$ be a homomorphism and $J \subset H$ is a left ideal in H. Then $\phi^{-1}(J)$ is also a left ideal in K.*

Proof. Pick any $a \in \phi^{-1}(J)$, $x \in K$. Then $\phi(a) \in J$ and hence $supp(p_{\phi(x)} * p_{\phi(a)}) \subset J$ and $(p_{\phi(x)} * p_{\phi(a)})(J) = 1$. Define a function f on H by

$$f \equiv \begin{cases} 1 & \text{on } J. \\ 0 & \text{elsewhere }. \end{cases}$$

Now for almost all $z \in supp(p_x * p_a)$ using Lemma 5.2.1 we have that

$$\phi(z) \in supp(p_{\phi(x)} * p_{\phi(a)}) \subset J.$$

Hence $f \circ \phi(z) = 1$ for almost all $z \in supp(p_x * p_a)$. Hence the result follows:

$$\begin{aligned}
(p_x * p_a)(\phi^{-1}(J)) &= \int_{\phi^{-1}(J)} 1 \ d(p_x * p_a) \\
&= \int_J f \circ \phi \ d(p_x * p_a) \\
&= \int_K f \circ \phi \ d(p_x * p_a) \\
&= f \circ \phi(x * a) \\
&= f(\phi(x) * \phi(a)) \\
&= \int_H f \ d(p_{\phi(x)} * p_{\phi(a)}) \\
&= \int_J 1 \ d(p_{\phi(x)} * p_{\phi(a)})
\end{aligned}$$

$$= \quad (p_{\phi(x)} * p_{\phi(a)})(J) = 1.$$

□

5.3 Minimal Ideals

From now onwards and throughout the rest of this chapter, all the (minimal) left and right ideals in question are assumed to be closed in K, unless otherwise mentioned.

In this section, we define and investigate left and right minimal ideals on a (semitopological) semihypergroup. We start off with examining some equivalence criteria for the minimality of a left ideal. Next we examine some basic properties of minimal left (resp. right) ideals that hold trivially for semigroups, for reasons explained in the previous section.

We see that again, most of these properties hold true for the semihypergroup case as well. Before we proceed to the results, let us briefly state the definition of minimal left (resp. right and two-sided) ideals on a semitopological semihypergroup.

Definition 5.3.1. *Let K be a (semitopological) semihypergroup. A left ideal $I \subset K$ is called a minimal left ideal of K if I does not contain any proper left ideal of K.*

Similarly, we can define a minimal right ideal of K. An ideal I of K which is *both minimal left and right ideal, is called a minimal ideal of K.*

Proposition 5.3.2. *For any left ideal $I \subset K$ the following are equivalent:*

1. *I is a minimal left ideal.*

2. *$K * \{a\} = I$ for any $a \in I$.*

3. *$I * \{a\} = I$ for any $a \in I$.*

Proof. (1) $\Rightarrow$ (3): Let I be a minimal left ideal of K and $a \in I$.

Then $I * \{a\} \subset I * I \subset I$ since I is a left ideal. Also, $I * \{a\}$ is a left ideal since

$$K * (I * \{a\}) = (K * I) * \{a\} \subset I * \{a\}$$

where the last inclusion follows since I is a left ideal.

Hence the minimality of I gives us that $I * \{a\} = I$.

(3) $\Rightarrow$ (2): Since I is a left ideal, for each $a \in I$ we have that

$$I = I * \{a\} \subset K * \{a\} \subset I$$

which forces that $K * \{a\} = I$.

(2) $\Rightarrow$ (1): Let J be a left ideal contained in I. Pick any $b \in J$. Since $b \in I$ as

well, (2) gives us that

$$I = K * \{b\} \subset K * J \subset J.$$

Therefore we get that $I = J$ implying the minimality of I in K. $\square$

Proposition 5.3.3. *For any (semitopological) semihypergroup K the following assertions hold.*

1. *If I_1, I_2 are minimal left ideals in K, then either $I_1 = I_2$ or $I_1 \cap I_2 = \emptyset$.*
2. *Let I be a minimal left ideal in K. Then any minimal left ideal in K will be of the form $I * \{x\}$ for some $x \in K$.*

   Moreover, we have that $K * J = J$ for any minimal left ideal J in K.
3. *K can have at most one minimal ideal, namely the intersection of all ideals in K.*

Proof. (1): Let $J := I_1 \cap I_2 \neq \emptyset$. We know that J is a left ideal since for any $x \in K$, $a \in J$ we have that $supp(p_x * p_a)$ lies in both I_1 and I_2 and hence in J.

Thus by the minimality of both I_1 and I_2 we have that $I_1 = J = I_2$.

(2): Pick any $x_0 \in K$ and set

$$I_0 := I * \{x_0\}.$$

Since I is a left ideal, I_0 has to be a left ideal as well, as

$$K * I_0 = K * (I * \{x_0\}) = (K * I) * \{x_0\} \subset I * \{x_0\} = I_0.$$

Now from Proposition 5.3.2 we see that

$$\begin{aligned} K * I &= \cup_{x \in K, a \in I} \;\; supp(p_x * p_a) \\ &= \cup_{a \in I} \cup_{x \in K} supp(p_x * p_a) \\ &= \cup_{a \in I} K * \{a\} \\ &= \cup_{a \in I} I = I \end{aligned}$$

Therefore we have:

$$\begin{aligned} K * I_0 &= K * (I * \{x_0\}) \\ &= (K * I) * \{x_0\} \\ &= I * \{x_0\} = I_0. \end{aligned}$$

Now let J be any minimal left ideal of K. Pick $y \in J$. Then

$$I * \{y\} \subset I * J \subset J$$

since J is a left ideal. The minimality of J then forces that $J = I * \{y\}$.

(3): Let I denote the intersection of all ideals in K. If $I \neq \emptyset$, then we know that I is a minimal ideal.

Let J be any minimal ideal in K. Then

$$J * I \subset I \cap J.$$

Hence $I \cap J$ is non-empty and the minimality of I gives us that $I = J$. $\square$

Remark 5.3.4. Unlike semigroups, the family of sets $\{I * \{x\} : x \in K\}$ does not serve as an exhaustive family of minimal left ideals for a semihypergroup, where I is a minimal left ideal in K. For any $x \in K$ the set $I * \{x\}$ is of course a left ideal of K, but it need not be minimal.

For example, simply consider the finite semihypergroup $(K, *)$ where $K = \{a, b, c\}$ and the operaion $*$ is given in the following table.

$*$	p_a	p_b	p_c
p_a	p_a	$\frac{1}{2}(p_a + p_b)$	$\frac{1}{2}(p_a + p_c)$
p_b	p_a	$\frac{1}{2}(p_a + p_b)$	$\frac{1}{2}(p_a + p_c)$
p_c	p_a	p_b	p_c

Set $I = \{a\}$. Then I is a minimal left ideal as the left multiplication by $*$ leaves p_a fixed. But $I * \{b\} = \{a, b\}$ and $I * \{c\} = \{a, c\}$, both of which are left ideals, but fail to be minimal.

Theorem 5.3.5. *Let K be a compact semitopological semihypergroup. Then each left ideal of K contains at least one minimal left ideal and each right ideal contains at least one minimal right ideal of K.*

Moreover, each minimal left and right ideal of K is closed.

Proof. We will prove the statement only for minimal left ideals, and the case for minimal right ideals will follow symmetrically.

Let I be a left ideal of K. Consider the following collection of left ideals in K

$$\mathcal{Q} := \{J \subset K : J \text{ is a left ideal in } K \text{ and } J \subset I\}.$$

If $a \in I$, then

$$K * \{a\} \subset K * I \subset I$$

is a left ideal in K, by Proposition 5.2.4. Hence $\mathcal{Q}$ is non-empty and non-trivial.

Equip $\mathcal{Q}$ with the partial order of reverse inclusion. Let $\mathcal{C}$ be a linearly ordered sub-collection of $\mathcal{Q}$. Since K is compact, the ideal $\cap_{J \in \mathcal{C}} J$ is non-empty. Hence we can use Zorn's Lemma to deduce that there exists a minimal element J_0 in $\mathcal{Q}$, which serves as a minimal left ideal of K contained in I.

Now let $a \in J_0$. Then $K * \{a\}$ is a left ideal in K contained in J_0. Hence by minimality of J_0 we get that

$$J_0 = K * \{a\}$$

which is closed as K is compact [18]. □

5.4 Kernel of a Semihypergroup

Here in the last section of this chapter, we define and explore the characteristics of the kernel of a (semitopological) semihypergroup. We restrict ourselves to the case where the underlying space is compact and investigate the interplay between the structures of a kernel and the exhaustive set of minimal left (resp. right) ideals. We conclude with a result outlining the relation between amenability and the existence of a unique minimal left ideal for a compact semihypergroup.

Let us first define the kernel of a semihypergroup, in the same light as the classical case of semigroups.

Definition 5.4.1. *For any (semitopological) semihypergroup K, the kernel of K denoted as $Ker(K)$ is defined to be the intersection of all (two-sided) ideals in K, i.e, we define*

$$Ker(K) := \bigcap_{\substack{I \subset K \\ \text{is a closed ideal}}} I.$$

Consider the set of all minimal left and right ideals and denote them as the following:

$$\begin{aligned} L(K) &:= \text{Set of all minimal left ideals in } K \\ R(K) &:= \text{Set of all minimal right ideals in } K \end{aligned}$$

Finally, the following theorems give us an explicit idea on the structure of the kernel of a compact semihypergroup.

Theorem 5.4.2. *Let K be a compact semitopological semihypergroup and I be a minimal left ideal in K. Then we have that*

$$\bigcup_{M \in L(K)} M \subset Ker(K) \subset \bigcup_{x \in K} (I * \{x\})$$

Proof. Let $I \in L(K)$ and consider the following union

$$I_0 := \cup_{x \in K} I * \{x\}.$$

We know from Proposition 5.3.3 that I_0 is a union of left ideals and hence is a left ideal of K itself. Now pick any $a \in I_0$. There exists some $x_0 \in K$ such that $a \in I * \{x_0\}$. Thus for any $y \in K$ we have

$$\begin{aligned} supp(p_a * p_y) &= \{a\} * \{y\} \\ &\subset (I * \{x_0\}) * \{y\} \\ &= I * (\{x_0\} * \{y\}) \\ &= \cup_{x \in \{x_0\} * \{y\}} (I * \{x\}) \\ &\subset \cup_{x \in K} (I * \{x\}) = I_0. \end{aligned}$$

Hence I_0 is a right ideal of K as well and therefore $Ker(K) \subset I_0$.

Now let J be any ideal in K and $M \in L(K)$. Then

$$K * (J * M) = (K * J) * M \subset J * M \subset M.$$

Thus $J * M$ is a left ideal of K contained in M and hence by minimality of M, we have that $M = J * M$. But J is a two-sided ideal and hence

$$M = J * M \subset J.$$

This is true for any $M \in L(K)$ and any ideal J in K. Hence finally we see that $\cup_{I \in L(K)} I \subset Ker(K)$ as required. □

The result holds true similarly for minimal right ideals as well, *i.e,* for any minimal right ideal $J \in R(K)$ we have that

$$\bigcup_{N \in R(K)} N \subset Ker(K) \subset \bigcup_{x \in K} (\{x\} * J)$$

Note that the above result implies that the kernels of semihypergroups can essentially be larger than the kernels of (topological) semigroups in general.

Corollary 5.4.3. *Let K be a compact semitopological semihypergroup. Then $Ker(K)$ is non-empty.*

Proof. We know by Proposition 5.2.4 that $K * \{a\}$ is a left ideal for any $a \in K$.

Since K is compact, it follows from Theorem 5.3.5 that it contains at least one minimal left ideal. Hence the result follows immediately from the above theorem. □

Note that the above Corollary only implies that K does not contain any two disjoint ideals. But it may very well be the case that $Ker(K) = K$ as demonstrated in Examples 5.3.4 and 2.2.4.

We now conclude this section by examining the relation between minimal left ideals and amenability of a compact semihypergroup.

Theorem 5.4.4. *Let K be a compact semihypergroup. If K is right amenable, i.e, if $C(K)$ admits a right invariant mean, then K has a unique minimal left ideal.*

Proof. Let m be a right invariant mean on $C(K)$. Since K is compact, it follows from Theorem 5.3.5 that there exists at least one minimal left ideal of K.

If possible, let I_1 and I_2 be two distinct minimal left ideals of K. Then by Proposition 5.3.3 and Theorem 5.3.5 we have that $I_1 \cap I_2 = \varnothing$ and both I_1 and I_2 are closed in K. Since K is compact, it is normal. Hence we can use Urysohn's Lemma to get a continuous function $f \in C(K)$ such that $f \equiv 0$ on I_1 and $f \equiv 1$ on I_2.

Now pick $a \in I_1$. For any $x \in K$ we have that

$$R_a f(x) \quad = \quad f(x * a)$$

$$
\begin{aligned}
&= \int_K f \ d(p_x * p_a) \\
&= \int_{supp(p_x * p_a)} f \ d(p_x * p_a) = 0.
\end{aligned}
$$

where the last equality follows since $supp(p_x * p_a) \subset I_1$ and hence

$$f \equiv 0 \text{ on } supp(p_x * p_a).$$

Similarly for any $b \in I_2$, $x \in K$ we have that

$$
\begin{aligned}
R_b f(x) &= f(x * b) \\
&= \int_K f \ d(p_x * p_b) \\
&= \int_{supp(p_x * p_b)} f \ d(p_x * p_b) \\
&= \int_{supp(p_x * p_b)} 1 \ d(p_x * p_b) = (p_x * p_b)(K) = 1.
\end{aligned}
$$

where as before, the fourth equality follows since $supp(p_x * p_b) \subset I_2$ and hence

$$f \equiv 1 \text{ on } supp(p_x * p_b).$$

Thus for any $a \in I_1$, $b \in I_2$ we have that $R_a f \equiv 0$ and $R_b f \equiv 1$. This leads to a contradiction as we have that

$$1 = m(1) = m(R_b f) = m(f) = m(R_a f) = m(0).$$

Hence the minimal left ideal of K must be unique. $\square$

Chapter 6

Free Structure on Semihypergroups

In this chapter, we initiate the study of a free structure on semihypergroups. We introduce a free product structure and a specific topology and convolution to a family of semihypergroups such that the resulting semihypergroup abides by an universal property equivalent to the universal property for free product of topological groups [24]. The contents of this chapter is based on [2, Section 6].

6.1 Introduction

Free groups and free products of topological groups or semigroups have been an useful tool for a number of reasons. Providing specific examples or counter-examples, constructing an unified group with specific properties of a number of groups and studying the problem of (topologically) embedding a topological semigroup into a topological group are a few of the areas where the use of a free structure proves to be very helpful.

Before we proceed to the definition and construction of a free product, let us first briefly recall some basic formal structures on an arbitrary set and examine how a homomorphism between two semihypergroups (Definition 5.1.5) translates to a specific homomorphism between their measure algebras, in the classical sense.

Definition 6.1.1. *Let A be any nonempty set. Then a* ***word*** *or string on A is a finite sequence $a_1a_2\dots a_n$ which is either void or $a_i \in A$ for each $1 \leq i \leq n$.*

Similarly, given a family $\{A_\alpha\}$ of nonempty sets, a word or string on $\{A_\alpha\}$ is a finite sequence $a_1a_2\dots a_n$ which is either void or $a_i \in A_\alpha$ for some α, for each $1 \leq i \leq n$.

For convenience of notation and in order to avoid confusion for some specific cases, we write the word $a_1a_2\dots a_n$ simply as $a_1a_2\dots a_n$ or $(a_1a_2\dots a_n)$.

Definition 6.1.2. *Let $\{K_\alpha\}$ be a family of semihypergroups and $w = a_1a_2\dots a_n$*

is a word on $\{K_\alpha\}$ where $a_i \in K_{\alpha_i}$ for $1 \leq i \leq n$. Then w is called a ***reduced word*** *if either w is empty or if w satisfies the following two conditions:*

1. *$a_i \neq 1_{K_\alpha}$ for $1 \leq i \leq n$ where 1_{K_α} is the identity element of K_α, if exists.*

2. *$\alpha_i \neq \alpha_{i+1}$ for $1 \leq i \leq n-1$.*

Definition 6.1.3. *Given a reduced word $a_1 a_2 \dots a_n$ on a family of semihypergroups $\{K_\alpha\}$, the* ***length*** *of the word is defined to be the number of elements in the word, and is written as $l(a_1 a_2 \dots a_n) = n$.*

Theorem 6.1.4. *Given a finite set A of cardinality n, we can form the free semihypergroup $(F_A, *)$ generated by A, where F_A is the set of all words (finite sequences) consisting of 0 or more elements from A.*

Proof. Note that given any finite set of cardinality n, we can easily form a free semigroup $(F, \cdot)$ generated by A where where F is the set of all words (finite sequences) consisting of 0 or more elements from A and the multiplication of two words $a_1 a_2 \dots a_n$, $b_1 b_2 \dots b_m$ are simple concatenation given by

$$(a_1 a_2 \dots a_n).(b_1 b_2 \dots b_m) = a_1 a_2 \dots a_n b_1 b_2 \dots b_m$$

Note that equipping F with the discrete topology makes it into a topological semigroup. Then as outlined in Example 2.2.1 we can consider $(F, \cdot)$ as a semihypergroup $(F_A, *)$ where as sets, we have $F = F_A$. $\square$

Theorem 6.1.5. *Let K and H be two semihypergroups. Then a homomorphism $\phi : K \to H$ induces a positive continuous homomorphism*

$$\Gamma_\phi : M(K) \to M(H).$$

Moreover, Γ_ϕ is unique for any such homomorphism ϕ.

Proof. First recall that $C(K)$ and $C(H)$ denote the spaces of bounded continuous functions on K and H respectively. Define $T_\phi : C(H) \to C(K)$ by

$$T_\phi f(x) := f(\phi(x))$$

for each $f \in C(H), x \in K$.

Then the adjoint operator $T_\phi^* : C(K)^* \to C(H)^*$ is simply the push-forward operator induced by ϕ as

$$T_\phi^*(m)(f) := m(T_\phi f)$$

for each $m \in C(K)^*, f \in C(H)$.

Note that on point-mass measures p_x for each $x \in K$, we have that

$$T_\phi^*(p_x)(f) = p_x(T_\phi f) = f(\phi(x)) = p_{\phi(x)}(f)$$

for any $f \in C(H)$. Hence in particular, we have that $T_\phi^*(M(X)) \subset M(Y)$.

Thus we have Γ_ϕ in the following way. For each $x \in K$ define

$$\Gamma_\phi(p_x) := p_{\phi(x)}$$

and then extend Γ_ϕ linearly to $M_F^+(K)$.

Since $M_F^+(K)$ is dense in $M^+(K)$ we can extend Γ_ϕ to $M^+(K)$ and then finally extend Γ_ϕ linearly to $M(K)$.

Thus by construction, we get that Γ_ϕ is a positive continuous linear map on $M(K)$. Now to see if it is a homomorphism, it suffices to show that

$$\Gamma_\phi(p_x * p_y) = \Gamma_\phi(p_x) * \Gamma_\phi(p_y)$$

for any $x, y \in K$.

Pick any $x, y \in K$. Then $p_x * p_y \in M^+(K)$. Hence there exists a net $\{m_\alpha\}$ in $M_F^+(K)$ such that

$$m_\alpha \to (p_x * p_y).$$

For each α let

$$m_\alpha = \sum_{i=1}^{k_\alpha} c_i^\alpha p_{x_i^\alpha}$$

for some $k_\alpha \in \mathbb{N}, c_i^\alpha \in \mathbb{F}, x_i^\alpha \in K$. Thus we have

$$\begin{aligned}
\Gamma_\phi(p_x * p_y) &= \Gamma_\phi(\lim_\alpha m_\alpha) \\
&= \lim_\alpha \Gamma_\phi\big(\sum_{i=1}^{k_\alpha} c_i^\alpha p_{x_i^\alpha}\big) \\
&= \lim_\alpha \sum_{i=1}^{k_\alpha} c_i^\alpha \Gamma_\phi(p_{x_i^\alpha}) \\
&= \lim_\alpha \sum_{i=1}^{k_\alpha} c_i^\alpha p_{\phi(x_i^\alpha)} \\
&= \lim_\alpha \tilde{m}_\alpha
\end{aligned}$$

where for each α we set

$$\tilde{m}_\alpha := \sum_{i=1}^{k_\alpha} c_i^\alpha p_{\phi(x_i^\alpha)}.$$

Now pick any $f \in C(H)$. We have

$$\begin{aligned}
(p_{\phi(x)} * p_{\phi(y)})(f) &= \int_H f \; d(p_{\phi(x)} * p_{\phi(y)}) \\
&= f(\phi(x) * \phi(y)) \\
&= (f \circ \phi)(x * y) \\
&= \int_K (f \circ \phi) \; d(p_x * p_y) \\
&= \int_K (f \circ \phi) \; d(\lim_\alpha m_\alpha) \\
&= \lim_\alpha \int_K (f \circ \phi) \; d(m_\alpha)
\end{aligned}$$

where the third equality holds since ϕ is a homomorphism and the last equality follows since $f \circ \phi \in C(K)$.

Thus we have that

$$
\begin{aligned}
(p_{\phi(x)} * p_{\phi(y)})(f) &= \lim_{\alpha} \sum_{i=1}^{k_\alpha} c_i^\alpha \int_K (f \circ \phi)\ d(p_{x_i^\alpha}) \\
&= \lim_{\alpha} \sum_{i=1}^{k_\alpha} c_i^\alpha f(\phi(x_i^\alpha)) \\
&= \lim_{\alpha} \sum_{i=1}^{k_\alpha} c_i^\alpha p_{\phi(x_i^\alpha)}(f) = \lim_{\alpha} \tilde{m}_\alpha(f).
\end{aligned}
$$

Since $f \in C(H)$ was chosen arbitrarily, it follows that

$$
\lim_{\alpha} \tilde{m}_\alpha = p_{\phi(x)} * p_{\phi(y)}
$$

Hence for any $x, y \in K$ we have that

$$
\begin{aligned}
\Gamma_\phi(p_x * p_y) &= p_{\phi(x)} * p_{\phi(y)} \\
&= \Gamma_\phi(p_x) * \Gamma_\phi(p_y)
\end{aligned}
$$

as required. Also the fact that Γ_ϕ is unique, follows immediately from its construction. $\square$

6.2 Free Product

Now we define the free product of a family of (semitopological) semihypergroups, following the standard norms of defining a free product on a category of objects. Note that the definition follows the definition of free product for topological groups [24, Definition 2.1]. The first two structural conditions essentially remain the same as in the case of topological groups, whereas the universal property is naturally defined in terms of measure spaces, generalizing the product to semihypergroups.

Definition 6.2.1. *Let $\{K_\alpha\}$ be a family of semihypergroups. Then a semihypergroup F is called a free (topological) product of $\{K_\alpha\}$, if the following conditions are satisfied:*

1. *For each α, K_α is a sub-semihypergroup of F.*
2. *F is algebraically generated by $\cup_\alpha K_\alpha$ (in terms of finite reduced words).*
3. *Given any semihypergroup H, if we have a continuous homomorphism $\phi_\alpha : K_\alpha \to H$ for each α, then there exists a unique positive continuous homomorphism $\Gamma : M(F) \to M(H)$ such that $\Gamma|_{M(K_\alpha)} = \Gamma_{\phi_\alpha}$ for each α.*

We denote F as $\prod_\alpha^* K_\alpha$.

Now before we prove that such products exists for mostly all the interesting examples we discussed in the third section, let us first define a specific type of semihypergroups.

Definition 6.2.2. *Let $(K, *)$ be a (semitopological) semihypergroup. We say that K is of* ***Type I*** *if for any $x, y \in K \setminus \{e\}$ we have that $supp(p_x * p_y)$ is not singleton whenever $e \in supp(p_x * p_y)$. Here e is the identity of K.*

Equivalently, we can say that K is of ***Type I*** *if we have that $p_x * p_y \neq p_e$ for any $x, y \in K \setminus \{e\}$.*

Note that if a semihypergroup does not have an identity, then the above conditions are vacuously true and so K is a semihypergroup of Type I. Also, note that all the examples discussed in Example 2.2.2, 2.2.3, 2.2.4, 2.2.5 and 2.2.6 are of Type I.

Theorem 6.2.3. *Let $\{K_\alpha\}$ be a family of semihypergroups such that if more than one of the K_α's has an identity element, then all the K_α's are of Type I. Then there exists a semihypergroup F such that $F = \prod^*_\alpha K_\alpha$.*

Moreover, if more than one of the K_α has an identity element, then F is also a semihypergroup with identity.

Proof. Set $\Lambda := \{\alpha : K_\alpha \text{ has an identity element}\}$. If $|\Lambda| \leq 1$, set

$$K := \{a_1 a_2 \ldots, a_n : n \in \mathbb{N}, a_1 a_2 \ldots a_n \text{ is a reduced word on } \{K_\alpha\}\}.$$

Note that if $|\Lambda| = 1$, then we treat the identity as any other arbitrary element in that semihypergroup. For convenience of notations, we rename the idenity element as x_e in this case, so that it can be included in finite sequences of reduced words.

If $|\Lambda| > 1$, let e denote the empty word on $\{K_\alpha\}$ and set $e = 1_{K_\alpha}$ for each $\alpha \in \Lambda$ where 1_{K_α} denote the identity element of K_α. Now set

$$K := \{e\} \,\cup\, \{a_1 a_2 \ldots a_n : n \in \mathbb{N},\ a_1 a_2 \ldots a_n \text{ is a reduced word on } \{K_\alpha\}\}.$$

For any $x_1 x_2 \ldots x_n,\ y_1 y_2 \ldots y_m \in K$ where $x_i \in K_{\alpha_i}, y_j \in K_{\beta_j}$ for $i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m$ and any subset $A \subset K_{\alpha_0}$ where $\alpha_0 \neq \alpha_n, \beta_1$ and $e \notin A$ we define that:

$$(x_1 x_2 \ldots x_n) A (y_1 y_2 \ldots y_m) := \{x_1 x_2 \ldots x_n a y_1 y_2 \ldots y_m : a \in K_{\alpha_0}\}.$$

On the other hand if $e \in A$ we define that

$$\begin{aligned}(x_1 x_2 \ldots x_n) A (y_1 y_2 \ldots y_m) \quad &:= \quad \{(x_1 x_2 \ldots x_n y_1 y_2 \ldots y_m)\} \cap K \\ &\qquad \cup \{(x_1 x_2 \ldots x_n a y_1 y_2 \ldots y_m) : a \in K_{\alpha_0}, a \neq e\}.\end{aligned}$$

Now for any two sets $A_1 \subset K_{\alpha_1}$, $A_2 \subset K_{\alpha_2}$ where $\alpha_1 \neq \alpha_2$ define

$$\begin{aligned}
A_1 A_2 \quad &:= \quad \{a_1 a_2 : a_1 \in A_1, a_2 \in A_2\},\ \text{if } e \notin A_1, A_2. \\
A_1 A_2 \quad &:= \quad \{a_1 a_2 : a_1 \in A_1 \setminus \{e\}, a_2 \in A_2\} \cup \{a_2 : a_2 \in A_2\},\ \text{if } e \in A_1, e \notin A_2. \\
A_1 A_2 \quad &:= \quad \{a_1 a_2 : a_1 \in A_1, a_2 \in A_2 \setminus \{e\}\} \cup \{a_1 : a_1 \in A_1\},\ \text{if } e \notin A_1, e \in A_2. \\
A_1 A_2 \quad &:= \quad \{a_1 a_2 : a_1 \in A_1 \setminus \{e\}, a_2 \in A_2 \setminus \{e\}\} \cup \{a_1 : a_1 \in A_1 \setminus \{e\}\} \\
&\qquad \cup \{a_2 : a_2 \in A_2 \setminus \{e\}\} \cup \{e\},\ \text{if } e \in A_1, A_2.
\end{aligned}$$

Similarly for $A_1 \subset K_{\alpha_1}$, $A_2 \subset K_{\alpha_2}$, $A_3 \subset K_{\alpha_3}$ where $\alpha_1 \neq \alpha_2 \neq \alpha_3$, we define

$$
\begin{aligned}
A_1A_2A_3 &:= \{a_1a_2a_3 : a_1 \in A_1, a_2 \in A_2, a_3 \in A_3\}, \text{ if } e \notin A_1, A_2, A_3. \\
A_1A_2A_3 &:= \{a_1a_2a_3 : a_1 \in A_1 \setminus \{e\}, a_2 \in A_2, a_3 \in A_3\} \\
&\quad \cup \{a_2a_3 : a_2 \in A_2, a_3 \in A_3\}, \text{ if } e \in A_1, \ e \notin A_2, A_3. \\
A_1A_2A_3 &:= \{a_1a_2a_3 : a_1 \in A_1, a_2 \in A_2 \setminus \{e\}, a_3 \in A_3\} \\
&\quad \cup \{a_1, a_3 : a_1 \in A_1, a_3 \in A_3\}, \text{ if } e \notin A_1, A_3; e \in A_2. \\
A_1A_2A_3 &:= \{a_1a_2a_3 : a_1 \in A_1, a_2 \in A_2, a_3 \in A_3 \setminus \{e\}\} \\
&\quad \cup \{a_1a_2 : a_1 \in A_1, a_2 \in A_2\}, \text{ if } e \notin A_1, A_2, \ e \in A_3. \\
A_1A_2A_3 &:= \{a_1a_2a_3 : a_1 \in A_1 \setminus \{e\}, a_2 \in A_2 \setminus \{e\}, a_3 \in A_3\} \cup \{a_2a_3 : a_2 \in A_2 \setminus \{e\}, \\
&\quad a_3 \in A_3\} \cup \{a_1a_3 : a_1 \in A_1 \setminus \{e\}, a_3 \in A_3\} \cup \{a_3 : a_3 \in A_3\}, \\
&\quad \text{ if } e \in A_1, A_2, \ e \notin A_3. \\
A_1A_2A_3 &:= \{a_1a_2a_3 : a_1 \in A_1 \setminus \{e\}, a_2 \in A_2, a_3 \in A_3 \setminus \{e\}\} \cup \{a_2a_3 : a_2 \in A_2, \\
&\quad a_3 \in A_3 \setminus \{e\}\} \cup \{a_1a_2 : a_1 \in A_1 \setminus \{e\}, a_2 \in A_2\} \cup \{a_2 : a_2 \in A_2\}, \\
&\quad \text{ if } e \in A_1, A_3, e \notin A_2. \\
A_1A_2A_3 &:= \{a_1a_2a_3 : a_1 \in A_1, a_2 \in A_2 \setminus \{e\}, a_3 \in A_3 \setminus \{e\}\} \cup \{a_1a_2 : a_1 \in A_1, \\
&\quad a_2 \in A_2 \setminus \{e\}\} \cup \{a_1a_3 : a_1 \in A_1, a_3 \in A_3 \setminus \{e\}\} \cup \{a_1 : a_1 \in A_1\}, \\
&\quad \text{ if } e \in A_2, A_3; e \notin A_1. \\
A_1A_2A_3 &:= \{a_1a_2a_3 : a_1 \in A_1 \setminus \{e\}, a_2 \in A_2 \setminus \{e\}, a_3 \in A_3 \setminus \{e\}\} \\
&\quad \cup \{a_1a_2 : a_1 \in A_1 \setminus \{e\}, a_2 \in A_2 \setminus \{e\}\} \cup \{a_2, a_3 : a_2 \in A_2 \setminus \{e\}, \\
&\quad a_3 \in A_3 \setminus \{e\}\} \cup \{a_1, a_3 : a_1 \in A_1 \setminus \{e\}, a_3 \in A_3 \setminus \{e\}\}
\end{aligned}
$$

$$\cup\{a_1 : a_1 \in A_1 \setminus \{e\}\} \cup \{a_2 : a_2 \in A_2 \setminus \{e\}\}$$
$$\cup\{a_3 : a_3 \in A_3 \setminus \{e\}\} \cup \{e\}, \text{ if } e \in A_1, A_2, A_3.$$

Similarly for any $n \in \mathbb{N}$ and subsets $A_i \subset K_{\alpha_i}$ for $i = 1, 2, \ldots, n,$ where $\alpha_l \neq \alpha_{l+1}$ for $l = 1, 2, \ldots, (n-1)$, let $e \in A_{i_k}$ for $k = 1, 2, \ldots, m$ and $e \notin A_l$ whenever $l \neq i_k$ for any $k = 1, 2, \ldots, m$. Then continuing in the same manner as above, we get that

$$A_1 A_2 A_3 \ldots A_n := \{a_1 a_2 \ldots a_n \in K : a_i \in A_i, i = 1, 2, \ldots, n\} \cup (F_n \cap K),$$

where F_n is defined as the following:

$$F_n := \bigcup_{l=1}^{m} \bigcup_{\substack{k_1, k_2, \ldots k_l = 1, \\ i_{k_s} < i_{k_{s+1}}, \\ s = 1, 2, \ldots, (l-1)}}^{m} \{(a_1 a_2 \ldots a_{i_{k_1}-1} a_{i_{k_1}+1} \ldots a_{i_{k_2}-1} a_{i_{k_2}+1} \ldots a_{i_{k_l}-1} a_{i_{k_l}+1} \ldots a_n)$$
$$: a_j \in A_j, j = 1, 2, \ldots, n\}.$$

Note that from now on unless otherwise mentioned, whenever we write a product $A_1 A_2 \ldots A_n$ of subsets $A_i \subset K_{\alpha_i}$, we assume that $\alpha_i \neq \alpha_{i+1}$ for $i = 1, 2, \ldots, (n-1)$. Also it follows immediately from the above construction that for any such sets $A_1, A_2, \ldots, A_n$ we have

$$A_1 A_2 \ldots A_n = A_1(A_2 A_3 \ldots A_n) = A_1(A_2 A_3 \ldots A_{(n-1)}) A_n = (A_1 A_2 \ldots A_{(n-1)}) A_n.$$

Now we first introduce a certain topology τ on K. Afterwards we will define a binary operation '$*$' on the measure space $M(K)$ and then investigate how the operation gives rise to an associative algebra on $M(K)$ and how the Borel sets act under the said operation and the cone topology on $M^+(K)$ induced by τ.

First, we set

$$\mathcal{B} := \{U_1U_2, \ldots, U_n : U_i \text{ is an open subset of } K_{\alpha_i}, i = 1, 2, \ldots n; n \in \mathbb{N}\}.$$

We will show that $\mathcal{B}$ serves as a base for a topology on K. Pick any element $x_1x_2 \ldots x_n \in K \setminus \{e\}$ where $x_i \in K_{\alpha_i}$, $i = 1, 2, \ldots, n$. Since $x_1x_2 \ldots x_n$ is a reduced word, $x_i \neq 1_{K_{\alpha_i}}$, even if $\alpha_i \in \Lambda$. Thus we can find an open neighbourhood V_{x_i} of x_i in K_{α_i} such that $e \notin V_{x_i}$. Then

$$x_1x_2 \ldots x_n \in V_{x_1}V_{x_2} \ldots V_{x_n} \in \mathcal{B}.$$

Also note that if $e \in K$, then for any open set U in K_α where $\alpha \in \Lambda$, we have that $e \in U \in \mathcal{B}$. Now let $U_1U_2 \ldots U_n, V_1V_2 \ldots V_m \in \mathcal{B}$ where $U_i \subset K_{\alpha_i}$ and $V_j \subset K_{\beta_j}$ for $i = 1, 2, \ldots n$; $j = 1, 2, \ldots, m$. Pick any element $x_1x_2 \ldots x_l \in U_1U_2 \ldots U_n \cap V_1V_2 \ldots V_m$. Note that $l \leq \min(n, m)$ and hence for each $k = 1, 2, \ldots l$, there exists some $i_k \in \{1, 2, \ldots, n\}$ and $j_k \in \{1, 2, \ldots, m\}$

such that $i_1 < i_2 < \ldots < i_l,\ \ j_1 < j_2 < \ldots < j_l$ and

$$x_k \in U_{i_k} \cap V_{j_k}$$

for each $k = 1, 2, \ldots, l$.

Set $W_k := U_{i_k} \cap V_{j_k}$ for each $k = 1, 2, \ldots, l$. Since each W_k is open in $K_{\alpha_{i_k}} = K_{\beta_{j_k}}$ and $x_k \neq e$ for each k, we have that $x_1 x_2 \ldots x_l \in W_1 W_2 \ldots W_l \in \mathcal{B}$. Now to show that $W_1 W_2 \ldots W_l \subset U_1 U_2 \ldots U_n \cap V_1 V_2 \ldots V_m$, pick any

$$x_1 x_2 \ldots x_t \in W_1 W_2 \ldots W_l$$

where $x_k \in W_{s_k} = U_{i_{s_k}} \cap V_{j_{s_k}}$, $k = 1, 2, \ldots t$.

Since $x_1 x_2 \ldots x_n \in U_1 U_2 \ldots U_n$ we must have that $e \in U_i$ whenever $i \in \{1, 2, \ldots n\} \setminus \{i_k : k = 1, 2, \ldots, l\}$. Again since $x_1 x_2 \ldots x_t \in W_1 W_2 \ldots W_l$, we must have that

$$e \in W_s = U_{i_s} \cap V_{j_s} \subset U_{i_s}$$

whenever $s \in \{1, 2, \ldots, l\} \setminus \{s_k : k = 1, 2, \ldots, t\}$. Combining these two statements gives us that $e \in U_i$ whenever $i \in \{1, 2, \ldots, n\} \setminus \{i_{s_k} : k = 1, 2, \ldots, t\}$. Also, we know that $x_k \in U_{i_{s_k}}$ for each $k = 1, 2, \ldots, t$. Hence it follows that

$$x_1 x_2 \ldots x_t \in U_1 U_2 \ldots U_n.$$

Similarly since $x_1 x_2 \ldots x_l \in V_1 V_2 \ldots V_m$, we must have that $e \in V_j$ whenever

$j \in \{1, 2, \ldots m\} \setminus \{j_k : k = 1, 2, \ldots, l\}$ and since $x_1x_2 \ldots x_t \in W_1W_2 \ldots W_l$, we must have that

$$e \in W_s = U_{i_s} \cap V_{j_s} \subset V_{j_s}$$

whenever $s \in \{1, 2, \ldots, l\} \setminus \{s_k : k = 1, 2, \ldots, t\}$, we can proceed in the same manner to get that $x_1x_2 \ldots x_t \in V_1V_2 \ldots V_m$.

Thus finally we see that

$$x_1x_2 \ldots x_l \in W_1W_2 \ldots W_l \subset U_1U_2 \ldots U_n \cap V_1V_2 \ldots V_m$$

where $W_1W_2 \ldots W_l \in \mathcal{B}$, implying that $\mathcal{B}$ serves as a base for a topology on K. We denote the topology on K generated by $\mathcal{B}$ as τ.

Now as pointed out before in the first section, in order to define a binary operation on $M(K)$ it is sufficient to define a binary operation on the point-mass measures on $M(K)$. Firstly if $\Lambda \neq \varnothing$ then for any $x_1x_2 \ldots x_n \in K$ we define

$$p_e * p_{x_1x_2\ldots x_n} = p_{x_1x_2\ldots x_n} = p_{x_1x_2\ldots x_n} * p_e.$$

Now for any two elements $x = x_1x_2 \ldots x_n, y = y_1y_2 \ldots y_m \in K \setminus \{e\}$ where $x_i \in K_{\alpha_i}, y_j \in K_{\beta_j}$ for $i = 1, 2, \ldots, n;\;\; j = 1, 2, \ldots, m$, define:

$$p_x * p_y = \begin{cases} p_{(x_1 x_2 \ldots x_n y_1 y_2 \ldots y_m)}, & \text{if } \alpha_n \neq \beta_1 \ . \\ \\ m_{[x,y]}, & \text{if } \alpha_n = \beta_1 \ . \\ \\ \end{cases}$$

where for any measurable set $E' \subset K$ the measure $m_{[x,y]}$ is defined in the following way.

$$m_{[x,y]}(E') = (p_{x_n} * p_{y_1})(E),$$

where E is the largest measurable set in K_{α_n} such that

$$(x_1 \ldots x_{n-1})E(y_2 \ldots y_m) = E' \cap \big((x_1 \ldots x_{n-1})K_{\alpha_n}(y_2 \ldots y_m)\big).$$

In particular, we have that:

$$m_{[x,y]}(E') = \begin{cases} (p_{x_n} * p_{y_1})(E), & \text{if } E' = (x_1 \ldots x_{n-1})E(y_2 \ldots y_m) \\ & \quad \text{for some measurable set } E \subset K_{\alpha_n} = K_{\beta_1}. \\ \\ 0, & \quad \text{if } E' \not\supseteq (x_1 \ldots x_{n-1})E(y_2 \ldots y_m) \\ & \quad \text{for any measurable set } E \subset K_{\alpha_n} = K_{\beta_1}. \end{cases}$$

Note that $m_{[x,y]}$ mimics the behaviour of the probability measure $(p_{x_n} * p_{y_1})$ and hence is a probability measure with support $(x_1 \ldots x_{n-1})\ \mathrm{supp}(p_{x_n} * p_{y_1})(y_2 \ldots y_m)$.

Also due to the construction of τ, for any two closed subsets $C_1 \subset K_{\alpha_1}$ and $C_2 \subset K_{\alpha_2}$ where $\alpha_1 \neq \alpha_2$, we have that C_1C_2 is a closed subset of K. This is true since $K_{\alpha_1}K_{\alpha_2}$ is open in K and we have that

$$K_{\alpha_1}K_{\alpha_2} \setminus C_1C_2 = (K_{\alpha_1} \setminus C_1).K_{\alpha_2} \cup K_{\alpha_1}.(K_{\alpha_2} \setminus C_2),$$

which is open in $K_{\alpha_1}K_{\alpha_2}$. Similarly, we can show that $C_1C_2 \ldots C_n$ is a closed set in K for any $n \in \mathbb{N}$ where $C_i \in K_{\alpha_i}$ for $1 \leq i \leq n$ and $\alpha_i \neq \alpha_{i+1}$ for $1 \leq i \leq n-1$. In fact, it is now easy to see that the set

$$\{C_1C_2 \ldots C_n : n \in \mathbb{N}, C_i \text{ is a closed set in some } K_{\alpha_i}\}$$

is the family of basic closed sets in K.

Now let both C_1 and C_2 are compact subsets of K_{α_1} and K_{α_2} respectively and let $\{A_\gamma\}_{\gamma \in I}$ be a family of closed subsets of C_1C_2 with finite intersection property. Then we must have that $A_\gamma = B_1^\gamma B_2^\gamma$ for some closed subsets $B_i^\gamma \subset C_i \subset K_{\alpha_i}$ for each γ, $i = 1, 2$. Now for any $\{\gamma_1, \gamma_2, \ldots, \gamma_n\} \subset I$ we have that

$$\cap_1^n B_1^{\gamma_i} B_2^{\gamma_i} = \cap_1^n A_{\gamma_i} \neq \varnothing.$$

Hence in particular, $\cap_1^n B_1^{\gamma_i} \neq \varnothing$ for any such finite subset $\{\gamma_1, \gamma_2, \ldots, \gamma_n\} \subset I$.

So $\{B_1^\gamma\}_{\gamma\in I}$ is a family of subsets in C_1 with the finite intersection property. But C_1 is compact in K_{α_1} and hence there exists some $x_1 \in C_1$ such that $x_1 \in \cap_{\gamma\in I} B_1^\gamma$. Similarly, since C_2 is also compact in K_{α_2}, there exists some $x_2 \in C_2$ such that $x_2 \in \cap_{\gamma\in I} B_2^\gamma$.

Thus we see that there exists an element $x \in C_1C_2$ such that $x \in \cap_{\gamma\in I} A_\gamma$ where x is given as the following:

$$x = \begin{cases} (x_1x_2), & \text{if } x_1, x_2 \neq e \ . \\ (x_1), & \text{if } x_1 \neq e, x_2 = e \ . \\ (x_2), & \text{if } x_1 = e, x_2 \neq e \ . \\ e, & \text{if } x_1 = x_2 = e \ . \end{cases}$$

Thus we see that C_1C_2 has to be compact too. Now let C_1, C_2, C_3 be compact subsets of $K_{\alpha_1}, K_{\alpha_2}, K_{\alpha_3}$ respectively. Note that if $\alpha_i \neq \alpha_j$ for any $i, j \in \{1, 2, 3\}$ or if $e \notin \cap_{i=1}^3 C_i$ then we can proceed similarly as above to show that $C_1C_2C_3$ gives us a compact set in K. Otherwise, first consider the following case.

Let C be a compact set in K_{α_0} and $(a_1a_2\ldots a_s), (b_1b_2\ldots b_t) \in K \setminus \{e\}$ where $a_i \in K_{\gamma_i}$ and $b_j \in K_{\delta_j}$ for $1 \leq i \leq s, 1 \leq j \leq t$ such that $\alpha_0 \neq \gamma_s, \delta_1$. Consider the set

$$C_0 := (a_1\ldots a_s)C(b_1\ldots b_t).$$

We want to show that any such C_0 is compact in (K, τ). Let $\{U_\alpha\}$ be any basic open cover of C_0. Hence $U_\alpha = U_1^\alpha U_2^\alpha \dots U_{n_\alpha}^\alpha$ for some $n_\alpha \in \mathbb{N}$ where each U_k^α is an open set in some K_α such that the product exists. Now for any $x \in C \setminus \{e\}$ there exists α_x such that

$$(a_1 \dots a_s x b_1 \dots b_t) \in U_{\alpha_x}.$$

Therefore there exists $i_x \in \{1, 2, \dots, n_{\alpha_x}\}$ such that $x \in U_{i_x}^{\alpha_x}$ where $U_{i_x}^{\alpha_x}$ is an open set in K_{α_0}. Thus $\mathcal{U} := \{U_{i_x}^{\alpha_x}\}_{x \in C}$ serves an open cover of the set $C \setminus \{e\}$.

If $e \notin C$, then $\mathcal{U}$ serves as an open cover of C and hence has a finite subcover $\{U_1, U_2, \dots, U_p\}$ where $U_k = U_{i_{x_k}}^{\alpha_{x_k}}$ for some $x_k \in C \setminus \{e\}$ for $1 \le k \le p$. Thus $\{U_{\alpha_{x_1}}, U_{\alpha_{x_2}}, \dots, U_{\alpha_{x_p}}\}$ serves as a finite subcover of $\{U_\alpha\}$.

Now let $e \in C$. Pick $\alpha' \in \Lambda$ such that $\alpha' \neq \alpha_0$. Since $K_{\alpha'}$ is locally compact, we can find a compact neighborhood of e in $K_{\alpha'}$. Let $\mathcal{V} := \{V_\beta\}$ be an open cover of F. Then $C \cup F$ is a compact set in $K_{\alpha_0} \cup K_{\alpha'}$ with open cover $\mathcal{U} \cup \mathcal{V}$. Thus we get a finite subcover of the form

$$\{U_1, U_2, \dots, U_p, V_{\beta_1}, V_{\beta_2}, \dots V_{\beta_q}\}$$

where $U_k = U_{i_{x_k}}^{\alpha_{x_k}}$ for some $x_k \in C \setminus \{e\}$ for $1 \le k \le p$. Note that $e \in B_{\beta_l}$ for some $l \in \{1, 2, \dots, q\}$ here and hence $\{U_{i_{x_k}}\}_{k=1}^{p}$ covers $C \setminus \{e\}$.

Thus if $\gamma_s = \delta_1$, then $\{U_{\alpha_{x_1}}, U_{\alpha_{x_2}}, \dots, U_{\alpha_{x_p}}\}$ covers C_0, as desired. Otherwise if $\gamma_s \neq \delta_1$, then there exists some α' such that $(a_1 \dots a_s b_1 \dots b_t) \in U_{\alpha'}$. Hence in this case, $\{U_{\alpha_{x_1}}, U_{\alpha_{x_2}}, \ldots, U_{\alpha_{x_p}}, U_{\alpha'}\}$ serves as a finite subcover of C_0. Hence we see that any set of the form C_0 is compact in (K, τ). Using similar argument we can immediately see that sets of the form $C_1C_2C_3$ are also compact where $C_i \in K_{\alpha_i}$ is compact.

In particular, for any two words $x = (x_1x_2\ldots x_n), y = (y_1y_2\ldots y_m) \in K$ where $x_i \in K_{\alpha_i}$, $y_j \in K_{\beta_j}$ and $\alpha_n = \beta_1$, we have that $\operatorname{supp}(p_x * p_y) = \operatorname{supp} m_{[x,y]} = (x_1 \ldots x_{n-1}) \operatorname{supp}(p_{x_n} * p_{y_1})(y_2 \ldots y_m)$ is compact since $\operatorname{supp}(p_{x_n} * p_{y_1})$ is compact in $K_{\alpha_n} = K_{\beta_1}$. Thus it follows from the construction of convolution products that for any two words $x, y \in K$ we have that $(p_x * p_y)$ is a probability measure with compact support. Also since convolution is associative in each K_α, it follows from the construction that convolution products are associative.

Now we proceed further to show that $(K, *)$ indeed gives us a semihypergroup. First note that since each K_α is Hausdorff, for any two elements $x_1x_2\ldots x_n, y_1y_2\ldots y_m \in K$ where $x_i \in K_{\alpha_i}$, $y_j \in K_{\beta_j}$, we can find sets $A_i \subset K_{\alpha_i}$, $B_j \subset K_{\beta_j}$ for $1 \leq i \leq n$, $1 \leq j \leq m$ such that $x_i \in A_i$, $y_j \in B_j$ and $A_i \cap B_j = \varnothing$ for all i, j. Hence we have that $x_1x_2\ldots x_n \in A_1A_2\ldots A_n$, $y_1y_2\ldots y_m \in B_1B_2\ldots B_m$ where $A_1A_2\ldots A_n \cap B_1B_2\ldots B_m = \varnothing$.

Also if $e \in K$, then for any such $x_1x_2\ldots x_n$ since $x_i \neq e$ for any i, we can choose A_i's such that $e \notin A_i$ for any i. Then again using the fact that each K_α is Hausdorff, we can choose a neighborhood V around e such that $V \cap A_i = \varnothing$ for all i and hence $V \cap A_1A_2\ldots A_n = \varnothing$. Thus we can easily see that the topology τ on K is Hausdorff.

Next pick any element $x_1 x_2 \ldots x_n \in K$ where $x_i \in K_{\alpha_i}$ for $1 \leq i \leq n$. Since each K_{α_i} is locally compact, we can find a compact neighborhood C_i of x_i in K_{α_i} such that $e \notin C_i$. Let U_i be an open set in K_{α_i} such that $C_i \subset U_i$ and $e \notin U_i$. Now consider the map $\phi : U_1 \times U_2 \times \ldots \times U_n \to U_1 U_2 \ldots U_n$ given by

$$\phi((a_1, a_2, \ldots, a_n)) := (a_1 a_2 \ldots a_n)$$

for any $(a_1, a_2, \ldots, a_n) \in U_1 \times U_2 \times \ldots \times U_n$.

Note that since $e \notin U_i$ for any i, the map ϕ is bijective. Moreover, for any basic open set $V_1 V_2 \ldots V_n \subset U_1 U_1 \ldots U_n$ we have $\phi^{-1}(V_1 V_2 \ldots V_n) = V_1 \times V_2 \times \ldots \times V_n$ which is open in $U_1 \times U_2 \times \ldots \times U_n$ in the product topology. Hence ϕ is continuous granting us that $C_1 C_2 \ldots C_n = \phi(C_1 \times C_2 \times \ldots \times C_n)$ is a compact neighborhood of $x_1 x_2 \ldots x_n$ in (K, τ). Also if $e \in K$ then there exists some α_0 such that $e \in K_{\alpha_0}$. Since K_{α_0} is locally compact we can always find a compact neighborhood for e. Thus we see that (K, τ) is indeed a locally compact Hausdorff space.

Now in order to see if the map $(x, y) \mapsto supp(p_x * p_y) : K \times K \to \mathfrak{C}(K)$ is continuous, first recall [18] that the map $\pi : X \to M(X)$ given by $x \mapsto p_x$ is a homemorphism onto its image for any locally compact Hausdorff space X. Now let $\{(x_\alpha, y_\alpha)\}$ be a net in $K \times K$ that converges to some (x, y) where $x = x_1 x_2 \ldots x_n$, $y = y_1 y_2 \ldots y_m \in K$ such that $x_i \in K_{\gamma_i}$ and $y_j \in K_{\delta_j}$ for all

i, j. Then $\{x_\alpha\}$ and $\{y_\alpha\}$ converges to x and y respectively. Then for each α we have that $x_\alpha = x_1^\alpha x_2^\alpha \ldots x_{n_\alpha}^\alpha$ for some $n_\alpha \in \mathbb{N}$ where $x_i^\alpha \in K_{\alpha_i}$ for $1 \leq i \leq n_\alpha$. But x_α eventually lies in any basic open set of the form $U_1U_2 \ldots U_n$ around x where U_i is an open neighborhood of x_i in K_{γ_i} that does not contain e. Hence without loss of generality for each α we can assume that $n_\alpha = n$ and $\alpha_i = \gamma_i$ for $1 \leq i \leq n$. Similarly we can assume that $y_\alpha = y_1^\alpha y_2^\alpha \ldots y_m^\alpha$ where $y_j^\alpha \in K_{\delta_j}$ for $1 \leq j \leq m$.

Now consider the Michael topology on $\mathfrak{C}(K)$ and recall [25] that the map $x \mapsto \{x\} : K \to \mathfrak{C}(K)$ is a homeomorphism onto its image which is also closed in $\mathfrak{C}(K)$. If $\gamma_n \neq \delta_1$ then using the same technique as above, we can easily see that the net $\{(x_1^\alpha \ldots x_n^\alpha y_1^\alpha \ldots y_m^\alpha)\}_\alpha$ converges to $(x_1 \ldots x_n y_1 \ldots y_m)$ in K. Hence using the homeomorphism mentioned above we have that

$$supp(p_{x_\alpha} * p_{y_\alpha}) = \{(x_1^\alpha \ldots x_n^\alpha y_1^\alpha \ldots y_m^\alpha)\}$$

converges to $\{(x_1 \ldots x_n y_1 \ldots y_m)\} = supp(p_x * p_y)$ in $\mathfrak{C}(K)$.

Now let $\gamma_n = \delta_1$. For any $i_0 \in \{1, 2, \ldots n\}$, pick and fix open neighborhoods U_i around x_i in K_{γ_i} for all $i \neq i_0$. Then for any open neighborhood V around x_{i_0} in $K_{\gamma_{i_0}}$ we have that $x_1^\alpha x_2^\alpha \ldots x_n^\alpha$ lie eventually in $U_1U_2 \ldots U_{i_0-1}VU_{i_0+1} \ldots U_n$. Hence we must have that $\{x_{i_0}^\alpha\}$ lie eventually in V, *i.e,* $\{x_{i_0}^\alpha\}$ converges to x_{i_0} in $K_{\gamma_{i_0}}$. Thus $\{x_i^\alpha\}$ converges to x_i in K_{γ_i} and $\{y_j^\alpha\}$ converges to y_j in K_{δ_j} for each i, j. In particular, we have that the net $\{(x_n^\alpha, y_1^\alpha)\}_\alpha$ converges to (x_n, y_1)

in $K_{\gamma_n} \times K_{\gamma_n}$. Since the map

$$(x, y) \mapsto supp(p_x * p_y) : K_{\gamma_n} \times K_{\gamma_n} \to \mathfrak{C}(K_{\gamma_n})$$

is continuous, we have that $supp(p_{x_n^\alpha} * p_{y_1^\alpha})$ converges to $supp(p_{x_n} * p_{y_1})$ in $\mathfrak{C}(K_{\gamma_n})$.

As discussed before, we also have that the nets $\left\{\{x_i^\alpha\}\right\}_\alpha$ and $\left\{\{y_j^\alpha\}\right\}_\alpha$ converge to $\{x_i\}$ and $\{y_j\}$ in $\mathfrak{C}(K_{\gamma_i})$ and $\mathfrak{C}(K_{\delta_j})$ respectively for $1 \leq i < n$ and $1 < j \leq m$. Hence it follows from the construction of basic open sets of K that $supp(p_{x_\alpha} * p_{y_\alpha}) = \{(x_1^\alpha \ldots x_{n-1}^\alpha)\} supp(p_{x_n^\alpha} * p_{y_1^\alpha})\{(y_2^\alpha \ldots y_m^\alpha)\}$ converges to $\{(x_1 \ldots x_{n-1})\} supp(p_{x_n} * p_{y_1})\{(y_2 \ldots y_m)\} = supp(p_x * p_y)$ in $\mathfrak{C}(K)$.

Finally, it follows immediately from the construction of convolution products that the map $* : M(K) \times M(K) \to M(K)$ is positive bilinear. Using the same technique as above we can see that the restricted map $*|_{M^+(K)\times M^+(K)} : M^+(K) \times M^+(K) \to M^+(K)$ is continuous. Hence we have that the pair $(K, *)$ indeed forms a semihypergroup.

Note that for each α the map $i_\alpha : K_\alpha \to K$ given by

$$i_\alpha(x) = \begin{cases} (x) & \text{if } x \neq e. \\ e & \text{if } x = e. \end{cases}$$

where (x) is a word of length 1 in K, enables us to view K_α as a subsemihypergroup of K since $i_\alpha(K_\alpha) * i_\alpha(K_\alpha) \subset i_\alpha(K_\alpha)$ and i_α is a homeomorphism onto its image.

Now let H be a semihypergroup and $\phi_\alpha : K_\alpha \to H$ be a homomorphism for each α. For each $x = x_1 x_2 \ldots x_n \in K$ where $x_i \in K_{\alpha_i}$ for $1 \leq i \leq n$ define

$$\Gamma(p_x) := p_{\phi_{\alpha_1}(x_1)} * p_{\phi_{\alpha_2}(x_2)} * \ldots * p_{\phi_{\alpha_n}(x_n)} \ .$$

We can then extend Γ to $M(K)$ as in the proof of Theorem 6.1.5 to get a positive continuous linear map $\Gamma : M(K) \to M(H)$.

Since for any $x \in K_\alpha$ we have that $\Gamma(p_(x)) = p_{\phi_\alpha(x)}$, it follows immediately from Theorem 6.1.5 that $\Gamma|_{M(K_\alpha)} = \Gamma \circ \Gamma_{i_\alpha} = \Gamma_{\phi_\alpha}$ for each α. Now to show that Γ is a homomorphism, pick any two words $x = x_1 x_2 \ldots x_n$, $y = y_1 y_2 \ldots y_m$ in K where $x_i \in K_{\alpha_i}$, $y_j \in K_{\beta_j}$ for $1 \leq i \leq n$, $1 \leq j \leq m$. Note that if $\alpha_n \neq \beta_1$ then we have that

$$\begin{aligned}
\Gamma(p_x * p_y) &= \Gamma(p_{x_1 \ldots x_n y_1 \ldots y_m}) \\
&= p_{\phi_{\alpha_1}(x_1)} * \ldots * p_{\phi_{\alpha_n}(x_n)} * p_{\phi_{\beta_1}(y_1)} * \ldots * p_{\phi_{\beta_m}(y_m)} \\
&= \big(p_{\phi_{\alpha_1}(x_1)} * \ldots * p_{\phi_{\alpha_n}(x_n)}\big) * \big(p_{\phi_{\beta_1}(y_1)} * \ldots * p_{\phi_{\beta_m}(y_m)}\big) \\
&= \Gamma(p_{x_1 x_2 \ldots x_n}) * \Gamma(p_{y_1 y_2 \ldots y_m}) = \Gamma(p_x) * \Gamma(p_y)
\end{aligned}$$

as required.

Now pick any $x \in K_\gamma$ and $y, z \in K_\beta$ where $\gamma \neq \beta$. Then there exists a net

$\{m_\alpha\}$ in $M_F^+(K_\beta)$ such that m_α converges to $(p_y * p_z)$ in $M^+(K_\beta)$. Hence $(p_x * m_\alpha)$ converges to $(p_x * (p_y * p_z)) = (p_x * p_y * p_z)$ in $M^+(K)$. Then for each α we have that $m_\alpha = \sum_{i=1}^{n_\alpha} c_i^\alpha p_{x_i^\alpha}$ for some $n_\alpha \in \mathbb{N}$, $c_i^\alpha \in \mathbb{C}$ and $x_i^\alpha \in K_\beta$.

Since Γ is continuous on $M^+(K)$, we have that

$$
\begin{aligned}
\Gamma(p_x * p_y * p_z) &= \Gamma\big(\lim_\alpha (p_x * m_\alpha)\big) \\
&= \lim_\alpha \Gamma(p_x * m_\alpha) \\
&= \lim_\alpha \Gamma\Big(p_x * \sum_{i=1}^{n_\alpha} c_i^\alpha p_{x_i^\alpha}\Big) \\
&= \lim_\alpha \Gamma\Big(\sum_{i=1}^{n_\alpha} c_i^\alpha (p_x * p_{x_i^\alpha})\Big) \\
&= \lim_\alpha \sum_{i=1}^{n_\alpha} c_i^\alpha \; \Gamma\big(p_x * p_{x_i^\alpha}\big) \\
&= \lim_\alpha \sum_{i=1}^{n_\alpha} c_i^\alpha \Big(\Gamma(p_x) * \Gamma(p_{x_i^\alpha})\Big) \\
&= \lim_\alpha \Gamma(p_x) * \Big(\sum_{i=1}^{n_\alpha} c_i^\alpha \Gamma(p_{x_i^\alpha})\Big) \\
&= \lim_\alpha \Gamma(p_x) * \Gamma\Big(\sum_{i=1}^{n_\alpha} c_i^\alpha p_{x_i^\alpha}\Big) \\
&= \Gamma(p_x) * \lim_\alpha \Gamma(m_\alpha) \\
&= \Gamma(p_x) * \Gamma\big(\lim_\alpha m_\alpha\big) = \Gamma(p_x) * \Gamma(p_y * p_z).
\end{aligned}
$$

where the fourth equality follows since the convolution $(\mu, \nu) \mapsto \mu * \nu : M(K) \times M(K) \to M(K)$ is bilinear, the fifth and eighth equality follows from the linearity of Γ on $M(K)$ and finally the seventh equality follows since the map $(\mu, \nu) \mapsto \mu * \nu : M(H) \times M(H) \to M(H)$ is bilinear.

Similarly for any $x, y \in K_\gamma$ and $z \in K_\beta$ where $\gamma \neq \beta$ we have that

$$\Gamma(p_x * p_y * p_z) = \Gamma(p_x * p_y) * \Gamma(p_z) \ .$$

Using these two equalities simultaneously for any two words $x = x_1 x_2 \ldots x_n$, $y = y_1 y_2 \ldots y_m$ in K where $x_i \in K_{\alpha_i}$, $y_j \in K_{\beta_j}$ for $1 \leq i \leq n$, $1 \leq j \leq m$ and $\alpha_n = \beta_1$, we have that

$$\begin{aligned}
\Gamma(p_x * p_y) &= \Gamma(p_{x_1 x_2 \ldots x_{n-1}} * p_{x_n} * p_{y_1} * p_{y_2 y_3 \ldots y_m}) \\
&= \Gamma(p_{x_1 x_2 \ldots x_{n-1}}) * \Gamma(p_{x_n} * p_{y_1}) * \Gamma(p_{y_2 y_3 \ldots y_m}) \\
&= \Gamma(p_{x_1 x_2 \ldots x_{n-1}}) * \Gamma_{\phi_{\alpha_n}}(p_{x_n} * p_{y_1}) * \Gamma(p_{y_2 y_3 \ldots y_m}) \\
&= \Gamma(p_{x_1 \ldots x_{n-1}}) * \Big(\Gamma_{\phi_{\alpha_n}}(p_{x_n}) * \Gamma_{\phi_{\alpha_n}}(p_{y_1})\Big) * \Gamma(p_{y_2 \ldots y_m}) \\
&= \Gamma(p_{x_1 \ldots x_{n-1}}) * \Big(p_{\phi_{\alpha_n}(x_n)} * p_{\phi_{\alpha_n}(y_1)}\Big) * \Gamma(p_{y_2 \ldots y_m}) \\
&= \Big(p_{\phi_{\alpha_1}(x_1)} * \ldots * p_{\phi_{\alpha_{n-1}}(x_{n-1})}\Big) * \Big(p_{\phi_{\alpha_n}(x_n)} * p_{\phi_{\beta_1}(y_1)}\Big) \\
&\quad * \Big(p_{\phi_{\beta_2}(y_2)} * \ldots * p_{\phi_{\beta_m}(y_m)}\Big) \\
&= \Big(p_{\phi_{\alpha_1}(x_1)} * \ldots * p_{\phi_{\alpha_{n-1}}(x_{n-1})} * p_{\phi_{\alpha_n}(x_n)}\Big) \\
&\quad * \Big(p_{\phi_{\beta_1}(y_1)} * p_{\phi_{\beta_2}(y_2)} * \ldots * p_{\phi_{\beta_m}(y_m)}\Big) \\
&= \Gamma\Big(p_{x_1 x_2 \ldots x_n}\Big) * \Gamma\Big(p_{y_1 y_2 \ldots y_m}\Big) \\
&= \Gamma(p_x) * \Gamma(p_y)
\end{aligned}$$

as required. Note that here the fourth equality follows since

$$\Gamma_{\phi_{\alpha_n}} : M(K_{\alpha_n}) \to M(H)$$

induced by ϕ_{α_n} is a homomorphism and the fifth equality follows from the construction of $\Gamma_{\phi_{\alpha_n}}$ as shown in Theorem 6.1.5.

Finally, note that if

$$\Theta : M(K) \to M(H)$$

is another positive linear continuous homomorphism such that

$$\Theta|_{M(K_\alpha)} = \Gamma_{\phi_\alpha}$$

then for any $x_1 x_2 \ldots x_n \in K$ where $x_i \in K_{\alpha_i}$ for $1 \le i \le n$ we have that

$$\begin{aligned}
\Theta(p_{x_1 x_2 \ldots x_n}) &= \Theta(p_{x_1} * p_{x_2} * \ldots * p_{x_n}) \\
&= \Theta(p_{x_1}) * \Theta(p_{x_2}) * \ldots * \Theta(p_{x_n}) \\
&= \Theta|_{M(K_{\alpha_1})}(p_{x_1}) * \Theta|_{M(K_{\alpha_2})}(p_{x_2}) * \ldots * \Theta|_{M(K_{\alpha_n})}(p_{x_n}) \\
&= \Gamma_{\phi_{\alpha_1}}(p_{x_1}) * \Gamma_{\phi_{\alpha_2}}(p_{x_2}) * \ldots * \Gamma_{\phi_{\alpha_n}}(p_{x_n}) \\
&= \Gamma|_{M(K_{\alpha_1})}(p_{x_1}) * \Gamma|_{M(K_{\alpha_2})}(p_{x_2}) * \ldots * \Gamma|_{M(K_{\alpha_n})}(p_{x_n}) \\
&= \Gamma(p_{x_1}) * \Gamma(p_{x_2}) * \ldots * \Gamma(p_{x_n}) \\
&= \Gamma(p_{x_1} * p_{x_2} * \ldots * p_{x_n}) = \Gamma(p_{x_1 x_2 \ldots x_n}) \ .
\end{aligned}$$

Thus we see that the map Γ constructed above is unique and hence $(K, *)$ satisfies all the conditions of Definition 6.2.1 giving us that $K = \prod_\alpha^* K_\alpha$. $\square$

Remark 6.2.4. Let $\{K_\alpha\}$ be a family of semihypergroups and F is a semihypergroup such that $F = \prod_\alpha^* K_\alpha$.

If any one of the K_α has a non-discrete topology, then the locally compact topology on F is also non-discrete.

Theorem 6.2.5. *Let $\{K_\alpha\}$ be a family of semihypergroups and F_1 and F_2 be semihypergroups such that $F_i = \prod_\alpha^* K_\alpha$ for $i = 1, 2$.*

Then there exists a continuous isomorphism

$$\psi : F_1 \to F_2.$$

Thus the free product $\prod_\alpha^* K_\alpha$ we constructed above is unique.

Proof. The proof follows readily from the universal property outlined in the definition of free product for semihypergroups (see Definition 6.2.1). $\square$

Chapter 7

Open Questions and Further Work

The lack of extensive prior research on semihypergroups in general and the significant examples it contains, make way to a number of new intruiging areas of research on this subject. Here we list some of the potential problems and areas of study on semihypergroups that we are currently working on and/or intend to work on in near future.

Problem 1: Use the algebraic structure imposed on $AP(K)^*$ to acquire a general compactification of semihypergroups.

Problem 2: Investigate the set of minimal ideals on a (semitopological) semihypergroup more closely, finally to explore its relation to the space of almost periodic and weakly almost periodic functions.

Problem 3: Investigate the idempotents of a compact semihypergroup and explore their relation to the space of minimal left ideals, kernel and amenability of a semihypergroup.

Problem 4: Explore if results equivalent to isomorphism theorems hold true for a semihypergroup, in addition to exploring the structure of the kernel of a homomorphism for a (semitopological) semihypergroup with identity, as in Example 2.2.4 and Example 2.2.5.

Problem 5: Investigate the behavior of free products on semihypergroups for compact case and generalize the construction for amalgamated products and for hypergroups.

Problem 6: Investigate the space of left/right uniformly continuous functions on a semitopological semihypergroup and explore its relation to other function spaces.

Problem 7: Investigate the measure algebra of a semihypergroup and use its duality to acquire certain information about the underlying semihypergroup.

Problem 8: Study the notion of amenability on (semitopological) semihypergroups, specially for the non-commutative case.

Problem 9: Investigate the F-algebraic structure (as defined by A. T. Lau in [19]) on the measure algebra of a semihypergroup.

Problem 10: Investigate Banach algebras on discrete semihypergroups [21].

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