THE UNIVERSITY OF ALBERTA

THE IDENTIFICATION OF A CLASS OF ORTHOGONAL POLYNOMIAL SETS

by

© WILLIAM ROY ALLAWAY

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled "THE IDENTIFICATION OF A CLASS OF ORTHOGONAL POLYNOMIAL SETS", submitted by William Roy Allaway in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

Waleed A. Al-Salam
Supervisor

Z. D. Izzau
G. A. Chandler
. . . .
. . . .
. . A. Sharma . . . .

. . . .
External Examiner

Date May 1, 1972. .

ABSTRACT

This thesis studies polynomial sets $\{A_n(x)\}_{n=0}^{\infty}$ satisfying a Fourier sine series expansion of the form:

$$(1) \qquad A_n(\cos\theta)\, w(\cos\theta) \sim \sum_{k=0}^{\infty} f_{k,n} \sin(n+2k+1)\theta$$

on $[0,\pi]$, where $n = 0, 1, 2 \ldots$. Szegö showed that the ultraspherical polynomials have this form.

If the leading coefficient of $A_n(x)$ is c_n, we show that a necessary and sufficient condition for $\{A_n(x)\}_{n=0}^{\infty}$ to be a symmetric orthogonal polynomial set on $[-1,1]$ with respect to the weight function $w(x)$, is that $c_n f_{0,n} > 0$ for $n = 0,1,2 \ldots$. We find all orthogonal polynomial sets satisfying relation (1) such that there exists a sequence of real numbers $\{\alpha_n\}_{n=1}^{\infty}$, with $f_{k,n} = \alpha_k f_{k-1,n+1}$ for $k \geq 1$; $n \geq 0$. We call this class of polynomial sets Σ .

The following two ideas are used to find the three term recursion relation for each polynomial set in Σ :

First, we find that a necessary condition for a polynomial set to be in Σ , is the existence of a set of sequences $\{\{\gamma_i\}_{i=1}^{\infty}$,

$\{b_j\}_{j=1}^{\infty}$, $\{\alpha_\ell\}_{\ell=1}^{m}\}$ satisfying

(2) $$\gamma_{n+k}(b_n^{-1} - \alpha_k) + \alpha_k \gamma_n - 1 = 0 \qquad (1 \leq n;\ 1 \leq k \leq m),$$

where m is the smallest integer such that $\alpha_m = 0$. To solve this finite difference equation we show that we can extend $\{\alpha_\ell\}_{\ell=1}^{m}$ to $\{\bar{\alpha}_\ell\}_{\ell=1}^{\infty}$ (i.e. $\bar{\alpha}_\ell = \alpha_\ell$ for $\ell = 1,2 .. m$) in such a way that $\{\gamma_i\}_{i=1}^{\infty}$, $\{b_j\}_{j=1}^{\infty}$ and $\{\bar{\alpha}_\ell\}_{\ell=1}^{\infty}$ satisfy

(3) $$\gamma_{n+k}(b_n^{-1} - \bar{\alpha}_k) + \bar{\alpha}_k \gamma_n - 1 = 0 \qquad (n=1,2;k\geq 1:k=1;n\geq 1).$$

We then show that if $\{\gamma_i\}_{i=1}^{\infty}$, $\{b_j\}_{j=1}^{\infty}$ and $\{\alpha_\ell\}_{\ell=1}^{\infty}$ satisfy (3), they also satisfy

$$\gamma_{n+k}(b_n^{-1} - \bar{\alpha}_k) + \bar{\alpha}_k \gamma_n - 1 = 0 \qquad (n \geq 1;\ k \geq 1).$$

From these results we find explicitly the common solution set of Equations (2) and (3). A number of examples show that not all solutions of (2) give rise to a polynomial set in Σ .

Second, in order to find sufficient conditions for a set of sequences satisfying (2) to yield a polynomial set in Σ , we introduce the following modified moment problem. Given an infinite sequence $\{g_k\}_{k=0}^{\infty}$; it is required to find a weight function $w(x)$ such that if $\{U_n(x)\}_{n=0}^{\infty}$ is the Chebychev polynomial set of the second kind then

$$\int_{-1}^{1} w(x)\, U_n(x)\, dx = g_n \qquad (n = 0,1,2 \ldots) .$$

We find sufficient conditions on $\{g_n\}_{n=0}^{\infty}$ to insure that the solution of this moment problem has the form:

$$w(x) \equiv \frac{2}{\pi} \sqrt{1-x^2} \sum_{k=0}^{\infty} g_k\, U_k(x) \qquad x\varepsilon(-1,1).$$

We also give sufficient conditions on $\{g_n\}_{n=0}^{\infty}$ to make $w(x)$ continuously differentiable on $(-1,1)$ and satisfy a Lipschitz condition on $[-1,1]$.

By using these two ideas we identify each polynomial set in Σ by giving its three term recursion formula. Not only does this modified moment problem help to find all the polynomial sets in Σ , but also we use its results in Chapter V to study some of the properties of the weight functions of the polynomial sets in Σ .

One of the polynomial sets in Σ satisfies the same three term recursion formula as does the Chebychev polynomial sets, but with a parameter in the boundary conditions. This suggests generalizing a result of T. S. Chihara. We investigate the relationships between two polynomial sets $\{B_n(x)\}_{n=0}^{\infty}$ and $\{C_n(x)\}_{n=0}^{\infty}$ having the same three term recursion formula but different boundary conditions. We show

a) how the zeros of $B_n(x)$ interlace those of $C_n(x)$, for $n = 1,2,3 \ldots$,

b) the relative positions of their true intervals of orthogonality and

c) relationships between their distribution functions.

In the last chapter we use these results to explore properties of one of the polynomial sets in Σ .

At the end of the thesis we study in detail two of the polynomial sets in Σ , finding their generating functions, orthogonality relations, expansions in terms of well known special functions, etc.

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CHAPTER I

PRELIMINARIES

1.1 ORTHOGONAL POLYNOMIAL SETS. An infinite sequence of real numbers $\{s_k\}_{k=0}^{\infty}$ is called *positive* if all the quadratic forms

$$\sum_{i,k=0}^{m} s_{i+k} x_i x_k \qquad (m = 0, 1, 2 \ldots)$$

are positive definite. One may associate with such a sequence a linear functional L defined on the linear space of all real polynomials by means of

$$L(\pi(x)) = \sum_{i=0}^{n} c_i s_i \tag{1.1.1}$$

where $\pi(x) = \sum_{i=0}^{n} c_i x^i$.

In the sequel we will always use $\pi(x)$ for an arbitrary real polynomial. One can show (see [2, p. 2]) that L is a *positive* functional (i.e. if $\pi(x) \geq 0$, $-\infty < x < \infty$ and $\pi(x) \not\equiv 0$ then $L(\pi(x)) > 0$) if and only if $\{s_k\}_{k=0}^{\infty}$ is positive. s_k is called the k^{th} *moment* associated with the functional L. For any positive functional L it is possible to construct (e.g., by using the Gram-Schmidt process) a sequence of polynomials $\{A_n(x)\}_{n=0}^{\infty}$ such that $A_n(x)$ is of exact degree n, its leading coefficient is positive and the following orthogonality relation holds

$$L\{A_m(x)\, A_n(x)\} = k_n \delta_{m,n} \qquad (n = 0, 1, 2 \ldots) \tag{1.1.2}$$

where $k_n \neq 0$. The polynomials $A_n(x)$, $n = 0, 1, 2 \ldots$ are then called *orthogonal with respect to* L or *orthonormal with respect to* L if $k_n = 1$, for all $n \geq 0$. Sometimes we will call $\{A_n(x)\}_{n=0}^{\infty}$ an *orthogonal polynomial set* and shorten this to O.P.S. It is well known that $\{A_n(x)\}_{n=0}^{\infty}$ is orthogonal with respect to the linear functional L if and only if for all $n \geq 0$

$$L(x^m A_n(x)) = \begin{cases} 0 & (m = 0, 1, 2 \ldots n-1) \\ k_n & (m = n), \end{cases}$$

where $k_n \neq 0$.

We will call $\sigma(t)$ a *distribution function* if $\sigma(t)$ is monotonically nondecreasing and of bounded variation on $(-\infty, \infty)$ such that

$$\int_{-\infty}^{\infty} x^n d\sigma(x) < \infty \qquad (n = 0, 1, 2 \ldots).$$

For our purpose we will always require $\sigma(t)$ to have an infinite number of points of increase.

If $w(x) \geq 0$ almost everywhere on $(-\infty, \infty)$,

$$\int_{-\infty}^{\infty} w(x)dx > 0$$

and

$$\int_{-\infty}^{\infty} x^n w(x)dx < \infty \qquad (n = 0, 1, 2 \ldots),$$

then $w(x)$ is said to be a *weight function.*

We shall have several occasions to refer to the *Hamburger's moment problem* (we will abbreviate this to H.M.P.) which may be stated as follows: given an infinite sequence of real numbers $\{s_k\}_{k=0}^{\infty}$ $s_0 = 1$ is it possible to find a distribution function $\sigma(t)$ satisfying the equations

$$s_k = \int_{-\infty}^{\infty} t^k d\sigma(t) \qquad (k = 0, 1, 2 \ldots)? \tag{1.1.3}$$

Hamburger [21] showed that the H.M.P. has a solution if and only if $\{s_k\}_{k=0}^{\infty}$ is positive. Thus, a positive moment functional L defined on the linear space of all polynomials may be represented by

$$L(\pi(x)) = \int_{-\infty}^{\infty} \pi(t)d\sigma(t)$$

where $\sigma(t)$ is a distribution function.

The H.M.P. is said to be *determinate* if for any two of its solutions $\sigma_1(t)$ and $\sigma_2(t)$, there exists a constant c such that $\sigma_1(t) - \sigma_2(t) = c$ at all points where the difference is continuous; otherwise, it is called *indeterminate*.

If in Equation (1.1.3) the interval of integration is finite the corresponding moment problem is called the *Hausdorff moment problem*. When a solution exists it is known that it is determinate.

Let $[\alpha, \beta]$ be the smallest closed interval that contains all the points of increase of $\sigma(t)$. Then $A_n(x)$, $n = 0, 1, 2 \ldots,$ in Equation (1.1.2) are said to be orthogonal with respect to the distribution $\sigma(t)$ on the interval (α, β). If $\sigma(t)$ is absolutely continuous on (α, β) then there exists a weight function $w(x)$ such that

$$\int_{\alpha}^{\beta} \pi(t)d\sigma(t) = \int_{\alpha}^{\beta} \pi(t)w(t)dt.$$

In this case, $A_n(x)$, $n = 0, 1, 2 \ldots$, in Equation (1.1.2) are said to be orthogonal on (α, β) with respect to the weight function $w(x)$.

Let $x_{n,i}$, $i = 1, 2 \ldots n$, be the n zeros of $A_n(x)$. It is well known (Szegö [35]) that $x_{n,i}$ are distinct, real and lie in (α, β). They have the separation property

$$x_{n+1,i} < x_{n,i} < x_{n+1,i+1} \qquad (i = 1, 2 \ldots n).$$

So that if

(1.1.4) $\quad \xi_i = \lim_{n\to\infty} x_{n,i}$ and $\eta_j = \lim_{n\to\infty} x_{n,n-j+1}$,

then ξ_i and η_i exist in the extended real number system for $i = 1, 2 \ldots$. We call (ξ_1, η_1) the *true interval of orthogonality* of $A_n(x)$, $n = 0, 1, 2 \ldots$. It is known (see Szegö [35]) that $\alpha = \xi_1$ and $\beta = \eta_1$ and thus the true interval of orthogonality is the smallest interval containing all the points of increase of $\sigma(t)$.

Favard [14] showed that a necessary and sufficient condition for a set of polynomials $\{A_n(x)\}_{n=0}^{\infty}$ to be orthogonal on $(-\infty, \infty)$ with respect to a distribution function $\sigma(t)$ having an infinite number of points of increase is the existence of real sequences $\{c_n\}_{n=1}^{\infty}$ and $\{\lambda_n\}_{n=2}^{\infty}$ such that

$$A_0(x) = 1 \qquad A_1(x) = x - c_1$$

(1.1.5) $A_n(x) = (x - c_n)\, A_{n-1}(x) - \lambda_n A_{n-2}(x)$ $(n = 2, 3 \ldots)$

where $\lambda_n > 0$ for $n \geq 2$. We call Equation (1.1.5) the *three term recursion relation associated with* $\{A_n(x)\}_{n=0}^{\infty}$.

It is easy to show the equivalence of the following three properties:

1) $A_n(x) = (-1)^n A_n(-x)$

2) $s_{2k+1} = 0$

3) $A_0(x) = 1$ $\qquad A_1(x) = a_1 x$

$$A_n(x) = a_n x\, A_{n-1}(x) - \lambda_n A_{n-2}(x)$$

where $\lambda_n > 0$ for $n \geq 2$ and $a_n > 0$ for $n \geq 1$. An Orthogonal polynomial set having any one of these three properties is called *symmetric*. In the case when $\{A_n(x)\}_{n=0}^{\infty}$ is symmetric and orthogonal with respect to the weight function $w(x)$, then $w(x)$ is even and the interval of orthogonality is symmetric about the origin.

Chihara [11] defines a *chain sequence* as any sequence that can be written in the form

$$(1 - g_0)g_1,\ (1 - g_1)g_2,\ (1 - g_2)g_3 \ldots$$

where $0 \leq g_0 < 1$, $0 < g_p < 1$ for $p = 1, 2, 3 \ldots.$

Lemma (1.1.1) (Chihara [10]). *Let* ξ_1, η_1, c_n *and* λ_n *be as defined in Equations* (1.1.4) *and* (1.1.5). *Let*

$$\alpha_n = \frac{\lambda_{n+1}}{(c_n-x)(c_{n+1}-x)} \qquad (n = 1, 2, \ldots).$$

A necessary and sufficient condition for $x \le \xi_1$ $(x \ge \eta_1)$ *is that* $x < c_n$ $(x > c_n)$ *for all* $n > 0$ *and* $\{\alpha_n(x)\}_{n=1}^{\infty}$ *is a chain sequence.*

Let

(1.1.6) $$K_1(z) = \frac{1|}{|z-c_1} - \frac{\lambda_2|}{|z-c_2} - \frac{\lambda_3|}{|z-c_3} \cdots$$

(1.1.7) $$K_n(z,t) = \frac{1|}{|z-c_1} - \frac{\lambda_2|}{|z-c_2} \cdots \frac{\lambda_n|}{|z-c_n+t}.$$

It is easy to see that the n^{th} convergent for $n \ge 1$, of the continued fraction (1.1.6) equals $Q_{n-1}(x)/A_n(x)$, where $\{Q_n(x)\}_{n=0}^{\infty}$ is a polynomial set having the three term recursion relation

$$Q_{-1}(x) = 0 \qquad\qquad Q_0(x) = 1$$

$$Q_n(x) = (x - c_{n+1})\, Q_{n-1}(x) - \lambda_{n+1}Q_{n-2}$$

and is called the *numerator* polynomial set of $\{A_n(x)\}_{n=0}^{\infty}$. Also $Q_n(x)$ is the denominator of the n^{th} convergents of the continued fraction

$$K_2(z) = \frac{1|}{|z-c_2} - \frac{\lambda_3|}{|z-c_3} - \frac{\lambda_4|}{|z-c_4} - \frac{\lambda_5|}{|z-c_5} \cdots$$

<u>Definition (1.1.1)</u> (Hamburger [21]). The continued fraction $K_1(z)$ *converges completely* to a function $F(z) \equiv \int_{-\infty}^{\infty} \frac{d\sigma(t)}{z-t}$ if, for arbitrary small $\varepsilon > 0$ and for every finite closed region Ω that does not contain points of the real axis, there exists a positive integer N depending only on ε and Ω such that

$$|K_n(z,t) - F(z)| < \varepsilon$$

for $z \in \Omega$, $n = N, N + 1 \ldots,$ and t an arbitrary extended real number.

Sherman [32] showed that the complete convergence of $K_1(z)$ implies the complete convergence of $K_2(z)$. Hamburger [20] showed that $K_1(z)$ converges completely if and only if the moment problem associated with $\{A_n(x)\}_{n=0}^{\infty}$ is determinate. Thus we have the following.

Lemma (1.1.2) (Sherman [32]). *If* $\{A_n(x)\}_{n=0}^{\infty}$ *is an orthogonal polynomial set associated with a determinate moment problem, then* $\{Q_n(x)\}_{n=0}^{\infty}$ *is also associated with a determinate moment problem.*

We also need the following two results that can be found in Akhiezer [2].

Lemma (1.1.3). *Let* $\sigma(t)$ *be a solution of a Hamburger moment problem. If* $\{p_n(x)\}_{n=0}^{\infty}$ *is orthonormal with respect to the distribution* $\sigma(t)$ *on the interval* $(-\infty, \infty)$ *then the maximum jump that* $\sigma(t)$ *may have at a point* x *is* $(\sum_{i=0}^{\infty} |p_i(x)|^2)^{-1}$.

Lemma (1.1.4). *Let* $\{p_n(x)\}_{n=0}^{\infty}$ *and* $\{q_n(x)\}_{n=0}^{\infty}$ *be two orthonormal polynomial sets such that* $\{q_n(x)\}_{n=0}^{\infty}$ *is the numerator polynomial set of* $\{p_n(x)\}_{n=0}^{\infty}$. *The Hamburger moment problem is determinate if and only if for all real* x *at least one of* $\sum_{n=0}^{\infty} |p_n(x)|^2$ *or* $\sum_{n=0}^{\infty} |q_n(x)|^2$ *is divergent.*

1.2 HISTORICAL BACKGROUND. A number of papers have appeared concerning the following type of problem. Find all orthogonal polynomial sets that satisfy a given condition.

W. Hahn [20] showed that if $\{A_n(x)\}_{n=0}^{\infty}$ is an orthogonal polynomial set and if $\{\frac{d^r A_n(x)}{dx^r}\}_{n=r}^{\infty}$, r a positive integer, is also an orthogonal polynomial set, then $\{A_n(x)\}_{n=0}^{\infty}$ must be the Jacobi, Hermite or Laguerre polynomial set.

Angelesco [5], among others, showed that if $\frac{dA_n(x)}{dx} = A_{n-1}(x)$ and $\{A_n(x)\}_{n=0}^{\infty}$ is an orthogonal polynomial set then $\{A_n(x)\}_{n=0}^{\infty}$ is the Hermite polynomial set.

Meixner [28] and Sheffer [31] determined all orthogonal polynomial sets with the property that there exists a differential operator

$$J(D) = \sum_{k=0}^{\infty} a_k D^{k+1} \qquad a_0 \neq 0, \qquad D \equiv \frac{d}{dx}$$

where $a_i (i = 0,1,2...)$ are real constants, such that $J(D)P_n(x) = P_{n-1}(x)$. They found that the Laguerre, Hermite, Charlier, and Meixner polynomial sets were the only polynomial sets that enjoy this property.

In 1968 Chihara [9] found all orthogonal polynomial sets that have a generating function of the form

$$A(w)\ B(x,w) = \sum_{n=0}^{\infty} P_n(x)\ w^n$$

where

$$A(w) = \sum_{k=0}^{\infty} a_k w^k$$

$$B(w) = \sum_{k=0}^{\infty} b_k w^k$$

where $b_n \neq 0$ for all $n \geq 0$ and $a_0 \neq 0$.

Let $C = \{\{A_n(x)\}_{n=0}^{\infty} | A_n(x)$ is a real polynomial of degree $n\}$. Let R be a binary relation on C defined by $\{P_n(x)\}_{n=0}^{\infty} R \{Q_n(x)\}_{n=0}^{\infty}$ if and only if there extists $\{\alpha_n\}_{n=0}^{\infty}$ $\alpha_n \neq 0$ for all $n \geq 0$, $a \neq 0$, and b such that $P_n(x) = \alpha_n Q_n(ax + b)$. Obviously R is an equivalence relation. When we say we have identified an orthogonal polynomial set $\{R_n(x)\}_{n=0}^{\infty}$ with a given property we will mean that we have found its three term recursion relation. Not all polynomial sets equivalent to $\{R_n(x)\}_{n=0}^{\infty}$ will necessarily have the given property. On the other hand, when we say that we have found all the orthogonal polynomial sets enjoying a given property, we mean we have found the three term recursion relation of at least one polynomial set in each of the equivalence classes.

Throughout this work we shall denote by $U_n(x)$, $n = 0, 1, 2 \ldots$, the Chebychev polynomials of the second kind defined by means of

$$U_n(x) = \frac{\sin (n+1)\theta}{\sin \theta}$$

where $x = \cos \theta$. We note that these polynomials have the orthogonality relationship

$$\int_{-1}^{1} (1-x^2)^{\frac{1}{2}} U_n(x) U_m(x) dx = \delta_{n,m} \frac{\pi}{2}$$

We shall denote by $T_n(x)$, $n = 0, 1, 2 \ldots$ the Chebychev polynomials of the first kind defined by means of

$$T_n(x) = \cos n\theta$$

where $x = \cos\theta$. These polynomials have the orthogonality relationship

$$\int_{-1}^{1} (1-x^2)^{\frac{1}{2}} T_n(x)T_m(x)dx = \begin{cases} \frac{\pi}{2}\,\delta_{n,m} & n \neq 0 \\ \pi\,\delta_{0,m} \end{cases}$$

Also, on a number of occasions throughout the thesis we encounter the Ultraspherical polynomial set of order λ. We shall denote then by $P_n^{\lambda}(x)$, $n = 0, 1, 2 \ldots.$ They satisfy the orthogonality relationship

$$\int_{-1}^{1} (1-x^2)^{\lambda-\frac{1}{2}} P_n^{\lambda}(x)P_m^{\lambda}(x)dx = \delta_{n,m}\,\frac{2^{1-2\lambda}\,\pi\Gamma(n+2\lambda)}{(\Gamma(\lambda))^2(n+\lambda)\Gamma(n+1)}\,.$$

We note that $P_n^{\frac{1}{2}}(x)$ is the Legendre polynomial of degree n. We will denote the Legendre Polynomial set by $\{P_n(x)\}_{n=0}^{\infty}$. They satisfy the orthogonality relation

$$\int_{-1}^{1} P_n(x)P_m(x)dx = \frac{2\delta_{n,m}}{2n+1}.$$

CHAPTER II

$\mathcal{F}$ POLYNOMIAL SETS

2.1 INTRODUCTION. Heine [35, p. 93] gave the following representation for the Legendre polynomials:

$$(2.1.1) \qquad P_n(\cos\theta) = \frac{4}{\pi}\,\frac{2\cdot 4\cdots 2n}{3\cdot 5\cdots(2n+1)}\sum_{k=0}^{\infty} f_{k,n}\sin(n+2k+1)\theta$$

where $f_{0,n} = 1,\ f_{k,n} = \dfrac{1\cdot 3\cdots(2k-1)}{2\cdot 4\cdots 2k}\,\dfrac{(n+1)\cdots(n+k)}{(n+\frac{3}{2})(n+\frac{5}{2})\cdots(n+k+\frac{1}{2})}$.

Szegö [35, p. 96] generalized this formula to

$$(2.1.2) \qquad (\sin\theta)^{2\lambda-1}P_n^{\lambda}(\cos\theta) = \sum_{k=0}^{\infty} f_{k,n}^{\lambda}\sin(n+2k+1)\theta$$

where $f_{k,n}^{\lambda} = \dfrac{2^{2-2\lambda}\,\Gamma(n+2\lambda)(1-\lambda)_k(n+1)_k}{\Gamma(\lambda)\Gamma(n+\lambda+1)k!\,(n+\lambda+1)_k}$, $\lambda > 0$, λ not an integer* .

Equations (2.1.1) and (2.1.2) are the Fourier sine expansion of the Legendre polynomial $P_n(\cos\theta)$ and $(\sin\theta)^{2\lambda-1}P_n^{\lambda}(\cos\theta)$ respectively. Because the coefficients are eventually monotonic and they have a limit zero, it follows that each of these series are pointwise convergent in $(0,\pi)$ and uniformly convergent in $[\varepsilon,\pi-\varepsilon]$ where $0 < \varepsilon < \frac{\pi}{2}$.

If we let $x = \cos\theta$ then Equations (2.1.1) and (2.1.2) may be written in the form,

*We show in appendix II that λ could be a positive integer. In this case the infinite sum reduces to a finite sum.

$$(2.1.3)\qquad P_n(x) = \frac{4}{\pi}\,\frac{2\cdots 2n}{3\cdots(2n+1)}\,\sqrt{1-x^2}\,\sum_{k=0}^{\infty} f_{k,n}U_{n+2k}(x)$$

where $f_{0,n} = 1$, $f_{k,n} = \dfrac{1\cdots(2k-1)}{2\cdots 2k}\,\dfrac{(n+1)\cdots(n+k)}{(n+\frac{3}{2})\cdots(n+k+\frac{1}{2})}$ and

$$(2.1.4)\qquad (1-x^2)^{\lambda-\frac{1}{2}}P_n^{\lambda}(x) = \sqrt{1-x^2}\,\sum_{k=0}^{\infty} f_{k,n}^{\lambda}U_{n+2k}(x)$$

where $f_{k,n} = \dfrac{2^{2-2\lambda}\Gamma(n+2\lambda)(1-\lambda)_k(n+1)_k}{\Gamma(\lambda)\Gamma(n+\lambda+1)k!\,(n+\lambda+1)_k}$.

The $\{U_n(x)\}_{n=0}^{\infty}$ is orthogonal on $[-1,1]$ with weight function $\sqrt{1-x^2}$. The Legendre polynomial set is orthogonal on $[-1,1]$ with weight function 1. Also, $\{P_n^{\lambda}(x)\}_{n=0}^{\infty}$ is orthogonal on $[-1,1]$ with weight function $(1-x^2)^{\lambda-\frac{1}{2}}$. We also note that if $\{A_n(x)\}_{n=0}^{\infty}$ is orthogonal on (α,β), a finite interval, we can always consider an equivalent polynomial set which is orthogonal on $(-1,1)$ [see Sec. (1.2)]. Thus we can assume without loss of generality that the interval of orthogonality is $(-1,1)$.

Let $w(x)$ be a weight function on $(-1,1)$ [see Sec. (1.1)] then we have the formal "Chebychev expansion"

$$w(x) \sim \sqrt{1-x^2}\,\sum_{k=0}^{\infty} a_{k,0}U_k(x)$$

where

$$a_{k,0} = \frac{2}{\pi}\int_{-1}^{1} w(x)U_k(x)dx \qquad\qquad (k = 0,\ 1,\ \ldots).$$

It is clear from the definition of a weight function that $a_{k,0}$, $k = 0, 1, 2 \ldots,$ exist.

Similarily, if $\{A_n(x)\}_{n=0}^{\infty}$ is an arbitrary polynomial set, such that the degree of $A_n(x)$ is n, then the following formal expansions

$$A_n(x)\ w(x) \sim \sqrt{1-x^2}\sum_{k=0}^{\infty} a_{k,n}U_k(x) \qquad (n = 0, 1 \ldots) \tag{2.1.5}$$

where, of course,

$$a_{k,n} = \frac{2}{\pi}\int_{-1}^{1} A_n(x)U_k(x)w(x)dx \qquad (k \geq 0;\ n \geq 0)$$

exist.

We may now state the following theorem:

<u>Theorem (2.1.1)</u>. *Let* $w(x)$ *be a weight function on* $[-1,1]$ *The polynomial set* $\{A_n(x)\}_{n=0}^{\infty}$ *in Relation* (2.1.5) *constitutes an* O.P.S. *with respect to the weight function* $w(x)$ *on* $(-1,1)$ *if and only if for all* $n \geq 0$

$$\begin{cases} a_{k,n} = 0 & (k = 0, 1, 2 \ldots, n-1) \\ a_{n,n} \neq 0\ . & \end{cases} \tag{2.1.6}$$

<u>Proof</u>: For all $m \geq 0$ there exists $\alpha_{m,k}$ $k = 0, 1, 2 \ldots m$ such that

$$x^m = \sum_{i=0}^{m} \alpha_{m,i}U_i(x)$$

where $\alpha_{m,m} \neq 0$. Therefore,

$$\int_{-1}^{1} x^m A_n(x)w(x)dx = \frac{\pi}{2} \sum_{i=0}^{m} \alpha_{m,i} a_{i,n}.$$

Thus, for all $n \geq 0$

$$\int_{-1}^{1} x^m A_n(x)w(x)dx = \begin{cases} 0 & (m = 0, 1, 2 \ldots n - 1) \\ \alpha_{n,n} a_{n,n} \neq 0 & \end{cases}$$

if and only if Equations (2.1.6) are satisfied. Therefore, $\{A_n(x)\}_{n=0}^{\infty}$ is an O.P.S. with respect to the weight function $w(x)$ if and only if Equations (2.1.6) are satisfied.

Q.E.D.

Being motivated by this theorem we shall define the class $\mathcal{F}$ to be the class of all the O.P.S. with weight function on (-1,1). In other words, if $\{R_n(x)\}$ belongs to $\mathcal{F}$ then there exists a weight function $w(x)$ on (-1,1) such that for all $n \geq 0$

(2.1.7) $$R_n(x)w(x) \sim \sqrt{1-x^2} \sum_{k=0}^{\infty} r_{k,n} U_{k+n}(x)$$

where

$$r_{k,n} = \frac{2}{\pi} \int_{-1}^{1} w(x)R_n(x)U_{k+n}(x)dx.$$

We immediately obtain the following result.

<u>Theorem (2.1.2)</u>. *Let* $\{R_n(x)\}_{n=0}^{\infty}$ *be as given in Relation (2.1.7).* $\{R_n(x)\}_{n=0}^{\infty}$ *is symmetric if and only if* $r_{2k+1,0} = 0$.

<u>Proof</u>: Since $\{U_n(x)\}_{n=0}^{\infty}$ is a symmetric polynomial set then

$$\int_{-1}^{1} x^{2k+1} \sqrt{1-x^2}\, dx = 0 \qquad (k = 0, 1, 2 \ldots),$$

and

$$x^{2k+1} = \sum_{i=0}^{k} \alpha_{i,k} U_{2i+1}(x).$$

But by hypothesis

$$\int_{-1}^{1} w(x) U_{2k+1}(x) dx = 0.$$

Therefore, for all $n \geq 0$,

$$\int_{-1}^{1} x^{2n+1} w(x) dx = 0.$$

Thus, $\{R_n(x)\}_{n=0}^{\infty}$ is symmetric.

Q.E.D.

We shall have occasion to consider the O.P.S. in $\mathcal{F}$ which are symmetric. We shall denote the class of such polynomial sets by $\mathcal{S}$.

2.2 A MODIFIED MOMENT PROBLEM. In this section we shall consider a problem which is related to the Hausdorff Moment Problem [see Sec. (1.1)].

Given a sequence of constants $\{g_k\}_{k=0}^{\infty}$. We find sufficient conditions on $\{g_k\}_{k=0}^{\infty}$ to insure the existence of a Lebesgue integrable function $w(x)$ on $(-1,1)$ with the property that

$$\int_{-1}^{1} U_k(x)w(x)dx = g_k \qquad (k = 0, 1, 2 \ldots). \tag{2.2.1}$$

Equation (2.2.1) may be written as

$$\int_{0}^{\pi} \sin(k+1)\theta \; w(\cos\theta)d\theta = g_k \qquad (k = 0, 1 \ldots), \tag{2.2.2}$$

from which we see that $\frac{2}{\pi} g_k$ is the $(k+1)^{st}$ Fourier coefficient in the sine expansion of $w(\cos\theta)$.

A similar moment problem was given by Akhiezer and Krein [3 p. 65]. They showed that a necessary and sufficient condition for the existence of a measurable function $f(\theta)$, which satisfies the conditions

$$- L \leq f(\theta) \leq L$$

and

$$b_k = \int_{0}^{\pi} f(\theta) \sin(k\theta)d\theta \qquad (k = 0, 1 \ldots),$$

is that the sequence

$$\beta_0 = 2, \beta_1, \beta_2, \ldots$$

defined by

$$\exp\{-\frac{1}{L}\sum_{k=1}^{\infty} b_k z^k\} = 1 + \sum_{k=1}^{\infty} \beta_k z^k$$

be non-negative definite on the circumference.

The following two lemmas are required in order to prove our first result. The proofs are easy and we shall omit them.

Lemma (2.2.1). *If we define*

$$G_N(x) = \frac{2}{\pi}\sqrt{1-x^2}\sum_{k=0}^{N} U_{2k}(x) \qquad (N = 0, 1 \ldots), \tag{2.2.3}$$

then

$$G_N(x) = \frac{2}{\pi}\sqrt{1-x^2}\,(U_N(x))^2, \qquad (N = 0, 1 \ldots). \tag{2.2.4}$$

For x *belonging to* $(-1,1)$ *we have*

$$|G_N(x)| \le \frac{2}{\pi(1-x^2)^{\frac{1}{2}}} \qquad (N = 0, 1, 2, \ldots) \tag{2.2.5}$$

and if $-1 + \varepsilon \le x \le 1 - \varepsilon$, *then as* $N \to \infty$

$$G_N'(x) = O(N) \tag{2.2.6}$$

where $G_N'(x) = \dfrac{dG_N(x)}{dx}$.

Lemma (2.2.2). *If we define*

$$H_N(x) = \frac{2}{\pi}\sqrt{1-x^2}\sum_{k=0}^{N} U_{2k+1}(x) \qquad (N = 0, 1 \ldots), \tag{2.2.7}$$

then

$$(2.2.8)\qquad H_N(x) = \frac{x-T_{2N+3}(x)}{\pi\sqrt{1-x^2}} \qquad (N = 0, 1 \ldots),$$

$$(2.2.9)\qquad |H_N(x)| \le \frac{2}{\pi(1-x^2)^{\frac{1}{2}}} \qquad (N = 0, 1 \ldots),$$

and if $-1 + \varepsilon \le x \le 1 - \varepsilon$, *then as* $N \to \infty$

$$(2.2.10)\qquad |H_N'(x)| = O(N),$$

where $$H_N'(x) = \frac{dH_N(x)}{dx}\ .$$

<u>Theorem (2.2.1)</u>. *If* $\sum_{n=0}^{\infty} (g_n - g_{n+2})$ *is absolutely convergent with* $g_n \to 0$ *as* $n \to \infty$, *then there exists a function* $w(x)$ *continuous on* $(-1,1)$ *with the properties:*

(a) $w(x) = \frac{2}{\pi}\sqrt{1-x^2}\sum_{k=0}^{\infty} g_k U_k(x) \qquad x \in (-1,1),$

(b) $\int_{-1}^{1} w(x)U_n(x)dx = g_n \qquad (n = 0, 1, 2 \ldots),$

(c) *if* $\sum_{n=0}^{\infty} n|g_n - g_{n+2}| < \infty$, *then* $w(x)$ *is continuously differentiable on* $(-1,1)$.

<u>Proof</u>: Let

$$w_n(x) = \frac{2}{\pi}\sqrt{1-x^2}\sum_{k=0}^{n} g_k U_k(x) \qquad (n = 0, 1 \ldots),$$

so that

$$(2.2.11)\qquad w_n(x) = \frac{2}{\pi}\sqrt{1-x^2}\left\{\sum_{k=0}^{[\frac{n}{2}]} g_{2k}U_{2k}(x) + \sum_{k=0}^{[\frac{n-1}{2}]} g_{2k-1}U_{2k+1}(x)\right\}$$

$$= \omega_n(x) + \Omega_n(x),$$

where $\omega_n(x)$ is the first summation that appears on the right hand side of Equation (2.2.11) and $\Omega_n(x)$ is the second.

By applying Abel's Transformation to $\omega_n(x)$ we get, by putting $m = [\frac{n}{2}]$,

$$(2.2.12)\qquad \omega_n(x) = \sum_{k=0}^{m-1} (g_{2k} - g_{2k+2})G_k(x) + g_{2m}G_m(x)$$

where $G_N(x)$ is defined by Equation (2.2.3). For any ε such that $0 < \varepsilon < 1$ we have by Lemma (2.2.1) $G_m(x)$ is uniformly bounded for $x \in [-1 + \varepsilon, 1 - \varepsilon]$. By using this fact and the fact that $g_{2m} \to 0$ as $m \to \infty$ we see that $g_{2m}G_m(x)$ converges uniformly to 0 on $[-1 + \varepsilon, 1 - \varepsilon]$. On the other hand by Equation (2.2.5) the terms in the sum

$$\sum_{k=0}^{\infty} (g_{2k} - g_{2k+2})G_k(x)$$

are majorized by

$$M_k = \frac{2|g_{2k} - g_{2k+2}|}{\pi(1-\varepsilon^2)^{\frac{1}{2}}} \qquad (k = 0, 1, 2 \ldots).$$

By the hypothesis of the theorem $\sum_{k=0}^{\infty} M_k$ converges and thus

$$\sum_{k=0}^{\infty} (g_{2k} - g_{2k+2})G_k(x)$$

converges uniformly on $[-1 + \varepsilon, 1 - \varepsilon]$. Hence $\{\omega_n(x)\}_{n=0}^{\infty}$ converges uniformly on the same interval.

In a similar manner we can show that $\{\Omega_n(x)\}_{n=0}^{\infty}$ converges uniformly on $[-1 + \varepsilon, 1 - \varepsilon]$.

By combining these two results it follows that $w_n(x)$ converges uniformly to $w(x)$ on $[-1 + \varepsilon, 1 - \varepsilon]$. From this it follows that $w(x)$ is continuous on $(-1,1)$ and

$$(2.2.13) \qquad w(x) = \sum_{k=0}^{\infty} (g_{2k} - g_{2k+2})G_k(x) + \sum_{k=0}^{\infty} (g_{2k+1} - g_{2k+3})H_k(x).$$

To prove part (c) one can show by a similar consideration that $\{w_n'(x)\}_{n=0}^{\infty}$ converges uniformly on $[-1 + \varepsilon, 1 - \varepsilon]$, so that in this case

$$\lim_{n\to\infty} w_n'(x) = w'(x) \qquad x \in (-1,1).$$

Now to prove (b) we proceed as follows. For all positive integers k, $\{U_k(x)w_n(x)\}_{n=0}^{\infty}$ is a sequence of Lebesgue measurable functions such that $\{U_k(x)w_n(x)\}_{n=0}^{\infty}$ converges pointwise to $U_k(x)w(x)$ on $(-1,1)$. And by Lemmas (2.2.1) and (2.2.2):

$$|U_k(x)w_n(x)| \le \frac{2(k+1) \sum_{i=0}^{\infty} |g_i - g_{i+2}|}{\pi(1-x^2)^{\frac{1}{2}}}.$$

Also,

$$\frac{2(k+1)}{\pi} \sum_{i=0}^{\infty} |g_i - g_{i+2}| \int_{-1}^{1} \frac{dx}{(1-x^2)^{\frac{1}{2}}} = (k+1) \sum_{i=0}^{\infty} |g_i - g_{i+2}|$$

which is finite by the hypothesis. Thus, by the Lebesgue Dominated Convergence Theorem

$$\int_{-1}^{1} U_k(x)w(x)dx = \lim_{n\to\infty} \int_{-1}^{1} U_k(x)w_n(x)dx$$

So we have,

$$\int_{-1}^{1} U_k(x)w(x)dx = \lim_{n\to\infty} \frac{2}{\pi} \int_{-1}^{1} \sqrt{1-x^2}\, U_k(x) \sum_{\ell=0}^{n} g_\ell U_\ell(x)dx$$

But because of the orthogonality of $\{U_n(x)\}_{n=0}^{\infty}$, we have

$$\int_{-1}^{1} U_k(x)w(x)dx = g_k \qquad (k = 0, 1, 2, \ldots).$$

Q.E.D.

In Chapter III we will also need the following Corollary.

<u>Corollary I</u>. *Let* $\{g_{2n}\}_{n=0}^{\infty}$ *and* $\{g_{2n+1}\}_{n=0}^{\infty}$ *be eventually monotonic and* $\lim_{n\to\infty} g_n = 0$. *If* $\left| \sum_{k=1}^{\infty} \frac{g_k}{k} \right| < \infty$, *then there exists a continuous function* $w(x)$ *such that* $w(\cos\theta)$ *belongs to* $L'[0,\pi]$ *and*

$$\int_{-1}^{1} w(x)U_n(x)dx = g_n.$$

<u>**Proof**</u>: It is easy to see that $\{g_n\}_{n=0}^{\infty}$ satisfies the conditions of Theorem (2.2.1); thus we need only show that $w(\cos\theta)$ belongs to $L'[0,\pi]$.

Without loss of generality we may assume $\{g_{2n}\}_{n=0}^{\infty}$ and $\{g_{2n+1}\}_{n=0}^{\infty}$ are both monotonically decreasing to zero.

We recall from the proof of Theorem (2.2.1) that

$$\text{(2.2.14)} \qquad w(\cos\theta) = \lim_{n\to\infty}[\omega_n^*(\cos\theta) + \Omega_n^*(\cos\theta)]$$

where

$$\text{(2.2.15)} \qquad \omega_n^*(x) = \sum_{k=0}^{m-1}(g_{2k} - g_{2k+2})G_k(x) \qquad (n = 0, 1 \ldots),$$

$$\text{(2.2.16)} \qquad \Omega_n^*(x) = \sum_{k=0}^{m-1}(g_{2k+1} - g_{2k+3})H_k(x) \qquad (n = 0, 1 \ldots),$$

and $m = [\frac{n}{2}]$.

Now since

$$G_n(\cos\theta) = \frac{2}{\pi}\sum_{k=0}^{n}\sin(2k+1)\theta \qquad (n = 0, 1 \ldots),$$

and

$$H_n(\cos\theta) = \frac{2}{\pi}\sum_{k=0}^{n}\sin(2k+2)\theta \qquad (n = 0, 1 \ldots),$$

then

$$H_n(\cos\theta) = \frac{1}{\pi}\,(D_{2n+2}(\theta) - D_{2n+2}(\pi-\theta))$$

and

$$G_n(\cos\theta) = \frac{1}{\pi}\,(D_{2n+1}(\theta) - D_{2n+1}(\pi-\theta))$$

where

$$D_n(\theta) = \sum_{k=1}^{n} \sin k\theta.$$

It is well known [37, p. 68] that as $n \to \infty$

$$\int_0^\pi |D_n(\theta)|\,d\theta \approx \log n.$$

Therefore, as $k \to \infty$

$$\int_0^\pi |G_k(\cos\theta)|\,d\theta = O(\log k)$$

and

$$\int_0^\pi |H_k(\cos\theta)|\,d\theta = O(\log k).$$

From (2.2.15)

$$|\omega_n^*(\cos\theta)| \le S(\theta)$$

where

$$S(\theta) = \sum_{k=0}^{\infty} (g_{2k} - g_{2k+2})|G_k(\cos\theta)|.$$

To show that $S(\theta)$ is Lebesgue Integrable we note from the above that

$$\left\{\sum_{k=0}^{m} (g_{2k} - g_{2k+2})|G_k(\cos\theta)|\right\}_{m=0}^{\infty}$$

is a monotonically increasing sequence of functions and

$$\begin{aligned}\sum_{k=0}^{\infty} (g_{2k} - g_{2k+2})\int_0^{\pi} |G_k(\cos\theta)|d\theta &\le C_1 \sum_{k=0}^{\infty} (g_{2k} - g_{2k+2})\log(k+1) \\ &\le C_2 \sum_{k=1}^{\infty} g_{2k}[\log(k+1) - \log(k)] \\ &< C_3 \sum_{k=0}^{\infty} \frac{g_{2k}}{k} < \infty\end{aligned}$$

where C_1, C_2, and C_3 are constants.

Thus by a theorem of Lebesgue (see[6, p. 27]), we see that $S(\theta) \in L'[0,\pi]$. Thus by the Lebesgue Dominated Convergence Theorem $\omega^*(\cos\theta) \in L'[0,\pi]$.

In exactly the same manner we can show that $\Omega^*(\cos\theta) \in L'[0,\pi]$. Thus $\omega(\cos\theta)$ belongs to $L'[0,\pi]$.

Q.E.D.

In conclusion we wish to find necessary and sufficient conditions on $\{g_k\}_{k=0}$ so that $w(x)$ satisfies a Lipschitz condition of order α.

We first note the following Lemma:

Lemma (2.2.3). (a) *If* $w(x) \in Lip(\alpha)$ *on* $[-1,1]$, *then* $w(\cos\theta) \in Lip(\alpha)$ *on* $[0,\pi]$.

(b) *If* $w(\cos\theta) \in Lip(\alpha)$ *on* $[0,\pi]$, *then* $w(x) \in Lip(\alpha/2)$ *on* $[-1,1]$.

Proof: To show that (a) is true we see that the inequality

$$|w(x) - w(y)| \le M|x - y|^{\alpha}$$

where $x, y \in [-1,1]$ and M is a constant implies

$$|w(\cos\theta_1) - w(\cos\theta_2)| \le |M \cos\theta_1 - \cos\theta_2|^{\alpha}$$
$$\le M\cdot M_1^{\alpha}|\theta_1 - \theta_2|^{\alpha}$$

To prove statement (b) we note first that $f(x) = \cos^{-1}x$ is $Lip(\frac{1}{2})$ on $[-1,1]$*. We then consider

$$|w(\cos\theta_1) - w(\cos\theta_2)| \le M|\theta_1 - \theta_2|^{\alpha}.$$

Hence by putting $x_1 = \cos(\theta_1)$ and $x_2 = \cos(\theta_2)$ we obtain

$$|w(x_1) - w(x_2)| \le M|\cos^{-1}x_1 - \cos^{-1}x_2| \le \pi^2 2^{-1} M|x_1 - x_2|^{\alpha/2},$$

If $0 < \alpha < 1$, then it is known [6, vol. II, p. 230-231] that if $g_n \downarrow 0$ then $w(\cos\theta)$ satisfies a Lipschitz condition of order α on $[o,\pi]$ if and only if $g_n = O(\frac{1}{n^{\alpha+1}})$. By a slight modification in the proof of this Theorem one can easily show that the results still hold if

*We omit the proof that $\cos^{-1}x$ is $Lip(\frac{1}{2})$ on $[-1,1]$ as it is an easy exercise.

$g_n \downarrow 0$ is replaced by $g_{2n} \downarrow 0$ and $g_{2n+1} \downarrow 0$. By using this fact and Lemma (2.2.3) we have the following theorem.

Theorem (2.2.2). *Let* $0 < \alpha < 1$, $g_{2n} \downarrow 0$ *and* $g_{2n+1} \downarrow 0$. *If* $w(x)$ *is* $\mathrm{Lip}(\alpha)$ *on* $[-1,1]$ *then as* $n \to \infty$

$$g_n = O(n^{-\alpha-1}). \tag{2.2.17}$$

Conversely, if Equation (2.2.17) *holds then* $w(x)$ *is* $\mathrm{Lip}(\frac{\alpha}{2})$ *on* $[-1,1]$.

2.3 PROPERTIES OF POLYNOMIAL SETS IN $\mathcal{S}$. In this section and throughout the rest of the Thesis we shall be mainly concerned with elements in $\mathcal{S}$. If $\{A_n(x)\}_{n=0}^{\infty}$ belongs to $\mathcal{S}$ then $\{A_n(x)\}_{n=0}^{\infty}$ is orthogonal with respect to some even weight function $w(x)$ on $(-1,1)$ and

$$\frac{A_n(x)w(x)}{(1-x^2)^{1/2}} \sim \sum_{k=0}^{\infty} f_{k,n} U_{n+2k}(x). \tag{2.3.1}$$

In order to make some of the further caluclations simpler we shall use the convention.

$$\begin{cases} U_n(x) = -U_{-n-2}(x) & (n < 0) \\ U_{-1}(x) = 0. \end{cases} \tag{2.3.2}$$

so that,

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta} \qquad (n = 0, \pm 1, \pm 2 \ldots)$$

and

$$\text{(2.3.3)} \qquad U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x) \qquad (n = 0, \pm 1, \pm 2 \ldots).$$

<u>Lemma (2.3.1)</u>. *If* $\{U_n(x)\}_{n=0}^{\infty}$ *is as defined by Equation* (2.3.2), *then*

$$\text{(2.3.4)} \qquad (2x)^r U_n(x) = \sum_{i=0}^{r} \binom{r}{i} U_{n+r-2i}(x)$$

and

$$\text{(2.3.5)} \qquad (2x)^r U_n(x) = \begin{cases} \displaystyle\sum_{i=0}^{\frac{n+r}{2}} \binom{r}{i} U_{n+r-2i}(x) - \sum_{i=0}^{\frac{r-n}{2}-1} \binom{r}{\frac{n+r}{2}+1+i} U_{2i}(x) & \text{if } n+r \text{ is even,} \\[2ex] \displaystyle\sum_{i=0}^{\frac{n+r-1}{2}} \binom{r}{i} U_{n+r-2i}(x) - \sum_{i=0}^{\frac{r-n-1}{2}} \binom{r}{\frac{n+r+1}{2}+i} U_{2i-1}(x) & \text{if } n+r \text{ is odd,} \end{cases}$$

where the void sum is zero and $\binom{r}{i} = 0$ *if* $i > r$.

Proof: For any integer n the validity of Equation (2.3.4) is easily proven by mathematical induction on r. Equation (2.3.5) follows from Equations (2.3.4) and (2.3.2).

Q.E.D.

Theorem (2.3.1). *Let* $\{U_n(x)\}_{n=0}^{\infty}$ *be as defined in Equation* (2.3.2), *and* $\{A_n(x)\}_{n=0}^{\infty}$ *and* $w(x)$ *be as defined in relation* (2.3.1). *For all integers* k *and all non-negative integers* n *and* r.

$$\int_{-1}^{1} x^r U_{n+k-r}(x)\, A_n(x) w(x) dx = 0$$

if k *is odd, and*

$$\int_{-1}^{1} x^r U_{n+k-r}(x)\, A_n(x) w(x) dx = \frac{\pi}{2^{r+1}} \sum_{i=0}^{\frac{k}{2}} \binom{r}{i} f_{\frac{k}{2}-i,n} - \sum_{i=n+\frac{k}{2}+1}^{r} \binom{r}{i} f_{i-(n+\frac{k}{2}+1),n}$$

if k *is even, where* $\binom{r}{i} = 0$ *if* $i > r$ *or* $i < 0$ *and* $f_{k,n} = 0$ *if* $k < 0$.

Proof: Since $\{A_n(x)\}_{n=0}^{\infty}$ belongs to $\mathcal{S}$ and $w(x)$ is even, then for all $n \geq 0$ we have,

$$\frac{2}{\pi}\int_{-1}^{1} U_{n+k-2i}(x)\, A_n(x) w(x) dx = \begin{cases} 0, & \text{if } k \text{ is odd} \\ 0, & \text{if } \frac{k}{2} < i < n + \frac{k}{2} + 1, \ k \text{ even} \\ f_{\frac{k}{2}-i,n} & \text{if } i \leq \frac{k}{2}, \ k \text{ even} \\ -f_{i-(n+\frac{k}{2}+1),n} & \text{if } n + \frac{k}{2} + 1 \leq i, \ k \text{ even.} \end{cases}$$

Now by using Lemma (2.3.1) we obtain

$$\frac{2}{\pi}\int_{-1}^{1}(2x)^r U_{n+k-r}(x)A_n(x)w(x)dx = \sum_{i=0}^{r}\binom{r}{i}\frac{2}{\pi}\int_{-1}^{1}U_{n+k-2i}(x)\,A_n(x)w(x)dx$$

$$= \begin{cases} 0 & \text{if } k \text{ is odd} \\ \sum_{i=0}^{\frac{k}{2}}\binom{r}{i} f_{\frac{k}{2}-i,n} - \sum_{i=n+\frac{k}{2}+1}^{r}\binom{r}{i} f_{i-(n+\frac{k}{2}+1),n} & \end{cases}$$

where k is even and $\binom{r}{i} = 0$ if $i > r$ or $i < 0$.

Q.E.D.

<u>Corollary I.</u>

(2.3.6) $$\int_{-1}^{1}A_m(x)A_n(x)w(x)dx = \frac{\delta_{m,n}\,\pi\,c_n f_{0,n}}{2^{n+1}}$$

where c_n *is the leading coefficient of* $A_n(x)$ *and*

(2.3.7) $$\int_{-1}^{1}x^r w(x)dx = \begin{cases} 0 & \text{if } r \text{ is odd} \\ \frac{\pi}{2^{r+1}}\left[\binom{r}{\frac{r}{2}} f_{0,0} + \sum_{i=0}^{\frac{r}{2}-1}\binom{r}{i}\left[f_{\frac{r}{2}-i,0} - f_{\frac{r}{2}-i-1,0}\right]\right] & \end{cases}.$$

__Proof:__ In Theorem (2.3.1) let $k = r - n$ and $0 \le r \le n$. If $r - n$ is odd or $0 \le r < n$, then

$$\int_{-1}^{1} x^r A_n(x) w(x) dx = 0.$$

If $r = n$, then

$$\int_{-1}^{1} x^r A_n(x) w(x) dx = \frac{\pi}{2^{n+1}} f_{0,n}.$$

Therefore,

$$\int_{-1}^{1} A_r(x) A_n(x) w(x) dx = \frac{\pi}{2^{r+1}} f_{0,n}\, c_n\, \delta_{r,n}$$

where c_n is the leading coefficient of $A_n(x)$.

Since $w(x)$ is even we have

$$\int_{-1}^{1} x^{2r+1} w(x) dx = 0$$

and

$$\int_{-1}^{1} x^{2m} w(x) dx = \frac{\pi}{2^{2m+1}} \sum_{i=0}^{m} \binom{2m}{i} f_{m-i,0} - \sum_{i=m+1}^{r} \binom{2m}{i} f_{i-m-1,0}$$

$$= \frac{\pi}{2^{2m+1}} \left[\binom{2m}{m} f_{0,0} + \sum_{i=0}^{m-1} \binom{2m}{i} \left[f_{m-i,0} - f_{m-i-1,0}\right]\right].$$

Q.E.D.

Because $\{A_n(x)\}_{n=0}^{\infty}$ is a symmetric orthogonal polynomial set there exists real non-zero sequences $\{b_n\}_{n=1}^{\infty}$ and $\{\lambda_n\}_{n=2}^{\infty}$ such that

$$(2.3.8)\qquad \begin{cases} A_0(x) = 1 \qquad\qquad A_1(x) = 2b_1x \\ \\ A_n(x) = 2b_nxA_{n-1}(x) - \lambda_nA_{n-2}(x) \qquad (n = 1, 2, 3 \ldots), \end{cases}$$

where $b_n > 0$ for $n \geq 1$ and $\lambda_1 = 0$, $\lambda_n > 0$ for all $n \geq 2$.

<u>Theorem (2.3.2)</u>. *If* $\{A_n(x)\}_{n=0}^{\infty}$ *belongs to the class* $\mathcal{S}$ *and satisfies* (2.3.1) *and* (2.3.8), *then*

$$(2.3.9)\qquad b_nf_{0,n-1} = \lambda_nf_{0,n-2} \qquad (n = 2, 3 \ldots),$$

$$(2.3.10)\qquad f_{k,1} = b_1(f_{k,0} + f_{k+1,0}) \qquad (k = 0, 1, 2 \ldots),$$

$$(2.3.11)\qquad f_{k,n} = b_n(f_{k,n-1} + f_{k+1,n-1}) - \lambda_nf_{k+1,n-2}$$

$$(n = 2, 3 \ldots; k = 0, 1 \ldots).$$

<u>Proof</u>: We use the three term recursion relation (2.3.3) for $\{U_n(x)\}_{n=0}^{\infty}$ and the fact that $\{A_n(x)\}_{n=0}^{\infty}$ belongs to the class $\mathcal{S}$. Let us consider.

$$0 = \int_{-1}^{1} w(x)A_n(x)U_{n-2}(x)dx \qquad (n = 2, 3 \ldots)$$

$$= \int_{-1}^{1} w(x)[2b_n xA_{n-1}(x) - \lambda_n A_{n-2}(x)]U_{n-2}(x)dx$$

$$= \int_{-1}^{1} w(x)[b_n A_{n-1}(x)(U_{n-1}(x) + U_{n-3}(x)) - \lambda_n A_{n-2}(x)U_{n-2}(x)]dx$$

$$= \frac{\pi}{2}(b_n f_{0,n-1} - \lambda_n f_{0,n-2}) \qquad (n = 2, 3 \ldots).$$

i.e., $b_n f_{0,n-1} = \lambda_n f_{0,n-2}$. Also,

$$f_{k,n} = \frac{2}{\pi}\int_{-1}^{1} w(x)A_n(x)U_{n+2k}(x)dx \qquad (n = 0, 1, 2 \ldots; k = 0, 1 \ldots)$$

$$= \frac{2}{\pi}\int_{-1}^{1} w(x)(2b_n xA_{n-1}(x) - \lambda_n A_{n-2}(x))U_{n+2k}(x)dx$$

$$= \frac{2}{\pi}\int_{-1}^{1} w(x)[b_n A_{n-1}(x)(U_{n+2k-1}(x)) + U_{n+2k+1}(x)) - \lambda_n A_{n-2}(x)U_{n+2k}(x)]dx.$$

That is,

$$f_{k,n} = b_n(f_{k,n-1} + f_{k+1,n-1}) - \lambda_n f_{k+1,n-2} \qquad (n = 1, 2 \ldots; k = 0, 1 \ldots).$$

Equations (2.3.10) follow from this last equation by letting $n = 0$ and noting $\lambda_1 = 0$.

Q.E.D.

Theorem (2.3.3). *Let* $w(x) \in L'[-1,1]$, $A_n(x)\, w(x)$ *satisfy relation* (2.3.1) *for* $n = 0, 1, 2 \ldots$, *and the leading coefficient of* $A_n(x)$ *be* c_n. $\{A_n(x)\}_{n=0}^{\infty}$ *belongs to* $\mathcal{S}$ *if and only if* $c_n f_{0,n} > 0$, *for* $n = 0, 1, 2, \ldots$.

Proof: If $\{A_n(x)\}_{n=0}^{\infty}$ belongs to $\mathcal{S}$ then $\{A_n(x)\}_{n=0}^{\infty}$ is orthogonal on $[-1,1]$ with respect to the weight function $w(x)$. By Favard's Theorem $\{A_n(x)\}_{n=0}^{\infty}$ satisfies a three term recursion relation of the form (2.3.8) with $\lambda_n/b_n > 0$ for all $n \geq 2$, $b_1 > 0$ and $\lambda_1 = 0$. But by Theorem (2.3.2) this implies that $c_n f_{0,n} > 0$ for all $n \geq 0$.

Conversely, if $c_n f_{0,n} > 0$, then by Corollary I of Theorem (2.3.1)

$$\int_{-1}^{1} [A_n(x)]^2\, w(x)\, dx > 0 \qquad (n \geq 0).$$

It is known (See Akhiezer [2] p. 2) that any non negative polynomial $\pi(x) \not\equiv 0$ on $(-\infty,\infty)$ can be written as the sum of squares of two polynomials. Thus

$$\int_{-1}^{1} \pi(x)\, w(x)\, dx > 0.$$

From this it is easy to show that $w(x) \geq 0$ almost everywhere and $w(x) \not\equiv 0$. Thus $w(x)$ is a weight function. From Theorems (2.1.1) and (2.1.2) and the definition of $\mathcal{S}$ it follows that $\{A_n(x)\}_{n=0}^{\infty}$ belongs to $\mathcal{S}$.

CHAPTER III

Σ - POLYNOMIAL SETS

3.1 INTRODUCTION. If in Equation (2.1.4) we put

$$P_n^{\lambda}(x) = \frac{(1+\lambda)_n E_n^{\lambda}(x)}{n!}$$

we see that

$$(1 - x^2)^{\lambda-\frac{1}{2}} \; E_n^{\lambda}(x) = \frac{2^{2-2\lambda}\Gamma(n+2\lambda)}{\Gamma(\lambda)\Gamma(n+\lambda+1)} \sqrt{1-x^2} \sum_{k=0}^{\infty} g_{k,n}^{\lambda} U_{n+2k}(x)$$

where

$$g_{k,n}^{\lambda} = \frac{k-\lambda}{k} g_{k-1,n+1}^{\lambda}$$

That is there exists a sequence $\{\alpha_k\}_{k=1}^{\infty}$ such that

$$g_{k,n}^{\lambda} = \alpha_k g_{k-1,n+1}^{\lambda} \qquad (k \geq 1;\ n \geq 1)$$

This suggests the problem of finding the subclass Σ of $\mathcal{S}$ consisting of all polynomial sets $\{A_n(x)\}_{n=0}^{\infty}$ which have a weight function $w(\cos\Theta) \in L'(0,\pi)$, such that there exists a sequence of real numbers $\{\alpha_k\}_{k=1}^{\infty}$ with the property

$$f_{k,n} = \alpha_k f_{k-1,n+1} \qquad (1 \leq k;\ n \geq 0), \tag{3.1.1}$$

so that,

$$f_{k,n} = \left(\prod_{i=1}^{k} \alpha_i\right) f_{0,n+k} \qquad (0 \leq k;\ 0 \leq n) \tag{3.1.2}$$

where

$$\frac{A_n(x)w(x)}{\sqrt{1-x^2}} \sim \sum_{k=0}^{\infty} f_{k,n} U_{n+2k}(x). \tag{3.1.3}$$

We note that the Ultraspherical polynomial set $\{P_n^{\lambda}(x)\}_{n=0}^{\infty}$ for $\lambda > 0$ is equivalent to $\{E_n^{\lambda}(x)\}_{n=0}^{\infty}$ which are elements of Σ. In this chapter we find all the elements of Σ.

3.2 <u>NECESSARY CONDITIONS</u>. We first find some necessary conditions for a polynomial set $\{A_n(x)\}_{n=0}^{\infty}$ to be in Σ.

Let $\{A_n(x)\}_{n=0}^{\infty}$ be in Σ and be associated with $\{\alpha_k\}_{k=1}^{\infty}$ so that Equations (3.1.1) and (3.1.3) are satisfied. Also let $\{A_n(x)\}_{n=0}^{\infty}$ have the three term recursion relation

$$\begin{cases} A_0(x) = 1, \qquad A_1(x) = 2b_1\, x \\ \\ A_n(x) = 2b_n x A_{n-1}(x) - \lambda_n A_{n-2}(x);\ b_n\lambda_n > 0; \quad (n = 2, 3 \ldots). \end{cases} \tag{3.2.1}$$

Throughout the rest of this chapter we will let $\gamma_n = \lambda_n b_n^{-1}$ for $n \geq 1$ and $\lambda_1 = 0$.

<u>Theorem (3.2.1)</u>. If $\{\gamma_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\{f_{k,n}\}_{k=0}^{\infty} \mid n = 0, 1 \ldots\}$ *are as defined above then,*

1) $b_n > 0$ *for* $n \geq 1$ *and* $\gamma_n > 0$ *for* $n \geq 2$,

2) *the following equations hold:*

$$f_{0,n} = \left(\prod_{i=2}^{n+1} \gamma_i\right) f_{0,0} \qquad (n = 1, 2 \ldots), \tag{3.2.2}$$

$$f_{k,n} = \left(\prod_{i=1}^{k} \alpha_i\right)\left(\prod_{j=2}^{n+k+1} \gamma_j\right) f_{0,0} \qquad (1 \leq k;\ 1 \leq n), \tag{3.2.3}$$

$$\gamma_{n+k}(b_n^{-1} - \alpha_k) + \alpha_k\gamma_n - 1 = 0 \qquad (1 \leq n;\ 1 \leq k \leq m) \tag{3.2.4}$$

where m *is the smallest integer such that* $\alpha_m = 0$,

3) $\{\gamma_n(4b_{n-1})^{-1}\}$ *is a chain sequence,*

4) $\lim_{k\to\infty} \prod_{i=1}^{k} \alpha_i\gamma_{i+1} = 0.$

Proof: 1) We can always choose an equivalent polynomial set (see Sec. 1.2) such that $b_n > 0$ for $n \geq 1$ and therefore by Equations (3.2.1) $\gamma_n > 0$ for $n \geq 2$.

2) Equations (3.2.2) follow from Equations (2.3.9) and Equations (3.2.3) follow from Equations (3.1.1) and (3.2.2).

From Equations (2.3.11) and (3.2.3) we obtain

$$\left(\prod_{i=1}^{k} \alpha_i\right)\left(\prod_{j=2}^{n+k+1} \gamma_j\right) f_{0,0} = b_n\left\{\left(\prod_{i=1}^{k} \alpha_i\right)\left(\prod_{j=2}^{n+k} \gamma_j\right) f_{0,0} + \left(\prod_{i=1}^{k+1} \alpha_i\right)\left(\prod_{j=2}^{n+k+1} \gamma_j\right) f_{0,0}\right\}$$
$$- \lambda_n\left(\prod_{i=1}^{k+1} \alpha_i\right)\left(\prod_{j=2}^{n+k} \gamma_j\right) f_{0,0}$$

for $n \geq 1$ and $k \geq 0$. This becomes

$$\gamma_{n+k}(b_n^{-1} - \alpha_k) + \gamma_n\alpha_k - 1 = 0 \qquad (1 \leq k \leq m;\ n \geq 1),$$

where m is the smallest integer such that $\alpha_m = 0$.

3) Because $\{A_n(x)\}_{n=0}^{\infty}$ is in Σ it follows that $\{A_n(x)\}_{n=0}^{\infty}$ is orthogonal on $(-1,1)$ with respect to some weight function $w(x)$. Therefore, by Lemma (1.1.1) $\{\gamma_n(4b_{n-1})^{-1}\}_{n=2}^{\infty}$ is a chain sequence.

4) Because $w(\cos\theta)$ belongs to $L'(0,\pi)$ and

$$\frac{2}{\pi}\int_0^{\pi} w(\cos\theta)\sin(2k+1)\theta\, d\theta = \left(\prod_{i=1}^{k} \alpha_i\gamma_{i+1}\right) f_{0,0}$$

therefore by the Riemann-Lebesgue Lemma

$$\lim_{k\to\infty} \prod_{i=1}^{k} \alpha_i \gamma_{i+1} = 0,$$

Q.E.D.

In order to find all the polynomial sets in Σ we will find sequences $\{\gamma_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ and $\{\alpha_n\}_{n=0}^{\infty}$ which satisfy Equations (3.2.4). We make an investigation into some of their properties in the next section. In the last section of this chapter we find those that give rise to elements in Σ.

3.3 PROPERTIES OF A CLASS OF SEQUENCES. From Equations (3.1.1) and (3.2.3) we see that if m is the smallest integer such that $\alpha_m = 0$, then for all $k \geq m$ and for all $n \geq 0$, $f_{k,n} = 0$. This is the case when the infinite sum in Relation (3.1.3) reduces to a finite sum. That is,

$$A_n(x)w(x) = \sqrt{1 - x^2} \sum_{k=0}^{m} f_{k,n} U_{n+2k}(x).$$

An example of this is the Ultraspherical polynomial set of integer order.* Thus the values α_k for $k > m$ are arbitrary and do not influence relation (3.1.3) since we have $f_{k,n} = 0$ for $k > m$. Consequently we may assign convenient values for α_k $(k > m)$ and this we shall do so that relation (3.2.4) can be extended for all k. Towards this end we put

*See Appendix II.

$$\bar{\alpha}_k = \begin{cases} \alpha_k & (1 \le k \le m) \\ \frac{1}{b_1} - \frac{1}{b_{k-m+1}} & (m < k), \end{cases} \tag{3.3.1}$$

where m is the smallest integer such that $f_{m,0} = 0$ or $\alpha_m = 0$. By extending $\{\alpha_k\}_{k=1}^{m}$ in this manner we will show that it does not change the class of sequences that satisfy Equations (3.2.4).

Theorem (3.3.1). *Let* m *be the smallest integer such that* $f_{m,0} = 0$ *and* $m > 1$. *If* $\{b_k\}_{k=1}^{\infty}$, $\{\bar{\alpha}_k\}_{k=1}^{\infty}$ *and* $\{\gamma_k\}_{k=1}^{\infty}$ *satisfies Equation* (3.2.4) *and* $\{\bar{\alpha}_k\}_{k=1}^{\infty}$ *is defined by Equation* (3.3.1) *then*

$$\bar{\alpha}_k = \frac{1}{b_1} - \frac{1}{\gamma_{k+1}} \qquad (k \ge 1) \tag{3.3.2}$$

$$\frac{1}{b_k} = \frac{1}{b_1} + \frac{\gamma_k}{\gamma_2}\left(\frac{1}{b_2} - \frac{1}{b_1}\right) \qquad (k \ge 1) \tag{3.3.3}$$

$$\begin{cases} \gamma_{n+k}\left(\frac{1}{b_n} - \bar{\alpha}_k\right) + \gamma_n\bar{\alpha}_k - 1 = 0 \\ \text{for (i) } n = 1, 2;\ k \ge 1 \qquad :\text{(ii) } k = 1;\ n \ge 1. \end{cases} \tag{3.3.4}$$

Proof: By letting $n = 1, 2$ in Equations (3.2.4) and recalling that $\gamma_1 = 0$ we obtain

$$\bar{\alpha}_k = \frac{1}{b_1} - \frac{1}{\gamma_{k+1}} \qquad (1 \le k \le m) \tag{3.3.5}$$

and

$$\gamma_{k+2}\left(\frac{1}{b_2} - \bar{\alpha}_k\right) + \gamma_2\bar{\alpha}_k - 1 = 0 \qquad (1 \le k \le m). \tag{3.3.6}$$

By using Equations (3.3.5) to substitute for $\bar{\alpha}_k$ in Equations (3.3.6) we obtain

$$(3.3.7) \qquad \frac{1}{b_1} + \frac{\gamma_n}{\gamma_2}\left(\frac{1}{b_2} - \frac{1}{b_1}\right) = \frac{1}{b_1} - \frac{1}{\gamma_2} + \frac{\gamma_n}{\gamma_2\gamma_{n+1}} - \frac{\gamma_n}{\gamma_{n+1}b_1} + \frac{1}{\gamma_{n+1}}$$

for $2 \le n \le m + 1$.

By letting $k = 1$ in Equations (3.2.4) we obtain

$$(3.3.8) \qquad \gamma_{n+1}\left(\frac{1}{b_n} - \bar{\alpha}_1\right) + \gamma_n\bar{\alpha}_1 - 1 = 0 \qquad (n \ge 1).$$

We use Equations (3.3.5) with $k = 1$ to eliminate $\bar{\alpha}_1$ from Equations (3.3.8) to obtain

$$(3.3.9) \qquad \frac{1}{b_n} = \frac{1}{b_1} - \frac{1}{\gamma_2} - \frac{\gamma_n}{\gamma_{n+1}b_1} + \frac{\gamma_n}{\gamma_{n+1}\gamma_2} + \frac{1}{\gamma_{n+1}} \qquad (n \ge 1).$$

By combining Equations (3.3.7) and (3.3.9) we obtain

$$(3.3.10) \qquad \frac{1}{b_n} = \frac{1}{b_1} + \frac{\gamma_n}{\gamma_2}\left(\frac{1}{b_2} - \frac{1}{b_1}\right) \qquad (1 \le n \le m + 1).$$

In Equations (3.2.4) let $k = m$, we have

$$(3.3.11) \qquad \gamma_{n+m} = b_n \qquad (n \ge 1).$$

Therefore by using Equations (3.3.5), (3.3.1) and (3.3.11) we obtain

$$(3.3.12) \qquad \gamma_{n+k}\left(\frac{1}{b_n} - \bar{\alpha}_k\right) + \gamma_n\bar{\alpha}_k - 1 = 0$$

for $n = 1$, $k \ge 1$. Due to Equations (3.2.4) we obtain Equations (3.3.12) for $k = 1$, $n \ge 1$.

To show that $\{\bar{\alpha}_k\}_{k=1}^{\infty}$, $\{\gamma_k\}_{k=1}^{\infty}$ and $\{b_k\}_{k=1}^{\infty}$ satisfy Equations (3.3.12) for $n = 2$ and $k \ge 1$ is more involved.

We first show that if there exists a smallest integer $m > 1$ such that $\alpha_m = 0$ then $b_1 \neq b_2$. Assume $b_1 = b_2$. From Equations (3.3.10) we obtain $b_m = b_1$ and from Equations (3.3.11) we obtain $\gamma_{m+1} = b_1$. By letting $n = m$ and $k = 1$ in Equations (3.2.4) we obtain

$$\gamma_{m+1} \left(\frac{1}{b_m} - \bar{\alpha}_1\right) + \gamma_m \bar{\alpha}_1 - 1 = 0.$$

That is,

$$(\gamma_m - b_1)\, \bar{\alpha}_1 = 0.$$

Thus $\bar{\alpha}_1 = 0$ or $\bar{\alpha}_{m-1} = 0$. This contradicts the fact that $m > 1$ and is the smallest integer such that $\bar{\alpha}_m = 0$.

By multiplying both sides of the Equations (3.3.9) by γ_{n+1} we obtain

$$\text{(3.3.13)} \qquad \gamma_{n+1} \left(\frac{1}{b_n} - \frac{1}{b_1} + \frac{1}{\gamma_2}\right) + \gamma_n \left(\frac{1}{b_1} - \frac{1}{\gamma_2}\right) - 1 = 0 \qquad (n \geq 1).$$

Next multiply both sides of this equation by $\left(\frac{1}{b_1} - \frac{1}{b_2}\right)$ and use Equations (3.3.10) to eliminate γ_{n+1} and γ_n, to obtain

$$\frac{1}{b_{n+1}} - \frac{1}{b_2} = \gamma_2 \left(\frac{1}{b_1} - \frac{1}{b_n}\right)\left(\frac{1}{b_{n+1}} - \frac{1}{\gamma_2}\right) \qquad (1 \leq n \leq m).$$

This may be rewritten in the form

$$\text{(3.3.14)} \qquad b_{n+1} \left(\frac{1}{b_2} - \frac{1}{b_1} + \frac{1}{b_n}\right) + \gamma_2 \left(\frac{1}{b_1} - \frac{1}{b_n}\right) - 1 = 0 \qquad (1 \leq n \leq m).$$

Now use Equation (3.3.11) to obtain

$$\gamma_{n+m+1}\left(\frac{1}{b_2}-\frac{1}{b_1}+\frac{1}{\gamma_{n+m}}\right)+\gamma_2\left(\frac{1}{b_1}-\frac{1}{\gamma_{n+m}}\right)-1=0,\ (1\le n\le m).$$

And finally by Equations (3.3.12) with $n = 1$ and $k = n + m$ we obtain

$$\gamma_{n+m+2}\left(\frac{1}{b_2}-\bar{\alpha}_{n+m}\right)+\gamma_2\bar{\alpha}_{n+m}-1=0 \qquad (0\le n\le m-1).$$

Thus by hypothesis and from these last equations we have that (3.3.12) is satisfied for $n = 2$ and $1 \le k \le 2m - 1$. By applying the above argument with m replaced by $2m - 1$ we obtain the validity of Equations (3.3.10) for $1 \le n \le 2m$ and Equations (3.3.12) for $n = 2$ and $1 \le k \le 3m - 2$. The results now follows by an induction argument and the fact that $m > 1$.

Q.E.D.

For future reference we state the following obvious Corollaries.

Corollary I. *If* $\{b_k\}_{k=1}^{\infty}$, $\{\bar{\alpha}_k\}_{k=1}^{\infty}$ *and* $\{\gamma_k\}_{k=1}^{\infty}$ *satisfy Equation* (3.3.4) *then they also satisfy Equation* (3.3.2) *and* (3.3.3).

Corollary II. *If* $\{b_k\}_{k=1}^{\infty}$, $\{\alpha_k\}_{k=1}^{\infty}$ *and* $\{\gamma_k\}_{k=1}^{\infty}$ *satisfy*

$$\gamma_{n+k}\left(\frac{1}{b_n}-\alpha_k\right)+\gamma_n\alpha_k-1=0 \qquad (n=1,\,2;\ 1\le k)$$

and

$$\frac{1}{b_n} = \frac{1}{b_1} + \frac{\gamma_n}{\gamma_2}\left(\frac{1}{b_2} - \frac{1}{b_1}\right) \qquad (n = 1, 2 \ldots)$$

then $\{b_k\}_{k=1}^{\infty}$, $\{\alpha_k\}_{k=1}^{\infty}$ *and* $\{\gamma_k\}_{k=1}^{\infty}$ *satisfy Equation* (3.3.4).

If $m = 1$, that is $\bar{\alpha}_1 = 0$, and $\gamma_{n+1} = b_n$, then $\lambda_n = b_n b_{n-1}$. It is easy to show that $\{A_n(x)\}_{n=0}^{\infty}$ in this case, is equivalent to $\{U_n(x)\}_{n=0}^{\infty}$. Actually $A_n(x) = (\prod_{i=1}^{n} b_i)\, U_n(x)$.

From Theorem (3.3.1) it follows that regardless of whether or not there exists an integer m such that $f_{m,0} = 0$, if $\{\gamma_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ and $\{\alpha_n\}_{n=1}^{\infty}$ satisfy Equations (3.2.4) then they satisfy Equations (3.3.4). It turns out that we can solve Equations (3.3.4). Before we do this, we show that if $\{\gamma_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ and $\{\alpha_n\}_{n=1}^{\infty}$ satisfy Equation (3.3.4), then they also satisfy

$$\text{(3.3.15)} \qquad \gamma_{n+k}\left(\frac{1}{b_n} - \bar{\alpha}_k\right) + \gamma_n\bar{\alpha}_k - 1 = 0 \qquad (n \geq 1,\ k \geq 1).$$

First, we need the following lemma.

<u>Lemma (3.3.1)</u>. *Let* $\{\gamma_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ *and* $\{\bar{\alpha}_n\}_{n=1}^{\infty}$ *satisfy Equations* (3.3.4). *If there exists an integer* $s \geq 2$ *such that*

$$\text{(3.3.16)} \qquad \gamma_{n+s-1}\left(\frac{1}{b_n} - \bar{\alpha}_{s-1}\right) + \gamma_n\bar{\alpha}_{s-1} - 1 = 0 \quad (k = s - 1;\ n \geq 1),$$

then

$$\text{(3.3.17)} \qquad (1 - \gamma_n\bar{\alpha}_s)\left(\frac{1}{b_{n+s-1}} - \bar{\alpha}_1\right) = \left(\frac{1}{b_n} - \bar{\alpha}_s\right)(1 - \gamma_{n+s-1}\bar{\alpha}_1) \quad (n \geq 1).$$

Proof: Consider the following identity for $n \geq 1$, and $s \geq 1$

(3.3.18) $$(\gamma_{n+s-1} - \gamma_n)[\frac{\gamma_2 - \gamma_{s+1}}{\gamma_s} + 1 - \frac{\gamma_2}{b_1} - 1]$$

$$= (\gamma_{s+1} - \gamma_2)[\frac{\gamma_n - \gamma_{n+s-1}}{\gamma_s} + 1 - \frac{\gamma_n}{b_1} - 1] + \frac{1}{b_1}[\gamma_{s+1}\gamma_n - \gamma_2\gamma_{n+s-1}].$$

We use Equations (3.3.2) and (3.3.3) to eliminate $\frac{1}{b_n}$ and $\bar{\alpha}_{s-1}$ from Equation (3.3.16) to obtain

(3.3.19) $$\frac{\gamma_{n+s-1}\gamma_n}{\gamma_2}(\frac{1}{b_2} - \frac{1}{b_1}) = \frac{\gamma_n - \gamma_{n+s-1}}{\gamma_s} + 1 - \frac{\gamma_n}{b_1} \qquad (n \geq 1).$$

Next let us use Equation (3.3.19) in Equation (3.3.18) to obtain for $n \geq 1$,

$$(\gamma_{n+s-1} - \gamma_n)[(\frac{1}{b_2} - \frac{1}{b_1})\frac{1}{\gamma_2} - \frac{1}{\gamma_{s+1}\gamma_2}]$$

$$= (\frac{1}{\gamma_2} - \frac{1}{\gamma_{s+1}})[\frac{\gamma_{n+s-1}\gamma_n}{\gamma_2}(\frac{1}{b_2} - \frac{1}{b_1}) - 1] + \frac{1}{b_1}[\frac{\gamma_n}{\gamma_2} - \frac{\gamma_{n+s-1}}{\gamma_{s+1}}].$$

By subtracting $\frac{\gamma_{n+s-1}\gamma_n}{b_1\gamma_2}(\frac{1}{b_2} - \frac{1}{b_1})$ from each side of this equation and rearranging the terms, we obtain for $n \geq 1$,

$$[1 - \gamma_n(\frac{1}{b_1} - \frac{1}{\gamma_{s+1}})][\frac{\gamma_{n+s-1}}{\gamma_2}(\frac{1}{b_2} - \frac{1}{b_1}) + \frac{1}{\gamma_2}]$$

$$= [1 - \gamma_{n+s-1}(\frac{1}{b_1} - \frac{1}{\gamma_2})][\frac{\gamma_n}{\gamma_2}(\frac{1}{b_2} - \frac{1}{b_1}) + \frac{1}{\gamma_{s+1}}].$$

By using Equations (3.3.2) and (3.3.3) we obtain the required results.

Q.E.D.

<u>Theorem (3.3.2)</u>. *If the sequences* $\{\gamma_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ *and* $\{\bar{\alpha}_n\}_{n=1}^{\infty}$ *satisfy Equations* (3.3.4) *and* $b_2 \neq \gamma_2$ *then they satisfy Equations* (3.3.15).

<u>Proof</u>: The proof is by mathematical induction on k. By hypothesis,

$$\gamma_{n+k}\left(\frac{1}{b_n} - \bar{\alpha}_k\right) + \gamma_n\bar{\alpha}_k - 1 = 0 \qquad (n \geq 1,\ k = 1).$$

Assume

$$\gamma_{n+k}\left(\frac{1}{b_n} - \bar{\alpha}_k\right) + \gamma_n\bar{\alpha}_k - 1 = 0 \quad (n \geq 1,\ k = 1, 2 \ldots s - 1),$$

where $s \geq 2$.

Therefore,

$$\gamma_{n+s}\left(\frac{1}{b_{n+s-1}} - \bar{\alpha}_1\right) + \gamma_{n+s-1}\bar{\alpha}_1 - 1 = 0 \qquad (n \geq 1).$$

That is,

$$\gamma_{n+s}\left(\frac{1}{b_{n+s-1}} - \bar{\alpha}_1\right)\left(\frac{1}{b_n} - \bar{\alpha}_s\right) + (\gamma_{n+s-1}\bar{\alpha}_1 - 1)\left(\frac{1}{b_n} - \bar{\alpha}_s\right) = 0 \quad (n \geq 1).$$

By Lemma (3.3.1) this becomes,

$$\left(\frac{1}{b_{n+s-1}} - \bar{\alpha}_1\right)\left[\gamma_{n+s}\left(\frac{1}{b_n} - \bar{\alpha}_s\right) + \gamma_n\bar{\alpha}_s - 1\right] = 0 \qquad (n \geq 1).$$

Therefore for each integer $n \geq 1$,

$$\frac{1}{b_{n+s-1}} - \bar{\alpha}_1 = 0$$

or

$$\gamma_{n+s}\left(\frac{1}{b_n} - \bar{\alpha}_s\right) + \gamma_n \bar{\alpha}_s - 1 = 0.$$

Assume there exists an integer m such that

$$\frac{1}{b_{m+s-1}} - \bar{\alpha}_1 = 0.$$

By using Equations (3.3.2) and (3.3.3) we obtain

$$\frac{\gamma_{m+s-1}}{\gamma_2}\left(\frac{1}{b_2} - \frac{1}{b_1}\right) + \frac{1}{\gamma_2} = 0. \tag{3.3.20}$$

By hypothesis we have

$$\gamma_{2+k}\left(\frac{1}{b_2} - \bar{\alpha}_k\right) + \gamma_2 \bar{\alpha}_k - 1 = 0 \qquad (k \geq 1)$$

and

$$\bar{\alpha}_k = \frac{1}{b_1} - \frac{1}{\gamma_{k+1}} \qquad (k \geq 1).$$

Use the latter to eliminate $\bar{\alpha}_k$ from the former to obtain

$$\gamma_{k+2}\left[\gamma_{k+1}\left(\frac{1}{b_2} - \frac{1}{b_1}\right) + 1\right] = \gamma_{k+1}\left[1 - \frac{\gamma_2}{b_1}\right] + \gamma_2$$

for $k \geq 1$. Thus from Equation (3.3.20) we obtain

$$\gamma_{m+s-1}\left(\frac{1}{\gamma_2} - \frac{1}{b_1}\right) + 1 = 0$$

By comparing this with Equation (3.3.20) and recalling that $\gamma_{m+s-1} \neq 0$ we obtain $\gamma_2 = b_2$, which contradicts the hypothesis.

Therefore,

$$\gamma_{n+s}\left(\frac{1}{b_n} - \bar{\alpha}_s\right) + \gamma_n\bar{\alpha}_s - 1 = 0 \qquad (n \geq 1).$$

Q.E.D.

We next find explicitly all sets of sequences $\{\gamma_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, $\{\bar{\alpha}_n\}_{n=1}^{\infty}$ that satisfy Equations (3.3.4) and thus by Theorem (3.3.2) in the case $\gamma_2 \neq b_2$ they must satisfy Equations (3.3.15). In Theorem (3.3.5) we find all sequence that satisfy Equations (3.3.4) when $\gamma_2 = b_2$. By direct substitution it is easy to show that they satisfy Equations (3.3.15).

<u>Theorem (3.3.3)</u>. *If the sequences* $\{\gamma_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ *and* $\{\bar{\alpha}_n\}_{n=1}^{\infty}$ *satisfy Equations* (3.3.4) *and* $b_1 = b_2$ *then,*

(a) *the following equations hold:*

(3.3.21) $$\gamma_n = b_1\left[1 - (1 - \gamma_2(b_1)^{-1})^{n-1}\right] \qquad (n = 1, 2 \ldots),$$

(3.3.22) $$b_n = b_1 \qquad (n = 1, 2 \ldots),$$

(3.3.23) $$\bar{\alpha}_n = (1 - \gamma_2(b_1)^{-1})^n\{b_1[(1 - \gamma_2(b_1)^{-1})^n - 1]\}^{-1} \quad (n = 1, 2 \ldots),$$

(b) *these sequences satisfy Equations* (3.3.15).

<u>Proof:</u> (a) In Equations (3.3.4) let $n = 2$ to obtain

$$\gamma_{k+2}\left(\frac{1}{b_2} - \bar{\alpha}_k\right) + \gamma_2\bar{\alpha}_k - 1 = 0 \qquad (k \geq 1).$$

We eliminate $\bar{\alpha}_k$ from these Equations by using Equations (3.3.2) and using the fact that $b_1 = b_2$ to obtain

$$(3.3.24) \qquad \gamma_{k+2} + (\gamma_2(b_1)^{-1} - 1)\, \gamma_{k+1} = \gamma_2 \qquad (k \geq 0).$$

This is a first order non-homogeneous finite difference equation with constant coefficients. A particular solution is $\gamma_k = b_1$ $k \geq 2$, $\gamma_1 = 0$. The general solution of the corresponding homogeneous equation is,

$$\gamma_k = A\,(1 - \frac{\gamma_2}{b_1})^{k-2}\gamma_2 \qquad (k \geq 2).$$

Thus the general solution of Equations (3.3.24) is

$$\gamma_k = b_1 + A\,(1 - \frac{\gamma_2}{b_1})^{k-2}\gamma_2 \qquad (k \geq 2).$$

By applying the boundary conditions we obtain

$$\gamma_k = b_1\,[1 - (1 - \frac{\gamma_2}{b_1})^{k-1}] \qquad (k \geq 1).$$

By using Equations (3.3.2) we obtain (3.3.23), and by using Equations (3.3.3) and the fact that $b_1 = b_2$ we obtain Equations (3.3.22).

(b) By Theorem (3.3.2) this set of sequences satisfy Equations (3.3.15) if $b_2 \neq \gamma_2$. If $b_2 = \gamma_2$ then $b_1 = \gamma_2$ and from Equations (3.3.23) $\alpha_n = 0$ for all $n \geq 1$ and $\gamma_n = b_1$ for all $n \geq 1$. By direct substitution we see that $\alpha_n = 0$ $n \geq 1$; $b_n = b_1$ $n \geq 1$; $\gamma_n = b_1$ $n \geq 2$; satisfied Equations (3.3.15).

Q.E.D.

Theorem (3.3.4). *If the sequences* $\{\gamma_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ and $\{\bar{\alpha}_n\}_{n=1}^{\infty}$ *satisfy the hypotheses of Theorem* (3.3.1) *and* $2b_1 = \gamma_2$ *then* (1) *either*

(a) $\gamma_n = 2b_1$, $b_n = 2b_1$ *for all* $n \geq 2$ *and* $\bar{\alpha}_n = (2b_1)^{-1}$ *for all* $n \geq 1$ *or,*

(b) *there exists a smallest integer* $r > 2$ *such that* $\gamma_r \neq 2b_1$ *and*

(3.3.25) $$\gamma_{kr-k+1} = \frac{2kb_1\gamma_r}{2b_1+(k-1)\gamma_r} \qquad (k = 1, 2, 3 \ldots),$$

(3.3.26) $$\gamma_n = 2b_1 \qquad (n \geq 2;\ n \neq kr - k + 1,\ k = 1, 2, 3 \ldots),$$

(3.3.27) $$b_{kr-k+1} = 2b_1 \left[\frac{2b_1-\gamma_r+k\gamma_r}{2(2b_1-\gamma_r)+k\gamma_r}\right] \qquad (k = 1, 2 \ldots),$$

(3.3.28) $$b_n = 2b_1 \qquad (n \geq 2;\ n \neq kr - k + 1,\ k = 1, 2 \ldots),$$

(3.3.29) $$\bar{\alpha}_{kr-k} = \frac{k\gamma_r-2b_1+\gamma_r}{2b_1k\gamma_r} \qquad (k = 1, 2 \ldots),$$

(3.3.30) $$\bar{\alpha}_n = \frac{1}{2b_1} \qquad (n \geq 1;\ n \neq kr - k,\ k = 1, 2 \ldots).$$

(2) *These sequences satisfy Equations* (3.3.15).

Proof: It is easy to show by direct substitution that $\gamma_n = 2b_1$, $b_n = 2b_1$ $n \geq 2$ and $\bar{\alpha}_n = (2b_1)^{-1}$ $n \geq 1$ satisfy Equations (3.2.4). Thus we have (1a).

(1b) Let r be the smallest integer such that $\gamma_r \neq 2b_1$. We use

$$\bar{\alpha}_k = \frac{1}{b_1} - \frac{1}{\gamma_{k+1}} \qquad (k \geq 1), \tag{3.3.31}$$

to eliminate γ_{k+2} from Equations (3.3.4) with $n = 2$ to obtain

$$\frac{b_1}{1-b_1\bar{\alpha}_{k+1}} \left(\frac{1}{b_2} - \bar{\alpha}_k\right) + \gamma_2\bar{\alpha}_k - 1 = 0 \qquad (k \geq 1). \tag{3.3.32}$$

By letting $k = 1$ in the Equations (3.3.31) and (3.3.32) and using the fact that $2b_1 = \gamma_2$ we get

$$\gamma_2 = b_2 = 2b_1 = (\alpha_1)^{-1}. \tag{3.3.33}$$

In Equations (3.2.4) let $k = 1$ to obtain

$$\gamma_{n+1} \left(\frac{1}{b_n} - \bar{\alpha}_1\right) + \gamma_n\bar{\alpha}_1 - 1 = 0 \qquad (n \geq 1).$$

By using Equations (3.3.3) we can eliminate $(b_n)^{-1}$ from these Equations to obtain

$$\left(1 - \frac{\gamma_n}{2b_1}\right)\left(1 - \frac{\gamma_{n+1}}{2b_1}\right) = 0 \qquad (n \geq 1).$$

Therefore for each integer $n \geq 1$ $\gamma_n = 2b_1$ or $\gamma_{n+1} = 2b_1$.

Because $r - 1 \leq m$, we have from Equations (3.2.4)

$$\gamma_{n+r-1} \left(\frac{1}{b_n} - \bar{\alpha}_{r-1}\right) + \gamma_n\bar{\alpha}_{r-1} - 1 = 0 \qquad (n \geq 1).$$

By the fact that $\gamma_n = 2b_1$ for $2 \leq n < r$ we obtain

$$\left(\frac{\gamma_{n+r-1}}{2b_1} - 1\right) \left(\frac{2b_1}{\gamma_r} - 1\right) = 0 \qquad (2 \leq n < r).$$

By hypothesis $\gamma_r \neq 2b_1$, therefore $\gamma_n = 2b_1$ for $n \neq r$ and $2 \leq n < 2r - 1$. It is easy to prove by mathematical induction on k that $\gamma_n = 2b_1$ for $n \geq 2$ and $n \neq kr - k + 1$, $k = 1, 2 \ldots$. Next we consider the case when $n = kr - k + 1$ for $k = 1, 2, 3 \ldots$. When we replace k by $r - 1$ and n by $kr - k + 1$ in Equations (3.2.4) we obtain

$$\gamma_{(k+1)r-k}\left(\frac{1}{b_{kr-k+1}} - \alpha_{r-1}\right) + \alpha_{r-1}\gamma_{kr-k+1} - 1 = 0 \quad (k = 1, 2 \ldots).$$

We use Equations (3.3.31) and (3.3.3) to obtain

$$(3.3.34)\quad \gamma_{(k+1)r-k}\left(\frac{1}{\gamma_r} - \frac{\gamma_{kr-k+1}}{4b_1^2}\right) + \left(\frac{1}{b_1} - \frac{1}{\gamma_r}\right)\gamma_{kr-k+1} - 1 = 0 \quad (k = 1, 2 \ldots).$$

If $\gamma_r^{-1} - \gamma_{kr-(k-1)}(2b_1)^{-1} = 0$ then it follows from Equations (3.3.34) that $\gamma_r = 2b_1$. But this contradicts the hypothesis, therefore

$$\frac{1}{\gamma_r} - \frac{\gamma_{kr-k+1}}{4b_1^2} \neq 0 \quad (k = 1, 2 \ldots).$$

By letting $\beta_k = \frac{1}{\gamma_r} - \frac{\gamma_{kr-(k-1)}}{4b_1^2}$ we obtain from Equations (3.3.34)

$$\beta_{k+1} = 2\left(\frac{1}{\gamma_r} - \frac{1}{2b_1}\right) - \frac{\left(\frac{1}{\gamma_r} - \frac{1}{2b_1}\right)^2}{\beta_k} \quad (k = 1, 2 \ldots).$$

By Appendix I Lemma (A.1.1) we thus have

$$\beta_k = \frac{((\gamma_r)^{-1} + k(2b_1)^{-1})((\gamma_r)^{-1} - (2b_1)^{-1})}{(\gamma_r)^{-1} + (k-1)(2b_1)^{-1}} \quad (k = 1, 2 \ldots).$$

Therefore,

$$\gamma_{kr-k+1} = \frac{2kb_1\gamma_r}{2b_1+(k-1)\gamma_r} \quad (k = 1, 2 \ldots).$$

By direct substitution of this in Equations (3.3.2) and (3.3.3) we obtain the required results.

(2) It may be shown by direct substitution that these sequences satisfy Equation (3.3.15).

Q.E.D.

It is interesting to note that in the case $\gamma_2 = 2b_1$, and there exists a smallest integer $r \geq 3$ such that $\gamma_r \neq 2b_1$, Equations (3.3.4) are not sufficient to uniquely define a set of sequences in terms of b_1, b_2, γ_2, and γ_r. To see this, let us try and find γ_{r+2} by using only Equations (3.3.4). We only have the following two equations

$$\gamma_{r+2}(b_2^{-1} - \alpha_r) + \alpha_r\gamma_2 - 1 = 0$$

and

$$\gamma_{r+2}(b_{r+1}^{-1} - \alpha_1) + \gamma_{r+1}\alpha_1 - 1 = 0$$

By using the results of Theorem (3.3.4) we obtain

$$\frac{1}{b_2} - \alpha_r = \frac{1}{b_{r+1}} - \alpha_1 = 0.$$

Thus γ_{r+2} is not uniquely determined by Equations (3.3.4). This is the only case we find in which the sequences that satisfy Equation (3.3.4) cannot be written in a closed form in terms of b_1, b_2 and γ_2.

<u>Theorem (3.3.5)</u>. (a) *If the real sequences* $\{\bar{\alpha}_n\}_{n=1}^{\infty}$, $\{\gamma_n\}_{n=1}^{\infty}$ *and* $\{b_n\}_{n=1}^{\infty}$ *satisfy Equations* (3.3.4) *and* $\gamma_2 \neq 2b_1$, $b_1 \neq b_2$ *and* $\gamma_2 = b_2$ *then*

(3.3.35) $$\gamma_n = b_2 \qquad (n \geq 2)$$

(3.3.36) $$b_n = b_2 \qquad (n \geq 2)$$

(3.3.37) $$\bar{\alpha}_n = \frac{1}{b_1} - \frac{1}{b_2} \qquad (n \geq 1).$$

(b) *These sequences satisfy Equations* (3.3.15).

<u>Proof</u>: Assume not all γ_n are equal to b_2. Let s be the smallest integer such that $\gamma_s \neq b_2$. By hypothesis $s \geq 3$. In Equations (3.3.4) let $k = s - 2$ and $n = 2$ to obtain

$$\gamma_s(\frac{1}{b_2} - \bar{\alpha}_{s-2}) + \gamma_2\bar{\alpha}_{s-2} - 1 = 0.$$

We eliminate $\bar{\alpha}_{s-2}$ from this Equation by using Equations (3.3.2) to obtain

$$\gamma_s(\frac{1}{b_2} - \frac{1}{b_1} + \frac{1}{\gamma_{s-1}}) + \gamma_2(\frac{1}{b_1} - \frac{1}{\gamma_{s-1}}) - 1 = 0.$$

$\gamma_{s-1} = b_2$ by hypothesis. Therefore,

$$(\frac{2}{b_2} - \frac{1}{b_1})(\gamma_s - b_2) = 0$$

Therefore $2b_1 = b_2$ or $\gamma_s = b_2$. The former contradicts the hypothesis and latter contradicts the assumption. Therefore

$\gamma_n = b_2$ for all $n \geq 2$. Equations (3.3.36) and (3.3.37) follow directly from Equations (3.3.2) and (3.3.3).

(b) By direct substitution it is easily seen that these sequences satisfy Equations (3.3.15).

Q.E.D.

Now we have from Theorems (3.3.2), (3.3.3), (3.3.4) and (3.3.5) the following,

<u>Theorem (3.3.6)</u>. *If* $\{\gamma_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ *and* $\{\bar{\alpha}_n\}_{n=1}^{\infty}$ *satisfy Equations* (3.3.4), *then they satisfy Equations* (3.3.15).

We have thus shown that the set of solutions of Equations (3.3.15) and the set of solutions of Equations (3.3.4) are the same. Since Theorem (3.3.1) shows that given any solution $\{\alpha_n\}_{n=1}^{m}$, $\{b_n\}_{n=1}^{\infty}$, $\{\gamma_n\}_{n=1}^{\infty}$, where m is the smallest integer such that $\alpha_m = 0$, of Equations (3.2.4), then $\{\bar{\alpha}_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, $\{\gamma_n\}_{n=1}^{\infty}$ is a solution of Equation (3.3.4). Therefore, $\{\bar{\alpha}_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, $\{\gamma_n\}_{n=1}^{\infty}$ is a solution of Equation (3.3.15). But it is clear that the set of solutions of Equations (3.3.15) is a subset of the set of solutions of Equations (3.2.4). It follows that to find all solutions of Equations (3.2.4), we need only find all solutions $\{\bar{\alpha}_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, $\{\gamma_n\}_{n=1}^{\infty}$ of Equations (3.3.4).

Next we consider the case when $2b_1 \neq \gamma_2$, $b_1 \neq b_2$ and $\gamma_2 \neq b_2$. Let $\{h_n\}_{n=1}^{\infty}$ be a sequence of numbers satisfying the equations

$$h_n = 1 - \frac{c}{h_{n-1}} \qquad (n = 2, 3 \ldots) \tag{3.3.38}$$

where

$$c = b_1^2(b_2 - \gamma_2)(b_2(2b_1 - \gamma_2)^2)^{-1}. \tag{3.3.39}$$

<u>Theorem (3.3.7)</u>. *Let* $\{\alpha_k\}_{k=1}^{\infty}$, $\{\gamma_k\}_{k=1}^{\infty}$, $\{b_k\}_{k=1}^{\infty}$ *satisfy Equations* (3.3.4) *with* $2b_1 \neq \gamma_2$.

(i) *If* $\gamma_2 \neq b_2$, *then*

$$\alpha_n = \frac{1}{\gamma_2} - \frac{h_n(2b_1 - \gamma_2)}{b_1\gamma_2} \qquad (n = 1, 2 \ldots) \tag{3.3.40}$$

where $\{h_n\}_{n=1}^{\infty}$ *satisfies Equation* (3.3.38) *with* $h_1 = 1$.

(ii) *If* $b_1 \neq b_2$ *and* $\gamma_2 \neq b_2$, *then*

$$\gamma_k = \frac{h_k b_2(2b_1 - \gamma_2) - b_1 b_2}{b_1 - b_2} \qquad (k = 1, 2 \ldots) \tag{3.3.41}$$

where $\{h_n\}_{n=1}^{\infty}$ *satisfies Equation* (3.3.38) *with* $h_1 = b_1(2b_1 - \gamma_2)^{-1}$ *and*

$$b_k = \frac{h_k b_2(2b_1 - \gamma_2) - b_1 b_2}{b_1 - b_2} \tag{3.3.42}$$

where $\{h_n\}_{n=1}^{\infty}$ *satisfies Equations* (3.3.38) *with* $h_1 = b_1^2(b_2(2b_1 - \gamma_2))^{-1}$.

<u>Proof</u>: i) In Equations (3.3.4) let $n = 2$ to obtain

$$\gamma_{k+2}\left(\frac{1}{b_2} - \alpha_k\right) + \gamma_2\alpha_k - 1 = 0 \qquad (k \geq 1). \tag{3.3.43}$$

We eliminate γ_{k+2} from these Equations by using Equations (3.3.2) to obtain

$$b_2^{-1} - \alpha_k + (\gamma_2\alpha_k - 1)(b_1^{-1} - \alpha_{k+1}) = 0 \qquad (k \geq 1).$$

or

$$\alpha_{k+1}(1 - \gamma_2\alpha_k) + \alpha_k\left(\frac{\gamma_2}{b_1} - 1\right) + \frac{1}{b_2} - \frac{1}{b_1} = 0 \qquad (k \geq 1). \tag{3.3.44}$$

We see from these Equations that if there exists a positive integer m such that $1 - \gamma_2\alpha_m = 0$ then $\gamma_2 = b_2$. This contradicts the hypothesis. Therefore, for all integers k, $1 - \gamma_2\alpha_k \neq 0$. If we let $\beta_k = 1 - \gamma_2\alpha_k$, then Equations (3.3.44) become

$$\frac{\beta_{k+1}}{2-\gamma_2(b_1)^{-1}} = 1 - \cfrac{c}{\cfrac{\beta_k}{2-\gamma_2(b_1)^{-1}}} \qquad (k \geq 1).$$

By letting $h_k = b_1(2b_1 - \gamma_2)^{-1}\beta_k$ we obtain the required results.

ii) In Equations (3.3.43) eliminate α_k by using Equations (3.3.2) to obtain

$$\gamma_{k+2}[\gamma_{k+1}(b_2^{-1} - b_1^{-1}) + 1] = \gamma_{k+1}(1 - \gamma_2 b_1^{-1}) + \gamma_2 \quad (k \geq 1).$$

Because $b_1 \neq b_2$ we can rearrange this and write it as

$$\left(\gamma_{k+2} + \frac{b_1b_2}{b_1-b_2}\right)\left(\gamma_{k+1} + \frac{b_1b_2}{b_1-b_2}\right) \qquad (k \geq 1)$$

$$= \left(\gamma_{k+1} + \frac{b_1b_2}{b_1-b_2}\right)\frac{(2b_1-\gamma_2)b_2}{(b_1-b_2)} + \frac{b_1^2b_2(\gamma_2-b_2)}{(b_1-b_2)^2}$$

From these Equations we note that if there exists an m such that $\gamma_{m+1} + \dfrac{b_1b_2}{b_1-b_2} = 0$ then $\gamma_2 = b_2$ but this contradicts the hypothesis. If we let

$$\beta_k = \gamma_k + b_1b_2(b_1-b_2)^{-1} \qquad (k \geq 1),$$

we obtain

$$\beta_{k+2} = b_2(2b_1-\gamma_2)(b_1-b_2)^{-1} - \frac{b_1^2b_2(b_2-\gamma_2)(b_1-b_2)^{-2}}{\beta_{k+1}} \qquad (k \geq 1).$$

By hypothesis $\gamma_2 \neq 2b_1$, thus, if we let $h_k = \beta_k(b_1-b_2)[b_2(2b_1-\gamma_2)]^{-1}$ we obtain the required results.

From Equation (3.3.4) with $k = 1$ we obtain

$$\gamma_{n+1}(b_n^{-1}-\alpha_1) + \gamma_n\alpha_1 - 1 = 0$$

for $n \geq 1$. We next eliminate γ_{n+1} and γ_n from these equations by using Equations (3.3.3) to obtain

$$b_{n+1}[b_n(b_2^{-1}-b_1^{-1}) + 1] = b_n[1 - \gamma_2b_1^{-1}] + \gamma_2 \qquad (n \geq 1).$$

The results now follow by an argument similar to that used for $\{\gamma_n\}_{n=1}^{\infty}$.

Q.E.D.

From Theorem (3.3.7) and Lemma (A.1.1) or (A.1.2) it is easy to solve Equations (3.3.4) in terms of b_1, b_2 and γ_2 in the case where $2b_1 \neq \gamma_2$, $\gamma_2 \neq b_2$ and $b_1 \neq b_2$ are satisfied simultaneously. From Theorem (3.3.2) these sequences must satisfy Equations (3.3.15).

In table (3.3.1) on page 59 we give all possible solutions of Equations (3.3.15).

We are now interested in conditions that will tell us whether or not there exists a positive integer m such that $\alpha_m = 0$. From Theorem (3.3.3) we see that if $b_1 = b_2$, then there exists an integer m such that $\alpha_m = 0$ if and only if $\gamma_2 = b_1$. In this case $\gamma_n = b_1$, $b_n = b_1$, and $\alpha_n = 0$ for $n = 2, 3 \ldots$. For the case $2b_1 = \gamma_2$ we see from Theorem (3.3.4) that $\alpha_m \neq 0$ if and only if there exists a positive integer r and a real number γ_r such that $2b_1/\gamma_r$ is a positive integer. In this case $\{\gamma_n\}_{n=2}^{\infty}$, $\{b_n\}_{n=2}^{\infty}$ and $\{\alpha_n\}_{n=1}^{\infty}$ are given by Equations (3.3.25) to (3.3.30). It is obvious from Theorem (3.3.5) that if $\gamma_2 = b_2$, $\gamma_2 \neq 2b_1$ and $b_1 \neq b_2$, then there does not exist a positive integer m such that $\alpha_m = 0$.

<u>Theorem (3.3.8)</u>. *Let* $\{\alpha_k\}_{k=1}^{\infty}$, $\{\gamma_k\}_{k=1}^{\infty}$ *and* $\{b_k\}_{k=1}^{\infty}$ *satisfy Equations* (3.3.4) *and* c *be as defined by equation* (3.3.39). *If* $0 < 2b_1 < \gamma_2$, $0 < b_2 \neq \gamma_2$ *and* $c < 1/4$, *then there does not exist an integer* m *such that* $\alpha_m = 0$.

<u>Proof</u>: From Theorem (3.3.7) part (i) it follows that

$$\alpha_n = \frac{1}{\gamma_2} - \frac{h_n(2b_1 - \gamma_2)}{b_1\gamma_2} \qquad (n = 1, 2 \ldots)$$

where $\{h_n\}_{n=1}^{\infty}$ satisfies Equation (3.3.38) with $h_1 = 1$. Therefore, by Corollary I of Lemma (A.1.3) of Appendix I it follows that $h_n > 0$ for all $n \geq 1$. But by hypothesis $2b_1 - \gamma_2 < 0$. Thus $\alpha_n > \frac{1}{\gamma_2} > 0$ for $n = 1, 2 \ldots$.

Q.E.D.

In table (3.3.2) on page 60 we indicate the solutions obtained for all possible combinations of γ_2, b_2 and b_1. Also this table exhibits conditions for determining whether or not there exists a positive integer m such that $\alpha_m = 0$. We next wish to study the properties of these sequences in order to determine which of them give rise to polynomial sets in Σ.

<u>Theorem (3.3.9)</u>. *If* $\{\gamma_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, $\{\alpha_n\}_{n=1}^{\infty}$ *satisfy Equation* (3.3.4) *with* $b_1 \neq b_2$, $\gamma_2 \neq 2b_1$, $\gamma_2 \neq b_2$, $\alpha_1 \neq 0$ *and* $c > \frac{1}{4}$ *where* c *is given by Equation* (3.3.39), *then either there exists an integer* q *such that* $\gamma_q = 0$ *or* $\{\frac{\gamma_n}{b_{n-1}}\}_{n=2}^{\infty}$ *is an unbounded sequence.*

<u>Proof</u>: From Theorem (3.3.7) part 1, Equation (3.3.2) and Lemma (A.1.2) part (a) we obtain

TABLE (3.3.1)

ALL POSSIBLE SOLUTIONS OF EQUATIONS (3.2.4)

Ref. No.	Remarks	γ_n	b_n	$\bar{a}_n$
1	See Th. (3.3.3)	$b_1\,[1-(1-\gamma_2 b_1^{-1})^{n-1}]$	b_1	$\dfrac{(1-\gamma_2 b_1^{-1})^n}{b_1[(1-\gamma_2 b_1^{-1})^n-1]}$
2	See Th. (3.3.4)	$2b_1$ if $n \geq 2$	$2b_1$ if $n \geq 2$	$(2b_1)^{-1}$
3	For all integers $r > 2$ and γ_r real. See Th. (3.3.4)	If $n = kr - k + 1$, then $2kb_1\gamma_r(2b_1+(k-1)\gamma_r)^{-1}$; Otherwise $2b_1$	If $n = kr - k + 1$, then $2b_1[2b_1-\gamma_r+k\gamma_r][2(2b_1-\gamma_r)+k\gamma_r]^{-1}$; otherwise $2b_1$	If $n = kr - k$, then $[(k+1)\gamma_r-2b_1](2b_1k\gamma_r)^{-1}$; otherwise $(2b_1)^{-1}$
4	See Th. (3.3.5)	b_2 if $n \geq 2$	b_2 if $n \geq 2$	$\dfrac{1}{b_1}-\dfrac{1}{b_2}$
5	See Th. (3.3.7) and Th. (3.3.1)	$\dfrac{2(n-1)\gamma_2 b_2}{(n-2)\gamma_2+2b_1}$	$\dfrac{b_1 b_2[\gamma_2(n-2)+2b_1]}{(n-2)b_2(\gamma_2-2b_1)+2(n-1)b_1^2}$	$\dfrac{(n-1)\gamma_2-2b_1}{2nb_1\gamma_2}$
6	Let $h_n = \dfrac{(1+d)^n+(1-d)^n}{2\{(1+d)^{n-1}+(1-d)^{n-1}\}}$ where $d = \left[\dfrac{\gamma_2^2-4b_1\gamma_2(1-\frac{b_1}{b_2})}{(2b_1-\gamma_2)^2}\right]^{1/2}$ See Theorems (3.3.1), (3.3.7) and (A.1.1).	$\dfrac{\gamma_2 b_1}{(1-h_n)\gamma_2-b_1(1-2h_n)}$	$\dfrac{b_1 b_2[\gamma_2(1-h_n)-b_1(1-2h_n)]}{(1-h_n)b_2(\gamma_2-2b_1)+b_1^2}$	$\dfrac{b_1(1-2h_{n+1})+h_{n+1}\gamma_2}{\gamma_2 b_1}$

TABLE (3.3.2)

HOW THE SOLUTIONS OF $\gamma_{n+k}(b_n^{-1}-\alpha_k) + \alpha_k\gamma_n - 1 = 0$ ARISE

								Ref. No. for γ_n, b_n and α_n in Table 3.3.1
$b_1=\gamma_2$	$\alpha_m = 0$, for $m = 1, 2, 3 \ldots$							1
$b_1\neq\gamma_2$	$b_1=b_2$	$\nexists m$ s.t. $\alpha_m = 0$						1
	$b_1\neq b_2$	$2b_1=\gamma_2$	$\exists r$ s.t. $\gamma_r\neq 2b_1$	$2b_1(\gamma_r)^{-1}$ is an integer	$\exists m$ s.t. $\alpha_m = 0$			3
				$2b_1(\gamma_r)^{-1}$ is not an integer	$\exists m$ s.t. $\alpha_m = 0$			3
			$\forall r\geq 2$, $\gamma_r=2b_1$	$\nexists m$ s.t. $\alpha_m = 0$				2
		$2b_1\neq\gamma_2$	$\gamma_2=b_2$	$\nexists m$ s.t. $\alpha_m = 0$				4
			$\gamma_2\neq b_2$	$c>1/4$	$\exists$ integer n s.t. $b_1(1-2h_n) + h_n = 0$	$\exists m$ s.t. $\alpha_m = 0$		6
					$\nexists$ integer n s.t. $b_1(1-2h_n) + h_n\gamma_2 = 0$	$\nexists m$ s.t. $\alpha_m = 0$		6
				$c=1/4$	$\frac{2b_1}{\gamma_2}$ is an integer	$\exists m$ s.t. $\alpha_m = 0$		5
					$\frac{2b_1}{\gamma_2}$ is not an integer	$\nexists m$ s.t. $\alpha_m = 0$		5
				$c<1/4$	$2b_1>\gamma_2$	$\exists$ integer n s.t. $b_1(1-h_n)+h_n\gamma_2=0$	$\exists m$ s.t. $\alpha_m=0$	6
						$\nexists$ integer n s.t. $b_1(1-h_n)+h_n\gamma_2=0$	$\nexists m$ s.t. $\alpha_m=0$	6
					$2b<\gamma_2$	$\nexists$ integer m s.t. $\alpha_m = 0$		6

Note: 1) m is the smallest integer such that $(\alpha_i)_{i=1}^{m}$ is uniquely defined by $(f_{k,0})_{k=0}^{\infty}$ and $(f_{k,1})_{k=0}^{\infty}$

2) $c = b_1^2(b_2-\gamma_2)[b_2(2b_1-\gamma_2)^2]^{-1}$

3) $b_1 > 0$, $b_2 > 0$, $\gamma_2 > 0$

4) $h_n = \dfrac{(1+d)^n + (1-d)^n}{2((1+d)^{n-1}+(1-d)^{n-1})}$

where $d = \sqrt{1-4c}$.

$$(3.3.45) \quad \frac{1}{\gamma_{k+1}} = \frac{1}{b_1} - \frac{1}{\gamma_2}\left[1 - \frac{\sin[(k-1)\theta]}{\sin[(k-2)\theta]}\,\frac{\sqrt{c}\,(2b_1-\gamma_2)}{b_1}\right]$$

where $\tan\theta = \sqrt{4c-1}$.

θ is either a rational or irrational multiple of π.

If θ is a rational multiple of π, then there exists integers p and q such that p and q are relatively prime, $p < q$ and $\theta = \frac{p\pi}{q}$. We note that $q > 1$. Therefore, for $k = q + 2$ formula (3.3.45) shows that $\gamma_{q+3} = 0$.

The other possibility is that θ is an irrational multiple of π. By using Theorem (3.3.7) part 2, Equation (3.3.4) with $k = 1$, and Lemma (A.1.2) (Appendix I), we obtain,

$$\frac{\gamma_n}{b_{n-1}} = 1 - \frac{\alpha_1(4c-1)4^{-1}c^{-1/2}b_2(2b_1-\gamma_2)(b_1-b_2)^{-1}}{\cos(n\theta)\,\cos[(n-1)\theta-\lambda]} \qquad (n = 2, 3 \ldots)$$

where c is given by Equation (3.3.39), $\tan\theta = \sqrt{4c-1}$, $\cos\theta = (4c)^{-1/2}$ and $\tan\lambda = -\gamma_2(2b_1-\gamma_2)^{-1}(4c-1)^{-1/2}$. Because θ is an irrational multiple of 2π, then by Kronocker's Theorem [22, pp. 375-378] we have that $n\theta$ is dense in $(0,2\pi)$ modulo 2π, so that $\cos n\theta$ attains values arbitrarily close to zero. By hypothesis

$$\frac{\alpha_1(4c-1)b_2(2b_1-\gamma_2)}{4\sqrt{c}\,(b_1-b_2)} \neq 0$$

Thus $\{\frac{\gamma_n}{b_{n-1}}\}_{n=2}^{\infty}$ is an unbounded sequence.

Q.E.D.

Remark. It is clear therefore that this case does not lead to a polynomial set $\{A_n(x)\}_{n=0}^{\infty}$ in Σ. For we know from Theorem (3.2.1) that $\{\gamma_n(4b_{n-1})^{-1}\}_{n=2}^{\infty}$ must be a chain sequence and therefore must be bounded and from the fact that $\{A_n(x)\}_{n=0}^{\infty}$ is orthogonal on $(-1,1)$ it follows that $\gamma_n \neq 0$ for all $n \geq 2$.

We now investigate the case when $c < 1/4$.

Lemma (3.3.2). *Let* $\{\gamma_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ *and* $\{\alpha_n\}_{n=1}^{\infty}$ *satisfy Equations* (3.3.4), $2b_1 \neq \gamma_2$, $\gamma_2 \neq b_2$ and $b_2 \neq b_1$. *If* c *is as defined in Equation* (3.3.39) *and* $c < 1/4$, *then*

(a) $$\lim_{n\to\infty} \frac{\gamma_n}{b_1} = -\frac{1+\sqrt{1+2x}\ \operatorname{sgn}(\gamma_2-2b_1)}{x},$$

(b) $$\lim_{n\to\infty} \frac{b_n}{b_1} = \frac{2}{1-\sqrt{1+2x}\ \operatorname{sgn}(\gamma_2-2b_1)},$$

where $x = \frac{2b_1}{\gamma_2}\left(\frac{b_1}{b_2} - 1\right)$.

Proof: By Theorem (3.3.7) part 2 we obtain

$$\gamma_n = \frac{h_n b_2(2b_1-\gamma_2)-b_1b_2}{b_1-b_2} \tag{3.3.46}$$

where h_n satisfies Equation (3.3.38) and $h_1 = b_1(2b_1-\gamma_2)^{-1}$. It is easy to show that if $h_1 = 2^{-1}(1 - \sqrt{1-4c})$ then $b_1 = b_2$, but this contradicts the hypothesis. Thus by Corollary I of Lemma (A.1.1) we have

$$\lim_{n\to\infty} h_n = \frac{1+\sqrt{1-4c}}{2}$$

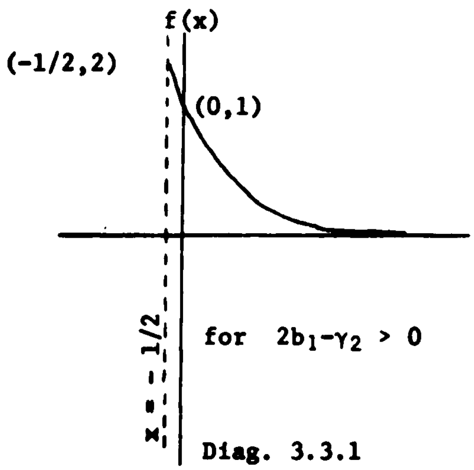
By using Equation (3.3.46) we obtain (a) and from (a) we obtain (b) by using Equation (3.3.3).

Q.E.D.

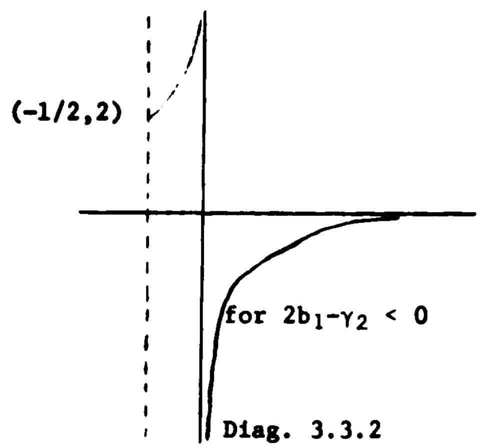
We are interested in seeing how the limit in Lemma (3.3.2) depends on x. Because $c < 1/4$, we have $x > -1/2$. If we let

$$f(x) = \lim_{k\to\infty} \frac{\gamma_k}{b_1} = -\frac{1+\sqrt{1+2x}\,\operatorname{sgn}(\gamma_2-2b_1)}{x} \tag{3.3.47}$$

then we have the following graphs.



Diag. 3.3.1



Diag. 3.3.2

Theorem (3.3.10). *If* $\{\gamma_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ *and* $\{\alpha_n\}_{n=1}^{\infty}$ *satisfy Equation* (3.3.4) *with* $\gamma_2 \neq b_2$, $b_1 \neq b_2$, $2b_1-\gamma_2 < 0$, *and* $c < 1/4$, *where* c *is given by Equation* (3.3.39), *then there does not exist a polynomial set in* $\sum$ *associated with these sequences.*

<u>Proof</u>: Assume there exists a polynomial set $\{A_n(x)\}_{n=0}^{\infty}$ with the properties given in the hypothesis. By Theorem (3.3.8) $\alpha_n \neq 0$ for $n = 1, 2 \ldots$. Thus $f_{k,0} = \prod_{i=1}^{k} \alpha_i \gamma_{i+1} \neq 0$. By using Equation (3.3.2) we have

$$\prod_{i=1}^{k} \alpha_i \gamma_{i+1} = \prod_{i=1}^{k} \left(\frac{\gamma_{i+1}}{b_1} - 1\right)$$

It follows from Lemma (3.3.2) that

$$\lim_{n\to\infty} \left| \frac{\gamma_{n+1}}{b_1} - 1 \right| > 1$$

(see Diag. (3.3.2)).

Thus

$$\text{(3.3.47)} \qquad \lim_{k\to\infty} \prod_{i=1}^{k} \alpha_i \gamma_{i+1} \neq 0$$

The results follow from Theorem (3.2.1).

3.4 <u>IDENTIFICATION OF ALL POLYNOMIAL SETS IN Σ</u>. In the last section we have found all sets of sequences that satisfy Equation (3.2.4) which could give rise to polynomial sets in Σ. If $\{A_n(x)\}_{n=0}^{\infty}$ is a polynomial set in Σ associated with $\{\alpha_k\}_{k=1}^{\infty}$ and $\{\{f_{k,n}\}_{k=0}^{\infty} | n = 0, 1, 2 \ldots\}$ and having the three term recursion relation

$$\text{(3.4.1)} \qquad \begin{cases} A_0(x) = 1 \qquad A_1(x) = 2b_1 x \\ \\ A_n(x) = 2b_n x A_{n-1}(x) - \lambda_n A_{n-2}(x) \end{cases}$$

then by Theorems (3.2.1), (3.3.1), (3.3.2) and (3.3.5) $\{\gamma_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ and $\{\alpha_n\}_{n=1}^{\infty}$ (where $\gamma_n = \lambda_n/b_n$) must satisfy

$$(3.4.2) \qquad \gamma_{n+k}\left(\frac{1}{b_n} - \alpha_k\right) + \gamma_n\alpha_k - 1 = 0 \qquad (n \geq 1,\ k \geq 1).$$

From the remark following Theorem (3.3.9) and Theorem (3.3.10) we see that not all sets of sequences $\{\gamma_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, $\{\alpha_k\}_{k=1}^{\infty}$ that satisfy Equations (3.4.2) give rise to polynomial sets in Σ. We wish to obtain some criterion that will allow us to identify those sets of sequences that lead to polynomial sets in Σ. Then we will consider each set of sequences that was obtained in section (3.3) and determine whether or not there is a corresponding polynomial set in Σ.

Theorem (3.4.1). *Let* $\{\gamma_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, $\{\alpha_n\}_{n=1}^{\infty}$ *satisfy Equation* (3.4.2) *and* $b_n > 0$ $n \geq 1$, $\gamma_1 = 0$ $\gamma_n > 0$, $n \geq 2$. *If there exists a weight function* $w(x)$ *such that* $w(\cos\theta)$ *belongs to* $L'[0,\pi]$ *and for all positive integers* n

$$\int_{-1}^{1} w(x)U_n(x)dx = \begin{cases} \prod_{i=1}^{k} (\alpha_i\gamma_{i+1}) & (n = 2k) \\ 0 & (n = 2k+1). \end{cases}$$

then $\{A_n(x)\}_{n=0}^{\infty}$ *as given by the three term recursion relation* (3.4.1) *with* $\gamma_n = \lambda_n/b_n$ *is a polynomial set in* Σ.

Proof: Because $w(\cos\theta) \in L'[0,\pi]$, it follows that for all $n \geq 0$, $x^n w(x) \in L'[-1,1]$. Let us define

$$\beta_{n,k} = \int_{-1}^{1} w(x)A_n(x)U_k(x)dx.$$

We will show by mathematical induction on n that

(a) $\beta_{n,k} = 0 \qquad (0 \leq k < n,\ n \geq 1)$,

(b) $\beta_{n,n+2k+1} = 0 \qquad (n \geq 0,\ k \geq 0)$,

and

(c) $\beta_{n,n+2k} = \left(\prod_{i=1}^{k} \alpha_i\right)\left(\prod_{j=2}^{n+k+1} \gamma_j\right) \qquad (n \geq 0,\ k \geq 0)$,

where the void product is taken to be 1.

The case $n = 0$ is just the hypothesis of the theorem.

For $n = 1$

(a)
$$\beta_{1,2} = \int_{-1}^{1} w(x)A_1(x)U_0(x)dx$$
$$= b_1\beta_{0,1}$$
$$= 0,$$

(b)
$$\beta_{1,2k+2} = \int_{-1}^{1} w(x)A_1(x)U_{2k+2}(x)dx$$
$$= b_1 \int_{-1}^{1} w(x)[U_{2k+1}(x) + U_{2k+3}(x)]dx$$
$$= b_1 [\beta_{0,2k+1} + \beta_{0,2k+3}]$$
$$= 0 \qquad (k \geq 0),$$

$$\text{(c)}\quad \beta_{1,2k+1} = \int_{-1}^{1} w(x)A_1(x)U_{2k+1}(x)dx$$

$$= b_1 \int_{-1}^{1} w(x)[U_{2k}(x) + U_{2k+2}(x)]dx$$

$$= b_1 \left[\left(\prod_{i=1}^{k} \alpha_i\right)\left(\prod_{j=2}^{k+1} \gamma_j\right) + \left(\prod_{i=1}^{k+1} \alpha_i\right)\left(\prod_{j=2}^{k+2} \gamma_j\right)\right]$$

$$= \left(\prod_{i=1}^{k} \alpha_i\right)\left(\prod_{j=2}^{k+1} \gamma_j\right) b_1[1 + \alpha_{k+1}\gamma_{k+2}] \qquad (k \geq 0).$$

But $b_1[1 + \alpha_{k+1}\gamma_{k+2}] = \gamma_{k+2}$, which follows from Equations (3.4.2) with $n = 1$. Hence we get

$$\beta_{1,2k+1} = \left(\prod_{i=1}^{k} \alpha_i\right)\left(\prod_{j=2}^{k+2} \gamma_j\right).$$

Assume for $r = 0, 1, 2 \ldots n$, $\beta_{r,k} = 0$ $0 \leq k < r$, $\beta_{r,r+2k+1} = 0$ for all $k \geq 0$, and $\beta_{r,r+2k} = \left(\prod_{i=1}^{k} \alpha_i\right)\left(\prod_{j=2}^{r+k+1} \gamma_j\right)$. Consider

$$\beta_{n+1,k} = \int_{-1}^{1} w(x)A_{n+1}(x)U_k(x)dx$$

$$= \int_{-1}^{1} w(x)[2b_{n+1}xA_n(x) - \lambda_{n+1}A_{n-1}(x)]U_k(x)dx$$

$$= \int_{-1}^{1} w(x)[b_{n+1}(U_{k+1}(x)+U_{k-1}(x))A_n(x) - \lambda_{n+1}A_{n-1}(x)U_k(x)]dx$$

for $k \geq 0$. That is,

$$(3.4.3)\qquad \beta_{n+1,k} = b_{n+1}\{\beta_{n,k+1} + \beta_{n,k-1}\} - \lambda_{n+1}\beta_{n-1,k} \qquad (k \geq 0).$$

From the induction hypothesis and Equations (3.4.3) it is clear that $\beta_{n+1,k} = 0$ if $0 \leq k \leq n - 2$ and $\beta_{n+1,n} = 0$. By putting $k = n - 1$ in Equations (3.4.3) we obtain

$$\beta_{n+1,n-1} = b_{n+1}\beta_{n,n} - \lambda_{n+1}\beta_{n-1,n-1}.$$

By the induction hypothesis $\beta_{n,n}/\beta_{n-1,n-1} = \gamma_{n+1}$, therefore $\beta_{n+1,n-1} = 0$. Hence (a) is satisfied for n replace by $n + 1$.

If $k \geq 0$,

$$\beta_{n+1,n+2k+2} = b_{n+1}(\beta_{n,n+2k+3} + \beta_{n,n+2k+1}) - \lambda_{n+1}\beta_{n-1,n+2k+2}.$$

Thus by using the induction hypothesis

$$\beta_{n+1,n+2k+2} = 0 \qquad (k \geq 0).$$

Hence (b) is satisfied for n replaced by $n + 1$.

From Equation (3.4.3) and the induction hypothesis

$$\beta_{n+1,n+2k+1} = b_{n+1}\left[\left(\prod_{i=1}^{k+1}\alpha_i\right)\left(\prod_{j=2}^{n+k+2}\gamma_j\right) + \left(\prod_{i=1}^{k}\alpha_i\right)\left(\prod_{j=2}^{n+k+1}\gamma_j\right)\right] - \lambda_{n+1}\left(\prod_{i=1}^{k+1}\alpha_i\right)\left(\prod_{j=2}^{n+k+1}\gamma_j\right)$$

$$= \left(\prod_{i=1}^{k}\alpha_i\right)\left(\prod_{j=2}^{n+k+1}\gamma_j\right)[b_{n+1}(\alpha_{k+1}\gamma_{n+k+2}+1) - \lambda_{n+1}\alpha_{k+1}].$$

From this and the fact that $\{\gamma_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, $\{\alpha_n\}_{n=1}^{\infty}$ satisfy Equation (3.4.2) we obtain

$$\beta_{n+1,n+2k+1} = \left(\prod_{i=1}^{k}\alpha_i\right)\left(\prod_{j=2}^{n+k+2}\gamma_j\right) \qquad (k \geq 0).$$

Thus (a), (b) and (c) are satisfied. From this it follows that for all $n \geq 0$,

$$(3.4.4) \qquad \int_{-1}^{1} w(x)x^m A_n(x)dx = 0 \qquad (0 \leq m < n),$$

(3.4.5) $$\int_{-1}^{1} w(x)\ x^n A_n(x)dx \neq 0,$$

and

(3.4.6) $$\int_{-1}^{1} w(x)\ x^{2n+1}dx = 0 \cdot$$

Also because $\{\frac{\gamma_{n+1}}{b_n}\}_{n=1}^{\infty}$ is a positive sequence, then it follows from Favard's Theorem (see section (1.1)) that there exists a distribution function $\sigma(t)$ such that

$$\int_{-\infty}^{\infty} A_n(x)A_m(x)d\sigma(x) = k_n\delta_{n,m} \qquad (n \geq 0,\ m \geq 0)$$

where $k_n \neq 0$ and $k_0 = 1$.

For all non-negative integers m there exists $\alpha_{m,k}$, $k = 0, 1, 2 \ldots m$, such that

$$x^m = \sum_{i=0}^{m} \alpha_{m,k}A_k(x).$$

Therefore

$$\int_{-1}^{1} x^m w(x)dx = \alpha_{m,0} \qquad (m \geq 0)$$

and

$$\int_{-1}^{1} x^m d\sigma(x) = \alpha_{m,0} \qquad (m \geq 0).$$

Therefore for all polynomials $\pi(x) \geq 0$ and $\pi(x) \not\equiv 0$

$$\int_{-1}^{1} \pi(x)w(x)dx = \int_{-\infty}^{\infty} \pi(x)d\sigma(x)$$
$$> 0.$$

Thus (see proof of Thm. (2.3.3)) $w(x) \geq 0$ almost everywhere on $(-1,1)$ and $w(x) \not\equiv 0$ on $(-1,1)$. From this fact and from Equations (3.4.4), (3.4.5) and (3.4.6) it follows that $\{A_n(x)\}_{n=0}^{\infty}$ belongs to $\mathcal{S}$ and $A_n(x)w(x)$ has the formal Chebychev expansion

$$A_n(x)w(x) \sim \sqrt{1-x^2} \sum_{k=0}^{\infty} f_{k,n} U_{n+2k}(x)$$

where $f_{k,n} = \prod_{i=1}^{k} \alpha_i \sum_{j=2}^{n+k+1} \gamma_j$. Thus there exists a real sequence $\{\alpha_k\}_{k=1}^{\infty}$ such that $f_{k,n} = \alpha_k f_{k-1,n+1}$. Therefore $\{A_n(x)\}_{n=0}^{\infty}$ is an element in Σ.

Q.E.D.

<u>Theorem (3.4.2)</u> *Let* $\{A_n(x)\}_{n=0}^{\infty}$ *have the three term recursion relation* (3.4.1) *with* $b_1 = b_2 > 0$. $\{A_n(x)\}_{n=0}^{\infty}$ *is in* Σ *and is associated with* $\{\alpha_k\}_{k=1}^{\infty}$ *and* $\{\{f_{k,n}\}_{k=0}^{\infty} | n = 0, 1 \ldots\}$ *if and only if* $2b_1 > \gamma_2 > 0$ *and*

(3.4.7) $$b_n = b_1 > 0 \qquad (n \geq 1),$$

(3.4.8) $$\lambda_{n+1} = b_1^2[1 - (1-\gamma_2/b_1)^n] \qquad (n \geq 0),$$

(3.4.9) $$\alpha_n = (1 - \gamma_2(b_1)^{-1})^n[b_1(1 - \gamma_2(b_1)^{-1})^n - 1]^{-1} \qquad (n \geq 0).$$

<u>Proof</u>: If $\{A_n(x)\}_{n=0}^{\infty}$ is an element of Σ with $b_1 = b_2$, then by Theorems (3.2.1) and (3.3.3) $\{b_n\}_{n=1}^{\infty}$ $\{\lambda_n\}_{n=1}^{\infty}$ and $\{\alpha_n\}_{n=1}^{\infty}$ are given by Equations (3.4.7), (3.4.8) and (3.4.9) respectively.

Also by Theorem (3.2.1)

$$\lim_{k\to\infty} \prod_{i=1}^{k} \alpha_i \gamma_{i+1} = 0.$$

That is,

$$\lim_{k\to\infty} (-1)^k (1-\gamma_2(b_1)^{-1})^{\frac{k(k+1)}{2}} = 0 .$$

Therefore, $-1 < 1 - \gamma_2(b_1)^{-1} < 1$. That is $0 < \gamma_2 < 2b_1$.

Conversely, we note that $b_n > 0$ for $n \geq 1$, $\gamma_1 = 0$ and $\gamma_n > 0$ for $n \geq 2$. By Theorem (3.3.3) $\{\gamma_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, $\{\alpha_n\}_{n=1}^{\infty}$ satisfy Equation (3.4.2). Because $|1 - \gamma_2(b_1)^{-1}| < 1$, therefore

$$\sum_{n=1}^{\infty} \left| \prod_{k=1}^{n} \alpha_k \gamma_{k+1} \right|^2 = \sum_{n=1}^{\infty} (1 - \gamma_2(b_1)^{-1})^{n(n+1)} < \infty.$$

Thus, by the Riesz-Fischer Theorem, there exists $w(\cos\theta) \in L^2[0,\pi]$ such that for all non negative integers m

$$\int_{-1}^{1} w(x)U_m(x)dx = \begin{cases} \prod_{i=1}^{k} \alpha_i \gamma_{i+1} & m = 2k \\ 0 & m = 2k+1 . \end{cases}$$

Thus $w(\cos\theta) \in L'[0,\pi]$ by Schwartz's Inequality. Therefore, by Theorem (3.4.1) $\{A_n(x)\}_{n=0}^{\infty}$ is an element in Σ.

Q.E.D.

The polynomial set $\{A_n(x)\}_{n=0}^{\infty}$ as defined in Theorem (3.4.2) is equivalent to $\{R_n^q(x)\}_{n=0}^{\infty}$ defined by

$$R_0^q(x) = 1 \qquad R_1^q(x) = 2x$$

$$R_n^q(x) = 2xR_{n-1}^q(x) - (1-q^{n-1})R_{n-2}^q(x) \qquad (n \geq 2)$$

where $|q| < 1$. We will find some of their properties in chapter V.

For the case $2b_1 = \gamma_2$ we have from Theorem (3.3.4) two possibilities. The first being

$$b_n = 2b_1 \qquad (n \geq 2),$$

$$\lambda_n = 4b_1^2 \qquad (n \geq 2),$$

$$\alpha_n = (2b_1)^{-1} \qquad (n \geq 1).$$

From this it follows that $\alpha_n\gamma_{n+1} = 1$, and therefore $\lim_{k\to\infty} \prod_{i=1}^{k} \alpha_i\gamma_{i+1} = 1$. Therefore, by Theorem (3.2.1) the polynomial set $\{A_n(x)\}_{n=0}^{\infty}$ defined by

$$A_0(x) = 1 \qquad A_1(x) = 2b_1x$$

$$A_n(x) = 4b_1x\, A_{n-1}(x) - 4b_1^2A_{n-2}(x)$$

is <u>not</u> a polynomial set in Σ. This polynomial set is equivalent to $\{T_n(x)\}_{n=0}^{\infty}$ the Chebychev polynomials of the first kind.

The second possibility is when there exists a positive integer r such that $\gamma_r \neq 2b_1$.

<u>Theorem (3.4.3)</u>. *Let* $\{A_n(x)\}_{n=0}^{\infty}$ *have the three term recursion relation* (3.4.1) *such that* $2b_1 = \gamma_2$ *and there exists a smallest*

positive integer r *such that* $2b_1 \neq \gamma_r$. $\{A_n(x)\}_{n=0}^{\infty}$ *is an element in* Σ *and is associated with* $\{\alpha_k\}_{k=1}^{\infty}$ *and* $\{\{f_{k,n}\}_{k=0}^{\infty} | n = 0, 1, 2 \ldots\}$ *if and only if* $2b_1 > \gamma_r > 0$ *and*

$$(3.4.10) \qquad b_n = \begin{cases} 2b_1 & (n \neq kr - k + 1) \\ \dfrac{(2b_1-\gamma_r+k\gamma_r)2b_1}{2(2b_1-\gamma_r)+k\gamma_r} & (n = kr - k + 1), \end{cases}$$

$$(3.4.11) \qquad \lambda_n = \begin{cases} 4b_1^2 & (n \neq 1,\ n \neq kr - k + 1) \\ 0 & (n = 1) \\ \dfrac{4b_1^2 k\gamma_r}{2(2b_1-\gamma_r)+k\gamma_r} & (n = kr - k + 1), \end{cases}$$

$$(3.4.12) \qquad \alpha_n = \begin{cases} (2b_1)^{-1} & (n \neq kr - k) \\ \dfrac{k\gamma_r-2b_1+\gamma_r}{2b_1 k\gamma_r} & (n = kr - k), \end{cases}$$

where k *is a positive integer.*

<u>Proof</u>: If $\{A_n(x)\}_{n=0}^{\infty}$ is a polynomial set in Σ with $2b_1 = \gamma_2$ and if there exists a smallest integer r such that $2b_1 \neq \gamma_r$, then by Theorems (3.2.1) and (3.3.4) $\{b_n\}_{n=1}^{\infty}$, $\{\lambda_n\}_{n=1}^{\infty}$, $\{\alpha_n\}_{n=1}^{\infty}$ are given by Equations (3.4.10), (3.4.11) and (3.4.12) respectively. Also,

$$\prod_{i=1}^{kr-k} \alpha_i \gamma_{i+1} = \prod_{i=1}^{k} \frac{i\gamma_r-(2b_1-\gamma_r)}{i\gamma_r+(2b_1-\gamma_r)} .$$

Therefore by Theorem (3.2.1)

$$\lim_{k\to\infty} \prod_{i=1}^{k} \frac{i\gamma_r-(2b_1-\gamma_r)}{i\gamma_r+(2b_1-\gamma_r)} = 0.$$

Therefore, $2b_1 > \gamma_r > 0$.

Conversely, by Theorem (3.3.4) $\{\gamma_n\}_{n=1}^{\infty} = \{\frac{\lambda_n}{b_n}\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, and $\{\alpha_n\}_{n=1}^{\infty}$ which are given by Equations (3.4.10), (3.4.11) and (3.4.12) satisfy Equation (3.4.2), and also $b_n > 0$ for all $n \geq 1$, $\gamma_1 = 0$ and $\gamma_n > 0$ for all $n \geq 2$.

By direct substitution we have that

(3.4.13) $$\sum_{k=0}^{\infty} \left(\prod_{i=1}^{k} \alpha_i \gamma_{i+1}\right) \sin(2k+1)\theta$$

$$= \sum_{s=1}^{\infty} \prod_{n=1}^{s-1} \left(\frac{n-(\frac{2b_1}{\gamma_r} -1)}{n+(\frac{2b_1}{\gamma_r} - 1)} \right) \sum_{e=(s-1)(r-1)}^{s(r-1)-1} \sin(2e+1)\theta$$

where the void product is 1.

There exists an n_0 such that for all $n \geq n_0 \geq 1$

$$\frac{n-(\frac{2b_1}{\gamma_r} - 1)}{n+(\frac{2b_1}{\gamma_r} - 1)} > 0$$

and

$$\left| \sum_{k=n_0(r-1)}^{n} \frac{\left(\prod_{i=1}^{k} \alpha_i \gamma_{i+1}\right)}{k} \right| \leq r \left| \sum_{k=n_0}^{n} \frac{\left(\prod_{i=1}^{k} \frac{(i-(\frac{2b_1}{\gamma_r}-1))}{(i+(\frac{2b_1}{\gamma_r}-1))}\right)}{k} \right|$$

If we let

$$a_s = \frac{1}{s} \prod_{n=1}^{s} \left(\frac{n-(\frac{2b_1}{\gamma_r}-1)}{n+(\frac{2b_1}{\gamma_r}-1)} \right),$$

then

$$\lim_{s\to\infty} s\left(\frac{a_s}{a_{s+1}} - 1\right) = 1 + 2\left(\frac{2b_1}{\gamma_r} - 1\right)$$

$$> 1.$$

Thus by Raabe's Test [7, p. 39]

$$\sum_{s=1}^{\infty} \frac{1}{s} \prod_{n=1}^{s} \left(\frac{n-(\frac{2b_1}{\gamma_r} - 1)}{n+(\frac{2b_1}{\gamma_r} - 1)} \right)$$

converges. Therefore, $\sum_{k=1}^{\infty} k^{-1} \left(\prod_{i=1}^{k} \alpha_i \gamma_{i+1}\right) < \infty$. That is, $\sum_{k=1}^{\infty} k^{-1} f_{k,0}$ converges and $\{f_{k,0}\}_{k=0}^{\infty}$ is eventually monotonically tending to zero. Therefore by Corollary I of Theorem (2.2.1), there exists $w(\cos\theta)$ belonging to $L'[0,\pi]$ such that for all $n \geq 0$

$$\int_{-1}^{1} U_n(x)w(x)dx = \begin{cases} \prod_{i=1}^{k} \alpha_i \gamma_{i+1} & (n = 2k) \\ 0 & (n = 2k+1). \end{cases}$$

Therefore by Theorem (3.4.1) $\{A_n(x)\}_{n=0}^{\infty}$ is in Σ.

Q.E.D.

Let us denote the polynomial set defined in Theorem (3.4.3) by $\{Q_n(x;\ \gamma_r,\ r)\}_{n=0}^{\infty}$. It is orthogonal on $(-1,1)$ with weight function

$$(3.4.14)\quad w(x) = \sqrt{1-x^2}\left[\sum_{s=1}^{\infty}\prod_{n=1}^{s-1}\left(\frac{n-(\frac{2b_1}{\gamma_r}-1)}{n+(\frac{2b_1}{\gamma_r}-1)}\right)\sum_{e=(s-1)(r-1)}^{s(r-1)-1} U_{2e}(x)\right]$$

where the void product is 1. The convergence in Equation (3.4.14) is pointwise on $(-1,1)$ and uniform on any closed subset of $(-1,1)$. We will study some more of its properties in Chapter 5.

<u>Theorem (3.4.4)</u>. *Let* $\{A_n(x)\}_{n=0}^{\infty}$ *have the three term recursion relation* (3.4.1) *with* $\gamma_2 \neq 2b_1$, $b_1 \neq b_2$ *and* $\gamma_2 = b_2$. $\{A_n(x)\}_{n=0}^{\infty}$ *is a polynomial set in* Σ *associated with* $\{\alpha_k\}_{k=1}^{\infty}$ *and* $\{\{f_{k,n}\}_{k=0}^{\infty} | n = 0, 1 \ldots\}$ *if and only if* $2b_1 > \gamma_2 > 0$ *and*

$$(3.4.15)\qquad b_n = b_2 > 0 \qquad (n \geq 2)$$

$$(3.4.16)\qquad \lambda_n = \begin{cases} b_2^2 & (n \geq 2) \\ 0 & (n = 1) \end{cases}$$

$$(3.4.17)\qquad \alpha_n = (b_1)^{-1} - (b_2)^{-1} \qquad (n \geq 1).$$

<u>Proof</u>: If $\{A_n(x)\}_{n=0}^{\infty}$ is in Σ then, by Theorems (3.2.1) and (3.3.5), the sequences $\{b_n\}_{n=1}^{\infty}$, $\{\lambda_n\}_{n=1}^{\infty}$ and $\{\alpha_n\}_{n=1}^{\infty}$ are given by Equations (3.4.15), (3.4.16) and (3.4.17) respectively. We also have from Theorem (3.2.1)

$$\lim_{k\to\infty} \prod_{i=1}^{k} \alpha_i \gamma_{i+1} = \lim_{k\to\infty} \left(\frac{\gamma_2}{b_1} - 1\right)^k$$

$$= 0.$$

Therefore, $0 < \gamma_2 < 2b_1$.

Conversely, we note that $b_n > 0$ for all $n \geq 1$, $\lambda_1 = 0$ and $\lambda_n > 0$ for all $n \geq 2$. If $0 < \gamma_2 < 2b_1$, then

$$\sum_{k=0}^{\infty} (f_{k,0})^2 = \sum_{k=0}^{\infty} \left(\frac{\gamma_2}{b_1} - 1\right)^{2k}$$

$$< \infty.$$

As before, by using the Riesz-Fischer Theorem, there exists $w(\cos\theta)$ belonging to $L'[0,\pi]$ such that for all $n \geq 0$

$$\int_{-1}^{1} U_n(x) w(x)\,dx = \begin{cases} 0 & \text{if } n = 2k+1 \\ \prod_{i=1}^{k} \alpha_i \gamma_{i+1} & \text{if } n = 2k \end{cases}$$

Therefore, by Theorem (3.4.1), $\{A_n(x)\}_{n=0}^{\infty}$ is an element of Σ.

Q.E.D.

If we let $s = b_1(b_2)^{-1}$ then the polynomial sets found in Theorem (3.4.4) are equivalent to $\{P_n(x:s,0)\}_{n=0}^{\infty *}$ defined by

$$P_0(x:s,0) = 1 \qquad P_1(x:s,0) = 2sx$$

$$P_n(x:s,0) = 2xP_{n-1}(x:s,0) - P_{n-2}(x:s,0)$$

where $s > 0$. $\{P_n(x:1/2,0)\}_{n=0}^{\infty}$ and $\{P_n(x:1,0)\}_{n=0}^{\infty}$ are the Chebychev polynomial sets of the first and second kind respectively. In order for $\{P_n(x:s,0)\}_{n=0}^{\infty}$ to be in Σ $s > 1/2$. Thus in this case we know that $\{P_n(x:s,0)\}_{n=0}^{\infty}$ is orthogonal on $(-1,1)$ with weight function

$$(3.4.18) \qquad w(x) = \sqrt{1-x^2} \sum_{k=0}^{\infty} \left(\frac{1-s}{s}\right)^k U_{2k}(x)$$

By Theorem (2.2.1) the convergence is pointwise in $(-1,1)$ and uniform in any closed subset of $(-1,1)$.

Now we are ready to consider the remaining case. That is, when $b_1 \neq b_2$, $b_1 \neq \gamma_2$, $2b_1 \neq \gamma_2$ and $\gamma_2 \neq b_2$ are simultaneously satisfied. Let c be as defined in Equation (3.3.39). We have seen in the remark following Theorem (3.3.9), that there does not exist a polynomial set in Σ corresponding to the case $c > 1/4$. For $c = 1/4$ we get from Theorem (3.3.7) and (A.1.1) the polynomial set $\{A_n(x)\}_{n=0}^{\infty}$ defined by

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*In chpater 4 we consider polynomial sets in which the second parameter is not zero.

$$A_0(x) = 1 \qquad\qquad A_1(x) = 2b_1x$$

$$A_n(x) = 2(2b_1)\left\{\frac{(n-1)\rho+1}{(n-1)\rho+2}\right\} xA_{n-1}(x) - 4b_1^2\left\{\frac{(n-1)\rho}{(n-1)\rho+2}\right\} A_{n-2}(x)$$

and

$$\alpha_n = \frac{n\rho-1}{2b_1n\rho}$$

where $\rho = \gamma_2(2b_1-\gamma_2)^{-1}$. From this it follows that

$$A_n(x) = \frac{(2b_1)^n n!P_n^{(\frac{1}{\rho})}(x)}{(\frac{2}{\rho})_n}$$

where $P_n^{(\lambda)}(x)$ is the Ultraspherical polynomial set of order λ. Thus for $\{A_n(x)\}_{n=0}^{\infty}$ to be in Σ, $1/\rho > 0$.

For $c < 1/4$, we know from Theorem (3.3.10) that in order to obtain a polynomial set in Σ, $2b_1 - \gamma_2 > 0$.

Theorem (3.4.5). *Let* $\{A_n(x)\}_{n=0}^{\infty}$ *have the three term recursion relation* (3.4.1) *with* $b_1 \neq b_2$, $b_1 \neq \gamma_2$, $2b_1 > \gamma_2$, $\gamma_2 \neq b_2$, $b_1 > 0$, $\gamma_2 > 0$, $b_2 > 0$ *and* c *is defined by Equation* (3.3.39).

If $c < 1/4$, *then* $\{A_n(x)\}_{n=0}^{\infty}$ *is a polynomial set in* Σ *if and only if*,

(3.4.19) $$\alpha_n = \frac{1-H_{n+1}}{b_1\gamma} \qquad (n = 1, 2 \ldots),$$

(3.4.20) $$b_n = \frac{\gamma b_1(\gamma-1+H_n)}{1-c(2-\gamma)^2+\gamma^2-2\gamma+\gamma H_n} \qquad (n = 1, 2 \ldots),$$

(3.4.21) $$\lambda_n = \frac{(b_1\gamma)^2}{1-c(2-\gamma)^2+\gamma^2-2\gamma+\gamma H_n} \qquad (n = 2, 3 \ldots),$$

where $\gamma = \gamma_2/b_1$ *and*

$$(3.4.22) \qquad H_n = \frac{[(1+\sqrt{1-4c})^n+(1-\sqrt{1-4c})^n](2-\gamma)}{2\{(1+\sqrt{1-4c})^{n-1}+(1-\sqrt{1-4c})^{n-1}\}} .$$

Proof: If $\{A_n(x)\}_{n=0}^{\infty}$ is in $\sum$ then by Theorems (3.3.1), (3.3.7) part (i), and Lemma (A.1.1) $\{\alpha_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ and $\{\lambda_n\}_{n=1}^{\infty}$ are given by Equations (3.4.19), (3.4.20) and (3.4.21) respectively.

Conversely, we have from Theorems (3.3.7) and (3.3.1) that $\{\alpha_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$, where $\gamma_n = \lambda_n/b_n$, satisfy Equation (3.3.4). Thus by Theorem (3.3.6) they also satisfy Equation (3.3.15). By using Theorem (3.3.7) we have

$$\gamma_k = \frac{h_k b_2(2b_1-\gamma_2)-b_1b_2}{b_1-b_2}$$

where $h_1 = b_1(2b_1-\gamma_2)^{-1}$ and $\{h_k\}_{k=1}^{\infty}$ satisfies Equations (3.3.38). It is easy to see that $h_1 > 2^{-1}(1-\sqrt{1-4c})$. Thus by Lemma (A.1.3) $\{\gamma_{2k}\}_{k=1}^{\infty}$ and $\{\gamma_{2k-1}\}_{k=1}^{\infty}$ are monotonic. By Lemma (3.3.2) (see Diag. (3.3.1)) they have a common positive limit. We note that $\gamma_1 = 0$ and $\gamma_2 > 0$, thus $\gamma_n > 0$, and γ_n is bounded above for $n = 2, 3, 4 \ldots$. In exactly the same manner we can show that $b_n > 0$ and b_n is bounded above for $n = 2, 3, 4 \ldots$.

Finally, it is easy to deduce from Theorem (3.3.1) and Lemma (3.3.2) that

$$\sum_{k=0}^{\infty} \left(\prod_{i=1}^{k} \alpha_i \gamma_{i+1} \right)^2 < \infty.$$

Thus by the Riesz-Fischer Theorem there exists a function $w(\cos\theta)$ belonging to $L'[0,\pi]$ such that for all $n \geq 0$

$$\int_{-1}^{1} U_n(x)w(x)dx = \begin{cases} 0 & (n = 1, 3 \ldots) \\ \prod_{i=1}^{k} \alpha_i\gamma_{i+1} & (n = 2, 4 \ldots) \end{cases}$$

Therefore by Theorem (3.4.1), $\{A_n(x)\}_{n=0}^{\infty}$ is an element of Σ.

Q.E.D.

It is obvious that the O.P.S. $\{A_n(x)\}_{n=0}^{\infty}$ identified in Theorem (3.4.5) is equivalent to $\{Z_n(x;c,\gamma)\}_{n=0}^{\infty}$ defined by

$$Z_0(x;c,\gamma) = 1 \qquad Z_1(x;c,\gamma) = 2x$$

$$Z_n(x;c,\gamma) = \frac{2\gamma(\gamma-1+H_n)}{1-c(2-\gamma)^2+\gamma^2-2\gamma+\gamma H_n} x\, Z_{n-1}(x;c,\gamma) - \frac{\gamma^2}{1-c(2-\gamma)^2+\gamma^2-2\gamma+\gamma H_n} Z_{n-2}(x;c,\gamma)$$

where $0 < \gamma < 2$, $c < 1/4$ and H_n is given by Equation (3.4.22). In fact, $A_n(x) = (b_1)^n Z_n(x;c,\gamma)$. We also note that $\{Z_n(x;c,\gamma)\}_{n=0}^{\infty}$ belongs to Σ. It can easily be shown that the sequences $\{\alpha_k\}_{k=1}^{\infty}$ associated with $\{Z_n(x;c,\gamma)\}_{n=0}^{\infty}$ is given by

$$\alpha_n = \gamma^{-1}(1-H_{n+1})$$

where H_n is given by (3.4.22).

In table (3.4.1) we give all possible elements of Σ. In table (3.4.2) we show under what conditions these polynomial sets arise for the case $m < \infty$ and in table (3.4.3) for the case $m = \infty$, where m is defined to be the smallest integer such that $\{\alpha_i\}_{i=1}^{m}$ is uniquely defined by $\{f_{k,0}\}_{k=0}^{\infty}$ and $\{f_{k,1}\}_{k=0}^{\infty}$.

Table (3.4.1)

ALL POLYNOMIAL SETS $\{A_n(x)\}_{n=0}^{\infty}$ IN Σ

$A_0(x) = 1 \qquad A_1(x) = 2b_1x$

$A_n(x) = 2b_n x A_{n-1}(x) - \lambda_n A_{n-2}(x)$

Ref. No.	Remarks	b_n	λ_n	α_n
1	$2 > \gamma > 0$ See Theorem (3.4.2)	1	$1-(1-\gamma)^{n-1}$	$\frac{(1-\gamma)^n}{(1-\gamma)^n-1}$
2	$s > 1/2$ See Theorem (3.4.4)	$b_1 = s$ $b_n = 1 \quad n \geq 2$	1	$\frac{1-s}{s}$
3	Equivalent to Ultraspherical Polynomials $\rho > 0$	$\frac{(n-1)\rho+1}{(n-1)\rho+2}$	$\frac{(n-1)\rho}{(n-1)\rho+2}$	$\frac{n\rho-1}{n\rho}$
4	r is any integer bigger than 2 $2 > \gamma_r > 0$ See Theorem (3.4.3)	For $k = 1, 2, \ldots$ $b_{rk-k+1} = \frac{1+\gamma_r(k-1)}{2(1-\gamma_r)+k\gamma_r}$; otherwise $b_n = 1$	$k = 0, 1, 2 \ldots$ $\lambda_{rk-k+1} = \frac{k\gamma_r}{2(1-\gamma_r)+k\gamma_r}$; otherwise $\lambda_n = 1$	$\alpha_{kr-k} = \frac{(k+1)\gamma_r-1}{k\gamma_r}$; otherwise $\alpha_n = 1$
5	$0 < \gamma < 2$ $c < 1/4$ $H_n = \frac{\{(1+\sqrt{1-4c})^n+(1-\sqrt{1-4c})^n\}(2-\gamma)}{2\{(1+\sqrt{1-4c})^{n-1}+(1-\sqrt{1-4c})^{n-1}\}}$	$\frac{2\gamma(\gamma-1+H_n)}{1-c(2-\gamma)^2+\gamma^2-2\gamma+\gamma H_n}$	$\frac{\gamma^2}{1-c(2-\gamma)^2+\gamma^2-2\gamma+\gamma H_n}$	$\frac{1-H_{n+1}}{\gamma}$

Table (3.4.2)
Conditions For The Existence of Polynomial Sets In Σ Having $m < \infty$

								Ref. No. for b_n, λ_n and a_n in Table (3.4.1)
$b_1=\gamma_2$	$a_m = 0$, for $m = 1, 2, 3 \ldots$ $\{U_n(x)\}_{n=0}^{\infty}$							1 with $\gamma=0$
$b_1\neq\gamma_2$	$b_1=b_2$	No Solution						
	$b_1\neq b_2$	$2b_1=\gamma_2$	$\exists r$ s.t. $\gamma_r\neq 2b_1$	$2b_1>\gamma_r>0$	$2b_1(\gamma_r)^{-1}$ is an integer	$\{Q_n(x;\gamma_r,r)\}_{n=0}^{\infty}$		4
					$2b_1(\gamma_r)^{-1}$ is not an integer	No Solution		
				$\gamma_r>2b$	No Solution			
			$\forall r$ $\gamma_r=2b_1$	No Solution				
		$2b_1\neq\gamma_2$	$\gamma_2=b_2$	No Solution				
			$\gamma_2\neq b_2$	$c > 1/4$	No Solution			
				$c = 1/4$	$\frac{2b_1}{\gamma_2}$ is an integer	Ultraspherical Polynomial Sets of Integer Order		3
					$\frac{2b_1}{\gamma_2}$ is not an integer	No Solution		
				$c < 1/4$	$2b_1<\gamma_2$	No Solutions		
					$2b_1>\gamma_2$	$\nexists$ integer n s.t. $b_1(1-h_n)+h_n\gamma_2=0$	No Solution	
						$\exists$ integer n s.t. $b_1(1-h_n)+h_n\gamma_2=0$	$\{Z_n(x;c,\gamma)\}_{n=0}^{\infty}$	5

Note: 1) m is the smallest integer such that $\{a_i\}_{i=1}^{m}$ is uniquely defined by $\{f_{k,0}\}_{k=0}^{\infty}$ and $\{f_{k,1}\}_{k=0}^{\infty}$.

2) $c = b_1^2(b_2-\gamma_2)[b_2(2b_1-\gamma_2)^2]^{-1}$.

3) $b_1 > 0$, $b_2 > 0$, $\gamma_2 > 0$

4) $h_n = \dfrac{(1+d)^n + (1-d)^n}{2((1+d)^{n-1}+(1-d)^{n-1})}$ where $d = \sqrt{1-4c}$

Table (3.4.3)
Conditions For The Existence of Polynomial Sets in Σ Having $m = \infty$

								Ref. No. for b_n, λ_n and a_n in Table (3.4.1)
$b_1=\gamma_2$	No Solution							
$b_1\neq\gamma_2$	$b_1=b_2$	$(R_n^q(x))_{n=0}^{\infty}$						1
	$b_1\neq b_2$	$2b_1=\gamma_2$	$\exists r$ s.t. $\gamma_r\neq 2b_1$	$2b_1>\gamma_r>0$	$2b_1(\gamma_r)^{-1}$ is not an integer	$(Q_n(x;\gamma_r,r))_{n=0}^{\infty}$		4
					$2b_1(\gamma_r)^{-1}$ is an integer	No Solution		
				$\gamma_r>2b_1$	No Solution			
			$\forall r$ $\gamma_r=2b_1$	No Solution				
		$2b_1\neq\gamma_2$	$\gamma_2=b_2$	$2b_1>\gamma_2>0$	$(P_n(x;s,0))_{n=0}^{\infty}$			2
				$\gamma_2>2b_1$	No Solution			
			$\gamma_2\neq b_2$	$c>1/4$	No Solution			
				$c=1/4$	$\frac{2b_1}{\gamma_2}$ is not an integer	Ultraspherical Polynomial Sets		3
					$\frac{2b_1}{\gamma_2}$ is an integer	No Solution		
				$c<1/4$	$2b_1<\gamma_2$	No Solutions		
					$2b_1>\gamma_2$	$\exists$ integer n s.t. $b_1(1-h_n)+h_n\gamma_2=0$	No Solution	
						$\nexists$ integer n s.t. $b\ (1-h_n)+h_n\gamma_2=0$	$(Z_n(x;c,\gamma))_{n=0}^{\infty}$	5

Note:
1) m is the smallest integer such that $(a_i)_{i=1}^{m}$ is uniquely defined by $(f_{k,0})_{k=0}^{\infty}$ and $(f_{k,1})_{k=0}^{\infty}$.
2) $c = b_1^2(b_2-\gamma_2)[b_2(2b_1-\gamma_2)^2]^{-1}$.
3) $b_1 > 0$, $b_2 > 0$, $\gamma_2 > 0$
4) $h_n = \dfrac{(1+d)^n + (1-d)^n}{2((1+d)^{n-1}+(1-d)^{n-1})}$ where $d = \sqrt{1-4c}$

CHAPTER IV

CO-RECURSIVE POLYNOMIALS

4.1 INTRODUCTION. We showed in Theorem (3.4.4) that if $\{A_n(x)\}_{n=0}^{\infty}$ is a polynomial set in Σ having the three term recursion relation (3.2.1) with $\gamma_2 < 2b_1$, $b_1 \neq b_2$ and $\gamma_2 = b_2$ then

$$A_n(x) = 2b_2xA_{n-1}(x) - b_2^2A_{n-2}(x) \qquad (n \geq 2),$$

$$A_0(x) = 1 \qquad A_1(x) = 2b_1x.$$

This polynomial set is equivalent to

$$p_n(x) = 2xp_{n-1}(x) - p_{n-2}(x) \qquad (n \geq 2)$$

$$p_0(x) = 1 \qquad p_1(x) = 2sx$$

where $s = b_1/b_2$.

This polynomial set satisfies the same recurrence relation that the Chebychev polynomials $U_n(x)$ and $T_n(x)$ do except that they enjoy different initial conditions. In fact we have

$$p_0(x) = U_0(x) \quad \text{and} \quad p_1(x) = sU_1(x).$$

Hence the system $\{p_n(x)\}_{n=0}^{\infty}$ constitutes a set of orthogonal polynomials which, as a function of the parameter s, reduces to the Chebychev polynomials of the first kind (if $s = \frac{1}{2}$) or to the Chebychev polynomials of the second kind (if $s = 1$).

In this chapter, instead of studing this system of orthogonal polynomials, we shall investigate a more general situation which at the same time generalizes a problem studied by T. S. Chihara [12].

Let the polynomial set $\{R_n(x)\}_{n=0}^{\infty}$ satisfy the recurrence relation

$$(4.1.1) \qquad y_n(x) = (x+c_n)y_{n-1}(x) - \lambda_n y_{n-2}(x) \qquad (n \geq 1),$$

$$(4.1.2) \qquad y_{-1}(x) = 0 \qquad y_0(x) = 1.$$

We define the polynomial set $\{R_n(x;s,r)\}_{n=0}^{\infty}$ so that they satisfy the relation (4.1.1) and the initial conditions

$$(4.1.3) \qquad R_0(x;s,r) = 1, \qquad R_1(x;s,r) = sR_1(x) - r$$

where s and r are fixed real constants with $s \neq 0$. Note that $R_n(x;1,0) = R_n(x)$, $(n = 0, 1, 2 \ldots)$, and that because we are dealing with orthogonality in the classical sense (see section 1.1) $s > 0$.

<u>Definition (4.1.1)</u>. We call the polynomial set $\{R_n(x;s,r)\}_{n=0}^{\infty}$ the (s,r) *co-recursive* polynomial set of $\{R_n(x)\}_{n=0}^{\infty}$.

The case $s = 1$ reduces the problem to that which Chihara studied in [12].

Before we study the polynomial set $\{R_n(x;s,r)\}_{n=0}^{\infty}$ we first establish an important relationship involving the $R_n(x)$, $R_n(x;s,r)$ and the numerator polynomial $Q_{n-1}(x)$.

Let $\{Q_n(x)\}_{n=0}^{\infty}$ be the numerator polynomial set of $\{R_n(x;s,r)\}_{n=0}^{\infty}$, so that they satisfy

$$(4.1.4)\qquad \begin{cases} Q_n(x) = (x+c_{n+1})Q_{n-1}(x) - \lambda_{n+1}Q_{n-2}(x) & (n \geq 1), \\ Q_{-1}(x) = 0, \qquad Q_0(x) = 1. \end{cases}$$

We assert that, for $n \geq 0$,

$$(4.1.5)\qquad R_n(x;s,r) = R_n(x) + [(s-1)R_1(x)-r]Q_{n-1}(x).$$

Indeed, since $\{R_n(x)\}_{n=0}^{\infty}$ and $\{Q_{n-1}(x)\}_{n=0}^{\infty}$ are two linearly independent solutions of Equation (4.1.1), so that the right hand side of Equation (4.1.5) satisfies the recurrence relation (4.1.1) (or is in the solution space of that difference equation). To prove the validity of Equation (4.1.5) we therefore only need to notice that the right hand and left hand sides are equal for $n = 0$ and $n = 1$.

4.2 THE ZEROS OF $R_n(x;s,r)$. Let us extend the polynomial sets $\{R_n(x)\}_{n=0}^{\infty}$, $\{R_n(x;s,r)\}_{n=0}^{\infty}$ and $\{Q_n(x)\}_{n=0}^{\infty}$ so that they are defined for all integers $(n = 0, \pm 1, \pm 2, \ldots)$. For $n \geq 2$, put $Q_{-n}(x) = -Q_{n-2}(x)$; let $R_{-n}(x)$ be defined arbitrarily, and let $R_n(x;s,r)$ be determined by Equation (4.1.5). With this extension it is easy to see that Equation (4.1.5) implies that if $\beta_{n,k}$, $(n = 0, 1, 2 \ldots; 0 \leq k \leq n)$ are complex numbers with the property that $\beta_{n,k} = \beta_{n,n-k}$ then we have for $n \geq 0$

$$(4.2.1)\qquad \sum_{k=0}^{n} \beta_{n,k} R_{n-2k}(x;s,r) = \sum_{k=0}^{n} \beta_{n,k} R_{n-2k}(x).$$

It is well known (see section (1.1)) that for $n = 1, 2 \ldots$ the zeros of $R_n(x)$ and $R_n(x;s,r)$ are all real and simple. Let $x_{n,j}$ and $x^*_{n,j}$ $n = 1, 2 \ldots; j = 1, 2, \ldots n$ denote the zeros of $R_n(x)$. and $R_n(x;s,r)$ respectively in increasing order of magnitude. For notational simplicity we will let $x_{n,0} = x^*_{n,0} = -\infty$, $x_{n,n+1} = x^*_{n,n+1} = +\infty$ for $n = 1, 2 \ldots;$ ζ be the zero of $(1-s)R_1(x) + r$, and the constant polynomial have a zero at $-\infty$. Our next results give the relative position of these zeros. First we need the following lemma.

<u>Lemma (4.2.1)</u>. *If* $x_{n,j} = \zeta$ *then* $\operatorname{sgn} R'_n(x_{n,j};s,r) = \operatorname{sgn} R'_n(x_{n,j}) \neq 0$. *where* $f'(x)$ *is the derivative of* $f(x)$ *at* x.

<u>Proof</u>. We obtain from formula (4.1.5) that there exists a k such that $x_{n,j} = x^*_{n,k} = \zeta$ and also

$$(4.2.2) \qquad R'_n(x_{n,j};s,r) = R'_n(x_{n,j}) + (s-1)Q_{n-1}(x_{n,j}).$$

Because $s > 0$ we see from Equation (4.2.2) that in order to prove the lemma we need only show the following two things:

(a) $\operatorname{sgn}\{R'_n(x_{n,j})\} = \operatorname{sgn}\{Q_{n-1}(x_{n,j})\}$

and

(b) $|R'_n(x_{n,j})| \geq |Q_{n-1}(x_{n,j})|$.

From Equations (4.1.1) and (4.1.3) we obtain

$$(4.2.3) \qquad R_n(x)R_{n-1}(x;s,r) - R_{n-1}(x)R_n(x;s,r) = \left(\prod_{i=2}^{n} \lambda_i\right)[(1-s)R_1(x) + r]$$

for $n \geq 1$, where the void product is equal to one. By evaluating the derivative of both sides of this equation at $x_{n,j}$ we obtain

$$(4.2.4) \qquad R_n'(x_{n,j})R_{n-1}(x_{n,j};s,r) - R_{n-1}(x_{n,j})R_n'(x_{n,j};s,r) = \left(\prod_{i=2}^{n} \lambda_i\right)(1-s).$$

From Equation (4.1.5)

$$R_{n-1}(x_{n,j};s,r) = R_{n-1}(x_{n,j}).$$

If we combine this with Equations (4.2.4) and (4.2.2) we obtain

$$R_n'(x_{n,j})R_{n-1}(x_{n,j}) - R_{n-1}(x_{n,j})(R_n'(x_{n,j}) + (s-1)Q_{n-1}(x_{n,j})) = \left(\prod_{i=2}^{n} \lambda_i\right)(1-s)$$

That is,

$$(4.2.5) \qquad R_{n-1}(x_{n,j})Q_{n-1}(x_{n,j}) = \prod_{i=2}^{n} \lambda_i.$$

It is well known (see Szegö [35, p. 46]) that the zeros of $R_{n-1}(x)$ interlace the zeros of $R_n(x)$. Thus $\operatorname{sgn}\{R_{n-1}(x_{n,j})\} = (-1)^{n+j}$. Also because the zeros of $R_n(x)$ are simple $\operatorname{sgn} R_n'(x_{n,j}) = (-1)^{n+j}$. From Equation (4.2.5) and the fact that $\lambda_n > 0$ for all $n \geq 2$

$\text{sgn}\,\{R_{n-1}(x_{n,j})\} = \text{sgn}\,\{Q_{n-1}(x_{n,j})\}$. Thus $\text{sgn}\,\{R_n'(x_{n,j})\} = \text{sgn}\,\{Q_{n-1}(x_{n,j})\}$, which proves part (a).

From the Christoffel - Darboux formula for $\{R_n(x)\}_{n=0}^{\infty}$ we obtain

$$R_n'(x)R_{n-1}(x) - R_n(x)R_{n-1}'(x) = \sum_{i=1}^{n} \left(\prod_{j=2}^{i} \lambda_{n-j+2}\right)[R_{n-i}(x)]^2.$$

Thus

$$R_n'(x_{n,j})R_{n-1}(x_{n,j}) \geq \prod_{i=2}^{n} \lambda_i.$$

By using Equation (4.2.5) we obtain the required results.

Q.E.D.

<u>Theorem (4.2.1)</u>. *Let* n *be a positive integer. If there exists an integer* j *such that* $0 \leq j \leq n$ *and* $\zeta \in [x_{n,j}, x_{n,j+1})$ *then*

(a) *if* $1 - s > 0$,

$$x_{n,1}^* < x_{n,1} < \ldots x_{n,j}^* \leq x_{n,j} < x_{n,j+1} < x_{n,j+1}^* < \ldots x_{n,n} < x_{n,n}^*,$$

(b) *if* $1 - s < 0$

$$x_{n,1} < x_{n,1}^* < \ldots x_{n,j} \leq x_{n,j}^* < x_{n,j+1}^* < x_{n,j+1} < \ldots x_{n,n}^* < x_{n,n}$$

and

(c) *if* $1 - s = 0$, *then if* $r > 0$, $x_{n,j} < x_{n,j}^*$ *and if* $r < 0$ *then* $x_{n,j} > x_{n,j}^*$ *for* $j = 1, 2 \ldots n$. *Where in* (a) *and* (b) $x_{n,j}^* = x_{n,j}$ *if and only if* $x_{n,j} = \zeta$.

<u>Proof</u>: From Equation (4.2.3) it follows that if $1 \leq j < n$ and $\zeta \notin [x_{n,j}, x_{n,j+1}]$ then $R_n(x;s,r)$ has one and only one zero in $(x_{n,j}, x_{n,j+1})$. Also in exactly the same manner we see that if $\zeta \notin [x^*_{n,j}, x^*_{n,j+1}]$ then $R_n(x)$ has one and only one zero in $(x^*_{n,j}, x^*_{n,j+1})$.

For the proof of (c) see (Chihara [12] Thm. 1). We note that $\{R_n(x)\}_{n=0}^{\infty}$ is the $(\frac{1}{s}, -\frac{r}{s})$ co-recursive polynomial set of $\{R_n(x;s,r)\}_{n=0}^{\infty}$. Therefore, (b) follows from (a). Thus we need only prove (a). In order to do this we consider the following three cases. First, if $\operatorname{sgn}[(1-s)R_1(x_{n,1}) + r] = -1$, then from Equation (4.2.3) we obtain $\operatorname{sgn}\{R_n(x_{n,1};s,r)\} = (-1)^{n+1}$. Thus $x^*_{n,1} < x_{n,1}$. By the hypothesis $x_{n,j} \leq \zeta < x_{n,j+1}$. Now from Lemma (4.2.1) and Equation (4.1.5) there exists a $\delta > 0$ such that $\operatorname{sgn}\{R_n(x)\} = \operatorname{sgn}\{R_n(x;s,r)\} \neq 0$ for all $x \in (\zeta-\delta, \zeta)$. Thus $R_n(x)$ and $R_n(x;s,r)$ both have the same number of zeros in $(-\infty, \zeta - \delta/2)$. Therefore,

$$x^*_{n,1} < x_{n,1} < \dots x^*_{n,j} \leq x_{n,j} < x_{n,j+1}$$

where the equality holds if and only if $x_{n,j} = \zeta$. From Equation (4.2.3) $\operatorname{sgn}\{R_n(x_{n,n};s,r)\} = -1$. Thus $x_{n,n} < x^*_{n,n}$ and therefore

$$x^*_{n,1} < x_{n,1} < \dots x^*_{n,j} \leq x_{n,j} < x_{n,j+1} < x^*_{n,j+1} < \dots x_{n,n} < x^*_{n,n}.$$

Second, if $\operatorname{sgn}[(1-s)R_1(x_{n,1}) + r] = 0$. From Equation (4.1.5) it follows that $R_n(\zeta;s,r) = 0$. From Equation (4.2.3) and the fact that $s - 1 > 0$ we obtain $\operatorname{sgn}\{R_n(x_{n,n};s,r)\} = -1$. Thus

$R_n(x;s,r)$ has a zero in $(x_{n,n}, \infty)$. Using these facts and the fact that $R_n(x;s,r)$ has one and only one zero in $(x_{n,j}, x_{n,j+1})$ for $2 \le j < n$ we obtain,

$$x^*_{n,1} = x_{n,1} < x_{n,2} < x^*_{n,2} < \dots < x_{n,n} < x^*_{n,n}.$$

Third, if $\operatorname{sgn}[(1-s)R_1(x_{n,1}) + r] = 1$, then from Equation (4.2.3) it follows that $\operatorname{sgn}\{R_n(x_{n,1};s,r)\} = (-1)^n$ and $\operatorname{sgn}\{R_n(x_{n,n};s,r)\} = -1$. Thus $R_n(x;s,r)$ has an odd number of zeros in $(x_{n,n}, \infty)$ and none or an even number of zeros in $(-\infty, x_{n,1})$. Thus

$$x_{n,1} < x^*_{n,1} < x_{n,2} < x^*_{n,2} < \dots x_{n,n} < x^*_{n,n} .$$

Q.E.D.

From Chihara's work [loc. cit.] it follows that for fixed s, the zeros of $R_n(x;s,r)$ are increasing functions of r and for $r_1 \neq r_2$ the zeros of $R_n(x;s,r_1)$ and $R_n(x;s,r_2)$ interlace.

Corollary I. *Let* r *be fixed.*

(a) *If* $x^*_{n,j} < -c_1$ $(x^*_{n,j} > -c_1)$, *then* $x^*_{n,j}$ *increases (decreases) with increasing* s.

(b) *The zeros, which are less than* $-c_1$ *or which are greater than* $-c_1$ *of two different co-recursive polynomial sets interlace.*

Proof: Let $\{R_{n,i}(x)\}_{n=0}^{\infty}$ be the (s_i,r) co-recursive polynomial set of $\{R_n(x)\}_{n=0}^{\infty}$. Thus $\{R_{n,2}(x)\}_{n=0}^{\infty}$ is the $(\frac{s_2}{s_1}, r(1 - \frac{s_2}{s_1}))$

co-recursive polynomial set of $\{R_{n,1}(x)\}_{n=0}^{\infty}$. If we let $x_{n,j}^{i}$ $j = 1, 2 \ldots n$ be the zeros of $R_{n,i}(x)$ and take $0 < s_1 < s_2$ then we obtain from Theorem (4.2.1)

$$x_{n,1}^{1} < x_{n,1}^{2} < \ldots x_{n,j}^{1} \le x_{n,j}^{2} \le -c_1 < x_{n,j+1}^{2} < x_{n,j+1}^{1} < \ldots x_{n,n}^{2} < x_{n,n}^{1}.$$

Q.E.D.

Corollary II. *Let* $\{Q_n(x)\}_{n=0}^{\infty}$ *be as defined in the three term recursion relation* (4.2.1) *and* $y_{n-1,i}$, $i = 1, 2 \ldots n-1$ *denote the zeros of* $Q_{n-1}(x)$. *If there exists an integer* j *such that* $0 \le j \le n$ *and* ζ *belongs to* $[x_{n,j}, x_{n,j+1})$ *then*

(a) *if* $1 - s > 0$

$$x_{n,1}^{*} < x_{n,1} < y_{n-1,1} < x_{n,2}^{*} < x_{n,2} \ldots x_{n,j}^{*} \le x_{n,j} < y_{n-1,j} < x_{n,j+1} < x_{n,j+1}^{*} < \ldots x_{n,n-1} < x^{*}_{n,n-1} < y_{n-1,n-1} < x_{n,n} < x_{n,n}^{*},$$

(b) *if* $1 - s < 0$

$$x_{n,1} < x_{n,1}^{*} < y_{n-1,1} < x_{n,2} < x_{n,2}^{*} < \ldots \quad x_{n,j} \le x_{n,j}^{*} < y_{n-1,j} < x_{n,j+1}^{*} < x_{n,j+1} < \ldots x_{n,n-1}^{*} < x_{n,n-1} < y_{n-1,n-1} < x_{n,n}^{*} < x_{n,n},$$

(c) *if* $s = 1$, *then if* $r > 0$ $x_{n,j} < x_{n,j}^{*} < y_{n-1,j}$ *and if* $r < 0$ *then* $x_{n,j}^{*} < x_{n,j} < y_{n-1,j}$ *for* $n = 2, 3 \ldots$ *and* $j = 1, 2, 3 \ldots n-1$.

Proof: The results follows directly from Theorem (4.2.1) and the well known fact (see Wall [35]) that for $1 \le j < n$,

$$x_{n,j} < y_{n-1,j} < x_{n,j+1}$$

and

$$x^*_{n,j} < y_{n-1,j} < x^*_{n,j+1}.$$

Q.E.D.

Let us denote the true interval of orthogonality (see Sect. (1.1)) of $\{R_n(x)\}_{n=0}^{\infty}$, $\{R_n(x;s,r)\}_{n=0}^{\infty}$ and $\{Q_n(x)\}_{n=0}^{\infty}$ by $[a,b]$ $[a^*,b^*]$ and $[a^1,b^1]$ respectively. That is, $a = \lim_{n\to\infty} x_{n,1}$, $b = \lim_{n\to\infty} x_{n,n}$ etc. From Corollary II of Theorem (4.2.1) we can obtain a number of interesting results about the relative position of a, a^*, a^1, b^1, b^* and b. First, if $\zeta \in (-\infty, a]$ and $1 - s > 0$; or $\zeta \in [b,\infty)$ and $1 - s < 0$; or $s = 1$ and $r > 0$ then

$$a \le a^* \le a^1 < b^1 \le b \le b^*.$$

Second, if $\zeta \in [b,\infty)$ and $1 - s > 0$; or $\zeta \in (-\infty, a]$ and $1 - s < 0$; or $s = 1$ and $r < 0$; then

$$a^* \le a \le a^1 < b^1 \le b^* \le b.$$

And third, if $\zeta \in (a,b)$, then if $1 - s > 0$

$$a^* \le a \le a^1 < b^1 \le b \le b^*$$

and if $1 - s < 0$

$$a \le a^* \le a^1 < b^1 \le b^* \le b.$$

We wish to obtain criterion that will enable us to replace $\le$ by either $<$ or $=$.

Theorem (4.2.2). *For all integers* $n \ge 0$,

(i) $R_n(x;s,r)$ *has all of its zeros in* (a,∞) *if and only if*

$$\lim_{n\to\infty} \frac{R_n(a)}{Q_{n-1}(a)} \equiv A \le r - (s-1)R_1(a),$$

and (ii) $R_n(x;s,r)$ *has all of its zeros in* $(-\infty,b)$ *if and only if*

$$r - (s-1)R_1(b) \le B \equiv \lim_{n\to\infty} \frac{R_n(b)}{Q_{n-1}(b)}$$

where $A(B)$ *must be replaced by* $-\infty(+\infty)$ *in the case* $a = -\infty(b = +\infty)$.

Proof: From Equations (4.1.1), (4.1.3) and (4.1.4) we obtain

$$(4.2.6) \qquad \frac{R_{n-1}(x)}{Q_{n-2}(x)} - \frac{R_n(x)}{Q_{n-1}(x)} = \frac{\prod_{i=2}^{n} \lambda_i}{Q_{n-2}(x)Q_{n-1}(x)} .$$

Thus for all $n \ge 2$ and $x \ge b$

$$\frac{R_{n-1}(x)}{Q_{n-2}(x)} > \frac{R_n(x)}{Q_{n-1}(x)} > 0 .$$

Thus B exists and by Equation (4.1.5)

$$\frac{R_n(b;s,r)}{Q_{n-1}(b)} = \frac{R_n(b)}{Q_{n-1}(b)} + (s-1)R_1(b) - r .$$

Therefore, if $B \geq r - (s-1)R_1(b)$, then for all $n \geq 1$ $R_n(b;s,r)/Q_{n-1}(b) > 0$ and therefore, $R_n(x;s,r)$ has all of its zeros in $(-\infty,b)$.

Conversely, if for all $n \geq 1$ $R_n(x;s,r)$ has all of its zeros in $(-\infty,b)$, then $R_n(b;s,r)/Q_{n-1}(b) > 0$ and therefore by Equation (4.1.5) for all $n \geq 1$

$$\frac{R_n(b)}{Q_{n-1}(b)} + (s-1)R_1(b) - r > 0.$$

Therefore $B \geq r - (s-1)R_1(b)$.

The proof of (i) is similar and is thus omitted.

Q.E.D.

4.3 DISTRIBUTION FUNCTIONS. From the theory of continued fractions it follows that $Q_{n-1}(z)/R_n(z)$ and $Q_{n-1}(z)/R_n(z;s,r)$ are the nth convergents of the continued fractions

$$(4.3.1) \qquad K(z) = \frac{1|}{|z+c_1} - \frac{\lambda_2|}{|z+c_2} - \frac{\lambda_3|}{|z+c_3} \cdots$$

and

$$(4.3.2) \qquad K^*(z) = \frac{1|}{|s(z+c_1)-r} - \frac{\lambda_2|}{|z+c_2} - \frac{\lambda_3|}{|z+c_3} \cdots$$

respectively. $Q_n(z)$ is the denominator of the nth convergent of the continued fraction

$$(4.3.3) \qquad K_1(z) = \frac{1|}{|z+c_2} - \frac{\lambda_3|}{|z+c_3} - \frac{\lambda_4|}{|z+c_4} \cdots .$$

We will denote by $\sigma(t)$, $\sigma^*(t)$ and $\sigma_1(t)$ the solution of the moment problem associated with $\{R_n(x)\}_{n=0}^{\infty}$, $\{R_n(x;s,r)\}_{n=0}^{\infty}$ and $\{Q_n(x)\}_{n=0}^{\infty}$ respectively.

Theorem (4.3.1). *If $\sigma(t)$ is the solution of a determinate moment problem, then so is $\sigma^*(t)$ and $\sigma_1(t)$. In this case if ζ is the zero of $(s-1)R_1(x) - r$ then,(a) no two of the three distributions have a common point of discontinuity except possibly at ζ; (b) $\sigma(t)$ has a discontinuity at ζ if and only if $\sigma^*(t)$ has a discontinuity at ζ. If $\sigma(t)$ has a discontinuity at ζ then $\sigma_1(t)$ does not have a discontinuity at ζ.*

Proof. Because $\sigma(t)$ is the solution of a determinate moment problem, then by Lemma (1.1.2) $\sigma_1(t)$ is also the solution of a determinate moment problem. Hamburger [21] showed that $K(z)$ converges completely if and only if $\sigma(t)$ is the solution of a determinate moment problem (see Shohat and Tamarkin [34, Thm. 2.10]). Thus $K(z)$ converges completely. By Carleman's Thm. (see Shohat and Tamarkin [34, pp. 59-60]), $K^*(z)$ also converges completely. Thus $\sigma^*(t)$ is the solution of a determinate moment problem.

Let us define,

$$q_{n-1}(x) = \left(\prod_{i=3}^{n+1} \lambda_i\right)^{1/2} Q_{n-1}(x)$$

$$p_n(x) = \left(\prod_{i=2}^{n+1} \lambda_i\right)^{1/2} R_n(x)$$

$$p_n^*(x) = \left(\prod_{i=2}^{n+1} \lambda_i\right)^{1/2} R_n(x;s,r).$$

$\{q_n(x)\}_{n=0}^{\infty}$, $\{p_n(x)\}_{n=0}^{\infty}$ and $\{p_n^*(x)\}_{n=0}^{\infty}$ are the polynomial sets orthonormal with respect to the distribution function $\sigma_1(t)$, $\sigma(t)$ and $\sigma^*(t)$ respectively. Because $\sigma(t)$ is the solution of a determinate moment problem then by Lemma (1.1.3) $\sum_{n=0}^{\infty} |p_n(x)|^2$ diverges at all points of continuity of $\sigma(t)$ and converges at all points of discontinuity of $\sigma(t)$. The same is true for $\sum_{n=0}^{\infty} |q_n(x)|^2$ and $\sum_{n=0}^{\infty} |p_n^*(x)|^2$ with respect to the distributions $\sigma_1(t)$ and $\sigma^*(t)$.

By Equation (4.1.5), it follows that if $(1-s)R_1(\xi) - r \neq 0$ and any two of $\sum_{n=0}^{\infty} |p_n(\xi)|^2$, $\sum_{n=0}^{\infty} |q_n(\xi)|^2$ or $\sum_{n=0}^{\infty} |p_n^*(\xi)|^2$ converge, then the other one must also converge. Thus if any two of $\sigma_1(t)$, $\sigma(t)$ or $\sigma^*(t)$ have a common discontinuity ξ and $(1-s)R_1(\xi) - r \neq 0$ then $\sum_{n=0}^{\infty} |p_n(\xi)|^2$ and $\sum_{n=0}^{\infty} |q_n(\xi)|^2$ both converge. This contradicts Lemma (1.1.4). Thus part (a) is proven.

By Lemma (1.1.3) $\sigma(t)$ has a discontinuity at ζ the zero of $(1-s)R_1(x) - r$ if and only if $\sum_{n=0}^{\infty} |p_n(\zeta)|^2 < \infty$. But by Equation (4.1.5) we have,

$$\sum_{n=0}^{\infty} |p_n(\zeta)|^2 = \sum_{n=0}^{\infty} |p_n^*(\zeta)|^2.$$

Therefore, $\sigma^*(t)$ has a discontinuity at ζ if and only if $\sigma(t)$ has a discontinuity at ζ.

By Lemma (1.1.4) $\sum_{n=0}^{\infty} |p_n(x)|^2$ and $\sum_{n=0}^{\infty} |q_n(x)|^2$ don't both converge together. Therefore, if $\sigma(t)$ has a discontinuity at z, then $\sigma_1(t)$ does not have a discontinuity at z.

Q.E.D.

Next we construct an integral representation of $\sigma^*(t)$. From here to the end of the Chapter we assume that $\sigma(t)$ is the solution of a determinate moment problem. Thus by Hamburger [21] $K(z)$, $K^*(z)$ and $K_1(z)$ converge completely to

(4.3.4) $$F(z) = \int_{-\infty}^{\infty} \frac{d\sigma(t)}{z - t}$$

(4.3.5) $$F^*(z) = \int_{-\infty}^{\infty} \frac{d\sigma^*(t)}{z-t}$$

(4.3.6) $$F_1(z) = \int_{-\infty}^{\infty} \frac{d\sigma_1(t)}{z - t}$$

respectively. From Equations (4.3.1), (4.3.2) and (4.3.3) we have

$$F(z) = [R_1(z) - \lambda_2 F_1(z)]^{-1}$$

and

$$F^*(z) = [sR_1(z) - r - \lambda_2 F_1(z)]^{-1}.$$

By eliminating $F_1(z)$ from these last two equations we obtain

(4.3.7) $$F^*(z) = \frac{F(z)}{1+[(s-1)R_1(z)-r]F(z)}.$$

We now use the same procedure as Sherman [29] to prove the following theorem.

Theorem (4.3.2). *If the analytic continuation of* $F^*(z)$ *is regular on* $[x_0,x]$ *then*

$$\sigma^*(x) - \sigma^*(x_0) = \frac{1}{\pi}\int_x^{x_0} \text{Im} \; \frac{F(u)}{1+[(s-1)R_1(u)-r]F(u)} \; du.$$

Proof: By using the Stieltjes Inversion Formula (see Shohat and Tamarkin [34, p. xiv]) we obtain.

$$\sigma^*(x) - \sigma^*(x_0) = \lim_{y\to 0^+} \frac{1}{\pi}\int_x^{x_0} \text{Im}\; F^*(u+iy)du \tag{4.3.8}$$

where $\sigma^*(t)$ has been normalized by

$$\sigma^*(t) = \frac{\sigma^*(x+0)-\sigma^*(x-0)}{2}$$

Now consider the rectangle R with vertices x_0, x, $x + iy$, $x_0 + iy$. $F^*(y)$ is known to be regular in any closed region of the complex plane that doesn't contain the real axis (see Hamburger [21]) and by hypothesis $F^*(z)$ is analytic on $[x_0,x]$. Thus by Cauchy's Integral Formula we have

$$\int_x^{x_0} F^*(u)du + \int_0^y F^*(x_0 + iv)dv + \int_{x_0}^x F^*(u+iy)du + \int_y^0 F^*(x+iv)dv = 0.$$

Because $F^*(z)$ is analytic in the rectangle R we have

$$\lim_{y\to 0}\int_0^y F^*(x_0+iv)dv = 0$$

and

$$\lim_{y\to 0}\int_y^0 F^*(x+iv)dv = 0.$$

Therefore, by Equation (4.3.7)

$$\begin{aligned}\sigma^*(x) - \sigma^*(x_0) &= \lim_{y\to 0^+} \frac{1}{\pi}\int_x^{x_0} \text{Im } F^*(u+iy)du \\ &= \frac{1}{\pi}\int_x^{x_0} \text{Im } F^*(u)du \\ &= \frac{1}{\pi}\int_x^{x_0} \text{Im } \left\{\frac{F(u)}{1+[(s-1)R_1(u)-r]F(u)}\right\} du.\end{aligned}$$

Q.E.D.

We obtain from this last theorem that if

$$\frac{F(u)}{1+[(s-1)R_1(u)-r]F(u)}$$

is analytic on $[a^*,b^*]$ then $\{R_n(x;s,r)\}_{n=0}^{\infty}$ has the weight function

$$\text{Im }\left\{\frac{F(u)}{1+[(s-1)R_1(u)-r]F(u)}\right\}.$$

Let us denote the *Spectrum* of $\sigma(t)$ by $Sp(\sigma(t))$ and define it by

$$Sp(\sigma(t)) = \{x \in R' \,|\, \sigma(x+\varepsilon) - \sigma(x-\varepsilon) > 0, \ \forall\, \varepsilon > 0\}.$$

It is easy to show that $Sp(\sigma(t))$ is a closed set in R_1. The next theorem gives us some properties of $Sp(\sigma^*(t))$.

__Theorem (4.3.3).__ *If* $\sigma(t)$ *is the solution of a determinate moment problem and is constant on the interval* $[\alpha,\beta]$, *then:*

(a) *If* ζ *the zero of* $(1-s)R_1(x) + r$ *does not belong to* (α,β), *then there is at most one point of* $Sp(\sigma^*(t))$ *in* (α,β); *if* ζ *does belong to* (α,β), *then there is at most two points of* $Sp(\sigma^*(t))$ *in* (α,β).

(b) *Let* $\xi \in (\alpha,\beta)$. $\xi \in Sp(\sigma^*(t))$ *if and only if* $\operatorname{Res}_{t=\xi} F^*(t) > 0$ *and* $F(\xi) = (r-(s-1)R_1(\xi))^{-1}$.

(c) *If* $\xi \in (\alpha,\beta) \cap Sp(\sigma^*(t))$ *then*

$$\sigma^*(\xi+0) - \sigma^*(\xi-0) = \operatorname*{Res}_{t=\xi} \frac{F(t)}{1+[(s-1)R_1(t)-r]F(t)} .$$

__Proof:__ By Theorem (4.3.1) $\sigma^*(t)$ is the solution of a determinate moment problem.

(a) The first and second part of (a) are proved similarily. We shall only prove the first part. That is, let ζ, the zero of $(1-s)R_1(x) + r$, not belong to (α,β). If the number of points in $(\alpha,\beta) \cap Sp(\sigma^*(t))$ is greater than one, then by noting the construction of the natural representation for $\sigma^*(t)$ we obtain the existence of a positive integer n_0 such that for all $n \geq n_0$ $R_n(x;s,r)$ has two or more zeros in (α,β). If we let $(\bar{\alpha},\bar{\beta})$ be the largest open interval containing (α,β) such that $\sigma(t)$ is constant, then by Theorem (4.2.1) and by noting the construction of the natural representation of $\sigma(t)$ we have that for all $n \geq n_0$ $R_n(x)$ has two or more zeros in $(\bar{\alpha},\bar{\beta})$. This is a contradiction (see Szegö [35] Thm. (3.4.12)).

(b) Let $\xi \in Sp(\sigma^*(t)) \cap (\alpha,\beta)$. Because $\sigma(t)$ is constant in $[\alpha,\beta]$ it is well known (See Szegö [35] Thm. (3.41.2)) that $R_n(x)$ has at most one zero in (α,β). Therefore by Theorem (4.2.1) $R_n^*(x)$ has at most 3 zeros in (α,β). Because $\sigma^*(t)$ is the solution of a determinate moment problem and by noting the construction of the natural representation of $\sigma^*(t)$ we have that $(\alpha,\beta) \cap Sp(\sigma^*(t))$ has at most three points. Thus ξ is an isolated point in $Sp(\sigma^*(t))$. Choose x_0 and x belonging to (α,β) such that $\{\xi\} = (x_0,x) \cap Sp(\sigma^*(t))$. Let $x + iy$, $x_0 + iy$, $x_0 - iy$, $x - iy$, where $y > 0$, be the corner points of a rectangle R. From Equation (4.3.5) we see that $F^*(z)$ is analytic everywhere in and on R except at ξ. Now apply Cauchy's residue theorem to $F^*(z)$ on the boundary of the rectangle R. We obtain

$$(4.3.9) \qquad \int_x^{x_0} [F^*(u+iy) - F^*(u-iy)]du + i\int_{-y}^{y} [F^*(x+iv) - F^*(x_0+iv)]dv$$

$$= 2\pi i \operatorname*{Res}_{z=\xi} F^*(z).$$

It follows from Equation (4.3.5) that $F^*(\bar{Z}) = \overline{F^*(Z)}$; thus Equation (4.3.9) becomes

$$\frac{1}{\pi}\int_x^{x_0} \operatorname{Im} F^*(u+iv)du = \operatorname*{Res}_{z=\xi} F^*(z) + \frac{1}{\pi}\int_0^{\pi} [F^*(x+iv)-F^*(x_0+iv)]dv.$$

By using Equation (4.3.8) and the fact that $F^*(z)$ is analytic on the rectangle R, we obtain

$$\text{(3.3.10)} \qquad \sigma^*(x) - \sigma(x_0) = \operatorname*{Res}_{z=\xi} \left\{ \frac{F(z)}{1+[(s-1)R_1(z)-r]F(z)} \right\} > 0.$$

Therefore, from Equation (4.3.7) we also have that

$$\text{(4.3.11)} \qquad F(\xi) = (r-(s-1)R_1(\xi))^{-1}.$$

This shows that if $\xi \in (\alpha,\beta) \cap Sp(\sigma^*(t))$ then Equation (4.3.10) is satisfied and $\operatorname*{Res}_{z=\xi} F^*(z) > 0$. The proof of the converse is similar and is thus omitted.

Part (c) follows directly from Equation (3.3.10).

Q.E.D.

From Theorem (4.2.2) we see that if $r - (s-1)R_1(b) > B \equiv \lim_{n\to\infty} R_n(b)/Q_{n-1}(b)$ then there exists an N such that the largest zero of $R_N(x;s,r)$ is in (b,∞). Therefore $b < b^*$ and from Theorem (4.2.1) for all $n \geq N$, $R_n(x;s,r)$ has one and only one of its zeros in (b,b^*). Thus $\sigma^*(t)$ is constant in (b,b^*) and therefore $\sigma^*(t)$ must have a jump at b^*. This jump can be found by using the results of the last Theorem.

We will use some of these results in the next chapter to study one of the Σ polynomial sets.

CHAPTER V

SOME PROPERTIES OF THE POLYNOMIAL SETS IN Σ

5.1 GENERAL PROPERTIES. In this chapter we wish to give some relationships that the polynomial sets in Σ satisfy. In the last part of the chapter we will study in some detail two of the polynimial sets in Σ.

We first give some general properties that a number of the polynomial sets in Σ satisfy. In Chapter III we have found for all polynomial sets $\{A_n(x)\}_{n=0}^{\infty}$ that belong to Σ the sequences $\{b_n\}_{n=1}^{\infty}$, $\{\lambda_n\}_{n=2}^{\infty}$ and $\{\alpha_n\}_{n=1}^{\infty}$ such that

$$(5.1.1)\quad \begin{cases} A_0(x) = 1 \quad A_1(x) = 2b_1x \\ A_n(x) = 2b_nxA_{n-1}(x) - \lambda_nA_{n-2}(x) & (n \geq 2) \end{cases}$$

and for $n \geq 0$

$$(5.1.2)\quad w(x)A_n(x) \sim \sum_{k=0}^{\infty} (\prod_{i=1}^{k} \alpha_i)(\prod_{j=2}^{n+k+1} \lambda_j/b_j)U_{n+2k}(x).$$

where $w(x)$ is the weight function for $\{A_n(x)\}_{n=0}^{\infty}$. If we take $n = 0$ in relation (5.1.2) we obtain

$$\int_{-1}^{1} U_n(x)w(x)dx = \begin{cases} f_k & \text{if } n = 2k \\ 0 & \text{if } n = 2k+1 \end{cases}$$

where $f_k = \prod_{i=1}^{k} \alpha_i \lambda_{i+1}/b_{i+1}$. By checking the sequence $\{f_k\}_{k=1}^{\infty}$ for each of the elements in Σ it is easy to see that $\{f_k\}_{k=0}^{\infty}$ is eventually monotonic and $\lim_{k\to\infty} f_k = 0$. Thus it follows that $\sum_{n=0}^{\infty} (f_n - f_{n+1})$ is absolutely convergent and therefore by Theorem (2.2.1) $w(x)$ is continuous and has the representation

$$w(x) \equiv \frac{2}{\pi} \sqrt{1-x^2} \sum_{k=0}^{\infty} f_k U_{2k}(x) \qquad (-1 \le x \le 1).$$

For some of the polynomial sets we can say more about their weight function. By checking the $\{f_k\}_{k=1}^{\infty}$ for each of the elements in Σ one finds that

$$\sum_{n=0}^{\infty} n|f_n - f_{n+1}| < \infty$$

for all elements in Σ except $\{Q_n(x;\, \gamma,\, r)\}_{n=0}^{\infty}$ where $4/3 \le \gamma < 2$ and $P_n^{\lambda}(x)$ where $0 < \lambda \le 1/2$. Thus by Theorem (2.2.1) part (c) it follows that if $\{A_n(x)\}_{n=0}^{\infty} \in \Sigma$ and $\{A_n(x)\}_{n=0}^{\infty}$ is neither $\{Q_n(x;\, \gamma,\, r)\}$ where $4/3 \le \gamma < 2$ nor $\{P_n^{\lambda}(x)\}_{n=0}^{\infty}$ where $0 < \lambda \le 1/2$ then the weight function for $\{A_n(x)\}_{n=0}^{\infty}$ is continuously differentiable on $(-1,1)$. But we know that the weight function for $\{P_n^{\lambda}(x)\}_{n=0}^{\infty}$ is $(1-x^2)^{\lambda-1/2}$ which is continuously differentiable on $(-1,1)$ for $0 < \lambda$. Thus we have the following theorem.

Theorem (5.1.1). (i) *If* $\{A_n(x)\}_{n=0}^{\infty} \in \Sigma$ *and* $\{A_n(x)\}_{n=0}^{\infty}$ *is not* $\{Q_n(x;\, \gamma,\, r)\}$ *where* $4/3 \le \gamma < 2$ *then the weight function for* $\{A_n(x)\}_{n=0}^{\infty}$ *is continuously differentiable on* $(-1,1)$.

(ii) *The weight function associated with* $\{Q_n(x;\ \gamma,\ r)\}_{n=0}^{\infty}$ *where* $4/3 \leq \gamma < 2$ *is continuous on* $(-1,1)$.

(iii) *For all* $\{A_n(x)\}_{n=0}^{\infty} \in \sum$ *its weight function* $w(x)$ *has a representation*

$$w(x) \equiv \frac{2}{\pi}\sqrt{1-x^2}\sum_{k=0}^{\infty} f_k U_{2k}(x) \qquad (-1 \leq x \leq 1),$$

where $f_k = \prod_{i=1}^{k} \alpha_i \lambda_{i+1}/b_{i+1}$.

This theorem gives us some properties of the weight function in $(-1,1)$. We wish to investigate what Lipschitz condition the weight function satisfies on $[-1,1]$. We first need the following Lemma.

Lemma (5.1.1). *If*

$$w(x) \equiv \sqrt{1-x^2}\sum_{k=0}^{\infty} f_k U_{2k}(x) \qquad (-1 \leq x \leq 1)$$

and

$$\sum_{k=0}^{\infty} k|f_k| < \infty$$

then $w(x)$ *satisfies Lipschitz condition of order* $1/2$ *on* $[-1,1]$.

Proof: From well known properties of the Trigonometric functions it follows that

$$\begin{aligned}|w(\cos(\theta+h)) - w(\cos(\theta-h)| &\leq 2\sum_{k=0}^{\infty} |f_k \cos(2k+1)\theta \sin(2k+1)h| \\ &\leq 2|h|\sum_{k=0}^{\infty} (2k+1)|f_k| \\ &\leq M|h|.\end{aligned}$$

Thus $w(\cos\theta)$ satisfies a Lipschitz condition of order 1 on $[0,\pi]$. Thus by Lemma (2.2.3) $w(x)$ satisfies a Lipschitz condition of order 1/2 on $[-1,1]$.

Q.E.D.

Again by checking the sequences $\{f_k\}_{k=0}^{\infty}$ for each of the elements in Σ it is easy to show that

$$\sum_{k=0}^{\infty} k|f_k| < \infty$$

for all elements of Σ except $\{Q_n(x;\ \gamma,\ r)\}_{n=0}^{\infty}$ with $\gamma > 1/2$ and $\{P_n^{\lambda}(x)\}_{n=0}^{\infty}$ with $0 < \lambda < 1$. Thus by Lemma (5.1.1) it follows that the weight function in these cases must satisfy Lipschitz condition of order 1/2 on $[-1,1]$. For $\{P_n^{\lambda}(x)\}_{n=0}^{\infty}$ we know that the weight function is $(1-x^2)^{\lambda-1/2}$. Thus, for $1/2 < \lambda < 1$, the weight function satisfie Lipschitz condition of order $\lambda - 1/2$ on $[-1,1]$ and for $\lambda = 1/2$ the weight function satisfies Lipschitz condition of order 1 on $[-1,1]$ and thus satisfies Lipschitz condition of order 1/2 on $[-1,1]$. For $\{Q_n(x;\ \gamma,\ r)\}_{n=0}^{\infty}$ it can be shown that

$$f_{n,0} = O(n^{-2\delta}) \qquad \text{as } n \to \infty,$$

where $\delta = \frac{1-\gamma}{\gamma}$. This follows from the fact that

$$f_n = \prod_{j=1}^{k} \frac{j-\delta}{j+\delta} \qquad (kr-k \le n < (k+1)(r-1))$$

and therefore

$$f_n \le -\frac{\Gamma(\delta)}{\Gamma(-\delta)} \frac{e^{2\delta\gamma}}{n^{2\delta}}$$

where $\Gamma(x)$ is the Gamma function and γ is the Euler constant. It can be shown that for all $\varepsilon > 0$

$$f_n \neq O\left(\frac{1}{n^{2(\delta+\varepsilon)}}\right) \qquad \text{as } n \to \infty.$$

Thus by Theorem (2.2.2) the weight function does not satisfy a Lipschitz condition of 1/2 on $[-1,1]$ for $\gamma > 1$. From these remarks we have the following theorem.

Theorem (5.1.2). *Let* $\{A_n(x)\}_{n=0}^{\infty}$ *be a polynomial set in* Σ. $\{A_n(x)\}_{n=0}^{\infty}$ *is neither* $\{Q_n(x;\ \gamma,\ r)\}_{n=0}^{\infty}$ *with* $\gamma > 1/2$ *nor* $\{P_n^{\lambda}(x)\}_{n=0}^{\infty}$ *with* $\lambda \in (0,1/2)\cup(1/2,1)$ *if and only if* $\{A_n(x)\}_{n=0}^{\infty}$ *has a weight function that satisfies Lipschitz condition of order* 1/2 *on* $[-1,1]$.

The results of this last theorem are not sharp. That is, there exists polynomial sets in Σ that are neither $\{Q_n(x;\ \gamma,\ r)\}_{n=0}^{\infty}$ with $\gamma > 1$ nor $\{P_n^{\lambda}(x)\}_{n=0}^{\infty}$ with $\lambda \in (0,1/2) \cup (1/2,1)$ whose weight function satisfies a Lipschitz condition of order greater than 1/2. For example $\{P_n^{\lambda}(x)\}_{n=0}^{\infty}$ with $\lambda > 1$. We do not pursue this problem any further.

From Chapter III we know the sequences $\{b_n\}_{n=1}^{\infty}$, $\{\lambda_n\}_{n=2}^{\infty}$ and $\{\alpha_n\}_{n=1}^{\infty}$ for each of the polynomial sets in Σ. Thus by using the results of section (2.3) one can easily find the moments, orthogonality relation (ie the value of $\int_{-1}^{1} (A_n(x))^2 w(x)dx$), and the three term recursion relation that each polynomial set in Σ satisfies.

5.2 THE POLYNOMIAL SET $\{R_n^q(x)\}_{n=0}^{\infty}$. We showed in Theorem (3.4.2) that if $\{A_n(x)\}_{n=0}^{\infty}$ is a polynomial set in Σ that satisfies the three term recursion relation (5.1.1) with $b_1 = b_2$ then

$$A_0(x) = 1 \qquad\qquad A_1(x) = 2b_1x$$

$$A_n(x) = 2b_1xA_{n-1}(x) - b_1^2[1-(1-\gamma_2/b_1)^{n-1}]A_{n-2}(x) \qquad (n \geq 2),$$

and

$$\alpha_k = \frac{(1-\gamma_2/b_1)^k}{b_1[(1-\gamma_2/b_1)^k-1]}$$

where $|1 - \gamma_2/b_1| < 1$ and $\gamma_2 = \lambda_2/b_2$. This polynomial set is equivalent to $\{R_n^q(x)\}_{n=0}^{\infty}$ defined by

$$R_n^q(x) = A_n(x)/(b_1)^n.$$

Thus $\{R_n^q(x)\}_{n=0}^{\infty}$ has the three term recursion relation

$$(5.2.1) \qquad \begin{cases} R_0^q(x) = 1 \qquad\qquad R_1^q(x) = 2x \\ R_n^q(x) = 2xR_{n-1}^q(x) - (1-q^{n-1})R_{n-2}^q(x) \qquad (n \geq 2). \end{cases}$$

$\{R_n^q(x)\}_{n=0}^{\infty}$ is a polynomial set in Σ with

$$\alpha_k = q^k/(q^k-1).$$

Theorem (5.2.1). $\{R_n^q(x)\}_{n=0}^{\infty}$ *has the following generating function,*

$$\sum_{n=0}^{\infty} \frac{R_n^q(x)t^n}{[q]_n} = \prod_{i=0}^{\infty} (1 - 2q^i xt + q^{2i}t^2)^{-1}$$

where $[q]_0 = 1$; $[q]_n = (1-q)(1-q^2) \cdots (1-q^n)$ *for* $n \geq 1$.

Proof: From the three term recursion relation (5.2.1) we have

$$\sum_{n=0}^{\infty} \frac{R_{n+1}^q(x)t^{n+1}}{[q]_n} = \sum_{n=0}^{\infty} \frac{2xR_n^q(x)t^{n+1}}{[q]_n} - \sum_{n=0}^{\infty} \frac{(1-q^n)R_{n-1}^q(x)t^{n+1}}{[q]_n} .$$

i.e.

$$\sum_{n=0}^{\infty} \frac{(1-q^{n+1})R_{n+1}^q(x)t^{n+1}}{[q]_{n+1}} = 2xt \sum_{n=0}^{\infty} \frac{R_n^q(x)t^n}{[q]_n} - t^2 \sum_{n=0}^{\infty} \frac{R_{n-1}^q(x)t^{n-1}}{[q]_{n-1}} .$$

i.e.

$$\sum_{n=0}^{\infty} \frac{(1-q^n)R_n^q(x)t^n}{[q]_n} = 2xt \sum_{n=0}^{\infty} \frac{R_n^q(x)t^n}{[q]_n} - t^n \sum_{n=0}^{\infty} \frac{R_n^q(x)t^n}{[q]_n} .$$

If we let

$$F(x,t) = \sum_{n=0}^{\infty} \frac{R_n^q(x)t^n}{[q]_n} ,$$

we obtain

$$F(x,t) = \frac{F(x,qt)}{1-2xt+t^2} .$$

This becomes

$$F(x,t) = \prod_{i=0}^{\infty} (1-2q^i xt+q^{2i}t^2)^{-1}$$
$$= \sum_{n=0}^{\infty} \frac{R_n^q(x)t^n}{[q]_n} .$$

Q.E.D.

If we let $x = \cos\theta$ in the generating function we obtain

$$\sum_{n=1}^{\infty} \frac{R_n^q(\cos\theta)t^n}{[q]_n} = \prod_{m=0}^{\infty} (1-q^m(e^{i\theta}+e^{-i\theta})t+q^{2m}t^2)^{-1}$$
$$= \prod_{m=0}^{\infty} (1-q^m e^{i\theta}t)^{-1}(1-q^m e^{-i\theta}t)^{-1}$$
$$= \sum_{n=0}^{\infty} \sum_{r=0}^{n} \frac{e^{(n-2r)\theta}}{[q]_{n-r}[q]_r} t^n.$$

Therefore,

$$R_n^q(\cos\theta) = \sum_{r=0}^{n} {n \brack r}_q T_{n-2r}(x)$$

where $\{T_n(x)\}_{n=0}^{\infty}$ is the Chebychev polynomial set of the first kind, $T_{-k}(x) = T_k(x)$ for $k \geq 1$, and ${n \brack r}_q = [q]_n/([q]_{n-r}[q]_r)$.

From Theorems (2.1.1), (3.4.2) and (5.1.1) it follows that $\{R_n^q(x)\}_{n=0}^{\infty}$ is orthogonal on $[-1,1]$ with respect to the continuously differentiable weight function $w(x)$ that has the following representation.

$$w(x) \equiv \sqrt{1-x^2} \sum_{k=0}^{\infty} (-1)^k q^{k(k+1)/2} U_{2k}(x) \qquad (-1 \leq x \leq 1).$$

By Theorem (2.3.2) and Corollary I of Theorem (2.3.1) we have the following orthogonality relationship for $\{R_n^q(x)\}_{n=0}^{\infty}$

$$\int_{-1}^{1} P_n^q(x)P_n^q(x)w(x)dx = \delta_{m,n}\frac{\pi}{2}[q]_n$$

and the moments associated with $w(x)$ are given by,

$$\int_{-1}^{1} x^r w(x)dx = \begin{cases} 0 & \text{if } r = 2m+1 \\ \frac{\pi}{2^{2m+1}}\left\{\binom{2m}{m} + \sum_{i=0}^{m-1}\binom{2m}{i}(-1)^{m-i}[q]_{m-i-1}(1+q^{m-i})\right\} & \text{if } r = 2m. \end{cases}$$

5.3 <u>SOME PROPERTIES OF $\{P_n(x;\ s,\ r)\}_{n=0}^{\infty}$</u>. We showed in Theorem (3.4.4) that if $\{A_n(x)\}_{n=0}^{\infty}$ is a polynomial set in Σ and $\{A_n(x)\}_{n=0}^{\infty}$ satisfies the three term recursion relation (5.1.5) with $b_1 \neq b_2$ and $b_2 = \lambda_2/b_2 < 2b_1$ then

$$A_0(x) = 1 \qquad A_1(x) = 2b_1x$$

$$A_n(x) = 2b_2xA_{n-1}(x) - b_2^2A_{n-2}(x) \qquad (n \geq 2),$$

and

$$\alpha_n = (b_1)^{-1} - (b_2)^{-1}.$$

If we define the polynomial set $\{P_n(x;\ s,\ r)\}_{n=0}^{\infty}$ by

$$P_0(x;\ s,\ r) = 1 \qquad P_1(x;\ s,\ r) = 2sx - r$$

$$P_n(x;\ s,\ r) = 2xP_{n-1}(x;\ s,\ r) - P_{n-2}(x;\ s,\ r) \qquad (n \geq 2),$$

we have that $\{A_n(x)\}_{n=0}^{\infty}$ is equivalent to $\{P_n(x; s, r)\}_{n=0}^{\infty}$ if $r = 0$ and $s > 1/2$. In fact $A_n(x) = (b_2)^n P_n(x; b_1/b_2, 0)$ for $n = 0, 1, 2 \dots$. Thus if $s > 1/2$, then $\{P_n(x; s, 0)\}_{n=0}^{\infty}$ is orthogonal on $[-1,1]$ with respect to the weight function given by (see Theorem (2.1.1), (3.4.4) and (5.1.1))

$$(5.3.1) \qquad w(x) = \sqrt{1-x^2} \sum_{k=0}^{\infty} \left(\frac{1-s}{s}\right)^k U_{2k}(x) \qquad (-1 < x < 1).$$

From Corallary I of Theorem (2.3.1) we have the orthogonality relation

$$(5.3.2) \qquad \int_{-1}^{1} P_n(x; s, 0) P_m(x; s, 0)\, w(x)dx = \begin{cases} s\,\frac{\pi}{2}\,\delta_{n,m} & (n, m > 0) \\ \frac{\pi}{2}\,\delta_{m,0} & (m \geq 0), \end{cases}$$

and the moments are given by

$$\int_{-1}^{1} x^r w(x)dx = \begin{cases} 0 & (r = 2m + 1) \\ \frac{\pi}{2^{2m+1}} \left[\binom{2m}{m} + \frac{1-2s}{s} \sum_{i=0}^{m-1} \binom{2m}{i} \left(\frac{s-1}{s}\right)^{m-i-1} \right], & (r = 2m) \end{cases}$$

where $m = 0, 1, 2 \dots$ and $s > 1/2$.

Horadam (see [24], [25] and [26]) introduced a sequence of numbers $\{w_n(a, b{:}p, q)\}$ defined by

$$w_0 = a \qquad w_1 = b$$

$$w_n = pw_{n-1} - qw_{n-2}.$$

Therefore $P_n(x; s, r) = w_n(1, 2sx - r:2x, 1)$. From Horadam's work (Loc. cit.) one can easily deduce a number of identities that $\{P_n(x; s, r)\}_{n=0}^{\infty}$ satisfy.

For example,

$$(5.3.3) \qquad \sum_{n=0}^{\infty} P_n(x; s, r)t^n = \frac{1+(2(s-1)x-r)t}{1-2xt+t^2},$$

$$(5.3.4) \qquad P_{n+m}(x; s, r) - [2x(s-1)-r]P_{n+m-1}(x; s, r)$$
$$= P_m(x; s, r)P_n(x; s, r) - P_{m-1}(x; s, r)P_{n-1}(x; s, r),$$

$$(5.3.5) \qquad P_n(x; s, r) = \frac{1}{2}\left\{(x + \sqrt{x^2-1})^n + (x - \sqrt{x^2-1})^n + \frac{(2s-1)x-r}{\sqrt{x^2-1}}[(x + \sqrt{x^2-1})^n - (x - \sqrt{x^2-1})^n]\right\}.$$

By letting $x = \cos\theta$ we obtain from Equation (5.3.5)

$$(5.3.6) \qquad P_n(x; s, r) = sU_n(x) - rU_{n-1}(x) + (s-1)U_{n-2}(x).$$

Of course Equations (5.3.3), (5.3.4), (5.3.5) and (5.3.6) are valid for all values of s and r. We know that $P_n(x; s, r)$ is orthogonal on $(-1,1)$ if $r = 0$ and $s > 1/2$. Thus by using Equation (5.3.6) and Equation (A.2.4) we find the weight function for $\{P_n(x; s, 0)\}_{n=0}^{\infty}$, $s > 1/2$, to be

$$w(x) = s(1-x^2)^{1/2}[4s(s-1)x^2 + 1]^{-1}.$$

By combining this with Equation (5.3.1) we obtain

$$s(4s(s-1)x^2 + 1)^{-1} = \sum_{k=0}^{\infty} ((1-s)/s)^k \, U_{2k}(x)),$$

which converges uniformily on $[-1,1]$.

We now wish to relax the condition $r = 0$ and $s > 1/2$ and use some of the results of Chapter IV to obtain the weight function and the true interval of orthogonality of $\{P_n(x;\ s,\ r)\}_{n=0}^{\infty}$. Let us adopt the notation used in Chapter IV. Thus, $R_n(x) = 2^{-n}U_n(x) = Q_n(x)$,

$$R_0(x;\ s,\ r) = 1, \qquad R_1(x;\ s,\ r) = sx - \frac{r}{2},$$

$$R_n(x;\ s,\ r) = xR_{n-1}(x;\ s,\ r) - \frac{1}{4} R_{n-2}(x;\ s,\ r) \qquad (n \geq 2),$$

(5.3.7) $$F(z) = \lim_{n\to\infty} \frac{R_{n-1}(z)}{R_n(z)} = \lim_{n\to\infty} \frac{2U_{n-1}(z)}{U_n(z)}$$

and

$$F^*(z) = F(z)[1+[(s-1)z - \frac{r}{2}]F(z)]^{-1}$$

where $z \notin [-1,1]$. Because $R_n(x;\ s,\ r) = 2^{-n}P_n(x;\ s,\ r)$, therefore the true interval of orthogonality of $\{P_n(x;\ s,\ r)\}_{n=0}^{\infty}$ and $\{R_n(x;\ s,\ r)\}$ are equal and also they are orthogonal with respect to the same distribution function. By Theorem (4.2.2) and because the set of all zeros of $U_n(x)$ for all $n \geq 1$ are dense in $[-1,1]$ a necessary and sufficient condition for the true interval of orthogonality of $\{P_n(x;\ s,\ r)\}_{n=0}^{\infty}$ to be $[-1,1]$ is that

$$\lim_{n\to\infty} \frac{R_n(-1)}{Q_{n-1}(-1)} \leq \frac{r}{2} + (s-1)$$

and

$$\frac{r}{2} - (s-1) \le \lim_{n\to\infty} \frac{R_n(1)}{Q_{n-1}(1)}$$

must be satisfied simultaneously. Thus the true interval of orthogonality of $\{R_n(x;\, s,\, r)\}_{n=0}^{\infty}$ is $[-1,1]$ if and only if

$$(5.3.8) \qquad -(2s-1) \le r \le 2s - 1.$$

Let us first consider the case where the true interval of orthogonality of $R_n(x;\, s,\, r)$ is $[-1,1]$. That is relation (5.3.8) is satisfied. In order to find the distribution function $\sigma^*(t)$ we analyse $F^*(z)$. From Equation (5.3.7) and the well known formula $U_n(z) = \sin(n+1)\theta/\sin\theta$, where $\cos\theta = z$, we obtain

$$F(z) = 2 \lim_{n\to\infty} \frac{[z+(z^2-1)^{1/2}]^n-[z-(z^2-1)^{1/2}]^n}{[z+(z^2-1)^{1/2}]^{n+1}-[z-(z^2-1)^{1/2}]^{n+1}} .$$

It is easy to show by letting $z = \cos\theta$ that if $z \notin [-1,1]$, then $|z + \sqrt{z^2-1}| \neq |z-\sqrt{z^2-1}|$. Therefore $F(z) = 2(z+(z^2-1)^{1/2})^{-1}$ where the sign of the radical is plus if $|z + \sqrt{z^2-1}| > |z - \sqrt{z^2-1}|$ and minus if $|z + \sqrt{z^2-1}| < |z - \sqrt{z^2-1}|$. That is, the Riemann Surface for the function $z + (z^2-1)^{1/2}$ has two sheets; we will adopt the convention that for all $z \notin [-1,1]$, $z + (z^2-1)^{1/2}$ will represent the branch with maximum modulus. Therefore, we have from Equation (4.3.7)

(5.3.9) $$F^*(z) = 2[2(s-1)z - r + z + (z^2-1)^{1/2}]^{-1}.$$

The analytic continuation of $F^*(z)$ is regular on $[-1,1]$. Thus by Theorem (4.3.2) the weight function of $\{P_n(x;\ s,\ r)\}_{n=0}^{\infty}$ in $[-1,1]$ is

$$\mathrm{Im}\{2[(2s-1)x - r + (x^2-1)^{1/2}]^{-1}\},$$

which equals

$$2\sqrt{1-x^2}\ [4s(s-1)x^2 - 2(2s-1)rx + r^2 + 1]^{-1}.$$

Therefore, $\{P_n(x;\ s,\ r)\}_{n=0}^{\infty}$ is equivalent to one of the Bernstein-Szego polynomial sets, (See Szegö [35], pp. 31-33).

Now let us consider the case when the true interval of orthogonality $[a^*,b^*]$ of $\{R_n(x;\ s,\ r)\}_{n=0}^{\infty}$ is not $[-1,1]$. We know from Theorem (4.2.1) that there can be at most one zero of $P_n(x;\ s,\ r)$ greater than 1 and at most one zero less than -1. Therefore, we obtain from Theorem (4.3.3) $\sigma^*(t)$ has at most one point of its spectrum in each of the sets $(-\infty,\ -1)$ and $(1,\ \infty)$. By using the same technique as we used to obtain Equation (5.3.8) we have that the spectrum of $\sigma^*(t)$ has a point in $(-\infty,\ -1)$ if and only if

$$1 - 2s > r$$

and it has a point in $(1,\ \infty)$ if and only if

$$r > 2s - 1\ .$$

From Theorem (4.3.3) we know that the points in the spectrum of $\sigma^*(t)$ that are not in $[-1,1]$ are located at the poles of order one of $F^*(z)$. First let us consider the case $s \neq 1$. The possible poles of $F^*(z)$ are

$$z_1 = \frac{(2s-1)r + \sqrt{r^2-4s(s-1)}}{4s(s-1)} \tag{5.3.10}$$

and

$$z_2 = \frac{(2s-1)r - \sqrt{r^2-4s(s-1)}}{4s(s-1)}. \tag{5.3.11}$$

By noting that

$$(2s-1)z_i - r + (z_i^2-1)^{1/2} = 0 \qquad (i = 1, 2)$$

and by using Equations (5.3.10) and (5.3.11) we obtain that $z_i + (z_i-1)^{1/2}$ equals either

$$-2(s-1)z_i + r = \begin{cases} \dfrac{r - \sqrt{r^2-4s(s-1)}}{2s} & \text{if } i = 1 \\[2ex] \dfrac{r + \sqrt{r^2-4s(s-1)}}{2s} & \text{if } i = 2 \end{cases} \tag{5.3.12}$$

or

$$2sz_i - r = \begin{cases} \dfrac{r + \sqrt{r^2-4s(s-1)}}{2(s-1)} & \text{if } i = 1 \\[2ex] \dfrac{r - \sqrt{r^2-4s(s-1)}}{2(s-1)} & \text{if } i = 2. \end{cases} \tag{5.3.13}$$

If we rationalize the denominator of $F^*(z)$ we obtain from Equation (5.3.9)

$$F^*(z) = \frac{2[2sz-r-(z+(z^2-1)^{1/2})]}{4s(s-1)z^2-2(2s-1)zr+r^2+1} .$$

By comparing this with Equation (5.3.12) and (5.3.13) we see that $F^*(z)$ has a pole of order one at $z_i (i = 1, 2)$ if and only if

$$z_i + (z_i-1)^{1/2} = -2(s-1)z_i + r,$$

Now by using the fact that $F^*(z)$ is the branch of

$$\frac{2[2sz-r-(z+(z^2-1)^{1/2})]}{4s(s-1)z^2-2(2s-1)zr+r^2+1} \tag{5.3.14}$$

which corresponds to the maximum modulus of $z + (z^2-1)^{1/2}$ we obtain that $F^*(z)$ has a pole of order one at z_1 if ard only if

$$\left| \frac{r + \sqrt{r^2-4s(s-1)}}{r - \sqrt{r^2-4s(s-1)}} \, \frac{s}{s-1} \right| < 1 \tag{5.3.15}$$

and a pole of order 1 at z_2 if and only if

$$\left| \frac{r - \sqrt{r^2-4s(s-1)}}{r + \sqrt{r^2-4s(s-1)}} \, \frac{s}{s-1} \right| < 1. \tag{5.3.16}$$

By using standard techniques it is easy to show that if z_1 is a pole of $F^*(z)$, then

$$\operatorname*{Res}_{z=z_1} F^*(z) = \frac{r + (2s-1)\sqrt{r^2-4s(s-1)}}{2s(s-1)\sqrt{r^2-4s(s-1)}}$$

and if z_2 is a pole of $F^*(z)$, then

$$\operatorname*{Res}_{z=z_2} F^*(z) = \frac{-r + (2s-1)\sqrt{r^2-4s(s-1)}}{2s(s-1)\sqrt{r^2-4s(s-1)}} .$$

Thus by Theorem (4.3.3) $\sigma^*(t)$ has a jump at z_1 equal to the $\operatorname*{Res}_{z=z_1} F^*(z)$ if and only if s and r satisfy Equation (5.3.15) and it has a jump at z_2 equal to the $\operatorname*{Res}_{z=z_2} F^*(z)$ if and only if s and r satisfy Equation (5.3.16).

Next let us consider the case when $s = 1$. In this case the only possible pole is at

$$z_3 = 2^{-1}(r + r^{-1}).$$

$z_3 + (z_3^2-1)^{1/2}$ is equal to r or r^{-1}. From the fact that $F^*(z)$ is given by (5.3.14) with $s = 1$ we see that $F^*(z)$ has a pole of order one at z_3 if and only if $|r| > |\frac{1}{r}|$, that is $|r| > 1$. If z_3 is a pole of $F^*(z)$ then

(5.3.17) $$\operatorname*{Res}_{z=z_3} F^*(z) = 1 - \frac{1}{r^2} .$$

Thus by Theorem (4.3.3) $\sigma^*(t)$ has a jump at $\frac{1}{2}(r + \frac{1}{r})$ equal to $1 - \frac{1}{r^2}$ if and only if

$$|r| > 1.$$

By Equations (5.3.8), (5.3.15) and (5.3.16) we have the following two criterions for the true interval of orthogonality of $\{P_n(x;\ s,\ r)\}_{n=0}^{\infty}$ to be $[-1,\ 1]$;

(i) $-(2s-1) \le r \le 2s - 1$

and

(ii) $$\left|\frac{(r - \sqrt{r^2-4s(s-1)})\ (s-1)}{(r + \sqrt{r^2-4s(s-1)})\ \ s}\right| \le 1$$

and

$$\left|\frac{(r + \sqrt{r^2-4s(s-1)})\ (s-1)}{(r - \sqrt{r^2-4s(s-1)})\ \ s}\right| \le 1.$$

In (ii) for the case $s = 1$ we take the limit as $s \to 1$ to obtain the required criterion. It can be shown directly by considering a number of cases $(0 < s < 1/2,\ s = 1/2,\ 1/2 < s < 1,\ s = 1,\ s > 1;$ and $r < 0,\ r = 0,\ r > 0)$ that criterion (i) and (ii) are equivalent.

Next we wish to obtain the differential equation which $P_n(x;\ s,\ r)$ satisfies. By using the well known differential equation and three term recursion relation the $U_n(x)$ satisfies and by Equation (5.3.6) we obtain for $n \ge 2$,

(5.3.19) $$\begin{cases} (1-x^2)P''_n(x;\ s,\ r) - 3xP'_n(x;\ s,\ r) \\ \\ = [r(n^2-1) - 2x(sn^2+2sn)]U_{n-1}(x) + n(4s+n-2)U_{n-2}(x). \end{cases}$$

By using the well known equation

$$(1-x^2)U'_n(x) + nxU_n(x) = (n+1)U_{n-1}(x) \qquad (n \ge 1)$$

we obtain

$$(5.3.20)\quad \begin{cases} (1-x^2)P_n'(x;\ s,\ r) + nxP_n(x;\ s,\ r) = (2s-1-rx+n)U_{n-1}(x) \\ + n((s-1)2x-r)U_{n-2}(x) \qquad (n \geq 2). \end{cases}$$

From the three term recursion relation that $\{U_n(x)\}_{n=0}^{\infty}$ satisfies and Equation (5.3.6) we obtain

$$(5.3.21)\qquad P_n(x;\ s,\ r) = (2sx-r)U_{n-1}(x) - U_{n-2}(x) \qquad (n \geq 2).$$

Next we combine Equations (5.3.20) and (5.3.21) to obtain,

$$(5.3.22)\quad \begin{cases} (1-x^2)P_n'(x;\ s,\ r) + n[(2s-1)x-r]P_n(x;\ s,\ r) \\ = [2s-1-rx+n+n(2sx-r)[2(s-1)x-r]]U_{n-1}(x), \end{cases}$$

and by combining Equations (5.3.19) and (5.3.21) we obtain

$$(5.3.23)\quad \begin{cases} (1-x^2)P_n''(x;\ s,\ r) - 3xP_n'(x;\ s,\ r) + n(n+4s-2)P_n(x;\ s,\ r) \\ = -\ [r(1+4sn-2n) + 8(1-s)nsx]U_{n-1}(x). \end{cases}$$

Let $2s - 1 = v$. By combining Equations (5.3.22) and (5.3.23) we conclude that $P_n(x;\ s,\ r)$ satisfies the differential equation

$$(1-x^2)[n+v+n(v^2-1)x^2-r(x+2nvx-nr)]y''$$

$$- \{x(3v+2nv^2+n(v^2-1)x^2) - r[(1+2x^2)(1+2nv) - 3nrx]\}y'$$

$$+ n\{(n+2v)(n+v) + n^2(v^2-1)x - r[3nx+2n^2vx-r(n^2-1)]\}y=0.$$

We next find a Rodrigues' type formula for $P_n(x;\ s,\ r)$. The Schlaefli's representation for $U_n(x)$ is

$$U_n(x) = \frac{(-1)^n(n+1)!}{2^{n+1}(2^{-1})_{n+1}(1-x^2)^{1/2}2\pi i} \oint_\Gamma \frac{(1-z^2)^{n+1/2}}{(z-x)^{n+1}}\,dz$$

where Γ is any simple closed curve containing x. By using Equation (5.3.6) we obtain the Schlaefli's formula for $P_n(x;\ s,\ r)$ to be

$$P_n(x;\ s,\ r) = \left[\frac{(-1)^n}{2^{n-1}(2^{-1})_{n-1}(1-x^2)^{1/2}}\right] \frac{n!}{2\pi i} \oint_\Gamma \left[\frac{(n+1)s(1-z^2)^{n+1/2}}{(2n+1)(2n-1)(z-x)^{n+1}} + \frac{r(1-z^2)^{n-1/2}}{(2n-1)(z-x)^n} + \frac{(s-1)(1-z^2)^{n-3/2}}{n(z-x)^{n-1}}\right] dz$$

where Γ is any simple closed curve containing x. Thus the Rodrigous type formula is

$$P_n(x;\ s,\ r) = \frac{(-1)^n}{2^{n-1}(2^{-1})_{n-1}(1-x^2)^{1/2}} \frac{d^n}{dz^n} \left[\frac{s(n+1)(1-z^2)^{n+1/2}}{(2n+1)(2n-1)} + \frac{r(1-z^2)^{n-1/2}(z-x)}{(2n-1)} + \frac{(s-1)(1-z^2)^{n-3/2}(z-x)^2}{n}\right]\Bigg|_{z\,=\,x}.$$

By letting $t = z^{-1}$ in the generating function (5.3.3) we obtain the integral representation

$$P_n(x;\ s,\ r) = \frac{1}{2\pi i} \oint_C \frac{t^n[t+2x(s-1)-r]}{1-2xt+t^2}\, dt$$

where C is any simple closed contour about the origin oriented in the positive direction and not containing $x \pm \sqrt{x^2-1}$.

It is easy to deduce from Equation (5.3.6) that

$$(5.3.24) \qquad P_n(x;\ s,\ r) = T_n(x) + [x(2s-1)-r]U_{n-1}(x).$$

Thus,

$$|P_n(x;\ s,\ r)| \le 1 + |x(2s-1) - r|\, n \qquad (-1 \le x \le 1).$$

If we let $\theta_{n,v}$, $v = 1, 2 \ldots, n$, be the n zeros of $P_n(\cos\theta;\ s,\ r)$ then from Equation (5.3.6) we obtain

$$\cot n\theta_{n,v} = r \csc\theta_{n,v} - (2s-1)\cot\theta_{n,v}.$$

As a final result we obtain a separation property for the zeros of a special class of polynomials expanded in terms of $\{P_n(x;\ s,\ r)\}_{n=0}^{\infty}$. Let us extend $\{P_n(x;\ s,\ r)\}_{n=0}^{\infty}$ to $\{P_n(x;\ s,\ r)\}_{n=-\infty}^{\infty}$ by extending $\{T_n(x)\}_{n=0}^{\infty}$ to $\{T_n(x)\}_{n=-\infty}^{\infty}$ by $T_n(x) = T_{-n}(x)$; $\{U_n(x)\}_{n=0}^{\infty}$ to $\{U_n(x)\}_{n=-\infty}^{\infty}$ by $U_{-1}(x) = 0$ $U_{-n}(x) = -U_{n-2}(x)$ and by using Equation (5.3.24). Let

$$F_n(\cos\theta) = \sum_{k=0}^{n} \alpha_k \alpha_{n-k} P_{n-2k}(\cos\theta;\ s,\ r).$$

If $\alpha_n > 0$ and α_1/α_0, $\alpha_2/\alpha_1 \ldots \alpha_n/\alpha_{n-1}$ are increasing, then each interval $(\frac{v-1/2}{n+1}\pi, \frac{v+1/2}{n+1}\pi)$, $v = 1, 2 \ldots n$, contains exactly one zero of $F_n(\cos\theta)$. If $\alpha_n > 0$ and α_n is completely monotonic then the v^{th} zero θ_v of $F_n(\cos\theta)$ satisfies

$$(v-1/2)\frac{\pi}{n} \le \theta_v \le \frac{v\pi}{n+1} \qquad (v = 1, 2 \ldots [\tfrac{n}{2}]).$$

The first result follows from Equation (4.1.5) and a Theorem due to Szegö (See Szegö [35] Theorem (6.5.1)). The second follows from Equation (4.1.5) and a Theorem due to Fejér (See Szegö [35], Theorem (6.5.2)).

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APPENDIX I

SOLUTION OF A FINITE DIFFERENCE EQUATION

In this Appendix we wish to exhibit some of the properties of the solution of

$$\text{(A.1.1)} \qquad \begin{cases} h_{n+1} = 1 - \dfrac{c}{h_n} & (n \geq 1) \\ h_1 = f. \end{cases}$$

Lemma (A.1.1). *If* $\{h_n\}_{n=1}^{\infty}$ *satisfies Equation* (A.1.1) *then*

(a) *if* $c \neq 1/4$

$$\text{(A.1.2)} \qquad h_n = \frac{(d-1+2f)(1+d)^n+(d+1-2f)(1-d)^n}{2\{(d-1+2f)(1+d)^{n-1}+(d+1-2f)(1-d)^{n-1}\}} ,$$

where $d = \sqrt{1-4c}$;

(b) *if* $c = 1/4$

$$\text{(A.1.3)} \qquad h_n = \frac{1}{2}\,\frac{[1+(2f-1)n]}{[1+(2f-1)(n-1)]} .$$

Proof: Let $h_n = A_n/A_{n-1}$. Therefore, we obtain from Equation (A.1.1)

$$A_{n+1} = A_n - cA_{n-1} \qquad (n \geq 1)$$

where $A_0 = 1$ and $A_1 = f$. By the standard techniques for solving finite difference equations we obtain the desired results.

Q.E.D.

Corollary I. *Let* $\{h_n\}_{n=1}^{\infty}$ *and* c *be as defined in Lemma* (A.1.1). *If* $c < 1/4$ *then* $\lim_{n\to\infty} h_n = 2^{-1}(1-\sqrt{1-4c})$ *if* $h_1 = 2^{-1}(1-\sqrt{1-4c})$ *and* $\lim_{n\to\infty} h_n = 2^{-1}(1+\sqrt{1-4c})$ *otherwise.*

Proof: The result follows directly from Lemma (A.1.1).

Lemma (A.1.2). *If* $\{h_n\}_{n=1}^{\infty}$ *satisfies Equation* (A.1.1) *then*

(a) *if* $c > 1/4$, *then*

$$h_n = r[\cos(n\theta - \lambda)]/\cos((n-1)\theta-\lambda),$$

where $r = \sqrt{c}$, $\tan\theta = \sqrt{4c-1}$, *and* $\tan\lambda = \frac{1-2f}{\sqrt{4c-1}}$;

(b) *if* $0 < c < 1/4$, *then*

$$h_n = r\,\frac{\cosh(n\theta-\lambda)}{\cosh((n-1)\theta-\lambda)},$$

where $r = \sqrt{c}$, $\tanh\theta = \sqrt{1-4c}$ *and* $\tanh\lambda = \frac{1-2f}{\sqrt{1-4c}}$;

(c) *if* $c < 0$, *then*

$$h_{2n} = r\,\frac{\cosh(2n\theta-\lambda)}{\sinh((2n-1)\theta-\lambda)}$$

and

$$h_{2n+1} = r\,\frac{\sinh((2n+1)\theta-\lambda)}{\cosh(2n\theta-\lambda)}$$

where $r = \sqrt{-c}$, $\coth\theta = \sqrt{1-4c}$ *and* $\tanh\lambda = \frac{1-2f}{\sqrt{1-4c}}$.

Proof: Let

$$B_n = (d-1+2f)\left(\frac{1+d}{2}\right)^n + (d+1-2f)\left(\frac{1-d}{2}\right)^n.$$

This may be rewritten in the form

(A.1.4) $$B_n = 2d\{\frac{1}{2}[(\frac{1+d}{2})^n + (\frac{1-d}{2})^n] - \frac{(1-2f)}{2d}[(\frac{1+d}{2})^n - (\frac{1-d}{2})^n]\}.$$

(a) If we let $\tan\theta = \sqrt{4c-1}$, $c > 1/4$, $d = \sqrt{1-4c}$ and $r = \sqrt{c}$ we get

$$B_n = 2d\{\frac{1}{2} r^n(e^{in\theta} + e^{-in\theta}) - \frac{(1-2f)}{2d} r^n(e^{in\theta} - e^{-in\theta})\}$$

$$= 2dr^n\{\cos n\theta + \frac{1-2f}{\sqrt{4c-1}} \sin n\theta\}.$$

If we let $\tan\lambda = \frac{1-2f}{\sqrt{4c-1}}$ and $k = \frac{1}{\cos\lambda}$ we obtain

$$B_n = 2kdr^n\cos(n\theta-\lambda).$$

By using Equation (A.1.2) we obtain the required results.

(b) If we let $0 < c < 1/4$, $d = \sqrt{1-4c}$, $r = \sqrt{c}$ and $\tanh\theta = \sqrt{1-4c}$, then

$$B_n = 2d\{\frac{1}{2} r^n(e^{n\theta}+e^{-n\theta}) - \frac{(1-2f)}{2d}(e^{n\theta}-e^{-n\theta})\}$$

$$= 2dr^n\{\cosh n\theta - \frac{(1-2f)}{d} \sinh n\theta\}.$$

If we let $\tanh\lambda = d^{-1}(1-2f)$ and $\sinh\lambda = k^{-1}$, then

$$B_n = 2kdr^n\cosh(n\theta-\lambda).$$

By using Equation (A.1.2) we obtain the required results.

(c) If we let $c < 0$, $d = \sqrt{1-4c}$, $r = \sqrt{-c}$ and $\coth\theta = \sqrt{1-4c}$, then

$$B_n = 2d\{\frac{1}{2}\, r^n(e^{n\theta}+(-1)^n e^{-n\theta}) - r^n \frac{(1-2f)}{2d} (e^{n\theta}-(-1)^n e^{-n\theta})\}$$

$$B_{2n} = 2dr^{2n}\{\cosh 2n\theta - \frac{(1-2f)}{d} \sinh 2n\theta\}$$

$$B_{2n+1} = 2dr^{2n+1}\{\sinh (2n+1)\theta - \frac{(1-2f)}{d} \cosh (2n+1)\theta\}.$$

If we let $\tanh \lambda = \frac{1-2f}{d}$, $k = \frac{1}{\cosh \lambda}$, then

$$B_{2n} = 2kdr^{2n} \cosh(2n\theta-\lambda)$$

$$B_{2n+1} = 2kdr^{2n+1} \sinh((2n+1)\theta-\lambda).$$

By using Equation (A.1.2) we obtain the required results.

Q.E.D.

<u>Lemma (A.1.3)</u>. *Let* $c < 1/4$, x_1 *and* x_2 *be the smallest and largest root of* $x^2 - x + c = 0$, *and* $\{h_n\}_{n=1}^{\infty}$ *satisfy Equations* (A.1.1).

(i) *If* $0 < c < 1/4$ *then* (a) *if* $x_1 < h_1 < x_2$, *then* $h_n < h_{n+1}$ *and* $x_1 < h_n < x_2$ *for all integers* $n \geq 1$, *and* (b) *if* $h_1 > x_2$ *then* $h_n > h_{n+1}$ *and* $x_2 < h_n$ *for all integers* $n \geq 1$.

(ii) *If* $c < 0$ *then* (a) *if* $0 < h_1 < x_2$, *then* $h_{2n-1} < h_{2n+1}$, $h_{2n} > h_{2n+2}$, $0 < h_{2n+1} < x_2$ *and* $h_{2n} > x_2$ *for all positive integers* n; *and* (b) *if* $h_1 > x_2$, *then* $h_{2n-1} > h_{2n+1}$, $h_{2n} < h_{2n+2}$, $h_{2n+1} > x_2$ *and* $0 < h_{2n} < x_2$ *for all positive integers* n.

<u>Proof</u>: The proofs of part (a) and (b) in (i) and (ii) are similar. We will only prove part (a) of (ii).

We first note that $x_1x_2 = c$ and $x_1 + x_2 = 1$. Let $0 < h_1 < x_2$.

$$\begin{aligned} h_2 - x_2 &= 1 - \frac{c}{h_1} - \frac{c}{x_1} \\ &> 1 - c\left(\frac{1}{x_2} + \frac{1}{x_1}\right) \\ &= 1 - c\left(\frac{x_1 + x_2}{x_1x_2}\right) \\ &= 0. \end{aligned}$$

Thus $h_2 > x_2$. In a similar manner it is easy to show that $x_1 < h_3 < x_2$. Also,

$$\begin{aligned} h_3 - h_1 &= 1 - \frac{c}{h_2} - h_1 \\ &= \frac{h_2 - c - h_1h_2}{h_2} > 0. \end{aligned}$$

Thus $h_3 > h_1$. From this we obtain

$$\begin{aligned} h_4 - h_2 &= -c\left(\frac{1}{h_3} - \frac{1}{h_1}\right) \\ &< 0. \end{aligned}$$

Therefore $h_4 < h_2$.

The results follows by an induction argument suggested by the above.

Q.E.D.

<u>Corollary I</u>. *Let* $\{h_n\}_{n=1}^{\infty}$ *satisfy Equation* (A.1.1).

(a) *If* $0 < c < 1/4$ *then* $h_n > 0$ *for all* $n \geq 1$ *if and only if* $h_1 \geq 2^{-1}(1-\sqrt{1-4c})$.

(b) *If* $c < 0$ *then* $h_n > 0$ *for all* $n \geq 1$ *if and only if* $h_1 > 0$.

<u>Proof</u>: Let $0 < c < 1/4$ and $h_n > 0$ for all $n \geq 1$. Assume

$$h_1 < 2^{-1}(1-\sqrt{1-4c}).$$

Let x_1 and x_2 be as defined in Lemma (A.1.3). That is $h_1 < x_1$.

$$h_2 - h_1 = \frac{h_1 - c - h_1^2}{h_1}$$

$$< 0.$$

Thus $h_2 < h_1$. But

$$h_n - h_{n-1} = \frac{c(h_{n-1} - h_{n-2})}{h_{n-1}h_{n-2}} .$$

Because $h_n > 0$ for $n = 1, 2 \ldots,$ and $h_2 < h_1$, therefore $\{h_n\}_{n=1}^{\infty}$ is a monotonically decreasing sequence. But this contradicts Corollary I of Lemma (A.1.1) that says

$$\lim_{n\to\infty} h_n = x_2$$

if $h_1 \neq x_1$.

The other parts of the Corollary follow directly from Lemma (A.1.3).

APPENDIX II

AN EXTENSION OF SOME RESULTS OF GERONIMUS

Let $\{A_n(x)\}_{n=0}^{\infty}$ and $\{B_n(x)\}_{n=0}^{\infty}$ be two polynomial sets orthogonal on an interval (α,β) finite or infinite with weight function $w(x)$ and $r(x)$ respectively. Geronimus [17] showed that

$$B_n(x) = \sum_{i=0}^{s} \alpha_{n,i} A_{n-i}(x) \qquad (n = 0, 1, 2 \ldots), \tag{A.2.1}$$

where s is independent of n and $A_{-k}(x) \equiv 0$ for $k = 1, 2 \ldots,$ if and only if there exists a real polynomial $P(x)$ of degree s such that

$$w(x) = r(x)P(x). \tag{A.2.2}$$

We shall now investigate the relationship between $w(x)$ and $r(x)$ in the case where s may depend on n, so that a generalization of Geronimus' result may be found.

Suppose,

$$B_n(x) = \sum_{i=0}^{s(n)} \alpha_{n,i} A_{n-i}(x) \qquad (n = 0, 1, 2 \ldots), \tag{A.2.3}$$

where $0 < s(n) \leq n$ and $\alpha_{n,s(n)} \neq 0$. Because $\{A_n(x)\}_{n=0}^{\infty}$ and $\{B_n(x)\}_{n=0}^{\infty}$ are O.P.S. on (α,β) with weight functions $w(x)$ and $r(x)$ respectively, it follows that $w(x)A_n(x)(r(x))^{-1}$ has the "formal $\{B_n(x)\}_{n=0}^{\infty}$ expansion"

(A.2.4) $$w(x)A_n(x)(r(x))^{-1} \sim \sum_{k=0}^{\infty} \frac{\omega_n}{r_{n+k}} \alpha_{n+k,k} B_{n+k}(x),$$

where

(A.2.5) $$r_n = \int_{\alpha}^{\beta} (B_n(x))^2 r(x)dx \qquad (n = 0, 1 \ldots),$$

(A.2.6) $$\omega_n = \int_{\alpha}^{\beta} (A_n(x))^2 w(x)dx \qquad (n = 0, 1 \ldots),$$

and $\alpha_{n+k,k} = 0$ if $k > s(n+k)$.

Theorem (A.2.1). *If* m *is the smallest integer such that for all* $n \geq m$, $s(n) < n$, *then* $s(n) \leq m$ *for all* $n \geq m$.

Proof: By letting $n = 0$ in Equation (A.2.4), it is easy to show that $w(x)(r(x))^{-1}$ is a polynomial of degree m almost everywhere. Thus,

(A.2.7) $$A_n(x)w(x)(r(x))^{-1} = \sum_{k=0}^{m} \omega_n (r_{n+k})^{-1} \alpha_{n+k,k} B_{n+k}(x)$$

where $\alpha_{n+m,m} \neq 0$ and $\alpha_{n+k,k} = 0$ if $k > m$. Thus

$$B_n(x) = \sum_{k=0}^{m} \alpha_{n,i} A_{n-i}(x) \qquad (n = 0, 1 \ldots),$$

where $A_{-k}(x) = 0$ for $k = 1, 2, \ldots$.

Q.E.D.

As an example of Equation (A.2.7) we shall show that if λ is a positive integer then

$$(1-x^2)^{\lambda-1/2}P_n^\lambda(x) = \sqrt{1-x^2}\sum_{k=0}^{\lambda-1} f_{k,n}^\lambda U_{n+2k}(x)$$

where

$$\text{(A.2.8)} \qquad f_{k,n} = \frac{2^{2-2\lambda}\Gamma(n+2\lambda)(1-\lambda)_k(n+1)_k}{\Gamma(\lambda)\Gamma(n+\lambda+1)k!(n+\lambda+1)_k}$$

and $\{U_n(x)\}_{n=0}^\infty$, $\{P_n^\lambda(x)\}_{n=0}^\infty$ are defined in section (1.2). Because $(1-x^2)^{\lambda-1/2}$ is the weight function for $\{P_n^\lambda(x)\}_{n=0}^\infty$ and both $\{P_n^\lambda(x)\}_{n=0}^\infty$ and $\{U_n(x)\}_{n=0}^\infty$ are symmetric, we have

$$(1-x^2)^{\lambda-1/2}P_n^\lambda(x) = \sqrt{1-x^2}\sum_{k=0}^{\lambda-1} f_{k,n}^\lambda U_{n+2k}(x) \qquad (n = 0, 1 \ldots),$$

where

$$f_{k,n}^\lambda = \frac{2}{\pi}\int_{-1}^{1}(1-x^2)^{\lambda-1/2}P_n^\lambda(x)U_{n+2k}(x)dx.$$

By using mathematical induction on λ and the two well known equations [35, p. 84],

$$\frac{dP_n^\lambda(x)}{dx} = 2\lambda P_{n-1}^{\lambda+1}(x)$$

and

$$(1-x^2)\,\frac{dU_n(x)}{dx} = \frac{1}{2}\{(n+2)U_{n-2}(x) - nU_{n+1}(x)\},$$

it is easy to show that $f_{k,n}^{\lambda}$ is as given by Equation (A.2.8).

Is there an example of when $s(n) = n$ for an infinite number of integers n but not for all n? Suppose $w(x)A_n(x)(r(x))^{-1}$ has the "formal $\{B_n(x)\}_{n=0}^{\infty}$ expansion"

$$w(x)A_n(x)(r(x))^{-1} \sim \sum_{k=0}^{\infty} f_{k,n}B_{n+k}(x) \tag{A.2.9}$$

where,

$$f_{k,n} = (r_{n+k})^{-1}\int_{\alpha}^{\beta} w(x)A_n(x)B_{n+k}(x)dx \qquad (k \geq 0, \quad n \geq 0).$$

Then

$$B_n(x) = \sum_{k=0}^{n} r_n f_{k,n-k}(\omega_{n-k})^{-1}A_{n-k}(x) \qquad (n = 0, 1 \ldots),$$

where r_n and ω_n are given by Equations (A.2.5) and (A.2.6). By noting that

$$\int_{-1}^{1}(1-x^2)^{\lambda-1/2}\{P_n^{\lambda}(x)\}^2dx = \frac{2^{1-2\lambda}\pi\ \Gamma(n+2\lambda)}{(\Gamma(\lambda))^2(n+\lambda)\Gamma(n+1)},$$

we obtain

$$U_n(x) = \begin{cases} \sum_{k=0}^{n} h_{k,n}P_{n-k}^{\lambda}(x) & (n = 0, 2, 4 \ldots) \\ \\ \sum_{k=0}^{n-1} h_{k,n}P_{n-k}^{\lambda}(x) & (n = 1, 3, 5 \ldots), \end{cases}$$

where, if we put $m = [\frac{k}{2}]$,

$$h_{k,n} = \begin{cases} 0 & (k = 1,3,4 \ldots;\ n = 0,1,2 \ldots) \\ \dfrac{(1-\lambda)_m \Gamma(\lambda)(n-2m+\lambda)\Gamma(n-m+1)}{m!\Gamma(n-m+\lambda+1)} & (k = 0,2, \ldots;\ n = 0,1 \ldots). \end{cases}$$

In this case λ is not an integer we see that $r(n) = n$ if n is even and $r(n) = n - 1$ if n is odd.

If instead of starting with Equation (A.2.3) we start with the "formal $\{B_n(x)\}_{n=0}^{\infty}$ expansion"

$$w(x)A_n(x)(r(x))^{-1} \sim \sum_{k=0}^{r(n)} f_{k,n}B_{n+k}(x),$$

where $f_{r(n),n} \neq 0$, $f_{0,n} \neq 0$ and

$$f_{k,n} = (r_{n+k})^{-1} \int_{\alpha}^{\beta} w(x)A_n(x)B_{n+k}(x)dx$$

for $n \geq 0$ and $k \geq 0$; r_n is given by Equation (A.2.5).

Theorem (A.2.2). *If there exists integers* m *and* q *such that* $r(m) = q$, *then for all* $n \geq 0$

(A.2.10) $$A_n(x)w(x)(r(x))^{-1} = \sum_{k=0}^{q} f_{k,n}B_{n+k}(x) \qquad (n \geq 0).$$

Proof: By hypothesis Equation (A.2.10) is true for $n = m$. Thus $w(x)(r(x))^{-1}$ is a polynomial of degree q almost everywhere. Thus Equation (A.2.10) follows.

Q.E.D.

Theorem (A.2.3). *If there exists a positive integer* s *such that Equation* (A.2.1) *is satisfied, then there does not exist an integer* s_1*, independent of* n *such that*

$$A_n(x) = \sum_{i=0}^{s_1} \beta_{n,i} B_{n-i}(x)$$

where $B_{-k}(x) = 0$ *for* $k = 1, 2 \ldots$.

Proof: If such an s_1 did exist then from the above $w(x)(r(x))^{-1}$ and $(w(x))^{-1}r(x)$ would both be polynomials almost everywhere. This is a contradiction.

Q.E.D.

A simple example of the results of the above Theorem is provided by the following well known relations for the Chebychev polynomial sets of the first and second kind.

$$\sqrt{1-x^2}\, U_n(x) = \frac{T_n(x) - T_{n+2}(x)}{2\sqrt{1-x^2}}$$

and

$$T_n(x) = 2^{-1}(U_n(x) - U_{n-2}(x)).$$

By noting that $\{U_n(x)\}_{n=0}^{\infty}$ and $\{T_n(x)\}_{n=0}^{\infty}$ have weight functions $(1-x^2)^{1/2}$ and $(1-x^2)^{-1/2}$ respectively and both are orthogonal on $(-1,1)$, then there does not exist a positive integer s independent of n such that

$$U_n(x) = \sum_{k=0}^{s} \alpha_{n,k} T_{n-k}(x).$$

Indeed, it is known [10] that

$$U_n(x) = \begin{cases} 3T_0(x) + 2 \sum_{k=1}^{\frac{n}{2}} T_{2k}(x) & (n = 0, 2 \ldots) \\ 2 \sum_{k=0}^{\frac{n-1}{2}} T_{2k+1}(x) & (n = 1, 3 \ldots). \end{cases}$$