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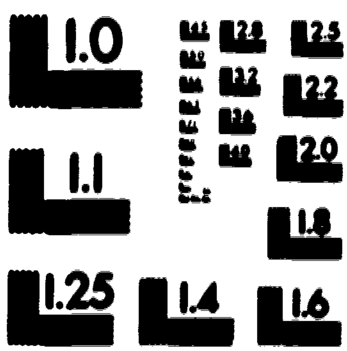

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UNIVERSITY OF ALBERTA

GEOMETRICAL OPTICS FOR NONLINEAR CONSERVATION LAWS AND SHOCK WAVE DYNAMICS

by

YUANPING HE

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PERMANENT ADDRESS:
Dept. of Mathematics
University of Alberta
Edmonton, AB, Canada T6G 2G1

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UNIVERSITY OF ALBERTA

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled GEOMETRICAL OPTICS FOR NONLINEAR CONSERVATION LAWS AND SHOCK WAVE DYNAMICS submitted by YUANPING HE in partial fulfillment of the degree of Doctor of Philosophy

Supervisor: T. Bryant Moodie

G. E. Swaters

R. R. Rosales

A. Z. Capri

H. Van Roessel

S. S. Shen

Date Dec. 30, 1993

To my parents, my wife, Hongying, and daughters, Jiaxue and Jasmine

ABSTRACT

Weakly nonlinear hyperbolic waves arising fro[illegible] or boundary disturbances in systems of one dimensional cons[illegible] considered. The study undertaken is divided into two parts. T[illegible]ludes a relatively complete ***single-wave-mode geometric optics*** the[illegible] investigate weakly nonlinear hyperbolic waves subject to small-am[illegible] frequency boundary disturbances of single-wave-mode type. By introducing a nonlinear phase from the outset, an asymptotic solution to the signaling problem is constructed. A rational scheme is designed to adjust the small-amplitude to high-frequency relation according to the order of local linear degeneracy. The transition process from smooth wave breaking to the generation of shock waves is carefully studied via a bifurcation analysis. We show that wave-breaking will lead to the generation of entropy admissible shock waves. Shock fitting and tracking are also accomplished. As a prototypical example, this process is demonstrated in a transparent fashion for scalar conservation laws in the large.

The second part is devoted to a non-resonant ***two-wave interaction*** theory which generalizes a characteristic method originally introduced by Lin [44] and Fox [15] by directly introducing two nonlinear phases and no longer demands ***a priori*** knowledge of the associated Riemann invariants. Both initial and signaling problems are investigated for systems of conservation laws through the deployment of asymptotic analysis. We apply this theory to compute the interaction and propagation of two weak sound waves in one dimensional gas dynamics. The theory is also applied to study an interesting problem arising from the context of geophysical fluid dynamics, that is, nonlinear Kelvin waves confined to a channel.

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Introduction

Nonlinear hyperbolic systems of conservation laws have occupied a central position in research relating to nonlinear wave theory for several decades. This is as a result of the fact that they are capable of encompassing a multitude of physical phenomena in their formulation and also owing to the rich mathematical structure which their solutions exhibit. One of their most prominent features, which serves to distinguish nonlinear hyperbolic systems from other evolutionary systems, is their tendency to shock formation – these shocks being formed as a result of the focusing of a characteristic family. Owing to this shock formation, solutions to nonlinear hyperbolic conservation laws, or *hyperbolic waves* as they are called, are, in general, not globally smooth and must be interpreted in the sense of weak solutions [39,71]. However, the engineer's intuitive belief that these nonlinear waves are piecewise smooth is more or less the case [53]. It is for this reason that various methods devoted to locating these discontinuities, or shocks, have arisen as an important research issue in the recent past. These are lumped together under the general title of *shock wave tracking methods.*

The mathematical theory for nonlinear systems of conservation laws has undergone substantial advancement over the past few decades. The seminal work of Lax [36–39] and the invention of the random choice scheme by Glimm [16] have led to many important results relating to existence, uniqueness, large-time behaviour, and stability of solutions to hyperbolic systems. These recent results have been achieved through the efforts of such mathematicians as DiPerna, Dafermos, Liu, and Majda [9, 11–14, 45–55, 55–58] to name but a few.

In spite of the inherently complex underlying structure of these nonlinear systems embracing, as they do, so many diverse physical phenomena, one aspires to a simplified approach which can provide qualitative understanding as well as quantitative information in a relatively transparent manner. Indeed, such a simplified approach is needed in order to reveal new phenomena and supply models that capture their salient features. The motivation behind the present study is an attempt to provide an approach with the desired attributes for unravelling some of the complex phenomena inherent to gas dynamics, elasticity, and fluid mechanics.

In the 1950's, Whitham proposed a nonlinearization technique in his studies of the flow pattern around a supersonic projectile [73] and the propagation of weak spherical shocks in stars [74]. By this method, the geometrical effects are accepted unchanged from linear theory for weakly nonlinear disturbances but the crucial influence on the characteristics of the nonlinear self interaction of the flow field is corrected for. This technique is capable of determining the position of shock initiation as well as providing an approximate representation for the solution to nonlinear hyperbolic systems. This approach is now generally known as *Whitham's nonlinearization technique*. Similar ideas can also be found in the earlier works of Landau [35] and Lighthill [43]. From the time of Whitham's 1950's work onwards, interest in devising systematic perturbation schemes for the analysis of nonlinear hyperbolic waves has been high. For example, in a series of papers published during the early 1970's, Taniuti *et. al.* [1–2, 71] developed a method which they called the *reductive perturbation method*. By this approach, one perturbs a nonlinear system about a constant state employing stretched linear phase and slow space variables to arrive at a single evolution equation of canonical form. At about the same time Choquet-Bruhat [6] published her influential paper in which she constructed a formal asymptotic expansion for small amplitude, high-frequency solutions of a nonlinear hyperbolic system in multi-dimensional space. In fact, the literature encompassing research in this direction in the period prior to 1980 is abundant. For comprehensive reviews of this earlier literature we refer the reader to Seymour and Mortell [70], Nayfeh [66], and Kluwick [33].

These earlier studies, especially that of Choquet-Bruhat, stimulated further systematic studies of perturbation procedures for hyperbolic systems by Keller, Hunter, Majda, Rosales, their students, and others. In a series of papers [5, 24–26, 61–63] they developed a procedure which is often referred to as *weakly nonlinear geometrical optics*(WNGO). By this procedure they employ a perturbation scheme combined with a multiple scale analysis for small amplitude, high-frequency waves to incorporate nonlinear corrections into geometrical optics in a systematic fashion.

It is well known that linear geometrical optics, which deals with high-frequency waves for linear problems, is an ideal tool for the analysis of linear hyperbolic systems. See, for example, Keller [31] or Whitham [75]. The ideas which led to such a mathematical theory for treating wave propagation had their origins in physics and, in particular, in physical geometrical optics. Waves of high frequency behave like particles, as, for example, do light waves. This is because we can associate a definite direction with such waves by the rays of geometrical optics and view them locally as plane waves. This provides an immediate and intuitive visualization for linear wave propagation processes. However, when nonlinearities come into play, the situation is considerably more complex requiring more than just a small adjustment to linear geometrical optics. A widely used device for incorporating nonlinearity into a solution scheme is to employ a perturbation analysis using the method of multiple scales. Designing such a perturbation scheme with the correct scaling to reflect the basic nonlinear features of the system is a delicate matter.

In the theory of weakly nonlinear geometrical optics (WNGO) an asymptotic solution for small amplitude, high-frequency waves is derived using linear phase variables together with a slow time variable. Most presentations of the analysis are based upon the tacit assumption that only smooth solutions are allowed [62]. However, the approach is valid even when shocks form. Indeed, DiPerna and Majda [14], in a remarkable paper, proved the validity of weakly nonlinear geometrical optics for general systems of conservation laws in a single space variable subject to the condition that the initial profile is of compact support. Furthermore, in [5], Cehelsky and Rosales presented a derivation of the equations of WNGO that explicitly deals with nonsmooth weak solutions of a very general kind. Also in [24–25], Hunter and Keller verify directly that the shock conditions provided by the asymptotic equations agree with the first term in the amplitude-expansion of the shock conditions for the full equations. One of the advantages that weakly nonlinear geometrical optics has over the method of characteristics is that it is capable of handling multi-wave interactions and resonance so that all wave modes may come into play simultaneously in the initial configuration.

However, the use of linear phases has its limitations. Notably the transition process from wave-breaking to the generation of shock waves is not accounted for. This is due to the fact that the linear phase(s), even after nonlinear correction, does not represent exactly the nonlinear characteristic family it indicates. Also one can notice this disadvantage by observing the fact that when the characteristic field experiences a *local linear degeneracy* about the steady state under consideration, the associated Burgers-type equation for the nonlinear evolution of amplitude also degenerates. Though such a degeneracy can be avoided by redesigning the small amplitude to high frequency relation, this is done *a posteriori* on a trial and see basis. It is due to this consideration that our investigation of nonlinear hyperbolic waves and their interactions will differ from the approach of weakly nonlinear geometrical optics (WNGO) outlined above in that we introduce nonlinear phase variables from the outset. Indeed, in the study of nonresonant hyperbolic waves, in particular the case of decoupled single wave-mode hyperbolic waves, the direct use of a nonlinear phase has the advantage of being clear and straightforward, providing asymptotic solutions as well as the characteristic family in a conceptually clear and simple fashion.

We concentrate on two basic studies for hyperbolic conservation laws. In the first part, a relatively complete theory of geometrical optics involving one nonlinear phase variable, that is, a *single wave-mode theory*, is developed. We study weakly nonlinear hyperbolic waves arising from the action of small amplitude, high-frequency single-wave-mode initial or boundary disturbances. As the work of John [29–30] suggests, the *lifespan* T of a smooth solution to such nonlinear hyperbolic conservation laws is inversely proportional to its amplitude ϵ, that is, $T = O(\epsilon^{-1})$. The appearance of certain forms of linear degeneracy has the effect of extending the lifespan to at least $T = O(\epsilon^{-2})$.

In our analysis, a rational approach to scaling is developed which enables us to find the time of breakdown for smooth solutions to systems having degeneracies. To this end we have employed the concept of *order of local linear degeneracy* [17,20] and demonstrated how it may be used as an index for determining the relation of amplitude to frequency that is appropriate for a given problem. We point out here that this concept was proposed by Rosales [69]. We then focus on the transition process from wave-breaking to the generation and propagation of shock waves. Our approach is based on an analysis of the associated characteristic family cast in terms of a bifurcation problem. This analysis provides a transparent picture of the transition from wave-breaking to shock generation. A shock criterion is established and the shock tracking is accomplished. The analysis here ascertains the after-shock validity of the asymptotic solution. Indeed, this direct use of a nonlinear phase is closely related to the method of characteristics and the characteristic family under consideration is parametrised by the nonlinear phase variable.

In an effort to accommodate applications in a wider class of problem involving spatial inhomogeneity, we allow for an explicit spatial dependence in the flux function, that is, $\mathbf{f} = \mathbf{f}(\mathbf{u}, x)$. It turns out that such inhomogeneities in the media lead to some interesting phenomena that appear not to have been noted previously. For instance, the asymptotic solution consists of two parts, one parallels that of the flux function without spatial dependence, whereas the other part, which appears at an order higher than the order of the local linear degeneracy, is induced solely by the spatial inhomogeneity of the flux function and it is continuous across the shock front thereby invalidating the resolution of higher order terms.

Another study undertaken here is an analysis of the important issue of two-wave interactions for hyperbolic conservation laws. By two-wave interactions here we mean the evolving pattern of a propagating disturbance arising out of the action of initial or boundary disturbances consisting of two wave-modes. Here we take an approach which differs from weakly nonlinear geometrical optics [25, 62] and its use of linear phases but is close in spirit to the use of Riemann invariants in the early studies of Lin [44] and Fox [15]. The method due to Lin and Fox was based on the method of characteristics and you may refer to either Kluwick [33] or Nayfeh [66] for an extensive review of the subject and list of the relevant references. We do not seek an explicit construction of the Riemann invariants, a task which may only be carried out with certainty on at most two-by-two systems, but rather we introduce two nonlinear phase variables and transform the space-time coordinates into nonlinear phase coordinates. When the system of conservation laws is then perturbed, non-resonant two-wave interactions can be handled in an explicit fashion in the sense that our analysis provides an asymptotic solution as well as the perturbed space-time coordinates explicitly.

We apply the above type of analysis to gas dynamics computing the propagation and interaction of two sound waves. Then, as a variation on the theme of hyperbolic conservation laws, we examine a system that is hyperbolic only to leading order in the small perturbation parameter. This particular system describes nonlinear Kelvin waves confined to a channel and its solution has several interesting features.

We now outline the organization of this thesis. In the next chapter, Chapter 1, we start by considering a general scalar conservation law having spatial variability in the flux function and deal with both initial and boundary problems via direct use of a nonlinear phase[19]. Although our purpose here is largely motivational for what is to follow in subsequent chapters and although the analysis is relatively straightforward, the results are suggestive and some are new. For instance, we examine the focusing of the characteristic family and prove a criterion for the generation and propagation of a shock wave by means of a bifurcation analysis. Then we treat a transonic model problem proposed by Liu [51, 55]. This model consists of a scalar conservation law with a strong source term and exhibits some interesting nonlinear phenomena in the transonic regime. The direct use of a nonlinear phase provides an efficient tool for unravelling these phenomena. References on this subject, including the more complicated transonic gas flow through a nozzle, can be found in [50, 52]. Although this transonic problem is in a sense outside the strict purview of this thesis its treatment here illustrates the utility of the nonlinear phase variable in treating a variety of situations. For a more detailed analysis of this interesting problem the reader can refer to the recent article by He and Moodie [18].

In Chapter 2, we develop a complete single wave-mode geometrical optics theory for systems of hyperbolic conservation laws involving n state variables. Many of the ideas employed here in the context of weakly nonlinear asymptotics have been introduced in the less technical exact solution context of Chapter 1. In this chapter we define the signaling problem and employ asymptotic analysis to analyze the solution in regions of smoothness as well as give a complete description of the shock generation process. This analysis is carried out in the presence of linear degeneracies. Our theoretical results are then deployed to give a full treatment of an interesting technical problem involving nonlinear waves in deformable fluid lines whose elastic behaviour is modelled by means of a strain energy function. We are able to recover some former results of Moodie and Swaters [64], as well as, provide a complete shock-tracking procedure.

Chapter 3 is devoted to an analysis of two-wave interaction and propagation of weakly nonlinear hyperbolic waves. We do this by superimposing initial or boundary disturbances consisting of two wave modes and seeking asymptotic expansion solution in the corresponding nonlinear phase configuration. The quadratic interaction of these two wave modes generates, on the next order, all other wave modes and can be resolved fully and systematically. Together with the resolution of the perturbed spatial and temporal coordinates in the nonlinear phase configuration, we derive a full asymptotic expansion solution describing the non-resonant evolving pattern of two-wave interactions. The analysis is conducted for both initial and signaling problems. As for the latter problem, the class of admissible boundary disturbances is also specified in the course of solution. We then make an application to one dimensional gas dynamics, in which the interaction and propagation of two weak sound waves are computed.

The last chapter deals with an interesting problem arising from the context of geophysical fluid dynamics, that is, the nonlinear Kelvin waves confined to a channel. This model is described by the three dimensional shallow water equations in a rotating channel. As it is well known [40, 67], linear Kelvin waves are hyperbolic and travelling in the along channel direction with definite cross channel profiles. We study the nonlinear version of Kelvin waves for two reasons. The first reason is that there are exactly two nonlinear phases involved in the analysis, which provides a suitable background for the application of the two-wave interaction theory furnished in the previous chapter. The second reason is that the model includes another spatial dimension, namely the cross channel dimension. This fact not only admits interesting phenomena to surface but also expands the range of application for the two-wave interaction theory. We shall present an asymptotic analysis in the nonlinear phase configuration and derive asymptotic solutions for along and cross channel velocities as well as surface elevation. These solutions are characterised by a solvable linear canonical initial boundary value problem, which arises as a result of quadratic interactions. Examples which exhibit the evolution pattern of nonlinear Kelvin waves will be given.

CHAPTER 1.

Scalar Conservation Laws, Spatially Dependent Flux Functions, and A Source Problem

In the past few years, interest in hyperbolic conservation laws in which the flux functions admit explicit spatial dependencies has been on the rise [17, 27, 64]. Examples of applications of this theory includes the Buckley-Leverett equations for multiphase flow in porous media [27], the equations for flows in distensible hydraulic lines [17, 27], as well as those for isothermal gas flow through a variable area duct [27]. These problems are closely related to conservation laws with spatially distributed sources [48] in the sense that both no longer preserve self-similar solutions, and their Riemann problems are not, in general, exactly solvable. They do, however, also differ from the latter in that they may be supplemented by an additional conservation law to form a classical system of hyperbolic conservation laws [27].

In this chapter, we shall study the scalar conservation law

$$u_t + f(u, x)_x = 0,$$

where u is the state variable, $f = f(u, x)$ is the flux function which has explicit spatial variability, and x and t, as usual, represent spatial and temporal variables, respectively. We shall conduct a detailed analysis of the signaling and initial value problems associated with the above equation.

The approach adopted here is to introduce a nonlinear phase variable directly into the problem and carry out the analysis in the space-phase configuration rather than in space-time. The use of a nonlinear phase variable here is equivalent to the standard characteristic method such as in Zauderer[76]. This facilitates our efforts to create, through a bifurcation analysis, a clear picture of the process by which a smooth wave breaks to generate a propagating shock wave. The presentation in this chapter is mainly motivational setting the scene for direct use of nonlinear phase variables in later discussions of systems of conservation laws.

We organize the contents of this chapter as follows. The next two sections are devoted to signaling and initial value problems, respectively. After a direct introduction of a nonlinear phase variable, the problems are cast into the space-phase configuration and solved implicitly. Then we examine the caustics which are formed through the focusing of the characteristic family. The point on a caustic which has minimum time is a cusp, and is referred to as a *shock initiation point*, indicating the time and position at which a smooth wave first breaks. Proper conditions guaranteeing the existence of such caustics are found and the caustic structure is analyzed. We then carry out a local bifurcation analysis enabling us to prove that, for uniformly convex (or concave) flux functions (with respect to u), a breaking wave will generate an admissible propagating shock wave. Hence, for the case of scalar conservation laws, a transparent picture depicting the transition process from wave-breaking to shock generation is provided. We digress slightly in the last section of this chapter to demonstrate the utility of the nonlinear phase by discussing a scalar transonic model problem due to Liu [51, 55].

1.1. The Signaling Problem

1.1.1 *Problem defined*

Consider the scalar conservation law

$$u_t + f(u, x)_x = 0. \tag{1.1.1}$$

We require that $\lambda(u, x) \triangleq \partial f(u, x)/\partial u$ not vanish anywhere and be bounded away from zero. For definiteness, we assume that

$$B \geq \lambda(u,x) \geq A > 0, \ \forall \, (u,x) \in \Omega \subset \mathbb{R}^2, \tag{1.1.2}$$

where Ω is the domain of discussion and A, B are constants.

We call a smooth time-independent solution of (1.1.1) a *steady state* of (1.1.1) [48]. Let $u = u_0(x)$ be such a steady state of (1.1.1). Then $u_0(x)$ solves

$$f(u_0(x), x)_x = 0, \tag{1.1.3}$$

or

$$f(u_0(x), x) = f(u_0(0), 0), \tag{1.1.4}$$

and a steady state $u = u_0(x)$ is fully determined by its value at $x = 0$ (or any other given point).

Now we suppose that the scalar conservation law (1.1.1) is initially in a steady state which we may assume, without loss of generality, to be $u_0(x) \equiv 0$. Otherwise we simply replace u by $u + u_0(x)$ in (1.1.1). When a boundary condition is imposed at $x = 0$, we have the following *signaling problem*:

$$u_t + f(u,x)_x = 0, \ x > 0, \ t > 0, \tag{1.1.1}$$

$$u(0,t) = g(t), \ t \geq 0, \tag{1.1.5}$$

$$u(x,0) = 0, \ x \geq 0. \tag{1.1.6}$$

Here $g(t)$ is a smooth function satisfying

$$g(0) = g'(0) = 0. \tag{1.1.7}$$

The mixed initial and boundary conditions of (1.1.5) and (1.1.6) are equivalent to

$$u(0,t) = g(t), \; -\infty < t < \infty, \tag{1.1.8}$$

where $g(t) \equiv 0$ for all $t \leq 0$.

1.1.2 *Nonlinear phase and eikonal transformation*

Now, in order to resolve (1.1.1), (1.1.5), and (1.1.6), we introduce a nonlinear phase θ which remains constant along each characteristic and is parameterized on the t-axis. The approach we take here is equivalent to the standard *characteristic method* [76] and can also be cast in that fashion. The notion of a nonlinear phase can also be extended to systems of nonlinear conservation laws.

The nonlinear phase $\theta = \theta(x,t)$ is defined as the solution of

$$\theta_t + \lambda(u,x)\theta_x = 0, \tag{1.1.9}$$

$$\theta|_{x=0} = t, \; t \in (-\infty, \infty). \tag{1.1.10}$$

The existence and smoothness of $\theta = \theta(x,t)$ is guaranteed so long as $u = u(x,t)$, the solution to the signaling problem (1.1.1), (1.1.5), and (1.1.6) exists and remains smooth. The inverse function of $\theta = \theta(x,t)$ gives the so-called *arrival time formula* [17, 64, 70], that is, the time required for the wave of phase θ to arrive at the position x and we write this as

$$t = T(x,\theta). \tag{1.1.11}$$

In fact, $\{t = T(x,\theta) : -\infty < \theta < \infty\}$ is the family of characteristics for the signaling problem.

We now transform the space-time coordinates into space-phase ones. That is,

$$(x,t) \longmapsto (x,\theta), \tag{1.1.12}$$

by

$$x = x, \tag{1.1.13a}$$

$$\theta = \theta(x,t). \tag{1.1.13b}$$

We point out here that in his study of developing singularities along the leading wavefront, Jeffrey [28] considered a transformation of the form (1.1.12).

Rewriting $u(x,t) = \tilde{u}(x,\theta)$, we have

$$\partial_x \longmapsto \partial_x - \frac{\theta_t}{\lambda}\partial_\theta, \tag{1.1.14a}$$

$$\partial_t \longmapsto \theta_t \partial_\theta. \tag{1.1.14b}$$

Transforming (1.1.1) into (x,θ) coordinates and dropping the tilde for notational convenience we obtain

$$\frac{\partial}{\partial x} f(u(x,\theta),x) \equiv 0, \tag{1.1.15}$$

which integrates to

$$f(u(x,\theta),x) = f(u(0,\theta),0) = f(g(\theta),0). \tag{1.1.16}$$

Meanwhile, the identity

$$t \equiv T(x,\theta(x,t)), \tag{1.1.17}$$

gives

$$\theta_x = -T_x/T_\theta, \ \theta_t = 1/T_\theta, \tag{1.1.18}$$

when (1.1.17) is differentiated with respect to x and t, respectively. Then a substitution of (1.1.18) into (1.1.9) results in

$$T_x = 1/\lambda(u,x), \tag{1.1.19}$$

$$T|_{x=0} = \theta. \tag{1.1.20}$$

We have therefore upon integration that

$$t = T(x,\theta) = \theta + \int_0^x \frac{ds}{\lambda(u(s,\theta),s)}. \tag{1.1.21}$$

Combining (1.1.16) and (1.1.21) provides the complete solution to the signaling problem (1.1.1), (1.1.5), and (1.1.6) as

$$f(u,x) = f(g(\theta),0) \triangleq F(\theta), \tag{1.1.22}$$

$$t = \theta + \int_0^x \frac{ds}{\lambda(u(s,\theta),s)}, \tag{1.1.21}$$

where the unique smooth value for $u = u(x,\theta)$ is obtained by solving (1.1.22).

In particular, we observe that $\theta \leq 0$ corresponds to the steady state region, for which

$$f(u,x) = f(g(\theta),0) = f(0,0),$$

and hence $u \equiv 0$ there and accordingly

$$t = \theta + \int_0^x \frac{ds}{\lambda(0,s)} = \theta + \int_0^x \frac{ds}{\lambda_0(s)}, \ \theta \leq 0, \tag{1.1.23}$$

where $\lambda(0,x) \triangleq \lambda_0(x)$. When $\theta = 0$, we then have

$$t = \int_0^x \frac{ds}{\lambda_0(s)}, \tag{1.1.24}$$

which is called the *leading wavefront* or *leading characteristic* and represents the time for the first boundary disturbance to arrive at position x. This leading wavefront provides us with a boundary separating the disturbed region from the undisturbed steady state region. This is depicted in Fig. 1.1 below.

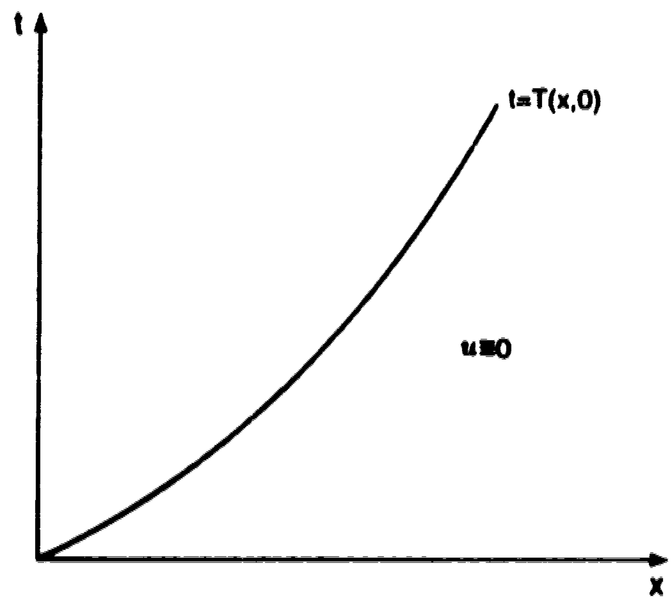

Fig. 1.1: The leading wavefront.

When there is no explicit spatial dependence in the flux function, that is, $f = f(u)$ so that $\lambda = \lambda(u)$, (1.1.21) and (1.1.22) reduce to

$$u = g(\theta), \tag{1.1.25}$$

and

$$t = \theta + \frac{1}{\lambda(g(\theta))} x. \tag{1.1.26}$$

In this case, the characteristics are straight lines, and along each characteristic u preserves its value. This result is classical and well known, see, for example, Whitham [75].

1.1.3 *Wave breaking and caustic structure*

The validity of the eikonal transformation (1.1.12) rests on the condition that the Jacobian

$$J \triangleq \frac{\partial(x,\theta)}{\partial(x,t)} = \theta_t \neq 0, \tag{1.1.27}$$

and

$$T_\theta = \theta_t^{-1} \neq 0. \tag{1.1.28}$$

It is apparent that the transformation (1.1.12) is valid near the boundary $x = 0$, since

$$0 < \theta_t|_{x=0} = 1 < +\infty. \tag{1.1.29}$$

However, the characteristic family may focus so that individual characteristics intersect thereby leading to breakdown of the smooth solution and the formation of shocks. Indeed, the characteristic family focuses when [17]

$$T_\theta = 0, \tag{1.1.30}$$

which, in turn, signals the breakdown of the transformation (1.1.12). As a consequence the smooth solution breaks down since (1.1.22) can be differentiated with respect to t to give

$$\lambda(u,x)u_t = \lambda(g(\theta),0)g'(\theta)\theta_t = \lambda(g(\theta),0)g'(\theta)/T_\theta, \tag{1.1.31}$$

and hence for $g'(\theta) \neq 0$,

$$u_t \longrightarrow \infty \quad \text{as} \quad T_\theta \longrightarrow 0. \tag{1.1.32}$$

That is, a smooth solution to (1.1.1) will break down as its derivative 'blows up'.

When it exists, we call the point which has the minimum time among those for which

$$T_\theta = 0,$$

the ***shock initiation point***. In fact, the envelope (a caustic) Γ for the characteristic family is specified by

$$\Gamma : \begin{cases} L(x,t,\theta) = T(x,\theta) - t = 0, & (1.1.33a) \\ L_\theta(x,t,\theta) = T_\theta = 0. & (1.1.33b) \end{cases}$$

From (1.1.21) we obtain

$$T_\theta = 1 + \int_0^x (-\lambda^{-2})\lambda_u u_\theta ds. \tag{1.1.34}$$

Then, differentiating with respect to θ in (1.1.22), we obtain

$$\lambda u_\theta = \lambda(g(\theta),0)g'(\theta), \tag{1.1.35}$$

which, when combined with (1.1.34), gives

$$\Gamma : \begin{cases} x = x(\theta) : \ \lambda(g(\theta),0)g'(\theta)\int_0^x \lambda^{-3}\lambda_u ds = 1, & (1.1.36a) \\ t = t(\theta) : \ t = \theta + \int_0^{x(\theta)} \frac{ds}{\lambda}. & (1.1.36b) \end{cases}$$

It is clear from the above that Γ exists for the set $\Theta^+ = \{\theta \in (-\infty, \infty) : g'(\theta) > 0\}$ when $\lambda_u > 0$ and the set $\Theta^- = \{\theta \in (-\infty, \infty) : g'(\theta) < 0\}$ when $\lambda_u < 0$. For definiteness in all of our future discussions we shall assume that $K_2 \geq \lambda_u \geq K_1 > 0$, where K_1 and K_2 are positive constants. This is equivalent to

$$K_2 \geq f_{uu}(u, x) \geq K_1 > 0, \tag{1.1.37}$$

that is, the flux function is uniformly convex with respect to u.

The following lemma addresses more than just the existence of the shock initiation point.

LEMMA 1.1. Suppose $g'(\theta) > 0$ for $\theta \in (0, \theta_0)$ and $g'(\theta_0) = 0$. Then the envelope (or caustic) Γ for the characteristic family exists for $\theta \in (0, \theta_0)$. Also, there exists $\theta_s \in (0, \theta_0)$ which gives the shock initiation point (x_s, t_s) : $x_s = x(\theta_s)$, $t_s = t(\theta_s)$. In addition, we have

1°.
$$x_s = x(\theta_s) = \min_{\theta \in (0, \theta_0)} \{x(\theta)\}, \tag{1.1.38a}$$

$$t_s = t(\theta_s) = \min_{\theta \in (0, \theta_0)} \{t(\theta)\}, \tag{1.1.38b}$$

2°. (x_s, t_s) is a cusp for the caustic Γ and along Γ

$$\lim_{\theta \to \theta_s^+} \frac{dt}{dx} = \lim_{\theta \to \theta_s^-} \frac{dt}{dx} = \frac{1}{\lambda(u(x_s, \theta_s), x_s)}. \tag{1.1.39}$$

Proof. Since $g'(0) = g'(\theta_0) = 0$, $g'(\theta) > 0$, $\forall\, \theta \in (0, \theta_0)$ and λ is positive and bounded away from zero, it follows from (1.1.36) that Γ exists for $0 < \theta < \theta_0$, and

$$\lim_{\theta \to 0^+} x(\theta) = \lim_{\theta \to \theta_0^-} x(\theta) = +\infty, \tag{1.1.40a}$$

$$\lim_{\theta \to 0^+} t(\theta) = \lim_{\theta \to \theta_0^-} t(\theta) = +\infty. \tag{1.1.40b}$$

This suggests that the minima for $x(\theta)$ and $t(\theta)$ both exist. Let θ_s be that value of θ for which $t(\theta)$ has its minimum value. Thus (1.1.38b) holds. That is, the shock initiation point (x_s, t_s) exists, where $x_s = x(\theta_s)$, $t_s = t(\theta_s)$. Now we need only to show that (1.1.38a) holds, that is, the same θ_s produces minima for both $x(\theta)$ and $t(\theta)$.

Let $x^* = x(\theta^*)$ be the minimum for $x(\theta)$. If $x^* < x_s$ then there are two cases: $t^* = t_s$ or $t^* > t_s$, where $t^* = t(\theta^*)$. In the first case we simply shift the shock initiation point to (x^*, t^*) and adopt the convention that θ_s is always chosen such that x_s is also the minimum of $x(\theta)$. In the second case, by connecting (x_s, t_s) to (x^*, t^*) by means of a line segment, we notice that its slope

$$k = \frac{t^* - t_s}{x^* - x_s} < 0,$$

so that $\exists\, \hat{\theta} \in (0, \theta_0)$ such that $t = T(x, \hat{\theta})$ has a negative tangent at $(x(\hat{\theta}), t(\hat{\theta}))$. This is, however, impossible for the characteristic family under consideration. Therefore, we have shown that

$$x_s = x(\theta_s) = \min_{\theta \in (0, \theta_0)} \{x(\theta)\}. \tag{1.1.38a}$$

Now to show (1.1.39), we differentiate (1.1.36b) with respect to θ to get

$$\begin{aligned} dt/d\theta &= 1 + \int_0^{x(\theta)} \frac{\partial}{\partial \theta}\left(\frac{1}{\lambda}\right) ds + \frac{1}{\lambda(u(x(\theta), \theta), x(\theta))} \frac{dx}{d\theta} \\ &= 1 + \int_0^{x(\theta)} (-\lambda^{-2}\lambda_u) u_\theta \, ds + \frac{1}{\lambda(u(x(\theta), \theta), x(\theta))} \frac{dx}{d\theta} \\ &= \frac{1}{\lambda(u(x(\theta), \theta), x(\theta))} \frac{dx}{d\theta}, \end{aligned} \tag{1.1.41}$$

in light of (1.1.33b) and (1.1.34). Thus along Γ we have

$$dt/dx = \frac{1}{\lambda(u(x(\theta), \theta), x(\theta))}. \tag{1.1.42}$$

In particular, letting $\theta \longrightarrow \theta_*^{\pm}$ we have

$$\lim_{\theta\to\theta_*^-} \frac{dt}{dx} = \lim_{\theta\to\theta_*^+} \frac{dt}{dx} = \frac{1}{\lambda(u(x_*,\theta_*),x_*)}, \tag{1.1.39}$$

and hence (x_*, t_*) is a cusp.
This completes the proof.

Apparently, for any interval (t', t'') on the t-axis for which $g'(t') = g'(t'') = 0$ and $g'(t) > 0$, $\forall\, t \in (t', t'')$, the corresponding part of the characteristic family has an envelope that retains the property stated in Lemma 1.1.

As a result of Lemma 1.1, we can separate the envelope Γ into two branches, namely

$$\Gamma_1 :\ x = x(\theta),\ t = t(\theta),\ 0 < \theta < \theta_*, \tag{1.1.43a}$$

$$\Gamma_2 :\ x = x(\theta),\ t = t(\theta),\ \theta_* < \theta < \theta_0. \tag{1.1.43b}$$

Here we suppose that the conditions of Lemma 1.1 for g are fulfilled. We denote the cusped region enclosed by Γ_1, Γ_2, in some neighbourhood of (x_*, t_*), as $\mathcal{D}$.

There are, in general, two cases.

Case 1. Γ_1 lies above Γ_2, or Γ_1 and Γ_2 coincide in the neighbourhood of (x_*, t_*).

Case 2. Γ_2 is above Γ_1.

1.1.4 *Relative positions of two caustic branches*

We have specified two cases for the relative position of two branches of the caustic Γ in a neighbourhood of the shock initiation point. We now examine this issue in some detail. Indeed, we shall prove that, under the uniform convexity condition (1.1.37), Case 2 is impossible and, in particular, Γ_1 is always above Γ_2 and as a result, the breaking of a smooth wave will always generate an admissible shock.

To begin, we note from (1.1.42) that Γ has slope

$$dx/dt = \lambda(u(x(\theta),\theta),x(\theta)) \triangleq G(\theta). \tag{1.1.44}$$

$G(\theta)$ is a smooth function of θ and we expand it about $\theta = \theta_s$ to get

$$G(\theta) = G(\theta_s) + G'(\theta_s)(\theta - \theta_s) + R(\theta)(\theta - \theta_s)^2, \tag{1.1.45}$$

where $R(\theta)$ is smooth and

$$\lim_{\theta \to \theta_s} R(\theta) = \frac{1}{2}G''(\theta_s). \tag{1.1.46}$$

The next lemma determines the sign of $G'(\theta_s)$ which plays a crucial role in ascertaining the relative positions of the branches Γ_1 and Γ_2.

LEMMA 1.2. Suppose $g'(\theta) > 0$, $\forall\, \theta \in (0,\theta_0)$ and $g'(0) = g'(\theta_0) = 0$ so that the caustic Γ and the shock initiation point exists. Then at $\theta = \theta_s$ we have

$$G'(\theta_s) = g'(\theta_s)\lambda_u(g(\theta_s),0)\lambda_u(u(x_s,\theta_s),x_s)/G(\theta_s) > 0. \tag{1.1.47}$$

Proof. We differentiate

$$f(u,x) = F(\theta), \tag{1.1.22}$$

with respect to θ to get

$$\lambda(u,x)u_\theta = F'(\theta). \tag{1.1.35}$$

Differentiate (1.1.35) with respect to θ to yield

$$\lambda_u u_\theta^2 + \lambda u_{\theta\theta} = F''(\theta). \tag{1.1.48}$$

Now (1.1.22) and (1.1.35) are independent of x and, in particular, we evaluate (1.1.35) at $x = x(\theta)$ obtaining

$$G(\theta)u_\theta = F'(\theta), \tag{1.1.49}$$

and

$$G'(\theta)u_\theta + G(\theta)\frac{d}{d\theta}u_\theta(x(\theta),\theta) = F''(\theta), \tag{1.1.50}$$

after a differentiation with respect to θ.

Now

$$\frac{d}{d\theta}u_\theta(x(\theta),\theta) = u_{\theta x}(x(\theta),\theta)x'(\theta) + u_{\theta\theta}(x(\theta),\theta). \tag{1.1.51}$$

We note that in the space-phase coordinates, u_θ, u_x, as well as $u_{\theta x}$, are all smooth functions. We may evaluate (1.1.48), (1.1.50), and (1.1.51) at $\theta = \theta_s$ to obtain

$$\lambda_u(u(x_s,\theta_s),x_s)u_\theta^2 + G(\theta_s)u_{\theta\theta} = F''(\theta_s), \tag{1.1.52}$$

and

$$G'(\theta_s)u_\theta + G(\theta_s)u_{\theta\theta} = F''(\theta_s), \tag{1.1.53}$$

where we have employed the fact that $x'(\theta_s) = 0$.

Subtracting (1.1.52) from (1.1.53) gives

$$G'(\theta_s)u_\theta = \lambda_u(u(x_s,\theta_s),x_s)u_\theta^2, \tag{1.1.54}$$

or

$$\begin{aligned} G'(\theta_s) &= F'(\theta_s)\lambda_u(u(x_s,\theta_s),x_s)/G(\theta_s) \\ &= g'(\theta_s)\lambda_u(g(\theta_s),0)\lambda_u(u(x_s,\theta_s),x_s)/G(\theta_s), \end{aligned} \tag{1.1.55}$$

where (1.1.49) has been noted. Thus $G'(\theta_s) > 0$.
This completes the proof.

We are now in a position to prove the following theorem.

THEOREM 1.1. Suppose $g'(\theta) > 0$, $\forall\, \theta \in (0, \theta_0)$ and $g'(0) = g'(\theta_0) = 0$. Also the flux function satisfies (1.1.2) and (1.1.37). Then for the two branches of the caustic defined by (1.1.36), (1.1.43), Γ_1 is always above Γ_2.

Proof. The existence of the caustic Γ follows from Lemma 1.1. Let (x_s, t_s) be the shock initiation point, where $x_s = x(\theta_s)$, $t_s = t(\theta_s)$ and $\theta = \theta_s$ provides the minima for both $x = x(\theta)$ and $t = t(\theta)$. Hence $t'(\theta_s) = 0$ and $t''(\theta_s) \geq 0$. Without loss of generality, we assume $t''(\theta_s) > 0$.

We now expand $t = t(\theta)$ about $\theta = \theta_s$ to get

$$\begin{aligned} t &= t_s + \frac{1}{2}t''(\theta_s)(\theta - \theta_s)^2 + O((\theta - \theta_s)^3) \\ &= t_s + Q(\theta)(\theta - \theta_s)^2, \end{aligned} \tag{1.1.56}$$

where $Q(\theta)$ is smooth and

$$\lim_{\theta \to \theta_s} Q(\theta) = \frac{1}{2}t''(\theta_s) > 0. \tag{1.1.57}$$

Rewrite (1.1.56) as

$$Q(\theta)(\theta - \theta_s)^2 = t - t_s. \tag{1.1.58}$$

One can invert $t = t(\theta)$ by an application of the implicit function theorem in (1.1.58) to get two solutions in a neighbourhood of $\theta = \theta_s$ when $t > t_s$. We have two cases:

1°. When $\theta < \theta_s$, the first solution $\theta = \theta_1(t)$ solves

$$\theta - \theta_s = -\{(t - t_s)/Q(\theta)\}^{1/2} \triangleq -p(t, \theta). \tag{1.1.59}$$

2°. When $\theta > \theta_s$, the second solution $\theta = \theta_2(t)$ solves

$$\theta - \theta_s = \{(t - t_s)/Q(\theta)\}^{1/2} = p(t, \theta). \tag{1.1.60}$$

Therefore $\theta - \theta_s = O((t - t_s)^{1/2})$ as $t \to t_s^+$ in both cases.

Substituting (1.1.45), (1.1.59) and (1.1.60) into (1.1.44) yields, respectively, that

$$\text{on } \Gamma_1 : \frac{dx}{dt} = G(\theta_s) - G'(\theta_s)p(t, \theta) + R(\theta)p^2(t, \theta), \tag{1.1.61}$$

and

$$\text{on } \Gamma_2 : \frac{dx}{dt} = G(\theta_s) + G'(\theta_s)p(t, \theta) + R(\theta)p^2(t, \theta). \tag{1.1.62}$$

Now, in order to ascertain the relative positions of Γ_1 and Γ_2, we compare their slopes. In the neighbourhood of (x_s, t_s) we denote Γ_1 and Γ_2 by $x = x_1(t)$ and $x = x_2(t)$, respectively. Fixing t and subtracting (1.1.62) from (1.1.61), we have that

$$\frac{d}{dt}(x_1 - x_2) = -G'(\theta_s)(p(t, \theta_1) + p(t, \theta_2)) + R(\theta_1)p^2(t, \theta_1) - R(\theta_2)p^2(t, \theta_2), \tag{1.1.63}$$

where $\theta_1 = \theta_1(t)$ and $\theta_2 = \theta_2(t)$. When t is close to t_s, we know from above that

$$p(t, \theta_1),\ p(t, \theta_2) = O((t - t_s)^{1/2}),$$

and

$$\lim_{t \to t_s} R(\theta_1(t)) = \lim_{t \to t_s} R(\theta_2(t)) = \frac{1}{2}G''(\theta_s), \tag{1.1.64}$$

are finite. Hence the sign of $\frac{d}{dt}(x_1 - x_2)$ is determined by the first term on the right of (1.1.63). Hence, as a result of Lemma 1.2, we have

$$\frac{d}{dt}(x_1 - x_2) < 0. \tag{1.1.65}$$

Thus, in the neighbourhood of (x_s, t_s), Γ_1 is above Γ_2.

This completes the proof.

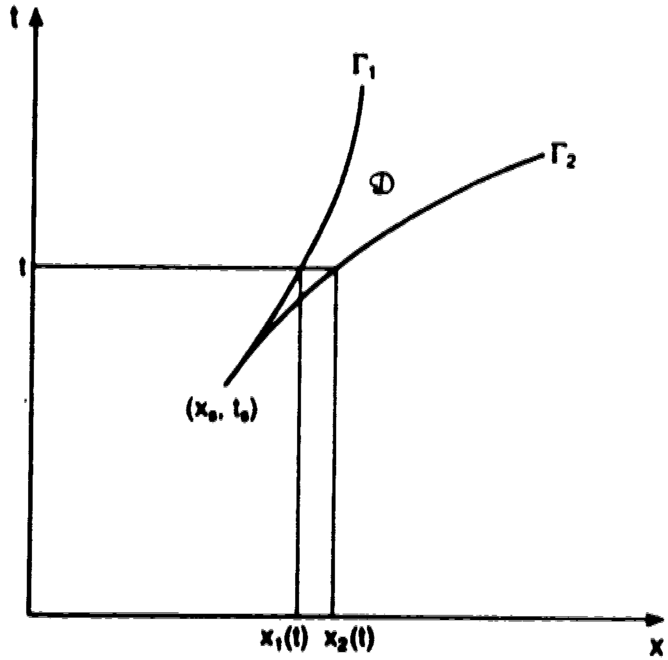

Fig. 1.2: Γ_1, Γ_2 in a neighbourhood of (x_s, t_s).

1.1.5 *Bifurcation and shock generation*

We have shown above that of the two branches of caustic Γ, Γ_1 is always above Γ_2. As a result the generation and propagation of a shock wave is predicted. We shall show this by first proving a lemma.

LEMMA 1.3. When Γ_1 is above Γ_2 then, $\forall\,(x,t) \in \mathcal{D}$, there are precisely two θ's, that is, $\theta = \theta_\ell(x,t)$, $\theta_r(x,t)$ with

$$0 < \theta_r(x,t) < \theta_1(t),\ \theta_2(t) < \theta_\ell(x,t) < \theta_0, \tag{1.1.66}$$

satisfying (1.1.21). Namely, there are two characteristics, with their phases θ_ℓ, θ_r, subject to (1.1.66), passing through each $(x,t) \in \mathcal{D}$. Here $\theta = \theta_1(t)$, $\theta_2(t)$ with

$\theta_1(t) < \theta_s < \theta_2(t)$ are two branches of the inversion of $t = t(\theta)$ near t_s, defined by (1.1.59) and (1.1.60) respectively.

Proof. When Γ_1 is above Γ_2, for any fixed θ : $0 < \theta < \theta_s$, the characteristic emanating from the boundary will cross Γ_2 and intersect Γ_1 at the point $(x(\theta), t(\theta))$. At this point of intersection the characteristic is tangent to Γ_1. As a convention, we terminate the characteristic at $(x(\theta), t(\theta))$ and denote it by $C(\theta)$. Then $\{C(\theta) : 0 < \theta < \theta_s\}$ constitutes a family of characteristic curves which vary smoothly with θ. In other words, $\forall\ (x,t) \in \mathcal{D}$, $\exists$ a unique $\theta(0 < \theta < \theta_s)$, such that $C(\theta)$ passes through (x,t). Similarly, for θ : $\theta_s < \theta < \theta_0$, we also have a family of characteristic segments $\{C(\theta) : \theta_s < \theta < \theta_0\}$ which has the same property as the above one. That is, $\forall\ (x,t) \in \mathcal{D}$, $\exists$ a unique θ : $\theta_s < \theta < \theta_0$ which passes through (x,t).

Thus, we have shown that $\forall\ (x,t) \in \mathcal{D}$, $\exists$ two θ's from $0 < \theta < \theta_s$ and $\theta_s < \theta < \theta_0$, respectively, and the two corresponding characteristic segments pass through (x,t). We denote these two θ's by $\theta_\ell = \theta_\ell(x,t)$ and $\theta_r = \theta_r(x,t)$, where $0 < \theta_r < \theta_s < \theta_\ell < \theta_0$.

To be more precise, we further note that $\forall\ (x,t) \in \mathcal{D}$,

$$0 < \theta_r(x,t) < \theta_1(t),\ \theta_2(t) < \theta_\ell(x,t) < \theta_0, \tag{1.1.66}$$

where $\theta = \theta_1(t)$, $\theta_2(t)$ with $\theta_1(t) < \theta_s < \theta_2(t)$ are two branches of the inversion of $t = t(\theta)$ on the caustic Γ, introduced in (1.1.59) and (1.1.60) respectively. See Fig. 1.3 below.

To see that $\forall\ (x,t) \in \mathcal{D}$, there are exactly two θ's, that is, θ_r and θ_ℓ with $0 < \theta_r(x,t) < \theta_1(t)$, $\theta_2(t) < \theta_\ell(x,t) < \theta_0$, satisfying (1.1.21). We suppose $\theta = \mathring{\theta}(x,t)$: $0 < \mathring{\theta} < \theta_0$ provides a third solution to (1.1.21). As we have shown above, there are only two characteristic segments from $\{C(\theta) : 0 < \theta < \theta_0\}$ passing through (x,t), providing the two θ's: θ_ℓ and θ_r, so the characteristic associated to $\mathring{\theta}$ and passing through (x,t) must correspond to the extension of $C(\mathring{\theta})$, thereby leading to $\theta_1(t) < \mathring{\theta} < \theta_2(t)$ since the characteristic has a positive slope everywhere.

This completes the proof.

Remark 1.1. Without the restriction (1.1.66), (1.1.21) may admit more than two solutions for θ. This has been demonstrated for the inviscid Burger's equation, see [3,4] for more details.

Apparently, the above analysis applies to the case when Γ_1 coincides with Γ_2 and $\mathcal{D}$ collapses to the caustic Γ itself. The cusped caustic and the region $\mathcal{D}$ are depicted in Fig. 1.3 below.

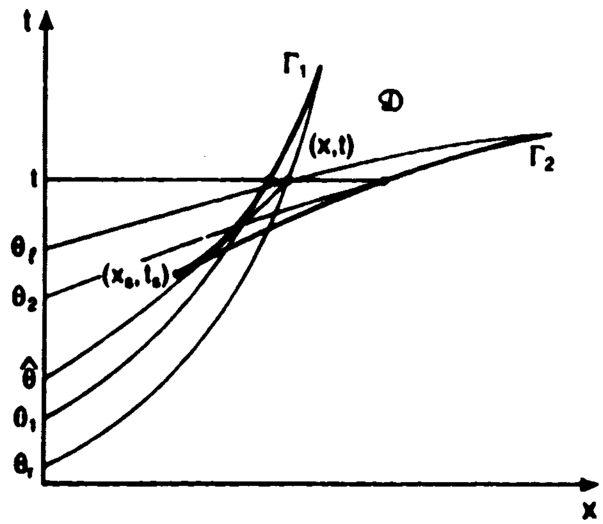

Fig. 1.3: Cusped envelope in the (x, t) plane.

The next theorem illustrates the generation and propagation of a shock wave in the neighbourhood of the shock initiation point and completes the shock-fitting procedure.

THEOREM 1.2. (Shock Propagation Rule). If Γ_1 lies above or is overlapping with Γ_2 in the neighbourhood of the shock initiation point (x_0, t_0) then an entropy admissible shock wave will be generated and propagate along the shock front

$$\frac{dt}{dx} = \frac{[u]}{[F]} = \frac{u_\ell - u_r}{f(g(\theta_\ell),0) - f(g(\theta_r),0)}, \; x > x_s, \tag{1.1.67a}$$

$$t(x_s) = t_s, \tag{1.1.67b}$$

with the shock strength $[u] = u_\ell - u_r$. Here $u_\ell = u(x,\theta_\ell)$, $u_r = u(x,\theta_r)$ and $\theta_\ell = \theta_\ell(x,t)$, $\theta_r = \theta_r(x,t)$ solve (1.1.21) with $0 < \theta_r < \theta_1(t)$, $\theta_2(t) < \theta_\ell < \theta_0$.

Proof. Clearly $\mathcal{D}$, according to Lemma 1.3, is a multi-valued region. We resolve this situation by introducing a curve of discontinuity, or, in other words, a shock front Σ in $\mathcal{D}$ and terminating at Σ any characteristic which enters $\mathcal{D}$. This curve Σ is initiated at the shock initiation point $t(x_s) = t_s$.

Invoking the Rankine-Hugoniot condition across the shock front Σ gives

$$-\frac{dx}{dt}[u] + [f] = 0, \tag{1.1.68}$$

which in turn provides

$$\begin{aligned}\frac{dt}{dx} &= \frac{[u]}{[f]} \\ &= (u_\ell - u_r)/(f(u_\ell,x) - f(u_r,x)) \\ &= (u_\ell - u_r)/(f(g(\theta_\ell),0) - f(g(\theta_r),0)).\end{aligned} \tag{1.1.67a}$$

Here $u_\ell = u(x,\theta_\ell)$, $u_r = u(x,\theta_r)$ with $\theta_\ell = \theta_\ell(x,t)$, $\theta_r = \theta_r(x,t)$ being solutions of (1.1.21) confirmed by Lemma 1.3.

Now we need to show that the shock front Σ, which is determined by (1.1.67a,b) exists and remains in $\mathcal{D}$ in a neighbourhood of (x_s,t_s). In fact, (1.1.67) constitutes an initial value problem for an ordinary differential equation. We notice that

$$\lim_{\substack{(x,t)\in\Sigma\subset\mathcal{D} \\ (x,t)\to(x_s,t_s)}} \frac{dt}{dx} = \frac{1}{\lambda(u(x_s,\theta_s),x_s)}, \tag{1.1.69}$$

from (1.1.67a) and the fact that

$$\theta_\ell(x,t), \theta_r(x,t) \longrightarrow \theta_s \text{ as } \mathcal{D} \ni (x,t) \longrightarrow (x_s, t_s).$$

The initial value problem (1.1.67) is well-posed and its solution exists if and only if Σ remains in $\mathcal{D}$. To show that Σ remains in $\mathcal{D}$ we need only show that at any point (x,t) on the boundary of $\mathcal{D}$, the tangent direction of Σ points into $\mathcal{D}$.

In fact, we have

$$\frac{dt}{dx} = \frac{[u]}{[f]} = \frac{1}{\lambda(u^*, x)}, \tag{1.1.70}$$

where $u^* \in (u_\ell, u_r)$ or (u_r, u_ℓ). This, in turn, leads to

$$\frac{dt}{dx} \in \left(\frac{1}{\lambda(u_\ell, x)}, \frac{1}{\lambda(u_r, x)}\right) \text{ or } \left(\frac{1}{\lambda(u_r, x)}, \frac{1}{\lambda(u_\ell, x)}\right), \tag{1.1.71}$$

since $\lambda(u,x)$ is monotonically increasing in u. It is obvious that on Γ the two end-points in the above interval correspond to directions that are either aligned with the direction of Γ or point into the interior of $\mathcal{D}$. Hence, dt/dx being between these two directions will always point into $\mathcal{D}$.

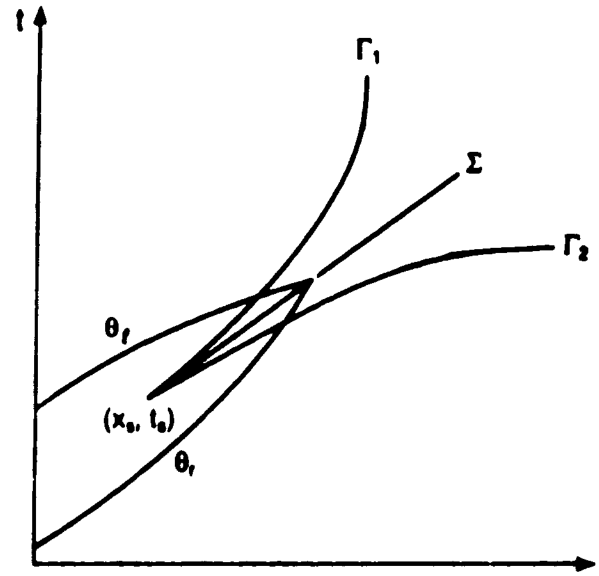

Fig. 1.4: Shock front Σ in $\mathcal{D}$.

Now we show that the shock wave we have introduced is admissible in the sense that it satisfies an appropriate entropy condition.

From (1.1.22) we have

$$\lambda u_\theta = \lambda(g(\theta), 0) g'(\theta),$$

and since $g'(\theta) > 0$, that $u_\theta > 0$, $\forall\, \theta \in (0, \theta_0)$. Thus $u(x, \theta)$ is strictly increasing in θ and

$$u_r = u(x, \theta_r) < u(x, \theta_\ell) = u_\ell, \ \forall\, (x, t) \in \mathcal{D}, \tag{1.1.72}$$

which, in turn, implies

$$\lambda(u_r, x) < \frac{[f]}{[u]} = \lambda(u^*, x) < \lambda(u_\ell, x), \tag{1.1.73}$$

where u^* is between u_r and u_ℓ. Equation (1.1.73) is just Lax's entropy condition in the scalar case [8,9,30], showing that the shock wave is compressive and admissible.

This completes the proof.

We note here that when $\theta = \theta_s$ provides a single minimum for $x(\theta)$ and $t(\theta)$, the shock strength grows from an initial zero value. Otherwise the shock wave may be initiated with a nonzero strength. This issue will be examined in the next section.

1.2. The Initial Value Problem

Now for the scalar conservation law introduced in the last section, that is

$$u_t + f(u, x)_x = 0, \tag{1.1.1}$$

we consider the associated initial problem. Here the flux function has the same properties as described in the preceding section.

We prescribe the initial condition

$$u|_{t=0} = g(x), \ x \in (-\infty, \infty), \tag{1.2.1}$$

where $g(x)$ is a smooth function.

1.2.1 *Nonlinear phase and eikonal transformation*

As before, we introduce the nonlinear phase variable θ with the difference that now θ is parametrised on the x-axis rather than on the boundary. Hence

$$\theta_t + \lambda(u, x)\theta_x = 0, \tag{1.1.9}$$

$$\theta|_{t=0} = x. \tag{1.2.2}$$

The existence and smoothness of $\theta = \theta(x, t)$ is inherited from the existence and smoothness of $u = u(x, t)$. As before, the inversion of $\theta = \theta(x, t)$ gives the arrival time formula

$$t = T(x, \theta), \tag{1.2.3}$$

which represents a family of characteristics emanating from the x-axis.

We now carry out the nonlinear eikonal transformation

$$(x, t) \longmapsto (x, \theta), \tag{1.2.4}$$

by

$$x = x, \tag{1.2.5a}$$

$$\theta = \theta(x, t), \tag{1.2.5b}$$

so that the scalar conservation law becomes

$$\frac{\partial}{\partial x} f(u(x, \theta), x) = 0, \tag{1.2.6}$$

or

$$f(u(x, \theta), x) = f(g(\theta), \theta) \triangleq F(\theta), \tag{1.2.7}$$

in light of the initial condition. Hence, as a result of the implicit function theorem, $u(x, \theta)$ is fully determined by (1.2.7).

Meanwhile we also have for the arrival time formula that

$$T_x = 1/\lambda. \tag{1.2.8}$$

Integrating and noting that $x = \theta$ when $t = 0$ we have

$$t = T(x, \theta) = \int_\theta^x \frac{ds}{\lambda(u(s, \theta), s)}. \tag{1.2.9}$$

Now (1.2.7) and (1.2.9) give a full solution to the initial value problem (1.1.1), (1.2.1).

In particular, when there is no explicit spatial dependence in the flux function, the solution becomes simplified to

$$u = g(\theta), \tag{1.2.10a}$$

$$t = \frac{1}{\lambda(g(\theta))}(x - \theta), \tag{1.2.10b}$$

or

$$u = g(\theta), \tag{1.2.11a}$$

$$x = \theta + \lambda(g(\theta))t, \tag{1.2.11b}$$

which can be found in Whitham [78].

1.2.2 *Wave-breaking and caustic structure*

The eikonal transformation (1.2.4) is valid when

$$\frac{\partial(x,\theta)}{\partial(x,t)} = \theta_t \neq 0,$$

and $T_\theta = \theta_t^{-1} \neq 0$. Therefore the transformation is justified near the x-axis on which we have

$$-\infty < \theta_t|_{t=0} = -\lambda(g(x)) < 0. \tag{1.2.12}$$

Now, as in the previous section, we discuss the mechanism for the generation of shocks. That is, the focusing of the characteristic family leads to the breakdown of the smooth solution by giving rise to infinite derivatives in the solution. Such a 'blow up' of the derivative and the breakdown of the solution can be resolved by introducing discontinuities into the solution and extending the meaning of 'solution' in a weak sense. This was observed for the signaling problem in the preceding section.

Again, we consider the family of characteristics (1.2.9). Their envelope is

$$\Gamma : \begin{cases} L(x,t,\theta) = T(x,\theta) - t = 0, & (1.2.13a) \\ L_\theta(x,t,\theta) = T_\theta = 0. & (1.2.13b) \end{cases}$$

Note that, upon differentiating in (1.2.9) with respect to θ, we have

$$\begin{aligned} T_\theta &= -\frac{1}{\lambda(u(\theta,\theta),\theta)} + \int_\theta^x \frac{\partial}{\partial\theta}\left(\frac{1}{\lambda}\right) ds \\ &= -\frac{1}{\lambda(g(\theta),\theta)} + \int_\theta^x (-\lambda^{-2})\lambda_u u_\theta ds. \end{aligned} \tag{1.2.14}$$

Differentiating in (1.2.7) with respect to θ we obtain

$$\lambda(u,x)u_\theta = F'(\theta), \tag{1.2.15}$$

and then combining this with (1.2.14) gives

$$\Gamma : \begin{cases} x = x(\theta) : F'(\theta) \int_\theta^x \lambda^{-3}\lambda_u ds = -\frac{1}{\lambda(g(\theta),\theta)}, & (1.2.16a) \\ t = t(\theta) : t = \int_\theta^{x(\theta)} \frac{ds}{\lambda(u(s,\theta),s)}, & (1.2.16b \end{cases}$$

for the envelope.

The next lemma assures the existence and describes the structure of Γ.

LEMMA 1.4. Suppose $F'(\theta) < 0$, $\forall\, \theta \in (\theta_a, \theta_b)$ and $F'(\theta_a) = F'(\theta_b) = 0$, then Γ exists. In addition, Γ has the following properties:

1°. $\exists\, \theta_s \in (\theta_a, \theta_b)$ which gives the shock initiation point. That is,

$$t_s = t(\theta_s) = \min_{\theta \in (\theta_a, \theta_b)} \{t(\theta)\}, \tag{1.2.17a}$$

$$x_s = x(\theta_s) = \min_{\theta \in (\theta_a, \theta_b)} \{x(\theta)\}. \tag{1.2.17b}$$

2°. (x_s, t_s) is a cusp for the caustic Γ and along Γ

$$\lim_{\theta \to \theta_s^-} \frac{dt}{dx} = \lim_{\theta \to \theta_s^+} \frac{dt}{dx} = \frac{1}{\lambda(g(\theta_s), \theta_s)}. \tag{1.2.18}$$

3°.
$$\lim_{\theta \to \theta_a^+} (x(\theta), t(\theta)) = (+\infty, +\infty), \tag{1.2.19a}$$

$$\lim_{\theta \to \theta_b^-} (x(\theta), t(\theta)) = (+\infty, +\infty). \tag{1.2.19b}$$

Proof. Since $F'(\theta) < 0$, $\forall\, \theta \in (\theta_a, \theta_b)$ and λ_u is bounded away from zero, the expressions (1.2.16) guarantee the existence of Γ. Also, 3° follows from (1.2.16) and the conditions that $F'(\theta_a) = F'(\theta_b) = 0$. This, in turn, suggests the existence of $\theta_s \in (\theta_a, \theta_b)$ such that

$$t_s = t(\theta_s) = \min_{\theta \in (\theta_a, \theta_b)} \{t(\theta)\}. \tag{1.2.17a}$$

Hence the shock initiation point exists.

By employing the same argument as that used in the proof of Lemma 1.1, we can show that

$$x_s = x(\theta_s) = \min_{\theta \in (\theta_a, \theta_b)} \{x(\theta)\}. \tag{1.2.17b}$$

Now to show 2°, we differentiate (1.2.16b) with respect to θ to get

$$\begin{aligned}\frac{dt}{d\theta} &= \frac{1}{\lambda(u(x(\theta),\theta),x(\theta))}\frac{dx}{d\theta} \\ &\quad - \frac{1}{\lambda(u(\theta,\theta),\theta)} + \int_{\theta}^{x(\theta)} \frac{\partial}{\partial\theta}\left(\frac{1}{\lambda}\right) ds \\ &= \frac{1}{\lambda(u(x(\theta),\theta),x(\theta))}\frac{dx}{d\theta},\end{aligned} \tag{1.2.20}$$

where (1.2.13b) and (1.2.14) have been noted. Therefore along Γ we have

$$\frac{dt}{dx} = \frac{1}{\lambda(u(x(\theta),\theta),x(\theta))}. \tag{1.2.21}$$

In particular, letting $\theta \longrightarrow \theta_s^{\pm}$ in (1.2.21) recovers (1.2.18).
This completes the proof.

Under the conditions of Lemma 1.4, Γ can be separated into two branches, that is,

$$\Gamma_1 : x = x(\theta),\ t = t(\theta),\ \theta_a < \theta < \theta_s, \tag{1.2.22a}$$

$$\Gamma_2 : x = x(\theta),\ t = t(\theta),\ \theta_s < \theta < \theta_b. \tag{1.2.22b}$$

As before we denote the region enclosed by Γ_1 and Γ_2 in a neighbourhood of (x_s, t_s) by $\mathcal{D}$.

1.2.3 *The relative positions of the caustic branches*

Now, as in Section 1.1, we examine the relative positions of Γ_1 and Γ_2. We do this by comparing the slopes of Γ_1 and Γ_2.

From (1.2.21) we know that Γ has a slope

$$\frac{dx}{dt} = \lambda(u(x(\theta),\theta),x(\theta)) \triangleq G(\theta). \tag{1.2.23}$$

We require a lemma which ascertains the sign of $G'(\theta)$ at $\theta = \theta_s$. This is parallel to Lemma 1.2 above.

LEMMA 1.5. Suppose $F'(\theta) < 0$, $\forall\, \theta \in (\theta_a, \theta_b)$ and $F'(\theta_a) = F'(\theta_b) = 0$ so that the caustic Γ as well as the shock initiation point exists. Then at $\theta = \theta_s$, we have

$$G'(\theta_s) = F'(\theta_s)\lambda_u(u(x_s, \theta_s), x_s)/G(\theta_s) < 0. \tag{1.2.24}$$

We omit the proof which mimics that of Lemma 1.2.

Our next theorem ascertains the relative positions of Γ_1 and Γ_2 and ensures that for initial problems having smooth data, a shock wave will be generated whenever the characteristic family focuses.

THEOREM 1.3. Suppose that $F'(\theta) < 0$, $\forall\, \theta \in (\theta_a, \theta_b)$ and $F'(\theta_a) = F'(\theta_b) = 0$. Also assume that the flux function satisfies (1.1.2) and (1.1.37). Then for the two branches of the caustic defined by (1.2.16) and (1.2.22), Γ_2 is always above Γ_1.

We omit the proof which parallels that for Theorem 1.1.

1.2.4 *Bifurcation and shock propagation*

As in Lemma 1.3, the following result suggests that the shock initiation point is a bifurcation point for the arrival time formula when Γ_2 is above Γ_1.

LEMMA 1.6. When Γ_2 is above Γ_1 then, $\forall\, (x,t) \in \mathcal{D}$ there are precisely two values of θ, namely $\theta_\ell(x,t)$ and $\theta_r(x,t)$, such that

$$\theta_a < \theta_\ell(x,t) < \theta_1(t), \;\; \theta_2(t) < \theta_r(x,t) < \theta_b, \tag{1.2.25}$$

satisfying (1.2.9). That is, there are two characteristics with phases θ_ℓ, θ_r subject to (1.2.25) passing through (x,t). Here $\theta = \theta_1(t)$, $\theta_2(t)$ with $\theta_1(t) < \theta_s < \theta_2(t)$ are two branches of the inversion of $t = t(\theta)$ on Γ near t_s.

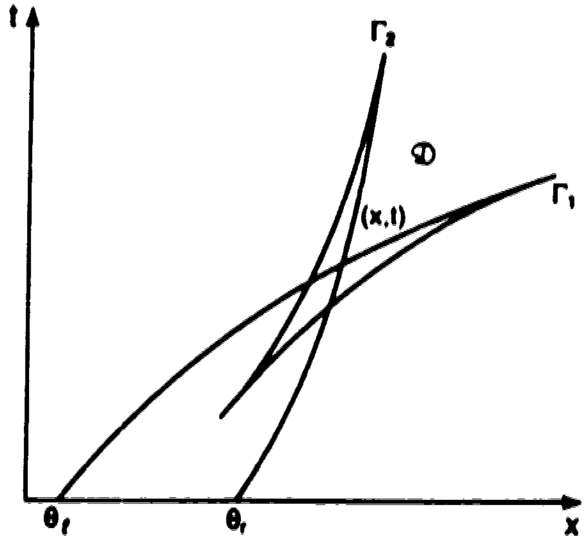

Fig. 1.5: Cusped envelope for initial problem.

We omit the proof since it parallels that for Lemma 1.3.

We can now establish the following shock propagation rule for the initial value problem.

THEOREM 1.4. (Shock propagation rule). If Γ_2 is above or overlaps Γ_1 in the neighbourhood of the shock initiation point (x_s, t_s) then an entropy admissible shock wave will be generated and propagate along the shock front

$$\frac{dt}{dx} = \frac{[u]}{[f]} = \frac{[u]}{[F]} = \frac{u(x,\theta_\ell) - u(x,\theta_r)}{F(\theta_\ell) - F(\theta_r)}, \; x > x_s, \qquad (1.2.26a)$$

$$t(x_s) = t_s, \qquad (1.2.26b)$$

with the shock strength $[u] = u_\ell - u_r = u(x,\theta_\ell) - u(x,\theta_r)$. Here $\theta_\ell = \theta_\ell(x,t)$ and $\theta_r = \theta_r(x,t)$ solve the arrival time formula (1.2.9), $\forall\,(x,t) \in \mathcal{D}$ and $\theta_a < \theta_\ell < \theta_1(t)$, $\theta_2(t) < \theta_r < \theta_b$.

Proof. We need only show that the above shock wave satisfies the appropriate entropy condition as all other aspects of the proof are similar to those of Theorem 1.2.

We note that (1.2.7) leads to

$$\lambda(u,x)u_\theta = F'(\theta) < 0, \ \forall\, \theta \in (\theta_a, \theta_b). \tag{1.2.15}$$

Then, since $\lambda(u,x) > 0$, $u_\theta < 0$ in (θ_a, θ_b) and therefore

$$u_\ell = u(x,\theta_\ell) > u(x,\theta_r) = u_r, \tag{1.2.27}$$

since $\theta_a < \theta_\ell < \theta_r < \theta_b$. Therefore, in light of $\lambda_u > 0$, we have

$$\lambda(u_\ell, x) > \frac{[f]}{[u]} = \lambda(u^*, x) > \lambda(u_r, x), \tag{1.2.28}$$

where $u^* \in (u_r, u_\ell)$ and the shock wave is therefore compressible and hence admissible by Lax's entropy inequality [39].

This completes the proof.

Now, we shall provide some detailed calculations of caustic structure by considering five particular cases employing the simplified flux function having no spatial dependence, that is, $f = f(u)$.

1.2.5 *Investigations of caustic structure*

For the initial value problem, the full solution is

$$u = g(\theta), \tag{1.2.11a}$$

$$x = \theta + F(\theta)t, \tag{1.2.11b}$$

where $F(\theta) \triangleq \lambda(g(\theta))$ and $\lambda = f'(u)$. The caustic Γ for the characteristic family is therefore

$$\Gamma : \begin{cases} x = \theta - \dfrac{F(\theta)}{F'(\theta)}, & (1.2.29a) \\ t = -\dfrac{1}{F'(\theta)}, & (1.2.29b) \end{cases}$$

so that Γ exists for $\theta \in (\theta_a, \theta_b)$, $F'(\theta) < 0$, or equivalently $g'(\theta) < 0$ owing to the fact that $\lambda_u > 0$.

Now on Γ we have

$$\frac{dt}{d\theta} = \frac{F''(\theta)}{(F'(\theta))^2}, \tag{1.2.30}$$

and

$$\frac{dx}{d\theta} = \frac{F(\theta)F''(\theta)}{(F'(\theta))^2}. \tag{1.2.31}$$

It is easy to see that $x = x(\theta)$ and $t = t(\theta)$ have the same critical points on (θ_a, θ_b). In addition, (1.2.29), (1.2.30) further indicate that $x = x(\theta)$ and $t = t(\theta)$ achieve local maxima or minima simultaneously for the same critical point $(F(\theta) = \lambda(g(\theta)) > 0)$. Each local minima or maxima for $F'(\theta)$ on (θ_a, θ_b) gives a cusp for the caustic Γ.

In order to illustrate these ideas in a concrete fashion, we shall explore the caustic structure in detail for several specific cases.

Case 1. We choose

$$F'(\theta) = \frac{1}{2}(\cos\theta - 1), \ \theta \in [0, 2\pi], \tag{1.2.32}$$

so that

$$F(\theta) = \frac{1}{2}(a_1 + \sin\theta - \theta), \ \theta \in [0, 2\pi], \tag{1.2.33}$$

and we must have $a_1 > 2\pi$ in order that $F(\theta)$ remain positive. The plot of $F'(\theta)$ together with its accompanying caustic structure are depicted in Fig. 1.6 below. The arrow indicates the direction of increasing θ.

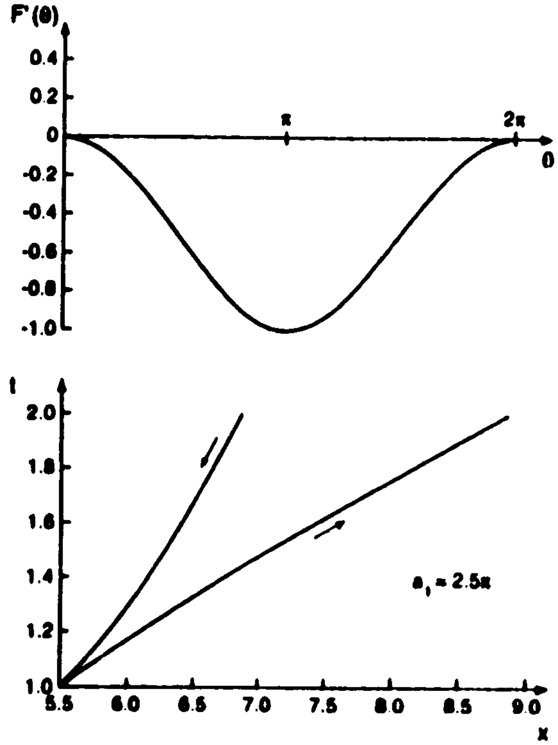

Fig. 1.6: Case 1 – $F'(\theta)$ with corresponding caustic structure.

Case 2. We choose

$$F'(\theta) = \frac{1}{3}(\cos\theta - 2), \ \theta \in [0, 4\pi], \tag{1.2.34}$$

which integrates to

$$F(\theta) = \frac{1}{3}(a_2 + \sin\theta - 2\theta), \ \theta \in [0, 4\pi], \tag{1.2.35}$$

with $a_2 > 8\pi$. In Fig. 1.7 below are displayed plots of $F'(\theta)$ and the corresponding caustic structure with its three cusps.

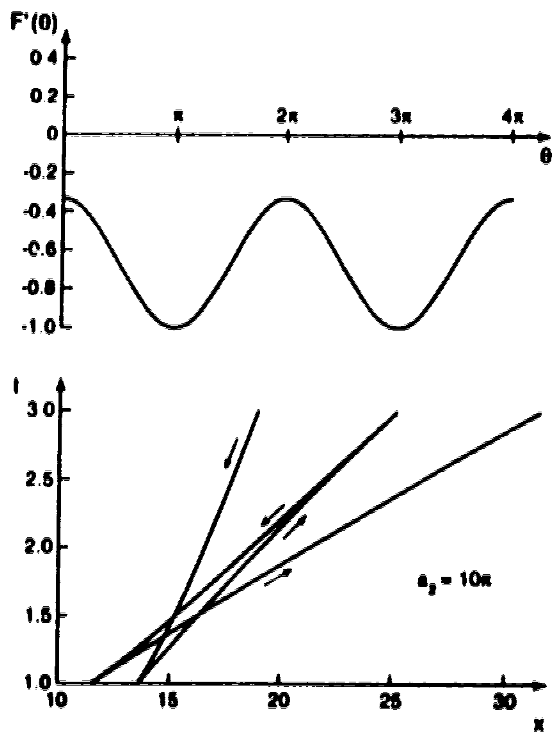

Fig. 1.7: Case 2 - $F'(\theta)$ with corresponding caustic structure.

Case 3. Choosing

$$F'(\theta) = \frac{1}{3}(\cos\theta - 2), \ \theta \in [0, 2n\pi], \tag{1.2.36}$$

we have that

$$F(\theta) = \frac{1}{3}(a_3 + \sin\theta - 2\theta), \ \theta \in [0, 2n\pi], \tag{1.2.37}$$

with $a_3 > 4n\pi$. In Fig. 1.8 we have plotted $F'(\theta)$ and its corresponding caustic structure in the case $n = 3$.

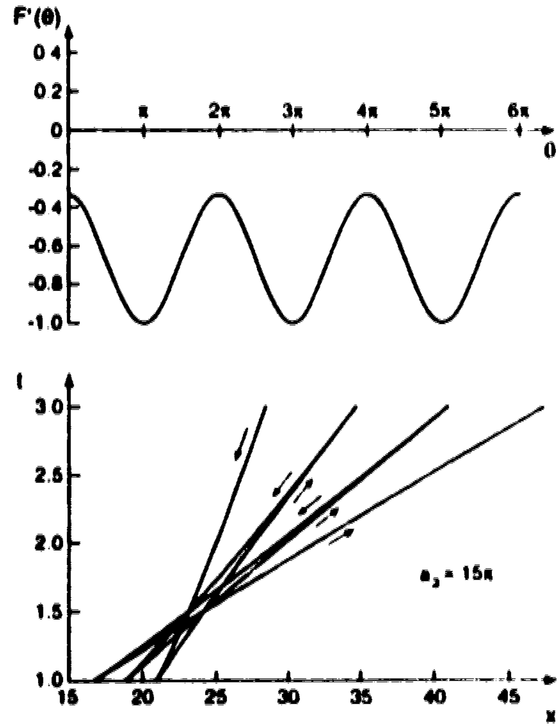

Fig. 1.8: Case 3 - $F'(\theta)$ with corresponding caustic structure.

Case 4. Choosing

$$F'(\theta) = \begin{cases} \frac{1}{3}(\cos\theta - 2), & \theta \in [0, 2\pi], \\ \frac{1}{3}(2\cos\theta - 3), & \theta \in [2\pi, 4\pi], \\ \frac{1}{3}(\cos\theta - 2), & \theta \in [4\pi, 6\pi], \end{cases} \tag{1.2.38}$$

we then have

$$F(\theta) = \begin{cases} \frac{1}{3}(a_4 + \sin\theta - 2\theta), & \theta \in [0, 2\pi], \\ \frac{1}{3}(a_4 + 2\pi + 2\sin\theta - 3\theta), & \theta \in [2\pi, 4\pi], \\ \frac{1}{3}(a_4 - 2\pi + \sin\theta - 2\theta), & \theta \in [4\pi, 6\pi], \end{cases} \tag{1.2.39}$$

with the conditions $a_4 > 4\pi$, $a_4 + 2\pi > 12\pi$, and $a_4 - 2\pi > 12\pi$ implying that $a_4 > 14\pi$. In Fig. 1.9 are plotted $F'(\theta)$ and the corresponding caustic structure.

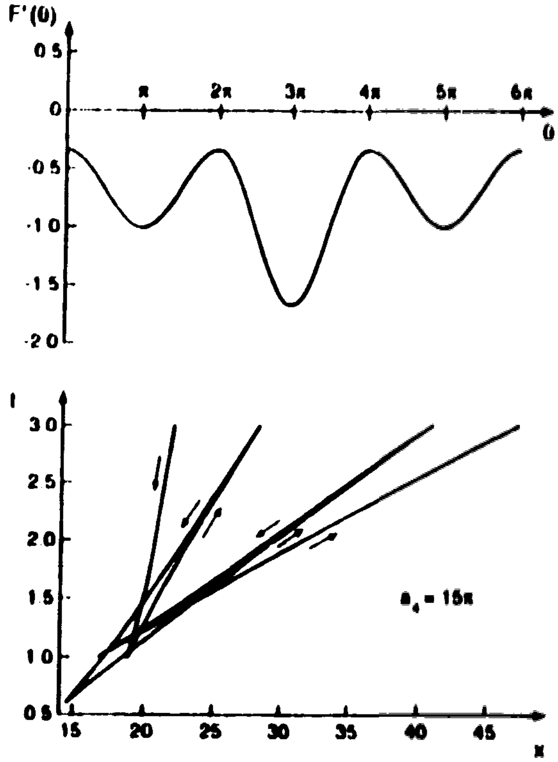

Fig. 1.9: Case 4 - $F'(\theta)$ with corresponding caustic structure.

Case 5. With

$$F'(\theta) = \begin{cases} \frac{1}{2}(\cos\theta - 1), & \theta \in [0, \pi], \\ -1 & , \theta \in [\pi, 3\pi], \\ \frac{1}{2}(\cos\theta - 1), & \theta \in [3\pi, 4\pi], \end{cases} \tag{1.2.40}$$

we have

$$F(\theta) = \begin{cases} \frac{1}{2}(a_5 + \sin\theta - \theta), & \theta \in [0, \pi], \\ \frac{1}{2}(a_5 + \pi - 2\theta), & \theta \in [\pi, 3\pi], \\ \frac{1}{2}(a_5 - 2\pi + \sin\theta - \theta), & \theta \in [3\pi, 4\pi], \end{cases} \tag{1.2.41}$$

with $a_5 > \pi$, $a_5 + \pi > 6\pi$, and $a_5 - 2\pi - 4\pi > 0$ implying that $a_5 > 6\pi$. In Fig. 1.10 are plotted $F'(\theta)$ together with the corresponding caustic structure.

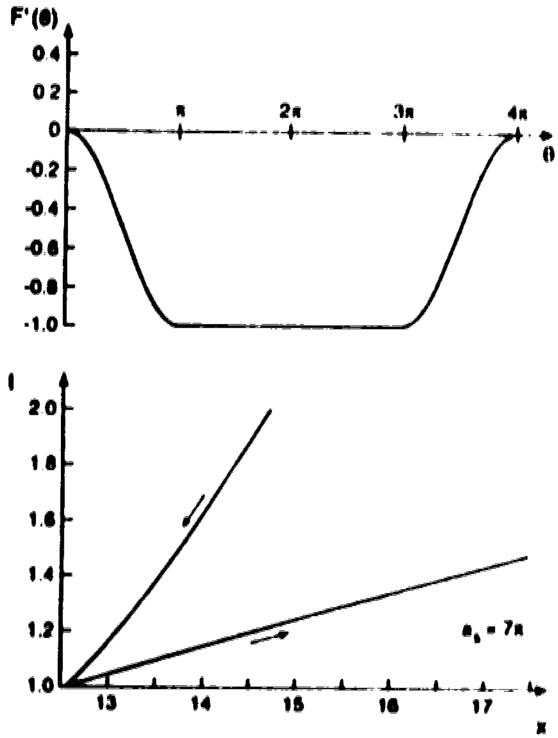

Fig. 1.10: Case 5 – $F'(\theta)$ with corresponding caustic structure.

It is interesting to note that in Case 5 depicted in Fig. 1.10, θ_s is no longer a single point but rather encompasses the interval $[\pi, 3\pi]$. In this situation we have a whole bunch of characteristic converging at a single point and forming a so-called 'compression fan', a shock is initiated with non-zero strength $[u] = u(\pi) - u(3\pi) = g(\pi) - g(3\pi)$ at (x_s, t_s) rather than one initiated with zero strength.

We end our discussion with some remarks about general scalar conservation laws with spatially dependent flux functions.

Remark 1.2. As in Fig. 1.10, when θ_s is no longer a single point but rather encompasses an interval $[\theta^*, \theta^{**}]$, the results in Theorems 1.1-1.4 remain valid. One can remedy the analysis by breaking θ_s into two points, $\theta_s^- = \theta^*$, $\theta_s^+ = \theta^{**}$, and carrying out the proofs on the left neighbourhood of θ_s^- for Γ_1 and the right neighbourhood of θ_s^+ for Γ_2, respectively.

Remark 1.3. Apparently, for smooth initial and boundary data, characteristic focusing is the only mechanism that leads to the breaking of smooth waves and hence generating shock waves. In the nonfocusing case, smooth waves are expansive and remain smooth. Discussions for flux functions which are either concave or of negative characteristic speed are similar.

Remark 1.4. We require the characteristic speed $\lambda(u,x)$ to be strictly positive (or negative) in order to avoid the transonic case in which $\lambda(u,x)$ varies around zero. The uniform convexity condition (1.1.37) (or concavity condition) is applied to guarantee the existence of a caustic. When the caustic is known to exist, we need $f(u,x)$ only to be strictly convex (or concave) in order to ensure the relative positions of Γ_1 and Γ_2 ascertained in Theorems 1.2 and 1.4. Therefore, the situation when $f(u,x)$ may change convexity, that is, f has an inflection point at $\theta = \theta_0$ poses an interesting problem which merits further investigation.

1.3. A Transonic Model Problem

In this section we shall employ the nonlinear phase variable to discuss a transonic model problem proposed by Liu [51, 55] and based on the equation

$$u_t + f(u)_x = c(x)h(u), \tag{1.3.1}$$

where the flux function $f(u)$ is a smooth convex function, and the source term on the right is stationary.

Equation (1.3.1) is perhaps the simplest model equation for hyperbolic conservation laws with a moving source. Indeed, (1.3.1) is a simplified version of

$$u_t + \bar{f}(u)_x = c(x - \alpha t)h(u), \tag{1.3.2}$$

after the change of variables $x - \alpha t \longmapsto x$, $\bar{f}(u) \longmapsto \bar{f}(u) - \alpha u \triangleq f(u)$, where α is a constant, representing the source speed. Examples in application often involve a system of conservation laws, such as those representing gas flow through a nozzle, and so will exhibit more complex structure and behaviour [50].

For hyperbolic conservation laws with a moving source term, the source speed is a key parameter affecting the behaviour of the system. In particular, resonance occurs when the source speed is close to one of the characteristic speeds of the system. Equation (1.3.1), however, provides the simplest model which inherits many of the features found in more complicated problems (see [51, 55]).

Liu [55] carried out a qualitative analysis for the model problem (1.3.1) revealing many interesting phenomena such as nonlinear stability, instability, and changing wave types.

For the other case, that is, when the source speed differs significantly from all the characteristic speeds, systems of conservation laws with source terms behave in a more regular fashion. We refer the reader to the significant paper of Liu [48] in which he studied the existence of global solutions as well as their large time behaviour via a generalized Glimm scheme. Dafermos and Hsiao [10] may also be consulted on these issues.

Here we concentrate on the transonic model (1.3.1) and analyze its associated initial value problem by taking advantage of the direct use of a nonlinear phase as before.

For definiteness, we impose the same conditions as those in [55]. These are:

(i). $c(x)$, which represents the source strength, is piecewise smooth and of compact support,

$$c(x) \equiv 0, \forall x \notin [0,1]. \tag{1.3.3}$$

(ii). $h(u)$, which couples the source with the hyperbolic conservation law, satisfies

$$h(u) \neq 0, \; h'(u) \neq 0. \tag{1.3.4}$$

In particular, $h(0) \neq 0$, $h'(0) \neq 0$. Thus a strong coupling is assumed.

(iii). We are interested in the transonic case, that is, the characteristic speed and the source speed are close. For our transformed stationary source problem (1.3.1) this means that $f'(u)$ is around zero so that we assume

$$f(0) = f'(0) = 0. \tag{1.3.5}$$

In addition, the convexity of f provides

$$f''(u) > 0, \tag{1.3.6}$$

so that $f'(u) > 0$ for $u > 0$ and $f'(u) < 0$ for $u < 0$. Thus $u > 0$ refers to a *supersonic* state, $u < 0$ to a *subsonic* state, and $u = 0$ to the *sonic* state.

1.3.1 *The initial value problem*

Our initial value problem then consists of (1.3.1) together with

$$u\,|_{t=0} = u_0(x), \;\; x \in (-\infty, \infty)\,, \tag{1.3.7}$$

where $u_0(x)$ is a smooth function.

To begin with, we introduce the nonlinear phase $\theta = \theta(x,t)$ as before, where θ solves

$$\theta_t + f'(u)\theta_x = 0, \tag{1.3.8a}$$

$$\theta\,|_{t=0} = x. \tag{1.3.8b}$$

We invert $\theta = \theta(x,t)$ to obtain the arrival time formula

$$t = T(x,\theta), \tag{1.3.9}$$

which gives, upon differentiation and substitution into (1.3.8), that

$$T_x = \frac{1}{f'(u)}. \tag{1.3.10}$$

An integration then leads to

$$t = T(x, \theta) = \int_\theta^x \frac{ds}{f'(u)}, \tag{1.3.11}$$

and it appears that the sonic state $u = 0$ provides a singularity for the arrival time formula.

We now transform the spatial-temporal coordinates into space-phase coordinates, that is,

$$(x, t) \longmapsto (x, \theta), \tag{1.3.12}$$

by means of

$$x = x, \tag{1.3.13a}$$

$$\theta = \theta(x, t). \tag{1.3.13b}$$

Thus $\partial_x \longmapsto \partial_x + \theta_x \partial_\theta$, $\partial_t \longmapsto \theta_t \partial_\theta$, so that (1.3.1) is transformed to

$$\frac{\partial}{\partial x} f(u) = c(x) h(u), \tag{1.3.14}$$

or,

$$f'(u) \frac{du}{dx} = c(x) h(u), \tag{1.3.15}$$

for any fixed value of θ. Meanwhile, the initial condition yields

$$u\,|_{x=\theta} = u_0(\theta), \tag{1.3.16}$$

and so (1.3.15) and (1.3.16) together pose an initial value problem for an ordinary differential equation that can be integrated directly giving

$$\int_{u_0(\theta)}^{u} \frac{f'(u)du}{h(u)} = \int_{\theta}^{x} c(s)ds, \tag{1.3.17}$$

or

$$G(u) = G(u_0(\theta)) + \int_{\theta}^{x} c(s)ds, \tag{1.3.18}$$

where

$$G(u) = \int_0^u \frac{f'(u)}{h(u)}du.$$

Therefore, the solution to the initial value problem (1.3.1), (1.3.7) is provided by

$$G(u) = G(u_0(\theta)) + \int_{\theta}^{x} c(s)ds \triangleq F(x,\theta), \tag{1.3.18}$$

$$t = T(x,\theta) = \int_{\theta}^{x} \frac{ds}{f'(u)}. \tag{1.3.11}$$

1.3.2 *The sonic line and transonic hyperbolic waves*

From (1.3.5) we see that (1.3.11) is singular at $u = 0$. From (1.3.18), we may observe that the value $u = 0$ is attained when

$$F(x,\theta) = G(u_0(\theta)) + \int_{\theta}^{x} c(s)ds = 0. \tag{1.3.19}$$

The solution to the above equation gives rise to the position of a *sonic state*. As θ varies, this sonic position also varies. We call the solution to (1.3.19) the *sonic line* and write it as

$$x = X(\theta), \tag{1.3.20a}$$

$$t = T(\theta), \tag{1.3.20b}$$

where $T(\theta) \triangleq T(X(\theta),\theta)$.

The next lemma states the conditions under which a sonic line exists.

LEMMA 1.7.

1°. If $c(x) > 0$ in $(0,1)$, then the sonic line $x = X(\theta), t = T(\theta)$ exists for those $\theta \in (-\infty,\infty)$ satisfying

$$G(u_0(\theta)) \leq \int_0^\theta c(s)ds, \quad u_0(\theta) \leq 0. \tag{1.3.21}$$

2°. If $c(x) < 0$ in $(0,1)$, then the sonic line $x = X(\theta)$ exists for those $\theta \in (-\infty,\infty)$ satisfying

$$G(u_0(\theta)) \leq \int_1^\theta c(s)ds, \quad u_0(\theta) \geq 0. \tag{1.3.22}$$

Hence $x = X(\theta)$ solves

$$\int_0^x c(s)ds = \int_0^\theta c(s)ds - G(u_0(\theta)). \tag{1.3.23}$$

In addition, $0 \leq X(\theta) \leq 1$.

The proof, being straightforward, is omitted.

We are interested in the behaviour of hyperbolic waves near sonic states. Here we study this behaviour near the sonic line and, as a result of Lemma 1.7, we may assume the sonic line to be contained in $0 \leq X(\theta) \leq 1$.

We now expand $F(x,\theta)$ about $x = X(\theta)$ to get

$$\begin{aligned} F(x,\theta) &= \left(\frac{\partial F}{\partial x}\right)_{x=X(\theta)} (x - X(\theta)) + O((x - X(\theta))^2) \\ &= c(X(\theta))(x - X(\theta)) + O((x - X(\theta))^2). \end{aligned} \tag{1.3.24}$$

On the other hand,

$$G(0) = G'(0) = 0, \tag{1.3.25}$$

and

$$\begin{aligned} G''(0) &= [f'(u)/h(u)]'_{u=0} \\ &= f''(0)/h(0), \end{aligned} \tag{1.3.26}$$

so that, combining (1.3.18) with (1.3.24), we find

$$\frac{1}{2}\,\frac{f''(0)}{h(0)}u^2 + O(u^3) = c(X(\theta))(x - X(\theta)) + O((x - X(\theta))^2),$$

or, equivalently,

$$u^2 = \frac{2c(X(\theta))h(0)}{f''(0)}(x - X(\theta)) + O((x - X(\theta))^2) + O(u^3). \tag{1.3.27}$$

From (1.3.18), when $x = X(\theta)$, we have that

$$u(X(\theta), \theta) = 0, \tag{1.3.28}$$

which is the sonic state. Now, letting $x \longrightarrow X(\theta)$ in (1.3.27), we obtain the following asymptotic relations:

When $\mathrm{sgn}(c(X(\theta))h(0)) > 0$,

$$u \sim \pm\sqrt{\frac{2c(X(\theta))h(0)}{f''(0)}}(x - X(\theta))^{1/2}, \; x > X(\theta), \; x \longrightarrow X(\theta). \tag{1.3.29}$$

When $\mathrm{sgn}(c(X(\theta)h(0)) < 0$,

$$u \sim \pm\sqrt{\frac{-2c(X(\theta))h(0)}{f''(0)}}(X(\theta) - x)^{1/2}, \; x < X(\theta), \; x \longrightarrow X(\theta). \tag{1.3.30}$$

Correspondingly, the characteristics take the following asymptotic forms

$$\frac{dt}{dx} = \frac{1}{f'(u)} \sim \begin{cases} \pm k^+(\theta)(x - X(\theta))^{-1/2}, \ x > X(\theta), \ x \longrightarrow X(\theta), \\ \qquad \text{if } \operatorname{sgn}(c(X(\theta))h(0)) > 0, & (1.3.31) \\ \pm k^-(\theta)(-x + X(\theta))^{-1/2}, \ x < X(\theta), \ x \longrightarrow X(\theta), \\ \qquad \text{if } \operatorname{sgn}(c(X(\theta))h(0) < 0, & (1.3.32) \end{cases}$$

where

$$k^{\pm}(\theta) = 1/\sqrt{\pm 2c(X(\theta))h(0)f''(0)}. \tag{1.3.33}$$

Accordingly, we have that

$$t - T(\theta) = \int_{X(\theta)}^{x} \frac{ds}{f'(u)} \sim \begin{cases} \pm 2k^+(\theta)(x - X(\theta))^{1/2}, \ x > X(\theta), \ x \longrightarrow X(\theta), & (1.3.34) \\ \qquad \text{if } \operatorname{sgn}(c(X(\theta))h(0)) > 0, \\ \pm 2k^-(\theta)(X(\theta) - x)^{1/2}, \ x < X(\theta), \ x \longrightarrow X(\theta), & (1.3.35) \\ \qquad \text{if } \operatorname{sgn}(c(X(\theta))h(0)) < 0, \end{cases}$$

where

$$T(\theta) = \int_{\theta}^{X(\theta)} \frac{ds}{f'(u)},$$

exists.

The above analysis suggests some interesting aspects to wave propagation near the sonic state. For example, suppose we consider the case wherein $\operatorname{sgn}(c(X(\theta))h(0)) > 0$. In this instance, a subsonic state corresponding to $u = u(x,\theta) < 0$ propagates along a characteristic $t = T(x,\theta)$ in the direction of decreasing x reaching the sonic state at $x = X(\theta)$ where it is then reflected back along another branch of the characteristic becoming supersonic ($u(x,\theta) > 0$). This characteristic can never cross $x = X(\theta)$ into the left region. Therefore, $\forall\theta \in (-\infty,\infty)$, when $0 \le X(\theta) \le 1$ exists, $(X(\theta), T(\theta))$ is a turning point for the characteristic of phase θ along which the nonlinear hyperbolic wave changes type. This interesting property was observed previously in [55]. See Fig. 1.11.

We note that if the sonic state is reached at any time (even initially) within the support of the source which comprises the spatial interval [0, 1] the above analysis still applies. Hence the singularity in (1.3.11) corresponding to this sonic state is, in fact, removable. The case in which a sonic state is reached outside the spatial interval [0, 1] is trivial since this sonic state is stationary and (1.3.11) reduces to a vertical line $x = \theta$ and we have initially $u_0(\theta) = 0$ at this point.

A more complete discussion of this problem is to be found in the recent article of He and Moodie [18].

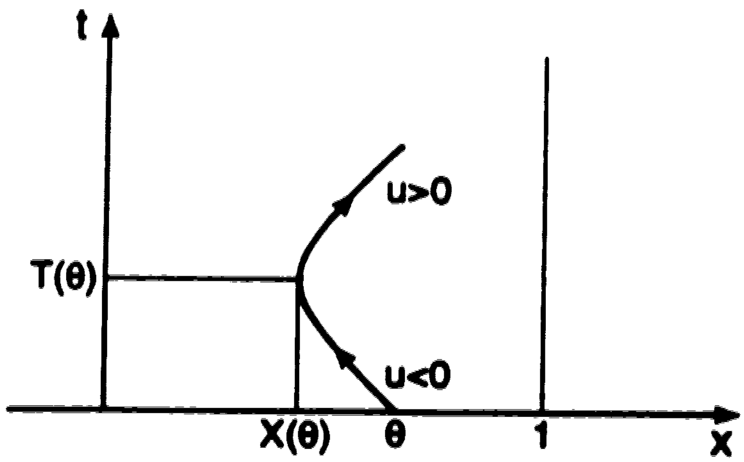

Fig. 1.11: Turning point for characteristic of phase θ.

CHAPTER 2.

Signaling Problem for Systems of Hyperbolic Conservation Laws

In this chapter we present a detailed study of weakly nonlinear hyperbolic waves arising from the action of small amplitude, high-frequency, and single wave-mode boundary disturbances. The main tools deployed in this study are the nonlinear phase introduced in Chapter 1 and an asymptotic analysis. Our objective is to develop a relatively complete geometrical optics theory involving only one nonlinear phase [17, 20]. As we have already shown in Chapter 1, a direct use of this nonlinear phase provides a significant advantage for the analysis of nonlinear hyperbolic waves. Notably, a transparent picture detailing the process from wave-breaking to shock generation and propagation can be obtained and the after shock admissiblity of the solution justified.

In the more complicated case involving systems of hyperbolic conservation laws, such a feature of the analysis is preserved when a nonlinear phase is incorporated into an asymptotic analysis. This leads to the formulation of a *single wave-mode geometrical optics theory*.

Asymptotic expansion methods have been employed by many authors to investigate weakly nonlinear hyperbolic waves and among these investigations are the recent works of Hunter, Keller, Majda, and Rosales [5, 24–26, 61–63]. Therein, they presented a systematic approach to handle both non-resonantly and resonantly interacting weakly nonlinear hyperbolic waves. The interested reader should consult the survey articles of Majda [59–60] and Rosales [68]. An earlier study of the signaling problem that employs a nonlinear phase can be found in Seymour and Mortell [70].

We arrange the contents of this chapter as follows. After presenting some preliminaries and notational conventions, the so-called signaling problem is introduced in Section 2.1. In Sections 2.2, 2.3, and 2.4 we carry out a detailed asymptotic analysis by means of which the signaling problem is solved in the regime of smooth solutions. In addition, we are able to prove an associated solvability condition. In Sections 2.5 and 2.6, we examine the wave-breaking phenomenon and devise a rational scheme to study the transition process from wave-breaking to shock generation. This result is made possible by arranging the small amplitude and high-frequency relation according to the order of local linear degeneracy and carrying out a bifurcation analysis. Similar to the scalar conservation laws discussed in Chapter 1, two typical features appear here. For the first feature, wave-breaking leads to shock generation – a fact that is well known and documented in, for example, Courant and Friedrichs [8] and Whitham [75]. The second feature, which is excluded in Chapter 1, is entirely new. That is, the asympototic solution consists of two parts, one parallels that of the flux function without spatial dependence and the other part, which appears at an order higher than the order of local linear degeneracy. This second part is induced solely by the spatial inhomogeneity of the flux function and it is continuous across the shock front thereby invalidating the resolution of higher order terms. The final section of this chapter will be devoted to an application. We consider a fluid-filled hyperelastic tube problem studied earlier by Moodie and Swaters [64]. The characteristic fields associated with this problem are locally linearly degenerate. We employ the above theory not only to recover the results in [64] but also to show that a shock wave is generated at the shock initiation point and complete the shock tracking procedure.

2.1. Preliminaries and Formulation of Signaling Problem

We are interested in the system of hyperbolic conservation laws in one space dimension given by

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u}, x)_x = \mathbf{0}, \tag{2.1.1}$$

where x and t, as usual, represent spatial and temporal variables, respectively, $\mathbf{u} = (u_1, \ldots, u_n)^T$ is the vector of n state variables and $\mathbf{f}(\mathbf{u}, x) = (f_1(\mathbf{u}, x), \ldots, f_n(\mathbf{u}, x))^T$

is the vector-valued flux function of n smooth nonlinear functions.

As in the case of Chapter 1, the flux function is allowed to admit explicit spatial dependence so as to accommodate applications from a wider class of problem involving spatial inhomogeneity in the transmitting medium.

In this and the following two sections, instead of (2.1.1), we shall consider the equivalent general quasi-linear and strictly hyperbolic system of the form

$$\mathbf{u}_t + \mathbf{A}(\mathbf{u}, x)\mathbf{u}_x = \mathbf{b}(\mathbf{u}, x), \tag{2.1.2}$$

where the notations bear the same meaning as for (2.1.1) and $\mathbf{A}(\mathbf{u}, x)$ and $\mathbf{b}(\mathbf{u}, x)$ are smooth matrix and vector functions of their arguments, respectively.

We assume the system (2.1.2) admits a steady state solution which we write, without loss of generality, as $\mathbf{u} \equiv \mathbf{0}$. As a result, one observes that

$$\mathbf{b}(\mathbf{0}, x) \equiv \mathbf{0}. \tag{2.1.3}$$

The strict hyperbolicity of (2.1.2) requires that $\mathbf{A}(\mathbf{u}, x)$ have n distinct and real eigenvalues $\{\lambda_i(\mathbf{u}, x)\}_{i=1}^{n}$. We further assume that

$$\lambda_1(\mathbf{u}, x) < \lambda_2(\mathbf{u}, x) < \cdots < \lambda_p(\mathbf{u}, x) \leq 0 < \lambda_{p+1}(\mathbf{u}, x) < \cdots < \lambda_n(\mathbf{u}, x), \tag{2.1.4}$$

when positive and negative eigenvalues are distinguished.

We denote by $\boldsymbol{\ell}^{(i)}(\mathbf{u}, x)$ and $\mathbf{r}^{(i)}(\mathbf{u}, x)$ the left and right eigenvectors associated with $\lambda_i(\mathbf{u}, x)$ $(i = 1, 2, \ldots, n)$, respectively. They satisfy the orthonormality condition

$$\boldsymbol{\ell}^{(i)}\mathbf{r}^{(j)} = \delta_{ij}, \; i, j = 1, 2, \ldots, n, \tag{2.1.5}$$

where δ_{ij} is the Kronecker function. In addition, each $\mathbf{r}^{(i)}(\mathbf{0}, x) \triangleq \mathbf{r}_0^{(i)}(x)$ is called a *wave mode* of the system (2.1.2).

If $\mathbf{u} = \mathbf{u}_0(x,t)$ is a solution of (2.1.2), the ith characteristic field is called *locally linearly degenerate* about $\mathbf{u} = \mathbf{u}_0$ if

$$(\text{grad}_{\mathbf{u}}\lambda_i)\mathbf{r}^{(i)} = 0, \tag{2.1.6}$$

when evaluated at $\mathbf{u} = \mathbf{u}_0$. This is a modification to the globally defined linear degeneracy condition first introduced by Lax [37, 39]. This local linear degeneracy concept appears in the work of Hunter, Majda, and Rosales [26, 62], although the terminology is not explicitly specified there. See also the recent paper of Rosales [69].

Now supposing the boundary $x = 0$ to be perturbed by a small amplitude, high-frequency boundary disturbance, one may expect a weakly nonlinear hyperbolic wave to be generated and propagate into the steady state region. If the discussion is restricted to the first quarter plane $x \geq 0$, $t \geq 0$, the ***signaling problem*** is formulated as

$$\mathbf{u}_t + \mathbf{A}(\mathbf{u},x)\mathbf{u}_x = \mathbf{b}(\mathbf{u},x),\ x > 0,\ t > 0, \tag{2.1.2}$$

$$\mathbf{u} \equiv \mathbf{0},\ x \geq 0,\ t = 0, \tag{2.1.7}$$

$$\mathbf{u} = \varepsilon\mathbf{g}_\varepsilon(t/\delta),\ x = 0,\ t \geq 0, \tag{2.1.8}$$

where ε and δ are two small but not independent positive parameters introduced to describe the small amplitude and high-frequency feature of the boundary perturbation. Also, $\mathbf{g}_\varepsilon(\cdot)$ is a smooth vector function appropriately restricted and satisfying

$$\mathbf{g}_\varepsilon(0) = \mathbf{g}'_\varepsilon(0) = \mathbf{0}. \tag{2.1.9}$$

In addition, $\mathbf{g}_\varepsilon(\cdot)$ is supposed Taylor-expandable about the small parameter ε, that is,

$$\mathbf{g}_\varepsilon(\cdot) = \mathbf{g}^{(0)}(\cdot) + \varepsilon\mathbf{g}^{(1)}(\cdot) + \frac{1}{2!}\varepsilon^2\mathbf{g}^{(2)}(\cdot) + \dots. \tag{2.1.10}$$

As is known, the well-posedness of the signaling problem relies on $\mathbf{g}_\epsilon(\cdot)$, which cannot be arbitrarily assigned. The reader may refer to [42, Chapter 4, Theorem 3.1] for a well-established sufficient condition which ensures the existence and uniqueness of a smooth solution for the signaling problem in a finite time layer. However, in our discussion, $\mathbf{g}_\epsilon(\cdot)$ is not specified but rather determined in the course of the solution.

If $\mathbf{g}_\epsilon(\cdot)$ is a wave mode, that is, $\mathbf{g}_\epsilon(\cdot)$ is parallel to $\mathbf{r}_0^{(i)}(0)$ for some i, we shall call $\mathbf{g}_\epsilon(\cdot)$ a *single wave-mode*. Since this will not, in general, happen, as we shall see later, we adopt the convention that $\mathbf{g}_\epsilon(\cdot)$ is of a single wave-mode if its leading term $\mathbf{g}^{(0)}(\cdot)$ is.

In order that our mathematical formulation agree with most physical applications of the boundary condition, we impose an additional constraint, that is, one component of $\mathbf{u}$, say u_1, satisfies

$$u_1(0,t) = \epsilon g_1^{(0)}(t/\delta), \tag{2.1.11}$$

where the $O(\epsilon^2)$ term vanishes.

2.2. Nonlinear Phase, Eikonal Transformation, and Transport Equations

2.2.1 *Nonlinear phase*

Let us pick a positive eigenvalue from among $\lambda_{p+1}(\mathbf{u},x),\ldots,\lambda_n(\mathbf{u},x)$, and for simplicity denote it by

$$\lambda = \lambda(\mathbf{u},x), \tag{2.2.1}$$

with $\boldsymbol{\ell} = \boldsymbol{\ell}(\mathbf{u},x)$ and $\mathbf{r} = \mathbf{r}(\mathbf{u},x)$ as the corresponding left and right eigenvectors, respectively.

We supply an additional boundary condition for the signaling problem (2.1.2), (2.1.7), and (2.1.8) by assuming $t = T(x)$, which is defined by

$$\frac{dt}{dx} = \frac{1}{\lambda(0,x)} \triangleq \frac{1}{\lambda_0(x)}, \tag{2.2.2a}$$

$$t(0) = 0, \tag{2.2.2b}$$

to be the leading wavefront. It gives the time for the first boundary disturbance (2.1.8) to arrive at a position x so that ahead of this leading wavefront is the steady state, namely $\mathbf{u} \equiv 0$. This notion of specifying the leading wavefront as separating the disturbed region from the steady state is, of course, valid only prior to times when the characteristics first focus causing the smooth solution to break down.

We introduce a nonlinear phase variable associated with $\lambda = \lambda(\mathbf{u}, x)$ and define it as the solution of

$$\theta_t + \lambda(\mathbf{u}, x)\theta_x = 0, \; x > 0, \, t > 0, \tag{2.2.3}$$

$$\theta|_{x=0} = t/\delta(\epsilon), \;\; t \geq 0. \tag{2.2.4}$$

The existence and smoothness of $\theta = \theta(x,t)$ is guaranteed by the existence and smoothness of $\mathbf{u} = \mathbf{u}(x,t)$. The inverse function of $\theta = \theta(x,t)$ gives the so-called *arrival time formula*, that is, the time for the wave of phase θ to arrive at a position x, which we write as

$$t = T(x, \theta; \epsilon). \tag{2.2.5}$$

Indeed, (2.2.5) is a representation of the characteristic family associated with $\lambda = \lambda(\mathbf{u}, x)$. As a result of (2.2.2), we see that, subject to the restrictions stated above, $\theta = 0$ in (2.2.5) corresponds to the leading wavefront $t = T(x)$.

2.2.2 *Bihonal transformation*

We regard θ as an independent variable and transform the signaling problem (2.1.2), (2.1.7), and (2.1.8) into space-phase coordinates, that is,

$$(x,t) \longmapsto (x,\theta), \tag{2.2.6}$$

by

$$x = x, \tag{2.2.7a}$$

$$\theta = \theta(x,t). \tag{2.2.7b}$$

Rewriting

$$\mathbf{u}(x,t) = \hat{\mathbf{u}}(x,\theta;\epsilon), \tag{2.2.8}$$

then

$$\partial_x \longmapsto \partial_x - \frac{\theta_t}{\lambda}\partial_\theta, \tag{2.2.9a}$$

$$\partial_t \longmapsto \theta_t \partial_\theta. \tag{2.2.9b}$$

Transforming (2.1.2), (2.1.7), and (2.1.8) into (x,θ) coordinates and dropping the tilde for notational convenience, we obtain

$$\theta_t \left\{ \mathbf{I} - \frac{\mathbf{A}(\mathbf{u},x)}{\lambda(\mathbf{u},x)} \right\} \mathbf{u}_\theta = \mathbf{b}(\mathbf{u},x) - \mathbf{A}(\mathbf{u},x)\mathbf{u}_x, \tag{2.2.10}$$

$$\mathbf{u} \equiv 0,\ \theta = 0,\ x \geq 0, \tag{2.2.11}$$

$$\mathbf{u} = \epsilon \mathbf{g}_\epsilon(\theta),\ x = 0,\ \theta \geq 0, \tag{2.2.12}$$

where ahead of the leading wavefront $\mathbf{u} \equiv \mathbf{0}$ is noted.

Meanwhile, the identity

$$t \equiv T(x,\theta(x,t);\epsilon), \tag{2.2.13}$$

gives

$$\theta_t = 1/T_\theta,\ \theta_x = -T_x/T_\theta \tag{2.2.14}$$

when differentiated with respect to t and x, respectively. A substitution from (2.2.14) into (2.2.3) and (2.2.4) results in

$$T_x = \frac{1}{\lambda(\mathbf{u}, x)}, \tag{2.2.15a}$$

$$T\,|_{x=0} = \delta(\epsilon)\theta, \tag{2.2.15b}$$

or

$$\begin{aligned} t &= T(x, \theta; \epsilon) \\ &= \delta(\epsilon)\theta + \int_0^x \frac{ds}{\lambda(\mathbf{u}(s, \theta; \epsilon), s)}. \end{aligned} \tag{2.2.16}$$

As in the case of scalar conservation laws discussed in Chapter 1, the validity of the eikonal transformation rests on the condition that the Jacobian

$$J \triangleq \frac{\partial(x, \theta)}{\partial(x, t)} = \theta_t \neq 0, \tag{2.2.17}$$

and

$$T_\theta = \frac{1}{\theta_t} \neq 0. \tag{2.2.18}$$

It is easy to check that the transformation is valid in the neighbourhood of the boundary $x = 0$ since

$$0 < \theta_t\,|_{x=0} = \frac{1}{\delta(\epsilon)} < +\infty. \tag{2.2.19}$$

One of the most important features of nonlinear hyperbolic systems is that the solution, no matter how smooth initially, may develop shocks. A mechanism underlying such a phenomenon is due to the focusing of a characteristic family, which in turn leads to the breakdown of a smooth solution by giving rise to an infinite derivative. Such a 'blow-up' of the derivative may introduce a discontinuity into the solution which then grows and propagates as a shock wave. This was explained in the simpler context of scalar conservation laws in Chapter 1. A criterion for determining this focusing effect is

$$T_\theta = 0, \tag{2.2.20}$$

which, in turn, signals the breakdown of the eikonal transformation (2.2.6).

2.2.3 *Weakly nonlinear waves and the transport equations*

We will now construct the asymptotic solution for the signaling problem (2.2.10)-(2.2.12) in the form

$$\mathbf{u}(x,\theta;\varepsilon) = \varepsilon \sum_{k=0}^{\infty} \frac{\varepsilon^k}{k!} \mathbf{u}^{(k)}(x,\theta). \tag{2.2.21}$$

Such expansions are often referred to as *weakly nonlinear (hyperbolic) waves*. Here we require

$$\mathbf{u}^{(k)}(x,\theta),\ \mathbf{u}_x^{(k)}(x,\theta),\ \mathbf{u}_\theta^{(k)}(x,\theta) = O(1), k = 0,1,2\dots, \tag{2.2.22}$$

that is, $\mathbf{u}^{(k)}(x,\theta)$ ($k = 0,1,2,\dots$) are independent of ε when θ is viewed as an independent variable.

We proceed to construct an asymptotic representation for the solution of the signalling problem in the form (2.2.21) by first employing (2.2.14) in order to rewrite (2.2.10) as

$$\left(\mathbf{I} - \frac{\mathbf{A}}{\lambda}\right)\mathbf{u}_\theta = (\mathbf{b} - \mathbf{A}\mathbf{u}_x)\,T_\theta. \tag{2.2.23}$$

From (2.2.23) the transport equation is arrived at by applying $\boldsymbol{\ell}$ to both sides of this equation to obtain

$$\boldsymbol{\ell}\mathbf{b} - \boldsymbol{\ell}\mathbf{A}\mathbf{u}_x = 0. \tag{2.2.24}$$

Based upon (2.2.23), (2.2.24), and the arrival time formula (2.2.16), we are able to construct a weakly nonlinear wave solution of (2.2.10)-(2.2.12) in the form of the asymptotic expansion (2.2.21), subject to (2.2.22), when $\mathbf{g}_\epsilon(\cdot)$ is of a single wave-mode.

Before we can proceed with our construction of the asymptotic expansion in terms of the small parameter ϵ, we need to Taylor-expand the following (scalar, matrix, and vector) functions about $\mathbf{u} \equiv \mathbf{0}$. That is,

$$\frac{1}{\lambda(\mathbf{u},x)} = \frac{1}{\lambda_0(x)} + \sum_{k=1}^{\infty} \Lambda^{(k)}(\mathbf{u},\ldots,\mathbf{u})/k!, \tag{2.2.25}$$

$$\mathbf{A}(\mathbf{u},x) = \sum_{k=0}^{\infty} \mathbf{A}^{(k)}(\mathbf{u},\ldots,\mathbf{u})/k!, \tag{2.2.26}$$

$$\mathbf{b}(\mathbf{u},x) = \sum_{k=1}^{\infty} \mathbf{b}^{(k)}(\mathbf{u},\ldots,\mathbf{u})/k!, \tag{2.2.27}$$

$$(\lambda\boldsymbol{\ell})(\mathbf{u},x) \triangleq \mathbf{a}(\mathbf{u},x) = \sum_{k=0}^{\infty} \mathbf{a}^{(k)}(\mathbf{u},\ldots,\mathbf{u})/k!, \tag{2.2.28}$$

$$(\boldsymbol{\ell}\mathbf{b})(\mathbf{u},x) \triangleq c(\mathbf{u},x) = \sum_{k=0}^{\infty} c^{(k)}(\mathbf{u},\ldots,\mathbf{u})/k!, \tag{2.2.29}$$

$$\left(\frac{1}{\lambda}\mathbf{A}\right)(\mathbf{u},x) \triangleq \mathbf{D}(\mathbf{u},x) = \sum_{k=0}^{\infty} \mathbf{D}^{(k)}(\mathbf{u},\ldots,\mathbf{u})/k!, \tag{2.2.30}$$

where

$$\mathbf{A}^{(0)} = (a_{ij}(\mathbf{0},x))_{n\times n},$$

and

$$\mathbf{A}^{(k)}(\mathbf{u},\ldots,\mathbf{u}) = \left(\sum_{k_1+k_2+\cdots+k_n=k} \left(\frac{\partial^k a_{ij}}{\partial u_1^{k_1}\ldots\partial u_n^{k_n}}\right)_{\mathbf{0}} u_1^{k_1}\ldots u_n^{k_n}\right)_{n\times n},$$

$$\mathbf{b}^{(k)}(\mathbf{u},\ldots,\mathbf{u}) = \left(\sum_{k_1+k_2+\cdots+k_n=k} \left(\frac{\partial^k b_i}{\partial u_1^{k_1}\ldots\partial u_n^{k_n}}\right)_{\mathbf{0}} u_1^{k_1}\ldots u_n^{k_n}\right)_{n\times 1},$$

$$\mathbf{a}^{(k)}(\mathbf{u},\ldots,\mathbf{u}) = \left(\sum_{k_1+k_2+\cdots+k_n=k}\left(\frac{\partial^k(\lambda\ell_i)}{\partial u_1^{k_1}\ldots\partial u_n^{k_n}}\right)_0 u_1^{k_1}\ldots u_n^{k_n}\right)_{1\times n},$$

$$c^{(k)}(\mathbf{u},\ldots,\mathbf{u}) = \sum_{k_1+k_2+\cdots+k_n=k}\left(\frac{\partial^k c}{\partial u_1^{k_1}\ldots\partial u_n^{k_n}}\right)_0 u_1^{k_1}\ldots u_n^{k_n},$$

are matrix-, vector-, and scalar-valued k-linear forms, respectively.

Now, substituting (2.2.21) into (2.2.25)-(2.2.30) and rewriting these as

$$\frac{1}{\lambda(\mathbf{u},x)} = \frac{1}{\lambda_0(x)} + \sum_{j=1}^{\infty}\frac{\varepsilon^j}{j!}\Lambda_j(x,\theta), \tag{2.2.31}$$

$$\mathbf{A}(\mathbf{u},x) = \sum_{j=0}^{\infty}\frac{\varepsilon^j}{j!}\mathbf{A}_j(x,\theta), \tag{2.2.32}$$

$$\mathbf{b}(\mathbf{u},x) = \sum_{j=1}^{\infty}\frac{\varepsilon^j}{j!}\mathbf{b}_j(x,\theta), \tag{2.2.33}$$

$$\mathbf{a}(\mathbf{u},x) = \sum_{j=0}^{\infty}\frac{\varepsilon^j}{j!}\mathbf{a}_j(x,\theta), \tag{2.2.34}$$

$$c(\mathbf{u},x) = \sum_{j=1}^{\infty}\frac{\varepsilon^j}{j!}c_j(x,\theta), \tag{2.2.35}$$

$$\mathbf{D}(\mathbf{u},x) = \sum_{j=0}^{\infty}\frac{\varepsilon^j}{j!}\mathbf{D}_j(x,\theta), \tag{2.2.36}$$

we have the following lemma:

LEMMA 2.1. For $j \geq 1$,

$$\Lambda_j = \sum_{k=1}^{j}\sum_{i_1+\cdots+i_k=j-k}\binom{j}{k\ i_1\ldots i_k}\Lambda^{(k)}\left(\mathbf{u}^{(i_1)},\ldots,\mathbf{u}^{(i_k)}\right), \tag{2.2.37}$$

$$\mathbf{A}_j = \sum_{k=1}^{j}\sum_{i_1+\cdots+i_k=j-k}\binom{j}{k\ i_1\ldots i_k}\mathbf{A}^{(k)}\left(\mathbf{u}^{(i_1)},\ldots,\mathbf{u}^{(i_k)}\right), \tag{2.2.38}$$

$$\mathbf{b}_j = \sum_{k=1}^{j} \sum_{i_1+\cdots+i_k=j-k} \binom{j}{k\ i_1 \ldots i_k} \mathbf{b}^{(k)}\left(\mathbf{u}^{(i_1)}, \ldots, \mathbf{u}^{(i_k)}\right), \tag{2.2.39}$$

$$\mathbf{a}_j = \sum_{k=1}^{j} \sum_{i_1+\cdots+i_k=j-k} \binom{j}{k\ i_1 \ldots i_k} \mathbf{a}^{(k)}\left(\mathbf{u}^{(i_1)}, \ldots, \mathbf{u}^{(i_k)}\right), \tag{2.2.40}$$

$$c_j = \sum_{k=1}^{j} \sum_{i_1+\cdots+i_k=j-k} \binom{j}{k\ i_1 \ldots i_k} c^{(k)}\left(\mathbf{u}^{(i_1)}, \ldots, \mathbf{u}^{(i_k)}\right), \tag{2.2.41}$$

$$\mathbf{D}_j = \sum_{k=1}^{j} \sum_{i_1+\cdots+i_k=j-k} \binom{j}{k\ i_1 \ldots i_k} \mathbf{D}^{(k)}\left(\mathbf{u}^{(i_1)}, \ldots, \mathbf{u}^{(i_k)}\right), \tag{2.2.42}$$

where

$$\binom{j}{k\ i_1, \ldots i_k} = \frac{j!}{k! i_1! \ldots i_k!},$$

subject to $k + i_1 + \cdots + i_k = j$.

Proof. We need only prove (2.2.37) as the proofs for (2.2.38)-(2.2.42) will be identical. Employing (2.2.21) in (2.2.25) we obtain

$$\begin{aligned}
\frac{1}{\lambda(\mathbf{u}, x)} &= \frac{1}{\lambda_0(x)} + \sum_{k=1}^{\infty} \frac{1}{k!} \Lambda^{(k)}(\mathbf{u}, \ldots, \mathbf{u}) \\
&= \frac{1}{\lambda_0(x)} + \sum_{k=1}^{\infty} \frac{1}{k!} \Lambda^{(k)}\left(\epsilon \sum_{i_1=0}^{\infty} \frac{\epsilon^{i_1}}{i_1!} \mathbf{u}^{(i_1)}, \ldots, \epsilon \sum_{i_k=0}^{\infty} \frac{\epsilon^{i_k}}{i_k!} \mathbf{u}^{(i_k)}\right) \\
&= \frac{1}{\lambda_0(x)} + \sum_{k=1}^{\infty} \sum_{i_1, \ldots, i_k=0}^{\infty} \frac{\epsilon^{k+i_1+\cdots+i_k}}{k! i_1! \ldots i_k!} \Lambda^{(k)}\left(\mathbf{u}^{(i_1)}, \ldots, \mathbf{u}^{(i_k)}\right) \\
&= \frac{1}{\lambda_0(x)} + \sum_{j=1}^{\infty} \frac{\epsilon^j}{j!} \sum_{\substack{k+i_1+\cdots+i_k=j \\ k \geq 1}} \binom{j}{k\ i_1 \ldots i_k} \Lambda^{(k)}\left(\mathbf{u}^{(i_1)}, \ldots, \mathbf{u}^{(i_k)}\right),
\end{aligned}$$

and hence that

$$\begin{aligned}
\Lambda_j &= \sum_{k+i_1+\cdots+i_k=j} \binom{j}{k\ i_1 \ldots i_k} \Lambda^{(k)} \left(\mathbf{u}^{(i_1)}, \ldots, \mathbf{u}^{(i_k)}\right) \\
&= \sum_{k=1}^{j} \sum_{i_1+\cdots+i_k=j-k} \binom{j}{k\ i_1 \ldots i_k} \Lambda^{(k)} \left(\mathbf{u}^{(i_1)}, \ldots, \mathbf{u}^{(i_k)}\right).
\end{aligned}$$

This completes the proof.

We now proceed to construct the asymptotic expansion of the solution to the signaling problem in the form given by (2.2.21). We first substitute (2.2.31) into the arrival time formula (2.2.16) and then insert the derived equation together with (2.2.21), (2.2.31)-(2.2.36) into the governing equations (2.2.23) and the transport equation (2.2.24) to form the $O(1)$, $O(\varepsilon)$, and $O(\varepsilon^k)$ problems.

Upon substituting (2.2.31) into (2.2.16) we obtain

$$\begin{aligned}
t &= \int_0^x \frac{ds}{\lambda_0(s)} + \delta(\varepsilon)\theta + \sum_{j=1}^{\infty} \frac{\varepsilon^j}{j!} \int_0^x \Lambda_j(s,\theta)ds \\
&\triangleq \sum_{j=0}^{\infty} \frac{\varepsilon^j}{j!} T^{(j)}(x,\theta).
\end{aligned} \tag{2.2.43}$$

To match the order of $\delta(\varepsilon)$ with ε, we require that

$$\delta(\varepsilon) = \varepsilon^m, \tag{2.2.44}$$

where $m \geq 1$ is an integer. The exponent m is chosen to be an integer in order to balance the two sides of the governing equations (2.2.23), and the stipulation that $m \geq 1$ is required so that the, as yet to be derived, $O(1)$ problem will be homogeneous. As was explained briefly above, this freedom to choose m will now be shown in detail as providing a particular advantage in the asymptotic analysis of signaling problems.

Equating powers of ϵ in (2.2.43) we find that

$$T^{(0)}(x,\theta) = \int_0^x \frac{ds}{\lambda_0(s)}, \tag{2.2.45a}$$

$$T^{(j)}(x,\theta) = \int_0^x \Lambda_j(s,\theta)ds, \; j \neq m, \tag{2.2.45b}$$

$$T^{(m)}(x,\theta) = m!\theta + \int_0^x \Lambda_m(s,\theta)ds. \tag{2.2.45c}$$

Then upon substituting (2.2.21), (2.2.32)-(2.2.36), and (2.2.43) into the governing equations (2.2.23) and the transport equation (2.2.24), and equating like powers of ϵ, we obtain, through somewhat tedious but relatively straightforward calculations, the following:

$O(1)$ problem:

$$\left(\mathbf{I} - \frac{\mathbf{A}_0}{\lambda_0}\right)\mathbf{u}_\theta^{(0)} = \mathbf{0}, \tag{2.2.46}$$

$$c^{(1)}(\mathbf{u}^{(0)}) - \mathbf{a}_0\mathbf{u}_x^{(0)} = 0. \tag{2.2.47}$$

$O(\epsilon)$ problem:

$$\left(\mathbf{I} - \frac{\mathbf{A}_0}{\lambda_0}\right)\mathbf{u}_\theta^{(1)} = \mathbf{M}_1(x,\theta), \tag{2.2.48}$$

$$c^{(1)}(\mathbf{u}^{(1)}) - \mathbf{a}_0\mathbf{u}_x^{(1)} = N_1(x,\theta), \tag{2.2.49}$$

where

$$\mathbf{M}_1(x,\theta) = \mathbf{D}_1\mathbf{u}_\theta^{(0)} + T_\theta^{(1)}(\mathbf{b}_1 - \mathbf{A}_0\mathbf{u}_x^{(0)}), \tag{2.2.50}$$

$$N_1(x,\theta) = \mathbf{a}_1\mathbf{u}_x^{(0)} - c^{(2)}(\mathbf{u}^{(0)},\mathbf{u}^{(0)}). \tag{2.2.51}$$

$O(\epsilon^k)$ problem:

$$\left(\mathbf{I}-\frac{\mathbf{A}_0}{\lambda_0}\right)\mathbf{u}_\theta^{(k)} = \mathbf{M}_k(x,\theta), \tag{2.2.52}$$

$$c^{(1)}(\mathbf{u}^{(k)}) - \mathbf{a}_0\mathbf{u}_x^{(k)} = N_k(x,\theta), \tag{2.2.53}$$

where

$$\mathbf{M}_k(x,\theta) = \sum_{\substack{k_1+k_2=k\\k_1>0}}\binom{k}{k_1 k_2}\mathbf{D}_{k_1}\mathbf{u}_\theta^{(k_2)} - \sum_{\substack{k_1+k_2+k_3=k\\k_1>0}}\binom{k}{k_1 k_2 k_3}T_\theta^{(k_1)}\mathbf{A}_{k_2}\mathbf{u}_x^{(k_3)} + \frac{1}{k+1}\sum_{\substack{k_1+k_2=k+1\\k_1,k_2>0}}\binom{k+1}{k_1 k_2}\mathbf{b}_{k_1}T_\theta^{(k_2)}, \tag{2.2.54}$$

$$N_k(x,\theta) = \sum_{\substack{k_1+k_2=k\\k_1>0}}\binom{k}{k_1 k_2}\mathbf{a}_{k_1}\mathbf{u}_x^{(k_2)} - \frac{1}{k+1}\sum_{j=2}^{k+1}\sum_{i_1+\cdots+i_j=k+1-j}\binom{k+1}{j\ i_1\ldots i_j}c^{(j)}(\mathbf{u}^{(i_1)},\ldots,\mathbf{u}^{(i_j)}). \tag{2.2.55}$$

2.3. Solutions to the $O(1)$, $O(\epsilon)$, and $O(\epsilon^k)(k < m)$ Problems

2.3.1 *The $O(1)$ problem*

First we solve the $O(1)$ problem. Equation (2.2.46) shows that $\mathbf{u}_\theta^{(0)}$ must be parallel to $\mathbf{r}_0(x)$, so that $\mathbf{u}^{(0)}$ can be written as

$$\mathbf{u}^{(0)}(x,\theta) = \sigma(x,\theta)\mathbf{r}_0(x), \tag{2.3.1}$$

where the condition

$$\mathbf{u}^{(k)}(x,\theta)|_{\theta=0} = \mathbf{0},\ k=0,1,\ldots, \tag{2.3.2}$$

arising from

$$\mathbf{u}(x,\theta;\varepsilon)\,|_{\theta=0} = 0, \tag{2.2.11}$$

has been noted and applied.

Denoting $\sigma(x,\theta)\,|_{x=0} = \sigma_0(\theta)$, the boundary condition (2.2.12) at $x = 0$ then leads to

$$\mathbf{g}^{(0)}(\theta) = \sigma_0(\theta)\mathbf{r}_0(0),$$

or

$$\mathbf{g}^{(0)}(\cdot) = \sigma_0(\cdot)\mathbf{r}_0(0). \tag{2.3.3}$$

Hence we come to the conclusion that only a single-wave-mode boundary perturbation such as that in (2.3.3) will admit the asymptotic expansion (2.2.21). If the leading term of the boundary perturbation is not of a single-wave-mode, (2.2.21) and (2.2.22) will fail and wave interactions may come into play. In other words, (2.2.21) and (2.2.22) enable us to distinguish a class of (nonresonant) nonlinear waves.

The function $\sigma(x,\theta)$ can be determined from the transport equation (2.2.47). Inserting $\mathbf{u}^{(0)}$ from (2.3.1) into (2.2.47) gives

$$\sigma_x - \Gamma_0(x)\sigma = 0, \tag{2.3.4}$$

wherein

$$\Gamma_0(x) = \frac{\mathbf{c}^{(1)}(\mathbf{r}_0) - \mathbf{a}_0\mathbf{r}_0'}{\mathbf{a}_0\mathbf{r}_0}. \tag{2.3.5}$$

Integrating in (2.3.4), we obtain

$$\sigma(x,\theta) = \sigma_0(\theta)\exp\left\{\int_0^x \Gamma_0(s)ds\right\}. \tag{2.3.6}$$

For the sake of clarity in the following discussions, we may "normalize" $\mathbf{r}_0$ by requiring that $\mathbf{r}_0$ satisfy the condition

$$\Gamma_0(x) \equiv 0, \tag{2.3.7}$$

so that

$$\sigma(x,\theta) = \sigma_0(\theta). \tag{2.3.8}$$

In fact, this procedure can be carried out simply by replacing $\mathbf{r}_0$ by

$$\tilde{\mathbf{r}}_0 = \mathbf{r}_0 \exp\left\{\int_0^x \Gamma_0(s)ds\right\}. \tag{2.3.9}$$

It is straightforward to check that

$$\tilde{\Gamma}_0(x) = \frac{c^{(1)}(\tilde{\mathbf{r}}_0) - \mathbf{a}_0\tilde{\mathbf{r}}_0'}{\mathbf{a}_0\tilde{\mathbf{r}}_0} \equiv 0. \tag{2.3.10}$$

We shall always assume that $\mathbf{r}_0$ has been "normalized" unless stated otherwise. Therefore, we have that

$$\mathbf{u}^{(0)}(x,\theta) = \sigma_0(\theta)\mathbf{r}_0(x). \tag{2.3.11}$$

2.3.2 *The $O(\epsilon)$ problem*

We consider the two cases, that is, $m > 1$ and $m = 1$.

Case 1. $m > 1$. Substituting (2.3.11) into (2.2.50) and (2.2.51) we obtain

$$\begin{aligned}\mathbf{M}_1(x,\theta) &= \mathbf{D}^{(1)}(\mathbf{u}^{(0)})\mathbf{u}_\theta^{(0)} + [\mathbf{b}^{(1)}(\mathbf{u}^{(0)}) - \mathbf{A}_0\mathbf{u}_x^{(0)}]\int_0^x \Lambda^{(1)}(\mathbf{u}_\theta^{(0)})ds \\ &= \left\{\mathbf{D}^{(1)}(\mathbf{r}_0)\mathbf{r}_0 + [\mathbf{b}^{(1)}(\mathbf{r}_0) - \mathbf{A}_0\mathbf{r}_0']\int_0^x \Lambda^{(1)}(\mathbf{r}_0)ds\right\}\sigma_0(\theta)\sigma_0'(\theta) \\ &\triangleq \mathbf{p}_1(x)\sigma_0(\theta)\sigma_0'(\theta),\end{aligned} \tag{2.3.12}$$

$$\begin{aligned} N_1(x,\theta) &= \mathbf{a}_1 \mathbf{u}_x^{(0)} - c^{(2)}(\mathbf{u}^{(0)}, \mathbf{u}^{(0)}) \\ &= \left\{ \mathbf{a}^{(1)}(\mathbf{r}_0)\mathbf{r}_0' - c^{(2)}(\mathbf{r}_0, \mathbf{r}_0) \right\} \sigma_0^2(\theta) \\ &\triangleq q_1(x)\sigma_0^2(\theta). \end{aligned} \tag{2.3.13}$$

Thus separation of variables holds and the $O(\epsilon)$ problem now assumes the form

$$\left(\mathbf{I} - \frac{\mathbf{A}_0}{\lambda_0}\right) \mathbf{u}_\theta^{(1)} = \mathbf{p}_1(x)\sigma_0(\theta)\sigma_0'(\theta), \tag{2.3.14}$$

$$c^{(1)}(\mathbf{u}^{(1)}) - \mathbf{a}_0 \mathbf{u}_x^{(1)} = q_1(x)\sigma_0^2(\theta). \tag{2.3.15}$$

We exclude the cases in which σ_0 or $\sigma_0' \equiv 0$ in order to avoid trivial solutions. Letting $\mathbf{r}_1(x)$ be a particular solution of

$$\left(\mathbf{I} - \frac{\mathbf{A}_0}{\lambda_0}\right) \mathbf{r}_1 = \mathbf{p}_1(x), \tag{2.3.16}$$

which is identically zero when $\mathbf{p}_1(x)$ vanishes, we may write

$$\mathbf{u}_\theta^{(1)} = \sigma_0(\theta)\sigma_0'(\theta) \left\{ \sigma_1(x,\theta)\mathbf{r}_0(x) + \mathbf{r}_1(x) \right\}, \tag{2.3.17}$$

where $\sigma_1(x,\theta)$ is to be determined from the transport equation (2.3.15). Differentiating in (2.3.15) with respect to θ and applying (2.3.17) gives

$$(\sigma_1)_x - \Gamma_0(x)\sigma_1 = K_1(x), \tag{2.3.18}$$

wherein

$$K_1(x) = \frac{c^{(1)}(\mathbf{r}_1) - \mathbf{a}_0\mathbf{r}_1' - 2q_1}{\mathbf{a}_0\mathbf{r}_0}. \tag{2.3.19}$$

Since $\mathbf{r}_0$ is "normalized", $\Gamma_0 \equiv 0$ and hence

$$(\sigma_1)_x = K_1(x),$$

or,

$$\sigma_1(x,\theta) = \sigma_1(0,\theta) + \int_0^x K_1(s)ds$$
$$\triangleq \sigma_1(\theta) + \int_0^x K_1(s)ds. \tag{2.3.20}$$

We now apply (2.1.11) in order to determine $\sigma_1(\theta)$. From (2.1.11) it follows that

$$u_1^{(k)}|_{x=0} = 0, \ k = 1, 2, \ldots,$$

and hence

$$u_{1,\theta}^{(k)}|_{x=0} = 0, \ k = 1, 2, \ldots. \tag{2.3.21}$$

Applying the case $k = 1$ in (2.3.17), (2.3.20) gives

$$\sigma_1(\theta) r_{0,1}(0) + r_{1,1}(0) \equiv 0,$$

or, since $r_{0,1}(0) \neq 0$,

$$\sigma_1(\theta) \equiv -\frac{r_{1,1}(0)}{r_{0,1}(0)}, \tag{2.3.22}$$

that is, $\sigma_1(\theta)$ is a constant, where $r_{0,1}(x)$ and $r_{1,1}(x)$ are the first components of $\mathbf{r}_0(x)$ and $\mathbf{r}_1(x)$, respectively.

Now, (2.3.17) becomes

$$\mathbf{u}_\theta^{(1)} = \sigma_0(\theta)\sigma_0'(\theta)\left\{\mathbf{r}_0(x)\left[-\frac{r_{1,1}(0)}{r_{0,1}(0)} + \int_0^x K_1(s)ds\right] + \mathbf{r}_1(x)\right\}. \tag{2.3.23}$$

We may "normalize" $\mathbf{r}_1(x)$ in the sense that

$$K_1(x) \equiv 0, \ r_{1,1}(0) = 0. \tag{2.3.24}$$

This procedure may be carried out by simply replacing $\mathbf{r}_1(x)$ with

$$\hat{\mathbf{r}}_1(x) = \mathbf{r}_0(x)\left[-\frac{r_{1,1}(0)}{r_{0,1}(0)} + \int_0^x K_1(s)ds\right] + \mathbf{r}_1(x). \tag{2.3.25}$$

We always assume that $\mathbf{r}_1(x)$ is "normalized" unless stated otherwise. Thus

$$\mathbf{u}_\theta^{(1)} = \sigma_0(\theta)\sigma_0'(\theta)\mathbf{r}_1(x). \tag{2.3.26}$$

Integrating in (2.3.26) gives

$$\mathbf{u}^{(1)} = \frac{1}{2}\sigma_0^2(\theta)\mathbf{r}_1(x), \tag{2.3.27}$$

where again (2.3.2) has been noted.

Remark 2.1. As we shall see, in order to proceed to the resolution of higher-order problems we need $\mathbf{u}_\theta^{(1)}$ to take the form (2.3.26), which in turn requires only that $\sigma_1(\theta) \equiv$ constant. While the condition (2.1.11) guarantees that $\sigma_1(\theta)$ is a constant, other similar conditions, such as the j-th component of $\mathbf{u}^{(k)}$ vanishing at $x = 0$ for all $k \geq 1$, are also available and especially in the instance when $r_{0,1}(0) = 0$.

Case 2. $m = 1$. Now, (2.2.45c) gives

$$T_\theta^{(1)} = 1 + \sigma'(\theta)\int_0^x \Lambda^{(1)}(\mathbf{r}_0(s))ds,$$

and $\mathbf{M}_1(x,\theta)$ no longer has a form in which the variables are separated. Although $\mathbf{u}^{(1)}(x,\theta)$ may be solved for by a similar procedure to that employed in Case 1, the further successive solutions of the $O(\varepsilon^k)$ problems will be too complicated to resolve.

However, we now have

$$\mathbf{u}(x,\theta;\varepsilon) = \varepsilon\sigma_0(\theta)\mathbf{r}_0(x) + O(\varepsilon^2), \tag{2.3.28}$$

and

$$t = T(x,\theta;\epsilon) = \int_0^x \frac{ds}{\lambda_0(s)} + \epsilon\left\{\theta + \sigma_0(\theta)\int_0^x \Lambda^{(1)}(\mathbf{r}_0(s))ds\right\} + O(\epsilon^2), \quad (2.3.29)$$

which constitutes a full asymptotic solution to the signalling problem (2.1.2), (2.1.7), and (2.1.8), subject to a single-wave-mode boundary disturbance of the form

$$\mathbf{u}\big|_{x=0} = \epsilon\sigma_0(t/\epsilon)\mathbf{r}_0(0) + O(\epsilon^2)\,. \quad (2.3.30)$$

One can observe that the asymptotic solution (2.3.28) retains the same expression as in WNGO [6,25], with the difference that θ represents a nonlinear phase rather than a linear one. Also the arrival time formula (2.3.29) indicates the exact characteristic family under consideration, thus we can examine the focusing of the characteristic family, which predicts wave breaking and hence possible shock generation. We do this by examining the caustic Γ of the characteristic family (2.3.29); namely, we consider

$$\Gamma:\ t = T(x,\theta;\epsilon),\ T_\theta = 0,$$

or, approximately

$$\Gamma:\begin{cases} x = x(\theta):\ 1 + \sigma_0'(\theta)\displaystyle\int_0^x \Lambda^{(1)}(\mathbf{r}_0(s))ds = 0, & (2.3.31\text{a})\\ t = t(\theta) = \displaystyle\int_0^{x(\theta)} \frac{ds}{\lambda_0(s)} + \epsilon\left\{\theta + \sigma_0(\theta)\int_0^x \Lambda^{(1)}(\mathbf{r}_0(s))ds\right\}. & (2.3.31\text{b})\end{cases}$$

We state the result as:

Shock initiation condition ($m = 1$). If there exists (x,θ), $x > 0$, $\theta > 0$, such that (2.3.31a) holds, then Γ is nonempty and a smooth solution (2.3.28) will break at (x_s, t_s) behind the leading wavefront

$$t = \int_0^x \frac{ds}{\lambda_0(s)},$$

where $(x_s, t_s) = (x(\theta_s), t(\theta_s)) \in \Gamma$ and

$$t_s = \min_{\theta > 0}\{t(\theta)\}. \tag{2.3.32}$$

Two immediate observations from this shock initiation are:

(i) If $\Lambda^{(1)}(\mathbf{r}_0) \not\equiv 0$, a function $\sigma_0(\cdot)$ is always constructable such that the smooth solution will break behind the leading wavefont.

(ii) If $\Lambda^{(1)}(\mathbf{r}_0) \equiv 0$, we have $(\text{grad}_{\mathbf{u}}\lambda)\mathbf{r} \equiv 0$ about $\mathbf{u} \equiv 0$, that is, the characteristic field $\lambda(\mathbf{u}, x)$ is locally linearly degenerate about the steady state. In this case the asymptotic expansion method formulated above under the small amplitude, high-frequency relation

$$m = 1, \ \delta = \epsilon,$$

fails to capture the essential nonlinear behaviour of characteristic focusing according to (2.3.29) or (2.3.31). As we shall see, this aspect of degeneracy will motivate our introduction of the concept of order of local linear degeneracy and our employing this concept to determine the relation of small amplitude to high-frequency required in the signalling data in order to capture the nonlinear behaviour of the characteristic field.

2.3.3 *The $O(\epsilon^k)$ problem ($k < m$)*

By the method of induction, we assume that

$$\mathbf{u}^{(j)}(x, \theta) = \frac{1}{j+1}\sigma_0^{j+1}(\theta)\mathbf{r}_j(x), \ j = 0, 1, \ldots, k-1, \tag{2.3.33}$$

and proceed to show that $\mathbf{u}^{(k)}$ has the same form.

Now, considering $\mathbf{M}_k(x, \theta)$ and $N_k(x, \theta)$, we first show that they may be expressed with variables separated. Since

$$\begin{aligned}
\mathbf{D}_j &= \sum_{s=1}^{j} \sum_{i_1+\cdots+i_s=j-s} \binom{j}{s\; i_1 \ldots i_s} \mathbf{D}^{(s)}\left(\mathbf{u}^{(i_1)}, \ldots, \mathbf{u}^{(i_s)}\right) \\
&= \sum_{s=1}^{j} \sum_{i_1+\cdots+i_s=j-s} \sigma_0^j(\theta) \binom{j}{s\; i_1 \ldots i_s} \mathbf{D}^{(s)}\left(\frac{1}{i_1+1}\mathbf{r}_{i_1}, \ldots, \frac{1}{i_s+1}\mathbf{r}_{i_s}\right) \qquad (2.3.34) \\
&= \sigma_0^j(\theta)[\mathbf{D}_j]_{\sigma_0=1} \\
&= \sigma_0^j(\theta)\langle \mathbf{D}_j \rangle,
\end{aligned}$$

with the same argument yielding

$$\begin{aligned}
\mathbf{A}_j &= \sigma_0^j(\theta)\langle \mathbf{A}_j \rangle, \\
\mathbf{b}_j &= \sigma_0^j(\theta)\langle \mathbf{b}_j \rangle, \\
\mathbf{a}_j &= \sigma_0^j(\theta)\langle \mathbf{a}_j \rangle, \\
T^{(j)} &= \sigma_0^j(\theta)\langle T^{(j)} \rangle, \; (j < m), \\
\wedge_j &= \sigma_0^j(\theta)\langle \wedge_j \rangle,
\end{aligned}$$

we have that

$$\begin{aligned}
\mathbf{M}_k(x,\theta) = \Bigg\{ &\sum_{\substack{k_1+k_2=k \\ k_1>0}} \binom{k}{k_1\; k_2} \langle \mathbf{D}_{k_1} \rangle \mathbf{r}_{k_2} \\
&- \sum_{\substack{k_1+k_2+k_3=k \\ k_1>0}} \binom{k}{k_1\; k_2\; k_3} k_1 \langle T^{(k_1)} \rangle \langle \mathbf{A}_{k_2} \rangle \frac{1}{k_3+1} \mathbf{r}'_{k_3} \qquad (2.3.35) \\
&+ \frac{1}{k+1} \sum_{\substack{k_1+k_2=k+1 \\ k_1,k_2>0}} \binom{k+1}{k_1\; k_2} \langle \mathbf{b}_{k_1} \rangle k_2 \langle T^{(k_2)} \rangle \Bigg\} \sigma_0^k \sigma_0' \\
\triangleq\; &\mathbf{p}_k(x) \sigma_0^k(\theta) \sigma_0'(\theta),
\end{aligned}$$

and

$$
\begin{aligned}
N_k(x,\theta) = \Bigg\{ & \sum_{k_1+k_2=k} \binom{k}{k_1\ k_2} \langle \mathbf{a}_{k_1} \rangle \frac{1}{k_2+1} \mathbf{r}'_{k_2} \\
& - \frac{1}{k+1} \sum_{j=2}^{k+1} \sum_{i_1+\cdots+i_j=k+1-j} \binom{k+1}{j\ i_1 \ldots i_j} c^{(j)} \\
& \left(\frac{1}{i_1+1} \mathbf{r}_{i_1}, \ldots, \frac{1}{i_j+i} \mathbf{r}_{i_j} \right) \Bigg\} \sigma_0^{k+1}(\theta) \\
\triangleq\ & q_k(x) \sigma_0^{k+1}(\theta),
\end{aligned}
\tag{2.3.36}
$$

so that the $O(\epsilon^k)$ problem is simplified to

$$
\left(\mathbf{I} - \frac{\mathbf{A}_0}{\lambda_0} \right) \mathbf{u}_\theta^{(k)} = \mathbf{p}_k(x) \sigma_0^k(\theta) \sigma_0'(\theta), \tag{2.3.37}
$$

$$
c^{(1)}(\mathbf{u}^{(k)}) - \mathbf{a}_0 \mathbf{u}_x^{(k)} = q_k(x) \sigma_0^{k+1}(\theta). \tag{2.3.38}
$$

In the above, the notation

$$
\langle \cdot \rangle \triangleq [\cdot]_{\sigma_0 \equiv 1},
$$

is employed.

We can now solve (2.3.37) and (2.3.38) as before. That is, we let $\mathbf{r}_k(x)$ be a particular solution of

$$
\left(\mathbf{I} - \frac{\mathbf{A}_0}{\lambda_0} \right) \mathbf{r}_k = \mathbf{p}_k, \tag{2.3.39}
$$

so that $\mathbf{r}_k(x)$ is identically zero when $\mathbf{p}_k$ vanishes. Hence we then have

$$
\mathbf{u}_\theta^{(k)} = \sigma_0^k(\theta) \sigma_0'(\theta) \left\{ \sigma_k(x,\theta) \mathbf{r}_0(x) + \mathbf{r}_k(x) \right\}. \tag{2.3.40}
$$

If we now differentiate the transport equation (2.3.38) with respect to θ and apply (2.3.40) we obtain

$$
(\sigma_k)_x - \Gamma_0(x) \sigma_k = K_k(x), \tag{2.3.41}
$$

where

$$K_k(x) = \frac{c^{(1)}(\mathbf{r}_k) - \mathbf{a}_0\mathbf{r}_k' - (k+1)q_k}{\mathbf{a}_0\mathbf{r}_0}. \tag{2.3.42}$$

Again we note that $\Gamma_0(x) \equiv 0$, so that

$$(\sigma_k)_x = K_k(x),$$

or, upon integrating,

$$\begin{aligned} \sigma_k(x,\theta) &= \sigma_k(0,\theta) + \int_0^x K_k(s)ds \\ &\triangleq \sigma_k(\theta) + \int_0^x K_k(s)ds. \end{aligned} \tag{2.3.43}$$

Now apply (2.3.21) to the first component of (2.3.40) [using (2.3.43)] in order to determine $\sigma_k(\theta)$. We find that

$$\sigma_k(\theta)r_{0,1}(0) + r_{k,1}(0) = 0,$$

or,

$$\sigma_k(\theta) = -\frac{r_{k,1}(0)}{r_{0,1}(0)}, \tag{2.3.44}$$

where $r_{k,1}(x)$ is the first component of $\mathbf{r}_k(x)$. Hence (2.3.40) becomes

$$\mathbf{u}_\theta^{(k)} = \sigma_0^k(\theta)\sigma_0'(\theta)\left\{\mathbf{r}_0(x)\left[-\frac{r_{k,1}(0)}{r_{0,1}(0)} + \int_0^x K_k(s)ds\right] + \mathbf{r}_k(x)\right\}. \tag{2.3.45}$$

Also, we "normalize" $\mathbf{r}_k(x)$ in the sense that

$$K_k(x) \equiv 0, \ r_{k,1}(0) = 0. \tag{2.3.46}$$

This is accomplished by replacing $\mathbf{r}_k(x)$ by

$$\hat{\mathbf{r}}_k(x) = \mathbf{r}_k(x) + \mathbf{r}_0(x)\left[-\frac{r_{k,1}(0)}{r_{0,1}(0)} + \int_0^x K_k(s)ds\right]. \tag{2.3.47}$$

As before, we assume that $\mathbf{r}_k(x)$ is "normalised" unless stated otherwise.

Now (2.3.40) becomes

$$\mathbf{u}_\theta^{(k)}(x,\theta) = \sigma_0^k(\theta)\sigma_0'(\theta)\mathbf{r}_k(x), \tag{2.3.48}$$

which, upon integrating, gives

$$\mathbf{u}^{(k)}(x,\theta) = \frac{1}{k+1}\sigma_0^{k+1}(\theta)\mathbf{r}_k(x), \tag{2.3.49}$$

and hence the proof by induction of the form taken by $\mathbf{u}^{(k)}$ is complete.

2.3.4 *Discussion*

By means of the method of induction, we have ascertained that the asymptotic expansion for the solution to the signalling problem takes the form

$$\mathbf{u}(x,\theta;\epsilon) = \sum_{k=1}^{m} \frac{\epsilon^k}{k!}\sigma_0^k(\theta)\mathbf{r}_{k-1}(x) + O(\epsilon^{m+1}),\ \theta \geq 0,\ x \geq 0, \tag{2.3.50}$$

with the arrival time determined from

$$\begin{aligned} t = \int_0^x \frac{ds}{\lambda_0(s)} &+ \sum_{k=1}^{m-1} \frac{\epsilon^k}{k!}\sigma_0^k(\theta)\int_0^x \langle \Lambda_k \rangle ds \\ &+ \epsilon^m\left\{\theta + \frac{1}{m!}\sigma_0^m(\theta)\int_0^x \langle \Lambda_m \rangle ds\right\} + O(\epsilon^{m+1}). \end{aligned} \tag{2.3.51}$$

The $\mathbf{r}_k(x)$ $(k = 1,2,\ldots,m-1)$ in (2.3.50) are all particular solutions of the algebraic systems

$$\left(\mathbf{I} - \frac{\mathbf{A}_0}{\lambda_0}\right)\mathbf{r}_k = \mathbf{p}_k,\ k = 1,2,\ldots,m-1, \tag{2.3.30}$$

and

$$\Lambda_k(x,\theta) = \sigma_0^k(\theta)\langle\Lambda_k\rangle,\ k = 1,2,\ldots,m, \tag{2.3.52}$$

$$\langle\Lambda_k\rangle = \sum_{j=1}^{k} \sum_{i_1+\cdots+i_j=k-j} \binom{k}{j\ i_1 \ldots i_j} \Lambda^{(j)}\left(\frac{1}{i_1+1}\mathbf{r}_{i_1},\ldots,\frac{1}{i_j+1}\mathbf{r}_{i_j}\right), \quad k = 1,2,\ldots,m, \tag{2.3.53}$$

are noted.

Here we note that $\mathbf{r}_k$ and $\langle\Lambda_k\rangle$ $(k = 1,2,\ldots,m)$ are independent of $\sigma_0(\theta)$ and m. In fact, $\{\mathbf{r}_k\}$, $\{\langle\Lambda_k\rangle\}$, and $\{\mathbf{p}_k\}$ constitute three sequences defined algebraically as functions of x, depending only on the local behaviour of the quasilinear hyperbolic system about its steady state solution.

As we have seen,

$$\langle\Lambda_1\rangle = \Lambda^{(1)}(\mathbf{r}_0) = \left(\text{grad}_{\mathbf{u}}\left(\frac{1}{\lambda}\right)\right)\mathbf{r}\,|_{\mathbf{u}=\mathbf{0}}\,, \tag{2.3.54}$$

so that

$$\langle\Lambda_1\rangle \equiv 0 \Longleftrightarrow (\text{grad}_{\mathbf{u}}\lambda)\mathbf{r}\,|_{\mathbf{u}=\mathbf{0}} \equiv 0, \tag{2.3.55}$$

which defines the local linear degeneracy of $\lambda(\mathbf{u},x)$ at $\mathbf{u} = \mathbf{0}$. In general, if q is the lowest index such that $\langle\Lambda_k\rangle$ does not vanish identically, namely

$$\langle\Lambda_k\rangle \equiv 0,\ k = 1,2,\ldots,q-1, \tag{2.3.56a}$$

$$\langle\Lambda_q\rangle \not\equiv 0, \tag{2.3.56b}$$

Here we follow a recently introduced definition by Rosales [60] and refer to the associated characteristic field $\lambda(\mathbf{u},x)$ as having a *local linear degeneracy of order* $q-1$ about $\mathbf{u} = \mathbf{0}$. Genuine nonlinearity in the sense of Lax [30] thus refers to a zeroth order degeneracy.

If we let $q-1$ be the order of local linear degeneracy for the characteristic field $\lambda(\mathbf{u},x)$ about the steady state $\mathbf{u}=\mathbf{0}$ and if we choose

$$m=q,\ \delta=\varepsilon^{q}, \tag{2.3.57}$$

then the arrival time formula becomes

$$t=\int_0^x \frac{ds}{\lambda_0(s)}+\varepsilon^q\left\{\theta+\frac{1}{q!}\sigma_0^q(\theta)\int_0^x \langle\Lambda_q\rangle ds\right\}+O(\varepsilon^{q+1}). \tag{2.3.58}$$

This leads us to the following shock initiation condition which includes the case $m=1$ as a particular case.

Let Γ again denote the caustic of the characteristic family (2.3.58), that is,

$$\Gamma:\ t=T(x,\theta;\varepsilon),\ T_\theta=0, \tag{2.3.59}$$

exactly, or

$$\Gamma:\begin{cases} x=x(\theta):\ 1+\dfrac{1}{q!}(\sigma_0^q(\theta))'\displaystyle\int_0^x\langle\Lambda_q\rangle ds=0, & (2.3.60a)\\ t=t(\theta)=\displaystyle\int_0^{x(\theta)}\frac{ds}{\lambda_0(s)}+\varepsilon^q\left\{\theta+\frac{1}{q!}\sigma_0^q(\theta)\int_0^{x(\theta)}\langle\Lambda_q\rangle ds\right\}, & (2.3.60b)\end{cases}$$

approximately when dropping the $O(\varepsilon^{q+1})$ term in (2.3.58). We therefore have the

THEOREM 2.1. (Shock initiation condition). Let the characteristic field $\lambda(\mathbf{u},x)$ have an order of local linear degeneracy about the steady state $\mathbf{u}\equiv\mathbf{0}$ equal to $q-1$, that is,

$$\langle\Lambda_k\rangle\begin{cases}\equiv 0,\ k=1,2,\dots,q-1,\\ \not\equiv 0,\ k=q.\end{cases} \tag{2.3.56}$$

Then, when $m=q$ is chosen, the corresponding smooth solution for the signaling problem, given by (2.3.50), will break behind the leading wavefront

$$t = \int_0^x \frac{ds}{\lambda_0(s)},$$

if there exists (x, θ), $x > 0$, $\theta > 0$, such that (2.3.60a) holds and Γ is nonempty. The smooth solution (2.3.50) breaks at (x_s, t_s), where

$$\begin{aligned} (x_s, t_s) &= (x(\theta_s), t(\theta_s)) \in \Gamma, \\ t_s &= t(\theta_s) = \min_{\theta > 0} \{t(\theta)\}\,. \end{aligned} \tag{2.3.61}$$

2.4. A Solvability Condition

It has, no doubt, become clear to the reader at this point that throughout our derivations carried out in the last section, we have either implicitly or explicitly employed the fact that for each $O(\epsilon^k)$ problem the associated algebraic system

$$\left(\mathbf{I} - \frac{\mathbf{A}_0}{\lambda_0}\right) \mathbf{r}_k = \mathbf{p}_k, \; k = 1, 2, \ldots, m-1, \tag{2.3.39}$$

is solvable. Namely, we have assumed recursively that

$$\boldsymbol{\ell}_0 \mathbf{p}_k = 0, \; k = 1, 2, \ldots, \tag{2.4.1}$$

or,

$$\boldsymbol{\ell}_0 \mathbf{M}_k = 0, \; k = 1, 2, \ldots, \tag{2.4.2}$$

as $\mathbf{p}_k$, $\mathbf{r}_k$ are independent of m (assume formally $m = +\infty$).

We shall now verify these solvability conditions as they are required to make legitimate our analysis of the previous section.

First we prove

LEMMA 2.2.

$$\ell_0 \mathbf{p}_1 = 0. \tag{2.4.3}$$

Proof. From (2.3.12) we have

$$\begin{aligned} \mathbf{p}_1 &= \mathbf{D}^{(1)}(\mathbf{r}_0)\mathbf{r}_0 + [\mathbf{b}^{(1)}(\mathbf{r}_0) - \mathbf{A}_0\mathbf{r}_0'] \int_0^s \langle \Lambda_1 \rangle ds \\ &\triangleq \mathbf{p}_1^{(1)} + \mathbf{p}_1^{(2)}, \end{aligned} \tag{2.4.4}$$

so that

$$\begin{aligned} \ell_o \mathbf{p}_1^{(2)} &= \ell_0 [\mathbf{b}^{(1)}(\mathbf{r}_0) - \mathbf{A}_0\mathbf{r}_0'] \int_0^s \langle \Lambda_1 \rangle ds \\ &= [c^{(1)}(\mathbf{r}_0) - \mathbf{a}_0\mathbf{r}_0'] \int_0^s \langle \Lambda_1 \rangle ds \\ &= 0, \end{aligned} \tag{2.4.5}$$

upon noting (2.3.5) and (2.3.7).

We need now only show that $\ell_o \mathbf{p}_1^{(1)} = 0$, or, equivalently,

$$\ell_0 \mathbf{D}^{(1)}(\mathbf{r}_0)\mathbf{r}_0 = 0. \tag{2.4.6}$$

We proceed by employing the identity $\mathbf{A}\mathbf{r} = \lambda\mathbf{r}$, or, equivalently,

$$\mathbf{D}\mathbf{r} = \mathbf{r}. \tag{2.4.7}$$

Expanding $\mathbf{D}$ and $\mathbf{r}$ about $\mathbf{u} = \mathbf{0}$ gives

$$\mathbf{D} = \mathbf{D}_0 + \mathbf{D}^{(1)}\mathbf{u} + O(\|\mathbf{u}\|^2), \tag{2.4.8}$$

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{r}^{(1)}(\mathbf{u}) + O(\|\mathbf{u}\|^2), \tag{2.4.9}$$

which, upon substitution into (2.4.7), we have for the left and right sides of that equation

$$\begin{aligned}\mathbf{Dr} &= \left[\mathbf{D}_0 + \mathbf{D}^{(1)}(\mathbf{u}) + O(\|\mathbf{u}\|^2)\right]\left[\mathbf{r}_0 + \mathbf{r}^{(1)}(\mathbf{u}) + O(\|\mathbf{u}\|^2)\right] \\ &= \mathbf{D}_0\mathbf{r}_0 + \mathbf{D}_0\mathbf{r}^{(1)}(\mathbf{u}) + \mathbf{D}^{(1)}(\mathbf{u})\mathbf{r}_0 + O(\|\mathbf{u}\|^2) \\ &= \mathbf{r}_0 + \mathbf{D}_0\mathbf{r}^{(1)}(\mathbf{u}) + \mathbf{D}^{(1)}(\mathbf{u})\mathbf{r}_0 + O(\|\mathbf{u}\|^2), \end{aligned} \tag{2.4.10}$$

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{r}^{(1)}(\mathbf{u}) + O(\|\mathbf{u}\|^2). \tag{2.4.9}$$

Equating these terms we have

$$\mathbf{D}_0\mathbf{r}^{(1)}(\mathbf{u}) + \mathbf{D}^{(1)}(\mathbf{u})\mathbf{r}_0 = \mathbf{r}^{(1)}(\mathbf{u}). \tag{2.4.11}$$

Applying $\boldsymbol{\ell}_0$ to both sides gives

$$\boldsymbol{\ell}_0\mathbf{D}^{(1)}(\mathbf{u})\mathbf{r}_0 = 0 \tag{2.4.12}$$

which, in particular, provides

$$\boldsymbol{\ell}_0\mathbf{D}^{(1)}(\mathbf{r}_0)\mathbf{r}_0 = 0, \tag{2.4.6}$$

when $\mathbf{u}$ is taken to be $\mathbf{r}_0$.

This completes the proof.

As for the most general case, we do not intend to prove (2.4.1) owing to the severe complexity of the calculations involved and so will leave this as an open question. Instead, we shall focus on the situation in which we are most interested. That is, we prove the following solvability condition.

THEOREM 2.2. (Solvability Condition). For any $k \geq 1$, when $\langle\Lambda_1\rangle = \langle\Lambda_2\rangle = \cdots = \langle\Lambda_k\rangle \equiv 0$, the associated algebraic system

$$\left(\mathbf{I} - \frac{\mathbf{A}_0}{\lambda_0}\right)\mathbf{r}_k = \mathbf{p}_k, \tag{2.3.39}$$

2

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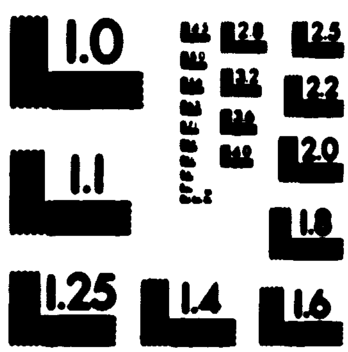

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is solvable. In addition, we also have that

$$\ell_0 \mathbf{M}_{k+1} = 0, \tag{2.4.13}$$

regardless of whether Λ_{k+1} vanishes or not.

Proof. It is obvious that $\langle \Lambda_1 \rangle = \Lambda^{(1)}(\mathbf{r}_0)$ is always defined and that, as a result of Lemma 2.2,

$$\left(\mathbf{I} - \frac{\mathbf{A}_0}{\lambda_0}\right)\mathbf{r}_1 = \mathbf{p}_1, \tag{2.3.16}$$

is solvable. We now use the induction assumption, that is, we assume that

$$\left(\mathbf{I} - \frac{\mathbf{A}_0}{\lambda_0}\right)\mathbf{r}_j = \mathbf{p}_j, \; j = 1, 2, \ldots, s-1, \tag{2.4.14}$$

are all solvable and $\langle \Lambda_{j+1} \rangle$, $j = 0, \ldots, s-1$, are all defined in a recursive manner. In addition, we have that

$$\langle \Lambda_1 \rangle = \langle \Lambda_2 \rangle = \cdots = \langle \Lambda_s \rangle \equiv 0. \tag{2.4.15}$$

In order to complete the induction proof we need show that, when $s \leq k$,

$$\left(\mathbf{I} - \frac{\mathbf{A}_0}{\lambda_0}\right)\mathbf{r}_s = \mathbf{p}_s, \tag{2.4.16}$$

is solvable, that is,

$$\ell_0 \mathbf{p}_s = 0,$$

or,

$$\ell_0 \mathbf{M}_s = 0. \tag{2.4.17}$$

Now, from (2.2.54) we have that

$$\mathbf{M}_s = \sum_{\substack{s_1+s_2=s \\ s_1>0}} \binom{s}{s_1\ s_2} \mathbf{D}_{s_1} \mathbf{u}_\theta^{(s_2)}, \tag{2.4.18}$$

since (2.4.15) gives

$$T^{(1)} = \cdots = T^{(s)} = 0. \tag{2.4.19}$$

In addition, we have also that $T_\theta = O(\varepsilon^{s+1})$ and hence from the governing equation (2.2.23) we find

$$(\mathbf{I} - \mathbf{D})\mathbf{u}_\theta = T_\theta(\mathbf{b} - \mathbf{A}\mathbf{u}_x) = O(\varepsilon^{s+2}). \tag{2.4.20}$$

Comparing (2.4.20) with the identity

$$(\mathbf{I} - \mathbf{D})\mathbf{r} = \mathbf{0}, \tag{2.4.21}$$

and noting that $\mathbf{u}_\theta = O(\varepsilon)$ while $\mathbf{r} = O(1)$, we have

$$\mathbf{u}_\theta = \varepsilon w \mathbf{r} + O(\varepsilon^{s+2}), \tag{2.4.22}$$

where w is a scalar function.

Expanding both sides in (2.4.22) we obtain

$$\begin{aligned} \varepsilon \sum_{j=0}^{s} \frac{\varepsilon^j}{j!} \mathbf{u}_\theta^{(j)} &= \varepsilon \sum_{j_1=0}^{s} \frac{\varepsilon^{j_1}}{j_1!} w_{j_1} \sum_{j_2=0}^{s} \frac{\varepsilon^{j_2}}{j_2!} \mathbf{r}_{,j_2} + O(\varepsilon^{s+2}) \\ &= \varepsilon \sum_{j=0}^{s} \frac{\varepsilon^j}{j!} \sum_{j_1+j_2=j} \binom{j}{j_1\ j_2} w_{j_1} \mathbf{r}_{,j_2} + O(\varepsilon^{s+2}), \end{aligned} \tag{2.4.23}$$

and hence that

$$\mathbf{u}_\theta^{(j)} = \sum_{j_1+j_2=j} \binom{j}{j_1\ j_2} w_{j_1} \mathbf{r}_{,j_2}, \ j = 0, 1, \ldots, s-1. \tag{2.4.24}$$

Now, expanding the identity

$$(\mathbf{I}-\mathbf{D})\mathbf{r}=\mathbf{0}, \tag{2.4.24}$$

about $\mathbf{u}=\mathbf{0}$, employing (2.2.30), (2.2.36), and (2.3.33) (for $j=0,1,\ldots,s-1$) in the result, and then equating like powers of ε, we obtain

$$\sum_{j_1+j_2=j}\binom{j}{j_1\ j_2}\mathbf{D}_{,j_1}\mathbf{r}_{,j_2}=\mathbf{r}_{,j}\,,\ j=0,1,\ldots,s-1,s. \tag{2.4.25}$$

In the above we have denoted the expansion for $\mathbf{r}$ formally as

$$\mathbf{r}=\sum_{j=0}^{\infty}\frac{\varepsilon^j}{j!}\mathbf{r}_{,j}\,. \tag{2.4.26}$$

Now, rewriting (2.4.25) by transposing terms to get

$$\left(\mathbf{I}-\frac{\mathbf{A}_0}{\lambda_0}\right)\mathbf{r}_{,j}=\sum_{\substack{j_1+j_2=j\\ j_1>0}}\binom{j}{j_1\ j_2}\mathbf{D}_{,j_1}\mathbf{r}_{,j_2}\,,j=0,1,\ldots,s, \tag{2.4.27}$$

and applying $\boldsymbol{\ell}_0$ to both sides of (2.4.27), we have

$$\boldsymbol{\ell}_0\sum_{\substack{j_1+j_2=j\\ j_1>0}}\binom{j}{j_1\ j_2}\mathbf{D}_{,j_1}\mathbf{r}_{,j_2}=0,\ j=0,1,\ldots,s. \tag{2.4.28}$$

Now by means of (2.4.18), (2.4.24), and (2.4.28) we obtain

$$
\begin{aligned}
\ell_0 \mathbf{M}_s &= \ell_0 \sum_{\substack{s_1+s_2=s \\ s_1>0}} \binom{s}{s_1\ s_2} \mathbf{D}_{s_1} \mathbf{u}_\theta^{(s_2)} \\
&= \ell_0 \sum_{\substack{s_1+s_2=s \\ s_1>0}} \binom{s}{s_1\ s_2} \mathbf{D}_{s_1} \sum_{j_1+j_2=s_2} \binom{s_2}{j_1\ j_2} w_{j_1} \mathbf{r}_{,j_2} \\
&= \ell_0 \sum_{\substack{s_1+j_1+j_2=s \\ s_1>0}} \binom{s}{s_1\ j_1\ j_2} \mathbf{D}_{s_1} w_{j_1} \mathbf{r}_{,j_2} \\
&= \ell_0 \sum_{\substack{s_1+j_1+j_2=s \\ j_1>0}} \binom{s}{s_1\ j_1\ j_2} w_{s_1} \mathbf{D}_{j_1} \mathbf{r}_{,j_2} \\
&= \sum_{s_1+s_2=s} \binom{s}{s_1\ s_2} w_{s_1} \ell_0 \sum_{\substack{j_1+j_2=s_2 \\ j_1>0}} \binom{s_2}{j_1\ j_2} \mathbf{D}_{j_1} \mathbf{r}_{,j_2} \\
&= 0. \qquad (2.4.17)
\end{aligned}
$$

This proves (2.4.17) and completes the induction process. Therefore we have

$$\ell_0 \mathbf{M}_k = 0$$

and (2.3.39) is solvable.

In order to prove (2.4.13), we notice that

$$
\begin{aligned}
\mathbf{M}_{k+1} &= \sum_{k_1+k_2=k+1} \binom{k+1}{k_1\ k_2} \mathbf{D}_{k_1} \mathbf{u}_\theta^{(k_2)} + T_\theta^{(k+1)} \left(\mathbf{b}_1 - \mathbf{A}_0 \mathbf{u}_x^{(0)} \right) \\
&\triangleq \mathbf{M}_{k+1}^{(1)} + \mathbf{M}_{k+1}^{(2)}.
\end{aligned}
$$

Now,

$$
\begin{aligned}
\ell_0 \mathbf{M}_{k+1}^{(2)} &= T_\theta^{(k+1)} \ell_0 \left(\mathbf{b}_1 - \mathbf{A}_0 \mathbf{u}_x^{(0)} \right) \\
&= T_\theta^{(k+1)} \left(\mathbf{c}^{(1)}(\mathbf{r}_0) - \mathbf{A}_0 \mathbf{r}_0' \right) \sigma_0(\theta) \\
&= 0,
\end{aligned}
$$

by virtue of our normalization condition $\Gamma_0 \equiv 0$, and hence we need only show that $\boldsymbol{\ell}_0 \mathbf{M}^{(1)}_{k+1} = 0$.

The representation given in (2.4.24) can be extended to provide

$$\mathbf{u}_\theta^{(j)} = \sum_{j_1+j_2=j} \binom{j}{j_1\ j_2} w_{j_1} \mathbf{r}_{,j_2},\ j = 0, 1, \dots, k$$

since $\mathbf{u}_\theta^{(k)}$ is defined. Also from (2.4.28) we have

$$\boldsymbol{\ell}_0 \sum_{\substack{j_1+j_2=j \\ j_1>0}} \binom{j}{j_1\ j_2} \mathbf{D}_{j_1} \mathbf{r}_{,j_2} = 0,\ j = 0, 1, \dots, k,$$

so that

$$\begin{aligned}
\boldsymbol{\ell}_0 \mathbf{M}_{k+1} &= \boldsymbol{\ell}_0 \mathbf{M}^{(1)}_{k+1} \\
&= \boldsymbol{\ell}_0 \sum_{\substack{k_1+k_2=k+1 \\ k_1>0}} \binom{k+1}{k_1\ k_2} \mathbf{D}_{k_1} \mathbf{u}_\theta^{(k_2)} \\
&= \boldsymbol{\ell}_0 \sum_{\substack{k_1+k_2=k+1 \\ k_1>0}} \binom{k+1}{k_1\ k_2} \mathbf{D}_{k_1} \sum_{j_1+j_2=k_2} \binom{k_2}{j_1\ j_2} w_{j_1} \mathbf{r}_{,j_2} \\
&= \boldsymbol{\ell}_0 \sum_{k_1+j_1+j_2=k+1} \binom{k+1}{k_1\ j_1\ j_2} \mathbf{D}_{k_1} w_{j_1} \mathbf{r}_{,j_2} \\
&= \boldsymbol{\ell}_0 \sum_{\substack{k_1+j_1+j_2=k+1 \\ j_1>0}} \binom{k+1}{k_1\ j_1\ j_2} w_{k_1} \mathbf{D}_{j_1} \mathbf{r}_{,j_2} \\
&= \sum_{k_1+k_2=k+1} \binom{k+1}{k_1\ k_2} w_{k_1} \boldsymbol{\ell}_0 \sum_{\substack{j_1+j_2=k_2 \\ j_1>0}} \binom{k_2}{j_1\ j_2} \mathbf{D}_{j_1} \mathbf{r}_{,j_2} \\
&= 0.
\end{aligned}$$

This completes the proof for the solvability condition.

2.5. Shock Waves

In this section we shall study the propagation of shock waves associated with the signaling problem for the system of conservation laws

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u}, x)_x = 0, \tag{2.1.1}$$

by first assuming that a shock wave is generated at the shock initiation point (x_s, t_s) and tracking its subsequent position in space-time. The more fundamental issues surrounding the generation and admissibility of the shock wave will be deferred until the following section.

2.5.1 *Shock wave problem and some identities*

The system of conservation laws (2.1.1) may, for smooth solutions, be written as

$$\mathbf{u}_t + \mathbf{A}(\mathbf{u}, x)\mathbf{u}_x = \mathbf{b}(\mathbf{u}, x),$$

where

$$\mathbf{A}(\mathbf{u}, x) \triangleq (\partial f_i / \partial u_j)_{n \times n},$$
$$\mathbf{b}(\mathbf{u}, x) \triangleq (-\partial f_i / \partial x)_{n \times 1},$$

and $\partial / \partial x$ designates differentation with respect to x only while treating $\mathbf{u}$ as formally independent of x.

Throughout our discussions in this and the following section, the characteristic field $\lambda = \lambda(\mathbf{u}, x)$ will be assumed to possess a local linear degeneracy of order $q - 1$ about the steady state $\mathbf{u} \equiv \mathbf{0}$, that is,

$$\langle \Lambda_k \rangle \equiv 0, \; k = 1, 2, \ldots, q - 1; \; \langle \Lambda_q \rangle \not\equiv 0.$$

Also the small amplitude to high frequency r

$$\delta = \varepsilon^{q}$$

We denote, as before, (x_s, t_s) as the shock . . . , indicating the position and time for the first blow-up of the deri . . . a weakly nonlinear hyperbolic wave solution. Such blow-up may, as was . . . ed in Chapter 1, induce a finite jump into the hyperbolic wave solution thereby initiating a shock wave which then propagates along a shock front. We assume for the present that this is the case and denote this shock front by Σ where

$$\Sigma : \begin{cases} \dfrac{dt}{dx} = S(x,t), & (2.5.1) \\ t(x_s) = t_s. & (2.5.2) \end{cases}$$

Invoking the Rankine-Hugoniot jump conditions then gives

$$[\mathbf{u}] - S[\mathbf{f}(\mathbf{u}, x)] = \mathbf{0}, \tag{2.5.3}$$

where $[\cdot]$ represents the jump of a quantity across Σ.

As our starting point, we will assume that the weakly nonlinear hyperbolic wave solution (2.3.50) (with $m = q$) is valid on both sides of Σ both prior to and after shock initation.

Before we can proceed with the derivation of the expression for the reciprocal of the shock speed, S, we require a number of algebraic relations. These we now develop.

First we Taylor expand the flux function $\mathbf{f}(\mathbf{u}, x)$ about $\mathbf{u} \equiv \mathbf{0}$. This gives

$$\mathbf{f}(\mathbf{u}, x) = \mathbf{f}(\mathbf{0}, x) + \sum_{k=1}^{\infty} \mathbf{f}^{(k)}(\mathbf{u}, \dots, \mathbf{u})/k!, \tag{2.5.4}$$

wherein

$$\mathbf{f}^{(k)}(\mathbf{u},\ldots,\mathbf{u}) = \sum_{k_1+\cdots+k_n=k} \left(\partial^k \mathbf{f}/\partial u_1^{k_1}\ldots\partial u_n^{k_n}\right)_0 u_1^{k_1}\ldots u_n^{k_n}$$
$$= \sum_{1\le i_1,\ldots,i_k\le n} (\partial^k \mathbf{f}/\partial u_{i_1}\partial u_{i_2}\ldots\partial u_{i_k})_0\, u_{i_1}\ldots u_{i_k}, \tag{2.5.5}$$

is a vector valued k-linear form ($k = 0,1,2,\ldots$), and the subscript '0' implies, as before, quantities evaluated at $\mathbf{u}\equiv\mathbf{0}$.

In order to facilitate certain future manipulations we write formally that

$$\mathbf{u}(x,\theta;\varepsilon) = \varepsilon\sum_{k=0}^{\infty}\frac{\varepsilon^k}{k!}\mathbf{u}^{(k)}(x,\theta), \tag{2.5.6}$$

although we have specified only

$$\mathbf{u}^{(k)}(x,\theta) = \frac{1}{k+1}\sigma_0^{k+1}(\theta)\mathbf{r}_k(x),\ k = 0,1,\ldots,q-1. \tag{2.5.7}$$

Substituting (2.5.6) into (2.5.4) and performing some relatively straightforward operations, we obtain

$$\mathbf{f}(\mathbf{u},x) = \mathbf{f}(\mathbf{0},x) + \sum_{j=1}^{\infty}\frac{\varepsilon^j}{j!}\mathbf{f}_j(x,\theta), \tag{2.5.8}$$

with

$$\mathbf{f}_j(x,\theta) = \sum_{k+i_1+\cdots+i_k=j}\binom{j}{k\ i_1\ldots i_k}\mathbf{f}^{(k)}(\mathbf{u}^{(i_1)},\ldots,\mathbf{u}^{(i_k)})$$
$$= \sum_{k=1}^{j}\sum_{i_1+\cdots+i_k=j-k}\binom{j}{k\ i_1\ldots i_k}\mathbf{f}^{(k)}(\mathbf{u}^{(i_1)},\ldots,\mathbf{u}^{(i_k)}). \tag{2.5.9}$$

Now, assume the expansion for S given by

$$S = \sum_{k=0}^{\infty}\frac{\varepsilon^k}{k!}S^{(k)}, \tag{2.5.10}$$

and substitute this together with (2.5.6), (2.5.8) into the Rankine-Hugoniot jump condition (2.5.3) to get

$$S^{(j)}[\mathbf{f}_1] = [\mathbf{u}^{(j)}] - \frac{1}{j+1} \sum_{\substack{j_1+j_2=j+1 \\ j_1<j}} \binom{j+1}{j_1\ j_2} S^{(j_1)}[\mathbf{f}_{j_2}]. \tag{2.5.11}$$

In order to solve (2.5.11) for each $S^{(j)}$ we must first prove several lemmas concerning algebraic properties of the related k-linear forms.

LEMMA 2.3. $\forall\, k \geq 1$ and any $\mathbf{v}^{(1)}, \ldots, \mathbf{v}^{(k)} \in \mathbb{R}^n$,

$$\mathbf{f}^{(k)}(\mathbf{v}^{(1)}, \ldots, \mathbf{v}^{(k)}) = \mathbf{A}^{(k-1)}(\mathbf{v}^{(1)}, \ldots, \mathbf{v}^{(k-1)})\mathbf{v}^{(k)}. \tag{2.5.12}$$

Proof. Noting that

$$\mathbf{A}(\mathbf{u}, x) = (\partial f_i/\partial u_j) \triangleq a_{ij},$$

we have

$$\begin{aligned}
&\mathbf{A}^{(k-1)}(\mathbf{v}^{(1)}, \ldots, \mathbf{v}^{(k-1)})\mathbf{v}^{(k)} \\
&= \sum_{j=1}^{n} \sum_{1\leq i_1,\ldots,i_{k-1}\leq n} ((\partial^{k-1} a_{ij}/\partial u_{i_1} \ldots \partial u_{i_{k-1}})_0)\, v_{i_1}^{(1)} \ldots v_{i_{k-1}}^{(k-1)} v_j^{(k)} \\
&= \sum_{j=1}^{n} \sum_{1\leq i_1,\ldots,i_{k-1}\leq n} ((\partial^{k} fi/\partial u_{i_1} \ldots \partial u_{i_{k-1}} \partial u_j)_0)\, v_{i_1}^{(1)} \ldots v_{i_{k-1}}^{(k-1)} v_j^{(k)} \\
&= \sum_{1\leq i_1,\ldots i_k\leq n} ((\partial^{k} f_i/\partial u_{i_1} \ldots \partial u_{i_k})_0)\, v_{i_1}^{(1)} \ldots v_{i_k}^{(k)} \\
&= \mathbf{f}^{(k)}(\mathbf{v}^{(1)}, \ldots, \mathbf{v}^{(k)}).
\end{aligned}$$

This completes the proof.

LEMMA 2.4. $\forall\, j \geq 0$ and any $\mathbf{v}^{(0)}, \ldots, \mathbf{v}^{(j)} \in \mathbb{R}^n$,

$$\sum_{k+i_1+\cdots+i_k=j+1} \binom{j+1}{k\; i_1 \ldots i_k} \mathbf{A}^{(k-1)} \left(\frac{1}{i_1+1}\mathbf{v}^{(i_1)}, \ldots, \frac{1}{i_{k-1}+1}\mathbf{v}^{(i_{k-1})} \right) \frac{1}{i_k+1}\mathbf{v}^{(i_k)}$$
$$= \sum_{k+i_1+\cdots+i_k=j+1} \binom{j}{k-1\; i_1 \ldots i_k} \mathbf{A}^{(k-1)} \left(\frac{1}{i_1+1}\mathbf{v}^{(i_1)}, \ldots, \frac{1}{i_{k-1}+1}\mathbf{v}^{(i_{k-1})} \right) \mathbf{v}^{(i_k)}. \tag{2.5.13}$$

Proof. For the right hand side of (2.5.13) we have

$$\begin{aligned}
\text{RHS} &= \sum_{k=1}^{j+1} \frac{1}{(k-1)!} \sum_{k+i_1+\cdots+i_k=j+1} \binom{j}{i_1+1, \ldots, i_{k-1}+1, i_k} \\
&\qquad \mathbf{A}^{(k-1)} \left(\mathbf{v}^{(i_1)}, \ldots, \mathbf{v}^{(i_{k-1})} \right) \mathbf{v}^{(i_k)} \\
&= \sum_{k=1}^{j+1} \frac{1}{k!} \sum_{i_1+\cdots+i_k+k=j+1} k \binom{j}{i_1+1, \ldots, i_{k-1}+1,\; i_k} \\
&\qquad \mathbf{A}^{(k-1)} \left(\mathbf{v}^{(i_1)}, \ldots, \mathbf{v}^{(i_{k-1})} \right) \mathbf{v}^{(i_k)} \\
&= \sum_{k=1}^{j+1} \frac{1}{k!} \sum_{s=1}^{k} \sum_{i_1+\cdots+i_k+k=j+1} \binom{j}{i_1+1, \ldots, i_{s-1}+1,\; i_s,\; i_{s+1}+1, \ldots, i_k+1} \\
&\quad \times \mathbf{A}^{(k-1)}(\mathbf{v}^{(i_1)}, \ldots, \mathbf{v}^{(i_{k-1})})\mathbf{v}^{(i_k)},
\end{aligned} \tag{2.5.14}$$

where the fact that for any permutation $(i_1', \ldots, i_k')$ of $(i_1, \ldots, i_k)$,

$$\begin{aligned}
\mathbf{A}^{(k-1)}(\mathbf{v}^{(i_1')}, \ldots, \mathbf{v}^{(i_{k-1}')})\mathbf{v}^{(i_k')} &= \mathbf{f}^{(k)}(\mathbf{v}^{(i_1')}, \ldots, \mathbf{v}^{(i_k')}) \\
&= \mathbf{f}^{(k)}(\mathbf{v}^{(i_1)}, \ldots, \mathbf{v}^{(i_k)}) \\
&= \mathbf{A}^{(k-1)}(\mathbf{v}^{(i_1)}, \ldots, \mathbf{v}^{(i_{k-1})})\mathbf{v}^{(i_k)},
\end{aligned} \tag{2.5.15}$$

has been used.

Interchanging the order of the two inner summations in (2.5.14) we then obtain

$$\begin{aligned}
\text{RHS} &= \sum_{k=1}^{j+1} \frac{1}{k!} \sum_{i_1+\cdots+i_k+k=j+1} \sum_{s=1}^{k} \binom{j}{i_1+1,\ldots,i_{s-1}+1,\ i_s,\ i_{s+1}+1,\ldots,i_k+1} \\
&\quad \times \mathbf{A}^{(k-1)}(\mathbf{v}^{(i_1)},\ldots,\mathbf{v}^{(i_{k-1})})\mathbf{v}^{(i_k)} \\
&= \sum_{k=1}^{j+1} \frac{1}{k!} \sum_{i_1+\cdots+i_k+k=j+1} \binom{j+1}{i_1+1,\ldots,i_s+1,\ i_k+1} \\
&\qquad \mathbf{A}^{(k-1)}\left(\mathbf{v}^{(i_1)},\ldots,\mathbf{v}^{(i_{k-1})}\right)\mathbf{v}^{(i_k)} \\
&= \sum_{i_1+\cdots+i_k+k=j+1} \binom{j+1}{k\ i_1\ldots i_k} \\
&\qquad \mathbf{A}^{(k-1)}\left(\frac{1}{i_1+1}\mathbf{v}^{(i_1)},\ldots,\frac{1}{i_{k-1}+1}\mathbf{v}^{(i_{k-1})}\right)\frac{1}{i_k+1}\mathbf{v}^{(i_k)} \\
&= \text{LHS},
\end{aligned}$$

where we have employed the identity

$$\binom{j+1}{i_1+1,\ldots,i_k+1} = \sum_{s=1}^{k} \binom{j}{i_1+1,\ldots,i_{s-1}+1,\ i_s,\ i_{s+1}+1,\ldots,i_k+1}$$

which follows upon noting that

$$(i_1+1)+\cdots+(i_k+1) = j+1.$$

This completes the proof.

LEMMA 2.5. For $0 \le j < q$, we have

$$\mathbf{r}_j - \frac{1}{\lambda_0}\langle \mathbf{f}_{j+1} \rangle \equiv 0. \tag{2.5.16}$$

Proof. From (2.3.29), $\mathbf{r}_j$ is a particular solution of

$$\left(\mathbf{I} - \frac{\mathbf{A}_0}{\lambda_0}\right)\mathbf{r}_j = \mathbf{p}_j, \; j = 1, 2, \ldots, q-1, \tag{2.5.17}$$

with $\mathbf{p}_j$ defined in (2.3.35) as

$$\mathbf{p}_j = \sum_{\substack{k_1+k_2=j \\ k_1>0}} \binom{j}{k_1 \; k_2} \langle \mathbf{D}_{k_1} \rangle \mathbf{r}_{k_2} - \sum_{\substack{k_1+k_2+k_3=j \\ k_1>0}} \binom{j}{k_1 \; k_2 \; k_3} k_1 \langle T^{(k_1)} \rangle \times \langle \mathbf{A}_{k_2} \rangle \frac{1}{k_3+1} \mathbf{r}'_{k_3} + \frac{1}{j+1} \sum_{\substack{k_1+k_2=j+1 \\ k_1,\, k_2>0}} \binom{j+1}{k_1 \; k_2} \langle \mathbf{b}_{k_1} \rangle k_2 \langle T^{(k_2)} \rangle. \tag{2.5.18}$$

Noting that

$$\langle T^{(k)} \rangle = \int_0^x \langle \Lambda_k \rangle ds \equiv 0, \; k = 1, 2, \ldots, q-1,$$

reduces the expression for $\mathbf{p}_j$ in (2.5.18) to

$$\mathbf{p}_j = \sum_{\substack{k_1+k_2=j \\ k_1>0}} \binom{j}{k_1 \; k_2} \langle \mathbf{D}_{k_1} \rangle \mathbf{r}_{k_2}. \tag{2.5.19}$$

Since

$$\begin{aligned} \mathbf{D}(\mathbf{u}, x) &= \sum_{j=1}^{\infty} \frac{\varepsilon^j}{j!} \mathbf{D}_j \\ &= \left(\frac{1}{\lambda} \mathbf{A}\right)(\mathbf{u}, x) \\ &= \left(\sum_{j=0}^{\infty} \frac{\varepsilon^j}{j!} \Lambda_j\right) \left(\sum_{j=0}^{\infty} \frac{\varepsilon^j}{j!} \mathbf{A}_j\right) \\ &= \sum_{j=0}^{\infty} \frac{\varepsilon^j}{j!} \sum_{j_1+j_2=j} \binom{j}{j_1 \; j_2} \Lambda_{j_1} \mathbf{A}_{j_2}, \end{aligned}$$

we have, upon equating coefficients of like powers of ε, that

$$\mathbf{D}_j = \sum_{j_1+j_2=j} \binom{j}{j_1\ j_2} \Lambda_{j_1} \mathbf{A}_{j_2}.$$

Hence, for $j = 1, 2, \ldots, q-1$, we have

$$\begin{aligned}\langle \mathbf{D}_j \rangle &= \sum_{j_1+j_2=j} \binom{j}{j_1\ j_2} \langle \Lambda_{j_1} \rangle \langle \mathbf{A}_{j_2} \rangle \\ &= \langle \Lambda_0 \rangle \langle \mathbf{A}_j \rangle \\ &= \frac{1}{\lambda_0} \langle \mathbf{A}_j \rangle,\end{aligned}$$

so that

$$\mathbf{p}_j = \frac{1}{\lambda_0} \sum_{\substack{k_1+k_2=j \\ k_1>0}} \binom{j}{k_1\ k_2} \langle \mathbf{A}_{k_1} \rangle \mathbf{r}_{k_2}.$$

Now, using the representation for $\mathbf{A}_j$ given by

$$\mathbf{A}_j = \sum_{k+i_1,\cdots+i_k=j} \binom{j}{k\ i_1 \ldots i_k} \mathbf{A}^{(k)}(\mathbf{u}^{(i_1)}, \ldots, \mathbf{u}^{(i_k)}),$$

together with (2.3.49) (while noting the meaning attached to the notation $\langle\cdot\rangle$), gives us that

$$\begin{aligned}\mathbf{p}_j &= \frac{1}{\lambda_0} \sum_{\substack{k_1+k_2=j \\ k_1>0}} \binom{j}{k_1\ k_2} \sum_{k+i_1+\cdots+i_k=k_1} \binom{k_1}{k\ i_1 \ldots i_k} \\ &\quad \times \mathbf{A}^{(k)} \left(\frac{1}{i_1+1} \mathbf{r}_{i_1}, \ldots, \frac{1}{i_k+1} \mathbf{r}_{i_k} \right) \mathbf{r}_{k_2} \\ &= \frac{1}{\lambda_0} \sum_{k+i_1+\cdots+i_k+k_2=j} \binom{j}{k\ i_1 \ldots i_k\ k_2} \mathbf{A}^{(k)} \left(\frac{1}{i_1+1} \mathbf{r}_{i_1}, \ldots, \frac{1}{i_k+1} \mathbf{r}_{i_k} \right) \mathbf{r}_{k_2} \\ &= \frac{1}{\lambda_0} \sum_{\substack{k+i_1+\cdots+i_k=j+1 \\ k>1}} \binom{j}{k-1\ i_1 \ldots i_k} \mathbf{A}^{(k-1)} \left(\frac{1}{i_1+1} \mathbf{r}_{i_1}, \ldots, \frac{1}{i_{k-1}+1} \mathbf{r}_{i_{k-1}} \right) \mathbf{r}_{i_k}.\end{aligned} \tag{2.5.20}$$

On the other hand, it follows from (2.5.9) and (2.5.12) that

$$\mathbf{f}_{j+1} = \sum_{k+i_1+\cdots+i_k=j+1} \binom{j+1}{k\ i_1 \ldots i_k} \mathbf{A}^{(k-1)} \left(\mathbf{u}^{(i_1)}, \ldots, \mathbf{u}^{(i_{k-1})}\right) \mathbf{u}^{(i_k)}, \qquad (2.5.21)$$

so that

$$\langle \mathbf{f}_{j+1} \rangle = \sum_{k+i_1+\cdots+i_k=j+1} \binom{j+1}{k\ i_1 \ldots i_k} \mathbf{A}^{(k)} \left(\frac{1}{i_1+1}\mathbf{r}_{i_1}, \ldots, \frac{1}{i_{k-1}+1}\mathbf{r}_{i_{k-1}}\right) \frac{1}{i_k+1}\mathbf{r}_{i_k}. \qquad (2.5.22)$$

Now, comparing (2.5.20) with (2.5.22), and applying Lemma 2.4, we obtain

$$\mathbf{p}_j = \frac{1}{\lambda_0}\langle \mathbf{f}_{j+1} \rangle - \frac{1}{\lambda_0}\mathbf{A}_0 \mathbf{r}_j,$$

so that from (2.5.17) we have

$$\mathbf{r}_j - \frac{1}{\lambda_0}\langle \mathbf{f}_{j+1} \rangle \equiv \mathbf{0}, \ j = 1, 2, \ldots, q-1. \qquad (2.5.16)$$

This completes the proof.

2.5.2 *Shock tracking procedure*

We shall now track the shock wave in (x,t)-space. This will be accomplished by determining $S(x,t)$ to the appropriate order in ε. As a byproduct of these calculations we find that the arrival time formula of (2.3.58) is recovered from the shock path formula in the limit as the nonlinear phase variable approaches its shock-initiation value θ_s.

In order to derive $S^{(j)}$, $j = 0, 1, \ldots, q$, we impose the Rankine-Hugoniot conditions in the form

$$S^{(j)}[\mathbf{f}_1] = [\mathbf{u}^{(j)}] - \frac{1}{j+1} \sum_{\substack{j_1+j_2=j+1 \\ j_1<j}} \binom{j+1}{j_1\ j_2} S^{(j_1)}[\mathbf{f}_{j_2}], \ 0 \le j \le q. \qquad (2.5.11)$$

and consider each case in turn.

Case: j = 0. In this instance, (2.5.11) reduces to

$$S^{(0)}[\mathbf{f}_1] = [\mathbf{u}^{(0)}].$$

Noting that

$$[\mathbf{f}_1] = [\sigma_0]\langle \mathbf{f}_1 \rangle = [\sigma_0]\mathbf{A}_0\mathbf{r}_0 = [\sigma_0]\lambda_0\mathbf{r}_0, \tag{2.5.23}$$

and $[\mathbf{u}^{(0)}] = [\sigma_0]\mathbf{r}_0$ gives

$$S^{(0)} = 1/\lambda_0. \tag{2.5.24}$$

Case: 1 ≤ j < q. For $j = 1$, (2.5.11) reduces to

$$\begin{aligned} S^{(1)}[\mathbf{f}_1] &= [\mathbf{u}^{(1)}] - \frac{1}{2}\binom{2}{0\ 2} S^{(0)}[\mathbf{f}_2] \\ &= \frac{1}{2}[\sigma_0^2]\left(\mathbf{r}_1 - \frac{1}{\lambda_0}\langle \mathbf{f}_2 \rangle\right) \\ &= 0, \end{aligned}$$

where Lemma 2.5 has been used in the last step above. We thus have that $S^{(1)} = 0$. Then upon assuming that

$$S^{(1)} = S^{(2)} = \cdots = S^{(j-1)} = 0, \; j < q, \tag{2.5.25}$$

we have from (2.5.11) and an application of Lemma 2.5, that

$$\begin{aligned} S^{(j)}[\mathbf{f}_1] &= [\mathbf{u}^{(j)}] - \frac{1}{j+1}\binom{j+1}{0,\ j+1} S^{(0)}[\mathbf{f}_{j+1}] \\ &= \frac{1}{j+1}[\sigma_0^{j+1}]\mathbf{r}_j - \frac{1}{j+1}\,\frac{1}{\lambda_0}[\sigma_0^{j+1}]\langle \mathbf{f}_{j+1}\rangle \\ &= \frac{1}{j+1}[\sigma_0^{j+1}]\left(\mathbf{r}_j - \frac{1}{\lambda_0}\langle \mathbf{f}_{j+1}\rangle\right) \\ &= 0. \end{aligned} \tag{2.5.26}$$

We have therefore proved, by the method of induction, that

$$S^{(1)} = S^{(2)} = \cdots = S^{(q-1)} = 0. \tag{2.5.27}$$

Case: j = q. In this instance, (2.5.11) reduces to

$$S^{(q)}[\mathbf{f}_1] = [\mathbf{u}^{(q)}] - \frac{1}{q+1}\,\frac{1}{\lambda_0}[\mathbf{f}_{q+1}], \tag{2.5.28}$$

so that, in order to find $S^{(q)}$, we require the explicit representation for $\mathbf{u}^{(q)}$ and this is not given by (2.5.7) and so must now be obtained.

From (2.2.52) and (2.2.54) we have

$$\left(I - \frac{A_0}{\lambda_0}\right)\mathbf{u}_0^{(q)} = \mathbf{M}_q(x,\theta), \tag{2.5.29}$$

wherein

$$\begin{aligned}
\mathbf{M}_q(x,\theta) = & \sum_{\substack{k_1+k_2=q \\ k_1>0}} \binom{q}{k_1\ k_2} \mathbf{D}_{k_1} \mathbf{u}_\theta^{(k_2)} \\
& - \sum_{\substack{k_1+k_2+k_3=q \\ k_1>0}} \binom{q}{k_1\ k_2\ k_3} T_\theta^{(k_1)} \mathbf{A}_{k_2} \mathbf{u}_x^{(k_3)} \\
& + \frac{1}{q+1} \sum_{\substack{k_1+k_2=q+1 \\ k_1,\ k_2>0}} \binom{q+1}{k_1\ k_2} \mathbf{b}_{k_1} T_\theta^{(k_2)} \\
= & \sum_{\substack{k_1+k_2=q \\ k_1>0}} \binom{q}{k_1\ k_2} \mathbf{D}_{k_1} \mathbf{u}_\theta^{(k_2)} + T_\theta^{(q)}(\mathbf{b}_1 - \mathbf{A}_0 \mathbf{u}_x^{(0)}).
\end{aligned} \qquad (2.5.30)$$

In the development of the final form of $\mathbf{M}_q$ given in (2.5.30), we employed the fact that the characteristic field is locally linearly degenerate to order $q-1$. The form for $\mathbf{M}_q$ can be made more explicit yet by noting that

$$\begin{aligned}
\langle \mathbf{D}_j \rangle &= \frac{1}{\lambda_0} \langle \mathbf{A}_j \rangle,\ j = 1, 2, \ldots, q-1, \\
\langle \mathbf{D}_q \rangle &= \frac{1}{\lambda_0} \langle \mathbf{A}_q \rangle + \mathbf{A}_0 \langle \Lambda_q \rangle,
\end{aligned}$$

and (from (2.3.58))

$$T_\theta^{(q)} = q! + q\sigma_0^{q-1} \sigma_0' \int_0^x \langle \Lambda_q \rangle ds$$

to give

$$\begin{aligned}
\mathbf{M}_q(x,\theta) = & \sigma_0^q(\theta) \sigma_0'(\theta) \frac{1}{\lambda_0} \sum_{\substack{k_1+k_2=q \\ k_1>0}} \binom{q}{k_1\ k_2} \langle \mathbf{A}_{k_1} \rangle \mathbf{r}_{k_2} \\
& + T_\theta^{(q)}(\mathbf{b}_1 - \mathbf{A}_0 \mathbf{u}_x^{(0)}) + \sigma_0^q(\theta) \sigma_0'(\theta) \langle \Lambda_q \rangle \lambda_0 \mathbf{r}_0.
\end{aligned}$$

Integrating with respect to θ in (2.5.29) we obtain

$$\left(\mathbf{I} - \frac{\mathbf{A}_0}{\lambda_0}\right)\mathbf{u}^{(q)} = \frac{1}{q+1}\sigma_0^{q+1}(\theta)\mathbf{p}_q + \left(\int_0^\theta \sigma_0(s)T_\theta^{(q)}(x,s)ds\right)\left(\mathbf{b}^{(1)}(\mathbf{r}_0) - \mathbf{A}_0\mathbf{r}_0'\right)$$
$$+ \frac{1}{q+1}\sigma_0^{q+1}(\theta)\langle\Lambda_q\rangle\lambda_0\mathbf{r}_0, \qquad (2.5.31)$$

wherein

$$\mathbf{p}_q = \frac{1}{\lambda_0}\sum_{\substack{k_1+k_2=q\\ k_1>0}}\binom{q}{k_1\ k_2}\langle\mathbf{A}_{k_1}\rangle\mathbf{r}_{k_2}. \qquad (2.5.32)$$

Also, noting that

$$\frac{1}{\lambda_0}\mathbf{f}_{q+1} = \frac{1}{\lambda_0}\sum_{k+i_1+\cdots+i_k=q+1}\binom{q+1}{k\ i_1\ldots i_k}\mathbf{f}^{(k)}(\mathbf{u}^{(i_1)},\ldots,\mathbf{u}^{(i_k)})$$
$$= \frac{1}{\lambda_0}\sum_{k+i_1+\cdots+i_k=q+1}\binom{q+1}{k\ i_1\ldots i_k}\mathbf{A}^{(k-1)}(\mathbf{u}^{(i_1)},\ldots,\mathbf{u}^{(i_k-1)})\mathbf{u}^{(i_k)}$$
$$= (q+1)\frac{\mathbf{A}_0}{\lambda_0}\mathbf{u}^{(q)} + \frac{1}{\lambda_0}\sigma_0^{q+1}\sum_{\substack{k+i_1+\cdots+i_k=q+1\\ k>1}}\binom{q+1}{k\ i_1\ldots i_k}$$
$$\times\mathbf{A}^{(k-1)}\left(\frac{1}{i_1+1}\mathbf{r}_{i_1},\ldots,\frac{1}{i_{k-1}+1}\mathbf{r}_{i_{k-1}}\right)\frac{1}{i_k+1}\mathbf{r}_{i_k}, \qquad (2.5.33)$$

we find for the right hand side of (2.5.28)

$$[\mathbf{u}^{(q)}] - \frac{1}{q+1}\,\frac{1}{\lambda_0}[\mathbf{f}_{q+1}]$$
$$= \left(\mathbf{I} - \frac{\mathbf{A}_0}{\lambda_0}\right)[\mathbf{u}^{(q)}] - \frac{1}{q+1}[\sigma_0^{q+1}]\frac{1}{\lambda_0}\sum_{\substack{k+i_1+\cdots+i_k=q+1\\ k>1}}\binom{q+1}{k\ i_1\ldots i_k}$$
$$\times\mathbf{A}^{(k-1)}\left(\frac{1}{i_1+1}\mathbf{r}_{i_1},\ldots,\frac{1}{i_{k-1}+1}\mathbf{r}_{i_{k-1}}\right)\frac{1}{i_k+1}\mathbf{r}_{i_k}$$

$$= \frac{1}{q+1}[\sigma_0^{q+1}]\langle\Lambda_q\rangle\lambda_0\mathbf{r}_0 + \left[\int_0^s \sigma_0 T_s^{(q)} ds\right](\mathbf{b}^{(1)}(\mathbf{r}_0) - \mathbf{A}_0\mathbf{r}_0')$$

$$+ \frac{1}{q+1}[\sigma_0^{q+1}]\left\{\mathbf{p}_q - \frac{1}{\lambda_0}\sum_{\substack{k+i_1+\cdots+i_k=q+1\\ k>1}}\binom{q+1}{k\ i_1\ldots i_k}\mathbf{A}^{(k-1)}\right.$$

$$\left.\left(\frac{1}{i_1+1}\mathbf{r}_{i_1},\ldots,\frac{1}{i_{k-1}+1}\mathbf{r}_{i_{k-1}}\right)\frac{1}{i_k+1}\mathbf{r}_{i_k}\right\}. \qquad (2.5.34)$$

However, since from (2.5.20) and Lemma 2.4 one can show that

$$\mathbf{p}_q = \frac{1}{\lambda_0}\sum_{\substack{k+i_1+\cdots+i_k=q+1\\ k>1}}\binom{q+1}{ki_1\ldots i_k}$$

$$\mathbf{A}^{(k-1)}\left(\frac{1}{i_1+1}\mathbf{r}_{i_1},\ldots,\frac{1}{i_{k-1}+1}\mathbf{r}_{i_{k-1}}\right)\frac{1}{i_k+1}\mathbf{r}_{i_k}, \qquad (2.5.35)$$

(2.5.34) may now be simplified to give

$$\left[\mathbf{u}^{(q)}\right] - \frac{1}{q+1}\frac{1}{\lambda_0}[\mathbf{f}_{q+1}] = \frac{1}{q+1}\left[\sigma_0^{q+1}\right]\langle\Lambda_q\rangle\lambda_0\mathbf{r}_0 - \left[\int_0^s \sigma_0 T_s^{(q)} ds\right]\mathbf{d}, \qquad (2.5.36)$$

wherein

$$\mathbf{d} = -\mathbf{b}^{(1)}(\mathbf{r}_0) + \mathbf{A}_0\mathbf{r}_0'. \qquad (2.5.37)$$

For the left hand side of (2.5.28) we have

$$S^{(q)}[\mathbf{f}_1] = S^{(q)}[\sigma_0]\lambda_0\mathbf{r}_0, \qquad (2.5.38)$$

so that if $\mathbf{d} \equiv 0$ we may solve (2.5.28) and (2.5.36) for $S^{(q)}$ obtaining

$$S^{(q)} = \frac{1}{q+1}\frac{\left[\sigma_0^{q+1}\right]}{[\sigma_0]}\langle\Lambda_q\rangle. \qquad (2.5.39)$$

This, in turn, provides

$$S = \frac{1}{\lambda_0} + \frac{\varepsilon^q}{(q+1)!}\frac{\left[\sigma_0^{q+1}\right]}{[\sigma_0]}\langle\Lambda_q\rangle + O(\varepsilon^{q+1}). \tag{2.5.40}$$

In general, we see from (2.5.36) and (2.5.38) that in order for (2.5.28) to be valid, $\mathbf{d}$ must be parallel to $\mathbf{r}_0$. We now show that this is not the case.

Decompose

$$\mathbf{d} = \rho\mathbf{r}_0 + \mathbf{d}^\perp,$$

where $\mathbf{d}^\perp \in \{\mathbf{r}_0\}^\perp$ and multiply both sides by $\boldsymbol{\ell}_0$ obtaining

$$\begin{aligned}\rho = \rho\boldsymbol{\ell}_0\mathbf{r}_0 &= \boldsymbol{\ell}_0\mathbf{d}\\ &= \boldsymbol{\ell}_0(-\mathbf{b}^{(1)}(\mathbf{r}_0) + \mathbf{A}_0\mathbf{r}_0')\\ &= -c^{(1)}(\mathbf{r}_0) + \mathbf{a}_0\mathbf{r}_0'\\ &= -\Gamma_0\mathbf{a}_0\mathbf{r}_0.\end{aligned} \tag{2.5.41}$$

Thus $\rho = -\Gamma_0\mathbf{a}_0\mathbf{r}_0 = 0$ since $\mathbf{r}_0$ has been normalized by requiring it to satisfy $\Gamma_0(x) \equiv 0$. Hence the only way in which (2.5.28) can be valid is for $\mathbf{d}$, which is perpendicular to $\mathbf{r}_0$, to vanish. Since

$$\begin{aligned}\mathbf{d} &= -\mathbf{b}^{(1)}(\mathbf{r}_0) + \mathbf{A}_0\mathbf{r}_0'\\ &= -\left(\frac{\partial}{\partial u_j}\left(-\frac{\partial f_i}{\partial x}\right)\right)_0\mathbf{r}_0 + \mathbf{A}_0\mathbf{r}_0'\\ &= \left(\frac{\partial}{\partial x}\left(\frac{\partial f_i}{\partial u_j}\right)\right)_0\mathbf{r}_0 + \mathbf{A}_0\mathbf{r}_0'\\ &= \mathbf{A}_0'\mathbf{r}_0 + \mathbf{A}_0\mathbf{r}_0'\\ &= (\lambda_0\mathbf{r}_0)',\end{aligned} \tag{2.5.42}$$

it follows that

$$\mathbf{d} \equiv \mathbf{0} \Longleftrightarrow \lambda_0\mathbf{r}_0 \equiv \text{ a constant vector.} \tag{2.5.43}$$

We may therefore conclude that if $\lambda_0 \mathbf{r}_0$ is not a constant vector, (2.5.28) will be invalid thereby signalling the breakdown of our formulation at $O(\varepsilon^{q+1})$, namely the $O(\varepsilon^q)$ problem (2.5.29), (2.5.30) cannot both be solvable and preserve the jump condition (2.5.28). In this situation, the inhomogeneity of the medium, that is, the explicit dependence of $\mathbf{f}$ on x excites wave modes orthogonal to $\mathbf{r}_0$. To deal with this, instead of demanding the validity of the asymptotic formulation to $O(\varepsilon^{q+1})$ on both sides of the shock front, we require that (2.5.28) be preserved.

For convenience, we denote by $\mathbf{u}_*^{(q)}$ the solution to the $O(\varepsilon^q)$ problem in the sense that it satisfies (2.5.28). Hence

$$S^{(q)}[\mathbf{f}_1] = [\mathbf{u}_*^{(q)}] - \frac{1}{q+1}\,\frac{1}{\lambda_0}\,[\mathbf{f}_{q+1}^*]\,, \qquad (2.5.44)$$

where $\mathbf{f}_{q+1}^*$ has the same representation as in (2.5.21) except that $\mathbf{u}^{(q)}$ is replaced by $\mathbf{u}_*^{(q)}$. Then writing

$$\frac{1}{\lambda_0}\mathbf{f}_{q+1}^* = \frac{1}{\lambda_0}\mathbf{f}_{q+1} - (q+1)\frac{\mathbf{A}_0}{\lambda_0}\mathbf{u}^{(q)} + (q+1)\frac{\mathbf{A}_0}{\lambda_0}\mathbf{u}_*^{(q)}, \qquad (2.5.45)$$

and substituting this into (2.5.44) yields

$$S^{(q)}[\mathbf{f}_1] = \left(\mathbf{I} - \frac{\mathbf{A}_0}{\lambda_0}\right)\left[\mathbf{u}_*^{(q)}\right] + \frac{\mathbf{A}_0}{\lambda_0}[\mathbf{u}^{(q)}] - \frac{1}{q+1}\,\frac{1}{\lambda_0}[\mathbf{f}_{q+1}]. \qquad (2.5.46)$$

Multiplying both sides by $\boldsymbol{\ell}_0$ and taking (2.5.36) and the implication following (2.5.41) into account, gives

$$S^{(q)} = \frac{1}{q+1}\,\frac{[\sigma_0^{q+1}]}{[\sigma_0]}\langle\Lambda_q\rangle. \qquad (2.5.47)$$

This is precisely the same result as in (2.5.39).

Now we further investigate the shock structure. Our analysis suggests that at the ε^{q+1}-st order, the asymptotic formulation cannot remain valid on both sides of the shock front in the presence of inhomogeneity. We therefore conclude that the asymptotic solution is valid only on the right of the shock front, that is, prior to the arrival of the shock wave and then breaks down on the left in order to preserve the Rankine-Hugoniot jump condition. It then follows from (2.5.31) that

$$\left(\mathbf{I} - \frac{\mathbf{A}_0}{\lambda_0}\right) \mathbf{u}_r^{(q)} = \frac{1}{q+1}\sigma_0^{q+1}(\theta_r)\mathbf{p}_q + \frac{1}{q+1}\sigma_0^{q+1}(\theta_r)\langle\Lambda_q\rangle\lambda_0\mathbf{r}_0 - \left(\int_0^{\theta_r} \sigma_0 T_\theta^{(q)} ds\right) \mathbf{d}, \tag{2.5.48}$$

and from (2.5.46), (2.5.33), and (2.5.35) that

$$S^{(q)}[\mathbf{f}_1] = \left(\mathbf{I} - \frac{\mathbf{A}_0}{\lambda_0}\right) \left(\mathbf{u}_\ell^{(q)} - \mathbf{u}_r^{(q)}\right) - \frac{1}{q+1}[\sigma_0^{q+1}]\mathbf{p}_q, \tag{2.5.49}$$

where subscripts ℓ, r indicate quantities evaluated on the left and right sides of the shock front, respectively.

From (2.5.48) and (2.5.49) we obtain

$$\left(\mathbf{I} - \frac{\mathbf{A}_0}{\lambda_0}\right) \mathbf{u}_\ell^{(q)} = \frac{1}{q+1}\sigma_0^{q+1}(\theta_\ell)\mathbf{p}_q + \frac{1}{q+1}\sigma_0^{q+1}(\theta_\ell)\langle\Lambda_q\rangle\lambda_0\mathbf{r}_0 - \left(\int_0^{\theta_r} \sigma_0 T_\theta^{(q)} ds\right) \mathbf{d}. \tag{2.5.50}$$

Letting

$$\mathbf{u}_r^{(q)} = \frac{1}{q+1}\sigma_0^{q+1}(\theta_r)\mathbf{r}_q + \mathbf{v}_r^{(q)}, \tag{2.5.51}$$

$$\mathbf{u}_\ell^{(q)} = \frac{1}{q+1}\sigma_0^{q+1}(\theta_\ell)\mathbf{r}_q + \mathbf{v}_r^{(q)}, \tag{2.5.52}$$

with $\mathbf{r}_q$, $\mathbf{v}_r^{(q)}$ satisfying

$$\left(\mathbf{I} - \frac{\mathbf{A}_0}{\lambda_0}\right) \mathbf{r}_q = \mathbf{p}_q + \langle\Lambda_q\rangle\lambda_0\mathbf{r}_0, \tag{2.5.53}$$

$$\left(\mathbf{I} - \frac{\mathbf{A}_0}{\lambda_0}\right) \mathbf{v}_r^{(q)} = -\left(\int_0^{\theta_r} \sigma_0 T_\theta^{(q)} ds\right) \mathbf{d}, \tag{2.5.54}$$

it is clear that the contribution from the inhomogeneity introduces a vector $\mathbf{v}_r^{(q)}$ at the ϵ^{q+1}-st order. Moreover, $\mathbf{v}_r^{(q)}$ is continuous across the shock front thereby invalidating the $O(\epsilon^q)$ problem (2.5.29), (2.5.30) on the left side of the shock front.

Therefore, the asymptotic solution derived in the previous sections is further distinguished on both sides of the shock front as

$$\mathbf{u}_\ell = \sum_{k=1}^{q+1} \frac{\varepsilon^k}{k!}\sigma_0^k(\theta_\ell)\mathbf{r}_{k-1} + \frac{\varepsilon^{q+1}}{q!}\mathbf{v}_r^{(q)} + O(\varepsilon^{q+2}), \tag{2.5.55}$$

$$\mathbf{u}_r = \sum_{k=1}^{q+1} \frac{\varepsilon^k}{k!}\sigma_0^k(\theta_r)\mathbf{r}_{k-1} + \frac{\varepsilon^{q+1}}{q!}\mathbf{v}_r^{(q)} + O(\varepsilon^{q+2}). \tag{2.5.56}$$

In summary, we have the following

THEOREM 2.3. (Shock Propagation Rule). Assuming a shock wave is induced at the shock-initiation point (x_s, t_s), this shock wave will propagate along the shock front Σ defined by

$$\frac{dt}{dx} = \frac{1}{\lambda_0(x)} + \frac{\varepsilon^q}{(q+1)!}\frac{[\sigma_0^{q+1}]}{[\sigma_0]}\langle\Lambda_q\rangle + O(\varepsilon^{q+1}), \tag{2.5.57}$$

$$t(x_s) = t_s, \tag{2.5.58}$$

with shock strength given by

$$[\mathbf{u}] = \mathbf{u}_\ell - \mathbf{u}_r = \sum_{k=1}^{q+1} \frac{\varepsilon^k}{k!}[\sigma_0^k]\mathbf{r}_{k-1}(x) + O(\varepsilon^{q+2}). \tag{2.5.59}$$

Here $[\sigma_0^k] = \sigma_0^k(\theta_\ell) - \sigma_0^k(\theta_r)$, $k = 1, 2, \ldots, q+1$, with $\theta_\ell = \theta_\ell(x,t)$, $\theta_r = \theta_r(x,t)$ satisfying the arrival time formula

$$t = \int_0^x \frac{ds}{\lambda_0(s)} + \varepsilon^q \left\{\theta_i + \frac{1}{q!}\sigma_0^q(\theta_i)\int_0^x \langle\Lambda_q\rangle ds\right\} + O(\varepsilon^{q+1}),\ \theta_i = \theta_\ell,\ \theta_r. \tag{2.5.60}$$

Subscripts ℓ, r bear the same meaning as mentioned above. In addition, $\mathbf{u}_\ell$, $\mathbf{u}_r$ take the forms in (2.5.55), (2.5.56), respectively, with $\mathbf{v}_r^{(q)}$ being a vector induced by the explicit dependence of f on x and remaining continuous across the shock front Σ.

It is interesting to note that our shock front formula (2.5.57) reduces to the arrival time formula (2.3.58) in the limit $\theta_\ell,\ \theta_r \longmapsto \theta_s$. This is readily seen by writing (2.5.57) as

$$\frac{dt}{dx} = \frac{1}{\lambda_0(x)} + \frac{\varepsilon^q}{(q+1)!}\,\frac{(\sigma_0^{q+1}(\theta_\ell) - \sigma_0^{q+1}(\theta_r))/(\theta_\ell - \theta_r)}{(\sigma_0(\theta_\ell) - \sigma_0(\theta_r))/(\theta_\ell - \theta_r)}\langle\Lambda q\rangle + O(\varepsilon^{q+1}),$$

taking the appropriate limit and integrating over x.

2.6. Admissibility and Existence of Shock Waves

2.6.1 *Admissible shock waves*

The shock wave, when it exists, propagates at a speed $1/S$ with

$$S = \frac{1}{\lambda_0(x)} + \frac{\varepsilon^q}{(q+1)!}\,\frac{[\sigma_0^{q+1}]}{[\sigma_0]}\langle\Lambda_q\rangle + O(\varepsilon^{q+1}).$$

Throughout our discussion we have not distinguished the subscript associated with λ. Let it be ν $(p < \nu \le n)$. The strict hyperbolicity condition (2.1.4) demands that

$$\lambda_{\nu-1}(\mathbf{u}, x) < \lambda_\nu(\mathbf{u}, x) = \lambda < \lambda_{\nu+1}(\mathbf{u}, x). \tag{2.6.1}$$

This inequality is preserved for small perturbations about $\mathbf{u} \equiv 0$, and hence it follows that

$$\lambda_{\nu-1}(\mathbf{u}_\ell, x) < \frac{1}{S} < \lambda_{\nu+1}(\mathbf{u}_r, x), \tag{2.6.2}$$

suggesting that the shock wave can only be a ν-shock [39,71]. We shall call it a λ-shock to avoid having to distinguish the subscript ν of λ.

Employing Lax's entropy inequality [39,71], we find that the shock wave should satisfy

$$\lambda_{\nu-1}(\mathbf{u}_\ell, x) < 1/S < \lambda(\mathbf{u}_\ell, x), \tag{2.6.3}$$

$$\lambda(\mathbf{u}_r, x) < 1/S < \lambda_{\nu+1}(\mathbf{u}_r, x), \tag{2.6.4}$$

or, equivalently,

$$\frac{1}{\lambda(\mathbf{u}_\ell, x)} < S < \frac{1}{\lambda(\mathbf{u}_r, x)}, \tag{2.6.5}$$

in light of (2.6.2).

Noting that

$$\frac{1}{\lambda(\mathbf{u}, x)} = \frac{1}{\lambda_0(x)} + \frac{\varepsilon^q}{q!}\sigma_0^q(\theta)\langle\Lambda_q\rangle + O(\varepsilon^{q+1}),$$

we then have

$$\sigma_0^q(\theta_\ell)\langle\Lambda_q\rangle < \frac{1}{q+1}\,\frac{[\sigma_0^{q+1}]}{[\sigma_0]}\langle\Lambda_q\rangle < \sigma_0^q(\theta_r)\langle\Lambda_q\rangle, \tag{2.6.6}$$

as a sufficiency condition for the determination of an admissible λ-shock wave satisfying Lax's entropy condition for small ε.

Observing that (see Fig. 2.1 below)

$$\theta_\ell(x,t) > \theta_r(x,t) \tag{2.6.7}$$

on Σ, (2.6.6) can be replaced by a stronger yet simpler condition. Let

$$\Theta = \{\theta_\ell(x,t),\ \theta_r(x,t) | (x,t) \in \Sigma\},$$

and let $J \subset \mathbb{R}^+$ be an interval. Then if $\Theta \subset J$ and

$$(\sigma_0^q(\theta))'\langle\Lambda_q\rangle < 0,\ \forall\theta \in J;\ \forall x : (x,t) \in \Sigma \text{ for some } t, \tag{2.6.8}$$

then (2.6.6) holds, that is, the shock wave is a λ-shock wave satisfying Lax's entropy condition.

In addition, we observe that (2.6.8) is further simplified when $\langle \Lambda_q \rangle$ keeps a definite sign. That is, (2.6.8) becomes

$$(\sigma_0^q(\theta))' < 0, \tag{2.6.9}$$

when $\langle \Lambda_q \rangle > 0$, $\forall x \in \mathbb{R}^+$ and

$$(\sigma_0^q(\theta))' > 0, \tag{2.6.10}$$

when $\langle \Lambda_q \rangle < 0$, $\forall x \in \mathbb{R}^+$. As we shall see in the next subsection, these are also conditions ensuring that the corresponding characteristic family focuses.

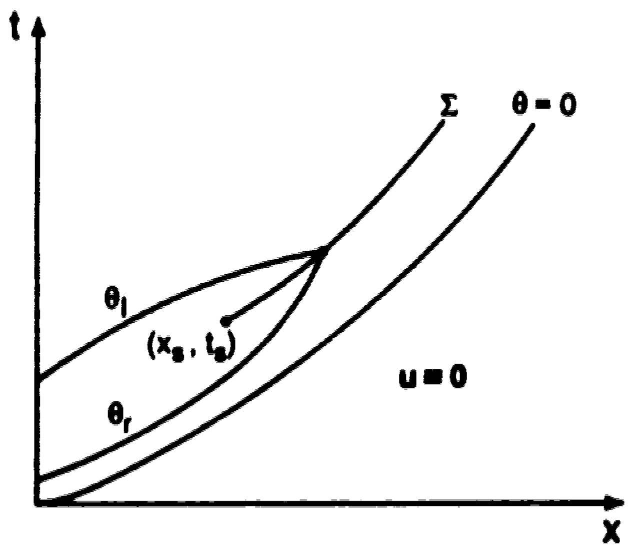

Fig. 2.1: Shock front Σ in (x, t) plane.

2.6.2 *Existence and caustic structure*

The *Shock Propagation Rule* suggests that when a shock wave is generated, the arrival time formula

$$t = T(x, \theta),$$

will admit bifurcating solutions for θ, that is, the shock-initiation point is a bifurcation point. Therefore, the shock fitting problem has an interpretation in terms of a bifurcation problem. The resolution of such a problem, as we shall see, provides an understanding of the transition process from wave breaking to the generation of shock waves.

A subtle feature here is that the shock initiation point is always a cusp for the caustic of the characteristic family when represented by the arrival time formula. This fact makes it possible to separate the branches of the caustic and determine the existence of a region where θ is multi-valued.

We commence by considering the caustic Γ of the characteristic family $t = T(x, \theta)$. In order to simplify the analysis and rule out uncertainty, we neglect the $O(\epsilon^{q+1})$ term and assume that

$$t = T(x, \theta) = \int_0^x \frac{ds}{\lambda_0(s)} + \epsilon^q \left\{ \theta + \frac{1}{q!} \sigma_0^q(\theta) \int_0^x \langle \Lambda_q \rangle ds \right\}, \tag{2.6.11}$$

is exact.

Writing $F(x, t, \theta) = T(x, \theta) - t \equiv 0$, the caustic Γ of the characteristic family is specified by

$$\Gamma : F(x, t, \theta) = 0, \; F_\theta(x, t, \theta) = 0, \tag{2.6.12}$$

which, in turn, provides

$$\Gamma : \begin{cases} x = x(\theta) : \displaystyle\int_0^x \langle \Lambda_q \rangle ds = -\frac{q!}{(\sigma_0^q)'}, & (2.6.13a) \\ t = t(\theta) = \displaystyle\int_0^{x(\theta)} \frac{ds}{\lambda_0(s)} + \epsilon^q \left\{ \theta - \frac{\sigma_0^q}{(\sigma_0^q)'} \right\}. & (2.6.13b) \end{cases}$$

In order to avoid more complications and also to ensure the existence of Γ, we shall assume that

$$\langle \Lambda_q \rangle (x) > 0, \ \forall x \in \mathbb{R}^+, \tag{2.6.14}$$

and

$$\int_0^\infty \langle \Lambda_q \rangle ds = +\infty. \tag{2.6.15}$$

The next lemma assures the existence of the caustic Γ, as well as, describes its structure.

LEMMA 2.6. Suppose $\langle \Lambda_q \rangle$ satisfies (2.6.14), (2.6.15), and also that $(\sigma_0^q(\theta))' < 0$, $\forall \theta \in (0, \theta_0)$, with $d\sigma_0^q(0)/d\theta = d\sigma_0^q(\theta_0)/d\theta = 0$. Then the caustic Γ for the characteristic family $t = T(x, \theta)$ exists for $\theta \in (0, \theta_0)$. In addition, Γ has the following properties:

1°. $\exists\, \theta_s \in (0, \theta_0)$ which gives the shock initiation point, and θ_s provides the minima for both $x = x(\theta)$ and $t = t(\theta)$, that is,

$$x_s = x(\theta_s) = \min_{\theta \in (0, \theta_0)} \{x(\theta)\}, \tag{2.6.16a}$$

$$t_s = t(\theta_s) = \min_{\theta \in (0, \theta_0)} \{t(\theta)\}. \tag{2.6.16b}$$

2°. (x_s, t_s) is a cusp for the caustic Γ and along Γ

$$\lim_{\theta \to \theta_s^-} \frac{dt}{dx} = \lim_{\theta \to \theta_s^+} \frac{dt}{dx} = \frac{1}{\lambda_0(x_s)} + \frac{\epsilon^q}{q!} \sigma_0^q(\theta_s) \langle \Lambda_q \rangle (x_s). \tag{2.6.17}$$

In addition

$$(\sigma_0^q)''|_{\theta = \theta_s} = 0, \tag{2.6.18}$$

and Γ has zero curvature at (x_s, t_s).

3°. $$\lim_{\theta\to 0^+}(x(\theta), t(\theta)) = (+\infty, +\infty), \tag{2.6.19a}$$

$$\lim_{\theta\to\theta_0^-}(x(\theta),\ t(\theta)) = (+\infty, +\infty). \tag{2.6.19b}$$

Proof. The existence of Γ follows directly from (2.6.13) and the conditions for $\sigma_0(\theta)$ and $\langle\Lambda_q\rangle$. It is also straightforward to verify (2.6.19) from (2.6.13) and (2.6.15) and hence to conclude that $t = t(\theta)$ must have a minimum at $\theta_s \in (0, \theta_0)$ such that

$$t_s = t(\theta_s) = \min_{\theta\in(0,\theta_0)}\{t(\theta)\}. \tag{2.6.16b}$$

Now we need to show that θ_s also provides the minimum for $x = x(\theta)$. The fact that $x = x(\theta)$ has a minimum is obvious from (2.6.19). We use an argument similar to that formulated for the proof of Lemma 1.1 to show that θ_s provides this minimum.

Let $x^* = x(\theta^*)$ be the minimum. Then there are two cases: $t^* = t_s$ or $t^* > t_s$, where $t^* = t(\theta^*)$. In the first case we simply shift the shock-initiation point to (x^*, t^*) and adopt the convention that θ_s is always chosen such that x_s is also the minimum of $x = x(\theta)$. In the second case, by connecting (x_s, t_s) to (x^*, t^*) by means of a line segment, we note that its slope

$$k = \frac{t^* - t_s}{x^* - x_s} < 0,$$

so that $\exists\hat{\theta} \in (0, \theta_0)$ such that $t = T(x, \hat{\theta})$ has a negative tangent at $(x(\hat{\theta}), t(\hat{\theta}))$. This, however, is impossible for the characteristic family under consideration. Therefore, we have shown that

$$x_s = x(\theta_s) = \min_{\theta\in(0,\theta_0)}\{x(\theta)\}. \tag{2.6.16a}$$

This establishes 1°.

Lastly, we need to show 2°. Indeed, by differentiating in (2.6.13), we obtain

$$\frac{dx}{d\theta} = \frac{q!}{\langle \Lambda_q \rangle} \frac{(\sigma_0^q)''}{((\sigma_0^q)')^2}, \tag{2.6.20a}$$

$$\frac{dt}{d\theta} = \left(\frac{1}{\lambda_0(x)} \frac{q!}{\langle \Lambda_q \rangle} + \varepsilon^q \sigma_0^q \right) \frac{(\sigma_0^q)''}{((\sigma_0^q)')^2}. \tag{2.6.20b}$$

Thus

$$(\sigma_0^q(\theta))''\big|_{\theta=\theta_s} = 0, \tag{2.6.18}$$

as θ_s provides a minimum for both $x = x(\theta)$ and $t = t(\theta)$. From (2.6.20), we have that along Γ

$$\frac{dt}{dx} = \frac{1}{\lambda_0(x)} + \frac{\varepsilon^q}{q!}\sigma_0^q(\theta)\langle \Lambda_q \rangle,$$

so that

$$\lim_{\theta \to \theta_s^-} \frac{dt}{dx} = \lim_{\theta \to \theta_s^+} \frac{dt}{dx} = \frac{1}{\lambda_0(x_s)} + \frac{\varepsilon^q}{q!}\sigma_0^q(\theta_s)\langle \Lambda_q \rangle(x_s), \tag{2.6.17}$$

and hence (x_s, t_s) is a cusp for Γ. In addition, we can compute d^2t/dx^2 in order to establish the zero curvature assertion. From (2.6.21) we have

$$\begin{aligned}
\frac{d^2t}{dx^2} &= \frac{d}{dx}\left(\frac{1}{\lambda_0(x)} + \frac{\varepsilon^q}{q!}\sigma_0^q(\theta)\langle \Lambda_q \rangle \right) \\
&= -\frac{\lambda_0'(x)}{\lambda_0^2(x)} + \frac{\varepsilon^q}{q!}\sigma_0^q(\theta)\langle \Lambda_q \rangle'(x) + \frac{\varepsilon^q}{q!} \frac{(\sigma_0^q(\theta))'\langle \Lambda_q \rangle}{dx/d\theta} \\
&= -\frac{\lambda_0'(x)}{\lambda_0^2(x)} + \frac{\varepsilon^q}{q!}\sigma_0^q(\theta)\langle \Lambda_q \rangle'(x) + \frac{\varepsilon^2}{(q!)^2} \frac{((\sigma_0^q(\theta))')^2 \, \Lambda_q'(x)}{\Lambda_q(x)(\sigma_0^q(\theta))''},
\end{aligned} \tag{2.6.22}$$

from which, by noting that $(\sigma_0^q)''\big|_{\theta=\theta_s} = 0$, we may deduce that

$$\lim_{\theta \to \theta_s} \frac{d^2t}{dx^2} = \infty \tag{2.6.23}$$

and hence that Γ has zero curvature at (x_s, t_s).

This completes the proof.

The above results establish a caustic structure similar to those discussed in Chapter 1. In particular, we can separate Γ into two branches in the neighbourhood of (x_s, t_s) :

$$\Gamma_1 : \; x = x(\theta), \; t = t(\theta), \; 0 < \theta \leq \theta_s, \tag{2.6.24a}$$

$$\Gamma_2 : \; x = x(\theta), \; t = t(\theta), \; \theta_s \leq \theta < \theta_0. \tag{2.6.24b}$$

As before, we denote the sharp region enclosed by Γ_1 and Γ_2 as $\mathcal{D}$.

The relative positions of Γ_1 and Γ_2 in the neighbourhood of the shock-initiation point (x_s, t_s) is ascertained by the following result.

LEMMA 2.7. Suppose that the conditions of Lemma 2.6 pertain so that Γ exists and can be separated into two branches Γ_1, Γ_2 as defined by (2.6.24). Then Γ_1 is above Γ_2 in a neighbourhood of the shock-initiation point (x_s, t_s).

Proof. The technique we employ here mimics that used to prove Theorem 1.2.

We begin by noting that

$$\frac{dt}{dx} = \frac{1}{\lambda_0(x)} + \frac{\varepsilon^q}{q!}\sigma_0^q(\theta)(\Lambda_q)(x), \tag{2.6.21}$$

where θ may be expressed in terms of x by inverting $x = x(\theta)$. To accomplish this inversion, we notice that θ_s is the value of θ providing a minimum for $x = x(\theta)$ so that $x'(\theta_s) = 0$ and $x''(\theta_s) \geq 0$. Without loss of generality, we assume $x''(\theta_s) > 0$. Now, by expanding $x = x(\theta)$ about $\theta = \theta_s$, we obtain

$$\begin{aligned} x &= x_s + \frac{1}{2}x''(\theta_s)(\theta - \theta_s)^2 + O((\theta - \theta_s)^3), \\ &= x_s + Q(\theta)(\theta - \theta_s)^2, \end{aligned} \tag{2.6.25}$$

where $Q(\theta)$ is smooth and

$$\lim_{\theta \to \theta_s} Q(\theta) = \frac{1}{2} x''(\theta_s) > 0. \tag{2.6.26}$$

Rewriting (2.6.25) as

$$Q(\theta)(\theta - \theta_s)^2 = x - x_s, \tag{2.6.27}$$

one can then invert $x = x(\theta)$ by an application of the implicit function theorem in (2.6.27) to obtain two solutions in a neighbourhood of $\theta = \theta_s$ when $x > x_s$. Indeed, each solution corresponds to a caustic branch:

1°. When $\theta < \theta_s$, the first solution, $\theta = \theta_1(x)$, solves

$$\theta - \theta_s = -\{(x - x_s)/Q(\theta)\}^{1/2}. \tag{2.6.28}$$

2°. When $\theta > \theta_s$, the second solution, $\theta = \theta_2(x)$, solves

$$\theta - \theta_s = \{(x - x_s)/Q(\theta)\}^{1/2}. \tag{2.6.29}$$

Thus

$$\theta_1(x) - \theta_2(x) < 0, \ \forall x > x_s, \tag{2.6.30}$$

in a neighbourhood of (x_s, t_s).

Therefore we have, respectively, that

$$\text{on} \quad \Gamma_1 : \frac{dt}{dx} = \frac{1}{\lambda_0(x)} + \frac{\epsilon^q}{q!} \sigma_0^q(\theta_1(x)) \langle \Lambda_q \rangle (x), \tag{2.6.31a}$$

and

$$\text{on} \quad \Gamma_2 : \frac{dt}{dx} = \frac{1}{\lambda_0(x)} + \frac{\epsilon^q}{q!} \sigma_0^q(\theta_2(x)) \langle \Lambda_q \rangle (x). \tag{2.6.31b}$$

Now, in order to determine the relative positions of Γ_1 and Γ_2, we compare their slopes. In the neighbourhood of (x_*, t_*), we denote Γ_1 and Γ_2 by $t = t_1(x)$ and $t = t_2(x)$, respectively (see Fig. 2.2 below). Fixing x and subtracting (2.6.31b) from (2.6.31a) we obtain

$$\begin{aligned} \frac{d}{dx}(t_1 - t_2) &= \frac{\varepsilon^q}{q!}\{\sigma_0^q(\theta_1(x)) - \sigma_0^q(\theta_2(\theta))\}\langle\Lambda_q\rangle(x) \\ &= \frac{\varepsilon^q}{q!}(\sigma_0^q(\xi))'(\theta_1(x) - \theta_2(x))\langle\Lambda_q\rangle(x), \end{aligned} \tag{2.6.32}$$

where $\xi \in (\theta_1(x),\ \theta_2(x))$. Thus

$$\frac{d}{dx}(t_1 - t_2) > 0, \tag{2.6.33}$$

since $(\sigma_0^q(\theta))' < 0$, $\forall\theta \in (0, \theta_0)$ and $\langle\Lambda_q\rangle(x) > 0$, $\forall x \in \mathbb{R}^+$. Therefore, in the neighbourhood of (x_*, t_*), Γ_1 is above Γ_2.

This completes the proof.

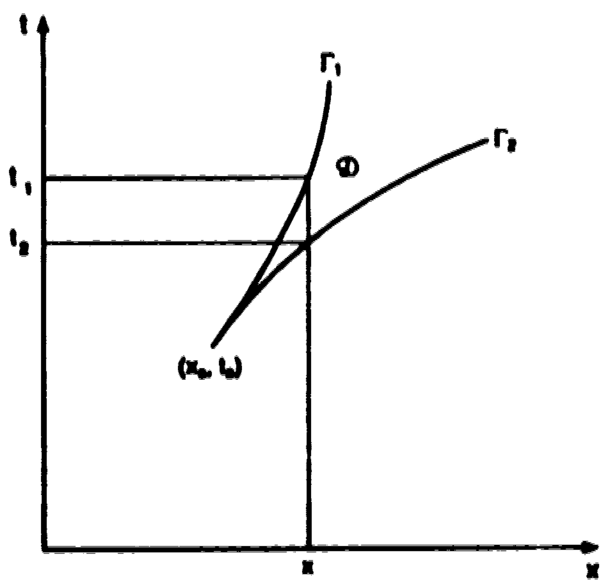

Fig. 2.2: Γ_1, Γ_2 in a neighbourhood of (x_*, t_*).

As a result of Lemma 2.7, we can prove that the shock-initiation point is a bifurcation point for the arrival time formula. More precisely, we have the following.

LEMMA 2.8. Suppose the conditions of Lemma 2.6 hold so that the caustic Γ exists and has two branches Γ_1 and Γ_2 with $\mathcal{D}$ being the enclosed sharp region. Then $\forall(x,t) \in \mathcal{D}$, there are two solutions $\theta = \theta_\ell(x,t)$, $\theta_r(x,t)$ which solve the arrival time formula

$$t = \int_0^x \frac{ds}{\lambda_0(s)} + \varepsilon^q \left\{ \theta + \frac{1}{q!}\sigma_0^q(\theta) \int_0^x \langle \Lambda_q \rangle ds \right\}, \qquad (2.6.11)$$

with

$$0 < \theta_r(x,t) < \theta_1(x), \quad \theta_2(x) < \theta_\ell(x,t) < \theta_0. \qquad (2.6.34)$$

Namely, $\forall(x,t) \in \mathcal{D}$, there are two characteristics with phases $\theta_\ell(x,t)$ and $\theta_r(x,t)$, respectively, passing through it. Here $\theta = \theta_1(x)$, $\theta_2(x)$ with $\theta_1(x) < \theta_s < \theta_2(x)$ are two branches of the inversion of $x = x(\theta)$ defined by (2.6.28), (2.6.29) respectively.

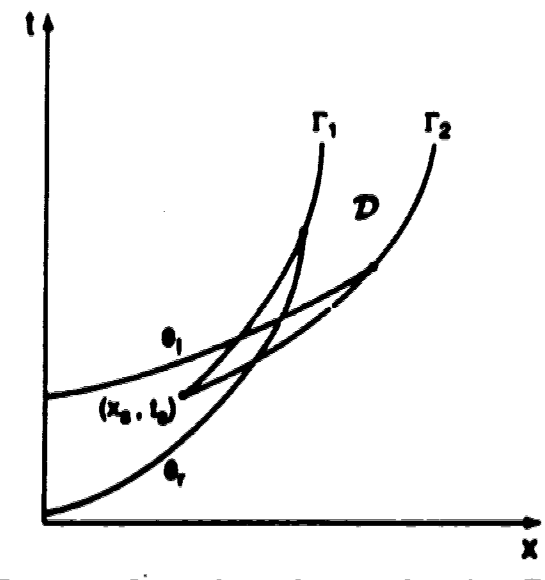

Fig. 2.3: Cusped envelope and region $\mathcal{D}$.

We omit the proof which parallels the proof of Lemma 1.2 of the last chapter.

We are now in the position to formulate the

THEOREM 2.4. (Shock Criterion). Suppose $\langle \Lambda_q \rangle(x) > 0$, $\forall x \in \mathbb{R}^+$, and $(\sigma_0^q(\theta))' < 0$, $\forall \theta \in (0, \theta_0)$, $d\sigma_0^q(0)/d\theta = d\sigma_0^q(\theta_0)/d\theta = 0$, then at the shock-initiation point, a λ-shock wave is generated in $\mathcal{D}$ and its propagation described by the *Shock Propagation Rule*. In addition, the shock wave is admissible in the sense that it satisfies Lax's entropy condition.

Proof. The conditions $\langle \Lambda_q \rangle(x) > 0$, $\forall x \in \mathbb{R}^+$ and $(\sigma_0^q(\theta))' < 0$, $\forall \theta \in (0, \theta_0)$, $d\sigma_0^q(0)/d\theta = d\sigma_0^q(\theta_0)/d\theta = 0$, ensure the existence of a caustic Γ for the characteristic family $\{t = T(x, \theta) | \theta \in (0, \theta_0)\}$ as well as the shock-initiation point (x_s, t_s), which provides a cusp for the caustic. According to Lemma 2.8, the sharp region $\mathcal{D}$ enclosed by Γ is a multi-valued region for the nonlinear phase θ. We resolve the situation by introducing a curve of discontinuity, or in other words, a shock front Σ in $\mathcal{D}$ and terminating at Σ any characteristic which enters $\mathcal{D}$. This curve Σ is initiated at the shock-initiation point (x_s, t_s) and described by the Shock Propagation Rule which is obtained from the Rankine-Hugoniot condition. Hence we have a shock wave being generated at the shock-initiation point. The shock wave is a λ-shock wave and admissible as it satisfies (2.6.8) – an interpretation of Lax's entropy condition.
This completes the proof.

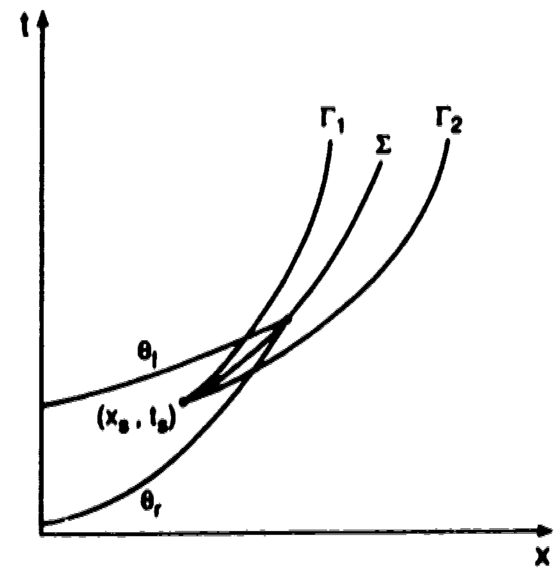

Fig. 2.4: Shock front Σ and caustic Γ.

One may observe that

$$\langle \Lambda_q \rangle(x) > 0, \ \forall x \in \mathbb{R}^+, \tag{2.6.14}$$

plays a role that parallels the one played by the convexity condition of Chapter 1. In particular, when it is used in conjunction with the boundary disturbance conditions

$$(\sigma_0^q(\theta))' < 0, \forall \theta \in (0, \theta_0),$$
$$d\sigma_0^q(0)/d\theta = d\sigma_0^q(\theta_0)/d\theta = 0,$$

in

$$\frac{dx}{d\theta} = \frac{q!}{\langle \Lambda_q \rangle} \frac{(\sigma_0^q)''}{((\sigma_0^q)')^2}, \tag{2.6.20a}$$

and

$$\frac{dt}{d\theta} = \left\{ \frac{1}{\lambda_0(x)} \frac{q!}{\langle \Lambda_q \rangle} + \varepsilon^q \sigma_0^q \right\} \frac{(\sigma_0^q)''}{((\sigma_0^q)')^2}, \tag{2.6.20b}$$

it is easy to see that $x = x(\theta)$ and $t = t(\theta)$ have the same critical numbers in $(0, \theta_0)$. In addition, (2.6.20) demonstrates that $x = x(\theta)$, $t = t(\theta)$ achieve local maxima or minima simultaneously for the same critical number. Each local minima or maxima for $(\sigma_0^q)'$ gives a cusp for the caustic Γ where the curvature is zero. The caustic Γ again takes the form displayed in Figs. 1.6-1.10.

Remark 2.2. Apparently, when $\langle \Lambda_q \rangle$ is strictly positive (or negative), for smooth boundary data, characteristic focusing is the only mechanism that leads to wave breaking and hence generates shock waves. In the nonfocusing case, smooth waves are expansive and remain smooth.

2.7. An Application: Shock Waves in Fluid-filled Hyperelastic Tubes.

Moodie and Swaters [64] considered the propagation of weakly nonlinear waves in fluid-filled, hyperelastic, tethered tubes subjected to axial strain. The one dimensional model they employed was originally developed by Moodie and Haddow [65] in the context of mathematical haemodynamics and is given by

$$A_t + (Au)_x = 0, \tag{2.7.1}$$

$$u_t + uu_x + p_x = 0, \tag{2.7.2}$$

or

$$A_t + (Au)_x = 0, \tag{2.7.1}$$

$$u_t + \left(\frac{1}{2}u^2 + p\right)_x = 0, \tag{2.7.3}$$

when the conservation form is retained. These are non-dimensional equations where $x \geq 0$ is the axial variable, $t \geq 0$ is time, u is the fluid velocity in the axial direction, p is the transmural pressure, and A is the tube's cross sectional area.

The governing equation (2.7.1) arises from mass conservation and (2.7.2) (or (2.7.3)) from momentum balance. The fluid is assumed to be incompressible.

The constitutional relation derived in [64] gives

$$\begin{aligned} A &= A(x,p) \\ &= A_0(x) + \varphi_0(x)p + \varphi_1(x)p^2 + \varphi_2(x)p^3 + O(p^4), \end{aligned} \tag{2.7.4}$$

where

$$\varphi_0(x) = A_0^{3/2}(x) / \left[2h_0(x)(W_1^0/(1+e)^2 + W_2^0)\right], \tag{2.7.5}$$

$$\varphi_1(x) = 3\varphi_0^2(x)/2A_0(x), \tag{2.7.6}$$

$$\varphi_2(x) = [5/2 - \beta(x)]\, \varphi_0^3(x)/A_0^2(x), \tag{2.7.7}$$

$$\beta(x) = \frac{W_{11}^0/(1+e) + 2(1+e)W_{12}^0 + (1+e)^3 W_{22}^0}{W_1^0 + (1+e)^2 W_2^0}, \tag{2.7.8}$$

are all known functions involving the strain energy function W, the tube wall thickness h, and the axial strain e. In particular, $A_0(0) = 1$ and $\varphi_0(0) = 1/2$.

The system admits a steady state solution

$$p = u = 0, \qquad A = A_0(x).$$

Now if the boundary is perturbed by

$$p\big|_{x=0} = \epsilon g(t/\delta), \quad t \geq 0,$$

we have the mixed initial and boundary problem prescribed by

$$p = u = 0, \quad A = A_0(x), \quad t = 0, \ x \geq 0, \tag{2.7.9}$$

$$p\big|_{x=0} = \epsilon g(t/\delta), \quad t \geq 0. \tag{2.7.10}$$

It was proved in [64] that a shock on the leading wavefront is not possible. A solution which leads to wave-breaking was constructed and the shock-initiation time and distance calculated for the interior shock. We shall now reconsider this example employing the ideas developed in this chapter. However, we mention here that in the presence of wave-breaking the mathematical model itself may no longer be a valid description of the physical correspndence, hence the discussion here is purely a mathematical treatise.

In a recent paper [17] we constructed the asymptotic solution of the mixed initial boundary problem prescribed by (2.7.1), (2.7.2) together with (2.7.9), (2.7.10) in the form

$$\mathbf{u} = \begin{pmatrix} p \\ u \end{pmatrix} = \epsilon 2^{-1/4} g(\theta) A_0^{-3/4} \varphi_0^{1/4} \begin{pmatrix} (A_0\varphi_0)^{1/2} \\ 1 \end{pmatrix} + \epsilon^2 2^{-3/2} g^2(\theta) A_0^{-3/2} \varphi_0^{1/2} \begin{pmatrix} 0 \\ (A_0/\varphi_0)^{-1/2} \end{pmatrix} + O(\epsilon^3), \tag{2.7.11}$$

with the arrival time formula

$$t = \int_0^x \varphi_0^{1/2}(\eta) A_0^{-1/2}(\eta) d\eta + \epsilon^2 \left\{ \theta - \frac{3}{2^{3/2}} g^2(\theta) \int_0^x \beta(\eta) \varphi_0^2(\eta) A_0^{-3}(\eta) d\eta \right\} + O(\epsilon^3), \tag{2.7.12}$$

where $\sigma_0(\theta) = 2^{-1/2} g(\theta)$, and $\delta = \epsilon^2$.

Thus, when a shock wave is induced at the shock-initiation point, and after applying our shock propagation rule, we expect the shock front to be described by

$$\frac{dt}{dx} = \varphi_0^{1/2}(x) A_0^{-1/2}(x) - \frac{\epsilon^2}{4} \frac{[g^3]}{[g]} \beta(x) \varphi^2(x) A_0^{-3}(x) + O(\epsilon^3), \tag{2.7.13}$$

$$t(x_s) = t_s, \tag{2.7.14}$$

where $[g^3] = g^3(\theta_\ell) - g^3(\theta_r)$, $[g] = g(\theta_\ell) - g(\theta_r)$, and θ_ℓ, θ_r ($\theta_\ell > \theta_r$) are bifurcation solutions of the arrival time formula (2.7.12).

2.7.1 *Arrival time formula*

The above approach needs some justification since the state variables we have chosen in [17] are $\mathbf{u} = (p, u)^T$, and with this choice the system (2.7.1), (2.7.3) fails to preserve the form of the conservation law (2.1.2) considered throughout the previous sections of this chapter.

We take another approach. Let

$$\mathbf{u} = \begin{pmatrix} A - A_0 \\ u \end{pmatrix} \triangleq \begin{pmatrix} \tilde{A} \\ u \end{pmatrix} \tag{2.7.15}$$

so that (2.7.1), (2.7.3) now become

$$\begin{pmatrix} \tilde{A} \\ u \end{pmatrix}_t + \begin{pmatrix} Au \\ \frac{1}{2}u^2 + p \end{pmatrix}_x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{2.7.16}$$

Then rewriting (2.7.16) as

$$\begin{pmatrix} \tilde{A} \\ u \end{pmatrix}_t + \begin{pmatrix} u & A \\ p_A & u \end{pmatrix} \begin{pmatrix} \tilde{A} \\ u \end{pmatrix}_x = \begin{pmatrix} -A_0' u \\ -\frac{\partial p}{\partial x} - p_A A_0' \end{pmatrix}, \tag{2.7.17}$$

we have

$$\mathbf{A} = \begin{pmatrix} u & A \\ p_A & u \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} -A_0' u \\ -\frac{\partial p}{\partial x} - p_A A_0' \end{pmatrix}.$$

We need only to show that the arrival time formula (2.7.12) is invariant under the current manipulation.

The eigenvalues of $\mathbf{A}$ are

$$\lambda = u \pm (p_A A)^{1/2}. \tag{2.7.18}$$

We choose the positive eigenvalue and write

$$\lambda = u + (p_A A)^{1/2}. \tag{2.7.19}$$

The left and right eigenvectors associated with λ are

$$\mathbf{l} = \frac{1}{2}\left((p_A/A)^{1/2}, 1\right), \qquad \mathbf{r} = \begin{pmatrix} (A/p_A)^{1/2} \\ 1 \end{pmatrix}. \tag{2.7.20}$$

In particular, $\mathbf{lr} = 1$ is satisfied and

$$\mathbf{l}_0 = \frac{1}{2}\left((A_0\varphi_0)^{-1/2}, 1\right), \qquad \mathbf{r}_0 = \begin{pmatrix} (A_0\varphi_0)^{1/2} \\ 1 \end{pmatrix}. \tag{2.7.21}$$

We Taylor expand $(1/\lambda)$ to get

$$\frac{1}{\lambda} = \frac{1}{\lambda_0} + \Lambda^{(1)}(\mathbf{u}) + \frac{1}{2}\Lambda^{(2)}(\mathbf{u}, \mathbf{u}) + O(\|\mathbf{u}\|^3), \tag{2.7.22}$$

where

$$\Lambda^{(1)}(\mathbf{u}) = \left((A_0/\varphi_0)^{-3/2}\varphi_0^{-1},\ -(A_0/\varphi_0)^{-1}\right)\begin{pmatrix} \bar{A} \\ u \end{pmatrix}, \tag{2.7.23}$$

$$\Lambda^{(2)}(\mathbf{u}, \mathbf{u}) = (\bar{A}, u)\begin{pmatrix} -3\beta\varphi_0^{-2}(A_0/\varphi_0)^{-5/2} & -2\varphi_0^{-1}(A_0/\varphi_0)^{-2} \\ -2\varphi_0^{-1}(A_0/\varphi_0)^{-2} & 2(A_0/\varphi_0)^{-3/2} \end{pmatrix}\begin{pmatrix} \bar{A} \\ u \end{pmatrix}. \tag{2.7.24}$$

We then have

$$\begin{aligned} \Lambda^{(1)}(\mathbf{r}_0) &= \left((A_0\varphi_0)^{-3/2}\varphi_0^{-1},\ -(A_0/\varphi_0)^{-1}\right)\begin{pmatrix} (A_0\varphi_0)^{1/2} \\ 1 \end{pmatrix} \\ &\equiv 0, \end{aligned} \tag{2.7.25}$$

so that the order of local linear degeneracy is one and with $\delta = \varepsilon^2$ we have the arrival time formula

$$t = \int_0^x \frac{ds}{\lambda_0(s)} + \varepsilon^2\left\{\theta + \frac{1}{2}\sigma_0^2\int_0^x \langle\Lambda_2\rangle ds\right\} + O(\varepsilon^3), \tag{2.7.26}$$

where $\langle\Lambda_2\rangle = \Lambda^{(2)}(\mathbf{r}_0, \mathbf{r}_0) + \Lambda^{(1)}(\mathbf{r}_1)$.

It remains to find $\mathbf{r}_1$, a particular solution of

$$\left(\mathbf{I} - \frac{\mathbf{A}_0}{\lambda_0}\right)\mathbf{r}_1 = \frac{1}{\lambda_0}\mathbf{A}^{(1)}(\mathbf{r}_0)\mathbf{r}_0, \tag{2.7.27}$$

wherein

$$\mathbf{A}^{(1)}(\mathbf{r}_0)\mathbf{r}_0 = \begin{pmatrix} r_0^{(1)}(\partial_A u)_0 + r_0^{(2)}(\partial_u u)_0 & r_0^{(1)}(\partial_A A)_0 + r_0^{(2)}(\partial_u A)_0 \\ r_0^{(1)}(\partial_A p_A)_0 + r_0^{(2)}(\partial_u p_A)_0 & r_0^{(1)}(\partial_A u)_0 + r_0^{(2)}(\partial_u u)_0 \end{pmatrix} \begin{pmatrix} r_0^{(1)} \\ r_0^{(2)} \end{pmatrix}$$

$$= \begin{pmatrix} 2(A_0\varphi_0)^{1/2} \\ -2 \end{pmatrix}. \qquad (2.7.28)$$

A direct check shows that

$$\mathbf{r}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} 2\varphi_0 \qquad (2.7.29)$$

is a particular solution.

It remains for us to "normalize" $\mathbf{r}_0$. Now

$$\Gamma_0(x) = \frac{1}{\lambda_0} c^{(1)}(\mathbf{r}_0) - \mathbf{l}_0 \mathbf{r}_0' = -\frac{3}{4}\frac{A_0'}{A_0} + \frac{1}{4}\frac{\varphi_0'}{\varphi_0}$$

so that

$$\exp\left\{\int_0^x \Gamma_0(s)\,ds\right\} = \left(\frac{A_0}{A_0(0)}\right)^{-3/4} \left(\frac{\varphi_0}{\varphi_0(0)}\right)^{1/4}$$

and hence $\mathbf{r}_0$ is normalized to

$$\mathbf{r}_0 = \begin{pmatrix} (A_0\varphi_0)^{1/2} \\ 1 \end{pmatrix} \left(\frac{A_0}{A_0(0)}\right)^{-3/4} \left(\frac{\varphi_0}{\varphi_0(0)}\right)^{1/4}, \qquad (2.7.30)$$

and

$$\mathbf{r}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} 2\varphi_0 \left(\frac{A_0}{A_0(0)}\right)^{-3/2} \left(\frac{\varphi_0}{\varphi_0(0)}\right)^{1/2}. \qquad (2.7.31)$$

Therefore

$$\langle \Lambda_2 \rangle = \Lambda^{(2)}(\mathbf{r}_0, \mathbf{r}_0) + \Lambda^{(1)}(\mathbf{r}_1) = -3\beta \left(\frac{\varphi_0}{A_0}\right)^{-3/2} \left(\frac{A_0}{A_0(0)}\right)^{-3/2} \left(\frac{\varphi_0}{\varphi_0(0)}\right)^{1/2}, \qquad (2.7.32)$$

which is precisely the same result as we found in [17]. Substituting (2.7.32) and $\sigma_0(\theta) = 2^{-1/2}g(\theta)$ into (2.7.26) results in the arrival time formula in (2.7.12) as required.

2.7.2 *Shock waves*

In the situation when a shock wave is induced at the shock-initiation point, the shock front is given by (2.7.13), (2.7.14). In the following discussion we make an example calculation and show that for the particular boundary disturbance function chosen in [64], a shock wave will be generated.

To be specific, we take a particular strain energy function [65]

$$W(I_1, I_2) = b(I_1 - 3) + (1 - b)(I_2 - 3) + \gamma(I_1 - 3)^2, \tag{2.7.33}$$

where b and γ are real constants. Then it follows from (2.7.5), (2.7.8) that

$$\varphi_0 \equiv 1/2, \qquad \beta = 2\gamma, \tag{2.7.34}$$

where $A_0 \equiv 1$ because of homogeneity. Choosing $\gamma = 1$ the arrival time formula reduces to

$$t = \frac{x}{\sqrt{2}} + \epsilon^2 \left\{ \theta - \left(\frac{3}{\sqrt{2}} \right) g^2(\theta) x \right\}, \tag{2.7.35}$$

where the $O(\epsilon^3)$ term is omitted for simplicity.

Apparently (2.7.35) represents a family of characteristics. The caustic, that is, the envelope formed by this family is defined by

$$x = \left(\frac{2\sqrt{2}}{3} \right) \Big/ g(\theta) g'(\theta), \tag{2.7.36}$$

$$t = \left(\frac{2}{3} \right) (g(\theta) g'(\theta))^{-1} + \epsilon^2 \left\{ \theta - \frac{g(\theta)}{2g'(\theta)} \right\}. \tag{2.7.37}$$

The boundary disturbance function chosen in [64] is

$$g(\theta) = \alpha\theta(1+\theta^2)e^{-\theta^2}, \quad 0 \le \theta < 1, \tag{2.7.38}$$

where α is a nondimensional amplitude parameter and $g(\theta)$ is cut off and appropriately smoothed for large θ so as to be of compact support.

It was shown [64] that the shock-initiation point (x_s, t_s) is given by

$$x_s \doteq (2.1466)\alpha^{-2}, \tag{2.7.39}$$

$$t_s \doteq (0.5179)\alpha^{-2} + \epsilon^2(0.2309), \tag{2.7.40}$$

with $\theta_s \doteq 0.5724$. The envelopes (2.7.36), (2.7.37) are generated and displayed in Figs. 2.5–2.7 below for different choices of α. These figures show that (x_s, t_s) is a cusp. The arrow indicates the direction in which θ is increasing and hence we see that Γ_1 lies above Γ_2. By the *Shock Criterion*, a shock wave is predicted and the shock front takes the form

$$\frac{dt}{dx} = \frac{1}{\sqrt{2}} - \frac{\epsilon^2}{8}\frac{[g^3]}{[g]}, \tag{2.7.41}$$

$$t(x_s) = t_s, \tag{2.7.42}$$

with $[g^3] = g^3(\theta_\ell) - g^3(\theta_r)$, $[g] = g(\theta_\ell) - g(\theta_r)$ and θ_ℓ, θ_r $(\theta_\ell > \theta_s > \theta_r > 0)$ solving

$$t = \frac{x}{\sqrt{2}} - \epsilon^2\left\{\theta - \frac{3}{4\sqrt{2}}g^2(\theta)x\right\}, \quad \forall (x,t) \in \mathcal{D}.$$

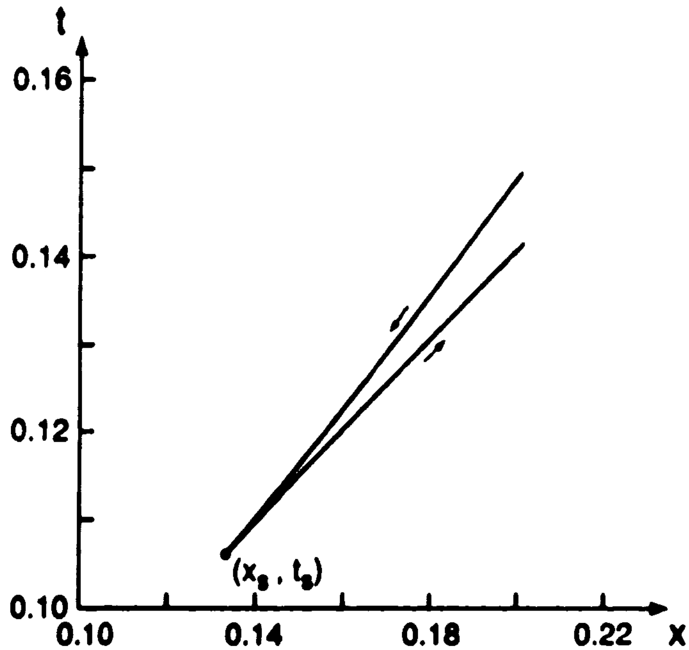

Fig. 2.5: Characteristic envelope for $\alpha = 4$, $\epsilon^2 = 0.05$, and $0.1 < \theta < 0.8$.

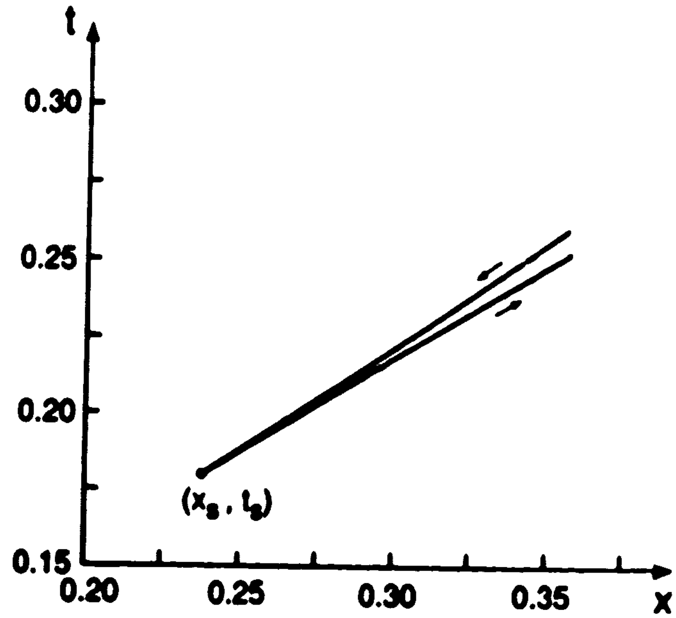

Fig. 2.6: Characteristic envelope for $\alpha = 3$, $\epsilon^2 = 0.05$, and $0.1 < \theta < 0.8$.

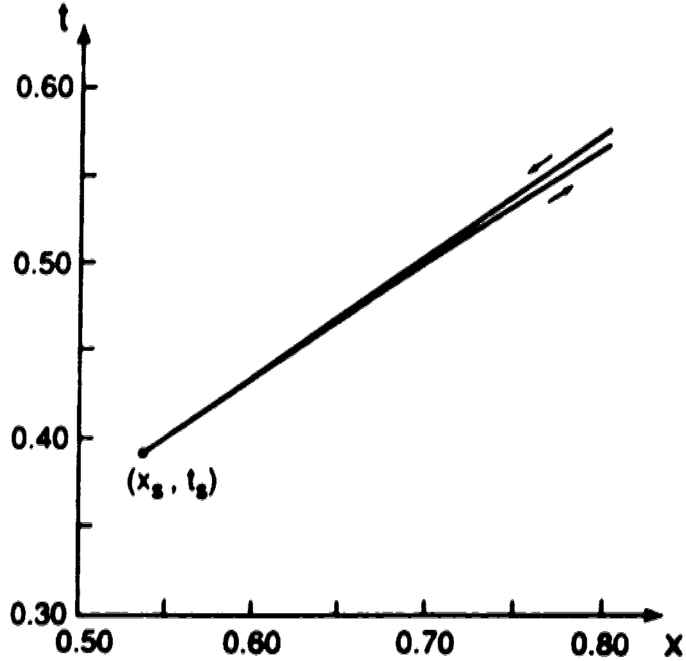

Fig. 2.7: Characteristic envelope for $\alpha = 2$, $\epsilon^2 = 0.05$, and $0.1 < \theta < 0.8$.

CHAPTER 3.

Two Wave Interactions for Weakly Nonlinear Hyperbolic Waves

It is well known that nonlinear hyperbolic waves whose propagation is governed by systems of conservation laws can be decomposed into a linear combination of wave modes, that is, a linear combination of eigenvectors associated with each characteristic speed. As time evolves, these wave modes tend to interact with each other as well as undergo self interaction. This process is nonlinear and may lead to complicated wave pattern. A detailed analysis of these interactions is essential for a deeper understanding of systems of conservation laws. For instance, the convergence of the celebrated Glimm scheme is based upon an estimate of the wave-wave interaction through a nonlinear functional in which the wave decomposition is carried out by employing Lax's construction of the solution to the Riemann problem (Smoller [70]).

In this chapter we focus our attention on two-wave interactions for hyperbolic systems of conservation laws. By two-wave interactions here we mean the non-resonant evolving pattern of a propagating disturbance arising from the action of initial or boudary disturbances consisting of two wave modes. We take an approach which is close in spirit to the use of Riemann invariants in the early studies of Lin [44] and Fox [15]. A summary of this approach is to be found in Kluwick [33] and also Nayfeh [66]. The reader is referred to Kevorkian and Cole [32] for more applications(We point out that the comments in [32] about the Lin-Fox method is unfounded as the argument made there is flawed).

Here we introduce directly the nonlinear phases instead of attempting to deal with Riemann invariants, which can be found with certainty only for 2×2 systems. Our asymptotic analysis is carried out in the phase coordinates rather than in spatial-temporal coordinates. This involves a nonlinear phase transformation from spatial-temporal coordinates to phase coordinates. We restrict our analysis here to two-wave interactions because for multi-waves (more than two) it is not clear how to select an appropriate set of phases to replace (x, t) coordinates. Furthermore, the nonlinearity imposes restrictions and couplings between the nonlinear phases whose nature is hard to ascertain. No doubt this difficulty is related to the *completeness, coherence and nondegeneracy* condition that appears–and some what restricts the applicability–in the analysis of [26].

However, for systems of hyperbolic conservation laws in one space dimension, when the number of nonlinear phases involved is restricted to two, an asymptotic theory for the nonresonant two-wave interactions can be established in a general setting.

We shall deal with systems of hyperbolic conservation laws in one space dimension which have a slightly different form from those studied in the previous chapter and which exhibit no explicit spatial dependence in their flux functions. Specifically, we deal with

$$\mathbf{G}(\mathbf{u})_t + \mathbf{F}(\mathbf{u})_x = 0, \tag{3.0.1}$$

where $\mathbf{u} = \mathbf{u}(x, t)$ is the vector of n state variables, $\mathbf{G}(\mathbf{u})$, $\mathbf{F}(\mathbf{u})$ are smooth vector-valued functions of $\mathbf{u}$, while x and t, as usual, represent space and time variables, respectively.

The system (3.0.1) is always assumed to admit a steady state solution, which, without loss of generality, we take to be $\mathbf{u} \equiv \mathbf{0}$. For smooth solutions, the system (3.0.1) can be written as

$$\mathbf{B}\mathbf{u}_t + \mathbf{A}\mathbf{u}_x = 0, \tag{3.0.2}$$

where $\mathbf{A} = (\partial F_i/\partial u_j)_{n\times n} \triangleq (a_{ij})_{n\times n}$ and $\mathbf{B} = (\partial G_i/\partial u_j)_{n\times n} \triangleq (b_{ij})_{n\times n}$. We further require that (3.0.1) be strictly hyperbolic, that is, $\mathbf{B}$ is nonsingular and

$$\det(\mathbf{A} - \lambda\mathbf{B}) = 0, \tag{3.0.3}$$

has n distinct real roots $\{\lambda_i(\mathbf{u})\}_{i=1}^{n}$. In particular, we assume positive and negative eigenvalues are distinguished, that is,

$$\lambda_1(\mathbf{u}) < \lambda_2(\mathbf{u}) < \cdots < \lambda_m(\mathbf{u}) \leq 0 < \lambda_{m+1}(\mathbf{u}) < \cdots < \lambda_n(\mathbf{u}). \tag{3.0.4}$$

For each i $(1 \leq i \leq n)$, we denote by $\boldsymbol{\ell}_i$ and $\mathbf{r}_i$ the left and right eigenvectors associated with $\lambda_i^{(0)} \triangleq \lambda_i(\mathbf{0})$, respectively, that is,

$$\boldsymbol{\ell}_i(\mathbf{A}_0 - \lambda_i^{(0)}\mathbf{B}_0) = \mathbf{0}, \; (\mathbf{A}_0 - \lambda_i^{(0)}\mathbf{B}_0)\mathbf{r}_i = \mathbf{0}, \tag{3.0.5}$$

where $\mathbf{A}_0 \triangleq \mathbf{A}(\mathbf{0})$, $\mathbf{B}_0 \triangleq \mathbf{B}(\mathbf{0})$. Also, we require the orthonormality condition

$$\boldsymbol{\ell}_i\mathbf{B}_0\mathbf{r}_j = \delta_{ij}; \; i,j = 1,2,\ldots,n, \tag{3.0.6}$$

where δ_{ij} is the Kronecker symbol. In addition, each right eigenvector $\mathbf{r}_i$ is called a *wave mode* of the system (3.0.1) (at the steady state $\mathbf{u} = \mathbf{0}$).

The remainder of this chapter has the following organization. In Section 1, we consider the initial problem in which the system of hyperbolic conservation laws (3.0.1) is perturbed initially from its base steady state by two superimposed wave modes. After introducing the nonlinear phases and carrying out the nonlinear phase transformation, we construct an explicit asymptotic solution together with the perturbed spatial-temporal coordinates. The next section is then devoted to the signaling problem. We perturb the base steady state by superimposing two wave modes on the boundary. The corresponding nonlinear phases are introduced and the asymptotic solution constructed along with perturbed space-time coordinates. In particular, for each wave mode a *separation line* is defined to distinguish the path of integration along each bicharacteristic as well as the boundary separating ranges of influence. As an application of the results in Sections 1 and 2, the last section includes a study of gas dynamics in one space dimension. We compute the propagation and interaction of two weak sound waves generated initially as well as on the boundary.

3.1. The Initial Value Problem

3.1.1 *Problem defined*

We perturb the system of hyperbolic conservation laws

$$\mathbf{G}(\mathbf{u})_t + \mathbf{F}(\mathbf{u})_x = 0, \tag{3.0.1}$$

from its base steady state $\mathbf{u} \equiv \mathbf{0}$ by imposing an initial disturbance of the form

$$\mathbf{u}\Big|_{t=0} = \varepsilon\sigma_p(x)\mathbf{r}_p + \varepsilon\sigma_q(x)\mathbf{r}_q, \ -\infty < x < \infty, \tag{3.1.1}$$

where ε is the small perturbation parameter, and $\sigma_p(\cdot)$, $\sigma_q(\cdot)$ are smooth scalar functions. The subscripts p, q are any fixed pair of integers satisfying $1 \le p < q \le n$.

3.1.2 *Nonlinear phase transformation and asymptotic expansion*

We introduce two nonlinear phase variables $\theta_p = \theta_p(x,t)$, $\theta_q = \theta_q(x,t)$ defining them as solutions to

$$(\theta_p)_t + \lambda_p(\mathbf{u})(\theta_p)_x = 0, \tag{3.1.2}$$

$$\theta_p\Big|_{t=0} = x, \tag{3.1.3}$$

and

$$(\theta_q)_t + \lambda_q(\mathbf{u})(\theta_q)_x = 0, \tag{3.1.4}$$

$$(\theta_q)\Big|_{t=0} = x, \tag{3.1.5}$$

respectively.

The space-time coordinates can be transformed into nonlinear phase coordinates

$$(x,t) \longmapsto (\theta_p, \theta_q), \tag{3.1.6}$$

by

$$x = x(\theta_p, \theta_q), \; t = t(\theta_p, \theta_q), \tag{3.1.7}$$

where $x(\theta_p, \theta_q)$ and $t(\theta_p, \theta_q)$ are determined from the relations given in (3.1.2)-(3.1.5).

We differentiate (3.1.7) with respect to x and t taking (3.1.2) and (3.1.4) into account to obtain

$$1 = x_{\theta_p}(\theta_p)_x + x_{\theta_q}(\theta_q)_x, \tag{3.1.8a}$$

$$0 = x_{\theta_p}[-\lambda_p(\theta_p)_x] + x_{\theta_q}[-\lambda_q(\theta_q)_x], \tag{3.1.8b}$$

and

$$0 = t_{\theta_p}(\theta_p)_x + t_{\theta_q}(\theta_q)_x, \tag{3.1.9a}$$

$$1 = t_{\theta_p}[-\lambda_p(\theta_p)_x] + t_{\theta_q}[-\lambda_q(\theta_q)_x], \tag{3.1.9b}$$

which, in turn, provide

$$x_{\theta_p} = \lambda_q(\lambda_q - \lambda_p)^{-1}/(\theta_p)_x, \; x_{\theta_q} = \lambda_p(\lambda_p - \lambda_q)^{-1}/(\theta_q)_x, \tag{3.1.10}$$

$$t_{\theta_p} = (\lambda_q - \lambda_p)^{-1}/(\theta_p)_x, \; t_{\theta_q} = (\lambda_p - \lambda_q)^{-1}/(\theta_q)_x. \tag{3.1.11}$$

It therefore follows from the above that

$$x_{\theta_p} - \lambda_q t_{\theta_p} = 0, \tag{3.1.12}$$

$$x_{\theta_q} - \lambda_p t_{\theta_q} = 0. \tag{3.1.13}$$

That is,

$$\frac{dx}{dt} = \lambda_q \text{along} \theta_q = \text{constant}, \frac{dx}{dt} = \lambda_p \text{along} \theta_p = \text{constant}.$$

The validity of the nonlinear phase transformation (3.1.6) depends upon the condition that the Jacobian

$$\frac{\partial(\theta_p, \theta_q)}{\partial(x,t)} = (\theta_p)_x(\theta_q)_x(\lambda_p - \lambda_q) \neq 0, \tag{3.1.14}$$

and that

$$t_{\theta_p},\ t_{\theta_q} \neq 0. \tag{3.1.15}$$

The condition (3.1.15) follows from (3.1.11) and the requirement that the Jacobian cannot become infinite.

Since the initial disturbance (3.1.1) is always assumed smooth, the theory of nonlinear hyperbolic equations [42, 58] ensures the existence and smoothness of $\mathbf{u}(x,t)$ and hence the existence and smoothness of $\theta_p = \theta_p(x,t)$ and $\theta_q = \theta_q(x,t)$. Throughout the derivation process in this chapter we shall tacitly assume, as in [62], that $\mathbf{u}$ remains a smooth function (of x,t). Indeed, as the work of F. John [29–30] suggests, $\mathbf{u}$ remains smooth at least up to $O(\varepsilon^{-1})$.

Now we consider $\mathbf{u}$ and the related (matrix, vector, and scalar) functions to be functions of the new phase variables (θ_p, θ_q) with the change of derivatives according to

$$\partial_x \longmapsto (\theta_p)_x\partial_{\theta_p} + (\theta_q)_x\partial_{\theta_q}, \tag{3.1.16a}$$

$$\partial_t \longmapsto (\theta_p)_t\partial_{\theta_p} + (\theta_q)_t\partial_{\theta_q}, \tag{3.1.16b}$$

or, upon using (3.1.10), (3.1.11), as

$$\partial_x \longmapsto (\lambda_q - \lambda_p)^{-1}t_{\theta_p}^{-1}\partial_{\theta_p} + (\lambda_p - \lambda_q)^{-1}t_{\theta_q}^{-1}\partial_{\theta_q}, \tag{3.1.17a}$$

$$\partial_t \longmapsto \lambda_p(\lambda_p - \lambda_q)^{-1}t_{\theta_p}^{-1}\partial_{\theta_p} + \lambda_q(\lambda_q - \lambda_p)^{-1}t_{\theta_q}^{-1}\partial_{\theta_q}. \tag{3.1.17b}$$

The governing equation (3.0.2) is then transformed to

$$t_{\theta_p}(\mathbf{A} - \lambda_q \mathbf{B})\mathbf{u}_{\theta_q} - t_{\theta_q}(\mathbf{A} - \lambda_p \mathbf{B})\mathbf{u}_{\theta_p} = \mathbf{0}. \tag{3.1.18}$$

Now, based upon (3.1.12), (3.1.13), and (3.1.18), which constitute the full system transformed into (θ_p, θ_q) coordinates, we shall seek an asymptotic solution of (3.0.1) in the form

$$\mathbf{u} = \epsilon \mathbf{u}^{(0)}(\theta_p, \theta_q) + \epsilon^2 \mathbf{u}^{(1)}(\theta_p, \theta_q) + O(\epsilon^3), \tag{3.1.19}$$

when (3.0.1) is subject to an initial disturbance having the particular structure of (3.1.1). Also, the space-time coordinates are assumed to be perturbed in the form

$$x = x^{(0)}(\theta_p, \theta_q) + \epsilon x^{(1)}(\theta_p, \theta_q) + O(\epsilon^2), \tag{3.1.20}$$

$$t = t^{(0)}(\theta_p, \theta_q) + \epsilon t^{(1)}(\theta_p, \theta_q) + O(\epsilon^2). \tag{3.1.21}$$

Throughout our discussion, the nonlinear phase variables θ_p, θ_q will be considered as being independent of the perturbation parameter ϵ.

In order to construct the asymptotic solution (3.1.19)–(3.1.21), we first expand (3.1.12), (3.1.13), and (3.1.18) to formulate the requisite $O(1)$ and $O(\epsilon)$ problems. Thus, in order to implement this procedure, we first need to expand the following (matrix and scalar) functions about the steady state $\mathbf{u} = \mathbf{0}$:

$$\mathbf{A}(\mathbf{u}) = \mathbf{A}_0 + \mathbf{A}^{(1)}(\mathbf{u}) + \frac{1}{2!}\mathbf{A}^{(2)}(\mathbf{u}, \mathbf{u}) + O(\|\mathbf{u}\|^3), \tag{3.1.22}$$

$$(\lambda_i \mathbf{B})(\mathbf{u}) \triangleq \mathbf{B}_i(\mathbf{u}) = \mathbf{B}_i^{(0)} + \mathbf{B}_i^{(1)}(\mathbf{u}) + \frac{1}{2!}\mathbf{B}_i^{(2)}(\mathbf{u}, \mathbf{u}) + O(\|\mathbf{u}\|^3), \tag{3.1.23}$$

$$\lambda_i(\mathbf{u}) = \lambda_i^{(0)} + \lambda_i^{(1)}(\mathbf{u}) + \frac{1}{2!}\lambda_i^{(2)}(\mathbf{u}, \mathbf{u}) + O(\|\mathbf{u}\|^3), \tag{3.1.24}$$

$$\mathbf{B}(\mathbf{u}) = \mathbf{B}_0 + \mathbf{B}^{(1)}(\mathbf{u}) + \frac{1}{2!}\mathbf{B}^{(2)}(\mathbf{u}, \mathbf{u}) + O(\|\mathbf{u}\|^3), \tag{3.1.25}$$

where

$$
\begin{aligned}
\mathbf{A}^{(k)}(\mathbf{u},\ldots,\mathbf{u}) &= \left(\sum_{k_1+\cdots+k_n=k} \left(\partial^k a_{ij}/\partial u_1^{k_1}\ldots\partial u_n^{k_n}\right)_0 u_1^{k_1}\ldots u_n^{k_n} \right)_{n\times n}, \\
\mathbf{B}_i^{(k)}(\mathbf{u},\ldots,\mathbf{u}) &= \left(\sum_{k_1+\cdots+k_n=k} \left(\partial^k (\lambda_i b_{jm})/\partial u_1^{k_1}\ldots\partial u_n^{k_n}\right)_0 u_1^{k_1}\ldots u_n^{k_n} \right)_{n\times n}, \\
\lambda_i^{(k)}(\mathbf{u},\ldots,\mathbf{u}) &= \sum_{k_1+\cdots+k_n=k} \left(\partial^k \lambda_i/\partial u_1^{k_1}\ldots\partial u_n^{k_n}\right)_0 u_1^{k_1}\ldots u_n^{k_n}, \\
\mathbf{B}^{(k)}(\mathbf{u},\ldots,\mathbf{u}) &= \left(\sum_{k_1+\cdots+k_n=k} \left(\partial^k b_{ij}/\partial u_1^{k_1}\ldots\partial u_n^{k_n}\right)_0 u_1^{k_1}\ldots u_n^{k_n} \right)_{n\times n},
\end{aligned}
$$

are matrix and scalar valued k-linear forms.

We now substitute (3.1.19) into (3.1.22)–(3.1.25) and regroup like powers of ε obtaining

$$\mathbf{A}(\mathbf{u}) = \varepsilon\mathbf{A}_0 + \varepsilon^2\mathbf{A}_1 + O(\varepsilon^3), \tag{3.1.26}$$

$$\mathbf{B}_i(\mathbf{u}) = \varepsilon\mathbf{B}_{i,0} + \varepsilon^2\mathbf{B}_{i,1} + O(\varepsilon^3), \tag{3.1.27}$$

$$\lambda_i(\mathbf{u}) = \varepsilon\lambda_{i,0} + \varepsilon\lambda_{i,1} + O(\varepsilon^3), \tag{3.1.28}$$

$$\mathbf{B}(\mathbf{u}) = \varepsilon\mathbf{B}_{0,0} + \varepsilon^2\mathbf{B}_{0,1} + O(\varepsilon^3), \tag{3.1.29}$$

with

$$
\begin{aligned}
\mathbf{A}_0 &= \mathbf{A}(0), \\
\mathbf{A}_1 &= \mathbf{A}^{(1)}(\mathbf{u}^{(0)}), \\
\mathbf{A}_2 &= \mathbf{A}^{(1)}(\mathbf{u}^{(1)}) + \frac{1}{2!}\mathbf{A}^{(2)}(\mathbf{u}^{(0)}, \mathbf{u}^{(0)}),
\end{aligned} \tag{3.1.30}
$$

$$
\begin{aligned}
\mathbf{B}_{i,0} &= \mathbf{B}_i(0), \\
\mathbf{B}_{i,1} &= \mathbf{B}_i^{(1)}(\mathbf{u}^{(0)}), \\
\mathbf{B}_{i,2} &= \mathbf{B}_i^{(1)}(\mathbf{u}^{(1)}) + \frac{1}{2!}\mathbf{B}_i^{(2)}(\mathbf{u}^{(0)}, \mathbf{u}^{(0)}),
\end{aligned} \tag{3.1.31}
$$

$$
\begin{aligned}
\lambda_{i,0} &= \lambda_i(0), \\
\lambda_{i,1} &= \lambda_i^{(1)}(\mathbf{u}^{(0)}), \\
\lambda_{i,2} &= \lambda_i^{(1)}(\mathbf{u}^{(1)}) + \frac{1}{2!}\lambda^{(2)}(\mathbf{u}^{(0)}, \mathbf{u}^{(0)}),
\end{aligned} \tag{3.1.32}
$$

$$
\begin{aligned}
\mathbf{B}_{0,j} &= \mathbf{B}(0), \\
\mathbf{B}_{1,j} &= \mathbf{B}^{(1)}(\mathbf{u}^{(0)}), \\
\mathbf{B}_{2,j} &= \mathbf{B}^{(1)}(\mathbf{u}^{(1)}) + \frac{1}{2!}\mathbf{B}^{(2)}(\mathbf{u}^{(0)}, \mathbf{u}^{(0)}),
\end{aligned} \tag{3.1.33}
$$

where the relation connecting $\mathbf{B}_{i,j}$ and $\mathbf{B}_{0,j}$ is given by

$$
\mathbf{B}_{i,j} = \sum_{j_1+j_2=j} \lambda_{i,j_1}\mathbf{B}_{0,j_2}. \tag{3.1.34}
$$

We now insert (3.1.26)–(3.1.29) together with (3.1.19)–(3.1.21) into the transformed system (3.1.12), (3.1.13), and (3.1.18), and equate like powers of ϵ to obtain the $O(1)$ and $O(\epsilon)$ problems.

$O(1)$ problem:

$$
t_{\theta_p}^{(0)}(\mathbf{A}_0 - \lambda_q^{(0)}\mathbf{B}_0)\mathbf{u}_{\theta_q}^{(0)} - t_{\theta_q}^{(0)}(\mathbf{A}_0 - \lambda_p^{(0)}\mathbf{B}_0)\mathbf{u}_{\theta_p}^{(0)} = 0, \tag{3.1.35}
$$

$$
(x^{(0)} - \lambda_p^{(0)}t^{(0)})_{\theta_q} = 0, \tag{3.1.36}
$$

$$
(x^{(0)} - \lambda_q^{(0)}t^{(0)})_{\theta_p} = 0. \tag{3.1.37}
$$

$O(\varepsilon)$ problem :

$$t^{(0)}_{\theta_p}(\mathbf{A}_0 - \lambda^{(0)}_q \mathbf{B}_0)\mathbf{u}^{(1)}_{\theta_q} - t^{(0)}_{\theta_q}(\mathbf{A}_0 - \lambda^{(0)}_p \mathbf{B}_0)\mathbf{u}^{(1)}_{\theta_p} = \mathbf{M}_1, \tag{3.1.38}$$

$$(x^{(1)} - \lambda^{(0)}_p t^{(1)})_{\theta_q} = H_{p,1}, \tag{3.1.39}$$

$$(x^{(1)} - \lambda^{(0)}_q t^{(1)})_{\theta_p} = H_{q,1}, \tag{3.1.40}$$

where

$$\mathbf{M}_1 = \sum_{k_1+k_2=1} \left\{ t^{(k_1)}_{\theta_q} (\mathbf{A}_{k_2} - \mathbf{B}_{p,k_2}) \mathbf{u}^{(0)}_{\theta_p} - t^{(k_1)}_{\theta_p} (\mathbf{A}_{k_2} - \mathbf{B}_{q,k_2}) \mathbf{u}^{(0)}_{\theta_q} \right\}, \tag{3.1.41}$$

$$H_{p,1} = \lambda_{p,1} t^{(0)}_{\theta_q}, \tag{3.1.42}$$

$$H_{q,1} = \lambda_{q,1} t^{(0)}_{\theta_p}. \tag{3.1.43}$$

We are now ready to solve the $O(1)$ and $O(\varepsilon)$ problems corresponding to an initial disturbance of the form

$$\mathbf{u}\big|_{t=0} = \varepsilon\sigma_p(x)\mathbf{r}_p + \varepsilon\sigma_q(x)\mathbf{r}_q. \tag{3.1.1}$$

Before doing this, we note that the parametrization of the nonlinear phases θ_p, θ_q on the x-axis, that is, (3.1.3) and (3.1.5) indicate that

$$t = 0, \ x = \theta_p = \theta_q. \tag{3.1.44}$$

This condition can be interpreted, after using (3.1.21) and (3.1.22), as

$$t^{(k)}(\theta_p, \theta_p) = t^{(k)}(\theta_q, \theta_q) = 0, \ k = 0, 1, 2, \ldots, \tag{3.1.45a}$$

$$x^{(0)}(\theta_p, \theta_p) = \theta_p, \ x^{(0)}(\theta_q, \theta_q) = \theta_q, \tag{3.1.45b}$$

$$x^{(k)}(\theta_p, \theta_p) = x^{(k)}(\theta_q, \theta_q) = 0, \ k = 1, 2, \ldots. \tag{3.1.45c}$$

We are now in a position to solve the $O(1)$ problem.

3.1.3 *Solution of the $O(1)$ problem*

First we integrate (3.1.36) and (3.1.37) over θ_q and θ_p, respectively, obtaining

$$x^{(0)} - \lambda_p^{(0)} t^{(0)} = f(\theta_p),$$
$$x^{(0)} - \lambda_q^{(0)} t^{(0)} = g(\theta_q).$$

Applying (3.1.45) gives

$$x^{(0)} - \lambda_p^{(0)} t^{(0)} = \theta_p, \; x^{(0)} - \lambda_q^{(0)} t^{(0)} = \theta_q, \tag{3.1.46}$$

or

$$x^{(0)} = \lambda_p^{(0)}(\lambda_p^{(0)} - \lambda_q^{(0)})^{-1}\theta_q + \lambda_q^{(0)}(\lambda_q^{(0)} - \lambda_p^{(0)})^{-1}\theta_p, \tag{3.1.47}$$
$$t^{(0)} = (\lambda_p^{(0)} - \lambda_q^{(0)})^{-1}\theta_q + (\lambda_q^{(0)} - \lambda_p^{(0)})^{-1}\theta_p. \tag{3.1.48}$$

Thus (3.1.35) reduces to

$$(\mathbf{A}_0 - \lambda_q^{(0)}\mathbf{B}_0)\mathbf{u}_{\theta_q}^{(0)} + (\mathbf{A}_0 - \lambda_p^{(0)}\mathbf{B}_0)\mathbf{u}_{\theta_p}^{(0)} = 0. \tag{3.1.49}$$

Denoting $\mathbf{u}^{(0)} = \sum_{i=1}^{n} \sigma_i^{(0)}(\theta_p, \theta_q)\mathbf{r}_i$, and substituting it into (3.1.49), we obtain

$$\sum_{i=1}^{n} \left\{ (\lambda_i^{(0)} - \lambda_q^{(0)}) \frac{\partial \sigma_i^{(0)}}{\partial \theta_q} + (\lambda_i^{(0)} - \lambda_p^{(0)}) \frac{\partial \sigma_i^{(0)}}{\partial \theta_p} \right\} \mathbf{B}_0 \mathbf{r}_i = 0,$$

or

$$(\lambda_i^{(0)} - \lambda_q^{(0)}) \frac{\partial \sigma_i^{(0)}}{\partial \theta_q} + (\lambda_i^{(0)} - \lambda_p^{(0)}) \frac{\partial \sigma_i^{(0)}}{\partial \theta_p} = 0, \; i = 1, 2, \ldots, n. \tag{3.1.50}$$

The general solutions to (3.1.50) possess the form

$$\sigma_i^{(0)}(\theta_p,\theta_q) = f_i((\lambda_i^{(0)} - \lambda_p^{(0)})\theta_q - (\lambda_i^{(0)} - \lambda_q^{(0)})\theta_p),\ \forall\ 1 \le i \le n,\ i \ne p,q, \tag{3.1.51a}$$

$$\sigma_p^{(0)}(\theta_p,\theta_q) = f_p(\theta_p), \tag{3.1.51b}$$

$$\sigma_q^{(0)}(\theta_p,\theta_q) = f_q(\theta_q), \tag{3.1.51c}$$

where $f_i(\cdot)$ $(i = 1,2,\ldots,n)$ are arbitrary C^1 functions. Applying the initial conditions (3.1.1) it follows immediately that

$$f_i(\cdot) \equiv 0,\ \forall\ 1 \le i \le n,\ i \ne p,q, \tag{3.1.52a}$$

$$f_p(\theta_p) = \sigma_p(\theta_p),\ f_q(\theta_q) = \sigma_q(\theta_q), \tag{3.1.52b}$$

and hence that

$$\mathbf{u}^{(0)}(\theta_p,\theta_q) = \sigma_p(\theta_p)\mathbf{r}_p + \sigma_q(\theta_q)\mathbf{r}_q. \tag{3.1.53}$$

It is clear that $(\mathbf{A}_0 - \lambda_q^{(0)}\mathbf{B}_0)\mathbf{u}_{\theta_q}^{(0)}$ and $(\mathbf{A}_0 - \lambda_p^{(0)}\mathbf{B}_0)\mathbf{u}_{\theta_p}^{(0)}$ vanish separately in (3.1.49).

3.1.4 *Solution of the $O(\epsilon)$ problem*

First we note from (3.1.41) that

$$\begin{aligned}
\mathbf{M}_1 &= t_{\theta_q}^{(0)}(\mathbf{A}_1 - \mathbf{B}_{p,1})\mathbf{u}_{\theta_p}^{(0)} - t_{\theta_p}^{(0)}(\mathbf{A}_1 - \mathbf{B}_{q,1})\mathbf{u}_{\theta_q}^{(0)} \\
&= (\lambda_p^{(0)} - \lambda_q^{(0)})^{-1}\left[\mathbf{A}^{(1)}(\mathbf{u}_0) - \mathbf{B}_p^{(1)}(\mathbf{u}_0)\right]\sigma_p'\mathbf{r}_p \\
&\quad + (\lambda_p^{(0)} - \lambda_q^{(0)})^{-1}\left[\mathbf{A}^{(1)}(\mathbf{u}_0) - \mathbf{B}_q^{(1)}(\mathbf{u}_0)\right]\sigma_q'\mathbf{r}_q \\
&= (\lambda_p^{(0)} - \lambda_q^{(0)})^{-1}\Big\{\left[\mathbf{A}^{(1)}(\mathbf{r}_p) - \mathbf{B}_p^{(1)}(\mathbf{r}_p)\right]\mathbf{r}_p\sigma_p\sigma_p' \\
&\qquad + \left[\mathbf{A}^{(1)}(\mathbf{r}_q) - \mathbf{B}_p^{(1)}(\mathbf{r}_q)\right]\mathbf{r}_p\sigma_q\sigma_p' \\
&\qquad + \left[\mathbf{A}^{(1)}(\mathbf{r}_p) - \mathbf{B}_q^{(1)}(\mathbf{r}_q)\right]\mathbf{r}_q\sigma_p\sigma_q' \\
&\qquad + \left[\mathbf{A}^{(1)}(\mathbf{r}_q) - \mathbf{B}_q^{(1)}(\mathbf{r}_q)\right]\mathbf{r}_q\sigma_q\sigma_q'\Big\}.
\end{aligned} \tag{3.1.54}$$

Similarly, from (3.1.42) and (3.1.43), we obtain

$$\begin{aligned} H_{p,1} &= \lambda_{p,1} t_{\theta_q}^{(0)} \\ &= (\lambda_p^{(0)} - \lambda_q^{(0)})^{-1} \lambda_p^{(1)}(\mathbf{u}^{(0)}) \\ &= (\lambda_p^{(0)} - \lambda_q^{(0)})^{-1} \left\{ \lambda_p^{(1)}(\mathbf{r}_p)\sigma_p + \lambda_p^{(1)}(\mathbf{r}_q)\sigma_q \right\}, \end{aligned} \tag{3.1.55}$$

$$\begin{aligned} H_{q,1} &= \lambda_{q,1} t_{\theta_p}^{(0)} \\ &= (\lambda_q^{(0)} - \lambda_p^{(0)})^{-1} \lambda_q^{(1)}(\mathbf{u}^{(0)}) \\ &= (\lambda_q^{(0)} - \lambda_p^{(0)})^{-1} \left\{ \lambda_q^{(1)}(\mathbf{r}_p)\sigma_p + \lambda_q^{(1)}(\mathbf{r}_q)\sigma_q \right\}. \end{aligned} \tag{3.1.56}$$

Writing

$$\mathbf{u}^{(1)}(\theta_p, \theta_q) = \sum_{s=1}^{n} \sigma_s^{(1)}(\theta_p, \theta_q)\mathbf{r}_s, \tag{3.1.57}$$

and applying, for each $1 \le i \le n$, the left eigenvector ℓ_i to (3.1.38) (with $k = 1$), we reduce the $O(\epsilon)$ problem to

$$\begin{aligned} (\lambda_i^{(0)} - \lambda_q^{(0)})\frac{\partial \sigma_i^{(1)}}{\partial \theta_q} + (\lambda_i^{(0)} - \lambda_p^{(0)})\frac{\partial \sigma_i^{(1)}}{\partial \theta_p} &= \Gamma_{pp}^i \sigma_p(\theta_p)\sigma_p'(\theta_p) + \Gamma_{qp}^i \sigma_q(\theta_q)\sigma_p'(\theta_p) \\ &\quad + \Gamma_{pq}^i \sigma_p(\theta_p)\sigma_q'(\theta_q) + \Gamma_{qq}^i \sigma_q(\theta_q)\sigma_q'(\theta_q),\ 1 \le i \le n, \end{aligned} \tag{3.1.58}$$

$$(x^{(1)} - \lambda_p^{(0)} t^{(1)})_{\theta_q} = (\lambda_p^{(0)} - \lambda_q^{(0)})^{-1} \left\{ \lambda_p^{(1)}(\mathbf{r}_p)\sigma_p(\theta_p) + \lambda_p^{(1)}(\mathbf{r}_q)\sigma_q(\theta_q) \right\}, \tag{3.1.59}$$

$$(x^{(1)} - \lambda_q^{(0)} t^{(1)})_{\theta_p} = (\lambda_q^{(0)} - \lambda_p^{(0)})^{-1} \left\{ \lambda_q^{(1)}(\mathbf{r}_p)\sigma_p(\theta_p) + \lambda_q^{(1)}(\mathbf{r}_q)\sigma_q(\theta_q) \right\}, \tag{3.1.60}$$

wherein Γ_{jk}^i is defined by

$$\Gamma_{jk}^i = \ell_i[\mathbf{A}^{(1)}(\mathbf{r}_j) - \mathbf{B}_k^{(1)}(\mathbf{r}_j)]\mathbf{r}_k,\ 1 \le i, j, k \le n, \tag{3.1.61}$$

and is called, as suggested by Majda and Rosales [62], the (asymmetric) interaction coefficient and represents the contribution to the i-th wave mode due to the nonlinear interaction of the j-th and k-th wave modes. In fact, (3.1.61) is the nonlinear phase version corresponding to the definition introduced in [62] and reduces to that form when λ_k is a constant eigenvalue.

The next lemma states a basic property of Γ^i_{jk}.

LEMMA 3.1. For conservation laws (3.0.1),

$$\Gamma^i_{ji} = 0, \ \forall\ 1 \le i, j \le n. \tag{3.1.62}$$

Proof. Let $\mathbf{L}_i(\mathbf{u})$, $\mathbf{R}_i(\mathbf{u})$ be the left and right eigenvectors associated with $\lambda_i(\mathbf{u})$, that is,

$$\mathbf{L}_i(\mathbf{A} - \mathbf{B}_i) = 0, \ (\mathbf{A} - \mathbf{B}_i)\mathbf{R}_i = 0. \tag{3.1.63}$$

In particular, we require that

$$\mathbf{L}_i(0) = \boldsymbol{\ell}_i, \ \mathbf{R}_i(0) = \mathbf{r}_i \tag{3.1.64}$$

It therefore follows that

$$\begin{aligned} 0 &= \mathbf{L}_i(\mathbf{A} - \mathbf{B}_i)\mathbf{R}_i \\ &= \left\{\boldsymbol{\ell}_i + \mathbf{L}_i^{(1)}(\mathbf{u})\right\}\left\{(\mathbf{A}_0 - \mathbf{B}_{i,0}) + (\mathbf{A}^{(1)} - \mathbf{B}_i^{(1)})(\mathbf{u})\right\}\left\{\mathbf{r}_i + \mathbf{R}_i^{(1)}(\mathbf{u})\right\} \\ &\qquad + O(\|\mathbf{u}\|^2) \\ &= \boldsymbol{\ell}_i[\mathbf{A}^{(1)}(\mathbf{u}) - \mathbf{B}_i^{(1)}(\mathbf{u})]\mathbf{r}_i + O(\|\mathbf{u}\|^2), \end{aligned} \tag{3.1.65}$$

and hence that

$$\boldsymbol{\ell}_i[\mathbf{A}^{(1)}(\mathbf{u}) - \mathbf{B}_i^{(1)}(\mathbf{u})]\mathbf{r}_i = 0, \ \forall\ \mathbf{u} \in \mathbb{R}^n, \ \forall\ 1 \le i \le n. \tag{3.1.66}$$

In particular, we have

$$\Gamma^i_{ji} = \boldsymbol{\ell}_i[\mathbf{A}^{(1)}(\mathbf{r}_j) - \mathbf{B}_i^{(1)}(\mathbf{r}_j)]\mathbf{r}_i = 0, \ \forall\ 1 \le i, j \le n.$$

This completes the proof.

If we use $\hat{\Gamma}^i_{jk}$ to denote the interaction coefficient introduced in [62], then it is straightforward to check that

$$\hat{\Gamma}^i_{jk} = \ell_i[\mathbf{A}^{(1)}(\mathbf{r}_j) - \lambda_k^{(0)}\mathbf{B}_0^{(1)}(\mathbf{r}_j)]\mathbf{r}_k. \tag{3.1.67}$$

The following lemma describes the relation between Γ^i_{jk} and $\hat{\Gamma}^i_{jk}$.

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$$\Gamma^i_{jk} = \hat{\Gamma}^i_{jk} - \lambda_k^{(1)}(\mathbf{r}_j)\delta_{ik}, \ \forall\, 1 \le i,j,k \le n, \tag{3.1.68}$$

where δ_{ik} is the Kronecker symbol.

Remark 3.1. It follows from Lemmas 3.1 and 3.2 that

$$\text{(i).}\quad \Gamma^i_{jk} = \hat{\Gamma}^i_{jk},\ \forall\, 1 \le i,j,k \le n,\ i \ne k, \tag{3.1.69}$$

$$\text{(ii).}\quad \lambda_i^{(1)}(\mathbf{r}_j) = \hat{\Gamma}^i_{ji},\ \forall\, i \le i,j \le n. \tag{3.1.70}$$

Proof. Since

$$\mathbf{B}_k^{(1)}(\mathbf{r}_j) = \lambda_k^{(0)}\mathbf{B}_0^{(1)}(\mathbf{r}_j) + \lambda_k^{(1)}(\mathbf{r}_j)\mathbf{B}_0, \tag{3.1.71}$$

we have

$$\begin{aligned}
\Gamma^i_{jk} &= \ell_i[\mathbf{A}^{(1)}(\mathbf{r}_j) - \mathbf{B}_k^{(1)}(\mathbf{r}_j)]\mathbf{r}_k \\
&= \ell_i[\mathbf{A}^{(1)}(\mathbf{r}_j) - \lambda_k^{(0)}\mathbf{B}_0^{(1)}(\mathbf{r}_j)]\mathbf{r}_k - \lambda_k^{(1)}(\mathbf{r}_j)\ell_0\mathbf{B}_0\mathbf{r}_k \\
&= \hat{\Gamma}^i_{jk} - \lambda_k^{(1)}(\mathbf{r}_j)\delta_{ik}.
\end{aligned}$$

This completes the proof.

Next we integrate (3.1.58) to yield an explicit solution for $\sigma_i^{(1)}(\theta_p, \theta_q)$. This is accomplished using the method of characteristics.

For all $1 \leq i \leq n,\ i \neq p, q$, the bicharacteristic

$$\frac{d\theta_p}{ds} = \lambda_i^{(0)} - \lambda_p^{(0)}, \ \frac{d\theta_q}{ds} = \lambda_i^{(0)} - \lambda_q^{(0)}, \tag{3.1.72}$$

implies that

$$\theta_p = \theta_p^0 + (\lambda_i^{(0)} - \lambda_p^{(0)})s_{pq}, \tag{3.1.73a}$$

$$\theta_q = \theta_q^0 + (\lambda_i^{(0)} - \lambda_q^{(0)})s_{pq}. \tag{3.1.73b}$$

This corresponds to the fact that the bicharacteristic when traced backwards from (θ_p, θ_q) intersects $t = 0$ at (θ_p^0, θ_q^0). Thus, $\theta_p^0 = \theta_q^0$ and

$$\theta_p^0 = \theta_q^0 = \frac{\lambda_i^{(0)} - \lambda_q^{(0)}}{\lambda_p^{(0)} - \lambda_q^{(0)}}\theta_p + \frac{\lambda_i^{(0)} - \lambda_p^{(0)}}{\lambda_q^{(0)} - \lambda_p^{(0)}}\theta_q \triangleq \varphi_{pq}^i, \tag{3.1.74}$$

$$s_{pq} = \frac{\theta_q - \theta_p}{\lambda_p^{(0)} - \lambda_q^{(0)}}. \tag{3.1.75}$$

As there is no contribution from the initial disturbance to any wave mode at $O(\varepsilon^2)$, integration along the bicharacteristic (3.1.72) from $(\varphi_{pq}^i, \varphi_{pq}^i)$ to (θ_p, θ_q) yields

$$\begin{aligned}
\sigma_i^{(1)}(\theta_p, \theta_q) = & \frac{1}{2}\Gamma_{pp}^i(\lambda_i^{(0)} - \lambda_p^{(0)})^{-1}[\sigma_p^2(\theta_p) - \sigma_p^2(\varphi_{pq}^i)] \\
& + \frac{1}{2}\Gamma_{qq}^i(\lambda_i^{(0)} - \lambda_q^{(0)})^{-1}[\sigma_q^2(\theta_q) - \sigma_q^2(\varphi_{pq}^i)] \\
& + \int_0^{s_{pq}} \Big\{\Gamma_{pq}^i \sigma_q[\varphi_{pq}^i + (\lambda_i^{(0)} - \lambda_q^{(0)})s]\sigma_p'[\varphi_{pq}^i + (\lambda_i^{(0)} - \lambda_p^{(0)})s] \\
& + \Gamma_{qp}^i \sigma_p[\varphi_{pq}^i + (\lambda_i^{(0)} - \lambda_p^{(0)})s]\sigma_q'[\varphi_{pq}^i + (\lambda_i^{(0)} - \lambda_q^{(0)})s]\Big\}\, ds, \\
& \forall\, 1 \leq i \leq n,\ i \neq p, q.
\end{aligned} \tag{3.1.76}$$

Similarly, we have

$$\sigma_p^{(1)}(\theta_p, \theta_q) = \frac{1}{2}\Gamma^p_{qq}(\lambda_p^{(0)} - \lambda_q^{(0)})^{-1}[\sigma_q^2(\theta_q) - \sigma_q^2(\theta_p)] + \Gamma^p_{pq}(\lambda_p^{(0)} - \lambda_q^{(0)})^{-1}\sigma_p'(\theta_p)\int_{\theta_p}^{\theta_q} \sigma_q(s)ds, \tag{3.1.77}$$

$$\sigma_q^{(1)}(\theta_p, \theta_q) = \frac{1}{2}\Gamma^q_{pp}(\lambda_q^{(0)} - \lambda_p^{(0)})^{-1}[\sigma_p^2(\theta_p) - \sigma_p^2(\theta_q)] + \Gamma^q_{qp}(\lambda_q^{(0)} - \lambda_p^{(0)})^{-1}\sigma_q'(\theta_q)\int_{\theta_q}^{\theta_p} \sigma_p(s)ds. \tag{3.1.78}$$

Meanwhile, we integrate (3.1.59), (3.1.60), obtaining

$$x^{(1)}(\theta_p, \theta_q) - \lambda_p^{(0)}t^{(1)}(\theta_p, \theta_q) = (\lambda_p^{(0)} - \lambda_q^{(0)})^{-1}\left\{\lambda_p^{(1)}(\mathbf{r}_p)\sigma_p(\theta_p)(\theta_q - \theta_p) + \lambda_p^{(1)}(\mathbf{r}_q)\int_{\theta_p}^{\theta_q} \sigma_q(s)ds\right\}, \tag{3.1.79}$$

$$x^{(1)}(\theta_p, \theta_q) - \lambda_q^{(0)}t^{(1)}(\theta_p, \theta_q) = (\lambda_q^{(0)} - \lambda_p^{(0)})^{-1}\left\{\lambda_q^{(1)}(\mathbf{r}_q)\sigma_q(\theta_q)(\theta_p - \theta_q) + \lambda_q^{(1)}(\mathbf{r}_p)\int_{\theta_q}^{\theta_p} \sigma_p(s)ds\right\}, \tag{3.1.80}$$

or,

$$x^{(1)}(\theta_p, \theta_q) = (\lambda_p^{(0)} - \lambda_q^{(1)})^{-2}\left\{\left[\lambda_q^{(0)}\lambda_p^{(1)}(\mathbf{r}_p)\sigma_p(\theta_p) - \lambda_p^{(0)}\lambda_q^{(1)}(\mathbf{r}_q)\sigma_q(\theta_q)\right](\theta_p - \theta_q) + \int_{\theta_q}^{\theta_p}\left[\lambda_q^{(0)}\lambda_p^{(1)}(\mathbf{r}_q)\sigma_q(s) - \lambda_p^{(0)}\lambda_q^{(1)}(\mathbf{r}_p)\sigma_p(s)\right]ds\right\}, \tag{3.1.81}$$

$$t^{(1)}(\theta_p, \theta_q) = (\lambda_p^{(0)} - \lambda_q^{(0)})^{-2}\left\{\left[\lambda_p^{(1)}(\mathbf{r}_p)\sigma_p(\theta_p) - \lambda_q^{(1)}(\mathbf{r}_q)\sigma_q(\theta_q)\right](\theta_p - \theta_q) + \int_{\theta_q}^{\theta_p}\left[\lambda_p^{(1)}(\mathbf{r}_q)\sigma_q(s) - \lambda_q^{(1)}(\mathbf{r}_p)\sigma_p(s)\right]ds\right\}, \tag{3.1.82}$$

The $O(\varepsilon)$ problem is thus completely solved.

In summary, we have shown that for the system of hyperbolic conservation laws (3.0.1) with the initial condition of the form

$$\mathbf{u}\big|_{t=0} = \varepsilon\sigma_p(x)\mathbf{r}_p + \varepsilon\sigma_q(x)\mathbf{r}_q, \tag{3.1.1}$$

representing a small amplitude two-wave-mode disturbance, the asymptotic solution takes the form

$$\mathbf{u} = \varepsilon\sigma_p(\theta_p)\mathbf{r}_p + \varepsilon\sigma_q(\theta_q)\mathbf{r}_q + \varepsilon^2\sum_{i=1}^{n}\sigma_i^{(1)}(\theta_p,\theta_q)\mathbf{r}_i + O(\varepsilon^3), \tag{3.1.83}$$

where $\sigma_i^{(1)}$ $(i = 1, 2, \ldots, n)$ are expressed explicitly by (3.1.76)–(3.1.78) and θ_p, θ_q are nonlinear phase variables defined by (3.1.2)–(3.1.5).

Meanwhile, the spatial-temporal coordinates take the perturbed form

$$x = x^{(0)}(\theta_p,\theta_q) + \varepsilon x^{(1)}(\theta_p,\theta_q) + O(\varepsilon^2), \tag{3.1.84}$$

$$t = t^{(0)}(\theta_p,\theta_q) + \varepsilon t^{(1)}(\theta_p,\theta_q) + O(\varepsilon^2), \tag{3.1.85}$$

where $x^{(0)}$, $t^{(0)}$, $x^{(1)}$, $t^{(1)}$ are expressed in an explicit fashion by (3.1.47), (3.1.48), (3.1.81), and (3.1.82), respectively.

3.2. The Signaling Problem

Now we turn to the signaling problem. Suppose the system of hyperbolic conservation laws (3.0.1) is perturbed from its base steady state $\mathbf{u} \equiv \mathbf{0}$ by the boundary disturbance of the form

$$\mathbf{u}\big|_{x=0} = \varepsilon\sigma_p(t)\mathbf{r}_p + \varepsilon\sigma_q(t)\mathbf{r}_q + O(\varepsilon^2),\ t \geq 0, \tag{3.2.1}$$

where, as before, ε is the perturbation parameter and $\sigma_p(\cdot)$, $\sigma_q(\cdot)$ are smooth scalar functions with $\sigma_p(0) = \sigma_q(0) = 0$, $\sigma_p'(0) = \sigma_q'(0) = 0$. The subscripts p, q are any fixed pair of integers satisfying $m < p < q \leq n$ so that $\lambda_q > \lambda_p > 0$. The $O(\varepsilon^2)$ term is added to accomodate well-posedness of the signalling problem, as well as to specify the class of all admissible boundary disturbances of the form (3.2.1).

We restrict the domain of discussion to the quarter plane $x \geq 0,\ t \geq 0$ so that the signaling problem is formulated as follows:

$$\mathbf{G}(\mathbf{u})_t + \mathbf{F}(\mathbf{u})_x = 0,\ x > 0,\ t > 0, \tag{3.0.1}$$

$$\mathbf{u}\big|_{x=0} = \varepsilon\sigma_p(t)\mathbf{r}_p + \varepsilon\sigma_q(t)\mathbf{r}_q + O(\varepsilon^2),\ t \geq 0, \tag{3.2.1}$$

$$\mathbf{u}\big|_{t=0} = 0,\ x \geq 0. \tag{3.2.2}$$

For clarity in our future analysis, we impose the additional requirement that

$$\frac{dx}{dt} = \lambda_q^{(0)},\ x(0) = 0, \tag{3.2.3}$$

or, equivalently, $x = \lambda_q^{(0)}t$ describes the leading wavefront and ahead of it the steady state is preserved.

3.2.1 *Nonlinear phase transformation and asymptotic expansion*

We first introduce nonlinear phase variables θ_p, θ_q associated with $\lambda_p = \lambda_p(\mathbf{u})$ and $\lambda_q = \lambda_q(\mathbf{u})$ defining them as solutions to

$$(\theta_p)_t + \lambda_p(\mathbf{u})(\theta_p)_x = 0, \tag{3.2.4}$$

$$\theta_p\big|_{x=0} = t,\ -\infty < t < \infty, \tag{3.2.5}$$

and

$$(\theta_q)_t + \lambda_q(\mathbf{u})(\theta_q)_x = 0, \tag{3.2.6}$$

$$\theta_q\big|_{x=0} = t,\ -\infty < t < \infty, \tag{3.2.7}$$

respectively.

In defining nonlinear phases θ_p, θ_q here, the main difference from the initial problem, as one can see, is that θ_p, θ_q are parametrized on the boundary rather than on the x-axis.

The spatial-temporal coordinates are then transformed into nonlinear phase coordinates, that is,

$$(x,t) \longmapsto (\theta_p, \theta_q) \tag{3.2.8}$$

by

$$x = x(\theta_p, \theta_q), \ t = t(\theta_p, \theta_q), \tag{3.2.9}$$

which are inverses of the solutions to (3.2.4)–(3.2.7), that is, of $\theta_p = \theta_p(x,t)$ and $\theta_q = \theta_q(x,t)$.

As was shown in the previous section, when we differentiate (3.2.9) with respect to x and t, and take advantage of (3.2.4) and (3.2.6), it follows that

$$x_{\theta_p} = \lambda_q(\lambda_q - \lambda_p)^{-1}/(\theta_p)_x, \ x_{\theta_q} = \lambda_p(\lambda_p - \lambda_q)^{-1}/(\theta_q)_x, \tag{3.2.10}$$

$$t_{\theta_p} = (\lambda_q - \lambda_p)^{-1}/(\theta_p)_x, \ t_{\theta_q} = (\lambda_p - \lambda_q)^{-1}/(\theta_q)_x. \tag{3.2.11}$$

These, in turn, provide

$$x_{\theta_p} - \lambda_q t_{\theta_p} = 0, \tag{3.2.12}$$

$$x_{\theta_q} - \lambda_p t_{\theta_q} = 0. \tag{3.2.13}$$

Since the boundary disturbance (3.2.1) is always assumed smooth, the theory of nonlinear hyperbolic equations (see [42]) ensures the existence and smoothness of $u(x,t)$ and hence the existence and smoothness of $\theta_p = \theta_p(x,t)$ and $\theta_q = \theta_q(x,t)$. Again, throughout the derivations in this section, u will be assumed to be a smooth function of x and t.

Now, under the new phase coordinates, the governing equation (3.0.?) is transformed, as in the previous section, to

$$t_{\theta_p}(\mathbf{A} - \lambda_q \mathbf{B})\mathbf{u}_{\theta_q} - t_{\theta_q}(\mathbf{A} - \lambda_p \mathbf{B})\mathbf{u}_{\theta_p} = 0. \tag{3.2.14}$$

We now employ (3.2.12), (3.2.13), and (3.2.14), which constitute the full system transformed into (θ_p, θ_q) coordinates, and seek an asymptotic solution of the signalling problem in the form

$$\mathbf{u} = \varepsilon \mathbf{u}^{(0)}(\theta_p, \theta_q) + \varepsilon^2 \mathbf{u}^{(1)}(\theta_p, \theta_q) + O(\varepsilon^3). \tag{3.2.15}$$

Also, the spatial-temporal coordinates are assumed to take the perturbed form

$$x = x^{(0)}(\theta_p, \theta_q) + \varepsilon x^{(1)}(\theta_p, \theta_q) + O(\varepsilon^2), \tag{3.2.16}$$

$$t = t^{(0)}(\theta_p, \theta_q) + \varepsilon t^{(1)}(\theta_p, \theta_q) + O(\varepsilon^2). \tag{3.2.17}$$

To find the asymptotic solution (3.2.15)–(3.2.17), we expand (3.2.12), (3.2.13), and (3.2.14) to formulate the requisite $O(1)$ and $O(\varepsilon)$ problems. This was accomplished in the last section and gave

$O(1)$ problem:

$$t^{(0)}_{\theta_p}(\mathbf{A}_0 - \lambda^{(0)}_q \mathbf{B}_0)\mathbf{u}^{(0)}_{\theta_q} - t^{(0)}_{\theta_q}(\mathbf{A}_0 - \lambda^{(0)}_p \mathbf{B}_0)\mathbf{u}^{(0)}_{\theta_p} = 0, \tag{3.2.18}$$

$$(x^{(0)} - \lambda^{(0)}_p t^{(0)})_{\theta_q} = 0, \tag{3.2.19}$$

$$(x^{(0)} - \lambda^{(0)}_q t^{(0)})_{\theta_p} = 0. \tag{3.2.20}$$

$O(\epsilon)$ problem:

$$t_{\theta_p}^{(0)}(\mathbf{A}_0 - \lambda_q^{(0)}\mathbf{B}_0)\mathbf{u}_{\theta_q}^{(1)} - t_{\theta_q}^{(0)}(\mathbf{A}_0 - \lambda_p^{(0)}\mathbf{B}_0)\mathbf{u}_{\theta_p}^{(1)} = \mathbf{M}_1, \tag{3.2.21}$$

$$(x^{(1)} - \lambda_p^{(0)}t^{(1)})_{\theta_q} = H_{p,1}, \tag{3.2.22}$$

$$(x^{(1)} - \lambda_q^{(0)}t^{(1)})_{\theta_p} = H_{q,1}, \tag{3.2.23}$$

where

$$\mathbf{M}_1 = \sum_{k_1+k_2=1} \left\{ t_{\theta_q}^{(k_1)} (\mathbf{A}_{k_2} - \mathbf{B}_{p,k_2})\, \mathbf{u}_{\theta_p}^{(0)} - t_{\theta_p}^{(k_1)} (\mathbf{A}_{k_2} - \mathbf{B}_{q,k_2})\, \mathbf{u}_{\theta_q}^{(0)} \right\}, \tag{3.2.24}$$

$$H_{p,1} = \lambda_{p,1} t_{\theta_q}^{(0)}, \tag{3.2.25}$$

$$H_{q,1} = \lambda_{q,1} t_{\theta_p}^{(0)}. \tag{3.2.26}$$

3.2.2 *Solution of the $O(1)$ problem*

We are now in the position to solve the signalling problem. For the convenience of our discussion, we adopt the convention that

$$\sigma_p(t) = \sigma_q(t) \equiv 0, \quad t < 0. \tag{3.2.27}$$

This convention allows us to extend our domain of discussion to $x \geq 0$, $-\infty < t < \infty$.

We further note that the parametrization of the nonlinear phases θ_p, θ_q on the boundary, that is,

$$x = 0, \quad \theta_p = \theta_q = t, \tag{3.2.28}$$

can be written as

$$x^{(k)}(\theta_p,\theta_p) = x^{(k)}(\theta_q,\theta_q) = 0, \quad k = 0,1,2,\ldots, \tag{3.2.29a}$$

$$t^{(0)}(\theta_p,\theta_p) = \theta_p, \quad t^{(0)}(\theta_q,\theta_q) = \theta_q, \tag{3.2.29b}$$

$$t^{(k)}(\theta_p,\theta_p) = t^{(k)}(\theta_q,\theta_q) = 0, \quad k = 1,2,\ldots, \tag{3.2.29c}$$

when (3.2.16) and (3.2.17) are employed.

Now let us consider the $O(1)$ problem.

First we integrate (3.2.19) and (3.2.20) over θ_p and θ_q, respectively, to obtain

$$x^{(0)} - \lambda_p^{(0)} t^{(0)} = f(\theta_p),$$
$$x^{(0)} - \lambda_q^{(0)} t^{(0)} = g(\theta_q),$$

which, upon using (3.2.29), give

$$x^{(0)} - \lambda_p^{(0)} t^{(0)} = -\lambda_p^{(0)}\theta_p, \quad x^{(0)} - \lambda_q^{(0)} t^{(0)} = -\lambda_q^{(0)}\theta_q,$$

or, alternatively,

$$x^{(0)} = \left(\frac{\lambda_p^{(0)}\lambda_q^{(0)}}{\lambda_q^{(0)} - \lambda_p^{(0)}}\right)(\theta_q - \theta_p), \tag{3.2.30}$$

$$t^{(0)} = \lambda_p^{(0)}(\lambda_p^{(0)} - \lambda_q^{(0)})^{-1}\theta_p + \lambda_q^{(0)}(\lambda_q^{(0)} - \lambda_p^{(0)})^{-1}\theta_q. \tag{3.2.31}$$

It then follows that (3.2.18) reduces to

$$\lambda_p^{(0)}(\mathbf{A}_0 - \lambda_q^{(0)}\mathbf{B}_0)\mathbf{u}_{\theta_q}^{(0)} + \lambda_q^{(0)}(\mathbf{A}_0 - \lambda_p^{(0)}\mathbf{B}_0)\mathbf{u}_{\theta_p}^{(0)} = \mathbf{0}. \tag{3.2.32}$$

Denoting

$$\mathbf{u}^{(0)} = \sum_{i=1}^{n} \sigma_i^{(0)}(\theta_p,\theta_q)\mathbf{r}_i,$$

and substituting it into (3.2.32), we obtain

$$\lambda_p^{(0)}(\lambda_i^{(0)} - \lambda_q^{(0)})\frac{\partial \sigma_i^{(0)}}{\partial \theta_q} + \lambda_q^{(0)}(\lambda_i^{(0)} - \lambda_p^{(0)})\frac{\partial \sigma_i^{(0)}}{\partial \theta_p} = 0, \quad i = 1, 2, \ldots, n. \qquad (3.2.33)$$

The general solution to (3.2.33) takes the form

$$\sigma_i^{(0)}(\theta_p, \theta_q) = f_i(\lambda_p^{(0)}(\lambda_i^{(0)} - \lambda_q^{(0)})\theta_p - \lambda_q^{(0)}(\lambda_i^{(0)} - \lambda_p^{(0)})\theta_q),$$
$$\forall\, 1 \le i \le n, \quad i \ne p, q, \qquad (3.2.34a)$$

$$\sigma_p^{(0)}(\theta_p, \theta_q) = f_p(\theta_p), \qquad (3.2.34b)$$

$$\sigma_q^{(0)}(\theta_p, \theta_q) = f_q(\theta_q), \qquad (3.2.34c)$$

where $f_i(\cdot)$ $(i = 1, 2, \ldots, n)$ are arbitrary C^1 functions. Upon invoking the boundary condition (3.2.1), it follows immediately that

$$f_i(\cdot) \equiv 0, \quad \forall\, 1 \le i \le n, \; i \ne p, q \qquad (3.2.35a)$$

$$f_p(\theta_p) = \sigma_p(\theta_p), \; f_q(\theta_q) = \sigma_q(\theta_q), \qquad (3.2.35b)$$

so that we now have

$$\mathbf{u}^{(0)}(\theta_p, \theta_q) = \sigma_p(\theta_p)\mathbf{r}_p + \sigma_q(\theta_q)\mathbf{r}_q. \qquad (3.2.36)$$

Clearly, $(\mathbf{A}_0 - \lambda_q^{(0)}\mathbf{B}_0)\mathbf{u}_{\theta_q}^{(0)}$ and $(\mathbf{A}_0 - \lambda_p^{(0)}\mathbf{B}_0)\mathbf{u}_{\theta_p}^{(0)}$ vanish separately in (3.2.32) or (3.2.18).

Employing the above results enables us to simplify the $O(\epsilon)$ problem.

3.2.3 *The simplified $O(\epsilon)$ problem*

We note from (3.2.24) that

$$\begin{aligned}
\mathbf{M}_1 &= t_{\theta_q}^{(0)}(\mathbf{A}_1 - \mathbf{B}_{p,1})\mathbf{u}_{\theta_p}^{(0)} - t_{\theta_p}^{(0)}(\mathbf{A}_1 - \mathbf{B}_{q,1})\mathbf{u}_{\theta_q}^{(0)} \\
&= \left(\frac{\lambda_q^{(0)}}{\lambda_q^{(0)} - \lambda_p^{(0)}}\right)\left[\mathbf{A}^{(1)}(\mathbf{u}_o) - \mathbf{B}_p^{(1)}(\mathbf{u}_0)\right]\mathbf{u}_{\theta_p}^{(0)} \\
&\quad - \left(\frac{\lambda_p^{(0)}}{\lambda_p^{(0)} - \lambda_q^{(0)}}\right)\left[\mathbf{A}^{(1)}(\mathbf{u}_0) - \mathbf{B}_q^{(1)}(\mathbf{u}_0)\right]\mathbf{u}_{\theta_q}^{(0)} \\
&= (\lambda_q^{(0)} - \lambda_p^{(0)})^{-1}\left\{\lambda_q^{(0)}\left[\mathbf{A}^{(1)}(\mathbf{r}_p) - \mathbf{B}_p^{(1)}(\mathbf{r}_p)\right]\mathbf{r}_p\sigma_p(\theta_p)\sigma_p'(\theta_p)\right. \\
&\quad + \lambda_q^{(0)}\left[\mathbf{A}^{(1)}(\mathbf{r}_q) - \mathbf{B}_p^{(1)}(\mathbf{r}_q)\right]\mathbf{r}_p\sigma_q(\theta_q)\sigma_q'(\theta_p) \\
&\quad + \lambda_p^{(0)}\left[\mathbf{A}^{(1)}(\mathbf{r}_p) - \mathbf{B}_q^{(1)}(\mathbf{r}_p)\right]\mathbf{r}_q\sigma_p(\theta_p)\sigma_q'(\theta_q) \\
&\quad \left. + \lambda_p^{(0)}\left[\mathbf{A}^{(1)}(\mathbf{r}_q) - \mathbf{B}_q^{(1)}(\mathbf{r}_q)\right]\mathbf{r}_q\sigma_q(\theta_q)\sigma_q'(\theta_q)\right\}. \qquad (3.2.37)
\end{aligned}$$

Similarly, from (3.2.25) and (3.2.26) we obtain

$$\begin{aligned}
H_{p,1} &= \lambda_{p,1}t_{\theta_q}^{(0)} \\
&= \left(\frac{\lambda_q^{(0)}}{\lambda_q^{(0)} - \lambda_p^{(0)}}\right)\lambda_p^{(1)}(\mathbf{u}_0) \\
&= \left(\frac{\lambda_q^{(0)}}{\lambda_q^{(0)} - \lambda_p^{(0)}}\right)\left[\lambda_p^{(1)}(\mathbf{r}_p)\sigma_p(\theta_p) + \lambda_p^{(1)}(\mathbf{r}_q)\sigma_q(\theta_q)\right], \qquad (3.2.38) \\
H_{q,1} &= \lambda_{q,1}t_{\theta_p}^{(0)} \\
&= \left(\frac{\lambda_p^{(0)}}{\lambda_p^{(0)} - \lambda_q^{(0)}}\right)\lambda_q^{(1)}(\mathbf{u}_0) \\
&= \left(\frac{\lambda_p^{(0)}}{\lambda_p^{(0)} - \lambda_q^{(0)}}\right)\left[\lambda_q^{(1)}(\mathbf{r}_p)\sigma_p(\theta_p) + \lambda_q^{(1)}(\mathbf{r}_q)\sigma_q(\theta_q)\right]. \qquad (3.2.39)
\end{aligned}$$

Writing

$$\mathbf{u}^{(1)}(\theta_p, \theta_q) = \sum_{s=1}^{n} \sigma_s^{(1)}(\theta_p, \theta_q)\mathbf{r}_s,$$

and applying, for each $1 \le i \le n$, the left eigenvector ℓ_i to (3.2.21) (with $k = 1$), we reduce the $O(\epsilon)$ problem to the simplified form

$$\begin{aligned}\lambda_p^{(0)}(\lambda_i^{(0)} - \lambda_q^{(0)})\frac{\partial\sigma_i^{(1)}}{\partial\theta_q} + \lambda_q^{(0)}(\lambda_i^{(0)} - \lambda_p^{(0)})\frac{\partial\sigma_i^{(1)}}{\partial\theta_p} \\ = -\lambda_q^{(0)}\Gamma^i_{pp}\sigma_p(\theta_p)\sigma_p'(\theta_p) - \lambda_q^{(0)}\Gamma^i_{qp}\sigma_q(\theta_q)\sigma_p'(\theta_p) \\ - \lambda_p^{(0)}\Gamma^i_{pq}\sigma_p(\theta_p)\sigma_q'(\theta_q) - \lambda_p^{(0)}\Gamma^i_{qq}\sigma_q(\theta_q)\sigma_q'(\theta_q),\end{aligned} \tag{3.2.40}$$

$$(x^{(1)} - \lambda_p^{(0)}t^{(1)})_{\theta_q} = \lambda_q^{(0)}(\lambda_q^{(0)} - \lambda_p^{(0)})^{-1}\left[\lambda_p^{(1)}(\mathbf{r}_p)\sigma_p(\theta_p) + \lambda_p^{(1)}(\mathbf{r}_q)\sigma_q(\theta_q)\right], \tag{3.2.41}$$

$$(x^{(1)} - \lambda_q^{(0)}t^{(1)})_{\theta_p} = \lambda_p^{(0)}(\lambda_p^{(0)} - \lambda_q^{(0)})^{-1}\left[\lambda_q^{(1)}(\mathbf{r}_p)\sigma_p(\theta_p) + \lambda_q^{(1)}(\mathbf{r}_q)\sigma_q(\theta_q)\right], \tag{3.2.42}$$

where Γ^i_{jk} is the nonlinear interaction coefficient.

3.2.4 *Solution to the $O(\epsilon)$ problem*

Again, the $O(\epsilon)$ problem in its simplified form may be integrated to give explicit solutions. Specifically, the integration for each $\sigma_i^{(1)}$ can be carried out along the bicharacteristic

$$\frac{d\theta_p}{ds} = \lambda_q^{(0)}(\lambda_i^{(0)} - \lambda_p^{(0)}), \quad \frac{d\theta_q}{ds} = \lambda_p^{(0)}(\lambda_i^{(0)} - \lambda_q^{(0)}). \tag{3.2.43}$$

However, the current situation is somewhat intricate when compared to the $O(\epsilon)$ problem associated with the initial value problem studied in the previous section. A basic feature here is that the bicharacteristic passing through a point (θ_p, θ_q) may intersect either the boundary $x = 0$ ($t \ge 0$), or the leading wavefront $\theta_q = 0$, or both of them.

For our analysis, we divide the half plane $x > 0$, $-\infty < t < \infty$ into three regions specified as

$$\mathcal{D}_{\rm I} = \{(\theta_p, \theta_q): \quad \theta_p, \theta_q > 0\}, \tag{3.2.44a}$$

$$\mathcal{D}_{\rm II} = \{(\theta_p, \theta_q): \quad \theta_p < 0,\ \theta_q > 0\}, \tag{3.2.44b}$$

$$\mathcal{D}_{\rm III} = \{(\theta_p, \theta_q): \quad \theta_p, \theta_q < 0\}, \tag{3.2.44c}$$

and depicted in Fig. 3.1 below.

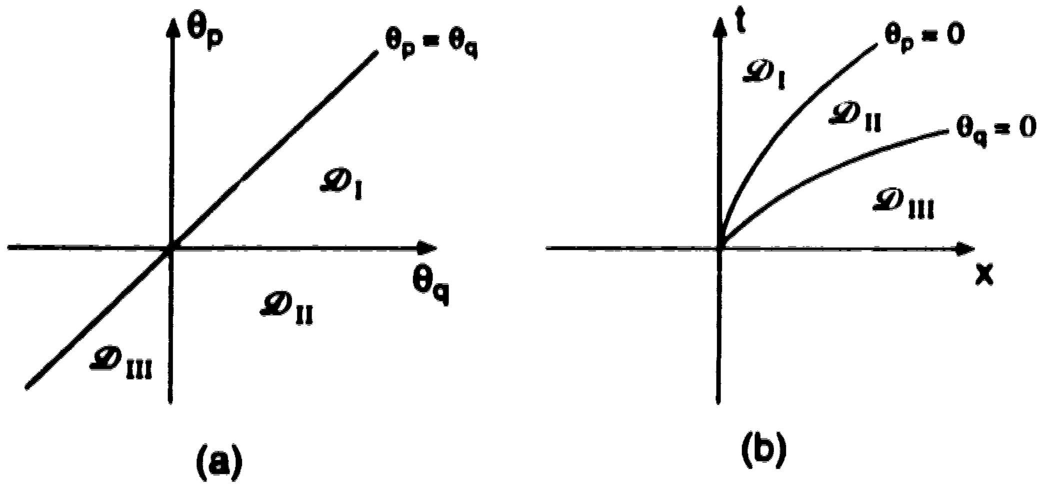


Fig. 3.1(a): The (θ_p, θ_q) plane. (b): The (x, t) plane.

In the (x, t) plane, the above specified regions as shown have clear geometrical interpretation. Indeed, $\mathcal{D}_{\rm III}$ is the steady state region and $\theta_q = 0$ is the leading characteristic for λ_q, whereas $\mathcal{D}_{\rm II}$ is a perturbed region which is ahead of the leading characteristic $\theta_p = 0$ of λ_p. As we shall see later, there is a simple wave region next to $\theta_q = 0$ in $\mathcal{D}_{\rm II}$. We also point out a simple but important fact that, in the half plane $x > 0$, we always have $\theta_p < \theta_q$.

The first case of interest to be examined is that for which the bicharacteristic passing through (θ_p, θ_q) intersects the leading wavefront $\theta_q = 0$ at $(\theta_p^*, 0)$. This implies that

$$\theta_p = \theta_p^* + \lambda_q^{(0)}(\lambda_i^{(0)} - \lambda_p^{(0)})s_{i1}, \tag{3.2.45a}$$

$$\theta_q = \lambda_p^{(0)}(\lambda_i^{(0)} - \lambda_q^{(0)})s_{i1}, \tag{3.2.45b}$$

or

$$\theta_p^* = \theta_p - \frac{\lambda_q^{(0)}(\lambda_i^{(0)} - \lambda_p^{(0)})}{\lambda_p^{(0)}(\lambda_i^{(0)} - \lambda_q^{(0)})}\theta_q \triangleq \beta_i, \tag{3.2.46}$$

$$s_{i1} = \frac{\theta_q}{\lambda_p^{(0)}(\lambda_i^{(0)} - \lambda_q^{(0)})}. \tag{3.2.47}$$

This case occurs only when $\beta_i \leq 0$.

The next case we note is that for which the bicharacteristic passing through (θ_p, θ_q) intersects the boundary $\theta_p = \theta_q$ (or, alternatively, $x = 0$, $t > 0$) at (θ_p^*, θ_q^*). This, in turn, implies that

$$\theta_p = \theta_p^* + \lambda_q^{(0)}(\lambda_i^{(0)} - \lambda_p^{(0)})s_{i2}, \tag{3.2.48a}$$

$$\theta_q = \theta_q^* + \lambda_p^{(0)}(\lambda_i^{(0)} - \lambda_q^{(0)})s_{i2}, \tag{3.2.48b}$$

or, for $\lambda_i^{(0)} \neq 0$, that

$$\theta_p^* = \theta_q^* = \frac{\lambda_p^{(0)}(\lambda_i^{(0)} - \lambda_q^{(0)})}{\lambda_i^{(0)}(\lambda_p^{(0)} - \lambda_q^{(0)})}\theta_p + \frac{\lambda_q^{(0)}(\lambda_i^{(0)} - \lambda_p^{(0)})}{\lambda_i^{(0)}(\lambda_q^{(0)} - \lambda_p^{(0)})}\theta_q \triangleq \alpha_i, \tag{3.2.49}$$

$$s_{i2} = -\frac{\theta_q - \theta_p}{\lambda_i^{(0)}(\lambda_q^{(0)} - \lambda_p^{(0)})}, \tag{3.2.50}$$

when $\theta_p^* = \theta_q^*$ is noted. Obviously the bicharacteristic passing through (θ_p, θ_q) intersects the boundary if and only if $\alpha_i = 0$.

Motivated by the above analysis, we now introduce the following definition: for each $1 \leq i \leq n$, $\lambda_i^{(0)} \neq 0$, denote

$$\rho_{pq}^i = \frac{\lambda_p^{(0)}(\lambda_i^{(0)} - \lambda_q^{(0)})}{\lambda_i^{(0)}(\lambda_p^{(0)} - \lambda_q^{(0)})}, \quad \rho_{qp}^i = \frac{\lambda_q^{(0)}(\lambda_i^{(0)} - \lambda_p^{(0)})}{\lambda_i^{(0)}(\lambda_q^{(0)} - \lambda_p^{(0)})}, \tag{3.2.51}$$

and call

$$\Gamma^i : \ \rho_{pq}^i \theta_p + \rho_{qp}^i \theta_q = 0, \tag{3.2.52}$$

the *separation line* for the ith mode. As particular cases, we find

$$\Gamma^p : \ \theta_p = 0 \quad \text{and} \quad \Gamma^q : \ \theta_q = 0.$$

These are boundaries separating $\mathcal{D}_I$ from $\mathcal{D}_{II}$ and $\mathcal{D}_{III}$, respectively.

The following lemma ascertains the position for each Γ^i ($i \neq p, q$; $\lambda_i^{(0)} \neq 0$).

LEMMA 3.3. (i). If $0 < \lambda_i^{(0)} < \lambda_p^{(0)}$, then $\Gamma^i \subset \mathcal{D}_I$, (ii). If $\lambda_p^{(0)} < \lambda_i^{(0)} < \lambda_q^{(0)}$, then $\Gamma^i \subset \mathcal{D}_{II}$, (iii). If $\lambda_q^{(0)} < \lambda_i^{(0)}$, or $\lambda_i^{(0)} < 0$, then $\Gamma^i \subset \mathcal{D}_{III}$.

Proof. Noting that $\theta_q > \theta_p$ in the half plane $x > 0$, and that

$$\rho_{pq}^i + \rho_{qp}^i = 1, \tag{3.2.53}$$

we shall prove (i)–(iii) separately.

(i). $0 < \lambda_i^{(0)} < \lambda_p^{(0)}$.

Now $\rho_{pq}^i > 1$ and $\rho_{qp}^i < 0$. On the boundary of $\mathcal{D}_I$ we have

$$\alpha_i = \rho_{pq}^i \theta_p + \rho_{qp}^i \theta_q = \theta_q > 0,$$

when $\theta_p = \theta_q$, and

$$\alpha_i = \rho_{pq}^i \theta_p + \rho_{qp}^i \theta_q = \rho_{qp}^i \theta_q < 0,$$

when $\theta_p = 0$. This suggests that α_i changes sign in $\mathcal{D}_{\rm I}$, that is, $\Gamma^i \subset \mathcal{D}_{\rm I}$.

(ii). $\lambda_p^{(0)} < \lambda_i^{(0)} < \lambda_q^{(0)}$.
Now $0 < \rho_{pq}^i, \rho_{qp}^i < 1$. On the boundary of $\mathcal{D}_{\rm II}$, we have

$$\alpha_i = \rho_{pq}^i\theta_p + \rho_{qp}^i\theta_q = \rho_{pq}^i\theta_p < 0,$$

when $\theta_p = 0$. That is, α_i changes sign in $\mathcal{D}_{\rm II}$ and hence $\Gamma^i \subset \mathcal{D}_{\rm II}$.

(iii). $\lambda_q^{(0)} < \lambda_i^{(0)}$ or $\lambda_i^{(0)} < 0$.
In both situations we have $\rho_{pq}^i < 0$, $\rho_{qp}^i > 1$. Thus $\forall\ (\theta_p, \theta_q) \in \mathcal{D}_{\rm I} \cup \mathcal{D}_{\rm II}$ we have

$$\begin{aligned}\alpha_i &= \rho_{pq}^i\theta_p + \rho_{qp}^i\theta_q \\ &= \theta_q + \rho_{pq}^i(\theta_p - \theta_q) > 0,\end{aligned}$$

and so Γ^i is not contained in $\mathcal{D}_{\rm I} \cup \mathcal{D}_{\rm II}$, and hence $\Gamma^i \subset \mathcal{D}_{\rm III}$.

This completes the proof.

The next lemma is in fact a corollary of Lemma 3.3 and specifies under what circumstances will a bicharacteristic intersect the boundary, the leading wavefront, or both of them.

LEMMA 3.4. $\forall\ (\theta_p, \theta_q) \in \mathcal{D}_{\rm I} \cup \mathcal{D}_{\rm III} \cup \{\theta_p = 0\}$, the bicharacteristic (3.2.43) passing through (θ_p, θ_q) will intersect
(i). either the boundary $x = 0$ ($t > 0$), or the leading wavefront $\theta_q = 0$, that is, $x = \lambda_q^{(0)}t$, but only one of them when $0 < \lambda_i^{(0)} < \lambda_q^{(0)}$,
(ii). both the boundary $x = 0$ ($t > 0$) and the leading wavefront $\theta_q = 0$ when $\lambda_i^{(0)} < 0$ or $\lambda_i^{(0)} > \lambda_q^{(0)}$.

We omit the proof since the results are evident from Fig. 3.2 wherein dotted lines represent the bicharacteristic passing through (θ_p, θ_q).

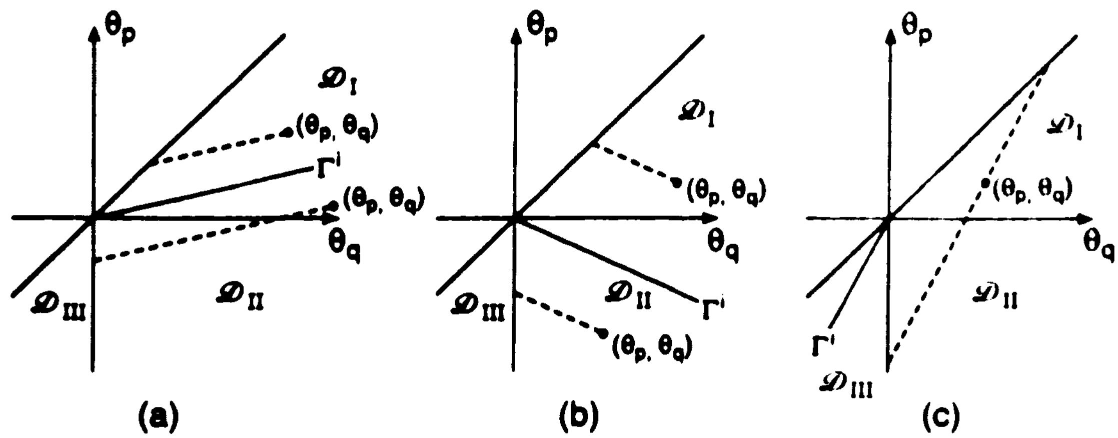


Fig. 3.2(a): The (θ_p, θ_q) plane with separation line $\Gamma^i \subset \mathcal{D}_{I}$.
(b): The (θ_p, θ_q) plane with separation line $\Gamma^i \subset \mathcal{D}_{II}$.
(c): The (θ_p, θ_q) plane with separation line $\Gamma^i \subset \mathcal{D}_{III}$.

We are now in a position to integrate (3.2.40) along the bicharacteristic according to the cases classified above.

Case 1. $0 < \lambda_i^{(0)} < \lambda_q^{(0)}$ and $i \neq p$.
Now $\Gamma^i \subset \mathcal{D}_{I} \cup \mathcal{D}_{II}$. There are two subcases.

(i). When $\alpha_i = \rho_{pq}^i \theta_p + \rho_{qp}^i \theta_q > 0$, we integrate (3.2.40) from (α_i, α_i) to (θ_p, θ_q) to obtain (see Fig. 3.2(a), (b))

$$\begin{aligned}\sigma_i^{(1)}(\theta_p, \theta_q) = \sigma_i^{(1)}(\alpha_i, \alpha_i) &- \frac{1}{2}\Gamma_{pp}^i(\lambda_i^{(0)} - \lambda_p^{(0)})^{-1}[\sigma_p^2(\theta_p) - \sigma_p^2(\alpha_i)] \\ &- \frac{1}{2}\Gamma_{qq}^i(\lambda_i^{(0)} - \lambda_q^{(0)})^{-1}[\sigma_q^2(\theta_q) - \sigma_q^2(\alpha_i)] \\ &+ \frac{1}{\lambda_i^{(0)}(\lambda_q^{(0)} - \lambda_p^{(0)})}\int_{\theta_p}^{\theta_q}\Big\{\lambda_q^{(0)}\Gamma_{qp}^i\sigma_p'[\rho_{pq}^i\theta_p + \rho_{qp}^i s]\sigma_q[\theta_q + \rho_{pq}^i(\theta_p - s)] \\ &+ \lambda_p^{(0)}\Gamma_{pq}^i\sigma_p[\rho_{pq}^i\theta_p + \rho_{qp}^i s]\sigma_q'[\theta_q + \rho_{pq}^i(\theta_p - s)]\Big\}\,ds. \qquad (3.2.54)\end{aligned}$$

(ii). When $\alpha_i = \rho^i_{pq}\theta_p + \rho^i_{qp}\theta_q \le 0$, we integrate (3.2.40) from $(\beta_i, 0)$ to (θ_p, θ_q) (see Fig. 3.2 (a),(b)) obtaining

$$\begin{aligned}
\sigma_i^{(1)}(\theta_p,\theta_q) = & -\frac{1}{2}\Gamma^i_{pp}(\lambda_i^{(0)} - \lambda_p^{(0)})^{-1}\sigma_p^2(\theta_p) \\
& -\frac{1}{2}\Gamma^i_{qq}(\lambda_i^{(0)} - \lambda_q^{(0)})^{-1}\sigma_q^2(\theta_q) \\
& + \frac{1}{\lambda_p^{(0)}(\lambda_q^{(0)} - \lambda_i^{(0)})}\int_0^{\theta_q}\Big\{\lambda_q^{(0)}\Gamma^i_{qp}\sigma_q(s) \\
& \times \sigma_p'\left[\theta_p + \frac{\lambda_q^{(0)}(\lambda_i^{(0)} - \lambda_p^{(0)})}{\lambda_p^{(0)}(\lambda_q^{(0)} - \lambda_i^{(0)})}(\theta_q - s)\right] \\
& + \lambda_p^{(0)}\Gamma^i_{pq}\sigma_p\left[\theta_p + \frac{\lambda_q^{(0)}(\lambda_i^{(0)} - \lambda_p^{(0)})}{\lambda_p^{(0)}(\lambda_q^{(0)} - \lambda_i^{(0)})}(\theta_q - s)\right]\sigma_q'(s)\Big\}\,ds.
\end{aligned} \tag{3.2.55}$$

Case 2. $\lambda_i^{(0)} < 0$ or $\lambda_i^{(0)} > \lambda_q^{(0)}$.

In this case Lemma 3.1 asserts that $\Gamma^i \subset \mathcal{D}_{III}$. As indicated by Lemma 3.2, the peculiar nature of this case resides in the fact that the bicharacteristic passing through (θ_p, θ_q) will intersect both the boundary $x = 0$ $(t > 0)$ and the leading wavefront $x = \lambda_q^{(0)}t$. Here we integrate (3.2.40) from $(\beta_i, 0)$ to (θ_p, θ_q) (see Fig. 3.2(c)) obtaining an expression which is the same as that in (3.2.55). In particular, we specify that on the boundary

$$\begin{aligned}
\sigma_i^{(1)}(\theta_p,\theta_q)\big|_{\theta_p=\theta_q=t} = & -\frac{1}{2}\Gamma^i_{pp}(\lambda_i^{(0)} - \lambda_p^{(0)})^{-1}\sigma_p^2(t) \\
& -\frac{1}{2}\Gamma^i_{qq}(\lambda_i^{(0)} - \lambda_q^{(0)})^{-1}\sigma_q^2(t) \\
& + \frac{1}{\lambda_p^{(0)}(\lambda_q^{(0)} - \lambda_i^{(0)})}\int_0^{t}\Big\{\lambda_q^{(0)}\Gamma^i_{qp}\sigma_q(s) \\
& \times \sigma_p'\left[t + \frac{\lambda_q^{(0)}(\lambda_i^{(0)} - \lambda_p^{(0)})}{\lambda_p^{(0)}(\lambda_i^{(0)} - \lambda_q^{(0)})}(s - t)\right] \\
& + \lambda_p^{(0)}\Gamma^i_{pq}\sigma_q'(s)\sigma_p\left[t + \frac{\lambda_q^{(0)}(\lambda_i^{(0)} - \lambda_p^{(0)})}{\lambda_p^{(0)}(\lambda_i^{(0)} - \lambda_q^{(0)})}(s - t)\right]\Big\}\,ds.
\end{aligned} \tag{3.2.56}$$

Case 3. $i = p, q$ or $\lambda_i^{(0)} = 0$.

(i). When $i = p$, we integrate (3.2.40) from (θ_p, θ_p) to (θ_p, θ_q) keeping θ_p a constant along the path to obtain

$$\sigma_p^{(1)}(\theta_p, \theta_q) = \sigma_p^{(1)}(\theta_p, \theta_p) - \frac{1}{2}\Gamma_{qq}^p(\lambda_p^{(0)} - \lambda_q^{(0)})^{-1}[\sigma_q^2(\theta_q) - \sigma_q^2(\theta_p)]$$
$$- \Gamma_{pq}^p(\lambda_p^{(0)} - \lambda_q^{(0)})^{-1}\sigma_p(\theta_p)[\sigma_q(\theta_q) - \sigma_q(\theta_p)]. \qquad (3.2.57)$$

Similarly, when $i = q$, we have

$$\sigma_q^{(1)}(\theta_p, \theta_q) = \sigma_q^{(1)}(\theta_q, \theta_q) - \frac{1}{2}\Gamma_{pp}^q(\lambda_q^{(0)} - \lambda_p^{(0)})^{-1}[\sigma_p^2(\theta_p) - \sigma_p^2(\theta_q)]$$
$$- \Gamma_{qp}^q(\lambda_q^{(0)} - \lambda_p^{(0)})^{-1}\sigma_q(\theta_q)[\sigma_p(\theta_p) - \sigma_p(\theta_q)]. \qquad (3.2.58)$$

(ii). When $\lambda_i^{(0)} = 0$, we have the situation in which the bicharacteristic degenerates to the boundary itself. Integrating (3.2.40) from 0 to t we obtain

$$\sigma_i^{(1)}(\theta_p, \theta_q)\big|_{\theta_p = \theta_q = t} = \frac{1}{2}\,\frac{\Gamma_{pp}^i}{\lambda_p^{(0)}}\sigma_p^2(t) + \frac{1}{2}\,\frac{\Gamma_{qq}^i}{\lambda_q^{(0)}}\sigma_q^2(t)$$
$$+ \int_0^t \left[(\lambda_q^{(0)})^{-1}\Gamma_{pq}^i\sigma_p(s)\sigma_q'(s) + (\lambda_p^{(0)})^{-1}\Gamma_{qp}^i\sigma_p'(s)\sigma_q(s)\right] ds. \qquad (3.2.59)$$

This completes the integration of (3.2.40).

Meanwhile, for the perturbed spatial-temporal coordinates, we integrate (3.2.41) and (3.2.42) with respect to θ_q and θ_p, respectively, obtaining after some simplifications that

$$x^{(1)} = (\lambda_q^{(0)} - \lambda_p^{(0)})^{-2} \left\{ \begin{vmatrix} \lambda_p^{(0)2} & \lambda_q^{(0)2} \\ \lambda_p^{(1)}(\mathbf{r}_p)\sigma_p(\theta_p) & \lambda_q^{(1)}(\mathbf{r}_q)\sigma_q(\theta_q) \end{vmatrix} (\theta_p - \theta_q) + \int_{\theta_q}^{\theta_p} \begin{vmatrix} \lambda_p^{(0)2}\sigma_p(s) & \lambda_q^{(0)2}\sigma_q(s) \\ \lambda_p^{(1)}(\mathbf{r}_q) & \lambda_q^{(1)}(\mathbf{r}_p) \end{vmatrix} ds \right\}, \quad (3.2.60)$$

$$t^{(1)} = (\lambda_q^{(0)} - \lambda_p^{(0)})^{-2} \left\{ \begin{vmatrix} \lambda_p^{(0)} & \lambda_q^{(0)} \\ \lambda_p^{(1)}(\mathbf{r}_p)\sigma_p(\theta_p) & \lambda_q^{(1)}(\mathbf{r}_q)\sigma_q(\theta_q) \end{vmatrix} (\theta_p - \theta_q) + \int_{\theta_q}^{\theta_p} \begin{vmatrix} \lambda_p^{(0)}\sigma_p(s) & \lambda_q^{(0)}\sigma_q(s) \\ \lambda_p^{(1)}(\mathbf{r}_q) & \lambda_q^{(1)}(\mathbf{r}_p) \end{vmatrix} ds \right\}. \quad (3.2.61)$$

The $O(\epsilon)$ problem has been completely and explicitly solved.

In summary, we have shown that for the system of hyperbolic conservation laws (3.0.1) subject to the boundary condition

$$\mathbf{u}\big|_{x=0} = \epsilon\sigma_p(t)\mathbf{r}_p + \epsilon\sigma_q(t)\mathbf{r}_q + O(\epsilon^2), \quad (3.2.1)$$

representing a small amplitude, two wave mode disturbance, the asymptotic solution takes the form

$$\mathbf{u} = \epsilon\sigma_p(\theta_p)\mathbf{r}_p + \epsilon\sigma_q(\theta_q)\mathbf{r}_q + \epsilon^2 \sum_{i=1}^{n} \sigma_i^{(1)}(\theta_p, \theta_q)\mathbf{r}_i + O(\epsilon^3), \quad (3.2.62)$$

where the $\sigma_i^{(1)}(i = 1, 2, \ldots, n)$ are expressed explicitly by (3.2.54)–(3.2.59) and θ_p, θ_q are nonlinear phases defined by (3.2.4)–(3.2.7). The spatial-temporal coordinates take the perturbed form

$$x = x^{(0)}(\theta_p, \theta_q) + \epsilon x^{(1)}(\theta_p, \theta_q) + O(\epsilon^2), \quad (3.2.62)$$

$$t = t^{(0)}(\theta_p, \theta_q) + \epsilon t^{(1)}(\theta_p, \theta_q) + O(\epsilon^2), \quad (3.2.63)$$

where $x^{(0)}, t^{(0)}, x^{(1)}$, and $t^{(1)}$ are given explicitly by (3.2.30), (3.2.31), (3.2.60), and (3.2.61), respectively.

It is well known that the signalling problem differs significantly from the initial value problem in the way the boundary or initial conditions are prescribed. The well-posedness of the signaling problem depends in a crucial way on how the boundary disturbance is assigned (see, for example, [23] and [34]). However, in the course of solution, we have specified the class of boundary disturbances admissible for the asymptotic solution constructed above. We state these findings as follows.

Remark 3.1. Under the requirement that ahead of the leading wavefront $\theta_q = 0$, or alternatively $x - \lambda_q^{(0)} t = 0$, there exists a steady state region, the class of boundary disturbances admissible for the asymptotic solution (3.2.62) is specified precisely. That is,

$$\mathbf{u}\big|_{x=0} = \varepsilon\sigma_p(t)\mathbf{r}_p + \varepsilon\sigma_q(t)\mathbf{r}_q + \varepsilon^2 \sum_{i=1}^{n} \sigma_i^{(1)}(t)\mathbf{r}_i + O(\varepsilon^3), \qquad (3.2.64)$$

wherein $\sigma_i^{(1)}(t) = \sigma_i^{(1)}(\theta_p, \theta_q)\big|_{\theta_p=\theta_q=t}$, and:
(i). $\forall\, 1 \le i \le n$, if $\lambda_i^{(0)} > \lambda_q^{(0)}$ or $\lambda_i^{(0)} < 0$, $\sigma_i^{(1)}(t)$ is specified by (3.2.56),
(ii). $\forall\, 1 \le i \le n$, if $\lambda_i^{(0)} = 0$, $\sigma_i^{(1)}(t)$ is specified by (3.2.59),
(iii). $\forall\, 1 \le i \le n$, if $0 < \lambda_i^{(0)} \le \lambda_q^{(0)}$, $\sigma_i^{(1)}(t)$ can be any C^1 functions satisfying $\sigma_i^{(1)}(0) = \frac{d\sigma_i^{(1)}}{dt}(0) = 0$.

Secondly, as we have already observed, at the $O(\varepsilon^2)$ order, each separation line Γ^i ($\forall\, 1 \le i \le n:\ 0 < \lambda_i^{(0)} \le \lambda_q^{(0)}$) separates the regions of dependence for the ith wave mode. We thus have the following remark.

Remark 3.2. $\forall\, 1 \le i \le n$ and $0 < \lambda_i^{(0)} \le \lambda_q^{(0)}$, Γ^i separates the region $\{(\theta_p, \theta_q):\ \alpha_i > 0\}$, in which $\sigma_i^{(1)}(\theta_p, \theta_q)$ depends on the boundary disturbance function $\sigma_i^{(1)}(\cdot)$, from the region $\{(\theta_p, \theta_q):\ \alpha_i \le 0\}$, in which $\sigma_i^{(1)}(\theta_p, \theta_q)$ is independent of $\sigma_i^{(1)}(\cdot)$. See Fig. 3.2.

Thirdly, we may note that there exists a region next to the leading wavefront such that the solution $\mathbf{u}(\theta_p, \theta_q)$ depends upon the single phase θ_q.

Remark 3.3. The region $\{(\theta_p, \theta_q) : \alpha_{q-1}(\theta_p, \theta_q) < 0, \theta_q > 0\}$ is a single phase region, that is, $\mathbf{u}(\theta_p, \theta_q)$ depends upon θ_q only.

In fact, this recovers, for 2×2 systems, the classical result of Courant and Friedrichs [8] which states that next to any constant region must be a *simple wave* region. The single phase region is depicted in Fig. 3.3 below.

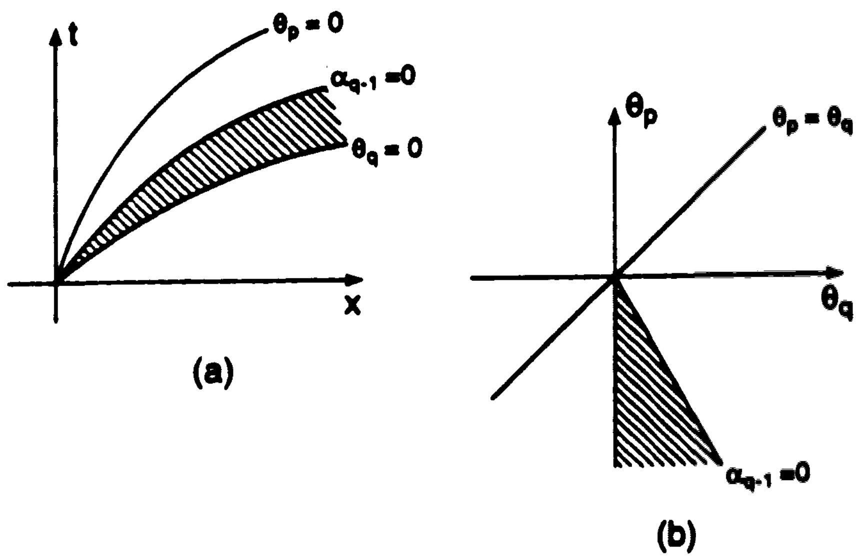


Fig. 3.3(a): The simple wave region in the (x, t) plane.
(b): The simple wave region in the (θ_p, θ_q) plane.

3.3. Propagation and Interaction of Weak Sound Waves in Gas Dynamics

As an application of the results derived in the previous two sections, we consider the equations of one dimensional gas dynamics which, in the absence of viscosity, have the following conservation laws [58,62,71,75]:

$$\rho_t + (\rho v)_x = 0, \tag{3.3.1}$$

$$(\rho v)_t + (\rho v^2 + p)_x = 0, \tag{3.3.2}$$

$$(\rho e + \frac{1}{2}\rho v^2)_t + (\rho e v + \frac{1}{2}\rho v^3 + p v)_x = 0, \tag{3.3.3}$$

where ρ is the mass density, v is the flow speed, p is the pressure, and e is the internal energy per unit of mass. In addition, we introduce the entropy S and the temperature T. We assume, as is conventional in thermodynamics, that given any two of ρ, p, e, T, and S, it is possible to obtain the remaining three through constitutive equations. We shall use ρ, v, and S as the dependent variables in the above equations. Moreover, we adopt the constitutive assumptions stated in [62] which are as follows:

(i). p, e, and T are smooth functions of ρ and S and they satisfy the thermodynamic relation

$$T dS = de + p d\rho^{-1}, \tag{3.3.4}$$

which, in turn, implies

$$e_\rho = p/\rho^2, \; e_S = T, \tag{3.3.5}$$

(ii). Clearly, $p > 0$, $\rho > 0$, and $T > 0$. Furthermore we assume that

$$c^2 = p_\rho > 0, \; p_S = \rho^2 T_\rho > 0, \tag{3.3.6}$$

where $c = \sqrt{p_\rho} > 0$ is the sound speed and $p_S = \rho^2 T_\rho$ follows from (3.3.5).

We now write (3.3.1)–(3.3.3) in matrix form as

$$\begin{pmatrix} 1 & 0 & 0 \\ v & \rho & 0 \\ h+\frac{1}{2}v^2 & \rho v & \rho T \end{pmatrix} \begin{pmatrix} \rho \\ v \\ S \end{pmatrix}_t + \begin{pmatrix} v & \rho & 0 \\ v^2 + p_\rho & 2\rho v & ps \\ vp_\rho + hv + \frac{1}{2}v^3 & \rho h + \frac{3}{2}\rho v^2 & vps + \rho vT \end{pmatrix} \begin{pmatrix} \rho \\ v \\ S \end{pmatrix}_x = 0 \qquad (3.3.7)$$

where we have introduced the enthalpy

$$h \triangleq e + p/\rho.$$

The system (3.3.1)–(3.3.3) or (3.3.7) admits the steady state solution

$$\rho \equiv \rho_0, \; v \equiv v_0, \; S \equiv S_0,$$

where ρ_0, v_0, and S_0 are constants.

Now let

$$\rho = \rho_0 + u_1, \; v = v_0 + u_2, \; S = S_0 + u_3, \qquad (3.3.8)$$

so that (3.3.7) may be written as

$$\mathbf{B}\mathbf{u}_t + \mathbf{A}\mathbf{u}_x = 0, \qquad (3.3.9)$$

with $\mathbf{u} = (u_1, u_2, u_3)^T$ and

$$\mathbf{A}(\mathbf{u}) = \begin{pmatrix} v & \rho & 0 \\ v^2 + p_\rho & 2\rho v & ps \\ vp_\rho + hv + \frac{1}{2}v^3 & \rho h + \frac{3}{2}\rho v^2 & vps + v\rho T \end{pmatrix}, \qquad (3.3.10)$$

$$\mathbf{B}(\mathbf{u}) = \begin{pmatrix} 1 & 0 & 0 \\ v & \rho & 0 \\ h+\frac{1}{2}v^2 & \rho v & \rho T \end{pmatrix}. \qquad (3.3.11)$$

Now, from $\det(\mathbf{A} - \lambda\mathbf{B}) = 0$, one can readily compute that the eigenvalues associated with (3.3.9) are

$$\lambda_1 = v - c, \ \lambda_2 = v, \ \lambda_3 = v + c.$$

After some straightforward calculations, we find that about the steady state $\mathbf{u} \equiv \mathbf{0}$, we have

$$\left.\begin{aligned} \lambda_1^{(0)} &= v_0 - c_0, \\ \boldsymbol{\ell}_1 &= \frac{1}{2\rho_0 c_0^2 T_0} \\ &\quad \times \left(c_0 T_0(v_0 + c_0) + \left(\frac{1}{2}v_0^2 - h_0\right)\rho_0(T_\rho)_0, -c_0 T_0 - \rho_0(T_\rho)_0 v_0, \rho_0(T_\rho)_0\right), \\ \mathbf{r}_1 &= \begin{pmatrix} \rho_0 \\ -c_0 \\ 0 \end{pmatrix}, \end{aligned}\right\} \tag{3.3.12}$$

$$\left.\begin{aligned} \lambda_2^{(0)} &= c_0, \\ \boldsymbol{\ell}_2 &= \frac{1}{\rho_0 c_0^2 T_0}\left(h_0 - \frac{1}{2}v_0^2, v_0, -1\right), \\ \mathbf{r}_2 &= \begin{pmatrix} (ps)_0 \\ 0 \\ -c_0^2 \end{pmatrix}, \end{aligned}\right\} \tag{3.3.13}$$

$$\left.\begin{aligned} \lambda_3^{(0)} &= v_0 + c_0, \\ \boldsymbol{\ell}_3 &= \frac{1}{2\rho_0 c_0^2 T_0} \\ &\quad \times \left(c_0 T_0(c_0 - v_0) + \left(\frac{1}{2}v_0^2 - h_0\right)\rho_0(T_\rho)_0, c_0 T_0 - \rho_0(T_\rho)_0 v_0, \rho_0(T_\rho)_0\right), \\ \mathbf{r}_3 &= \begin{pmatrix} \rho_0 \\ c_0 \\ 0 \end{pmatrix}, \end{aligned}\right\} \tag{3.3.14}$$

with the subscript "0" referring to the evaluation of the corresponding quantity at the steady state.

We now consider the propagation and interaction of weak sound waves generated by either initial or boundary disturbances.

3.3.1 *Propagation and interaction of two weak sound waves arising from initial disturbance*

We take the initial perturbation to be

$$\begin{aligned}\mathbf{u}\big|_{t=0} &= \varepsilon\sigma_1(x)\mathbf{r}_1 + \varepsilon\sigma_3(x)\mathbf{r}_3 \\ &= \varepsilon\sigma_1(x)\begin{pmatrix}\rho_0\\ -c_0\\ 0\end{pmatrix} + \varepsilon\sigma_3(x)\begin{pmatrix}\rho_0\\ c_0\\ 0\end{pmatrix},\end{aligned} \tag{3.3.15}$$

that is, we choose $p = 1$ and $q = 3$. $\sigma_1(\cdot)$ and $\sigma_3(\cdot)$ are smooth scalar functions.

The results derived in the first section of this chapter suggest that the perturbation solution will be

$$\mathbf{u} = \varepsilon\sigma_1(\theta_1)\mathbf{r}_1 + \varepsilon\sigma_3(\theta_3)\mathbf{r}_3 + \varepsilon^2\sum_{i=1}^{n}\sigma_i^{(1)}(\theta_1,\theta_3) + O(\varepsilon^3), \tag{3.3.16}$$

where $\sigma_i^{(1)}(\theta_1,\theta_3)$ satisfy (3.1.58).

We now carry out the computations to find the interaction coefficients Γ^i_{jk}, where by Lemma 3.2,

$$\Gamma^i_{jk} = \boldsymbol{\ell}_i\left[\mathbf{A}^{(1)}(\mathbf{r}_j) - \lambda_k^{(0)}\mathbf{B}_0^{(1)}(\mathbf{r}_j)\right]\mathbf{r}_k - \lambda_k^{(1)}(\mathbf{r}_j)\delta_{ik}. \tag{3.3.17}$$

First we expand λ_i $(i = 1,2,3)$ about $\mathbf{u} \equiv \mathbf{0}$ obtaining

$$\lambda_i(\mathbf{u}) = \lambda_i^{(0)} + \lambda_i^{(1)}(\mathbf{u}) + O(\|\mathbf{u}\|^2), \tag{3.3.18}$$

in which

$$\lambda_1^{(1)}(\mathbf{u}) = \left(-\frac{1}{2}c_0^{-1}(p_{\rho\rho})_0, 1, -\frac{1}{2}c_0^{-1}(p_\rho s)_0\right)\mathbf{u}, \tag{3.3.19}$$

$$\lambda_2^{(1)}(\mathbf{u}) = (0,1,0)\mathbf{u}, \tag{3.3.20}$$

$$\lambda_3^{(1)}(\mathbf{u}) = \left(\frac{1}{2}c_0^{-1}(p_{\rho\rho})_0, 1, \frac{1}{2}c_0^{-1}(p_\rho s)_0\right)\mathbf{u}. \tag{3.3.21}$$

Thus we have

$$\lambda_3^{(1)}(\mathbf{r}_3) = -\lambda_1^{(1)}(\mathbf{r}_1) = c_0 + \frac{1}{2}(\rho_0/c_0)(p_{\rho\rho})_0 \triangleq \alpha, \tag{3.3.22}$$

$$\lambda_1^{(1)}(\mathbf{r}_3) = -\lambda_3^{(1)}(\mathbf{r}_1) = c_0 - \frac{1}{2}(\rho_0/c_0)(p_{\rho\rho})_0 \triangleq \beta, \tag{3.3.23}$$

$$\lambda_1^{(1)}(\mathbf{r}_2) = -\lambda_3^{(1)}(\mathbf{r}_2) = \frac{1}{2}c_0(p_\rho s)_0 - \frac{1}{2}c_0^{-1}(ps)_0(p_{\rho\rho})_0 \triangleq \delta, \tag{3.3.24}$$

$$\lambda_2^{(1)}(\mathbf{r}_3) = -\lambda_2^{(1)}(\mathbf{r}_3) = c_0, \tag{3.3.25}$$

$$\lambda_2^{(1)}(\mathbf{r}_2) = 0. \tag{3.3.26}$$

The fact that $\lambda_2^{(1)}(\mathbf{r}_2) = 0$ implies the local linear degeneracy about the steady state solution of the characteristic field $\lambda_2 = \lambda_2(\mathbf{u}) = v_0 + u_2$.

Further detailed calculations give us the pair of relations required for the evaluation of the interaction coefficients, namely,

$$\mathbf{c}(\mathbf{v},\mathbf{w}) \triangleq \mathbf{B}_0^{(1)}(\mathbf{v})\mathbf{w} = \begin{pmatrix} 0 \\ v_1 w_2 + v_2 w_1 \\ \rho_0^{-1} c_0^2 v_1 w_1 + \rho_0 v_2 w_2 + (\rho T_\rho)_0 v_3 w_3 \\ + v_0(v_1 w_2 + v_2 w_1) + (\rho T_\rho)_0 (v_1 w_3 + v_3 w_1) \end{pmatrix} \tag{3.3.27}$$

and

$$\mathbf{D}(\mathbf{v},\mathbf{w}) \triangleq \mathbf{A}^{(1)}(\mathbf{v})\mathbf{w}$$
$$= \begin{pmatrix} v_1 w_2 + v_2 w_1 \\ (p_{\rho\rho})_0 v_1 w_1 + 2\rho_0 v_2 w_2 + (pss)_0 v_3 w_3 \\ +2v_0(v_1 w_2 + v_2 w_1) + (p_\rho s)_0(v_1 w_3 + v_3 w_1) \\ v_0(\rho^{-1}c^2 + p_{\rho\rho})_0 v_1 w_1 + 3\rho_0 v_0 v_2 w_2 + [v\rho(\rho T)_\rho s]_0 v_3 w_3 \\ +\left(h + c^2 + \frac{3}{2}v^2\right)_0 (v_1 w_2 + v_2 w_1) \\ +v_0[(\rho(\rho T)_\rho)_\rho]_0(v_1 w_3 + v_3 w_1) \\ +\rho_0[(\rho T_\rho)_\rho]_0(v_2 w_3 + v_3 w_2) \end{pmatrix}. \tag{3.3.28}$$

Now we compute the interaction coefficients to get

$$\Gamma^1_{11} = \Gamma^2_{11} = \Gamma^2_{33} = \Gamma^3_{33} = 0, \tag{3.3.29a}$$

$$\Gamma^2_{13} = \Gamma^2_{31} = \Gamma^3_{13} = \Gamma^1_{31} = 0, \tag{3.3.29b}$$

$$\Gamma^1_{13} = \Gamma^1_{33} = \beta,\ \Gamma^3_{11} = \Gamma^3_{31} = -\beta. \tag{3.3.29c}$$

We further simplify the problem by taking the base flow speed to be identically zero, that is, $v_0 = 0$. Thus (3.1.58) reduces to

$$\frac{\partial\sigma_1^{(1)}}{\partial\theta_3} = -\left(\frac{\beta}{2c_0}\right)\{\sigma_3(\theta_3)(\sigma_3'(\theta_3) + \sigma_1(\theta_1)\sigma_3'(\theta_3))\}, \tag{3.3.30a}$$

$$\frac{\partial\sigma_2^{(1)}}{\partial\theta_1} - \frac{\partial\sigma_2^{(1)}}{\partial\theta_3} = 0, \tag{3.3.30b}$$

$$\frac{\partial\sigma_3^{(1)}}{\partial\theta_1} = -\left(\frac{\beta}{2c_0}\right)\{\sigma_3(\theta_3)\sigma_1'(\theta_1) + \sigma_1(\theta_1)\sigma_1'(\theta_1)\}, \tag{3.3.30c}$$

whereas (3.1.59) and (3.1.60) reduce to

$$\frac{\partial}{\partial\theta_1}(x^{(1)} - c_0 t^{(1)}) = \left(\frac{1}{2c_0}\right)[\alpha\sigma_3(\theta_3) - \beta\sigma_1(\theta_1)], \tag{3.3.31}$$

and

$$\frac{\partial}{\partial\theta_3}(x^{(1)} + c_0 t^{(1)}) = \left(\frac{1}{2c_0}\right)[-\beta\sigma_3(\theta_3) + \alpha\sigma_1(\theta_1)]. \tag{3.3.32}$$

After integration we have

$$\begin{pmatrix}\rho\\ v\\ S\end{pmatrix} = \begin{pmatrix}\rho_0\\ 0\\ S_0\end{pmatrix} + \mathbf{u}$$
$$= \begin{pmatrix}\rho_0\\ 0\\ S_0\end{pmatrix} + \epsilon\sigma_1(\theta_1)\begin{pmatrix}\rho_0\\ -c_0\\ 0\end{pmatrix} + \epsilon\sigma_3(\theta_3)\begin{pmatrix}\rho_0\\ c_0\\ 0\end{pmatrix}$$
$$+ \epsilon^2\left(-\frac{\beta}{2c_0}\right)\left[\frac{1}{2}\sigma_3^2(\theta_3) - \frac{1}{2}\sigma_3^2(\theta_1) - \sigma_1(\theta_1)\sigma_3(\theta_1) + \sigma_1(\theta_1)\sigma_3(\theta_3)\right]\begin{pmatrix}\rho_0\\ -c_0\\ 0\end{pmatrix}$$
$$+ \epsilon^2\left(-\frac{\beta}{2c_0}\right)\left[\frac{1}{2}\sigma_1^2(\theta_1) - \frac{1}{2}\sigma_1^2(\theta_3) - \sigma_1(\theta_3)\sigma_3(\theta_3) + \sigma_1(\theta_1)\sigma_3(\theta_3)\right]\begin{pmatrix}\rho_0\\ c_0\\ 0\end{pmatrix}$$
$$+ O(\epsilon^3), \tag{3.3.33}$$

$$\left.\begin{aligned} x &= \frac{1}{2}(\theta_1 + \theta_3) + \epsilon\left(\frac{1}{4c_0}\right)\Big\{\alpha(\theta_3 - \theta_1)[\sigma_1(\theta_1) - \sigma_3(\theta_3)] \\ &\qquad + \beta\int_{\theta_1}^{\theta_3}[\sigma_1(s) - \sigma_3(s)]ds\Big\} + O(\epsilon^2), \\ t &= \left(\frac{1}{2c_0}\right)(\theta_1 - \theta_3) + \epsilon\left(\frac{1}{2c_0}\right)^2\Big\{\alpha(\theta_3 - \theta_1)[\sigma_1(\theta_1) + \sigma_3(\theta_3)] \\ &\qquad - \beta\int_{\theta_1}^{\theta_3}[\sigma_1(s) + \sigma_3(s)]ds\Big\} + O(\epsilon^2). \end{aligned}\right\} \tag{3.3.34}$$

In particular, we may note that

$$S = S_0 + O(\epsilon^3), \tag{3.3.35}$$

so that the entropy wave is a higher order effect.

As a particular example, we shall carry out some numerical experiments for a polytropic gas for which [70]

$$T = e^{S/c_v}\rho^{\gamma-1},\ p = Re^{S/c_v},\ e = c_vT, \tag{3.3.36}$$

where $R = c_p - c_v > 0$ is the gas constant, $c_v > 0$ and $c_p > 0$ are the specific heats of constant volume and pressure, respectively, and $1 < \gamma = c_p/c_v < 2$ is the ratio of specific heats. After some direct calculations we find that

$$\alpha = \frac{\gamma + 1}{2}, \ \beta = \frac{3 - \gamma}{2}, \tag{3.3.37}$$

and, as $1 < \gamma < 2$, we have $1 < \alpha < \frac{3}{2}$ and $\frac{1}{2} < \beta < 1$. The figures that follow represent calculations performed for air at a moderate temperature, for which [8] $\gamma = 1.4$ and hence $\alpha = 1.2$, and $\beta = 0.8$. For both the density waves of Fig. 3.4 and the velocity waves of Fig. 3.5, the impact at $t = 0$ is based upon the functions

$$\sigma_1(x) = \begin{cases} [(x-2)^2 - 1]^2, & |x - 2| \le 1, \\ 0, & |x - 2| > 1, \end{cases} \tag{3.3.38a}$$

and

$$\sigma_2(x) = \begin{cases} [(x+2)^2 - 1]^2, & |x + 2| \le 1, \\ 0, & |x + 2| > 1. \end{cases}, \tag{3.3.38b}$$

that is, $\sigma_2(x) = \sigma_1(-x)$.

In the sequence of time frames $t = 0, 1, 2, 3$, and 4 shown in Fig. 3.4, the propagation and nonlinear interaction of the density waves is depicted clearly. It is an advantage of our method that the solution is obtained explicitly so that numerical work is very straightforward. Shown also in Fig. 3.5 is the propagation and nonlinear interaction of the velocity waves.

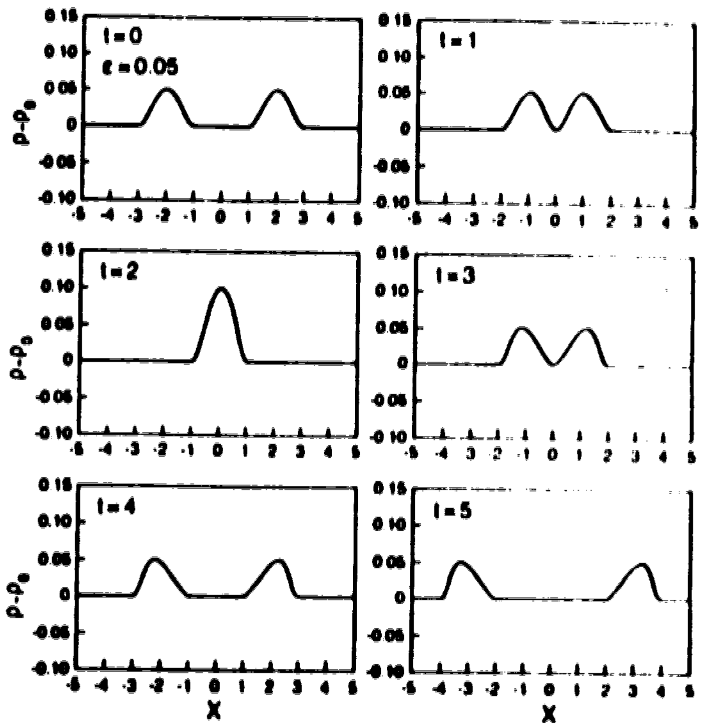


Fig. 3.4: Density waves at $t = 0, 1, 2, 3, 4$ for $\gamma = 1.4$ and $\epsilon = 0.05$.

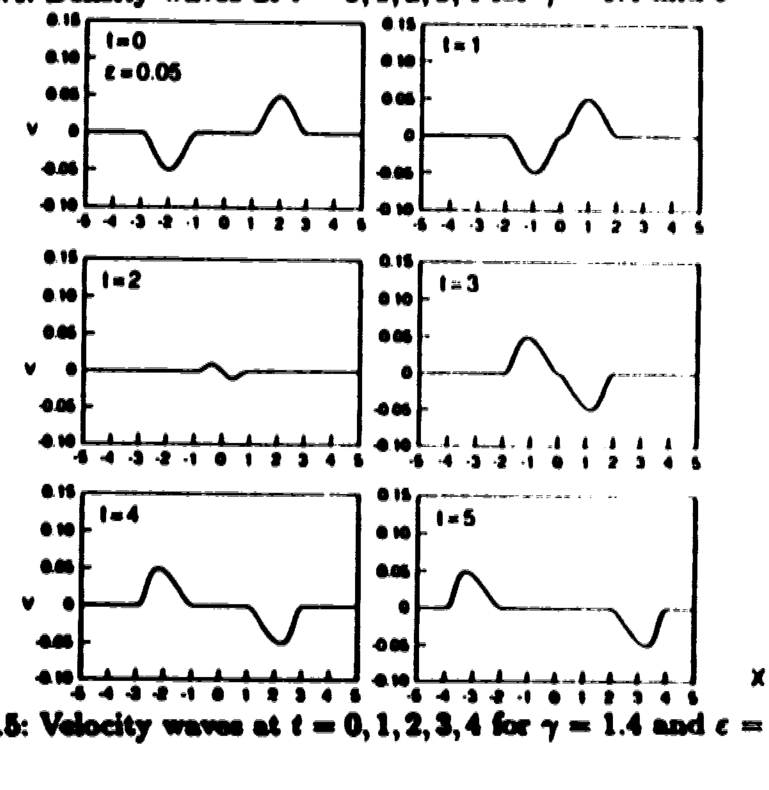


Fig. 3.5: Velocity waves at $t = 0, 1, 2, 3, 4$ for $\gamma = 1.4$ and $\epsilon = 0.05$.

3.3.2 *Propagation and interaction of two weak sound waves arising from boundary*

disturbance

We shall perturb two sound waves at the boundary, that is, we pick $p = 1$, $q = 3$. We therefore require that

$$\lambda_1^{(0)} = v_0 - c_0 > 0, \tag{3.3.39}$$

or, equivalently, that

$$M_0 = v_0/c_0 > 1, \tag{3.3.40}$$

where M_0 is the Mach number for the steady state. That is, our base steady state is a supersonic steady flow.

In light of Remark 3.2, we may take the boundary perturbation to be

$$\mathbf{u}\big|_{x=0} = \epsilon\sigma_1(t)\mathbf{r}_1 + \epsilon\sigma_3(t)\mathbf{r}_3, \tag{3.3.41}$$

where ϵ is the small perturbation parameter and $\sigma_1(\cdot)$, $\sigma_3(\cdot)$ are smooth scalar functions satisfying

$$\sigma_1(0) = \sigma_1'(0) = 0, \ \sigma_3(0) = \sigma_3'(0) = 0. \tag{3.3.42}$$

The results derived in the previous section suggest the perturbation solution to be

$$\mathbf{u} = \epsilon\sigma_1(\theta_1)\mathbf{r}_1 + \epsilon\sigma_3(\theta_3)\mathbf{r}_3 + \epsilon^2\sum_{i=1}^{3}\sigma_i^{(1)}(\theta_1,\theta_3)\mathbf{r}_i + O(\epsilon^3), \tag{3.3.43}$$

with $\sigma_i^{(1)}(\theta_1,\theta_3)$ satisfying (3.2.40).

We compute $\sigma_i^{(1)}(\theta_1,\theta_3)$ $(i = 1, 2, 3)$ by directly integrating (3.2.40). Here we note that the interaction coefficients, as have been detailed in the previous subsection, are

$$\Gamma^1_{11} = \Gamma^2_{11} = \Gamma^2_{33} = \Gamma^3_{33} = 0, \tag{3.3.29a}$$

$$\Gamma^2_{13} = \Gamma^2_{31} = \Gamma^3_{13} = \Gamma^1_{31} = 0, \tag{3.3.29b}$$

$$\Gamma^1_{13} = \Gamma^1_{33} = \beta,\ \Gamma^3_{11} = \Gamma^3_{31} = -\beta, \tag{3.3.29c}$$

where $\beta = c_0 - \frac{1}{2}(\rho_0/c_0)(p_{\rho\rho})_0$.

Therefore, (3.2.40) reduces to

$$\frac{\partial\sigma_1^{(1)}}{\partial\theta_3} = \left(\frac{\beta}{2c_0}\right)\left\{\sigma_1(\theta_1)\sigma_3'(\theta_3) + \sigma_3(\theta_3)\sigma_3'(\theta_3)\right\}, \tag{3.3.44a}$$

$$(1 + M_0)\frac{\partial\sigma_2^{(1)}}{\partial\theta_1} + (1 - M_0)\frac{\partial\sigma_2^{(1)}}{\partial\theta_3} = 0, \tag{3.3.44b}$$

$$\frac{\partial\sigma_3^{(1)}}{\partial\theta_1} = \left(\frac{\beta}{2c_0}\right)\left\{\sigma_1(\theta_1)\sigma_1'(\theta_1) + \sigma_3(\theta_3)\sigma_1'(\theta_1)\right\}. \tag{3.3.44c}$$

In the mean time we have from (3.3.22)–(3.3.26) that

$$\lambda_3^{(1)}(\mathbf{r}_3) = -\lambda_1^{(1)}(\mathbf{r}_1) = \alpha,$$
$$\lambda_1^{(1)}(\mathbf{r}_3) = -\lambda_3^{(1)}(\mathbf{r}_1) = \beta,$$
$$\lambda_1^{(1)}(\mathbf{r}_2) = -\lambda_3^{(1)}(\mathbf{r}_2) = \delta,$$
$$\lambda_2^{(1)}(\mathbf{r}_3) = -\lambda_2^{(1)}(\mathbf{r}_1) = c_0,\ \lambda_2^{(1)}(\mathbf{r}_2) = 0,$$

where $\alpha = c_0 + \frac{1}{2}(\rho_0/c_0)(p_{\rho\rho})_0$ and $\delta = \frac{1}{2}c_0(p_\rho s)_0 - \frac{1}{2}c_0^{-1}(ps)_0(p_{\rho\rho})_0$. Thus (3.2.41) and (3.2.42) reduce to

$$\left[x^{(1)} - c_0(M_0 - 1)t^{(1)}\right]_{\theta_3} = \left(\frac{M_0 + 1}{2}\right)\left\{-\alpha\sigma_1(\theta_1) + \beta\sigma_3(\theta_3)\right\}, \tag{3.3.45}$$

$$\left[x^{(1)} - c_0(M_0 + 1)t^{(1)}\right]_{\theta_1} = \left(\frac{M_0 - 1}{2}\right)\left\{\beta\sigma_1(\theta_1) - \alpha\sigma_3(\theta_3)\right\}. \tag{3.3.46}$$

Upon integrating we find

$$
\begin{pmatrix} \rho \\ v \\ S \end{pmatrix} = \begin{pmatrix} \rho_0 \\ v_0 \\ S_0 \end{pmatrix} + \mathbf{u}
$$

$$
\begin{aligned}
&= \begin{pmatrix} \rho_0 \\ v_0 \\ S_0 \end{pmatrix} + \varepsilon\sigma_1(\theta_1)\begin{pmatrix} \rho_0 \\ -c_0 \\ 0 \end{pmatrix} + \varepsilon\sigma_3(\theta_3)\begin{pmatrix} \rho_0 \\ c_0 \\ 0 \end{pmatrix} \\
&\quad + \varepsilon^2\left(\frac{\beta}{2c_0}\right)\left[\frac{1}{2}\sigma_3^2(\theta_3) - \frac{1}{2}\sigma_3^2(\theta_1) - \sigma_1(\theta_1)\sigma_3(\theta_1) + \sigma_1(\theta_1)\sigma_3(\theta_3)\right]\begin{pmatrix} \rho_0 \\ -c_0 \\ 0 \end{pmatrix} \\
&\quad + \varepsilon^2\left(\frac{\beta}{2c_0}\right)\left[\frac{1}{2}\sigma_1^2(\theta_1) - \frac{1}{2}\sigma_1^2(\theta_3) - \sigma_1(\theta_3)\sigma_3(\theta_3) + \sigma_1(\theta_1)\sigma_3(\theta_3)\right]\begin{pmatrix} \rho_0 \\ c_0 \\ 0 \end{pmatrix} \\
&\quad + O(\varepsilon^3).
\end{aligned} \tag{3.3.47}
$$

$$
\begin{aligned}
x &= \frac{1}{2}(M_0^2 - 1)c_0(\theta_3 - \theta_1) \\
&\quad + \frac{\varepsilon}{4}\Big\{-\alpha\left[(M_0+1)^2\sigma_1(\theta_1) + (M_0-1)^2\sigma_3(\theta_3)\right](\theta_3 - \theta_1) \\
&\quad + \beta\int_{\theta_1}^{\theta_3}\left[(M_0-1)^2\sigma_1(s) + (M_0+1)^2\sigma_3(s)\right]ds\Big\} + O(\varepsilon^2),
\end{aligned} \tag{3.3.48}
$$

$$
\begin{aligned}
t &= -\frac{1}{2}(M_0 - 1)\theta_1 + \frac{1}{2}(M_0 + 1)\theta_3 \\
&\quad + \frac{\varepsilon}{4c_0}\Big\{-\alpha\left[(M_0+1)\sigma_1(\theta_1) + (M_0-1)\sigma_3(\theta_3)\right](\theta_3 - \theta_1) \\
&\quad + \beta\int_{\theta_1}^{\theta_3}\left[(M_0-1)\sigma_1(s) + (M_0+1)\sigma_3(s)\right]ds\Big\} + O(\varepsilon^2).
\end{aligned} \tag{3.3.49}
$$

We also note in particular that

$$
S = S_0 + O(\varepsilon^3), \tag{3.3.50}
$$

indicating that the entropy wave is a higher order effect.

We carry out our example calculation for polytropic gas for which

$$
T = e^{S/c_v}\rho^{\gamma-1}, \; p = Re^{S/c_v}, \; e = c_vT, \tag{3.3.26}
$$

with the constants R, c_v, c_p as explained before. $1 < \gamma = c_p/c_v < 2$ and we have

$$\alpha = \frac{\gamma + 1}{2}, \ \beta = \frac{3 - \gamma}{2}. \tag{3.3.37}$$

Figure 3.6 and 3.7 below depict the density and velocity waves, respectively. The calculation is performed for air at moderate temperatures with $\gamma = 1.4$ and hence $\alpha = 1.2$, $\beta = 0.8$. We also choose, for simplicity, $M_0 > 1$ and $\rho_0 = c_0 = 1$. For both waves, the input functions at $x = 0$ are

$$\sigma_1(t) = \begin{cases} [1 - (t-1)^2]^2, & |t - 1| \leq 1, \\ 0, & |t - 1| > 1, \end{cases} \tag{3.3.51}$$

$$\sigma_3(t) = \begin{cases} [1 - (t-3)^2]^2, & |t - 3| \leq 1, \\ 0, & |t - 3| > 1. \end{cases}, \tag{3.3.52}$$

that is, $\sigma_3(t + 2) = \sigma_1(t)$.

In the sequence of time frames $t = 1, 2, \ldots, 9$ shown in Figures 3.6 and 3.7, the propagation and nonlinear interaction of both density and velocity waves is depicted clearly. We observe that, as $M_0 > 1$, the waveform associated with mode $\mathbf{r}_3$ overtakes the one associated with mode $\mathbf{r}_1$.

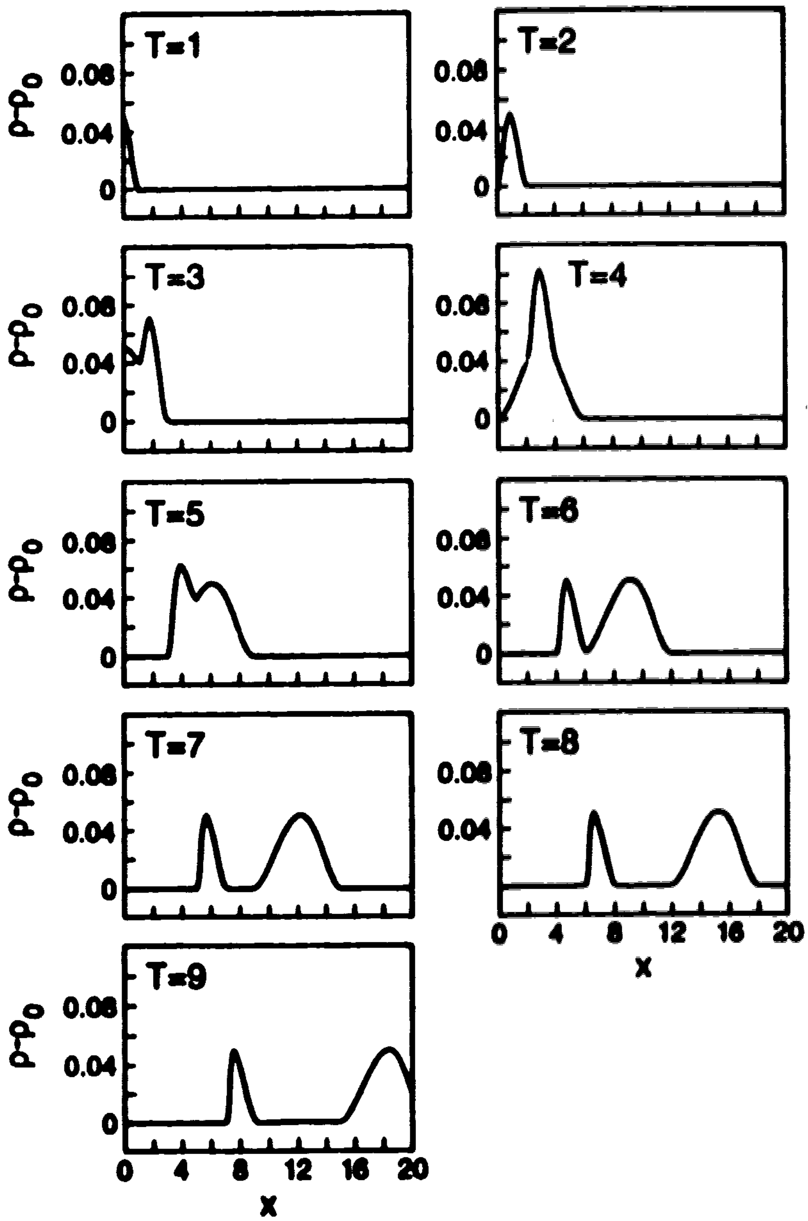


Fig. 3.6: Density waves at $t = 1, 2, \ldots, 9$ for $\gamma = 1.4$, $\epsilon = 0.05$, and $M_0 = 1.5$.

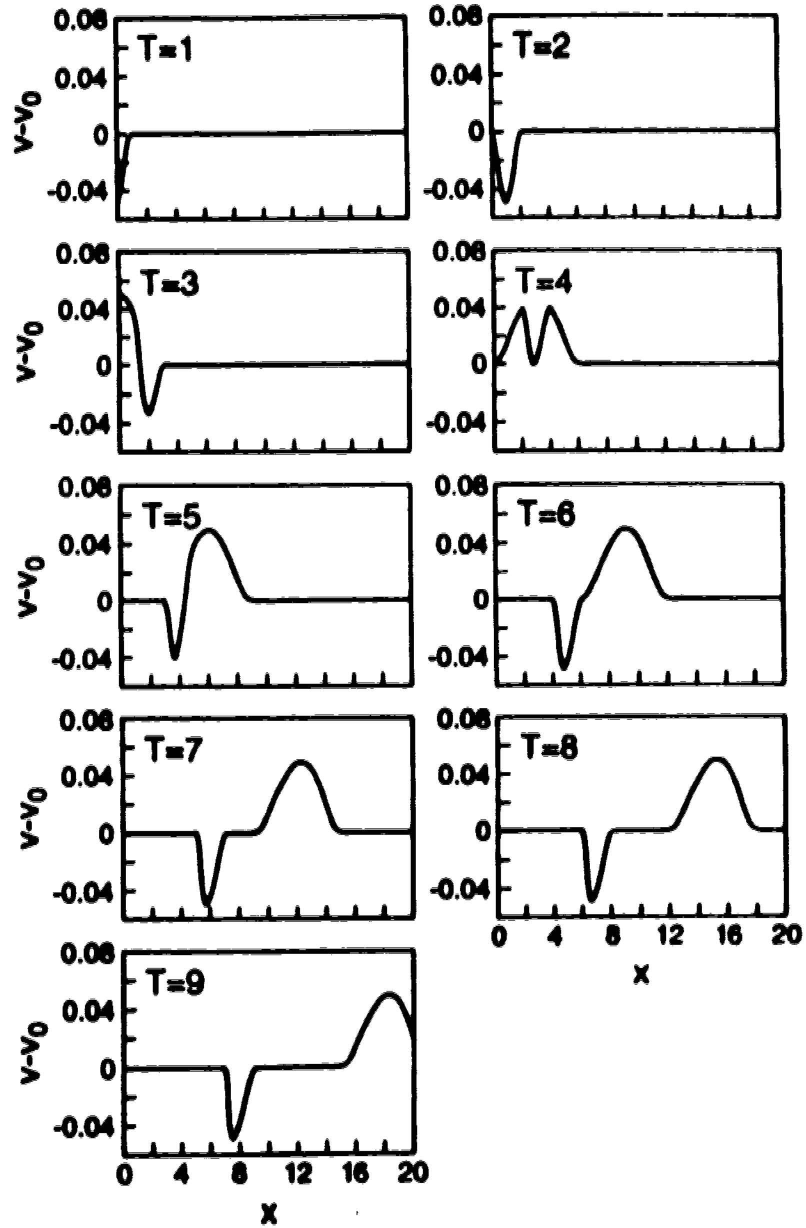
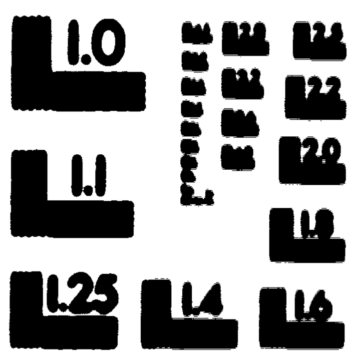


Fig. 3.7: Velocity waves at $t = 1, 2, \ldots, 9$ for $\gamma = 1.4$, $\varepsilon = 0.05$, and $M_0 = 1.5$.

3 of/de 3

PM-1 3½"x4" PHOTOGRAPHIC MICROCOPY TARGET
NBS 1010a ANSI/ISO #2 EQUIVALENT



PRECISION℠ RESOLUTION TARGETS

CHAPTER 4.

Nonlinear Kelvin Waves in a Channel

4.1. Introduction

In the context of geophysical fluid dynamics, the shallow water model, which describes the dynamics of a shallow, rotating layer of homogeneous, incompressible, and inviscid fluid, has long been studied and considered prototypical for being capable of capturing important aspects of atmospheric and oceanic motions as well as giving insight into the nature of the subject of geophysical fluid dynamics.

The shallow water equations we are interested in are

$$u_t + uu_x + vu_y - fv = -gh_x, \tag{4.1.1}$$

$$v_t + uv_x + vv_y + fu = -gh_y, \tag{4.1.2}$$

$$h_t + \{(h - h_B)u\}_x + \{(h - h_B)v\}_y = 0, \tag{4.1.3}$$

when a sheet of fluid with constant and uniform density (as shown below) is considered.

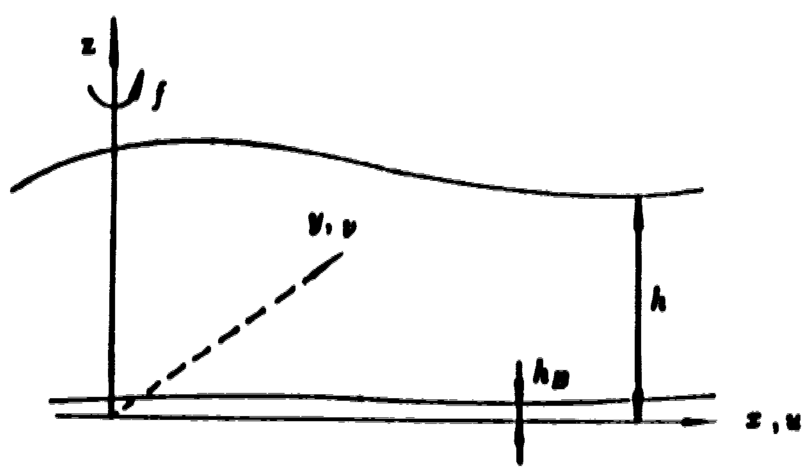


Fig. 4.1: Layer of fluid on a variable bottom.

The notations used are as follows: u, v represent flow velocity in the x and y directions, respectively, h is the height of the water surface, and h_B describes the bottom topography, while f is a constant reflecting the action of the Coriolis force due to the earth's rotation. Obviously, (4.1.1) and (4.1.2) follow from the momentum equations. (4.1.3) follows from the mass conservation . As a simplification, the pressure is assumed to be hydrostatic. See, for example, J. Pedlosky [67] for more detail.

The shallow water equations (4.1.1)–(4.1.3) can be rewritten in a slightly different form. Take the constant H to be the mean surface water height and η the surface elevation from the mean, then $h = H + \eta$ and (4.1.1)–(4.1.3) become

$$u_t + uu_x + vu_y - fv = -g\eta_x, \tag{4.1.4}$$

$$v_t + uv_x + vv_y + fu = -g\eta_y, \tag{4.1.5}$$

$$\eta_t + \{(h_0 + \eta)u\}_x + \{(h_0 + \eta)v\}_y = 0, \tag{4.1.6}$$

where $h_0 = H - h_B$.

When the shallow water flow is restricted to a partially bounded region, namely a channel of width $2L$ oriented parallel to the x-axis, as shown in Fig. 4.2, the linearized theory gives rise to Poincaré, Kelvin, and Rossby waves. Among them, Kelvin waves stand apart owing to the fact that they are nondispersive and exhibit purely hyperbolic behaviour along the x-direction. In addition, their cross-channel velocity vanishes identically.

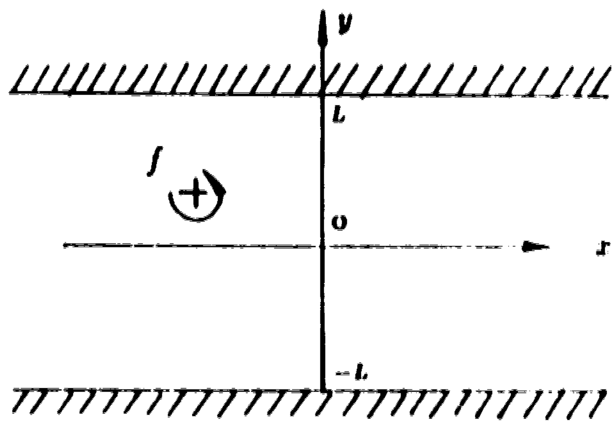


Fig. 4.2: The channel of width $2L$.

4.1.1 *Linear Kelvin waves*

Indeed, under the small amplitude regime, the linearized version of the shallow water equations (4.1.4)–(4.1.6) is

$$u_t - fv = -g\eta_x, \tag{4.1.7}$$

$$v_t + fu = -g\eta_y, \tag{4.1.8}$$

$$\eta_t + (h_0 u)_x + (h_0 v)_y = 0. \tag{4.1.9}$$

Taking $v \equiv 0$ and h_0 to be a constant, we reduce (4.0.7)–(4.0.9) to

$$u_t = -g\eta_x, \tag{4.1.10}$$

$$fu = -g\eta_y, \tag{4.1.11}$$

$$\eta_t = -h_0 u_x, \tag{4.1.12}$$

which gives for η

$$\eta_{tt} = c_0^2 \eta_{xx}, \tag{4.1.13}$$

where $c_0^2 = gh_0$.

It is straightforward to verify that the general solution to (4.1.10)–(4.1.12) takes the form

$$\eta(x, y, t) = e^{\frac{f}{c_0} y} \sigma_1(x + c_0 t) + e^{-\frac{f}{c_0} y} \sigma_2(x - c_0 t) + \gamma(y), \tag{4.1.14}$$

$$\begin{aligned} u(x, y, t) &= -\frac{g}{f} \eta_y(x, y, t) \\ &= -\frac{g}{c_0} e^{\frac{f}{c_0} y} \sigma_1(x + c_0 t) + \frac{g}{c_0} e^{-\frac{f}{c_0} y} \sigma_2(x - c_0 t) - \frac{g}{f} \gamma'(y), \end{aligned} \tag{4.1.15}$$

where $\sigma_1, \sigma_2 \in C^2(\mathbb{R})$ and $\gamma \in C^1[-L, L]$ are arbitrary functions. Apparently, when initial value problem is considered for the system (4.1.10)–(4.1.12), the equation $fu = -g\eta_y$ puts a severe restriction on the initial data that can be prescribed. From the general solution above, it is clear that only initial data of the form

$$\eta(x, y, 0) = e^{\frac{f}{c_0} y} \sigma_1(x) + e^{-\frac{f}{c_0} y} \sigma_2(x) + \gamma(y), \tag{4.1.16}$$

$$u(x, y, 0) = -\frac{g}{f} \eta_y(x, y, 0). \tag{4.1.17}$$

are allowed.

We call (4.1.16) the *admissible initial profile* of surface elevation for Kelvin waves.

The above analysis provides, in the linear context, the full solution for the Kelvin waves as well as some additional insights.

In fact, (4.1.14) indicates that, for any admissible initial profile of the surface elevation, the variation in the x-direction translates into travelling waves, and is counter-balanced by a *guidance* profile in the y-direction, of the definite forms, $e^{\frac{f}{c_0} y}$ or $e^{-\frac{f}{c_0} y}$. The sign determines, in turn, the direction of the generated travelling wave. For an observer riding with the travelling wave, the waveheight increases to the right. The constant

$$R = \frac{c_0}{f},$$

as pointed out by Pedlosky [67], is an intrinsic length scale and called the *Rossby radius of deformation*. It is the distance over which the tendency of gravity to render the free surface flat is balanced by the Coriolis acceleration deforming the surface. For the part of the initial surface elevation profile uniform in the x direction, namely $\gamma(y)$, no particular guidance in the y-direction is required, but is rather related to the uniform and stationary part of the initial along-channel velocity profile, namely $\gamma'(y)$, again reflecting the action of Coriolis force to a moving flow.

Our objective in this chapter is to study the nonlinear version of Kelvin waves. The basic idea behind our approach is as follows. We study the nonlinear shallow water equations (4.1.4)–(4.1.6) in the small amplitude regime by using an asymptotic perturbation method, insisting upon the fact that the cross-channel velocity appears only at the $O(\varepsilon^2)$ order. Nonlinear phases are introduced to take account of the nonlinear hyperbolic nature of the problem and the two-wave interaction theory developed in the previous chapter is applied.

Now we formulate the problem and summarize the main results for nonlinear Kelvin waves in the following two subsections.

4.1.2 *Formulation of the problem*

We are concerned with the nonlinear shallow water equations

$$u_t + uu_x + vu_y - fv = -g\eta_x, \tag{4.1.4}$$

$$v_t + uv_x + vv_y + fu = -g\eta_y, \tag{4.1.5}$$

$$\eta_t + \{(h_0 + \eta)u\}_x + \{(h_0 + \eta)v\}_y = 0. \tag{4.1.6}$$

The fact that the flow is confined to a channel oriented parallel to the x-axis, as shown in Figure 4.2, leads to the boundary conditions,

$$v\big|_{y=-L} = v\big|_{y=L} = 0. \tag{4.1.18}$$

The initial configuration of the flow is to be given later. We bear in mind that proper formulation of the initial condition must ensure that the cross-channel velocity v, as time evolves, appears only at $O(\epsilon^2)$. Here ϵ is the to-be-introduced perturbation parameter. Hence a mixed initial boundary value problem is prescribed.

The shallow water equations alluded to above can be nondimensionized. We choose the half-width of the channel, namely L, and the shallow water depth h_0 to be the typical horizontal and vertical length scales, respectively. The linear shallow water wave speed $c_0 = \sqrt{gh_0}$ is designated as the typical velocity scale. The following dimensionless variables are introduced

$$(x', y') = (x, y)/L, \quad \eta' = \eta/h_0, \tag{4.1.19a}$$

$$(u', v') = (u, v)/c_0, \tag{4.1.19b}$$

$$t' = t/\left(\frac{L}{c_0}\right). \tag{4.1.19c}$$

As a result, the nondimensional shallow water system takes the form

$$u_t + uu_x + vu_y - \nu v = -\eta_x, \tag{4.1.20}$$

$$v_t + uv_x + vv_y + \nu u = -\eta_y, \tag{4.1.21}$$

$$\eta_t + \{(1+\eta)u\}_x + \{(1+\eta)v\}_y = 0, \tag{4.1.22}$$

where, for notational convenience we have dropped all the primes after the nondimensionalization process. Here

$$\nu = fL/c_0 = L/R, \tag{4.1.23}$$

is a dimensionless parameter which gives the rati[illegible] ngth scale to the Rossby deformation radius. In the geoph[illegible] a length in the scale of hundreds or thousands of kilometers [60,67] [illegible] ove model is most proper for the description of oceanic motio[illegible] de, ν essentially vanishes.

Also, the boundary conditions (4.1.18) become[illegible] dimensionization,

$$v\big|_{y=-1} = v\big|_{y=1} = 0 \tag{4.1.24}$$

We rewrite (4.1.20), (4.1.22) in the form

$$\begin{aligned} u_t + uu_x + \eta_x &= \nu v - vu_y, \\ \eta_t + u\eta_x + (1+\eta)u_x &= -\left\{(1+\eta)v\right\}_y, \end{aligned}$$

or

$$\mathbf{u}_t + \mathbf{A}\mathbf{u}_x = \mathbf{B}(\mathbf{u}, v), \tag{4.1.25}$$

with

$$\mathbf{u} = \begin{pmatrix} u \\ \eta \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} u & 1 \\ 1+\eta & u \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} \nu(v - u_y) \\ -\left\{(1+\eta)v\right\}_y \end{pmatrix}. \tag{4.1.26}$$

Now (4.1.25) is coupled with (4.1.21) through $\mathbf{B}(\mathbf{u}, v)$, which is rewritten as

$$v_t + uv_x = -\nu u - \eta_y - vv_y \triangleq b. \tag{4.1.27}$$

We focus on (4.1.25) and introduce phase variables while regarding y as a parameter.

The eigenvalues of $\mathbf{A}$ are

$$\lambda_1 = u - \sqrt{1+\eta} < \lambda_2 = u + \sqrt{1+\eta}. \tag{4.1.28}$$

We expect that, as a result of the fact that cross-channel velocity is a higher order effect, $\mathbf{B}(\mathbf{u}, v)$ is also a higher order term. Thus (4.1.25) can be treated as a nonlinear hyperbolic system and the two nonlinear phase variables $\theta_1 = \theta_1(x, y, t)$, $\theta_2 = \theta_2(x, y, t)$ are introduced and defined as the solutions to

$$\theta_{1,t} + \lambda_1(\mathbf{u})\theta_{1,x} = 0, \tag{4.1.29}$$

$$\theta_1\big|_{t=0} = x, \tag{4.1.30}$$

and

$$\theta_{2,t} + \lambda_2(\mathbf{u})\theta_{2,x} = 0, \tag{4.1.31}$$

$$\theta_2\big|_{t=0} = x, \tag{4.1.32}$$

respectively, where here a subscript preceded by a comma will imply differentiation with respect to that variable.

The existence and smoothness of $\theta_1 = \theta_1(x, y, t)$, $\theta_2 = \theta_2(x, y, t)$ depend on, in turn, the existence and smoothness of the solution to the aforementioned mixed initial boundary value problem. This we shall assume to be the case.

We now transform the original spatial-temporal coordinates into phase-cross channel coordinates. This is indeed an extended version of the nonlinear phase transformation introduced in the previous chapter. That is

$$(x, y, t) \longmapsto (\theta_1, \theta_2, y^*), \tag{4.1.33}$$

by

$$\theta_1 = \theta_1(x, y, t), \tag{4.1.34a}$$

$$\theta_2 = \theta_2(x, y, t), \tag{4.1.34b}$$

$$y^* = y. \tag{4.1.34c}$$

We write the inversion of (4.0.34) as

$$x = x(\theta_1, \theta_2, y^*), \tag{4.1.35a}$$

$$t = t(\theta_1, \theta_2, y^*), \tag{4.1.35b}$$

$$y = y^*. \tag{4.1.35c}$$

As a result of the nonlinear phase transformation, we have the following differential relations

$$\partial_x \longmapsto \theta_{1,x}\partial_{\theta_1} + \theta_{2,x}\partial_{\theta_2}, \tag{4.1.36a}$$

$$\partial_t \longmapsto \theta_{1,t}\partial_{\theta_1} + \theta_{2,t}\partial_{\theta_2}, \tag{4.1.36b}$$

$$\partial_y \longmapsto \theta_{1,y}\partial_{\theta_1} + \theta_{2,y}\partial_{\theta_2} + \partial y^*. \tag{4.1.36c}$$

The shallow water equations(4.1.25), (4.1.27) can thus be transformed into representations in $(\theta_1, \theta_2, y^*)$ coordinates

$$t_{\theta_2}(\mathbf{A} - \lambda_1 \mathbf{I})\mathbf{u}_{\theta_1} - t_{\theta_1}(\mathbf{A} - \lambda_2 \mathbf{I})\mathbf{u}_{\theta_2} = (\lambda_2 - \lambda_1)t_{\theta_1}t_{\theta_2}(\mathbf{B}^* + \mathbf{D}), \tag{4.1.37}$$

and

$$t_{\theta_2}(u - \lambda_1)v_{\theta_1} - t_{\theta_1}(u - \lambda_2)v_{\theta_2} = (\lambda_2 - \lambda_1)t_{\theta_1}t_{\theta_2}(b^* + d), \tag{4.1.38}$$

respectively. Here $\mathbf{D}$ and d are related to $\theta_{1,y}$ and $\theta_{2,y}$.

$$\mathbf{D} = \begin{pmatrix} -v(\theta_{1,y}u_{\theta_1} + \theta_{2,y}u_{\theta_2}) \\ -\theta_{1,y}\{(1+\eta)v\}_{\theta_1} - \theta_{2,y}\{(1+\eta)v\}_{\theta_2} \end{pmatrix}, \tag{4.1.39}$$

$$d = -(\theta_{1,y}\eta_{\theta_1} + \theta_{2,y}\eta_{\theta_1}) - v(\theta_{1,y}v_{\theta_1} + \theta_{2,y}v_{\theta_2}). \tag{4.1.40}$$

Meanwhile, $\mathbf{B}^*$ and b^* are obtained by replacing all differentiations with respect to

y by those with respect to y^*, that is

$$\mathbf{B}^* = \begin{pmatrix} v(\nu - u_{y^*}) \\ -\{(1+\eta)v\}_{y^*} \end{pmatrix}, \tag{4.1.41}$$

$$b^* = -\nu u - \eta_{y^*} - v v_{y^*}. \tag{4.1.42}$$

We also note that, as in the previous chapter, the differentiation of (4.1.35) with respect to θ_1, θ_2 leads to, after employing the definitions of two phase variables (4.1.29) and (4.1.30) that

$$x_{\theta_1} - \lambda_2 t_{\theta_1} = 0, \tag{4.1.43}$$

$$x_{\theta_2} - \lambda_1 t_{\theta_2} = 0. \tag{4.1.44}$$

Now the equations (4.1.37), (4.1.38), supplemented by (4.1.43), (4.1.44), and the boundary condition (4.1.24), constitute governing equations in the new phase-cross channel coordinates.

4.1.3 *Main results*

We now perturb the shallow water equations from their steady state $(u, \eta, v) = (0, 0, 0)$ and assume that an asymptotic solution in the following form is admitted

$$u = \epsilon u^{(0)}(\theta_1, \theta_2, y^*) + \epsilon^2 u^{(1)}(\theta_1, \theta_2, y^*) + O(\epsilon^3), \tag{4.1.45a}$$

$$\eta = \epsilon \eta^{(0)}(\theta_1, \theta_2, y^*) + \epsilon^2 \eta^{(1)}(\theta_1, \theta_2, y^*) + O(\epsilon^3), \tag{4.1.45b}$$

$$v = \epsilon v^{(0)}(\theta_1, \theta_2, y^*) + \epsilon^2 v^{(1)}(\theta_1, \theta_2, y^*) + O(\epsilon^3), \quad v^{(0)} \equiv 0. \tag{4.1.45c}$$

Here ϵ is the perturbation parameter. The requirement that $v^{(0)} \equiv 0$ is imposed to reflect the nature of the Kelvin wave. In particular, the leading term solution $(u^{(0)}, \eta^{(0)}, v^{(0)})$ will be seen to recover the linear Kelvin waves when the shallow water equations are linearized at the steady state.

$$x = x^{(0)}(\theta_1, \theta_2, y^*) + \varepsilon x^{(1)}(\theta_1, \theta_2, y^*) + O(\varepsilon^2), \qquad (4.1.46a)$$

$$t = t^{(0)}(\theta_1, \theta_2, y^*) + \varepsilon t^{(1)}(\theta_1, \theta_2, y^*) + O(\varepsilon^2). \qquad (4.1.46b)$$

To derive the requisite $O(1)$ and $O(\varepsilon)$ problems, we need to expand the following nonlinear (matrix, vector and scalar) functions

$$\mathbf{A} = \mathbf{A}_0 + \varepsilon \mathbf{A}_1 + O(\varepsilon^2), \qquad (4.1.47)$$

$$\mathbf{B}^* = \varepsilon \mathbf{B}_0^* + \varepsilon^2 \mathbf{B}_1^* + O(\varepsilon^3), \qquad (4.1.48)$$

$$\lambda_i = \lambda_i^{(0)} + \varepsilon \lambda_{i,1} + O(\varepsilon^2), \qquad (4.1.49)$$

$$b^* = \varepsilon b_0^* + \varepsilon^2 b_1^* + O(\varepsilon^3). \qquad (4.1.50)$$

These are accomplished by first Taylor expanding the functions about the steady state, then substituting (4.1.45) into them and collecting like powers of ε.

Due to the appearance of $\theta_{1,y}$ and $\theta_{2,y}$, the expansions of $\mathbf{D}$ and d are more complicated. However, as we shall see, $\theta_{1,y}$ and $\theta_{2,y}$ have no contribution to the $O(1)$ problem. We write first

$$\mathbf{D} = \varepsilon \mathbf{D}_0 + \varepsilon^2 \mathbf{D}_1 + O(\varepsilon^3), \qquad (4.1.51)$$

$$d = \varepsilon d_0 + \varepsilon^2 d_1 + O(\varepsilon^3). \qquad (4.1.52)$$

Now, by substituting (4.1.45) and all the above expansions into the governing equations (4.1.37), (4.1.38), (4.1.43) and (4.1.44), we have, after some straightforward calculations, the $O(1)$ and $O(\varepsilon)$ equations as follows

O(1) *problem*

$$t^{(0)}_{\theta_2}(\mathbf{A}_0 - \lambda^{(0)}_1\mathbf{I})\mathbf{u}^{(0)}_{\theta_1} - t^{(0)}_{\theta_1}(\mathbf{A}_0 - \lambda^{(0)}_2\mathbf{I})\mathbf{u}^{(0)}_{\theta_2} = \mathbf{0},$$
$$\nu u^{(0)} + \eta^{(0)}_{y^*} = 0,$$
$$x^{(0)}_{\theta_1} - \lambda^{(0)}_2 t^{(0)}_{\theta_1} = 0,$$
$$x^{(0)}_{\theta_2} - \lambda^{(0)}_1 t^{(0)}_{\theta_2} = 0.$$

The general solution to the $O(1)$ problem takes the form

$$\begin{aligned}\mathbf{u}^{(0)} &= \begin{pmatrix} u^{(0)} \\ \eta^{(0)} \end{pmatrix} \\ &= \left\{ e^{\nu y^*}\sigma_1(\theta_1) + \gamma_1(y^*) \right\} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \\ &\quad + \left\{ e^{-\nu y^*}\sigma_2(\theta_2) + \gamma_2(y^*) \right\} \begin{pmatrix} 1 \\ 1 \end{pmatrix},\end{aligned}$$

with $v^{(0)} \equiv 0$. This recovers linear Kelvin wave solution when the phases are replaced by linear phases.

O(*ε*) *problem*

$$\begin{aligned}(\mathbf{A}_0 + \mathbf{I})\mathbf{u}^{(1)}_{\theta_1} + (\mathbf{A}_0 - \mathbf{I})\mathbf{u}^{(1)}_{\theta_2} &= -(\mathbf{A}_1 - \lambda_{1.1}\mathbf{I})\mathbf{u}^{(0)}_{\theta_1} \\ &\quad - (\mathbf{A}_1 - \lambda_{2.1}\mathbf{I})\mathbf{u}^{(0)}_{\theta_2} \\ &\quad + \mathbf{B}^*_1, \\ -v^{(1)}_{\theta_1} + v^{(1)}_{\theta_2} &= -(d_1 + b^*_1), \\ (x^{(1)} - t^{(1)})_{\theta_1} &= \frac{1}{2}\lambda_{2.1}, \\ (x^{(1)} - t^{(1)})_{\theta_2} &= -\frac{1}{2}\lambda_{1.1},\end{aligned}$$

After further reduction and the use of initial and boundary conditions, we arrive at a canonical initial boundary problem which describes the evolution of cross-channel velocity

$$\begin{aligned} \phi_{\alpha\alpha} - \phi_{\beta\beta} - \phi_{yy} + \nu^2\phi &= p(\beta+\alpha, \beta-\alpha, y), \\ &\qquad \alpha > 0,\ -\infty < \beta < \infty,\ -1 < y < 1, \\ \phi\big|_{y=-1} &= \phi\big|_{y=1} = 0, \\ \phi\big|_{\alpha=0} &= 0, \\ \phi_\alpha\big|_{\alpha=0} &= q(\beta, y), \end{aligned} \tag{4.3.53}$$

where the cross-channel velocity is related to ϕ by

$$v^{(1)}(\alpha, \beta, y) = -\int_0^\alpha \phi d\alpha.$$

where we have replaced by α, β, the nonlinear phases according to

$$\theta_1 = \beta - \alpha, \quad \theta_2 = \beta + \alpha,$$

and $p(\dot{)}, q(\dot{)}$ are all known functions.

Meanwhile, $u^{(1)}, \eta^{(1)}$ are all explicitly expressed in terms of $v^{(1)}$ and the perturbed spatial-temporal coordinates take the form

$$\begin{aligned} x &= \beta + \frac{1}{8}\epsilon\{6[r_2(\beta-\alpha, y) - r_1(\beta+\alpha, y)]\alpha \\ &\quad + \int_{\beta-\alpha}^{\beta+\alpha} [r_2(\theta, y) - r_1(\theta, y)]\,d\theta\} \\ &\quad + O(\epsilon^2), \\ y &= y, \\ t &= \alpha + \frac{1}{8}\epsilon\{-6[r_1(\beta+\alpha, y) + r_2(\beta-\alpha, y)]\alpha \\ &\quad + \int_{\beta-\alpha}^{\beta+\alpha} [r_1(\theta, y) + r_2(\theta, y)]\,d\theta\} \\ &\quad + O(\epsilon^2), \end{aligned}$$

Thus α, β have the implication as *unperturbed* x and t coordinates, respectively.

We point out here that it is only in the context of nonlinear Kelvin waves that this nonvanishing cross-channel velocity is expected and our treatment captures this important feature and provides a solvable linear initial boundary value problem describing it.

The remainder of this chapter is organized as follows. In Section 2, we perturb the shallow water system (4.1.4)-(4.1.6) from its steady state and seek an asymptotic expansion solution in terms of the nonlinear phases. Meanwhile, the spatial and temporal coordinates are also assumed to be perturbed admitting asymptotic expansions in nonlinear phases. We derive the requisite $O(1)$ and $O(\epsilon)$ problems and solve the $O(1)$ problem. The $O(\epsilon)$ problem is then discussed in the next section. We carry out a reduction procedure and derive, as a result, the linear canonical initial boundary value problem, thereby giving rise to the evolution pattern of nonlinear Kelvin waves. We close this chapter by further examining nonlinear Kelvin waves and giving some particularly interesting examples in which the surface elevation as well as flow pattern are presented graphically.

4.2. Asymptotic Expansions and the $O(1)$, $O(\epsilon)$ Problems

4.2.1 *Asymptotic expansions*

Now we carry on the detailed work outlined in the previous section. As it is indicated, we perturb the shallow water equations from their steady state $(u, \eta, v) = (0, 0, 0)$ and assume that an asymptotic solution in the following form is admitted

$$u = \epsilon u^{(0)}(\theta_1, \theta_2, y^*) + \epsilon^2 u^{(1)}(\theta_1, \theta_2, y^*) + O(\epsilon^3), \tag{4.1.45a}$$

$$\eta = \epsilon \eta^{(0)}(\theta_1, \theta_2, y^*) + \epsilon^2 \eta^{(1)}(\theta_1, \theta_2, y^*) + O(\epsilon^3), \tag{4.1.45b}$$

$$v = \epsilon v^{(0)}(\theta_1, \theta_2, y^*) + \epsilon^2 v^{(1)}(\theta_1, \theta_2, y^*) + O(\epsilon^3), \quad v^{(0)} \equiv 0. \tag{4.1.45c}$$

Here ϵ is the perturbation parameter. The requirement that $v^{(0)} \equiv 0$ is imposed to reflect the nature of the Kelvin wave. In particular, the leading term solution $(u^{(0)}, \eta^{(0)}, v^{(0)})$ will be seen to recover the linear Kelvin waves when the shallow water equations are linearized at the steady state.

In the meantime, the spatial coordinate in the along channel direction and the temporal coordinate are also assumed perturbed and they take the forms

$$x = x^{(0)}(\theta_1, \theta_2, y^*) + \epsilon x^{(1)}(\theta_1, \theta_2, y^*) + O(\epsilon^2), \tag{4.1.46a}$$

$$t = t^{(0)}(\theta_1, \theta_2, y^*) + \epsilon t^{(1)}(\theta_1, \theta_2, y^*) + O(\epsilon^2). \tag{4.1.46b}$$

Upon expanding all the releavent nonlinear functions, namely $\mathbf{A}, \mathbf{D}, \mathbf{B}^*, \lambda_i, b^*, d$ as expressed in the previous section(see (4.1.47)-(4.1.53)), we substitute (4.1.45)-(4.1.53) into (4.1.37), (4.1.38) and have, after some straightforward calculations, the $O(1)$ and $O(\epsilon)$ equations as follows

$O(1)$

$$t^{(0)}_{\theta_2}(\mathbf{A}_0 - \lambda_1^{(0)}\mathbf{I})\mathbf{u}^{(0)}_{\theta_1} - t^{(0)}_{\theta_1}(\mathbf{A}_0 - \lambda_2^{(0)}\mathbf{I})\mathbf{u}_{\theta_2} = (\lambda_2^{(0)} - \lambda_1^{(0)})t^{(0)}_{\theta_1}t^{(0)}_{\theta_2}(\mathbf{B}^*_0 + \mathbf{D}_0), \tag{4.2.1}$$

$$b^*_0 + d_0 = 0. \tag{4.2.2}$$

$O(\varepsilon)$

$$\begin{aligned}
t^{(0)}_{\theta_2}(\mathbf{A}_0 - \lambda^{(0)}_1\mathbf{I})\mathbf{u}^{(0)}_{\theta_1} - t^{(0)}_{\theta_1}(\mathbf{A}_0 - \lambda^{(0)}_2\mathbf{I})\mathbf{u}^{(0)}_{\theta_2}& \\
= -\Big\{ t^{(1)}_{\theta_2}(\mathbf{A}_0 - \lambda^{(0)}_1\mathbf{I})\mathbf{u}^{(0)}_{\theta_1} &- t^{(1)}_{\theta_1}(\mathbf{A}_0 - \lambda^{(0)}_2\mathbf{I})\mathbf{u}^{(0)}_{\theta_2} \\
+ t^{(0)}_{\theta_2}(\mathbf{A}_1 - \lambda_{1.1}\mathbf{I})\mathbf{u}^{(0)}_{\theta_1} &- t^{(0)}_{\theta_1}(\mathbf{A}_1 - \lambda_{2.1}\mathbf{I})\mathbf{u}^{(0)}_{\theta_2}\Big\} \\
&+ (\lambda^{(0)}_2 - \lambda^{(0)}_1)(t^{(1)}_{\theta_1}t^{(0)}_{\theta_2} + t^{(0)}_{\theta_1}t^{(1)}_{\theta_2})(\mathbf{B}^*_0 + \mathbf{D}_0) \\
&+ (\lambda^{(0)}_2 - \lambda^{(0)}_1)t^{(0)}_{\theta_1}t^{(0)}_{\theta_2}(\mathbf{B}^*_1 + \mathbf{D}_1) \\
&+ (\lambda_{2.1} - \lambda_{1.1})t^{(0)}_{\theta_1}t^{(0)}_{\theta_2}(\mathbf{B}^*_0 + \mathbf{D}_0), \qquad (4.2.3)
\end{aligned}$$

$$t^{(0)}_{\theta_2}(-\lambda^{(0)}_1)v^{(1)}_{\theta_1} - t^{(0)}_{\theta_1}(-\lambda^{(0)}_2)v^{(1)}_{\theta_2} = (\lambda^{(0)}_2 - \lambda^{(0)}_1)t^{(0)}_{\theta_1}t^{(0)}_{\theta_2}(b^*_1 + d_1). \qquad (4.2.4)$$

Also, by substituting (4.1.46) and (4.1.49) into (4.1.43), (4.1.44) and equating like powers of ε we obtain

$O(1)$

$$x^{(0)}_{\theta_1} - \lambda^{(0)}_2 t^{(0)}_{\theta_1} = 0, \qquad (4.2.5)$$

$$x^{(0)}_{\theta_2} - \lambda^{(0)}_1 t^{(0)}_{\theta_2} = 0. \qquad (4.2.6)$$

$O(\varepsilon)$

$$x^{(1)}_{\theta_1} - \lambda^{(0)}_2 t^{(1)}_{\theta_1} = \lambda_{2.1}t^{(0)}_{\theta_1}, \qquad (4.2.7)$$

$$x^{(1)}_{\theta_2} - \lambda^{(0)}_1 t^{(1)}_{\theta_2} = \lambda_{1.1}t^{(0)}_{\theta_2}. \qquad (4.2.8)$$

Now (4.2.1), (4.2.2), (4.2.5) and (4.2.6) constitute the $O(1)$ *problem*, (4.2.3), (4.2.4), (4.2.7) and (4.2.8) constitute the $O(\varepsilon)$ *problem*.

However, before we solve these problems, some simplification is necessary. In particular, we need to make clear the role played by $\theta_{1,y}$ and $\theta_{2,y}$ in order to express $\mathbf{D}$ and d explicitly in (4.1.51) and (4.1.52). We do this in the immediate subsection.

4.2.2 *Simplification of the $O(1)$ and $O(\varepsilon)$ problems.*

Consider

$$\theta_{1,t} + \lambda_1(\mathbf{u})\theta_{1,x} = 0, \tag{4.1.29}$$

$$\theta_1\Big|_{t=0} = x, \tag{4.1.30}$$

and

$$\theta_{2,t} + \lambda_2(\mathbf{u})\theta_{2,x} = 0, \tag{4.1.31}$$

$$\theta_2\Big|_{t=0} = x. \tag{4.1.32}$$

Differentiate (4.1.29)–(4.1.32) with respect to y to get

$$(\theta_{1,y})_t + \lambda_1(\theta_{1,y})_x = -\lambda_1^{(1)}(\mathbf{u}_y)\theta_{1,x}, \tag{4.2.9}$$

$$\theta_{1,y}\Big|_{t=0} = 0, \tag{4.2.10}$$

and

$$(\theta_{2,y})_t + \lambda_2(\theta_{2,y})_x = -\lambda_2^{(1)}(\mathbf{u}_y)\theta_{2,x}, \tag{4.2.11}$$

$$\theta_{2,y}\Big|_{t=0} = 0, \tag{4.2.12}$$

where $\theta_1 = \theta_1(x,y,t)$ and $\theta_2 = \theta_2(x,y,t)$ are supposed twice continuously differentiable so the order of differentiation can be rearranged.

Denoting

$$\theta_{1,y} = E_1(\theta_1,\theta_2,y^*), \tag{4.2.13a}$$

$$\theta_{2,y} = E_2(\theta_1,\theta_2,y^*), \tag{4.2.13b}$$

we have for (4.2.9)

$$
\begin{aligned}
-\lambda_1^{(1)}(\mathbf{u}_y)\theta_{1,x} &= E_{1,t} + \lambda_1 E_{1,x} \\
&= E_{1,\theta_1}\theta_{1,t} + E_{1,\theta_2}\theta_{2,t} + \lambda_1 E_{1,\theta_1}\theta_{1,x} + \lambda_1 E_{1,\theta_2}\theta_{2,x} \\
&= E_{1,\theta_1}(\theta_{1,t} + \lambda_1\theta_{1,x}) + E_{1,\theta_2}(\theta_{2,t} + \lambda_1\theta_{2,x}) \\
&= E_{1,\theta_2}(\lambda_1 - \lambda_2)\theta_{2,x},
\end{aligned}
$$

that is,

$$E_{1,\theta_2} = -\lambda_1^{(1)}(\mathbf{u}_y)(\lambda_1 - \lambda_2)^{-1}\frac{\theta_{1,x}}{\theta_{2,x}}. \tag{4.2.14}$$

Similarly, we have for (4.2.11) that

$$E_{2,\theta_1} = -\lambda_2^{(1)}(\mathbf{u}_y)(\lambda_2 - \lambda_1)^{-1}\frac{\theta_{2,x}}{\theta_{1,x}}. \tag{4.2.15}$$

By using (4.1.13) we rewrite (4.2.14) and (4.2.15) as

$$E_{1,\theta_2} = \lambda_1^{(1)}(\mathbf{u}_y)(\lambda_1 - \lambda_2)^{-1}\frac{t_{\theta_2}}{t_{\theta_1}}, \tag{4.2.16}$$

and

$$E_{2,\theta_1} = \lambda_1^{(1)}(\mathbf{u}_y)(\lambda_2 - \lambda_1)^{-1}\frac{t_{\theta_1}}{t_{\theta_2}}, \tag{4.2.17}$$

respectively. Then an expansion of the above relations about the steady state, in light of the differential relation (4.1.36c), yields

$$E_{1,\theta_2} = \epsilon\lambda_1^{(1)}(\mathbf{u}_{y^*}^{(0)})(\lambda_1^{(0)} - \lambda_2^{(0)})^{-1}\frac{t_{\theta_2}^{(0)}}{t_{\theta_1}^{(0)}} + O(\epsilon^2), \tag{4.2.18}$$

$$E_{2,\theta_1} = \epsilon\lambda_2^{(1)}(\mathbf{u}_{y^*}^{(0)})(\lambda_2^{(0)} - \lambda_1^{(0)})^{-1}\frac{t_{\theta_1}^{(0)}}{t_{\theta_2}^{(0)}} + O(\epsilon^2). \tag{4.2.19}$$

We then integrate the above relations with respect to θ_2 and θ_1 respectively, obtaining

$$\begin{aligned} E_1 &= E_1\big|_{\theta_2=\theta_1} + \int_{\theta_1}^{\theta_2} E_{1,\theta_2} d\theta_2 \\ &= \int_{\theta_1}^{\theta_2} E_{1,\theta_2} d\theta_2 \\ &= \epsilon E_1^{(0)} + O(\epsilon^2), \end{aligned} \tag{4.2.20}$$

$$\begin{aligned} E_2 &= E_2\big|_{\theta_1=\theta_2} + \int_{\theta_2}^{\theta_1} E_{2,\theta_1} d\theta_1 \\ &= \int_{\theta_2}^{\theta_1} E_{2,\theta_1} d\theta_1 \\ &= \epsilon E_2^{(0)} + O(\epsilon^2), \end{aligned} \tag{4.2.21}$$

since

$$E_1\big|_{\theta_2=\theta_1} = E_1\big|_{t=0} = 0, \quad E_2\big|_{\theta_1=\theta_2} = E_2\big|_{t=0} = 0.$$

Here

$$E_1^{(0)} = \int_{\theta_1}^{\theta_2} \lambda_1^{(1)}(\mathbf{u}_{y^*}^{(0)})(\lambda_1^{(0)} - \lambda_2^{(0)})^{-1} \frac{t_{\theta_2}^{(0)}}{t_{\theta_1}^{(0)}} d\theta_2, \tag{4.2.22}$$

$$E_2^{(0)} = \int_{\theta_2}^{\theta_1} \lambda_2^{(1)}(\mathbf{u}_{y^*}^{(0)})(\lambda_2^{(0)} - \lambda_1^{(0)})^{-1} \frac{t_{\theta_1}^{(0)}}{t_{\theta_2}^{(0)}} d\theta_1. \tag{4.2.23}$$

The analysis thus shows that, for $E_1 = \theta_{1,y}$ and $E_2 = \theta_{2,y}$, their leading order contributions in the asymptotic expansions are of $O(\epsilon)$. A further examination of $\mathbf{D}$ and d then suggests

$$\begin{aligned} \mathbf{D} &= \begin{pmatrix} -v(E_1 \mathbf{u}_{\theta_1} + E_2 \mathbf{u}_{\theta_2}) \\ -E_1\{(1+\eta)v\}_{\theta_1} - E_2\{(1+\eta)v\}_{\theta_2} \end{pmatrix} \\ &= O(\epsilon^3), \end{aligned} \tag{4.2.24}$$

$$\begin{aligned} d &= -(E_1 \eta_{\theta_1} + E_2 \eta_{\theta_2}) - v(E_1 v_{\theta_1} + E_2 v_{\theta_2}) \\ &= -\epsilon^2(E_1^{(0)} \eta_{\theta_1}^{(0)} + E_2^{(0)} \eta_{\theta_2}^{(0)}) + O(\epsilon^3). \end{aligned} \tag{4.2.25}$$

Therefore

$$\mathbf{D}_0 = \mathbf{D}_1 = \mathbf{0}, \tag{4.2.26}$$

$$d_0 = 0,\ d_1 = -(E_1^{(0)}\eta_{\theta_1}^{(0)} + E_2^{(0)}\eta_{\theta_2}^{(0)}). \tag{4.2.27}$$

This ends our discussion for the role played by $E_1 = \theta_{1,y},\ E_2 = \theta_{2,y}$.

In addition, we notice that

$$\begin{aligned}\mathbf{B}^* &= \begin{pmatrix} v(\nu - u_{y^*}) \\ -\{(1+\eta)v\}_{y^*} \end{pmatrix} \\ &= \epsilon^2 \begin{pmatrix} \nu v^{(1)} \\ -v_{y^*}^{(1)} \end{pmatrix} + O(\epsilon^3),\end{aligned} \tag{4.2.28}$$

$$\begin{aligned} b^* &= -\nu u - \eta_{y^*} - v v_{y^*} \\ &= \epsilon(-\nu u^{(0)} - \eta_{y^*}^{(0)}) + \epsilon^2(-\nu u^{(1)} - \eta_{y}^{(1)}) + O(\epsilon^3),\end{aligned} \tag{4.2.29}$$

hence

$$\mathbf{B}_0^* = \mathbf{0},\ \mathbf{B}_1^* = \begin{pmatrix} \nu v^{(1)} \\ -v_{y^*}^{(1)} \end{pmatrix}, \tag{4.2.30}$$

$$b_0^* = -\nu u^{(0)} - \eta_{y^*}^{(0)},\ b_1^* = -\nu u^{(1)} - \eta_{y^*}^{(1)}. \tag{4.2.31}$$

As a result of the above discussion, the $O(1)$ and $O(\epsilon)$ problems are simplified to

$O(1)$ problem

$$t_{\theta_2}^{(0)}(\mathbf{A}_0 - \lambda_1^{(0)}\mathbf{I})\mathbf{u}_{\theta_1}^{(0)} - t_{\theta_1}^{(0)}(\mathbf{A}_0 - \lambda_2^{(0)}\mathbf{I})\mathbf{u}_{\theta_2}^{(0)} = \mathbf{0}, \tag{4.2.32}$$

$$\nu u^{(0)} + \eta_{y^*}^{(0)} = 0, \tag{4.2.33}$$

$$x_{\theta_1}^{(0)} - \lambda_2^{(0)} t_{\theta_1}^{(0)} = 0, \tag{4.2.5}$$

$$x_{\theta_2}^{(0)} - \lambda_1^{(0)} t_{\theta_2}^{(0)} = 0. \tag{4.2.6}$$

$O(\epsilon)$ problem

$$
\begin{aligned}
t_{\theta_2}^{(0)}(\mathbf{A}_0 - \lambda_1^{(0)}\mathbf{I})\mathbf{u}_{\theta_1}^{(1)} - t_{\theta_1}^{(0)}(\mathbf{A}_0 - \lambda_2^{(0)}\mathbf{I})\mathbf{u}_{\theta_2}^{(1)} \\
= -\Big\{ t_{\theta_2}^{(1)}(\mathbf{A}_0 - \lambda_1^{(0)}\mathbf{I})\mathbf{u}_{\theta_1}^{(0)} - t_{\theta_1}^{(1)}(\mathbf{A}_0 - \lambda_2^{(0)}\mathbf{I})\mathbf{u}_{\theta_2}^{(0)} \\
+ t_{\theta_2}^{(0)}(\mathbf{A}_1 - \lambda_{1.1}\mathbf{I})\mathbf{u}_{\theta_1}^{(0)} - t_{\theta_1}^{(0)}(\mathbf{A}_0 - \lambda_{2.1}\mathbf{I})\mathbf{u}_{\theta_2}^{(0)} \Big\} \\
+ (\lambda_2^{(0)} - \lambda_1^{(0)}) t_{\theta_1}^{(0)} t_{\theta_2}^{(0)} \mathbf{B}_1^{*}, \qquad (4.2.34)
\end{aligned}
$$

$$
t_{\theta_2}^{(0)}\lambda_1^{(0)} v_{\theta_1}^{(1)} - t_{\theta_1}^{(0)}\lambda_2^{(0)} v_{\theta_2}^{(1)} = -(\lambda_2^{(0)} - \lambda_1^{(0)}) t_{\theta_1}^{(0)} t_{\theta_2}^{(0)} (d + b_1^{*}), \qquad (4.2.35)
$$

$$
x_{\theta_1}^{(1)} - \lambda_2^{(0)} t_{\theta_1}^{(1)} = \lambda_{2.1} t_{\theta_1}^{(0)}, \qquad (4.2.7)
$$

$$
x_{\theta_2}^{(1)} - \lambda_1^{(0)} t_{\theta_2}^{(1)} = \lambda_{1.1} t_{\theta_2}^{(0)}. \qquad (4.2.8)
$$

4.3. Solution to the $O(1)$ Problem and the Reduction of the $O(\epsilon)$ Problem

In this section we solve the $O(1)$ problem and then use the result to further simplify the $O(\epsilon)$ problem and reduce it to a linear canonical mixed initial and boundary value problem which is exactly solvable.

4.3.1 *Solution to the $O(1)$ problem*

To solve the $O(1)$ problem, we first note that (4.2.5), (4.2.6) are decoupled from (4.2.32), (4.2.33) and hence can be integrated independently.

The parametrization of θ_1 and θ_2 at $t = 0$ is

$$
\theta_1\big|_{t=0} = x, \quad \theta_2\big|_{t=0} = x, \qquad (4.3.1)
$$

which in turn provides,

$$t^{(k)}(\theta_1,\theta_1,y^*) = t^{(k)}(\theta_2,\theta_2,y^*) = 0,\ k = 0,1,2,\dots, \tag{4.3.2a}$$

$$x^{(0)}(\theta_1,\theta_1,y^*) = \theta_1,\ x^{(0)}(\theta_2,\theta_2,y^*) = \theta_2,\ x^{(k)}(\theta_1,\theta_1,y^*) = x^{(k)}(\theta_2,\theta_2,y^*) = 0,$$
$$k = 1,2,3,\dots. \tag{4.3.2b}$$

Thus we obtain, upon integrating (4.2.5), (4.2.6) over θ_1 and θ_2 respectively, that

$$x^{(0)} - \lambda_2^{(0)} t^{(0)} = \theta_2, \quad x^{(0)} - \lambda_1^{(0)} t^{(0)} = \theta_1,$$

or

$$x^{(0)} = \lambda_1^{(0)}(\lambda_1^{(0)} - \lambda_2^{(0)})^{-1}\theta_2 + \lambda_2^{(0)}(\lambda_2^{(0)} - \lambda_1^{(0)})^{-1}\theta_1,$$
$$t^{(0)} = (\lambda_1^{(0)} - \lambda_2^{(0)})^{-1}\theta_2 + (\lambda_2^{(0)} - \lambda_1^{(0)})^{-1}\theta_1,$$

or

$$x^{(0)} = \frac{1}{2}(\theta_1 + \theta_2), \quad t^{(0)} = \frac{1}{2}(\theta_1 - \theta_2), \tag{4.3.3}$$

as

$$\lambda_1^{(0)} = -1, \quad \lambda_2^{(0)} = 1, \tag{4.3.4}$$

are noted.

Now (4.2.32) reduces to

$$(\mathbf{A}_0 - \lambda_1^{(0)}\mathbf{I})\mathbf{u}_{\theta_1}^{(0)} + (\mathbf{A}_0 - \lambda_2^{(0)}\mathbf{I})\mathbf{u}_{\theta_2}^{(0)} = \mathbf{0}. \tag{4.3.5}$$

Let

$$\mathbf{u}^{(0)} = \begin{pmatrix} u^{(0)} \\ \eta^{(0)} \end{pmatrix} = \sigma_1^{(0)}(\theta_1,\theta_2,y^*)\mathbf{r}_1 + \sigma_2^{(0)}(\theta_1,\theta_2,y^*)\mathbf{r}_2, \tag{4.3.6}$$

where $\mathbf{r}_1, \mathbf{r}_2$ are right eigenvectors associated with $\lambda_1^{(0)}, \lambda_2^{(0)}$ respectively. In fact, we have

$$\lambda_1^{(0)} = -1: \quad \boldsymbol{\ell}_1 = \frac{1}{2}(-1,1), \quad \mathbf{r}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \tag{4.3.7a}$$

$$\lambda_2^{(0)} = 1: \quad \boldsymbol{\ell}_2 = \frac{1}{2}(1,1), \quad \mathbf{r}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{4.3.7b}$$

where $\boldsymbol{\ell}_i$ $(i = 1,2)$ are left eigenvectors associated with $\lambda_i^{(0)}$ $(i = 1,2)$. These eigenvectors as computed above satisfy the orthonormality condition

$$\boldsymbol{\ell}_i \mathbf{r}_j = \delta_{ij}, \quad i,j = 1,2. \tag{4.3.8}$$

A substitution of (4.3.6) into (4.3.5) yields

$$\frac{\partial \sigma_1^{(0)}}{\partial \theta_2} = \frac{\partial \sigma_2^{(0)}}{\partial \theta_1} \equiv 0,$$

or

$$\sigma_1^{(0)} = \tau_1(\theta_1, y^*), \quad \sigma_2^{(0)} = \tau_2(\theta_2, y^*), \tag{4.3.9}$$

where τ_1, τ_2 are arbitrary smooth functions. Therefore

$$\mathbf{u}^{(0)} = \begin{pmatrix} u^{(0)} \\ \eta^{(0)} \end{pmatrix} = \tau_1(\theta_1, y^*)\mathbf{r}_1 + \tau_2(\theta_2, y^*)\mathbf{r}_2. \tag{4.3.10}$$

Now equation (4.2.33) in the $O(1)$ problem imposes a constraint on the choices of τ_1 and τ_2. Insert (4.3.10) into (4.2.33) leading to

$$\begin{aligned} 0 &= \nu(-\tau_1 + \tau_2) + (\tau_1 + \tau_2)_{y^*} \\ &= (\tau_{1,y^*} - \nu\tau_1) + (\tau_{2,y^*} + \nu\tau_2), \end{aligned}$$

or

$$\tau_{1,y^*}(\theta_1, y^*) - \nu\tau_1(\theta_1, y^*) = -\{\tau_{2,y^*}(\theta_2, y^*) + \nu\tau_2(\theta_2, y^*)\}$$
$$= c(y^*), \tag{4.3.11}$$

which is a function of y^* only, since θ_1, θ_2 are independent variables. Integrating (4.3.11) with respect to y^* we obtain

$$\tau_1(\theta_1, y^*) = e^{\nu y^*}\sigma_1(\theta_1) + \gamma_1(y^*), \tag{4.3.12a}$$

$$\tau_2(\theta_2, y^*) = e^{-\nu y^*}\sigma_2(\theta_2) + \gamma_2(y^*), \tag{4.3.12b}$$

where σ_1, σ_2 are arbitrary functions of θ_1 and θ_2, respectively. γ_1, γ_2 are functions of y^* and they are related to each other as follows

$$\gamma_1'(y^*) - \nu\gamma_1(y^*) = -\{\gamma_2'(y^*) + \nu\gamma_2(y^*)\} = c(y^*), \tag{4.3.13a}$$

$$\gamma_1(0) = \gamma_2(0) = 0, \tag{4.3.13b}$$

where c is an arbitrary function ascertained by the initial wave profile.

We find thus, in particular, that the initial wave profile can be prescribed as

$$t = 0: \quad \begin{aligned} \tau_1(x, y) &= e^{\nu y}\sigma_1(x) + \gamma_1(y), \\ \tau_2(x, y) &= e^{-\nu y}\sigma_2(x) + \gamma_2(y). \end{aligned} \tag{4.3.14}$$

This apparently agrees with the linear Kelvin wave theory described previously.

In summary, for the $O(1)$ problem, the admissible initial condition is

$$\begin{aligned} t = 0: \quad \mathbf{u}^{(0)} &= \begin{pmatrix} u^{(0)} \\ \eta^{(0)} \end{pmatrix} \\ &= \{e^{\nu y}\sigma_1(x) + \gamma_1(y)\}\begin{pmatrix} -1 \\ 1 \end{pmatrix} \\ &\quad + \{e^{-\nu y}\sigma_2(x) + \gamma_2(y)\}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \end{aligned} \tag{4.3.15}$$

which ensures the cross-channel velocity vanishes to leading order.

The general solution to the $O(1)$ problem takes the form

$$\begin{aligned}\mathbf{u}^{(0)} &= \begin{pmatrix} u^{(0)} \\ \eta^{(0)} \end{pmatrix} \\ &= \left\{ e^{\nu y^*} \sigma_1(\theta_1) + \gamma_1(y^*) \right\} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \\ &\quad + \left\{ e^{-\nu y^*} \sigma_2(\theta_2) + \gamma_2(y^*) \right\} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \end{aligned} \tag{4.3.16}$$

with $v^{(0)} \equiv 0$.

We observe that the above solution recovers the linear Kelvin wave solution when θ_1, θ_2 are regarded as linear phases. In addition, we see that $(\mathbf{A}_0 - \lambda_1^{(0)}\mathbf{I})\mathbf{u}_{\theta_1}^{(0)}$ and $(\mathbf{A}_0 - \lambda_2^{(0)}\mathbf{I})\mathbf{u}_{\theta_2}^{(0)}$ vanish separately in (4.3.5).

4.3.2 *Reduction of the $O(\epsilon)$ problem*

For simplicity, we prescribe the initial condition as

$$t = 0: \quad \mathbf{u} = \epsilon\tau_1(x,y)\mathbf{r}_1 + \epsilon\tau_2(x,y)\mathbf{r}_2, \quad v = 0, \tag{4.3.17}$$

where τ_1, τ_2 are given by (4.3.14).

As a result, the initial conditions for the $O(\epsilon)$ problem read

$$t = 0: \quad u^{(1)} = \eta^{(1)} = v^{(1)} = 0, \quad \forall\, (x,y) \in \mathbb{R} \times [-1,1]. \tag{4.3.18}$$

These are accompanied by the boundary conditions

$$v^{(1)}\big|_{y=-1} = v^{(1)}\big|_{y=1} = 0, \ \forall\, (x,t) \in \mathbb{R} \times \mathbb{R}^+, \tag{4.3.19}$$

which follow from (4.1.24).

Now the $O(\epsilon)$ problem (4.2.34), (4.2.35), (4.2.7), and (4.2.8) reduces to

$$\begin{aligned}
(\mathbf{A}_0 + \mathbf{I})\mathbf{u}^{(1)}_{\theta_1} + (\mathbf{A}_0 - \mathbf{I})\mathbf{u}^{(1)}_{\theta_2} &= -(\mathbf{A}_1 - \lambda_{1.1}\mathbf{I})\mathbf{u}^{(0)}_{\theta_1} \\
&\quad - (\mathbf{A}_1 - \lambda_{2.1}\mathbf{I})\mathbf{u}^{(0)}_{\theta_2} \\
&\quad + \mathbf{B}^*_1, && (4.3.20) \\
-v^{(1)}_{\theta_1} + v^{(1)}_{\theta_2} &= -(d_1 + b^*_1), && (4.3.21) \\
(x^{(1)} - t^{(1)})_{\theta_1} &= \frac{1}{2}\lambda_{2.1}, && (4.3.22) \\
(x^{(1)} - t^{(1)})_{\theta_2} &= -\frac{1}{2}\lambda_{1.1}, && (4.3.23)
\end{aligned}$$

upon using (4.3.3), (4.3.4) and the fact that $(\mathbf{A}_0 - \lambda^{(0)}_1\mathbf{I})\mathbf{u}^{(0)}_{\theta_1}$ and $(\mathbf{A}_0 - \lambda^{(0)}_2\mathbf{I})\mathbf{u}^{(0)}_{\theta_2}$ vanish separately.

Employing the solution to the $O(1)$ problem, we have

$$\begin{aligned}
\mathbf{A}_1 - \lambda_{1.1}\mathbf{I} &= \mathbf{A}^{(1)}(\mathbf{u}^{(0)}) - \lambda^{(1)}_1(\mathbf{u}^{(0)})\mathbf{I} \\
&= (\mathbf{A}^{(1)}(\mathbf{r}_1) - \lambda^{(1)}_1(\mathbf{r}_1))\tau_1 + (\mathbf{A}^{(1)}(\mathbf{r}_2) - \lambda^{(1)}_1(\mathbf{r}_2))\tau_2, \\
\mathbf{A}_1 - \lambda_{2.1}\mathbf{I} &= \mathbf{A}^{(1)}(\mathbf{u}^{(0)}) - \lambda^{(1)}_2(\mathbf{u}^{(0)})\mathbf{I} \\
&= (\mathbf{A}^{(1)}(\mathbf{r}_1) - \lambda^{(1)}_2(\mathbf{r}_1))\tau_1 + (\mathbf{A}^{(1)}(\mathbf{r}_2) - \lambda^{(1)}_2(\mathbf{r}_2))\tau_2.
\end{aligned}$$

Write

$$\mathbf{u}^{(1)} = \sum_{j=1}^{2} \sigma^{(1)}_j(\theta_1, \theta_2, y^*)\mathbf{r}_j, \qquad (4.3.24)$$

and substitute into (4.3.20), then apply ℓ_1, ℓ_2 to both sides of (4.3.20) respectively, obtaining

$$-2\frac{\partial \sigma_1^{(1)}}{\partial \theta_2} = \ell_1 \mathbf{B}_1^* - \{\Gamma_{11}^1 r_1 r_{1,\theta_1} + \Gamma_{21}^1 r_2 r_{1,\theta_1} + \Gamma_{12}^1 r_1 r_{2,\theta_2} + \Gamma_{22}^1 r_2 r_{2,\theta_2}\}, \tag{4.3.25}$$

$$2\frac{\partial \sigma_2^{(1)}}{\partial \theta_1} = \ell_2 \mathbf{B}_1^* - \{\Gamma_{11}^2 r_1 r_{1,\theta_1} + \Gamma_{21}^2 r_2 r_{2,\theta_1} + \Gamma_{12}^2 r_1 r_{2,\theta_2} + \Gamma_{22}^2 r_2 r_{2,\theta_2}\}, \tag{4.3.26}$$

where

$$\Gamma_{jk}^i = \ell_i \left\{ \mathbf{A}^{(1)}(\mathbf{r}_j) - \lambda_k^{(1)}(\mathbf{r}_j) \right\} \mathbf{r}_k, \quad 1 \le i, j, k \le 2, \tag{4.3.27}$$

is the nonlinear interaction coefficient, describing the contribution to the i-th wave mode at the $O(\varepsilon)$ order owing to the interaction of the j-th and k-th wave modes.

According to Lemma 3 in the previous chapter,

$$\Gamma_{ji}^i = 0, \quad 1 \le i, j \le 2. \tag{4.3.28}$$

Also,

$$\begin{aligned} \ell_1 \mathbf{B}_1^* &= \frac{1}{2}(-1, 1)\begin{pmatrix} \nu v^{(1)} \\ v_{y^*}^{(1)} \end{pmatrix} \\ &= -\frac{1}{2}(\nu v^{(1)} + v_{y^*}^{(1)}), \\ \ell_2 \mathbf{B}_1^* &= \frac{1}{2}(1, 1)\begin{pmatrix} \nu v^{(1)} \\ -v_{y^*}^{(1)} \end{pmatrix} \\ &= \frac{1}{2}(\nu v^{(1)} - v_{y^*}^{(1)}), \end{aligned}$$

so (4.3.25) and (4.3.26) become

$$-2\frac{\partial\sigma_1^{(1)}}{\partial\theta_2} = -\frac{1}{2}(\nu v^{(1)} + v_{y^*}^{(1)}) - \left\{\Gamma_{12}^1 \tau_1 \tau_{2,\theta_2} + \Gamma_{22}^1 \tau_2 \tau_{2,\theta_2}\right\}, \qquad (4.3.29)$$

$$2\frac{\partial\sigma_2^{(1)}}{\partial\theta_1} = \frac{1}{2}(\nu v^{(1)} - v_{y^*}^{(1)}) - \left\{\Gamma_{11}^2 \tau_1 \tau_{1,\theta_1} + \Gamma_{21}^2 \tau_2 \tau_{1,\theta_1}\right\}. \qquad (4.3.30)$$

Now we turn to simplify (4.3.21).

It follows from (4.2.35) and (4.2.30), (4.2.31) that

$$d_1 = -(E_1^{(0)}\eta_{\theta_1}^{(0)} + E_2^{(0)}\eta_{\theta_2}^{(0)}), \qquad (4.2.35)$$

and

$$\begin{aligned}
E_1^{(0)} &= \int_{\theta_1}^{\theta_2} \lambda_1^{(1)}(u_{y^*}^{(0)})(\lambda_1^{(0)} - \lambda_2^{(0)})^{-1}\frac{t_{\theta_2}^{(0)}}{t_{\theta_1}^{(0)}}d\theta_2 \\
&= \frac{1}{2}\int_{\theta_1}^{\theta_2} \lambda_1^{(1)}(\tau_{1,y^*}\mathbf{r}_1 + \tau_{2,y^*}\mathbf{r}_2)d\theta_2 \\
&= \frac{1}{2}\left\{\lambda_1^{(1)}(\mathbf{r}_1)\tau_{1,y^*}(\theta_2 - \theta_1) + \lambda_1^{(1)}(\mathbf{r}_2)\int_{\theta_1}^{\theta_2} \tau_{2,y^*}d\theta_2\right\}, \\
E_2^{(0)} &= \int_{\theta_2}^{\theta_1} \lambda_2^{(1)}(u_{y^*}^{(0)})(\lambda_2^{(0)} - \lambda_1^{(0)})^{-1}\frac{t_{\theta_1}^{(0)}}{t_{\theta_2}^{(0)}}d\theta_1 \\
&= -\frac{1}{2}\int_{\theta_2}^{\theta_1} \lambda_2^{(1)}(\tau_{1,y^*}\mathbf{r}_1 + \tau_{2,y^*}\mathbf{r}_2)d\theta_1 \\
&= -\frac{1}{2}\left\{\lambda_2^{(1)}(\mathbf{r}_1)\int_{\theta_2}^{\theta_1} \tau_{1,y^*}d\theta_1 + \lambda_2^{(1)}(\mathbf{r}_2)\tau_{2,y^*}(\theta_1 - \theta_2)\right\}.
\end{aligned}$$

Direct calculation suggests

$$\begin{aligned}
\lambda_1^{(1)}(\mathbf{r}_1) &= ((\partial_u \lambda_1)_0, (\partial_\eta \lambda_1)_0)\mathbf{r}_1 \\
&= -\frac{3}{2}, \\
\lambda_1^{(1)}(\mathbf{r}_2) &= ((\partial_u \lambda_1)_0, (\partial_\eta \lambda_1)_0)\mathbf{r}_2 \\
&= \frac{1}{2}, \\
\lambda_2^{(1)}(\mathbf{r}_1) &= ((\partial_u \lambda_2)_0, (\partial_\eta \lambda_2)_0)\mathbf{r}_1 \\
&= -\frac{1}{2}, \\
\lambda_2^{(1)}(\mathbf{r}_2) &= ((\partial_u \lambda_2)_0, (\partial_\eta \lambda_2)_0)\mathbf{r}_2 \\
&= \frac{3}{2},
\end{aligned}$$

thereby giving

$$\begin{aligned}
d_1 &= -(E_1^{(0)}\eta_{\theta_1}^{(0)} + E_2^{(0)}\eta_{\theta_2}^{(0)}) \\
&= -(E_1^{(0)}\tau_{1,\theta_1} + E_2^{(0)}\tau_{2,\theta_2}) \\
&= -\frac{1}{4}\left\{\tau_{1,\theta_1}\left[-3\tau_{1,y^*}(\theta_2-\theta_1) + \int_{\theta_1}^{\theta_2} \tau_{2,y^*}\,d\theta_2\right]\right. \\
&\quad \left. +\tau_{2,\theta_2}\left[\int_{\theta_2}^{\theta_1} \tau_{1,y^*}\,d\theta_1 - 3\tau_{2,y^*}(\theta_1-\theta_2)\right]\right\}. \qquad (4.3.31)
\end{aligned}$$

Hence (4.3.21) reduces to

$$\begin{aligned}
v_{\theta_2}^{(1)} - v_{\theta_1}^{(1)} &= \nu u^{(1)} + \eta_{y^*}^{(1)} \\
&\quad + \frac{1}{4}\left\{\tau_{1,\theta_1}\left[-3\tau_{1,y^*}(\theta_2-\theta_1) + \int_{\theta_1}^{\theta_2} \tau_{2,y^*}\,d\theta_2\right]\right. \\
&\quad \left. +\tau_{2,\theta_2}\left[\int_{\theta_2}^{\theta_1} \tau_{1,y^*}\,d\theta_1 - 3\tau_{2,y^*}(\theta_1-\theta_2)\right]\right\}. \qquad (4.3.32)
\end{aligned}$$

As (4.3.22), (4.3.23) are decoupled from (4.3.20), (4.3.21), they can be integrated independently to give

$$\begin{aligned}
x^{(1)} - t^{(1)} &= \frac{1}{2}\int_{\theta_2}^{\theta_1} \lambda_2^{(1)}(\mathbf{u}^{(0)})d\theta_1 \\
&= \frac{1}{2}\int_{\theta_2}^{\theta_1} \left\{\lambda_2^{(1)}(\mathbf{r}_1)\tau_1 + \lambda_2^{(1)}(\mathbf{r}_2)\tau_2\right\} d\theta_1 \\
&= \frac{1}{4}\left\{-\int_{\theta_2}^{\theta_1} \tau_1(\theta_1, y^*)d\theta_1 + 3\tau_2(\theta_2, y^*)(\theta_1 - \theta_2)\right\}, \\
x^{(1)} + t^{(1)} &= -\frac{1}{2}\int_{\theta_1}^{\theta_2} \lambda_1^{(1)}(\mathbf{u}^{(0)})d\theta_2 \\
&= -\frac{1}{2}\int_{\theta_1}^{\theta_2} \left\{\lambda_1^{(1)}(\mathbf{r}_1)\tau_1 + \lambda_1^{(1)}(\mathbf{r}_2)\tau_2\right\} d\theta_2 \\
&= \frac{1}{4}\left\{3\tau_1(\theta_1, y^*)(\theta_2 - \theta_1) - \int_{\theta_1}^{\theta_2} \tau_2(\theta_2, y^*)d\theta_2\right\},
\end{aligned}$$

or

$$\begin{aligned}
x^{(1)} &= \frac{1}{8}\Big\{3(\tau_2(\theta_2, y^*) - \tau_1(\theta_1, y^*))(\theta_1 - \theta_2) \\
&\quad - \int_{\theta_2}^{\theta_1} \tau_1(\theta_1, y^*)d\theta_1 - \int_{\theta_1}^{\theta_2} \tau_2(\theta_2, y^*)d\theta_2\Big\}, \qquad (4.3.33) \\
t^{(1)} &= \frac{1}{8}\Big\{3(\tau_1(\theta_1, y^*) + \tau_2(\theta_2, y^*))(\theta_2 - \theta_1) \\
&\quad - \int_{\theta_1}^{\theta_2} \tau_2(\theta_2, y^*)d\theta_2 + \int_{\theta_2}^{\theta_1} \tau_1(\theta_1, y^*)d\theta_1\Big\}, \qquad (4.3.34)
\end{aligned}$$

where

$$x^{(1)} = t^{(1)} = 0,$$

when $\theta_1 = \theta_2$ (that is, $t = 0$) has been noted.

Returning to (4.3.29) and (4.3.30), we compute the nonlinear interaction coefficients to get

$$\Gamma_{12}^1 = \frac{1}{2}, \quad \Gamma_{22}^1 = \frac{1}{2}, \quad \Gamma_{11}^2 = -\frac{1}{2}, \quad \Gamma_{21}^2 = -\frac{1}{2}. \qquad (4.3.35)$$

In addition, we note that

$$\mathbf{u}^{(1)} = \begin{pmatrix} u^{(1)} \\ \eta^{(1)} \end{pmatrix} = \sigma_1^{(1)} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \sigma_2^{(1)} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

gives

$$\sigma_1^{(1)} = \frac{1}{2}(\eta^{(1)} - u^{(1)}), \quad \sigma_2^{(1)} = \frac{1}{2}(\eta^{(1)} + u^{(1)}). \tag{4.3.36}$$

Thus (4.3.29) and (4.3.30) can be written as

$$u^{(1)}_{\theta_2} - \eta^{(1)}_{\theta_2} + \frac{1}{2}(\nu v^{(1)} + v^{(1)}_{y^*}) = -\frac{1}{2}(\tau_1 + \tau_2)\tau_{2,\theta_2}, \tag{4.3.37}$$

$$u^{(1)}_{\theta_1} + \eta^{(1)}_{\theta_1} - \frac{1}{2}(\nu v^{(1)} - v^{(1)}_{y^*}) = \frac{1}{2}(\tau_1 + \tau_2)\tau_{1,\theta_1}. \tag{4.3.38}$$

In summary, we have a reduced $O(\epsilon)$ problem consisting of equations (4.3.32), (4.3.37), and (4.3.38). The associated initial and boundary conditions are as follows:

Initial conditions:

$$t = 0 \quad (\text{i.e. } \theta_1 = \theta_2): \; u^{(1)} = v^{(1)} = \eta^{(1)} = 0, \tag{4.3.18}$$

Boundary conditions: since $y^* = -1, 1$ when $y = -1, 1$ respectively, one has

$$v^{(1)}\big|_{y^*=-1} = v^{(1)}\big|_{y^*=1} = 0. \tag{4.3.39}$$

4.3.3 *A linear canonical mixed initial and boundary problem*

In order to solve the reduced $O(\epsilon)$ problem, that is, the 3×3 linear partial differential equations (4.3.32), (4.3.37) and (4.3.38) subject to initial and boundary conditions (4.3.18) and (4.3.39), we first eliminate $u^{(1)}$ and $\eta^{(1)}$ to get a single equation for

$v^{(1)}$. As a result, a linear canonical mixed initial and boundary value problem is established.

Indeed, we differentiate (4.3.32) with respect to θ_1, θ_2 obtaining

$$
\begin{aligned}
(v^{(1)}_{\theta_2} - v^{(1)}_{\theta_1})_{\theta_1\theta_2} - \nu u^{(1)}_{\theta_1\theta_2} - \eta^{(1)}_{y^*\theta_1\theta_2} = \\
&\frac{1}{4}\{\tau_{1,\theta_1\theta_1}[-3\tau_{1,y^*} + \tau_{2,y^*}] + \tau_{2,\theta_2\theta_2}[\tau_{1,y^*} - 3\tau_{2,y^*}] \\
&-3[\tau_{1,\theta_1}\tau_{1,y^*\theta_1} + \tau_{2,\theta_2}\tau_{2,y^*\theta_2}]\}. \qquad (4.3.40)
\end{aligned}
$$

Meanwhile, the differentiation of (4.3.37), (4.3.38) with respect to θ_1 and θ_2, respectively results in

$$
u^{(1)}_{\theta_2\theta_1} - \eta^{(1)}_{\theta_2\theta_1} + \frac{1}{2}\nu v^{(1)}_{\theta_1} + \frac{1}{2}v^{(1)}_{y^*\theta_1} = -\frac{1}{2}\tau_{1,\theta_1}\tau_{2,\theta_2}, \qquad (4.3.41)
$$

and

$$
u^{(1)}_{\theta_1\theta_2} + \eta^{(1)}_{\theta_1\theta_2} - \frac{1}{2}\nu v^{(1)}_{\theta_2} + \frac{1}{2}v^{(1)}_{y^*\theta_2} = \frac{1}{2}\tau_{1,\theta_1}\tau_{2,\theta_2}, \qquad (4.3.42)
$$

respectively.

The addition of (4.3.41) and (4.3.42) yields

$$
u^{(1)}_{\theta_1\theta_2} = \frac{1}{4}\nu(v^{(1)}_{\theta_2} - v^{(1)}_{\theta_1}) - \frac{1}{4}(v^{(1)}_{y^*\theta_2} + v^{(1)}_{y^*\theta_1}). \qquad (4.3.43)
$$

The subtraction of (4.3.41) from (4.3.42) yields

$$
\eta^{(1)}_{\theta_1\theta_2} = \frac{1}{4}\nu(v^{(1)}_{\theta_2} + v^{(1)}_{\theta_1}) - \frac{1}{4}(v^{(1)}_{y^*\theta_2} - v^{(1)}_{y^*\theta_1}) + \frac{1}{2}\tau_{1,\theta_1}\tau_{2,\theta_2}.
$$

Thereby substituting them into (4.3.40) we obtain

$$\begin{aligned}(v_{\theta_2}^{(1)} - v_{\theta_1}^{(1)})_{\theta_1\theta_2} - \frac{1}{4}\nu^2(v_{\theta_2}^{(1)} - v_{\theta_1}^{(1)}) + \frac{1}{4}(v_{\theta_2}^{(1)} - v_{\theta_1}^{(1)})_{y^*y^*} \\ = \frac{1}{4}\{\tau_{1,\theta_1\theta_1}[-3\tau_{1,y^*} + \tau_{2,y^*}] \\ + \tau_{2,\theta_2\theta_2}[\tau_{1,y^*} - 3\tau_{2,y^*}] \\ -3[\tau_{1,\theta_1}\tau_{1,y^*\theta_1} + \tau_{2,\theta_2}\tau_{2,y^*\theta_2}]\},\end{aligned} \tag{4.3.43}$$

where

$$(\tau_{1,\theta_1}\tau_{2,\theta_2})_{y^*} \equiv 0,$$

is noted.

Now we introduce a new dependent variable

$$v_{\theta_2}^{(1)} - v_{\theta_1}^{(1)} \triangleq \phi, \tag{4.3.44}$$

to write (4.3.43) as

$$\phi_{\theta_1\theta_2} + \frac{1}{4}\phi_{y^*y^*} - \frac{1}{4}\nu^2\phi = -4p(\theta_1, \theta_2, y^*), \tag{4.3.45}$$

where

$$\begin{aligned}p(\theta_1, \theta_2, y^*) &= 4d_{1,\theta_1,\theta_1} \\ &= \tau_{1,\theta_1\theta_1}[3\tau_{1,y^*} - \tau_{2,y^*}] \\ &\quad + \tau_{2,\theta_2\theta_2}[-\tau_{1,y^*} + 3\tau_{2,y^*}] \\ &\quad + 3[\tau_{1,\theta_1}\tau_{1,\theta_1y^*} + \tau_{2,\theta_2}\tau_{2,\theta_2y^*}].\end{aligned} \tag{4.3.46}$$

We rewrite (4.3.45) by introducing new independent variables. Let

$$\alpha = \frac{1}{2}(\theta_1 - \theta_2), \quad \beta = \frac{1}{2}(\theta_1 + \theta_2), \quad y^* = y^*, \tag{4.3.47}$$

thus

$$\theta_1 = \beta + \alpha, \quad \theta_2 = \beta - \alpha, \quad y^* = y^*, \tag{4.3.48}$$

and

$$\partial_{\theta_1} \longmapsto \frac{1}{2}(\partial_\alpha + \partial_\beta), \quad \partial_{\theta_2} \longmapsto \frac{1}{2}(\partial_\beta - \partial_\alpha), \quad y^* \longmapsto y^*. \tag{4.3.49}$$

Therefore, (4.3.45) becomes

$$\phi_{\alpha\alpha} - \phi_{\beta\beta} - \phi_{yy} + \nu^2\phi = p(\beta + \alpha, \beta - \alpha, y), \tag{4.3.50}$$

where we have dropped the star on y^* from now on for simplicity.

Meanwhile, $v^{(1)}$ is related to ϕ by

$$v_\alpha^{(1)} = -\phi. \tag{4.3.51}$$

Now we need to reinterpret existing initial and boundary conditions in order to formulate proper initial and boundary conditions for ϕ.

Boundary conditions. This is straightforward. (4.3.39) suggests

$$\phi\big|_{y=-1} = \phi\big|_{y=1} = 0. \tag{4.3.52}$$

Initial conditions. We employ (4.3.32) and the fact that

$$t = 0: \ u^{(1)} = \eta^{(1)} = 0 \quad (\Longrightarrow \eta_y^{(1)} = 0),$$

also noting that

$$t = 0 \Longleftrightarrow \theta_1 = \theta_2 \Longleftrightarrow \alpha = 0,$$

to obtain

$$\phi\big|_{\alpha=0} = 0. \tag{4.3.53}$$

The derivation for $\phi_\alpha\big|_{\alpha=0}$ is much more complicated. The idea is to make use of (4.3.32), (4.3.37) and (4.3.38) as well as the fact that

$$t = 0: \quad u^{(1)} = \eta^{(1)} = v^{(1)} = 0. \tag{4.3.18}$$

First we differentiate (4.3.32) with respect to θ_1, θ_2 respectively and evaluate them at $t = 0$ obtaining

$$\phi_{\theta_1}\big|_{\theta_1=\theta_2} = \nu u^{(1)}_{\theta_1}\big|_{\theta_1=\theta_2} + \eta^{(1)}_{y\theta_1}\big|_{\theta_1=\theta_2} - d_{1,\theta_1}\big|_{\theta_1=\theta_2}, \tag{4.3.54}$$

and

$$\phi_{\theta_2}\big|_{\theta_1=\theta_2} = \nu u^{(1)}_{\theta_2}\big|_{\theta_1=\theta_2} + \eta^{(1)}_{y\theta_2}\big|_{\theta_1=\theta_2} - d_{1,\theta_2}\big|_{\theta_1=\theta_2}, \tag{4.3.55}$$

where by (4.3.31) and (4.3.32)

$$\begin{aligned} d_{1,\theta_1}\big|_{\theta_1=\theta_2} &= -\frac{1}{4}\left\{\tau_{1,\theta_1}\left[3\tau_{1,y} - \tau_{2,y}\right] + \tau_{2,\theta_1}\left[\tau_{1,y} - 3\tau_{2,y}\right]\right\}\big|_{\theta_1=\theta_2} \\ &= d_{1,\theta_2}\big|_{\theta_1=\theta_2}. \end{aligned} \tag{4.3.56}$$

Also, letting $t = 0$ in (4.3.37), (4.3.38) we get

$$u^{(1)}_{\theta_2} - \eta^{(1)}_{\theta_2} = -\frac{1}{2}(\tau_1 + \tau_2)\tau_{2,\theta_2}\big|_{\theta_1=\theta_2}, \tag{4.3.57}$$

$$u^{(1)}_{\theta_1} + \eta^{(1)}_{\theta_1} = \frac{1}{2}(\tau_1 + \tau_2)\tau_{1,\theta_1}\big|_{\theta_1=\theta_2}. \tag{4.3.58}$$

We hope to solve for $u^{(1)}_{\theta_1}, u^{(1)}_{\theta_2}, \eta^{(1)}_{\theta_1}, \eta^{(1)}_{\theta_2}$ from above, thus we need to supplement some auxiliary relations. To this end, we notice that

$$\begin{aligned} u^{(1)}_{\theta_i} &= u^{(1)}_t t_{\theta_i} + u^{(1)}_x x_{\theta_i}, \\ \eta^{(1)}_{\theta_i} &= \eta^{(1)}_t t_{\theta_i} + \eta^{(1)}_x x_{\theta_i}, \end{aligned} \qquad i = 1, 2. \tag{4.3.59}$$

When $t = 0$, $u^{(1)} = \eta^{(1)} = 0 \Longrightarrow u_x^{(1)} = \eta_x^{(1)} = 0$.

In addition, t_{θ_i} $(i = 1, 2)$ have their leading terms in the asymptotic expansions as

$$t_{\theta_1} \sim t_{\theta_1}^{(0)} = \frac{1}{2}, \quad t_{\theta_2} \sim t_{\theta_2}^{(0)} = -\frac{1}{2}.$$

Thereby we conclude

$$u_{\theta_1}^{(1)} = -u_{\theta_2}^{(1)}, \quad \eta_{\theta_1}^{(1)} = -\eta_{\theta_2}^{(1)}, \tag{4.3.60}$$

in (4.3.59).

The above relation (4.3.60) when combined with (4.3.57) and (4.3.58) produces

$$\begin{aligned} u_{\theta_1}^{(1)}\big|_{\theta_1=\theta_2} &= -u_{\theta_2}^{(1)}\big|_{\theta_1=\theta_2} \\ &= \frac{1}{4}(r_1 + r_2)(r_{1,\theta_1} + r_{2,\theta_2})\big|_{\theta_1=\theta_2}, \end{aligned} \tag{4.3.61}$$

and

$$\begin{aligned} \eta_{\theta_1}^{(1)}\big|_{\theta_1=\theta_2} &= -\eta_{\theta_2}^{(1)}\big|_{\theta_1=\theta_2} \\ &= \frac{1}{4}(r_1 + r_2)(r_{1,\theta_1} - r_{2,\theta_2})\big|_{\theta_1=\theta_2}. \end{aligned} \tag{4.3.62}$$

Thus returning to (4.3.54), (4.3.55) and using the above results as well as (4.3.56) we have

$$
\begin{aligned}
\phi_\alpha\big|_{\alpha=0} &= (\phi_{\theta_1} - \phi_{\theta_2})\big|_{\theta_1=\theta_2} \\
&= \left\{\nu u^{(1)}_{\theta_1} + \eta^{(1)}_{y\theta_1} - d_{1,\theta_1}\right\}\Big|_{\theta_1=\theta_2} \\
&\quad - \left\{\nu u^{(1)}_{\theta_2} + \eta^{(1)}_{y\theta_2} - d_{1,\theta_2}\right\}\Big|_{\theta_1=\theta_2} \\
&= 2\left\{\nu u^{(1)}_{\theta_1} + \eta^{(1)}_{y\theta_1} - d_{1,\theta_1}\right\}\Big|_{\theta_1=\theta_2} \\
&= \left\{\frac{1}{2}\nu(\tau_1+\tau_2)(\tau_{1,\theta_1}+\tau_{2,\theta_2})\right. \\
&\quad + \frac{1}{2}\left[(\tau_1+\tau_2)(\tau_{1,\theta_1}+\tau_{2,\theta_2})\right]_y \\
&\quad \left. +\frac{1}{2}\left[\tau_{1,\theta_1}(3\tau_{1,y}-\tau_{2,y}) + \tau_{2,\theta_2}(\tau_{1,y}-3\tau_{2,y})\right]\right\}\Big|_{\theta_1=\theta_2} \\
&= \frac{1}{2}\nu\left[\tau_1(x,y)+\tau_2(x,y)\right]\left[\tau_{1,x}(x,y)+\tau_{2,x}(x,y)\right] \\
&\quad + \frac{1}{2}\left\{\left[\tau_1(x,y)+\tau_2(x,y)\right]\left[\tau_{1,x}(x,y)-\tau_{2,x}(x,y)\right]\right\}_y \\
&\quad + \frac{1}{2}\left\{\tau_{1,x}(x,y)\left[3\tau_{1,y}(x,y)-\tau_{2,y}(x,y)\right]\right. \\
&\quad \left. +\tau_{2,x}(x,y)\left[\tau_{1,y}(x,y)-3\tau_{2,y}(x,y)\right]\right\} \\
&\triangleq q(x,y) \\
&= q(\beta,y), \qquad\qquad (4.3.63)
\end{aligned}
$$

as $\beta = \theta_1 = x$ at $t = 0$.

In summary, we have formulated the following mixed initial and boundary value problem of a canonical form

$$
\phi_{\alpha\alpha} - \phi_{\beta\beta} - \phi_{yy} + \nu^2\phi = p(\beta+\alpha, \beta-\alpha, y),
$$

$$
\alpha > 0,\ -\infty < \beta < \infty,\ -1 < y < 1, \qquad (4.3.50)
$$

$$
\phi\big|_{y=-1} = \phi\big|_{y=1} = 0, \qquad (4.3.52)
$$

$$
\phi\big|_{\alpha=0} = 0, \qquad (4.3.53)
$$

$$
\phi_\alpha\big|_{\alpha=0} = q(\beta,y), \qquad (4.3.63)
$$

where p and q are expressed by (4.3.46) and (4.3.63), respectively.

Apparently, the above problem is solvable. In order to give a physically reliable interpretation, we point out that by comparing (4.3.3) and (4.3.47) one has

$$\alpha = t^{(0)}, \quad \beta = x^{(0)}. \tag{4.3.64}$$

So α and β can be properly regarded as *time* and *space* variables, respectively. Precisely, α, β are the unperturbed temporal variable and spatial variable in the along channel direction. Consequently, the *perturbed* spatial and temporal coordinates can be written as, according to (4.3.64), (4.3.33), and (4.3.34)

$$\begin{aligned} x = \beta + \frac{1}{8}\epsilon\{6\,[\tau_2(\beta-\alpha,y) - \tau_1(\beta+\alpha,y)]\,\alpha \\ + \int_{\beta-\alpha}^{\beta+\alpha} [\tau_2(\theta,y) - \tau_1(\theta,y)]\,d\theta\} \\ + O(\epsilon^2), \end{aligned} \tag{4.3.65a}$$

$$y = y, \tag{4.3.65b}$$

$$\begin{aligned} t = \alpha + \frac{1}{8}\epsilon\{-6\,[\tau_1(\beta+\alpha,y) + \tau_2(\beta-\alpha,y)]\,\alpha \\ + \int_{\beta-\alpha}^{\beta+\alpha} [\tau_1(\theta,y) + \tau_2(\theta,y)]\,d\theta\} \\ + O(\epsilon^2), \end{aligned} \tag{4.3.65c}$$

where we have replaced by α, β, the nonlinear phases according to

$$\theta_1 = \beta - \alpha, \quad \theta_2 = \beta + \alpha, \tag{4.3.66}$$

which also have an interpretation as the *unperturbed* (linear) phases.

We close this section by writing down $u^{(1)}, \eta^{(1)}$ in terms of $v^{(1)}$. This is accomplished by direct integration of (4.3.37) and (4.3.38).

$$
\begin{aligned}
u^{(1)}(\theta_1,\theta_2,y) = \frac{1}{4}\Bigg\{ & \int_{\theta_2}^{\theta_1}(\nu v^{(1)} - v_y^{(1)})d\theta_1 - \int_{\theta_1}^{\theta_2}(\nu v^{(1)} + v_y^{(1)})d\theta_2 \\
& + \frac{1}{2}\tau_1^2(\theta_1,y) - \frac{1}{2}\tau_1^2(\theta_2,y) + \tau_2(\theta_2,y)(\tau_1(\theta_1,y) - \tau_1(\theta_2,y)) \\
& - \frac{1}{2}\tau_2^2(\theta_2,y) + \frac{1}{2}\tau_2^2(\theta_1,y) - \tau_1(\theta_1,y)(\tau_2(\theta_2,y) - \tau_2(\theta_1,y))\Bigg\}, \\
\eta^{(1)}(\theta_1,\theta_2,y) = \frac{1}{4}\Bigg\{ & \int_{\theta_2}^{\theta_1}(\nu v^{(1)} - v_y^{(1)})d\theta_1 + \int_{\theta_1}^{\theta_2}(\nu v^{(1)} + v_y^{(1)})d\theta_2 \\
& + \frac{1}{2}\tau_1^2(\theta_1,y) - \frac{1}{2}\tau_1^2(\theta_2,y) + \tau_2(\theta_2,y)(\tau_1(\theta_1,y) - \tau_1(\theta_2,y)) \\
& + \frac{1}{2}\tau_2^2(\theta_2,y) - \frac{1}{2}\tau_2^2(\theta_1,y) + \tau_1(\theta_1,y)(\tau_2(\theta_2,y) - \tau_1(\theta_1,y))\Bigg\},
\end{aligned}
$$

where we keep using $\theta_1 = \beta - \alpha$, $\theta_2 = \beta + \alpha$ for notational convenience.

4.4. Nonlinear Kelvin Waves

4.4.1 *Nonlinear Kelvin waves*

For the linear canonical initial boundary value problem formulated in the previous section, one may observe that upon using (4.3.14), the initial wave profile,

$$
\tau_1(x,y) = e^{\nu y}\sigma_1(x) + \gamma_1(y), \quad \tau_2(x,y) = e^{-\nu y}\sigma_2(x) + \gamma_2(y),
$$

the functions p and q can further be written as

$$\begin{aligned} p(\theta_1,\theta_2,y) = {} & 3\nu e^{2\nu y}\left[\sigma_1(\theta_1)\sigma_1''(\theta_1)+\sigma_1'^2(\theta_1)\right] \\ & -3\nu e^{-2\nu y}\left[\sigma_2(\theta_2)\sigma_2''(\theta_2)+\sigma_2'^2(\theta_2)\right] \\ & +\nu\left[\sigma_1(\theta_1)\sigma_2''(\theta_2)-\sigma_1''(\theta_1)\sigma_2(\theta_2)\right] \\ & +e^{\nu y}\sigma_1''(\theta_1)\left[3\gamma_1'(y)-\gamma_2'(y)\right]+e^{-\nu y}\sigma_2''(\theta_2)\left[-\gamma_1'(y)+3\gamma_2'(y)\right], \end{aligned} \tag{4.4.1}$$

$$\begin{aligned} q(\beta,y) = {} & 3\nu e^{2\nu y}\sigma_1(\beta)\sigma_1'(\beta)+3\nu e^{-2\nu y}\sigma_2(\beta)\sigma_2'(\beta) \\ & +\nu(\sigma_1(\beta)\sigma_2(\beta))' \\ & +e^{\nu y}\sigma_1'(\beta)\left[(2\gamma_1'(y)+\nu\gamma_1(y))+\nu\gamma_2(y)\right] \\ & +e^{-\nu y}\sigma_2'(\beta)\left[\nu\gamma_1'(y)+(-2\gamma_2'(y)+\nu\gamma_2(y))\right], \end{aligned} \tag{4.4.2}$$

where, as before, the prime indicates differentiation with respect to corresponding variables.

When the initial profile takes a simpler form, namely putting $\gamma_1(y)\equiv\gamma_2(y)\equiv 0$ in (4.3.14), the above two functions become

$$\begin{aligned} p(\theta_1,\theta_2,y) = {} & 3\nu e^{2\nu y}\left[\sigma_1(\theta_1)\sigma_1''(\theta_1)+\sigma_1'^2(\theta_1)\right] \\ & -3\nu e^{-2\nu y}\left[\sigma_2(\theta_2)\sigma_2''(\theta_2)+\sigma_2'^2(\theta_2)\right] \\ & +\nu\left[\sigma_1(\theta_1)\sigma_2''(\theta_2)-\sigma_1''(\theta_1)\sigma_2(\theta_2)\right], \end{aligned} \tag{4.4.3}$$

$$\begin{aligned} q(\beta,y) = {} & 3\nu e^{2\nu y}\sigma_1(\beta)\sigma_1'(\beta)+3\nu e^{-2\nu y}\sigma_2(\beta)\sigma_2'(\beta) \\ & +\nu(\sigma_1(\beta)\sigma_2(\beta))'. \end{aligned} \tag{4.4.4}$$

For function $p=p(\theta_1,\theta_2,y)$, the first term arises from self-interaction of the r_1 mode, whereas the second term arises due to self interaction of the r_2 mode and the third term is due to quadratic interaction of the r_1, r_2 modes. The same can be said for (4.4.4), the expression for $q(\beta,y)$.

The solution to the linear canonical initial boundary value problem can be decomposed as

$$\phi = \phi^{(1)} + \phi^{(2)}, \tag{4.4.5}$$

where $\phi^{(1)}$ solves

$$\phi^{(1)}_{\alpha\alpha} - \phi^{(1)}_{\beta\beta} - \phi^{(1)}_{yy} + v^2\phi^{(1)} = 0, \quad \alpha > 0,\ -\infty < \beta < \infty,\ -1 < y < 1, \tag{4.4.6}$$

$$\phi^{(1)}\big|_{y=1} = \phi^{(1)}\big|_{y=1} = 0, \tag{4.4.7}$$

$$\phi^{(1)}\big|_{\alpha=0} = 0, \tag{4.4.8}$$

$$\phi^{(1)}_{\alpha}\big|_{\alpha=0} = q(\beta, y), \tag{4.4.9}$$

and $\phi^{(2)}$ solves

$$\phi^{(2)}_{\alpha\alpha} - \phi^{(2)}_{\beta\beta} - \phi^{(2)}_{yy} + v^2\phi^{(2)} = p(\beta+\alpha, \beta-\alpha, y),$$
$$\alpha > 0,\ -\infty < \beta < \infty,\ -1 < y < 1, \tag{4.4.10}$$

$$\phi^{(2)}\big|_{y=1} = \phi^{(2)}\big|_{y=1} = 0, \tag{4.4.11}$$

$$\phi^{(2)}\big|_{\alpha=0} = 0, \tag{4.4.12}$$

$$\phi^{(2)}_{\alpha}\big|_{\alpha=0} = 0. \tag{4.4.13}$$

However, we dwell awhile here to note that for smooth solutions, a consistency condition between (4.4.7) and (4.4.9) requires that

$$q(\beta, 0) = q(\beta, 1) \equiv 0, \tag{4.4.14}$$

which is equivalent to

$$3\sigma_1(\beta)\sigma_1'(\beta) + 3\sigma_2(\beta)\sigma_2'(\beta) + (\sigma_1(\beta)\sigma_2'(\beta))' \equiv 0, \tag{4.4.15a}$$

$$3e^{2v}\sigma_1(\beta)\sigma_1'(\beta) + 3e^{-2v}\sigma_2(\beta)\sigma_2'(\beta) + (\sigma_1(\beta)\sigma_2(\beta))' \equiv 0. \tag{4.4.15b}$$

That is

$$\frac{3}{2}\sigma_1^2(\beta) + \frac{3}{2}\sigma_2^2(\beta) + \sigma_1(\beta)\sigma_2(\beta) \equiv \text{constant},$$
$$\frac{3}{2}e^{2\nu}\sigma_1^2(\beta) + \frac{3}{2}e^{-2\nu}\sigma_2^2(\beta) + \sigma_1(\beta)\sigma_2(\beta) \equiv \text{constant},$$

or

$$\sigma_1(\beta),\ \sigma_2(\beta) \equiv \text{constant}, \tag{4.4.16}$$

upon integration. Therefore the consistency condition (4.4.14) will not be fulfilled except for the trivial case. As a result, we expect discontinuities for $\phi_\alpha^{(1)}$ will be generated at the boundary $y = -1, 1$ initially and propagate along linear characteristics. Thus in general $\phi^{(1)}$ and hence ϕ will be globally C^1 and piecewise C^∞ functions.

To this end, we may decompose ϕ as follows

$$\phi = \phi_{\rm I} + \phi_{\rm II} + \phi_{\rm III}, \tag{4.4.17}$$

where $\phi_{\rm I}, \phi_{\rm II}, \phi_{\rm III}$ solve

$$\phi_{i,\alpha\alpha} - \phi_{i,\beta\beta} - \phi_{i,yy} + \nu^2\phi_i = p_i(\beta + \alpha, \beta - \alpha, y), \tag{4.4.18}$$
$$\phi_i\big|_{y=-1} = \phi_i\big|_{y=1} = 0, \tag{4.4.19}$$
$$\phi_i\big|_{\alpha=0} = 0, \tag{4.4.20}$$
$$\phi_{i,\alpha}\big|_{\alpha=0} = q_i(\beta, y), \tag{4.4.21}$$

respectively, for i = I,II,III. Here

$$p_{\mathrm{I}} = p_{\mathrm{I}}(\theta_1, \theta_2, y) = 3\nu e^{2\nu y}\left[\sigma_1(\theta_1)\sigma_1''(\theta_1) + \sigma_1'^2(\theta_1)\right], \tag{4.4.22a}$$

$$p_{\mathrm{II}} = p_{\mathrm{II}}(\theta_1, \theta_2, y) = -3\nu e^{-2\nu y}\left[\sigma_2(\theta_2)\sigma_2''(\theta_2) + \sigma_2'^2(\theta_2)\right], \tag{4.4.22b}$$

$$p_{\mathrm{III}} = p_{\mathrm{III}}(\theta_1, \theta_2, y) = \nu\left[\sigma_1(\theta_1)\sigma_2''(\theta_2) - \sigma_1''(\theta_1)\sigma_2(\theta_2)\right], \tag{4.4.22c}$$

$$q_{\mathrm{I}} = q_{\mathrm{I}}(\beta, y) = 3\nu e^{2\nu y}\sigma_1(\beta)\sigma_1'(\beta), \tag{4.4.23a}$$

$$q_{\mathrm{II}} = q_{\mathrm{II}}(\beta, y) = 3\nu e^{-2\nu y}\sigma_2(\beta)\sigma_2'(\beta), \tag{4.4.23b}$$

$$q_{\mathrm{III}} = q_{\mathrm{III}}(\beta, y) = \nu(\sigma_1(\beta)\sigma_2(\beta))'. \tag{4.4.23c}$$

Thereby, $\phi_{\mathrm{I}}, \phi_{\mathrm{II}}, \phi_{\mathrm{III}}$ represent contributions to ϕ due to self-interactions of the $\mathbf{r}_1$ and $\mathbf{r}_2$ modes, and the interaction of $\mathbf{r}_1, \mathbf{r}_2$ modes, respectively.

Presumably, one can solve, for any given initial profile σ_1, σ_2, the corresponding $\phi_{\mathrm{I}}, \phi_{\mathrm{II}}, \phi_{\mathrm{III}}$ to obtain the evolution pattern of nonlinear Kelvin waves.

We end our discussion of weakly nonlinear Kelvin waves confined in a channel by considering some interesting examples.

4.4.2 *Single mode initial disturbance*

Take the $\mathbf{r}_1$ mode into consideration, that is, we put $\sigma_1(x) = \sigma(x), \sigma_2(x) \equiv 0$. Obviously, in this case $\phi_{\mathrm{II}} = \phi_{\mathrm{III}} \equiv 0$ and

$$\phi = \phi_{\mathrm{I}}. \tag{4.4.24}$$

In particular, we select

$$\sigma^2(x) = 2ax + k, \tag{4.4.25}$$

where $a, k > 0$ are constants. Thus $p \equiv 0, q = 3a\nu e^{2\nu y}$ and ϕ is determined by an initial boundary value problem for the Klein-Gordon equation

$$\phi_{\alpha\alpha} - \phi_{yy} + \nu^2\phi = 0, \tag{4.4.26}$$

$$\phi\big|_{y=-1} = \phi\big|_{y=1} = 0, \tag{4.4.27}$$

$$\phi\big|_{\alpha=0} = 0,\ \phi_\alpha\big|_{\alpha=0} = 3a\nu e^{2\nu y}. \tag{4.4.28}$$

Similarly, single r_2-mode can also be considered. That is, the case of $\sigma_1(x) \equiv 0,\ \sigma_2(x) = \sigma(x)$.

4.4.3 *Non-interacting nonlinear Kelvin waves*

One may observe that by letting

$$p_{III}(\theta_1, \theta_2, y) \equiv 0, \tag{4.4.29}$$

in (4.4.22c) it follows that

$$\frac{\sigma_1''(\theta_1)}{\sigma_1(\theta_1)} = \frac{\sigma_2''(\theta_2)}{\sigma_2(\theta_2)} = \mu^2, \tag{4.4.30}$$

where μ is an arbitrary number.

Solving (4.4.30) one has

$$\sigma_1(\theta_1) = c_{11}e^{\mu\theta_1} + c_{12}e^{-\mu\theta_1},\ \sigma_2(\theta_2) = c_{21}e^{\mu\theta_2} + c_{22}e^{-\mu\theta_2}. \tag{4.4.31}$$

Now we set the initial profile to be

$$\sigma_1(x) = a_1e^{\mu x},\ \sigma_2(x) = a_2e^{-\mu x}, \tag{4.4.32}$$

then it also yields

$$q_{III} = v(\sigma_1(\beta)\sigma_2(\beta))' \equiv 0. \tag{4.4.33}$$

Thus the two wave modes r_1, r_2 are noninteracting, namely $\phi_{III} \equiv 0$ and

$$\phi = \phi_I + \phi_{II}. \tag{4.4.34}$$

Here ϕ_I and ϕ_{II} solve (4.4.18)–(4.4.21) for i = I,II, respectively with

$$\begin{aligned} p_I(\theta_1, \theta_2, y) &= 6\nu a_1^2 \mu^2 e^{2\nu y + 2\mu\theta_1}, \\ p_{II}(\theta_1, \theta_2, y) &= -6\nu a_2^2 \mu^2 e^{-2\nu y - 2\mu\theta_2}, \\ q_I(\beta, y) &= 3\nu\mu a_1^2 e^{2\nu y + 2\mu\beta}, \\ q_{II}(\beta, y) &= -3\nu\mu a_2^2 e^{-2\nu y - 2\mu\beta}. \end{aligned}$$

Now let $\varphi = \varphi(\alpha, \beta, y)$ be the solution of

$$\varphi_{\alpha\alpha} - \varphi_{\beta\beta} - \varphi_{yy} + \nu^2\varphi = 2\mu^2 e^{2\nu y + 2\mu(\beta+\alpha)}, \tag{4.4.35}$$

$$\varphi\big|_{y=-1} = \varphi\big|_{y=1} = 0, \tag{4.4.36}$$

$$\varphi\big|_{\alpha=0} = 0,\ \varphi_\alpha\big|_{\alpha=0} = \mu e^{2\nu y + 2\mu\beta}. \tag{4.4.37}$$

Then it is straightforward to verify that

$$\begin{aligned} \phi_I(\alpha, \beta, y) &= 3\nu a_1^2 \varphi(\alpha, \beta, y), \\ \phi_{II}(\alpha, \beta, y) &= -3\nu a_2^2 \varphi(\alpha, -\beta, -y), \end{aligned}$$

and thereby

$$\phi(\alpha, \beta, y) = 3\nu a_1^2 \varphi(\alpha, \beta, y) - 3\nu a_2^2 \varphi(\alpha, -\beta, -y). \tag{4.4.38}$$

We carry a further reduction. Let $\psi = \psi(\alpha, y)$ be the solution of

$$\psi_{\alpha\alpha} - \psi_{yy} + \delta^2\psi = 2\mu^2 e^{2\nu y + 2\mu\alpha}, \tag{4.4.39}$$

$$\psi\big|_{y=-1} = \psi\big|_{y=1} = 0, \tag{4.4.40}$$

$$\psi\big|_{\alpha=0} = 0,\ \psi_\alpha\big|_{\alpha=0} = \mu e^{2\nu y}. \tag{4.4.41}$$

It can be shown that by properly choosing δ, φ and ψ are related as follows

$$\varphi(\alpha,\beta,y) = \psi(\alpha,y)e^{2\mu\beta}. \tag{4.4.42}$$

Indeed, substitute (4.4.42) into (4.4.35)–(4.4.37) obtaining

$$\begin{aligned}\varphi_{\alpha\alpha} - \varphi_{\beta\beta} - \varphi_{yy} + \nu^2\varphi &= (\psi_{\alpha\alpha} - \psi_{yy} + \nu^2\psi)e^{2\mu\beta} - 4\mu^2\psi e^{2\mu\beta} \\ &= \left\{\psi_{\alpha\alpha} - \psi_{yy} + (\nu^2 - 4\mu^2)\psi\right\} e^{2\mu\beta},\end{aligned}$$

thus one can see that (4.4.39) holds when

$$\delta^2 = \nu^2 - 4\mu^2.$$

(4.4.39) is an inhomogeneous linear Klein-Gordon equation and at this moment we observe that, for instance, when $\mu = \frac{1}{2}\nu$ is chosen, namely

$$\sigma_1(x) = a_1 e^{\frac{1}{2}\nu x}, \; \sigma_2(x) = a_2 e^{-\frac{1}{2}\nu x}, \tag{4.4.43}$$

then ψ is determined by an inhomogeneous wave equation with mixed initial boundary conditions

$$\psi_{\alpha\alpha} - \psi_{yy} = \frac{1}{2}\nu^2 e^{2\nu y+\nu\alpha}, \tag{4.4.44}$$

$$\psi\big|_{y=-1} = \psi\big|_{y=1} = 0, \tag{4.4.40}$$

$$\psi\big|_{\alpha=0} = 0, \; \psi_\alpha\big|_{\alpha=0} = \frac{1}{2}\nu e^{2\nu y}, \tag{4.4.41}$$

thereby providing a closed form solution.

Meanwhile, we have

$$\phi(\alpha,\beta,y) = 3\nu a_1^2 e^{\nu\beta}\psi(\alpha,y) - 3\nu a_2^2 e^{-\nu\beta}\psi(\alpha,-y), \tag{4.4.45}$$

and

$$
\begin{aligned}
v^{(1)}(\alpha,\beta,y) &= -\int_0^{\alpha} \phi\, d\alpha \\
&= -3va_1^2 e^{v\beta} \int_0^{\alpha} \psi(\alpha, y)\, d\alpha + 3va_2^2 e^{-v\beta} \int_0^{\alpha} \psi(\alpha, -y)\, d\alpha.
\end{aligned}
\tag{4.4.46}
$$

The solution ψ to (4.4.44), (4.4.40) and (4.4.41) can be explicitly integrated [76]. We divide the domain $\{(\alpha, y):\ \alpha > 0,\ -1 < y < 1\}$ as shown in Figure 4.3 and

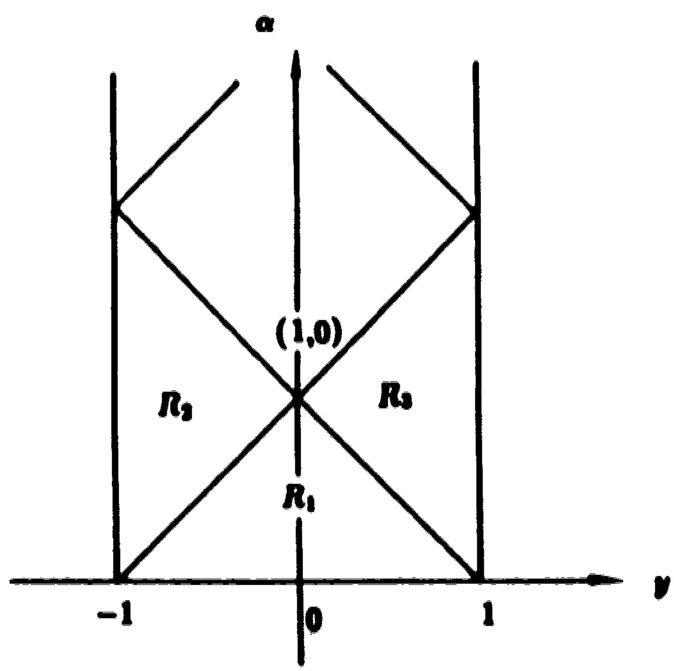


Fig. 4.3: The (α, y) domain.

after some extensive algebra to obtain

$$\psi(\alpha,y)=\begin{cases}\frac{1}{6}e^{2\nu y}\left\{sh(2\nu\alpha)+(e^{2\nu\alpha}-e^{\nu\alpha})\right\}, & (\alpha,y)\in R_1\\ \frac{1}{6}e^{2\nu(\alpha-1)}\left\{sh(2\nu(y+1))+(e^{2\nu(y+1)}-e^{\nu(y+1)})\right\} & \\ -\frac{1}{6}\left[e^{\frac{3}{2}\nu y}-e^{-\frac{3}{2}\nu(2+y)}\right]\left[e^{\frac{1}{2}\nu y+\nu\alpha}e^{-\frac{1}{2}\nu(y+2)+2\nu\alpha}\right], & (\alpha,y)\in R_2\\ \frac{1}{6}e^{2\nu(1-\alpha)}\left\{sh(2\nu(1-y))+(e^{2\nu(1-y)}-e^{\nu(1-y)})\right\} & \\ -\frac{1}{6}\left[e^{\frac{3}{2}\nu y+\nu\alpha}-e^{\frac{3}{2}\nu(2-y)-2\nu\alpha}\right]\left[e^{\frac{1}{2}\nu y}-e^{-\frac{1}{2}\nu(y-2)}\right], & (\alpha,y)\in R_3,\end{cases}$$

and thereby when $(\alpha,y)\in R_1,\ \beta\in(-\infty,+\infty)$,

$$v^{(1)}(\alpha,\beta,y)=\frac{1}{4}(a_2^2e^{-\nu\beta-2\nu y}-a_1^2e^{\nu\beta+2\nu y})(ch(2\nu\alpha)+e^{2\nu\alpha}-e^{\nu\alpha}),$$

when $(\alpha,y)\in R_2,\ \beta\in(-\infty,+\infty)$,

$$v^{(1)}(\alpha,\beta,y)=-a_1^2e^{-\nu\beta}\Phi_2(\alpha,y)+a_2^2e^{-\nu\beta}\Phi_3(\alpha,-y),$$

and when $(\alpha,y)\in R_3,\ \beta\in(-\infty,+\infty)$

$$v^{(1)}(\alpha,\beta,y)=-a_1^2e^{\nu\beta}\Phi_3(\alpha,y)+a_2^2e^{-\nu\beta}\Phi_2(\alpha,-y),$$

where

$$\begin{aligned}\Phi_2(\alpha,y)=&\ \frac{1}{4}e^{2\nu y}\left[ch(2\nu(1+y))+(e^{2\nu(1+y)}-2e^{\nu(1+y)})\right]\\ &+\frac{1}{4}\left[sh(2\nu(1+y))+(e^{2\nu(y+1)}-e^{\nu(y+1)})\right]\left[e^{2\nu(\alpha-1)}-e^{2\nu y}\right]\\ &-\frac{1}{4}\left[e^{\frac{3}{2}\nu y}-e^{-\frac{3}{2}\nu(2+y)}\right]\left[2e^{\frac{1}{2}\nu y}(e^{\nu\alpha}-e^{\nu(1+y)})\right.\\ &\left.-e^{-\frac{1}{2}\nu(2+y)}(e^{2\nu\alpha}-e^{2\nu(1+y)})\right],\end{aligned}$$

and

$$
\begin{aligned}
\Phi_3(\alpha, y) = \frac{1}{4}e^{2\nu y}\left[ch(2\nu(1-y)) + (e^{2\nu(1-y)} - 2e^{\nu(1-y)})\right] \\
-\frac{1}{4}\left[sh(2\nu(1-y)) + (e^{2\nu(1-y)} - e^{\nu(1-y)})\right]\left[e^{2\nu(1-\alpha)} - e^{2\nu y}\right] \\
-\frac{1}{4}\left[e^{\frac{1}{2}\nu y} - e^{\frac{1}{2}\nu(2-y)}\right]\Big[2e^{\frac{3}{2}\nu y}(e^{\nu\alpha} - e^{\nu(1-y)}) \\
+e^{\frac{3}{2}\nu(2-y)}(e^{-2\nu\alpha} - e^{-2\nu(1-y)})\Big].
\end{aligned}
$$

The figures that follow represent the time-progressive profiles for both along- and cross-channel velocities, as well as surface elevation. In the sequence of time frames $t = 0, 0.5, 1$ the propagation of these waves is depicted clearly. As it is shown in Fig. 4.4 and Fig. 4.6, along-channel velocity and surface elevation waves travel to the left near $y = -1$ and to the right near $y = 1$. In Fig. 4.6, cross-channel velocity is seen being generated as time evolves.

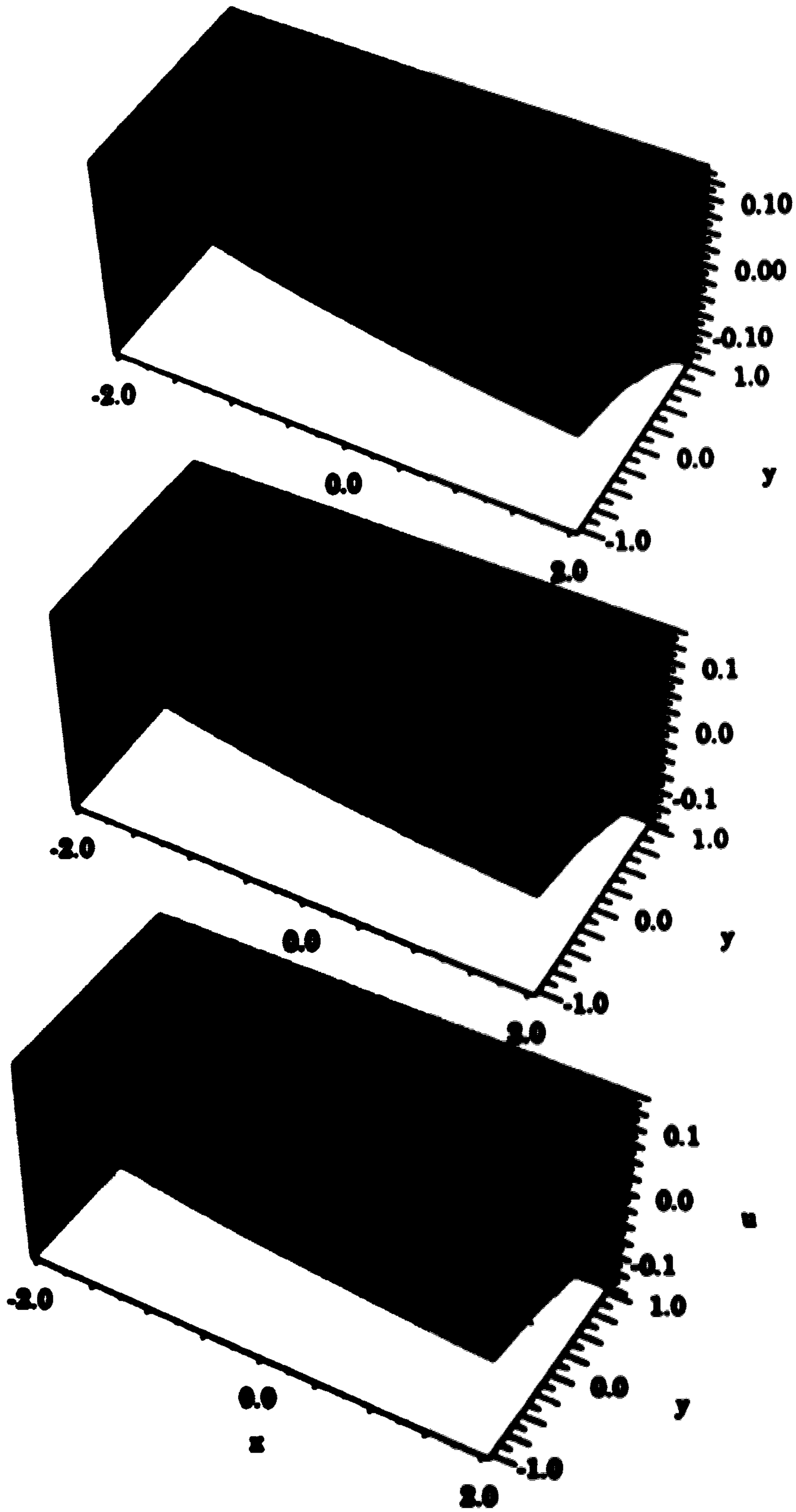


Fig. 4.4: Along-channel velocity at $t = 0, 0.5, 1$ for $\alpha_1 = \alpha_2 = 1$, $\nu = 1$ and $\epsilon = 0.02$

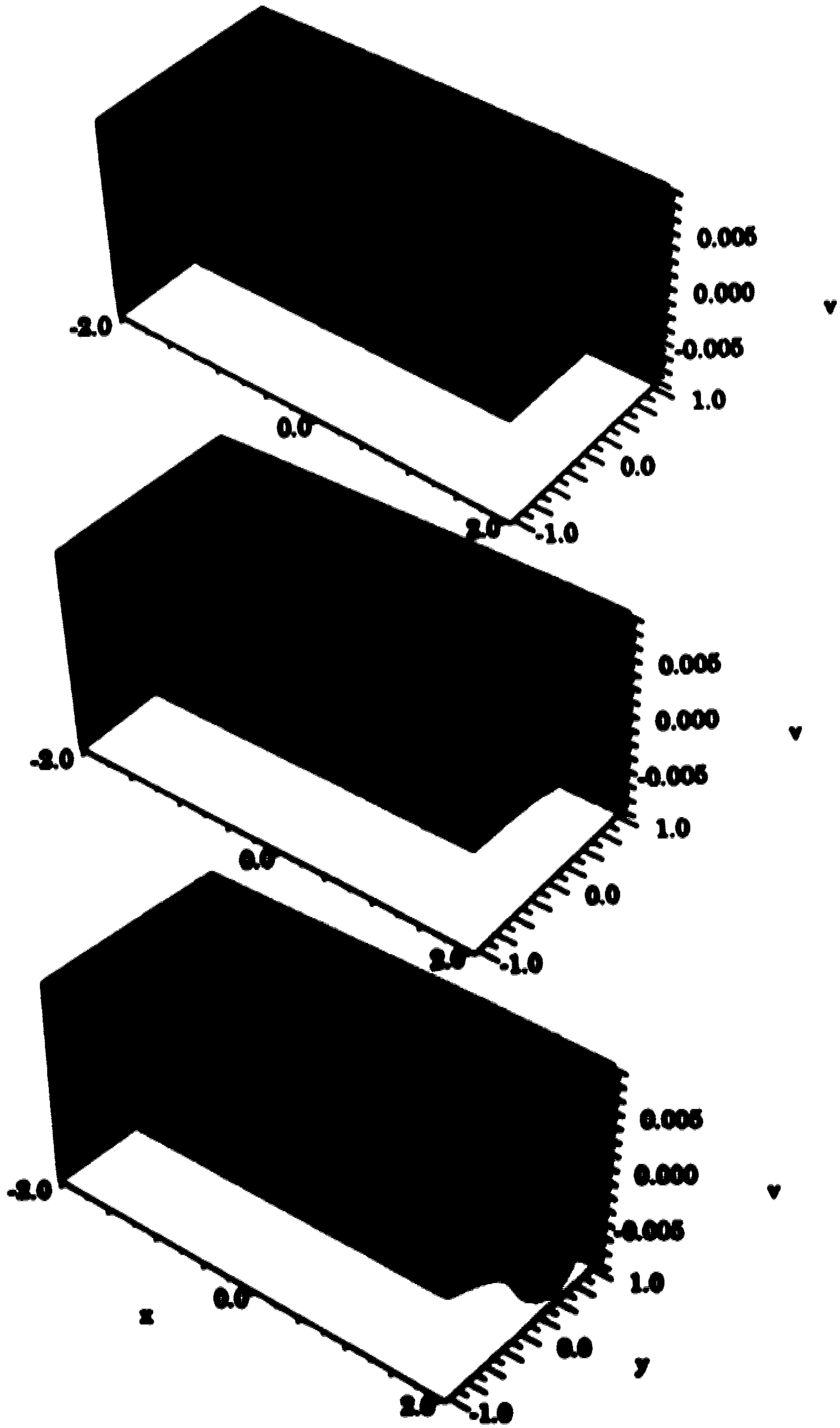


Fig. 4.5: Cross-channel velocity at $t = 0, 0.5, 1$ for $\alpha_1 = \alpha_2 = 1$, $\nu = 1$ and $\epsilon = 0.02$

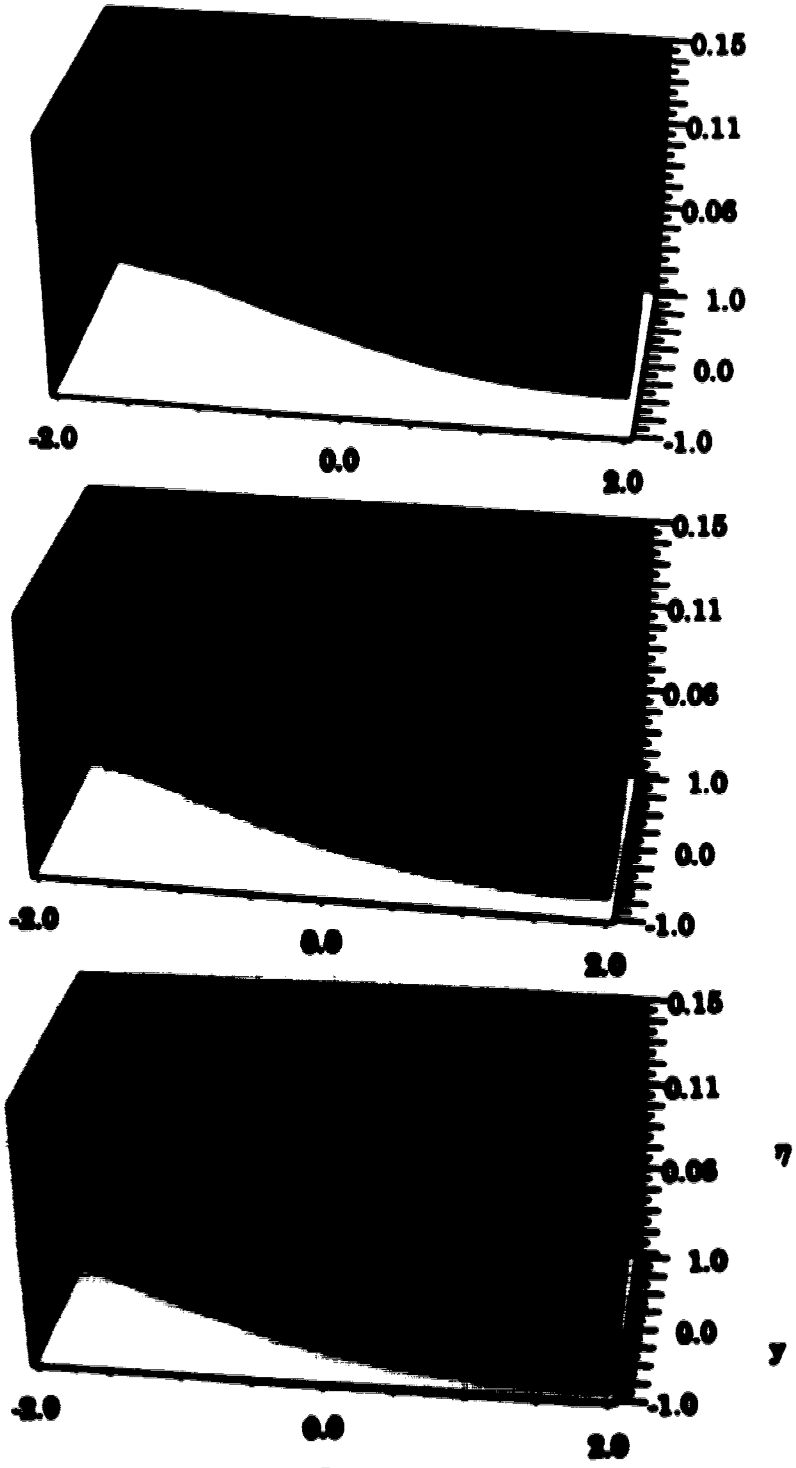


Fig. 4.6: Surface elevation at $t = 0, 0.5, 1$ for $a_1 = a_2 = 1$, $\nu = 1$ and $\epsilon = 0.02$

CHAPTER 5.

Concluding Remarks

We have presented in this thesis two studies of weakly nonlinear hyperbolic waves. The first part is a relatively complete single-wave-mode geometric optics theory based on the use of a single nonlinear phase. The second part includes an asymptotic analysis of two wave interaction theory, based on the use of two nonlinear phases.

These two studies are characterized notably by their employment of nonlinear phases and asymptotic analysis. However they have quite different perspectives. For example, the single wave-mode geometric optics theory as it stands here not only admits a rational choice of small-amplitude to high frequency relation, but also is capable of capturing shock waves and retains validity afterwards. Whereas the two wave interaction theory is basically a smooth solution approach and in general remains valid only before wave-breaking. On the other hand, the two wave interaction theory formulated in this thesis is a generalization to the so-called characteristic method initiated by Lin [44] and Fox [15] and provides an easy approach to assess the evolving pattern of weakly nonlinear hyperbolic waves after two wave interaction, a task which the single wave mode theory cannot handle.

As the single wave mode geometric optics theory show, the typical pattern for smooth hyperbolic waves to break and generate shock waves is a passage starting from characteristic focusing, which produces a multi- (phase) valued region in the $x-t$ plane, and a shock front and hence a shock wave results therein. Such characteristic focusing is found necessarily leading to the generation of entropy admissible shock waves, whereas in the nonfocusing case, nonlinear hyperbolic waves remain smooth and present no complication. The analysis conducted in Chapter 2 provides both qualitative and quantitative information and is suitable for practical application of weak shock computations.

The spatial inhomogeneity involved in the flux function has two implications on the general setting of the single wave-mode geometric optics theory. First of all, spatial inhomogeneity affects the geometric structure of the characteristic family and hence existence and shape of the caustic as well as shock waves. Indeed, this effect is a delicate issue and we do not try to address it in every detail. However, when the function $\langle \Lambda q \rangle$, which plays a key role in the arrival time formula, is not of a definite sign and in particular when it changes sign around the shock initiation point, a fact which amounts to the change of convexity in the flux function spatially, interesting qualitative as well as quantitative behaviours may happen. We leave this to a future analysis. Another phenomenon being discovered in this study of single wave-mode geometric optics is that at an order higher than the order of local linear degeneracy, the asympototic solution consists of two parts, one being the regular part and the other is induced solely by spatial inhomogeneity. This second part is continuous across the shock front, hence invalidating asymptotic formulation of higher order problems. This fact appears not to have been noted in the literature before.

The two wave interaction theory as it stands in Chapter 3 follows from a generalisation of the characteristic method of Lin [44] and Fox [15]. The basis of such a generalization is the introduction from the outset of two nonlinear phase variables. The theory, straightforward as it is, does find interesting applications in one dimensional sub- and supersonic gas dynamics as well as two dimensional nonlinear Kelvin waves in a channel.

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