Harmonic Analysis on Spherical h-harmonics and Dunkl Transforms

by

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Abstract

The thesis consists of two closely related parts: (i) Cesàro summability of the spherical h-harmonic expansions on the sphere $\mathbb{S}^{d-1}$, and (ii) Bochner-Riesz summability of the inverse Dunkl transforms on $\mathbb{R}^d$, both being studied with respect to the weight $h_\kappa^2(x) := \prod_{j=1}^d |x_j|^{2\kappa_j}$, which is invariant under the Abelian group $\mathbb{Z}_2^d$ in Dunkl analysis.

In the first part, we prove a weak type estimate of the maximal Cesàro operator of the spherical h-harmonics at the critical index. This estimate allows us to improve several known results on spherical h-harmonics, including the almost everywhere (a.e.) convergence of the Cesàro means at the critical index, the sufficient conditions in the Marcinkiewitcz multiplier theorem, and a Fefferman-Stein type inequality for the Cesàro operators. In particular, we obtain a new result on a.e. convergence of the Cesàro means of spherical h-harmonics at the critical index, which is quite surprising as it is well known that the same result is not true for the ordinary spherical harmonics. We also establish similar results for weighted orthogonal polynomial expansions on the ball and the simplex.

In the second part, we first prove that the Bochner-Riesz mean of each function in $L^1(\mathbb{R}^d; h_\kappa^2)$ converges almost everywhere at the critical index. This result is surprising due to the celebrated counter-example of Kolmogorov on a.e. convergence of the Fourier partial sums of integrable functions in one variable, and the counter-example of E.M. Stein in several variables showing that a.e. convergence does not hold at the critical index even for H^1-functions. Next, we study the critical index for the a.e. convergence of the Bochner-Riesz means in L^p-spaces with $p > 2$. We obtain results that are in full analogy with the classical result of M. Christ (*Proc. Amer. Math. Soc.* **95** (1985)) on estimates of the maximal Bochner-Riesz means of Fourier integrals and the classical result of A. Carbery, José L. Rubio De Francia and L. Vega (*J. London Math. Soc.* **38** (1988), no. 2, 513–524) on a.e. convergence of Fourier integrals. The proofs of these results for the Dunkl transforms are highly nontrivial since the underlying weighted space

is not translation invariant. We need to establish several new results in Dunkl analysis, including: (i) local restriction theorem for the Dunkl transform which is significantly stronger than the global one, but more difficult to prove; (ii) the weighted Littlewood Paley inequality with A_p weights in the Dunkl noncommutative setting; (iii) sharp local pointwise estimates of several important kernel functions.

Preface

Chapter 3 and Chapter 4 of this thesis has been published as F. Dai, S. Wang and W. Ye, "Maximal Estimates for the Cesáro Means of Weighted Orthogonal Polynomial Expansions on the Unit Sphere," Journal of Functional Analysis, vol. 265, issue 10, 2357-2387.

Chapter 6 of this thesis has been published as F. Dai and W. Ye, "Almost Everywhere Convergence of the Bochner-Riesz Means with the Dunkl Transform," Journal of Approximation Theory, vol. 29, 129-155.

Chapter 5 and Chapter 6-10 of this thesis will be published as a joint paper with Dr. Feng Dai soon.

All of the proofs in this thesis are joint work of Dr. Feng Dai and me.

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Contents

Chapter 1

Summary of the main results

1.1 Spherical h-harmonic analysis on the sphere

The first part of this thesis is to study the pointwise convergence of the Cesáro means of spherical h-harmonic expansions on the unit sphere. For a class of product weights that are invariant under the group $\mathbb{Z}_2^d$ on the sphere, estimates of the maximal Cesàro operator of the weighted orthogonal polynomial expansions at the critical index are proved, which allow us to improve several known results in this area, including the critical index for the almost everywhere convergence of the Cesàro means, the sufficient conditions in the Marcinkiewitcz multiplier theorem, and a Fefferman-Stein type inequality for the Cesàro operators. These results on the unit sphere also enable us to establish similar results on the unit ball and on the simplex.

The main results in this part are contained in my joint paper [10] with Feng Dai and Sheng Wang.

To be more precise, we need to introduce some necessary notations. Let $\mathbb{S}^{d-1} := \{x \in \mathbb{R}^d : \|x\| = 1\}$ denote the unit sphere of $\mathbb{R}^d$ equipped with the usual rotation-invariant measure $d\sigma$, where $\|x\|$ denotes the Euclidean norm. Let

$$h_\kappa(x) := \prod_{j=1}^{d} |x_j|^{\kappa_j}, \quad x = (x_1, \cdots, x_d) \in \mathbb{R}^d, \tag{1.1.1}$$

where $\kappa := (\kappa_1, \cdots, \kappa_d) \in \mathbb{R}^d$ and $\kappa_{\min} := \min_{1 \leqslant j \leqslant d} \kappa_j \ \geq 0$. Throughout the thesis, all functions and sets will be assumed to be Lebesgue measurable.

We denote by $L^p(h_\kappa^2; \mathbb{S}^{d-1}),\ 1 \leq p \leq \infty$, the L^p-space of functions defined on $\mathbb{S}^{d-1}$ with respect to the measure $h_\kappa^2(x)\, d\sigma(x)$. More precisely, $L^p(h_\kappa^2; \mathbb{S}^{d-1})$ is the space of functions on $\mathbb{S}^{d-1}$ with finite norm

$$\|f\|_{\kappa,p} := \left(\int_{\mathbb{S}^{d-1}} |f(y)|^p h_\kappa^2(y) d\sigma(y) \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty.$$

For $p = \infty$, $L^\infty(h_\kappa^2)$ is replaced by $C(\mathbb{S}^{d-1})$, the space of continuous functions on $\mathbb{S}^{d-1}$ with the usual uniform norm.

A spherical polynomial of degree at most n on $\mathbb{S}^{d-1}$ is the restriction to $\mathbb{S}^{d-1}$ of an algebraic polynomial in d variables of total degree n. We denote by Π_n^d the space of all spherical polynomials of degree at most n on $\mathbb{S}^{d-1}$.

We denote by $\mathcal{H}_n^d(h_\kappa^2)$ the orthogonal complement of Π_{n-1}^d in Π_n^d with respect to the norm of $L^2(h_\kappa^2; \mathbb{S}^{d-1})$, where it is agreed that $\Pi_{-1}^d = \{0\}$. Each element in $\mathcal{H}_n^d(h_\kappa^2)$ is then called a spherical h-harmonic polynomial of degree n on $\mathbb{S}^{d-1}$. In the case of $h_\kappa = 1$, a spherical h-harmonic is simply the ordinary spherical harmonic.

The theory of h-harmonics is developed by Dunkl (see [22, 23, 25]) for a family of weight functions invariant under a finite reflection group, of which h_κ in (1.1.1) is the example of the group $\mathbb{Z}_2^d$. Properties of h-harmonics are quite similar to those of ordinary spherical harmonics. For example, each $f \in L^2(h_\kappa^2; \mathbb{S}^{d-1})$ has an orthogonal expansion in h-harmonics, $f = \sum_{n=0}^{\infty} \operatorname{proj}_n(h_\kappa^2; f)$, converging in the norm of $L^2(h_\kappa^2; \mathbb{S}^{d-1})$, where $\operatorname{proj}_n(h_\kappa^2; f)$ denotes the orthogonal projection of f onto $\mathcal{H}_n^d(h_\kappa^2)$, which can be extended to all $f \in L^1(h_\kappa^2; \mathbb{S}^{d-1})$.

For $\delta > -1$, the *Cesàro* (C, δ)*- means* of the spherical h-harmonic expansions are defined by

$$S_n^\delta(h_\kappa^2; f) := \sum_{j=0}^{n} \frac{A_{n-j}^\delta}{A_n^\delta} \operatorname{proj}_j(h_\kappa^2; f), \ A_{n-j}^\delta = \binom{n-j+\delta}{n-j}, \ n = 0, 1, \cdots,$$

whereas the *maximal Cesàro operator* of order δ is defined by

$$S_*^\delta(h_\kappa^2; f)(x) := \sup_{n \in \mathbb{N}} |S_n^\delta(h_\kappa^2; f)(x)|, \quad x \in \mathbb{S}^{d-1}.$$

Our main goal in this first part of the thesis is to study the following weak type estimate of the maximal Cesàro operator: for $f \in L^1(h_\kappa^2; \mathbb{S}^{d-1})$,

$$\operatorname{meas}_\kappa\Big\{x \in \mathbb{S}^{d-1} : \quad S_*^\delta(h_\kappa^2; f)(x) > \alpha\Big\} \leqslant C \frac{\|f\|_{\kappa,1}}{\alpha}, \quad \forall \alpha > 0, \tag{1.1.2}$$

here, and in what follows, we write $\operatorname{meas}_\kappa(E) := \int_E h_\kappa^2(x)\, d\sigma(x)$ for a measurable subset $E \subset \mathbb{S}^{d-1}$. Such estimates have been playing crucial roles in spherical harmonic analysis on the sphere; for example, they can be used to establish a Marcinkiewicz type multiplier theorem for the spherical h-harmonic expansions (see [4, 15]).

The background for this problem is as follows. In the case of ordinary spherical harmonics (i.e., the case of $\kappa = 0$), it is known that (1.1.2) holds if and only if $\delta > \frac{d-2}{2}$. (See [4, 45]). Indeed, in this case, since the Cesàro operators are rotation-invariant, a well-known result of Stein [38] implies that for $h_\kappa(x) \equiv 1$, (1.1.2) holds if and only if

$$\lim_{n\to\infty} S_n^\delta(h_\kappa^2; f)(x) = f(x), \quad \text{a.e. } x \in \mathbb{S}^{d-1}, \quad \forall f \in L^1(h_\kappa^2; \mathbb{S}^{d-1}). \tag{1.1.3}$$

In the case of $\kappa \neq 0$ (i.e., the weighted case), while a standard density argument shows that (1.1.2) implies (1.1.3), the result of Stein [38] is not applicable to deduce the equivalence of (1.1.2) and (1.1.3), since the measure $h_\kappa^2 d\sigma$ is no longer rotation-invariant. In fact, an estimate much weaker than (1.1.2) was proved and used to study (1.1.3) for $\delta > \lambda_\kappa := \frac{d-2}{2} + \sum_{j=1}^d \kappa_j$ in [51], whereas (1.1.2) itself was later proved in [15] for $\delta > \lambda_\kappa$, where the results are also applicable to the case of more general weights invariant under a reflection group. Finally, for h_κ in (1.1.1), it was shown in [57] that (1.1.3) fails for $\delta < \sigma_\kappa$ with

$$\sigma_\kappa := \lambda_\kappa - \kappa_{\min} = \frac{d-2}{2} + \sum_{j=1}^d \kappa_j - \min_{1\leqslant j\leqslant d} \kappa_j. \tag{1.1.4}$$

Of related interest is the fact that σ_κ is the critical index for the summablity of the

Cesàro means in the space $L^1(h_\kappa^2;\mathbb{S}^{d-1})$. More precisely,

$$\lim_{N\to\infty}\|S_N^\delta(h_\kappa^2;f)-f\|_{\kappa,1}=0,\quad \forall f\in L^1(h_\kappa^2;\mathbb{S}^{d-1}) \tag{1.1.5}$$

if and only if $\delta>\sigma_\kappa$. (See [30, 16]).

In Chapter 3 of this thesis, we prove that if $\kappa\neq 0$, then (1.1.3) holds if and only if $\delta\geq\sigma_\kappa$, and moreover, if at most one of the κ_i is zero, then the weak estimate (1.1.2) holds if and only if $\delta\geq\sigma_\kappa$. Of special interest is the case of $\delta=\sigma_\kappa$, where our results are a little bit surprising in view of the facts that (1.1.5) fails at the critical index $\delta=\sigma_\kappa$, and the corresponding results in the case of $\kappa=0$ (i.e., the case of ordinary spherical harmonics) are known to be false at the critical index $\sigma_0:=\frac{d-2}{2}$.

Our results on the estimates of the maximal Cesàro operators also allow us to establish a Fefferman-Stein type inequality for the Cesàro operators and to weaken the conditions in the Marcinkiewitcz multiplier theorem that was established previously in [15]. The precise statements of our results on the sphere can be found in Theorem 3.1.1, and Corollaries 3.7.1-3.7.6 in the third chapter of the thesis.

We will also establish similar results for the weighted orthogonal polynomial expansions with respect to the weight function

$$W_\kappa^B(x):=\Big(\prod_{j=1}^d|x_j|^{\kappa_j}\Big)(1-\|x\|^2)^{\kappa_{d+1}-1/2},\qquad \min_{1\leqslant i\leqslant d+1}\kappa_i\geq 0 \tag{1.1.6}$$

on the unit ball $\mathbb{B}^d=\{x\in\mathbb{R}^d:\|x\|\leq 1\}$, as well as for the weighted orthogonal polynomial expansions with respect to the weight function

$$W_\kappa^T(x) := \Big(\prod_{i=1}^d x_i^{\kappa_i - 1/2}\Big)(1-|x|)^{\kappa_{d+1}-1/2}, \qquad \min_{1\leqslant i\leqslant d+1} \kappa_i \geq 0. \tag{1.1.7}$$

on the simplex $\mathbb{T}^d = \{x \in \mathbb{R}^d : x_1 \geq 0, \ldots, x_d \geq 0, 1-|x| \geq 0\}$, here, and in what follows, $|x| := \sum_{j=1}^d |x_j|$ for $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$. The precise statements of our results on $\mathbb{B}^d$ and $\mathbb{T}^d$ can be found in Theorem 4.1.1, Corollaries 4.1.2-4.1.5, Theorem 4.2.2, and Corollaries 4.2.3-4.2.6 in the fourth and the fifth chapters of the thesis.

It turns out that results on the unit ball $\mathbb{B}^d$ are normally easier to be deduced directly from the corresponding results on the unit sphere $\mathbb{S}^d$, whereas in most cases, results on the simplex are not able to be deduced directly from those on the ball and on the sphere due to the differences in their orthogonal structures. (See, for instance, [15, 16, 49, 52]). In the fifth chapter of this thesis, we will develop a new technique which allows one to deduce results on the Cesàro means on the simplex directly from the corresponding results on the unit ball.

Our main results on the unit sphere are stated and proved in the third chapter. After that, in the fourth chapter, similar results are established on the unit ball. These results are deduced directly from the corresponding results on the unit sphere. Finally, in the fourth chapter we also discuss how to deduce similar results on the simplex from the corresponding results on the unit ball. A new technique is developed.

1.2 Dunkl transforms and analysis on $\mathbb{R}^d$

Given $\kappa = (\kappa_1, \cdots, \kappa_d) \in [0, \infty)^d$, let

$$h_\kappa(x) := \prod_{j=1}^{d} |x_j|^{\kappa_j}, \quad x = (x_1, x_2, \cdots, x_d) \in \mathbb{R}^d. \tag{1.2.8}$$

Denote by $L^p(\mathbb{R}^d; h_\kappa^2) \equiv L^p(\mathbb{R}^d; h_\kappa^2 dx)$, $1 \leqslant p \leqslant \infty$, the L^p-space defined with respect to the measure $h_\kappa^2(x)dx$ on $\mathbb{R}^d$, and $\|\cdot\|_{\kappa,p}$ the norm of $L^p(\mathbb{R}^d; h_\kappa^2)$. For a set $E \subset \mathbb{R}^d$, we write

$$\text{meas}_\kappa(E) := \int_E h_\kappa^2(x)\, dx. \tag{1.2.9}$$

Let $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$ denote the Euclidean norm and the Euclidean inner product on $\mathbb{R}^d$, respectively.

The Dunkl transform $\mathcal{F}_\kappa f$ of $f \in L^1(\mathbb{R}^d; h_\kappa^2)$ is defined by

$$\mathcal{F}_\kappa f(x) = c_\kappa \int_{\mathbb{R}^d} f(y) E_\kappa(-ix, y) h_\kappa^2(y) dy, \quad x \in \mathbb{R}^d, \tag{1.2.10}$$

where $c_\kappa^{-1} = \int_{\mathbb{R}^d} h_\kappa^2(y) e^{-\|y\|^2/2}\, dy$, and $E_\kappa(-ix, y) = V_\kappa \left[e^{-i\langle x, \cdot \rangle} \right](y)$ is the weighted analogue of the character $e^{-i\langle x,y \rangle}$ on $\mathbb{R}^d$. Here, $V_\kappa :\ C(\mathbb{R}^d) \to C(\mathbb{R}^d)$ is the Dunkl intertwining operator associated with the weight $h_\kappa^2(x)$ and the reflection group $\mathbb{Z}_2^d$, whose precise definition will be given in Section 2. In the case of $\kappa = 0$ (i.e., the unweighted case), V_κ is simply the identity operator on $C(\mathbb{R}^d)$, and hence the Dunkl transform $\mathcal{F}_\kappa f$ becomes the classical Fourier transform.

The Dunkl transform has applications in physics for the analysis of quantum many body systems of Calogero-Moser-Sutherland type (see, for instance, [24, Section 11.6], [36]). From the mathematical analysis point of view, its importance lies in that it generalizes the classical Fourier transform, and plays the similar role as the Fourier transform in classical Fourier analysis.

The Dunkl transform enjoys many properties similar to those of the classical Fourier transform (see, for instance, [29, 47, 46]). For example, each function $f \in L^1(\mathbb{R}^d; h_\kappa^2)$ is uniquely determined by its Dunkl transform $\mathcal{F}_\kappa f$. A very useful tool to recover a function $f \in L^1(\mathbb{R}^d; h_\kappa^2)$ from its Dunkl transform is the Bochner-Riesz means of f, which, in the Dunkl setting, are defined as

$$B_R^\delta(h_\kappa^2; f)(x) = c_\kappa \int_{\|y\| \leqslant R} \left(1 - \frac{\|y\|^2}{R^2}\right)^\delta \mathcal{F}_\kappa f(y) E_\kappa(ix, y) h_\kappa^2(y)\, dy, \quad x \in \mathbb{R}^d,$$

where $R > 0$, $\delta > -1$ and $f \in L^1(\mathbb{R}^d; h_\kappa^2)$. As in classical Fourier analysis, $B_R^\delta(h_\kappa^2; f)(x)$ can be expressed as an integral

$$B_R^\delta(h_\kappa^2; f)(x) = c_\kappa \int_{\mathbb{R}^d} f(y) K_R^\delta(h_\kappa^2; x, y) h_\kappa^2(y)\, dy, \quad x \in \mathbb{R}^d, \tag{1.2.11}$$

which further extends $B_R^\delta(h_\kappa^2; f)$ to a bounded operator on $L^p(\mathbb{R}^d; h_\kappa^2)$ for all $1 \leqslant p < \infty$ and $R > 0$.

Summability of the Bochner-Riesz means $B_R^\delta(h_\kappa^2; f)$ in the spaces $L^p(\mathbb{R}^d; h_\kappa^2)$, $1 \leqslant p < \infty$ was studied by Thangavelu and Xu [46, Theorem 5.5], who showed that for $\delta > \lambda_\kappa := \frac{d-1}{2} + |\kappa|$,

$$\lim_{R \to \infty} \|B_R^\delta(h_\kappa^2; f) - f\|_{\kappa,p} = 0 \tag{1.2.12}$$

holds for all $f \in L^p(\mathbb{R}^d; h_\kappa^2)$ and $1 \leqslant p < \infty$, and that this result is no longer true when $p = 1$ and $\delta \leqslant \lambda_\kappa$. Here and throughout the paper, we write $|\kappa| = \sum_{j=1}^d \kappa_j$. This, in particular, means that $\delta = \lambda_\kappa$ is the critical index for the summability of the Bochner-Riesz means $B_R^\delta(h_\kappa^2; f)$ in the weighted space $L^1(\mathbb{R}^d; h_\kappa^2)$. For the critical index of $B_R^\delta(h_\kappa^2; f)$ in the spaces $L^p(\mathbb{R}^d; h_\kappa^2)$ with $1 \leqslant p \leqslant \infty$, we refer to [8, Theorem 4.3].

Thangavelu and Xu [46, Theorem 7.5] also studies almost everywhere convergence (a.e.) of the Bochner-Riesz means $B_R^\delta(h_\kappa^2; f)$, showing that for $\delta > \lambda_\kappa$ and $f \in L^p(\mathbb{R}^d; h_\kappa^2)$ with $1 \leqslant p < \infty$,

$$\lim_{R\to\infty} B_R^\delta(h_\kappa^2; f)(x) = f(x) \quad \text{a.e. } x \in \mathbb{R}^d. \tag{1.2.13}$$

Using Stein's interpolation theorem for analytic families of operators, one can easily deduce from this result that for $f \in L^p(\mathbb{R}^d; h_\kappa^2)$ with $1 < p < \infty$, (1.2.13) holds at the critical index $\delta = \lambda_\kappa$ as well (see, for instance, [42]). A natural question also arises here: what will happen when $\delta = \lambda_\kappa$ and $f \in L^1(\mathbb{R}^d; h_\kappa^2)$?

In the classical case of Fourier transform (i.e., the case when $\kappa = (0, \cdots, 0)$ and $h_\kappa(x) \equiv 1$), the answer to the above question is negative. Indeed, it is well-known that if $\delta = \lambda_0 := \frac{d-1}{2}$, then there exists a function $f \in L^1(\mathbb{R}^d)$ whose Bochner-Riesz mean $B_R^\delta(f)(x) \equiv B_R^{\frac{d-1}{2}}(h_0^2; f)(x)$ diverges a.e. on $\mathbb{R}^d$ as $R \to \infty$, (see, for instance, [41]).

In this thesis, we will show that in contrast to the classical case of Fourier transform, the above question has an affirmative answer in the weighted case (i.e., the case when $\kappa \neq 0$). More precisely, we have the following result:

If $\kappa \neq 0$, $1 \leqslant p < \infty$ and $f \in L^p(\mathbb{R}^d; h_\kappa^2)$, then the Bochner-Riesz mean $B_R^{\lambda_\kappa}(h_\kappa^2; f)(x)$ converges a.e. to $f(x)$ on $\mathbb{R}^d$ as $R \to \infty$.

The conclusion of the above result is a little bit surprising because of the following two reasons. First, as indicated above, for the classical Fourier transform (i.e., $\kappa = 0$), (1.2.13) fails for some $f \in L^1(\mathbb{R}^d)$ at the critical index $\delta = \lambda_0 = \frac{d-1}{2}$. Second, in the general case of $\kappa \in [0, \infty)^d$, there exists a function $f \in L^1(\mathbb{R}^d; h_\kappa^2)$ for which the Bochner-Riesz means $B_R^{\lambda_\kappa}(h_\kappa^2; f)(x)$ at the critical index diverges in the norm of $L^1(\mathbb{R}^d; h_\kappa^2)$, (see, for instance, [8]).

By a standard density argument, the proof of almost everywhere convergence can be reduced to showing a weak-type estimate of the following maximal Bochner-Riesz operator:

$$B_*^\delta(h_\kappa^2; f)(x) = \sup_{R>0} |B_R^\delta(h_\kappa^2; f)(x)|, \quad x \in \mathbb{R}^d. \tag{1.2.14}$$

Indeed, we just need to prove the following result:

Assume that $\kappa \neq 0$. If $\delta = \lambda_\kappa$ and $f \in L^1(\mathbb{R}^d; h_\kappa^2)$, then for any $\alpha > 0$,

$$\text{meas}_\kappa \left(\left\{ x \in \mathbb{R}^d : \ B_*^\delta(h_\kappa^2; f)(x) \geqslant \alpha \right\} \right) \leqslant c_\kappa \frac{\|f\|_{\kappa,1}}{\alpha}, \tag{1.2.15}$$

where we need to replace $\frac{\|f\|_{\kappa,1}}{\alpha}$ by $\frac{\|f\|_{\kappa,1}}{\alpha} \left| \log \frac{\|f\|_{\kappa,1}}{\alpha} \right|$ when $\min_{1 \leqslant j \leqslant d} \kappa_j = 0$.

Note that according to Theorem 7.5 of [46], (1.2.15) holds for $\delta > \lambda_\kappa$ as well, whereas by Theorem 4.3 of [8], it does not hold when $\delta < \lambda_\kappa$. We further point out that similar results for the spherical h-harmonic expansions on the unit sphere were recently established by the current authors and S. Wang [10].

One of the key steps in our proof of weak type estimates of the maximal Bochner-Riesz operator is to show the following sharp pointwise estimate of the integral kernel $K_R^\delta(h_\kappa^2; x, y)$ of $B_R^\delta(h_\kappa^2; f)(x)$: for $\delta > 0$, $R > 0$ and $x, y \in \mathbb{R}^d$,

$$|K_R^\delta(h_\kappa^2; x, y)| \leqslant CR^d \frac{\prod_{j=1}^d (|x_j y_j| + R^{-2} + R^{-1}\|\bar{x} - \bar{y}\|)^{-\kappa_j}}{(1 + R\|\bar{x} - \bar{y}\|)^{\frac{d+1}{2}+\delta}}, \tag{1.2.16}$$

where we write $\bar{x} = (|x_1|, \cdots, |x_d|)$ for $x = (x_1, \cdots, x_d)$. In the case when $\|x\| = \|y\|$, the estimate (1.2.16) can be deduced directly from Lemma 3.4 of [13]. However, for general $x, y \in \mathbb{R}^d$, this is a fairly nontrivial estimate. Of crucial importance in the proof of (1.2.16) is the explicit formula of Yuan Xu [50] for the Dunkl intertwining operator associated with the weight $h_\kappa^2(x)$ and the reflection group $\mathbb{Z}_2^d$, (see (2.2.3) in Section 2).

The proof of the weak type estimate of the maximal Bochner-Riesz operator will be given in Chapter 6 of the thesis. And all of the above results in this part were published in my joint paper [17] with Dr. Feng Dai.

Next, we consider the strong type estimates of the maximal Bochner-Riesz operator. Our first goal is to establish a result for the Dunkl transform that is analogue to a classical result of Michael Christ [6] on strong estimates of the maximal Bochner-Riesz means of the Fourier integrals under the critical index $\lambda = \frac{d-1}{2}$. Our main result in this direction can be stated as follows:

Let $\delta_\kappa(p) = (2\lambda_\kappa + 1)(\frac{1}{2} - \frac{1}{p}) - \frac{1}{2}$. If $p \geqslant 2 + \frac{2}{\lambda_\kappa}$ and $\delta > \max\{0, \delta_\kappa(p)\}$, then for all $f \in L^p(\mathbb{R}^d; h_\kappa^2)$,

$$\|B_*^\delta(h_\kappa^2; f)\|_{\kappa,p} \leqslant C\|f\|_{\kappa,p}.$$

It is worthwhile to point out that this last inequality is no longer true for $\delta \leqslant \delta_\kappa(p)$.

The proof of this result will be given in Chapter 9.

One of the most important tools in our proof of the above strong type estimates is the restriction theorem for Dunkl transforms. Let $\mathbb{S}^{d-1} := \{x \in \mathbb{R}^d : \|x\| = 1\}$ be the unit sphere in $\mathbb{R}^d$, and $d\sigma$ the Lebesgue measure on $\mathbb{S}^{d-1}$. We define Rf to be the restriction to the sphere $\mathbb{S}^{d-1}$ of the Dunkl transform $\mathcal{F}_\kappa f$ of $f \in L^1(\mathbb{R}^d; h_\kappa^2)$. In this thesis, we shall prove the following global restriction theorem.

If $1 \leqslant p \leqslant \frac{2\lambda_\kappa+2}{\lambda_\kappa+2}$, then R extends to a bounded operator from $L^p(\mathbb{R}^d, h_\kappa^2)$ to $L^2(\mathbb{S}^{d-1}, h_\kappa^2)$, and the dual operator R^* extends to a bounded operator from $L^2(\mathbb{S}^{d-1}, h_\kappa^2)$ to $L^{p'}(\mathbb{R}^d, h_\kappa^2)$.

Since the weight function h_k in Dunkl analysis is neither translation invariant nor rotation invariant, unlike the case of the classical Fourier transform, the global restriction theorem stated above is not enough for our purpose. The proof of our main results requires the following local restriction theorem, which is stronger than the global one but significantly more difficult to prove:

Let $c_0 \in (0,1)$ be a constant depending only on d and κ, and B the ball $B(\omega,\theta)$ centered at $\omega \in \mathbb{R}^d$ and having radius $\theta \geq c_0 > 0$. If $1 \leqslant p \leqslant p_\kappa := \frac{2+2\lambda_\kappa}{\lambda_\kappa+2}$, and $f \in L^p(\mathbb{R}^d; h_\kappa^2)$ is supported in the ball B, then

$$\|\widehat{f}\|_{L^2(\mathbb{S}^{d-1};h_\kappa^2)} \leqslant C\Big(\frac{\theta^{2\lambda_\kappa+1}}{\int_B h_\kappa^2(y)\,dy}\Big)^{\frac{1}{p}-\frac{1}{2}}\|f\|_{L^p(\mathbb{R}^d;h_\kappa^2)}.$$

The proof of the restriction theorem will be given in Chapter 7 of the thesis.

In addition to the restriction theorem, we also need to establish weighted Littlewood-Paley inequality in the Dunkl setting, which seems to be of independent interest. This inequality will be proved in Chapter 8 of the thesis.

Finally, we shall study the almost everywhere convergence of the Bochner-Riesz means of functions in $L^p(R^d; h_\kappa^2)$-spaces. Our main purpose in this part is to establish a result for the Dunkl transform that is in full analogy with a classical result of A. Carbery, Francia and L. Vega [5] on the Fourier transform. Our main result can be stated as follows.

Let $\delta_\kappa(p) = (2\lambda_\kappa + 1)(\frac{1}{2} - \frac{1}{p}) - \frac{1}{2}$. *If* $p \geqslant 2$ *and* $\delta > \max\{0, \delta_\kappa(p)\}$*, then for all* $f \in L^p(\mathbb{R}^d; h_\kappa^2)$,

$$\lim_{R\to\infty} B_R^\delta(h_\kappa^2; f)(x) = f(x), \quad a.e.\ x \in \mathbb{R}^d.$$

The proof of this result will be given in Chapter 10.

Our results on local restriction theorem and the maximal Bochner-Riesz Means for the Dunkl transforms will be published in my joint paper [18] with Dr. Feng Dai soon.

Chapter 2

Preliminaries

In this chapter, we will describe some necessary materials for weighted orthogonal polynomial expansions on the sphere, the ball and the simplex. Unless otherwise stated, the main reference for the materials in this chapter is the book [25].

2.1 Notations

In this section, we shall introduce some necessary notations that will be used frequently in the rest of the thesis. We use the notation $C_1 \sim C_2$ to mean that there exists a positive universal constant C, called the constant of equivalence, such that $C^{-1}C_1 \leqslant C_2 \leqslant CC_1$. And we note $C_1 \lesssim C_2 (C_1 \gtrsim C_2)$ if there exists a positive universal constant C such that $C_1 \leqslant CC_2 (C_1 \geqslant CC_2)$.

Let $\mathbb{R}^d$ denote the d-dimensional Euclidean space, and for $x \in \mathbb{R}^d$, we write $x = (x_1, x_2, \cdots, x_d)$. The norm of x is defined by $\|x\| := \sqrt{\sum_{j=1}^d x_j^2}$. The unit sphere

$\mathbb{S}^{d-1}$ and the unit ball $\mathbb{B}^d$ of $\mathbb{R}^d$ are defined by

$$\mathbb{S}^{d-1} := \{x : \ \|x\| = 1\}, \text{ and } \mathbb{B}^d := \{x : \ \|x\| \leqslant 1\}.$$

Given $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$, and $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_d) \in \mathbb{Z}_2^d := \{\pm 1\}^d$, we write $\bar{x} := (|x_1|, \cdots, |x_d|)$, $|x| := \sum_{j=1}^d |x_j|$, and $x\varepsilon := (x_1\varepsilon_1, \cdots, x_d\varepsilon_d)$. We denote by $\rho(x, y)$ the geodesic distance, $\arccos x \cdot y$, of $x, y \in \mathbb{S}^{d-1}$.

The simplex $\mathbb{T}^d$ of $\mathbb{R}^d$ is defined by

$$\mathbb{T}^d = \{x \in \mathbb{R}^d : x_1 \geq 0, \ldots, x_d \geq 0, 1 - |x| \geq 0\}$$

Let Ω denote a compact domain in $\mathbb{R}^d$ endowed with the usual Lebesgue measure dx, where in the case of $\Omega = \mathbb{S}^{d-1}$, we use $d\sigma(x)$ instead of dx to denote the Lebesgue measure. Given a nonnegative product weight function W on Ω, we denote by $L^p(W; \Omega)$ the usual L^p-space defined with respect to the measure $W dx$ on Ω. For each function $f \in L^p(W; \Omega)$, we define its $\|\cdot\|_{p,W}$ norm as following

$$\|f\|_{p,W} := \left(\int_\Omega |f(x)|^p W(x) dx \right)^{\frac{1}{p}}, \quad 1 \leqslant p < \infty,$$

and for $p = \infty$, we consider the space of continuous functions with the uniform norm

$$\|f\|_\infty := \operatorname*{ess\,sup}_{x \in \Omega} |f(x)|.$$

Let $\mathcal{S}(\mathbb{R}^d)$ denote the class of all Schwartz functions on $\mathbb{R}^d$, and $\mathcal{S}'(\mathbb{R}^d)$ its dual (i.e., the class of all tempered distributions on $\mathbb{R}^d$).

Finally, given a sequence of operators T_n, $n = 0, 1, \cdots$ on some L^p space, we denote by T_* the corresponding maximal operator defined by $T_* f(x) = \sup_n |T_n f(x)|$.

2.2 Dunkl operators and Dunkl intertwining operators

Recall that $\mathbb{Z}_2^d$ is the reflection group generated by the reflections $\sigma_1, \cdots, \sigma_d$, where σ_j denotes the reflection with respect to the coordinate plane $x_j = 0$; that is,

$$x\sigma_j = (x_1, \cdots, x_{j-1}, -x_j, x_{j+1}, \cdots, x_d), \quad x \in \mathbb{R}^d.$$

Define a family of difference operators E_j, $j = 1, \cdots, d$ by

$$E_j f(x) := \frac{f(x) - f(x\sigma_j)}{x_j}, \quad x \in \mathbb{R}^d.$$

Let ∂_j denote the partial derivative with respect to the j-th coordinate x_j. The Dunkl operators $\mathcal{D}_{\kappa,j}$, $j = 1, \cdots, d$ with respect to the weight $h_\kappa^2(x)$ and the group $\mathbb{Z}_2^d$ are defined by

$$\mathcal{D}_{\kappa,j} := \partial_j + \kappa_j E_j, \quad j = 1, \cdots, d. \tag{2.2.1}$$

A remarkable property of these operators is that they mutually commute, that is,

$\mathcal{D}_{\kappa,i}\mathcal{D}_{\kappa,j} = \mathcal{D}_{\kappa,j}\mathcal{D}_{\kappa,i}$ for $1 \leqslant i, j \leqslant d$. We denote by $\mathbb{P}_n^d$ the space of homogeneous polynomials of degree n in d variables, and by $\Pi^d := \Pi(\mathbb{R}^d)$ the $\mathbb{C}$-algebra of polynomial functions on $\mathbb{R}^d$. It is clear that the Dunkl operators $\mathcal{D}_{\kappa,i}$ map $\mathbb{P}_n^d$ to $\mathbb{P}_{n-1}^d$. A fundamental result in Dunkl theory states that there exists a linear operator $V_\kappa : \Pi^d \to \Pi^d$ determined uniquely by

$$V_\kappa(\mathbb{P}_n^d) \subset \mathbb{P}_n^d, \quad V_\kappa(1) = 1, \quad \text{and} \quad \mathcal{D}_{\kappa,i}V_\kappa = V_\kappa\partial_i, \quad 1 \leqslant i \leqslant d. \tag{2.2.2}$$

Such an operator is called the intertwining operator.

For the weight function $h_\kappa^2(x)$ given in (1.2.8) and the reflection group $\mathbb{Z}_2^d$, the following very useful explicit formula for V_κ was obtained by Xuan Xu [50]:

$$V_\kappa f(x) = c'_\kappa \int_{[-1,1]^d} f(x_1t_1, \cdots, x_dt_d) \prod_{j=1}^d (1+t_j)(1-t_j^2)^{\kappa_j-1}dt_j, \tag{2.2.3}$$

where $c'_\kappa = \prod_{j=1}^d c'_{\kappa_j} = \prod_{j=1}^d \frac{\Gamma(\kappa_j+1/2)}{\sqrt{\pi}\Gamma(\kappa_j)}$, and if any κ_j is equal to 0, the formula holds under the limits

$$\lim_{\mu\to 0} c'_\mu \int_{-1}^1 g(t)(1-t^2)^{\mu-1}dt = \frac{g(1)+g(-1)}{2}.$$

In particular, the formula (2.2.3) extends V_κ to a positive operator on the space of continuous functions on $\mathbb{R}^d$. This formula will play a crucial role in this thesis. It should be pointed out that such an explicit formula for V_κ is available only in the case of $\mathbb{Z}_2^d$. In the case of a general reflection group, a very deep result on the operator V_κ is due to Rösler, who, among other things, proved that V_κ extends to a positive operator on

$C(\mathbb{R}^d)$.

2.3 Spherical h-harmonic expansions on the unit sphere

We restrict our discussion to h_κ in (1.1.1), and denote the L^p norm of $L^p(h_\kappa^2;\mathbb{S}^{d-1})$ by $\|\cdot\|_{\kappa,p}$,

$$\|f\|_{\kappa,p} := \Big(\int_{\mathbb{S}^{d-1}} |f(y)|^p h_\kappa^2(y) d\sigma(y)\Big)^{1/p}, \quad 1 \leqslant p < \infty$$

with the usual change when $p=\infty$.

We denote $\mathcal{V}_n^d(h_\kappa^2)$ the space of orthogonal polynomials of degree n with respect to the weight function h_κ^2 on $\mathbb{S}^{d-1}$. Thus, if we denote by $\Pi_n(\mathbb{S}^{d-1})$ the space of all algebraic polynomials in d variables of degree at most n restricted on the domain $\mathbb{S}^{d-1}$, then $\mathcal{V}_n^d(h_\kappa^2)$ is the orthogonal complement of $\Pi_{n-1}(\mathbb{S}^{d-1})$ in the space $\Pi_n(\mathbb{S}^{d-1})$ with respect to the inner product of $L^2(h_\kappa^2;\mathbb{S}^{d-1})$, where it is agreed that $\Pi_{-1}(\mathbb{S}^{d-1}) = \{0\}$.

Since $\mathbb{S}^{d-1}$ is compact, each function $f \in L^2(h_\kappa^2;\mathbb{S}^{d-1})$ has a weighted orthogonal polynomial expansion on $\mathbb{S}^{d-1}$, $f = \sum_{n=0}^\infty \operatorname{proj}_n(h_\kappa^2; f)$, converging in the norm of $L^2(h_\kappa^2;\mathbb{S}^{d-1})$, where $\operatorname{proj}_n(h_\kappa^2; f)$ denotes the orthogonal projection of f onto the space $\mathcal{V}_n^d(h_\kappa^2)$. Let $P_n(h_\kappa^2;\cdot,\cdot)$ denote the reproducing kernel of the space $\mathcal{V}_n^d(h_\kappa^2)$; that is,

$$P_n(h_\kappa^2;x,y) := \sum_{j=1}^{a_n^d} \varphi_{n,j}(x)\overline{\varphi_{n,j}(y)}, \quad x,y \in \mathbb{S}^{d-1}$$

for an orthonormal basis $\{\varphi_{n,j} : \ 1 \leqslant j \leqslant a_n^d := \dim \mathcal{V}_n^d(h_\kappa^2)\}$ of the space $\mathcal{V}_n^d(h_\kappa^2)$.

The projection operator $\operatorname{proj}_n(h_\kappa^2) : L^2(h_\kappa^2; \mathbb{S}^{d-1}) \mapsto \mathcal{V}_n^d(h_\kappa^2)$ can be expressed as an integral operator

$$\operatorname{proj}_n(h_\kappa^2; f, x) = \int_{\mathbb{S}^{d-1}} f(y) P_n(h_\kappa^2; x, y) h_\kappa^2(y) dy, \quad x \in \mathbb{S}^{d-1}, \tag{2.3.4}$$

which also extends the definition of $\operatorname{proj}_n(h_\kappa^2; f)$ to all $f \in L(h_\kappa^2; \mathbb{S}^{d-1})$ since the kernel $P_n(W; x, y)$ is a polynomial in both x and y.

Let $S_n^\delta(h_\kappa^2; f)$, $n = 0, 1, \cdots$, denote the Cesàro (C, δ) means of the weighted orthogonal polynomial expansions of $f \in L^1(h_k^2; \mathbb{S}^{d-1})$. Each $S_n^\delta(h_\kappa^2; f)$ can be expressed as an integral against a kernel, $K_n^\delta(h_\kappa^2; x, y)$, called the Cesàro (C, δ) kernel,

$$S_n^\delta(h_\kappa^2; f, x) := \int_{\mathbb{S}^{d-1}} f(y) K_n^\delta(h_\kappa^2; x, y) h_\kappa^2(y) dy, \quad x \in \mathbb{S}^{d-1},$$

where

$$K_n^\delta(h_\kappa^2; x, y) := (A_n^\delta)^{-1} \sum_{j=0}^{n} A_{n-j}^\delta P_j(h_\kappa^2; x, y), \quad x, y \in \mathbb{S}^{d-1}.$$

An h-harmonic on $\mathbb{R}^d$ is a homogeneous polynomial P in d variables that satisfies the equation $\Delta_h P = 0$, where $\Delta_h := \mathcal{D}_{\kappa,1}^2 + \ldots + \mathcal{D}_{\kappa,d}^2$. The restriction of an h-harmonic on the sphere is called a spherical h-harmonic. A spherical h-harmonic is an orthogonal polynomial with respect to the weight function $h_\kappa^2(x)$ on $\mathbb{S}^{d-1}$, and we denote by $\mathcal{H}_n^d(h_\kappa^2)$ the space of spherical h-harmonics of degree n on $\mathbb{S}^{d-1}$. Thus, we have $\mathcal{H}_n^d(h_\kappa^2) \equiv \mathcal{V}_n^d(h_\kappa^2)$.

A fundamental result in the study of h-harmonic expansions is the following compact expression of the reproducing kernel (see [23, 49, 50]):

$$P_n(h_\kappa^2; x, y) = c_\kappa \frac{n+\lambda_\kappa}{\lambda_\kappa} \int_{[-1,1]^d} C_n^{\lambda_\kappa}(\sum_{j=1}^d x_i y_j t_j) \prod_{i=1}^d (1+t_i)(1-t_i^2)^{\kappa_i - 1} dt, \tag{2.3.5}$$

where C_n^λ is the Gegenbauer polynomial of degree n, and c_κ is a normalization constant depending only on κ and d. Here, and in what follows, if some $\kappa_i = 0$, then the formula holds under the limit relation

$$\lim_{\lambda \to 0} c_\lambda \int_{-1}^1 f(t)(1-t)^{\lambda-1} dt = \frac{f(1)+f(-1)}{2}.$$

The following pointwise estimates on the Cesàro (C, δ) kernels were proved in [16].

Theorem 2.3.1. *Let* $x = (x_1, \cdots, x_d) \in \mathbb{S}^{d-1}$ *and* $y = (y_1, \cdots, y_d) \in \mathbb{S}^{d-1}$. *Then for* $\delta > -1$,

$$|K_n^\delta(h_\kappa^2; x, y)| \le c n^{d-1} \left[\frac{\prod_{j=1}^d (|x_j y_j| + n^{-1}\rho(\bar{x}, \bar{y}) + n^{-2})^{-\kappa_j}}{(n\rho(\bar{x}, \bar{y}) + 1)^{\delta + d/2}} + \frac{\prod_{j=1}^d (|x_j y_j| + \rho(\bar{x}, \bar{y})^2 + n^{-2})^{-\kappa_j}}{(n\rho(\bar{x}, \bar{y}) + 1)^d} \right].$$

2.4 Orthogonal polynomial expansions on the unit ball and simplex

The weight function W_κ^B we consider on the unit ball $\mathbb{B}^d$ is given in (1.1.6) with $\kappa := (\kappa_1, \cdots, \kappa_{d+1}) \in \mathbb{R}_+^d$. It is related to the h_κ on the sphere $\mathbb{S}^d$ by

$$h_\kappa^2(x, \sqrt{1-\|x\|^2}) = W_\kappa^B(x)\sqrt{1-\|x\|^2}, \quad x \in \mathbb{B}^d, \tag{2.4.6}$$

in which h_κ is defined in (1.1.1) with $\mathbb{S}^d$ in place of $\mathbb{S}^{d-1}$. Furthermore, under the change of variables $y = \phi(x)$ with

$$\phi : x \in \mathbb{B}^d \mapsto (x, \sqrt{1-\|x\|^2}) \in \mathbb{S}_+^d := \{y \in \mathbb{S}^d : \ y_{d+1} \geq 0\}, \tag{2.4.7}$$

we have

$$\int_{\mathbb{S}^d} g(y) d\sigma(y) = \int_{\mathbb{B}^d} \left[g(x, \sqrt{1-\|x\|^2}\,) + g(x, -\sqrt{1-\|x\|^2}\,) \right] \frac{dx}{\sqrt{1-\|x\|^2}}. \tag{2.4.8}$$

The orthogonal structure is preserved under the mapping (2.4.7) and the study of orthogonal expansions for W_κ^B on $\mathbb{B}^d$ can be essentially reduced to that of h_κ^2 on $\mathbb{S}^d$. More precisely, we have

$$\begin{aligned} P_n(W_\kappa^B; x, y) = \frac{1}{2} \big[& P_n(h_\kappa^2; (x, x_{d+1}), (y, y_{d+1})) \\ & + P_n(h_\kappa^2; (x, x_{d+1}), (y, -y_{d+1})) \big] \end{aligned} \tag{2.4.9}$$

where $x, y \in \mathbb{B}^d$, and $x_{d+1} = \sqrt{1 - \|x\|^2}$, $y_{d+1} = \sqrt{1 - \|y\|^2}$. As a consequence, the orthogonal projection, $\operatorname{proj}_n(W_\kappa^B; f)$, of $f \in L^2(W_\kappa^B; \mathbb{B}^d)$ onto $\mathcal{V}_n^d(W_\kappa^B)$ can be expressed in terms of the orthogonal projection of $F(x, x_{d+1}) := f(x)$ onto $\mathcal{H}_n^{d+1}(h_\kappa^2)$:

$$\operatorname{proj}_n(W_\kappa^B; f, x) = \operatorname{proj}_n(h_\kappa^2; F, X), \qquad \text{with } \; X := (x, \sqrt{1 - \|x\|^2}). \tag{2.4.10}$$

This relation allows us to deduce results on the convergence of orthogonal expansions with respect to W_κ^B on $\mathbb{B}^d$ from that of h-harmonic expansions on $\mathbb{S}^d$.

For $d = 1$ the weight W_κ^B in (1.1.6) becomes the weight function

$$w_{\kappa_2,\kappa_1}(t) = |t|^{2\kappa_1}(1 - t^2)^{\kappa_2 - 1/2}, \qquad \kappa_i \geq 0, \quad t \in [-1, 1], \tag{2.4.11}$$

whose corresponding orthogonal polynomials, $C_n^{(\kappa_2,\kappa_1)}$, are called generalized Gegenbauer polynomials, and can be expressed in terms of Jacobi polynomials,

$$\begin{aligned} C_{2n}^{(\lambda,\mu)}(t) &= \frac{(\lambda+\mu)_n}{\left(\mu+\frac{1}{2}\right)_n} P_n^{(\lambda-1/2,\mu-1/2)}(2t^2 - 1), \\ C_{2n+1}^{(\lambda,\mu)}(t) &= \frac{(\lambda+\mu)_{n+1}}{\left(\mu+\frac{1}{2}\right)_{n+1}} t P_n^{(\lambda-1/2,\mu+1/2)}(2t^2 - 1), \end{aligned} \tag{2.4.12}$$

where $(a)_n = a(a+1)\cdots(a+n-1)$, and $P_n^{(\alpha,\beta)}$ denotes the usual Jacobi polynomial of degree n and index (α, β) defined as in [44].

The weight functions we consider on the simplex $\mathbb{T}^d$ are defined by (1.1.7), which are related to W_κ^B, hence to h_κ^2. In fact, W_κ^T is exactly the product of the weight function W_κ^B under the mapping

$$\psi : (x_1, \dots, x_d) \in \mathbb{B}^d \mapsto (x_1^2, \dots, x_d^2) \in \mathbb{T}^d \tag{2.4.13}$$

and the Jacobian of this change of variables. Furthermore, the change of variables shows

$$\int_{\mathbb{B}^d} g(x_1^2, \dots, x_d^2) dx = \int_{\mathbb{T}^d} g(x_1, \dots, x_d) \frac{dx}{\sqrt{x_1 \cdots x_d}}. \tag{2.4.14}$$

The orthogonal structure is preserved under the mapping (2.4.13). In fact, $R \in \mathcal{V}_n^d(W_\kappa^T)$ if and only if $R \circ \psi \in \mathcal{V}_{2n}^d(W_\kappa^B)$. The orthogonal projection, $\mathrm{proj}_n(W_\kappa^T; f)$, of $f \in L^2(W_\kappa^T; \mathbb{T}^d)$ onto $\mathcal{V}_n^d(W_\kappa^T)$ can be expressed in terms of the orthogonal projection of $f \circ \psi$ onto $\mathcal{V}_{2n}^d(W_\kappa^B)$:

$$\mathrm{proj}_n(W_\kappa^T; f, \psi(x)) = \frac{1}{2^d} \sum_{\varepsilon \in \mathbb{Z}_2^d} \mathrm{proj}_{2n}(W_\kappa^B; f \circ \psi, x\varepsilon), \quad x \in \mathbb{B}^d. \tag{2.4.15}$$

The fact that $\mathrm{proj}_n(W_\kappa^T)$ of degree n is related to $\mathrm{proj}_{2n}(W_\kappa^B)$ of degree $2n$ suggests that some properties of the orthogonal expansions on $\mathbb{B}^d$ cannot be transformed directly to those on $\mathbb{T}^d$.

2.5 Dunkl transforms

The classical Fourier transform, initially defined on $L^1(\mathbb{R}^d)$ extends to an isometry of $L^2(\mathbb{R}^d)$ and commutes with the rotation group. For a family of weight functions h_κ

invariant under a reflection group G, there is a similar isometry of $L^2(\mathbb{R}^d; h_\kappa^2)$, called the Dunkl transform, which enjoys properties similar to those of the classical Fourier transform (see [46, 47]).

Given $\alpha \in \mathbb{C}$ with Re $\alpha > -1$, let J_α denote the first kind Bessel function of order α:

$$J_\alpha(t) = \left(\frac{t}{2}\right)^\alpha \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\alpha+1)} \left(\frac{t}{2}\right)^{2n}, \quad t \in \mathbb{R}. \tag{2.5.16}$$

The Dunkl transform $\mathcal{F}_\kappa f$ of $f \in L^1(\mathbb{R}^d; h_\kappa^2)$ is defined by

$$\mathcal{F}_\kappa f(x) = \widehat{f}(x) = c_\kappa \int_{\mathbb{R}^d} f(y) E(-ix, y) h_\kappa^2(y) dy, \quad x \in \mathbb{R}^d, \tag{2.5.17}$$

where $c_\kappa^{-1} = \int_{\mathbb{R}^d} h_\kappa^2(y) e^{-\|y\|^2/2}\, dy$, and

$$E_\kappa(-ix, y) = V_\kappa\left[e^{-i\langle x, \cdot\rangle}\right](y) = \prod_{j=1}^{d} c_{\kappa_j} \left[\frac{J_{\kappa_j - \frac{1}{2}}(x_j y_j)}{(x_j y_j)^{\kappa_j - \frac{1}{2}}} - i x_j y_j \frac{J_{\kappa_j + \frac{1}{2}}(x_j y_j)}{(x_j y_j)^{\kappa_j + \frac{1}{2}}}\right].$$

We shall also consider the Dunkl transform on the space of finite Borel measures on $\mathbb{R}^d$:

$$\mathcal{F}_\kappa \mu(\xi) \equiv \widehat{\mu}(\xi) := c_\kappa \int_{\mathbb{R}^d} E(-i\xi, y) h_\kappa^2(y) d\mu(y), \quad \xi \in \mathbb{R}^d.$$

If $\kappa = 0$ then $V_\kappa = \text{id}$ and the Dunkl transform coincides with the usual Fourier transform.

The Dunkl transform $\mathcal{F}_\kappa$ on the Schwartz class $\mathcal{S}(\mathbb{R}^d)$ extends uniquely to an isometric isomorphism on $L^2(\mathbb{R}^d; h_\kappa^2)$, i.e., $\|f\|_{\kappa,2} = \|\mathcal{F}_\kappa f\|_{\kappa,2}$ for each $f \in L^2(\mathbb{R}^d; h_\kappa^2)$.

Since $L^p \subset L^1 + L^2$ for $1 \leqslant p \leqslant 2$, we can also define the Dunkl transform $\mathcal{F}_\kappa f$ for each $f \in L^p(\mathbb{R}^d; h_\kappa^2)$ with $1 \leqslant p \leqslant 2$.

Many properties of the Euclidean Fourier transform carry over to the Dunkl transform. The results listed below can be found in [21, 29, 35].

Lemma 2.5.1. *[21, 29, 35]*

(i) *If* $f \in L^1(\mathbb{R}^d; h_\kappa^2)$ *then* $\mathcal{F}_\kappa f \in C(\mathbb{R}^d)$ *and* $\lim_{\|\xi\| \to \infty} \mathcal{F}_\kappa f(\xi) = 0$.

(ii) *The Dunkl transform* $\mathcal{F}_\kappa$ *is an isomorphism of the Schwartz class* $\mathcal{S}(\mathbb{R}^d)$ *onto itself, and* $\mathcal{F}_\kappa^2 f(x) = f(-x)$.

(iii) *If* f *and* $\mathcal{F}_\kappa f$ *are both in* $L^1(\mathbb{R}^d; h_\kappa^2)$ *then the following inverse formula holds:*

$$f(x) = c_\kappa \int_{\mathbb{R}^d} \mathcal{F}_\kappa f(y) E_\kappa(ix, y) h_\kappa^2(y)\, dy, \quad x \in \mathbb{R}^d.$$

(iv) *If* $f, g \in L^1(\mathbb{R}^d; h_\kappa^2)$ *then*

$$\int_{\mathbb{R}^d} \mathcal{F}_\kappa f(x) g(x)\, h_\kappa^2(x)\, dx = \int_{\mathbb{R}^d} f(x) \mathcal{F}_\kappa g(x)\, h_\kappa^2(x)\, dx. \tag{2.5.18}$$

(v) *(Haussdorf-Young) If* $1 \leqslant p \leqslant 2$, *then*

$$\|\mathcal{F}_\kappa f\|_{\kappa,p'} \leqslant \|f\|_{\kappa,p}, \tag{2.5.19}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

(vi) *Given* $\varepsilon > 0$*, let* $f_\varepsilon(x) = \varepsilon^{-(2\lambda_\kappa+1)} f(\varepsilon^{-1}x)$ *with* $\lambda_\kappa := \frac{d-1}{2} + |\kappa|$*. Then* $\mathcal{F}_\kappa f_\varepsilon(\xi) = \mathcal{F}_\kappa f(\varepsilon\xi)$*.*

(vii) *If* f *is a Schwartz function on* $\mathbb{R}^d$*, then*

$$\mathcal{F}_\kappa(\mathcal{D}_\kappa^\alpha f)(x) = (-ix)^\alpha \mathcal{F}_\kappa f(x), \quad x \in \mathbb{R}^d.$$

where $\mathcal{D}_\kappa^\alpha = \mathcal{D}_{\kappa,1}^{\alpha_1} \cdots \mathcal{D}_{\kappa,d}^{\alpha_d}$ *and* $\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{Z}_+^d$*.*

(viii) *If* $f(x) = f_0(\|x\|)$ *is a radial function in* $L^p(\mathbb{R}^d; h_\kappa^2)$ *with* $1 \leqslant p \leqslant 2$*, then* $\mathcal{F}_\kappa f(\xi) = H_{\lambda_\kappa} f_0(\|\xi\|)$ *is again a radial function, where* H_α *denotes the Hankel transform defined by*

$$H_\alpha g(s) = \frac{1}{\Gamma(\alpha+1)} \int_0^\infty g(r) \frac{J_\alpha(rs)}{(rs)^\alpha} r^{2\alpha+1}\, dr, \quad \alpha > -\frac{1}{2}.$$

Many identities in this thesis have to be interpreted in a distributional sense. As a result, throughout the thesis, we identify a function f in $L^p(\mathbb{R}^d; h_\kappa^2)$, $1 \leqslant p \leqslant \infty$ with a tempered distribution in $\mathcal{S}'(\mathbb{R}^d)$ given by

$$(f, \varphi) := \int_{\mathbb{R}^d} f(x)\varphi(x) h_\kappa^2(x)\, dx, \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^d).$$

For $f \in L^p(\mathbb{R}^d, h_\kappa^2)$ with $2 < p \leqslant \infty$, by (2.5.18), we may define its distributional Dunkl transform $\mathcal{F}_\kappa f$ via

$$(\mathcal{F}_\kappa f, \varphi) := (f, \mathcal{F}_\kappa \varphi) \equiv \int_{\mathbb{R}^d} f(x)\mathcal{F}_\kappa\varphi(x)h_\kappa^2(x)\,dx, \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^d). \tag{2.5.20}$$

For more information on distributional Dunkl transform, we refer to [2, 9].

For later applications, we also record some useful facts about the Bessel functions in the following lemma:

Lemma 2.5.2. (i) ([44, (1.71.1), (1.71.5)]) *For each $\alpha \in \mathbb{C}$ with $Re\,\alpha > -1$, $z^{-\alpha}J_\alpha(z)$ is an even entire function of $z \in \mathbb{C}$ and*

$$\frac{d}{dz}\left[z^{-\alpha}J_\alpha(z)\right] = -z^{-\alpha}J_{\alpha+1}(z). \tag{2.5.21}$$

(ii) ([44, (1.71.1), (1.71.11)]) *For each $\alpha = \sigma + i\tau \in \mathbb{C}$ with $\sigma > -1$,*

$$|x^{-\alpha}J_\alpha(x)| \leqslant Ce^{c|\tau|}(1+|x|)^{-\sigma-\frac{1}{2}}, \quad x \in \mathbb{R}. \tag{2.5.22}$$

(iii) ([1, p. 218, (4.11.12)]) *If Re $\alpha > -1$ and Re $\beta > 0$, then*

$$\begin{aligned} H_\alpha(j_{\alpha+\beta})(t) &= \frac{1}{\Gamma(\alpha+1)}\int_0^\infty \frac{J_{\alpha+\beta}(s)}{s^{\alpha+\beta}}\frac{J_\alpha(st)}{(st)^\alpha}s^{2\alpha+1}\,ds \\ &= \frac{1}{2^{\beta-1}\Gamma(\beta)\Gamma(\alpha+1)}(1-t^2)_+^{\beta-1}, \quad t \in \mathbb{R}, \end{aligned} \tag{2.5.23}$$

where $j_{\alpha+\beta}(t) = \frac{J_{\alpha+\beta}(t)}{t^{\alpha+\beta}}$.

2.6 Generalized translations and convolutions with Schwartz functions

We first give the definition of generalized translation on the class of Schwartz functions:

Definition 2.6.1. Given $y \in \mathbb{R}^d$ and $f \in \mathcal{S}(\mathbb{R}^d)$, we define its generalized translation $T^y f$ by

$$T^y f(x) := c_\kappa \int_{\mathbb{R}^d} \widehat{f}(\xi) E_\kappa(-iy, \xi) E_\kappa(i\xi, x) h_\kappa^2(\xi)\, d\xi, \quad x \in \mathbb{R}^d. \tag{2.6.24}$$

By the inverse formula for Dunkl transforms, we have that for $f \in \mathcal{S}(\mathbb{R}^d)$ and $x, y \in \mathbb{R}^d$,

$$\mathcal{F}_\kappa(T^y f)(x) = E_\kappa(-ix, y)\mathcal{F}_\kappa f(x). \tag{2.6.25}$$

The following lemma collects some useful known results on generalized translations on $\mathcal{S}(\mathbb{R}^d)$.

Lemma 2.6.2. (i) *([35, Lemma 2.2]) If $f \in \mathcal{S}(\mathbb{R}^d)$ and $y \in \mathbb{R}^d$, then also $T^y f \in \mathcal{S}(\mathbb{R}^d)$. Thus, $T^y : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ is a linear operator on $\mathcal{S}(\mathbb{R}^d)$.*

(ii) *([46, Theorem 7.1]) For $y = (y_1, \cdots, y_d) \in \mathbb{R}^d$ and $f \in \mathcal{S}(\mathbb{R}^d)$,*

$$T^y f(x) = T_{1,y_1} T_{2,y_2} \cdots T_{d,y_d} f(x), \ x = (x_1, \cdots, x_d) \in \mathbb{R}^d, \tag{2.6.26}$$

where

$$\begin{aligned}T_{j,y_j}f(x) :=& c'_{\kappa_j}\int_{-1}^{1} f_{j,e}(u_j(x,y,t))(1-t^2)^{\kappa_j-1}(1+t)\,dt\\ &+ c'_{\kappa_j}\int_{-1}^{1} f_{j,o}(u_j(x,y,t))\frac{x_j-y_j}{\sqrt{x_j^2+y_j^2-2x_jy_jt}}(1-t^2)^{\kappa_j-1}(1+t)\,dt,\end{aligned} \tag{2.6.27}$$

$$u_j(x,y,t) = \Big(x_1,\cdots,x_{j-1},\sqrt{x_j^2+y_j^2-2x_jy_jt},x_{j+1},\cdots x_d\Big),$$

and

$$f_{j,e}(x) = \frac{1}{2}\Big(f(x)+f(x\sigma_j)\Big),\quad f_{j,o}(x) = \frac{1}{2}\Big(f(x)-f(x\sigma_j)\Big).$$

(iii) *For* $f \in \mathcal{S}(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} T^y f(x)\varphi(x)h_\kappa^2(x)\,dx = \int_{\mathbb{R}^d} f(x)T^{-y}\varphi(x)h_\kappa^2(x)\,dx,\quad \forall\varphi\in\mathcal{S}(\mathbb{R}^d). \tag{2.6.28}$$

Definition 2.6.3. The generalized convolution of $f, g \in \mathcal{S}(\mathbb{R}^d)$ is defined by

$$f *_\kappa g(x) = \int_{\mathbb{R}^d} f(y)T^y g(x)h_\kappa^2(y)\,dy,\quad x\in\mathbb{R}^d. \tag{2.6.29}$$

The generalized convolution has the following basic property: for $f, g \in \mathcal{S}(\mathbb{R}^d)$,

$$\mathcal{F}_\kappa(f *_\kappa g)(\xi) = \mathcal{F}_\kappa f(\xi)\mathcal{F}_\kappa g(\xi),\quad \xi\in\mathbb{R}^d. \tag{2.6.30}$$

Chapter 3

Maximal Cesàro operators for spherical h-harmonics on the sphere and their applications

3.1 Main results

Recall that the letter κ denotes a nonzero vector $\kappa := (\kappa_1, \cdots, \kappa_d)$ in

$$\mathbb{R}^d_+ := \Big\{(x_1, \cdots, x_d) \in \mathbb{R}^d : \quad x_i \geq 0, \quad i = 1, 2, \cdots, d\Big\},$$

and

$$\kappa_{\min} := \min_{1 \leqslant j \leqslant d} \kappa_j, \quad |\kappa| = \sum_{j=1}^{d} \kappa_j, \quad \sigma_\kappa := \frac{d-2}{2} + |\kappa| - \kappa_{\min}. \tag{3.1.1}$$

We will keep these notations throughout this chapter. Some of our results and estimates below are not true if $\kappa = 0$.

Our main result on the unit sphere can be stated as follows:

Theorem 3.1.1. (i) *If $\delta \geq \sigma_\kappa$, then for $f \in L^1(h_\kappa^2; \mathbb{S}^{d-1})$ with $\|f\|_{\kappa,1} = 1$,*

$$\text{meas}_\kappa \Big\{ x \in \mathbb{S}^{d-1} : \quad S_*^\delta(h_\kappa^2; f)(x) > \alpha \Big\} \leqslant C \frac{1}{\alpha}, \quad \forall \alpha > 0$$

with $\alpha^{-1} |\log \alpha|$ in place of α^{-1} in the case when $\delta = \sigma_\kappa$ and at least two of the κ_i are zero.

(ii) *If $\delta < \sigma_\kappa$, then there exists a function $f \in L^1(h_\kappa^2; \mathbb{S}^{d-1})$ of the form $f(x) = f_0(|x_{j_0}|)$ such that $S_*^\delta(h_\kappa^2; f)(x) = \infty$ for a.e. $x \in \mathbb{S}^{d-1}$, where $1 \leqslant j_0 \leqslant d$ and $\kappa_{j_0} = \kappa_{\min}$.*

3.2 Proof of Theorem 3.1.1: Part(i)

Let us first introduce several necessary notations for the proofs in the next few subsections. Recall that $\rho(x, y)$ denotes the geodesic distance $\arccos x \cdot y$ between two points $x, y \in \mathbb{S}^{d-1}$. We denote by $B(x, \theta)$ the spherical cap $\{y \in \mathbb{S}^{d-1} : \ \rho(x, y) \leqslant \theta\}$ centered at $x \in \mathbb{S}^{d-1}$ of radius $\theta \in (0, \pi]$. It is known that for any $x \in \mathbb{S}^{d-1}$ and $\theta \in (0, \pi)$

$$V_\theta(x) := \text{meas}_\kappa(B(x, \theta)) = \int_{B(x,\theta)} h_\kappa^2(y) d\sigma(y) \sim \theta^{d-1} \prod_{j=1}^{d} (x_j + \theta)^{2\kappa_j}, \tag{3.2.2}$$

which, in particular, implies that h_κ^2 is a doubling weight on $\mathbb{S}^{d-1}$ (see [7, 5.3]). And we denote that:

$$V(x, y) := \text{meas}_\kappa(B(x, \rho(x, y))).$$

For $f \in L^1(h_\kappa^2; \mathbb{S}^{d-1})$, we define

$$M_\kappa f(x) := \sup_{0<\theta\leqslant\pi} \frac{1}{\mathrm{meas}_\kappa(B(x,\theta))} \int_{\{y\in\mathbb{S}^{d-1}:\ \rho(\bar{x},\bar{y})\leqslant\theta\}} |f(y)| h_\kappa^2(y)\, d\sigma(y).$$

Since the weight h_κ^2 satisfies the doubling condition and is invariant under the group $\mathbb{Z}_2^d$, the usual properties of the Hardy-Littlewood maximal functions imply that for $f \in L^1(h_\kappa^2; \mathbb{S}^{d-1})$,

$$\mathrm{meas}_\kappa\{x \in \mathbb{S}^{d-1} :\ M_\kappa f(x) > \alpha\} \leqslant C\frac{\|f\|_{\kappa,1}}{\alpha}, \quad \forall \alpha > 0. \tag{3.2.3}$$

For the proof of the first assertion in Theorem 3.1.1, we use Theorem 2.3.1 to obtain

$$|K_n^\delta(h_\kappa^2; x, y)| \leqslant C E_n^\delta(h_\kappa^2; x, y) + C R_n(h_\kappa^2; x, y), \tag{3.2.4}$$

where

$$E_n^\delta(h_\kappa^2; x, y) := n^{d-1} \frac{\prod_{j=1}^d (|x_j y_j| + n^{-1}\rho(\bar{x},\bar{y}) + n^{-2})^{-\kappa_j}}{(n\rho(\bar{x},\bar{y}) + 1)^{\delta+d/2}}, \tag{3.2.5}$$

$$R_n(h_\kappa^2; x, y) := n^{d-1} \frac{\prod_{j=1}^d (|x_j y_j| + \rho(\bar{x},\bar{y})^2 + n^{-2})^{-\kappa_j}}{(n\rho(\bar{x},\bar{y}) + 1)^{d}}. \tag{3.2.6}$$

Thus,

$$|S_n^\delta(h_\kappa^2; f, x)| \leqslant C|E_n^\delta(h_\kappa^2; f, x)| + C|T_n^\delta(h_\kappa^2; f, x)| + C|R_n(h_\kappa^2; f, x)|,$$

where

$$E_n^\delta(h_\kappa^2; f, x) := \int_{\{y\in\mathbb{S}^{d-1}:\rho(\bar{x},\bar{y})\leqslant\frac{1}{2\sqrt{d}}\}} E_n^\delta(h_\kappa^2; x, y) f(y) h_\kappa^2(y)\, d\sigma(y), \tag{3.2.7}$$

$$T_n^\delta(h_\kappa^2; f, x) := \int_{\{y\in\mathbb{S}^{d-1}:\ \ \rho(\bar{x},\bar{y})\geq\frac{1}{2\sqrt{d}}\}} E_n^\delta(h_\kappa^2; x, y) f(y) h_\kappa^2(y)\, d\sigma(y), \tag{3.2.8}$$

$$R_n(h_\kappa^2; f, x) := \int_{\mathbb{S}^{d-1}} R_n(h_\kappa^2; x, y) f(y) h_\kappa^2(y)\, d\sigma(y). \tag{3.2.9}$$

This implies that

$$\begin{aligned}\mathrm{meas}_\kappa\{x\in\mathbb{S}^{d-1}: S_*^\delta(h_\kappa^2; f, x) > \alpha\} \leqslant & \,\mathrm{meas}_\kappa\{x\in\mathbb{S}^{d-1}: E_*^\delta(h_\kappa^2; f, x) > \frac{\alpha}{3C}\} \\ & + \mathrm{meas}_\kappa\{x\in\mathbb{S}^{d-1}: T_*^\delta(h_\kappa^2; f, x) > \frac{\alpha}{3C}\} \\ & + \mathrm{meas}_\kappa\{x\in\mathbb{S}^{d-1}: R_*(h_\kappa^2; f, x) > \frac{\alpha}{3C}\},\end{aligned}$$

where

$$E_*^\delta(h_\kappa^2; f, x) := \sup_{n\in\mathbb{N}} |E_n^\delta(h_\kappa^2; f, x)|, \quad T_*^\delta(h_\kappa^2; f, x) := \sup_{n\in\mathbb{N}} |T_n^\delta(h_\kappa^2; f, x)|$$

$$R_*(h_\kappa^2; f, x) := \sup_{n\in\mathbb{N}} |R_n(h_\kappa^2; f, x)|.$$

Thus, for the proof of the stated weak estimates of $S_*^\delta(h_\kappa^2; f, x)$ in Theorem 3.1.1, it will suffice to establish the corresponding weak estimates for the maximal operators E_*^δ, T_*^δ and R_*. Namely, it suffices to prove the following three propositions:

Proposition 3.2.1. *For $\delta \geq \sigma_\kappa$ and each $f \in L^1(h_\kappa^2; \mathbb{S}^{d-1})$, we have that*

$$R_*(h_\kappa^2; f, x) \leqslant C M_\kappa f(x), \quad x \in \mathbb{S}^{d-1}, \tag{3.2.10}$$

and

$$\text{meas}_\kappa(\{x \in \mathbb{S}^{d-1} : \ R_*(h_\kappa^2; f, x) > \alpha\}) \leq C \frac{\|f\|_{\kappa,1}}{\alpha}, \quad \forall \alpha > 0. \tag{3.2.11}$$

Proposition 3.2.2. *For $\delta \geq \sigma_\kappa$ and $f \in L^1(h_\kappa^2; \mathbb{S}^{d-1})$,*

$$\text{meas}_\kappa(\{x \in \mathbb{S}^{d-1} : T_*^\delta(h_\kappa^2; f, x) > \alpha\}) \leq C \frac{\|f\|_{\kappa,1}}{\alpha}, \quad \forall \alpha > 0.$$

Proposition 3.2.3. *If either $\delta > \sigma_\kappa$ or $\delta = \sigma_\kappa$ and at most one of the κ_i is zero, then*

$$\text{meas}_\kappa\Big\{x \in \mathbb{S}^{d-1} : \quad E_*^\delta(h_\kappa^2; f, x) > \alpha\Big\} \leqslant C \frac{\|f\|_{\kappa,1}}{\alpha}, \quad \forall \alpha > 0. \tag{3.2.12}$$

Furthermore, if $\delta = \sigma_\kappa$ and at least two of the κ_i are zero, then

$$\text{meas}_\kappa\Big\{x \in \mathbb{S}^{d-1} : \quad E_*^\delta(h_\kappa^2; f)(x) > \alpha\Big\} \leqslant C \frac{\|f\|_{\kappa,1}}{\alpha} \log \frac{\|f\|_{\kappa,1}}{\alpha}, \quad \forall \alpha > 0.$$

The proofs of these three propositions will be given in Sections 3.3, 3.4, 3.5 respectively.

3.3 Proof of Proposition 3.2.1

For the proof of Proposition 3.2.1, we need the following two simple lemmas.

Lemma 3.3.1. *For* $x, y \in \mathbb{S}^{d-1}$,

$$R_n(h_\kappa^2, x, y) \sim \frac{1}{1 + n\rho(\bar{x}, \bar{y})} \cdot \frac{1}{V(\bar{x}, \bar{y}) + V_{n^{-1}}(\bar{x})} \tag{3.3.13}$$

Proof. By (3.2.6), it is sufficient to show that for each $1 \leqslant j \leqslant d$,

$$J_j(x, y) := (|x_j y_j| + \rho(\bar{x}, \bar{y})^2 + n^{-2})^{-\kappa_j} \sim (|x_j| + \rho(\bar{x}, \bar{y}) + n^{-1})^{-2\kappa_j}. \tag{3.3.14}$$

In fact, let't consider the following two cases:

Case 1. If $|x_j| \geqslant 2\rho(\bar{x}, \bar{y})$, since $|x_j| \geqslant 2\rho(\bar{x}, \bar{y}) \geqslant 2||x_j| - |y_j||$, we have that

$$|x_j| \sim |y_j|,$$

thus

$$J_j(x, y) \sim (|x_j|^2 + n^{-2} + \rho(\bar{x}, \bar{y})^2)^{-\kappa_j} \sim (|x_j| + n^{-1} + \rho(\bar{x}, \bar{y}))^{-2\kappa_j}.$$

Case 2. If $|x_j| \leqslant 2\rho(\bar{x}, \bar{y})$, then since $|y_j| - |x_j| \leqslant \rho(\bar{x}, \bar{y})$,

$$|y_j| \leqslant \rho(\bar{x}, \bar{y}) + |x_j| < 3\rho(\bar{x}, \bar{y}),$$

thus

$$J_j(x,y) \sim (\rho(\bar{x},\bar{y})+n^{-1})^{-2\kappa_j} \sim (|x_j|+\rho(\bar{x},\bar{y})+n^{-1})^{-2\kappa_j}.$$

Hence, in either case, we have have proven (3.3.14).

It follows that

$$\begin{aligned}\prod_{j=1}^{d} J_j(x,y) &\sim \prod_{j=1}^{d}(|x_j|+\rho(\bar{x},\bar{y})+n^{-1})^{-2\kappa_j}\\ &\sim \frac{n^{-(d-1)}+\rho(\bar{x},\bar{y})^{d-1}}{V_{n^{-1}}(\bar{x})+V(\bar{x},\bar{y})}\end{aligned}$$

Then

$$R_n(h_\kappa^2,x,y) = n^{d-1}\cdot\frac{\prod_{j=1}^{d} J_j(x,y)}{(n\rho(\bar{x},\bar{y})+1)^d} \sim \frac{1}{1+n\rho(\bar{x},\bar{y})}\cdot\frac{1}{V(\bar{x},\bar{y})+V_{n^{-1}}(\bar{x})}$$

□

Lemma 3.3.2. *For $x,y \in \mathbb{S}^{d-1}$ and $\alpha \geq 0$, let*

$$A_n^\alpha(x,y) := \frac{n^{d-1}}{(1+n\rho(\bar{x},\bar{y}))^\alpha}\prod_{j=1}^{d}(|x_j|+\rho(\bar{x},\bar{y})+n^{-1})^{-2\kappa_j}.$$

If $\alpha > d-1$ and $f \in L^1(h_\kappa^2;\mathbb{S}^{d-1})$, then

$$\int_{\mathbb{S}^{d-1}} |f(y)|A_n^\alpha(x,y)h_\kappa^2(y)\,d\sigma(y) \leqslant CM_\kappa f(x),$$

where the constant C is independent of n, f and x. Furthermore, if $\alpha = d-1$ and $\varepsilon > 0$

then

$$\int_{\{y\in\mathbb{S}^{d-1}:\quad \rho(\bar{x},\bar{y})\geq\varepsilon\}} |f(y)|A_n^\alpha(x,y)h_\kappa^2(y)\,d\sigma(y) \leqslant C\left|\log\frac{1}{\varepsilon}\right|M_\kappa f(x).$$

Proof. For $x\in\mathbb{S}^{d-1}$, by the last Lemma we have

$$\begin{aligned} A_n^\alpha(x,y) &= \frac{R_n^\alpha(x,y)}{(1+n\rho(\bar{x},\bar{y}))^{\alpha-d}} \\ &\sim \frac{(1+n\rho(\bar{x},\bar{y}))^{d-\alpha-1}}{V(\bar{x},\bar{y})+V_{n^{-1}}(\bar{x})} \end{aligned}$$

Let

$$A_n^\alpha(h_\kappa^2;f,x) := \int_{\mathbb{S}^{d-1}} A_n^\alpha(h_\kappa^2;x,y)f(y)h_\kappa^2(y)\,d\sigma(y).$$

and

$$\tilde{A}_n^\alpha(h_\kappa^2;f,x) := \int_{\{y\in\mathbb{S}^{d-1}:\ \rho(\bar{x},\bar{y})\geq\varepsilon\}} A_n^\alpha(h_\kappa^2;x,y)f(y)h_\kappa^2(y)\,d\sigma(y).$$

Then if $\alpha > d-1$,

$$\begin{aligned}
|A_n^\alpha(h_\kappa^2; f, x)| \leqslant & \int_{\mathbb{S}^{d-1}} \frac{|f(y)|h_\kappa^2(y)}{V(\bar{x},\bar{y})+V_{n^{-1}}(\bar{x})}(1+n\rho(\bar{x},\bar{y}))^{\alpha-d+1}d\sigma(y) \\
\leqslant & \int_{B(\bar{x},n^{-1})} \frac{|f(y)|h_\kappa^2(y)}{V(\bar{x},\bar{y})+V_{n^{-1}}(\bar{x})}(1+n\rho(\bar{x},\bar{y}))^{\alpha-d+1}d\sigma(y) \\
& + \sum_{j=0}^{\infty} \int_{\{y:\frac{2^j}{n}<\rho(\bar{x},\bar{y})\leqslant\frac{2^{j+1}}{n}\}} \frac{|f(y)|h_\kappa^2(y)}{V(\bar{x},\bar{y})+V_{n^{-1}}(\bar{x})}(1+n\rho(\bar{x},\bar{y}))^{\alpha-d+1}d\sigma(y) \\
\leqslant & \int_{B(\bar{x},n^{-1})} \frac{|f(y)|}{V_{n^{-1}}(\bar{x})}h_\kappa^2(y)d\sigma(y) \\
& + \sum_{j=0}^{\infty} 2^{(d-\alpha-1)j} \int_{\{y:\frac{2^j}{n}<\rho(\bar{x},\bar{y})\leqslant\frac{2^{j+1}}{n}\}} \frac{|f(y)|}{V(\bar{x},\bar{y})}h_\kappa^2(y)d\sigma(y) \\
\lesssim & M_\kappa(f)(x) + \sum_{j=0}^{\infty} \frac{2^{-j}}{\mathrm{meas}_\kappa(B(\bar{x},\frac{2^j}{n}))} \int_{B(\bar{x},\frac{2^{j+1}}{n})} |f(y)|h_\kappa^2(y)d\sigma(y) \\
\lesssim & M_\kappa(f)(x) + \sum_{j=0}^{\infty} 2^{-j} M_\kappa(f)(x) \\
\lesssim & M_\kappa(f)(x)
\end{aligned}$$

If $\alpha = d-1$, then

$$\begin{aligned}
|\tilde{A}_n^\alpha(h_\kappa^2; f, x)| \leqslant & \int_{\{y\in\mathbb{S}^{d-1}:\ \rho(\bar{x},\bar{y})\geq\varepsilon\}} \frac{|f(y)|}{V(\bar{x},\bar{y})}h_\kappa^2(y)d\sigma(y) \\
\lesssim & \sum_{j=1}^{\lceil \log_2 \frac{\pi}{\varepsilon} \rceil} \frac{1}{\mathrm{meas}_\kappa(B(\bar{x},2^j\varepsilon))} \int_{B(\bar{x},2^j\varepsilon)} |f(y)|h_\kappa^2(y)d\sigma(y) \\
\lesssim & \left|\log\frac{1}{\varepsilon}\right| M_\kappa f(x).
\end{aligned}$$

□

Proof of Proposition 3.2.1. The pointwise (3.2.10) follows directly from (3.2.9), Lemma 3.3.1 and Lemma 3.3.2, while the weak estimate (3.2.11) is an immediate consequence of (3.2.10) and (3.2.3). □

3.4 Proof of Proposition 3.2.2

Without loss of generality, we may assume that $\|f\|_{1,\kappa} = 1$ and $\alpha > 1$. Let $\mathbb{S}_j^{d-1} := \{x \in \mathbb{S}^{d-1} : \ |x_j| \geq \frac{1}{2\sqrt{d}}\}$ for $1 \leqslant j \leqslant d$. Since for each $x \in \mathbb{S}^{d-1}$,

$$\max_{1\leqslant j\leqslant d} |x_j| \geq \frac{1}{\sqrt{d}}\|x\| = \frac{1}{\sqrt{d}},$$

it follows that $\mathbb{S}^{d-1} = \cup_{j=1}^{d} \mathbb{S}_j^{d-1}$. By (3.2.8), this implies that

$$\begin{aligned}
|T_n^\delta(h_\kappa^2; f, x)| \lesssim & n^{\frac{d-2}{2}-\delta} \cdot \int_{\substack{\rho(\bar{x},\bar{y})\geqslant\frac{1}{2\sqrt{d}} \\ y\in\mathbb{S}^{d-1}}} |f(y)| \prod_{j=1}^{d} (|x_j y_j| + n^{-1}\rho(\bar{x},\bar{y}) + n^{-2})^{-\kappa_j} h_\kappa^2(y) d\sigma(y) \\
\leqslant & \sum_{m=1}^{d} n^{\frac{d-2}{2}-\delta} \cdot \int_{\substack{\rho(\bar{x},\bar{y})\geqslant\frac{1}{2\sqrt{d}} \\ |y_m|\geqslant\frac{1}{\sqrt{d}}}} |f(y)| \prod_{j=1}^{d} (|x_j y_j| + n^{-1}\rho(\bar{x},\bar{y}) + n^{-2})^{-\kappa_j} h_\kappa^2(y) d\sigma(y) \\
\leqslant & C \sum_{j=1}^{d} T_{n,j}^\delta(h_\kappa^2; f, x),
\end{aligned}$$

where

$$T_{n,j}^\delta(h_\kappa^2; f, x) := \int_{\{y\in\mathbb{S}_j^{d-1}:\rho(\bar{x},\bar{y})\geqslant\frac{1}{2\sqrt{d}}\}} \frac{n^{\frac{d-2}{2}-\delta}|f(y)|}{\prod_{i=1}^{d}(|x_i y_i| + n^{-1}\rho(\bar{x},\bar{y}) + n^{-2})^{\kappa_i}} h_\kappa^2(y) d\sigma(y).$$

Thus, it suffices to establish the weak estimates of

$$T^{\delta}_{*,j}(h^2_\kappa; f, x) := \sup_{n\in\mathbb{N}} T^{\delta}_{n,j}(h^2_\kappa; f, x)$$

for each $1 \leqslant j \leqslant d$. By symmetry, we only need to consider the case of $j = 1$.

Take $\varepsilon > 0$ such that $\varepsilon^{-\kappa_1} = c\alpha$ for some absolute constant c to be specified later. Set $F_\varepsilon = \{x \in \mathbb{S}^{d-1} : |x_1| \leqslant \varepsilon\}$. A straightforward calculation then shows that

$$\begin{aligned}\operatorname{meas}_\kappa(F_\varepsilon) &= \int_{-\varepsilon}^{\varepsilon} |x_1|^{2\kappa_1}(1-x_1^2)^{\frac{d-3}{2}+|\kappa|-\kappa_1} dx_1 \int_{\mathbb{S}^{d-2}} |y_2|^{2\kappa_2}\cdots|y_d|^{\kappa_d} d\sigma(y) \\ &\sim \varepsilon^{2\kappa_1+1} \leqslant C\varepsilon^{\kappa_1} \leqslant C\alpha^{-1}.\end{aligned}$$

On the other hand, if $x \in \mathbb{S}^{d-1} \setminus F_\varepsilon$, $y \in \mathbb{S}^{d-1}_1$ and $\rho(\bar{x}, \bar{y}) \geq \frac{1}{2\sqrt{d}}$, then

$$\prod_{i=1}^{d}(|x_i y_i| + n^{-1}\rho(\bar{x},\bar{y}) + n^{-2})^{\kappa_i} \geqslant C\varepsilon^{\kappa_1} n^{-|\kappa|+\kappa_1},$$

which implies that

$$\begin{aligned}|T^{\delta}_{n,1}(h^2_\kappa; f, x)| &\leqslant Cn^{\frac{d-2}{2}-\sigma_\kappa}\varepsilon^{-\kappa_1} n^{|\kappa|-\kappa_1}\|f\|_{1,\kappa} \\ &= Cn^{\frac{d-2}{2}+|\kappa|-\kappa_1-\sigma_\kappa}\varepsilon^{-\kappa_1} \leqslant C\varepsilon^{-\kappa_1} = Cc\alpha.\end{aligned}$$

Therefore, choosing $c > 0$ so that $Cc = \frac{1}{2}$, we deduce that

$$\operatorname{meas}_\kappa\Big\{x \in \mathbb{S}^{d-1} : T^{\delta}_{*,1}(h^2_\kappa; f, x) > \alpha\Big\} \leqslant \operatorname{meas}_\kappa(F_\varepsilon) \leqslant C\frac{1}{\alpha},$$

which is as desired.

3.5 Proof of Proposition 3.2.3

The proof of Proposition 3.2.3 relies on the following lemma.

Lemma 3.5.1. *Let $x, y \in \mathbb{S}^{d-1}$ be such that $\rho(\bar{x}, \bar{y}) \leqslant \frac{1}{2\sqrt{d}}$. If i is a positive integer such that $i \leqslant d$ and $|x_i| \geq \frac{1}{\sqrt{d}}$, then*

$$\prod_{j=1}^{d} I_j(x, y) \leqslant C(1 + n\rho(\bar{x}, \bar{y}))^{|\kappa|-\kappa_i} \prod_{j=1}^{d} (|x_j| + \rho(\bar{x}, \bar{y}) + n^{-1})^{-2\kappa_j}, \tag{3.5.15}$$

where

$$I_j(x, y) := (|x_j y_j| + n^{-1}\rho(\bar{x}, \bar{y}) + n^{-2})^{-\kappa_j}. \tag{3.5.16}$$

Proof. By symmetry, we may assume that $i = 1$. Consider the following two cases:

Case 1. $\rho(\bar{x}, \bar{y}) \leqslant n^{-1}$.

In this case, note that $I_j(x, y) \sim (n^{-2} + |x_j y_j|)^{-\kappa_j}$. If $|x_j| \geqslant 2n^{-1} > 2\rho(\bar{x}, \bar{y})$, then $|x_j| \sim |y_j|$ and

$$I_j(x, y) \sim |x_j|^{-2\kappa_j} \sim (|x_j| + \rho(\bar{x}, \bar{y}) + n^{-1})^{-2\kappa_j}.$$

If $|x_j| < 2n^{-1}$, then $|y_j| < 3n^{-1}$ and

$$I_j(x, y) \sim n^{2\kappa_j} \sim (|x_j| + \rho(\bar{x}, \bar{y}) + n^{-1})^{-2\kappa_j}.$$

Thus, we conclude that

$$\prod_{j=1}^{d} I_j(x,y) \sim \prod_{j=1}^{d} (|x_j| + \rho(\bar{x},\bar{y}) + n^{-1})^{-2\kappa_j},$$

which clearly implies (3.5.15).

Case 2. $\rho(\bar{x},\bar{y}) > n^{-1}$.

In this case, note first that if $|x_j| \geqslant 2\rho(\bar{x},\bar{y})$, then

$$I_j(x,y) \sim (|x_j|^2 + n^{-1}\rho(\bar{x},\bar{y}))^{-\kappa_j} \sim |x_j|^{-2\kappa_j} \sim (|x_j| + \rho(\bar{x},\bar{y}) + n^{-1})^{-2\kappa_j};$$

while if $|x_j| < 2\rho(\bar{x},\bar{y})$, then

$$\begin{aligned} I_j(x,y) &\leqslant (n^{-1}\rho(\bar{x},\bar{y}) + n^{-2})^{-\kappa_j} \\ &\sim (1 + n\rho(\bar{x},\bar{y}))^{\kappa_j} (\rho(\bar{x},\bar{y}) + |x_j| + n^{-1})^{-2\kappa_j} \end{aligned}$$

This means that for all $1 \leqslant j \leqslant d$,

$$I_j(x,y) \leqslant C(1 + n\rho(\bar{x},\bar{y}))^{\kappa_j} (\rho(\bar{x},\bar{y}) + |x_j| + n^{-1})^{-2\kappa_j}.$$

On the other hand, however, recalling that $|x_1| \geqslant \frac{1}{\sqrt{d}} \geqslant 2\rho(\bar{x},\bar{y})$, we have that $|x_1| \sim |y_1| \sim 1$, and hence

$$I_1(x,y) \sim (|x_1| + \rho(\bar{x},\bar{y}) + n^{-1})^{-2\kappa_1}.$$

Therefore, putting the above together, we conclude that

$$\prod_{j=1}^{d} I_j(x,y) = I_1(x,y)\prod_{j=2}^{d} I_j(x,y)$$
$$\leqslant C(1+n\rho(\bar{x},\bar{y}))^{|\kappa|-\kappa_1}\prod_{j=1}^{d}(|x_j|+\rho(\bar{x},\bar{y})+n^{-1})^{-2\kappa_j},$$

which is as desired. □

Now we are in a position to prove Proposition 3.2.3.

Proof of Proposition 3.2.3. Without loss of generality, we may assume that $\|f\|_{\kappa,1}=1$ and $\alpha>1$. As in the proof of Proposition 3.2.2, we have $\mathbb{S}^{d-1}=\bigcup_{i=1}^{d}\mathbb{S}_i^{d-1}$ with

$$\mathbb{S}_i^{d-1} := \{x\in\mathbb{S}^{d-1}:\ |x_i|\geq \frac{1}{\sqrt{d}}\}.$$

Thus, it is enough to prove that for each $1\leqslant i\leqslant d$,

$$\mathrm{meas}_\kappa(\{x\in\mathbb{S}_i^{d-1}:\quad E_*^\delta(h_\kappa^2;f,x)>\alpha\})\leqslant C\alpha^{-1}, \tag{3.5.17}$$

with $\alpha^{-1}\log\alpha^{-1}$ in place of α^{-1} in the case when $\delta=\sigma_\kappa$ and at least two of the κ_i are zero.

To prove (3.5.17), we consider the following cases:

Case 1. $\kappa_i>\kappa_{\min}$ or $\delta>\sigma_\kappa$

In this case, we shall prove that

$$E_*^\delta(h_\kappa^2;f,x)\leqslant CM_\kappa f(x),\quad \forall x\in\mathbb{S}_i^{d-1}, \tag{3.5.18}$$

from which (3.5.17) will follow by (3.2.3).

By Lemma 3.5.1, if $x \in \mathbb{S}_j^{d-1}$, $y \in \mathbb{S}^{d-1}$ and $\rho(\bar{x}, \bar{y}) \leqslant \frac{1}{2\sqrt{d}}$, then

$$\begin{aligned}|E_n^\delta(h_\kappa^2; x, y)| &\leqslant C n^{d-1}(1 + n\rho(\bar{x}, \bar{y}))^{-\delta - \frac{d}{2}} \prod_{j=1}^d I_j(x, y) \\ &\leqslant C n^{d-1}(1 + n\rho(\bar{x}, \bar{y}))^{-(d-1-\kappa_{\min}+\kappa_i+\delta-\sigma_\kappa)} \prod_{j=1}^d (|x_j| + \rho(\bar{x}, \bar{y}) + n^{-1})^{-2\kappa_j}.\end{aligned}$$

Since $\kappa_i - \kappa_{\min} + \delta - \sigma_\kappa > 0$ in this case, the estimate (3.5.18) then follows by Lemma 3.3.2.

Case 2. $\kappa_i = \kappa_{\min}$ and $\min_{j \neq i} \kappa_j > 0$.

Without loss of generality, we may assume that $i = 1$ in this case. Let $\varepsilon > 0$ be such that $\varepsilon^{d-1+2|\kappa|-2\kappa_1} = c_1^{-1}\alpha^{-1}$, where $c_1 > 0$ is an absolute constant to be specified later. Set

$$F_\varepsilon = \{x \in \mathbb{S}^{d-1} : 1 - \varepsilon^2 \leqslant |x_1| \leqslant 1\}.$$

A straightforward calculation shows that

$$\text{meas}_\kappa(F_\varepsilon) = c_\kappa \int_{1-\varepsilon^2}^1 x_1^{2\kappa_1}(1 - x_1^2)^{\frac{d-3}{2}+|\kappa|-\kappa_1}\, dx_1 \sim \varepsilon^{d-1+2|\kappa|-2\kappa_1} \sim \alpha^{-1}.$$

Next, for $x \in \mathbb{S}_1^{d-1} \setminus F_\varepsilon$, and $y \in \mathbb{S}^{d-1}$, we set

$$\begin{aligned}J &:= J(x, y) = \{j : 2 \leqslant j \leqslant d,\ |x_j| < 2\rho(\bar{x}, \bar{y})\}, \\ J' &:= J'(x, y) = \{2, 3, \cdots, d\} \setminus J.\end{aligned}$$

Recall that $I_j(x,y)$ is defined in (3.5.16). From the proof of Lemma 3.5.1, it is easily seen that if $|x_j| \geqslant 2\rho(\bar{x},\bar{y})$,

$$I_j(x,y) \le C(|x_j| + \rho(\bar{x},\bar{y}) + n^{-1})^{-2\kappa_j} \tag{3.5.19}$$

and that if $|x_j| < 2\rho(\bar{x},\bar{y})$,

$$I_j(x,y) \le C(1 + n\rho(\bar{x},\bar{y}))^{\kappa_j}(\rho(\bar{x},\bar{y}) + |x_j| + n^{-1})^{-2\kappa_j}. \tag{3.5.20}$$

Note also that if $x \in \mathbb{S}_1^{d-1}$ and $\rho(\bar{x},\bar{y}) \leqslant \frac{1}{2\sqrt{d}}$, then $|x_1| \ge \frac{1}{2\sqrt{d}}$ and $|y_1| \ge |x_1| - \rho(\bar{x},\bar{y}) \ge \frac{1}{2\sqrt{d}}$.

Thus, under the condition $x \in \mathbb{S}_1^{d-1}$ and $\rho(\bar{x},\bar{y}) \leqslant \frac{1}{2\sqrt{d}}$,

$$\prod_{j=1}^{d} I_j(x,y) \le C(1 + n\rho(\bar{x},\bar{y}))^{\sum_{j\in J}\kappa_j} \prod_{j=1}^{d}(|x_j| + \rho(\bar{x},\bar{y}) + n^{-1})^{-2\kappa_j},$$

which, in turn, implies that

$$\begin{aligned} &|E_n^{\delta}(h_\kappa^2; x, y)| \\ &\leqslant Cn^{d-1}(1 + n\rho(\bar{x},\bar{y}))^{-\delta - \frac{d}{2} + \sum_{j\in J}\kappa_j} \prod_{j=1}^{d}(|x_j| + \rho(\bar{x},\bar{y}) + n^{-1})^{-2\kappa_j}. \end{aligned} \tag{3.5.21}$$

If $J \subsetneqq \{2, 3, \cdots, d\}$, then $\sum_{j\in J}\kappa_j \leqslant |\kappa| - \kappa_1 - \min_{2\leqslant j\leqslant d}\kappa_j$ and

$$\delta + \frac{d}{2} - \sum_{j\in J}\kappa_j \ge d - 1 + \min_{2\leqslant j\leqslant d}\kappa_j > d - 1.$$

On the other hand, however, if $J=\{2,3,\cdots,d\}$, and $x\in\mathbb{S}_1^{d-1}\setminus F_\varepsilon$, then

$$\delta+\frac{d}{2}-\sum_{j\in J}\kappa_j=\delta+\frac{d}{2}-|\kappa|+\kappa_1\geq d-1,$$

and moreover,

$$\rho(\bar{x},\bar{y})\geq\frac{1}{2}\max_{2\leqslant j\leqslant d}|x_j|\geq\frac{\sqrt{1-x_1^2}}{2\sqrt{d-1}}\geq\frac{\varepsilon}{2\sqrt{d-1}},$$

where the last step uses the fact that $1-|x_1|>\varepsilon^2$ for $x\notin F_\varepsilon$. Thus, using (3.5.21) and recalling that $\varepsilon^{-(d-1+2|\kappa|-2\kappa_1)}=c_1\alpha$, we conclude that if $x\in\mathbb{S}_1^{d-1}\setminus F_\varepsilon$ and $\rho(\bar{x},\bar{y})\leqslant\frac{1}{2\sqrt{d}}$, then

$$\begin{aligned}|E_n^\delta(h_\kappa^2;x,y)|\leqslant& C\frac{n^{d-1}}{(1+n\rho(\bar{x},\bar{y}))^{d-1+\min\limits_{2\leqslant j\leqslant d}\kappa_j}\prod_{j=1}^d(|x_j|+\rho(\bar{x},\bar{y})+n^{-1})^{2\kappa_j}}\\&+Cc_1\alpha.\end{aligned}$$

Since $\|f\|_{\kappa,1}=1$ and $\kappa_{\min}>0$, using Lemma 3.3.2, and choosing $c_1=(2C)^{-1}$, we deduce that for $x\in\mathbb{S}_1^{d-1}\setminus E_\varepsilon$,

$$E_*^\delta(h_\kappa^2;f,x)\leqslant CM_\kappa f(x)+\frac{1}{2}\alpha.$$

It follows that

$$\begin{aligned}
&\operatorname{meas}_\kappa(\{x \in \mathbb{S}_1^{d-1} : E_*^\delta(h_\kappa^2; f, x) > \alpha\}) \\
&\leqslant \operatorname{meas}_\kappa(F_\varepsilon) + \operatorname{meas}_\kappa(\{x \in \mathbb{S}_1^{d-1} \setminus F_\varepsilon : \quad E_*^\delta(h_\kappa^2; f, x) > \alpha\}) \\
&\leq C\frac{1}{\alpha} + \operatorname{meas}_\kappa(\{x \in \mathbb{S}^{d-1} : M_\kappa f(x) \geq \frac{\alpha}{2C}\}) \leq C\frac{1}{\alpha}.
\end{aligned}$$

Case 3. $\kappa_i = 0$, $\min_{j\neq i}\kappa_j = 0$ and $\delta = \sigma_\kappa$.

Since $\kappa \neq 0$, we may assume, without loss of generality, that $i = 2$ and $\kappa_1 > 0$. In this case, using (3.5.19) and (3.5.20), we have that for $x, y \in \mathbb{S}^{d-1}$,

$$\begin{aligned}
&|E_n^\delta(h_\kappa^2; x, y)| \\
\leqslant & Cn^{d-1}\frac{\prod_{j=1}^d (|x_j| + \rho(\bar{x}, \bar{y}) + n^{-1})^{-2\kappa_j}}{(1 + n\rho(\bar{x}, \bar{y}))^{d-1}} \chi_{\{y\in\mathbb{S}^{d-1}: \quad |x_1|\leqslant 2\rho(\bar{x},\bar{y})\}}(y) \\
&+ Cn^{d-1}\frac{\prod_{j=1}^d (|x_j| + \rho(\bar{x}, \bar{y}) + n^{-1})^{-2\kappa_j}}{(1 + n\rho(\bar{x}, \bar{y}))^{d-1+\kappa_1}},
\end{aligned}$$

where χ_F denotes the characteristic function of the set F. Thus, using Lemma 3.3.2, we conclude that

$$E_*^{\sigma_\kappa}(h_\kappa^2; f, x) \leqslant C\Big(\log \frac{1}{|x_1|}\Big) M_\kappa f(x).$$

Therefore, for $\|f\|_{\kappa,1} = 1$ and $\alpha > 0$,

$$\begin{aligned} &\operatorname{meas}_\kappa\{x \in \mathbb{S}^{d-1} : \quad E_*^{\sigma_\kappa}(h_\kappa^2; f, x) > \alpha\} \\ \leqslant &\operatorname{meas}_\kappa\{x \in \mathbb{S}^{d-1} : \ |x_1| \leqslant \alpha^{-1}\} \\ &+ \operatorname{meas}_\kappa\{x \in \mathbb{S}^{d-1} : \ M_\kappa f(x) > \alpha(\log \alpha)^{-1}\} \\ \leqslant &C\alpha^{-1}|\log \alpha|. \end{aligned}$$

3.6 Proof of Theorem 3.1.1: Part (ii)

The proof of Theorem 3.1.1 (ii) follows along the same idea as that of [32], where the Cantor-Lebesgue Theorem is combined with the Uniform Boundedness Principle to deduce a divergence result for the Cesàro means of spherical harmonic expansions. The result of [32] was later extended to the case of h-harmonic expansions in [57]. Our proof below is different from that of [57], and it leads to more information on the counterexample f, from which the corresponding results for weighted orthogonal polynomial expansions on the ball $\mathbb{B}^d$ and on the simplex $\mathbb{T}^d$ can be easily deduced.

The proof of Theorem 3.1.1 (ii) relies on several lemmas. The first lemma is a well known result on Cesàro means of general sequences (see, for instance, [58, Theorem 3.1.22, p. 78] and [58, Theorem 3.1.23, p. 78]).

Lemma 3.6.1. *Let $s_n^\delta := (A_n^\delta)^{-1} \sum_{j=0}^n A_{n-j}^\delta a_j$ denote the Cesàro (C, δ)-means of a* *sequence* $\{a_j\}_{j=0}^{\infty}$ *of real numbers. Then for* $\delta \geq 0$

$$|a_n| \leqslant C_\delta n^\delta \max_{0\leqslant j\leqslant n} |s_j^\delta|, \quad n = 0, 1, \cdots, \tag{3.6.22}$$

and for $0 \leqslant \delta_1 < \delta_2$,

$$|s_n^{\delta_1}| \leqslant C_{\delta_1,\delta_2} n^{\delta_2-\delta_1} \max_{1\leqslant j\leqslant n} |s_j^{\delta_2}|, \quad n = 0, 1, \cdots. \tag{3.6.23}$$

The second lemma was proved in [32, Section 3.3]. It follows from the asymptotics of the Jacobi polynomials and the Riemann-Lebesgue theorem.

Lemma 3.6.2. *Let* α, $\beta \geq -\frac{1}{2}$, *and let* F *be a subset of* $[-1, 1]$ *with positive Lebesgue measure. Then there exists a positive integer* N *depending on the set* F *for which*

$$\sup_{t\in F} |P_n^{(\alpha,\beta)}(t)| \geq C n^{-\frac{1}{2}}, \quad \forall n \geq N,$$

where the constant C *depends on the set* F*, but is independent of* n.

To state our next lemma, recall that the generalized Gegenbauer polynomial $C_n^{(\lambda,\mu)}$ is the weighted orthogonal polynomial of degree n with respect to the weight $|t|^{2\mu}(1 - t^2)^{\lambda-\frac{1}{2}}$ on $[-1, 1]$.

Lemma 3.6.3. *Let* $f \in L(w_\kappa; [0, 1])$ *with* $w_\kappa(t) = |t|^{2\kappa_1}(1 - t^2)^{\lambda_\kappa-\kappa_1-\frac{1}{2}}$. *Let* $\widetilde{f} : \mathbb{S}^{d-1} \to \mathbb{R}$ *be given by* $\widetilde{f}(x) = f(|x_1|)$. *Then* $\widetilde{f} \in L^1(h_\kappa^2; \mathbb{S}^{d-1})$ *and*

$$\mathrm{proj}_{2n}(h_\kappa^2; \widetilde{f}, x) = d_{2n}(f) C_{2n}^{(\lambda_\kappa-\kappa_1,\kappa_1)}(x_1), \quad x \in \mathbb{S}^{d-1}, \tag{3.6.24}$$

where

$$d_{2n}(f) := \frac{1}{\left\|C_{2n}^{(\lambda_\kappa-\kappa_1,\kappa_1)}\right\|^2_{L^2(w_\kappa;[0,1])}} \int_0^1 f(t) C_{2n}^{(\lambda_\kappa-\kappa_1,\kappa_1)}(t) w_\kappa(t)\, dt. \tag{3.6.25}$$

Proof. We need the following formula for the reproducing kernel $P_n(h_\kappa^2; \cdot, e_1)$ of the space $\mathcal{H}_n^d(h_\kappa^2)$ (see [16, proof of Theorem 2.2 (lower bound)]):

$$P_n(h_\kappa^2; x, e_1) = \frac{n+\lambda_\kappa}{\lambda_\kappa} C_n^{(\lambda_\kappa-\kappa_1,\kappa_1)}(x_1), \quad x \in \mathbb{S}^{d-1},\ n = 0, 1, \cdots, \tag{3.6.26}$$

where $e_1 = (1, 0, \cdots, 0) \in \mathbb{S}^{d-1}$.

By (2.4.12), it follows that $\{C_{2n}^{(\lambda_\kappa-\kappa_1,\kappa_1)}\}_{n=0}^\infty$ is an orthogonal polynomial basis with respect to the weight $w_\kappa(t)$ on $[0,1]$. Thus, each function $f \in L(w_\kappa; [0,1])$ has a weighted orthogonal polynomial expansion $\sum_{n=0}^\infty d_{2n}(f) C_{2n}^{(\lambda_\kappa-\kappa_1,\kappa_1)}(t)$ on $[0,1]$, which particularly implies that for each polynomial g of degree at most $2n$ on $[-1,1]$,

$$\int_{-1}^1 f(|t|) g(t) w_\kappa(t)\, dt = \sum_{j=0}^n d_{2j}(f) \int_{-1}^1 C_{2j}^{(\lambda_\kappa-\kappa_1,\kappa_1)}(t) g(t) w_\kappa(t)\, dt. \tag{3.6.27}$$

Next, we note that (3.6.26) implies that the term on the right hand side of (3.6.24) is an h-harmonic in $\mathcal{H}_{2n}^d(h_\kappa^2)$. Thus, for the proof of (3.6.24), it is sufficient to verify that for each $P \in \mathcal{H}_{2n}^d(h_\kappa^2)$,

$$\begin{aligned} &\int_{\mathbb{S}^{d-1}} \widetilde{f}(x) P(x) h_\kappa^2(x)\, d\sigma(x) \\ =& d_{2n}(f) \int_{\mathbb{S}^{d-1}} C_{2n}^{(\lambda_\kappa-\kappa_1,\kappa_1)}(x_1) P(x) h_\kappa^2(x)\, d\sigma(x). \end{aligned} \tag{3.6.28}$$

Indeed, for $P \in \mathcal{H}_{2n}^d(h_\kappa^2)$,

$$\begin{aligned}
&\int_{\mathbb{S}^{d-1}} \widetilde{f}(x) P(x) h_\kappa^2(x)\, d\sigma(x) \\
&= \int_{-1}^1 f(|x_1|) w_\kappa(x_1) \Big[\int_{\mathbb{S}^{d-2}} P(x_1, \sqrt{1-x_1^2} y) h_{\widetilde{k}}^2(y)\, d\sigma(y) \Big]\, dx_1,
\end{aligned}$$

where $h_{\widetilde{k}}(y) = \prod_{j=1}^{d-1} |y_j|^{\kappa_{j+1}}$ for $y = (y_1, \cdots, y_{d-1}) \in \mathbb{R}^{d-1}$. Since the weight $h_{\widetilde{k}}^2(y)$ is even in each y_j, it is easily seen that the integral over $\mathbb{S}^{d-2}$ of the last equation is an algebraic polynomial in x_1 of degree at most $2n$. Thus, it follows by (3.6.27) that

$$\begin{aligned}
&\int_{\mathbb{S}^{d-1}} \widetilde{f}(x) P(x) h_\kappa^2(x)\, d\sigma(x) \\
&= \sum_{j=0}^n d_{2j}(f) \int_{-1}^1 C_{2j}^{(\lambda_\kappa - \kappa_1, \kappa_1)}(x_1) w_\kappa(x_1) \Big[\int_{\mathbb{S}^{d-2}} P(x_1, \sqrt{1-x_1^2} y) h_{\widetilde{k}}^2(y)\, d\sigma(y) \Big]\, dx_1 \\
&= \sum_{j=0}^n d_{2j}(f) \int_{\mathbb{S}^{d-1}} C_{2j}^{(\lambda_\kappa - \kappa_1, \kappa_1)}(x_1) P(x) h_\kappa^2(x)\, d\sigma(x).
\end{aligned}$$

Since, by (3.6.26), $C_j^{(\lambda_\kappa - \kappa_1, \kappa_1)}(x_1) \in \mathcal{H}_j^d(h_\kappa^2)$, the desired equation (3.6.28) follows by the orthogonality of the spherical h-harmonics. $\square$

Now we are in a position to prove Theorem 3.1.1(ii).

Proof of Theorem 3.1.1(ii). Without loss of generality, we may assume that $\kappa_1 = \kappa_{\min}$. Assume that the stated conclusion were not true. This would mean that $S_*^\delta(h_\kappa^2; \widetilde{f}, x)$ is finite on a set $E_f \subset \mathbb{S}^{d-1}$ of positive measure for all $f \in L^1(w_\kappa; [0,1])$ and some $\delta < \sigma_\kappa$, where $\widetilde{f}(x) = f(|x_1|)$ for $x \in \mathbb{S}^{d-1}$, and $w_\kappa(t) = |t|^{2\kappa_1}(1-t^2)^{\sigma_\kappa - \frac{1}{2}}$. By

Lemma 3.6.1, this implies that

$$\sup_{n\in\mathbb{N}} n^{-\delta}|\operatorname{proj}_{2n}(h_\kappa^2;\widetilde{f},x)| < \infty, \quad \forall x\in E_f, \forall f\in L(w_\kappa;[0,1]). \tag{3.6.29}$$

We will show that (3.6.29) is impossible unless $\delta \geq \sigma_\kappa$.

In fact, by(3.6.29),

$$E_f = \bigcup_{N=1}^{\infty}\Big\{x\in E_f: \ \sup_{n\in\mathbb{N}} n^{-\delta}|\operatorname{proj}_{2n}(h_\kappa^2;\widetilde{f},x)| \leqslant N\Big\},$$

hence, there must exist a subset E_f' of E_f with positive Lebesgue measure such that

$$\sup_{x\in E_f'}\sup_{n\in\mathbb{N}} n^{-\delta}|\operatorname{proj}_{2n}(h_\kappa^2;\widetilde{f},x)| \leqslant N_f < \infty.$$

By Lemma 3.6.3, this in turn implies that

$$\sup_{x\in E_f'}\sup_{n\in\mathbb{N}} n^{-\delta}|d_{2n}(f)||C_{2n}^{(\sigma_\kappa,\kappa_1)}(x_1)| \leqslant N_f, \tag{3.6.30}$$

where $d_{2n}(f)$ is defined in (3.6.25). Note that by (2.4.12),

$$C_{2n}^{(\lambda_\kappa-\kappa_1,\kappa_1)}(x_1) = \frac{\Gamma(\lambda_\kappa+n)\Gamma(\kappa_1+\frac{1}{2})}{\Gamma(\lambda_\kappa)\Gamma(\kappa_1+\frac{1}{2}+n)} P_n^{(\sigma_\kappa-\frac{1}{2},\kappa_1-\frac{1}{2})}(2x_1^2-1).$$

Hence, using [44, (4.3.3)], we can rewrite (3.6.30) as

$$\sup_{n\in\mathbb{N}} n^{1-\delta}|\ell_n(f)|\sup_{t\in I_f}|P_n^{(\sigma_\kappa-\frac{1}{2},\kappa_1-\frac{1}{2})}(t)| \leqslant N_f, \tag{3.6.31}$$

where $I_f := \{2x_1^2 - 1 : \quad x \in E'_f\}$, and

$$\ell_n(f) := \int_0^1 f(t) P_n^{(\sigma_\kappa - \frac{1}{2}, \kappa_1 - \frac{1}{2})}(2t^2 - 1) w_\kappa(t)\, dt. \tag{3.6.32}$$

Since $E'_f \subset \mathbb{S}^{d-1}$ has a positive Lebesgue measure, it is easily seen that $I_f \subset [-1, 1]$ has a positive Lebesgue measure as well. Thus, (3.6.31) together with Lemma 3.6.3 implies that

$$\sup_{n \in \mathbb{N}} n^{\frac{1}{2} - \delta} |\ell_n(f)| < \infty, \quad \forall f \in L(w_\kappa; [0, 1]). \tag{3.6.33}$$

Since $\{n^{\frac{1}{2} - \delta} \ell_n(f)\}_{n=0}^\infty$ is a sequence of bounded linear functionals on the Banach space $L(w_\kappa; [0, 1])$, it follows by (3.6.33) and the uniform boundedness theorem that

$$\sup_n n^{\frac{1}{2} - \delta} \sup_{\|f\|_{L(w_\kappa; [0,1])} \leqslant 1} |\ell_n(f)| < \infty. \tag{3.6.34}$$

On the other hand, however, using (3.6.32) and [44, (7.32.2), p. 168], we have

$$\sup_{\|f\|_{L(w_\kappa; [0,1])} \leqslant 1} |\ell_n(f)| = \max_{t \in [0,1]} |P_{2n}^{(\sigma_\kappa - \frac{1}{2}, \kappa_1 - \frac{1}{2})}(2t^2 - 1)| = P_{2n}^{(\sigma_\kappa - \frac{1}{2}, \kappa_1 - \frac{1}{2})}(1) \sim n^{\sigma_\kappa - \frac{1}{2}}.$$

Thus, (3.6.34) implies that

$$\sup_{n \in \mathbb{N}} n^{\frac{1}{2} - \delta} n^{\sigma_\kappa - \frac{1}{2}} = \sup_{n \in \mathbb{N}} n^{\sigma_\kappa - \delta} < \infty,$$

which can not be true unless $\delta \geq \sigma_\kappa$. This completes the proof. □

3.7 Corollaries

3.7.1 The pointwise convergence

In this subsection, we devote to the investigation of almost everywhere convergence of *Cesàro* (C,δ)*-mean* S_n^δ of weighted orthogonal expansions on the unit sphere $\mathbb{S}^{d-1}$ by our weak-type estimation. What we have already known is for $\delta > \frac{d-2}{2} + |\kappa|$,

$$\lim_{n\to\infty} S_n^\delta(h_\kappa^2; f, x) = f(x), \quad a.e. x \in \mathbb{S}^{d-1},$$

And for $\delta < \frac{d-2}{2} + |\kappa| - \min\limits_{1\leqslant i\leqslant d} \kappa_i$, there exists a function $f \in L^1(h_\kappa^2; \mathbb{S}^{d-1})$ such that

$$\limsup_{n\to\infty} |S_n^\delta(h_\kappa^2; f, x)| = \infty, \quad a.e. x \in \mathbb{S}^d.$$

At here, we proved the critical index for the a.e. convergence of *Cesàro* (C,δ)*-mean* means, that is, for $f \in L^1(h_\kappa^2; \mathbb{S}^d)$, if $\delta \geqslant \frac{d-2}{2} + |\kappa| - \kappa_{\min}$, and $\kappa_{\min} > 0$, then

$$S_n^\delta(h_\kappa^2; f, x) = f(x), \quad a.e. x \in \mathbb{S}^{d-1},$$

Corollary 3.7.1. *In order that*

$$\lim_{n\to\infty} S_n^\delta(h_\kappa^2; f)(x) = f(x)$$

holds almost everywhere on $\mathbb{S}^{d-1}$ *for all* $f \in L^1(h_\kappa^2; \mathbb{S}^{d-1})$*, it is sufficient and necessary that* $\delta \geq \sigma_\kappa$*.*

Proof. For all $f \in L^1(h_\kappa^2; \mathbb{S}^{d-1})$ we can write

$$f(x) = g_m(x) + b_m(x),$$

where $g_m(x) \in \mathcal{H}_n^d$, and $\lim_{m\to\infty} \|b_m\|_{1,\kappa} = 0$. Set

$$\Lambda^\delta(f)(x) := \limsup_{n\to\infty} S_n^\delta(h_\kappa^2; f, x) - \liminf_{n\to\infty} S_n^\delta(h_\kappa^2; f, x),$$

Then by Theorem 3.1.1 (i), $\forall \varepsilon > 0$

$$\begin{aligned} \text{meas}_\kappa(\{x \in \mathbb{S}^{d-1} : \Lambda^\delta(f)(x) > \varepsilon\}) =& \text{meas}_\kappa(\{x \in \mathbb{S}^{d-1} : \Lambda^\delta(b_m)(x) > \varepsilon\}) \\ &\leqslant \text{meas}_\kappa(\{x \in \mathbb{S}^{d-1} : S_*^\delta(h_\kappa^2; |b_m|, x) \gtrsim \varepsilon\}) \\ &\lesssim \frac{\|b_m\|_{1,\kappa}}{\varepsilon} \to 0, \quad \text{as } m \to \infty. \end{aligned}$$

This implies that $\lim_{n\to\infty} S_n^\delta(h_\kappa^2; f, x)$ exists.

Then since $g_m(x) \in \mathcal{H}_n^d$,

$$\begin{aligned} &\text{meas}_\kappa(\{x \in \mathbb{S}^{d-1} : |\lim_{n\to\infty} S_n^\delta(h_\kappa^2; f, x) - f(x)| > \varepsilon\}) \\ \leqslant& \text{meas}_\kappa(\{x \in \mathbb{S}^{d-1} : S_*^\delta(h_\kappa^2; |b_m|, x) > \frac{\varepsilon}{2}\}) \\ &+ \text{meas}_\kappa(\{x \in \mathbb{S}^{d-1} : |b_m(x)| > \frac{\varepsilon}{2}\}) \\ \lesssim& \frac{\|b_m\|_{1,\kappa}}{\varepsilon} \end{aligned}$$

Let $m \to \infty$, we get

$$\text{meas}_\kappa(\{x \in \mathbb{S}^{d-1} : |\lim_{n\to\infty} S_n^\delta(h_\kappa^2; f, x) - f(x)| > \varepsilon\}) = 0$$

$$\lim_{n\to\infty} S_n^\delta(h_\kappa^2; f, x) = f(x), \quad a.e.\ x \in \mathbb{S}^{d-1}.$$

Then we finish the proof of sufficiency, whereas the necessity follows directly from Theorem 3.1.1 (ii). □

3.7.2 Strong estimates on L^p

Using Stein's interpolation theorem for analytic families of operators ([39]), we can deduce the following strong estimates for the maximal Cesàro operators:

Corollary 3.7.2. *If* $1 < p < \infty$ *and* $\delta > 2\sigma_\kappa|\frac{1}{2} - \frac{1}{p}|$*, then*

$$\|S_*^\delta(h_\kappa^2; f)\|_{\kappa,p} \leqslant C_p\|f\|_{\kappa,p}. \tag{3.7.35}$$

In particular,

$$\|S_*^{\sigma_\kappa}(h_\kappa^2; f)\|_{\kappa,p} \leqslant C_p\|f\|_{\kappa,p}, \quad 1 < p < \infty.$$

We first show S_*^δ is strong-type $(2, 2)$ for $\delta > 0$. It is sufficient to show the following lemmas. The idea of the proof is directly from the proof of Lemma 3.5 of [4].

Lemma 3.7.3. *If there exists a $\delta_0 > 0$ such that for all $f \in L^2(h_\kappa^2; \mathbb{S}^{d-1})$,*

$$\|S_*^{\delta_0}(h_\kappa^2; f, x)\|_{\kappa,2} \lesssim_p \|f\|_{\kappa,2}.$$

Then for all $\delta > 0$ and for all $f \in L^2(h_\kappa^2; \mathbb{S}^{d-1})$, we have

$$\|S_*^{\delta}(h_\kappa^2; f, x)\|_{\kappa,2} \lesssim_p \|f\|_{\kappa,2}.$$

Proof. Firstly, since for any $\alpha > 0$, and $\beta > \frac{1}{2}$,

$$\sum_{k=0}^{n} \left(\frac{A_k^\delta A_{n-k}^{\beta-1}}{A_n^{\delta+\beta}} \right)^2 \sim \sum_{k=0}^{n} \left(\frac{k^\delta (n-k)^{\beta-1}}{n^{\delta+\beta}} \right)^2 \sim n^{-1}$$

Then

$$\begin{aligned}
|S_n^{\delta+\beta}(h_\kappa^2; f, x)| &= \left| \sum_{k=0}^{n} \frac{A_k^\delta A_{n-k}^{\beta-1}}{A_n^{\delta+\beta}} S_k^\delta(h_\kappa^2; f, x) \right| \\
&\leqslant \sum_{k=0}^{n} \left| \frac{A_k^\delta A_{n-k}^{\beta-1}}{A_n^{\delta+\beta}} \right| \cdot |S_k^\delta(h_\kappa^2; f, x)| \\
&\leqslant \left(\sum_{k=0}^{n} \left| \frac{A_k^\delta A_{n-k}^{\beta-1}}{A_n^{\delta+\beta}} \right|^2 \right)^{\frac{1}{2}} \cdot \left(\sum_{k=0}^{n} |S_k^\delta(h_\kappa^2; f, x)|^2 \right)^{\frac{1}{2}} \\
&\lesssim \left(\sum_{k=0}^{n} |S_k^\delta(h_\kappa^2; f, x)|^2 \cdot n^{-1} \right)^{\frac{1}{2}}
\end{aligned}$$

Hence

$$S_*^{\delta+\beta}(h_\kappa^2; f, x) \leqslant \sup_n \left(\sum_{k=0}^{n} |S_k^\delta(h_\kappa^2; f, x)|^2 \cdot n^{-1} \right)^{\frac{1}{2}}$$

Therefore, we just need to show that for all $\delta > -\frac{1}{2}$, and for all $f \in L^2(h_\kappa^2; \mathbb{S}^{d-1})$,

$$\| \sup_n (\sum_{k=0}^{n} |S_k^\delta(h_\kappa^2; f, x)|^2 \cdot n^{-1})^{\frac{1}{2}} \|_{\kappa,2} \lesssim \|f\|_{\kappa,2}$$

In fact, on one side, we know that for all $f \in L^2(h_\kappa^2; \mathbb{S}^{d-1})$,

$$\| \sup_n (\sum_{k=0}^{n} |S_k^{\delta_0}(h_\kappa^2; f, x)|^2 \cdot n^{-1})^{\frac{1}{2}} \|_{\kappa,2} \leqslant \|S_*^{\delta_0}(h_\kappa^2; f, x)\|_{\kappa,2} \lesssim \|f\|_{\kappa,2}$$

On the other side, since $(A_{n-k}^\delta)(A_n^\delta)^{-1} = \prod_{j=0}^{k} (n-j+\delta)^{-1}$ is a decreasing function of δ,

$$\begin{aligned}
\sum_{k=0}^{n} |S_k^\delta(h_\kappa^2; f, x) - S_k^{\delta_0}(h_\kappa^2; f, x)|^2 k^{-1} &\leqslant \sum_{k=0}^{n} \frac{1}{n} |S_k^\delta(h_\kappa^2; f, x) - S_k^{\delta_0}(h_\kappa^2; f, x)|^2 \\
&\leqslant \sum_{n=0}^{\infty} \frac{1}{n} |S_n^\delta(h_\kappa^2; f, x) - S_n^{\delta_0}(h_\kappa^2; f, x)|^2 \\
&\sim \sum_{n=0}^{\infty} \frac{1}{n} \left| \sum_{k=0}^{n} \left(\frac{A_{n-k}^{\delta+1}}{A_n^{\delta+1}} - \frac{A_{n-k}^{\delta}}{A_n^{\delta}} \right) \mathrm{proj}_k(h_\kappa^2; f, x) \right|^2 \\
&= \sum_{n=0}^{\infty} \frac{(A_n^{\delta+1})^{-2}}{n(\delta+1)^2} | \sum_{k=0}^{n} k A_{n-k}^\delta \, \mathrm{proj}_k(h_\kappa^2; f, x)|^2
\end{aligned}$$

we can get

$$
\begin{aligned}
&\|\sup_n(\sum_{k=0}^{n}|S_k^{\delta}(h_\kappa^2;f,x)-S_k^{\delta_0}(h_\kappa^2;f,x)|^2k^{-1})^{\frac{1}{2}}\|_{\kappa,2}^2\\
\lesssim&\|(\delta+1)^{-1}(\sum_{n=0}^{\infty}n^{-1}(A_n^{\delta+1})^{-2}|\sum_{k=0}^{n}kA_{n-k}^{\delta}\operatorname{proj}_k(h_\kappa^2;f,x)|^2)^{\frac{1}{2}}\|_{\kappa,2}^2\\
=&(\delta+1)^{-2}\sum_{n=0}^{\infty}n^{-1}(A_n^{\delta+1})^{-2}\sum_{k=0}^{n}k^2(A_{n-k}^{\delta})^2\|\operatorname{proj}_k(h_\kappa^2;f,x)\|_{\kappa,2}^2\\
=&(\delta+1)^{-2}\sum_{k=0}^{\infty}\|\operatorname{proj}_k(h_\kappa^2;f,x)\|_{\kappa,2}^2\cdot k^2\sum_{n=k}^{\infty}n^{-1}(A_{n-k}^{\delta})^2(A_n^{\delta+1})^{-2}
\end{aligned}
$$

Since

$$
k^2\sum_{n=k}^{\infty}n^{-1}(A_{n-k}^{\delta})^2(A_n^{\delta+1})^{-2}\sim k^2\sum_{n=k}^{\infty}n^{-1}n^{-2(\delta+1)}(n-k)^{2\delta}\sim 1,
$$

we have

$$
\|\sup_n(\sum_{k=0}^{n}|S_k^{\delta}(h_\kappa^2;f,x)-S_k^{\delta_0}(h_\kappa^2;f,x)|^2k^{-1})^{\frac{1}{2}}\|_{\kappa,2}^2\lesssim\|f\|_{\kappa,2}
$$

Then by using triangle inequality,

$$
\begin{aligned}
&\|\sup_n(\sum_{k=0}^{n}|S_k^{\delta}(h_\kappa^2;f,x)|^2\cdot n^{-1})^{\frac{1}{2}}\|_{\kappa,2}\\
\leqslant&\|\sup_n(\sum_{k=0}^{n}|S_k^{\delta_0}(h_\kappa^2;f,x)|^2\cdot n^{-1})^{\frac{1}{2}}\|_{\kappa,2}\\
&+\|\sup_n(\sum_{k=0}^{n}|S_k^{\delta}(h_\kappa^2;f,x)-S_k^{\delta_0}(h_\kappa^2;f,x)|^2k^{-1})^{\frac{1}{2}}\|_{\kappa,2}^2\\
\lesssim&\|f\|_{\kappa,2}
\end{aligned}
$$

□

By this lemma, we can get the following Lemma.

Lemma 3.7.4. *For $\delta > 0$ and $f(x) \in L^2(h_\kappa^2; \mathbb{S}^{d-1})$, $\|S_*^\delta(h_\kappa^2; f, x)\|_{\kappa,2} \lesssim \|f\|_{\kappa,2}$.*

Proof of Theorem 3.7.2. Firstly, recalling that ([30])

$$\|S_*^\delta(h_\kappa^2; f)\|_\infty \leqslant C\|f\|_\infty, \quad \delta > \sigma_\kappa,$$

we deduce from Theorem 3.1.1 and the Marcinkiewitcz interpolation theorem that

$$\|S_*^\delta(h_\kappa^2; f)\|_{\kappa,p} \leqslant C_p\|f\|_{\kappa,p}, \quad 1 < p < \infty, \quad \delta > \sigma_\kappa. \tag{3.7.36}$$

Secondly, in Lemma 3.7.4, we have already get

$$\|S_*^\delta(h_\kappa^2; f)\|_{\kappa,2} \leqslant C\|f\|_{\kappa,2}, \ \delta > 0. \tag{3.7.37}$$

Thirdly, the index δ of the Cesàro (C, δ)-means can be extended analytically to $\delta \in \mathbb{C}$ with $\operatorname{Re}\delta > -1$, as can be easily seen from the definition. Furthermore, it is well known (see [4]) that for $\delta > 0$, $\varepsilon > 0$ and $y \in \mathbb{R}$,

$$S_n^{\delta+\varepsilon+iy}(h_\kappa^2; f) = (A_n^{\delta+\varepsilon+iy})^{-1} \sum_{j=0}^{n} A_{n-j}^{\varepsilon-1+iy} A_j^\delta S_j^\delta(h_\kappa^2; f), \tag{3.7.38}$$

and

$$|A_n^{\delta+\varepsilon+iy}|^{-1} \sum_{j=0}^{n} |A_{n-j}^{\varepsilon-1+iy}| A_j^\delta \leqslant C(\varepsilon) e^{cy^2}. \tag{3.7.39}$$

It follows that for $\delta > 0$, $\varepsilon > 0$ and $y \in \mathbb{R}$,

$$S_*^{\delta+\varepsilon+iy}(h_\kappa^2; f, x) \leqslant C(\varepsilon) e^{c(\varepsilon)y^2} S_*^\delta(h_\kappa^2; f, x). \tag{3.7.40}$$

Finally, for each measurable function $N: \ \mathbb{S}^{d-1} \to \{0, 1, \cdots\}$, define $Q_N^\alpha f(x) := S_{N(x)}^\alpha(h_\kappa^2; f, x)$ for $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha > 0$. It can be easily verified that $\{Q_N^\alpha : \quad \alpha \in \mathbb{C}, \quad \operatorname{Re}\alpha > 0\}$ is a sequence of analytic operators in the sense of [39]. On one hand, since $2|\frac{1}{p} - \frac{1}{2}| \in (0, 1)$ for $p \neq 2$, it follows that for any $\delta > 2\sigma_\kappa|\frac{1}{p} - \frac{1}{2}|$, we can always find $\theta \in [0, 1]$ such that $2|\frac{1}{p} - \frac{1}{2}| < 1 - \theta < \frac{\delta}{\sigma_\kappa}$, and two numbers $\varepsilon, \varepsilon' > 0$ satisfying $\delta = \theta\varepsilon + (1 - \theta)(\sigma_\kappa + \varepsilon)$, and $\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{p_{\varepsilon'}}$, where $p_{\varepsilon'} = 1 + \varepsilon'$ if $p < 2$, and $p_{\varepsilon'} = 2 + (\varepsilon')^{-1}$ if $p > 2$. On the other hand, however, using (3.7.36),(3.7.37), (3.7.40), we have that for any $y \in \mathbb{R}$,

$$\|Q_N^{\varepsilon+iy} f\|_{\kappa,2} \leqslant C(\varepsilon) e^{cy^2} \|f\|_{\kappa,2},$$
$$\|Q_N^{\sigma_\kappa+\varepsilon+iy} f\|_{\kappa,p_{\varepsilon'}} \leqslant C(\varepsilon) e^{cy^2} \|f\|_{\kappa,p_{\varepsilon'}}.$$

Thus, applying Stein's interpolation theorem [39], we conclude that

$$\|Q_N^\delta f\|_{\kappa,p} \leqslant C\|f\|_{\kappa,p}, \quad \delta > 2\sigma_\kappa|\frac{1}{p} - \frac{1}{2}|.$$

Since the constant C in this last equation is independent of the function N, the stated estimate (3.7.35) follows. □

3.7.3 Marcinkiewitcz multiplier theorem

We can also deduce the following vector-valued inequalities for the Cesàro operators.

Corollary 3.7.5. *For $1 < p < \infty$, $\delta > 2\sigma_\kappa|\frac{1}{p} - \frac{1}{2}|$ and any sequence $\{n_j\}$ of positive integers,*

$$\left\| \left(\sum_{j=0}^{\infty} \left| S_{n_j}^{\delta}(h_\kappa^2; f_j) \right|^2 \right)^{1/2} \right\|_{\kappa,p} \le c \left\| \left(\sum_{j=0}^{\infty} \left| f_j \right|^2 \right)^{1/2} \right\|_{\kappa,p}. \tag{3.7.41}$$

Proof. Note first that (3.7.41) for $\delta > 0$ and $p = 2$ is a direct consequence of Corollary 3.7.2. Next, we prove (3.7.41) for $\delta > \sigma_\kappa$ and $1 < p < \infty$. Define the following positive operators:

$$\widetilde{S}_n^{\delta}(h_\kappa^2; f, x) := \int_{\mathbb{S}^{d-1}} f(y) \left| K_n^{\delta}(h_\kappa^2; x, y) \right| h_\kappa^2(y)\, d\sigma(y), \quad x \in \mathbb{S}^{d-1}, \quad n = 0, 1, \cdots .$$

It is easily seen from the proofs of Theorem 3.1.1 and Corollary 3.7.2 that

$$\|\widetilde{S}_*^{\delta}(h_\kappa^2; f)\|_{\kappa,p} \leqslant C\|f\|_{\kappa,p}, \quad 1 < p \leqslant \infty, \quad \delta > \sigma_\kappa. \tag{3.7.42}$$

We shall follow the approach of [41, p.104-105] that uses a generalization of the Riesz convexity theorem for sequences of functions. Let $L^p(\ell^q)$ denote the space of all sequences $\{f_k\}$ of functions for which

$$\|(f_k)\|_{L^p(\ell^q)} := \left(\int_{\mathbb{S}^{d-1}} \left(\sum_{j=0}^{\infty} |f_j(x)|^q \right)^{p/q} h_\kappa^2(x) d\sigma(x) \right)^{1/p} < \infty.$$

If T is a bounded operator on both $L^{p_0}(\ell^{q_0})$ and $L^{p_1}(\ell^{q_1})$ for some

$1 \leqslant p_0, q_0, p_1, q_1 \leqslant \infty$, then the generalized Riesz convexity theorem (see [3]) states that T is also bounded on $L^{p_t}(\ell^{q_t})$, where

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1}, \quad \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}, \quad 0 \le t \le 1.$$

We apply this theorem to the operator T that maps the sequence $\{f_j\}$ to the sequence $\{S_{n_j}^\delta(h_\kappa^2; f_j)\}$. By Corollary 3.7.2, T is bounded on $L^p(\ell^p)$. By (3.7.42), it is also bounded on $L^p(\ell^\infty)$ as

$$\left\| \sup_{j\geq 0} \left| S_{n_j}^\delta(h_\kappa^2; f_j) \right| \right\|_{\kappa,p} \leq \left\| \widetilde{S}_*^\delta \Big(h_\kappa^2; \sup_{j\geq 0} |f_j| \Big) \right\|_{\kappa,p} \leq c \left\| \sup_{j\geq 0} |f_j| \right\|_{\kappa,p}.$$

Thus, the Riesz convexity theorem shows that T is bounded on $L^p(\ell^q)$ if $1 < p \le q \le \infty$. In particular, T is bounded on $L^p(\ell^2)$ if $1 < p \leqslant 2$. The case $2 < p < \infty$ follows by the standard duality argument, since the dual space of $L^p(\ell^2)$ is $L^{p'}(\ell^2)$, where $1/p + 1/p' = 1$, under the paring

$$\langle (f_j), (g_j) \rangle := \int_{\mathbb{S}^{d-1}} \sum_j f_j(x) g_j(x) h_\kappa^2(x) d\sigma(x)$$

and T is self-adjoint under this paring.

Finally, we prove that (3.7.41) for the general case follows by the Stein interpolation theorem ([39]). Without loss of generality, we may assume that there are only finitely many nonzero functions f_j in (3.7.41). Using (3.7.38), (3.7.39), the Cauchy-Schwartz inequality, and applying the above already proven case of (3.7.41), we obtain that for $\delta > 0$ and $p = 2$ or $\delta > \sigma_\kappa$ and $1 < p < \infty$,

$$\left\| \left(\sum_{j=0}^{\infty} \left| S_{n_j}^{\delta+\varepsilon+iy}(h_\kappa^2; f_j) \right|^2 \right)^{1/2} \right\|_{\kappa,p} \le C(\varepsilon) e^{cy^2} \left\| \left(\sum_{j=0}^{\infty} |f_j|^2 \right)^{1/2} \right\|_{\kappa,p}, \tag{3.7.43}$$

where $y \in \mathbb{R}$ and $\varepsilon > 0$. (3.7.41) then follows from (3.7.43) via applying Stein's interpolation theorem to the family of analytic operators,

$$T^\alpha f := \sum_{j=0}^{\infty} S_{n_j}^{\alpha}(h_\kappa^2; f) g_j, \quad \operatorname{Re} \alpha > 0,$$

where (g_j) is a sequence of functions on $\mathbb{S}^{d-1}$ with $\sum_j |g_j(x)|^2 = 1$ for $x \in \mathbb{S}^{d-1}$. □

Corollary 3.7.5 allows us to weaken the condition of the Marcinkiewitcz multiplier theorem established in [15].

Corollary 3.7.6. *Let $\{\mu_j\}_{j=0}^{\infty}$ be a sequence of complex numbers that satisfies*

(i) $\sup_j |\mu_j| \le c < \infty$,

(ii) $\sup_j 2^{j(n_0-1)} \sum_{l=2^j}^{2^{j+1}} |\Delta^{n_0} u_l| \le c < \infty$,

where n_0 is the smallest integer $\ge \sigma_\kappa + 1$, $\Delta\mu_j = \mu_j - \mu_{j+1}$, and $\Delta^{\ell+1} = \Delta^\ell \Delta$.

Then $\{\mu_j\}$ defines an $L^p(h_\kappa^2; \mathbb{S}^{d-1})$, $1 < p < \infty$, multiplier; that is,

$$\left\| \sum_{j=0}^{\infty} \mu_j \operatorname{proj}_j(h_\kappa^2; f) \right\|_{\kappa,p} \le c \|f\|_{\kappa,p}, \qquad 1 < p < \infty,$$

where c is independent of μ_j.

In the case when the weights are invariant under a general reflection group, Corollary 3.7.6 was proved in [15] under a stronger assumption that n_0 is the smallest integer $\geq \sigma_\kappa + 2 + \kappa_{\min}$. The proof of Corollary 3.7.6 is based on Corollary 3.7.5 and runs along the same line as that of [4].

Chapter 4

Maximal Cesàro estimates for weighted orthogonal polynomial expansions on the unit ball and simplex

4.1 Maximal estimates on the unit ball

Analysis in weighted spaces on the unit ball $\mathbb{B}^d = \{x \in \mathbb{R}^d : \ \|x\| \leqslant 1\}$ can often be deduced from the corresponding results on the unit sphere $\mathbb{S}^d$, due to the close connection between the weighted orthogonal polynomial expansions on $\mathbb{B}^d$ and $\mathbb{S}^d$, as described in Section 2.3, see [25, 51, 52, 54] and the reference therein. In this section, we shall develop results on $\mathbb{B}^d$ that are analogous to those on $\mathbb{S}^d$.

Throughout this section, we will use a slight abuse of notations. The letter κ denotes a fixed, nonzero vector $\kappa := (\kappa_1, \cdots, \kappa_{d+1})$ in $\mathbb{R}_+^{d+1}$ rather than in $\mathbb{R}_+^d$, and h_κ denotes the weight function $h_\kappa(x) := \prod_{j=1}^{d+1} |x_j|^{\kappa_j}$ on $\mathbb{S}^d$ rather than the weight on $\mathbb{S}^{d-1}$. Accordingly, we write

$$\kappa_{\min} := \min_{1\leqslant j\leqslant d+1} \kappa_j, \quad |\kappa| = \sum_{j=1}^{d+1} \kappa_j, \quad \sigma_\kappa := \frac{d-1}{2} + |\kappa| - \kappa_{\min}. \tag{4.1.1}$$

For a set $E \subset \mathbb{B}^d$, we write $\text{meas}_\kappa^B(E) := \int_E W_\kappa^B(x)\, dx$. Finally, recall that $S_n^\delta(W_\kappa^B; f)$ denotes the (C, δ)-means for the orthogonal polynomial expansions with respect to the weight function W_κ^B on $\mathbb{B}^d$ that is given in (1.1.6).

Theorem 4.1.1. (i) *If $\delta \geq \sigma_\kappa := \frac{d-1}{2} + |\kappa| - \kappa_{\min}$, then for $f \in L(W_\kappa^B; \mathbb{B}^d)$ with $\|f\|_{L(W_\kappa^B;\mathbb{B}^d)} = 1$,*

$$\text{meas}_\kappa^B\Big\{x \in \mathbb{B}^d : \quad S_*^\delta(W_\kappa^B; f)(x) > \alpha\Big\} \leqslant C\frac{1}{\alpha}, \quad \forall \alpha > 0,$$

with $\alpha^{-1}|\log \alpha|$ in place of α^{-1} in the case when $\delta = \sigma_\kappa$ and at least two of the κ_i are zero.

(ii) *If $\delta < \sigma_\kappa$, then there exists a function $f \in L(W_\kappa^B; \mathbb{B}^d)$ of the form $f(x) = f_0(|x_{j_0}|)$ such that $S_*^\delta(W_\kappa^B; f)(x) = \infty$ for a.e. $x \in \mathbb{B}^d$, where $1 \leqslant j_0 \leqslant d+1$ is the integer such that $\kappa_{j_0} = \kappa_{\min}$, and $x_{d+1} = \sqrt{1 - \|x\|^2}$.*

Proof. Given $f \in L^p(W_\kappa^B; \mathbb{B}^d)$, define $\widetilde{f} : \mathbb{S}^d \to \mathbb{R}$ by $\widetilde{f}(X) = f(x)$ for $X = (x, x_{d+1}) \in \mathbb{S}^d$. Clearly, $\widetilde{f} \circ \phi = f$, where $\phi : \ \mathbb{B}^d \to \mathbb{S}_+^d$ is defined in (2.4.7), which, using (2.4.8), is measure-preserving in the sense that for each

$\mathrm{meas}_\kappa(E) = c_\kappa \,\mathrm{meas}_\kappa^B(\phi^{-1}(E))$ for each $E \subset \mathbb{S}^d_+$. Using (2.4.8), we also have that $\widetilde{f} \in L^p(h_\kappa^2; \mathbb{S}^d)$ and $\|\widetilde{f}\|_{L^p(h_\kappa^2;\mathbb{S}^d)} = c\|f\|_{L^p(W_\kappa^B;\mathbb{B}^d)}$. Furthermore, by (2.4.10),

$$S_n^\delta(h_\kappa^2; \widetilde{f}, X) = S_n^\delta(W_\kappa^B; f, x), \quad X = (x, x_{d+1}) \in \mathbb{S}^d, \quad n = 0, 1, \cdots .$$

Thus, we may identify each function $f \in L^p(W_\kappa^B; \mathbb{B}^d)$ with a function $\widetilde{f} \in L^p(h_\kappa^2; \mathbb{S}^d)$ under the measure-preserving mapping ϕ, and such an identification preserves the Cesàro means of the corresponding weighted orthogonal polynomial expansions. Consequently, the stated conclusions of Theorem 4.1.1 follow directly from the corresponding results on the sphere $\mathbb{S}^d$ that are stated in Theorem 3.1.1. $\square$

We can also deduce the following corollaries from the corresponding results on the sphere $\mathbb{S}^d$, using a similar approach.

Corollary 4.1.2. *In order that*

$$\lim_{n\to\infty} S_n^\delta(W_\kappa^B; f)(x) = f(x)$$

holds almost everywhere on $\mathbb{B}^d$ for all $f \in L(W_\kappa^B; \mathbb{B}^d)$, it is sufficient and necessary that $\delta \geq \sigma_\kappa$.

Corollary 4.1.3. *If $1 < p < \infty$ and $\delta > 2\sigma_\kappa|\frac{1}{2} - \frac{1}{p}|$, then*

$$\|S_*^\delta(W_\kappa^B; f)\|_{L^p(W_\kappa^B;\mathbb{B}^d)} \leqslant C_p\|f\|_{L^p(W_\kappa^B;\mathbb{B}^d)}. \tag{4.1.2}$$

In particular,

$$\|S_*^{\sigma_\kappa}(W_\kappa^B; f)\|_{L^p(W_\kappa^B;\mathbb{B}^d)} \leqslant C_p \|f\|_{L^p(W_\kappa^B;\mathbb{B}^d)}, \quad 1 < p < \infty.$$

Corollary 4.1.4. *For $1 < p < \infty$, $\delta > 2\sigma_\kappa|\frac{1}{p} - \frac{1}{2}|$ and any sequence $\{n_j\}$ of positive integers,*

$$\left\| \left(\sum_{j=0}^{\infty} \left| S_{n_j}^{\delta}(W_\kappa^B; f_j) \right|^2 \right)^{1/2} \right\|_{L^p(W_\kappa^B;\mathbb{B}^d)} \le c \left\| \left(\sum_{j=0}^{\infty} |f_j|^2 \right)^{1/2} \right\|_{L^p(W_\kappa^B;\mathbb{B}^d)}. \tag{4.1.3}$$

Corollary 4.1.5. *Let $\{\mu_j\}_{j=0}^{\infty}$ be a sequence of complex numbers that satisfies*

(i) $\sup_j |\mu_j| \le c < \infty$,

(ii) $\sup_j 2^{j(n_0-1)} \sum_{l=2^j}^{2^{j+1}} |\Delta^{n_0} u_l| \le c < \infty$,

where n_0 is the smallest integer $\ge \sigma_\kappa + 1$. Then $\{\mu_j\}$ defines an $L^p(W_\kappa^B; \mathbb{B}^d)$, $1 < p < \infty$, multiplier; that is,

$$\left\| \sum_{j=0}^{\infty} \mu_j \operatorname{proj}_j(W_\kappa^B; f) \right\|_{L^p(W_\kappa^B;\mathbb{B}^d)} \le c\|f\|_{L^p(W_\kappa^B;\mathbb{B}^d)}, \qquad 1 < p < \infty,$$

where c is independent of μ_j.

In the case when the weights are invariant under a general reflection group, Corollary 4.1.5 was proved in [15] under a stronger assumption that n_0 is the smallest integer $\ge \sigma_\kappa + 2 + \kappa_{\min}$.

4.2 Maximal estimates on the simplex

In this section, we will show how to deduce similar results on the simplex $\mathbb{T}^d$ from those on the ball $\mathbb{B}^d$. Recall that $S_n^\delta(W_\kappa^T; f)$ denotes the (C,δ)-means of the orthogonal polynomial expansions with respect to the weight function W_κ^T on $\mathbb{T}^d$ that is given in (1.1.7). Our argument in this section is based on the following proposition.

Proposition 4.2.1. *Let* $\psi: \mathbb{B}^d \to \mathbb{T}^d$ *be the mapping defined in* (2.4.13)*. Then for each* $f \in L(W_\kappa^T; \mathbb{T}^d)$ *and* $\delta \geq 0$,

$$S_*^\delta(W_\kappa^B; f\circ\psi, x) \sim S_*^\delta(W_\kappa^T; f, \psi(x)), \quad x \in \mathbb{B}^d.$$

Proof. For simplicity, we set $F = f\circ\psi$. Clearly, $F \in L(W_\kappa^B; \mathbb{B}^d)$ and $F(x\varepsilon) = F(x)$ for all $\varepsilon \in \mathbb{Z}_2^d$, and $x \in \mathbb{B}^d$. In particular, this implies that

$$\operatorname{proj}_{2n+1}(W_\kappa^B; F) = 0, \quad n = 0, 1, \cdots. \tag{4.2.4}$$

We further claim that

$$\operatorname{proj}_n(W_\kappa^T; f, \psi(x)) = \operatorname{proj}_{2n}(W_\kappa^B; F, x). \tag{4.2.5}$$

Indeed, using (2.3.5) and (2.4.9), we have

$$P_n(W_\kappa^B; x\varepsilon, y\varepsilon) = P_n(W_\kappa^B; x, y), \quad x, y \in \mathbb{B}^d, \quad \varepsilon \in \mathbb{Z}_2^d, \tag{4.2.6}$$

and hence, for each $\varepsilon \in \mathbb{Z}_2^d$,

$$\begin{aligned}
\operatorname{proj}_{2n}(W_\kappa^B; F, x\varepsilon) &= \int_{\mathbb{B}^d} F(y) P_{2n}(W_\kappa^B; x\varepsilon, y) W_\kappa^B(y)\, dy \\
&= \int_{\mathbb{B}^d} F(y\varepsilon) P_{2n}(W_\kappa^B; x\varepsilon, y\varepsilon) W_\kappa^B(y)\, dy \\
&= \int_{\mathbb{B}^d} F(y) P_{2n}(W_\kappa^B; x, y) W_\kappa^B(y)\, dy \\
&= \operatorname{proj}_{2n}(W_\kappa^B; F, x),
\end{aligned}$$

where we used the $\mathbb{Z}_2^d$-invariance of the measure $W_\kappa^B(x)dx$ in the second step, (4.2.6) and the fact that $F(\cdot\varepsilon) = F(\cdot)$ in the third step. (4.2.5) then follows by (2.4.15).

Next, we prove the inequality

$$S_*^\delta(W_\kappa^T; f, \psi(x)) \leqslant C S_*^\delta(W_\kappa^B; F, x), \quad x \in \mathbb{B}^d. \tag{4.2.7}$$

To this end, we set

$$A_x^\delta := \frac{\Gamma(x+\delta+1)}{\Gamma(x+1)} \frac{1}{\Gamma(\delta+1)}, \quad x \geq 0.$$

Using asymptotic expansions for ratios of gamma functions (see [1, p.616]), we have that for $\ell = 0, 1, \cdots$,

$$\Big(\frac{d}{dx}\Big)^\ell A_x^\delta = \frac{\Gamma(\delta+\ell)}{\delta(\Gamma(\delta))^2}(x+1)^{\delta-\ell} + O\Big((x+1)^{\delta-\ell-1}\Big), \quad x \geq 0. \tag{4.2.8}$$

Define the operator

$$\tau_{2n}^{\delta}(W_{\kappa}^{B}; g, x) = \sum_{j=0}^{2n} \Phi_n(j) \operatorname{proj}_j(W_{\kappa}^{B}; g, x), \quad g \in L(W_{\kappa}^{B}; \mathbb{B}^d),$$

where

$$\Phi_n(x) = \begin{cases} \frac{A_{n-x/2}^{\delta}}{A_n^{\delta}} - \frac{A_{2n-x}^{\delta}}{A_{2n}^{\delta}}, & 0 \leqslant x \leqslant 2n, \\ 0, & x > 2n. \end{cases}$$

Let ℓ be an integer such that $\delta - 1 < \ell \leqslant \delta$. It is easily seen from (7.2.22) that for $0 < x < 2n$,

$$|\Phi_n^{(m)}(x)| \leqslant C n^{-\delta}(n - \frac{x}{2} + 1)^{\delta - m - 1}, \quad m = 0, 1, \cdots, \ell + 1,$$

which, in turn, implies that

$$|\triangle^{\ell+1}\Phi_n(j)| \leqslant C n^{-\delta}(n - \frac{x}{2} + 1)^{\delta - \ell - 2}, \quad 0 \leqslant j \leqslant 2n - 1, \tag{4.2.9}$$

and $\triangle^m \Phi_n(2n) = 0$ for $m = 0, 1, \cdots, \ell - 1$. Thus, using summation by parts ℓ times, we obtain

$$|\tau_{2n}^{\delta}(W_{\kappa}^{B}; g)| \leqslant C \sum_{j=0}^{2n-1} |\Delta^{\ell+1}\Phi_n(j)| j^{\ell} |S_j^{\ell}(W_{\kappa}^{B}; g)| + C|\Delta^{\ell}\Phi_n(2n)| n^{\ell} |S_{2n}^{\ell}(W_{\kappa}^{B}; g)|,$$

which, using Lemma 3.6.1, is controlled by

$$Cn^{-\delta}\sum_{j=0}^{2n}(2n-j+1)^{\delta-\ell-2}j^{\ell}j^{\delta-\ell}S_*^\delta(W_\kappa^B;g)\leqslant CS_*^\delta(W_\kappa^B;g). \tag{4.2.10}$$

On the other hand, however, using (4.2.4) and (4.2.5), we have

$$\begin{aligned} S_n^\delta(W_\kappa^T;f,\psi(x)) &= (A_n^\delta)^{-1}\sum_{j=0}^{n}A_{n-j}^\delta \operatorname{proj}_{2j}(W_\kappa^B;F,x) \qquad (4.2.11)\\ &= (A_n^\delta)^{-1}\sum_{j=0}^{2n}A_{n-j/2}^\delta \operatorname{proj}_{j}(W_\kappa^B;F,x)\\ &= \sum_{j=0}^{2n}\Big[\frac{A_{n-j/2}^\delta}{A_n^\delta}-\frac{A_{2n-j}^\delta}{A_{2n}^\delta}\Big]\operatorname{proj}_j(W_\kappa^B;F,x)+S_{2n}^\delta(W_\kappa^B;F,x)\\ &= \tau_{2n}^\delta(W_\kappa^B;F,x)+S_{2n}^\delta(W_\kappa^B;F,x). \qquad (4.2.12)\end{aligned}$$

Thus, combing (4.2.9) with (4.2.12), we deduce the estimate (4.2.7).

Finally, we show the converse inequality

$$S_*^\delta(W_\kappa^B;F,x)\leqslant CS_*^\delta(W_\kappa^T;f,\psi(x)),\quad x\in\mathbb{B}^d. \tag{4.2.13}$$

The proof is similar to that of (4.2.7), and we sketch it as follows.

Let m be the integer such that $2m\leqslant n<2m+1$. Then by (4.2.4) and (4.2.5),

$$\begin{aligned} S_n^\delta(W_\kappa^B;F,x) &= \sum_{j=0}^{m}\frac{A_{n-2j}^\delta}{A_n^\delta}\operatorname{proj}_{2j}(W_\kappa^B;F,x)=\sum_{j=0}^{m}\frac{A_{n-2j}^\delta}{A_n^\delta}\operatorname{proj}_j(W_\kappa^T;f,\psi(x))\\ &= \sum_{j=0}^{m}\mu_j\operatorname{proj}_j(W_\kappa^T;f,\psi(x))+S_m^\delta(W_\kappa^T;f,\psi(x)),\end{aligned}$$

where

$$\mu_j = \begin{cases} \frac{A_{n-2j}^\delta}{A_n^\delta} - \frac{A_{m-j}^\delta}{A_m^\delta}, & 0 \leqslant j \leqslant m, \\ 0, & j > m. \end{cases}$$

Using (4.2.8) and similar to the proof of (4.2.9), one can easily verify that for $0 \leqslant j \leqslant m$,

$$|\Delta^i \mu_j| \leqslant C m^{-\delta}(m-j+1)^{\delta-i-1}, \quad i = 0, 1, \cdots. \tag{4.2.14}$$

Let ℓ be an integer such that $\delta - 1 < \ell \leqslant \delta$. Summation by parts ℓ times shows that

$$\begin{aligned} \Big|\sum_{j=0}^{m} \mu_j \operatorname{proj}_j(W_\kappa^T; f, \psi(x))\Big| \leqslant & C\sum_{j=0}^{m-\ell} |\Delta^{\ell+1}\mu_j|(j+1)^\ell |S_j^\ell(W_\kappa^T; f, \psi(x))| \\ & + Cm^\ell \max_{0\leqslant i\leqslant \ell} |\Delta^i \mu_{m-i}||S_{m-i}^\ell(W_\kappa^T; f, \psi(x))|, \end{aligned}$$

which, using Lemma 3.6.1, and (4.2.14), is controlled by $CS_*^\delta(W_\kappa^T; f, \psi(x))$. The desired inequality (4.2.13) then follows. □

Recall that $\kappa_{\min}$, $|\kappa|$ and σ_κ are defined in (4.1.1). For a set $E \subset \mathbb{T}^d$, we write $\operatorname{meas}_\kappa^T(E) := \int_E W_\kappa^T(x)\,dx$. The following result is a simple consequence of Proposition 4.2.1, Theorem 4.1.1, and (2.4.14).

Theorem 4.2.2. (i) *If $\delta \geq \sigma_\kappa := \frac{d-1}{2} + |\kappa| - \kappa_{\min}$, then for $f \in L(W_\kappa^T; \mathbb{T}^d)$ with $\|f\|_{L(W_\kappa^T;\mathbb{T}^d)} = 1$,*

$$\operatorname{meas}_\kappa^T\Big\{x \in \mathbb{T}^d : \quad S_*^\delta(W_\kappa^T; f)(x) > \alpha\Big\} \leqslant C\frac{1}{\alpha}, \quad \forall \alpha > 0,$$

with $\alpha^{-1}|\log\alpha|$ in place of α^{-1} in the case when $\delta = \sigma_\kappa$ and at least two of the κ_i

are zero.

(ii) *If* $\delta < \sigma_\kappa$*, then there exists a function* $f \in L(W_\kappa^T; \mathbb{T}^d)$ *of the form* $f(x) = f_0(|x_{j_0}|)$ *such that* $S_*^\delta(W_\kappa^T; f)(x) = \infty$ *for a.e.* $x \in \mathbb{T}^d$*, where* $1 \leqslant j_0 \leqslant d+1$ *is the integer such that* $\kappa_{j_0} = \kappa_{\min}$*, and* $x_{d+1} = \sqrt{1-|x|}$*.*

As a consequence of Theorem 4.2.2, we obtain

Corollary 4.2.3. *In order that*

$$\lim_{n\to\infty} S_n^\delta(W_\kappa^T; f)(x) = f(x)$$

holds almost everywhere on $\mathbb{T}^d$ *for all* $f \in L(W_\kappa^T; \mathbb{T}^d)$*, it is sufficient and necessary that* $\delta \geq \sigma_\kappa$*.*

Corollary 4.2.4. *If* $1 < p < \infty$ *and* $\delta > 2\sigma_\kappa|\frac{1}{2} - \frac{1}{p}|$*, then*

$$\|S_*^\delta(W_\kappa^T; f)\|_{L^p(W_\kappa^T;\mathbb{T}^d)} \leqslant C_p\|f\|_{L^p(W_\kappa^T;\mathbb{T}^d)}. \tag{4.2.15}$$

In particular,

$$\|S_*^{\sigma_\kappa}(W_\kappa^T; f)\|_{L^p(W_\kappa^T;\mathbb{T}^d)} \leqslant C_p\|f\|_{L^p(W_\kappa^T;\mathbb{T}^d)}, \quad 1 < p < \infty.$$

Corollary 4.2.5. *For* $1 < p < \infty$*,* $\delta > 2\sigma_\kappa|\frac{1}{p} - \frac{1}{2}|$ *and any sequence* $\{n_j\}$ *of positive* *integers,*

$$\left\| \left(\sum_{j=0}^{\infty} \left| S_{n_j}^{\delta}(W_\kappa^T; f_j) \right|^2 \right)^{1/2} \right\|_{L^p(W_\kappa^T;\mathbb{T}^d)} \le c \left\| \left(\sum_{j=0}^{\infty} \left| f_j \right|^2 \right)^{1/2} \right\|_{L^p(W_\kappa^T;\mathbb{T}^d)}. \tag{4.2.16}$$

Using Corollary 4.1.4, and following the approach of [4], we have

Corollary 4.2.6. *Let $\{\mu_j\}_{j=0}^{\infty}$ be a sequence of complex numbers that satisfies*

(i) $\sup_j |\mu_j| \le c < \infty$,

(ii) $\sup_j 2^{j(n_0-1)} \sum_{l=2^j}^{2^{j+1}} |\Delta^{n_0} u_l| \le c < \infty$,

where n_0 is the smallest integer $\ge \sigma_\kappa + 1$. Then $\{\mu_j\}$ defines an $L^p(W_\kappa^T; \mathbb{T}^d)$, $1 < p < \infty$, multiplier; that is,

$$\left\| \sum_{j=0}^{\infty} \mu_j \operatorname{proj}_j(W_\kappa^T; f) \right\|_{L^p(W_\kappa^T;\mathbb{T}^d)} \le c \|f\|_{L^p(W_\kappa^T;\mathbb{T}^d)}, \qquad 1 < p < \infty,$$

where c is independent of μ_j.

Corollary 4.2.6 was proved in [15] under a stronger assumption that n_0 is the smallest integer $\ge \sigma_\kappa + 2 + \kappa_{\min}$.

Chapter 5

Generalized translations for Dunkl transforms on $\mathbb{R}^d$

5.1 Integral representation of generalized translations

The generalized translation T^y, initially defined on the space of Schwartz functions, extends to a bounded operator on the space $L^2(\mathbb{R}^d; h_\kappa^2)$, as can be easily seen from (2.6.25). On the other hand, Thangavelu and Xu [46] proved that the integral representation (2.6.26) of T^y defines a bounded operator on $L^\infty(\mathbb{R}^d; h_\kappa^2)$. It is, therefore, very natural to ask whether the generalized translation T^y given by (2.6.25) has the integral representation (2.6.26) on the space $L^\infty \cap L^2(\mathbb{R}^d; h_\kappa^2)$. This question is fairly nontrivial as $\mathcal{S}(\mathbb{R}^d)$ is not dense in L^∞, but is important for the extension of T^y to general $L^p(\mathbb{R}^d; h_\kappa^2)$-spaces with $1 \leqslant p \leqslant \infty$.

Our main purpose in this section is to clarify the definition of generalized translations on L^p-spaces and some related facts, which will be needed in later sections. We will show that the expression on the right hand side of the integral representation (2.6.26) defines a bounded operator on $L^p(\mathbb{R}^d; h_\kappa^2)$ for all $1 \leqslant p \leqslant \infty$, which, in particular, implies that the formula (2.6.26) is applicable to all $f \in L^2(\mathbb{R}^d; h_\kappa^2)$. More precisely, we have

Theorem 5.1.1. *The integral representation* (2.6.26) *extends T^y to a bounded operator on the spaces $L^p(\mathbb{R}^d; h_\kappa^2)$ for all $1 \leqslant p \leqslant \infty$ with*

$$\sup_{y\in\mathbb{R}^d} \|T^y f\|_{\kappa,p} \leqslant C_d \|f\|_{\kappa,p}, \quad 1 \leqslant p \leqslant \infty. \tag{5.1.1}$$

In other words, for each $1 \leqslant j \leqslant d$ and a.e. $x \in \mathbb{R}^d$, the expression on the right hand side of (2.6.27) *is well defined for all $f \in L^p(\mathbb{R}^d; h_\kappa^2)$ with $1 \leqslant p \leqslant \infty$, and moreover, it defines a bounded operator T_{j,y_j} on $L^p(\mathbb{R}^d; h_\kappa^2)$ which satisfies*

$$\|T_{j,y_j} f\|_{\kappa,p} \leqslant C_d \|f\|_{\kappa,p}, \quad 1 \leqslant p \leqslant \infty. \tag{5.1.2}$$

Note that we may rewrite the integral representation (2.6.26) in the form

$$T^y f(x) := \int_{\mathbb{R}^d} f(z)\, d\mu_{x,y}(z), \quad x, y \in \mathbb{R}^d, \tag{5.1.3}$$

where $d\mu_{x,y}$ is a signed Borel measure supported on

$$\Big\{z = (z_1, \cdots, z_d) \in \mathbb{R}^d : \quad \big||x_i| - |y_i|\big| \leqslant |z_i| \leqslant |x_i| + |y_i|, \quad i = 1, 2, \cdots, d\Big\}.$$

As a direct consequence of Theorem 5.1.1, we obtain the following integral representation of generalized translations for radial functions, which will play a crucial role in this paper:

Corollary 5.1.2. *If $f(x) = f_0(\|x\|)$ is a radial function in $L^p(\mathbb{R}^d; h_\kappa^2)$ with $1 \leqslant p \leqslant \infty$, then for each $y \in \mathbb{R}^d$ and a.e. $x \in \mathbb{R}^d$,*

$$T^y f(x) = c_\kappa \int_{[-1,1]^d} f_0\big(z(x,y,t)\big) \prod_{j=1}^{d} (1 + t_j)(1 - t_j^2)^{\kappa_j - 1} \, dt_j, \tag{5.1.4}$$

where $z(x, y, t) = \sqrt{\|x\|^2 + \|y\|^2 - 2\sum_{j=1}^{d} x_j y_j t_j}$.

In the case when f is a radial Schwartz function, (5.1.4) is a direct consequence of a more general formula of Rösler [35] and the explicit expression (2.2.3) of V_κ. That (5.1.4) holds under the relaxed condition $f = f_0(\|\cdot\|) \in L^p(\mathbb{R}^d; h_\kappa^2)$ is very important in the proofs of the main results of this thesis. A standard limit argument doesn't seem to give this result.

Now we are in a position to show Theorem 5.1.1.

Proof of Theorem 5.1.1. By Fubini's theorem, it is enough to show the result for $d = 1$. Assume that $x, y \in \mathbb{R}$, $\kappa \geq 0$ and $f \in L^p(\mathbb{R}; |x|^{2\kappa})$. Without loss of generality, we may assume that $\kappa > 0$ and $xy \neq 0$. Note that for $t \in [-1, 1]$,

$$\sqrt{x^2 + y^2 - 2xyt} \geq \max\{|x - yt|, |xt - y|\} \geq \frac{1}{2}|x - y|(1 + t). \tag{5.1.5}$$

It then follows from (2.6.27) that

$$|T^y f(x)| \leqslant C \int_{-1}^{1} \Big[|f_e(u(x, y, t))| + |f_o(u(x, y, t))|\Big](1 - t^2)^{\kappa - 1}\, dt,$$

where $u(x, y, t) = \sqrt{x^2 + y^2 - 2xyt}$. Thus, we reduce to showing that the integral

$$\widetilde{T^y} f(x) := \int_{-1}^{1} f(\sqrt{x^2 + y^2 - 2xyt})(1 - t^2)^{\kappa - 1}\, dt, \quad f \in L^p(\mathbb{R}; |x|^{2\kappa}), \tag{5.1.6}$$

defines a bounded operator on $L^p(\mathbb{R}; |x|^{2\kappa})$:

$$\|\widetilde{T^y} f\|_{L^p(\mathbb{R}; |x|^{2\kappa} dx)} \leqslant C_\kappa \|f\|_{L^p(\mathbb{R}; |x|^{2\kappa} dx)}, \quad 1 \leqslant p \leqslant \infty. \tag{5.1.7}$$

Performing a change of variable $z = \sqrt{x^2 + y^2 - 2xyt}$ in (5.1.6), we obtain that

$$\begin{aligned} \widetilde{T^y} f(x) &= \int_{\left||x| - |y|\right|}^{|x| + |y|} f(z) \Big[1 - \Big(\frac{x^2 + y^2 - z^2}{2xy}\Big)^2\Big]^{\kappa - 1} \frac{z}{|xy|}\, dz \\ &= \int_{\mathbb{R}} f(z) W(x, y, z) |z|^{2\kappa}\, dz, \end{aligned} \tag{5.1.8}$$

where

$$W(x,y,z)=\begin{cases}\dfrac{\Big(\Delta(x,y,z)\Big)^{\kappa-1}}{2^{2\kappa-2}|xyz|^{2\kappa-1}}, & \text{if } \big||x|-|y|\big|\leqslant z\leqslant |x|+|y|,\\[2ex] 0, & \text{otherwise},\end{cases} \tag{5.1.9}$$

$$\Delta(x,y,z)=(|x|+|y|+z)(|x|+|y|-z)(z+|x|-|y|)(z-|x|+|y|). \tag{5.1.10}$$

We claim that

$$\sup_{y,z\in\mathbb{R}\setminus\{0\}}\int_{\mathbb{R}} W(x,y,z)|x|^{2\kappa}\,dx\leqslant C<\infty, \tag{5.1.11}$$

from which the assertion (5.1.7) will follow by Hölder's inequality.

To show (5.1.11), we first note that

$$\big||x|-|y|\big|\leqslant z\leqslant |x|+|y| \Leftrightarrow \big||y|-z\big|\leqslant |x|\leqslant |y|+z,\quad z\geq 0.$$

Also, it is easily seen from (5.1.9) and (5.1.10) that

$$W(x,y,z)=W(|x|,|y|,z)=W(|x|,z,|y|),\quad \forall x,y\in\mathbb{R},\quad \forall z\geq 0.$$

Thus,

$$\sup_{y,z\in\mathbb{R}\setminus\{0\}}\int_{\mathbb{R}} W(x,y,z)|x|^{2\kappa}\,dx=\sup_{y\geq z>0}\int_{y-z}^{y+z} W(x,y,z)x^{2\kappa}\,dx.$$

A straightforward calculation shows that for $y \geq z > 0$ and $x \in [y-z, y+z]$,

$$W(x,y,z) \sim \Big[xy\big(x-(y-z)\big)(y+z-x)\Big]^{\kappa-1}(xyz)^{1-2\kappa}$$
$$\leqslant C_\kappa (xyz)^{-\kappa}\Big[\big(x-(y-z)\big)^{\kappa-1}+(y+z-x)^{\kappa-1}\Big].$$

It follows that for $y \geq z > 0$

$$\sup_{y\geq z>0}\int_{y-z}^{y+z} W(x,y,z)x^{2\kappa}\,dx$$
$$\leqslant C_\kappa (yz)^{-\kappa}(y+z)^\kappa \int_{y-z}^{y+z}\Big[\big(x-(y-z)\big)^{\kappa-1}+(y+z-x)^{\kappa-1}\Big]\,dx$$
$$\leqslant C_\kappa,$$

which completes the proof of Theorem 5.1.1.

□

The multiplier property (2.6.25) of the generalized translation operators plays a crucial role in our argument. This property carries over to L^p spaces:

Proposition 5.1.3. *Let $y \in \mathbb{R}^d$ and $f \in L^p(\mathbb{R}^d; h_\kappa^2)$ with $1 \leqslant p \leqslant \infty$.* (2.6.25) *holds a.e. on $\mathbb{R}^d$ for $1 \leqslant p \leqslant 2$, and in a distributional sense for $p > 2$; that is,*

$$\Big(\mathcal{F}_\kappa(T^y f), \varphi\Big) = \Big(\widehat{f}, E_\kappa(-iy,\cdot)\varphi\Big),\ \ \forall \varphi \in \mathcal{S}(\mathbb{R}^d). \tag{5.1.12}$$

Proof. That (2.6.25) holds a.e. on $\mathbb{R}^d$ for $1 \leqslant p \leqslant 2$ follows by the Hausdorff-Young inequality (2.5.19) and a standard density argument. For $p > 2$, and any $\varphi \in \mathcal{S}(\mathbb{R}^d)$, we have

$$\begin{aligned}(\mathcal{F}_\kappa T^y f, \varphi) &= (T^y f, \mathcal{F}_\kappa \varphi) = \int_{\mathbb{R}^d} f(x) T^{-y} \mathcal{F}_\kappa \varphi(x) h_\kappa^2(x)\, dx \\ &= \int_{\mathbb{R}^d} f(x) \mathcal{F}_\kappa \Big(E_\kappa(-iy, \cdot)\varphi \Big)(x) h_\kappa^2(x)\, dx = \Big(\widehat{f}, E_\kappa(-iy, \cdot)\varphi \Big),\end{aligned}$$

where we used (2.6.28) and (5.1.1) in the second step, the identity

$$T^y \widehat{\varphi}(x) = \mathcal{F}_\kappa \Big(\varphi E_\kappa(iy, \cdot) \Big)(x), \quad \varphi \in \mathcal{S}(\mathbb{R}^d)$$

in the third step. This proves (5.1.12). □

5.2 Generalized convolution

Finally, we give a few comments on generalized convolutions on L^p-spaces. Recall that the generalized convolution $f *_\kappa g$ is defined in (2.6.29) for $f, g \in \mathcal{S}(\mathbb{R}^d)$. Since the generalized translation operators are uniformly bounded on L^p-spaces with $1 \leqslant p \leqslant \infty$, the following Young's inequality for the generalized convolution can be established (see [46, Proposition 7.2]):

$$\|f *_\kappa g\|_{\kappa,r} \leqslant \|f\|_{\kappa,p} \|g\|_{\kappa,q}, \tag{5.2.13}$$

where $1 \leqslant p, q, r \leqslant \infty$ and $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. This, in particular, implies that the generalized convolution $f *_\kappa g$ can be defined for $f \in L^p(\mathbb{R}^d; h_\kappa^2)$ and $g \in L^q(\mathbb{R}^d; h_\kappa^2)$ with $1 \leqslant p, q \leqslant \infty$ and $\frac{1}{p} + \frac{1}{q} \geq 1$.

The generalized convolution has the following basic property:

Corollary 5.2.1. *Let* $f \in L^p(\mathbb{R}^d; h_\kappa^2)$, $1 \leqslant p \leqslant \infty$ *and* $g \in \mathcal{S}(\mathbb{R}^d)$. *Then*

$$\mathcal{F}_\kappa(f *_\kappa g)(\xi) = \mathcal{F}_\kappa f(\xi)\mathcal{F}_\kappa g(\xi) \tag{5.2.14}$$

holds in a distributional sense.

Proof. A straightforward calculation shows that for any $\varphi \in \mathcal{S}(\mathbb{R}^d)$,

$$(f *_\kappa g, \widehat{\varphi}) = (f, \widehat{\varphi} *_\kappa \widetilde{g}) = (f, (\varphi\widehat{g})^\wedge) = (\widehat{f}, \widehat{g}\varphi) = (\widehat{f}\widehat{g}, \varphi).$$

□

Chapter 6

Almost everywhere convergence of the Bochner-Riesz means of the inverse Dunkl transforms of L^1-functions at the critical index

6.1 Sharp Pointwise estimates of the Bochner-Riesz kernels

Recall that the Bochner-Riesz means of order δ of $f \in L^1(\mathbb{R}^d; h_\kappa^2)$ are defined by

$$B_R^\delta(h_\kappa^2; f)(x) = c \int_{\mathbb{R}^d} \mathcal{F}_\kappa f(\xi) E_\kappa(ix, \xi) \Phi^\delta\big(R^{-1}\xi\big) h_\kappa^2(\xi)\, d\xi, \qquad R > 0,$$

where $\Phi^\delta(x) := (1-\|x\|^2)_+^\delta$. It is known (see [46, p. 44]) that $\Phi^\delta(x) = \mathcal{F}_\kappa \phi^\delta(x)$, where

$$\phi^\delta(x) = 2^{\lambda_\kappa}\|x\|^{-\lambda_\kappa-\delta-\frac{1}{2}} J_{\lambda_\kappa+\delta+\frac{1}{2}}(\|x\|) =: \phi^{\delta,0}(\|x\|). \tag{6.1.1}$$

Setting $\phi_R^\delta(x) := R^{2\lambda_\kappa+1}\phi^\delta(Rx)$ for $R>0$, we obtain from Lemma 2.5.1 (vi) that $\widehat{\phi_R^\delta}(\xi) = \widehat{\phi^\delta}(R^{-1}\xi) = \Phi^\delta(\xi/R)$, and hence, $\widehat{f}(\xi)\Phi^\delta(\xi/R) = \left(f *_\kappa \phi_R^\delta\right)^\wedge(\xi)$. By (5.2.14), this implies that $B_R^\delta(h_\kappa^2; f)(x) = f *_\kappa \phi_R^\delta(x)$. Thus, using (2.6.29) and (5.1.4), we obtain

$$B_R^\delta(h_\kappa^2; f)(x) = c_\kappa \int_{\mathbb{R}^d} f(y) K_R^\delta(h_\kappa^2; x, y) h_\kappa^2(y)\, dy, \tag{6.1.2}$$

where

$$K_R^\delta(h_\kappa^2; x, y) : = T^x \phi_R^\delta(y) = V_\kappa\left(\phi_R^{\delta,0}\big(\sqrt{\|x\|^2+\|y\|^2-2\|x\|\langle y,\cdot\rangle}\big)\right)(x/\|x\|), \tag{6.1.3}$$

$$\phi^{\delta,0}(t) = 2^{\lambda_\kappa} t^{-\lambda_\kappa-\delta-\frac{1}{2}} J_{\lambda_\kappa+\delta+\frac{1}{2}}(t), \qquad \phi_R^{\delta,0}(t) = R^{-2\lambda_\kappa-1}\phi^{\delta,0}(R^{-1}t). \tag{6.1.4}$$

The main goal in this section is to show the following pointwise estimate of the kernel $K_R^\delta(h_\kappa^2; x, y)$:

Theorem 6.1.1. *For $\delta>0$, $R>0$ and $x, y \in \mathbb{R}^d$,*

$$|K_R^\delta(h_\kappa^2; x, y)| \leqslant C \frac{R^d \prod_{j=1}^d (|x_j y_j| + R^{-2} + R^{-1}\|\bar{x}-\bar{y}\|)^{-\kappa_j}}{(1+R\|\bar{x}-\bar{y}\|)^{\frac{d+1}{2}+\delta}}, \tag{6.1.5}$$

where we write $\bar{x} = (|x_1|, \cdots, |x_d|)$ for $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$.

As pointed out in the introduction, this is a fairly nontrivial estimate since an application of Lemma 3.4 of [13] would only give the estimate for the case of $\|x\| = \|y\|$. The idea of our proof is from the paper [12].

Proof. By (2.2.3) and (6.1.3), it is easily seen that

$$K_R^\delta(h_\kappa^2; x, y) = R^{2\lambda_\kappa+1} K_1^\delta(h_\kappa^2; Rx, Ry), \quad x, y \in \mathbb{R}^d, \quad R > 0. \tag{6.1.6}$$

Thus, it suffices to show (6.1.5) for $R = 1$. For simplicity, we write $K(x, y) = K_1^\delta(h_\kappa^2; x, y)$. Using (2.2.3), we have

$$K(x, y) = c_\kappa \int_{[-1,1]^d} \phi^{\delta,0}\big(z(x, y, t)\big) \prod_{j=1}^{d} (1 - t_j^2)^{\kappa_j - 1}(1 + t_j)\, dt_j, \tag{6.1.7}$$

where $z(x, y, t) = \sqrt{\|x\|^2 + \|y\|^2 - 2\sum_{j=1}^d x_j y_j t_j}$ and $t = (t_1, \cdots, t_d)$.

Let $\xi_0 \in C^\infty(\mathbb{R})$ be such that $\xi_0(s) = 1$ for $|s| \leqslant 1/4$ and $\xi_0(s) = 0$ for $|s| \geq 1$, and let $\xi(s) = \xi_0(s/4) - \xi_0(s)$. Clearly, supp $\xi \subset \{s : \ \frac{1}{4} \leqslant |s| \leqslant 4\}$, and $\sum_{n=0}^\infty \xi(s/4^n) = 1$ for $s \geq 1$. Thus, by (6.1.7), we may decompose $K(x, y)$ as $K(x, y) = \sum_{n=0}^\infty K_n(x, y)$, where

$$\begin{aligned} K_n(x, y) := &\int_{[-1,1]^d} \phi^{\delta,0}\big(z(x, y, t)\big)\xi\left(\frac{1 + z(x, y, t)^2}{4^n}\right) \times \\ &\times \prod_{j=1}^{d} (1 - t_j^2)^{\kappa_j - 1}(1 + t_j)\, dt_j. \end{aligned} \tag{6.1.8}$$

Next, fix $x, y \in \mathbb{R}^d$ and let $n_0 = n_0(x, y) \in \mathbb{Z}_+$ be such that

$$2^{2n_0} \leqslant 1 + \|\bar{x} - \bar{y}\|^2 \leqslant 2^{2n_0+2}. \tag{6.1.9}$$

Note that for $t = (t_1, \cdots, t_d) \in [-1,1]^d$,

$$z(x,y,t)^2 = \|x\|^2 + \|y\|^2 - 2\sum_{j=1}^{d} x_j y_j t_j \geq \|\bar{x} - \bar{y}\|^2,$$

hence, $\xi\left(\frac{1+z(x,y,t)^2}{4^n}\right)$ is zero unless

$$4 > \frac{1+z(x,y,t)^2}{4^n} \geq \frac{1+\|\bar{x}-\bar{y}\|^2}{4^n} \geq 4^{n_0-n}.$$

This means that

$$K(x,y) = \sum_{n=n_0}^{\infty} K_n(x,y). \tag{6.1.10}$$

The following lemma gives an estimate of the kernel $K_n(x,y)$:

Lemma 6.1.2. *For $n \geq n_0 = n_0(x,y)$,*

$$|K_n(x,y)| \leqslant C 2^{-n(\frac{d+1}{2}+\delta)} \prod_{j=1}^{d} (|x_j y_j| + 2^n)^{-\kappa_j}, \tag{6.1.11}$$

where $C > 0$ is independent of n, x and y.

For the moment, we take Lemma 6.1.2 for granted and proceed with the proof of Theorem 6.1.1. Indeed, once Lemma 6.1.2 is proved, then using (6.1.10) and (6.1.11), we have that

$$
\begin{aligned}
|K(x,y)| &\leqslant \sum_{n=n_0}^{\infty} |K_n(x,y)| \leqslant C \sum_{n=n_0}^{\infty} 2^{-n(\frac{d+1}{2}+\delta)} \prod_{j=1}^{d} (|x_j y_j| + 2^n)^{-\kappa_j} \\
&\leqslant C \Big(\prod_{j=1}^{d} (|x_j y_j| + 2^{n_0})^{-\kappa_j} \Big) \sum_{n=n_0}^{\infty} 2^{-n(\frac{d+1}{2}+\delta)} \leqslant C \Big(\prod_{j=1}^{d} (|x_j y_j| + 2^{n_0})^{-\kappa_j} \Big) 2^{-n_0(\frac{d+1}{2}+\delta)} \\
&\sim (1 + \|\bar{x} - \bar{y}\|)^{-(\frac{d+1}{2}+\delta)} \prod_{j=1}^{d} (|x_j y_j| + 1 + \|\bar{x} - \bar{y}\|)^{-\kappa_j},
\end{aligned}
$$

which proves the desired estimate (6.1.5) for $R = 1$.

□

To complete the proof of Theorem 6.1.1, it remains to prove Lemma 6.1.2.

Proof of Lemma 6.1.2. Let $G_\alpha(u) = (\sqrt{u})^{-\alpha} J_\alpha(\sqrt{u})$ and $F_\alpha(t) = G_\alpha\Big(u(x,y,t)\Big)$, where

$$
u(x,y,t) = \|x\|^2 + \|y\|^2 - 2\sum_{j=1}^{d} x_j y_j t_j = z(x,y,t)^2.
$$

By (2.5.21) and (2.5.22), it is easily seen that for $\alpha \in \mathbb{R}$,

$$
\frac{\partial}{\partial t_j} F_{\alpha-1}(t) = x_j y_j F_\alpha(t), \quad t = (t_1, \cdots, t_d) \in [-1,1]^d, \tag{6.1.12}
$$

and

$$
|F_\alpha(t)| \leqslant C(1 + u(x,y,t))^{-\frac{\alpha}{2}-\frac{1}{4}}, \quad t \in [-1,1]^d. \tag{6.1.13}
$$

Also, note that

$$\phi^{\delta,0}\big(z(x,y,t)\big) = 2^{\lambda_\kappa}\big(\sqrt{u(x,y,t)}\big)^{-\lambda_\kappa-\delta-\frac{1}{2}} J_{\lambda_\kappa+\delta+\frac{1}{2}}\big(\sqrt{u(x,y,t)}\big) = C_\kappa F_{\lambda_\kappa+\delta+\frac{1}{2}}(t).$$

Thus, by (6.1.8), we may write

$$K_n(x,y) = c_\kappa \int_{[-1,1]^d} F_{\lambda_\kappa+\delta+\frac{1}{2}}(t)\xi\left(\frac{1+u(x,y,t)}{4^n}\right)\prod_{j=1}^{d}(1-t_j^2)^{\kappa_j-1}(1+t_j)\,dt_j.$$

Without loss of generality, we may assume that $|x_jy_j| \geq 2^n$, $j = 1,\cdots,m$ and $|x_jy_j| < 2^n$, $j = m+1,\cdots,d$ for some $1 \leqslant m \leqslant d$ (otherwise, we re-index the sequence $\{x_jy_j\}_{j=1}^d$). Fix temporarily $t_{m+1},\cdots,t_d \in [-1,1]$, and set

$$\begin{aligned} &I(t_{m+1},\cdots,t_d) \\ &:= c_\kappa \int_{[-1,1]^m} F_{\lambda_\kappa+\delta+\frac{1}{2}}(t)\xi\left(\frac{1+u(x,y,t)}{4^n}\right)\prod_{j=1}^{m}(1-t_j^2)^{\kappa_j-1}(1+t_j)\,dt_j. \end{aligned} \tag{6.1.14}$$

By Fubini's theorem, we then have

$$K_n(x,y) = \int_{[-1,1]^{d-m}} I(t_{m+1},\cdots,t_d)\prod_{j=m+1}^{d}(1-t_j^2)^{\kappa_j-1}(1+t_j)\,dt_j. \tag{6.1.15}$$

Thus, for the proof of (6.1.11), it suffices to show that for each $t_{m+1},\cdots,t_d \in [-1,1]$,

$$|I(t_{m+1},\cdots,t_d)| \leqslant C2^{-n(\sum_{j=m+1}^d \kappa_j + \frac{d+1}{2}+\delta)}\prod_{j=1}^{m}|x_jy_j|^{-\kappa_j}. \tag{6.1.16}$$

To show (6.1.16), let $\eta_0 \in C^\infty(\mathbb{R})$ be such that $\eta_0(s) = 1$ for $|s| \leqslant \frac{1}{2}$ and $\eta_0(s) = 0$

for $|s| \geq 1$, and let $\eta_1(s) = 1 - \eta_0(s)$. Set $B_j := \frac{2^n}{|x_j y_j|}$ for $j = 1, \cdots, m$. Given $\varepsilon := (\varepsilon_1, \cdots, \varepsilon_m) \in \{0,1\}^m$, we define $\psi_\varepsilon : \ [-1,1]^m \to \mathbb{R}$ by

$$\psi_\varepsilon(t) := \xi\left(\frac{1+u(x,y,t)}{4^n}\right)\prod_{j=1}^m \eta_{\varepsilon_j}\Big(\frac{1-t_j^2}{B_j}\Big)(1+t_j)(1-t_j^2)^{\kappa_j-1},$$

where $t = (t_1, \cdots, t_m)$. We then split the integral in (6.1.14) into a finite sum to obtain

$$I(t_{m+1}, \cdots, t_d) = \sum_{\varepsilon \in \{0,1\}^m} \int_{[-1,1]^m} F_{\lambda_\kappa+\delta+\frac{1}{2}}(t)\psi_\varepsilon(t)\, dt_1 \cdots dt_m =: \sum_{\varepsilon\in\{0,1\}^m} J_\varepsilon,$$

where

$$J_\varepsilon \equiv J_\varepsilon(t_{m+1}, \cdots, t_d) := \int_{[-1,1]^m} F_{\lambda_\kappa+\delta+\frac{1}{2}}(t)\psi_\varepsilon(t)\, dt_1 \cdots dt_m. \tag{6.1.17}$$

Thus, it suffices to prove the estimate (6.1.16) with $I(t_{m+1}, \cdots, t_d)$ replaced by J_ε for each $\varepsilon \in \{0,1\}^m$, namely,

$$|J_\varepsilon(t_{m+1}, \cdots, t_d)| \leqslant C 2^{-n(\sum_{j=m+1}^d \kappa_j + \frac{d+1}{2} + \delta)} \prod_{j=1}^m |x_j y_j|^{-\kappa_j}. \tag{6.1.18}$$

By symmetry and Fubini's theorem, we need only to prove (6.1.18) for the case of $\varepsilon_1 = \cdots = \varepsilon_{m_1} = 0$ and $\varepsilon_{m_1+1} = \cdots = \varepsilon_m = 1$ with m_1 being an integer in $[0, m]$. Write

$$\psi_\varepsilon(t) = \varphi(t)\prod_{j=1}^{m_1} \eta_0\Big(\frac{1-t_j^2}{B_j}\Big)(1+t_j)(1-t_j^2)^{\kappa_j-1} \tag{6.1.19}$$

with

$$\varphi(t) := \xi\left(\frac{1+u(x,y,t)}{4^n}\right)\prod_{j=m_1+1}^m \eta_1\Big(\frac{1-t_j^2}{B_j}\Big)(1+t_j)(1-t_j^2)^{\kappa_j-1}.$$

Since the support set of each $\eta_1\left(\frac{1-t_j^2}{B_j}\right)$ is a subset of $\{t_j : |t_j| \leqslant 1 - \frac{1}{4}B_j\}$, we can use (6.1.12) and integration by parts $|\mathbf{l}| = \sum_{j=m_1+1}^{m} \ell_j$ times to obtain

$$\begin{aligned}
&\Big|\int_{[-1,1]^{m-m_1}} F_{\lambda_\kappa+\delta+\frac{1}{2}}(t)\varphi(t)\,dt_{m_1+1}\cdots dt_m\Big| \\
&= c\prod_{j=m_1+1}^{m} |x_j y_j|^{-\ell_j}\Big|\int_{[-1,1]^{m-m_1}} F_{\lambda_\kappa+\delta+\frac{1}{2}-|\mathbf{l}|}(t)\frac{\partial^{|\mathbf{l}|}\varphi(t)}{\partial^{\ell_{m_1+1}}t_{m_1+1}\cdots\partial^{\ell_m}t_m}\,dt\Big| \\
&\leqslant c\prod_{j=m_1+1}^{m} |x_j y_j|^{-\ell_j}\int_{[-1,1]^{m-m_1}} \Big|F_{\lambda_\kappa+\delta+\frac{1}{2}-|\mathbf{l}|}(t)\Big|\Big|\frac{\partial^{|\mathbf{l}|}\varphi(t)}{\partial^{\ell_{m_1+1}}t_{m_1+1}\cdots\partial^{\ell_m}t_m}\Big|\,dt,
\end{aligned}$$

where $\mathbf{l} = (\ell_{m_1+1}, \cdots, \ell_m) \in \mathbb{N}^{m-m_1}$ satisfies $\ell_j > \kappa_j$ for all $m_1 < j \leqslant m$. Since ξ is supported in $[\frac{1}{4}, 1]$, $\varphi(t)$ is zero unless

$$\begin{aligned}
4^{n+1} &\geq 1 + \|x\|^2 + \|y\|^2 - 2\sum_{j=1}^{d} |x_j y_j t_j| \qquad (6.1.20)\\
&\geq 1 + \|\bar{x} - \bar{y}\|^2 + 2|x_j y_j|(1 - |t_j|) \geq 2|x_j y_j|(1 - |t_j|),
\end{aligned}$$

for all $m_1 + 1 \leqslant j \leqslant m$; that is, $\frac{|x_j y_j|}{4^n} \leqslant 2(1 - |t_j|)^{-1}$ for $j = m_1 + 1, \cdots, m$. On the other hand, note that the derivative of the function $\eta_1\left(\frac{1-t_j^2}{B_j}\right)$ in the variable t_j is supported in $\{t_j : \ \frac{1}{2}B_j \leqslant 1 - t_j^2 \leqslant B_j\}$.

Consequently, by the Leibniz rule, we conclude

$$\Big|\frac{\partial^{|\mathbf{l}|}\varphi(t)}{\partial^{\ell_{m_1+1}}t_{m_1+1}\cdots\partial^{\ell_m}t_m}\Big| \leqslant c\prod_{j=m_1+1}^{m} (1 - |t_j|)^{\kappa_j-\ell_j-1}.$$

Finally, note that in the support of $\varphi(t)$, $1+u(x,y,t)\sim 4^n$, and hence by (6.1.13),

$$\left|F_{\lambda_\kappa+\delta+\frac{1}{2}-|\mathbf{l}|}(t)\right| \leqslant c(1+u(x,y,t))^{-\frac{\lambda_\kappa+\delta-|\mathbf{l}|}{2}-\frac{1}{2}} \sim 2^{n(-\lambda_\kappa-\delta+|\mathbf{l}|-1)}.$$

It follows that

$$\begin{aligned}
&\int_{[-1,1]^{m-m_1}} \left|F_{\lambda_\kappa+\delta+\frac{1}{2}-|\mathbf{l}|}(t)\right| \left|\frac{\partial^{|\mathbf{l}|}\varphi(t)}{\partial^{\ell_{m_1+1}} t_{m_1+1}\cdots\partial^{\ell_m} t_m}\right| dt_{m_1+1}\cdots dt_m \\
&\qquad \leqslant c2^{n(-\lambda_\kappa-\delta-1+|\mathbf{l}|)} \prod_{j=m_1+1}^{m} \int_0^{1-\frac{B_j}{4}} (1-t_j)^{\kappa_j-\ell_j-1}\, dt_j \\
&\qquad \leqslant c2^{n(-\lambda_\kappa-\delta-1+|\mathbf{l}|)} \prod_{j=m_1+1}^{m} B_j^{\kappa_j-\ell_j} \\
&\qquad \leqslant c2^{n(\alpha-\lambda_\kappa-1-\delta)} \prod_{j=m_1+1}^{m} |x_j y_j|^{\ell_j-\kappa_j},
\end{aligned} \tag{6.1.21}$$

where $\alpha=\sum_{j=m_1+1}^m \kappa_j$. Thus, using (6.1.17) and Fubini's theorem, we obtain that

$$\begin{aligned}
|J_\varepsilon| &\leqslant \int_{[-1,1]^{m_1}} \left|\int_{[-1,1]^{m-m_1}} F_{\lambda_\kappa+\delta+\frac{1}{2}}(t)\varphi(t)\, dt_{m_1+1}\cdots dt_m\right| \\
&\qquad\quad \times \prod_{j=1}^{m_1} \eta_0\left(\frac{1-t_j^2}{B_j}\right)(1+t_j)(1-t_j^2)^{\kappa_j-1} dt_j \\
&\leqslant c2^{n(\alpha-\lambda_\kappa-1-\delta)} \prod_{j=m_1+1}^{m} |x_j y_j|^{-\kappa_j} \prod_{j=1}^{m_1} \int_{1-B_j\leqslant|t_j|\leqslant 1} (1-|t_j|)^{\kappa_j-1}\, dt_j \\
&\leqslant c2^{n(\sum_{j=1}^m \kappa_j-\lambda_\kappa-1-\delta)} \prod_{j=1}^{m} |x_j y_j|^{-\kappa_j},
\end{aligned}$$

where we used (6.1.21) and the fact that $\eta_0\left(\frac{1-t_j^2}{B_j}\right)$ is supported in $\{t_j:\ \ 1-B_j\leqslant|t_j|\leqslant 1\}$ for $1\leqslant j\leqslant m_1$ in the second step. This yields the desired estimate (6.1.18) and hence completes the proof of Lemma 6.1.2. □

6.2 Proof of the main results

As mentioned in the introduction, we want to prove the almost everywhere convergence of Bochner-Riesz mean.

Theorem 6.2.1. *If* $\kappa \neq 0$, $1 \leqslant p < \infty$ *and* $f \in L^p(\mathbb{R}^d; h_\kappa^2)$, *then the Bochner-Riesz mean* $B_R^{\lambda_\kappa}(h_\kappa^2; f)(x)$ *converges a.e. to* $f(x)$ *on* $\mathbb{R}^d$ *as* $R \to \infty$.

However, Theorem 6.2.1 follows directly from weak type estimates of the maximal Bochner-Riesz operator. Thus, we only need to prove the following theorem in this section.

Theorem 6.2.2. *Assume that* $\kappa \neq 0$. *If* $\delta = \lambda_\kappa$ *and* $f \in L^1(\mathbb{R}^d; h_\kappa^2)$, *then for any* $\alpha > 0$,

$$\operatorname{meas}_\kappa \left(\left\{x \in \mathbb{R}^d : \ B_*^\delta(h_\kappa^2; f)(x) \geqslant \alpha\right\}\right) \leqslant c_\kappa \frac{\|f\|_{\kappa,1}}{\alpha}, \tag{6.2.22}$$

where we need to replace $\frac{\|f\|_{\kappa,1}}{\alpha}$ *by* $\frac{\|f\|_{\kappa,1}}{\alpha}\left|\log \frac{\|f\|_{\kappa,1}}{\alpha}\right|$ *when* $\min_{1\leqslant j\leqslant d} \kappa_j = 0$.

We first describe several necessary notations. Let $B_r(x) := \{y \in \mathbb{R}^d : \ \|x-y\| \leqslant r\}$ denote the ball centered at $x \in \mathbb{R}^d$ and having radius $r > 0$. Define

$$V_r(x) := \operatorname{meas}_\kappa \left(B_r(x)\right) = \int_{B_r(x)} h_\kappa^2(z)\, dz, \quad r > 0, \quad x \in \mathbb{R}^d,$$

and

$$V(x,y) = V_{\|\bar{x}-\bar{y}\|}(x) = \int_{\|z-x\| \leqslant \|\bar{x}-\bar{y}\|} h_\kappa^2(z)\, dz, \quad x, y \in \mathbb{R}^d.$$

Since the measure $h_\kappa^2(x)dx$ is $\mathbb{Z}_2^d$-invariant, it is easily seen that $V(x,y) = V(\bar{x},\bar{y})$ and

$V_r(x) = V_r(\bar{x})$. Furthermore, a straightforward calculation shows that

$$V_r(x) \sim r^d \prod_{j=1}^{d} (|x_j| + r)^{2\kappa_j}, \quad V(x,y) \sim \|\bar{x} - \bar{y}\|^d \prod_{j=1}^{d} (|x_j| + \|\bar{x} - \bar{y}\|)^{2\kappa_j}. \tag{6.2.23}$$

Here and elsewhere in the paper, the notation $A \sim B$ means that $c^{-1}A \leqslant B \leqslant cB$ for some positive constant c depending only on κ and d.

The proof of Theorem 6.2.2 relies on a series of lemmas.

Lemma 6.2.3. *For $x, y \in \mathbb{R}^d$, set*

$$I_j(x,y) := (|x_j y_j| + R^{-1}\|\bar{x} - \bar{y}\| + R^{-2})^{-\kappa_j}, \qquad j = 1, \cdots, d.$$

Then

$$R^d \prod_{j=1}^{d} I_j(x,y) \leqslant C \frac{(1 + R\|\bar{x} - \bar{y}\|)^{d+|\kappa|_J}}{V_{R^{-1}}(x) + V(x,y)}, \tag{6.2.24}$$

where $|\kappa|_J = \sum_{j \in J} \kappa_j$ and $J = J(x,y) := \Big\{ j \in \{1, \cdots, d\} : |x_j| < 2\|\bar{x} - \bar{y}\| \Big\}$.

Proof. If $j \notin J = J(x,y)$, then $|x_j| \geq 2\|\bar{x} - \bar{y}\|$, $|x_j| \sim |y_j|$ and hence

$$\begin{aligned} I_j(x,y) &\sim (|x_j|^2 + R^{-1}\|\bar{x} - \bar{y}\| + R^{-2})^{-\kappa_j} \sim (|x_j|^2 + R^{-2})^{-\kappa_j} \\ &\sim (|x_j| + \|\bar{x} - \bar{y}\| + R^{-1})^{-2\kappa_j}. \end{aligned}$$

If $j \in J(x,y)$, then $|x_j| < 2\|\bar{x}-\bar{y}\|$, and

$$I_j(x,y) \leqslant (R^{-2}+R^{-1}\|\bar{x}-\bar{y}\|)^{-\kappa_j} = (1+R\|\bar{x}-\bar{y}\|)^{\kappa_j}(R^{-1}+\|\bar{x}-\bar{y}\|)^{-2\kappa_j}$$
$$\sim (1+R\|\bar{x}-\bar{y}\|)^{\kappa_j}(\|\bar{x}-\bar{y}\|+|x_j|+R^{-1})^{-2\kappa_j}.$$

Thus, using (6.2.23), we obtain

$$R^d\prod_{j=1}^d I_j(x,y) \leqslant C(1+R\|\bar{x}-\bar{y}\|)^{\sum_{j\in J}\kappa_j} R^d\prod_{j=1}^d(|x_j|+\|\bar{x}-\bar{y}\|+R^{-1})^{-2\kappa_j}$$
$$\sim (1+R\|\bar{x}-\bar{y}\|)^{|\kappa|_J}\frac{1+R^d\|\bar{x}-\bar{y}\|^d}{V_{R^{-1}}(\bar{x})+V(\bar{x},\bar{y})},$$

where the last step can be obtained by considering the cases $\|\bar{x}-\bar{y}\| \leqslant R^{-1}$ and $\|\bar{x}-\bar{y}\| \geq R^{-1}$ separately. This yields the desired estimate (6.2.24). □

A combination of Lemma 6.2.3 and Theorem 6.1.1 yields the following estimate of the Bochner-Riesz kernels $K_R^\delta(h_\kappa^2;x,y)$:

Lemma 6.2.4. *For $\delta>0, x,y\in\mathbb{R}^d$ and $R>0$,*

$$|K_R^\delta(h_\kappa^2;x,y)| \leqslant C\frac{(1+R\|\bar{x}-\bar{y}\|)^{\frac{d-1}{2}+|\kappa|_J-\delta}}{V_{R^{-1}}(x)+V(x,y)}, \quad x,y\in\mathbb{R}^d, \quad R>0, \tag{6.2.25}$$

where

$$|\kappa|_J := \sum_{j\in J(x,y)}\kappa_j = \sum_{j:\ \ |x_j|\leqslant 2\|\bar{x}-\bar{y}\|}\kappa_j.$$

Next, recall that the weighted Hardy-Littlewood maximal function is defined for

$f \in L^1_{\text{loc}}(\mathbb{R}^d; h_\kappa^2)$ by

$$M_\kappa f(x) := \sup_{r>0} \frac{1}{V_r(x)} \int_{B_r(x)} |f(y)| h_\kappa^2(y) dy, \quad x \in \mathbb{R}^d.$$

Our proof will also use a modified weighted Hardy-Littlewood maximal function, defined as follows:

$$\widetilde{M}_\kappa f(x) := \sup_{r>0} \frac{1}{V_r(x)} \int_{\|\bar{x}-\bar{y}\| \leqslant r} |f(y)| h_\kappa^2(y) dy, \quad x \in \mathbb{R}^d.$$

Since $\|\bar{x} - \bar{y}\| = \|x\sigma - y\| \leqslant \|x - y\|$ for some $\sigma \in \mathbb{Z}_2^d$ that depends on x, y and since the measure $h_\kappa^2(x)\, dx$ is $\mathbb{Z}_2^d$-invariant, it follows that

$$M_\kappa f(x) \leqslant \widetilde{M}_\kappa f(x) \leqslant \sum_{\sigma \in \mathbb{Z}_2^d} M_\kappa f(x\sigma), \quad x \in \mathbb{R}^d.$$

Since h_κ^2 is a doubling weight on $\mathbb{R}^d$, this implies that

$$\text{meas}_\kappa\{x \in \mathbb{R}^d : \ \widetilde{M}_\kappa f(x) \geq \alpha\} \leqslant C \frac{\|f\|_{\kappa,1}}{\alpha}, \quad \forall \alpha > 0, \tag{6.2.26}$$

and

$$\|\widetilde{M}_\kappa f\|_{\kappa,p} \leqslant C_p \|f\|_{\kappa,p}, \quad 1 < p \leqslant \infty. \tag{6.2.27}$$

Finally, the following lemma can be verified straightforwardly:

Lemma 6.2.5. *Let $\beta > 0$ and $f \in L^1_{loc}(\mathbb{R}^d; h_\kappa^2)$. Then for $x \in \mathbb{R}^d$ and $R > 0$,*

$$\int_{\mathbb{R}^d} |f(y)| \frac{(1 + R\|\bar{x} - \bar{y}\|)^{-\beta}}{V_{R^{-1}}(x) + V(x,y)} h_\kappa^2(y)\, dy \leqslant C \widetilde{M}_\kappa f(x), \tag{6.2.28}$$

where C depends only on β, κ and d.

We are now in a position to show Theorem 6.2.2.

Proof of Theorem 6.2.2 . First, we prove Theorem 6.2.2 for the case of $\kappa_{\min} > 0$; namely, we prove that if $\kappa_{\min} > 0$, then for any $f \in L^1(\mathbb{R}^d; h_\kappa^2)$,

$$\operatorname{meas}_\kappa\left(\left\{x \in \mathbb{R}^d : \ B_*^{\lambda_\kappa}(h_\kappa^2; f)(x) \geqslant \alpha\right\}\right) \leqslant c_\kappa \frac{\|f\|_{\kappa,1}}{\alpha}, \quad \forall \alpha > 0.$$

To this end, without loss of generality, we may assume that $\|f\|_{\kappa,1} = 1$. Given $\alpha > 0$, let $\varepsilon > 0$ be such that $\varepsilon^{2|\kappa|+d} =: \frac{C_1}{\alpha}$, where $C_1 > 0$ is a large constant to be specified later. Setting $F_\varepsilon = [-\varepsilon, \varepsilon]^d$, we have

$$\operatorname{meas}_\kappa(F_\varepsilon) = \int_{F_\varepsilon} h_\kappa^2(y)dy \sim \varepsilon^{2|\kappa|+d} = \frac{C_1}{\alpha}. \tag{6.2.29}$$

Next, we split the integral in (1.2.11) into two parts $\int_{E_\varepsilon} \cdots + \int_{\mathbb{R}^d \setminus E_\varepsilon} \cdots$, where $E_\varepsilon = E_\varepsilon(x) := \{y \in \mathbb{R}^d : \ \|\bar{x} - \bar{y}\| \geqslant \varepsilon/2\}$. We then write

$$B_R^{\lambda_\kappa}(h_\kappa^2; f)(x) = T_{R,\varepsilon}^{\lambda_\kappa}(h_\kappa^2; f)(x) + S_{R,\varepsilon}^{\lambda_\kappa}(h_\kappa^2; f)(x), \tag{6.2.30}$$

where

$$T_{R,\varepsilon}^{\lambda_\kappa}(h_\kappa^2; f)(x) = c_\kappa \int_{E_\varepsilon} f(y) K_R^{\lambda_\kappa}(h_\kappa^2; x, y) h_\kappa^2(y)\, dy,$$
$$S_{R,\varepsilon}^{\lambda_\kappa}(h_\kappa^2; f)(x) = c_\kappa \int_{\mathbb{R}^d \setminus E_\varepsilon} f(y) K_R^{\lambda_\kappa}(h_\kappa^2; x, y) h_\kappa^2(y)\, dy.$$

To estimate $T^{\lambda_\kappa}_{R,\varepsilon}(h^2_\kappa; f)(x)$, we note that by Theorem 6.1.1, for $y \in E_\varepsilon(x)$,

$$|K^{\lambda_\kappa}_R(h^2_\kappa; x, y)| \leqslant CR^d(R\varepsilon)^{-(\frac{d+1}{2}+\lambda_\kappa)}(R^{-1}\varepsilon)^{-|\kappa|} = C\varepsilon^{-2|\kappa|-d} = \frac{C\alpha}{C_1}.$$

Thus, choosing C_1 such that $C_1 = 2c_\kappa C$, we get that

$$|T^{\lambda_\kappa}_{R,\varepsilon}(h^2_\kappa; f)(x)| \leqslant \frac{Cc_\kappa \alpha}{C_1}\|f\|_{1,\kappa} = \frac{\alpha}{2}. \tag{6.2.31}$$

Next, we estimate $S^{\lambda_\kappa}_{R,\varepsilon}(h^2_\kappa; f)(x)$ for $x \notin F_\varepsilon$. If $x \in \mathbb{R}^d \setminus F_\varepsilon$ and $y \in \mathbb{R}^d \setminus E_\varepsilon$, then

$$|x_{j_0}| := \max_{1\leqslant j\leqslant d} |x_j| > \varepsilon \geqslant 2\|\bar{x} - \bar{y}\|,$$

and hence

$$|\kappa|_J := \sum_{j:|x_j|\leqslant 2\|\bar{x}-\bar{y}\|} \kappa_j \leqslant |\kappa| - \kappa_{j_0} \leqslant |\kappa| - \kappa_{\min}.$$

It then follows by Lemma 6.2.4 that for $x \in \mathbb{R}^d \setminus F_\varepsilon$ and $y \in \mathbb{R}^d \setminus E_\varepsilon$,

$$|K^{\lambda_\kappa}_R(h^2_\kappa; x, y)| \leqslant C\frac{(1 + R\|\bar{x} - \bar{y}\|)^{|\kappa|_J-|\kappa|}}{V_{R^{-1}}(x) + V(x, y)} \leqslant C\frac{(1 + R\|\bar{x} - \bar{y}\|)^{-\kappa_{\min}}}{V_{R^{-1}}(x) + V(x, y)}.$$

Since $\kappa_{\min} > 0$, by Lemma 6.2.5, this implies that

$$S^{\lambda_\kappa}_{R,\varepsilon}(h^2_\kappa; f)(x) \leqslant C\widetilde{M}_\kappa f(x), \quad x \in \mathbb{R}^d \setminus F_\varepsilon. \tag{6.2.32}$$

Now combining (6.2.30), (6.2.31) with (6.2.32), we obtain that for $x \in \mathbb{R}^d \setminus F_\varepsilon$,

$$B_*^{\lambda_\kappa}(h_\kappa^2; f)(x) \leqslant C\widetilde{M}_\kappa(f)(x) + \frac{\alpha}{2}, \quad x \in \mathbb{R}^d \setminus F_\varepsilon. \tag{6.2.33}$$

Finally, using (6.2.26), (6.2.29) and (6.2.33), we obtain

$$\begin{aligned}
&\mathrm{meas}_\kappa\{x \in \mathbb{R}^d : \ B_*^{\lambda_\kappa}(h_\kappa^2; f)(x) \geq \alpha\} \\
\leqslant& \mathrm{meas}_\kappa(F_\varepsilon) + \mathrm{meas}_\kappa\{x \in \mathbb{R}^d \setminus F_\varepsilon : \ B_*^{\lambda_\kappa}(h_\kappa^2; f)(x) \geq \alpha\} \\
\leqslant& \frac{C}{\alpha} + \mathrm{meas}_\kappa\{x \in \mathbb{R}^d : \ \widetilde{M}_\kappa f(x) \geq \frac{\alpha}{2C}\} \leqslant \frac{C}{\alpha}.
\end{aligned}$$

This completes the proof of Theorem 6.2.2 for the case of $\kappa_{\min} > 0$.

Next, we show Theorem 6.2.2 for the case of $\kappa_{\min} = 0$, namely, we show that if $|\kappa| > 0$, $\kappa_{\min} = 0$ and $\|f\|_{1,\kappa} = 1$, then for all $\alpha > 0$,

$$\mathrm{meas}_\kappa\left(\{x \in \mathbb{R}^d : \ B_*^{\lambda_\kappa}(h_\kappa^2; f)(x) \geqslant \alpha\}\right) \leqslant c_\kappa \frac{1 + |\ln \alpha|}{\alpha}. \tag{6.2.34}$$

For the proof of (6.2.34), we claim that it is enough to show that for $f \in L^1(\mathbb{R}^d; h_\kappa^2)$ with $\|f\|_{1,\kappa} = 1$,

$$\mathrm{meas}_\kappa\left(\{x \in [-1,1]^d : \ B_*^{\lambda_\kappa}(h_\kappa^2; f)(x) \geqslant \alpha\}\right) \leqslant c_\kappa \frac{1 + |\ln \alpha|}{\alpha}. \tag{6.2.35}$$

To see this, set $f_t(x) = f(tx)$ for each $t > 0$, and observe that

$\|f\|_{\kappa,1} = t^{2\lambda_\kappa+1}\|f_t\|_{\kappa,1}$. Furthermore, using (6.1.6), we obtain that for $t > 0$,

$$B_R^{\lambda_\kappa}(h_\kappa^2; f)(x) = c_\kappa t^{-2\lambda_\kappa-1}\int_{\mathbb{R}^d} f(y)K_{tR}^{\lambda_\kappa}(h_\kappa^2; \frac{x}{t}, \frac{y}{t})h_\kappa^2(y)dy$$
$$= c_\kappa \int_{\mathbb{R}^d} f(tv)K_{tR}^{\lambda_\kappa}(h_\kappa^2; x/t, v)h_\kappa^2(v)dv = B_{tR}^{\lambda_\kappa}(h_\kappa^2; f_t)(x/t),$$

which implies that

$$B_*^{\lambda_\kappa}(h_\kappa^2; f)(x) = B_*^{\lambda_\kappa}(h_\kappa^2; f_t)(x/t), \quad t > 0, \quad x \in \mathbb{R}^d. \tag{6.2.36}$$

Next, set

$$D_{N,\alpha}(f) = \{x \in [-N, N]^d : \; B_*^{\lambda_\kappa}(h_\kappa^2; f)(x)| \geqslant \alpha\}, \quad \alpha > 0, \quad N > 1.$$

It's easily seen from (6.2.36) that $x \in D_{N,\alpha}(f)$ if and only if $x/N \in [-1, 1]^d$ and $B_*^\delta(h_\kappa^2; f_N)(x/N) \geqslant \alpha$, namely, $x/N \in D_{1,\alpha}(f_N)$. Thus, once (6.2.35) is proved, then

$$\text{meas}_\kappa\left(D_{N,\alpha}(f)\right) = \int_{\mathbb{R}^d} \chi_{D_{1,\alpha}(f_N)}(N^{-1}x)h_\kappa^2(x)dx = N^d \int_{D_{1,\alpha}(f_N)} h_\kappa^2(Ny)dy$$
$$= N^{d+2|\kappa|}\,\text{meas}_\kappa\left(\{x \in [-1, 1]^d : \sup_{R>0} |B_R^\delta(h_\kappa^2; f_N)(x)| \geqslant \alpha\}\right)$$
$$\leqslant c_\kappa \frac{1 + |\ln \alpha|}{\alpha} N^{d+2|\kappa|}\|f_N\|_{\kappa,1} = c_\kappa \frac{1 + |\ln \alpha|}{\alpha}.$$

The desired estimate (6.2.34) will then follow by Levi's monotone convergence theorem.

It remains to prove the estimate (6.2.35). Without loss of generality, we may assume that $\alpha > 2C_2$, where C_2 is a sufficiently large constant. Denote by $G(x)$ the set of all $y \in \mathbb{R}^d$ for which there exists $i \in \{1, \cdots, d\}$ such that $\kappa_i > 0$ and $|x_i| > 2\|\bar{x} - \bar{y}\|$. We then write $B_R^{\lambda_\kappa}(h_\kappa^2; f)(x) = G_{1,R}^{\lambda_\kappa}(h_\kappa^2; f)(x) + G_{2,R}^{\lambda_\kappa}(h_\kappa^2; f)(x)$, where

$$G_{1,R}^{\lambda_\kappa}(h_\kappa^2; f)(x) = c_\kappa \int_{G(x)} f(y) K_R^{\lambda_\kappa}(h_\kappa^2; x, y) h_\kappa^2(y)\, dy,$$
$$G_{2,R}^{\lambda_\kappa}(h_\kappa^2; f)(x) = c_\kappa \int_{\mathbb{R}^d \setminus G(x)} f(y) K_R^{\lambda_\kappa}(h_\kappa^2; x, y) h_\kappa^2(y)\, dy.$$

By Lemma 6.2.4 and Lemma 6.2.5, it is easily seen that

$$|G_{1,R}^{\lambda_\kappa}(h_\kappa^2; f)(x)| \leqslant C \widetilde{M}_\kappa f(x).$$

To estimate $G_{2,R}^{\lambda_\kappa}(h_\kappa^2; f)(x)$, assume that $\kappa_l > 0$ for some $l \in \{1, \cdots, d\}$. By the definition of $G(x)$, we conclude that for each $y \in \mathbb{R}^d \setminus G(x)$, $|x_l| \leqslant 2\|\bar{x} - \bar{y}\|$. Thus, by Lemma 6.2.4, for $0 < |x_l| \leqslant 1$,

$$\begin{aligned}
|G_{2,R}^{\lambda_\kappa}(h_\kappa^2; f)(x)| &\leqslant C \int_{|x_l| \leqslant 2\|\bar{x}-\bar{y}\| \leqslant 1} \frac{|f(y)|}{V(\bar{x}, \bar{y})} h_\kappa^2(y) dy + C \int_{\|\bar{x}-\bar{y}\| \geq \frac{1}{2}} |f(y)| h_\kappa^2(y)\, dy \\
&\leqslant C \sum_{j=1}^{\lceil \log_2 \frac{1}{|x_l|} \rceil} \frac{1}{\mathrm{meas}_\kappa \left(B(\bar{x}, 2^j |x_l|)\right)} \int_{B(\bar{x}, 2^j |x_l|)} |f(y)| h_\kappa^2(y) dy + C_2 \\
&\leqslant C \Big(1 + \ln \frac{1}{|x_l|}\Big) \widetilde{M}_\kappa f(x) + \frac{\alpha}{2},
\end{aligned}$$

where the last step uses the assumption that $\alpha > 2C_2$.

Putting the above together, we conclude that for $x \in [-1, 1]^d$,

$$B_*^{\lambda_\kappa}(h_\kappa^2; f)(x) \leqslant C \Big(1 + \ln \frac{1}{|x_l|}\Big) \widetilde{M}_\kappa f(x) + \frac{\alpha}{2}.$$

Recalling that $\alpha > 2$, we deduce

$$\begin{aligned}
&\operatorname{meas}_\kappa\Big\{x \in [-1,1]^d:\ |B_*^{\lambda_\kappa}(h_\kappa^{\lambda_\kappa}; f)(x)| > \alpha\Big\} \\
\leqslant &\operatorname{meas}_\kappa\Big\{x \in [-1,1]^d:\ \big(1+\ln\alpha\big)\widetilde{M}_\kappa f(x) > \frac{\alpha}{2C}\Big\} + \operatorname{meas}_\kappa\{x \in [-1,1]^d:\ |x_l| \leqslant \alpha^{-1}\} \\
\leqslant &C\alpha^{-1}\ln\alpha + C\alpha^{-2\kappa_l - 1} \leqslant C\alpha^{-1}(1+\ln\alpha).
\end{aligned}$$

This shows the estimate (6.2.35). The proof of Theorem 6.2.2 is then completed. □

Chapter 7

Restriction theorem for the Dunkl transform

7.1 Global restriction theorem

Let $d\sigma$ denote the Lebesgue measure on the unit sphere $\mathbb{S}^{d-1} := \{x \in \mathbb{R}^d : \ \|x\| = 1\}$ of $\mathbb{R}^d$. Recall that for $f \in L^1(\mathbb{S}^{d-1}, h_\kappa^2 d\sigma)$, we write

$$\mathcal{F}_\kappa(f d\sigma)(\xi) \equiv \widehat{f d\sigma}(\xi) := \int_{\mathbb{S}^{d-1}} f(y) E_\kappa(-i\xi, y) h_\kappa^2(y)\, d\sigma(y), \quad \xi \in \mathbb{R}^d. \tag{7.1.1}$$

In particular,

$$\mathcal{F}_\kappa(d\sigma)(\xi) \equiv \widehat{d\sigma}(\xi) := \int_{\mathbb{S}^{d-1}} E_\kappa(-i\xi, y) h_\kappa^2(y)\, d\sigma(y), \quad \xi \in \mathbb{R}^d. \tag{7.1.2}$$

It is known that (see, for instance, [11, Proposition 6.1.9] and [46, Proposition 2.3]))

$$\widehat{d\sigma}(\xi) = c_{\kappa,d}\|\xi\|^{-(\frac{d-2}{2}+|\kappa|)} J_{\frac{d-2}{2}+|\kappa|}(\|\xi\|) = c j_{\lambda_\kappa-\frac{1}{2}}(\|\xi\|), \tag{7.1.3}$$

where $\lambda_\kappa = \frac{d-1}{2} + |\kappa|$.

Our main result in this section can be stated as follows:

Theorem 7.1.1. *Let* $p_\kappa = \frac{2\lambda_\kappa+2}{\lambda_\kappa+2}$. *Then for* $1 \leqslant p \leqslant p_\kappa$,

$$\|f *_\kappa \widehat{d\sigma}\|_{\kappa,p'} \leqslant C\|f\|_{\kappa,p}.$$

Proof. By (2.5.16), the function,

$$K_z(x) := j_{\lambda_\kappa-\frac{1}{2}+z}(\|x\|) = \frac{J_{\lambda_\kappa-\frac{1}{2}+z}(\|x\|)}{\|x\|^{\lambda_\kappa-\frac{1}{2}+z}}, \quad x \in \mathbb{R}^d,$$

is analytic in z on the domain $\{z \in \mathbb{C} : \text{ Re } z > -\frac{1}{2} - \lambda_\kappa\}$, whereas by (2.5.22),

$$|K_{\sigma+i\tau}(x)| \leqslant C_\sigma e^{c|\tau|}(1+\|x\|)^{-(\lambda_\kappa+\sigma)}, \quad x \in \mathbb{R}^d, \quad \lambda_\kappa + \sigma \geq 0. \tag{7.1.4}$$

Furthermore, according to (2.6.24), and by analytic continuation, the function K_z has the following distributional Dunkl transform:

$$\widehat{K_z}(\xi) = \frac{c_\kappa}{2^z\Gamma(z+1)}(1-\|\xi\|^2)_+^{z-1}, \quad \text{Re } z > 0, \quad \xi \in \mathbb{R}^d. \tag{7.1.5}$$

Namely, the following holds for all $z \in \mathbb{C}$ with Re$z > 0$ and $\varphi \in \mathcal{S}(\mathbb{R}^d)$.

$$\int_{\mathbb{R}^d} K_z(x)\widehat{\varphi}(x)h_\kappa^2(x)\,dx = \frac{c_\kappa}{2^z\Gamma(z+1)}\int_{\|x\|\leqslant 1}(1-\|x\|^2)^{z-1}\varphi(x)h_\kappa^2(x)\,dx.$$

Define

$$R_z f(x) := f *_\kappa K_z(x) = c\int_{\mathbb{R}^d} f(y)T^y K_z(x)h_\kappa^2(y)\,dy, \quad \forall f \in \mathcal{S}(\mathbb{R}^d).$$

It is easily seen that $R_z f$ is analytic in z on the domain $\{z \in \mathbb{C}: \quad \text{Re } z > -\frac{1}{2} - \lambda_\kappa\}$.

On the one hand, by (7.1.4) and Young's inequality (5.2.13) , we have that

$$\|R_{-\lambda_\kappa + i\tau} f\|_\infty \leqslant \|K_z\|_\infty \|f\|_{\kappa,1} \leqslant Ce^{c|\tau|}\|f\|_{\kappa,1}. \tag{7.1.6}$$

On the other hand, by (7.1.5) and (5.2.14), it follows that

$$\|R_{1+i\tau} f\|_{\kappa,2} \leqslant \|\widehat{K_{1+i\tau}}\|_\infty \|f\|_{\kappa,2} \leqslant Ce^{c|\tau|}\|f\|_{\kappa,2}.$$

Thus, by Stein's interpolation theorem of analytic families of operators, we conclude that

$$\|f *_\kappa \widehat{d\sigma}\|_{\kappa,p'_\kappa} = c\|R_0 f\|_{\kappa,p'_\kappa} \leqslant C\|f\|_{\kappa,p_\kappa}. \tag{7.1.7}$$

Finally, by (7.1.3) and Young's inequality (5.2.13),

$$\|f *_\kappa \widehat{d\sigma}\|_\infty \leqslant C\|f\|_{\kappa,1}\|\widehat{d\sigma}\|_\infty \leqslant C\|f\|_{\kappa,1}.$$

Theorem 7.1.1 then follows by the Riesz-Thorin theorem. $\square$

Next, we define Rf to be the the restriction to the sphere $\mathbb{S}^{d-1}$ of the Dunkl transform $\mathcal{F}_\kappa f$ of $f \in L^1(\mathbb{R}^d; h_\kappa^2)$, and R^*g to be the inverse Dunkl transform of the measure $gd\sigma$ for a function $g \in L^1(\mathbb{S}^{d-1}, h_\kappa^2 d\sigma)$. Thus, $Rf = \mathcal{F}_\kappa f\Big|_{\mathbb{S}^{d-1}}$, and

$$R^*g(x) = \int_{\mathbb{S}^{d-1}} g(\xi) E_\kappa(ix, \xi) h_\kappa^2(\xi)\, d\sigma(\xi), \quad x \in \mathbb{R}^d.$$

A straightforward calculation shows that for $f \in \mathcal{S}(\mathbb{R}^d)$

$$R^*Rf(x) = c \int_{\mathbb{R}^d} f(z) T^x \mathcal{F}_\kappa(d\sigma)(z)\, h_\kappa^2(z)\, dz = c_\kappa' f *_\kappa \widehat{d\sigma}(x). \tag{7.1.8}$$

By Theorem 7.1.1, R^*R extends to a bounded operator from $L^p(\mathbb{R}^d; h_\kappa^2)$ to $L^{p'}(\mathbb{R}^d; h_\kappa^2)$ with $1 \leqslant p \leqslant p_\kappa$. On the other hand, it is easy to verify that

$$\langle Rf, g \rangle_{L^2(\mathbb{S}^{d-1}, h_\kappa^2)} = \langle f, R^*g \rangle_{L^2(\mathbb{R}^d, h_\kappa^2)}, \quad \forall f \in \mathcal{S}(\mathbb{R}^d), \quad \forall g \in C(\mathbb{S}^{d-1}),$$

where the notation $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ denotes the inner product of a given Hilbert space $\mathcal{H}$. Thus, observing that

$$\begin{aligned} \|R\|^2_{L^p(\mathbb{R}^d, h_\kappa^2) \to L^2(\mathbb{S}^{d-1}, h_\kappa^2)} &= \|R^*\|^2_{L^2(\mathbb{S}^{d-1}, h_\kappa^2) \to L^{p'}(\mathbb{R}^d, h_\kappa^2)} \\ &= \|R^*R\|_{L^p(\mathbb{R}^d, h_\kappa^2) \to L^{p'}(\mathbb{R}^d, h_\kappa^2)}. \end{aligned}$$

we conclude

Corollary 7.1.2. *If $1 \leqslant p \leqslant p_\kappa$, then R extends to a bounded operator from $L^p(\mathbb{R}^d, h_\kappa^2)$ to $L^2(\mathbb{S}^{d-1}, h_\kappa^2)$, and R^* extends to a bounded operator from $L^2(\mathbb{S}^{d-1}, h_\kappa^2)$ to $L^{p'}(\mathbb{R}^d, h_\kappa^2)$.*

7.2 Local restriction theorem

The global restriction theorem proved in Section 7.1 will not be enough for our purpose. In order to show the main results in this thesis, we need the following local restriction theorem.

Theorem 7.2.1. *Let $c_0 \in (0,1)$ be a parameter depending only on d and κ, and let B denote a ball $B(\omega,\theta)$ centered at $\omega \in \mathbb{R}^d$ and having radius $\theta \geq c_0$. If $f \in L^p(\mathbb{R}^d; h_\kappa^2)$ is supported in the ball B, $d \geqslant 2$ and $1 \leqslant p \leqslant p_\kappa := \frac{2+2\lambda_\kappa}{\lambda_\kappa+2}$, then*

$$\Big(\int_B |f *_\kappa \widehat{d\sigma}(x)|^{p'} h_\kappa^2(x)\,dx\Big)^{\frac{1}{p'}} \leqslant C\Big(\frac{1}{\theta^{2\lambda_\kappa+1}}\int_B h_\kappa^2(y)\,dy\Big)^{1-\frac{2}{p}}\|f\|_{\kappa,p}, \tag{7.2.9}$$

where $\frac{1}{p}+\frac{1}{p'}=1$.

The local estimate (7.2.9) is, in general, stronger than the global estimate in Theorem 7.1.1. To see this, we write $\omega = (\omega_1, \cdots, \omega_d)$ and observe that

$$\Big(\frac{1}{\theta^{2\lambda_\kappa+1}}\int_B h_\kappa^2(y)\,dy\Big)^{-1} \sim \prod_{j=1}^d \big(\frac{|\omega_j|}{\theta}+1\big)^{-2\kappa_j} \leqslant C_\kappa.$$

The proof of Theorem 7.2.1 is much more involved than that of the global restriction theorem. Indeed, a direct application of Stein's interpolation theorem for analytic families of operators or the real technique used in the proof of the Stein-Tomas restriction theorem would yield (7.2.9) for a smaller p only.

The proof of Theorem 7.2.1 will be given in the next few subsections. For the moment, we take it for granted and deduce a useful corollary from it.

Corollary 7.2.2. *Let $c_0 \in (0,1)$ be a constant depending only on d and κ, and B the ball $B(\omega,\theta)$ centered at $\omega \in \mathbb{R}^d$ and having radius $\theta \geq c_0 > 0$.*

(i) *If $1 \leqslant p \leqslant p_\kappa := \frac{2+2\lambda_\kappa}{\lambda_\kappa+2}$, and $f \in L^p(\mathbb{R}^d; h_\kappa^2)$ is supported in the ball B, then*

$$\|\widehat{f}\|_{L^2(\mathbb{S}^{d-1};h_\kappa^2)} \leqslant C\Big(\frac{\theta^{2\lambda_\kappa+1}}{\int_B h_\kappa^2(y)\,dy}\Big)^{\frac{1}{p}-\frac{1}{2}} \|f\|_{L^p(\mathbb{R}^d;h_\kappa^2)}. \tag{7.2.10}$$

(ii) *If $2+\frac{2}{\lambda_\kappa} \leqslant q \leqslant \infty$, and $f \in L^2(\mathbb{S}^{d-1}; h_\kappa^2)$, then*

$$\Big\|\int_{\mathbb{S}^{d-1}} f(\xi) E_\kappa(i\xi,\cdot) h_\kappa^2(\xi)\,d\sigma(\xi)\Big\|_{L^q(B;h_\kappa^2)} \leqslant C\Big(\frac{\theta^{2\lambda_\kappa+1}}{\int_B h_\kappa^2(y)\,dy}\Big)^{\frac{1}{2}-\frac{1}{q}} \|f\|_{L^2(\mathbb{S}^{d-1};h_\kappa^2)}, \tag{7.2.11}$$

where $L^q(B;h_\kappa^2)$ denotes the L^q-space defined with respect to the measure $h_\kappa^2(x)dx$ on the ball B.

Proof. Consider the operator $Tf := \Big((f\chi_B) *_\kappa \widehat{d\sigma}\Big)\chi_B$. According to Theorem 7.2.1, T is a bounded operator from $L^p(\mathbb{R}^d; h_\kappa^2)$ to $L^{p'}(\mathbb{R}^d; h_\kappa^2)$ satisfying

$$\|Tf\|_{\kappa,p'} \leqslant C\Big(\frac{1}{\theta^{2\lambda_\kappa+1}} \int_B h_\kappa^2(y)\,dy\Big)^{1-\frac{2}{p}} \|f\|_{\kappa,p} \tag{7.2.12}$$

for $1 \leqslant p \leqslant p_\kappa$. Next, define

$$Rf(x) := c_\kappa \int_B f(y) E_\kappa(-ix, y) h_\kappa^2(y)\, dy, \quad x \in \mathbb{S}^{d-1}, \quad f \in L^1(B; h_\kappa^2),$$

and

$$R^* f(x) = c_\kappa \int_{\mathbb{S}^{d-1}} f(y) E_\kappa(ix, y) h_\kappa^2(y)\, d\sigma(y), \quad x \in B, \quad f \in L^1(h_\kappa^2; \mathbb{S}^{d-1}).$$

Namely, $Rf = \mathcal{F}_\kappa(f\chi_B)\Big|_{\mathbb{S}^{d-1}}$ and $R^* f = (f d\sigma)^\vee\Big|_B$. A straightforward calculation shows that

$$\langle Rf, g\rangle_{L^2(\mathbb{S}^{d-1}; h_\kappa^2)} = \langle f, R^* g\rangle_{L^2(B; h_\kappa^2)}, \ \forall f \in L^1(B; h_\kappa^2), \ g \in L^1(\mathbb{S}^{d-1}; h_\kappa^2). \tag{7.2.13}$$

We further claim that

$$R^* R f(x) = c_\kappa' T f(x), \quad x \in B, \quad f \in L^1(B; h_\kappa^2), \tag{7.2.14}$$

where c_κ' is a positive constant depending only on d and κ. Indeed, for $f \in L^1(B; h_\kappa^2)$ and $x \in B$,

$$\begin{aligned} R^* R f(x) &= c_\kappa \int_{\mathbb{S}^{d-1}} Rf(z) E_\kappa(ix, z) h_\kappa^2(z)\, d\sigma(z) \\ &= c_\kappa^2 \int_B f(y) h_\kappa^2(y) \Big[\int_{\mathbb{S}^{d-1}} E_\kappa(x, iz) E_\kappa(-y, iz) h_\kappa^2(z)\, d\sigma(z)\Big]\, dy. \end{aligned}$$

However, it is known that (see, for instance, [11, p.77])

$$\int_{\mathbb{S}^{d-1}} E_\kappa(x, iz)E_\kappa(-y, iz)h_\kappa^2(z)\, d\sigma(z)$$
$$= c''_\kappa V_\kappa\Big[j_{\lambda_\kappa-\frac{1}{2}}(\sqrt{\|x\|^2+\|y\|^2-2\langle x,\cdot\rangle})\Big](y) = c''_\kappa T^y(\widehat{d\sigma})(x), \quad x, y \in \mathbb{R}^d.$$

Thus, it follows that

$$R^*Rf(x) = c'_\kappa \int_B f(y)T^y(\widehat{d\sigma})(x)h_\kappa^2(y)\, dy = c'_\kappa Tf(x), \quad x \in B,$$

which proves the claim (7.2.14).

Now using (7.2.12),(7.2.13), (7.2.14) and a standard duality argument, we obtain that for $1 \leqslant p \leqslant p_\kappa$,

$$\|R\|^2_{L^p(B;h_\kappa^2)\to L^2(\mathbb{S}^{d-1};h_\kappa^2)} = \|R^*\|^2_{L^2(\mathbb{S}^{d-1};h_\kappa^2)\to L^{p'}(B;h_\kappa^2)}$$
$$= c'_\kappa\|T\|_{L^p(B;h_\kappa^2)\to L^{p'}(B;h_\kappa^2)} < \infty,$$

which yields the assertions stated in the corollary. □

7.2.1 Proof of Theorem 7.2.1

We write $\omega = (\omega_1, \cdots, \omega_d)$. Set $I = \{j : \ |\omega_j| \leqslant 4\theta\}$ and $I' = \{1, \cdots, d\} \setminus I$. Let $\gamma = \gamma_B := \sum_{j\in I}\kappa_j$. We consider the following two cases:

Case 1: $\gamma = |\kappa|$

In this case, $\kappa_j = 0$ whenever $|\omega_j| > 4\theta$. Thus,

$$\int_B h_\kappa^2(y)\,dy \sim \theta^d \prod_{j=1}^{d}(|\omega_j| + \theta)^{2\kappa_j} \sim \theta^{2\lambda_\kappa+1}.$$

which implies that

$$\Big(\frac{1}{\theta^{2\lambda_\kappa+1}}\int_B h_\kappa^2(y)\,dy\Big)^{1-\frac{2}{p}} \sim 1.$$

Thus, the stated estimate in this case follows directly from Theorem 7.1.1, the global restriction theorem, which is proved in the last section.

Case 2: $\gamma < |\kappa|$.

In this case, there exists $1 \leqslant j \leqslant d$ such that $|\omega_j| \geq 4\theta$ and $\kappa_j > 0$. The proof in this case is more involved. Our goal is to show the estimate (7.2.12) with $Tf := \Big((f\chi_B) *_\kappa \widehat{d\sigma}\Big)\chi_B$.

Let ξ_0 be an even C^∞-function on $\mathbb{R}$ that equals 1 on $[-1, 1]$ and equals zero outside the interval $[-2, 2]$. Let $\xi(x) = \xi_0(x) - \xi_0(2x)$. Define $\xi_j(x) = \xi(2^{-j}x) = \xi_0(2^{-j}x) - \xi_0(2^{-j+1}x)$ for $j \geq 1$ and $x \in \mathbb{R}$. Then $\sum_{j=0}^{\infty} \xi_j(x) = 1$ for all $x \in \mathbb{R}$.

Recall that

$$Tf(x) = c_\kappa \int_B f(y)K(x, y)h_\kappa^2(y)\,dy, \quad x \in B,$$

where

$$K(x, y) = T^y(\mathcal{F}_\kappa(d\sigma))(x) = c_\kappa' T^y\Big[j_{\lambda_\kappa-\frac{1}{2}}(\|\cdot\|)\Big](x).$$

Thus, we may decompose the operator T as $T = \sum_{n=0}^{\infty} T_n$, where

$$T_n f(x) = \Big[\int_B f(y) K_n(x,y) h_\kappa^2(y)\, dy\Big] \chi_B(x), \tag{7.2.15}$$

and

$$K_n(x,y) = T^y \Big[(j_{\lambda_\kappa - \frac12} \xi_n)(\|\cdot\|)\Big](x). \tag{7.2.16}$$

First, we show that

$$\|T_n f\|_\infty \leqslant C 2^{-n(\frac{d-1}{2}+\gamma)} \theta^{2\gamma+d} \Big(\int_B h_\kappa^2(y)\, dy\Big)^{-1} \|f\|_{\kappa,1}. \tag{7.2.17}$$

To this end, we need the following kernel estimates:

Lemma 7.2.3. *For $\alpha > -1$ and $n = 0, 1, \cdots$, set*

$$K_{\alpha,n}(x,y) := T^y \Big[(j_\alpha \xi_n)(\|\cdot\|)\Big](x), \quad x, y \in \mathbb{R}^d.$$

Then for $x, y \in \mathbb{R}^d$,

$$|K_{\alpha,n}(x,y)| \leqslant C 2^{-n(\alpha + \frac12 - |\kappa|)} \prod_{j=1}^d (|x_j y_j| + 2^n)^{-\kappa_j}. \tag{7.2.18}$$

The proof of Lemma 7.2.3 is long, so we postpone it until the next subsection. For the moment, we take it for granted and proceed with the proof of (7.2.17).

To show (7.2.17), we note that $|y_j| \sim |\omega_j|$ for $j \in I'$ whenever $y \in B$. Thus, using

Lemma 7.2.3 with $\alpha = \lambda_\kappa - \frac{1}{2}$, we obtain that for $x, y \in B$,

$$|K_n(x,y)| \leqslant C 2^{-n(\frac{d-1}{2})} \prod_{j=1}^{d} (|x_j y_j| + 2^n)^{-\kappa_j} \leqslant C 2^{-n\frac{d-1}{2}} \Big[\prod_{j\in I'} (|\omega_j|^2 + \theta^2)^{-\kappa_j}\Big] \Big(\prod_{j\in I} 2^{-n\kappa_j}\Big)$$
$$\leqslant C 2^{-n(\frac{d-1}{2}+\gamma)} \theta^{2\gamma+d} \Big(\int_B h_\kappa^2(z)\, dz\Big)^{-1}.$$

(7.2.17) then follows by (7.2.15).

Next, we show that for $n \geq 0$,

$$\|T_n f\|_{\kappa,2} \leqslant C 2^n \|f\|_{\kappa,2}. \tag{7.2.19}$$

To this end, we write

$$T_n f(x) = \Big[(f\chi_B) *_\kappa G_n\Big]\chi_B,$$

where

$$G_n(x) = c j_{\lambda_\kappa - \frac{1}{2}}(\|x\|)\xi_n(\|x\|) = \widehat{d\sigma}(x)\xi_n(\|x\|).$$

Let ψ be a radial Schwartz function on $\mathbb{R}^d$ such that $\widehat{\psi_{2^{-n}}}(x) = \xi_n(x)$, where $\psi_{2^{-n}}(x) := 2^{n(2\lambda_\kappa+1)}\psi(2^n x)$. Then

$$\mathcal{F}_\kappa G_n(x) = c_\kappa \int_{\mathbb{S}^{d-1}} T^y(\psi_{2^{-n}})(x) h_\kappa^2(y)\, d\sigma(y). \tag{7.2.20}$$

The proof of (7.2.19) relies the following lemma, which gives an estimate of this last integral.

Lemma 7.2.4. *Assume that $\varphi(x) = \varphi_0(\|x\|)$ is a radial Schwartz function on $\mathbb{R}^d$, and* *let* $\varphi_{2^{-n}}(x) = 2^{n(2\lambda_\kappa+1)}\varphi(2^n x)$ *for* $n \in \mathbb{N}$. *Then for a.e.* $x \in \mathbb{R}^d$,

$$\Big|\int_{\mathbb{S}^{d-1}} \Big[T^y \varphi_{2^{-n}}(x)\Big] h_\kappa^2(y)\, d\sigma(y)\Big| \leqslant C2^n.$$

The proof of Lemma 7.2.4 will be given in Section.

By (7.2.20) and Lemma 7.2.4, it follows that for a.e. $x \in \mathbb{R}^d$,

$$|\mathcal{F}_\kappa G_n(x)| = c_\kappa \Big|\int_{\mathbb{S}^{d-1}} T^y \psi_{2^{-n}}(x) h_\kappa^2(y)\, d\sigma(y)\Big| \leqslant C2^n.$$

Thus,

$$\|T_n f\|_{\kappa,2} \leqslant \|\widehat{f\chi_B}\widehat{G_n}\|_{\kappa,2} \leqslant C2^n \|f\|_{\kappa,2}.$$

On one hand, using (7.2.17), (7.2.19) and the Riesz-Thorin interpolation theorem, we obtain that

$$\|T_n f\|_{\kappa,p'} \leqslant C2^{-n\left((\frac{d+1}{2}+\gamma)t-1\right)}\theta^{(2\gamma+d)t}A^{-t}\|f\|_{\kappa,p}, \tag{7.2.21}$$

where $A = \int_B h_\kappa^2(y)\, dy$, $t = \frac{1}{1+\lambda_\kappa} = \frac{2}{p} - 1$ and $p = p_\kappa = \frac{2+2\lambda_\kappa}{\lambda_\kappa+2}$.

On the other hand, using (7.2.17) and Hölder's inequality, we obtain that

$$\begin{aligned}\|T_n f\|_{\kappa,p'} &\leqslant A^{\frac{1}{p'}}\|T_n f\|_\infty \leqslant C2^{-n(\frac{d-1}{2}+\gamma)}\theta^{2\gamma+d}A^{-\frac{1}{p}}\|f\|_{\kappa,1}\\ &\leqslant C2^{-n(\frac{d-1}{2}+\gamma)}\theta^{2\gamma+d}A^{1-\frac{2}{p}}\|f\|_{\kappa,p}.\end{aligned} \tag{7.2.22}$$

Finally, recalling that $Tf = \sum_{n=0}^{\infty} T_n f$, we obtain

$$\begin{aligned}\|Tf\|_{\kappa,p'} &\leqslant \sum_{n=0}^{\infty} \|T_n f\|_{\kappa,p'} = \sum_{2^n \leqslant \theta^2} \cdots + \sum_{2^n > \theta^2} \cdots \\ &=: \Sigma_1 + \Sigma_2.\end{aligned}$$

For the first sum Σ_1, noticing that

$$1 - (\frac{d+1}{2} + \gamma)t = \frac{1}{1+\lambda_\kappa}(|\kappa| - \gamma) > 0,$$

we use (7.2.21) to obtain

$$\begin{aligned}\Sigma_1 &\leqslant C\theta^{(2\gamma+d)t} A^{-t} \|f\|_{\kappa,p} \sum_{2^n \leqslant \theta^2} 2^{n\left(-(\frac{d+1}{2}+\gamma)t+1\right)} \\ &\leqslant C\theta^{\frac{2}{1+\lambda_\kappa}(|\kappa|-\gamma)} \theta^{\frac{2\gamma+d}{1+\lambda_\kappa}} A^{-\frac{1}{1+\lambda_\kappa}} \|f\|_{\kappa,p} = C\theta^{\frac{2\lambda_\kappa+1}{1+\lambda_\kappa}} A^{1-\frac{2}{p}} \|f\|_{\kappa,p}.\end{aligned}$$

For the second sum Σ_2, we use (7.2.22) and obtain

$$\begin{aligned}\Sigma_2 &\leqslant C \sum_{2^n > \theta^2} 2^{-n(\frac{d-1}{2}+\gamma)} \theta^{2\gamma+d} A^{1-\frac{2}{p}} \|f\|_{\kappa,p} \\ &\leqslant C\theta A^{1-\frac{2}{p}} \|f\|_{\kappa,p} \leqslant C\theta^{\frac{2\lambda_\kappa+1}{1+\lambda_\kappa}} A^{1-\frac{2}{p}} \|f\|_{\kappa,p},\end{aligned}$$

where the last step uses the assumption $\theta \geq c_0 > 0$. This completes the proof of Theorem 7.2.1.

7.2.2 Proof of Lemma 7.2.3

Let η denote either the function ξ_0 or the function ξ on $\mathbb{R}$ depending on whether $n = 0$ or $n \geq 1$. Then η is an even C_c^∞-function on $\mathbb{R}$ which is constant near the origin. According to (2.2.3) and (5.1.4), we have

$$K_{\alpha,n}(x,y) = c \int_{[-1,1]^d} j_\alpha(z(x,y,t))\eta(2^{-n}z(x,y,t)) \prod_{j=1}^d (1-t_j^2)^{\kappa_j-1}(1+t_j)\, dt_j, \quad (7.2.23)$$

where

$$z(x,y,t) = \sqrt{\|x\|^2 + \|y\|^2 - 2\sum_{j=1}^d x_j y_j t_j}.$$

Next, let $G_\alpha(u) = (\sqrt{u})^{-\alpha} J_\alpha(\sqrt{u}) = j_\alpha(\sqrt{u})$. Fix $x, y \in \mathbb{R}^d$ and set $F_\alpha(t) = G_\alpha\Big(u(x,y,t)\Big) = j_\alpha(z(x,y,t))$, where $u(x,y,t) = z(x,y,t)^2$ and $t = (t_1, \cdots, t_d) \in [-1,1]^d$. By (2.5.21) and (2.5.22), it is easily seen that for $\alpha \in \mathbb{R}$,

$$\frac{\partial}{\partial t_j} F_{\alpha-1}(t) = x_j y_j F_\alpha(t), \quad t = (t_1, \cdots, t_d) \in [-1,1]^d, \quad (7.2.24)$$

and

$$|F_\alpha(t)| \leqslant C(1 + u(x,y,t))^{-\frac{\alpha}{2}-\frac{1}{4}}, \quad t \in [-1,1]^d. \quad (7.2.25)$$

By (7.2.23), we may write

$$K_{\alpha,n}(x,y) = c_\kappa \int_{[-1,1]^d} F_\alpha(t)\widetilde{\eta}\left(\frac{u(x,y,t)}{4^n}\right) \prod_{j=1}^d (1-t_j^2)^{\kappa_j-1}(1+t_j)\, dt_j,$$

where $\widetilde{\eta}(z) = \eta(\sqrt{|z|})$ for $z \in \mathbb{R}$. Since η is constant near the origin, it is easily seen that $\widetilde{\eta} \in C_c^\infty(\mathbb{R})$. Without loss of generality, we may assume that $|x_j y_j| \geq 2^n$, $j = 1, \cdots, m$ and $|x_j y_j| < 2^n$, $j = m+1, \cdots, d$ for some $1 \leqslant m \leqslant d$ (otherwise, we re-index the sequence $\{x_j y_j\}_{j=1}^d$). Fix temporarily $t_{m+1}, \cdots, t_d \in [-1, 1]$, and set

$$\begin{aligned} &I(t_{m+1}, \cdots, t_d) \\ &:= c_\kappa \int_{[-1,1]^m} F_\alpha(t) \widetilde{\eta}\left(\frac{u(x,y,t)}{4^n}\right) \prod_{j=1}^m (1-t_j^2)^{\kappa_j - 1}(1+t_j)\, dt_j. \end{aligned} \tag{7.2.26}$$

By Fubini's theorem, we then have

$$K_{\alpha,n}(x,y) = \int_{[-1,1]^{d-m}} I(t_{m+1}, \cdots, t_d) \prod_{j=m+1}^d (1-t_j^2)^{\kappa_j - 1}(1+t_j)\, dt_j.$$

Thus, for the proof of (7.2.18), it suffices to show that for each $t_{m+1}, \cdots, t_d \in [-1, 1]$,

$$|I(t_{m+1}, \cdots, t_d)| \leqslant C 2^{-n(\alpha + \frac{1}{2} - \sum_{j=1}^m \kappa_j)} \prod_{j=1}^m |x_j y_j|^{-\kappa_j}. \tag{7.2.27}$$

To show (7.2.27), let $\eta_0 \in C^\infty(\mathbb{R})$ be such that $\eta_0(s) = 1$ for $|s| \leqslant \frac{1}{2}$ and $\eta_0(s) = 0$ for $|s| \geq 1$, and let $\eta_1(s) = 1 - \eta_0(s)$. Set $B_j := \frac{2^n}{|x_j y_j|}$ for $j = 1, \cdots, m$. Given $\varepsilon := (\varepsilon_1, \cdots, \varepsilon_m) \in \{0,1\}^m$, we define $\psi_\varepsilon : \ [-1,1]^m \to \mathbb{R}$ by

$$\psi_\varepsilon(t) := \widetilde{\eta}\left(\frac{u(x,y,t)}{4^n}\right) \prod_{j=1}^m \eta_{\varepsilon_j}\Big(\frac{1-t_j^2}{B_j}\Big)(1+t_j)(1-t_j^2)^{\kappa_j - 1},$$

where $t = (t_1, \cdots, t_m)$. We then split the integral in (7.2.26) into a finite sum to obtain

$$I(t_{m+1}, \cdots, t_d) = \sum_{\varepsilon \in \{0,1\}^m} \int_{[-1,1]^m} F_\alpha(t) \psi_\varepsilon(t)\, dt_1 \cdots dt_m =: \sum_{\varepsilon \in \{0,1\}^m} J_\varepsilon,$$

where

$$J_\varepsilon \equiv J_\varepsilon(t_{m+1}, \cdots, t_d) := \int_{[-1,1]^m} F_\alpha(t) \psi_\varepsilon(t)\, dt_1 \cdots dt_m. \tag{7.2.28}$$

Thus, it suffices to prove the estimate (7.2.27) with $I(t_{m+1}, \cdots, t_d)$ replaced by J_ε for each $\varepsilon \in \{0,1\}^m$, namely,

$$|J_\varepsilon(t_{m+1}, \cdots, t_d)| \leqslant C 2^{-n(\alpha + \frac{1}{2} - \sum_{j=1}^m \kappa_j)} \prod_{j=1}^m |x_j y_j|^{-\kappa_j}. \tag{7.2.29}$$

By symmetry and Fubini's theorem, we need only to prove (7.2.29) for the case of $\varepsilon_1 = \cdots = \varepsilon_{m_1} = 0$ and $\varepsilon_{m_1+1} = \cdots = \varepsilon_m = 1$ with m_1 being an integer in $[0, m]$. Write

$$\psi_\varepsilon(t) = \varphi(t) \prod_{j=1}^{m_1} \eta_0\Big(\frac{1 - t_j^2}{B_j}\Big)(1 + t_j)(1 - t_j^2)^{\kappa_j - 1} \tag{7.2.30}$$

with

$$\varphi(t) := \widetilde{\eta}\left(\frac{u(x, y, t)}{4^n}\right) \prod_{j=m_1+1}^{m} \eta_1\Big(\frac{1 - t_j^2}{B_j}\Big)(1 + t_j)(1 - t_j^2)^{\kappa_j - 1}.$$

Since the support set of each $\eta_1\left(\frac{1-t_j^2}{B_j}\right)$ is a subset of $\{t_j : |t_j| \leqslant 1 - \frac{1}{4}B_j\}$, we can use

(7.2.24) and integration by parts $|\mathbf{l}| = \sum_{j=m_1+1}^{m} \ell_j$ times to obtain

$$
\begin{aligned}
&\Big|\int_{[-1,1]^{m-m_1}} F_\alpha(t)\varphi(t)\,dt_{m_1+1}\cdots dt_m\Big| \\
&= c\prod_{j=m_1+1}^{m}|x_jy_j|^{-\ell_j}\Big|\int_{[-1,1]^{m-m_1}} F_{\alpha-|\mathbf{l}|}(t)\frac{\partial^{|\mathbf{l}|}\varphi(t)}{\partial^{\ell_{m_1+1}}t_{m_1+1}\cdots\partial^{\ell_m}t_m}\,dt\Big| \\
&\leqslant c\prod_{j=m_1+1}^{m}|x_jy_j|^{-\ell_j}\int_{[-1,1]^{m-m_1}}\Big|F_{\alpha-|\mathbf{l}|}(t)\Big|\Big|\frac{\partial^{|\mathbf{l}|}\varphi(t)}{\partial^{\ell_{m_1+1}}t_{m_1+1}\cdots\partial^{\ell_m}t_m}\Big|\,dt,
\end{aligned}
$$

where $\mathbf{l} = (\ell_{m_1+1},\cdots,\ell_m)\in\mathbb{N}^{m-m_1}$ satisfies $\ell_j > \kappa_j$ for all $m_1 < j \leqslant m$. Since $\widetilde{\eta}$ is supported in $(-4,4)$, $\varphi(t)$ is zero unless

$$
\begin{aligned}
4^{n+1} &\geq \|x\|^2 + \|y\|^2 - 2\sum_{j=1}^{d}|x_jy_jt_j| \qquad (7.2.31)\\
&\geq \|\bar{x}-\bar{y}\|^2 + 2|x_jy_j|(1-|t_j|) \geq 2|x_jy_j|(1-|t_j|),
\end{aligned}
$$

for all $m_1+1\leqslant j\leqslant m$; that is, $\frac{|x_jy_j|}{4^n}\leqslant 2(1-|t_j|)^{-1}$ for $j = m_1+1,\cdots,m$. On the other hand, note that the derivative of the function $\eta_1\Big(\frac{1-t_j^2}{B_j}\Big)$ in the variable t_j is supported in $\{t_j:\ \ \frac{1}{2}B_j\leqslant 1-t_j^2\leqslant B_j\}$. Consequently, by the Lebnitz rule, we conclude

$$
\Big|\frac{\partial^{|\mathbf{l}|}\varphi(t)}{\partial^{\ell_{m_1+1}}t_{m_1+1}\cdots\partial^{\ell_m}t_m}\Big| \leqslant c\prod_{j=m_1+1}^{m}(1-|t_j|)^{\kappa_j-\ell_j-1}.
$$

Finally, recall that $\widetilde{\eta}\in C_c^\infty(\mathbb{R})$ for all $n\geq 0$, and that $\widetilde{\eta}$ is zero near the origin for $n\geq 1$. This implies that for all $n\geq 0$, $\widetilde{\eta}\left(\frac{u(x,y,t)}{4^n}\right) = 0$ unless $c_14^n < 1+u(x,y,t) < c_24^n$ for some absolute constants $c_1, c_2 > 0$. It then follows by (7.2.25) that

$$\left|F_{\alpha-|\mathbf{l}|}(t)\right| \leqslant c(1+u(x,y,t))^{-\frac{\alpha-|\mathbf{l}|}{2}-\frac{1}{4}} \sim 2^{-n(\alpha-|\mathbf{l}|+\frac{1}{2})}.$$

Thus,

$$\begin{aligned}
&\int_{[-1,1]^{m-m_1}} \left|F_{\alpha-|\mathbf{l}|}(t)\right|\left|\frac{\partial^{|\mathbf{l}|}\varphi(t)}{\partial^{\ell_{m_1+1}}t_{m_1+1}\cdots\partial^{\ell_m}t_m}\right| dt_{m_1+1}\cdots dt_m \\
&\quad \leqslant c2^{-n(\alpha-|\mathbf{l}|+\frac{1}{2})}\prod_{j=m_1+1}^{m}\int_0^{1-\frac{B_j}{4}}(1-t_j)^{\kappa_j-\ell_j-1}\,dt_j \\
&\quad \leqslant c2^{-n(\alpha-|\mathbf{l}|+\frac{1}{2})}\prod_{j=m_1+1}^{m} B_j^{\kappa_j-\ell_j} \\
&\quad \leqslant c2^{-n(\alpha+\frac{1}{2}-\sum_{j=m_1+1}^{m}\kappa_j)}\prod_{j=m_1+1}^{m}|x_jy_j|^{\ell_j-\kappa_j}. \qquad (7.2.32)
\end{aligned}$$

Thus, using (7.2.28) and Fubini's theorem, we obtain that

$$\begin{aligned}
|J_\varepsilon| &\leqslant \int_{[-1,1]^{m_1}}\left|\int_{[-1,1]^{m-m_1}} F_\alpha(t)\varphi(t)\,dt_{m_1+1}\cdots dt_m\right| \\
&\qquad \times\prod_{j=1}^{m_1}\eta_0\left(\frac{1-t_j^2}{B_j}\right)(1+t_j)(1-t_j^2)^{\kappa_j-1}dt_j \\
&\leqslant c2^{-n(\alpha+\frac{1}{2}-\sum_{j=m_1+1}^{m}\kappa_j)}\prod_{j=m_1+1}^{m}|x_jy_j|^{-\kappa_j}\prod_{j=1}^{m_1}\int_{1-B_j\leqslant|t_j|\leqslant 1}(1-|t_j|)^{\kappa_j-1}\,dt_j \\
&\leqslant c2^{-n(\alpha+\frac{1}{2}-\sum_{j=1}^{m}\kappa_j)}\prod_{j=1}^{m}|x_jy_j|^{-\kappa_j},
\end{aligned}$$

where we used (7.2.32) and the fact that $\eta_0\left(\frac{1-t_j^2}{B_j}\right)$ is supported in $\{t_j :\ 1-B_j\leqslant|t_j|\leqslant 1\}$ for $1\leqslant j\leqslant m_1$ in the second step. This yields the desired estimate (7.2.29) and hence completes the proof of Lemma 7.2.3.

We conclude this subsection with the following useful corollary.

Corollary 7.2.5. *For $\alpha > |\kappa| - \frac{1}{2}$ and a.e. $x, y \in \mathbb{R}^d$,*

$$\left| T^y \big(j_\alpha(\|\cdot\|) \big)(x) \right| \leqslant C \frac{\prod_{j=1}^d (|x_j y_j| + 1 + \|\bar{x} - \bar{y}\|)^{-\kappa_j}}{(1 + \|\bar{x} - \bar{y}\|)^{\alpha + \frac{1}{2} - |\kappa|}}.$$

Proof. Set $K_\alpha(x, y) = T^y \Big[j_\alpha(\|\cdot\|) \Big](x)$. We then write

$$K_\alpha(x, y) = \sum_{n=0}^{\infty} T^y \Big[(j_\alpha \xi_n)(\|\cdot\|) \Big](x) =: \sum_{n=0}^{\infty} K_{\alpha,n}(x, y). \tag{7.2.33}$$

It is easily seen that $K_{\alpha,n}(x, y)$ is supported in $\{(x, y) : \ \|\bar{x} - \bar{y}\| \leqslant 2^{n+1}\}$. Thus, by (7.2.33) and (7.2.18), it follows that

$$\begin{aligned} |K_\alpha(x, y)| &\leqslant C \sum_{2^{n+1} \geq \max\{\|\bar{x} - \bar{y}\|, 1\}} 2^{-n(\alpha + \frac{1}{2} - |\kappa|)} \prod_{j=1}^d (|x_j y_j| + 2^n)^{-\kappa_j} \\ &\leqslant C (1 + \|\bar{x} - \bar{y}\|)^{-(\alpha + \frac{1}{2} - |\kappa|)} \prod_{j=1}^d (1 + \|\bar{x} - \bar{y}\| + |x_j y_j|)^{-\kappa_j}, \end{aligned}$$

where the last step uses the assumption that $\alpha > |\kappa| - \frac{1}{2}$.

□

7.2.3 Proof of Lemma 7.2.4

For the proof of Lemma 7.2.4, we need an additional lemma:

Lemma 7.2.6. *Assume that $\varphi(x) = \varphi_0(\|x\|)$ is a radial Schwartz function on $\mathbb{R}^d$, and* *let* $\varphi_t(x) = t^{-2\lambda_\kappa-1}\varphi(t^{-1}x)$ *for* $t > 0$*. Then for a.e.* $x, y \in \mathbb{R}^d$*, any* $t > 0$ *and* $\ell > 0$*,*

$$|T^y\varphi_t(x)| \leqslant \frac{C}{\big(1 + t^{-1}\big\|\bar{x} - \bar{y}\big\|\big)^\ell \operatorname{meas}_\kappa(B(y,t))}.$$

Proof. Clearly, it is enough to show that for any $\ell > 0$,

$$|T^y\varphi_t(x)| \leqslant C\frac{t^{-d}\prod_{j=1}^d(|y_j| + t)^{-2\kappa_j}}{\big(1 + t^{-1}\big\|\bar{x} - \bar{y}\big\|\big)^\ell}, \tag{7.2.34}$$

where $\varphi(x) = \varphi_0(\|x\|)$ is a radial Schwartz function on $\mathbb{R}^d$, $\varphi_t(x) = t^{-2\lambda_\kappa-1}\varphi(t^{-1}x)$ for $t > 0$.

Note that for $t > 0$,

$$\begin{aligned}(T^y\varphi_t)(x) &= t^{-2\lambda_\kappa-1}V_\kappa\Big[\varphi\big(\sqrt{\|t^{-1}x\|^2 + \|t^{-1}y\|^2 - 2\langle t^{-1}y, \cdot\rangle}\big)\Big](t^{-1}x) \\ &= t^{-2\lambda_\kappa-1}\Big(T^{t^{-1}y}\varphi\Big)(t^{-1}x). \end{aligned} \tag{7.2.35}$$

Thus, it suffices to show (7.2.34) for $t = 1$.

We claim that for any $\ell > 0$,

$$|T^y\varphi(x)| \leqslant C(1 + \|\bar{x} - \bar{y}\|)^{-\ell}\prod_{j=1}^d(1 + \|\bar{x} - \bar{y}\|^2 + |x_jy_j|)^{-\kappa_j}, \tag{7.2.36}$$

which will imply (7.2.34) for $t = 1$. Indeed,

$$
\begin{aligned}
|T^y\varphi(x)| &= \Big|V_\kappa\Big[\varphi(\sqrt{\|x\|^2+\|y\|^2-2\langle y,\cdot\rangle})\Big](x)\Big| \\
&= c_\kappa\Big|\int_{[-1,1]^d}\varphi\Big(\sqrt{\|x\|^2+\|y\|^2-2\sum_{j=1}^d x_jy_jt_j}\Big)\prod_{j=1}^d(1-t_j^2)^{\kappa_j-1}(1+t_j)\,dt_j\Big|.
\end{aligned}
$$

And if any κ_j is equal to 0, the above formula holds under the limits

$$\lim_{\mu\to 0} c_\mu\int_{-1}^1 g(t)(1-t^2)^{\mu-1}dt = \frac{g(1)+g(-1)}{2}.$$

Since $\varphi(x)$ is a radial Schwartz function on $\mathbb{R}^d$, there exists $\ell' > |\kappa|$ such that

$$|T^y\varphi(x)| \leqslant C\int_{[-1,1]^d}\Big(1+\|x\|^2+\|y\|^2-2\sum_{j=1}^d x_jy_jt_j\Big)^{-2\ell'}\prod_{j=1}^d(1-t_j^2)^{\kappa_j-1}(1+t_j)\,dt_j.$$

Since for each fixed $t=(t_1,\cdots,t_d)\in[-1,1]^d$,

$$
\begin{aligned}
\|x\|^2+\|y\|^2-2\sum_{j=1}^d x_jy_jt_j &\geq \|x\|^2+\|y\|^2-2\sum_{j=1}^d |x_jy_j||t_j| = \|\bar{x}-\bar{y}\|^2+2\sum_{j=1}^d(1-|t_j|)|x_jy_j| \\
&\geq \|\bar{x}-\bar{y}\|^2+2\max_{1\leqslant j\leqslant d}|x_jy_j|(1-|t_j|),
\end{aligned}
$$

it follows that

$$
\begin{aligned}
&\Big(1+\|x\|^2+\|y\|^2-2\sum_{j=1}^d x_jy_jt_j\Big)^{-2\ell'} \\
&\leqslant C(1+\|\bar{x}-\bar{y}\|^2)^{-\ell'}\prod_{j=1}^d\Big(1+\|\bar{x}-\bar{y}\|^2+2|x_jy_j|(1-|t_j|)\Big)^{-\ell'}.
\end{aligned}
$$

This implies that

$$|T^y\varphi(x)| \leqslant C(1+\|\bar{x}-\bar{y}\|^2)^{-\ell'}\prod_{j=1}^{d}\int_{-1}^{1}\left[1+\|\bar{x}-\bar{y}\|^2+2|x_jy_j|(1-|t_j|)\right]^{-\ell'}(1-t_j^2)^{\kappa_j-1}(1+t_j)dt_j.$$

If there are some $\kappa_j = 0$, then

$$\lim_{\kappa_j\to 0}\int_{-1}^{1}\left[1+\|\bar{x}-\bar{y}\|^2+2|x_jy_j|(1-|t_j|)\right]^{-\ell'}(1-t_j^2)^{\kappa_j-1}(1+t_j)dt_j = C_\kappa(1+\|\bar{x}-\bar{y}\|^2)^{-\ell'} \leqslant C_\kappa.$$

If $\kappa_j \neq 0$, letting $s = 1-|t_j|$, we get that

$$|T^y\varphi(x)| \leqslant C(1+\|\bar{x}-\bar{y}\|^2)^{-\ell'}\prod_{j=1}^{d}\int_{0}^{1}(1+\|\bar{x}-\bar{y}\|^2+2|x_jy_j|s)^{-\ell'}s^{\kappa_j-1}\,ds.$$

It is easy to see

$$\int_0^1(1+\|\bar{x}-\bar{y}\|^2+2|x_jy_j|s)^{-\ell'}s^{\kappa_j-1}\,ds = c|x_jy_j|^{-\kappa_j}\int_0^{|x_jy_j|}(1+\|\bar{x}-\bar{y}\|^2+2s)^{-\ell'}s^{\kappa_j-1}\,ds.$$

Case 1. If $|x_jy_j|^{-\kappa_j} > 1$, then $|x_jy_j| < 1$. And

$$\int_0^{|x_jy_j|}(1+\|\bar{x}-\bar{y}\|^2+2s)^{-\ell'}s^{\kappa_j-1}\,ds \leqslant \int_0^{|x_jy_j|}s^{\kappa_j-1}\,ds = C_\kappa|x_jy_j|^{\kappa_j}.$$

Case 2. If $|x_jy_j|^{-\kappa_j} < 1$, then $|x_jy_j| > 1$. And

$$\int_0^{|x_jy_j|}(1+\|\bar{x}-\bar{y}\|^2+2s)^{-\ell'}s^{\kappa_j-1}\,ds \leqslant \int_0^{\infty}(1+s)^{-\ell'}s^{\kappa_j-1}\,ds = C_\kappa.$$

Hence

$$\int_0^1 (1+\|\bar{x}-\bar{y}\|^2+2|x_jy_j|s)^{-\ell'} s^{\kappa_j-1}\,ds = c|x_jy_j|^{-\kappa_j}\int_0^{|x_jy_j|}(1+\|\bar{x}-\bar{y}\|^2+2s)^{-\ell'} s^{\kappa_j-1}\,ds$$
$$\leqslant C\min\Big\{1,|x_jy_j|^{-\kappa_j}\Big\}\leqslant C(1+|x_jy_j|)^{-\kappa_j}.$$

And letting $\ell=\ell'-|\kappa|>0$, we get that

$$\begin{aligned}|T^y\varphi(x)| &\leqslant C(1+\|\bar{x}-\bar{y}\|^2)^{-\ell'}\prod_{j=1}^d(1+|x_jy_j|^{-\kappa_j})\\ &= C(1+\|\bar{x}-\bar{y}\|^2)^{-\ell}\prod_{j=1}^d(1+\|\bar{x}-\bar{y}\|^2)^{-\kappa_j}(1+|x_jy_j|^{-\kappa_j})\\ &\leqslant C(1+\|\bar{x}-\bar{y}\|^2)^{-\ell}\prod_{j=1}^d(1+\|\bar{x}-\bar{y}\|^2+|x_jy_j|)^{-\kappa_j}.\end{aligned}$$

This completes the proof of the inequality (7.2.36).

Then let us prove the inequality (7.2.34) for $t=1$.

Case 1. If $\|\bar{x}-\bar{y}\|\leqslant 2|y_j|$, then $|x_j|\sim|y_j|$. And

$$1+\|\bar{x}-\bar{y}\|^2+|x_jy_j|\sim 1+|y_j|^2\sim(1+|y_j|)^2.$$

Case 2. If $\|\bar{x}-\bar{y}\|>2|y_j|$, then

$$1+\|\bar{x}-\bar{y}\|^2+|x_jy_j|\geqslant C(1+|y_j|^2)\sim(1+|y_j|)^2.$$

Therefore, for all $\ell > 0$,

$$|T^y \varphi(x)| \leqslant C(1 + \|\bar{x} - \bar{y}\|^2)^{-\ell} \prod_{j=1}^{d} (1 + |y_j|)^{-2\kappa_j}.$$

We got the inequality that we desired. $\square$

We are now in a position to show Lemma 7.2.4.

Proof of Lemma 7.2.4. By Lemma 7.2.6, for any $\ell \in \mathbb{N}$ and $x \in \mathbb{R}^d$,

$$\begin{aligned}
&\Big| \int_{\mathbb{S}^{d-1}} \Big[T^y \varphi_{2^{-n}}(x) \Big] h_\kappa^2(y)\, d\sigma(y) \Big| \\
&\leqslant C \sum_{\sigma \in \mathbb{Z}_2^d} \int_{\mathbb{S}^{d-1}} \Big(1 + 2^n \|x\sigma - y\| \Big)^{-\ell} 2^{nd} \Big(\prod_{j=1}^{d} (|y_j| + 2^{-n})^{-2\kappa_j} \Big)\, h_\kappa^2(y)\, d\sigma(y) \\
&\leqslant C \sum_{\sigma \in \mathbb{Z}_2^d} 2^{nd} \int_{\mathbb{S}^{d-1}} \Big(1 + 2^n \|x\sigma - y\| \Big)^{-\ell} d\sigma(y).
\end{aligned}$$

Thus, it is sufficient to show that for a sufficiently large ℓ, (say, $\ell \geq d + 1$), and any $x \in \mathbb{R}^d$,

$$2^{nd} \int_{\mathbb{S}^{d-1}} \Big(1 + 2^n \|x - y\| \Big)^{-\ell} d\sigma(y) \leqslant C 2^n. \tag{7.2.37}$$

Without loss of generality, we may assume that $\frac{1}{2} \leqslant \|x\| \leqslant 2$, since otherwise the desired estimate (7.2.37) holds trivially. Writing $x = \|x\| x'$, we have that for $y \in \mathbb{S}^{d-1}$,

$$\|x - y\|^2 = (\|x\| - 1)^2 + 2\|x\|(1 - \langle x', y \rangle) \geq 1 - \langle x', y \rangle.$$

Thus,

$$2^{nd} \int_{\mathbb{S}^{d-1}} \Big(1 + 2^n \|x - y\|\Big)^{-\ell} d\sigma(y) \leqslant C 2^{nd} \int_{\mathbb{S}^{d-1}} \Big(1 + 4^n (1 - \langle x', y \rangle)\Big)^{-\ell/2} d\sigma(y)$$

$$\leqslant C 2^n.$$

□

Chapter 8

Weighted Littlewood-Paley theory in Dunkl analysis

8.1 Weighted Littlewood-Paley inequality

Given a ball $B = B(x, r)$, we write

$$\widetilde{B} = \{y \in \mathbb{R}^d : \quad \|\bar{x} - \bar{y}\| \leqslant r\}.$$

Recall that

$$M_\kappa f(x) := \sup_B \frac{1}{\text{meas}_\kappa(B)} \int_{\widetilde{B}} |f(y)| h_\kappa^2(y)\, dy,$$

where the supremum is taken over all balls B such that $x \in \widetilde{B}$.

Definition 8.1.1. Let Ψ be a radial Schwartz function such that

$\text{supp}\hat{\Psi} \subseteq \{\xi \in \mathbb{R}^d : \quad \frac{1}{16} \leqslant \|\xi\| \leqslant 16\}$. Let $\Psi_j(x) = 2^{j(2\lambda_\kappa+1)}\Psi(2^j x)$ for $j \in \mathbb{Z}$. Define the square function $L(f)$ by

$$L(f)(x) := \Big(\sum_{j\in\mathbb{Z}} |f *_\kappa \Psi_j(x)|^2\Big)^{\frac{1}{2}}. \tag{8.1.1}$$

The operator $L(f)$ can be viewed as a vector-valued convolution operator

$$Tf(x) = \{\Psi_j *_\kappa f(x)\}_{j=-\infty}^{\infty} = \Big\{\int_{\mathbb{R}^d} f(y) T^y \Psi_j(x) h_\kappa^2(y)\, dy\Big\}_{j=-\infty}^{\infty}.$$

The norm of $L(f)$ is $\|Tf\|_{\ell^2}$.

Lemma 8.1.2. (i) *For $x \neq y \in \mathbb{R}^d$,*

$$\Big\|\{T^y(\Psi_j)(x)\}_{j=-\infty}^{\infty}\Big\|_{\ell^2} \leqslant \frac{C}{\mathrm{meas}_\kappa(B(x, \|\bar{x} - \bar{y}\|))}. \tag{8.1.2}$$

(ii) *If $x \neq y$ and $\|x - z\| \leqslant \frac{1}{2}\|\bar{x} - \bar{y}\|$, then*

$$\Big\|\{T^y(\Psi_j)(z) - T^y(\Psi_j)(x)\}_{j=-\infty}^{\infty}\Big\|_{\ell^2} \leqslant \frac{\|x - z\|}{\|\bar{x} - \bar{y}\|} \cdot \frac{C}{\mathrm{meas}_\kappa(B(x, \|\bar{x} - \bar{y}\|))}. \tag{8.1.3}$$

Proof. (i) By the Lemma 7.2.6,

$$|T^y\Psi_j(x)| = |T^{-x}\Psi_j(-y)| \leqslant C \cdot \frac{2^{dj} \prod_{i=1}^{d} (|x_i| + 2^{-j})^{-2\kappa_i}}{(1 + 2^j\|\bar{x} - \bar{y}\|)^l}.$$

If $\rho := \|\bar{x}-\bar{y}\| < 2^{-j}$, then

$$\sum_{2^j\rho<1} \frac{2^{dj}\prod_{i=1}^{d}(|x_i|+2^{-j})^{-2\kappa_i}}{(1+2^j\|\bar{x}-\bar{y}\|)^l} \leqslant \rho^{-d}\prod_{i=1}^{d}(|x_i|+\rho)^{-2\kappa_i}\sum_{2^j\rho<1}(1+2^j\rho)^l \sim \frac{C}{\mathrm{meas}_\kappa(B(x,\|\bar{x}-\bar{y}\|))}.$$

If $\rho := \|\bar{x}-\bar{y}\| \geqslant 2^{-j}$, let $J = \{j:\ 1\leqslant j\leqslant d,\ |x_j|\geqslant \rho\}$ and $J^c = \{1,2,\cdots,d\}\setminus J$. Then

$$\begin{aligned}
\sum_{2^j\rho\geqslant 1} \frac{2^{dj}\prod_{i=1}^{d}(|x_i|+2^{-j})^{-2\kappa_i}}{(1+2^j\|\bar{x}-\bar{y}\|)^l} &\leqslant \sum_{2^j\rho\geqslant 1} 2^{dj}(2^j\rho)^l\left(\prod_{i\in J}|x_i|^{-2\kappa_i}\right)\left(\prod_{i\in J}(2^{-j})^{-2\kappa_i}\right)\\
&\leqslant C\rho^{-d}\cdot\rho^{2|\kappa|_{J^c}}\prod_{i\in J}(|x_i|+\rho)^{-2\kappa_i}\\
&\leqslant C\rho^{-d}\prod_{i=1}^{d}(|x_i|+\rho)^{-2\kappa_i} \sim \frac{C}{\mathrm{meas}_\kappa(B(x,\|\bar{x}-\bar{y}\|))}.
\end{aligned}$$

Therefore,

$$\Big\|\{T^y(\Psi_j)(x)\}_{j=-\infty}^{\infty}\Big\|_{\ell^2} \leqslant C\Big\|\{T^y(\Psi_j)(x)\}_{j=-\infty}^{\infty}\Big\|_{\ell^1} \leqslant \frac{C}{\mathrm{meas}_\kappa(B(x,\|\bar{x}-\bar{y}\|))}.$$

(ii) Let $\Phi(x) = \Psi(\|x\|)$, then $\Phi\in\mathcal{S}(\mathbb{R})$. Let $u(x,y,t) = \sqrt{\|x\|^2+\|y\|^2-2x_jy_jt_j}$, then for $1\leqslant n\leqslant d$,

$$\frac{\partial}{\partial x_n}T^y(\Psi_0)(x) = c_\kappa\int_{[-1,1]^d}\Phi'\big(u(x,y,t)\big)\cdot\frac{x_n-y_nt_n}{u}\prod_{j=1}^{d}(1-t_j^2)^{\kappa_j-1}(1+t_j)dt_j.$$

Since for each fixed $t = (t_1, \cdots, t_d) \in [-1,1]^d$,

$$u^2(x,y,t) = \sum_{j=1}^{d}(x_j^2 + y_j^2 - 2x_j y_j t_j) \geqslant x_n^2 + y_n^2 - 2x_n y_n t_n \geqslant (x_n - y_n t_n)^2,$$

we get $|x_n - y_n t_n| \leqslant |u(x,y,t)|$. And by Lemma 7.2.6, for all $l > 0$,

$$\begin{aligned} \left|\frac{\partial}{\partial x_n} T^y(\Psi_0)(x)\right| &\leqslant c_\kappa \int_{[-1,1]^d} |\Phi'\big(u(x,y,t)\big)| \prod_{j=1}^{d}(1-t_j^2)^{\kappa_j - 1}(1+t_j)dt_j \\ &\leqslant C(1 + \|\bar{x} - \bar{y}\|)^{-l} \prod_{i=1}^{d}(|y_i| + 1)^{-2\kappa_i}. \end{aligned}$$

And by (7.2.35), we get

$$\left|\frac{\partial}{\partial x_n} T^y(\Psi_j)(x)\right| \leqslant C \cdot 2^{-j(d+1)} \left(1 + \frac{\|\bar{x} - \bar{y}\|}{2^j}\right)^{-l} \prod_{i=1}^{d}(|y_i| + 2^j)^{-2\kappa_i}.$$

Then by the mean value theorem, there exist $\theta \in (0,1)$ such that

$$\begin{aligned} |T^y(\Psi_j)(z) - T^y(\Psi_j)(x)| &\leqslant \|x - z\| \cdot |\nabla T^y(\Psi_j)(\xi)| \\ &\leqslant C \cdot 2^{-j(d+1)} \|x - z\| \left(1 + 2^{-j}\|\bar{\xi} - \bar{y}\|\right)^{-l} \prod_{i=1}^{d}(|y_i| + 2^j)^{-2\kappa_i}, \end{aligned}$$

where $\xi = \theta x + (1-\theta)z$.

Since $\|x - z\| \leqslant \frac{1}{2}\|\bar{x} - \bar{y}\|$,

$$\|\bar{\xi} - \bar{y}\| \geqslant \|\bar{x} - \bar{y}\| - \|\bar{x} - \bar{\xi}\| \geqslant \|\bar{x} - \bar{y}\| - \|x - \xi\| \geqslant \|\bar{x} - \bar{y}\| - \|x - z\| \geqslant \frac{1}{2}\|\bar{x} - \bar{y}\|.$$

Then

$$|T^y(\Psi_j)(z) - T^y(\Psi_j)(x)| \leqslant C\|x - z\| \cdot 2^{-j(d+1)} \left(1 + 2^{-j}\|\bar{x} - \bar{y}\|\right)^{-l} \prod_{i=1}^{d}(|y_i| + 2^j)^{-2\kappa_i}.$$

Let $\rho = \|\bar{x} - \bar{y}\|$ and $I_j(x, y) = 2^{-j(d+1)} (1 + 2^{-j}\rho)^{-l} \prod_{i=1}^{d}(|y_i| + 2^j)^{-2\kappa_i}$.

If $\rho < 2^j$, then

$$\sum_{2^j < \rho} I_j(x, y) \leqslant \rho^{-d-1} \prod_{i=1}^{d}(|y_i| + \rho)^{-2\kappa_i} \sim \frac{C}{\|\bar{x} - \bar{y}\|} \cdot \frac{1}{\operatorname{meas}_\kappa(B(y, \|\bar{x} - \bar{y}\|))}.$$

If $\rho \geqslant 2^j$, let $J = \{j:\ 1 \leqslant j \leqslant d,\ |y_j| \geqslant \rho\}$ and $J^c = \{1, 2, \cdots, d\} \setminus J$. Then

$$\begin{aligned}
\sum_{2^j \geqslant \rho} I_j(x, y) &\leqslant \sum_{2^j \geqslant \rho} 2^{-j(d+1)}(2^{-j}\rho)^{-l} \left(\prod_{i \in J} |y_i|^{-2\kappa_i}\right) \left(\prod_{i \in J^c} 2^{-2j\kappa_i}\right) \\
&\leqslant C\rho^{-d-1} \left(\prod_{i \in J} |y_i|^{-2\kappa_i}\right) \left(\prod_{i \in J^c} \rho^{-2\kappa_i}\right) \leqslant C\rho^{-d-1} \prod_{i=1}^{d}(|y_i| + \rho)^{-2\kappa_i} \\
&\sim \frac{C}{\|\bar{x} - \bar{y}\|} \cdot \frac{1}{\operatorname{meas}_\kappa(B(y, \|\bar{x} - \bar{y}\|))}.
\end{aligned}$$

Since $B(\bar{x}, \|\bar{x} - \bar{y}\|) \subseteq B(\bar{y}, 2\|\bar{x} - \bar{y}\|)$,

$$\begin{aligned}
\operatorname{meas}_\kappa(B(y, \|\bar{x} - \bar{y}\|)) = \operatorname{meas}_\kappa(B(\bar{y}, \|\bar{x} - \bar{y}\|)) &= C_\kappa \operatorname{meas}_\kappa(B(\bar{y}, 2\|\bar{x} - \bar{y}\|)) \\
&\geqslant C_\kappa \operatorname{meas}_\kappa(B(\bar{x}, \|\bar{x} - \bar{y}\|)).
\end{aligned}$$

Therefore,

$$\begin{aligned}\Big\|\{T^y(\Psi_j)(z)-T^y(\Psi_j)(x)\}_{j=-\infty}^{\infty}\Big\|_{\ell^2} &\leqslant C\Big\|\{T^y(\Psi_j)(z)-T^y(\Psi_j)(x)\}_{j=-\infty}^{\infty}\Big\|_{\ell^1}\\ &\leqslant \frac{\|x-z\|}{\|\bar{x}-\bar{y}\|}\cdot\frac{C}{\operatorname{meas}_\kappa(B(x,\|\bar{x}-\bar{y}\|))}.\end{aligned}$$

□

Definition 8.1.3. Let $w(x)$ be a non-negative, locally integrable function on $\mathbb{R}^d$. We say w is an A_p weight for some $1 < p < \infty$, if

$$\sup_{B\subseteq\mathbb{R}^d}\left(\frac{1}{\operatorname{meas}_\kappa(B)}\int_B w(x)d\mu_\kappa(x)\right)\left(\frac{1}{\operatorname{meas}_\kappa(B)}\int_B w(x)^{\frac{1}{1-p}}d\mu_\kappa(x)\right)^{p-1}\leqslant C,$$

where $d\mu_\kappa = h_\kappa^2(x)dx$ and the supremum is taken over all balls $B \subseteq \mathbb{R}^d$. We say that w is an A_1 weight, if

$$\sup_{B\subseteq\mathbb{R}^d}\left(\frac{1}{\operatorname{meas}_\kappa(B)}\int_B w(x)d\mu_\kappa(x)\right)\leqslant w(x),\quad \text{a.e. } x\in B.$$

Theorem 8.1.4. *Suppose $L(f)$ is the square function defined by* (8.1.1), *and w is an A_p weight for some $1 < p < \infty$. If $w(\sigma x) = w(x)$ for all $\sigma \in \mathbb{Z}_2^d$, then*

$$\|L(f)\|_{L^p(w)} \leqslant C\|f\|_{L^p(w)}.$$

Proof. Let ε_j be independent and identically distributed random variables with

$P(\varepsilon_j = \pm 1) = \frac{1}{2}$ for every $|j| \leqslant n$. We define

$$T_n f(x) = \int_{\mathbb{R}^d} K_n(x,y) f(y) w(y) d\mu_\kappa(y),$$

where $K_n(x,y) = \sum_{j=-n}^{n} T^y \Psi_j(x) \cdot \varepsilon_j$.

Let $R_\sigma \subseteq \mathbb{R}^d$ be the subspace such that $\sigma R_\sigma = \mathbb{R}^d_+$ for some $\sigma \in \mathbb{Z}_2^d$. Then

$$\begin{aligned} T_n f(x) &= \sum_{\sigma' \in \mathbb{Z}_2^d} \sum_{\sigma \in \mathbb{Z}_2^d} \int_{R_\sigma} K_n(x,y) f(y) w(y) d\mu_\kappa(y) \cdot \chi_{R_{\sigma'}}(x) \\ &= \sum_{\sigma' \in \mathbb{Z}_2^d} \sum_{\sigma \in \mathbb{Z}_2^d} \int_{\mathbb{R}^d_+} K_n(x, \sigma y) f(\sigma y) w(\sigma y) d\mu_\kappa(y) \cdot \chi_{R_{\sigma'}}(x). \end{aligned}$$

Let $f_\sigma(y) = f(\sigma y)$, then

$$T_n f(x) = \sum_{\sigma' \in \mathbb{Z}_2^d} \sum_{\sigma \in \mathbb{Z}_2^d} \int_{\mathbb{R}^d_+} K_n(\sigma' x, \sigma y) f_\sigma(y) w(y) d\mu_\kappa(y) \cdot \chi_{\mathbb{R}^d_+}(x).$$

For $x \in \mathbb{R}^d_+$, set

$$T_{n,\sigma,\sigma'} f(x) = \int_{\mathbb{R}^d_+} K_n(\sigma' x, \sigma y) f_\sigma(y) w(y) d\mu_\kappa(y).$$

By Plancherel's Theorem,

$$\|T_{n,\sigma,\sigma'} f_\sigma\|_{L^2(w,\mathbb{R}^d_+)} \leqslant C \|f_\sigma\|_{L^2(w,\mathbb{R}^d_+)}.$$

Thus, the Lemma 8.1.2 implies that $T_{n,\sigma,\sigma'}$ are Calderón-Zygmund operators. Hence

$$\|T_{n,\sigma,\sigma'}f_\sigma\|_{L^p(w,\mathbb{R}^d_+)} \leqslant C\|f_\sigma\|_{L^p(w,\mathbb{R}^d_+)}.$$

Then the Minkowski's inequality gives that

$$\|T_n f\|_{L^p(w,\mathbb{R}^d)} \leqslant C\|f_\sigma\|_{L^p(w,\mathbb{R}^d)}.$$

And by Khintchine's inequality and the dominated convergence theorem, we get

$$\|L(f)\|_{L^p(w,\mathbb{R}^d)} \leqslant C\|f\|_{L^p(w,\mathbb{R}^d)}.$$

□

8.2 An important corollary

Lemma 8.2.1. *Let $f \in L^1_{loc}(\mathbb{R}^d; h^2_\kappa)$ be such that $M_\kappa f(x) < \infty$ for a.e. $x \in \mathbb{R}^d$. If $0 < \delta < 1$, then for every ball $B \subset \mathbb{R}^d$,*

$$\frac{1}{\mathrm{meas}_\kappa(B)} \int_{\widetilde{B}} M_\kappa f(y)^\delta h^2_\kappa(y)\,dy \leqslant CM_\kappa f(x)^\delta, \quad \forall x \in \widetilde{B}.$$

Proof. Fix a ball B and decompose f as $f = f_1 + f_2$, where $f_1 = f\chi_{\widetilde{2B}}$. Then for $0 < \delta < 1$,

$$M_\kappa f(y)^\delta \leqslant M_\kappa f_1(y)^\delta + M_\kappa f_2(y)^\delta.$$

Since M_κ is weak $(1,1)$, it follows by Kolmogorov's inequality that

$$\begin{aligned}\frac{1}{\mathrm{meas}_\kappa(B)}\int_{\widetilde{B}}(M_\kappa f_1(z))^\delta h_\kappa^2(z)\,dz &\leqslant C_\delta\Big(\frac{1}{\mathrm{meas}_\kappa(2B)}\int_{\widetilde{2B}}|f(z)|h_\kappa^2(z)\,dz\Big)^\delta\\ &\leqslant CM_\kappa f(x)^\delta, \quad \forall x\in\widetilde{B}.\end{aligned}$$

Next, we claim that

$$M_\kappa f_2(y)\leqslant CM_\kappa f(x), \quad \forall x,y\in\widetilde{B}. \tag{8.2.4}$$

Indeed, fix $x\in\widetilde{B}$ and $y\in\widetilde{B}$, and let B_1 be a ball such that $y\in\widetilde{B_1}$. Since f_2 is supported in $\mathbb{R}^d\setminus\widetilde{2B}$, in order that $\int_{\widetilde{B_1}}|f_2(z)|h_\kappa^2(z)\,dz>0$, one must have that $2\mathrm{rad}(B_1)\geq\mathrm{rad}(B)$, which implies that $x\in\widetilde{B}\subset\widetilde{5B_1}$. Thus,

$$\frac{1}{\mathrm{meas}_\kappa(B_1)}\int_{B_1}|f_2(z)|h_\kappa^2(z)\,dz\leqslant\frac{C}{\mathrm{meas}_\kappa(5B_1)}\int_{\widetilde{5B_1}}|f(z)|h_\kappa^2(z)\,dz\leqslant CM_\kappa f(x).$$

This shows the claim.

Now using (8.2.4), we obtain that

$$\frac{1}{\mathrm{meas}_\kappa(B)}\int_{\widetilde{B}}M_\kappa f_2(y)^\delta h_\kappa^2(y)\,dy\leqslant CM_\kappa f(x)^\delta.$$

□

Lemma 8.2.1 implies that $(M_\kappa)^\delta d\mu_\kappa$ satisfy the A_1-condition for all $\delta\in(0,1)$. And it follows that $(M_\kappa)^\delta d\mu_\kappa$ are A_p weights for all $\delta\in(0,1)$ and $p\in(1,\infty)$. Thus, by the

Theorem 8.1.4, we can get the following corollary.

Corollary 8.2.2. *Let* $g \in L^1_{loc}(\mathbb{R}^d; h^2_\kappa)$ *be such that* $M_\kappa g(x) < \infty$ *for a.e.* $x \in \mathbb{R}^d$, *and* $L(f)$ *be the square function defined by* (8.1.1). *If* $0 < \delta < 1$ *and* $1 < p < \infty$, *then*

$$\|L(f)\|_{L^p(M_\kappa g d\mu_\kappa)} \leqslant C\|f\|_{L^p(M_\kappa g d\mu_\kappa)}.$$

Chapter 9

Strong estimates of the maximal Bochner-Riesz means of the Dunkl transforms

9.1 Main results

The Bochner-Riesz means of f of order $\delta > -1$ in the Dunkl setting are defined by

$$B_R^\delta(h_\kappa^2; f)(x) = c_\kappa \int_{\|y\| \leqslant R} \left(1 - \frac{\|y\|^2}{R^2}\right)^\delta \mathcal{F}_\kappa f(y) E_\kappa(ix, y) h_\kappa^2(y)\, dy, \quad x \in \mathbb{R}^d, \quad R > 0,$$

whereas the maximal Bochner-Riesz operators are defined by

$$B_*^\delta(h_\kappa^2; f)(x) = \sup_{R>0} |B_R^\delta(h_\kappa^2; f)(x)|, \quad x \in \mathbb{R}^d. \tag{9.1.1}$$

Theorem 9.1.1. *Let $\delta_\kappa(p) = (2\lambda_\kappa + 1)(\frac{1}{2} - \frac{1}{p}) - \frac{1}{2}$. If $p \geqslant 2 + \frac{2}{\lambda_\kappa}$ and $\delta > \max\{0, \delta_\kappa(p)\}$, then for all $f \in L^p(\mathbb{R}^d; h_\kappa^2)$,*

$$\|B_*^\delta(h_\kappa^2; f)\|_{\kappa,p} \leqslant C\|f\|_{\kappa,p}.$$

Let $\delta > \varepsilon > 0$. Since

$$\left(1 - \frac{|\xi|^2}{R^2}\right)^\delta = \frac{2\Gamma(\delta)}{\Gamma(\varepsilon + \frac{1}{2})\Gamma(\delta - \varepsilon + \frac{1}{2})} \cdot R^{-2\delta} \int_{|\xi|}^R (R^2 - t^2)^{\delta - \varepsilon - \frac{1}{2}}\ t^{2\varepsilon} \left(1 - \frac{|\xi|^2}{t^2}\right)^{\varepsilon - \frac{1}{2}}\ dt,$$

we get that

$$B_R^\delta(h_\kappa^2; f)(x) = C_{\varepsilon,\delta} R^{-2\delta} \int_0^R (R^2 - t^2)^{\delta - \varepsilon - \frac{1}{2}}\ t^{2\varepsilon}\ B_t^{\varepsilon - \frac{1}{2}}(h_\kappa^2; f)(x)\ dt.$$

By Cauchy-Schwartz inequality,

$$|B_R^\delta(h_\kappa^2; f)(x)| \leqslant C_{\varepsilon,\delta} \left(\int_0^1 [(1 - t^2)^{\delta - \varepsilon - \frac{1}{2}}\ t^{2\varepsilon}]^2\ dt\right)^{\frac{1}{2}} \left(\frac{1}{R}\int_0^R |B_t^{\varepsilon - \frac{1}{2}}(h_\kappa^2; f)(x)|^2\ dt\right)^{\frac{1}{2}} \tag{9.1.2}$$

and the first integral above is bounded under the condition $\delta > \varepsilon > 0$.

Let γ be a C_c^∞ function supported in $[0, \frac{1}{2}]$ such that

$$\sum_{k=1}^\infty \gamma_k(t) = 1, \qquad t \in [0, \frac{1}{2}],$$

where $\gamma_k(t) = \gamma(2^k t)$. And define $\gamma_0(t) = 1 - \sum_{k=1}^\infty \gamma_k(t)$ for $\frac{1}{2} < t \leqslant 1$ and $\gamma_0(t) = 0$

otherwise. Then

$$
\begin{aligned}
(1-|\xi|^2)_+^\delta &= \sum_{k=0}^{\infty}(1-|\xi|^2)^\delta\gamma_k(1-|\xi|^2)\\
&= (1-|\xi|^2)\gamma_0(1-|\xi|^2) + \sum_{k=1}^{\infty} 2^{-k\delta}\left[\Big(2^k(1-|\xi|^2)\Big)^\delta \gamma\Big(2^k(1-|\xi|^2)\Big)\right].
\end{aligned}
$$

Define $\phi^0(\xi) = (1-|\xi|^2)\gamma_0(1-|\xi|^2)$. And for $\lambda \in (0, 1/2]$, define

$$
\phi^\lambda(\xi) = \Big(\frac{1-|\xi|^2}{\lambda}\Big)^\delta \gamma\Big(\frac{1-|\xi|^2}{\lambda}\Big), \qquad \xi \in \mathbb{R}^d.
$$

Clearly,

$$
\operatorname{supp}\phi^\lambda \subset \{\xi : \;\; 1-\frac{\lambda}{2} \leqslant \|\xi\| \leqslant 1+\frac{\lambda}{2}\},
$$

and

$$
|\nabla^\ell \phi^\lambda(\xi)| \leqslant C_\ell \lambda^{-\ell}, \;\; \xi \in \mathbb{R}^d, \;\; \ell = 0, 1, \cdots. \tag{9.1.3}
$$

Set $\phi_t^\lambda(\xi) = \phi^\lambda(\xi/t)$ for $t > 0$, then

$$
\Big(1-\frac{|\xi|^2}{t^2}\Big)_+^\delta = \phi_t^0(\xi) + \sum_{k=1}^{\infty} 2^{-k\delta}\phi_t^{2^{-k}}(\xi). \tag{9.1.4}
$$

It follows that

$$
B_t^{\varepsilon-\frac{1}{2}}(h_\kappa^2; f)(x) = f *_\kappa \widehat{\phi_t^0}(x) + \sum_{k=1}^{\infty} 2^{-k(\varepsilon-\frac{1}{2})} f *_\kappa \widehat{\phi_t^{2^{-k}}}(x).
$$

By triangle inequality and (9.1.2),

$$|B_R^\delta(h_\kappa^2; f)(x)| \leqslant C\left(\frac{1}{R}\int_0^R |f *_\kappa \widehat{\phi_t^0}(x)|^2\, dt\right)^{\frac{1}{2}} + C\sum_{k=1}^\infty 2^{-k(\varepsilon-\frac{1}{2})}\left(\frac{1}{R}\int_0^R |f *_\kappa \widehat{\phi_t^{2^{-k}}}(x)|^2\, dt\right)^{\frac{1}{2}}.$$

Define

$$G_\lambda f(x) = \Big(\int_0^\infty |f *_\kappa \widehat{\phi_t^\lambda}(x)|^2 \frac{dt}{t}\Big)^{1/2}.$$

Then

$$B_*^\delta(h_\kappa^2; f)(x) = \sup_{R>0}|B_R^\delta(h_\kappa^2; f)(x)| \leqslant CM_\kappa f(x) + C\sum_{k=1}^\infty 2^{-k(\varepsilon-\frac{1}{2})}G_{2^{-k}}f(x).$$

Therefore, to show Theorem 9.1.1, we just need to prove for all $p \geqslant p_0 := 2 + \frac{2}{\lambda_\kappa}$,

$$\|G_\lambda f\|_{\kappa,p} \leqslant C\lambda^{\frac{1}{2\lambda_\kappa+2}}\|f\|_{\kappa,p}$$

which is a consequence of the following theorem.

Theorem 9.1.2. *Let $p_0 = 2 + \frac{2}{\lambda_\kappa}$ and $r = (\frac{1}{2}p_0)' = \lambda_\kappa + 1$. Then for any nonnegative function g on $\mathbb{R}^d$,*

$$\int_{\mathbb{R}^d} |G_\lambda f(x)|^2 g(x) h_\kappa^2(x)\, dx \leqslant C\lambda^{\frac{1}{\lambda_\kappa+1}}\int_{\mathbb{R}^d}|f(x)|^2 M_{\kappa,r}(g)(x)h_\kappa^2(x)\, dx,$$

where $M_{\kappa,r}g = (M_\kappa(g^r))^{1/r}$.

9.2 A locality lemma

Denote by $\mathcal{D}_j$ the collection of all dyadic cubes in $\mathbb{R}^d$ with side length 2^j. Let T be a sublinear operator with the following local property: for any function f supported in a cube $Q \in \mathcal{D}_j$, Tf is supported in a fixed dilate $Q^* = c\widetilde{Q}$ of $\widetilde{Q} = \bigcup_{\varepsilon \in \mathbb{Z}_2^d} Q\varepsilon$. By (5.1.3), it is easily seen that if K is a kernel supported in $B(0, c2^j)$, then $Tf = f *_\kappa K$ has the above local property.

Lemma 9.2.1. *Suppose T has the above local property, $p_0 > 2$ and $r = (p_0/2)' = \frac{p_0}{p_0-2}$. Suppose further that for any $Q \in \mathcal{D}_j$, and any function f supported in Q,*

$$\|Tf\|_{\kappa,p_0} \leqslant A\Big(\frac{2^{j(2\lambda_\kappa+1)}}{\mathrm{meas}_\kappa(Q)}\Big)^{\frac{1}{2}-\frac{1}{p_0}} \|f\|_{\kappa,2}.$$

Then for any f defined on $\mathbb{R}^d$ and any testing function $g \geq 0$,

$$\int_{\mathbb{R}^d} |Tf(x)|^2 g(x) h_\kappa^2(x)\, dx \leqslant CA^2 2^{j(2\lambda_\kappa+1)/r} \int_{\mathbb{R}^d} |f(x)|^2 M_{\kappa,r} g(x) h_\kappa^2(x)\, dx. \tag{9.2.5}$$

Proof. First, we show (9.2.5) for f supported in a cube $Q \in \mathcal{D}_j$. Indeed, by the local property of T,

$$\begin{aligned}\int_{\mathbb{R}^d} |Tf(x)|^2 g(x) h_\kappa^2(x)\,dx &= \int_{Q^*} |Tf(x)|^2 g(x) h_\kappa^2(x)\,dx\\ &\leqslant \left\| |Tf|^2 \right\|_{\kappa,\frac{p_0}{2}} \Big(\int_{Q^*} |g(x)|^r h_\kappa^2(x)\,dx\Big)^{1/r}\\ &\leqslant C(\mathrm{meas}_\kappa(Q))^{\frac{1}{r}} \|Tf\|_{\kappa,p_0}^2 \inf_{x\in Q} M_{\kappa,r} g(x)\\ &\leqslant CA^2 2^{j(2\lambda_\kappa+1)/r} \Big(\int_Q |f(x)|^2\, h_\kappa^2(x) dx\Big) \inf_{x\in Q} M_{\kappa,r} g(x)\\ &\leqslant CA^2 2^{j(2\lambda_\kappa+1)/r} \int_Q |f(x)|^2 M_{\kappa,r} g(x) h_\kappa^2(x)\,dx.\end{aligned}$$

Next, we show (9.2.5) for a general f. Write

$$f = \sum_{Q\in\mathcal{D}_j} f\chi_Q = \sum_{Q\in\mathcal{D}_j} f_Q.$$

Since T is sublinear, we have, by the local property of T,

$$|Tf| \leqslant \sum_{Q\in\mathcal{D}_j} |T(f_Q)|\chi_{Q^*},$$

which implies

$$|Tf|^2 \leqslant C \sum_{Q\in\mathcal{D}_j} |T(f_Q)|^2.$$

Thus,

$$\begin{aligned}\int_{\mathbb{R}^d} |Tf(x)|^2 g(x) h_\kappa^2(x)\,dx &\leqslant C \sum_{Q\in\mathcal{D}_j} \int_{\mathbb{R}^d} |T(f_Q)|^2 g(x) h_\kappa^2(x)\,dx \\ &\leqslant CA^2 2^{j(2\lambda_\kappa+1)/r} \sum_{Q\in\mathcal{D}_j} \int_{\mathbb{R}^d} |f_Q|^2 M_{\kappa,r} g(x) h_\kappa^2(x)\,dx \\ &= CA^2 2^{j(2\lambda_\kappa+1)/r} \int_{\mathbb{R}^d} |f|^2 M_{\kappa,r}(g)(x) h_\kappa^2(x)\,dx.\end{aligned}$$

□

Remark 9.2.2. Note that (9.2.5) implies that for $2 < p < p_0$, and $\widetilde{r} = (p/2)' > r$,

$$\begin{aligned}\|Tf\|_{\kappa,p}^2 = \||Tf|^2\|_{\kappa,p/2} &= \sup_{\|g\|_{\kappa,\widetilde{r}}\leqslant 1} \int_{\mathbb{R}^d} |Tf(x)|^2 g(x) h_\kappa^2(x)\,dx \\ &\leqslant CA^2 2^{j(2\lambda_\kappa+1)/r} \sup_{\|g\|_{\kappa,\widetilde{r}}\leqslant 1} \int_{\mathbb{R}^d} |f(x)|^2 M_{\kappa,r} g(x) h_\kappa^2(x)\,dx \\ &\leqslant CA^2 2^{j(2\lambda_\kappa+1)/r} \||f|^2\|_{\kappa,p/2} \sup_{\|g\|_{\kappa,\widetilde{r}}\leqslant 1} \|M_{\kappa,r} g\|_{\kappa,\widetilde{r}} \\ &\leqslant CA^2 2^{j(2\lambda_\kappa+1)/r} \|f\|_{\kappa,p}^2.\end{aligned}$$

9.3 A pointwise kernel estimate

Assume that $2^{-i-1} \leqslant \lambda < 2^{-i}$ for some $i \in \mathbb{N}$. Let $\eta \in C_c^\infty(\mathbb{R}^d)$ be a radial function such that $\eta(x) = 1$ for $\|x\| \leqslant 1$, and $\eta(x) = 0$ for $\|x\| \geq 2$. Set $\eta_i(x) = \eta(2^{-i}x)$ and $\eta_j(x) = \eta(2^{-j}x) - \eta(2^{-j+1}x)$ for $j > i$. Then

$$\sum_{j=i}^{\infty} \eta_j(\xi) = \lim_{j\to\infty} \eta(2^{-j}\xi) = 1, \quad \xi \in \mathbb{R}^d \setminus \{0\}. \tag{9.3.6}$$

Lemma 9.3.1. *For* $\frac{1}{2} \leqslant t \leqslant 4$, $j \geq i$ *and any* $N \in \mathbb{N}$,

$$|\widehat{\eta_j} *_\kappa \phi_t^\lambda(x)| \leqslant \begin{cases} C_N 2^{(i-j)N}\big(1+2^i\big|\|x\|-t\big|\big)^{-N}, & \text{if } \frac{1}{4} \leqslant \|x\| \leqslant 8; \\ C_N 2^{(i-j)N} 2^{-iN}(1+\|x\|)^{-N}, & \text{otherwise,} \end{cases} \tag{9.3.7}$$

where $\phi_t^\lambda(x) = \phi^\lambda(t^{-1}x)$.

The proof of Lemma 9.3.1 is long and technical, so we postpone it until Section 9.5.

9.4 Proof of Theorem 9.1.2

For $2^{-1} \leqslant t \leqslant 4$, write

$$Tf(x,t) = f *_\kappa \widehat{\phi^\lambda}_t(x) = \sum_{j=i}^\infty f *_\kappa (\widehat{\phi^\lambda}_t \eta_j)(x) =: \sum_{j=i}^\infty T_j f(x,t), \quad x \in \mathbb{R}^d, \tag{9.4.8}$$

where $T_j f(x,t) = f *_\kappa (\eta_j \widehat{\phi^\lambda}_t)(x)$. Each T_j will be considered as a vector-valued operator $T_j:\ L^2(\mathbb{R}^d; h_\kappa^2) \to L^{p_0}(L^2[2^{-1},4])$ with

$$\|T_j f\|_{L^{p_0}(L^2[2^{-1},4])} := \Big(\int_{\mathbb{R}^d} \|T_j f(x,\cdot)\|_{L^2[2^{-1},4]}^{p_0} h_\kappa^2(x)\,dx\Big)^{1/p_0},$$

and

$$\|T_j f(x,\cdot)\|_{L^2[2^{-1},4]} := \Big(\int_{2^{-1}}^4 |T_j f(x,t)|^2\,dt\Big)^{\frac{1}{2}}.$$

Note that

$$\Big(T_j f(\cdot,t)\Big)^{\wedge}(\xi) = \widehat{f}(\xi)(\eta_j \widehat{\phi_t^\lambda})^{\wedge}(\xi) = \widehat{f}(\xi)(\widehat{\eta_j} *_\kappa \phi_t^\lambda)(\xi). \tag{9.4.9}$$

Thus, by the Fourier inverse formula, we have that for $f \in \mathcal{S}(\mathbb{R}^d)$,

$$\begin{aligned} T_j f(x,t) &= c_\kappa \int_{\mathbb{R}^d} \widehat{f}(\xi)(\widehat{\eta_j} *_\kappa \phi_t^\lambda)(\xi) E_\kappa(i\xi, x) h_\kappa^2(\xi)\, d\xi \\ &= c \int_0^\infty \int_{\mathbb{S}^{d-1}} \widehat{f}(\rho\xi)(\widehat{\eta_j} *_\kappa \phi_t^\lambda)(\rho\xi) E_\kappa(i\rho\xi, x) h_\kappa^2(\xi)\, d\sigma(\xi) \rho^{2\lambda_\kappa}\, d\rho. \end{aligned} \tag{9.4.10}$$

Recall that $p_0 = 2 + \frac{2}{\lambda_\kappa}$.

Lemma 9.4.1. *Let $B = B(\omega, c2^j)$ denote a ball centered at $\omega \in \mathbb{R}^d$ having radius $c2^j$ for some $c > 0$, and let $\widetilde{B} = \bigcup_{\sigma \in \mathbb{Z}_2^d} B(\sigma(\omega), c2^j)$. Then for $j \geq i \geq 1$,*

$$\Big\| \Big(\int_{2^{-1}}^4 |T_j f(\cdot,t)|^2\, dt \Big)^{1/2} \Big\|_{L^{p_0}(\widetilde{B}, h_\kappa^2)} \leqslant C 2^{-j} \Big(\frac{2^{j(2\lambda_\kappa+1)}}{\mathrm{meas}_\kappa(B)} \Big)^{\frac{1}{2} - \frac{1}{p_0}} \|f\|_{\kappa,2}.$$

Proof. Write $f = f_1 + f_2$, where $\widehat{f_1}(\xi) = \widehat{f}(\xi)\chi_{4^{-1} \leqslant \|\xi\| \leqslant 8}(\xi)$, $\widehat{f_2}(\xi) = \widehat{f}(\xi)\chi_I(\xi)$ and $I := [0, \frac{1}{4}) \bigcup (8, \infty)$. We then reduce to showing that for $k = 1, 2$,

$$\Big\| \Big(\int_{2^{-1}}^4 |T_j f_k(\cdot,t)|^2\, dt \Big)^{1/2} \Big\|_{L^{p_0}(\widetilde{B}, h_\kappa^2)} \leqslant C 2^{-j} \Big(\frac{2^{j(2\lambda_\kappa+1)}}{\mathrm{meas}_\kappa(B)} \Big)^{\frac{1}{2} - \frac{1}{p_0}} \|f_k\|_{\kappa,2}. \tag{9.4.11}$$

First, we show (9.4.11) for $k = 1$. Using (9.4.10) and Minkowski's inequality, we obtain that for $t \in [2^{-1}, 4]$,

$$\|T_j f_1(\cdot,t)\|^2_{L^{p_0}(\widetilde{B},h_\kappa^2)} \leqslant C\Big\|\int_{4^{-1}}^8 \int_{\mathbb{S}^{d-1}} \widehat{f}(\rho\xi)(\widehat{\eta_j} *_\kappa \phi_t^\lambda)(\rho\xi)E_\kappa(i\rho\cdot,\xi)h_\kappa^2(\xi)\,d\sigma(\xi)\,d\rho\Big\|^2_{L^{p_0}(\widetilde{B},h_\kappa^2)}$$
$$\leqslant C\Big(\int_{4^{-1}}^8 \Big\|\int_{\mathbb{S}^{d-1}} \widehat{f}(\rho\xi)(\widehat{\eta_j} *_\kappa \phi_t^\lambda)(\rho\xi)E_\kappa(i\cdot,\xi)h_\kappa^2(\xi)\,d\sigma(\xi)\Big\|_{L^{p_0}(\widetilde{B}_\rho,h_\kappa^2)}\,d\rho\Big)^2,$$

where $\widetilde{B}_\rho = \bigcup_{\sigma\in\mathbb{Z}_2^d} B(\rho\sigma(\omega)\omega, 2^j\rho)$. By the Cauchy-Schwartz inequality, the term on the right hand side of this last inequality is controlled by a constant multiple of

$$2^{-i}\Big(\int_{4^{-1}}^8 \Big\|\int_{\mathbb{S}^{d-1}} \widehat{f}(\rho\xi)(\widehat{\eta_j} *_\kappa \phi_t^\lambda)(\rho\xi)E_\kappa(i\cdot,\xi)h_\kappa^2(\xi)\,d\sigma(\xi)\Big\|^2_{L^{p_0}(\widetilde{B}_\rho,h_\kappa^2)}(1+2^i|\rho-t|)^2\,d\rho\Big),$$

which, using the restriction theorem (Corollary 7.2.2 (ii)), is bounded above by

$$C2^{-i}\Big(\frac{2^{(2\lambda_\kappa+1)j}}{\mathrm{meas}_\kappa(B)}\Big)^{1-\frac{2}{p_0}} \int_{4^{-1}}^8 (1+2^i|\rho-t|)^2 \int_{\mathbb{S}^{d-1}} |\widehat{\eta_j} *_\kappa \phi_t^\lambda(\rho\xi)|^2 |\widehat{f}(\rho\xi)|^2 h_\kappa^2(\xi)\,d\sigma(\xi)\,d\rho.$$

Here we used the fact that for $B_\rho = B(\rho\omega, c2^j\rho)$ and any $\rho > 0$,

$$\frac{(2^j\rho)^{2\lambda_\kappa+1}}{\mathrm{meas}_\kappa(\widetilde{B}_\rho)} \sim \frac{(2^j\rho)^{2\lambda_\kappa+1}}{\mathrm{meas}_\kappa(B_\rho)} \sim \frac{2^{j(2\lambda_\kappa+1)}}{\mathrm{meas}_\kappa(B)} \sim \frac{2^{2j|\kappa|}}{\prod_{n=1}^d(|\omega_n|+2^j)^{2\kappa_n}} \tag{9.4.12}$$

Thus, by Lemma 9.3.1, it follows that for any $t \in [2^{-1}, 4]$,

$$\|T_j f_1(\cdot,t)\|^2_{L^{p_0}(\widetilde{B},h_\kappa^2)}$$
$$\leqslant C2^{-i}4^{i-j}\Big(\frac{2^{(2\lambda_\kappa+1)j}}{\mathrm{meas}_\kappa(B)}\Big)^{1-\frac{2}{p_0}} \int_{4^{-1}}^8 (1+2^i|\rho-t|)^{-N} \int_{\mathbb{S}^{d-1}} |\widehat{f}(\rho\xi)|^2 h_\kappa^2(\xi)\,d\sigma(\xi)\,d\rho.$$

Here and throughout the proof, N denotes a sufficiently large number depending only on κ and d. Now using Minkowski's inequality again, we deduce

$$\begin{aligned}&\Big\|\Big(\int_{2^{-1}}^4 |T_jf_1(\cdot,t)|^2\,dt\Big)^{\frac12}\Big\|^2_{L^{p_0}(\widetilde{B},h_\kappa^2)} \leqslant \int_{2^{-1}}^4 \Big\|T_jf_1(\cdot,t)\Big\|^2_{L^{p_0}(\widetilde{B},h_\kappa^2)}\,dt\\
&\leqslant C2^{-i}4^{i-j}\Big(\frac{2^{(2\lambda_\kappa+1)j}}{\mathrm{meas}_\kappa(B)}\Big)^{1-\frac{2}{p_0}}\int_{4^{-1}}^8\int_{\mathbb{S}^{d-1}}|\widehat{f}(\rho\xi)|^2h_\kappa^2(\xi)\,d\sigma(\xi)\int_{2^{-1}}^4(1+2^i|\rho-t|)^{-N}\,dt\,d\rho\\
&\leqslant C4^{-j}\Big(\frac{2^{(2\lambda_\kappa+1)j}}{\mathrm{meas}_\kappa(B)}\Big)^{1-\frac{2}{p_0}}\int_{4^{-1}}^8\rho^{2\lambda_\kappa}\int_{\mathbb{S}^{d-1}}|\widehat{f}(\rho\xi)|^2h_\kappa^2(\xi)\,d\sigma(\xi)\,d\rho\\
&\leqslant C4^{-j}\Big(\frac{2^{(2\lambda_\kappa+1)j}}{\mathrm{meas}_\kappa(B)}\Big)^{1-\frac{2}{p_0}}\|f\|^2_{\kappa,2}.\end{aligned}$$

This shows (9.4.11) for $k=1$.

Next, we show (9.4.11) for $k=2$. By Minkowski's inequality, it suffices to show that for any $t\in[2^{-1},4]$ and $p=p_0$,

$$\|T_jf_2(\cdot,t)\|_{L^p(\widetilde{B},h_\kappa^2)}\leqslant C2^{-iN}2^{i-j}\Big(\frac{2^{(2\lambda_\kappa+1)j}}{\mathrm{meas}_\kappa(B)}\Big)^{\frac12-\frac1p}\|f_2\|_{\kappa,2}. \tag{9.4.13}$$

Note that the log-convexity of the L^p-norm implies

$$\|T_jf_2(\cdot,t)\|_{L^{p_0}(\widetilde{B},h_\kappa^2)}\leqslant \|T_jf_2(\cdot,t)\|^{1-\frac{2}{p_0}}_{L^\infty(\widetilde{B},h_\kappa^2)}\|T_jf_2(\cdot,t)\|^{\frac{2}{p_0}}_{L^2(\widetilde{B},h_\kappa^2)}.$$

Thus, it suffices to show (9.4.13) for $p=\infty$ and $p=2$.

To show (9.4.13) for $p=\infty$, we observe that by (9.4.10), for $t\in[2^{-1},4]$ and $x\in\widetilde{B}$,

$$|T_jf_2(x,t)| = \Big|\int_I \rho^{2\lambda_\kappa}\int_{\mathbb{S}^{d-1}} \widehat{f}(\rho\xi)(\widehat{\eta_j}*_\kappa \phi_t^\lambda)(\rho\xi)E_\kappa(i\rho x,\xi)h_\kappa^2(\xi)\,d\sigma(\xi)\,d\rho\Big|$$
$$\leqslant \int_I \rho^{2\lambda_\kappa}\sup_{y\in\widetilde{B}_\rho}\Big|\int_{\mathbb{S}^{d-1}} \widehat{f}(\rho\xi)(\widehat{\eta_j}*_\kappa \phi_t^\lambda)(\rho\xi)E_\kappa(iy,\xi)h_\kappa^2(\xi)\,d\sigma(\xi)\Big|\,d\rho,$$

which, using the fact (9.4.12) and Corollary 7.2.2 (ii) with $q=\infty$, is estimated above by

$$C\Big(\frac{2^{j(2\lambda_\kappa+1)}}{\mathrm{meas}_\kappa(B)}\Big)^{\frac12}\int_I \rho^{2\lambda_\kappa}\Big(\int_{\mathbb{S}^{d-1}}|\widehat{\eta_j}*_\kappa\phi_t^\lambda(\rho\xi)|^2|\widehat{f}(\rho\xi)|^2h_\kappa^2(\xi)\,d\sigma(\xi)\Big)^{\frac12}\,d\rho.$$

Thus, by Lemma 9.3.1, it follows that

$$\sup_{x\in\widetilde{B}}|T_jf_2(x,t)| \leqslant C2^{-iN}2^{i-j}\Big(\frac{2^{(2\lambda_\kappa+1)j}}{\mathrm{meas}_\kappa(B)}\Big)^{\frac12}\int_I(1+|\rho|)^{-N}\rho^{2\lambda_\kappa}\Big(\int_{\mathbb{S}^{d-1}}|\widehat{f}(\rho\xi)|^2h_\kappa^2(\xi)\,d\sigma(\xi)\Big)^{\frac12}\,d\rho$$
$$\leqslant C2^{-iN}2^{i-j}\Big(\frac{2^{(2\lambda_\kappa+1)j}}{\mathrm{meas}_\kappa(B)}\Big)^{\frac12}\Big(\int_I\rho^{2\lambda_\kappa}\int_{\mathbb{S}^{d-1}}|\widehat{f}(\rho\xi)|^2h_\kappa^2(\xi)\,d\sigma(\xi)\,d\rho\Big)^{\frac12}$$
$$\leqslant C2^{-iN}2^{i-j}\Big(\frac{2^{(2\lambda_\kappa+1)j}}{\mathrm{meas}_\kappa(B)}\Big)^{\frac12}\|f\|_{\kappa,2},$$

where the second step uses the Cauchy-Schwartz inequality. This shows (9.4.13) for $p=\infty$.

Finally, we show (9.4.13) for $p=2$. Indeed, by Plancherel's theorem and Lemma 9.3.1,

$$\|T_jf_2(\cdot,t)\|_{\kappa,2}^2 = c\int_{\mathbb{R}^d}|\widehat{f_2}(\xi)|^2|\widehat{\eta_j}*_\kappa\phi_t^\lambda(\xi)|^2h_\kappa^2(\xi)\,d\xi$$
$$\leqslant c4^{-iN}4^{i-j}\int_{\mathbb{R}^d}|\widehat{f_2}(\xi)|^2(1+\|\xi\|)^{-2N}h_\kappa^2(\xi)\,d\xi \leqslant C4^{-iN}4^{i-j}\|f_2\|_{\kappa,2}^2.$$

This completes the proof of the lemma. □

Lemma 9.4.2. *For any $g \geq 0$,*

$$\int_{2^{-1}}^{4} \int_{\mathbb{R}^d} |f *_\kappa \widehat{\phi_t^\lambda}(x)|^2 g(x) h_\kappa^2(x)\, dx\, dt \leqslant C\lambda^{\frac{1}{\lambda_\kappa+1}} \int_{\mathbb{R}^d} |f(x)|^2 M_{\kappa,r} g(x) h_\kappa^2(x)\, dx.$$

Proof. Recall that $T_j: \quad L^2(\mathbb{R}^d; h_\kappa^2) \to L^{p_0}(L^2[2^{-1}, 4])$ is a vector-valued operator given by

$$T_j f(x,t) = f *_\kappa (\eta_j \widehat{\phi^\lambda}_t)(x) = c_\kappa \int_{\mathbb{R}^d} f(y) T^y(\widehat{\phi^\lambda}_t \eta_j)(x) h_\kappa^2(y)\, dy, \quad t \in [2^{-1}, 4].$$

Since η_j is supported in the ball $B_{2^{j+2}}(0)$, we conclude from (5.1.3) that $T^y(\eta_j \widehat{\phi_t^\lambda})(x) = 0$ unless $\big||x_n| - |y_n|\big| \leqslant 2^{j+2}$ for $n = 1, 2, \cdots, d$. This means that for each fixed $y \in \mathbb{R}^d$, the function $T^y(\eta_j \widehat{\phi_t^\lambda})$ is supported in the set $\bigcup_{\sigma \in \mathbb{Z}_2^d} B(\sigma y, \sqrt{d} 2^{j+2})$. Thus, for any function f supported in a ball $B = B(\omega, c2^j)$, $T_j f(\cdot, t)$ is supported in the set

$$\widetilde{B} = \bigcup_{\sigma \in \mathbb{Z}_2^d} B(\sigma\omega, (c + \sqrt{d}) 2^j).$$

Thus, the operator T_j has the locality property stated before Lemma 9.2.1. On the other hand, however, by Lemma 9.4.1,

$$\Big(\int_{\mathbb{R}^d} \|T_j f(x,\cdot)\|_{L^2([2^{-1},4])}^{p_0} h_\kappa^2(x)\, dx\Big)^{\frac{1}{p_0}} = \Big(\int_{\widetilde{B}} \|T_j f(x,\cdot)\|_{L^2([2^{-1},4])}^{p_0} h_\kappa^2(x)\, dx\Big)^{\frac{1}{p_0}}$$
$$\leqslant C 2^{-j} \Big(\frac{2^{j(2\lambda_\kappa+1)}}{\mathrm{meas}_\kappa(B)}\Big)^{\frac{1}{2} - \frac{1}{p_0}} \|f\|_{\kappa,2}.$$

Thus, using Lemma 9.2.1, we conclude that for $j = i, i+1, \cdots,$

$$\int_{\mathbb{R}^d} \|T_j f(x,\cdot)\|^2_{L^2([2^{-1},4])} g(x) h^2_\kappa(x)\,dx$$
$$\leqslant C4^{-j} 2^{j(2\lambda_\kappa+1)/r} \int_{\mathbb{R}^d} |f(x)|^2 M_{\kappa,r} g(x) h^2_\kappa(x)\,dx, \tag{9.4.14}$$

where $r = (p_0/2)' = \lambda_\kappa + 1$. Now let $\varepsilon \in (0,1)$ be such that $0 < \varepsilon < 2 - \frac{2\lambda_\kappa+1}{r} = \frac{1}{\lambda_\kappa+1}$. We obtain from (9.4.8) that

$$\|Tf(x,\cdot)\|^2_{L^2([2^{-1},4])} \leqslant \Big(\sum_{j=i}^{\infty} \|T_j f(x,\cdot)\|_{L^2([2^{-1},4])}\Big)^2 \leqslant C2^{-i\varepsilon} \sum_{j=i}^{\infty} 2^{j\varepsilon} \|T_j f(x,\cdot)\|^2_{L^2([2^{-1},4])}.$$

Thus, by (9.4.14),

$$\begin{aligned}\int_{\mathbb{R}^d} \|Tf(x,\cdot)\|^2_{L^2([2^{-1},4])} g(x) h^2_\kappa(x)\,dx &\leqslant C2^{-i\varepsilon} \sum_{j=i}^{\infty} 2^{j\varepsilon} \int_{\mathbb{R}^d} \|T_j f(x,\cdot)\|^2_{L^2([2^{-1},4])} g(x) h^2_\kappa(x)\,dx \\ &\leqslant C2^{-i\varepsilon} \sum_{j=i}^{\infty} 2^{j\varepsilon} 2^{-2j} 2^{j(2\lambda_\kappa+1)/r} \int_{\mathbb{R}^d} |f(x)|^2 M_{\kappa,r} g(x) h^2(x)\,dx \\ &\leqslant C\lambda^{\frac{1}{1+\lambda_\kappa}} \int_{\mathbb{R}^d} |f(x)|^2 M_{\kappa,r} g(x) h^2_\kappa(x)\,dx.\end{aligned}$$

□

Lemma 9.4.3. *For $k \in \mathbb{Z}$ and any function $g \geq 0$,*

$$\int_{2^{k-1}}^{2^{k+2}} \int_{\mathbb{R}^d} |f *_\kappa \widehat{\phi^\lambda}_t(x)|^2 g(x) h^2_\kappa(x)\,dx \frac{dt}{t} \leqslant C\lambda^{\frac{1}{1+\lambda_\kappa}} \int_{\mathbb{R}^d} |f(x)|^2 M_{\kappa,r} g(x) h^2_\kappa(x)\,dx,$$

where $\phi^\lambda_t(x) = \phi^\lambda(x/t)$.

Proof. For $1 \leqslant p \leqslant \infty$, set

$$\mathrm{Dil}_u^p f(x) = u^{-(2\lambda_\kappa+1)/p} f(x/u), \quad u > 0, \quad x \in \mathbb{R}^d.$$

Note that for any $u > 0$,

$$\begin{aligned}(f *_\kappa \widehat{\phi_t^\lambda})^\wedge(\xi) &= \widehat{f}(\xi)\phi^\lambda(t^{-1}\xi) = \mathrm{Dil}_{u^{-1}}^\infty \Big[(\mathrm{Dil}_u^\infty \widehat{f})(\mathrm{Dil}_{ut}^\infty \phi^\lambda)\Big](\xi) \\ &= \mathcal{F}_\kappa \Big[\mathrm{Dil}_u^1(\mathrm{Dil}_{u^{-1}}^1 f *_\kappa \widehat{\phi_{ut}^\lambda})\Big](\xi).\end{aligned}$$

It follows that for any $x \in \mathbb{R}^d$ and any $u > 0$,

$$\begin{aligned}f *_\kappa \widehat{\phi_t^\lambda}(x) &= \mathrm{Dil}_u^1(\mathrm{Dil}_{u^{-1}}^1 f) *_\kappa (\mathcal{F}_\kappa \phi_{ut}^\lambda)(x) = (\mathrm{Dil}_{u^{-1}}^\infty f) *_\kappa (\mathcal{F}_\kappa \phi_{ut}^\lambda)(u^{-1}x) \\ &= f_{u^{-1}} *_\kappa \widehat{\phi_{ut}^\lambda}(u^{-1}x),\end{aligned}$$

where, for convenience, we set $f_u(y) = f(u^{-1}y) = \mathrm{Dil}_u^\infty f(y)$. It follows by Lemma 9.4.2 that

$$\begin{aligned}&\int_{2^{k-1}}^{2^{k+2}} \int_{\mathbb{R}^d} |f *_\kappa \widehat{\phi_t^\lambda}(x)|^2 g(x) h_\kappa^2(x)\, dx\, \frac{dt}{t} = \int_{2^{-1}}^{4} \int_{\mathbb{R}^d} |f *_\kappa \widehat{\phi_{2^k t}^\lambda}(x)|^2 g(x) h_\kappa^2(x)\, dx \frac{dt}{t} \\ &= \int_{2^{-1}}^{4} \int_{\mathbb{R}^d} |f_{2^k} *_\kappa \widehat{\phi_{2^{-k}2^k t}^\lambda}(2^k x)|^2 g(x) h_\kappa^2(x)\, dx\, \frac{dt}{t} \\ &= 2^{-k(2\lambda_\kappa+1)} \int_{2^{-1}}^{4} \int_{\mathbb{R}^d} |f_{2^k} *_\kappa \widehat{\phi_t^\lambda}(x)|^2 g(2^{-k}x) h_\kappa^2(x)\, dx\, \frac{dt}{t} \\ &\leqslant C\lambda^{\frac{1}{1+\lambda_\kappa}} 2^{-k(2\lambda_\kappa+1)} \int_{\mathbb{R}^d} |f(2^{-k}x)|^2 M_{\kappa,r} g(2^{-k}x) h_\kappa^2(x)\, dx \\ &= C\lambda^{\frac{1}{1+\lambda_\kappa}} \int_{\mathbb{R}^d} |f(x)|^2 M_{\kappa,r} g(x) h_\kappa^2(x)\, dx,\end{aligned}$$

where we used the fact that $M_{\kappa,r}(g(u\cdot))(x) = M_{\kappa,r}g(ux)$ for any $u > 0$ in the fourth step. □

Now we are in a position to prove Theorem 9.1.2.

Proof of Theorem 9.1.2. Let φ be a C^∞-radial function on $\mathbb{R}^d$ with the properties that $\operatorname{supp}\varphi \subset \{\xi \in \mathbb{R}^d : \ 1 \leqslant |\xi| \leqslant 2\}$ and $\sum_{j\in\mathbb{Z}} \varphi(2^j\xi) = 1$ for all $\xi \in \mathbb{R}^d \setminus \{0\}$. Define the operators L_j by

$$(L_j f)^\wedge(\xi) = \varphi(2^j\xi)\widehat{f}(\xi), \quad j \in \mathbb{Z}.$$

Thus,

$$f *_\kappa \widehat{\phi_t^\lambda}(x) = \sum_{j\in\mathbb{Z}} (L_j f) *_\kappa \widehat{\phi_t^\lambda}(x).$$

Note that

$$((L_j f) *_\kappa \widehat{\phi_t^\lambda})^\wedge(\xi) = \varphi(2^j\xi)\widehat{f}(\xi)\phi^\lambda(t^{-1}\xi),$$

which is zero unless

$$2^{-1-j} \leqslant \frac{4}{5}\cdot 2^{-j} \leqslant \frac{4}{5}\|\xi\| \leqslant t \leqslant \frac{4}{3}\|\xi\| \leqslant \frac{4}{3}\cdot 2^{1-j} \leqslant 2^{2-j}.$$

This implies that

$$f *_\kappa \widehat{\phi_t^\lambda}(x) = \sum_{j\in\mathbb{Z}:\ \ 2^{-1-j}\leqslant t\leqslant 2^{2-j}} (L_j f) *_\kappa \widehat{\phi_t^\lambda}(x),$$

and hence

$$|f *_\kappa \widehat{\phi_t^\lambda}(x)|^2 \leqslant C \sum_{j\in\mathbb{Z}:\ \ 2^{-1-j}\leqslant t\leqslant 2^{2-j}} |(L_j f) *_\kappa \widehat{\phi_t^\lambda}(x)|^2.$$

It follows that

$$\int_{\mathbb{R}^d}\Big(\int_0^\infty |f *_\kappa \widehat{\phi_t^\lambda}(x)|^2 \frac{dt}{t}\Big) g(x) d\mu_\kappa(x) \leqslant C \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^d} \int_{2^{-1-j}}^{2^{2-j}} |(L_j f) *_\kappa \widehat{\phi_t^\lambda}(x)|^2 \frac{dt}{t} g(x) d\mu_\kappa(x)$$

$$\leqslant C\lambda^{\frac{1}{1+\lambda_\kappa}} \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^d} |L_j f|^2 M_{\kappa,r} g(x) d\mu_\kappa(x) = \lambda^{\frac{1}{1+\lambda_\kappa}} \Big\|\big(\sum_{j\in\mathbb{Z}} |L_j f|^2\big)^{1/2}\Big\|^2_{L^2(\mathbb{R}^d, M_{\kappa,r} g d\mu_\kappa)},$$

where $d\mu_\kappa(x) = h_\kappa^2(x)\,dx$. Without loss of generality, we may assume that $M_\kappa(g^r)(x) < \infty$ a.e.. Since $r > 1$, this implies that the weight $M_{\kappa,r} g = (M_\kappa(g^r))^{1/r}$ satisfies the A_2-condition with respect to the measure $d\mu_\kappa$. It then follows by the weighted Paley-Littlewood inequality that

$$\Big\|\big(\sum_{j\in\mathbb{Z}} |L_j f|^2\big)^{1/2}\Big\|^2_{L^2(M_{\kappa,r} g d\mu_\kappa)} \leqslant C\|f\|^2_{L^2(M_{\kappa,r} g d\mu_\kappa)} = C\int_{\mathbb{R}^d} |f(x)|^2 M_{\kappa,r} g(x)\, d\mu_\kappa(x).$$

This completes the proof. □

9.5 Proof of Lemma 9.3.1

Denote by ψ the radial Schwartz function on $\mathbb{R}^d$ whose Dunkl transform is either the function η or the function $\eta(\xi) - \eta(2\xi)$ depending on whether $j = i$ or $j > i$. Then $\widehat{\eta_j}(x) = 2^{j(2\lambda_\kappa+1)}\psi(2^j x) =: \psi_j(x)$ and

$$\widehat{\eta_j} *_\kappa \phi_t^\lambda(x) = \psi_j *_\kappa \phi_t^\lambda = c2^{j(2\lambda_\kappa+1)} \int_{\mathbb{R}^d} \psi(2^j y) T^y \phi_t^\lambda(x) h_\kappa^2(y)\, dy.$$

Since $T^y\phi_t^\lambda(x) = T^{y/t}\phi^\lambda(x/t)$, we have

$$\widehat{\eta}_j *_\kappa \phi_t^\lambda(x) = c(2^j t)^{(2\lambda_\kappa+1)} \int_{\mathbb{R}^d} \psi(2^j ty) T^y\phi^\lambda(x/t) h_\kappa^2(y)\, dy = (\psi_m *_\kappa \phi^\lambda)(x/t),$$

where $2^m = 2^j t$. Thus, it suffices to prove the stated estimates for $t = 1$ (for the cases $\|x\| \in [\frac{1}{4}, 8]$ and $\|x\| \notin [\frac{1}{4}, 8]$).

Firstly, by Lemma 7.2.6, we have that

$$|T^y\phi^\lambda(x)| \leqslant C \prod_{j=1}^{d} (1 + |x_j|)^{-2\kappa_j} \sim \frac{1}{\text{meas}_\kappa(B(x,1))}, \quad x \in \mathbb{R}^d. \tag{9.5.15}$$

Then we turn to the proof of the estimates (9.3.7) with $t = 1$. Assume that $T^y\phi^\lambda(x) \neq 0$. As stated before, we need to consider the following cases:

Case 1. $\|x\| \leqslant \frac{1}{4}$.

Recall first that

$$1 - 2^{-i-1} \leqslant \|\bar{x} + \bar{y}\| \ \text{ and } \ \|\bar{x} - \bar{y}\| \leqslant 1 + 2^{-i-1}, \tag{9.5.16}$$

which in turn implies that

$$\Big| \|y\| - 1 \Big| \leqslant \|x\| + 2^{-i-1}. \tag{9.5.17}$$

Hence, if $\|x\| \leqslant \frac{1}{4}$, then

$$\|y\| = \|x\| + \|y\| - \|x\| \geq 1 - 2^{-i-1} - \frac{1}{4} \geq \frac{1}{4}.$$

It follows that

$$
\begin{aligned}
|\psi_j *_\kappa \phi^\lambda(x)| &\leqslant c2^{j(2\lambda_\kappa+1)} \int_{\mathbb{R}^d} |\psi(2^j y)||T^y \phi^\lambda(x)| h_\kappa^2(y)\, dy \\
&\leqslant C2^{j(2\lambda_\kappa+1)} \int_{\|y\| \geq \frac{1}{4}} (2^j \|y\|)^{-2\lambda_\kappa-1-N} h_\kappa^2(y)\, dy \\
&\leqslant C2^{-jN} \int_{\mathbb{R}^d} (1+\|y\|)^{-2\lambda_\kappa-1-N} h_\kappa^2(y)\, dy \\
&\leqslant C2^{-jN} \sim 2^{(i-j)N} 2^{-iN} (1+\|x\|)^{-N}.
\end{aligned}
$$

Case 2. $\|x\| \geq 8$.

Note that by (9.5.16), $\|\bar{x}-\bar{y}\| \leqslant \frac{3}{2} \leqslant \frac{\|x\|}{2}$, which implies

$$
\frac{\|x\|}{2} \leqslant \|y\| \leqslant \frac{3}{2}\|x\|.
$$

It follows by (9.5.15) that

$$
\begin{aligned}
|\psi_j *_\kappa \phi^\lambda(x)| &\leqslant C2^{j(2\lambda_\kappa+1)} \int_{\|y\| \sim \|x\|} (2^j \|y\|)^{-2\lambda_\kappa-2-N} h_\kappa^2(y)\, dy \\
&\leqslant C(2^j\|x\|)^{-N} \int_{\mathbb{R}^d} (1+\|y\|)^{-2\lambda_\kappa-2} h_\kappa^2(y)\, dy \leqslant C2^{(i-j)N} 2^{-iN}(1+\|x\|)^{-N}.
\end{aligned}
$$

Case 3. $\frac{1}{4} \leqslant \|x\| \leqslant 8$.

In this case, we will show that for any $N \in \mathbb{Z}_+$,

$$
|\psi_j *_\kappa \phi^\lambda(x)| \leqslant C2^{(i-j)N} \Big(1 + 2^i \Big|1 - \|x\|\Big|\Big)^{-N}. \tag{9.5.18}
$$

First, we show

$$|\psi_j *_\kappa \phi^\lambda(x)| \leqslant C2^{(i-j)N}, \tag{9.5.19}$$

which will yield (9.5.18) for $|1-\|x\|| \leqslant 2^{-i+5}$.

If $j = i$, then (9.5.19) holds trivially since

$$|\psi_j *_\kappa \phi^\lambda(x)| \leqslant C\|\psi_j\|_{1,\kappa} = c' < \infty.$$

Now assume that $j > i$. Since $\widehat{\psi}$ is zero near the origin when $j > i$, it follows from (2.2.1) that $\mathcal{D}_\kappa^\alpha \widehat{\psi}(0) = 0$ for any $\alpha \in \mathbb{Z}_+^d$. Thus,

$$0 = \mathcal{D}_\kappa^\alpha \widehat{\psi}(0) = c\widehat{x^\alpha \psi}(0) = c\int_{\mathbb{R}^d} x^\alpha \psi(x) h_\kappa^2(x)\,dx, \quad \forall \alpha \in \mathbb{Z}_+^d.$$

This implies that for every polynomial P on $\mathbb{R}^d$,

$$\int_{\mathbb{R}^d} \psi_j(x) P(x) h_\kappa^2(x)\,dx = \int_{\mathbb{R}^d} \psi(x) P(2^{-j}x) h_\kappa^2(x)\,dx = 0. \tag{9.5.20}$$

Fix temporarily $x \in \{z \in \mathbb{R}^d : \frac{1}{32} \leqslant \|z\| \leqslant 16\}$ and $t = (t_1, \cdots, t_d) \in [-1,1]^d$. Set $F_{x,t}(y) = \phi^\lambda(u(x,y,t))$, where $u(x,y,t) := \sqrt{\|x\|^2 + \|y\|^2 - 2\sum_{j=1}^d x_j y_j t_j}$. It is easy to see

$$u(x,y,t)^2 \geq \|x\|^2 + \|y\|^2 - 2\sum_{j=1}^d |x_j y_j||t_j| = \|\bar{x}-\bar{y}\|^2 + 2\sum_{j=1}^d |x_j y_j|(1-|t_j|) \tag{9.5.21}$$

$$\geq \|\bar{x}-\bar{y}\|^2 + 2\max_{1\leqslant j\leqslant d} |x_j y_j|(1-|t_j|).$$

Since $\phi^\lambda(u(x,y,t)) = 0$ unless $1 - 2^{-i-1} \leqslant u(x,y,t) \leqslant 1 + 2^{-i-1}$, (9.5.17) and (9.5.21) give that $F_{x,t}(y)$ is a C^∞- function of y supported in the set where

$$\sum_{j=1}^{d} |x_j||y_j|(1 - |t_j|) < 2, \quad \big|\|x\| - 1\big| < \|y\| + 2^{-i} \text{and } \big|\|y\| - 1\big| < \|x\| + 2^{-i}, \tag{9.5.22}$$

Furthermore, by (9.1.3),

$$\|\nabla^n F_{x,t}\|_\infty \leqslant C 2^{in}, \quad \forall n = 0, 1, \cdots . \tag{9.5.23}$$

Now using Taylor's theorem, we obtain that given any $N \in \mathbb{Z}_+$,

$$\phi^\lambda(u(x,y,t)) = \sum_{|\alpha| \leqslant N-1} \frac{\partial^\alpha F_{x,t}(0)}{\alpha!} y^\alpha + \sum_{|\alpha| = N} \frac{\partial^\alpha F_{x,t}(\theta y)}{\alpha!} y^\alpha,$$

for some $\theta = \theta(x,y,t) \in [0,1]$. It then follows by (5.1.4) that

$$\begin{aligned} T^y \phi(x) = c \sum_{|\alpha| \leqslant N-1} \frac{y^\alpha}{\alpha!} \int_{[-1,1]^d} \partial^\alpha F_{x,t}(0) \prod_{j=1}^{d} (1 - t_j^2)^{\kappa_j - 1}(1 + t_j)\, dt_j \\ + c \sum_{|\alpha| = N} \frac{y^\alpha}{\alpha!} \int_{[-1,1]^d} \partial^\alpha F_{x,t}(\theta(x,y,t) y) \prod_{j=1}^{d} (1 - t_j^2)^{\kappa_j - 1}(1 + t_j)\, dt_j. \end{aligned} \tag{9.5.24}$$

Thus, using (9.5.20) and (9.5.23), we conclude that

$$
\begin{aligned}
|\psi_j *_\kappa \phi^\lambda(x)| &= \Big|\int_{\mathbb{R}^d} \psi_j(y) T^y \phi^\lambda(x) h_\kappa^2(y)\, dy\Big| \leqslant C \int_{\mathbb{R}^d} |\psi_j(y)| \|y\|^N \times \\
&\quad \times \Big[\int_{[-1,1]^d} \|\nabla^N F_{x,t}(\theta(x,y,t)y)\| \prod_{j=1}^d (1-t_j^2)^{\kappa_j - 1}(1+t_j) dt_j\Big] h_\kappa^2(y) dy \quad (9.5.25) \\
&\leqslant C 2^{iN} 2^{j(2\lambda_\kappa+1)} \int_{\mathbb{R}^d} (1+2^j\|y\|)^{-N-2\lambda_\kappa-2} \|y\|^N h_\kappa^2(y)\, dy \leqslant C 2^{(i-j)N}.
\end{aligned}
$$

This proves (9.5.19), and hence (9.5.18) for $\Big|1-\|x\|\Big| \leqslant 2^{-i+5}$.

Finally, we begin to prove (9.5.18) under the assumption $\Big|\|x\|-1\Big| \geq 2^{-i+5}$. First, we observe that if $\Big|\|x\|-1\Big| \geq 2^{-i+5}$, then by (9.5.22), $\partial^\alpha F_{x,t}(0) = 0$ for all $\alpha \in \mathbb{Z}_+^d$, and hence by (9.5.24), (9.5.25) holds for all $j \geq i$. Second, observe that if $\|y\| \leqslant \frac{1}{2}\Big|1-\|x\|\Big|$, then

$$
\Big|\|x\|-1\Big| > \|y\| + 2^{-i} \geq \|\theta(x,y,t)y\| + 2^{-i},
$$

which, by (9.5.22), implies that $\partial^\alpha F_{x,t}(\theta(x,y,t)y) = 0$ for all $\alpha \in \mathbb{Z}_+^d$. Thus, (9.5.25) implies that for all $j \geq i$,

$$
\begin{aligned}
|\psi_j *_\kappa \phi^\lambda(x)| &\leqslant C \int_{\|y\|\geq \frac{1}{2}\left|1-\|x\|\right|} |\psi_j(y)| \|y\|^N \times \\
&\quad \times \Big[\int_{[-1,1]^d} \|\nabla^N F_{x,t}(\theta(x,y,t)y)\| \prod_{j=1}^d (1-t_j^2)^{\kappa_j - 1}(1+t_j) dt_j\Big] h_\kappa^2(y) dy \\
&\leqslant C 2^{iN} 2^{j(2\lambda_\kappa+1)} \int_{\|y\|\geq \frac{1}{2}\left|1-\|x\|\right|} (2^j\|y\|)^{-2N-2\lambda_\kappa-1} \|y\|^N h_\kappa^2(y)\, dy \\
&\leqslant C 2^{iN} 2^{j(2\lambda_\kappa+1)} 2^{-j(2N+2\lambda_\kappa+1)} \Big(2^{-i} + \big|1-\|x\|\big|\Big)^{-N} \\
&\leqslant C 2^{(i-j)N} \Big(1 + 2^i\big|1-\|x\|\big|\Big)^{-N}.
\end{aligned}
$$

Chapter 10

Almost everywhere convergence of Bochner-Riesz means for the Dunkl transforms of functions in L^p-spaces

10.1 Main results

In this section, we want to prove the almost everywhere convergence of Bochner-Riesz means of functions in $L^p(\mathbb{R}^d; h_\kappa^2)$ which is stated in the following theorem.

For convenience, we do not distinguish $|x|$ and $\|x\|$ in this chapter. Namely,

$$|x| = \|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_d^2}, \quad \forall\ x = (x_1, x_2, \cdots, x_d) \in \mathbb{R}^d.$$

Theorem 10.1.1. *Let* $\delta_\kappa(p) = (2\lambda_\kappa + 1)(\frac{1}{2} - \frac{1}{p}) - \frac{1}{2}$. *If* $p \geqslant 2$ *and* $\delta > \max\{0, \delta_\kappa(p)\}$, *then for all* $f \in L^p(\mathbb{R}^d; h_\kappa^2)$,

$$\lim_{R\to\infty} B_R^\delta(h_\kappa^2; f)(x) = f(x), \quad a.e.\ x \in \mathbb{R}^d.$$

Recall the definition of Bochner-Riesz means of $f \in L^p(\mathbb{R}^d; h_\kappa^2)$ in the last chapter, we know that

$$\widehat{B_R^\delta}(h_\kappa^2; f)(x) = \big(1 - \frac{|\xi|^2}{R^2}\big)_+^\delta \hat{f}(\xi).$$

Then by (9.1.4),

$$B_R^\delta(h_\kappa^2; f)(x) = \sum_{k=0}^\infty 2^{-k\delta} f *_\kappa \widehat{\phi_R^{2^{-k}}}(x).$$

Definition 10.1.2. For $t > 0$, define $S_t^\lambda f(x) = f *_\kappa \widehat{\phi_t^\lambda}(x)$, and $S_*^\lambda f(x) = \sup_{t>0} |S_t^\lambda f(x)|$.

By definition, we get that $\widehat{S_t^\lambda} f(\xi) = \hat{f}(\xi)\phi_t^\lambda(\xi)$. And $B_R^\delta(h_\kappa^2; f)(x) = \sum_{k=0}^\infty 2^{-k\delta} S_R^{2^{-k}} f(x)$. Thus,

$$B_*^\delta(h_\kappa^2; f)(x) \leqslant \sum_{k=0}^\infty 2^{-k\delta} S_*^{2^{-k}} f(x). \tag{10.1.1}$$

To prove Theorem 10.1.1, let us first show the following theorem.

Theorem 10.1.3. *For* $0 \leqslant \lambda < 2\lambda_\kappa + 1$,

$$\int_{\mathbb{R}^d} |S_*^\lambda f(x)|^2 h_\kappa^2(x) \frac{dx}{|x|^\alpha} \leqslant C_\alpha A_\alpha(\lambda) \int_{\mathbb{R}^d} |f(x)|^2 h_\kappa^2(x) \frac{dx}{|x|^\alpha},$$

where

$$A_\alpha(t) = \begin{cases} 1, & \textit{if } 0 \leqslant \alpha < 1, \\ |\ln t|, & \textit{if } \alpha = 1, \\ t^{1-\alpha}, & \textit{if } 1 < \alpha < 2\lambda_\kappa + 1. \end{cases}$$

The proof of Theorem 10.1.3 is long and technical, so we postpone it until Section 10.2.

Proof of Theorem 10.1.3 Suppose $f \in L^p(\mathbb{R}^d; h_\kappa^2)$ with $2 \leqslant p < \frac{2\lambda_\kappa+1}{\lambda_\kappa-\delta}$. Then we get that $\delta \leqslant \lambda_\kappa$ and

$$(2\lambda_\kappa + 1)(1 - \frac{2}{p}) < 1 + 2\delta. \tag{10.1.2}$$

Choose $\alpha \in \Big((2\lambda_\kappa + 1)(1 - \frac{2}{p}), 1 + 2\delta\Big)$, and let $f_1(x) = f(x)\chi_{|x|\leqslant 1}$, $f_2(x) = f(x)\chi_{|x|>1}$. Then by Hölder inequality, $\|f_1\|_{L^2(\mathbb{R}^d;h_\kappa^2)} \leqslant C\|f\|_{L^p(\mathbb{R}^d;h_\kappa^2)}$, which implies that $f_1 \in L^2(\mathbb{R}^d; h_\kappa^2)$. Thus, (10.3.13) and Theorem 10.1.3 give that

$$\begin{aligned} \|B_*^\delta(h_\kappa^2; f_1)\|_{L^2(\mathbb{R}^d;h_\kappa^2)} &\leqslant \sum_{k=0}^\infty 2^{-k\delta}\|S_*^{2^{-k}} f_1\|_{L^2(\mathbb{R}^d;h_\kappa^2)} \\ &\leqslant C_\alpha \sum_{k=0}^\infty 2^{-k\delta}\|f_1\|_{L^2(\mathbb{R}^d;h_\kappa^2)} \leqslant C\|f_1\|_{L^2(\mathbb{R}^d;h_\kappa^2)}. \end{aligned}$$

This implies that

$$\lim_{R\to\infty} B_R^\delta(h_\kappa^2; f_1)(x) = f_1(x), \quad \text{a.e. } x \in \mathbb{R}^d.$$

Since $\alpha > (2\lambda_\kappa + 1)(1 - \frac{2}{p})$, $2\lambda_\kappa + \frac{\alpha p}{2-p} < -1$. Thus, by Hölder inequality,

$$\begin{aligned}
\int_{|x|>1} |f_2(x)|^2 h_\kappa^2(x) \frac{dx}{|x|^\alpha} &\leqslant \left(\int_{\mathbb{R}^d} |f(x)|^p h_\kappa^2(x) dx \right)^{\frac{2}{p}} \left(\int_{|x|>1} |x|^{\frac{\alpha p}{2-p}} h_\kappa^2(x) dx \right)^{1-\frac{2}{p}} \\
&\leqslant \|f_2\|^2_{L^p(\mathbb{R}^d;h_\kappa^2)} \left(\int_1^\infty r^{2\lambda_\kappa + \frac{\alpha p}{2-p}} \int_{\mathbb{S}^{d-1}} h_\kappa^2(x') d\sigma(x') dr \right)^{1-\frac{2}{p}} \\
&\leqslant C \|f\|^2_{L^p(\mathbb{R}^d;h_\kappa^2)}.
\end{aligned}$$

This means that $f_2 \in L^2(\mathbb{R}^d; |x|^{-\alpha} h_\kappa^2(x))$.

Since $\alpha < 1 + 2\delta < 1 + 2\lambda_\kappa$, (10.3.13) and Theorem 10.1.3 give that

$$\begin{aligned}
\|B_*^\delta(h_\kappa^2; f_2)\|_{L^2(\mathbb{R}^d;|x|^{-\alpha}h_\kappa^2)} &\leqslant \sum_{k=0}^\infty 2^{-k\delta} \|S_*^{2^{-k}} f_2\|_{L^2(\mathbb{R}^d;|x|^{-\alpha}h_\kappa^2(x))} \\
&\leqslant C_\alpha \sum_{k=0}^\infty \frac{\sqrt{A_\alpha(2^{-n})}}{2^{k\delta}} \|f_2\|_{L^2(\mathbb{R}^d;|x|^{-\alpha}h_\kappa^2(x))} \\
&\leqslant C \|f_2\|_{L^2(\mathbb{R}^d;|x|^{-\alpha}h_\kappa^2(x))}.
\end{aligned}$$

This implies that

$$\lim_{R\to\infty} B_R^\delta(h_\kappa^2; f_2)(x) = f_2(x), \quad \text{a.e. } x \in \mathbb{R}^d.$$

Since $f = f_1 + f_2$, we get that

$$\lim_{R\to\infty} B_R^\delta(h_\kappa^2; f)(x) = f(x), \quad \text{a.e. } x \in \mathbb{R}^d.$$

10.2 Proof of Theorem 10.1.3

To proof Theorem 10.1.3, we need to first get an estimate of $|S_1^\lambda f(x)|$.

Lemma 10.2.1. *For $f \in L^2(|x|^{-\alpha} h_\kappa^2(x))$ and $0 < \alpha < d$,*

$$\int_{\mathbb{R}^d} |S_1^\lambda f(x)|^2 h_\kappa^2(x) \frac{dx}{|x|^\alpha} \leqslant C_\alpha A_\alpha(\lambda) \int_{\mathbb{R}^d} |f(x)|^2 h_\kappa^2(x) \frac{dx}{|x|^\alpha}.$$

Remark 10.2.2. Lemma 10.2.1 and a duality argument imply that for $0 \leqslant \alpha < 2\lambda_\kappa + 1$ and $t > 0$,

$$\int_{\mathbb{R}^d} |S_t^\lambda f(x)|^2 |x|^\alpha h_\kappa^2(x)\, dx \leqslant C_\alpha A_\alpha(\lambda) \int_{\mathbb{R}^d} |f(x)|^2 |x|^\alpha h_\kappa^2(x)\ dx. \tag{10.2.3}$$

Here, we first assume that Lemma 10.2.1 is true. The proof of this lemma will be given in the next section.

Definition 10.2.3. Let $\ell \in \mathbb{N}$ and $0 < \alpha < \ell$. Define

$$D^\alpha f(x) := \Big(\int_{\mathbb{R}^d} \frac{|\triangle_y^\ell f(x)|^2}{|y|^{2\lambda_\kappa+1+2\alpha}} h_\kappa^2(y)\, dy \Big)^{\frac{1}{2}},$$

where

$$\triangle_y^\ell f(x) = \sum_{j=0}^{\ell} \binom{\ell}{j} (-1)^j T^{jy} f(x).$$

By (5.1.3),

$$\triangle_y^\ell f(x) = \int_{\mathbb{R}^d} f(z) d\mu_{x,y,\ell}(z), \tag{10.2.4}$$

where $d\mu_{x,y,\ell}$ is a signed Borel measure supported in

$$\{z \in \mathbb{R}^d : \quad |\bar{z} - \bar{x}| \leqslant \ell|y|\}.$$

Lemma 10.2.4. *For $\ell \in \mathbb{N}$ and $0 < \alpha < \ell$,*

$$\|\Delta_\kappa^{\alpha/2} f\|_{\kappa,2} \sim \|D^\alpha f\|_{\kappa,2}.$$

Proof. By definition, we have

$$(\triangle_y^\ell f)^\wedge(x) = A_\ell(x,y)\widehat{f}(x),$$

where

$$\begin{aligned}
A_\ell(x,y) &= \sum_{j=0}^{\ell} \binom{\ell}{j}(-1)^j V_\kappa\Big[e^{-ij\langle y,\cdot\rangle}\Big](x) = V_\kappa\Big[\sum_{j=0}^{\ell} \binom{\ell}{j}(-1)^j (e^{-i\langle y,\cdot\rangle})^j\Big](x) \\
&= V_\kappa\Big[(1 - e^{-i\langle y,\cdot\rangle})^\ell\Big](x) \\
&= c_\kappa \int_{[-1,1]^d} \Big(1 - \exp\big(-i\sum_{j=1}^{d} t_j x_j y_j\big)\Big)^\ell \prod_{j=1}^{d}(1 - t_j^2)^{\kappa_j - 1}(1 + t_j)\, dt_j.
\end{aligned}$$

It is easily seen that the function $A_\ell(x,y)$ has the following properties:

(i) For $x, y \in \mathbb{R}^d$,

$$|A_\ell(x,y)| \leqslant 2^\ell.$$

(ii) As $|y|\to 0$,

$$A_\ell(x,y)=q_\ell(x,y)+O(|y|^{\ell+1}) \tag{10.2.5}$$

holds uniformly in $x\in\mathbb{S}^{d-1}$, where

$$\begin{aligned} q_\ell(x,y)&=i^\ell V_\kappa\Big[(\langle y,\cdot\rangle)^\ell\Big](x)\\ &=c_\kappa i^\ell\int_{[-1,1]^d}\Big(\sum_{j=1}^d t_jx_jy_j\Big)^\ell\prod_{j=1}^d(1-t_j^2)^{\kappa_j-1}(1+t_j)\,dt_j. \end{aligned}$$

(iii) For any $r>0$ and any $x,y\in\mathbb{R}^d$, $A_\ell(rx,y)=A_\ell(x,ry)$.

On the other hand, using Plancherel's formula, we obtain

$$\begin{aligned} \|D^\alpha f\|_{\kappa,2}^2&=\int_{\mathbb{R}^d}\frac{h_\kappa^2(y)}{|y|^{2\lambda_\kappa+1+2\alpha}}\int_{\mathbb{R}^d}\Big|A_\ell(\frac{\xi}{|\xi|},|\xi|y)\Big|^2|\widehat{f}(\xi)|^2h_\kappa^2(\xi)\,d\xi\,dy\\ &=\int_{\mathbb{R}^d}|\widehat{f}(\xi)|^2|\xi|^{2\alpha}h_\kappa^2(\xi)B(\xi/|\xi|)\,d\xi, \end{aligned}$$

where

$$B(x)=\int_{\mathbb{R}^d}\frac{|A_\ell(x,y)|^2}{|y|^{2\lambda_\kappa+1+2\alpha}}h_\kappa^2(y)\,dy.$$

Thus, it suffices to show that

$$B(x)\sim_{\kappa,d}1,\quad \forall x\in\mathbb{S}^{d-1}. \tag{10.2.6}$$

To this end, let $\varepsilon = \varepsilon_{d,\kappa} \in (0,1)$ be a small constant depending only on d and κ, and set

$$B(x,\varepsilon) := \int_{|y|\leqslant\varepsilon} \frac{|A_\ell(x,y)|^2}{|y|^{2\lambda_\kappa+1+2\alpha}} h_\kappa^2(y)\,dy.$$

Clearly,

$$B(x,\varepsilon) \leqslant B(x) \leqslant B(x,\varepsilon) + O_{\varepsilon,\ell}(1). \quad \forall x \in \mathbb{S}^{d-1}.$$

Thus, for the proof of (10.2.6), it is sufficient to show that

$$B(x,\varepsilon) \sim_\varepsilon 1, \quad \forall x \in \mathbb{S}^{d-1}. \tag{10.2.7}$$

Indeed, using (10.2.5), we obtain that for $x \in \mathbb{S}^{d-1}$,

$$\begin{aligned} B(x,\varepsilon) &= \int_{|y|\leqslant\varepsilon} \frac{|q_\ell(x,y)|^2}{|y|^{2\lambda_\kappa+1+2\alpha}} h_\kappa^2(y)\,dy + O(1) \int_{|y|\leqslant\varepsilon} \frac{|y|^{2\ell+1}}{|y|^{2\lambda_\kappa+1+2\alpha}} h_\kappa^2(y)\,dy \\ &= \int_0^\varepsilon r^{2\ell-2\alpha} \int_{\mathbb{S}^{d-1}} |q_\ell(x,y')|^2 h_\kappa^2(y')\,d\sigma(y')dr + O(1)\varepsilon^{2\ell+2-2\alpha} \\ &= \frac{M_\ell(x)}{2\ell-2\alpha+1}\varepsilon^{2\ell-2\alpha+1} + O(1)\varepsilon^{2\ell+2-2\alpha}, \end{aligned}$$

where

$$M_\ell(x) := \int_{\mathbb{S}^{d-1}} |q_\ell(x,y)|^2 h_\kappa^2(y)\,d\sigma(y).$$

Thus, (10.2.7) is a consequence of the following estimate:

$$M_\ell(x) \sim_\ell 1, \quad \forall x \in \mathbb{S}^{d-1}.$$

Since M_ℓ is a homogeneous polynomial of degree 2ℓ on $\mathbb{S}^{d-1}$ and $\mathbb{S}^{d-1}$ is compact, it is enough to show that

$$M_\ell(x) := \int_{\mathbb{S}^{d-1}} |q_\ell(x,y)|^2 h_\kappa^2(y)\, d\sigma(y) > 0, \quad \forall x \in \mathbb{S}^{d-1}. \tag{10.2.8}$$

Assume (10.2.8) were not true, that is, there exists $x = (x_1, \cdots, x_d) \in \mathbb{S}^{d-1}$ such that $M_\ell(x) = 0$. This would imply that $q_\ell(x,y) = 0$ for all $y \in \mathbb{S}^{d-1}$. However, this is impossible because for $y = e_{j_0}$ with $j_0 \in \{1, \cdots, d\}$ satisfying $|x_{j_0}| = \max_{1\leqslant j\leqslant d} |x_j|$, we have

$$\begin{aligned} |q_\ell(x,y)| &= c_{\kappa_{j_0}} \Big| \int_{-1}^{1} (t x_{j_0})^\ell (1-t^2)^{\kappa_{j_0}-1}(1+t)\, dt \Big| \\ &\geq c_{\kappa_{j_0}} d^{\ell/2} \int_{-1}^{1} t^\ell (1-t^2)^{\kappa_{j_0}-1}(1+t)\, dt > 0. \end{aligned}$$

□

Note that by Lemma 10.2.4,

$$\int_{\mathbb{R}^d} |f(x)||x|^\alpha\, dx \sim \|D^{\alpha/2}\widehat{f}\|_{\kappa,2}^2 = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\triangle_y^r f(x)|^2}{|y|^{2\lambda_\kappa+1+\alpha}} h_\kappa^2(y)\, dy h_\kappa^2(x)\, dx, \tag{10.2.9}$$

where

$$\triangle_y^r f(x) = \sum_{j=0}^{r} \binom{r}{j} (-1)^j T^{jy} f(x)$$

and r is the smallest integer bigger than $\alpha/2$. Note that if f is supported in a set $\widetilde{B}(x_0,t) = \bigcup_{\sigma\in\mathbb{Z}_2^d} B(\sigma x_0, t)$, then $\triangle_y^r f(x)$ is supported in the domain $\widetilde{B}(x_0, t + r|y|)$.

Lemma 10.2.5. *For $0 \leqslant \alpha < 2\lambda_\kappa + 1$ and $k \in \mathbb{Z}$,*

$$\int_{2^k}^{2^{k+1}} \int_{\mathbb{R}^d} |S_t^\lambda f(x)|^2 h_\kappa^2(x) \frac{dxdt}{t|x|^\alpha} \leqslant C_\alpha \lambda A_\alpha(\lambda) \int_{\mathbb{R}^d} |f(x)|^2 h_\kappa^2(x) \frac{dx}{|x|^\alpha}. \tag{10.2.10}$$

Proof. It suffices to prove the asserted conclusion for $k = 0$. For simplicity, we denote by $L^2_{x,t}$ the Hilbert L^2-space defined with respect to the measure $t^{-1}|x|^{-\alpha} h_\kappa^2(x) dtdx$ on $\mathbb{R}^d \times [1,2]$. Consider the operator $T : L^2(|x|^{-\alpha} h_\kappa^2(x)) \to L^2_{x,t}$ given by $f \mapsto S_t^\lambda f(x)$. Its dual $T^* : L^2_{x,t} \to L^2(|x|^{-\alpha} h_\kappa^2(x))$ can be obtained as follows: for $\{g_t\}_{1 \leqslant t \leqslant 2} \in L^2_{x,t}$,

$$\begin{aligned}
\langle S_t^\lambda f, g_t \rangle_{L^2_{x,t}} &= \int_{\mathbb{R}^d} \int_1^2 S_t^\lambda f(x) g_t(x) h_\kappa^2(x) \frac{dtdx}{t|x|^\alpha} \\
&= \int_1^2 \int_{\mathbb{R}^d} f(x) \Big[|x|^\alpha S_t^\lambda \Big(|\cdot|^{-\alpha} g_t(\cdot) \Big)(x) \Big] h_\kappa^2(x) \frac{dxdt}{t|x|^\alpha} \\
&= \Big\langle f(x), \int_1^2 \Big[|x|^\alpha S_t^\lambda \Big(|\cdot|^{-\alpha} g_t(\cdot) \Big)(x) \Big] \frac{dt}{t} \Big\rangle_{L^2(|x|^{-\alpha} h_\kappa^2(x))}.
\end{aligned}$$

Thus,

$$T^*(g_t)(x) = \int_1^2 \Big[|x|^\alpha S_t^\lambda \Big(|\cdot|^{-\alpha} g_t(\cdot) \Big)(x) \Big] \frac{dt}{t},$$

and it is sufficient to show that

$$\Big(\int_{\mathbb{R}^d} \Big| \int_1^2 S_t^\lambda \big(|\cdot|^{-\alpha} g_t(\cdot) \big)(x) dt \Big|^2 |x|^\alpha h_\kappa^2(x) dx \Big)^{\frac{1}{2}} \leqslant C \lambda^{\frac{1}{2}} A_\alpha(\lambda)^{\frac{1}{2}} \Big(\int_1^2 \int_{\mathbb{R}^d} |g_t(x)|^2 h_\kappa^2(x) \frac{dxdt}{|x|^\alpha} \Big)^{\frac{1}{2}}.$$

Setting $f_t(x) = |x|^{-\alpha} g_t(x)$, we then conclude that (10.2.10) is a consequence of the following estimate:

$$\int_{\mathbb{R}^d}\Big|\int_1^2 S_t^\lambda f_t(x)dt\Big|^2|x|^\alpha h_\kappa^2(x)\,dx\leqslant C\lambda A_\alpha(\lambda)\int_1^2\int_{\mathbb{R}^d}|f_t(x)|^2|x|^\alpha h_\kappa^2(x)\,dxdt. \qquad (10.2.11)$$

Note that by (10.2.9),

$$\int_{\mathbb{R}^d}\Big|\int_1^2 S_t^\lambda f_t(x)dt\Big|^2|x|^\alpha h_\kappa^2(x)\,dx=C\int_{\mathbb{R}^d}\Big|D^{\alpha/2}\int_1^2\phi_t^\lambda(x)\widehat{f}_t(x)dt\Big|^2h_\kappa^2(x)dx$$
$$=C\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|y|^{-2\lambda_\kappa-1-\alpha}\Big|\int_1^2\triangle_y^r(\phi_t\widehat{f})(x)\,dt\Big|^2h_\kappa^2(y)h_\kappa^2(x)\,dydx$$

Since for $t\in[1,2]$, ϕ_t^λ is supported in $\{z\in\mathbb{R}^d:\ \ (1-\frac{\lambda}{2})t\leqslant|z|\leqslant(1+\frac{\lambda}{2})t\}$, $\triangle_y^r(\phi_t^\lambda\widehat{f}_t)(x)=0$ unless

$$\{z\in\mathbb{R}^d:\ \ |x|-r|y|\leqslant|z|\leqslant|x|+r|y|\}\cap\{z:\ \ (1-\frac{\lambda}{2})t\leqslant|z|\leqslant(1+\frac{\lambda}{2})t\}\neq\emptyset,$$

which holds either

$$(1-\frac{\lambda}{2})t\leqslant|x|-r|y|\leqslant(1+\frac{\lambda}{2})t$$

or

$$(1-\frac{\lambda}{2})t\leqslant\big||x|+r|y|\big|\leqslant(1+\frac{\lambda}{2})t$$

In the first case, setting $a_{x,y}=\frac{|x|-r|y|}{1+\frac{\lambda}{2}}$, we have that

$$0\leqslant t-a_{x,y}\leqslant\big||x|-r|y|\big|\Big(\frac{1}{1-\frac{\lambda}{2}}-\frac{1}{1+\frac{\lambda}{2}}\Big)\leqslant(1+\frac{\lambda}{2})t\lambda(1-\frac{1}{4}\lambda^2)^{-1}$$
$$\leqslant 2(1+\frac{\lambda}{2})\lambda(1-\frac{1}{4}\lambda^2)^{-1}\leqslant c\lambda.$$

Similarly, in the second case, $0 \leqslant t - a'_{x,y} \leqslant c\lambda$, where $a'_{x,y} = \frac{|x|+r|y|}{1+\frac{\lambda}{2}}$. Thus, for fixed $x, y \in \mathbb{R}^d$, the function $A_{x,y}(t) := \triangle_y^r(\phi_t^\lambda \widehat{f}_t)(x)$ is supported in the set

$$I_{x,y} = \Big([a_{x,y}, a_{x,y} + c\lambda] \bigcup [a'_{x,y}, a'_{x,y} + c\lambda]\Big) \bigcap [1, 2].$$

Using Hölder's inequality, we then deduce that

$$\Big|\int_1^2 \triangle_y^r(\phi_t^\lambda \widehat{f}_t)(x)\, dt\Big|^2 \leqslant C\lambda \int_1^2 |\triangle_y^r(\phi_t^\lambda \widehat{f}_t)(x)|^2\, dt.$$

This implies that

$$\begin{aligned}
&\int_{\mathbb{R}^d} \Big|\int_1^2 S_t^\lambda f_t(x) dt\Big|^2 |x|^\alpha h_\kappa^2(x)\, dx \\
&\leqslant C\lambda \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |y|^{-2\lambda_\kappa - 1 - \alpha} \int_1^2 \big|\triangle_y^r(\phi_t^\lambda \widehat{f}_t)(x)\big|^2\, dt h_\kappa^2(x) h_\kappa^2(y)\, dy dx \\
&= C\lambda \int_1^2 \int_{\mathbb{R}^d} |D^{\alpha/2}(\phi_t^\lambda \widehat{f}_t)(x)|^2\, h_\kappa^2(x) dx dt \\
&= C\lambda \int_1^2 \int_{\mathbb{R}^d} |S_t^\lambda f_t(x)|^2 |x|^\alpha h_\kappa^2(x) dx dt,
\end{aligned}$$

which, by (10.2.3), is estimated above by

$$C\lambda A_\alpha(\lambda) \int_1^2 \int_{\mathbb{R}^d} |f_t(x)|^2 |x|^\alpha h_\kappa^2(x) dx dt.$$

This shows the asserted estimate (10.2.11). □

Lemma 10.2.6. *For $0 \leqslant \alpha < 2\lambda_\kappa + 1$,*

$$\int_0^\infty \int_{\mathbb{R}^d} |S_t^\lambda f(x)|^2 h_\kappa^2(x) \frac{dx}{|x|^\alpha} \frac{dt}{t} \leqslant C_\alpha \lambda A_\alpha(\lambda) \int_{\mathbb{R}^d} |f(x)|^2 h_\kappa^2(x) \frac{dx}{|x|^\alpha}.$$

Proof. Recall that $(L_j f)^\wedge(\xi) = \widehat{f}(\xi)\varphi(2^j\xi)$, where $\varphi \in C_c^\infty(\mathbb{R}^d)$ is supported in $[1, 2]$ and satisfies that $\sum_{j\in\mathbb{Z}} \varphi(2^j\xi) = 1$ for all $\xi \in \mathbb{R}^d \setminus \{0\}$. Thus,

$$S_t^\lambda f(x) = f *_\kappa \widehat{\phi_t^\lambda}(x) = \sum_{j\in\mathbb{Z}} (L_j f) *_\kappa \widehat{\phi_t^\lambda}(x),$$

where

$$((L_j f) *_\kappa \widehat{\phi_t^\lambda})^\wedge(\xi) = \varphi(2^j\xi)\widehat{f}(\xi)\phi^\lambda(t^{-1}\xi).$$

Note that $\varphi(2^j\xi)\phi^\lambda(t^{-1}\xi)$ is identically zero unless

$$2^{-j} \leqslant \|\xi\| \leqslant 2 \cdot 2^{-j}, \quad \text{and} \quad \frac{5}{4}\|\xi\| \leqslant t \leqslant \frac{4}{3}\|\xi\|,$$

which also implies that

$$2^{-j-1} \leqslant \frac{4}{5} \cdot 2^{-j} \leqslant t \leqslant \frac{8}{3} \cdot 2^{-j} \leqslant 2^{2-j}.$$

This means that

$$S_t^\lambda f(x) = f *_\kappa \widehat{\phi_t^\lambda}(x) = \sum_{j\in\mathbb{Z}:\ \ 2^{-j-1} \leqslant t \leqslant 2^{2-j}} (L_j f) *_\kappa \widehat{\phi_t^\lambda}(x),$$

and hence

$$|S_t^\lambda f(x)|^2 \leqslant C \sum_{j\in\mathbb{Z}:\ \ 2^{-j-1}\leqslant t\leqslant 2^{2-j}} |(L_j f) *_\kappa \widehat{\phi_t^\lambda}(x)|^2.$$

It follows by Lemma 10.2.5 that

$$\int_{\mathbb{R}^d}\Big(\int_0^\infty |S_t^\lambda f(x)|^2\frac{dt}{t}\Big)h_\kappa^2(x)\frac{dx}{|x|^\alpha} \leqslant C\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^d}\int_{2^{-j-1}}^{2^{j-2}} |S_t^\lambda L_j f(x)|^2\frac{dt}{t}h_\kappa^2(x)\frac{dx}{|x|^\alpha}$$
$$\leqslant CA_\alpha(\lambda)\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^d}|L_j f|^2 h_\kappa^2(x)\frac{dx}{|x|^\alpha} = CA_\alpha(\lambda)\Big\|\big(\sum_{j\in\mathbb{Z}}|L_j f|^2\big)^{1/2}\Big\|^2_{L^2(|x|^{-\alpha}h_\kappa^2(x))}.$$

Since for $0\leqslant\alpha<2\lambda_\kappa+1$, $w(x)=|x|^{-\alpha}\in A_1$, it follows by the weighted

Paley-Littlewood inequality that

$$\Big\|\big(\sum_{j\in\mathbb{Z}}|L_j f|^2\big)^{1/2}\Big\|^2_{L^2(|x|^{-\alpha}h_\kappa^2(x))} \leqslant C\|f\|^2_{L^2(|x|^{-\alpha})} = C\int_{\mathbb{R}^d}|f(x)|^2 h_\kappa^2(x)\frac{dx}{|x|^\alpha}.$$

This completes the proof. □

Let $\widetilde{\phi}(u)=\lambda\phi'(u)u$ for $u\geq 0$. Define $\widetilde{S_t^\lambda}f(x)$ by

$$\Big(\widetilde{S_t^\lambda}f\Big)^\wedge(\xi)=\widetilde{\phi}_t(|\xi|)\widehat{f}(\xi),\quad t>0,$$

where $\widetilde{\phi}_t^\lambda(|\xi|)=\widetilde{\phi}^\lambda(t^{-1}|\xi|)$. It is easily seen that

$$\frac{d}{dt}S_t^\lambda f(x) = -\frac{1}{t\lambda}\widetilde{S_t^\lambda}f(x). \tag{10.2.12}$$

Lemma 10.2.7. *For* $0 \leqslant \alpha < 2\lambda_\kappa + 1$,

$$\int_0^\infty \int_{\mathbb{R}^d} |\widetilde{S_t^\lambda} f(x)|^2 h_\kappa^2(x) \frac{dx}{|x|^\alpha} \frac{dt}{t} \leqslant C_\alpha \lambda A_\alpha(\lambda) \int_{\mathbb{R}^d} |f(x)|^2 h_\kappa^2(x) \frac{dx}{|x|^\alpha}.$$

Now, we are in a position to proof Theorem 10.1.3.

Proof of Theorem 10.1.3

Proof. For $t > 0$,

$$\begin{aligned}
\Big| |S_t^\lambda f(x)|^2 - |S_1^\lambda f(x)|^2 \Big| &= 2\Big| \int_1^t (S_u^\lambda f(x)) \big(\frac{d}{du} S_u^\lambda f(x) \big)\, dx \Big| = \frac{2}{\lambda} \Big| \int_1^t (S_u^\lambda f(x)) (\widetilde{S_u^\lambda} f(x)) \frac{du}{u} \Big| \\
&\leqslant \frac{2}{\lambda} \int_0^\infty |S_u^\lambda f(x)| |\widetilde{S_u^\lambda} f(x)| \frac{du}{u} \\
&\leqslant \frac{2}{\lambda} \Big(\int_0^\infty |S_u^\lambda f(x)|^2 \frac{du}{u} \Big)^{\frac{1}{2}} \Big(\int_0^\infty |\widetilde{S_u^\lambda} f(x)|^2 \frac{du}{u} \Big)^{\frac{1}{2}} \\
&=: \frac{2}{\lambda} G^\lambda f(x) \widetilde{G^\lambda} f(x),
\end{aligned}$$

where

$$\begin{aligned}
G^\lambda f(x) &= \Big(\int_0^\infty |S_t^\lambda f(x)|^2 \frac{dt}{t} \Big)^{\frac{1}{2}}, \\
\widetilde{G^\lambda} f(x) &= \Big(\int_0^\infty |\widetilde{S_t^\lambda} f(x)|^2 \frac{dt}{t} \Big)^{\frac{1}{2}}.
\end{aligned}$$

It follows that

$$\begin{aligned}\int_{\mathbb{R}^d} |S_*^\lambda f(x)|^2 h_\kappa^2(x)\frac{dx}{|x|^\alpha} &\leqslant C\int_{\mathbb{R}^d} |S_1^\lambda f(x)|^2 h_\kappa^2(x)\frac{dx}{|x|^\alpha} + C\lambda^{-1}\int_{\mathbb{R}^d} G^\lambda f(x)\widetilde{G^\lambda} f(x) h_\kappa^2(x)\frac{dx}{|x|^\alpha}\\ &\leqslant CA_\alpha(\lambda)\int_{\mathbb{R}^d} |f(x)|^2 h_\kappa^2(x)\frac{dx}{|x|^\alpha}\\ &+ C\lambda^{-1}\Big(\int_{\mathbb{R}^d} |G^\lambda f(x)|^2 h_\kappa^2(x)\frac{dx}{|x|^\alpha}\Big)^{\frac12}\Big(\int_{\mathbb{R}^d} |\widetilde{G^\lambda} f(x)|^2 h_\kappa^2(x)\frac{dx}{|x|^\alpha}\Big)^{\frac12}\\ &\leqslant CA_\alpha(\lambda)\int_{\mathbb{R}^d} |f(x)|^2 h_\kappa^2(x)\frac{dx}{|x|^\alpha}.\end{aligned}$$

□

10.3 Proof of Lemma 10.2.1

To prove Lemma 10.2.1, we need the following Lemma.

Lemma 10.3.1. *For $0<\varepsilon<100d$ and $0\leqslant\alpha<2\lambda_\kappa+1$, we have*

$$\int_{\big|1-|\xi|\big|\leqslant\varepsilon} |\widehat{f}(\xi)|^2 h_\kappa^2(\xi)\,d\xi \leqslant C_\alpha A_\alpha(\varepsilon)\varepsilon^\alpha \int_{\mathbb{R}^d} |f(x)|^2|x|^\alpha h_\kappa^2(x)dx. \tag{10.3.13}$$

Furthermore, for any $M>2\lambda_\kappa+1$ and $\alpha>0$,

$$\int_{\mathbb{R}^d} |\widehat{f}(\xi)|^2(1+|\xi|)^{-M} h_\kappa^2(\xi)\,d\xi \leqslant C_{M,d,\alpha}\int_{\mathbb{R}^d} |f(x)|^2|x|^\alpha h_\kappa^2(x)\,dx. \tag{10.3.14}$$

The proof of Lemma 10.3.1 is technical, we postpone it in Section 10.4. Now, let us prove Lemma 10.2.1 by using Lemma 10.3.1.

Proof of Lemma 10.2.1

Proof. We follow the notations of Section 9.4. According to (9.4.8), we have

$$S_1^\lambda f(x) := \sum_{j=i}^{\infty} f *_\kappa (\widehat{\phi^\lambda}\eta_j)(x) = \sum_{j=i}^{\infty} f *_\kappa K_j(x), \tag{10.3.15}$$

where $K_j(x) = \widehat{\phi^\lambda}(x)\eta_j(x)$ and $2^{-i} \sim \delta$. By Lemma 9.3.1, for any $N \ge 1$,

$$|\widehat{K_j}(x)| = |\phi^\lambda * \widehat{\eta_j}(x)| \leqslant \begin{cases} C2^{-(j-i)N}\big(1+\lambda^{-1}\big|1-|x|\big|\big)^{-N}, & \frac{1}{4} \leqslant |x| \leqslant 8, \\ C2^{-(j-i)N}\lambda^N(1+|x|)^{-N}, & \text{otherwise.} \end{cases}$$

In particular, this implies that

$$|\widehat{K_j}(x)| \leqslant \begin{cases} C2^{-(j-i)N}, & \text{for all } x \in \mathbb{R}^d, \\ C2^{-(j-i)N}2^{-kN}, & \text{if } \big|1-|x|\big| \ge 2^k\lambda \text{ and } k \ge 4, \\ C2^{-(j-i)N}\lambda^N(1+|x|)^{-N}, & \text{if } |x| \leqslant \frac{1}{4} \text{ or } |x| \ge 8. \end{cases} \tag{10.3.16}$$

For $j \ge i$, we set $T_j f = f *_\kappa K_j$, and claim that

$$\int_{\mathbb{R}^d} |T_j f(x)|^2 h_\kappa^2(x)\,dx \leqslant C2^{-N(j-i)}\lambda^\alpha A_\alpha(\lambda) \int_{\mathbb{R}^d} |f(x)|^2 |x|^\alpha h_\kappa^2(x)\,dx. \tag{10.3.17}$$

Here and throughout the proof, N is a large positive integer whose exact value is not important. Indeed,

$$\begin{aligned}\int_{\mathbb{R}^d} |T_j f(x)|^2 h_\kappa^2(x)\,dx &= \int_{\mathbb{R}^d} |\widehat{f}(x)|^2 |\widehat{K_j}(x)|^2 h_\kappa^2(x)\,dx \\ &= \int_{\left|1-|x|\right|\leqslant 16\lambda} \cdots + \int_{16\lambda<\left|1-|x|\right|\leqslant 9} \cdots + \int_{\left|1-|x|\right|>9} \cdots \\ &=: I_1 + I_2 + I_3.\end{aligned}$$

For the first integral I_1, using (10.3.16) and Lemma 10.3.1, we have

$$\begin{aligned}I_1 &\leqslant C 2^{-N(j-i)} \int_{\left|1-|x|\right|\leqslant 16\lambda} |\widehat{f}(x)|^2 h_\kappa^2(x)\,dx \\ &\leqslant C\lambda^\alpha A_\alpha(\lambda) 2^{-N(j-i)} \int_{\mathbb{R}^d} |f(x)|^2 |x|^\alpha h_\kappa^2(x)\,dx.\end{aligned}$$

For I_2, using (10.3.16) and Lemma 10.3.1 again, we obtain

$$\begin{aligned}I_2 &= \sum_{16\lambda\leqslant 2^k\lambda\leqslant 9} \int_{2^k\lambda\leqslant\left|1-|x|\right|\leqslant 2^{k+1}\lambda} |\widehat{f}(x)|^2 |\widehat{K_j}(x)|^2 h_\kappa^2(x)\,dx \\ &\leqslant C 2^{-N(j-i)} \sum_{16\leqslant 2^k\leqslant 9/\lambda} 2^{-kN} \int_{\left|1-|x|\right|\leqslant 2^{k+1}\lambda} |\widehat{f}(x)|^2 h_\kappa^2(x)\,dx \\ &\leqslant C 2^{-N(j-i)} \sum_{16\leqslant 2^k\leqslant 9/\lambda} 2^{-kN} A_\alpha(2^{k+1}\lambda)(2^{k+1}\lambda)^\alpha \int_{\mathbb{R}^d} |f(x)|^2 |x|^\alpha h_\kappa^2(x)\,dx \\ &\leqslant C 2^{-N(j-i)} \lambda^\alpha A_\alpha(\lambda) \int_{\mathbb{R}^d} |f(x)|^2 |x|^\alpha h_\kappa^2(x)\,dx \sum_{k=1}^\infty 2^{-kN} 2^{k(\alpha+1)} \\ &\leqslant C 2^{-N(j-i)} \lambda^\alpha A_\alpha(\lambda) \int_{\mathbb{R}^d} |f(x)|^2 h_\kappa^2(x) |x|^\alpha\,dx.\end{aligned}$$

For I_3, we have

$$\begin{aligned}
I_3 &\leqslant C\lambda^N 2^{-(j-i)N} \int_{\left|1-|x|\right|\geq 9} |\widehat{f}(x)|^2 (1+|x|)^{-N} h_\kappa^2(x)\, dx \\
&\leqslant C\lambda^N 2^{-(j-i)N} \int_{\mathbb{R}^d} |f(x)|^2 |x|^\alpha h_\kappa^2(x)\, dx \\
&\leqslant C 2^{-(j-i)N} \lambda^\alpha A_\alpha(\lambda) \int_{\mathbb{R}^d} |f(x)|^2 |x|^\alpha h_\kappa^2(x)\, dx.
\end{aligned}$$

Now combining the above estimates of I_1, I_2, I_3, we deduce the estimate (10.3.17).

Second, we show that (10.3.17) implies that for $j \geq i$,

$$\int_{\mathbb{R}^d} |T_j f(x)|^2 h_\kappa^2(x) \frac{dx}{|x|^\alpha} \leqslant C 2^{-N(j-i)} \lambda^\alpha A_\alpha(\lambda) \int_{\mathbb{R}^d} |f(x)|^2 h_\kappa^2(x)\, dx. \tag{10.3.18}$$

Indeed, the dual operator $T_j^* : L^2(h_\kappa^2) \to L^2(|x|^\alpha h_\kappa^2(x))$ of the operator $T_j : L^2(|x|^\alpha h_\kappa^2(x)) \to L^2(h_\kappa^2)$ can be obtained as follows:

$$\langle T_j f, g\rangle_{L^2(h_\kappa^2)} = \langle f, T_j g\rangle_{L^2(h_\kappa^2)} = \langle f, T_j^* g\rangle_{L^2(|x|^\alpha h_\kappa^2(x))},$$

where $T_j^* g(x) = |x|^{-\alpha} T_j g(x)$. The asserted estimate (10.3.18) then follows by (10.3.17) and duality.

Third, we show that for $j \geq i$,

$$\int_{\mathbb{R}^d} |T_j f(x)|^2 h_\kappa^2(x) \frac{dx}{|x|^\alpha} \leqslant C 2^{-N(j-i)} A_\alpha(\lambda) \int_{\mathbb{R}^d} |f(x)|^2 h_\kappa^2(x) \frac{dx}{|x|^\alpha}. \tag{10.3.19}$$

To this end, we recall the following local property of the operator T_j: If f is supported in a cube Q of side length $c2^j$, then $T_j f$ is supported in the set

$$Q^* = \bigcup_{\sigma \in \mathbb{Z}_2^d} \sigma\Big(c_d Q\Big).$$

We decompose $\mathbb{R}^d$ as an almost pairwise disjoint union of cubes Q_k, $k \in \mathbb{Z}$, where Q_0 is a cube centered at the origin of side length $2^\ell 2^j$, and the Q_k, $k \neq 0$ are cubes of side length 2^j. We choose $\ell \in \mathbb{N}$ large enough so that $Q_k^* \cap \frac{1}{2}Q_0 = \emptyset$ for all $k \neq 0$. With this decomposition, we have

$$\begin{aligned}
&|x| \leqslant C2^j, \quad \text{for all } x \in Q_0^*,\\
&|x| \sim |x'| \ \text{ for all } x, x' \in Q_k^* \text{ and } k \neq 0.
\end{aligned}$$

Set $f_k = f\chi_{Q_k}$ for $k \in \mathbb{Z}$. Then $f = \sum_{k\in\mathbb{Z}} f_k$. For $k = 0$, we use (10.3.18) to obtain

$$\begin{aligned}
\int_{\mathbb{R}^d} |T_j f_0(x)|^2 |x|^{-\alpha} h_\kappa^2(x)\, dx &\leqslant C2^{-N(j-i)} \lambda^\alpha A_\alpha(\lambda) \int_{Q_0} |f(x)|^2 h_\kappa^2(x)\, dx\\
&\leqslant C2^{-N(j-i)} \lambda^\alpha A_\alpha(\lambda) 2^{j\alpha} \int_{Q_0} |f(x)|^2 h_\kappa^2(x)\, \frac{dx}{|x|^\alpha}\\
&\leqslant C2^{-(j-i)(N-\alpha)} A_\alpha(\lambda) \int_{Q_0} |f(x)|^2 h_\kappa^2(x) \frac{dx}{|x|^\alpha}.
\end{aligned}$$

For $k \neq 0$, we use Plancerel's theorem to obtain

$$\int_{Q_k^*} |T_j f_k(x)|^2 h_\kappa^2(x)\, dx = \int_{\mathbb{R}^d} |\widehat{f}_k(x)|^2 |\widehat{K_j}(x)|^2 h_\kappa^2(x)\, dx \leqslant C2^{-(j-i)N} \int_{Q_k} |f(x)|^2 h_\kappa^2(x)\, dx.$$

Since $|x| \sim |x'|$ for all $x, x' \in Q_k^*$, it follows that for $k \neq 0$,

$$\int_{Q_k^*} |T_j f_k(x)|^2 h_\kappa^2(x) \frac{dx}{|x|^\alpha} \leqslant C 2^{-(j-i)N} \int_{Q_k} |f(x)|^2 h_\kappa^2(x) \frac{dx}{|x|^\alpha}$$
$$\leqslant C 2^{-(j-i)N} A_\alpha(\lambda) \int_{Q_k} |f(x)|^2 h_\kappa^2(x) \frac{dx}{|x|^\alpha}.$$

On the other hand, since supp $T_j f_k \subset Q_k^*$ for all $k \in \mathbb{Z}$ and $\sum_{k\in\mathbb{Z}} \chi_{Q_k^*}(x) \leqslant C_d$, we have

$$|T_j f(x)|^2 \leqslant \Big(\sum_{k\in\mathbb{Z}} |T_j f_k(x)| \chi_{Q_k^*}(x)| \Big)^2 \leqslant C_d^2 \sum_{k\in\mathbb{Z}} |T_j f_k(x)|^2 \chi_{Q_k^*}(x).$$

It follows that

$$\int_{\mathbb{R}^d} |T_j f(x)|^2 h_\kappa^2(x) \frac{dx}{|x|^\alpha} \leqslant C \sum_{k\in\mathbb{Z}} \int_{Q_k^*} |T_j f_k(x)|^2 h_\kappa^2(x) \frac{dx}{|x|^\alpha}$$
$$\leqslant C A_\alpha(\lambda) 2^{-(j-i)N} \sum_{k\in\mathbb{Z}} \int_{Q_k} |f_k(x)|^2 h_\kappa^2(x) \frac{dx}{|x|^\alpha}$$
$$= C A_\alpha(\lambda) 2^{-(j-i)N} \int_{\mathbb{R}^d} |f(x)|^2 h_\kappa^2(x) \frac{dx}{|x|^\alpha}.$$

This proves (10.3.19).

Finally, using (10.3.15) and Minkowski's inequality, we obtain

$$\|S_1^\lambda f\|_{L^2(|x|^{-\alpha} h_\kappa^2(x))} \leqslant \sum_{j=i}^\infty \|T_j f\|_{L^2(|x|^{-\alpha} h_\kappa^2(x))} \leqslant C \sqrt{A_\alpha(\lambda)} \sum_{j=i}^\infty 2^{-(j-i)} \|f\|_{L^2(|x|^{-\alpha} h_\kappa^2(x))}$$
$$\leqslant C \sqrt{A_\alpha(\lambda)} \|f\|_{L^2(|x|^{-\alpha} h_\kappa^2(x))}.$$

□

10.4 Proof of Lemma 10.3.1

Definition 10.4.1. For $0 < \alpha < 2\lambda_\kappa + 1$, define

$$I_\kappa^\alpha f(x) = c(\kappa, \alpha)^{-1} \int_{\mathbb{R}^d} T^y f(x) |y|^{\alpha - 2\lambda_\kappa - 1} h_\kappa^2(y)\, dy,$$

where

$$c(\kappa, \alpha) := 2^{\alpha - \lambda_\kappa - \frac{1}{2}} \frac{\Gamma(\alpha/2)}{\Gamma(\lambda_\kappa + \frac{1}{2} - \frac{\alpha}{2})}.$$

Remark 10.4.2. For $0 < \alpha < 2\lambda_\kappa + 1$, the identity [47, Proposition 4.1]

$$\widehat{I_\kappa^\alpha f}(x) = |x|^{-\alpha} \widehat{f}(x)$$

holds in a distributional sense. We also define the operator $(-\Delta_\kappa)^\alpha$ in a distributional sense by

$$\Big((-\Delta_\kappa)^\alpha f\Big)^\wedge(\xi) = |\xi|^{2\alpha} \widehat{f}(\xi).$$

Then $I_\kappa^\alpha (-\Delta_\kappa)^{\alpha/2} f = f$.

Lemma 10.4.3. *[47, Lemma 4.1] If $0 < \alpha < 2\lambda_\kappa + 1$, then*

$$\mathcal{F}_\kappa\Big(|\cdot|^{\alpha - 2\lambda_\kappa - 1}\Big)(\xi) = c(\kappa, \alpha) |\xi|^{-\alpha}$$

holds in a distributional sense.

Lemma 10.4.4. *[47, Theorem 4.3] Let $1 < p < q < \infty$ and $0 < \alpha < 2\lambda_\kappa + 1$ be such* *that* $\alpha = (2\lambda_\kappa + 1)(\frac{1}{p} - \frac{1}{q})$. *Then*

$$\|I_\kappa^\alpha f\|_{\kappa,q} \leqslant C\|f\|_{\kappa,p}, \quad \forall f \in L^p(h_\kappa^2).$$

Lemma 10.4.5. *Let* $0 < \alpha < 2\lambda_\kappa + 1$ *and* $0 < \varepsilon < 100d$. *Set* $u_\alpha(x) = |x|^{\alpha-2\lambda_\kappa-1}$ *and* $E = \{x \in \mathbb{R}^d : \quad ||x| - 1| \leqslant \varepsilon$. *Then for* $x, y \in E$,

$$h_\kappa^2(x)|T^y u_\alpha(x)| \leqslant \begin{cases} C|\bar{x} - \bar{y}|^{\alpha-d}, & \text{if } 0 < \alpha < d, \\ C\Big|\ln|\bar{x} - \bar{y}|\Big|, & \text{if } \alpha = d, \\ C, & \text{if } d < \alpha < 2\lambda_\kappa + 1. \end{cases}$$

Proof. Let $\psi \in C_c^\infty(\mathbb{R}^d)$ be a radial function supported in $[\frac{1}{2}, 4]$ such that $\sum_{k\in\mathbb{Z}} \psi(2^k x) = 1, \quad \forall x \neq 0$. Then setting $\varphi(x) = |x|^{\alpha-2\lambda_\kappa-1}\psi(x)$, we obtain that for $x \neq 0$,

$$u_\alpha(x) = \sum_{k\in\mathbb{Z}} |x|^{\alpha-2\lambda_\kappa-1}\psi(2^k x) = \sum_{k\in\mathbb{Z}} 2^{k(2\lambda_\kappa+1-\alpha)}\varphi(2^k x) = \sum_{k\in\mathbb{Z}} 2^{-k\alpha}\varphi_{2^{-k}}(x),$$

where $\varphi_t(x) = t^{-2\lambda_\kappa-1}\varphi(x/t)$. It follows by Lemma 7.2.6 that for $\ell > 2\lambda_\kappa + 3$,

$$\begin{aligned} |T^y u_\alpha(x)| = |\sum_{k\in\mathbb{Z}} 2^{-k\alpha} T^y \varphi_{2^{-k}}(x)| &\leqslant C \sum_{k\in\mathbb{Z}} 2^{-k\alpha} \frac{1}{\big(1 + 2^k\|\bar{x} - \bar{y}\|\big)^\ell \int_{B(x,2^{-k})} h_\kappa^2(z)\,dz} \\ &\leqslant C \sum_{k\in\mathbb{Z}} 2^{k(d-\alpha)}\big(1 + 2^k\|\bar{x} - \bar{y}\|\big)^{-\ell} \prod_{j=1}^d (|x_j| + 2^{-k})^{-2\kappa_j}. \end{aligned}$$

For simplicity, we set $\rho = |\bar{x} - \bar{y}|$. If $0 < \alpha < d$, then

$$h_\kappa^2(x)|T^y u_\alpha(x)| \leqslant C \sum_{2^k\rho \leqslant 1} 2^{k(d-\alpha)} + C \sum_{2^k\rho > 1} 2^{k(d-\alpha)}(2^k\rho)^{-\ell} \leqslant C\rho^{\alpha-d}.$$

If $\alpha = d$, then

$$h_\kappa^2(x)|T^y u_\alpha(x)| \leqslant C \sum_{2^k\rho \leqslant 1} 1 + C \sum_{2^k\rho > 1} (2^k\rho)^{-\ell} \leqslant C|\ln \rho|.$$

Finally, if $d < \alpha < 2\lambda_\kappa + 1$, then

$$\begin{aligned} & h_\kappa^2(x)|T^y u_\alpha(x)| \\ & \leqslant Ch_\kappa^2(x) \sum_{k \leqslant 0} 2^{k(2\lambda_\kappa+1-\alpha)} + C \sum_{\substack{2^k\rho \leqslant 1 \\ k \geq 0}} 2^{k(d-\alpha)} + C \sum_{2^k\rho \geq 1} 2^{k(d-\alpha)}(2^k\rho)^{-\ell} \\ & \leqslant Ch_\kappa^2(x) + C + C\rho^{\alpha-d} \leqslant C. \end{aligned}$$

□

Now, we are in a position to proof Lemma 10.3.1.

Proof of Lemma 10.3.1

Proof. We start with the proof of (10.3.13). Without loss of generality, we may assume $\alpha > 0$. Set $E := \{x \in \mathbb{R}^d : \ \big|1 - |x|\big| \leqslant \varepsilon\}$, and denote by $L^2(E; h_\kappa^2)$ the subspace of $L^2(h_\kappa^2)$ consisting of all functions supported in the set E. We first claim that (10.3.13), is a consequence of the following estimate: for any function $g \in L^2(E; h_\kappa^2)$,

$$\int_{\mathbb{R}^d} |\widehat{g}(x)|^2 |x|^{-\alpha} h_\kappa^2(x)\, dx \leqslant C_\alpha A_\alpha(\varepsilon)\varepsilon^\alpha \int_E |g(x)|^2 h_\kappa^2(x)\, dx. \tag{10.4.20}$$

To see this, consider the operator $T : L^2(|x|^\alpha h_\kappa^2(x)) \to L^2(E; h_\kappa^2)$ given by $Tf = \widehat{f}\Big|_E$, and note that for any $g \in L^2(E; h_\kappa^2)$,

$$\langle Tf, \bar{g}\rangle_{L^2(E;h_\kappa^2)} = \int_E \widehat{f}(\xi) g(\xi) h_\kappa^2(\xi)\, d\xi = \int_{\mathbb{R}^d} f(x)\Big(|x|^{-\alpha}\widehat{g}(x)\Big)|x|^\alpha h_\kappa^2(x) dx.$$

This means that $T^*\bar{g}(x) = |x|^{-\alpha}\overline{\widehat{g}(x)}$ where $T^* : L^2(E; h_\kappa^2) \to L^2(|x|^\alpha h_\kappa^2(x))$ denotes the dual operator of T. The claim then follows.

By the standard density argument, it suffices to show (10.4.20) for $g \in L^2(E; h_\kappa^2) \cap C_c^\infty(\mathbb{R}^d)$. Indeed, invoking Lemma 10.4.3, we obtain that for $g \in L^2(E; h_\kappa^2) \cap C_c^\infty(\mathbb{R}^d)$,

$$\begin{aligned}
&\int_{\mathbb{R}^d} |\widehat{g}(x)|^2 |x|^{-\alpha} h_\kappa^2(x)\, dx \\
&= \int_{\mathbb{R}^d} (g *_\kappa \overline{\overline{g}})^\wedge(\xi)|\xi|^{-\alpha} h_\kappa^2(\xi)\, d\xi = \int_{\mathbb{R}^d} (g *_\kappa \overline{\overline{g}})(\xi)\mathcal{F}_\kappa(|\cdot|^{-\alpha})(\xi) h_\kappa^2(\xi)\, d\xi \\
&= c(\kappa,\alpha)\int_{\mathbb{R}^d} g *_\kappa \overline{\overline{g}}(x)|x|^{\alpha-2\lambda_\kappa-1} h_\kappa^2(x)\, dx = c(\kappa,\alpha)\int_E g(y)\overline{g *_\kappa u_\alpha(y)}\, h_\kappa^2(y) dy \\
&= c(\kappa,\alpha)\langle g, Lg\rangle_{L^2(E;h_\kappa^2)} \leqslant C\|Lg\|_{L^2(E;h_\kappa^2)}\|g\|_{L^2(E;h_\kappa^2)},
\end{aligned}$$

where

$$Lg(x) := \int_E g(y) T^y u_\alpha(x)\, dy, \quad x \in E.$$

Clearly,

$$\|Lg\|_{L^2(E;h_\kappa^2)} \leqslant B_\alpha \|g\|_{L^2(E;h_\kappa^2)},$$

where B_α is the Lebesgue constant of the operator L; that is,

$$B_\alpha = \sup_{x\in E} \int_E T^y u_\alpha(x) h_\kappa^2(y)\, dy.$$

According to Lemma 10.4.5, if $0 < \alpha < d$, then

$$\begin{aligned} B_\alpha &\leqslant C \sup_{x\in E} \int_E |\bar{x} - \bar{y}|^{\alpha-d}\, dy \leqslant C \sum_{\sigma\in\mathbb{Z}_2^d} \sup_{x\in E} \int_E |\sigma x - y|^{\alpha-d}\, dy \\ &\leqslant C \sup_{x\in E} \int_E |x - y|^{\alpha-d}\, dy \leqslant C_\alpha \varepsilon^\alpha A_\alpha(\varepsilon); \end{aligned}$$

if $\alpha = d$, then

$$\begin{aligned} B_\alpha &\leqslant C \sup_{x\in E} \int_E \Big| \ln |\bar{x} - \bar{y}| \Big|\, dy \leqslant C \sup_{x\in E} \int_E \Big| \ln |x - y| \Big|\, dy \\ &\leqslant C \sup_{x\in E} \int_{\{y\in E:\ \ |y-x|\leqslant 4\varepsilon\}} \Big| \ln |x - y| \Big|\, dy + C \sup_{x\in E} \int_{\{y\in E:\ \ |y-x|>4\varepsilon\}} \Big| \ln |x - y| \Big|\, dy \\ &\leqslant C \int_0^{4\varepsilon} r^{d-1} \ln r\, dr + C \int_{1-\varepsilon}^{1+\varepsilon} r^{d-1} dr \max_{x'\in\mathbb{S}^{d-1}} \int_{\{y'\in\mathbb{S}^{d-1}:\ \ |x'-y'|\geq 2\varepsilon\}} \Big| \ln |x' - y'| \Big| \\ &\leqslant C\varepsilon \int_{2\varepsilon}^{\pi} |\ln\theta| \sin^{d-2}\theta\, d\theta \leqslant C\varepsilon; \end{aligned}$$

and finally, if $d < \alpha < 2\lambda_\kappa + 1$, then

$$B_\alpha \leqslant C \sup_{x\in E} \int_E dy \leqslant C\varepsilon.$$

Now we turn to the proof of (10.3.14). Note that

$$\int_{\mathbb{R}^d} |f(x)|^2 |x|^\alpha h_\kappa^2(x)\,dx = C\int_{\mathbb{R}^d} \big|(-\Delta_{\kappa,0})^{\alpha/2}\widehat{f}(\xi)\big|^2 h_\kappa^2(\xi)\,d\xi = \Big\|(-\Delta_{\kappa,0})^{\alpha/2}\widehat{f}\|_{\kappa,2}^2.$$

Thus, it suffices to show that

$$\int_{\mathbb{R}^d} |f(x)|^2 (1+|x|)^{-M} h_\kappa^2(x)\,dx \leqslant C\|(-\Delta_\kappa)^{\alpha/2} f\|_{\kappa,2}^2. \tag{10.4.21}$$

(10.4.21) is an easy consequence of Lemma 10.4.4. Indeed, let $q \geq 2$ be such that $(2\lambda_\kappa+1)(\frac{1}{2}-\frac{1}{q}) = \frac{\alpha}{2}$. Then using Hölder's inequality and Lemma 10.4.4, we obtain

$$\int_{\mathbb{R}^d} |f(x)|^2 (1+|x|)^{-M} h_\kappa^2(x)\,dx \leqslant C\|f\|_{\kappa,q}^2 = C\Big\|I_\kappa^\alpha(-\Delta_\kappa)^{\alpha/2} f\Big\|_{\kappa,q}^2 \leqslant C\Big\|(-\Delta_\kappa)^{\alpha/2} f\Big\|_{\kappa,2}^2.$$

□

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