University of Alberta

TOPOLOGICAL CENTER OF DUAL BANACH ALGEBRAS ASSOCIATED TO HYPERGROUPS

BY

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A THESIS

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to my Parents, teachers, my wife and my children

Atiyeh and Ata

Abstract

A hypergroup is essentially a locally compact Hausdorff space in which the product of two elements is a probability measure with compact support. Such spaces have been studied by Dunkl, Jewett and Spector. Hypergroups naturally arise as double coset spaces of locally compact groups by compact subgroups.

Let K be a (commutative) locally compact hypergroup with a left Haar measure. Let $L^1(K)$ be the hypergroup algebra of K and $UC_r(K)$ be the Banach space of bounded left uniformly continuous complex-valued functions on K.

In this thesis, we show, among other things, that the topological (algebraic) centers of the Banach algebras $L^1(K)^{**}$ and $UC_r(K)^*$ are $L^1(K)$ and $M(K)$, the measure algebra of K, respectively. Some applications to $L^1(K)$, $L^1(K)^{**}$ and $M(K)^{**}$ are given.

Let $P^1(K) = \{f \in L^1(K) : f \geq 0, \|f\|_1 = 1\}$. Then $P^1(K)$ is a topological semigroup under the convolution product of $L^1(K)$ induced in $P^1(K)$. We say that K has property (p) if there exists a left invariant mean on $C(P^1(K))$, the space of bounded continuous functions on $P^1(K)$. We study hypergroups with property (p) and its relation with amenability of K.

A short proof of the existence of Haar measure for commutative hypergroups is also given.

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Contents

Chapter 1

INTRODUCTION

The theory of hypergroups was initiated by Dunkl [12], Jewett [33] and Spector [57] in the early 1970's and has received a good deal of attention from harmonic analysts. Hypergroups are sufficiently general to cover a variety of important examples including double coset spaces, yet have enough structure to allow a substantial theory to develop. Although the definitions of hypergroup given by the three authors are not identical, the ideas are essentially the same and all interesting examples are hypergroups by all the definitions. Jewett calls hypergroups "convos" in his paper [33]. In [46], Pym also considers convolution structures which are close to hypergroups. A fairly complete history is given in Ross's survey article [51] (see also [52]).

Let $L^1(G)^{**}$ be the second dual algebra of the group algebra $L^1(G)$ of a locally compact group G. Civin and Yood were the first authors who made in [6] an extensive study of $L^1(G)^{**}$.

In [65], Young proved that $L^1(G)$ is Arens regular if and only if G is a finite group.

Another proof of Young's result was later given by Ülger in [58]. This result was extended to hypergroups by Skantharajah in [56, Theorem 5.2.3].

Watanabe proved that $L^1(G)$ is an ideal of $L^1(G)^{**}$ if and only if G is compact [64]. Later, Grosser proved this result in [25] using functional analytic methods.

The results on the second dual algebras obtained prior to the year 1979 were surveyed in the article by Duncan and Hosseiniun [10].

Renewed interest in the second dual algebra of the group algebra $L^1(G)$ emerged by publication of [31] by Isik, Ülger and Pym, where among other things, it was shown that if G is a compact group, then the topological center of $L^1(G)^{**}$ is precisely $L^1(G)$.

Lau and Losert in [38] showed that $L^1(G)$ is the topological center of $L^1(G)^{**}$ for any locally compact group G. In [39], Lau and Ülger study related results on weakly sequentially complete Banach algebras with bounded approximate identity.

Let $UC_r(G)$ be the Banach space of all bounded left uniformly continuous functions on G and $UC_r(G)^*$ be its dual Banach space. Then $UC_r(G)^*$ under the restriction of the multiplication on $L^1(G)^{**}$ is a Banach algebra. In [37], Lau showed that the topological center of $UC_r(G)^*$ is $M(G)$, the algebra of bounded regular Borel measures on G.

Let K be hypergroup with a left Haar measure λ. Let $L^1(K)$ denotes the hypergroup algebra of K (see [21] and [22]) i.e. all Borel measurable functions ϕ on K with $\|\phi\|_1 = \int_K |\phi(x)|\, d\lambda(x) < \infty$ (with functions equal almost everywhere identified).

and the multiplication defined by

$$\phi * \psi(x) = \int_K \phi(x * y)\psi(\check{y})\, d\lambda(y) \qquad \text{(see [33, §5.5])}.$$

Let the second dual $L^1(K)^{**}$ $(= L^\infty(K)^*)$ of $L^1(K)$ be equipped with the first Arens product [10]. Then $L^1(K)^{**}$ is a Banach algebra with this product. The *topological center* of $L^1(K)^{**}$ is defined by

$$Z(L^1(K)^{**}) = \{m \in L^1(K)^{**} : \text{the mapping } n \longmapsto mn \text{ is}$$
$$w^* - \text{continuous on } L^1(K)^{**}\}.$$

Let $UC_r(K)$ be the Banach space of all bounded left uniformly continuous complex-valued functions on K and $UC_r(K)^*$ be its dual Banach space. Then there is a natural multiplication on $UC_r(K)^*$ under which it is a Banach algebra. More specifically, for $m, n \in UC_r(K)^*$, $f \in UC_r(K)$, and $x \in K$

$$\langle mn, f\rangle = \langle m, nf\rangle \text{ where } nf(x) = \langle n, {}_xf\rangle,$$

where ${}_xf(y) = \int_K f(t)\, d\delta_x * \delta_y(t)$. This product is in fact the restriction of the first Arens product on $L^1(K)^{**}$ to $UC_r(K)^*$. The *topological center* of $UC_r(K)^*$ is defined by

$$Z(UC_r(K)^*) = \{m \in UC_r(K)^* : \text{the mapping } n \longmapsto mn \text{ is}$$
$$w^* - \text{continuous on } UC_r(K)^*\}.$$

Note that when K is commutative, then $Z(L^1(K)^{**})$ and $Z(UC_r(K)^*)$ are precisely the algebraic centers of $L^1(K)^{**}$ and $UC_r(K)^*$ respectively.

The main purpose of this thesis is to establish the results of Lau [37] and Lau and Losert [38] for hypergroups. The method of the proof which we shall use also provides a new proof of Lau's result in [37, Theorem 1] in the group case. Many of our results are on hypergroups with left Haar measures. It is still unknown if an arbitrary hypergroup admits a left Haar measure, but all the known examples of hypergroups do.

This thesis consists of six chapters and one appendix. Chapter 2, section 2.2 contains a summary of definitions and notations used throughout the thesis. Section 2.3 deals with some well-known results on the second duals of Banach algebras, while section 2.4 concerns with some technical lemmas necessary to prove our main results on chapters 3 and 4. In particular, the technical lemma 2.4.5 in this section plays a key role in proving our main results (Theorem 4.2.5 and Theorem 3.4.3).

In chapter 3, we prove that the topological center of the Banach algebra $UC_r(K)^*$ is $M(K)$. The results of this chapter generalize the corresponding ones for locally compact groups in [37].

Motivated by Lau and Losert in [38] in chapter 4, we show that the topological center of the Banach algebra $L^1(K)^{**}$ is $L^1(K)$ and as a corollary we have, $L^1(K)$ is Arens regular if and only if K is finite [56].

Some applications of the main theorems are given in chapter 5. First we generalize a result of Watanabe in [64] that $L^1(K)$ is an ideal of $L^1(K)^{**}$ if and only if K is compact. Here we show, among other things, that the compactness of K is equivalent to $UC_r(K) = WAP(L^1(K))$ (weakly almost periodic functionals on $L^1(K)$)

and K is compact if and only if $L^1(K)$ is an ideal of $M(K)^{**}$, and that $L^\infty(K)$ has a unique topological left invariant mean if and only if K is compact.

Let $P^1(K) = \{f \in L^1(K) : f \geq 0, \quad \|f\|_1 = 1\}$. Then $P^1(K)$ is a topological semigroup (a semigroup with jointly continuous multiplication and Huasdorff topology) under the convolution product in $L^1(K)$ and when equipped with the norm topology. Let $C(P^1(K))$ denote the space of all bounded continuous functions on $P^1(K)$. By a left invariant mean on $C(P^1(K))$ denoted by LIM, we mean a linear functional $m \in C(P^1(K))^*$ such that $\|m\| = m(1_{P^1(K)}) = 1$ with $m({}_f\theta) = m(\theta)$ where ${}_f\theta(g) = \theta(f * g)$ ('$*$' is the convolution product on $P^1(K)$) for all $f \in P^1(K)$, $\theta \in C(P^1(K))$. We say that K has property (p) *if there exists a LIM on* $C(P^1(K))$.

Motivated by the work of Ganesan in [16], we study hypergroups with property (p) in chapter 6. Examples of hypergroups with property (p), are given in section 6.2. Here, we also study the relationship between amenability of the semigroup $P^+(K)$ and $P^1(K)$.

In appendix A, we give a proof on the existence of Haar measure for commutative hypergroups, by using Markov-Kakutani fixed-point theorem, based on an idea of Izzo [32]. A proof of this result was originally given by Ross [50] with a completely different method.

Chapter 2

Notations and Preliminary results

2.1 Introduction

In this chapter, we include some definitions and notations used throughout the thesis. Section 2.2 is an introduction to locally compact hypergroups, associated functions spaces and invariant means. Section 2.3 deals with some notation and preliminary results on second duals of Banach algebras. In section 2.4, we establish some properties of $Z(L^1(K)^{**}$ necessary to prove our main results in chapters 3 and 4. In particular, the technical Lemma 2.4.5 in this section plays a key role in proving the main results. Also in this section, we collect some facts about the Arens product on $L^1(K)^{**}$ which will be used in the sequel.

2.2 Locally compact hypergroups

Let X be a locally compact Hausdorff space. The following notations are used throughout the thesis:

C(X)	The bounded continuous functions on X
$C_{00}(X)$	The set of continuous functions with compact support on X
$C_{00}^{+}(X)$	The members of $C_{00}(X)$ which are non-negative
$C_0(X)$	The set of continuous functions vanishing at ∞
1_D	The characteristic function of the nonempty set $D \subseteq X$
$\bar{D}$	The closure of $D \subseteq X$
$M(X) = C_0(X)^*$	The regular Borel measure on X
$M^+(X)$	The members of $M(X)$ which are non-negative
δ_x	The Dirac measure concentrated at $x \in X$
sptμ	The support of measure $\mu \in M(X)$
A^{**}	The second dual of Banach algebra A equipped with the first Arens product
$Z(A^{**})$	The topological center of Banach algebra A^{**}
S	Topological semigroup

Always, an unspecified topology on $M^+(X)$ is the cone topology. This is the weak topology induced on $M^+(X)$ by the family $C_{00}^+(X) \cup \{1\}$.

Definition 2.2.1 *A hypergroup is a non-void locally compact Hausdorff space K which satisfies the following conditions:*

1. *$M(K)$ admits a binary operation $*$ under which it is a complex algebra.*

2. *The binary mapping $* : M(K) \times M(K) \longrightarrow M(K)$ given by $(\mu, \nu) \longmapsto \mu * \nu$ is non-negative ($\mu * \nu \geq 0$, whenever $\mu, \nu \geq 0$) and continuous on $M^+(K) \times M^+(K)$.*

3. *If $x, y \in K$, then $\delta_x * \delta_y$ is a probability measure with compact support.*

4. *The mapping $(x, y) \longmapsto spt(\delta_x * \delta_y)$ of $K \times K$ into the space $\mathcal{E}(K)$ of compact subsets of K is continuous, where $\mathcal{E}(K)$ is given the topology with subbasis given by all*

$$\mathcal{E}_U(V) = \{A \in \mathcal{E}(K) : A \cap U \neq \emptyset, A \subseteq V\}$$

   where U, V are open subsets of K. This topology is studied by E. Michael [41]. See [33, §2.5] also.

5. *There exists a (necessarily unique) element e, called the identity, in K such that*

$$\delta_x * \delta_e = \delta_e * \delta_x = \delta_x \text{ for all } x \in K.$$

6. *There exists a (necessarily unique) involution $x \longmapsto \check{x}$, a homeomorphism of K onto itself such that $(\check{x})\check{} = x$ for all $x \in K$, satisfying:*

(a) *For $x, y \in K$, $e \in spt(\delta_x * \delta_y)$ if and only if $x = \check{y}$*

(b) *If $\check{\mu}$ is defined for $\mu \in M(K)$ by $\check{\mu}(A) = \mu(\check{A})$ for all Borel subsets A of K, where $\check{A} = \{\check{x} : x \in A\}$, then $(\mu * \nu)\check{} = \check{\nu} * \check{\mu}$.*

The definition of hypergroups above is the one given by Jewett [33] who called them convos. A survey of the subject appears in Ross [51].

We shall denote the maximal subgroup of K by

$$G(K) = \{x \in K : \delta_x * \delta_{\check{x}} = \delta_e\}.$$

For a Borel function f on K and $x \in K$, ${}_xf$ denotes the left translation

$${}_xf(y) = f(x * y) = \int_K f(t)\ d\delta_x * \delta_y(t).$$

and f_x the right translation

$$f_x(y) = f(y * x) = \int_K f(t)\ d\delta_y * \delta_x(t).$$

if the integrals exist. We write ${}_{x*y}f$ and f_{x*y} for ${}_y({}_xf)$ and $(f_y)_x$ respectively.

Many of our results require the existence of a left Haar measure. Throughout, K will denote a hypergroup with a left Haar measure λ. Hence, by definition, λ is a non-negative regular Borel (not necessarily bounded) measure on K such that $\delta_x * \lambda = \lambda$ for every $x \in K$. The modular function Δ is defined on K by the identity $\lambda * \delta_{\check{x}} = \Delta(x)\lambda$. Unless otherwise stated, we use the definitions and notations of Jewett [33]

If K is compact, then it admits a left Haar measure and all the known examples also do [33, §5]. Compact hypergroups are unimodular [33, §7].

A hypergroup K is called *commutative* if $\delta_x * \delta_y = \delta_y * \delta_x$ for all $x, y \in K$.

A *subhypergroup* is a non-empty closed subset H of K with $\check{H} = H$ and $H * H \subseteq H$.

Example 2.2.2 [13] Let Z^+ be the non-negative integers and $Z^+ \cup \{\infty\}$ its one-point compactification. Let $0 < a \leq 1/2$. Let δ_∞ be the identity element and define

$$\delta_m * \delta_n = \delta_{min(m,n)} \quad \text{if } m, n \in Z^+, \quad m \neq n$$

$$\delta_n * \delta_n(\{t\}) = \begin{cases} 0 & \text{if } t < n, \\ \frac{1-2a}{1-a} & \text{if } t = n, \\ a^k & \text{if } t = n + k > n. \end{cases}$$

and $\check{n} = n$ for all n.

The compact commutative hypergroup obtained this way is denoted by H_a which studied by Dunkl and Ramirez. The nomalized Haar measure on H_a is given by

$$\lambda(\{k\}) = \begin{cases} (1-a)a^k & \text{if } k \neq \infty, \\ 0 & \text{otherwise;} \end{cases}$$

and $H = \{1, 2, \cdots, \infty\}$ is a subhypergroup of H_a. For more examples see [33, Chapter 9, PP. 49-60].

For non-empty sets C, D of K, write

$$C * D = \cup_{x \in C, y \in D} \, spt(\delta_x * \delta_y).$$

For $C \subseteq K$ and $y \in K$, let $C * y$ denote the subset $C * \{y\}$ in K. For a subhypergroup H, let

$$K/H = \{x * H : x \in K\}$$

and

$$K//H = \{H * x * H : x \in K\}$$

equipped with the quotient topology.

Lemma 2.2.3 *Let K be a locally compact non-compact hypergroup. Then there exists a family $\{C_i : i \in I\}$ of compact subsets of K, indexed by I, $y_i, z_i \in K$, $i \in I$ such that C_i° (the interior of C_i) is non-empty, $\cup_{i\in I} C_i^\circ = K$, $\{C_i : i \in I\}$ is closed under finite unions, and*

(a) the families $\{C_i * y_i : i \in I\}$ and $\{C_i * z_i : i \in I\}$ are pairwise disjoint.

(b) $C_i * y_i * \check{y}_j \cap C_p * z_p * \check{z}_q = \emptyset$, $i \neq j$ and $p \neq q$, $i, j, p, q \in I$.

Proof: Let $\{C_i : i \in I\}$ be a family of compact subsets of K with C_i° nonempty, $K = \cup_{i\in I} C_i^\circ$ and that the index set I has minimal cardinality among all such families. By taking finite union of such sets, we may assume that $\{C_i : i \in I\}$ is closed under finite unions We may also assume that I is well ordered in such a way that each nontrivial order segment $\{i \in I : i \leq j\}$, $j \in I$, of I has smaller cardinality than I. We now proceed with the selection of y_i, z_i, $i \in I$ by transfinite induction. Assume that y_j, z_j have been selected for $j < i$, then y_i has to meet the following requirements:

from (a) for $p < i$

$$C_i * y_i \cap C_p * y_p = \emptyset,$$

from (b) for any $p \neq q$, $j < i$

$$((C_i * y_i) * \check{y}_j) \cap ((C_p * z_p) * \check{z}_q) = \emptyset.$$

and (by changing i with j in (b))

$$((C_j * y_j) * \check{y}_i) \cap ((C_p * z_p) * \check{z}_q) = \emptyset.$$

Now by using 4.1B in [33] they are equivalent to

$$
\begin{aligned}
y_i &\notin \check{C}_i * (C_p * y_p) & (1)\\
y_i &\notin (\check{C}_i * (((C_p * z_p) * \check{z}_q) * y_j)) & (2)\\
y_i &\notin ((C_p * z_p) * \check{z}_q)\check{} * (C_j * y_j) \text{ for } j,p,q < i \text{ and } p \neq q & (3)
\end{aligned}
$$

respectively.

Such choice of y_i is possible, since the collection of compact sets (see [33, 3.2B]) on the right hand side of (1), (2) and (3) do not cover K, by minimality of I. Similarly, we can find z_i such that

$$
\begin{aligned}
z_i &\notin \check{C}_i * (C_p * z_p) \qquad p < i,\\
z_i &\notin \check{C}_i * (((C_p * y_p) * \check{y}_q) * z_j)\\
\text{and } z_i &\notin ((C_p * y_p) * \check{y}_q)\check{} * (C_j * z_j) \qquad \text{for } j,p,q < i \text{ and } p \neq q.
\end{aligned}
$$

Now by transfinite induction the proof is complete. □

Lemma 2.2.4 *Let K be a hypergroup and U a symmetric neighborhood of the identity $e \in K$. Then there exists a subset M of K such that for any finite subset $\{g_1, g_2, \cdots g_n\}$ of K, the set $g_1 * g_2 * \cdots * g_n * U * U$ contains at least one element of M and the set $g_1 * g_2 * \cdots * g_n * U$ contains at most one element of M.*

Proof: Let

$$\mathcal{A} = \{T \subseteq K : \text{for any} \quad p \neq q \in T, \quad \text{there is a finite subset} \quad \{g_1, g_2, \cdots, g_n\}$$

$$\text{of} \quad K \quad \text{such that} \quad p \notin q * A * \check{A}, \quad \text{where} \quad A = U * \check{g}_n * \cdots * \check{g}_1\}.$$

Then $\mathcal{A}$ is non-empty and any chain $\{T_\alpha\}_{\alpha \in I}$ in $\mathcal{A}$ has an upper bound $\cup_{\alpha \in I} T_\alpha$. So by Zorn's Lemma $\mathcal{A}$ has a maximal element M. By using [33, 4.1A, 4.1B], we have $M \cap g * U * U \neq \emptyset$ for all $g \in K$. Now for $\{g_1, g_2, \cdots, g_n\}$ an arbitrary finite subset of K, we have

$$M \cap g_1 * g_2 * \cdots * g_n * U * U = M \cap (\cup_{x \in g_1 * g_2 * \cdots * g_n} x * U * U) =$$

$$\cup_{x \in g_1 * g_2 * \cdots * g_n} (M \cap x * U * U) \neq \emptyset.$$

To show that M intersects $g_1 * g_2 * \cdots * g_n * U$ at most at one point, let there be s_1 and s_2 in M such that $s_1 \neq s_2$ and $s_i \in g_1 * g_2 * \cdots * g_n * U$ for $i = 1, 2$. Then by using [33, 4.1A, 4.1B] we have $s_1 \in s_2 * A * \check{A}$, where A is $U * \check{g}_n * \cdots * \check{g}_2$ and this contradicts $M \in \mathcal{A}$. So the proof of the Lemma is complete. □

The function $\check{f}$ is given by $\check{f}(x) = f(\check{x})$. The integral $\int \ldots d\lambda(x)$ is often denoted by $\int \ldots dx$.

Let $(L^P(K), \|.\|_p), 1 \leq p \leq \infty$, denote the usual Banach spaces of Borel functions on K [33, §6.2]. Then $L^\infty(K)$ is a Banach algebra with the essential supremum norm $\|\cdot\|_\infty$, $L^\infty(K) = L^1(K)^*$ [33, §6.2.], and $C(K)$ is a norm-closed subspace of $L^\infty(K)$ in a natural way. We say that $X \subseteq L^\infty(K)$ is left [right] translation invariant if

${}_xf \in X$ $[f_x \in X]$ and is topologically left [right] translation invariant if $\phi * f \in X$ $[f * \check{\phi} \in X]$ for $f \in X, \phi \in P^1(K) = \{\phi \in L^1(K) : \phi \geq 0, \| \phi \|_1 = 1\}$.

In addition, we make use of the following abbreviations:

$$UC_r(K) = \{f \in C(K) : x \longmapsto {}_xf \text{ is continuous from K into } (C(K), \| \cdot \|_\infty)\}.$$

$$UC_l(K) = \{f \in C(K) : x \longmapsto f_x \text{ is continuous from K into } (C(K), \| \cdot \|_\infty)\}.$$

It is known that

$$UC_r(K) = \{f \in C(K) : x \longmapsto {}_xf \text{ is continuous from K into } C(K) \text{ with the weak-topology}\}$$

[56, Theorem 4.2.2, p 88].

Each of the spaces $UC_r(K)$ and $UC_l(K)$ is a normed closed, conjugate closed, translation invariant and topologically translation invariant subspace of $C(K)$ containing the constant functions and $C_0(K)$ [55, Lemma 2.2.] and

(i) $UC_r(K) = L^1(K) * UC_r(K) = L^1(K) * L^\infty(K)$;

(ii) $UC_l(K) = UC_l(K) * L^1(K)\check{} = L^\infty(K) * L^1(K)\check{}$ [55, Lemma 2.2.]

Note that $UC_r(K)$ and $UC_l(K)$ are in general not algebras [55, Remark 2.3(b)].

For $\phi \in L^1(K)$, we write $\tilde{\phi}(x) = \Delta(\check{x})\phi(\check{x})$ then $\|\tilde{\phi}\| = \|\phi\|_1$. If $f \in L^P(K), 1 \leq p \leq \infty, x \in K$, then $\| {}_xf\|_p \leq \|f\|$, and this is in general not an isometry [33, §3.3]. The mapping $x \longmapsto {}_xf$ is continuous from K to $(L^P(K), \|.\|_p), 1 \leq p < \infty$, [33, 2.2B and 5.4H].

It is easy to show that $L^1(K)$ has a bounded approximate identity(B.A.I) $\{e_i : i \in I\} \subseteq C_{00}^+(K)$ such that $\|e_i\|_1 = 1$ (see [55, Lemma 2.1]).

Let X be a subspace of $L^\infty(K)$ containing the constant functions and closed under conjugation. A complex linear functional m on X is called a *mean* on X if we have:

(i) $\operatorname{ess\,inf} f \leq m(f) \leq \operatorname{ess\,sup} f$

for $f \in X_R$, the real valued functions in X.

Like the group case, it can be shown that a complex linear functional m on X is a mean if and only if each pair of the following conditions holds:

(ii) $(f \in X, f \geq 0)$ implies $m(f) \geq 0$.

(iii) $m(1_K) = 1$.

(iv) $\| m \| = 1$.

See [44, Proposition 3.2, p. 23] or [43, Proposition 0.1].

Let X be a subspace of $L^\infty(K)$ with $1_K \in X$ that is closed under complex conjugation and is left translation invariant. A mean on X is called a *left invariant mean* [LIM] if $m({}_xf) = m(f)$ for all $f \in X$, $x \in K$. A mean on X is called a *topological left invariant mean* [TLIM] (*topological right invariant mean* [TRIM]) if $m(\phi * f) = m(f)$ $(m(f * \check{\phi}) = m(f))$ for all $f \in X$, $\phi \in P^1(K)$.

2.3 Second duals of Banach algebras

From now on, for a Banach space X, we denote by X^* and X^{**} its first and second dual, respectively.

Let A be a Banach algebra. For any $f \in A^*$ and $a \in A$ we may define a linear functional fa on A by $\langle fa, b\rangle = \langle f, ab\rangle, (b \in A)$. One can check that $fa \in A^*$ and $\|fa\| \leq \|f\|\|a\|$. Now for $n \in A^{**}$ we may define $nf \in A^*$ by $\langle nf, a\rangle = \langle n, fa\rangle$; clearly we have $\|nf\| \leq \|n\|\|f\|$. Next for $m \in A^{**}$, define $mn \in A^{**}$ by $\langle mn, f\rangle = \langle m, nf\rangle$. We have $\|mn\| \leq \|m\|\|n\|$, and A^{**} becomes a Banach algebra with the multiplication mn, just defined, referred to as the first *Arens product*, versus another multiplication on A^{**} called the second Arens product, which is denoted by $m \circ n$ and defined successively as follows:

$\langle m \circ n, f\rangle = \langle n, fm\rangle$, in which $\langle fm, a\rangle = \langle m, af\rangle$, where $\langle af, b\rangle = \langle f, ba\rangle$,

herein m, n, f, a, and b are taken as above.

From now on A^{**} will always be regarded as a Banach algebra with the first Arens product.

Let $Z(A^{**})$ denote all $m \in A^{**}$ such that

$$mn = m \circ n$$

for all $n \in A^{**}$. We call $Z(A^{**})$ the *topological center* of A^{**}.

Proposition 2.3.1 *$Z(A^{**})$ is a closed subalgebra of A^{**} containing A.*

For a proof see [10, p.310] or [38, Lemma 1].

Proposition 2.3.2 *Let $m \in A^{**}$. The following are equivalent:*

(a) $m \in Z(A^{**})$;

(b) the map $n \to mn$ from A^{**} into A^{**} is w^* -w^* continuous;

(c) the map $n \to mn$ from A^{**} into A^{**} is w^* -w^* continuous on norm bounded subsets of A^{**}.

For a proof see [10, p.313].

Note that for n fixed in A^{**}, the mapping $m \longmapsto mn$ is always w^* -w^* continuous.

2.4 Some properties of $\mathbf{Z}\left(\mathbf{L}^1(\mathbf{K})^{**}\right)$

In this section, we shall show some properties about topological center of $L^1(K)^{**}$ which will be used to prove the main result of chapters 3 and 4. In particular, the technical Lemma 2.4.5 in this section plays a key role in proving the main results.

Lemma 2.4.1 *Let $\phi, \psi \in L^1(K)$, $f \in L^\infty(K)$ and $m, n \in L^1(K)^{**}$. Then*

(i) $\langle \psi f, \phi \rangle = \langle f\phi, \psi \rangle$.

(ii) $\psi f = f * \check{\psi} \in UC_l(K)$, $f\phi = \tilde{\phi} * f \in UC_r(K)$ *where* $\tilde{\phi}(x) = \Delta(\check{x})\phi(\check{x})$ *and* $\check{\psi}(x) = \psi(\check{x})$.

(iii) ${}_a(\psi f) = \psi({}_a f)$, $(f\phi)_a = (f_a)\phi$ *and* $({}_a f)\phi = f({}_{\check{a}}\phi)$ *for* $a \in K$.

(iv) If m, n are means, then so is mn.

(v) If m is a TLIM and n is a mean, then mn is also a TLIM.

(vi) If n is a TLIM and m is a mean on $L^\infty(K)$, then $mn = n$.

Proof: (i), (ii) and (iii) immediate.

(iv) For $\phi \geq 0$ in $L^1(K)$, we have $1\phi = \tilde{\phi} * 1 = \|\phi\|_1 1$ and $\langle n1, \phi\rangle = \langle n, 1\phi\rangle = \langle n, \|\phi\|_1 1\rangle = \| \phi \|_1 = \langle 1, \phi\rangle$ where 1 is the one function on K. Hence, by linearity $n1 = 1$. So $\langle mn, 1\rangle = \langle m, n1\rangle = \langle m, 1\rangle = 1$ and since $\|mn\| \leq \|m\| \|n\|$, we are done.

By using (iv), one can easily check (v).

(vi) Let $\{\phi_\alpha\} \subset P^1(K)$ be a net such that $\phi_\alpha \to m$ in the w^*-topology of $L^\infty(K)^*$, then $\phi_\alpha n \to mn$ in the w^*-topology of $L^\infty(K)^*$. But $\langle \phi_\alpha n, f\rangle = \langle n, f\phi_\alpha\rangle = \langle n, f\rangle$. Hence, $mn = n$. □

Lemma 2.4.2 *Let $0 \neq m \in L^\infty(K)^*$, then there is a net $\{u_\alpha\}$ in $L^1(K)$ such that $\|u_\alpha\| \leq \|m\|$, all u_α have compact support and $u_\alpha \to m$ in the w^*-topology of $L^\infty(K)^*$.*

Proof: Follows from Goldstine's theorem and the density of $C_{00}(K)$ in $L^1(K)$. □

Lemma 2.4.3 *If $m \in Z(L^1(K)^{**})$ and $f \in L^\infty(K)$, then $fm \in UC_r(K)$ and $(fm)(x * y) = \langle m, f_{x*y}\rangle$.*

Proof: We may assume that $m \neq 0$. Let $\{u_\alpha\}$ be the net in the Lemma 2.4.2. Then

$$\langle n, fm\rangle = \langle m \circ n, f\rangle = \langle mn, f\rangle = \langle m, nf\rangle = \lim_\alpha \langle u_\alpha, nf\rangle =$$

$$\lim_\alpha \langle u_\alpha n, f\rangle = \lim_\alpha \langle n, fu_\alpha\rangle.$$

for all $n \in L^\infty(K)^*$. That is, $fu_\alpha \to fm$ weakly for all $f \in L^\infty(K)$. Note that $fu_\alpha = \tilde{u_\alpha} * f \in UC_r(K)$ (Lemma 2.4.1(ii)). It follows that $fm \in UC_r(K)$.

Furthermore, if $f \in L^\infty(K)$, $y \in K$ then

$$fm(y) = \lim_\alpha \tilde{u_\alpha} * f(y) = \lim_\alpha \int_K \tilde{u_\alpha}(x) f(\check{x} * y)dx =$$

$$\lim_\alpha \int_K u_\alpha(x) f(x * y)dx = \lim_\alpha \langle u_\alpha, f_y\rangle = \langle m, f_y\rangle$$

(using [33, 5.5A]).

Now let $\phi \in L^1(K)$, $a \in K$, then by what we have seen above

$$\langle \phi, (fm)_a\rangle = (fm)\phi(a) = f(m \circ \phi)(a) =$$

$$f(m\phi)(a) = \langle m\phi, f_a\rangle = \langle m \circ \phi, f_a\rangle = \langle \phi, (f_a)m\rangle$$

since $m \in Z(L^1(K)^{**})$; i.e. $(fm)_a = (f_a)m$.

Now for $x, y \in K$,

$$\begin{aligned} fm(x * y) &= (fm)_y(x) = (f_y)m(x) \\ &= \langle m, (f_y)_x\rangle = \langle m, f_{x*y}\rangle. \end{aligned}$$

Hence, $(fm)(x * y) = \langle m, f_{x*y}\rangle$. □

Note 2.4.4 That for $m \in L^\infty(K)^*$, $f \in L^\infty(K)$ and $x \in K$, $m({}_xf) = {}_x(mf)$. To see this, first note that if $\{\phi_\alpha\}$ be an approximate identity of $L^1(K)$, then for any $g \in L^\infty(K)$ $g\phi_\alpha \longmapsto g$ in w^*- topology of $L^\infty(K)$. Hence by using 2.4.1(iii),

$$m({}_af) = w^* - \lim_\alpha m(({}_af)\phi_\alpha) = w^* - \lim_\alpha m(f({}_{\check{a}}\phi_\alpha)) =$$

$$w^* - \lim_\alpha (mf)({}_{\check{a}}\phi_\alpha) = w^* - \lim_\alpha {}_a(mf)\phi_\alpha = {}_a(mf).$$

Lemma 2.4.5 *Let $n \in Z(L^1(K)^{**})$ and $u \in L^1(K)$ be such that $(n-u)(f) = 0$ for all $f \in C_0(K)$, then $n = u$.*

Proof: By Proposition 2.3.1, it is enough to show that any element of $Z(L^1(K)^{**})$ vanishing on $C_0(K)$ is zero. So let $n \in Z(L^1(K)^{**})$ such that $n(f) = 0$ for all $f \in C_0(K)$. First we show that $n(f) = 0$ for all $f \in L^\infty(K)$ vanishing outside a compact subset of K. Let $\varepsilon > 0$ be given, since fn is continuous (Lemma 2.4.3) we can find $V \subseteq \{x : |fn(x) - fn(e)| < \varepsilon\}$ such that V is open with compact closure. Put $\nu = \frac{1_V}{\lambda(V)}$, then $f * \check{\nu} \in C_0(K)$(see 2.4.1(ii) and [33, 4.2E]); hence by using 2.4.1(ii),

$$fn * \check{\nu}(e) = \nu(fn)(e) = (\nu f)n(e) = (f * \check{\nu})n(e) = \langle n, f * \check{\nu}\rangle = 0.$$

It follows from Lemma 2.4.3 that

$$|n(f)| = |fn(e) - (fn * \check{\nu}(e))| = \frac{1}{\lambda(V)}\left|\int_V (fn(e) - fn(x))dx\right| \leq \frac{\varepsilon}{\lambda(V)}\int_V dx = \varepsilon.$$

Hence, we may assume that K is non-compact. Now let $\{u_\alpha\}$ be the net in Lemma 2.4.2, then replacing $\{u_\alpha\}$ by a convex combination of $\{u_\alpha\}$ if necessary, we may assume that for any $f \in L^\infty(K)$:

$$\|fu_\alpha - fn\| \to 0. \tag{2.1}$$

(see the proof of Lemma 2.4.3)

If $n \neq 0$ we may assume that n positive and $\|n\| = 1$. Now we claim that for any probability measure $\mu \in M(K)$, $\|n\mu\| = 1$.

Indeed, $\mu 1 = 1$ where 1 is the one function on K. So

$$\langle n\mu, 1\rangle = \langle n, \mu 1\rangle = 1$$

and since

$$\|n\mu\| \leq \|n\|\,\|\mu\| = 1.$$

thus

$$\|n\mu\| = 1.$$

Let $0 < \varepsilon < 1/6$. then there exists $f \in L^\infty(K)$ such that

$$|\langle n\mu, f\rangle| > 1 - \varepsilon, \tag{2.2}$$

(using separation theorem for locally convex spaces) for any probability measure $\mu \in M(K)$ with compact support. Let $\{C_i : i \in I\}$ be a family of compact subsets of K and let $y_i, z_i \in K$, and $i \in I$ satisfying the conditions of Lemma 2.2.3. For each i, define

$$f_i'(x) = \begin{cases} f(x * \check{y}_i) & \text{if } x \in C_i * y_i, \\ 0 & \text{otherwise:} \end{cases}$$

and

$$f_i''(x) = \begin{cases} f(x * z_i) & \text{if } x \in C_i * z_i, \\ 0 & \text{otherwise.} \end{cases}$$

For any finite subset σ of I, let

$$f_\sigma' = \Sigma_{i\in\sigma} f_i' \quad \text{and} \quad f_\sigma'' = \Sigma_{i\in\sigma} f_i''.$$

Since the positive and negative parts of $Re(f_\sigma')$, $Im(f_\sigma')$, $Re(f_\sigma'')$ and $Im(f_\sigma'')$ are monotonically increasing bounded nets of positive functions in $L^\infty(K)$, there exists f', f'' in $L^\infty(K)$ such that

$$f_\sigma' \to f' \qquad \text{and} \qquad f_\sigma'' \to f'' \qquad \text{(see [2, Example 15, P. 64])}$$

in the w^*-topology of $L^\infty(K)$.

By (2.1) we may choose α such that

$$\|fu_\alpha - fn\| < \varepsilon, \qquad \|f'u_\alpha - f'n\| < \varepsilon \quad \text{and} \quad \|f''u_\alpha - f''n\| < \varepsilon.$$

Since each measure u_α has compact support, the family $\{C_i : i \in I\}$ is closed under finite unions and $K = \cup\{C_i^\circ : i \in I\}$, there exists $i_0 \in I$ such that $spt\ u_\alpha \subseteq C_{i_0}$. Let $g' = f_{y_{i_0}}'$ and $g'' = f_{z_{i_0}}''$, then

$$|n(g')| > 1 - 3\varepsilon \qquad \text{and} \qquad |n(g'')| > 1 - 3\varepsilon \qquad (3).$$

Indeed, since $1_{C_{i_0}}(f'_\sigma)_{y_{i_0}} \to 1_{C_{i_0}}(f'_{y_{i_0}})$ in the w^*-topology of $L^\infty(K)$ and $1_{C_{i_0}}(f'_\sigma)_{y_{i_0}} = 1_{C_{i_0}} f_{y_{i_0} * \check{y}_{i_0}}$ for σ containing i_0, it follows that $1_{C_{i_0}} g' = 1_{C_{i_0}} f_{y_{i_0} * \check{y}_{i_0}}$, namely $g' = f_{y_{i_0} * \check{y}_{i_0}}$ on C_{i_0}. Hence by Lemma 2.4.3,

$$\begin{aligned} \varepsilon &> |f'u_\alpha(y_{i_0}) - f'n(y_{i_0})| = |u_\alpha(f'_{y_{i_0}}) - n(f'_{y_{i_0}})| \\ &= |u_\alpha(f_{y_{i_0} * \check{y}_{i_0}}) - n(g')| = |fu_\alpha(y_{i_0} * \check{y}_{i_0}) - n(g')|. \end{aligned}$$

Now since

$$|fn(y_{i_0} * \check{y}_{i_0})| = |n(f_{y_{i_0} * \check{y}_{i_0}})| > 1 - \varepsilon \qquad \text{(by (2.2))}$$

and

$$|fn(y_{i_0} * \check{y}_{i_0}) - fu_\alpha(y_{i_0} * \check{y}_{i_0})| = |n(f_{y_{i_0} * \check{y}_{i_0}}) - u_\alpha(f_{y_{i_0} * \check{y}_{i_0}})| < \varepsilon,$$

we have

$$\begin{aligned} &|fu_\alpha(y_{i_0} * \check{y}_{i_0})| \geq \\ &\qquad |fn(y_{i_0} * \check{y}_{i_0})| - |fu_\alpha(y_{i_0} * \check{y}_{i_0}) - fn(y_{i_0} * \check{y}_{i_0})| > 1 - \varepsilon - \varepsilon = 1 - 2\varepsilon. \end{aligned}$$

Consequently,

$$|n(g')| \geq |fu_\alpha(y_{i_0} * \check{y}_{i_0})| - |fu_\alpha(y_{i_0} * \check{y}_{i_0}) - n(g')| > 1 - 3\varepsilon.$$

Similarly, $|n(g'')| > 1 - 3\varepsilon$.

By (a) and (b) of Lemma 2.2.3, the support of $g'g''$ is contained in the compact set $D_0 = (C_{i_0} * y_{i_0} * \check{y}_{i_0}) \cap (C_{i_0} * z_{i_0} * \check{z}_{i_0})$. Let $h \in C_0(K)$ with $h = 1$ on D_0 and $\|1 - h\| \leq \frac{1}{\|f\|+1}$ (see [37, Lemma 4]), we have by (3) and what we have shown first,

$$\langle |g'(1-h)|, n\rangle \geq |n(g' - g'h)| = |n(g')| > 1 - 3\varepsilon$$

and similarly, $\langle |g''(1-h)|, n\rangle > 1 - 3\varepsilon$.

Hence, by adding these inequalities,we get

$$\langle (|g'| + |g''|)(|1-h|), n\rangle > 2 - 6\varepsilon > 1.$$

But $\|(|g'| + |g''|)(|1-h|)\| \leq 1$. This contradicts the assumption $\|n\| = 1$ and we are done. $\square$

Chapter 3

Topological Center of $\mathbf{UC}_r\,(\mathbf{K})^*$

3.1 Introduction

Let K be a locally compact hypergroup with left Haar measure. $UC_r(K)$ be the Banach subspace of all bounded left uniformly continuous functions on K and $UC_r(K)^*$ be its dual Banach space. For $f \in UC_r(K)$ and $m \in UC_r(K)^*$, we define the function mf on K by $mf(x) = \langle m, {}_xf\rangle$ where ${}_xf(y) = f(x * y)$ for $x, y \in K$. Then $mf \in UC_r(K)$.

Indeed, it is easy too see that $mf \in C(K)$. Also

$$
\begin{aligned}
{}_x(mf)(y) = mf(x * y) = \int_{x*y} mf(t)\, d\delta_x * \delta_y(t) = \\
\int_{x*y} \langle m, {}_tf\rangle d\delta_x * \delta_y(t) = \langle m, \int_{x*y} {}_tf\, d\delta_x * \delta_y(t)\rangle \qquad (*)
\end{aligned}
$$

But the Bochner's integral $\int_{x*y} {}_tf\, d\delta_x * \delta_y(t)$ is ${}_y({}_xf)$ since

$$\begin{aligned}
\int_{x*y} {}_tf \, d\delta_x * \delta_y(t)(\xi) &= \langle \delta_\xi, \int_{x*y} {}_tf \, d\delta_x * \delta_y(t) \rangle \\
&= \int_{x*y} \langle \delta_\xi, {}_tf \rangle d\delta_x * \delta_y(t) \\
&= \int_{x*y} {}_tf(\xi) d\delta_x * \delta_y(t) \\
&= \int_{x*y} f_\xi(t) d\delta_x * \delta_y(t) \\
&= f_\xi(x*y) = {}_y({}_xf)(\xi).
\end{aligned}$$

So $(*)$ implies that

$${}_x(mf)(y) = \langle m, {}_y({}_xf) \rangle = m({}_xf)(y).$$

i.e.

$${}_x(mf) = m({}_xf). \tag{3.1}$$

Hence

$$\| {}_x(mf) - {}_y(mf) \| \leq \| m({}_xf) - m({}_yf) \| \leq \|m\| \| {}_xf - {}_yf \|.$$

Note that if $m = \delta_a$ for some $a \in K$, then $\delta_a f = f_a$.

Now we may define a product on $UC_r(K)^*$ by $\langle nm, f \rangle = \langle n, mf \rangle$ for $m, n \in UC_r(K)^*$ and $f \in UC_r(K)$. With this product, $UC_r(K)^*$ is a Banach algebra.

In section 3.2, we show that this product is the restriction of the first Arens product on $L^\infty(K)^*$ to $UC_r(K)^*$.

Motivated by the work of Ghahramani, Lau and Losert in [20], we show that $UC_r(K)^* = M(K) \oplus C_0(K)^\perp$ and that $C_0(K)^\perp$ is a closed ideal of $UC_r(K)^*$ where

$C_0(K)^{\perp} = \{f \in UC_r(K)^* : f|_{C_0(K)} = 0\}$. Then by using this, we show that the members of $G(K)$ are the only invertible means with positive inverse.

Section 3.3 consists of definition and properties of topological center of $UC_r(K)^*$. We show that the topological center of $UC_r(K)^*$ contains $M(K)$ and that the members of it are precisely those elements m of $UC_r(K)^*$ that the linear operator

$$n \longmapsto mn \quad \text{on} \quad UC_r(K)^*$$

is $w^* - w^*$ continuous.

Section 3.4 contains the main results of this chapter. Here we prove that the topological center of $UC_r(K)^*$ is exactly $M(K)$. Our proof also provides a new proof in group case given by Lau in [37].

3.2 On the Banach algebra $\mathbf{UC}_r(\mathbf{K})^*$

In this section, we prove some lemmas which will be used to prove the main result of this chapter. We also show that the elements of $G(K)$ are the only invertible means with positive inverse.

Lemma 3.2.1 *The product on $UC_r(K)^*$ is the restriction of the first Arens product on $L^\infty(K)^*$ to $UC_r(K)^*$.*

Proof: For $m \in UC_r(K)^*$, let $\tilde{m}$ denote an extension of m to $L^\infty(K)$. Then for $f \in UC_r(K)$ it suffices to show that $\langle \tilde{m}f, \phi\rangle = \langle mf, \phi\rangle$ for any $\phi \in L^1(K)$ with compact support.

First, let $m = \delta_a$ for some $a \in K$, then using Lemma 2.4.1(iii)

$$\langle \bar{m}f, \phi\rangle = \langle \bar{m}, f\phi\rangle = \langle \delta_a, f\phi\rangle = f\phi(a) = (f\phi)_a(e) =$$

$$(f_a)\phi(e) = \langle f_a, \phi\rangle = \langle \delta_a f, \phi\rangle = \langle mf, \phi\rangle.$$

If m is positive and $\|m\| = 1$, then there exists a net $m_\beta = \Sigma_{i=1}^{n_\beta} \lambda_i \delta_{x_i}$ of convex combinations of point evaluations with $m_\beta \to m$ in the w^*-topology of $UC_r(K)^*$. So,

$$\langle \bar{m}f, \phi\rangle = \langle \bar{m}, f\phi\rangle = \langle m, f\phi\rangle = \lim_\beta \langle m_\beta, f\phi\rangle =$$

$$\lim_\beta \langle m_\beta f, \phi\rangle = \lim_\beta \int_K m_\beta f(x)\phi(x)dx \qquad (*)$$

But $m_\beta f \in UC_r(K)$ and

$$|m_\beta f(s) - m_\beta f(t)| = |\langle m_\beta, {}_s f - {}_t f\rangle| \leq \| {}_s f - {}_t f\|_\infty.$$

So by [34, p 232], the family $\{m_\beta f\}$ in $UC_r(K)$ is equicontinuous. Since $m_\beta f \to mf$ pointwise, $m_\beta f \to mf$ uniformly on every compact set in K [34, Theorem 7.15]. Hence from $(*)$ we have

$$\langle \bar{m}f, \phi\rangle = \int_K mf(x)\phi(x)\, dx = \langle mf, \phi\rangle. \qquad \square$$

Note that we can even identify $UC_r(K)^*$ as a closed right ideal of the Banach algebra $L^\infty(K)^*$ with the first Arens product (see [39, p 13]).

Let $C_0(K)^\perp = \{m \in UC_r(K)^* : m|_{C_0(K)} = 0\}$.

Lemma 3.2.2 *$UC_r(K)^* = C_0(K)^\perp \oplus M(K)$. If $m \in UC_r(K)^*$ and $m = m_1 + \mu$ for $m_1 \in C_0(K)^\perp$ and $\mu \in M(K)$, then $\|m\| = \|m_1\| + \|\mu\|$ and $C_0(K)^\perp$ is a closed ideal in $UC_r(K)^*$.*

Proof: We need to show that $C_0(K)^\perp$ is an ideal. The proof of the other parts is the same as [20, Lemma 1.1].

Let $h \in C_0(K), \phi \in L^1(K)$, then $h\phi \in C_0(K)$ (see Lemma 2.4.1(ii) and [33, 4.2E]). Hence for $n \in C_0(K)^\perp$ it follows that $\langle nh, \phi\rangle = \langle n, h\phi\rangle = 0$ i.e. $nh = 0$. Consequently, for $m \in UC_r(K)^* \ \langle mn, h\rangle = \langle m, nh\rangle = 0$. Hence $C_0(K)^\perp$ is a left ideal in $UC_r(K)^*$. Let $\mu \in M(K)$ and $h \in C_0(K)$. Then by using [33, §4.2] we have

$$\begin{aligned} h * \mu^*(x) = \int_K h(x * \check{y}) d\mu^*(y) &= \int_K ({}_x h)(\check{y}) d\mu^*(y) \\ &= \int_K ({}_x h\check{)}(\check{y})\, d\mu(y) \\ &= \int_K ({}_x h)(y)\, d\mu(y) \\ &= \langle \mu, {}_x h\rangle = \mu h(x). \end{aligned}$$

where $\int_K f d\mu^* = \int_K f(\check{x}) d\mu(x)$ for $f \in C_0(K)$. In particular, $\mu h \in C_0(K)$(see[33, 4.2.E]). Hence $n \in C_0(K)^\perp$ implies that $n\mu \in C_0(K)^\perp$. Now let $m \in UC_r(K)^*$ be arbitrary. Then $m = \mu + m_1$ with $\mu \in M(K)$ and $m_1 \in C_0(K)^\perp$. If $n \in C_0(K)^\perp$, then $nm_1 \in C_0(K)^\perp$. So $nm = n\mu + nm_1 \in C_0(K)^\perp$ i.e. $C_0(K)^\perp$ is a right ideal in $UC_r(K)^*$. □

Proposition 3.2.3 *Let $m \in UC_r(K)^*$ and $G(K)$ be the maximal subgroup of hypergroup K. Then m is an invertible mean with positive inverse on $UC_r(K)$, if and only if $m = \delta_x$ for some $x \in G(K)$.*

Proof: Sufficiency is clear. For necessity, let $m \in UC_r(K)^*$ be an invertible mean with positive inverse. Then m^{-1} is a mean, since as the proof of Lemma 2.4.1(iv)

$m1 = 1$ and $\langle m^{-1}, 1\rangle = \langle m^{-1}, m1\rangle = \langle m^{-1}m, 1\rangle = \langle \delta_e, 1\rangle = 1$. Now let $m = \mu + m_1$ and $m^{-1} = \nu + m_2$ with $\mu, \nu \in M(K)$ and $m_1, m_2 \in C_0(K)^{\perp}$ (Lemma 3.2.2), then $\delta_e = mm^{-1} = \mu * \nu + \mu m_2 + m_1\nu + m_1 m_2$ and $\mu m_2 + m_1\nu + m_1 m_2 \in C_0(K)^{\perp}$ (Lemma 3.2.2). Hence, $\mu m_2 + m_1\nu + m_1 m_2 = 0$ since $\delta_e \in M(K)$ and $1 = \|\delta_e\| = \|\mu * \nu\| \leq \|\mu\|\,\|\nu\| \leq 1$. Consequently, $\|\mu\| = \|\nu\| = 1$ and $\|m_1\| = \|m_2\| = 0$ i.e. $m = \mu$ and $m^{-1} = \nu$. Hence $\{e\} = spt(\mu * \nu) = \overline{\{spt(\mu) * spt(\nu)\}}$ [33, 3.2F]. In particular, μ consists of a single point in $G(K)$ i.e. $\mu = \delta_x$ for some $x \in G(K)$. □

3.3 Some properties of $Z(UC_r(K)^*)$

For $m \in UC_r(K)^*$ and $f \in UC_r(K)$, we may define a bounded complex function fm on K by $fm(x) = \langle m, f_x\rangle$. Generally fm is not in $UC_r(K)$ but for $m = \delta_a$ $(a \in K)$ $fm = f\delta_a = {}_af \in UC_r(K)$. If $n \in UC_r(K)^*$ and $fm \in UC_r(K)$ for all $f \in UC_r(K)$, then we may define another product on $UC_r(K)^*$ by $\langle m \circ n, f\rangle = \langle n, fm\rangle$.

Let $Z(UC_r(K)^*)$ denotes the set of all $m \in UC_r(K)^*$ such that $fm \in UC_r(K)$ for all $f \in UC_r(K)$ and $mn = m \circ n$ for all $n \in UC_r(K)^*$. One can check that $Z(UC_r(K)^*)$ contains all point evaluation functionals δ_x, $x \in K$.

Note 3.3.1 For $m \in UC_r(K)^*$, define the linear operator L_m from $UC_r(K)^*$ into itself by

$$L_m(n) = mn, \quad n \in UC_r(K)^*.$$

Put

$$C = \{m \in UC_r(K)^* : L_m \text{ is } w^* - w^*\text{continuous}$$

$$\text{on norm bounded subsets of } UC_r(K)^*\}$$

Lemma 3.3.2 $M(K) \subseteq C$.

Proof: Let $\mu \in M(K)$. we need to show that the map $m \to \mu m$ is w^*-w^* continuous on any norm bounded subset of $UC_r(K)^*$. Let $\{m_\alpha\}$ be a net in $UC_r(K)^*$ with $\|m_\alpha\| \leq c$ converging to $m \in UC_r(K)^*$ in the w^*-topology of $UC_r(K)^*$ for some constant c. Then for any $f \in UC_r(K)$ and $s, t \in K$ we have $|m_\alpha f(s) - m_\alpha f(t)| = |\langle m_\alpha, {}_sf - {}_tf\rangle| \leq c\| {}_sf - {}_tf\|$. Hence by [34, p. 232] the family $\{m_\alpha f\}$ in $UC_r(K)$ is equicontinuous. Since $m_\alpha f \to mf$ pointwise on K, the convergence is uniform on every compact set in K [34, Theorem 7.15]. Let $\mu \in M(K)$ be with compact support, then $\langle \mu m_\alpha - \mu m, f\rangle = \langle \mu, m_\alpha f - mf\rangle = \int_K (m_\alpha f - mf)(x) d\mu(x) \to 0$. Since the measures with compact supports are norm dense in $M(K)$ and $\|m_\alpha f\| \leq c\|f\|$, it follows that $\mu m_\alpha \to \mu m$ in the w^*-topology of $UC_r(K)^*$ and we are done. □

Lemma 3.3.3 *If $m \in C$ and $f \in UC_r(K)$, then $fm \in C(K)$ and $fm(x * y) = \langle m, f_{x*y}\rangle$ for all $x, y \in K$.*

Proof: Let $\{x_\alpha\}$ be a net in K converging to x; then the net $\{\delta_{x_\alpha}\}$ converging to δ_x in the w^*-topology of $UC_r(K)^*$ (see [33, Lemma 2.2B] and Lemma 3.2.2). Hence

$$fm(x_\alpha) = \langle m, f_{x_\alpha}\rangle = \langle m, \delta_{x_\alpha} f\rangle = \langle m\delta_{x_\alpha}, f\rangle \to$$

$$\langle m\delta_x, f\rangle = \langle m, \delta_x f\rangle = \langle m, f_x\rangle = fm(x),$$

since $m \in C$ and $\{\delta_{x_\alpha}\}$ is bounded. Furthermore, we know that fm is also bounded: consequently $fm \in C(K)$. Now note that for any $x, y \in K$, the Bochner's integral $\int_{x*y} f_t \, d\delta_x * \delta_y$ exists. Indeed the map $t \to f_t$ from the compact subset $x*y$ of K into $UC_r(K)$ is continuous in the topology $\sigma(UC_r(K), C)$ of $UC_r(K)$, and C separates the points of $UC_r(K)$ (C contains the point evaluations). Hence for any $m \in C$

$$\langle m, \int_{x*y} f_t \, d\delta_x * \delta_y(t)\rangle = \int_{x*y} \langle m, f_t\rangle \, d\delta_x * \delta_y(t) =$$

$$\int_{x*y} fm(t) \, d\delta_x * \delta_y(t) = fm(x*y) \qquad (*)$$

On the other hand, the Bochner's integral $\int_{x*y} f_t \, d\delta_x * \delta_y$ is equal to f_{x*y}. Indeed for any $\phi \in L^1(K) \subset C$ (Lemma 3.3.2), by using Lemma 2.4.1(iii), $(*)$ implies that

$$\begin{aligned}\langle \phi, \int_{x*y} f_t \, d\delta_x * \delta_y(t)\rangle &= f\phi(x*y) = (f\phi)_y(x) \\ &= ((f_y)\phi)(x) = ((f_y)\phi)_x(e) = (f_y)_x\phi(e) = \langle \phi, (f_y)_x\rangle.\end{aligned}$$

Hence from $(*)$ we have $\langle m, f_{x*y})\rangle = fm(x*y)$. □

Lemma 3.3.4 *For $m \in UC_r(K)^*$ the following are equivalent:*

(a) $m \in Z(UC_r(K)^*)$,

(b) The operator L_m is w^*-w^* continuous.

(c) $m \in C$.

Proof: First we show that (a) implies (b). Let $\{n_\alpha\}$ be a net in $UC_r(K)^*$ converges to $n \in UC_r(K)^*$ in the w^*-topology of $UC_r(K)^*$. Then for any $f \in UC_r(K)$

$$\begin{aligned}
\lim_\alpha mn_\alpha(f) &= \lim_\alpha \langle mn_\alpha, f\rangle \\
&= \lim_\alpha \langle m \circ n_\alpha, f\rangle \\
&= \lim_\alpha \langle n_\alpha, fm\rangle \\
&= \langle m \circ n, f\rangle \\
&= mn(f).
\end{aligned}$$

(b) implies (c) is trivial.

For (c) implies (a), let $m \in C$ and $f \in UC_r(K)$. Then by 3.3.3, $fm \in C(K)$. To see that $fm \in UC_r(K)$, we first show that if $\theta \in C(K)^*$, $a \in K$, then

$$\langle \theta, {}_a(fm)\rangle = \langle m\delta_a\theta, f\rangle. \qquad (**)$$

Indeed for $\theta = \delta_x$ $(x \in K)$, by using 3.3.3, we have

$$\begin{aligned}
\langle \delta_x, {}_a(fm)\rangle = {}_a(fm)(x) &= fm(a * x) = \langle m, f_{a*x}\rangle = \langle m, (f_x)_a\rangle \\
&= \langle m, \delta_a(f_x)\rangle = \langle m\delta_a, f_x\rangle = \langle m\delta_a, \delta_x f\rangle \\
&= \langle m\delta_a\delta_x, f\rangle.
\end{aligned}$$

If θ is a mean on $C(K)$, then there is $\theta_\beta = \Sigma_{i=1}^{n_\beta} \lambda_i \delta_{x_i}$, convex combinations of point evaluations, such that $\theta_\beta \to \theta$ in the w^*-topology of $C(K)^*$. Hence

$$\begin{aligned}
\langle \theta, {}_a(fm)\rangle &= \lim_\beta \langle \theta_\beta, {}_a(fm)\rangle \\
&= \lim_\beta \langle m\delta_a\theta_\beta, f\rangle \\
&= \langle m\delta_a\theta, f\rangle
\end{aligned}$$

by w^*-w^* continuity of L_m on norm bounded subsets of $UC_r(K)^*$. Consequently $(**)$ holds.

Now to see that $fm \in UC_r(K)$, by [56, Theorem 4.2.2. p 88], it is enough to show that the map $x \to {}_x(fm)$ from K to $C(K)$ is weakly continuous. Let $\{x_\alpha\}$ be a net in K converging to x and $\theta \in C(K)^*$, then by using $(**)$

$$\begin{aligned}
\lim_\alpha \langle \theta, {}_{x_\alpha}(fm)\rangle &= \lim_\alpha \langle m\delta_{x_\alpha}\theta, f\rangle \\
&= \langle m\delta_x\theta, f\rangle = \langle \theta, {}_x(fm)\rangle.
\end{aligned}$$

by w^*-w^* continuity of L_m on norm bounded subsets of $UC_r(K)^*$. Hence, $fm \in UC_r(K)$.

If n is a mean on $UC_r(K)$, there exists a net $n_\alpha = \Sigma_{i=1}^{l_\alpha}\lambda_i\delta_{x_i}$ in $Z(UC_r(K)^*)$ (see page 32) where $\lambda_i > 0$ and $\Sigma_{i=1}^{l_\alpha}\lambda_i = 1$ such that $n_\alpha \to n$ in the w^*-topology of $UC_r(K)^*$; then for each $f \in UC_r(K)$,

$$\begin{aligned}
\langle m \circ n, f\rangle &= \langle n, fm\rangle \\
&= \lim_\alpha \langle n_\alpha, fm\rangle \\
&= \lim_\alpha \langle m \circ n_\alpha, f\rangle \\
&= \lim_\alpha \langle mn_\alpha, f\rangle \qquad \text{(see page 32)} \\
&= \langle mn, f\rangle
\end{aligned}$$

by continuity of L_m. Now by linearity, we have $m \circ n = mn$ for all $n \in UC_r(K)^*$. i.e. $m \in Z(UC_r(K)^*)$. □

3.4 The topological center of $UC_r(K)^*$

The main result of this chapter is contained in this section.

Let $\phi \in L^1(K)$ and $m \in UC_r(K)^*$. Then the product ϕm makes sense both as an element of $UC_r(K)^*$ and as an element of $L^\infty(K)^*$ (see [39, §3, p 13]).

Lemma 3.4.1 *Let $\pi : L^\infty(K)^* \to UC_r(K)^*$ be the adjoint of the inclusion map of $UC_r(K)$ into $L^\infty(K)$. Then π is w^*-w^* continuous and $mn = m\pi(n)$ for all $m, n \in L^\infty(K)^*$.*

Proof: It is easy to check that π is w^*-w^* continuous. For the second part, we first define a continuous map $f \longmapsto Ff$ of $L^\infty(K)$ into itself for each $F \in UC_r(K)^*$. Note that for $f \in L^\infty(K), \phi \in L^1(K)$, we know $f\phi \in UC_r(K)$ (Lemma 2.4.1(ii)), so $\phi \longmapsto \langle F, f\phi\rangle$ is a continuous linear functional on $L^1(K)$ and therefore an element Ff of $L^\infty(K)$. The adjoint of $f \longmapsto fF$ is a continuous and w^*-continuous map $m \longmapsto mF$ of $L^\infty(K)^*$ into itself.

Thus for $\phi \in L^1(K), f \in L^\infty(K), F \in UC_r(K)^*$, and $m \in L^\infty(K)^*$,

$$\langle Ff, \phi\rangle = \langle F, f\phi\rangle \quad (*), \quad \langle mF, f\rangle = \langle m, Ff\rangle \quad (**).$$

Let $\{\phi_i\} \subseteq L^1(K)$ be a net converging to m in the w^*-topology of $L^\infty(K)^*$. Then for any $f \in L^\infty(K)$, by using $(*)$ and $(**)$, we have

$$\langle mn, f\rangle = \lim_i \langle \phi_i n, f\rangle = \lim_i \langle \phi_i \circ n, f\rangle = \lim_i \langle n, f\phi_i\rangle = \lim_i \langle \pi(n), f\phi_i\rangle =$$

$$\lim_i \langle \pi(n) f, \phi_i\rangle = \lim_i \langle \phi_i, \pi(n) f\rangle = \langle m, \pi(n) f\rangle = \langle m\pi(n), f\rangle. \qquad \square$$

Lemma 3.4.2

$$Z(UC_r(K)^*) = \{m \in UC_r(K)^* : \phi m \in Z(L^\infty(K)^*) \quad \text{for all} \quad \phi \in L^1(K)\}.$$

Proof: Let $\phi \in L^1(K)$ and $m \in Z(UC_r(K)^*)$. By page 37 we can consider ϕm in $L^1(K)^{**}$. To prove that $\phi m \in Z(L^\infty(K)^*)$, using Lemma 2.3.2, it is enough to show that $n \to \phi mn$ is w^*-w^* continuous. So, let $n_\alpha \to n$ in the w^*-topology of $L^\infty(K)^*$, then $\pi(n_\alpha) \longmapsto \pi(n)$ (since π is w^*-w^* continuous) in the w^*-topology of $UC_r(K)^*$. Hence, by using Lemma 3.3.4, for any $f \in L^\infty(K)$, we have:

$$\langle \phi m n_\alpha, f\rangle = \langle \phi \circ (mn_\alpha), f\rangle = \langle mn_\alpha, f\phi\rangle = \langle m\pi(n_\alpha), f\phi\rangle \to$$

$$\langle m\pi(n), f\phi\rangle = \langle \phi \circ (m\pi(n)), f\rangle = \langle \phi m\pi(n), f\rangle = \langle \phi mn, f\rangle.$$

hence by Lemma 2.3.2 $\phi m \in Z(L^\infty(K)^*)$.

Conversely, let $m \in UC_r(K)^*$, $n_\alpha \to n$ in the w^*-topology of $UC_r(K)^*$. Then for $f \in UC_r(K)$, there exist $g \in UC_r(K)$ and $\phi \in L^1(K)$ such that $f = g\phi$ ([55, Lemma 2.2] and Lemma 2.4.1(ii)). Hence

$$\langle mn_\alpha, f\rangle = \langle mn_\alpha, g\phi\rangle = \langle \phi \circ (mn_\alpha), g\rangle = \langle \phi mn_\alpha, g\rangle \to (\text{using} \quad 2.3.2)$$

$$\langle \phi mn, g\rangle = \langle mn, g\phi\rangle = \langle mn, f\rangle. \qquad \square$$

Now we are ready for the main theorem of this section.

Theorem 3.4.3 $Z(UC_r(K)^*) = M(K)$

Proof: By Lemmas 3.3.2 and 3.3.4, it is enough to show that $Z(UC_r(K)^*) \subseteq M(K)$. Let $m \in Z(UC_r(K)^*)$. Then by Lemma 3.2.2, $m = \mu + m_1$ for $\mu \in M(K)$ and $m_1 \in C_0(K)^\perp$. It is enough to show that $m_1 = 0$. Let $\phi \in L^1(K)$. Since $C_0(K)^\perp$ is an ideal in $UC_r(K)^*$ (Lemma 3.2.2) $\phi m_1 \in C_0(K)^\perp$ and $\phi m_1 \in Z(L^1(K)^{**})$ by Lemma 3.4.2 as well. Hence $\phi m_1 = 0$ (Lemma 2.4.5). Let $f \in UC_r(K)$. Then $f = g\phi$ for some $g \in UC_r(K)$ and $\phi \in L^1(K)$ ([55, Lemma 2.2] and Lemma 2.4.1(ii)), and

$$\langle m_1, f\rangle = \langle m_1, g\phi\rangle = \langle \phi \circ m_1, g\rangle = \langle \phi m_1, g\rangle = 0.$$

Hence, $m_1 = 0$ as desired. □

Chapter 4

Topological Center of $\mathbf{L}^1(\mathbf{K})^{**}$

4.1 Introduction

Let K be a hypergroup with a left Haar measure λ and $L^1(K)$ be the hypergroup algebra of K with multiplication defined by

$$\phi * \psi(x) = \int_K \phi(x * y)\psi(\check{y})\, d\lambda(y) \qquad \text{(see [33, §5.5])}.$$

Let $L^1(K)^{**}$ be the second dual Banach algebra of $L^1(K)$ considered with the first Arens product. In this rather short chapter, we show that $Z(L^1(K)^{**}) = L^1(K)$ and as a corollary we have, $L^1(K)$ is Arens regular if and only if K is finite.

4.2 The topological center of $\mathbf{L}^1(\mathbf{K})^{**}$

In this section, we shall show that the topological center of $L^1(K)^{**}$ is $L^1(K)$.

Lemma 4.2.1 *Let $\{e_i\}_{i\in I}$ be a bounded approximate identity for $L^1(K)$ and $f \in UC_r(K)$, then $\int_K f(x)\, e_i(x)\, dx \longmapsto f(e)$.*

Proof: By Lemma 2.4.1(ii) and [55, Lemma 2.2], $f = g\phi$ for some $g \in UC_r(K)$ and $\phi \in L^1(K)$. Then by using Proposition 2.3.2, Lemma 2.4.3, and [55, Lemma 2.2(i)]

$$\int_K f(x)\, e_i(x)\, dx = \langle e_i, f\rangle = \langle e_i, g\phi\rangle = \langle \phi e_i, g\rangle \longmapsto \langle \phi, g\rangle = g\phi(e) = f(e). \qquad \square$$

Lemma 4.2.2 *Let H be a compact subhypergroup of K with the normalized Haar measure λ_H and $\{U_n\}$ be a decreasing sequence of relatively compact neighborhoods of H with $H = \cap_{n=1}^{\infty} U_n$. Put $\mu_n = \frac{1_{U_n}}{\lambda(U_n)}$, then $\mu_n \to \lambda_H$ in the $\sigma(M(K), C(K))$ topology of $M(K)$.*

Proof: Define $\{\mathcal{U}'_n\}_{n=1}^{\infty}$ in $L^1(K/H)$ by $\mathcal{U}'_n(x * H) = \mathcal{U}'_n(\dot{x}) = \int_H \mu_n(x * t)\, d\lambda_H(t)$, then by [35, Remark 2.5, p 180] we have

$$\int_{K/H} \mathcal{U}'_n(\dot{x})\, d\dot{x} = \int_{K/H} \int_H \mu_n(x * t)\, d\lambda_H(t)\, d\dot{x} = \int_K \mu_n(x)\, dx = 1,$$

so $\mathcal{U}'_n \in L^1(K/H)$. We have also $spt\, \mathcal{U}'_n \subseteq U_n/H$, since if $x * H = \dot{x} \notin U_n/H = \{a * H : a \in U_n\}$ then $x * t \cap U_n = \emptyset$, for all $t \in H$ (using [33, 10.3A]). Thus $\{\mathcal{U}'_n\}$ is a bouned approximate identity in $L^1(K/H)$. Now for $f \in C_{00}(K)$, by using [35, Remark 2.5] and Lemma 4.2.1

$$\int_K \mu_n(x)\, f(x)\, dx = \int_{K/H} f'(\dot{x})\mathcal{U}'_n(\dot{x})\, d\dot{x} \longmapsto f'(e * H) = \int_H f(t)\, d\lambda_H(t)$$

where $f'(\dot{x}) = f'(x * H) = \int_H f(x * t)\, d\lambda_H(t)$.

Hence $\langle \mu_n, f\rangle \longmapsto \langle \lambda_H, f\rangle$ for all $f \in C_{00}(K)$. On the other hand $\{\mu_n\}$ has a w^*-cluster point in $C(K)^*$, say μ (by Alaoglu theorem). Then $\mu = \lambda_H$ on $C_{00}(K)$ and therefore by continuity, on $C_0(K)$. Now since $\|\lambda_H\| = 1$ as an element of $M(K)$ and $\|\mu_n\| = 1 \longmapsto \|\lambda_H\| = 1$, so by [40, Theorem 3.9] we have $\mu = \lambda_H$ on $C(K)$. □

Lemma 4.2.3 *Let H be a compact subhypergroup of K such that K/H is metrizable and $m \in Z(L^\infty(K)^*)$, then $m\lambda_H \in L^1(K)$, where λ_H is the normalized Haar measure on H.*

Proof: First we show that if, for $\mu \in M(K)$, there exists a sequence $\{u_n\}$ in $L^1(K)$ converging to μ in the $\sigma(M(K), C(K))$ topology, then $m\mu \in L^1(K)$ for any $m \in Z(L^\infty(K)^*)$. Let $u \in L^1(K)$ and $m \in Z(L^\infty(K)^*)$ and $\nu = m|_{C_0(K)}$, then for $f \in C_0(K)$ by using Lemma 2.4.1(ii) we have

$$\langle mu, f\rangle = \langle m, uf\rangle = \langle \nu, uf\rangle = \langle \nu * u, f\rangle \text{ (see [33, 4.2E])}.$$

Since $mu \in Z(L^\infty(K)^*)$, it follows from Lemma 2.4.5 that $mu = \nu * u$. Now if $f \in L^\infty(K)$, $fm \in UC_r(K)$ (Lemma 2.4.3). Hence, $\langle m\mu, f\rangle = \langle \mu, fm\rangle = \lim_n\langle u_n, fm\rangle = \lim_n\langle mu_n, f\rangle$. We know that $mu_n \in L^1(K)$ for all n and that $L^1(K)$ is weakly sequentially complete; it follows $m\mu \in L^1(K)$. Now let $\{U_n\}$ be a decreasing sequence of relatively compact neighborhoods of H such that $H = \cap_{n=1}^{\infty} U_n$. Such a sequence exists, since the canonical map $\pi : K \to K/H$ is continuous, open and onto and K is locally compact Hausdorff and K/H is Hausdorff. Put $\mu_n = \frac{1_{U_n}}{\lambda(U_n)}$; then $\mu_n \in L^1(K)$ and $\mu_n \to \lambda_H$ in the $\sigma(M(K), C(K))$ topology of $M(K)$ (Lemma 4.2.2); so by what we have shown above, $m\lambda_H \in L^1(K)$. □

To show that $Z(L^{\infty}(K)^{*}) = L^{1}(K)$, we need one more lemma.

Lemma 4.2.4 *Let H be a compact subhypergroup of K and $m \in Z(L^{\infty}(K)^{*})$. If $f \in L^{\infty}(K)$ is right H-periodic (i.e. $f_x = f$ for all $x \in H$) then $\langle m, f\rangle = \langle m\lambda_H, f\rangle$.*

Proof: As the proof of Lemma 2.4.3 for any $a \in H$, $(fm)_a = (f_a)m$. Consequently, by Lemma 2.4.3,

$$\begin{aligned}\langle m, f\rangle = fm(e) &= \int_H fm(e)d\lambda_H(x)\\ &= \int_H fm(x)d\lambda_H(x)\\ &= \langle \lambda_H, fm\rangle = \langle m\lambda_H, f\rangle. \qquad \square\end{aligned}$$

Now we are ready to prove the main theorem of this section.

Theorem 4.2.5 *Let K be a locally compact hypergroup with left Haar measure, then $Z(L^{\infty}(K)^{*}) = L^{1}(K)$.*

Proof: We follow the proof of Theorem 1 in [38], by Proposition 2.3.1, it is enough to show that $Z(L^{\infty}(K)^{*}) \subseteq L^{1}(K)$. Let $m \in Z(L^{\infty}(K)^{*})$ and $\mu = m|_{C_0(K)}$. By Lemma 2.4.5, it is enough to show that $\mu \in L^{1}(K)$. Let B be a compact subset of K with $\lambda(B) = 0$. We may assume that B contains the identity of K. Then there exists a decreasing sequence of open relatively compact sets $U_n \supseteq B$ such that $(\lambda + |\mu|)(U_n \setminus B) \to 0$ (by regularity).

By induction, we construct a sequence ϕ_n in $C_0(K)$ such that $0 \leq \phi_n \leq 1$, $\phi_n = 1$ on B and that $\phi_n = 0$ outside of $U_n \cap V_{n-1}$, where $V_0 = K$ and $V_n = \{y \in K : \phi_n(y) \neq 0\}$ for all $n \in \mathbb{N}$.

For any $n \in \mathbb{N}$, $d_n(x, y) = \|(\phi_n)_x - (\phi_n)_y\|_{\infty}$ defines a continuous psedometric on

K and $C_n = \{x \in K : d_n(x, e) = 0\}$ is a compact subhypergroup of K.

Indeed C_n is closed, it contains the identity and $C_n * C_n \subseteq C_n$. We note that C_n is compact, since $C_n \subseteq \bar{U}_n$. Moreover, C_n is a compact subhypergroup of K by [33, 10.2F].

But if $C = \cap_{n=1}^{\infty} C_n$, then K/C is metrizable and hence by Lemma 4.2.3, $m\lambda_C \in L^1(K)$. Consequently, Lemma 4.2.4 implies $\langle \mu, f \rangle = \langle m\lambda_C, f \rangle$, for all right C-periodic functions f. Also, since $\{V_n\}$ is decreasing $\lambda(V_n) \to \lambda(B) = 0$, hence, $m\lambda_C(V_n) \to 0$ (since $m\lambda_C \in L^1(K)$). Since $B \subseteq V_n \subseteq U_n$, $\mu(V_n) \to \mu(B)$ consequently, $\mu(B) = 0$.

Now by regularity of μ, we have $\mu \ll \lambda$ i.e. $\mu \in L^1(K)$. □

Definition 4.2.6 *A Banach algebra A is called* Arens regular *if $mn = m \circ n$ for all $m, n \in A^{**}$.*

The following corollary was proved by Young in [65] and was cited as a corollary by Lau and Losert in [38] for locally compact groups. For hypergroups, it was shown by Skantharajah [56, Theorem 5.2.3].

Corollary 4.2.7 *$L^1(K)$ is Arens regular if and only if K is finite.*

Proof: If K is finite, then $L^1(K)$ is reflexive, hence Arens regular.

If $L^1(K)$ is Arens regular, by Theorem 4.2.5, $L^1(K)^{**} = L^1(K)$. Hence, $L^\infty(K)$ is reflexive, consequently, of finite dimension. Therefore, K is finite. Note also that then $L^1(K)^{**}$ has an identity (see [4, Corollary 8, P. 147]). □

Corollary 4.2.8 *If K is commutative then $L^1(K)$ is the algebraic center of the algebra $L^1(K)^{**}$.*

Definition 4.2.9 *Let A be a Banach algebra and T be a bounded linear operator on A. T is called a right multiplier if $T(ab) = aT(b)$ for $a, b \in A$. We denote by $RM(A)$ the set of all right multipliers of A which is a closed normed subalgebra of all bounded linear operators on A.*

Corollary 4.2.10 *$RM(L^1(K))$ is isometrically isomorphic to $M(K)$.*

Proof: This follows from Theorem 4.4 in [39], Lemma 3.4.2, Theorem 3.4.3, and Theorem 4.2.5. □

Chapter 5

Some Applications

5.1 Introduction

In this chapter, we extend a known result on the second dual of the group algebra $L^1(G)$ to hypergroups. S. Watanabe proved in [64] that if G is a locally compact group then $L^1(G)$ is an ideal in $L^1(G)^{**}$ if and only if G is compact. Later Grosser has given a functional-analytic proof of it in [25]. We show that this result remain valid for hypergroups. Some applications of the main theorems in chapters 3 and 4 are also given. Here we show, among other things, that the compactness of K is equivalent to $UC_r(K) = WAP(L^1(K))$ (weakly almost periodic functionals on $L^1(K)$) and we also show, K is compact if and only if $L^1(K)$ is an ideal of $M(K)^{**}$. Finally we prove that K is compact if and only if $L^\infty(K)$ has a unique topological left invariant mean.

5.2 On the Banach algebras $\mathbf{L}^1(\mathbf{K})^{**}$ and $\mathbf{M}(\mathbf{K})^{**}$

For a locally compact group G, the following lemma is due to S. Watanabe [64] (see also [25]).

Lemma 5.2.1 *Let K be a locally compact hypergroup. Then $L^1(K)$ is a left (right, two sided) ideal in $L^1(K)^{**}$ if and only if K is compact.*

Proof: Suppose that $L^1(K)$ is a left ideal of $L^1(K)^{**}$ but K is not compact. Then by regularity of λ and theorems 7.2A, 7.2B in [33], we can find a sequence $\{C_n\}$ of pairwise disjoint compact subsets of K such that $\lambda(C_n) > 1$ for all $n \in N$. Put $U_n = \cup_{i=1}^{n} C_i$ and $U_\infty = \cup_{i=1}^{\infty} C_i$ and let f_n, ϕ and f be respectively the characteristic functions of $U_n * C_1$, C_1 and $U_\infty * C_1$. Then $f_n, f \in L^\infty(K)$ and $\phi \in L^1(K)$. Also $f_n(x) \to f(x)$ for all $x \in K$. Hence by the Lebesgue's bounded convergence theorem, $f_n \to f$ in the w^*-topology of $L^\infty(K)$. Now for any $m \in L^\infty(K)^*$, $\langle m, \phi f_n\rangle = \langle m\phi, f_n\rangle \to \langle m\phi, f\rangle = \langle m, \phi f\rangle$ by our assumption. Namely, $\phi f_n \to \phi f$ in the weak topology of $L^\infty(K)$. However (using [33, 5.5A] and 2.4.1(ii)),

$$\phi f(x) = f*\check{\phi}(x) = \int_K f(x*y)\phi(y)dy = \int_{C_1} f(x*y)dy = \int_{C_1}\int_{x*y} f(t)\, d\delta_x * \delta_y(t)\, dy > 1$$

for all $x \in U_\infty$. This means that ϕf does not vanish at infinity.

On the other hand for any $x \notin (U_n * C_1 * \check{C_1})$ (compact set)

$$\phi f_n(x) = \int_{C_1} f_n(x*y)dy = \int_{C_1}\int_{x*y} f_n(t)\, d\delta_x * \delta_y(t)\, dy = 0$$

(using [33, 4.1B]). This means that the support of ϕf_n is compact: hence ϕf_n vanishes at infinity.

Consequently, $\phi f_n \in C_0(K)$ for $n \in \mathbb{N}$. Since $C_0(K)$ is norm closed, $\phi f \in C_0(K)$ and this is a contradiction.

Similarly when $L^1(K)$ is a right ideal, one can show that K is compact.

Conversely, by [56, Lemma 5.2.4(v), p 115] we have

$$L^\infty(K)^* = UC_r(K)^\perp \oplus UC_r(K)^*$$

where $UC_r(K)^\perp = \{\phi \in L^\infty(K)^* : \quad \phi|_{UC_r(K)} = 0\}$.

Let K be compact. Then $UC_r(K) = C_0(K)$ [33, 4.2F]. Thus for any $m \in L^\infty(K)^*$, there exists $m_1 \in UC_r(K)^\perp$ and $\mu \in C_0(K)^* = M(K)$ with

$$m = m_1 + \mu. \tag{5.1}$$

Now for $\phi \in L^1(K)$,

$$m\phi = m_1\phi + \mu\phi.$$

But by [56, Lemma 5.2.4(iv)] $m_1\phi = 0$. Hence $m\phi = \mu\phi \in L^1(K)$. Namely $L^1(K)$ is a left ideal in $L^\infty(K)^*$.

To show $L^1(K)$ is a right ideal, from (5.1), for any $\phi \in L^1(K)$ we have $\phi m = \phi m_1 + \phi\mu$. But for any $f \in L^\infty(K)$,

$$\langle \phi m_1, f\rangle = \langle m_1, f\phi\rangle = 0,$$

since $f\phi = \tilde{\phi} * f \in UC_r(K)$ (Lemma 2.4.1(ii)), hence $\phi m_1 = 0$. Thus $\phi m = \phi\mu \in L^1(K)$ and we are done. □

Now by using lemma 2.2 in [18] we have:

Corollary 5.2.2 *Let K_1 and K_2 be compact hypergroups. Then any continuous (algebra) isomorphism from $L^\infty(K_1)^*$ onto $L^\infty(K_2)^*$. maps $L^1(K_1)$ onto $L^1(K_2)$.*

Definition 5.2.3 *A function f in $L^1(K)^*$ is said to be weakly almost periodic if the set $\{f\phi : \phi \in L^1(K), \|\phi\|_1 \leq 1\}$ is relatively weakly compact. We denote by $WAP(L^1(K))$ the closed subspace of $L^\infty(K)$ consisting of all the weakly almost periodic functionals in $L^\infty(K)$(see [39, p 4]).*

Corollary 5.2.4 *$WAP(L^1(K)) = UC_r(K)$ if and only if K is compact.*

Proof: This corollary follows from Lemma 5.2.1, Theorem 4.2.5 and Corollary 3.7 in [39]. □

The following is a special case of our Theorem 4.2.5.

Corollary 5.2.5 *If K is compact and $L^1(K)$ has a sequential bounded approximate identity then $Z(L^\infty(K)^*) = L^1(K)$.*

Proof: This follows from Lemma 5.2.1 and Theorem [39, 3.4(a)]. □

Remark 5.2.6 Consider $M(K)^{**}$ with the first Arens product and let $i : L^1(K) \hookrightarrow M(K)$ denote the inclusion map. Then

> $i^{**} : L^1(K)^{**} \to M(K)^{**}$ is an isometric algebra homomorphism. (see [19, Lemma 1.1]) and $i^{**}(L^1(K)^{**})$ is an ideal of $M(K)^{**}$ (see [19, Proposition 1.3]).

If we consider $M(K)$ and $L^1(K)$ as subspaces of $M(K)^{**}$ and $L^1(K)^{**}$ respectively, under the canonical embedding, then we have:

Proposition 5.2.7 *Let K be a locally compact hypergroup. Then $i^{**}(L^1(K)^{**}) \cap M(K) = L^1(K)$*

Proof: Use Lemma 3.3 in [19] and Theorem 4.2.5. □

Lemma 5.2.8 *Let K be a locally compact hypergroup, then $L^1(K) * L^p(K) = L^p(K)$ for $1 \leq p < \infty$.*

Proof: By [33, 6.2C] $L^p(K)$ is a left Banach $L^1(K)$-module where $1 \leq p < \infty$, and we know that $L^1(K)$ has a two-sided approximate identity bounded by one [55, Lemma 2.1.]. Now we show $L^1(K) * L^p(K)$ is norm dense in $L^p(K)$, $1 \leq p < \infty$. Let $f \in L^p(K)$, $1 \leq p < \infty$ and $\varepsilon > 0$ be given. Then there exists $\psi \in P^1(K)$ such that $\|f - \psi * f\|_p < \varepsilon$.

Indeed, for $0 \leq \psi \in L^1(K)$ with $\|\psi\| = 1$ and $f \in L^p(K)$, $\psi * f \in L^p(K)$ [33, 6.2C]. Hence by Hölder's inequality, $\int_K (\psi * f - f)(x) g(x)\, dx \in L^1(K)$ where $g \in L^q(K)$ and $\frac{1}{p} + \frac{1}{q} = 1$. Now by using [33, 5.5A],

$$\begin{aligned}
\langle \psi * f - f, g \rangle &= \int_K ((\psi * f)(x) - f(x)) g(x) dx \\
&= \int_K \left(\int_K \psi(y) f(\check{y} * x) dy - f(x) \right) g(x) dx \qquad (*) \\
&= \int_K \int_K \psi(y) (f(\check{y} * x) - f(x)) g(x) dy dx
\end{aligned}$$

is a linear functional on $L^q(K)$. By using Fubini's theorem and Hölder's inequality, (*) implies

$$\begin{aligned}|\langle \psi * f - f, g\rangle| &= |\int_K \psi(y)(\int_K ({}_y f(x) - f(x))g(x)dx)dy| \\ &\leq \int_K \psi(y)(\int_K |({}_y f(x) - f(x))g(x)|dx)dy \\ &\leq \int_K \psi(y)\| {}_y f - f\|_p \| g\|_q dy.\end{aligned}$$

Now by considering the operator norm of a linear functional, we get

$$\|\psi * f - f\|_p \leq \int_K \psi(y)\| {}_y f - f\|_p dy.$$

By 2.2B and 5.4H in [33] there is a neighborhood V of the identity of K such that $\| {}_y f - f\|_p < \varepsilon$ for all $y \in V$. Thus if $\int_K \psi(x)dx = 1$ and $\psi(y) = 0$ for $y \notin V$, then $\|\psi * f - f\|_p < \varepsilon$. This completes the proof of this claim. Now by the Cohen's Factorization Theorem [28, Theorem 32.22] we are done. □

Corollary 5.2.9 *$L^1(K)$ is an ideal of $M(K)^{**}$ if and only if K is compact.*

Proof: If $L^1(K)$ is an ideal of $M(K)^{**}$, then it is an ideal in $i^{**}(L^1(K)^{**})$. Hence by Lemma 5.2.1, K is compact.

Conversely, if K is compact, then $L^1(K)$ is a two sided ideal in $L^1(K)^{**}$ (Lemma 5.2.1). Let $m \in M(K)^{**}$ and $\phi \in L^1(K)$. Hence by Lemma 5.2.8, $\phi = \phi_1 * \phi_2$ for some $\phi_1, \phi_2 \in L^1(K)$ and $\phi m = \phi_1\phi_2 m \in L^1(K)$, since $i^{**}(L^1(K)^{**})$ is an ideal in $M(K)^{**}$ (Remark 5.2.6). □

For the following corollary in the group case, see Corollary 5 in [37].

Corollary 5.2.10 *Let K be a locally compact hypergroup. Then K is compact if and only if $L^\infty(K)$ has a unique TLIM.*

Proof: If K is compact, then the normalized Haar measure is the unique left invariant mean on $UC_r(K) = C(K)$. Hence by [56, Remark 3.2.7(iii)(b)], $L^\infty(K)$ has a unique TLIM.

Conversely let m be the unique topological left invariant mean on $L^\infty(K)$. Then for any $n \in L^\infty(K)^*$, mn is also topologically left invariant (see Lemma 2.4.1(v)). Hence $mn = \lambda m$ for some complex number λ. Now let $\{n_\alpha\}$ be a net in $L^\infty(K)^*$ converging to n in the w^*-topology, $mn_\alpha = \lambda_\alpha m$ and $mn = \lambda m$, then

$$\lambda_\alpha = mn_\alpha(1) = n_\alpha(1) \to n(1) = \lambda.$$

Hence, for any $f \in L^\infty(K)$,

$$\langle mn_\alpha, f\rangle = \langle \lambda_\alpha m, f\rangle = \lambda_\alpha m(f) \longmapsto \lambda m(f) = \langle \lambda m, f\rangle = \langle mn, f\rangle.$$

i.e. L_m is w^*-w^* continuous. Now by Theorem 4.2.5 and Lemma 2.3.2, $m \in L^1(K)$ and by [33, 7.2B] K is compact. □

Chapter 6

Semigroup of Probability Measures in Hypergroup Algebra and Amenability

6.1 Introduction

Let K be a locally compact hypergroup with left Haar measure λ and

$$P^1(K) = \{f \in L^1(K) : f \geq 0, \|f\|_1 = 1\}.$$

Then $P^1(K)$ is a topological semigroup (a semigroup with jointly continuous multiplication and Hausdorff topology) under the convolution product in $L^1(K)$ and norm topology.

For a topological semigroup S, let $l^\infty(S)$ denote the space of all bounded real-valued functions on S with the supremum norm. For $f \in S$ and $\theta \in l^{\infty}(S)$, let ${}_f\theta$ and θ_f denote, respectively, the *left* and the *right translate* of θ by f, i.e. ${}_f\theta(g) = \theta(fg)$ and $\theta_f(g) = \theta(gf)$, $g \in S$. Let $C(S)$ denote the space of all bounded continuous real-valued functions on S and let $UC_r(S)$ denote the closed subalgebra of $C(S)$ consisting of all *left uniformly continuous* functions on S, i.e. all $F \in C(S)$ such that the mapping $f \longmapsto {}_fF$ from S into $C(S)$ is continuous when $C(S)$ has the supremum norm topology. Then both Banach algebras $UC_r(S)$ and $C(S)$ are invariant under translations and contain constant functions (see [42]). We call S is *(left) amenable* if there exists a LIM on $C(S)$.

We say that a hypergroup K has property (p) if $C(P^1(K))$ has a LIM, i.e. the topological semigroup $P^1(K)$ is left amenable. In this chapter we initiate a study of hypergroups with property (p) and generalize some of the results of [16]. In section 6.2, among other things, we show that if H be a strongly normal subhypergroup (for a definition see 6.2.3) of K, then K has property (p) implies that $K//H$ (see Remark 6.2.2) has property (p). In particular, we show that the hypergroup joins $K = H \vee J$ (see page 56) has property (p) if and only if J has property (p) where H is a compact hypergroup and J is a discrete hypergroup with $H \cap J = \{e\}$ (see Corollary 6.2.11) and a centeral hypergroup (see 6.2.13) has property (p) if $Z(K) \cap G(K)$ (see 6.2.13) is compact (see 6.2.14). We also show that existence of a LIM on $UC_r(P^1(K))$ is equivalent to existence of a LIM on $UC_r(P^+(K))$ (see 6.2.16).

6.2 Hypergroups with property (p)

We begin with some examples.

Example 6.2.1 (a) Every abelian hypergroup K has property (p). Indeed if K is abelian, then so is $P^1(K)$ and we know that every abelian topological semigroup is left amenable (see [59] and [8]).

(b) Every compact hypergroup K has property (p). To see this, first note that if K is compact then $1_K \in P^1(K)$. Let m on $l^\infty(P^1(K))$ be defined by $m(F) = F(1_K)$, for all $F \in l^\infty(P^1(K))$. Then m is a LIM on $l^\infty(P^1(K))$. Indeed, it is clear that m is a mean on $l^\infty(P^1(K))$. Now by using the fact that $f * 1_K = 1_K$ for all $f \in P^1(K)$, one can see easily that m is left invariant on $l^\infty(P^1(K))$. This implies that K has property (p).

Remark 6.2.2 Let H be a compact subhypergroup of K with the normalized Haar measure σ. As shown in [33, §14] the double coset space $K//H = \{H*x*H : x \in K\}$ is a hypergroup with convolution defined by

$$\int_{K//H} f \, d\delta_{H*x*H} * \delta_{H*y*H} = \int_H f \circ \pi \, d\delta_x * \sigma * \delta_y.$$

for all positive Borel measurable functions f on $K//H$ where π is the canonical projection of K onto $K//H$. A Haar measure on $K//H$ is given by $\dot{\lambda} = \int_K \delta_{H*x*H} \, dx$. By [33, 14.2H] the Haar measure $\dot{\lambda}$ on $K//H$ can be so chosen that

$$\int_{K//H} \int_K f \, d\sigma * \delta_x * \sigma \, d\dot{\lambda}(\dot{x}) = \int_K f \, dx \tag{6.1}$$

Let T_H be the mapping defined by $T_H f(H * x * H) = \int_K f \, d\sigma * \delta_x * \sigma$ for $f \in L^1(K)$. Then as shown in [35, Theorem 2.4.(ii)], T_H is a bounded linear map of $L^1(K)$ onto $L^1(K//H)$ with norm 1.

We next recall the definition of *hypergroup joins* which we shall use in this section. Let H be a compact hypergroup and J a discrete hypergroup with $H \cap J = \{e\}$, where e is the identity of both hypergroups. Let $H \cup J$ have the unique topology for which both H and J are closed subspaces of K. Let σ be the normalized Haar measure on H. Define the operation $\bullet$ on K as follows:

(i) If $s, t \in H$, then $\delta_s \bullet \delta_t = \delta_s * \delta_t$;

(ii) If $a, b \in J$, $a \neq \check{b}$, then $\delta_a \bullet \delta_b = \delta_a * \delta_b$;

(iii) If $s \in H$, $a \in J$ $(a \neq e)$, then $\delta_s \bullet \delta_a = \delta_a \bullet \delta_s = \delta_a$;

(iv) If $a \in J$, $a \neq e$, and $\delta_{\check{a}} * \delta_a = \Sigma_{b \in J} c_b \delta_b$, the c_b's are non-negative, only finitely many are non-zero and $\Sigma_{b \in J} c_b = 1$, then

$$\delta_{\check{a}} \bullet \delta_a = c_e \sigma + \Sigma_{b \in J \setminus \{e\}} c_b \delta_b.$$

We call the hypergroup K the *join* of H and J, and write $K = H \vee J$. Observe that H is a subhypergroup of K, but J is not a subhypergroup unless J or H is equal to $\{e\}$. The hypergroup joins always has a left Haar measure [61, Proposition 1.1] and $K//H \cong J$ as hypergroups [61, Proposition 1.3].

Definition 6.2.3 *A compact subhypergroup H of a locally compact hypergroup K is called strongly normal if $\delta_x * \sigma = \sigma * \delta_x$ for all $x \in K$ where σ is the normalized Haar measure of H. In this case, we have $x * H = H * x = H * x * H$ for each $x \in K$ and (6.1) takes the form*

$$\int_{K/H} \int_H f(x * \xi)\, d\sigma(\xi)\, d\dot{\lambda}(\dot{x}) = \int_K f(x)\, d\lambda(x).$$

Moreover, T_H is an algebra homomorphism [35, Theorem 2.4(iv)].

As an example, in the hypergroup joins $K = H \vee J$, where H is a compact hypergroup and J a discrete hypergroup with $H \cap J = \{e\}$, H is strongly normal in K [61, Proposition 1.2].

Theorem 6.2.4 *Let H be a strongly normal subhypergroup of K. Then if K has property (p), so is $K//H$.*

Proof: Let m be a LIM on $C(P^1(K))$. Define $\bar{m} : C(P^1(K//H)) \to I\!R$ by $\bar{m}(F) = m(\bar{F})$ $(F \in C(P^1(K//H)))$ where $\bar{F} : P^1(K) \to I\!R$ is defined by $\bar{F}(f) = F(T_H f)$ for $f \in P^1(K)$. Then clearly $\bar{F}$ is a bounded continuous function and $\bar{m}$ is a mean on $C(P^1(K//H))$. Indeed, it is easy to check that $\bar{m}$ is linear and positive. And for $F = 1_{P^1(K//H)}$, one can see $\bar{F} = 1_{P^1(K)}$ (see [26, Lemma 1.1 and Remark 1.5]), hence

$$\bar{m}(F) = m(\bar{F}) = 1.$$

If $g \in P^1(K//H)$, then there exists $f \in P^1(K)$ such that $T_H f = g$ (see [26, Lemma 1.1] and [35, Theorem 2.4]). Now observe that for $h \in P^1(K)$,

$$(\,_gF\bar{)}(h) = \,_gF(T_H h) = F(g * T_H h) =$$

$$F(T_H f * T_H h) = F(T_H(f * h)) = \bar{F}(f * h) = \,_f(\bar{F})(h).$$

So $(\,_gF\bar{)} = \,_f(\bar{F})$. Hence

$$\langle \bar{m}, \,_gF\rangle = \langle m, (\,_gF\bar{)}\rangle = \langle m, \,_f(\bar{F})\rangle = \langle m, \bar{F}\rangle = \langle \bar{m}, F\rangle.$$

Consequently $\bar{m}$ is a LIM on $C(P^1(K//H))$. □

Definition 6.2.5 *Let H be a compact subhypergroup of K. We say that H is supernormal in K if $\check{x} * H * x \subseteq H$.*

Corollary 6.2.6 *Let hypergroup K has property (p) and H is supernormal subhypergroup in K. Then K/H has property (p).*

Proof: This follows from the fact that any supernormal subhypergroup is strongly normal, and the Theorem 6.2.4. □

As an example, in hypergroup joins $K = H \vee G$, where H is a compact hypergroup and G is any discrete group with $H \cap G = \{e\}$, H is supernormal.

Definition 6.2.7 *A subgroup N of a locally compact hypergroup K is called normal if $xN = Nx$ for all $x \in K$. In this case K/N is a hypergroup with convolution defined by $\int_{K/N} f\, d\delta_{xN} * \delta_{yN} = \int_N f \circ \pi d\delta_x * \delta_y$ for all $x, y \in K$ and $f \in C_{00}(K/N)$ (see [26, p 84]).*

Note that any compact normal subgroup N of a hypergroup K is strongly normal since, $xN = Nx = NxN$ for all $x \in K$.

Corollary 6.2.8 *Let N be a normal subgroup of a locally compact hypergroup K. If K has property (p) then so does K/N.*

Theorem 6.2.9 *Let H be a strongly normal compact subhypergroup in hypergroup K. Then if there exists a left invariant mean on $l^\infty(P^1(K/H))$, then there exists a left invariant mean on $l^\infty(P^1(K))$.*

Proof: Let

$$I = \{f \in P^1(K) : f \quad \text{is constant on the cosets} \quad x * H = H * x, \quad \text{for all} \quad x \in K\}.$$

Then I is a left ideal in $P^1(K)$ i.e. $P^1(K) * I \subseteq I$. Indeed, for $f \in I, g \in P^1(K)$

$$g * f(x) = \int_K g(x * y)\, f(\check{y})\, dy = \int_K g(x * h * y)\, f(\check{y} * \check{h})\, dy = \int_K g(x * h * y)\, f(\check{y})\, dy = g * f(x * h).$$

Then by Theorem 2.4 in [35], $I \neq \emptyset$. It is enough to prove that I is a left amenable semigroup. Let θ be a real valued function on I. Define

$$\bar{\theta} : P^1(K/H) \to \mathbb{R} \quad \text{by} \quad \bar{\theta}(\bar{f}) = \theta(\bar{f} \circ \pi_H)$$

where $\pi_H : K \to K/H$ is the canonical map (see [33, §14.1]). Then by Theorem 2.4(i) in [35], $\bar{f} \circ \pi_H \in P^1(K)$ is constant on cosets and therefore belong to I. Hence.

$\bar{\theta}$ is well defined. Let $\bar{m}$ be a LIM on $l^\infty(P^1(K/H))$ and define a linear functional m on $l^\infty(I)$ by $\langle m, \theta\rangle = \langle \bar{m}, \bar{\theta}\rangle$. Then m is a mean on $l^\infty(I)$. Note that if $f \in I$, then $f = \bar{f} \circ \pi_H$ for some $\bar{f} \in P^1(K/H)$ (by definition of I). Now, by Theorem 2.4(i(c)) in [35], we have $\bar{({}_f\theta)} = {}_{\bar{f}}\bar{\theta}$ since for any $\bar{g} \in P^1(K/H)$,

$$\bar{({}_f\theta)}(\bar{g}) = {}_f\theta(\bar{g} \circ \pi_H) = \theta(f * \bar{g} \circ \pi_H) = \theta(\bar{f} \circ \pi_H * \bar{g} \circ \pi_H) =$$

$$\theta((\bar{f} * \bar{g}) \circ \pi_H) = \bar{\theta}(\bar{f} * \bar{g}) = {}_{\bar{f}}\bar{\theta}(\bar{g}).$$

Hence

$$\langle m, {}_f\theta\rangle = \langle \bar{m}, \bar{({}_f\theta)}\rangle = \langle \bar{m}, {}_{\bar{f}}\bar{\theta}\rangle = \langle \bar{m}, \bar{\theta}\rangle = \langle m, \theta\rangle,$$

for any $f \in I$, i.e. m is a LIM on $l^\infty(I)$. □

Remark 6.2.10 Note that the expression "right-ideal I" in the proof of Proposition 1.5 in [16] should be changed to "left -ideal". Indeed, let S be a semigroup with at least two elements and multiplication defined by:

$$ab = a, \qquad \text{for all} \quad a, b \in S;$$

let I be the right ideal $\{a\} = aS$ in S. Then I is left amenable but S is not.

The following Corollary shows that there are non-compact, non-abelian hypergroups with property (p).

Corollary 6.2.11 *Let $K = H \vee J$ be a hypergroup joins, where H is a compact hypergroup and J a discrete hypergroup with $H \cap J = \{e\}$. Then K has property (p) if and only if J has property (p).*

Proof: We know that $K//H \cong J$ ([61, Proposition 1.3]) and H is strongly normal compact subhypergroup of K [61, Proposition 1.2]. Now by Theorems 6.2.4 and 6.2.9, we are done. □

The following example shows that we can not totally remove the condition 'strongly normal' in Theorem 6.2.9.

Example 6.2.12 Let $SL(2, \mathbb{R})$ be the locally compact group (with the usual topology) of 2×2 matrices with determinant 1 and $SU(2)$ the compact subgroup of unitary matrices in $SL(2, \mathbb{R})$. Then $SL(2, \mathbb{R})$ contains the discrete (closed) free subgroup F_2 on two generators [44, Corollary 14.6], so it dose not have property (p). But the hypergroup $SL(2, \mathbb{R})/SU(2)$ is commutative [33, 15.5] and hence has property (p) (see also [33, §15.6]).

Definition 6.2.13 *Let K be a hypergroup, and let $Z(K) = \{x \in K : \delta_y * \delta_x = \delta_x * \delta_y$ for each $y \in K\}$. Then K is called a central hypergroup or Z-hypergroup if $K/(Z(K) \cap G(K))$ is compact where $G(K) = \{x \in K : \delta_x * \delta_{\check{x}} = \delta_e\}$ is the maximal subgroup of K [26]. Central hypergroups admit left Haar measures and are unimodular (see [26, p 93] and [49, §4]).*

Corollary 6.2.14 *Any central hypergroup K (Z-hypergroup) has property (p) if $Z(K) \cap G(K)$ is compact.*

Proof: This follows from Example 6.2.1(b) and Theorem 6.2.9. □

Discussion 6.2.15 If hypergroup K has property (p), then K is amenable. Indeed, if K is a hypergroup, then $L^1(K)$ is an Lau-algebra as defined in [45] (see [36]). Consequently it follows from Theorem 4.12 in [36] that K is amenable if and only if for any two right ideals $I_1, \quad I_2$ of $P^1(K)$, $d(I_1, I_2) = 0$ (where $d(I_1, I_2) = inf\{\|f_1 - f_2\|_1 : f_1 \in I_1, f_2 \in I_2\}$) (see [14, Theorem 1.7] for the group case).
Now if K has property (p), then $P^1(K)$ is *left reversible* i.e. $I_1 \cap I_2 \neq \emptyset$ for any closed right ideals I_1, I_2 in $P^1(K)$ (see [30, §4]). Consequently K is amenable.
The converse is not known even in the group case. However, there is an amenable locally compact group G for which $P^1(G)$ is not amenable as a discrete semigroup (see [16]).

Notation 6.2.16 Put $P^+(K) = \{f \in L^1(K) : \ f \geq 0, \|f\|_1 > 0\}$. Then $P^+(K)$ is a topological semigroup under the convolution product of $L^1(K)$ induced in $P^+(K)$.

Lemma 6.2.17 *Let K be a locally compact hypergroup. Define $\theta : P^+(K) \to P^1(K)$ by $\theta(f) = \frac{f}{\|f\|_1}$ for $f \in P^+(K)$. Then θ is a continuous, surjective homomorphism.*

Proof: Clearly θ is surjective and it is easy to check that θ is continuous. Now by using the fact that $\|f * g\|_1 = \|f\|_1 \, \|g\|_1$ for all $f, g \in P^+(K)$, we have

$$\theta(f * g) = \frac{f * g}{\|f * g\|_1} = \frac{f}{\|f\|_1} \frac{g}{\|g\|_1} = \theta(f) \; \theta(g).$$

This shows that θ is a homomorphism. □

Proposition 6.2.18 *Let K be a locally compact hypergroup. Then there exists a LIM on $UC_r(P^1(K))$ if and only if there exists a LIM on $UC_r(P^+(K))$.*

Proof: We follow an idea in [16, Proposition 1.3]. If $P^+(K)$ is left amenable then Lemma 6.2.17 shows that there exists a LIM on $UC_r(P^1(K))$. For the converse, let m be a LIM on $UC_r(P^1(K))$ and $\theta \in UC_r(P^+(K))$. Define $\tilde{\theta}$ on $P^+(K)$ by $\tilde{\theta}(g) = \langle m, {}_g\theta|_{P^1(K)}\rangle$, $g \in P^+(K)$. Then $\tilde{\theta}$ is continuous. Moreover if $f \in P^1(K)$, then

$$\tilde{\theta}(g * f) = \langle m, {}_{g*f}\theta|_{P^1(K)}\rangle = \langle m, {}_f({}_g\theta)|_{P^1(K)}\rangle =$$

$$\langle m, {}_f({}_g\theta|_{P^1(K)})\rangle = \langle m, {}_g\theta|_{P^1(K)}\rangle = \tilde{\theta}(g).$$

i.e.

$$\tilde{\theta}(g * f) = \tilde{\theta}(g) \quad \text{for all } f \in P^1(K) \quad \text{and} \quad g \in P^+(K). \qquad (*)$$

Also note that $\widetilde{({}_g\theta)} = {}_g\tilde{\theta}$ for all $g \in P^+(K)$. Let $\{\epsilon_\alpha\} \subseteq P^1(K)$ be an approximate identity for $L^1(K)$ (see [55, Lemma 2.1]). Then by using $(*)$ and continuity of $\tilde{\theta}$, for $g \in P^+(K)$, we have

$$\tilde{\theta}(g) = \lim_\alpha \tilde{\theta}(\epsilon_\alpha * g) = \lim_\alpha \tilde{\theta}(\|g\|_1\, \epsilon_\alpha * \frac{g}{\|g\|_1}) = \lim_\alpha \tilde{\theta}(\|g\|_1\, \epsilon_\alpha).$$

This shows that for any $\lambda > 0$, $\lim_\alpha \tilde{\theta}(\lambda\, \epsilon_\alpha)$ exists. Define $\bar{\theta}$ on $I\!R^+$ by $\bar{\theta}(\lambda) = \lim_\alpha \tilde{\theta}(\lambda\, \epsilon_\alpha)$. Let $\bar{m}$ be a LIM on $l^\infty(I\!R^+)$ and define $\bar{\mu}$ on $UC_r(P^+(K))$ by

$$\langle \bar{\mu}, \theta\rangle = \langle \bar{m}, \bar{\theta}\rangle.$$

Then one can check that $\bar{\mu}$ is a mean. Now we show that $\bar{\mu}$ is left invariant. For any $f, f_1 \in P^+(K)$,

$$\widetilde{({}_f\theta)}(f_1) = {}_f\tilde{\theta}(f_1) = \tilde{\theta}(f * f_1) = \lim_\alpha \tilde{\theta}(\epsilon_\alpha * f * f_1) = \lim_\alpha \tilde{\theta}(\|f_1\|\, \|f\|\, \epsilon_\alpha).$$

Hence, for any $\lambda > 0$.

$$(\,_f\theta\tilde{)}(\lambda) = \lim_\alpha (\,_f\theta\tilde{)}(\lambda\, e_\alpha) = \lim_\alpha \tilde{\theta}(\lambda \|f\|_1\, e_\alpha) = \tilde{\theta}(\lambda\|f\|_1) = \,_{\|f\|_1}[\tilde{\theta}](\lambda).$$

i.e. $(\,_f\theta\tilde{)} = \,_{\|f\|_1}[\tilde{\theta}]$. Therefore

$$\langle \bar{\mu}, \,_f\theta\rangle = \langle \bar{m}, (\,_f\theta\tilde{)}\rangle = \langle \bar{m}, \,_{\|f\|_1}[\tilde{\theta}]\rangle = \langle \bar{m}, \tilde{\theta}\rangle = \langle \bar{\mu}, \theta\rangle.$$

This proves $\bar{\mu}$ is a LIM on $UC_r(P^+(K))$. □

Remark 6.2.19 (1) It follows from Lemma 6.2.17 that if $P^+(K)$ is left amenable, then $P^1(K)$ is left amenable. However, we do not know if the converse is true. Indeed, if $\theta \in C(P^+(K))$, then the function $\tilde{\theta}(g) = \langle m, \,_g\theta|_{P^1(K)}\rangle$ defined in the proof of Proposition 6.2.18 may not be continuous on the topological semigroup $P^+(K)$, unless $\theta \in UC_r(P^+(K))$, even when K is a group. Consequently, there is a gap in the proof of Proposition 1.3 in [16].

(2) Both Lemma 6.2.17 and Proposition 6.2.18 remain valid for an Lau-algebra L.

Appendix A

Existence of Haar Measure on Commutative Hypergroups

A.1 Introduction

A fundamental open question about hypergroups is the existence of Haar measure for any hypergroup. If a hypergroup K is compact or discrete, then K possesses a Haar measure. All known examples have a Haar measure [33, §5]. Spector in [57] claims that any commutative hypergroup possesses a Haar measure but as Ross in [50] mentioned there are several technical problems in his proof. Ross in [50] has given a lengthy proof for existence of Haar measure on commutative hypergroups. Recently Izzo in [32] has given a short proof of the existence of Haar measure on a commutative locally compact group by using the Markov-Kakutani fixed-point theorem [7, pp. 155-156]. Based on his idea, we give a short proof of the existence of Haar measure on commutative hypergroups.

For the reader's convenience, we include the Markov-Kakutani fixed point theorem. Let $\mathcal{S}$ be a compact convex subset of a Hausdorff topological vector space and $\mathcal{F}$ be a commutative family of continuous affine mappings of $\mathcal{S}$ into $\mathcal{S}$. Then there exists $p \in \mathcal{S}$ such that $\Lambda(p) = p$ for all $\Lambda \in \mathcal{F}$ (for a proof see [7]).

Note A.1.1 For a vector space X, let $X^{\#}$ be the space of all linear functionals on X with the weak topology induced by X. Then, if C is a closed subset of $X^{\#}$ such that the set $\{\Lambda x : \Lambda \in C\}$ is bounded, for any $x \in X$, then C is compact (see [11, PP. 423-424]).

Theorem A.1.2 *Every commutative hypergroup K has a left Haar measure.*

Proof: Let $C_{00}(K)^{\#}$ be the space of all linear functionals on $C_{00}(K)$. We consider on $C_{00}(K)^{\#}$ the weak topology generated by $C_{00}(K)$. It is clear that if there exists a $\Lambda \in C_{00}(K)^{\#}$ such that $f(\Lambda) = 0$ for all $f \in C_{00}(K)$, then $\Lambda = 0$. So $C_{00}(K)^{\#}$ with this topology is a locally convex space (see[15, P. 50]) . Let U be a fixed symmetric neighborhood of the identity $e \in K$ with compact closure. Let $\mathcal{S}$ be the set of all positive linear functionals Λ on $C_{00}(K)$ that satisfy the following two conditions:

(i) $\Lambda(f) \leq 1$ whenever $f \leq 1$ in $C_{00}^{+}(K)$ and $spt f \subseteq a_1 * a_2 * \cdots * a_r * U$ for some finite subset $\{a_1, a_2, \cdots, a_r\}$ in K,

(ii) $\Lambda(f) \geq 1$ whenever $f \leq 1$ in $C_{00}^{+}(K)$ and $f = 1$ on $a_1 * a_2 * \cdots * a_r * U * U$ for some finite subset $\{a_1, a_2, \cdots, a_r\}$ in K.

Then one can easily check that $\mathcal{S}$ is closed and convex. Moreover, any $f \in C_{00}^{+}(K)$ can be written as a finite sum of non-negative continuous functions, each of which has support in $a * U$ for some $a \in K$. To see this, let $sptf = C$. (compact set). Then $C \subseteq \cup_{1 \leq i \leq n} a_i * U$ for some $a_i \in K$, $1 \leq i \leq n$. By the partition of unity on compact sets, there are $h_i \in C_{00}^{+}(K)$ such that $0 < \frac{h_i}{f} \leq 1$ on C. That is for any $x \in C$, $0 < h_i(x) \leq f(x)$ and $h_1(x) + h_2(x) + \cdots + h_n(x) = f(x)$. Now it follows from (i) that the set $\{\Lambda(f) : \Lambda \in \mathcal{S}\}$ is bounded. So by Note A.1.1, $\mathcal{S}$ is compact.

To see $\mathcal{S}$ is non-empty, let M be as in Lemma 2.2.4. Put $\Lambda(f) = \sum_{s \in M} f(s)$, then $\Lambda \in \mathcal{S}$. Indeed, if $f \in C_{00}^{+}(K)$ and $f \leq 1$ with $sptf \subseteq a_1 * a_2 * \cdots * a_n * U$ for some $a_i \in K$, $1 \leq i \leq n$, then by Lemma 2.2.4, M intersects $a_1 * a_2 * \cdots * a_n * U$ at most at one point. Hence $\Lambda(f) \leq 1$. If $f \in C_{00}^{+}(K)$ and $f = 1$ on $a_1 * a_2 * \cdots * a_n * U * U$ for some $a_i \in K$, $1 \leq i \leq n$, then again by Lemma 2.2.4, M intersects $a_1 * a_2 * \cdots * a_n * U * U$ at least at one point. So $\Lambda(f) \geq 1$.

For each $x \in K$, let $T_x : C_{00}(K)^{\#} \to C_{00}(K)^{\#}$ is defined by $T_x\Lambda(f) = \Lambda({}_xf)$ for $f \in C_{00}(K)$. Then it is easy to see that T_x is affine and $T_x(\mathcal{S}) \subseteq \mathcal{S}$. Indeed, let $\Lambda \in \mathcal{S}$. If $f \in C_{00}^{+}(K)$ and $f \leq 1$ with $sptf \subseteq a_1 * a_2 * \cdots * a_n * U$ for some $a_i \in K$, $1 \leq i \leq n$, then ${}_xf \in C_{00}^{+}(K)$ (see [33, 4.2E]) and ${}_xf \leq 1$ with $spt({}_xf) \subseteq \check{x} * a_1 * a_2 * \cdots * a_n * U$. So by (i) $\Lambda({}_xf) \leq 1$. If $f \in C_{00}^{+}(K)$ and $f = 1$ on $a_1 * a_2 * \cdots * a_n * U * U$ for some $a_i \in K$, $1 \leq i \leq n$, then ${}_xf \in C_{00}^{+}(K)$ and ${}_xf = 1$ on $\check{x} * a_1 * a_2 * \cdots * a_n * U * U$. So by (ii), $\Lambda({}_xf) \geq 1$.

Also T_x is continuous, since if $\lim_\alpha \Lambda_\alpha = \Lambda$ in $\mathcal{S}$, then for any $f \in C_{00}(K)$,

$$\lim_\alpha |T_x\Lambda_\alpha(f) - T_x\Lambda(f)| = \lim_\alpha |\Lambda_\alpha({}_xf) - \Lambda({}_xf)| = 0.$$

Moreover for $x, y \in K$,

$$T_x(T_y\Lambda) = T_{x*y}\,\Lambda = T_{y*x}\,\Lambda = T_y(T_x\Lambda)$$

for any $\Lambda \in C_{00}(K)^{\#}$. This shows that the family $\mathcal{F} = \{T_x : x \in K\}$ and $\mathcal{S}$ (as above) have all properties in Markov-Kakutani fixed-point theorem. So there exists $\Lambda_0 \in \mathcal{S}$ such that $T_x\Lambda_0 = \Lambda_0$ for all $x \in K$. In another words

$$T_x(\Lambda_0 f) = \Lambda_0({}_xf) = \Lambda_0(f) \qquad \text{for all} \quad a \in K \quad \text{and} \quad f \in C_{00}(K).$$

Now since all elements of $\mathcal{S}$ are non-zero positive linear functionals on $C_{00}(K)$, by [33, §5.2] the proof is complete. □

Remark A.1.3 We do not know if the above proof can be modified to show that every amenable hypergroup has a left Haar measure using Day's generalization of Markov-Kakutani fixed-point theorem [9, Theorem 1] (see also [48, Theorem 4.2] and [56, Theorem 3.3.1]).

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