University of Alberta

Topological Recursion and the Supereigenvalue Model

by

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A thesis submitted to the Faculty of Graduate Studies and Research
in partial fulfillment of the requirements for the degree of

Master of Science
in
Mathematical Physics

Department of Mathematical and Statistical Sciences

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Fall 2012
Edmonton, Alberta

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Abstract

This thesis is interested in the topological recursion first introduced in [11] and generalized to algebraic curves in [20, 21]. A presentation of the Hermitian matrix model is given and includes a derivation of this topological recursion. The second part introduces a supersymmetric analog of the Hermitian matrix model first derived in [3] and know as the Supereigenvalue model. The development of the Supereigenvalue model follows in close parallel with the discussion on the Hermitian matrix model and considers the possibility of finding a supersymmetric generalization of the recursion.

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CHAPTER 1

Introduction

Those who study matrix models are interested in a subfield of the extensive discipline of random matrix theory that is related to mathematical models of physical theories. To put it simply, matrix models are statistical models over fields of matrices. We can classify models by two types: convergent and formal. Of interest is the formal case, because convergent integrals occur as concrete evaluations of formal integrals. Formally speaking, we may interpret these integrals as path-integrals and thereby arrive at zero dimensional quantum field theory.

Historically, matrix models were 'solved' using combinatorial methods to count "fat-graphs". The idea behind this is simply perturbation theory about the convergent Gaussian integrals. However, the fat-graph notion is inherently related to two dimensional surfaces via an injective imbedding. Furthermore, each non-intersecting graph corresponds to a discretization of the surface it is imbedded in, and thus we are in the domain of 2D Euclidean quantum gravity. This brought matrix models to the attention of those engaged in mathematical physics [12, 30, 15, 13, 14, 28, 25].

Consistent with the interpretation of matrix models as representations of physical dynamics, the partition function – or vacuum state, to use physics diction – is invariant under infinitesimal changes in the dynamical variables within the model. In particular, the systems are invariant under operators that generate the Virasoro algebra. Naturally, this implies that the systems satisfy a hierarchal series of relations which allow for an iterative method of solving the model.

Motivated by the relation to Euclidean quantum gravity, the next step was to construct simple models of supersymmetric quantum gravity. Supermatrix models, completely analogous to matrix models but with supersymmetric matrices, disappointed after investigation showed that the models de-couple into standard matrix models. Success in the supersymmetric pursuit came by generalizing the eigenvalue representation of matrix models that follows from the symmetry of Hermitian matrices under the unitary group. Alvarez-Gaume et al. introduced the supereigenvalue model in [3]. In the supereigenvalue model the dynamic parameters are both complex and Grassmann numbers and the partition function of the model was constructed to be invariant under generators of the super-Virasoro algebra. The hierarchy suggested by the super-Virasoro algebra was explicitly utilized in [29] where an iterative procedure for solving the supereigenvalue model was presented for the one-cut case.

A reinvestigation of matrix models by Chekhov, Eynard, and Orantin, specifically the two Hermitian matrix model (2HMM) in [11], led to a new formulation of the recursive nature of the resolvents in terms of residues. The idea is to translate the resolvents into multilinear differential forms on a compact Riemann surface determined by the particular matrix model under consideration. Moreover, Eynard and Orantin were able to generalize the residue recursion for arbitrary algebraic curves and found that the recursion is able to compute numbers that are invariant under symplectomorphisms of the algebraic curve [20, 21]. This recursion was then applied to Calabi-Yau threefolds in the context of mirror symmetry and appears to generate some fascinating results – Hurwitz numbers, Gromov-Witten invariants, etc.[7, 8, 9, 25].

In this paper we are interested in the residue formulation of the topological recursion first introduced by Chekhov, Eynard, and Orantin. In the first half we introduce the 1 Hermitian matrix model (1HMM) and thoroughly work our way through to the derivation of the residue formulation of the recursion. Despite an extensive amount of literature on the topological recursion, there is no simple derivation that motivates the origin of this ubiquitous recursion. In the second half we introduce the neglected Supereigenvalue model (SEV) and provide complete proofs on the relationship between the SEV and 1HMM and on the expansion of the free energy in terms of the Grassmann coupling constants. After defining all the necessary objects and deriving the superloop equations, this thesis proves that the structure of the superloop resolvents is indeed recursive. We conclude by discussing the possibility of expressing this recursive structure in terms of residues of forms on some (perhaps super) algebraic curve.

CHAPTER 2

The Hermitian matrix model

The residue formulation of the topological recursion first presented itself as a solution to the 2 Hermitian matrix model (2HMM) in the work of [11]. This particular formulation of the recursion was found applicable to a variety of matrix models and was further generalized to a recursion over differential forms defined on a compact Riemann surface [20, 21]. In this section we introduce the 1 Hermitian matrix model (1HMM) and perform a thorough investigation that arrives at the residue formulation of the recursion. While the statement of the topological recursion is well known and documented, a simple derivation eludes the literature. Consequently, we hope the reader will value partaking in an explicit derivation.

2.1 1HMM

The Hermitian matrix model is a statistical model over $N \times N$ Hermitian matrices.

Definition 2.1.1 *Let $N \in \mathbb{N}$. Let $V(x) = \sum_{k\geq 0} g_k x^k$ be a polynomial of degree $d+1$. The* ***partition function*** *of the Hermitian matrix model is given by*

$$\bar{Z}_H(N, g_k, T) \equiv \int dM e^{-\frac{N}{T} TrV(M)} \tag{2.1}$$

where $dM = dM_{11} \cdots dM_{NN} \prod_{i<j} d\Re(M_{ij}) d\Im(M_{ij})$ is the Haar measure over the Hermitian matrices. T is the ***charge*** *of the model, typically set to unity.*

For our purposes it is more convenient to work in terms of eigenvalues (indeed this will allow us to further generalize to the supereigenvalue model). We can take N orthonormal eigenvectors of any Hermitian matrix M and write out a unitary matrix U which will allow us to diagonalize $M = U^\dagger \Lambda U$ where $\Lambda = \text{diagonal}(\lambda_1, \ldots, \lambda_N)$. Of course $TrM = Tr\Lambda$. What is dM in terms of Λ? First note that $U^\dagger U = \text{Id}$ implies that $(dU^\dagger)U + U^\dagger dU = 0$ which gives $dU^\dagger = -U^\dagger dU U^\dagger$.

$$\begin{aligned} dM = d(U^\dagger \Lambda U) &= -U^\dagger(dU)U^\dagger \Lambda U + U^\dagger(d\Lambda)U + U^\dagger \Lambda(dU) \\ &= -U^\dagger(dU)U^\dagger \Lambda U + U^\dagger(d\Lambda)U + U^\dagger \Lambda(dU)U^\dagger U \\ &= U^\dagger(-(dU)U^\dagger \Lambda + d\Lambda + \Lambda(dU)U^\dagger)U \\ &= U^\dagger(d\Lambda + [\Lambda, (dU)U^\dagger])U. \end{aligned}$$

In particular, around $U = \text{Id}$ we have $dM = d\Lambda + [\Lambda, dU]$ or $dM_{ii} = d\Lambda_{ii} = d\lambda_i$ and $dM_{i\neq j} = (\lambda_i - \lambda_j)dU_{ij}$. See [14, 28, 25]. Now we can express the partition function in terms of the eigenvalues:

Definition 2.1.2 *The eigenvalue representation of the partition function for the Hermitian matrix model is given by*

$$Z_H(N, g_k, T) = \int d\lambda_1 \cdots d\lambda_N \prod_{i<j} (\lambda_i - \lambda_j)^2 e^{-\frac{N}{T}\sum_{i=1}^N V(\lambda_i)}. \tag{2.2}$$

For conciseness we often write $\Delta^2(\lambda) = \prod_{i<j}(\lambda_i - \lambda_j)^2$. This is this square of the Vandermonde determinant. See Appendix A.1. Z and $\bar{Z}$ are proportional. However, as we are interested in calculating expectation values relative to the vacuum state defined by the partition function, i.e.,

$$\langle A \rangle = \frac{1}{Z_H} \int d\lambda_1 \cdots d\lambda_N A \Delta^2(\lambda) e^{-\frac{N}{T}\sum_i V(\lambda_i)}, \tag{2.3}$$

the (N dependent) constant of proportionality from the integral over the unitary group is irrelevant.

Definition 2.1.3 *The **free energy** of the Hermitian model, F_H, is defined by*

$$Z_H \equiv e^{(\frac{N}{T})^2 F_H(g_k)}. \tag{2.4}$$

2.1.1 Observables and the loop insertion operator

Our objective is to be able to calculate all observables in this model, i.e., the expectation value of any function of λ_i. The significance of F_H is that it contains all the information of the observables. To see this we introduce the loop **correlators** or **resolvents**

$$W(x_1,\dots,x_m) \equiv \Big(\frac{N}{T}\Big)^{m-2}\Big\langle \sum_{i_1=1}^{N}\frac{1}{x_1-\lambda_{i_1}}\cdots\sum_{i_m=1}^{N}\frac{1}{x_m-\lambda_{i_m}}\Big\rangle_c \tag{2.5}$$

where $\langle A\rangle_c$ means the connected part of $\langle A\rangle$. In particular, with $m=1$ we have

$$W(x)=\frac{T}{N}\Big\langle\sum_i\frac{1}{x-\lambda_i}\Big\rangle=\frac{T}{N}\sum_i\Big\langle\sum_{k\geq 0}\frac{\lambda_i^k}{x^{k+1}}\Big\rangle. \tag{2.6}$$

We see that the loop correlators are generating functionals of all observables in the model. The **loop insertion operator**

$$\frac{\partial}{\partial V(x)}\equiv-\sum_{k=0}^{\infty}\frac{1}{x^{k+1}}\frac{\partial}{\partial g_k} \tag{2.7}$$

acts on F_H to generate the loop correlators:

$$\begin{aligned}
W(x)&=\frac{\partial}{\partial V(x)}F_H,\\
W(x_1,\cdots,x_m)&=\frac{\partial}{\partial V(x_m)}\cdots\frac{\partial}{\partial V(x_1)}F_H,\\
\Longrightarrow\; W(x_1,\cdots,x_m)&=\frac{\partial}{\partial V(x_m)}\cdots\frac{\partial}{\partial V(x_2)}W(x_1),
\end{aligned} \tag{2.8}$$

thus solving $W(x)$ effectively solves all correlators. To see (2.8) observe that

$$\begin{aligned}
\frac{\partial}{\partial V(x_1)}F_H&=\frac{\partial}{\partial V(x_1)}\Big(\frac{T}{N}\Big)^2\ln Z_H=\Big(\frac{T}{N}\Big)^2\frac{1}{Z_H}\frac{\partial}{\partial V(x_1)}Z_H\\
&=\Big(\frac{T}{N}\Big)^2\frac{1}{Z_H}\int(\prod_{i=1}^{N}d\lambda_i)\Delta^2(\lambda)e^{-\frac{N}{T}\sum_{i=1}^{N}V(\lambda_i)}\frac{\partial}{\partial V(x_1)}\Big(-\frac{N}{T}\sum_{i=1}^{N}V(\lambda_i)\Big),
\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}
\Big(\frac{T}{N}\Big)^2\frac{\partial}{\partial V(x_1)}\Big(-\frac{N}{T}\sum_i V(\lambda_i)\Big)&=-\frac{T}{N}\sum_i\frac{\partial}{\partial V(x_1)}V(\lambda_i)=\frac{T}{N}\sum_i\sum_{k=0}^{\infty}\frac{1}{x^{k+1}}\frac{\partial}{\partial g_k}(\sum_j\lambda_i^j g_j)\\
&=\frac{T}{N}\sum_{i=1}^{N}\sum_{k\geq 0}\frac{\lambda_i^k}{x^{k+1}}=\frac{T}{N}\sum_i\frac{1}{x-\lambda_i}.
\end{aligned} \tag{2.10}$$

The infinite series in the definition of the loop insertion operator (2.7) is a formal necessity that allows the previous (and similar) calculation(s). When applying the loop insertion operator we imagine $V(x) = \sum_{k\geq 0} g_k x^k$ and after application set $g_j = 0$ for $j > d+1$. In anticipation we introduce

$$P(x_1, \ldots, x_m) \equiv \left(\frac{N}{T}\right)^{m-2} \Big\langle \sum_{i_1=1}^{N} \frac{V'(x_1) - V'(\lambda_{i_1})}{x_1 - \lambda_{i_1}} \sum_{i_2=1}^{N} \frac{1}{x_2 - \lambda_{i_2}} \cdots \sum_{i_m=1}^{N} \frac{1}{x_m - \lambda_{i_m}} \Big\rangle_c \tag{2.11}$$

which will permit us a nicer formulation of the **loop equations**, a series of differential equations that $W(\ldots)$ necessarily satisfies. Note that $P(x_1, \ldots, x_m)$ is a polynomial of degree $d-1$ in x_1 and can be computed by application of the loop insertion operator:

$$\frac{\partial}{\partial V(y)} P(x_1, \ldots, x_k) = P(x_1, \ldots, x_k, y) + \frac{d}{dy} \frac{W(x_2, \ldots, x_k, y)}{y - x_1}. \tag{2.12}$$

2.1.2 t'Hooft $\left(\frac{T}{N}\right)^2$ topological expansion

The Hermitian matrix model was initially dealt with by perturbation theory where the expectation values of non-quadratic-powers were calculated perturbatively in terms of fat-graph diagrams [15, 14]. The symmetry over the unitary group allows for a perturbative t'Hooft expansion in $1/N$ [31]. In particular we have that the free energy has a $\left(\frac{T}{N}\right)$ expansion

$$F_H = \sum_{g=0}^{\infty} \left(\frac{T}{N}\right)^{2g} F_g. \tag{2.13}$$

It follows that $W()$ and $P()$ have topological expansions:

$$\begin{aligned} W(x_1, \ldots, x_n) &= \sum_{g\geq 0} \left(\frac{T}{N}\right)^{2g} W_g(x_1, \ldots, x_n) \\ P(x_1, \ldots, x_n) &= \sum_{g\geq 0} \left(\frac{T}{N}\right)^{2g} P_g(x_1, \ldots, x_n). \end{aligned} \tag{2.14}$$

For the remainder of our discussion we set $T = 1$ unless otherwise noted.

2.2 Virasoro algebra

The invariance of Z_H under infinitesimal changes in λ_i implies the model obeys a set of constraints related to the Virasoro algebra. We perform the variation $\lambda_i \to \lambda_i + \varepsilon \lambda_i^{n+1}$ in (2.2) and find to first order in ε

1.

$$d(\lambda_1 + \varepsilon\lambda_1^{n+1}) \cdots d(\lambda_N + \varepsilon\lambda_i^{n+1}) = d\lambda_1 \cdots d\lambda_N \Big(1 + \varepsilon \sum_j (n+1)\lambda_j^n + O(\varepsilon^2)\Big), \quad (2.15)$$

2.

$$\begin{aligned} V(\lambda_i + \varepsilon\lambda_i^{n+1}) &= V(\lambda_i) + \varepsilon \sum_k k g_k \lambda_i^{k+n} + O(\varepsilon^2) \\ \implies e^{-N\sum_i V(\lambda_i + \varepsilon\lambda_i^{n+1})} &= e^{-N\sum V(\lambda_i)} \Big(1 - \varepsilon N \sum_i \sum_k k g_k \lambda_i^{k+n} + O(\varepsilon^2)\Big), \end{aligned} \quad (2.16)$$

3.

$$\begin{aligned} &\prod_{i<j} (\lambda_i + \varepsilon\lambda_i^{n+1} - \lambda_j - \varepsilon\lambda_j^{n+1})^2 = \prod_{i<j} (\lambda_i - \lambda_j)^2 \Big(1 + \frac{\varepsilon\lambda_i^{n+1} - \varepsilon\lambda_j^{n+1}}{\lambda_i - \lambda_j}\Big)^2 \\ &= \prod_{i<j} (\lambda_i - \lambda_j)^2 \Big(1 + 2\varepsilon \frac{\lambda_i^{n+1} - \lambda_j^{n+1}}{\lambda_i - \lambda_j} + O(\varepsilon^2)\Big) = \Delta^2 \Big(1 + \varepsilon \sum_{i \neq j} \frac{\lambda_i^{n+1} - \lambda_j^{n+1}}{\lambda_i - \lambda_j} + O(\varepsilon^2)\Big). \end{aligned} \quad (2.17)$$

Notice that

$$\sum_{i \neq j} \frac{\lambda_i^{n+1} - \lambda_j^{n+1}}{\lambda_i - \lambda_j} = \sum_{i \neq j} \lambda_j^n \frac{\left(\frac{\lambda_i}{\lambda_j}\right)^{n+1} - 1}{\left(\frac{\lambda_i}{\lambda_j} - 1\right)} = \sum_{i \neq j} \lambda_j^n \sum_{k=0}^n \lambda_i^k \lambda_j^{-k} = \sum_{i,j} \sum_{k=0}^n \lambda_i^k \lambda_j^{n-k} - (n+1)\lambda_i^n. \quad (2.18)$$

Combining the above, the condition that Z_H is invariant to first order in ε imposes

$$\Big\langle -N \sum_k \sum_i k g_k \lambda_i^{n+k} + \sum_{k=0}^n \sum_i \sum_j \lambda_i^{n-k} \lambda_j^k \Big\rangle = 0, \quad (2.19)$$

and is equivalent to

$$\begin{aligned} &\mathcal{L}_n Z_H = 0, \quad \text{for } n \geq -1 \text{ and where} \\ &\mathcal{L}_n \equiv \sum_{k \geq 0} k g_k \frac{\partial}{\partial g_{k+n}} + N^{-2} \sum_{k=0}^n \frac{\partial}{\partial g_k} \frac{\partial}{\partial g_{n-k}}. \end{aligned} \quad (2.20)$$

The operators $\mathcal{L}_n$ generate the Witt algebra. To see this we evaluate the commutator $[\mathcal{L}_a, \mathcal{L}_b]$. Consider

$$\mathcal{L}_a\mathcal{L}_b = \Big(\sum_{k\geq 0} kg_k \frac{\partial}{\partial g_{k+a}} + N^{-2}\sum_{k=0}^{a}\frac{\partial}{\partial g_k}\frac{\partial}{\partial g_{a-k}}\Big)\Big(\sum_{l\geq 0} lg_l \frac{\partial}{\partial g_{l+b}} + N^{-2}\sum_{l=0}^{b}\frac{\partial}{\partial g_l}\frac{\partial}{\partial g_{b-l}}\Big) \quad (2.21)$$

Distributing the operators we have

1.

$$\Big(\sum_{k\geq 0} kg_k \frac{\partial}{\partial g_{k+a}}\Big)\Big(\sum_{l\geq 0} lg_l \frac{\partial}{\partial g_{l+b}}\Big) = \sum_{k\geq 0} kg_k\Big((k+a)\frac{\partial}{\partial g_{k+a+b}} + \sum_{l\geq 0} lg_l \frac{\partial}{\partial g_{k+a}}\frac{\partial}{\partial g_{l+b}}\Big) \quad (2.22)$$

from which we subtract $(a \leftrightarrow b)$ giving

$$(a-b)\sum_{k\geq 0} kg_k \frac{\partial}{\partial g_{k+a+b}}. \quad (2.23)$$

2.

$$\Big(N^{-2}\sum_{k=0}^{a}\frac{\partial}{\partial g_k}\frac{\partial}{\partial g_{a-k}}\Big)\Big(N^{-2}\sum_{l=0}^{b}\frac{\partial}{\partial g_l}\frac{\partial}{\partial g_{b-l}}\Big) - \Big(a \leftrightarrow b\Big) = 0 \quad (2.24)$$

3.

$$\Big(\sum_{k\geq 0} kg_k \frac{\partial}{\partial g_{k+a}}\Big)\Big(N^{-2}\sum_{l=0}^{b}\frac{\partial}{\partial g_l}\frac{\partial}{\partial g_{b-l}}\Big) = N^{-2}\sum_{k\geq 0}\sum_{l=0}^{b} kg_k \frac{\partial}{\partial g_{k+a}}\frac{\partial}{\partial g_l}\frac{\partial}{\partial g_{b-l}} \quad (2.25)$$

4.

$$\begin{aligned}\Big(N^{-2}\sum_{k=0}^{a}\frac{\partial}{\partial g_k}\frac{\partial}{\partial g_{a-k}}\Big)\Big(\sum_{l\geq 0} lg_l \frac{\partial}{\partial g_{l+b}}\Big) &= N^{-2}\sum_{k=0}^{a}\frac{\partial}{\partial g_k}\Big((a-k)\frac{\partial}{\partial g_{a+b-k}} + \sum_{l\geq 0} lg_l \frac{\partial}{\partial g_{a-k}}\frac{\partial}{\partial g_{l+b}}\Big)\\ &= N^{-2}\sum_{k=0}^{a}(a-k)\frac{\partial}{\partial g_k}\frac{\partial}{\partial g_{a+b-k}} + N^{-2}\sum_{k=0}^{a} k\frac{\partial}{\partial g_{a-k}}\frac{\partial}{\partial g_{k+b}} + N^{-2}\sum_{l\geq 0}\sum_{k=0}^{a} lg_l \frac{\partial}{\partial g_k}\frac{\partial}{\partial g_{a-k}}\frac{\partial}{\partial g_{l+b}}\\ &= N^{-2}\sum_{k=0}^{a}(a-k)\frac{\partial}{\partial g_k}\frac{\partial}{\partial g_{a+b-k}} + N^{-2}\sum_{k=b}^{a+b}(k-b)\frac{\partial}{\partial g_{a+b-k}}\frac{\partial}{\partial g_k} + N^{-2}\sum_{l\geq 0}\sum_{k=0}^{a} lg_l \frac{\partial}{\partial g_k}\frac{\partial}{\partial g_{a-k}}\frac{\partial}{\partial g_{l+b}}.\end{aligned} \quad (2.26)$$

Combining (2.25) and (2.26) and subtracting ($a \leftrightarrow b$) will result in the final term(s) from (2.26) canceling with the term(s) in (2.25) and yielding

$$N^{-2}\sum_{k=0}^{a}(a-k)\frac{\partial}{\partial g_k}\frac{\partial}{\partial g_{a+b-k}} - N^{-2}\sum_{k=0}^{b}(b-k)\frac{\partial}{\partial g_k}\frac{\partial}{\partial g_{a+b-k}} + N^{-2}\sum_{k=b}^{a+b}(k-b)\frac{\partial}{\partial g_{a+b-k}}\frac{\partial}{\partial g_k}$$
$$- N^{-2}\sum_{k=a}^{a+b}(k-a)\frac{\partial}{\partial g_{a+b-k}}\frac{\partial}{\partial g_k} = N^{-2}(a-b)\sum_{k=0}^{a+b}\frac{\partial}{\partial g_k}\frac{\partial}{\partial g_{a+b-k}}. \tag{2.27}$$

Thus we have

$$[\mathcal{L}_a, \mathcal{L}_b] = (a-b)\sum_{k\geq 0} k g_k \frac{\partial}{\partial g_{k+a+b}} + N^{-2}(a-b)\sum_{k=0}^{a+b}\frac{\partial}{\partial g_k}\frac{\partial}{\partial g_{a+b-k}} = (a-b)\mathcal{L}_{a+b}, \tag{2.28}$$

which shows that the $\mathcal{L}_n$ are generators of the Virasoro algebra with zero central charge, hence the constraint

$$\mathcal{L}_n Z_H(N) = \left\langle -N\sum_{k\geq 0} k g_k \sum_i \lambda_i^{n+k} + \sum_{k=0}^{n}\sum_i\sum_j \lambda_i^{n-k}\lambda_j^k \right\rangle = 0 \tag{2.29}$$

is referred to as the **Virasoro constraint**. On the other hand, starting with the Virasoro algebra we may derive the partition function (2.2) by demanding its invariance under the generators (2.29) of the algebra. This method uses a correlator function in a conformal field theory and the fact that the modes of the energy-momentum tensor of a free scalar field obey the Virasoro algebra [30, 26, 3]. Indeed, working in this direction is exactly how the supereigenvalue model was derived.

2.3 The loop equations

The loop equations are a sequence of relations between the loop correlators that are a result of the vanishing of a total derivative. The continual application of the loop insertion operator (2.7) leads to a recursive hierarchy of equations.

2.3.1 The master equation

We can arrive at the loop equations by considering the following total derivative [24] (remember $T = 1$)

$$0 = \frac{1}{Z_H}\int(d\lambda_1\cdots d\lambda_N)\sum_{p=1}^{N}\frac{\partial}{\partial\lambda_p}\Big((\frac{1}{x-\lambda_p})\prod_{i<j}(\lambda_i-\lambda_j)^2 e^{-N\sum_i V(\lambda_i)}\Big)$$
$$= \sum_p\Big\langle\frac{1}{(x-\lambda_p)^2}\Big\rangle + N\Big\langle\frac{V'(x)-V'(\lambda_p)}{x-\lambda_p}\Big\rangle - NV'(x)\Big\langle\frac{1}{x-\lambda_p}\Big\rangle + 2\Big\langle\frac{1}{x-\lambda_p}\sum_{i\neq p}\frac{1}{\lambda_p-\lambda_i}\Big\rangle. \tag{2.30}$$

We simplify

1. $\sum_p\langle\frac{1}{(x-\lambda_p)^2}\rangle = -NW'(x)$,
2. $\sum_p N\langle\frac{V'(\lambda(x)-V'(\lambda_p)}{x-\lambda_p}\rangle - NV'(x)\langle\frac{1}{x-\lambda_p}\rangle = N^2P(x) - N^2V'(x)W(x)$,
3. $2\sum_p\langle\frac{1}{x-\lambda_p}\sum_{i\neq p}\frac{1}{\lambda_p-\lambda_i}\rangle = \sum_{p,i\neq p}\langle\frac{1}{x-\lambda_p}\frac{1}{\lambda_p-\lambda_i} - \frac{1}{x-\lambda_i}\frac{1}{\lambda_p-\lambda_i}\rangle = \sum_{p,i\neq p}\langle\frac{1}{(x-\lambda_p)(x-\lambda_i)}\rangle$
$= \sum_{p,i}\langle\frac{1}{(x-\lambda_p)(x-\lambda_i)}\rangle - \sum_p\langle\frac{1}{(x-\lambda_p)^2}\rangle = \sum_{p,i}\langle\frac{1}{(x-\lambda_p)(x-\lambda_i)}\rangle + NW'(x)$
$= \langle\sum_{p,i}\frac{1}{(x-\lambda_p)(x-\lambda_i)}\rangle_c + \langle\sum_p\frac{1}{(x-\lambda_p)}\rangle\langle\sum_i\frac{1}{(x-\lambda_i)}\rangle + NW'(x)$
$= W(x,x) + N^2W(x)^2 + NW'(x)$.

Combining the above and multiplying by $1/N^2$ we arrive at the **master loop equation**

$$\boxed{\frac{W(x,x)}{N^2} + W(x)^2 = V'(x)W(x) - P(x).} \tag{2.31}$$

Please note that we can derive the master loop equation (2.31) directly from the Virasoro constraints (2.29) which implies

$$0 = \sum_{n\geq -1}\frac{1}{x^{n+2}}\Big\langle -N\sum_{k\geq 0}kg_k\sum_i\lambda_i^{n+k} + \sum_{k=0}^{n}\sum_i\sum_j\lambda_i^{n-k}\lambda_j^k\Big\rangle. \tag{2.32}$$

We have

$$\begin{aligned}\sum_{n\geq -1}\frac{1}{x^{n+2}}N\sum_{k\geq 0}kg_k\sum_i\lambda_i^{n+k} &= N\sum_{n\geq -1}\sum_k kg_k\sum_i\frac{\lambda_i^{n+k}}{x^{n+2}}\\ &= N\sum_k\sum_i kg_k\frac{\lambda_i^{k-1}}{x-\lambda_i} = N\sum_i\frac{V'(\lambda_i)}{x-\lambda_i},\end{aligned} \tag{2.33}$$

and similarly

$$\sum_{n\geq -1}\sum_{k=0}^{n}\sum_i\sum_j\frac{\lambda_i^{n-k}\lambda_j^k}{x^{n+2}} = (\sum_i\frac{1}{x-\lambda_i})^2, \tag{2.34}$$

thus

$$0 = \left\langle -N\sum_i \frac{V'(\lambda_i)}{x-\lambda_i} + \left(\sum_i \frac{1}{x-\lambda_i}\right)^2 \right\rangle = -\left\langle N\sum_i \frac{V'(\lambda_i)}{x-\lambda_i}\right\rangle + N^2 W(x)^2 + W(x,x) \quad (2.35)$$

which gives (2.31). Intuitively it makes sense that performing the shift $\lambda_i \to \lambda_i + \varepsilon\lambda_i^{n+1}$ $\forall n \geq -1$ and considering the total derivative $\sum_i \partial/\partial\lambda_i (x-\lambda_i)^{-1}$ are equivalent. Regardless, we now see it explicitly.

2.3.2 The general loop equations

We can use the loop insertion operator to get loop equations for multi-correlators [18], i.e., we operate $\frac{\partial}{\partial V(x_1)}$ on $\frac{1}{N^2}W(x,x) + W(x)^2 - V'(x)W(x) = -P(x)$. Looking at each term,

1. $\frac{\partial}{\partial V(x_1)}\frac{1}{N^2}W(x,x) = \frac{1}{N^2}W(x,x,x_1)$,
2. $\frac{\partial}{\partial V(x_1)}W(x)^2 = 2W(x)W(x,x_1)$,
3. $\frac{\partial}{\partial V(x_1)}V'(x)W(x) = V'(x)W(x,x_1) - \frac{W(x)}{(x-x_1)^2} = V'(x)W(x,x_1) - \frac{\partial}{\partial x_1}\frac{W(x)}{(x-x_1)}$,
4. $\frac{\partial}{\partial V(x_1)}P(x) = P(x,x_1) + \frac{\partial}{\partial x_1}\frac{W(x_1)}{(x_1-x)} = P(x,x_1) - \frac{\partial}{\partial x_1}\frac{W(x_1)}{(x-x_1)}$,

we arrive at the next loop equation:

$$\frac{1}{N^2}W(x,x,x_1) + 2W(x)W(x,x_1) + \frac{\partial}{\partial x_1}\frac{W(x)-W(x_1)}{x-x_1} = V'(x)W(x,x_1) - P(x,x_1). \quad (2.36)$$

By continued application of the loop insertion operator [18] on the master loop equation (2.31) we find that the general equation by number of variables and order in $\frac{1}{N}$ is

$$\boxed{\begin{aligned} W_{g-1}(x,x,J) + \sum_{h=0}^{g}\sum_{I\subset J} W_h(x,I)W_{g-h}(x,J\backslash I) + \sum_{j=1}^{n}\frac{\partial}{\partial x_j}\frac{W_g(x,J\backslash x_j) - W_g(J)}{x-x_j} \\ = V'(x)W_g(x,J) - P_g(x,J), \quad \text{where } J = \{x_1,\ldots,x_n\}. \end{aligned}} \quad (2.37)$$

(2.37) is the foundation of the recursion for the correlators. If $2g+n+1 \geq 3$ we have

$$\begin{aligned} \Big(V'(x) - 2W_0(x)\Big)W_g(x,J) = W_{g-1}(x,x,J) + \sum_{h=1}^{g-1}\sum_{\emptyset\not\subseteq I\not\subseteq J} W_h(x,I)W_{g-h}(x,J\backslash I) \\ +2W_g(x)W_0(x,J) + \sum_{j=1}^{n}\frac{\partial}{\partial x_j}\frac{W_g(x,J\backslash x_j) - W_g(J)}{x-x_j} + P_g(x,J) \end{aligned} \quad (2.38)$$

with which we see explicitly the nature of the recursion in $2g + n$.

2.4 The planar limit

From (2.38) we are able to solve $W_g(x, J)$ recursively. In order to start building up correlators we must first investigate the planar limit, that is when $g = 0$.

2.4.1 The spectral curve

To leading order in the $1/N$ expansion we have $W_0(x)^2 - V'(x)W_0(x) + P_0(x) = 0$ with solution

$$W_0(x) = \frac{V'(x)}{2} - \sqrt{\left(\frac{V'(x)}{2}\right)^2 - P_0(x)} \tag{2.39}$$

where the minus sign is needed for this to agree with the asymptotic behavior $W_0(x)_{x\to\infty} \sim 1/x$ from the definition. If we introduce

$$y(x) = \frac{V'(x)}{2} - W_0(x) \tag{2.40}$$

we can interpret (2.39) as corresponding to a hyperelliptic Riemann surface Σ defined so that $\forall p \in \Sigma$

$$\boxed{E\big(x(p), y(p)\big) \equiv y(p)^2 - \left(\frac{V'(x(p))}{2}\right)^2 + P_0\big(x(p)\big) = 0.} \tag{2.41}$$

The algebraic relation (2.41) defines the **spectral curve**. The Vandermonde term in the partition function (2.2) can be expressed as an exponential, allowing one to write

$$Z_H(N, g_k, T = 1) = \int d\lambda_1 \cdots d\lambda_N e^{-N\sum_{i=1}^{N} V_{\text{effective}}(\lambda_i)} \tag{2.42}$$

where

$$V_{\text{effective}}(\lambda_i) = V(\lambda_i) - \frac{2}{N}\sum_{j\neq i} \ln|\lambda_i - \lambda_j|. \tag{2.43}$$

Thus, in the planar limit, which corresponds to $N \to \infty$, we have a statistical distribution of eigenvalues determined by the potential V. Since deg $V = d + 1$, $V'(x)$ has $s \leq d$ **distinct** real roots α_i. For finite N, the eigenvalues experience pair-wise repulsion from the logarithm term in $V_{\text{effective}}$ that causes them to spread out from each of the s roots of V. The result is that the eigenvalues will distribute around $s \leq d$ (distinct) regions on the real axis surrounding the α_i; these regions are referred to as **cuts**. We define the **filling**

fractions ϵ_i by

$$\epsilon_i = \frac{n_i}{N} \tag{2.44}$$

where n_i is the number of eigenvalues around α_i. Since we have $\sum_{i=1}^{s} \epsilon_i = 1$, we call them filling fractions because they represent the fractional distributions of the eigenvalues around the disjoint cuts. By expanding about the α_i, $W(x) = \frac{1}{N}\sum_i \langle \sum_{k\geq 0} \frac{(\lambda_i - \alpha_j)^k}{(x-\alpha_j)^{k+1}} \rangle$ and thus

$$\mathrm{Res}_{x\to\alpha_j} W(x) = \epsilon_j, \tag{2.45}$$

and, seen similarly, for $j \geq 2$

$$\mathrm{Res}_{x_1\to\alpha_k} W(x_1, x_2, \ldots, x_j) = 0. \tag{2.46}$$

Additionally, the filling fractions are assumed to be independent of N, which gives for $g \geq 1$

$$\mathrm{Res}_{x_1\to\alpha_k} W_g(x_1, x_2, \ldots, x_j) = 0. \tag{2.47}$$

For simplicity, let $s = d$. Since deg $V = d+1$, then by (2.40) deg $y^2 = 2d$ and Σ corresponds to a hyperelliptic curve of genus $d-1$:

$$y(p)^2 = \prod_{i=1}^{2d} (p - a_i). \tag{2.48}$$

The surface Σ is composed of a two-sheeted covering of $\mathbb{P}^1$: $\chi_+ = \{p \in \Sigma | y(p) \geq 0\}$ and $\chi_- = \{p \in \Sigma | y(p) \leq 0\}$. Thus the two sheets meet at $\chi_+ \cap \chi_- = \{a_i | i = 1, \ldots, 2d\}$, called **ramification** points. The ramification points are the complete solutions of $dx = 0$. For any $p \neq a_i$, $\exists\, !\ \bar{p} \neq p \in \Sigma$ such that $x(p) = x(\bar{p})$ and $y(p) = -y(\bar{p})$. We say that p and $\bar{p}$ are **conjugate**: p and $\bar{p}$ are in different sheets. Additionally, Σ has two poles $\infty_\pm$ corresponding to the x projection to ∞ on each sheet.

$$y(p)_{p\to\infty_\pm} \sim \pm \frac{V'\big(x(p)\big)}{2} \mp \frac{1}{x(p)}. \tag{2.49}$$

If $s < d$ we introduce a modified surface defined by $\tilde{y}(p)^2 = \prod_{i=1}^{2s}(p - a_i)$ [10].

We chose a homology basis for Σ (of genus $g = s-1 \leq d-1$) by having cycles $\mathcal{A}_i$ surrounding the cuts about the α_i and dual cycles $\mathcal{B}_i$ canonically normalized so that $\mathcal{A}_i \cap \mathcal{B}_j = \delta_{ij}$ and $\mathcal{A}_i \cap \mathcal{A}_j = 0 = \mathcal{B}_i \cap \mathcal{B}_j$.

2.4.2 Insertion operator in Σ

We translate our mathematical objects in $\mathbb{C}$ into objects defined over Σ. The following is motivated by (2.46) and (2.49):

Definition 2.4.1 *Let $p \in \Sigma$ then*

1. *for $p \in \chi_+$, $V'(p) \equiv V'\big(x(p)\big)$*
2. *for $p \in \chi_-$, $V'(p) \equiv -V'\big(x(p)\big)$.*

In particular, for $a_i \in \chi_+ \cap \chi_-$, $V'(a_i) = 0$. This shows that the ramification points and the optimal eigenvalue large N limits agree when paramaterized on Σ. In Σ consider a non-ramification point p and its conjugate $\bar{p}$. We have

$$y(\bar{p}) = \frac{V'\big(x(\bar{p})\big)}{2} - W_0\big(x(\bar{p})\big) = -y(p) = -\frac{V'\big(x(p)\big)}{2} + W_0\big(x(p)\big)$$

which motivates the following:

Definition 2.4.2 *The loop insertion operator on Σ is defined by*

1. *for $p \in \chi_+$, $\frac{\partial}{\partial V(p)} = \frac{\partial}{\partial V(x(p))}$*
2. *for $q \in \chi_-$, $\frac{\partial}{\partial V(q)} = -\frac{\partial}{\partial V(x(q))}$*

Definition 2.4.3 *Let $p_1, \ldots, p_n \in \Sigma$, $g \in \mathbb{Z}$ with $2g + n \geq 3$.*

$$\omega_g(p_1, \ldots, p_n) = dx(p_1) \cdots dx(p_n) \frac{\partial}{\partial V(p_1)} \cdots \frac{\partial}{\partial V(p_m)} F_g \tag{2.50}$$

It follows that $\omega_g(p_1, \ldots, p_i, \ldots, p_n) + \omega_g(p_1, \ldots, \bar{p}_i, \ldots, p_n) = 0$ when $2g + n \geq 3$. These differential forms are the main objects of interest and will be investigated shortly.

2.4.3 The differential $y(p)dx(p)$

In this section we work with general T and mention $\sum_{i=1}^{s} \epsilon_i = T$ and that the second term on the RHS of (2.49) is $\mp \frac{T}{x(p)}$. Following (2.45), one observes that

$$\oint_{\mathcal{A}_i} y(p)dx(p) = -\epsilon_i \tag{2.51}$$

for $i = 1, \dots, d-1$ (note $\epsilon_d = -T + \sum_{i=1}^{d-1} \epsilon_i$). From (2.49)

$$\begin{aligned} &\mathrm{Res}_{p\to\infty_\pm} y(p)dx(p) = \pm T, \text{ and} \\ &\mathrm{Res}_{p\to\infty_+} \frac{2y(p)dx(p)}{k\big(x(p)\big)^k} = g_k, \end{aligned} \tag{2.52}$$

which are the charge and the coupling constants respectively. The above show that $y(p)dx(p)$ gives T, ϵ_i, and g_k: these are precisely the moduli of the Hermitian matrix model. The least obvious of the moduli are the filling fractions ϵ_i. Recall that the leading order of the master loop equation depends on the polynomial $P_0(x)$ of degree $d-1$ in x.

$$P_0(x)_{x\to\infty} \sim V'(x)W_0(x) = \frac{T}{N}\sum_{i=1}^{N}\Big\langle V'(x)\sum_{k\geq 0}\frac{(\lambda_i-\alpha_i)^k}{(x-\alpha_i)^{k+1}}\Big\rangle \tag{2.53}$$

so if we look at leading order and sum i over the number of cuts s we have

$$P_0(x)_{x\to\infty} \sim \sum_{i=1}^{s}\frac{n_i T}{N}\Big\langle V'(x)\frac{1}{(x-\alpha_i)}\Big\rangle = \sum_{i=1}^{s}\epsilon_i\frac{V'(x)}{x-\alpha_i}. \tag{2.54}$$

The $d = s$ constraints from (2.54) determine $P_0(x)$. Indeed, since the location and fractional distribution of the eigenvalues is assumed fixed, the filling fractions are necessarily independent of the matrix size N. Thus the only relevant term for defining ϵ_i comes from the leading order of $P(x)$ in the T/N expansion. This shows that by picking values of the filling fractions ϵ_i is equivalent to defining $P_0(x)$ so that the expression of $W_0(x)$ in (2.39) is defined.

2.5 Special forms

In this section we introduce the three kinds of differential forms and relations between them. They will be used to derive the residue formulation of the topological recursion later.

2.5.1 3 differential types

We place ourselves on a Riemann surface of genus g. The set $\{\mathcal{A}_i, \mathcal{B}_j\}$ where $1 \leq i, j \leq g$, $\mathcal{A}_i \cap \mathcal{B}_j = \delta_{ij}$ and $\mathcal{A}_i \cap \mathcal{A}_j = 0$, $\mathcal{B}_i \cap \mathcal{B}_j = 0$ forms a canonical homology basis of the surface. We can choose g linearly independent holomorphic forms du_i normalized on the $\mathcal{A}_i$ cycles, i.e., $\oint_{\mathcal{A}_i} du_j = \delta_{ij}$. Holomorphic forms are sometimes referred to as differentials of the 1st kind.

Differential forms of the 2nd kind are meromorphic differentials **with poles but no residues**, normalized over the $\mathcal{A}_i$ cycles. A basis of such differentials is given by

$$\begin{aligned} d\Omega_n(p) &\sim \Big(z(p)^{-n-1} + O(1)\Big)dz(p), \quad n \geq 1, \\ \oint_{\mathcal{A}_j} d\Omega_n(p) &= 0. \end{aligned} \tag{2.55}$$

The fundamental bidifferential of the second kind, $B(p,q)$, is the unique meromorphic bidifferential with pole **only at** $p = q$ of order 2, zero residue and normalized over the $\mathcal{A}_i$ cycles:

$$\begin{aligned} B(p,q) &\sim_{p\to q} \frac{dz(p)dz(q)}{\big(z(p)-z(q)\big)^2} + \text{holomorphic}, \\ \oint_{\mathcal{A}_i} B(p,q) &= 0. \end{aligned} \tag{2.56}$$

If we expand $B(p,q)$ about q we get a generating functional for the differentials $d\Omega_n(p)$. The fundamental bidifferential is symmetric $B(p,q) = B(q,p)$, and on a hyperelliptic curve we have

$$B(p,q) + B(\bar{p},q) = \frac{dz(p)dz(q)}{\big(z(p)-z(q)\big)^2}. \tag{2.57}$$

Differential forms of the third kind are meromorphic differentials that have **poles only at 1st order**. We introduce a basis of such differentials $dS_{q,r}(p)$ with poles at $p = q, r$ with residues $+1$, -1 respectively and normalized over the $\mathcal{A}_i$ cycles:

$$\begin{aligned} dS_{q,r}(p) &\sim_{p\to q} \Big(\frac{1}{z(p)-z(q)} + O(1)\Big)dz(p), \\ dS_{q,r}(p) &\sim_{p\to r} \Big(\frac{-1}{z(p)-z(r)} + O(1)\Big)dz(p), \\ \oint_{\mathcal{A}_i} dS_{q,r}(p) &= 0. \end{aligned} \tag{2.58}$$

It follows from Stokes' theorem that $dS_{q,r}(p) = \int_{p'=r}^{q} B(p',p)$ and thus for meromophic f, $df(p) = \text{Res}_{q\to p} B(p,q) f(q)$.

2.6 $W_0(x, x_1)$ and the fundamental bidifferential

Let us consider $g = 0$ and $n = 1$ in (2.37). Using (2.40) we may write

$$2y(x)W_0(x,x_1) = \frac{\partial}{\partial x_1}\Big(\frac{W_0(x) - W_0(x_1)}{x - x_1}\Big) + P_0(x,x_1) \tag{2.59}$$

or

$$\begin{aligned}W_0(x,x_1) &= \frac{\frac{\partial}{\partial x_1}(\frac{W_0(x)-W_0(x_1)}{x-x_1}) + P_0(x,x_1)}{2y(x)} \\ &= \frac{\frac{1}{2}\frac{\partial}{\partial x_1}(\frac{V'(x)-V'(x_1)+2y(x_1)}{x-x_1}) + P_0(x,x_1)}{2y(x)} - \frac{1}{2(x-x_1)^2}\end{aligned} \tag{2.60}$$

Think of the points $x, y(x), x_1, y(x_1) \in \mathbb{C}$ as being paramaterized by our Riemann surface Σ, that is $\exists\ p, q \in \Sigma$ with $p = \big(x(p) = x, y(p) = y(x)\big)$ and $q = \big(x(q) = x_1, y(q) = y(x_1)\big)$ and write the correlator as a meromorphic bilinear form

$$\begin{aligned}W_0\big(x(p),x(q)\big)dx(p)dx(q) &= \left(\frac{\frac{\partial}{\partial x(q)}(\frac{W_0(x(p))-W_0(x(q))}{x(p)-x(q)}) + P_0(x(p),x(q))}{2y(p)}\right)dx(p)dx(q) \\ &= \left(\frac{\frac{1}{2}\frac{\partial}{\partial x(q)}(\frac{V'(x(p))-V'(x(q))+2y(q))}{x(p)-x(q)}) + P_0(x(p),x(q))}{2y(p)} - \frac{1}{2(x(p)-x(q))^2}\right)dx(p)dx(q).\end{aligned} \tag{2.61}$$

The first expression shows there is no pole as p approaches a ramification point because dx has a simple zero and y has a zero of order $1/2$ at such a point, hence $\lim_{p\to\alpha_i} dx(p)/y(p) = 0$. Similarly, there is no pole as p approaches q because

$$\begin{aligned}\frac{W(x(p)) - W(x(q))}{x(p)-x(q)} &= \frac{N^{-1}}{x(p)-x(q)}\Big\langle\sum_i \frac{1}{x(p)-\lambda_i} - \frac{1}{x(q)-\lambda_i}\Big\rangle \\ = \frac{1}{N}\sum_i\Big\langle\frac{x(q)-x(p)}{(x(p)-x(q))(x(p)-\lambda_i)(x(q)-\lambda_i)}\Big\rangle &= \frac{1}{N}\sum_i\Big\langle\frac{-1}{(x(p)-\lambda_i)(x(q)-\lambda_i)}\Big\rangle.\end{aligned} \tag{2.62}$$

However, the second line in (2.61) shows there is a double pole as $p \to \bar{q}$ with no residue; also, from (2.46) the form vanishes over the $\mathcal{A}_i$ cycles. The fundamental bidifferential of the 2nd kind is the unique meromorphic form with these properties. Recall

$$B(p,q) \sim_{p\to q} \frac{dx(p)dx(q)}{(x(p)-x(q))^2} + \text{holomorphic}, \tag{2.63}$$

and observe from the second expression in (2.61) that

$$W_0(x(p),x(q))dx(p)dx(q) + W_0(x(\bar{p}),x(q))dx(p)dx(q) = -\frac{dx(p)dx(q)}{(x(p)-x(q))^2},$$

and thus

$$W_0(x(p), x(q))dx(p)dx(q) = -B(p, \bar{q}) = B(p, q) - \frac{dx(p)dx(q)}{(x(p) - x(q))^2}. \tag{2.64}$$

2.7 Correlators to differential forms

Motivated by all we've seen so far, we turn the loop correlators into meromorphic forms on our surface Σ. Our interest is in working with objects that are single valued, and to do this we update the correlators to forms by defining

$$\begin{aligned} \omega_0(p) &\equiv -y(x(p))dx(p) \\ \omega_0(p_1, p_2) &\equiv B(p_1, p_2) \end{aligned} \tag{2.65}$$

and for $2g + n \geq 3$

$$\omega_g(p_1, \ldots, p_n) \equiv dx(p_1)) \cdots dx(p_n) \frac{\partial}{\partial V(p_1)} \cdots \frac{\partial}{\partial V(p_n)} F_g \tag{2.66}$$

Thus, for $(g, n) \neq (0, 2)$ we have $\omega_g(p_1, \ldots, p_i, \ldots, p_n) = -\omega_g(p_1, \ldots, \bar{p_i}, \ldots, p_n)$. Simply put, $\omega_g(p_1, \ldots, p_n) = \pm W_g(x(p_1), \ldots, x(p_n))dx(p_1) \cdots dx(p_n)$ with sign determined by the sheets to which $\{p_1, \ldots, p_n\}$ belong. It is straightforward how one may start with points on Σ and define forms.

Let $J = \{x_1, \ldots, x_n\}$ and let $2g + n + 1 \geq 3$. Recall (2.37) which using (2.40) says

$$\begin{aligned} 2W_g(x, J)y(x) = {} & W_{g-1}(x, x, J) + \sum_{h=1}^{g-1} \sum_{\emptyset \not\subseteq I \not\subseteq J} W_h(x, I) W_{g-h}(x, J \backslash I) \\ & + 2W_g(x) W_0(x, J) + \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \frac{W_g(x, J \backslash x_j) - W_g(J)}{x - x_j} + P_g(x, J). \end{aligned} \tag{2.67}$$

Now consider our form

$$\omega_g(p, p_1, \ldots, p_n) = \pm W_g(x(p), x(p_1), \ldots, x(p_n))dx(p)dx(p_1) \cdots dx(p_n)$$

which has two candidates for poles:

1. when $x(p) = x(p_j)$
2. when $y(x(p)) = 0$, i.e. iff $p = a_i$ where $dx(a_i) = 0$.

However we can disregard the first from (2.62). Thus, the only possible poles come from the ramification points. Indeed, below we show that $\omega_0(p, p_1, p_2)$ has poles only at ramification points.

2.7.1 $\omega_0(p, p_1, p_2)$

The recursion gives

$$\begin{aligned}2y(x)W_0(x, x_1, x_2) &= 2W_0(x, x_1)W_0(x, x_2) + P_0(x, x_1, x_2) \\ &+ \frac{\partial}{\partial x_1}\frac{W_0(x, x_2) - W_0(x_1, x_2)}{x - x_1} + \frac{\partial}{\partial x_2}\frac{W_0(x, x_1) - W_0(x_1, x_2)}{x - x_2}.\end{aligned} \tag{2.68}$$

Observe that

$$\frac{\partial}{\partial x_1}\frac{W_0(x, x_2) - W_0(x_1, x_2)}{x - x_1} = \frac{W_0(x, x_2)}{(x - x_1)^2} - \frac{W_0(x_1, x_2)}{(x - x_1)^2} - \frac{1}{x - x_1}\frac{\partial}{\partial x_1}W_0(x_1, x_2) \tag{2.69}$$

(and similarly for $x_1 \leftrightarrow x_2$). As $x \to x_1$ (or $x \to x_2$) the poles cancel. Therefore $W_0(x, x_1, x_2)$ – and consequently $\omega_0(p, p_1, p_2)$ – only has a pole when $y(p) = 0$, that is, when $p = a_i$ is a ramification point.

2.7.2 $\omega_1(p)$

The loop equation for $W_1(x)$ is

$$V'(x)W_1(x) - P_1(x) = W_0(x, x) + \sum_{h=0}^{1} W_h(x)W_{1-h}(x) = W_0(x, x) + 2W_1(x)W_0(x) \tag{2.70}$$

which can be written as

$$2W_1(x)y(x) = W_0(x, x) + P_1(x) \tag{2.71}$$

and therefore $\omega_1(p)$ only has a pole when $y(p) = 0$.

2.7.3 The pole structure

We have seen that $\omega_0(p)$ has no pole. $\omega_0(p, q)$ has a double pole as $p \to \bar{q}$. $\omega_0(p, p_1, p_2)$ and $\omega_1(p)$ have poles at the ramification points. In general, when $2g + n \geq 3$, $\omega_g(x_1, \ldots, x_n)$ has poles only at the ramification points a_i. This follows by looking at (2.67) and observing that the RHS has poles only at the ramification points and therefore the only poles of ω_g come from the ramification points.

We briefly address the polynomials $P(x_1, \ldots, x_n)$. On Σ we have $P(x(p)) = P(x(\bar{p}))$ since P is independent of y. Furthermore the differential, $\frac{P(x)dx}{y}$ has no pole as $y \to 0$, which follows immediately from the work done in 2.6 so it vanishes when we take residues.

2.8 The topological recursion

The majority of the tools are in place for us to derive the residue formulation of the topological recursion. The hyperelliptic structure allows for some very simple relations. While some of the steps may appear to be unnecessary, the end result is applicable to a broad range of algebraic curves, and the method of deriving the recursion in general is provided below. The recursion was first derived in [11].

Let $2g + n \geq 3$. We will use the Riemann bilinear identity as follows. Let ζ, η be two meromorphic forms on a Riemann surface of genus g, let o be a base point in the fundamental domain, and let $\Phi(p) = \int_o^p \zeta$. The Riemann bilinear identity gives

$$\mathrm{Res}_{p\to \text{all poles}} \Phi(p)\eta(p) = \frac{1}{2\pi i} \sum_{j=1}^{g} \oint_{\mathcal{A}_j} \zeta \oint_{\mathcal{B}_j} \eta - \oint_{\mathcal{B}_j} \zeta \oint_{\mathcal{A}_j} \eta. \tag{2.72}$$

For instance, let $\zeta(p) = B(p, q)$, then

$$\mathrm{Res}_{p\to \text{all poles}}\, dS_{p,o}(q)\omega(p) = -\sum_{j=1}^{g} du_j(q) \oint_{\mathcal{A}_j} \omega. \tag{2.73}$$

Now the only pole of $dS_{p,o}(q)$ (as a function of p) occurs as $p \to q$ with a residue of -1. It follows that

$$\omega(q) = \mathrm{Res}_{p\to \text{poles of } \omega}\, dS_{p,o}\omega(p) + \sum_{j=1}^{g} du_j(q) \oint_{\mathcal{A}_j} \omega. \tag{2.74}$$

Or, let $\eta(\hat{p}) = \omega_g(\hat{p}, p_1, \ldots, p_n)$ and let $\zeta(\hat{p}) = B(\hat{p}, q)$. Then $dS_{\hat{p},o}(p)\omega_g(\hat{p}, p_1, \ldots, p_n)$ has poles when $\hat{p}$ approaches ramification points (from ω) and when $\hat{p}$ approaches p (from $dS = \int B$). See section 2.5.1.

$$\begin{aligned}
&\mathrm{Res}_{\hat{p}\to p} dS_{\hat{p},o}(p)\omega_g(\hat{p}, p_1, \ldots, p_n) + \sum_i \mathrm{Res}_{\hat{p}\to a_i} dS_{\hat{p},o}(p)\omega_g(\hat{p}, p_1, \ldots, p_n) \\
&= \frac{1}{2\pi i} \sum_{j=1}^{g} \oint_{\mathcal{A}_j} B(p, \hat{p}) \oint_{\mathcal{B}_j} \omega_g(\hat{p}, p_1, \ldots, p_n) - \frac{1}{2\pi i} \sum_{j=1}^{g} \oint_{\mathcal{B}_j} B(p, \hat{p}) \oint_{\mathcal{A}_j} \omega_g(\hat{p}, p_1, \ldots, p_n) = 0
\end{aligned} \tag{2.75}$$

Thus

$$\omega_g(p,p_1,\ldots,p_n) = -\mathrm{Res}_{\hat{p}\to p} dS_{\hat{p},o}(p)\omega_g(\hat{p},p_1,\ldots,p_n) = \sum_i \mathrm{Res}_{\hat{p}\to a_i} dS_{\hat{p},o}(p)\omega_g(\hat{p},p_1,\ldots,p_n) \tag{2.76}$$

Now the first loop equation over Σ gives

$$\begin{aligned}2y(q)dx(q)\omega_g(q) + 2y(\bar{q})dx(q)\omega_g(\bar{q}) &= \omega_{g-1}(q,q) + \sum_{h=1}^{g-1}\omega_h(q)\omega_{g-h}(q) \\ &+ P_g(q)dx(q)dx(q) - \omega_{g-1}(\bar{q},q) - \sum_{h=1}^{g-1}\omega_h(\bar{q})\omega_{g-h}(q) + P_g(\bar{q})dx(q)dx(q).\end{aligned}$$

Of course,

$$2y(q)dx(q)\omega_g(q) + 2y(\bar{q})dx(q)\omega_g(\bar{q}) = 2\Big(y(q) - y(\bar{q})\Big)dx(q)\omega_g(q) \tag{2.77}$$

and

$$\begin{aligned}&\omega_{g-1}(q,q) + \sum_{h=1}^{g-1}\omega_h(q)\omega_{g-h}(q) + P_g(q)dx(q)dx(q) - \omega_{g-1}(\bar{q},q) - \sum_{h=1}^{g-1}\omega_h(\bar{q})\omega_{g-h}(q) + P_g(\bar{q})dx(q)dx(q) \\ &= -2\omega_{g-1}(q,\bar{q}) - 2\sum_{h=1}^{g-1}\omega_h(q)\omega_{g-h}(\bar{q}) + 2P_g(q)dx(q)dx(q).\end{aligned} \tag{2.78}$$

If we operate $\sum_i \mathrm{Res}_{q\to a_i} \frac{1}{2}\frac{\int_{\bar{q}}^{q} B(k,p)}{(y(q)-y(\bar{q}))dx(q)}$ on (2.77) we get $2\omega_g(p)$:

$$\begin{aligned}\sum_i \mathrm{Res}_{q\to a_i}\int_{k=\bar{q}}^{q} B(k,p)\omega_g(q) &= \sum_i \mathrm{Res}_{q\to a_i}\left(\int_{k=o}^{q} B(k,p)\omega_g(q) - \int_{k=o}^{\bar{q}} B(k,p)\omega_g(q)\right) \\ = \sum_i \mathrm{Res}_{q\to a_i}\left(\int_{k=o}^{q} B(k,p)\omega_g(q) - \int_{k=o}^{q} B(k,p)\omega_g(\bar{q})\right) &= 2\sum_i \mathrm{Res}_{q\to a_i}\int_{k=o}^{q} B(k,p)\omega_g(q) \\ &= 2\sum_i \mathrm{Res}_{q\to a_i} dS_{q,o}(p)\omega_g(q) = 2\omega_g(p).\end{aligned}$$

It follows that $\omega_g(p) =$

$$= -\sum_i \mathrm{Res}_{q\to a_i} \frac{1}{2} \frac{\int_{k=\bar{q}}^{q} B(k,p)}{\big(y(q)-y(\bar{q})\big)dx(q)} \Bigg(\omega_{g-1}(q,\bar{q}) + \sum_{h=1}^{g-1} \omega_h(q)\omega_{g-h}(\bar{q}) - P_g(q)dx(q)dx(q)\Bigg)$$
$$= -\sum_i \mathrm{Res}_{q\to a_i} \frac{1}{2} \frac{\int_{k=\bar{q}}^{q} B(k,p)}{\big(y(q)-y(\bar{q})\big)dx(q)} \Bigg(\omega_{g-1}(q,\bar{q}) + \sum_{h=1}^{g-1} \omega_h(q)\omega_{g-h}(\bar{q})\Bigg) \tag{2.79}$$

where P_g vanishes when we take the residue.

Definition 2.8.1 *The recursion kernel $K(p,q)$ is*

$$K(p,q) = -\frac{1}{2} \frac{\int_{k=\bar{q}}^{q} B(k,p)}{\big(y(q)-y(\bar{q})\big)dx(q)}. \tag{2.80}$$

The kernel allows (2.79) to be written concisely as

$$\boxed{\omega_g(p) = \sum_{a_i} \mathrm{Res}_{q\to a_i} K(p,q)\Bigg(\omega_{g-1}(q,\bar{q}) + \sum_{h=1}^{g-1} \omega_h(q)\omega_{g-h}(\bar{q})\Bigg)} \tag{2.81}$$

which solves $\omega_g(p)$ for all $g > 0$ starting from $\omega_0(p)$ and $\omega_0(p,q)$. Now to formulate an expression for higher correlators, recall (2.67) which can be expressed over Σ as

$$\begin{aligned}
2y(q)dx(q)\omega_g(q,J) - 2y(\bar{q})dx(q)\omega_g(q,J) &= 2\Big(y(q)-y(\bar{q})\Big)dx(q)\omega_g(q,J) \\
= \omega_{g-1}(q,q,J) + \sum_{h=1}^{g-1} \sum_{\emptyset \not\subseteq I \not\subseteq J} &\omega_h(q,I)\omega_{g-h}(q,J) + P_g(q,J)dx(q)dx(q) \\
-\omega_{g-1}(\bar{q},q,J) - \sum_{h=1}^{g-1} \sum_{\emptyset \not\subseteq I \not\subseteq J} &\omega_h(\bar{q},I)\omega_{g-h}(q,J) + P_g(\bar{q},J)dx(q)dx(q) \\
= -2\omega_{g-1}(q,\bar{q},J) - 2\sum_{h=1}^{g-1} \sum_{\emptyset \not\subseteq I \not\subseteq J} &\omega_h(q,I)\omega_{g-h}(\bar{q},J) + 2P_g(q,J)dx(q)dx(q)
\end{aligned} \tag{2.82}$$

where $J = (q_1, \ldots, q_n) \in \Sigma^n$. Again,

$$-\sum_i \mathrm{Res}_{q\to a_i} K(p,q) 2\Big(y(q)-y(\bar{q})\Big)dx(q)\omega_g(q,J) = 2\omega_g(p,J) \tag{2.83}$$

and therefore

$$\omega_g(p,J) = \sum_i \mathrm{Res}_{q\to a_i} K(p,q)\Big(\omega_{g-1}(q,\bar{q},J) + \sum_{h=1}^{g-1}\sum_{\emptyset\subsetneq I\subsetneq J}\omega_h(q,I)\omega_{g-h}(\bar{q},J\backslash I) + 2\omega_g(q)\omega_0(\bar{q},J)\Big)$$

$$\Rightarrow \boxed{\omega_g(p,J) = \sum_i \mathrm{Res}_{q\to a_i} K(p,q)\Big(\omega_{g-1}(q,\bar{q},J) + \sum_{h,I}^{*}\omega_h(q,I)\omega_{g-h}(\bar{q},J\backslash I)\Big)} \tag{2.84}$$

where the notation $\sum_{h,I}^{*}$ means we sum over all pairs (h,I) **excluding** $(0,\emptyset)$ and (g,J).

2.9 Inverting the loop insertion operator

Recall (2.50) which sends $\omega_g(p_1,\ldots,p_m)\to\omega_g(p_1,\ldots,p_{m-1})$:

$$\omega_g(p_1,\ldots,p_m) = \frac{\partial}{\partial V(p_m)}\omega_g(p_1,\ldots,p_{m-1}).$$

There is a manner in which we can go from m variables to $m-1$ variables.

Theorem 2.9.1 *Let $\Phi(q)$ satisfy $d\Phi(q) = y(q)dx(q)$. Then for $2g+n\geq 3$ we have*

$$\omega_g(p_1,\ldots,p_n) = \frac{1}{2-2g-n}\sum_i Res_{q\to a_i}\Phi(q)\omega_g(p_1,\ldots,p_n,q). \tag{2.85}$$

For a fixed g, this allows us to go backwards in the sequence of forms and turn the linear form in a complex number. For a proof of 2.9.1 see [20]. The RHS of 2.9.1 is the inverse of the insertion operator satisfying $\frac{\partial}{\partial V(p)}F_g = \omega_g(p)$. Thus

Corollary 2.9.2 *Let $g\geq 2$ and, for q near a branch-point, let $\Phi(q)$ satisfy $d\Phi(q) = y(q)dx(q)$. One may compute the free energy*

$$F_g = \frac{1}{2-2g}\sum_i Res_{q\to a_i}\Phi(q)\omega_g(q). \tag{2.86}$$

2.9.1 The topological recursion in general

Theorem 2.9.1 gives a way of computing the F_g from the correlators defined as forms over the surface. In [20, 21], 2.9.1 is used as a way of defining a class of invariants from a sequence of differential forms that satisfy the recursion shown in the Hermitian matrix model. The procedure goes as follows. The recursive expression in (2.84) can be used to define a doubly-infinite sequence of meromophic forms over a surface defined by an algebraic curve. From here, one is able to use 2.9.2 as a way to define the F_g of the curve, corresponding to the free energy of a (possibly theoretical) statistical model from which to derive the algebraic curve.

1. Begin with a algebraic curve $E(x, y)$, $x, y, \in \mathbb{C}$ and use this to define a (compact) Riemann surface Σ by
$$\Sigma = \{p : E\big(x(p), y(p)\big) = 0\}.$$
We assume x has simple branch points. If the curve is of genus g then we choose a canonical homology basis for Σ by picking $2g$ cycles $\mathcal{A}_i, \mathcal{B}_i$, $i = 1, \ldots, g$ such that $\mathcal{A}_i \cap \mathcal{A}_j = \mathcal{B}_i \cap \mathcal{B}_j = 0$ and $\mathcal{A}_i \cap \mathcal{B}_j = \delta_{ij}$.

2. Introduce the form $y(p)dx(p)$ and identify the fundamental bidifferential. Then we can define the recursion kernel
$$K(p, q) = -\frac{1}{2}\frac{\int_{k=\bar{q}}^{q} B(k, p)}{(y(q) - y(\bar{q}))dx(q)}. \tag{2.87}$$

3. Use the recursion in (2.84) as a way to define forms:
$$\begin{aligned} &\omega_0(p) \equiv -y(x(p))dx(p) \\ &\omega_0(p_1, p_2) \equiv B(p_1, p_2)\ , \quad \text{and for } 2g + n \geq 3 \\ &\omega_g(p, J) = \sum_i \mathrm{Res}_{q \to a_i} K(p, q)\Big(\omega_{g-1}(q, \bar{q}, J) + \sum_{h,I}^{*} \omega_h(q, I)\omega^{g-h}(\bar{q}, J \backslash I)\Big) \end{aligned} \tag{2.88}$$
where and we **exclude** $(0, \emptyset)$ and (g, J) from the sum over (h, I).

4. Use 2.9.1 to define complex numbers. Let $g \geq 2$, and, for q near a ramification point, let $\Phi(q)$ satisfy $d\Phi(q) = y(q)dx(q)$.
$$F_g = \frac{1}{2 - 2g}\sum_i \mathrm{Res}_{q \to a_i} \Phi(q) w^g(q) \tag{2.89}$$
For the definitions of F_0, F_1, see [20, 21].

The fascinating property of the F_g as defined by (2.89) via (2.88) is that they are invariant under the action of the group of symplectomorphisms on x, y in the algebraic curve $E(x, y)$ [20, 21, 19] and are thereby referred to as the symplectic invariants of the algebraic curve. Another fact worth mentioning is that in the case of matrix models, the F_g computed agree with the $1/N$ expansion of the free energy. However, the significance is that we can compute the invariants F_g for more general algebraic curves without need of any matrix models.

There are two applications to topological string theory which we briefly mention. The first is the Dijkgraaf-Vafa correspondence for type B topological strings. The second is the BKMP conjecture.

The Dijkgraaf-Vafa correspondence is for a particular class of non-compact Calabi-Yau manifolds described by $a^2+b^2+c^2=V'(x)^2-P(x)$ where both V and P are polynomials with $\deg(V)=\deg(P)+2$. This space has a holomorphic 3-form which we integrate over each of the $\deg(V)-1$ $\mathcal{A}_i$ cycles (around the cuts defined by V) which gives the so-called inhomogenous coordinates t_i^B related to deformations of the complex structure of the B model. The Dijkgraaf-Vafa correspondence states that $F_g(t_i^B)$ is given by the free energy from the matrix model $Z \propto \int dMe^{-\frac{1}{g_s}Tr(V(M))}$, which we know is equivalant to the free energy computed by the topological recursion applied to the spectral curve given by $Y(x)^2=V'(x)^2-P(x)$. This correspondence has been explicitly proven for $g=0,1$ and there exist heuristic arguments for why it ought to work for $g \geq 2$ [16, 17, 1].

A similar correspondence occurs with toric Calabi-Yau manifolds $\tilde{X}$ on the A model side. These occur in mirror pairs to manifolds X on the B side whose topological B theory is equivalent to the topological A theory on $\tilde{X}$. The geometry of X on the B side is of the form $a^2+b^2=P(x,y)$ where P is a polynomial in x and y. We can define an algebraic curve Σ from the variety $P=0$. The inhomogenous coordinates here are $t_i=\oint_{\mathcal{A}_i} \ln y \frac{dx}{x}$ where $\mathcal{A}_i$ surrounds the ith cut as defined by P. The BKMP conjecture says that the free energy of the topological B theory can be computed from the topological recursion on the spectral curve defined by P. Using mirror symmetry gives the free energy of the topological A theory which has an expansion where the coefficients are the Gromov-Witten invariants; hence, one can likely use the topological recursion to compute Gromov-Witten invariants [7, 8, 9].

In conclusion, the topological recursion is a immensely powerful computational tool that extends well beyond matrix models.

CHAPTER 3

Supereigenvalue model

The supereigenvalue model was constructed to describe two dimensional supersymmetric quantum gravity. The partition function was derived in [3] by imposing a set of constraints from the super-Virasoro algebra. Thus, the supereigenvalue model is effectively the non-trivial supersymmetric analog of the Hermitian matrix model. In addition to complex parameters, the model introduces a collection of Grassmann variables θ_i and parameters ξ_j. See Appendix B for details on Grassmann numbers.

Theorems 3.2.1 and 2.2 are of profound importance for studying the supereigenvalue model, and so we give special attention to providing proofs. The proof of 3.2.1 is motivated by the original work in [5] but is independent and complete. The proof of 2.2 closely follows and expands upon that found in [27].

Our motivation for the rigorous investigation of the supereigenaluve model is the question as to whether or not one can generalize the topological recursion to calculate the free energy of this model. This thesis investigates the viability of working with the meromorphic coefficients from the ξ expansion of the superloop equations. Unfortunately, it appears that this is an unreasonable approach, and future work will explore a method that involves the Grassmann parameters, perhaps on a supermanifold.

3.1 The partition function

Let $N \in \mathbb{N}$. Consider a supersymmetric field extension of $\mathbb{C}^{2N}$ by adding Grassmann variables $\theta_1, \ldots, \theta_{2N}$. We introduce an arbitrary polynomial potential with a *complex* (alt. *even*, *bosonic*) component given by $V(\lambda_i) = \sum_k g_k \lambda_i^k$ for $\lambda_i, g_k \in \mathbb{C}$ and a *Grassmann* (alt.

odd, *fermionic*) component given by $\Psi(\lambda_i) = \sum_{k\geq 1} \xi_k \lambda_i^k$ with Grassmann numbers ξ_k. We let deg (Ψ) = D+1. and deg(V)=d+1. For convenience we define

$$\Delta(\lambda, \theta) \equiv \prod_{1\leq i<j\leq 2N} (\lambda_i - \lambda_j - \theta_i\theta_j) \tag{3.1}$$

which will be refered to as the Vandermonde term by analogy with the Vandermonde determinant. The partition function and free energy are defined in complete analogy with matrix (and general statistical) models.

Definition 3.1.1 *The* ***supereigenvalue partition function*** Z_S *is given as*

$$Z_S \equiv \int (d\lambda_1 d\theta_1 \cdots d\lambda_{2N} d\theta_{2N}) \Delta(\lambda, \theta) e^{-2N\sum_i \left(V(\lambda_i) - \theta_i \Psi(\lambda_i)\right)}. \tag{3.2}$$

Definition 3.1.2 *The* ***free energy*** *of the supereigenvalue model is defined by*

$$Z_S = e^{(2N)^2 F_S}. \tag{3.3}$$

In fact, later we will see that superloop correlators are defined in terms of the free energy in a manner that is parallel to the 1HMM.

3.1.1 Super-Virasoro algebra

As noted, the supereigenvalue model is defined by the super-Virasoro algebra. The super algebra is generated by

$$\mathcal{G}_n = \sum_{k\geq 0} k g_k \partial_{\xi_{k+n}} + \sum_{k\geq 0} \xi_k \partial_{g_{k+n+1}} + \frac{1}{N^2}\sum_{k=0}^{n} \partial_{\xi_k}\partial_{g_{n-k}}, \quad \text{and}$$

$$\mathcal{L}_n = \sum_{k\geq 0} k g_k \partial_{g_{k+n}} + \frac{1}{2N^2}\sum_{k=0}^{n} \partial_{g_k}\partial_{g_{n-k}} + \sum_{k\geq 0}(k + \frac{n+1}{2})\xi_k \partial_{\xi_{k+n}} + \frac{1}{2N^2}\sum_{k=0}^{n-1} k\partial_{\xi_{n-k}}\partial_{\xi_k}. \tag{3.4}$$

It is straightforward (though tedious and thus omitted here) to show that the generators satisfy the following (anti-) commutation relations:

1. $[\mathcal{L}_a, \mathcal{L}_b] = (a-b)\mathcal{L}_{a+b}$
2. $[\mathcal{L}_a, \mathcal{G}_b] = (\frac{a}{2} - b)\mathcal{G}_{a+b}$
3. $\{\mathcal{G}_a, \mathcal{G}_b\} = 2\mathcal{L}_{a+b}$

1. follows directly from the commutation relation of the (non-super) Virasoro operators shown earlier. Most importantly, 3 implies that the super-Virasoro algebra can be generated by the $\mathcal{G}_n$ alone. Indeed, the supereigenvalue model partition function Z_S was originally constructed by requiring

$$\mathcal{G}_n Z_S = 0,\ n \geq -1. \tag{3.5}$$

For the derivation of the partition function we refer the reader to [3, 2, 30].

3.2 Expansion in Grassmann coupling constants

Our method of analyzing the supereigenvalue model will be to work in terms of the ξ-expansion. It was observed in [5] that there is a direct relation between the free energy of the 1HMM and the non-Grassmann component of the free energy of the supereigenvalue model. Further analysis of genus 0 behavior motivated a conjecture regarding the ξ expansion of F_S – namely F_S is at most quadratic in ξ [5]. This was verified in [27]. In this section we show explicitly the relation between F_S and F_H and compute the ξ expansion. We follow and expand upon the path presented in [27].

Note the Vandermonde term in (3.2) is

$$\Delta(\lambda,\theta) = \prod_{i<j}(\lambda_i - \lambda_j)(1 + \frac{\theta_i\theta_j}{\lambda_j - \lambda_i}) = \bar{\Delta}(\lambda)\prod_{i<j}(1 + \frac{\theta_i\theta_j}{\lambda_j - \lambda_i}), \tag{3.6}$$

where we introduced the notation $\bar{\Delta}(\lambda) = (-1)^{2N(2N-1)}\Delta(\lambda) = (-1)^N\Delta(\lambda)$ (see Appendix A.1). Furthermore, because of the anti-commutation relation $\{\theta_i, \theta_j\} = 0$ we can always expand Grassmann exponentials as $e^{\theta_i} = 1 + \theta_i$. Thus we may write (3.2) as

$$Z_S = \int \left(\prod_{i=1}^{2N} d\lambda_i d\theta_i e^{-2NV(\lambda_i)}\right)\bar{\Delta}(\lambda)\left(\prod_{1\leq i<j\leq 2N} 1 + \frac{\theta_i\theta_j}{\lambda_j - \lambda_i}\right)\left(\prod_{i=1}^{2N} 1 + 2N\theta_i\Psi(\lambda_i)\right). \tag{3.7}$$

The product $\prod 1+2N\theta_i\Psi(\lambda_i)$ allows Z_S to be expanded in terms of the Grassmann coupling constants

$$Z_S(g,\xi) = \sum_{k=0}^{N} Z_S^{(2k)} = \sum_{k=0}^{N}\sum_{i_1,\ldots i_{2k}=1}^{D+1} Z_S^{i_1,\ldots,i_{2k}}\xi_{i_1}\cdots\xi_{i_{2k}} \tag{3.8}$$

where the Vandermonde term forces all terms with a product of an odd number of Grassmann coupling constants to vanish. Please note that if $D+1 \geq 2N$ then the expansion terms $Z_S^{(2k)}$ are not trivially zero for $k \leq N$; however, if $2N > D+1$ then $Z_S^{(2k)} = 0$ for $2k > D+1$. Of course, it should be clear that the largest possible order of Z_S in terms of ξ is the degree of Ψ $(= D+1)$. In the following proofs we always assume $D+1 \geq 2N$ (if $D+1 < 2N$ we simply set $\xi_i = 0$ for $i = D+2, \ldots 2N$).

3.2.1 0th order in ξ

We now evaluate $Z_S^{(0)}$ and relate it to Z_H. To compute $Z_S^{(0)}$ we look at the the term with no Ψ, namely

$$Z_S^{(0)} = \int (\prod_{i=1}^{2N} d\lambda_i d\theta_i e^{-2NV(\lambda_i)})(\prod_{i<j} \lambda_i - \lambda_j)(\prod_{i<j} 1 + \frac{\theta_i\theta_j}{\lambda_j - \lambda_i}). \tag{3.9}$$

The only terms that contribute to the integral over the Grassmann numbers come from the rearrangements of $\frac{\theta_1\theta_2\cdots\theta_{2n-1}\theta_{2N}}{(\lambda_2-\lambda_1)\cdots(\lambda_{2N}-\lambda_{2N-1})}$ in multiplying out $(\prod_{i<j} 1 + \frac{\theta_i\theta_j}{\lambda_j-\lambda_i})$. See Appendix B for our convention of Grassmann integration. Thus

$$\int (\prod_{k=1}^{2N} d\theta_k)(\prod_{i<j} 1 + \frac{\theta_i\theta_j}{\lambda_j - \lambda_i}) = \frac{1}{2^N N!} \sum_{\sigma \in S_{2N}} \prod_{k=1}^{N} \frac{(-1)^\sigma}{\lambda_{\sigma(2k)} - \lambda_{\sigma(2k-1)}} \tag{3.10}$$

where the coefficient of $N!$ is present because the product over $i < j$ fixes the order of the N collection of pairs (i, j) and the 2^N is because the order of each pair (of which there are N) is determined by $i < j$. Now we can write

$$\begin{aligned} Z_S^{(0)} &= \frac{1}{2^N N!} \int (\prod_{i=1}^{2N} d\lambda_i e^{-2NV(\lambda_i)}) \bar{\Delta}(\lambda) \sum_{\sigma \in S_{2N}} \prod_{k=1}^{N} \frac{(-1)^\sigma}{\lambda_{\sigma(2k)} - \lambda_{\sigma(2k-1)}} \\ &= \frac{(-1)^N}{2^N N!} \int (\prod_{i=1}^{2N} d\lambda_i e^{-2NV(\lambda_i)}) \sum_{\tau,\sigma \in S_{2N}} (-1)^\tau (-1)^\sigma \prod_{i=1}^{2N} \prod_{k=1}^{N} \frac{\lambda_i^{\tau(i)-1}}{\lambda_{\sigma(2k)} - \lambda_{\sigma(2k-1)}}, \end{aligned} \tag{3.11}$$

where the second line follows by expanding Vandermonde determinant (see Appendix A.1). Since the λs are integrated, it matters not how we name them so we rewrite $\lambda_i \to \lambda_{\sigma(i)}$:

$$\begin{aligned} Z_S^{(0)} &= \frac{(-1)^N}{2^N N!} \int (\prod_{i=1}^{2N} d\lambda_i e^{-2NV(\lambda_i)}) \sum_{\tau,\sigma \in S_{2N}} (-1)^\tau (-1)^\sigma \prod_{k=1}^{N} \frac{\lambda_{\sigma(2k-1)}^{\tau(\sigma(2k-1))-1} \lambda_{\sigma(2k)}^{\tau(\sigma(2k))-1}}{\lambda_{\sigma(2k)} - \lambda_{\sigma(2k-1)}} \\ &= \frac{(-1)^N}{2^N N!} \int (\prod_{i=1}^{2N} d\lambda_i e^{-2NV(\lambda_i)}) \sum_{\tau \in S_{2N}} (-1)^\tau (2N)! \prod_{k=1}^{N} \frac{\lambda_{2k-1}^{\tau(2k-1)-1} \lambda_{2k}^{\tau(2k)-1}}{\lambda_{2k} - \lambda_{2k-1}} \\ &= \frac{(-1)^N (2N)!}{2^N N!} \sum_{\sigma \in S_{2N}} (-1)^\sigma \prod_{i=1}^{N} \int d\lambda_{2i-1} d\lambda_{2i} e^{-2NV(\lambda_{2i-1})} e^{-2NV(\lambda_{2i})} \frac{\lambda_{2i-1}^{\sigma(2i-1)-1} \lambda_{2i}^{\sigma(2i)-1}}{\lambda_{2i} - \lambda_{2i-1}}. \end{aligned}$$

If we introduce the skew-symmetric matrix

$$A \equiv A_{ij} \equiv \int d\lambda_1 d\lambda_2 e^{-2NV(\lambda_1)} e^{-2NV(\lambda_2)} \frac{\lambda_1^{i-1}\lambda_2^{j-1}}{\lambda_1 - \lambda_2}, \tag{3.12}$$

then we can express $Z_S^{(0)}$ in terms of a Pfaffian (see Appendix A.3):

$$\boxed{\begin{aligned} Z_S^{(0)} &= \frac{(-1)^N (2N)!}{2^N N!} \sum_{\sigma \in S_{2N}} (-1)^\sigma \prod_{i=1}^{N} (-1) A_{\sigma(2i-1)\sigma(2i)} \\ &= (2N)!\text{Pfaffian}(A). \end{aligned}} \tag{3.13}$$

From this result, it was noticed in [5] how to relate $Z_S^{(0)}$ to Z_H. We provide a complete proof of the following theorem.

Theorem 3.2.1 *The supersymmetric partition function with vanishing fermionic coupling constants can be expressed in terms of the Hermitian matrix model partition function as*

$$Z_S^{(0)}(2N, g_k) = \frac{1}{2^N} \binom{2N}{N} \big(Z_H(N, 2g_k)\big)^2. \tag{3.14}$$

Proof: First note

$$\begin{aligned} Z_H(n, 2g_k) &= \int (\prod_{i=1}^{n} d\lambda_i e^{-2nV(\lambda_i)}) (\prod_{i<j} \lambda_i - \lambda_j)^2 \\ &= (-1)^{n(n-1)/2} \sum_{\sigma^{-1} \in S_n} (-1)^{\sigma^{-1}} \int (\prod_{i=1}^{n} d\lambda_i e^{-2nV(\lambda_i)}) (\prod_{i=1}^{n} \lambda_i^{\sigma^{-1}(i)-1}) (\prod_{i<j} \lambda_i - \lambda_j) \\ &= (-1)^{n(n-1)/2} \sum_{\sigma^{-1} \in S_n} (-1)^{\sigma^{-1}} \int (\prod_{i=1}^{n} d\lambda_{\sigma(i)} e^{-2nV(\lambda_{\sigma(i)})}) (\prod_{i=1}^{n} \lambda_{\sigma(i)}^{i-1}) (\prod_{\sigma(i)<\sigma(j)} \lambda_{\sigma(i)} - \lambda_{\sigma(j)}) \\ &= (-1)^{n(n-1)/2} n! \int (\prod_{i=1}^{n} d\lambda_i e^{-2nV(\lambda_i)}) (\prod_{i<j} \lambda_i - \lambda_j) (\prod_{i=1}^{n} \lambda_i^{i-1}). \end{aligned} \tag{3.15}$$

Thus we have

$$\begin{aligned} \frac{1}{2^n} \binom{2n}{n} Z_H(n, 2g_k)^2 = \frac{(2n)!}{2^n} &\int (\prod_{i=1}^{n} d\lambda_i e^{-2nV(\lambda_i)} \lambda_i^{i-1}) (\prod_{1 \le i<j \le n} \lambda_i - \lambda_j) \\ \times &\int (\prod_{i=n+1}^{2n} d\lambda_i e^{-2nV(\lambda_i)} \lambda_i^{i-n-1}) (\prod_{n+1 \le i<j \le 2n} \lambda_i - \lambda_j). \end{aligned} \tag{3.16}$$

Recall we may write

$$\begin{aligned} Z_S^{(0)}(2n) &= \frac{(2n)!}{2^n n!}\int\Big(\prod_{i=1}^n d\lambda_{2i-1}d\lambda_{2i}\frac{e^{-2nV(\lambda_{2i-1})}e^{-2nV(\lambda_{2i})}}{\lambda_{2i-1}-\lambda_{2i}}\Big)\bar{\Delta}_{(1\le\lambda\le 2n)} \\ &= \frac{(2n)!}{2^n n!}\int(\prod_{i=1}^n d\mu_{2i-1}d\mu_{2i}\frac{1}{\lambda_{2i-1}-\lambda_{2i}})\bar{\Delta}_{(1\le\lambda\le 2n)}, \end{aligned} \tag{3.17}$$

where we introduced $d\mu_i = d\lambda_i e^{-2nV(\lambda_i)}$ to simplify notation. To prove theorem 3.2.1 we proceed by induction. For $n = 1$ we have

$$\begin{aligned} Z_S^{(0)}(2n=2,g_k) &= \int d\lambda_1 d\lambda_2 e^{-2V(\lambda_1)}e^{-2V(\lambda_2)}\frac{\lambda_1-\lambda_2}{\lambda_1-\lambda_2} \\ &= \int d\lambda_1 d\lambda_2 e^{-2V(\lambda_1)}e^{-2V(\lambda_2)} = \Big(\int d\lambda_1 e^{-2V(\lambda_1)}\Big)^2 \\ &= \Big(Z_H(n=1,2g_k)\Big)^2 = \frac{1}{2}\binom{2}{1}\Big(Z_H(1,2g_k)\Big)^2. \end{aligned} \tag{3.18}$$

We assume that for $k \le n$, $Z_S^{(0)}(2k) = \frac{1}{2^k}\binom{2k}{k}(Z_H(k))^2$. We start with the equality between

$$\int \frac{(2n)!}{2^n n!}(\prod_{i=1}^n d\mu_{2i-1}d\mu_{2i}\frac{1}{\lambda_{2i-1}-\lambda_{2i}})\bar{\Delta}_{(1\le\lambda\le 2n)} \quad \text{and} \tag{3.19}$$

$$\int \frac{(2n)!}{2^n}(\prod_{i=1}^n d\mu_i\lambda_i^{i-1})(\prod_{1\le i<j\le n}\lambda_i-\lambda_j)(\prod_{i=n+1}^{2n}d\mu_i\lambda_i^{i-n-1})(\prod_{n+1\le i<j\le 2n}\lambda_i-\lambda_j), \tag{3.20}$$

multiply the integrands of both (3.19) and (3.20) by

$$\frac{(2n+2)(2n+1)}{2(n+1)}dAdBe^{(-2n-2)V(A)}e^{(-2n-2)V(B)}\prod_{i=1}^{2n}(\lambda_i-A)(\lambda_i-B)e^{-2V(\lambda_i)} \tag{3.21}$$

and integrate over the λs and A, B. Letting $d\bar{\mu}_i = d\lambda_i e^{-2(n+1)V(\lambda_i)}$, equation (3.19) becomes

$$\int \frac{(2n+2)!}{2^{n+1}(n+1)!}d\bar{\mu}_A d\bar{\mu}_B(\prod_{i=1}^n d\bar{\mu}_{2i-1}d\bar{\mu}_{2i}\frac{1}{\lambda_{2i-1}-\lambda_{2i}})\bar{\Delta}_{(1\le\lambda\le 2n)}\prod_{i=1}^{2n}(\lambda_i-A)(\lambda_i-B) = Z_S^{(0)}(2n+2),$$

and equation (3.20) becomes

$$\begin{aligned}
&\int \frac{(2n+2)!}{2^{n+1}(n+1)} d\bar{\mu}_A d\bar{\mu}_B (\prod_{i=1}^{n} d\bar{\mu}_i \lambda_i^{i-1})(\prod_{1\le i<j\le n} \lambda_i - \lambda_j) \\
&\times (\prod_{i=n+1}^{2n} d\bar{\mu}_i \lambda_i^{i-n-1})(\prod_{n+1\le i<j\le 2n} \lambda_i - \lambda_j)\prod_{i=1}^{2n}(\lambda_i - A)(\lambda_i - B) \\
&= \int \frac{(2n+2)!}{2^{n+1}(n+1)} d\bar{\mu}_A d\bar{\mu}_B (\prod_{i=1}^{n} d\bar{\mu}_i \lambda_i^{i-1}) \left[(\prod_{1\le i<j\le n} \lambda_i - \lambda_j)(\prod_{j=1}^{n} \lambda_j - A) \right] \\
&\times (\prod_{i=n+1}^{2n} d\bar{\mu}_i \lambda_i^{i-n-1}) \left[(\prod_{n+1\le i<j\le 2n} \lambda_i - \lambda_j)(\prod_{j=n+1}^{2n} \lambda_j - B) \right] \prod_{i=1}^{n}(\lambda_{i+n} - A)(\lambda_i - B).
\end{aligned} \tag{3.22}$$

The terms in [brackets] in (3.22) are $\bar{\Delta}$ in our notation:

$$\begin{aligned}
\text{equation (3.22)} = \frac{(2n+2)!}{2^{n+1}(n+1)} &\int d\bar{\mu}_A (\prod_{i=1}^{n} d\bar{\mu}_i \lambda_i^{i-1}) \bar{\Delta}(\lambda_1, \ldots, \lambda_n, A) \prod_{i=1}^{n}(\lambda_{i+n} - A) \\
&\times d\bar{\mu}_B (\prod_{i=n+1}^{2n} d\bar{\mu}_i \lambda_i^{i-n-1}) \bar{\Delta}(\lambda_{n+1}, \ldots, \lambda_{2n}, B) \prod_{i=1}^{n}(\lambda_i - B).
\end{aligned} \tag{3.23}$$

Now it will make things nicer if we re-label

1. $A \to \lambda_{n+1}$,
2. $\lambda_i \to \lambda_{i+1}$ when $n+1 \le i \le 2n$,
3. $B \to \lambda_{2n+2}$

so that (3.22) is

$$\begin{aligned}
\frac{(2n+2)!}{2^{n+1}(n+1)} \int &(\prod_{i=1}^{n+1} d\bar{\mu}_i)(\prod_{i=1}^{n} \lambda_i^{i-1}) \bar{\Delta}(\lambda_1, \ldots, \lambda_{n+1}) \prod_{i=n+2}^{2n+1} (\lambda_i - \lambda_{n+1}) \\
&\times (\prod_{i=n+2}^{2n+2} d\bar{\mu}_i)(\prod_{i=n+2}^{2n+1} \lambda_i^{i-n-2}) \bar{\Delta}(\lambda_{n+2}, \ldots, \lambda_{2n+2}) \prod_{i=1}^{n}(\lambda_i - \lambda_{2n+2}).
\end{aligned} \tag{3.24}$$

We'll proceed by expressing our objects in terms of determinants (see Appendix A.1):

$$\begin{aligned}
\bar{\Delta}(\lambda_1, \ldots, \lambda_{n+1}) &= (-1)^{n+1} \det \left(\lambda_i^{j-1} \right)_{i,j=1}^{n+1} \\
\bar{\Delta}(\lambda_{n+2} \ldots, \lambda_{2n+2}) &= (-1)^{n+1} \det \left(\lambda_{i+n+1}^{j-1} \right)_{i,j=1}^{n+1}
\end{aligned} \tag{3.25}$$

hence

$$(\prod_{i=1}^{n}\lambda_i^{i-1})\bar{\Delta}(\lambda_1,\ldots,\lambda_{n+1}) = (\prod_{i=1}^{n}\lambda_i^{i-1})(-1)^{n+1}\det\left(\lambda_i^{j-1}\right)_{i,j=1}^{n+1} = (-1)^{n+1}\det(M)_{i,j=1}^{n+1} \tag{3.26}$$

where

$$\begin{aligned} &M_{i,j} = \lambda_i^{j+i-2} \text{ when } 1 \le i \le n \text{ and} \\ &M_{i=n+1,j} = \lambda_{n+1}^{j-1}. \end{aligned} \tag{3.27}$$

Similarly we have

$$(\prod_{i=1}^{n}\lambda_{n+i+1}^{i-1})\bar{\Delta}(\lambda_{n+2},\ldots,\lambda_{2n+2}) = (-1)^{n+1}\det(N)_{i,j=1}^{n+1} \tag{3.28}$$

where

$$\begin{aligned} &N_{i,j} = \lambda_{i+n+1}^{i+j-2} \text{ when } 1 \le i \le n \text{ and} \\ &N_{i=n+1,j} = \lambda_{2n+2}^{j-1}. \end{aligned} \tag{3.29}$$

Thus (3.22) becomes

$$\begin{aligned} &\frac{(2n+2)!}{2^{n+1}(n+1)}\int(\prod_{i=1}^{n+1}d\bar{\mu}_i)\det(M)\times(\prod_{i=n+2}^{2n+2}d\bar{\mu}_i)\det(N)\prod_{i=n+2}^{2n+1}(\lambda_i-\lambda_{n+1})\prod_{i=1}^{n}(\lambda_i-\lambda_{2n+2}) \\ &\quad= \frac{(2n+2)!}{2^{n+1}(n+1)}\sum_{J\subseteq\{1,\ldots n\}}\sum_{K\subseteq\{n+2,\ldots,2n+1\}}(-1)^{|J|+|K|} \\ &\quad\times\int(\prod_{i=1}^{n+1}d\bar{\mu}_i)\det(M)\lambda_{n+1}^{n-|K|}\lambda_J\times(\prod_{i=n+2}^{2n+2}d\bar{\mu}_i)\det(N)\lambda_{2n+2}^{n-|J|}\lambda_K, \end{aligned} \tag{3.30}$$

which allows us to observe how the integral splits into two integrals over $n+1$ variables. Consider

$$\int(\prod_{i=1}^{n+1}d\bar{\mu}_i)\det(M)\lambda_{n+1}^{n-|K|}\lambda_J. \tag{3.31}$$

The only non-vanishing terms come from $\lambda_{n+1}^{|J|}\lambda_{n-|J|}\ldots\lambda_n$. For all other terms, the matrix $M\lambda_J$ will have two rows of identical **form** and after integrating will vanish (see Appendix A.2). For instance, the term $\lambda_{n+1}^{n-1}\lambda_i$ where $i \neq n$ will not contribute because then we could multiply the ith row by λ_i so that the ith and $(i+1)$st row are of the same form (in λ_i and

λ_{i+1} respectively):

$$\begin{aligned} \text{row } i &\Rightarrow \lambda_i^{j+i-1} \\ \text{row } i+1 &\Rightarrow \lambda_{i+1}^{j+i-1} \end{aligned} \tag{3.32}$$

and the integral will vanish. But if we multiply by $\lambda_{n+1}^{n-1}\lambda_n$ then each row is of a different form and the integral does not vanish. This continues to apply when $|J| > 1$. Hence $|J| = |K|$, and for the same reason we only have contributions from $(\lambda_{n+1}\lambda_{2n+2})^k\lambda_{n-k}\dots\lambda_n\lambda_{2n+1-k}\dots\lambda_{2n+1}$. Consequently we have

$$\begin{aligned} &\sum_{k=0}^{n} \frac{(2n+2)!}{2^{n+1}(n+1)} \times \int (\prod_{i=1}^{n+1} d\bar{\mu}_i)\det(M)\lambda_{n+1}^{n-k}\lambda_{n-k}\dots\lambda_n \\ &\qquad \times (\prod_{i=n+2}^{2n+2} d\bar{\mu}_i)\det(N)\lambda_{2n+2}^{n-k}\lambda_{2n+1-k}\dots\lambda_{2n+1} \\ = &\sum_{k=0}^{n} \frac{(2n+2)!}{2^{n+1}(n+1)} \int (\prod_{i=1}^{n+1} d\bar{\mu}_i)(\prod_{i=1}^{n}\lambda_i^{i-1})\Delta(\lambda_1,\dots,\lambda_{n+1})\lambda_{n+1}^{n-k}\lambda_{n-k}\dots\lambda_n \\ &\qquad \times (\prod_{i=n+2}^{2n+2} d\bar{\mu}_i)(\prod_{i=n+2}^{2n+1}\lambda_i^{i-n-2})\Delta(\lambda_{n+2},\dots,\lambda_{2n+2})\lambda_{2n+2}^{n-k}\lambda_{2n+1-k}\dots\lambda_{2n+1} \\ = &\frac{(2n+2)!}{2^{n+1}} \int (\prod_{i=1}^{n+1} d\bar{\mu}_i)(\prod_{i=1}^{n}\lambda_i^{i-1})\Delta(\lambda_1,\dots,\lambda_{n+1})\lambda_{n+1}^{n-k} \\ &\qquad \times (\prod_{i=n+2}^{2n+2} d\bar{\mu}_i)(\prod_{i=n+2}^{2n+1}\lambda_i^{i-n-2})\Delta(\lambda_{n+2},\dots,\lambda_{2n+2})\lambda_{2n+2}^{n-k} \end{aligned} \tag{3.33}$$

where the last equality comes from re-labeling the variables in each of the $n+1$ terms in the summation and re-arranging the Vandermonde determinantes (in pairs the signs cancel). This final expression is exactly

$$\frac{1}{2^{n+1}}\binom{2(n+1)}{n+1} Z_H(n+1, 2g_k)^2. \tag{3.34}$$

□

We have just proved the precise relation between $Z_S^{(0)}$ and $(Z_H)^2$. This is a very important result for the supereigenvalue model that returns us in the familiar 1HMM.

Corollary 3.2.2 *Ignoring an additive constant, we have*

$$F_S^{(0)}(2N, g_k) = 2F_H(N, 2g_k). \tag{3.35}$$

Now we have $F_S^{(0)}$ within our grasp. In the following section we will compute higher orders of the ξ expansion of Z_S and then we will discover that the free energy has two components, $F_S = F_S^{(0)} + F_S^{(2)}$, the former of which is now familiar.

3.2.2 Higher orders in ξ

Recall that the Vandermonde term in the partition function forces all terms with an odd number of Grassmann coupling constants to vanish. Second order terms (in ξ) come from $\sum_{i<j}(2N)^2\theta_i\Psi(\lambda_i)\theta_j\Psi(\lambda_j)$ and the integral over the Grassmann variables gives

$$\begin{aligned}&\int(\prod_{k=1}^{2N} d\theta_k)(\prod_{i<j} 1+\frac{\theta_i\theta_j}{\lambda_j-\lambda_i})(\sum_{i<j}(2N)^2\theta_i\Psi(\lambda_i)\theta_j\Psi(\lambda_j))\\ &=\frac{1}{2!(N-1)!2^{N-1}}\sum_{\sigma\in S_{2N}}(2N)^2\prod_{k=2}^{N}\frac{(-1)^\sigma\Psi(\lambda_{\sigma(1)})\Psi(\lambda_{\sigma(2)})}{\lambda_{\sigma(2k)}-\lambda_{\sigma(2k-1)}}.\end{aligned} \tag{3.36}$$

Indeed, the $2k$ order term comes from

$$\sum_{1\le i_1<\cdots<i_{2k}\le 2N}(2N)^{2k}\theta_{i_1}\Psi(\lambda_{i_1})\dots\theta_{i_{2k}}\Psi(\lambda_{i_{2k}})$$

and the Grassmann integration gives

$$\begin{aligned}&\int(\prod_{k=1}^{2N} d\theta_k)(\prod_{i<j} 1+\frac{\theta_i\theta_j}{\lambda_j-\lambda_i})\sum_{i_1<\cdots<i_{2k}}(2N)^{2k}\theta_{i_1}\Psi(\lambda_{i_1})\dots\theta_{i_{2k}}\Psi(\lambda_{i_{2k}})\\ &=\frac{1}{(2k)!2^{N-k}(N-k)!}\sum_{\sigma\in S_{2N}}(2N)^{2k}\prod_{j=k+1}^{N}\frac{(-1)^\sigma\Psi(\lambda_{\sigma(1)})\dots\Psi(\lambda_{\sigma(2k)})}{\lambda_{\sigma(2j)}-\lambda_{\sigma(2j-1)}}.\end{aligned} \tag{3.37}$$

So the $2k$th term in the partition function is

$$\begin{aligned}Z_S^{(2k)} &= \frac{(2N)^{2k}}{(2k)!2^{N-k}(N-k)!}\int(\prod_{i=1}^{2N} d\lambda_i e^{-2NV(\lambda_i)})\bar\Delta(\lambda)\sum_{\sigma\in S_{2N}}\prod_{j=k+1}^{N}\frac{(-1)^\sigma\Psi(\lambda_{\sigma(1)})\dots\Psi(\lambda_{\sigma(2k)})}{\lambda_{\sigma(2j)}-\lambda_{\sigma(2j-1)}}\\ &=\frac{(-1)^N(2N)^{2k}}{(2k)!2^{N-k}(N-k)!}\int(\prod_{i=1}^{2N} d\lambda_i e^{-2NV(\lambda_i)})\sum_{\sigma,\tau\in S_{2N}}(-1)^\sigma(-1)^\tau\prod_{i=1}^{2N}\prod_{j=k+1}^{N}\frac{\lambda_i^{\tau(i)-1}\Psi(\lambda_{\sigma(1)})\dots\Psi(\lambda_{\sigma(2k)})}{\lambda_{\sigma(2j)}-\lambda_{\sigma(2j-1)}}\\ &=\frac{(-1)^N(2N)^{2k}(2N)!}{(2k)!2^{N-k}(N-k)!}\int(\prod_{i=1}^{2N} d\lambda_i e^{-2NV(\lambda_i)})\sum_{\tau\in S_{2N}}(-1)^\tau\frac{\lambda_1^{\tau(1)-1}\cdots\lambda_{2N}^{\tau(2N)-1}\Psi(\lambda_1)\dots\Psi(\lambda_{2k})}{(\lambda_{2k+2}-\lambda_{2k+1})\cdots(\lambda_{2N}-\lambda_{2N-1})}.\end{aligned} \tag{3.38}$$

We explicitly introduce the ξs through $\Psi(\lambda_i) = \sum_{p=1}^{D+1} \xi_p \lambda_i^p$,

$$\sum_{\tau \in S_{2N}} (-1)^\tau (\prod_{i=1}^{2k} \int d\lambda_i e^{-2NV(\lambda_i)} \lambda_i^{\tau(i)-1} \Psi(\lambda_i))(\prod_{j=k+1}^{N} \int d\lambda_{2j} d\lambda_{2j-1} e^{-2NV(\lambda_j)} \frac{\lambda_{2j}^{\tau(j)} \lambda_{2j-1}^{\tau(2j)-1}}{\lambda_{2j} - \lambda_{2j-1}})$$
$$= \sum_{\tau \in S_{2N}} (-1)^\tau (\prod_{i=1}^{2k} \int d\lambda_i e^{-2NV(\lambda_i)} \sum_{p=1}^{D+1} \xi_p \lambda_i^{\tau(i)-1+p})(\prod_{j=k+1}^{N} A_{\tau(2j),\tau(2j-1)}). \tag{3.39}$$

If we let

$$\zeta_i \equiv 2N \sum_{p=1}^{D+1} \xi_p \int d\lambda e^{-2NV(\lambda)} \lambda^{i-1+p} \tag{3.40}$$

we have concisely

$$\boxed{Z_S^{(2k)} = \frac{(-1)^N (2N)!}{(2k)! 2^{N-k} (N-k)!} \sum_{\sigma \in S_{2N}} (-1)^\sigma (\prod_{i=1}^{2k} \zeta_{\sigma(i)})(\prod_{j=k+1}^{N} A_{\sigma(2j),\sigma(2j-1)}).} \tag{3.41}$$

Now that there is an explicit form of the supereigenvalue model partition function, we direct our attention to the free energy.

Theorem 3.2.3 *Let $\zeta = (\zeta_1, \ldots, \zeta_{2N})^T$ be the column vector of ζs defined above. The full partition function may be written as*

$$Z_S = Z_S^{(0)} e^{\frac{1}{2}\zeta^T A^{-1} \zeta}. \tag{3.42}$$

A direct consequence of Theorem 3.2.3 is of profound significance and merits theorem status.

Theorem 3.2.4 *The free energy depends on the fermionic coupling constants ξ_k only up to quadratic order, that is*

$$\begin{aligned} F_S &= F_S^{(0)} + F_S^{(2)} \quad \textit{where} \\ F_S^{(2)} &= \sum_{j,k} F_{jk} \xi_j \xi_k. \end{aligned} \tag{3.43}$$

To show these we use the following relations for Grassmann integrations. See Appendix B for the conventions adopted in this paper.

Lemma 3.2.5 *If A is a skew-symmetric $2N \times 2N$ matrix with complex entries and θ is the column vector $(\theta_1, \ldots, \theta_{2N})^T$ of Grassmann variables, then*

$$\int (\prod_{i=1}^{2N} d\theta_i) e^{-\frac{1}{2}\theta^T A\theta} = (-1)^N \text{Pfaffian}(A). \tag{3.44}$$

Proof: We encourage the reader to see Appendix A.3 for information on Pfaffians.

$$\begin{aligned}
\int (\prod_{i=1}^{2N} d\theta_i) e^{-\frac{1}{2}\theta^T A\theta} &= \int (\prod_i d\theta_i) \prod_{j=1}^{2N} \prod_{k=1}^{2N} (1 - \frac{1}{2}\theta_j A_{j,k}\theta_k) \\
&= \int (\prod_i d\theta_i)(1 - \frac{1}{2}\theta_1 A_{1,2}\theta_2)(1 - \frac{1}{2}\theta_1 A_{1,3}\theta_3)\cdots(1 - \frac{1}{2}\theta_1 A_{1,2N}\theta_{2N}) \\
&\times (1 - \frac{1}{2}\theta_2 A_{2,1}\theta_1)(1 - \frac{1}{2}\theta_2 A_{2,3}\theta_3)\cdots(1 - \frac{1}{2}\theta_2 A_{2,2N}\theta_{2N}) \\
&\times \cdots \times (1 - \frac{1}{2}\theta_{2N} A_{2N,1}\theta_1)(1 - \frac{1}{2}\theta_{2N} A_{2N,3}\theta_3)\cdots(1 - \frac{1}{2}\theta_{2N} A_{2N,2N-1}\theta_{2N-1}) \\
&= \frac{(-1)^N}{2^N N!} \sum_{\sigma\in S_{2N}} (-1)^\sigma \prod_{i=1}^{N} A_{\sigma(2i-1),\sigma(2i)} = (-1)^N \text{Pfaffian}(A),
\end{aligned} \tag{3.45}$$

where the $N!$ is because the product fixes the orderings of the A_{ij}, hence when we sum over the symmetric group we over-count the $N!$ rearrangements. □

Lemma 3.2.6 *Let $\zeta = (\zeta_1, \ldots, \zeta_{2N})^T$ be the column vector of ζs and let θ be the column vector $(\theta_1, \ldots, \theta_{2N})^T$ of Grassmann variables. Then*

$$(-1)^N \text{Pfaffian}(A) e^{\frac{1}{2}\zeta^T A^{-1}\zeta} = \int (\prod_{i=1}^{2N} d\theta_i) e^{-\frac{1}{2}\theta^T A\theta + \theta^T\zeta}. \tag{3.46}$$

Proof: First let N =1 and $A_{12} = -A_{21} = a = \text{Pfaffian}(A) \in \mathbb{C}$. In this case

$$\begin{aligned}
&\int d\theta_1 d\theta_2 e^{-\frac{1}{2}\theta^T A\theta + \theta^T\zeta} = \int d\theta_1 d\theta_2 e^{-\theta_1 a\theta_2 + \theta_1\zeta_1 + \theta_2\zeta_2} \\
&= \int d\theta_1 d\theta_2 (1 - \theta_1 a\theta_2)(1 + \theta_1\zeta_1)(1 + \theta_2\zeta_2) = -a + \zeta_1\zeta_2 = (-1)a(1 - \frac{\zeta_1\zeta_2}{a}) \\
&= (-1)\text{Pfaffian}(A) e^{\frac{1}{2}\zeta^T A^{-1}\zeta}.
\end{aligned} \tag{3.47}$$

Now let N be arbitrary. We write $A = O^T BO$ where O is orthogonal and B is a block-diagonal skew-symmetric matrix. Let $x = O\theta$. Since O is orthogonal $dx = d\theta$ (simply a rotation). Then

$$\int(\prod_{i=1}^{2N} d\theta_i)e^{-\frac{1}{2}\theta^T A\theta+\theta^T\zeta} = \int(\prod_{i=1}^{2N} dx_i)e^{-\frac{1}{2}x^T Bx+x^T O\zeta}$$
$$= \int\prod_{i=1}^{N} dx_{2i-1}dx_{2i}e^{(-x_{2i-1}B_{2i-1,2i}x_{2i}+x_{2i-1}(O\zeta)_{2i-1}+x_{2i}(O\zeta)_{2i})}$$
$$= (-1)^N\prod_{i=1}^{N}\Big(B_{2i-1,2i}-(O\zeta)_{2i-1}(O\zeta)_{2i}\Big) = (-1)^N\text{Pfaffian}(B)\prod_{i=1}^{N}\Big(1+\frac{1}{B_{2i,2i-1}}(O\zeta)_{2i-1}(O\zeta)_{2i}\Big) \tag{3.48}$$

where the + comes from switching the indices of B in the product. Since Pfaffian(A) = Pfaffian(B), we have

$$\begin{aligned}&(-1)^N\text{Pfaffian}(A)\exp\Big(\sum_i\frac{1}{B_{2i,2i-1}}(O\zeta)_{2i-1}(O\zeta)_{2i}\Big)\\ =&(-1)^N\text{Pfaffian}(A)e^{\frac{1}{2}(O\zeta)^T B^{-1}(O\zeta)}\\ =&(-1)^N\text{Pfaffian}(A)e^{\frac{1}{2}\zeta^T A^{-1}\zeta}.\end{aligned} \tag{3.49}$$

□

Proof: of Theorem 2.2. Our previous results allow us to express

$$\begin{aligned}Z_S^{(0)}e^{\frac{1}{2}\zeta^T A^{-1}\zeta} &= (2N)!\text{Pfaffian}(A)e^{\frac{1}{2}\zeta^T A^{-1}\zeta} = (-1)^N(2N)!\int(\prod_{i=1}^{2N}d\theta_i)e^{-\frac{1}{2}\theta^T A\theta+\theta^T\zeta}\\ &= (-1)^N(2N)!\int\prod_{i=1}^{2N}\prod_{j=1}^{2N}\prod_{k=1}^{2N}d\theta_i(1-\frac{1}{2}\theta_j A_{j,k}\theta_k)(1+\theta_i\zeta_i).\end{aligned} \tag{3.50}$$

The only relevant parts of the integrand are the coefficients of the rearrangements $\theta_1\cdots\theta_{2N}$ which we get by taking $2J$ terms from the product over the ζ_is and $N-J$ terms from the product over the A_{ij}s with each θ_i distinct. For example, looking at the terms with $J=0$ gives $Z_s^{(0)}$. The term from $J=N$ gives

$$(-1)^N(2N)!\zeta_i\cdots\zeta_{2N} = (-1)^N\sum_{\sigma\in S_{2N}}(-1)^\sigma\zeta_{\sigma(1)}\cdots\zeta_{\sigma(2N)} = Z_S^{(2N)}, \tag{3.51}$$

which can be seen by comparing with (3.41). Now consider all terms with $2J$ ζ_is, which are of the form

$$(-1/2)^{N-J}(\prod_{i=J+1}^{N}\theta_{2i-1}A_{2i-1,2i}\theta_{2i})(\prod_{k=1}^{2J}\theta_k\zeta_k), \tag{3.52}$$

and after integrating give

$$\frac{1}{2^{N-J}}(\prod_{i=J+1}^{N} A_{2i,2i-1})(\prod_{k=1}^{2J}\zeta_k) \tag{3.53}$$

where the factor of $(-1)^{N-J}$ cancelled with transposing A in the product. Summing over all terms with exactly $2J$ ζ_is is summing over all rearrangements which leave the ordering of the location of the pairs from A_{ij} and the ordering of the ζ_ks fixed:

$$\frac{1}{(2J)!(N-J)!}\sum_{\sigma\in S_{2N}}\frac{1}{2^{N-J}}(\prod_{i=J+1}^{N} A_{\sigma(2i),\sigma(2i-1)})(\prod_{k=1}^{2J}\zeta_{\sigma(k)}) \tag{3.54}$$

where the coefficient comes from the fixed ordering. Hence

$$\begin{aligned} Z_S^{(0)}e^{\frac{1}{2}\zeta^T A^{-1}\zeta} &= (-1)^N(2N)!\int\prod_{i=1}^{2N}d\theta_i\prod_{j=1}^{2N}\prod_{k=1}^{2N}(1-\frac{1}{2}\theta_j A_{j,k}\theta_k)(1+\theta_i\zeta_i) \\ &= \sum_{J=0}^{N}\frac{(-1)^N(2N)!}{(2J)!(N-J)!2^{N-J}}\sum_{\sigma\in S_{2N}}(-1)^\sigma(\prod_{i=J+1}^{N}A_{\sigma(2i),\sigma(2i-1)})(\prod_{k=1}^{2J}\zeta_{\sigma(k)}), \end{aligned} \tag{3.55}$$

which by (3.41) is $\sum_{J=0}^{N} Z_S^{(2J)} = Z_S$. □

Theorem 3.2.4 follows immediately,

$$\begin{aligned} F_S &= \frac{1}{(2N)^2}\ln(Z_S) = \frac{1}{(2N)^2}\ln(Z_0 e^{\frac{1}{2}\zeta^T A^{-1}\zeta}) \\ &= \frac{1}{(2N)^2}\ln(Z_0) + \frac{1}{(2N)^2}\frac{1}{2}\zeta^T A^{-1}\zeta = F_S^{(0)} + F_S^{(2)}, \end{aligned} \tag{3.56}$$

and is critical in classifying the observables of the supereigenvalue model.

3.3 Superloop correlators etc.

Now we introduce objects generalized from those found in the Hermitian matrix model that will lead to a infinite sequence of loop equations. The observables come in two categories, odd and even. The one variable **superloop correlators** are

$$W(x|) \equiv \frac{1}{2N}\Big\langle\sum_i\frac{\theta_i}{x-\lambda_i}\Big\rangle \;;\; W(|x) \equiv \frac{1}{2N}\Big\langle\sum_i\frac{1}{x-\lambda_i}\Big\rangle. \tag{3.57}$$

Additionally, we have two **superloop operators**

$$\frac{\partial}{\partial V(x)} = -\sum_{k\geq 0} \frac{1}{x^{k+1}} \frac{\partial}{\partial g_k} \; ; \; \frac{\partial}{\partial \Psi(x)} = -\sum_{k\geq 1} \frac{1}{x^{k+1}} \frac{\partial}{\partial \xi_k}, \tag{3.58}$$

that allow us to generalize the correlators as follows:

$$\begin{aligned} W(x_1, \cdots, x_n | y_1, \cdots, y_m) &\equiv \frac{\partial}{\partial \Psi(x_1)} \cdots \frac{\partial}{\partial \Psi(x_n)} \frac{\partial}{\partial V(y_1)} \cdots \frac{\partial}{\partial V(y_m)} F \\ &= (2N)^{n+m-2} \Big\langle \sum_i \frac{\theta_i}{x_1 - \lambda_i} \cdots \sum_i \frac{\theta_i}{x_n - \lambda_i} \sum_i \frac{1}{y_1 - \lambda_i} \cdots \sum_i \frac{1}{y_m - \lambda_i} \Big\rangle_c \end{aligned} \tag{3.59}$$

where $\langle A \rangle_c$ means the connected part of $\langle A \rangle$. For convenience, we define

$$\begin{aligned} P_1(x_1, \ldots, x_n | y_1, \ldots, y_m) \equiv (2N)^{n+m-2} \Big\langle \sum_i \theta_i \frac{V'(x_1) - V'(\lambda_i)}{x_1 - \lambda_i} \sum_i \frac{\theta_i}{x_2 - \lambda_i} \cdots \sum_i \frac{\theta_i}{x_n - \lambda_i} \\ \times \sum_i \frac{1}{y_1 - \lambda_i} \cdots \sum_i \frac{1}{y_m - \lambda_i} \Big\rangle_c \end{aligned} \tag{3.60}$$

$$\begin{aligned} P_0(x_1, \ldots, x_n | y_1, \ldots, y_m) \equiv (2N)^{n+m-2} \Big\langle \sum_i \frac{\theta_i}{x - \lambda_i} \sum_i \frac{\theta_i}{x_2 - \lambda_i} \cdots \sum_i \frac{\theta_i}{x_n - \lambda_i} \\ \times \sum_i \frac{V'(y_1) - V'(\lambda_i)}{y_1 - \lambda_i} \sum_i \frac{1}{y_2 - \lambda_i} \cdots \sum_i \frac{1}{y_m - \lambda_i} \Big\rangle_c \end{aligned} \tag{3.61}$$

$$\begin{aligned} \Pi_1(x_1, \ldots, x_n | y_1, \ldots, y_m) = (2N)^{n+m-2} \Big\langle \sum_i \theta_i \frac{\Psi(x_1) - \Psi(\lambda_i)}{x_1 - \lambda_i} \sum_i \frac{\theta_i}{x_2 - \lambda_i} \cdots \sum_i \frac{\theta_i}{x_m - \lambda_i} \\ \times \sum_i \frac{1}{y_1 - \lambda_i} \cdots \sum_i \frac{1}{y_m - \lambda_i} \Big\rangle_c \end{aligned} \tag{3.62}$$

$$\begin{aligned} \Phi_1(x_1, \ldots, x_n | y_1, \ldots, y_m) = (2N)^{n+m-2} \Big\langle \sum_i \theta_i \frac{\Psi'(x_1) - \Psi'(\lambda_i)}{x_1 - \lambda_i} \sum_i \frac{\theta_i}{x_2 - \lambda_i} \cdots \sum_i \frac{\theta_i}{x_m - \lambda_i} \\ \times \sum_i \frac{1}{y_1 - \lambda_i} \cdots \sum_i \frac{1}{y_m - \lambda_i} \Big\rangle_c \end{aligned} \tag{3.63}$$

and Π_0 and Φ_0 similarly. The subscript denotes the placement of the potential term. Each of these functions can be gotten by repeated application of the insertion operators. They are polynomials in y_1 or x_1 depending on whether the subscript is 0 or 1 respectively.

3.3.1 t'Hooft expansion

When working with the supereigenvalue model it is standard to assume that the free energy and correlators (and related objects defined from Z_S or F_S) have a genus expansion:

$$F_S = \sum_g (2N)^{-2g} F_g$$

$$W(x_1, \cdots, x_n | y_1, \cdots, y_m) = \sum_g (2N)^{-2g} W_g(x_1, \cdots, x_n | y_1, \cdots, y_m).$$

However, we reiterate that the expansion is an assumption. It seems worthwhile to provide a formal proof of the expansion for the quadratic component although such a proof eludes this author.

3.3.2 ξ expansion

Recall theorem 3.2.4 which says the free energy is quadratic in ξ. From the definitions involving the insertion operators we have the following.

Corollary 3.3.1 *For $n \geq 3$, $W(x_1, \cdots, x_n | y_1, \cdots, y_m) = 0$.*

Corollary 3.3.2 *The super-loop correlators can be expanded in terms of the fermionic coupling constants ξ_k as*

$$\begin{aligned}
W(|\dots) &= W^{(0)}(|\dots) + W^{(2)}(|\dots) = \sum_g (2N)^{-2g} \left(W_g^{(0)}(\dots) + W_g^{(2)}(\dots) \right) \\
W(z|\dots) &= W^{(1)}(z|\dots) = \sum_g (2N)^{-2g} W_g^{(1)}(z|\dots) \\
W(x, z|\dots) &= W^{(0)}(x, z|\dots) = \sum_g (2N)^{-2g} W_g^{(0)}(x, z|\dots)
\end{aligned}$$

where the superscript denotes the order of ξ_k in the term.

Also observe that $P_1(x|)$ is the same order in ξ as $W(x|)$ and similarly with $P_0(|x)$ and $W(|x)$.

Corollary 3.3.3 *For $i = 0, 1$*

$$\begin{aligned}
P_0(|\dots) &= P_0^{(0)}(|\dots) + P_0^{(2)}(|\dots), \\
P_i(z|\dots) &= P_i^{(1)}(z|\dots), \\
P_i(x, z|\dots) &= P_i^{(0)}(x, z|\dots).
\end{aligned}$$

Additionally, the Π and Φ polynomials are one order higher in ξ then the correlators.

Corollary 3.3.4 *For* $\Lambda = \Pi, \Phi,$ *and* $i = 0, 1$

$$\begin{aligned}
\Lambda_i(|\dots) &= \Lambda_i^{(1)}(|\dots) + \Lambda_i^{(3)}(|\dots),\\
\Lambda_i(z|\dots) &= \Lambda_i^{(0)}(z|\dots) + \Lambda_i^{(2)}(z|\dots),\\
\Lambda_i(x,z|\dots) &= \Lambda_i^{(1)}(x,z|\dots).
\end{aligned}$$

3.4 Superloop equations

As in any field theory, we have that Z_s is invariant under infinitesimal changes in λ_i and θ_i. We can use the invariance to derive a pair of differential equations that, in complete analogy to the Euler-Lagrange equations, capture all the dynamics of our model. The pair of loop equations follow from the invariance of Z_s under infinitesimal changes in $\{\lambda_i, \theta_j\}$. To simplify some writing, in the following we work with N λs and N θs where $N \equiv 0$ mod 2 (instead of explicitly working with $2N$ of each type of parameter as was done previously). We compute total derivatives

$$0 = \sum_{p=1}^{N} \int (\prod_i d\lambda_i d\theta_i) K_p(\lambda_p, \theta_p) \Big(\prod_{i<j} (\lambda_i - \lambda_j - \theta_i\theta_j) e^{-N\sum_i (V(\lambda_i) - \theta_i \Psi(\lambda_i))} \Big) \qquad (3.64)$$

where K_p is a differential operator that represents an infinitesimal variation of λ_p or θ_p. Of course, it immediately follows that

$$0 = \sum_{p=1}^{N} \frac{1}{Z_S} \int (\prod_i d\lambda_i d\theta_i) K_p \Big(\prod_{i<j} (\lambda_i - \lambda_j - \theta_i\theta_j) e^{-N\sum_i (V(\lambda_i) - \theta_i \Psi(\lambda_i))} \Big). \qquad (3.65)$$

In order to express equations in terms of our superloop correlators, we use particular choices for the operators K_p. Just like with the 1HMM, there are conventional shifts that are used to derive the SEV superloop equations. Additionally, we note that the total derivative method has not been used elsewhere to determine the superloop equations: the literature always works with direct shifts of λ_i and θ_i. Of course we arrive at the same superloop equations [29], although they are expressed in a manner unique to us that is more convenient for our interests.

3.4.1 Odd loop equation

To get the odd-loop equation we take an odd total derivative of $\langle\frac{1}{x-\lambda_p}\rangle$:

$$K_p = \left(\theta_p \frac{\partial}{\partial\lambda_p} - \frac{\partial}{\partial\theta_p}\right)\left(\frac{1}{x-\lambda_p}\right)$$

and find (see 2.3.1)

$$0 = -\sum_p N\left\langle\frac{\theta_p V'(\lambda_p)}{x-\lambda_p}\right\rangle - \sum_p N\left\langle\frac{\Psi(\lambda_p)}{x-\lambda_p}\right\rangle + \sum_p\left\langle\frac{\theta_p}{(x-\lambda_p)^2}\right\rangle + \sum_{p,i\neq p}\left\langle\left(\frac{\theta_p}{x-\lambda_p}\right)\left(\frac{1}{x-\lambda_i}\right)\right\rangle$$

which we rewrite using connected pieces as

$$\sum_p N\left\langle\frac{\theta_p V'(\lambda_p)}{x-\lambda_p}\right\rangle + \sum_p N\left\langle\frac{\Psi(\lambda_p)}{x-\lambda_p}\right\rangle = N^2 W(x|)W(|x) + W(x|x), \tag{3.66}$$

and after dividing by N^2 becomes

$$\boxed{V'(x)W(x|) - P_1(x|) + \Psi(x)W(|x) - \Pi_0(|x) = W(x|)W(|x) + \frac{W(x|x)}{N^2}.} \tag{3.67}$$

3.4.2 Even loop equation

To get the even-loop equation we let $K_p^1 = [\frac{\partial}{\partial\lambda_p}\frac{1}{(x-\lambda_p)}]$ and $K_p^2 = [\frac{1}{2}\frac{\partial}{\partial\theta_p}\frac{\theta_p}{(x-\lambda_p)^2}]$ and apply $K_p^1 - K_p^2$. The operation of K_p^1 gives (see 2.3.1 which is nearly identical)

$$0 = \sum_p\left\langle\frac{1}{(x-\lambda_p)^2}\right\rangle - \sum_p N\left\langle\frac{V'(\lambda_p) - \theta_p\Psi'(\lambda_p)}{x-\lambda_p}\right\rangle + \sum_{p,i\neq p}\left\langle\left(\frac{1}{x-\lambda_p}\right)\left(\frac{1}{\lambda_p-\lambda_i-\theta_p\theta_i}\right)\right\rangle \tag{3.68}$$

and the operation of K_p^2 gives

$$0 = \frac{1}{2}\left(\sum_p\left\langle\frac{1}{(x-\lambda_p)^2}\right\rangle - \sum_p N\left\langle\frac{\theta_p\Psi(\lambda_p)}{(x-\lambda_p)^2}\right\rangle + \sum_{p,i\neq p}\left\langle\left(\frac{\theta_p}{(x-\lambda_p)^2}\right)\left(\frac{\theta_i}{\lambda_p-\lambda_i-\theta_p\theta_i}\right)\right\rangle\right) \tag{3.69}$$

Applying the operator $K = K_p^1 - K_p^2$, from (3.68) and (3.69) we re-group terms and find

$$\begin{aligned}
&\sum_p N\Big\langle \frac{V'(\lambda_p)}{x-\lambda_p}\Big\rangle - \sum_p N\Big\langle \frac{\theta_p \Psi'(\lambda_p)}{x-\lambda_p}\Big\rangle - \frac{1}{2}\sum_p N\Big\langle \frac{\theta_p \Psi(\lambda_p)}{(x-\lambda_p)^2}\Big\rangle \\
&= \frac{1}{2}\sum_p \Big\langle \frac{1}{(x-\lambda_p)^2}\Big\rangle + \sum_{p,i\neq p}\Big\langle \left(\frac{1}{x-\lambda_p}\right)\left(\frac{1}{\lambda_p-\lambda_i-\theta_p\theta_i}\right)\Big\rangle \\
&\qquad -\frac{1}{2}\sum_{p,i\neq p}\Big\langle \left(\frac{\theta_p}{(x-\lambda_p)^2}\right)\left(\frac{\theta_i}{\lambda_p-\lambda_i-\theta_p\theta_i}\right)\Big\rangle.
\end{aligned} \tag{3.70}$$

We Taylor-expand and (anti-)symmetrize the penultimate term:

$$\begin{aligned}
&\sum_{p,i\neq p}\Big\langle \left(\frac{1}{x-\lambda_p}\right)\left(\frac{1}{\lambda_p-\lambda_i-\theta_p\theta_i}\right)\Big\rangle = \sum_{p,i\neq p}\Big\langle \left(\frac{1}{x-\lambda_p}\right)\left(\frac{1}{\lambda_p-\lambda_i}+\frac{\theta_p\theta_i}{(\lambda_p-\lambda_i)^2}\right)\Big\rangle \\
&= \frac{1}{2}\sum_{p,i\neq p}\Big\langle \left(\frac{1}{x-\lambda_p}\right)\left(\frac{1}{x-\lambda_i}\right)\Big\rangle + \frac{1}{2}\sum_{p,i\neq p}\Big\langle \left(\frac{1}{\lambda_p-\lambda_i}\right)\left(\frac{\theta_p}{x-\lambda_p}\right)\left(\frac{\theta_i}{x-\lambda_i}\right)\Big\rangle,
\end{aligned} \tag{3.71}$$

and re-write the last term

$$\frac{1}{2}\sum_{p,i\neq p}\Big\langle \left(\frac{\theta_p}{(x-\lambda_p)^2}\right)\left(\frac{\theta_i}{\lambda_p-\lambda_i-\theta_p\theta_i}\right)\Big\rangle = \frac{1}{2}\sum_{p,i\neq p}\Big\langle \left(\frac{\theta_p\theta_i}{(x-\lambda_p)^2(\lambda_p-\lambda_i)}\right)\Big\rangle \tag{3.72}$$

and combining these and using connected components gives

$$\begin{aligned}
&\frac{1}{2}\sum_{p,i\neq p}\Big\langle \left(\frac{1}{\lambda_p-\lambda_i}\right)\left(\frac{\theta_p}{x-\lambda_p}\right)\left(\frac{\theta_i}{x-\lambda_i}\right)\Big\rangle - \frac{1}{2}\sum_{p,i\neq p}\Big\langle \left(\frac{\theta_p\theta_i}{(x-\lambda_p)^2(\lambda_p-\lambda_i)}\right)\Big\rangle \\
&= \frac{1}{2}\sum_{p,i\neq p}\Big\langle \frac{\theta_p\theta_i}{(x-\lambda_p)(\lambda_p-\lambda_i)}\left(\frac{1}{x-\lambda_i}-\frac{1}{x-\lambda_p}\right)\Big\rangle = -\frac{1}{2}\sum_{p,i\neq p}\Big\langle \frac{\theta_p\theta_i}{(x-\lambda_p)^2(x-\lambda_i)}\Big\rangle \\
&= \frac{1}{2}\sum_{p,i\neq p}\Big\langle \frac{\theta_i\theta_p}{(x-\lambda_i)(x-\lambda_p)^2}\Big\rangle = \frac{1}{2}\left(-N^2 W(x|)\frac{d}{dx}W(x|) - \frac{d}{dy}W(x,y|)|_{y=x}\right).
\end{aligned} \tag{3.73}$$

Putting everything together we arrive at the even loop equation:

$$\begin{aligned}
&\sum_p N\Big\langle \frac{V'(\lambda_p)}{x-\lambda_p}\Big\rangle - \sum_p N\Big\langle \frac{\theta_p \Psi'(\lambda_p)}{x-\lambda_p}\Big\rangle + \frac{1}{2}\sum_p N\Big\langle \frac{\theta_p \Psi(\lambda_p)}{(x-\lambda_p)^2}\Big\rangle \\
&= \frac{1}{2}\left(N^2 W(|x)^2 + W(|x,x) - N^2 W(x|)\frac{d}{dx}W(x|) - \frac{d}{dy}W(x,y|)|_{y=x}\right),
\end{aligned} \tag{3.74}$$

and after dividing by N^2

$$\boxed{\begin{aligned}&V'(x)W(|x) - P_0(|x) + \Phi_1(x|) + \Psi'(x)W(x|) + \frac{1}{2}\frac{d}{dx}\Big(\Pi_1(x|) + \Psi(x)W(x|)\Big)\\ &\qquad = \frac{1}{2}\Big(W(|x)^2 - W(x|)W'(x|) + \frac{W(|x,x)}{N^2} - \frac{d}{dy}\frac{W(x,y|)}{N^2}|_{y=x}\Big).\end{aligned}} \tag{3.75}$$

3.4.3 The general loop equations

By applying the two insertion operators to equations (3.67) and (3.75) we can get loop equations in multiple variables $x_1, \ldots, x_k$. Lets first apply $\partial/\partial V()$ an arbitrary number of times. Applying $\partial/\partial V(x_1)\cdots\partial/\partial V(x_k)$ to the odd loop equation gives

$$\boxed{\begin{aligned}&\sum_{i=1}^{k}\frac{d}{dx_i}\frac{W(x|J\backslash x_i) - W(x_i|J\backslash x_i)}{x_i - x} + V'(x)W(x|J) - P_1(x|J) + \Psi(x)W(|x,J) - \Pi_0(|x,J)\\ &= \sum_{I\subset J} W(x|I)W(|x,J\backslash I) + \frac{W(x|x,J)}{N^2}\ , \quad \text{where } J = \{x_1,\ldots,x_k\},\end{aligned}} \tag{3.76}$$

and to the even equation gives

$$\boxed{\begin{aligned}&V'(x)W(|x,J) - P_0(|x,J) + \Phi_1(x|J) + \Psi'(x)W(x|J)\\ &\qquad + \sum_i \frac{d}{dx_i}\frac{W(|x,J\backslash x_i) - W(|J)}{x_i - x} + \frac{1}{2}\frac{d}{dx}\Big(\Pi_1(x|J) + \Psi(x)W(x|J)\Big)\\ &= \frac{1}{2}\Big(\sum_{I\subset J} W(|x,I)W(|x,J\backslash I) - W(x|I)\frac{d}{dx}W(x|J\backslash I)\Big)\\ &\qquad + \frac{1}{2}\Big(\frac{W(|x,x,J)}{N^2} - \frac{d}{dz}\frac{W(x,z|J)}{N^2}|_{z=x}\Big), \quad \text{where } J = \{x_1,\ldots,x_k\}.\end{aligned}} \tag{3.77}$$

We can organize the loop equations by order in $1/N$ so that the odd equation is

$$\begin{aligned}&\sum_{i=1}^{k}\frac{d}{dx_i}\frac{W_g(x|J\backslash x_i) - W_g(x_i|J)}{x_i - x} + V'(x)W_g(x|J) - P_{1,g}(x|J) + \Psi(x)W_g(|x,J) - \Pi_{0,g}(|x,J)\\ &= \sum_{I\subset J}\sum_{h=0}^{g} W_h(x|I)W_{g-h}(|x,J\backslash I) + W_{g-1}(x|x,J),\end{aligned} \tag{3.78}$$

and the even equation is

$$\begin{aligned} &V'(x)W_g(|x,J) - P_{0,g}(|x,J) + \Phi_{1,g}(x|J) + \Psi'(x)W_g(x|J) \\ &\qquad + \sum_i \frac{d}{dx_i}\frac{W_g(|x,J\backslash x_i) - W_g(|J)}{x_i - x} + \frac{1}{2}\frac{d}{dx}\Big(\Pi_{1,g}(x|J) + \Psi(x)W_g(x|J)\Big) \\ &\quad = \frac{1}{2}\Big(\sum_{I\subset J}\sum_{h=0}^{g} W_h(|x,I)W_{g-h}(|x,J\backslash I) - W_h(x|I)\frac{d}{dx}W_{g-h}(x|J\backslash I)\Big) \\ &\qquad + \frac{1}{2}\Big(W_{g-1}(|x,x,J) - \frac{d}{dz}W_{g-1}(x,z|J)|_{z=x}\Big). \end{aligned} \tag{3.79}$$

3.5 ξ expansion

Furthermore, we may organize the loop equations by their order in the ξ expansion so that the even equation (3.77) splits into components of order 0 and order 2 in ξ and the odd equation (3.76) splits into components of order 1 and 3.

Order 0

$$\begin{aligned} &V'(x)W_g^{(0)}(|x,J) - P_{0,g}^{(0)}(|x,J) + \sum_i \frac{d}{dx_i}\frac{W_g^{(0)}(|x,J\backslash x_i) - W_g^{(0)}(|J)}{x_i - x} \\ &= \frac{1}{2}\Big(\sum_{I\subset J}\sum_{h=0}^{g} W_h^{(0)}(|x,I)W_{g-h}^{(0)}(|x,J\backslash I) + W_{g-1}^{(0)}(|x,x,J) - \frac{d}{dz}W_{g-1}^{(0)}(x,z|J)|_{z=x}\Big) \end{aligned} \tag{3.80}$$

Order 1

$$\begin{aligned} &\sum_{i=1}^{k}\frac{d}{dx_i}\frac{W_g^1(x|J\backslash x_i) - W_g^1(x_i|J\backslash x_i)}{x_i - x} + V'(x)W_g^1(x|J) - P_{1,g}^1(x|J) + \Psi(x)W_g^0(|x,J) - \Pi_{0,g}^1(|x,J) \\ &= \sum_{I\subset J}\sum_{h=0}^{g} W_h^1(x|I)W_{g-h}^0(|x,J\backslash I) + W_{g-1}^1(x|x,J) \end{aligned} \tag{3.81}$$

Order 2

$$\begin{aligned}
&V'(x)W_g^2(|x,J) - P_{0,g}^2(|x,J) + \Phi_{1,g}^2(x|J) - \Psi'(x)W_g^1(x|J) \\
&+ \sum_i \frac{d}{dx_i}\frac{W_g^2(|x,J\backslash x_i) - W_g^2(|J)}{x_i - x} + \frac{1}{2}\frac{d}{dx}\Big(\Pi_{1,g}^2(x|J) - \Psi(x)W_g^1(x|J)\Big) \\
&= \frac{1}{2}\Bigg(\sum_{I\subset J}\sum_{h=0}^{g} 2W_h^0(|x,I)W_{g-h}^2(|x,J\backslash I) - W_h^1(x|I)\frac{d}{dx}W_{g-h}^1(x|J\backslash I) + W_{g-1}^2(|x,x,J)\Bigg)
\end{aligned} \tag{3.82}$$

Order 3

$$\Psi(x)W_g^2(|x,J) - \Pi_{0,g}^3(|x,J) = \sum_{I\subset J}\sum_{h=0}^{g} W_h^1(x|I)W_{g-h}^2(|x,J\backslash I) \tag{3.83}$$

Our reason for splitting the two superloop equations into 4 equations determined by the order in ξ is so that later we may investigate the possibility of finding a residue formulation for the meromormphic coefficients of the ξ. But before we can get there we must see if these 4 equations allow for a recursion.

CHAPTER 4

The Recursive Nature of the Superloop Equations

In this section we lay the foundation for the recursive relations between the four superloop equations. The supercorrelators can be divided into four categories: $W^{(0)}(|J)$, $W^{(1)}(x|J)$, $W^{(2)}(|J)$ and $W^{(0)}(x, x_1|J)$. We show these superloop equations are sufficient to obtain all correlators and demonstrate how these four correlator types can be recursively derived. It turns out that we only need to use 3 of the equations. However, we are assuming that the functions P_i, Π_i, Φ_i are understood if not ultimately irrelevant.

4.1 The planar limit

We demonstrate the recursive relations between the superloop equations when $g = 0$.

4.1.1 Order 0 equation

When $g = 0$, (3.80) involves only the $W^{(0)}_{g=0}(|x, J)$ terms. Letting $J = \emptyset$ in (3.80) allows us to solve for

$$W_0^{(0)}(|x) = V'(x) - \sqrt{(V'(x))^2 - 2P_{0,0}^{(0)}(|x)}. \tag{4.1}$$

Motivated by our work with the Hermitian matrix model, let us introduce

$$Y(x) = V'(x) - W_0^{(0)}(|x). \tag{4.2}$$

Then we find

$$W_0^{(0)}(|x, x_1) = \left(\frac{\frac{\partial}{\partial x_1}\left(\frac{W_0^{(0)}(|x) - W_0^{(0)}(|x_1)}{x - x_1}\right) + P_{0,0}^{(0)}(x, x_1)}{Y(x)} \right). \tag{4.3}$$

Observe the similarity with the Hermitian matrix model. Now relation (3.80) at $g = 0$ provides a recursion for $W_0^{(0)}(|x, J)$:

$$\boxed{\begin{aligned} Y(x)W_0^{(0)}(|x, J) =& P_{0,0}^{(0)}(|x, J) + \sum_i \frac{d}{dx_i} \frac{W_0^{(0)}(|x, J\backslash x_i) - W_0^{(0)}(|J)}{x - x_i} \\ &+ \frac{1}{2}\Big(\sum_{\emptyset \nsubseteq I \nsubseteq J} W_0^{(0)}(|x, I)W_0^{(0)}(|x, J\backslash I)\Big). \end{aligned}} \tag{4.4}$$

As we progress keep in mind that we are now familiar with all terms $W_0^{(0)}(| \dots)$.

4.1.2 Order 1 equation

From (3.81) we have

$$\begin{aligned} W_0^{(1)}(x|) &= \frac{1}{Y(x)}\Big(P_{1,0}^{(1)}(x|) + \Pi_{0,0}^{(1)}(|x) - \Psi(x)W_0^{(0)}(|x)\Big) \\ &= \frac{1}{Y(x)}\Big(P_{1,0}^{(1)}(x|) + \Pi_{0,0}^{(1)}(|x) - \Psi(x)V'(x)\Big) + \Psi(x), \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} Y(x)W_0^{(1)}(x|x_1) = &\frac{d}{dx_1} \frac{W_0^{(1)}(x|) - W_0^{(1)}(x_1|)}{x - x_1} + P_{1,0}^{(1)}(x|x_1) + \Pi_{0,0}^{(1)}(|x, x_1) \\ &+ W_0^{(1)}(x|)W_0^{(0)}(|x, x_1) - \Psi(x)W_0^{(0)}(|x, x_1). \end{aligned} \tag{4.6}$$

Using $Y(x)$, we may write the recursion as

$$\boxed{\begin{aligned} &Y(x)W_0^{(1)}(x|J) = P_{1,0}^{(1)}(x|J) - \Psi(x)W_0^{(0)}(|x, J) + \Pi_{0,0}^{(1)}(|x, J) \\ &+ \sum_{I \nsubseteq J} W_0^{(1)}(x|I)W_0^{(0)}(|x, J\backslash I) + \sum_{i=1}^{k} \frac{d}{dx_i} \frac{W_0^{(1)}(x|J\backslash x_i) - W_0^{(1)}(x_i|J\backslash x_i)}{x - x_i}. \end{aligned}} \tag{4.7}$$

Hence we are able to compute all $W_0^{(1)}(x| \dots)$ terms recursively.

Order 0 from order 1

To determine $W_0^{(0)}(x, x_1|\dots)$ we apply the odd loop operator to (3.81) to eliminate the ξs and which gives

$$\sum_{i=1}^{j} \frac{d}{dx_i} \frac{W_g^{(0)}(x, z|J\backslash x_i) - W_g^{(0)}(x_i, z|J\backslash x_i)}{x_i - x} + V'(x) W_g^{(0)}(x, z|J) - P_{1,g}^{(0)}(x, z|J) - \Pi_0(z|x, J)$$
$$+ \frac{W_g^{(0)}(|x, J) - W_g^{(0)}(|z, J)}{x - z} = \sum_{I \subset J} \sum_{h=0}^{g} W_h^{(0)}(x, z|I) W_{g-h}^{(0)}(|x, J\backslash I) + W_{g-1}^{(0)}(x, z|x, J). \tag{4.8}$$

For $g = 0,\ J = \emptyset$ equation (4.8) becomes

$$Y(x) W_0^{(0)}(x, x_1|) = P_{1,0}^{(0)}(x, x_1|) - \frac{W_0^{(0)}(|x) - W_0^{(0)}(|x_1)}{x - x_1} + \Pi_{0,0}^{(0)}(x_1|x), \tag{4.9}$$

and when $J = \{z\}$

$$\begin{aligned} Y(x) W^{(0)}(x, x_1|z) &= P_1^{(0)}(x, x_1|z) + \frac{d}{dz} \frac{W^{(0)}(x, x_1|) - W^{(0)}(z, x_1|)}{x - z} \\ &+ \Pi_0^{(0)}(x_1|x, z) - \frac{W^{(0)}(|x, z) - W^{(0)}(|x_1, z)}{x - x_1} + W^{(0)}(x, x_1|) W^{(0)}(|x, z) \end{aligned} \tag{4.10}$$

We can express $W_0^{(0)}(x, z|J)$ through

$$\boxed{\begin{aligned} Y(x) W_0^{(0)}(x, z|J) &= P_{1,0}^{(0)}(x, z|J) + \sum_{i=1}^{j} \frac{d}{dx_i} \frac{W_0^{(0)}(x, z|J\backslash x_i) - W_0^{(0)}(x_i, z|J\backslash x_i)}{x - x_i} \\ &+ \Pi_{0,0}^{(0)}(z|x, J) - \frac{W_0^{(0)}(|x, J) - W_0^{(0)}(|z, J)}{x - z} + \sum_{I \not\subseteq J} W_0^{(0)}(x, z|I) W_0^{(0)}(|x, J\backslash I). \end{aligned}} \tag{4.11}$$

4.1.3 Order 3

To determine $W_0^{(2)}(|\dots)$ we may use either the order 2 or order 3 equation. For simplicity we work with the order 3 equation which gives

$$\left(\Psi(x) - W_0^{(1)}(x|)\right) W_0^{(2)}(|x) = \Pi_{0,0}^{(3)}(|x) \tag{4.12}$$

and

$$\Big(\Psi(x) - W_0^{(1)}(x|)\Big) W_0^{(2)}(|x, x_1) = \Pi_{0,0}^{(3)}(|x, x_1) + W_0^{(1)}(x|x_1) W_0^{(2)}(|x) \tag{4.13}$$

and the recursion

$$\boxed{\Big(\Psi(x) - W_0^{(1)}(x|)\Big) W_0^{(2)}(|x, J) = \Pi_{0,0}^{(3)}(|x, J) + \sum_{I \not\subseteq J} W_0^{(2)}(|x, I) W_0^{(1)}(x|J \backslash I)}. \tag{4.14}$$

At this point we have a method that determines all planar super-correlators.

4.2 $g = 1$ recursion

We assume all planar correlators are known and demonstrate how to theoretically evaluate all $g = 1$ superloop correlators. The process is exactly the same, hence this section is exceedingly terse.

4.2.1 order 0

Plugging $g = 1$ into (3.80) gives

$$Y(x) W_1^{(0)}(|x) = P_{0,1}^{(0)}(|x) + \frac{1}{2}\left(W_0^{(0)}(|x, x) - \frac{d}{dz} W_0^{(0)}(x, z|)|_{z=x} \right), \tag{4.15}$$

and

$$\begin{aligned} Y(x) W_1^{(0)}(|x, x_1) = P_{0,1}^{(0)}(|x, x_1) &+ \frac{d}{dx_1} \frac{W_1^{(0)}(|x) - W_1^{(0)}(|x_1)}{x - x_1} \\ &+ \frac{1}{2}\Big(W_0^{(0)}(|x, x_1) W_1^{(0)}(|x) + W_1^{(0)}(|x) W_0^{(0)}(|x, x_1) \\ &+ W_0^{(0)}(|x, x, x_1) - \frac{d}{dz} W_0^{(0)}(x, z|x_1)|_{z=x} \Big) \end{aligned} \tag{4.16}$$

and the recursion

$$\boxed{\begin{aligned}&Y(x)W_1^{(0)}(|x,J) = P_{0,1}^{(0)}(|x,J) + \sum_i \frac{d}{dx_i}\frac{W_1^{(0)}(|x,J\backslash x_i) - W_1^{(0)}(|J)}{x-x_i}\\&+\frac{1}{2}\left(\sum_{I,h}^{*}\left(W_h^{(0)}(|x,I)W_{g-h}^{(0)}(|x,J\backslash I)\right) + W_0^{(0)}(|x,x,J) - \frac{d}{dz}W_0^{(0)}(x,z|J)|_{z=x}\right)\\&\text{where } * \text{ means } (I,h)\neq(J,g),(\emptyset,0).\end{aligned}} \tag{4.17}$$

4.2.2 Order 1

From (3.81) we find

$$Y(x)W_1^{(1)}(x|) = W_0^{(1)}(x|)W_1^{(0)}(|x) + W_0^{(1)}(x|x) + P_{1,1}^{(1)}(x|) - \Psi(x)W_1^{(0)}(|x) + \Pi_{0,1}^{(1)}(|x) \tag{4.18}$$

and

$$\begin{aligned}Y(x)W_1^{(1)}(x|x_1) = {}& \frac{d}{dx_1}\frac{W_1^{(1)}(x|) - W_1^{(1)}(x_1|)}{x-x_1} + P_{1,1}^{(1)}(x|x_1) - \Psi(x)W_1^{(0)}(|x,x_1)\\&+\Pi_{0,1}^{(1)}(|x,x_1) + W_0^{(1)}(x|x_1)W_1^{(0)}(|x) + W_0^{(1)}(x|)W_1^{(0)}(|x,x_1)\\&+W_1^{(1)}(x|I)W_0^{(0)}(|x,x_1) + W_0^{(1)}(x|x,x_1)\end{aligned} \tag{4.19}$$

with recursion

$$\boxed{\begin{aligned}Y(x)W_g^{(1)}(x|J) = {}& \sum_{i=1}^{k}\frac{d}{dx_i}\frac{W_g^{(1)}(x|J\backslash x_i) - W_g^{(1)}(x_i|J\backslash x_i)}{x-x_i} + P_{1,g}^{(1)}(x|J) - \Psi(x)W_g^{(0)}(|x,J)\\&+\Pi_{0,g}^{(1)}(|x,J) + \sum_{I,h}^{**}W_h^{(1)}(x|I)W_{g-h}^{(0)}(|x,J\backslash I) + W_{g-1}^{(1)}(x|x,J),\\&\text{where } ** \text{ means } (I,h)\neq(J,g).\end{aligned}} \tag{4.20}$$

Order 0 from order 1

Again we use (4.8) now with $g = 1$ and find

$$Y(x)W_1^{(0)}(x,z|) = P_{1,1}^{(0)}(x,z|) + \Pi_{0,1}^{(0)}(z|x) + \frac{W_1^{(0)}(|x) - W_1^{(0)}(|z)}{x-z} + W_0^{(0)}(x,z|x) \tag{4.21}$$

and

$$\begin{aligned} Y(x)W_1^{(0)}(x,z|x_1) = {} & \frac{d}{dx_1}\frac{W_1^{(0)}(x,z|) - W_1^{(0)}(x_1,z|)}{x-x_1} + P_{1,1}^{(0)}(x,z|x_1) + \Pi_0(z|x,x_1) \\ & - \frac{W_1^{(0)}(|x,x_1) - W_1^{(0)}(|z,x_1)}{x-z} + W_0^{(0)}(x,z|x,x_1) \\ & + W_0^{(0)}(x,z|x_1)W_1^{(0)}(|x) + W_0^{(0)}(x,z|)W_1^{(0)}(|x,x_1) + W_1^{(0)}(x,z|)W_0^{(0)}(|x,x_1) \end{aligned} \tag{4.22}$$

and the recursion

$$\boxed{\begin{aligned} Y(x)W_1^{(0)}(x,z|J) = {} & \sum_{i=1}^{j}\frac{d}{dx_i}\frac{W_1^{(0)}(x,z|J\backslash x_i) - W_1^{(0)}(x_i,z|J\backslash x_i)}{x-x_i} + P_{1,1}^{(0)}(x,z|J) + \Pi_0(z|x,J) \\ & + \sum_{I\subset J} W_0^{(0)}(x,z|I)W_1^{(0)}(|x,J\backslash I) + W_0^{(0)}(x,z|x,J) \\ & + \sum_{I\not\subseteq J} W_1^{(0)}(x,z|I)W_0^{(0)}(|x,J\backslash I) - \frac{W_1^{(0)}(|x,J) - W_1^{(0)}(|z,J)}{x-z}. \end{aligned}} \tag{4.23}$$

4.2.3 Order 3

$$\left(\Psi(x) - W_0^{(1)}(x|)\right)W_1^{(2)}(|x) = \Pi_{0,1}^{(3)}(|x) + W_0^{(2)}(|x)W_1^{(1)}(x|) \tag{4.24}$$

$$\begin{aligned} \left(\Psi(x) - W_0^{(1)}(x|)\right)W_1^{(2)}(|x,x_1) = {} & \Pi_{0,1}^{(3)}(|x,x_1) + W_0^{(2)}(|x,x_1)W_1^{(1)}(x|) \\ & + W_0^{(2)}(|x)W_1^{(1)}(x|x_1) + W_1^{(2)}(|x)W_0^{(1)}(x|x_1) \end{aligned} \tag{4.25}$$

Recursion:

$$\boxed{\begin{aligned} \left(\Psi(x) - W_0^{(1)}(x|)\right)W_1^{(2)}(|x,J) = {} & \Pi_{0,1}^{(3)}(|x,J) + \left(\sum_{I\subset J} W_0^{(2)}(|x,I)W_1^{(1)}(x|J\backslash I)\right) \\ & + \left(\sum_{I\not\subseteq J} W_1^{(2)}(|x,I)W_0^{(1)}(x|J\backslash I)\right). \end{aligned}} \tag{4.26}$$

4.2.4 Recursion for arbitrary g

Assuming we know the supercorrelators at $g-1$, we are able to determine all correlators at g following the same procedure.

1. The order 0 equation allows us to solve for all $W_g^{(0)}(|\dots)$.
2. The order 1 equation with the previous results gives us all $W_g^{(1)}(x|\dots)$.
3. Using the odd operator on the order 1 equation then gives $W_g^{(0)}(x, x_1|\dots)$.
4. Finally, using either the order 2 or order 3 equation gives $W_g^{(2)}(|\dots)$.

Please note that the first 3 steps are enough to solve the order 0 and order 1 equations for $W_g^{(0)}(|\dots)$, $W_g^{(0)}(x, x_1|\dots)$, and $W_g^{(1)}(x|\dots)$. See (3.80) and (3.81). Furthermore, we have confirmed that the two (master) superloop equations are enough to permit us to theoretically determine all superloop correlators and that the superloop correlators can be built up recursively.

4.3 A residue formulation?

This section is primarily a discussion on how one may be able to describe these recursive relations with a residue formulation similar to that found for the 1HMM and generalized in [20, 21].

One thought is to work with the complex coefficients of the ξs: thus we completely disregard the Grassmann parameters. Our notation will be as follows. For each order i in ξ, $i = 0, \dots, 3$, we introduce an oriented basis of the Grassmann algebra $\mathbb{G}_i$ from the set $\{\xi_1, \dots, \xi_{D+1}\}$. We will use a bar to denote the coefficients when the order in ξ is non-zero, that is

$$A^{(1)} = \sum_{i=1}^{D+1} \bar{A}_i^{(1)} \xi_i \tag{4.27}$$

$$A^{(2)} = \sum_{i=1}^{D} \sum_{j=i+1} \bar{A}_{ij}^{(1)} \xi_i \xi_j \tag{4.28}$$

$$A^{(3)} = \sum_{i<j<k} \bar{A}_{ijk} \xi_i \xi_j \xi_k. \tag{4.29}$$

We use this notation to extract the meromorphic coefficients of the superloop equations. With matrix models the equations are typically such that the recursion over g can be found first and then by repeated use of the loop insertion operator one gets a recursion over (g, n). However, the supereigenvalue model involves three inter-related functions and consequently our approach will be to attempt to first construct the recursion over n and order by order in ξ before we add to that the recursion over g.

4.3.1 The planar limit

We know that the free energy at order 0 in ξ is proportional to the free energy of the Hermitian matrix model. Furthermore, looking at (3.80), the planar limit involves only the correlators $W_0^{(0)}(|J)$. It turns out that we can express $W_0^{(0)}(|J)$ recursively in $|J|$ using the standard residue topological recursion.

Order 0 equation

Recall that

$$2V'(x)W_0^{(0)}(|x) - 2P_{0,0}^{(0)}(|x) = \left(W_0^{(0)}(|x)\right)^2 \tag{4.30}$$

with solution

$$W_0^{(0)}(|x) = V'(x) - \sqrt{(V'(x))^2 - 2P_{0,0}^{(0)}(|x)} \tag{4.31}$$

where the minus sign is needed because $W_0^{(0)}(|x)_{x\to\infty} \sim \frac{1}{x}$. Note that $P_{0,0}^{(0)}$ is a polynomial in x. Previously we defined

$$Y(x) \equiv V'(x) - W_0^{(0)}(|x) \tag{4.32}$$

which from now on is known. From here we define the spectral curve Σ_0

$$E(x, Y) := \{(x, Y) : Y^2 - (V'(x))^2 + 2P_{0,0}^{(0)}(|x) = 0\} \tag{4.33}$$

which parameterizes a hyper-elliptic surface Σ_0 corresponding to solutions of $W_0^{(0)}(|x)$. We introduce a homology basis, and define and identify ramification points in the same manner as was done for the 1HMM.

$W_0^{(0)}(|x, x_1)$ and fundamental bidifferential

We have seen that

$$Y(x)W_0^{(0)}(|x, x_1) = \frac{d}{dx_1}\frac{W_0^{(0)}(|x) - W_0^{(0)}(|x_1)}{x - x_1} + P_{0,0}^{(0)}(|x, x_1) \tag{4.34}$$

and we noted that this is of the same form as the two point correlator in the 1HMM model, hence is related to the Bergmann bidifferential. Over $\mathbb{C}$, there is no residue. However, we can turn this into a form $W^{(0)}\big(|x(p), x(q)\big)dx(p)dx(q)$ over $\Sigma_0 \ni p, q$. We can analyze this in the same manner as was done with $W_0(x, x_1)$ for the 1HMM. The result is $W^{(0)}\big(|x(p), x(q)\big)dx(p)dx(q) = -B(p, \bar{q})$.

Moving forward,

$$\begin{aligned} Y(x)W_0^{(0)}(|x, x_1, x_2) =& P_{0,0}^{(0)}(|x, x_1, x_2) + \frac{d}{dx_1}\frac{W_0^{(0)}(|x, x_2) - W_0^{(0)}(|x_1, x_2)}{x - x_1} \\ &+ \frac{d}{dx_2}\frac{W_0^{(0)}(|x, x_1) - W_0^{(0)}(|x_1, x_2)}{x - x_2} + W_0^{(0)}(|x, x_1)W_0^{(0)}(|x, x_2), \end{aligned} \tag{4.35}$$

which, as we already saw, has no pole on the RHS. Thus $W_0^{(0)}(|x(p), x(p_1), x(p_2))$ only has a pole as p approaches a ramification point. The pole structure is clearly the same as in the 1HMM, and therefore for $|J| \geq 2$, $W(|x(p), X(J))$ has poles only at the ramification points of Σ_0. Note we introduced the notation for $J \in (\Sigma_0)^k$, $X(J) = x(p_1), \ldots, x(p_k)$.

If we define on Σ_0

$$\begin{aligned} \omega_0^{(0)}(|p) &= -y(p)dx(p) \\ \omega_0^{(0)}(|p, q) &= B(p, q) \\ \omega_0^{(0)}(|p_1, \ldots) &= dx(p_1)) \cdots \frac{\partial}{\partial V(p_1)} \cdots F_0^{(0)} \end{aligned}$$

where $\frac{\partial}{\partial V(p)} = \pm\frac{\partial}{\partial V(x(p))}$ for $x \in \chi_\pm \subset \Sigma_0$ then we have

$$\omega_0^{(0)}(|p, J) = \sum_{a_i} \mathrm{Res}_{q \to a_i} K(p, q)\Bigg(\sum_{\emptyset \not\subseteq I \not\subseteq J} W_0^{(0)}(q, I)W_0^{(0)}(\bar{q}, J\backslash I)\Bigg) \tag{4.36}$$

where $K(p, q)$ is the same as in (2.80).

Higher Order

The equation for $W_g^{(0)}(|x, J)$ (3.80) involves $W_{g-1}^{(0)}(x, z|J)$ which leads to difficulty. If we work with $g = 1$ we have

$$V'(x)W_1^{(0)}(|x, J) - P_{0,1}^{(0)}(|x, J) + \sum_i \frac{d}{dx_i}\frac{W_1^{(0)}(|x, J\backslash x_i) - W_1^{(0)}(|J)}{x_i - x}$$
$$= \frac{1}{2}\Bigg(\sum_{I\subset J} W_0^{(0)}(|x, I)W_1^{(0)}(|x, J\backslash I) + W_1^{(0)}(|x, I)W_0^{(0)}(|x, J\backslash I) \qquad (4.37)$$
$$+ W_0^{(0)}(|x, x, J) - \frac{d}{dz}W_0^{(0)}(x, z|J)|_{z=x}\Bigg).$$

Using (4.38) at g=0 gives

$$Y(x)W_0^{(0)}(x, z|J) = P_{1,0}^{(0)}(x, z|J) + \Pi_0(z|x, J) + \sum_{i=1}^{j} \frac{d}{dx_i}\frac{W_0^{(0)}(x, z|J\backslash x_i) - W_0^{(0)}(x_i, z|J\backslash x_i)}{x - x_i}$$
$$- \frac{W_0^{(0)}(|x, J) - W_0^{(0)}(|z, J)}{x - z} + \sum_{I\not\subseteq J} W_0^{(0)}(x, z|I)W_0^{(0)}(|x, J\backslash I). \qquad (4.38)$$

Observe that

$$Y(x)W_0^{(0)}(x, x_1|) = P_{1,0}^{(0)}(x, x_1|) - \frac{W_0^{(0)}(|x) - W_0^{(0)}(|x_1)}{x - x_1} + \Pi_{0,0}^{(0)}(x_1|x), \qquad (4.39)$$

implies $W_0^{(0)}(x, x_1|)$ has poles only at the zeroes of Y. However, if we attempt to parameterize this on Σ_0 there is a pole as $p \to \bar{p}_1$. Hence its existence in the expression for $W_g^{(0)}(|x, J)$ where its derivative is computed gives another fundamental bidifferential. How can we have two fundamental bidifferentials, because clearly the two classes of correlators $W(x, z|\dots)$ and $W(|\dots)$ are different?

Furthermore, upon analyzing the equations in higher orders in ξ we find the presence of the fundamental bidifferential again. Consequently, it is ambiguous how to interpret different classes of correlators as existing on the same algebraic surface.

4.3.2 No end in sight

Another example of the ambiguous nature comes from the order 1 equation. First we extract the coefficient of ξ_k in the order 1 equation:

$$Y(x)\bar{W}^{(1)}_{k,0}(x|J) = \bar{P}^{(1)}_{k,1,0}(x|J) - x^k W^{(0)}_0(|x,J) + \bar{\Pi}^{(1)}_{k,0,0}(|x,J) + \sum_{I \not\subseteq J} \bar{W}^{(1)}_{k,0}(x|I) W^{(0)}_0(|x, J\backslash I) + \sum_i \frac{d}{dx_i} \frac{\bar{W}^{(1)}_{k,0}(x|J\backslash x_i) - \bar{W}^{(1)}_{k,0}(x_i|J\backslash x_i)}{x - x_i}. \tag{4.40}$$

where $x^k W^{(0)}_0(|x,J) = \bar{\Psi}_k(x) W^{(0)}_0(|x,J)$. In one parameter we have

$$Y(x)\bar{W}^{(1)}_{k,0}(x|) = \bar{P}^{(1)}_{k,1,0}(x|) - x^k W^{(0)}_0(|x) + \bar{\Pi}^{(1)}_{k,0,0}(|x) := U(x). \tag{4.41}$$

If by complete analogy we define a surface corresponding to the solutions of $\bar{W}^{(1)}_{k,0}(x|)$, we would have

$$\Sigma_1 = \{p = (x, Y) : Y\bar{W}^{(1)}_{k,0}(x|) - U(x) = 0\}. \tag{4.42}$$

But this relation involves both Y and $W^{(0)}_0(|x)$ which have a relation that defines Σ_0. So it would appear that an element of Σ_1 is a point in Σ_0 that satisfies the additional relationship (4.41). Of course, this can be interpreted as intersecting the two relationships (4.41) and (4.33). However, both these objects are zero-sets, and therefore we have an infinite number of ways to combine them. Additionally, we really have $D+1$ different expressions from each coefficient corresponding to $D+1$ curves and it is unclear how to unify them all.

If we continue introducing higher order equations and extracting the coefficients, the basic integrated analytic structure makes identifying the poles very challenging. Furthermore, in the end it is not obvious how we could define an algebraic curve upon which could exist all four equations and the relationships between them. While we were hoping that working with the coefficients would provide clarity, the more we calculate and the further we journey, the more convinced we are that this approach is certainly not ideal.

CHAPTER 5

Conclusion

When first addressing the possibility of generalizing the topological recursion to calculate the free energy of the supereigenvalue model, one of the earliest thoughts was to work on a supermanifold. Maybe this seems like a daunting task, one characterized by little and non-uniform information regarding a supermanifold. However, there are many problems of interest related to supersymmetry, for example superstring theory. String theory is inherently related to the study of the moduli-space of curves, so perhaps superstring theory is related to the moduli superspace of supercurves, whatever they may be[1]!

With regard to the supereigenvalue model, we have seen explicitly the direct relationship between the free energy of the model with that from the 1HMM. In general, the non-Grassmann component of the partition function is so closely related to the 1HMM that there is strong reason to expect the topological recursion, or some variation of the recursion, ought to be applicable to the supereigenvalue model. Indeed, corollary 3.2.2 following theorem 3.2.1 gives $F_S^{(0)} = 2F_H$ from which we expect that the non-Grassmann correlators ought to be directly identified with the correlators from the 1HMM. This simple observation which suggests a simple relation is evidence that our approach of disregarding the Grassmann numbers is faulty. Somehow it seems that $W(\dots)$ in the 1HMM ought to match up with the two classes $W(x, z|\dots)$ and $W(|\dots)$ in the SEV model. While our work has demonstrated a vague relationship with the correlators, the difficulty stems from the fact that the class $W(x, z|\dots)$ ultimately comes from the odd loop equation.

The topological recursion requires the universe of an algebraic curve defined by whichever equation is sufficient to define the dynamics of the model under consideration. For the supereigenvalue model we would need some universe defined by our two superloop equations,

[1] Dr. Bouchard recently attended a conference on string theory where Dr. Witten gave a talk discussing his interest in defining these objects.

and it is natural to imagine that such a space would take the form of a supersymmetric surface. The objects generated by the recursion are differential forms, so the next step is to understand thoroughly what supersymmetric differential forms are and how they behave. Furthermore, the essential criteria of the recursion is the simple pole structure of these forms, but how can one generalize the notion of the pole of a Grassmann function?

In conclusion, in order to pursue a purely supersymmetric generalization of the topological recursion, one ought make well-defined the notions of integrability, differential forms, poles and residues of supersymmetric objects. While the formalism that would go into making these notions precise may be tedious at first, perhaps the end result would be a simple and elegant expression of some more general recursion that could collapse to the powerful, and now more familiar, topological recursion.

APPENDIX A

Matrix Theory

A.1 The Vandermonde matrix

The standard Vandermonde matrix X is given by

$$X = X_{ij} = (x_i^{j-1})_{i,j=1}^{N} \tag{A.1}$$

and its determinant is

$$\begin{aligned} \det(X) = \det(x_i^{j-1})_{i,j=1}^{N} &= \sum_{\sigma \in S_N} (-1)^{\sigma} x_i^{\sigma(i)-1} \\ &= \prod_{1 \le i < j \le N} (x_j - x_i) := \Delta(x) \end{aligned} \tag{A.2}$$

where the sum is over the symmetry group S_N and $(-1)^{\sigma} \equiv \text{sign}(\sigma)$. The eigenvalue representation of the 1HMM model includes the square of the Vandermonde determinant:

$$\prod_{1 \le i < j \le N} (\lambda_i - \lambda_j)^2 = \prod_{i<j} (\lambda_j - \lambda_i)^2 = \Delta^2(\lambda). \tag{A.3}$$

A.2 Vanishing integral

We know that the determinant of a matrix with repeated row (or columns) vanishes. Something similar happens when we integrate over the determinant of a matrix. Let X be an $n \times n$ matrix such that for some $k, i \neq k$, $X_{i,j} = f_j(x_i)$, and $X_{k,j} = f_j(x_k)$ for $j = 1, \ldots, n$.

To visualize this, without loss of generality let $i = 1$ and $k = 2$ so that

$$X = \begin{pmatrix} f_1(x_1) & f_2(x_1) & ... & f_{n-1}(x_1) & f_n(x_1) \\ f_1(x_2) & f_2(x_2) & ... & f_{n-1}(x_2) & f_n(x_2) \\ X_{3,1} & X_{3,2} & ... & X_{3,n-1} & X_{3,n} \\ X_{4,1} & X_{4,2} & ... & X_{4,n-1} & X_{4,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ X_{n,1} & X_{n,2} & ... & X_{n,n-1} & X_{n,n} \end{pmatrix}.$$

For such an X, we have

$$\int dx_1 \dots dx_n \det(X) = 0. \tag{A.4}$$

This follows immediately from the familiar Laplace's cofactor algorithm for computing a determinant. Let $M_{i,j}$ the the minor matrix created by removing the ith row and jth column from X.

$$\begin{aligned} \det(X) &= \sum_{i=1}^{n} (-1)^{i+1} f_i(x_1) M_{1,i} = \sum_{i=1}^{n} (-1)^{i+2} f_i(x_2) M_{2,i} \\ \Rightarrow \det(X) &= \frac{1}{2} \sum_{i=1}^{n} (-1)^{i+1} \Big(f_i(x_1) M_{1,i} - f_i(x_2) M_{2,i} \Big) \end{aligned} \tag{A.5}$$

Of course, $f_i(x_1)M_{1,i}$ and $f_i(x_2)M_{2,i}$ only differ by exchanging $x_1 \leftrightarrow x_2$. Therefore integrating $\det(X)$ over x_1 and x_2 yields zero. We make use of this result in our proof of Theorem 3.2.1.

A.3 Pfaffians

The Pfaffian is defined for a skew-symmetric $2N \times 2N$ matrix A.

$$\text{Pfaffian}(A) = \frac{1}{2^N N!} \sum_{\sigma \in S_{2N}} (-1)^{\sigma} \prod_{i=1}^{N} A_{\sigma(2i-1),\sigma(2i)}. \tag{A.6}$$

The Pfaffian is related to the determinant through

$$\det(A) = \text{Pfaffian}(A)^2. \tag{A.7}$$

The proof of theorem 2.2 makes use of some properties of Pfaffians. For any $2N \times 2N$ matrix M,

$$\text{Pfaffian}(MAM^T) = \det(M)\text{Pfaffian}(A). \tag{A.8}$$

Let B be a block diagonal skew-symmetric matrix, i.e., $B_{2i-1,2i} = -B_{2i,2i-1} \neq 0$ and $B_{ij} = 0$ for all other i, j. It is easy to compute the Pfaffian of a block diagonal matrix. This is analogous to calculating the determinant of a diagonal matrix.

$$\text{Pfaffian}(B) = \text{Pfaffian}\begin{pmatrix} 0 & a & 0 & \dots & \dots & \dots & \dots & 0 \\ -a & 0 & \dots & \dots & \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & b & 0 & \dots & \dots & \vdots \\ 0 & 0 & -b & 0 & \dots & \dots & \dots & \vdots \\ \vdots & \vdots & & \vdots & \ddots & \vdots & & \\ 0 & \dots & \dots & \dots & \dots & \dots & 0 & z \\ 0 & \dots & \dots & \dots & \dots & \dots & -z & 0 \end{pmatrix} = ab \dots z.$$

One result of the spectral theorem is that for a given skew-symmetric matrix A, there is a block diagonal skew-symmetric matrix B and an orthogonal matrix O such that

$$A = OBO^T. \tag{A.9}$$

By (A.8), it follows that Pfaffian(A) = Pfaffian(B).

APPENDIX B

Grassmann Numbers

Grassmann numbers anti-commute amongst themselves and commute with complex numbers. Following [23] , the Grassmann algebra has as basis $(\theta_1, \dots, \theta_{2N})$ with $\{\theta_i, \theta_j\} = 0$ and $[x, \theta_k] = 0$ for $x \in \mathbb{C}$. We adopt the following convention for Grassman integration. With one Grassmann parameter we have

$$\begin{aligned} \int d\theta &= 0 \\ \int d\theta\ \theta &= 1. \end{aligned} \tag{B.1}$$

For higher dimensional integrals:

$$\int \left(d\theta_1 \dots d\theta_n\right) \left(\theta_1 \dots \theta_n\right) = 1, \tag{B.2}$$

that is we think of $d\theta_1 \dots d\theta_n$ as $d\theta$ and, similarly, think of $\theta_1 \dots \theta_n$ as θ. Thus, for $\sigma \in S_n$,

$$\int \left(d\theta_1 \dots d\theta_n\right) \left(\theta_{\sigma(1)} \dots \theta_{\sigma(n)}\right) = (-1)^{\sigma}. \tag{B.3}$$

Another way of looking at it is that we write $\int d\theta_1 \dots d\theta_n\ A\ \theta_1\theta_n$ as a convenient form of $\int d\theta_n \dots d\theta_1\ A\ \theta_1 \dots \theta_n$ so that in the supereigenvalue model we have

$$\int \prod_{i=1}^{n} d\lambda_i d\theta_i\ f(\lambda_i)\theta_1 \dots \theta_n = \int \prod_i d\lambda_i f(\lambda_i). \tag{B.4}$$

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