UNIVERSITY OF ALBERTA

Steinberg characters for Chevalley groups over rings

by

Peter Steven Campbell ©

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To my wife

Abstract

Lees, and independently Hill, described an irreducible character of the general linear group over a finite local ring which can be considered as an analogue of the Steinberg character in the case of a finite field. The aim of this thesis is to extend the construction of the analogue to the Chevalley groups over finite local rings and to investigate the properties of the resultant character.

We begin by defining the analogue of the Steinberg character for the Chevalley groups extended by the diagonal automorphisms. The analogue is given as a virtual character with an expression as an alternating sum of permutation characters. However, we show that it is actually the character afforded by a particular module. Indeed, we prove that its alternating sum formula is a consequence of it being afforded by the top homology space of a simplicial complex similar to Solomon's combinatorial building. Further, by carefully examining the double cosets we are able to prove that the analogue is irreducible and has a characterisation identical to Curtis' characterisation of the Steinberg character. Additionally, we show that it can be described in terms of a linear character of the Hecke algebra.

Unfortunately, when we examine the analogue for the Chevalley groups themselves we find that it is irreducible only when the Chevalley group agrees with its extended version. Consequently, we determine its decomposition into distinct irreducible constituents. Finally we show that these constituents can be characterised by analogues of the Gelfand-Graev character.

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Contents

1 Introduction **1**

1.1 The Steinberg character 1

1.2 Analogues of the Steinberg character 3

1.3 Main results 4

1.4 Character theory of finite groups 9

2 Chevalley groups over fields **12**

2.1 Root systems 12

2.2 Simple Lie algebras over $\mathbb{C}$ 16

2.3 Chevalley groups over fields 19

2.4 Extension by the diagonal automorphisms 22

3 Chevalley groups over rings **27**

3.1 Local fields 27

3.2 Chevalley groups over finite local rings 31

3.3 Extended Chevalley groups over finite local rings 35

3.4 Congruence subgroups 37

4 Construction **41**

4.1 Parabolic subgroups 42

4.2 Module affording St_ℓ 45

4.3 Induction from a Steinberg character 49

5 Homology **53**

5.1 Definitions 53

5.2 Combinatorial building for $\widehat{G}$ 55

5.3 Homology spaces of $\Omega_{\widehat{H}}$ 58

6 Characterisation **62**

6.1 Example: $\mathrm{PGL}_2(R)$ 62

6.2 Double coset structure of $\widehat{H}$ 65

6.3 Characterisation of St_ℓ 71

6.4 Proof of Theorem 6.3.1 75

7 Hecke algebras **83**

7.1 Definitions and standard results 83

7.2 Example: $\mathrm{PGL}_2(R)$ 85

7.3 Hecke algebra of $\widehat{H}$ 88

7.4 Extending to the Hecke algebra of $\widehat{G}$ 91

8 Restriction **95**

8.1 Restriction to the Chevalley group 95

8.2 Example: $\mathrm{PSL}_2(R)$, char $\kappa \neq 2$ 98

8.3 Double coset structure of H 102

8.4 Reducibility of St'_ℓ 107

9 Decomposition **108**

9.1 Example: $\mathrm{PSL}_2(R)$, char $\kappa \neq 2$ 108

9.2 Clifford theory for χ' 113

9.3 Decomposition of St'_ℓ 116

9.4 Gelfand-Graev characters for G 117

A Irreducible root systems **125**

B Proof of Proposition 6.4.7 **129**

C Proof of Lemma 6.4.9 **136**

List of Tables

2.1 The Dynkin diagrams of the irreducible root systems 15

2.2 The determinants of the Cartan matrices 24

2.3 The structure of Λ/Λ_r 24

2.4 The index $d = [\widehat{T}(\mathbb{F}_q) : T(\mathbb{F}_q)]$ 24

B.1 The roots Σ_2, Σ_3, Σ_5, Σ_6 and Σ_7 for $\Sigma = F_4$ 134

B.2 The roots in $\Sigma_4 - \{\gamma'\}$ for $\Sigma = F_4$ 134

C.1 The decompositions and structure constants for $\mathcal{S} = \Sigma_4$ in F_4 138

C.2 The decompositions and structure constants for $\mathcal{S} = \Sigma_4$ in E_6 139

C.3 The decompositions and structure constants for $\mathcal{S} = \Sigma_9$ in E_7 140

C.4 The decompositions and structure constants for $\mathcal{S} = \Sigma_6$ in E_8 141

C.5 The decompositions and structure constants for $\mathcal{S} = \Sigma_{10}$ in E_8 142

C.6 The decompositions and structure constants for $\mathcal{S} = \Sigma_{15}$ in E_8 143

List of Symbols

A_n	15, 125
$A_{\alpha,\beta}$	14
ad_x	16
$B(K)$	22
$\widehat{B}(K)$	25
$\widehat{B}(\mathfrak{m}^i)$	38
$B(\mathfrak{o})$	35
$\widehat{B}(\mathfrak{o})$	37
$B(R) = B$	35
$\widehat{B}(R) = \widehat{B}$	37
B_i	100
$\widehat{B}_i$	42
B_n	15, 126
β_g	83
β_i	85
β_J	88
β_α	88
c_i	85
$c_{i,j,\alpha,\beta}$	20
$C_k(\Omega)$	53
$\widetilde{C}_k(\Omega)$	55
C_n	15, 126
χ	49
χ'	97
χ_i	115
d	24, 98
d_J	102
D_n	15, 126
$\mathfrak{D}_G(H, K)$	11
∂_k	53
$\widetilde{\partial}_k$	55
$\Delta(\Omega)$	53
e	48, 49
E_6	15, 127
E_7	15, 127
E_8	15, 128
e_H	45
e_α	18

ϵ	98
η	31
η_i	37
F_4	15, 128
G_2	15, 128
Γ_i	119
$G(K)$	19
$\widehat{G}(K)$	25
$G(\mathfrak{o})$	32
$G(R) = G$	32
$\widehat{G}(R) = \widehat{G}$	37
H	97
$\widehat{H}$	49
$h(\mu),\ \overline{h}(\overline{\mu})$	22, 36
H_J	96
$\widehat{H}_J$	44
$H_k(\Omega)$	54
$\widetilde{H}_k(\Omega)$	55
h_α	17
H_α	96
$\widehat{H}_\alpha$	42
$h_\alpha(r),\ \overline{h}_\alpha(r)$	21, 34
$\mathfrak{h}$	16
$\mathrm{ind}(g)$	84
J_i	55
K	28
K_i	37
$\widehat{K}_i$	38
k_α	13
$\mathcal{K}$	91
κ	29
ℓ	29
$\mathfrak{L}$	16
$\mathfrak{L}_K$	19
$\mathfrak{L}_\mathfrak{o}$	31
$\mathfrak{L}_R$	31
$\mathfrak{L}_\mathbb{Z}$	19
$\mathfrak{L}_\alpha$	17
Λ	23
λ_i	115
Λ_r	22
λ_r	108
λ_α	23
$\mathfrak{m}$	29
μ	22
$\widehat{N}$	50
$N(K)$	22
$n_w,\ \overline{n}_w$	22, 35

$n_\alpha(r),\ \overline{n}_\alpha(r)$	$21, 34$	σ_J	50
$N_{\alpha,\beta}$	18	Σ_k	14
$\mathfrak{o}$	29	σ_α	49
Ω	$53, 55$	$T(K)$	22
$\Omega_{\widehat{H}}$	57	$\widehat{T}(K)$	23
Ω_i	59	$T(\mathfrak{m}^i)$	37
$\Omega * \Omega'$	58	$\widehat{T}(\mathfrak{m}^i)$	38
P	95	$T(\mathfrak{o})$	35
$\widehat{P}$	42	$\widehat{T}(\mathfrak{o})$	35
$\mathfrak{p}$	29	$T(R) = T$	35
ϕ	90	$\widehat{T}(R) = \widehat{T}$	36
$\phi_\alpha,\ \overline{\phi}_\alpha$	$21, 34$	Θ	118
Π	13	θ	118
π	29	Θ_i	118
ψ	$87, 92$	θ_i	119
q	29	$U(K)$	20
R	29	$U^-(K)$	20
$R_2^\times$	98	$U(\mathfrak{m}^i)$	38
S	42	$U^-(\mathfrak{m}^i)$	38
St_ℓ	45	$U(\mathfrak{o})$	33
St'_ℓ	97	$U(R) = U$	33
Σ^+	14	$U^-(R) = U^-$	33
Σ	$13, 17$	$U_\alpha(K)$	20
Σ^-	14	$U_\alpha(\mathfrak{m}^i)$	37

$U_\alpha(\mathfrak{o})$	32
$U_\alpha(R) = U_\alpha$	32
V	66
$\widetilde{V}$	108, 113
W	13
w_0	14
w_α	12
X_J	44
x_J	67
$x_J^{(i)}$	103
X_α	42
$x_\alpha(r),\ \overline{x}_\alpha(r)$	19, 32
$x_\alpha(\xi)$	18
$\mathcal{X}$	113
$\mathcal{X}_i$	115
$\mathfrak{x}_J$	66
$\mathfrak{x}_J^{(i)}$	102
$\overline{y}_\alpha(r),\ y_\alpha(r)$	36
ζ_i	116
ζ_r	110

Chapter 1

Introduction

Let $\mathfrak{o}$ be the ring of integers of a non-archimedean local field K with maximal ideal $\mathfrak{p}$ and finite residue class field $\kappa = \mathfrak{o}/\mathfrak{p}$ of order q. For a fixed integer $\ell > 1$ the quotient ring $R = \mathfrak{o}/\mathfrak{p}^\ell$ is a finite local ring with unique maximal ideal $\mathfrak{m} = \mathfrak{p}/\mathfrak{p}^\ell$. Further, $\mathfrak{m}$ is principal and if π denotes its generator, then every ideal of R is of the form $\mathfrak{m}^i = \pi^i R$ for some $0 \leq i \leq \ell$.

We will be interested in the Chevalley group $G = G(R)$ over R and its extension $\widehat{G} = \widehat{G}(R)$ by the diagonal automorphisms. More specifically, we will be considering characters of G and $\widehat{G}$ which can be regarded as analogues of the Steinberg character in the finite field case.

1.1 The Steinberg character

In 1951, Steinberg [29] described all of the irreducible characters of the general linear group over a finite field appearing as constituents of the permutation character over the subgroup B of upper triangular matrices. One of these characters, now called the Steinberg character, was of particular interest as its degree was the same as the contribution of the field characteristic to the order of the group. Later, using different methods he constructed a character with this same property for the classical groups [30] and then the Chevalley groups [31] over finite fields.

In 1966, Curtis [4] then gave a description of an irreducible character for the finite groups with BN-pair which coincided with the Steinberg character in the case of the Chevalley groups over finite fields. In particular, this character was expressed as the alternating sum of permutation characters over parabolic subgroups corresponding to Solomon's alternating sum formula for the sign character of its Weyl group W [27]. Further, it had a characterisation as the unique irreducible constituent of the permutation character over B which was not a constituent of the permutation character over any strictly larger parabolic subgroup.

Solomon [28] also showed that the alternating sum formula for the Steinberg character had a homological origin by considering the combinatorial building of a Chevalley group over a finite field. This was the simplicial complex whose vertices were the left cosets of the maximal parabolic subgroups containing B, and whose simplices were sets of left cosets which had non-empty intersection. The permutation action of the group on the left cosets then gave rise to an action on the homology spaces. These were zero except for the bottom homology space which afforded the trivial character and the top homology space which afforded the Steinberg character. The alternating sum formula was then obtained from the Hopf Trace Formula. This approach was later refined and generalised by Curtis and Lehrer [8] and then Curtis, Lehrer and Tits [9].

An alternative construction of the Steinberg character was given by Curtis, Iwahori and Kilmoyer [7] in terms of the Hecke algebra over B, i.e. the endomorphism algebra of the permutation module over B. Iwahori [19] had shown for Chevalley groups over finite fields, and Matsumoto [23] for finite groups with BN-pair, that the Hecke algebra had a presentation in terms of generators and relations which was similar to the presentation of the Weyl group as a Coxeter group. Thus, it was possible to define a linear character of the Hecke algebra in the same way as the sign character of W. This linear character then corresponded to a unique irreducible constituent of the permutation character over B which was the Steinberg character.

For more details about the Steinberg character for Chevalley groups over finite fields see [17] or the note by Steinberg himself at the end of [33].

1.2 Analogues of the Steinberg character

If we now consider the general linear group $\mathrm{GL}_n(R)$ over R with $\ell > 1$, then the subgroup B of upper triangular matrices no longer forms part of a BN-pair for the group. Thus, the standard constructions of the Steinberg character do not hold in this situation. However, it is still possible to consider characters which are analogues of the Steinberg character.

The first such analogue S_G was constructed by Lees in [20] and [21]. He defined S_G inductively as an alternating sum of characters obtained from similar characters of general linear groups of smaller rank. This virtual character was then shown to be a constituent of the permutation character over B, except possibly when $q = 2$, which was expressible as an alternating sum of permutation characters. Further, the degree of S_G was the highest power of char κ dividing the order of the group. However, in general S_G was not irreducible.

The second analogue St_ℓ was described by Lees [20] and independently by Hill [15]. Although the degree of St_ℓ was not a power of char κ, it was an irreducible constituent of the permutation character over B. The definition given by Lees was as the character afforded by the top homology space of a simplicial complex analogous to the combinatorial building. Consequently, St_ℓ again had an expression as an alternating sum of permutation characters.

Hill defined the analogue in a different way, however. He was primarily interested in regular characters of $\mathrm{GL}_n(R)$ and an analogue of the Gelfand-Graev character Γ_0. The analogue St_ℓ of the Steinberg character was then the unique irreducible constituent of both Γ_0 and the permutation character over B. In particular, this meant that St_ℓ was the unique regular character in the permutation character over B.

We will be considering the second analogue St_ℓ of the Steinberg character. Our main aim will be to extend the construction of St_ℓ to the Chevalley and extended Chevalley groups over R and to investigate the properties of the resultant characters.

1.3 Main results

Let G be the Chevalley group over R corresponding to the irreducible root system Σ, and $\widehat{G}$ its extension by the diagonal automorphisms. Further, suppose that Π is a base for Σ with positive roots Σ^+ and negative roots Σ^-. For each $\alpha \in \Sigma$ and $r \in R$, denote by $x_\alpha(r)$ the usual generator of G with root subgroups $U_\alpha(\mathfrak{m}^i) = \{x_\alpha(r) : r \in \mathfrak{m}^i\}$ for $0 \le i \le \ell - 1$. Then we take $B = TU$ and $\widehat{B} = \widehat{T}U$ where T and $\widehat{T}$ are the diagonal subgroups of G and $\widehat{G}$ respectively and $U = \langle U_\alpha(R) : \alpha \in \Sigma^+ \rangle$.

We begin by defining the analogue St_ℓ of the Steinberg character for the extended Chevalley group $\widehat{G}$ over R. As both Steinberg's original construction [29] for the general linear group and Curtis' later extension [4] to all finite groups with BN-pair were given in terms of permutation characters over parabolic subgroups, we consider the parabolic subgroups in our situation. These are taken to be the subgroups which contain $\widehat{B}$ and can be constructed using the root subgroups of $\widehat{G}$.

In particular, we find that the minimal parabolic subgroups which strictly contain $\widehat{B}$ are of the form $\widehat{H}_\alpha = X_\alpha \widehat{B}$ (Proposition 4.1.4) for some $\alpha \in S = \{\beta \in \Sigma : -\beta \in \Pi\}$ where $X_\alpha = U_\alpha(\mathfrak{m}^{\ell-1})$. Then, if we set $\widehat{H}_J = \langle \widehat{H}_\alpha : \alpha \in J \rangle$ for non-empty subsets

$J \subseteq S$ and $\widehat{H}_\emptyset = \widehat{B}$, the analogue St_ℓ is defined to be the virtual character

$$\mathrm{St}_\ell = \sum_{J \subseteq S} (-1)^{|J|} (1_{\widehat{H}_J})^{\widehat{G}}.$$

In fact, St_ℓ is the character afforded by the $\mathbb{C}\widehat{G}$-module $\mathbb{C}\widehat{G}e$ where e is the idempotent

$$e = \sum_{J \subseteq S} (-1)^{|J|} e_{\widehat{H}_J}$$

with $e_{\widehat{H}_J} = |\widehat{H}_J|^{-1} \sum_{g \in \widehat{H}_J} g$ for each $J \subseteq S$ (Theorem 4.2.6).

Further, if we consider the parabolic subgroup $\widehat{H} = \widehat{H}_S$ of $\widehat{G}$, then when $q \neq 2$ we see that $\widehat{B}$ forms part of a BN-pair for $\widehat{H}$ (Theorem 4.3.4(i)) and St_ℓ is induced from the Steinberg character

$$\chi = \sum_{J \subseteq S} (-1)^{|J|} (1_{\widehat{H}_J})^{\widehat{H}}$$

of $\widehat{H}$ (Corollary 4.3.5).

In Chapter 5 we also show that the alternating sum formula for St_ℓ has a homological origin. We consider a simplicial complex $\Delta(\Omega)$ which is an analogue of the combinatorial building of $\widehat{G}$. Its vertices are the left cosets in $\widehat{G}$ of the parabolic subgroups $\widehat{H}_J$ for maximal proper subsets $J \subset S$ and a set of left cosets is a simplex if and only if the intersection of the cosets is non-empty. Then the permutation action of $\widehat{G}$ on the left cosets gives an action on the homology spaces.

In particular, if we consider the subcomplex $\Delta(\Omega_{\widehat{H}})$ obtained by taking only the left cosets in $\widehat{H}$, then we find that the homology spaces of $\Delta(\Omega)$ are induced from the homology spaces of $\Delta(\Omega_{\widehat{H}})$ (Theorem 5.2.5). Further, the k-th homology space of $\Delta(\Omega_{\widehat{H}})$ is zero except for $k = 0$ and $k = n - 1$ (Proposition 5.3.2). The bottom homology space affords the trivial character of $\widehat{H}$ (Lemma 5.3.3), while the Hopf Trace Formula implies that the top homology space affords χ (Theorem 5.3.4). Consequently, the top homology space of $\Delta(\Omega)$ affords St_ℓ (Corollary 5.3.5).

To prove that St_ℓ is irreducible we need to determine the inner product $(\mathrm{St}_\ell, \mathrm{St}_\ell)$. However, since St_ℓ is given by an alternating sum of permutation characters, we obtain

$$(\mathrm{St}_\ell, \mathrm{St}_\ell) = \sum_{I,J \subseteq S} (-1)^{|I|+|J|} |\mathcal{D}_{\widehat{G}}(\widehat{H}_I, \widehat{H}_J)|$$

where $D_{\widehat{G}}(\widehat{H}_I, \widehat{H}_J)$ denotes the set of $(\widehat{H}_I, \widehat{H}_J)$-double cosets in $\widehat{G}$. Thus, in Chapter 6 we examine the double coset structure of $\widehat{G}$.

There seems to be a natural distinction between the $(\widehat{B}, \widehat{B})$-double cosets of $\widehat{G}$ that are contained in $\widehat{H}$ and those that are not. We find that $(\widehat{B}, \widehat{B})$-double cosets in $\widehat{H}$ are of the form $\widehat{B} x_I \widehat{B}$ for some $I \subseteq S$ (Proposition 6.2.3) where

$$x_I = \prod_{\alpha \in I} x_\alpha(\pi^{\ell-1}).$$

Similarly, for $J, J' \subseteq S$ the $(\widehat{H}_J, \widehat{H}_{J'})$-double cosets are of the form $\widehat{H}_J x_I \widehat{H}_{J'}$ for some $I \subseteq S - (J \cup J')$ (Proposition 6.2.4). Thus, by counting the double cosets we see that for each $J \subseteq S$

$$(\chi, (1_{\widehat{H}_J})^{\widehat{H}}) = \begin{cases} 1 & \text{if } J = \emptyset; \\ 0 & \text{otherwise} \end{cases} \qquad \text{(Theorem 6.2.6)}$$

and so χ is an irreducible constituent of $(1_{\widehat{B}})^{\widehat{H}}$ (Corollary 6.2.7).

Unfortunately, the double cosets that are not contained in $\widehat{H}$ have a more complicated structure and it does not seem possible to explicitly describe them in general. However, it turns out that it suffices to show that each $(\widehat{B}, \widehat{B})$-double coset not contained in $\widehat{H}$ is actually an $(\widehat{H}_\alpha, \widehat{B})$-double coset for some $\alpha \in S$ which depends only on the corresponding $(\widehat{H}, \widehat{H})$-double coset (Theorem 6.3.1). This then allows us to pair up the double cosets in such a way that they cancel in the alternating sum (Theorem 6.3.3). Thus we find that for each $J \subseteq S$

$$(\mathrm{St}_\ell, (1_{\widehat{H}_J})^{\widehat{G}}) = (\chi, (1_{\widehat{H}_J})^{\widehat{H}}) \qquad \text{(Corollary 6.3.4)}$$

and St_ℓ is consequently an irreducible constituent of $(1_{\widehat{B}})^{\widehat{G}}$. In fact, this allows us to characterise St_ℓ as the unique irreducible constituent of $(1_{\widehat{B}})^{\widehat{G}}$ which is not a constituent of $(1_{\widehat{P}})^{\widehat{G}}$ for any parabolic subgroup $\widehat{P}$ strictly containing $\widehat{B}$ (Theorem 6.3.6).

The final construction of St_ℓ that we consider is in terms of the Hecke algebra $\mathcal{H}(\widehat{G},\widehat{B}) = e_{\widehat{B}}\mathbb{C}\widehat{G}e_{\widehat{B}}$ of $\widehat{G}$. For $\widehat{H}$, the Hecke algebra $\mathcal{H}(\widehat{H},\widehat{B})$ over $\widehat{B}$ is generated by the basis elements β_α corresponding to the $(\widehat{B},\widehat{B})$-double cosets $\widehat{B}x_\alpha(\pi^{\ell-1})\widehat{B}$ together with the quadratic relations

$$(\beta_\alpha - (q-1))(\beta_\alpha + 1) = 0$$

and the homogeneous relations $\beta_\alpha\beta_{\alpha'} = \beta_{\alpha'}\beta_\alpha$ (Corollary 7.3.2). Thus we can define a linear character ϕ of $\mathcal{H}(\widehat{H},\widehat{B})$ by

$$\phi(\beta_J) = (-1)^{|J|}$$

for each $J \subseteq S$, where β_J is the basis element corresponding to the $(\widehat{B},\widehat{B})$-double coset $\widehat{B}x_J\widehat{B}$ (Proposition 7.3.3). The linear character ϕ then extends uniquely to a linear character ψ of $\mathcal{H}(\widehat{G},\widehat{B})$ (Theorem 7.4.2) which in turn corresponds to a unique irreducible character of G. This irreducible character is the analogue St_ℓ of the Steinberg character (Lemma 7.4.3). Indeed, the idempotent e from Theorem 4.2.6 is exactly the idempotent in $\mathcal{H}(\widehat{G},\widehat{B})$ which affords the linear character ψ (Proposition 7.4.4).

Thus far we have only examined the analogue of the Steinberg character of the extended Chevalley groups. Consequently, in Chapter 8 we turn our attention to the Chevalley group G with an additional requirement on the characteristic of the residue field which ensures that the kernel K_i of the natural projection $\eta_i : \widehat{G}(R) \to \widehat{G}(\mathfrak{o}/\mathfrak{p}^i)$ is contained in G for each $1 \leq i \leq \ell$ (Proposition 8.1.5).

By considering the corresponding parabolic subgroups H_J of G we may repeat the definition from the extended case to obtain the character

$$\mathrm{St}'_\ell = \sum_{J \subseteq S} (-1)^{|J|} (1_{H_J})^G.$$

However, St'_ℓ is simply the restriction of St_ℓ to G (Lemma 8.1.4) and is induced from the restriction

$$\chi' = \sum_{J \subseteq S} (-1)^{|J|} (1_{H_J})^H.$$

of χ to $H = H_S$

To determine whether or not St'_ℓ is irreducible we again examine the double coset structure of G. For each $J \subseteq S$ let d_J denote the number of distinct (B, B)-double cosets contained in the intersection of the $(\widehat{B}, \widehat{B})$-double coset $\widehat{B} x_J \widehat{B}$ with G. In particular, we see that $(\widehat{B} x_S \widehat{B}) \cap G$ decomposes into the union of $d = [\widehat{G} : G]$ distinct (B, B)-double cosets (Lemma 8.3.1). Further, for each $J, J' \subseteq S$ we again find that the intersection of G with the $(\widehat{H}_J, \widehat{H}_{J'})$-double coset $\widehat{H}_J x_I \widehat{H}_{J'}$ with $I \subseteq S - (J \cup J')$ contains d_J distinct $(H_J, H_{J'})$-double cosets. Thus, by counting the double cosets we are able to determine that for each $J \subseteq S$

$$(\chi', (1_{H_J})^H) = \begin{cases} d & \text{if } J = \emptyset; \\ 0 & \text{otherwise.} \end{cases} \qquad \text{(Theorem 8.3.5)}$$

Moreover, we again see that each (B, B)-double coset of G not contained in H is actually an (H_α, B)-double coset for some $\alpha \in S$ (Theorem 8.4.1). Hence, the approach from Section 6.3 gives for each $J \subseteq S$

$$(\mathrm{St}'_\ell, (1_{H_J})^G) = (\chi, (1_{H_J})^H) \qquad \text{(Proposition 8.4.2)}$$

and so St'_ℓ is irreducible only when $\widehat{G} = G$ (Theorem 8.4.4).

Consequently, in the final chapter we decompose St'_ℓ into its irreducible constituents. We begin by examining the restriction of χ' to the normal subgroup $\widetilde{V} = X_S V$ of H where $X_S = \langle X_\alpha : \alpha \in S \rangle$ and $V = (T \cap K_1)U$. If we let $\mathfrak{X}$

denote the set of linear characters of $\widetilde{V}$ which are trivial on V, but non-trivial on X_α for each $\alpha \in S$ then

$$\chi'_{\widetilde{V}} = \sum_{\lambda \in \mathfrak{X}} \lambda. \qquad \text{(Lemma 9.2.2)}$$

Further, H permutes the linear characters in $\mathfrak{X}$ (Lemma 9.2.3) forming d orbits. If we choose a representative λ_i from each orbit we find that $\mathrm{Stab}_H(\lambda_i) = \widetilde{V}$ (Lemma 9.2.4). Thus, setting $\chi_i = \lambda_i^H$ for each i, Clifford theory implies the decomposition

$$\chi = \sum_{i=1}^{d} \chi_i$$

where $\chi_1, \ldots, \chi_d$ are the distinct irreducible constituents of χ' (Theorem 9.2.5). Indeed, we are also able to show that χ_i induces irreducibly for each i (Proposition 9.3.1). Thus, if we let $\zeta_i = \chi_i^G$, then we obtain the corresponding decomposition

$$\mathrm{St}'_\ell = \sum_{i=1}^{d} \zeta_i$$

where $\zeta_1, \ldots, \zeta_d$ are the distinct irreducible constituents of St'_ℓ (Theorem 9.3.2).

Finally, we characterise the irreducible constituents of St'_ℓ in terms of analogues of the Gelfand-Graev character. A linear character θ of U is non-degenerate if its restriction to $U_\alpha(\mathfrak{m}^{\ell-1})$ is non-trivial for each $\alpha \in \Pi$. The non-degenerate characters are then permuted by T and form d orbits which correspond to the d orbits of H on $\mathfrak{X}$. Thus, if we choose a representative θ_i from each orbit and set $\Gamma_i = \theta_i^G$, then we find that ζ_i is the unique common constituent of Γ_i and $(1_B)^G$ for each i (Theorem 9.4.9).

1.4 Character theory of finite groups

For completeness, we recall some elementary definitions and results from the character theory of finite groups that will be required later. Proofs can be found in [18], [10] or [25].

Let G be a finite group and M be a module for the group ring $\mathbb{C}G$ of G over the field $\mathbb{C}$ of complex numbers. The **character** ϕ afforded by M is the map $\phi : G \to \mathbb{C}$ where for each $g \in G$

$$\phi(g) = \mathrm{tr}(g, M),$$

i.e. $\phi(g)$ is the trace of the linear map on M given by multiplication by g. In particular, the **trivial character** 1_G of G is the character afforded by the trivial module of $\mathbb{C}G$ and so has $1_G(g) = 1$ for every $g \in G$.

Suppose that ϕ and ψ are characters of G. We say that ψ is a **constituent** of ϕ if ϕ can be expressed as $\phi = \psi + \psi'$ where ψ' is either a character of G or zero. Further, ϕ is called **irreducible** if its only constituent is ϕ itself.

An important tool used to show that a character is irreducible is the inner product. For characters ϕ and ψ of G, their inner product is defined to be the sum

$$(\phi, \psi) = \frac{1}{|G|} \sum_{g \in G} \phi(g)\overline{\psi(g)}.$$

A character ϕ is then irreducible if and only if $(\phi, \phi) = 1$. Moreover, if ϕ and ψ are distinct irreducible characters then we have $(\phi, \psi) = 0$.

Now suppose that H is a subgroup of G. For any character ψ of G, its **restriction** to H is the character ψ_H of H with $\psi_H(h) = \psi(h)$ for each $h \in H$. On the other hand if ϕ is a character of H, then the **induced character** ϕ^G on G is given by

$$\phi^G(g) = \frac{1}{|G|} \sum_{x \in G} \phi^\circ(xgx^{-1})$$

where $\phi^\circ(g) = \phi(g)$ for $g \in H$ and 0 otherwise. In particular, the **permutation character** over H is the induced character $(1_H)^G$. Further, induction and restriction are related in terms of the inner product via Frobenius reciprocity

$$(\phi^G, \psi) = (\phi, \psi_H).$$

Let H and K be subgroups of G and suppose that ϕ and ψ are characters of H and K respectively. For any $g \in G$ the conjugate character ψ^g is the character of gKg^{-1} defined by $\psi^g(gkg^{-1}) = \psi(k)$ for every $k \in K$. Mackey's Theorem then expresses $(\psi^G)_H$ as

$$(\psi^G)_H = \sum_{HgK \in \mathcal{D}_G(H,K)} (\psi^g_{H \cap gKg^{-1}})^H$$

where $\mathcal{D}_G(H,K)$ denotes the set of (H,K)-double cosets in G. Thus, combining Mackey's Theorem with Frobenius reciprocity we obtain the Intertwining Number Theorem,

$$(\phi^G, \psi^G) = \sum_{HgK \in \mathcal{D}_G(H,K)} (\phi_{gHg^{-1} \cap K}, \psi^g_{gHg^{-1} \cap K}).$$

Consequently, for the permutation characters $(1_H)^G$ and $(1_K)^G$ we have

$$((1_H)^G, (1_K)^G) = |\mathcal{D}_G(H,K)|.$$

Finally, suppose that H is a normal subgroup of G. For an irreducible character ϕ of H define

$$\mathrm{Stab}_G(\phi) = \{g \in G : \phi^g = \phi\}.$$

If $\mathrm{Stab}_G(\phi) = H$, then as a special case of Mackey's Theorem we obtain

$$(\phi^G)_H = \sum_{t \in \mathcal{T}} \phi^t$$

where $\mathcal{T}$ denotes a complete set of left coset representatives of H in G and $\{\phi^t : t \in \mathcal{T}\}$ are the distinct conjugates of ϕ in H. Further, this implies that

$$(\phi^G, \phi^G) = \sum_{t \in \mathcal{T}} (\phi, \phi^t) = (\phi, \phi) = 1$$

and so ϕ^G is an irreducible character of G.

Chapter 2

Chevalley groups over fields

In this chapter we briefly outline the construction of the Chevalley groups over an arbitrary field K. Following [2], these are defined as groups of automorphisms of Lie algebras over K corresponding to the non-abelian simple Lie algebras over the complex numbers. Further, we also consider the extension of the Chevalley groups by the diagonal automorphisms.

Proofs of the results can be found in [2] or [32] together with [16] for root systems and simple Lie algebras. The identification with certain classical groups is taken from [13]. A more general approach to the construction of the Chevalley groups can be found in [5] or [32].

2.1 Root systems

Let $\mathfrak{E}$ be a finite-dimensional real vector space together with an inner product $(\cdot,\cdot)$. To each non-zero $\alpha \in \mathfrak{E}$ there is an associated orthogonal transformation $w_\alpha : \mathfrak{E} \to \mathfrak{E}$ given by the reflection in the hyperplane orthogonal to α. More specifically,

$$w_\alpha(\beta) = \beta - 2\frac{(\alpha,\beta)}{(\alpha,\alpha)}\alpha$$

for each $\beta \in \mathfrak{E}$ so that $w_\alpha(\alpha) = -\alpha$ and $w_\alpha(\beta) = \beta$ if and only if $(\alpha,\beta) = 0$.

Definition 2.1.1. A **root system** Σ is a finite set of non-zero vectors in $\mathfrak{E}$ such that

(i) Σ spans $\mathfrak{E}$;

(ii) If $\alpha, \beta \in \Sigma$ then $w_\alpha(\beta) \in \Sigma$;

(iii) If $\alpha, c\alpha \in \Sigma$, then $c = \pm 1$; and

(iv) If $\alpha, \beta \in \Sigma$, then $2(\alpha, \beta)/(\alpha, \alpha) \in \mathbb{Z}$.

Further, the **Weyl group** W of Σ is the group of orthogonal transformations on $\mathfrak{E}$ generated by the corresponding set of reflections $\{w_\alpha : \alpha \in \Sigma\}$.

The most general definition of a root system includes only properties (i) and (ii). Root systems also satisfying (iii) and (iv) should properly be referred to as reduced crystallographic root systems. However, the only root systems that we will consider are those satisfying all of (i) - (iv).

Definition 2.1.2. A **base** Π of the root system Σ is a subset of Σ such that

(i) Π is a basis for $\mathfrak{E}$; and

(ii) Each $\beta \in \Sigma$ can be expressed as a linear combination

$$\beta = \sum_{\alpha \in \Pi} k_\alpha \alpha \tag{2.1}$$

where the coefficients k_α are integers which are either all non-negative or all non-positive.

The **rank** of Σ is the number of roots in Π.

Every root system must contain a base and the different bases are transitively permuted by the action of the Weyl group. The elements of Π are called **simple roots** and the corresponding **fundamental reflections** $\{w_\alpha : \alpha \in \Pi\}$ generate W.

Further, we say that a root $\beta \in \Sigma$ is **positive** if all of the coefficients k_α are non-negative and **negative** if they are all non-positive. The sets of positive and negative roots in Σ are denoted by Σ^+ and Σ^- respectively.

Additionally, we define the **height** of each $\beta \in \Sigma$ to be

$$\mathrm{ht}(\beta) = \sum_{\alpha \in \Pi} k_\alpha$$

where the k_α are as in (2.1). If we let Σ_k denote the set of roots of height k, then in particular $\Sigma_1 = \Pi$ and $\Sigma_{-1} = \{\beta : -\beta \in \Pi\}$.

Proposition 2.1.3. *Let W be the Weyl group of a root system Σ with base Π.*

(i) If $w(\alpha) \in \Pi$ for every $\alpha \in \Pi$, then $w = 1$.

(ii) For each $w \in W$ with $w \neq 1$ there is an $\alpha \in \Pi$ with $w(\alpha) \in \Sigma^-$.

(iii) There is a unique element $w_0 \in W$ so that $w_0(\alpha) \in \{\beta : -\beta \in \Pi\}$ for every $\alpha \in \Pi$. Further, w_0 has order 2.

We now describe the classification of the **irreducible** root systems, i.e. the root systems that cannot be expressed as a disjoint union of two mutually orthogonal non-empty subsets.

Definition 2.1.4. The **Cartan integers** of a root system Σ with base Π are the integers

$$A_{\alpha,\beta} = 2\frac{(\alpha,\beta)}{(\alpha,\alpha)}$$

for each $\alpha, \beta \in \Sigma$. The **Cartan matrix** is then the invertible matrix $A = [A_{\alpha,\beta}]_{\alpha,\beta\in\Pi}$.

Cartan matrices are independent of the choice of base up to a reordering of the simple roots and determine root systems up to **equivalence**, i.e. if two root systems possess the same Cartan matrix then there must be a bijection from one to the other which preserves the inner product.

Definition 2.1.5. The **Dynkin diagram** of a root system Σ with base Π is a graph whose vertices are indexed by the simple roots and has $A_{\alpha,\beta}A_{\beta,\alpha}$ edges between the vertices α and β with an arrow pointing in the direction of β if $(\alpha, \alpha) > (\beta, \beta)$.

Given the Dynkin diagram of a root system it is possible to recover its Cartan matrix and so Dynkin diagrams also determine root systems up to equivalence. Thus, by classifying the Dynkin diagrams of irreducible root systems we obtain a classification of the irreducible root systems themselves.

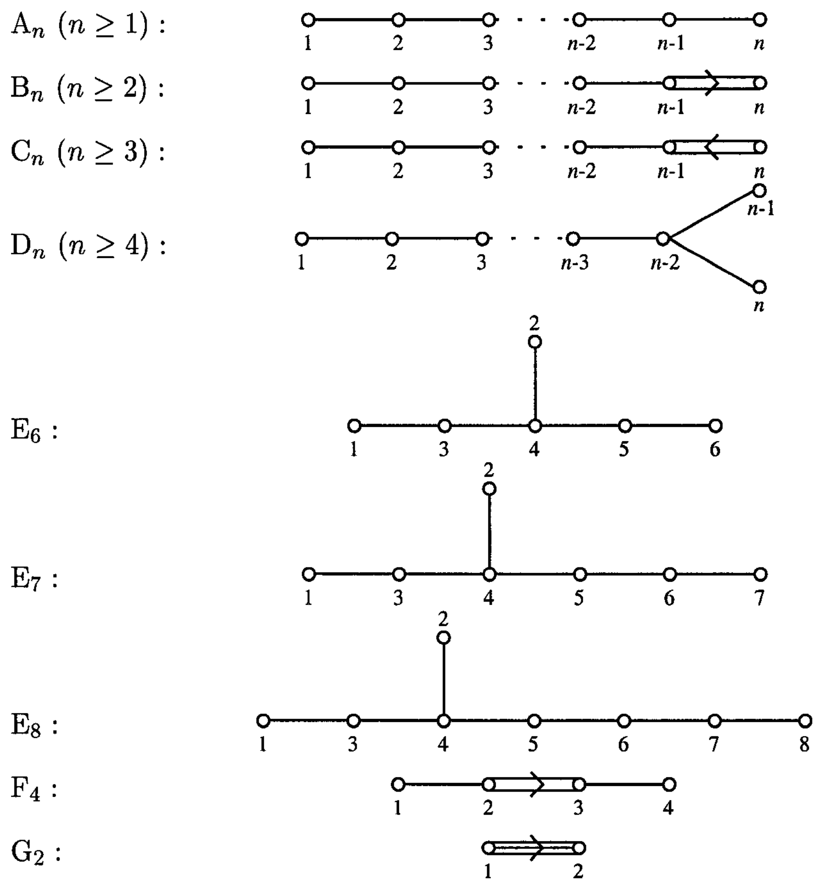

Table 2.1: The Dynkin diagrams of the irreducible root systems

Theorem 2.1.6 (Classification). *Let Σ be an irreducible root system with base $\Pi = \{\alpha_1, \ldots, \alpha_n\}$, then its Dynkin diagram is one of those contained in Table 2.1.*

It should be noted that each diagram in Table 2.1 is indeed the Dynkin diagram of an irreducible root system. A description of the irreducible root system of each type can be found in Appendix A.

2.2 Simple Lie algebras over $\mathbb{C}$

Definition 2.2.1. A **Lie algebra** $\mathfrak{L}$ over $\mathbb{C}$ is a complex vector space together with a bilinear map $[\cdot,\cdot] : \mathfrak{L} \times \mathfrak{L} \to \mathfrak{L}$ such that

(i) $[x, x] = 0$ for every $x \in \mathfrak{L}$; and

(ii) $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ for every $x, y, z \in \mathfrak{L}$.

For each $x \in \mathfrak{L}$ the **adjoint map** $\mathrm{ad}_x : \mathfrak{L} \to \mathfrak{L}$ is defined for every $y \in \mathfrak{L}$ by $\mathrm{ad}_x(y) = [x, y]$.

For subsets X and Y of $\mathfrak{L}$, let $[X, Y]$ denote the set of all linear combinations of commutators $[x, y]$ for $x \in X$ and $y \in Y$. A **subalgebra** $\mathfrak{M}$ of $\mathfrak{L}$ is then a subspace of $\mathfrak{L}$ such that $[\mathfrak{M}, \mathfrak{M}] \subseteq \mathfrak{M}$ and an **ideal** of $\mathfrak{L}$ is a subalgebra $\mathfrak{I}$ such that $[\mathfrak{L}, \mathfrak{I}] \subseteq \mathfrak{I}$. The Lie algebra $\mathfrak{L}$ is said to be **simple** if it contains no ideals other than itself and the zero ideal. Further, it is **abelian** if $[\mathfrak{L}, \mathfrak{L}] = 0$.

The first result that we will need is that each non-abelian simple Lie algebra has a decomposition corresponding to an irreducible root system. This is achieved by considering the adjoint action of a particular subalgebra.

Definition 2.2.2. A **Cartan subalgebra** $\mathfrak{H}$ of $\mathfrak{L}$ is a subalgebra of $\mathfrak{L}$ such that

(i) If we set $\mathfrak{L}^1 = [\mathfrak{L}, \mathfrak{L}]$ and $\mathfrak{L}^i = [\mathfrak{L}, \mathfrak{L}^{i-1}]$ for each i, then $\mathfrak{L}^k = 0$ for some k; and

(ii) If we set $N_{\mathfrak{L}}(\mathfrak{H}) = \{x \in \mathfrak{L} : [x, h] \in \mathfrak{H} \text{ for every } h \in \mathfrak{H}\}$, then $N_{\mathfrak{L}}(\mathfrak{H}) = \mathfrak{H}$.

Every Lie algebra must contain a Cartan subalgebra and any two Cartan subalgebras are conjugate. In the particular case of simple Lie algebras the Cartan subalgebra turns out to be abelian. Consequently, the associated adjoint maps commute and $\mathfrak{L}$ can be expressed as a direct sum of simultaneous eigenspaces.

More specifically, for each α in the dual space $\mathfrak{H}^*$ of $\mathfrak{H}$ define

$$\mathfrak{L}_\alpha = \{y \in \mathfrak{L} : \mathrm{ad}_x(y) = \alpha(y) \text{ for each } x \in \mathfrak{H}\}.$$

Then if we let Σ be the set of non-zero $\alpha \in \mathfrak{H}^*$ for which $\mathfrak{L}_\alpha \neq 0$ we obtain the following decomposition of $\mathfrak{L}$.

Theorem 2.2.3 (Cartan Decomposition). *Let $\mathfrak{L}$ be a non-abelian simple Lie algebra over $\mathbb{C}$, then*

$$\mathfrak{L} = \mathfrak{H} \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{L}_\alpha$$

where each $\mathfrak{L}_\alpha$ has dimension 1.

Further, if we consider the **Killing form** on $\mathfrak{L}$ given by $(x, y) = \mathrm{tr}(\mathrm{ad}_x \mathrm{ad}_y)$ for each $x, y \in \mathfrak{L}$, then this allows us to identify $\mathfrak{H}$ with its dual space. Thus we may consider Σ as a subset of $\mathfrak{H}$ and setting $\mathfrak{H}_\mathbb{R}$ to be the $\mathbb{R}$-span of Σ in $\mathfrak{H}$, we find that Σ forms an irreducible root system in $\mathfrak{H}_\mathbb{R}$ with inner product $(\cdot, \cdot)$. The classification of the irreducible root systems then gives rise to a classification of the non-abelian simple Lie algebras over $\mathbb{C}$.

Theorem 2.2.4. *For each irreducible root system Σ there is a simple Lie algebra over $\mathbb{C}$ with root system equivalent to Σ and any two Lie algebras with equivalent root systems are isomorphic.*

We now use the Cartan decomposition to describe a particular basis for $\mathfrak{L}$ which will be an important part of the construction of the Chevalley groups.

Fix a base Π of Σ. For each root $\alpha \in \Sigma$ the associated co-root $h_\alpha \in \mathfrak{H}$ is defined by

$$h_\alpha = \frac{2\alpha}{(\alpha, \alpha)}.$$

If for each positive root $\alpha \in \Sigma^+$ we pick a non-zero element $e_\alpha \in \mathfrak{L}_\alpha$, then there is a unique $e_{-\alpha} \in \mathfrak{L}_{-\alpha}$ with $[e_\alpha, e_{-\alpha}] = h_\alpha$. Moreover, the $\{e_\alpha : \alpha \in \Sigma^+\}$ may be chosen in such a way that whenever $\alpha, \beta \in \Sigma^+$ are such that $\alpha + \beta \in \Sigma$ then $[e_\alpha, e_\beta] = N_{\alpha,\beta} e_{\alpha+\beta}$ for some integer $N_{\alpha,\beta}$.

Theorem 2.2.5 (Chevalley Basis). *The set $\{h_\alpha, e_\beta : \alpha \in \Pi, \beta \in \Sigma\}$ is a basis for $\mathfrak{L}$ and the multiplication of basis elements is given by*

(i) $[h_\alpha, h_\beta] = 0$ *for* $\alpha, \beta \in \Pi$;

(ii) $[h_\alpha, e_\beta] = A_{\alpha,\beta} e_\beta$ *for* $\alpha \in \Pi, \beta \in \Sigma$;

(iii) $[e_\alpha, e_{-\alpha}] = h_\alpha$ *for* $\alpha \in \Sigma^+$;

(iv) $[e_\alpha, e_\beta] = 0$ *for* $\alpha, \beta \in \Sigma$ *with* $\alpha + \beta \notin \Sigma$; *and*

(v) $[e_\alpha, e_\beta] = N_{\alpha,\beta} e_{\alpha+\beta}$ *for* $\alpha, \beta \in \Sigma$ *with* $\alpha + \beta \in \Sigma$.

Finally, we use the Chevalley basis to define certain automorphisms of $\mathfrak{L}$. For each root $\alpha \in \Sigma$ we have $(\mathrm{ad}_{e_\alpha})^k = 0$ for all sufficiently large k. Consequently, we may define a linear map $x_\alpha(\xi) : \mathfrak{L} \to \mathfrak{L}$ for every $\xi \in \mathbb{C}$ by setting

$$x_\alpha(\xi) = \sum_{k=0}^{\infty} \frac{\xi^k (\mathrm{ad}_{e_\alpha})^k}{k!}.$$

Theorem 2.2.6. *Let $\alpha \in \Sigma$ and $\xi \in \mathbb{C}$, then $x_\alpha(\xi)$ is an automorphism of $\mathfrak{L}$ where*

(i) $x_\alpha(\xi) \cdot h_\beta = h_\beta - A_{\beta,\alpha} \xi e_\alpha$ *for* $\beta \in \Pi$;

(ii) $x_\alpha(\xi) \cdot e_\alpha = e_\alpha$;

(iii) $x_\alpha(\xi) \cdot e_{-\alpha} = e_{-\alpha} + \xi h_\alpha - \xi^2 e_\alpha$; *and*

(iv) $x_\alpha(\xi) \cdot e_\beta = \sum_{i=0}^{k} M_{\alpha,\beta,i} \xi^i e_{i\alpha+\beta}$ *for* $\beta \in \Sigma$ *with* $\alpha \neq \pm\beta$

for some integers $M_{\alpha,\beta,i}$ and k.

2.3 Chevalley groups over fields

Let $\{h_\alpha, e_\beta : \alpha \in \Pi, \beta \in \Sigma\}$ be the Chevalley basis of a non-abelian simple Lie algebra $\mathfrak{L}$ over $\mathbb{C}$ with root system Σ and base Π. Denote by $\mathfrak{L}_\mathbb{Z}$ the $\mathbb{Z}$-span of the Chevalley basis in $\mathfrak{L}$ and define

$$\mathfrak{L}_K = \mathfrak{L}_\mathbb{Z} \otimes K.$$

Then $\mathfrak{L}_K$ is a Lie algebra over K with basis

$$\{h_\alpha \otimes 1_K, e_\beta \otimes 1_K : \alpha \in \Pi, \beta \in \Sigma\}$$

and Lie bracket obtained by taking

$$[x \otimes 1_K, y \otimes 1_K] = [x, y] \otimes 1_K$$

for each $x, y \in \mathfrak{L}_\mathbb{Z}$ and extending linearly.

Further, the multiplication constants with respect to this basis are the multiplication constants of $\mathfrak{L}$ with respect to the Chevalley basis interpreted as elements of K. In particular, this means that if we use Theorem 2.2.6 to define a corresponding linear map $x_\alpha(r)$ on $\mathfrak{L}_K$ for each $\alpha \in \Sigma$ and $r \in K$, then it must automatically be an automorphism of $\mathfrak{L}_K$.

Definition 2.3.1. For each $\alpha \in \Sigma$ and $r \in K$ define $x_\alpha(r) \in \mathrm{Aut}(\mathfrak{L}_K)$ by

(i) $x_\alpha(r) \cdot (h_\beta \otimes 1_K) = (h_\beta \otimes 1_K) - A_{\beta,\alpha} r (e_\alpha \otimes 1_K)$ for $\beta \in \Pi$;

(ii) $x_\alpha(r) \cdot (e_\alpha \otimes 1_K) = (e_\alpha \otimes 1_K)$;

(iii) $x_\alpha(r) \cdot (e_{-\alpha} \otimes 1_K) = (e_{-\alpha} \otimes 1_K) + r(h_\alpha \otimes 1_K) - r^2(e_\alpha \otimes 1_K)$; and

(iv) $x_\alpha(r) \cdot (e_\beta \otimes 1_K) = \sum_{i=0}^{k} M_{\alpha,\beta,i} r^i (e_{i\alpha+\beta} \otimes 1_K)$ for $\beta \in \Sigma$ with $\alpha \neq \pm\beta$.

The **Chevalley group** of type Σ over the field K is then the subgroup

$$G(K) = \langle x_\alpha(r) : \alpha \in \Sigma, r \in K \rangle$$

of the automorphism group $\mathrm{Aut}(\mathfrak{L}_K)$ of $\mathfrak{L}_K$.

$G(K)$ is independent of the choice of Chevalley basis for $\mathfrak{L}$ and is determined up to isomorphism by the irreducible root system Σ and the field K. Further, it should be mentioned that this construction actually produces the adjoint Chevalley group of type Σ over K. The other Chevalley groups can be obtained by replacing $\mathfrak{L}_{\mathbb{Z}}$ with an admissible $\mathbb{Z}$-form for a different faithful $\mathfrak{L}$-module (cf. [5] or [32]). However, the resulting groups are merely central extensions of the adjoint group and so any representation constructed for the adjoint group can easily be pulled back to a representation of the more general group.

We now examine the structure of $G(K)$. For each $\alpha \in \Sigma$ and $r, s \in K$ we have

$$x_\alpha(r)x_\alpha(s) = x_\alpha(r+s).$$

Thus, for each $\alpha \in \Sigma$ the **root subgroup** $U_\alpha(K) = \{x_\alpha(r) : r \in K\}$ is isomorphic to the additive group K. The relation between the generators $x_\alpha(r)$ and $x_\beta(s)$ for $\alpha \neq \pm\beta$ is given by the following commutator formula. Here we are using the commutator $[a, b] = aba^{-1}b^{-1}$.

Theorem 2.3.2 (Chevalley Commutator Formula). *Let $\alpha, \beta \in \Sigma$ be such that $\alpha \neq \pm\beta$, then for every $r, s \in K$*

$$[x_\alpha(r), x_\beta(s)] = \prod_{i,j>0} x_{i\alpha+j\beta}(c_{i,j,\alpha,\beta}(-r)^i s^j)$$

where the product is taken over all positive i and j such that $i\alpha + j\beta \in \Sigma$. Further, the coefficients $c_{i,j,\alpha,\beta}$ are independent of r and s. More specifically, $c_{i,j,\alpha,\beta} \in \{\pm1\}$ except for $\Sigma = B_n, C_n$ or F_4 which have $c_{i,j,\alpha,\beta} \in \{\pm1, \pm2\}$ and $\Sigma = G_2$ which has $c_{i,j,\alpha,\beta} \in \{\pm1, \pm2, \pm3\}$.

Let $U(K) = \langle U_\alpha(K) : \alpha \in \Sigma^+\rangle$ and $U^-(K) = \langle U_\alpha(K) : \alpha \in \Sigma^-\rangle$. As an immediate consequence of the Chevalley Commutator Formula we obtain the following results.

Theorem 2.3.3. *(i) Each $g \in U(K)$ can be expressed uniquely as*

$$g = \prod_{\alpha \in \Sigma^+} x_\alpha(r_\alpha)$$

for some $r_\alpha \in K$ where the product is taken over the positive roots in a fixed, but arbitrary, order.

(ii) Similarly, each $g \in U^-(K)$ can be expressed uniquely as

$$g = \prod_{\alpha \in \Sigma^-} x_\alpha(r_\alpha)$$

for some $r_\alpha \in K$ where the product is taken over the negative roots in a fixed, but arbitrary, order.

The Chevalley Commutator Formula only gives the relationship between the generators when $\alpha \neq \pm\beta$. If $\alpha = -\beta$, then the situation is closely related to $\mathrm{SL}_2(K)$.

Proposition 2.3.4. *For each $\alpha \in \Sigma$ the map $\phi_\alpha : \mathrm{SL}_2(K) \to G(K)$ given by*

$$\phi_\alpha\left(\begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix}\right) = x_\alpha(r) \qquad \text{and} \qquad \phi_\alpha\left(\begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix}\right) = x_{-\alpha}(s)$$

for every $r, s \in K$ is a homomorphism.

In view of Proposition 2.3.4, for each $\alpha \in \Sigma$ and $r \in K^\times$ consider the elements

$$h_\alpha(r) = \phi_\alpha\left(\begin{bmatrix} r & 0 \\ 0 & r^{-1} \end{bmatrix}\right) \qquad \text{and} \qquad n_\alpha(r) = \phi_\alpha\left(\begin{bmatrix} 0 & r \\ -r^{-1} & 0 \end{bmatrix}\right).$$

In particular, $n_\alpha(r) = x_\alpha(r)x_{-\alpha}(-r^{-1})x_\alpha(r)$ and $h_\alpha(r) = n_\alpha(r)n_\alpha(1)^{-1}$.

Lemma 2.3.5. *Let $\alpha, \beta \in \Sigma$, $r, s \in K^\times$ and $t \in K$, then*

(i) $h_\alpha(r)h_\alpha(s) = h_\alpha(rs)$;

(ii) $h_\alpha(r)h_\beta(s) = h_\beta(s)h_\alpha(r)$; and

(iii) $h_\alpha(r)x_\beta(t)h_\alpha(r)^{-1} = x_\beta(r^{A_{\alpha,\beta}}t)$.

Thus, if we set $T(K) = \langle h_\alpha(r) : \alpha \in \Sigma, r \in K^\times \rangle$ and $B(K) = \langle U(K), T(K) \rangle$, then $U(K)$ is normal $B(K)$ and so $B(K) = T(K)U(K)$.

Lemma 2.3.6. *Let $\alpha, \beta \in \Sigma$, $r \in K$ and $s \in K^\times$, then*

(i) $n_\alpha(1)h_\beta(r)n_\alpha(1)^{-1} = h_{w_\alpha(\beta)}(r)$; *and*

(ii) $n_\alpha(1)x_\beta(r)n_\alpha(1)^{-1} = x_{w_\alpha(\beta)}(\eta_{\alpha,\beta}r)$

where $\eta_{\alpha,\beta} \in \{\pm 1\}$ *is independent of* r.

If we let $N(K) = \langle n_\alpha(r) : \alpha \in \Sigma, r \in K^\times \rangle$, then we find that there is a homomorphism from $N(K)$ onto W which has $T(K)$ as its kernel. Further, for each $\alpha \in \Sigma$ the element $n_\alpha(1) \in N(K)$ gets mapped to $w_\alpha \in W$ under this homomorphism. Consequently, if we write any $w \in W$ as the product $w = w_{\alpha_{i_1}} \cdots w_{\alpha_{i_k}}$ for some $\alpha_{i_1}, \ldots, \alpha_{i_k} \in \Pi$, then setting

$$n_w = n_{\alpha_{i_1}}(1) \cdots n_{\alpha_{i_k}}(1)$$

we see that the image of n_w is w.

Indeed, $B(K)$ and $N(K)$ form a BN-pair for $G(K)$ and so $G(K)$ can be expressed as the disjoint union

$$G(K) = \bigcup_{w \in W} B(K)n_wB(K).$$

2.4 Extension by the diagonal automorphisms

Let Λ_r denote the $\mathbb{Z}$-span of the roots Σ in $\mathfrak{H}$, then a **K-character** of Λ_r is a homomorphism $\mu : \Lambda_r \to K^\times$. Further, since Λ_r is a free abelian group generated by Π, it is clear that each K-character of Λ_r is completely determined by its value on the simple roots.

Every K-character μ of Λ_r gives a **diagonal automorphism** $h(\mu)$ of $\mathfrak{L}_K$ via

$$h(\mu) \cdot (h_\alpha \otimes 1_K) = (h_\alpha \otimes 1_K) \qquad \text{and} \qquad h(\mu) \cdot (e_\beta \otimes 1_K) = \mu(\beta)(e_\beta \otimes 1_K) \quad (2.2)$$

for each $\alpha \in \Pi$ and $\beta \in \Sigma$. Further, given any two K-characters μ_1 and μ_2 their product $\mu_1\mu_2$ is also a K-character and

$$h(\mu_1)h(\mu_2) = h(\mu_1\mu_2).$$

Hence the set $\widehat{T}(K) = \{h(\mu) : \mu \text{ is a } K\text{-character of } \Lambda_r\}$ of diagonal automorphisms forms a subgroup of $\mathrm{Aut}(\mathfrak{L}_K)$.

Now, for each $\alpha \in \Sigma$ and $r \in K^\times$ the element $h_\alpha(r) \in G(K)$ acts on the Chevalley basis via

$$h_\alpha(r) \cdot (h_\beta \otimes 1_K) = (h_\beta \otimes 1_K) \qquad \text{and} \qquad h_\alpha(r) \cdot (e_\beta \otimes 1_K) = r^{A_{\alpha,\beta}}(e_\beta \otimes 1_K). \quad (2.3)$$

Thus, if we define a K-character $\mu_{\alpha,r} : \Lambda_r \to K^\times$ by

$$\mu_{\alpha,r}(\beta) = r^{2(\alpha,\beta)/(\alpha,\alpha)}$$

then we see that $h_\alpha(r) = h(\mu_{\alpha,r})$. Consequently, $T(K)$ is a subgroup of $\widehat{T}(K)$ and it is natural to consider which K-characters μ give automorphisms $h(\mu)$ in $T(K)$.

Let Λ denote the set of all $\lambda \in \mathfrak{H}_\mathbb{R}$ for which $2(\lambda, \alpha)/(\alpha, \alpha) \in \mathbb{Z}$ for every $\alpha \in \Sigma$. Λ is called the **weight lattice** and its elements **weights**. In particular, for each $\alpha \in \Pi$ the **fundamental weight** λ_α is the unique element in Λ with

$$2\frac{(\lambda_\alpha, \beta)}{(\beta, \beta)} = \begin{cases} 1 & \text{if } \alpha = \beta; \\ 0 & \text{if } \alpha \neq \beta \end{cases}$$

for each $\beta \in \Pi$ and Λ is a free abelian group generated by $\{\lambda_\alpha : \alpha \in \Pi\}$. Further, Λ_r is clearly a subgroup of Λ and each simple root $\alpha \in \Pi$ can be expressed as the linear combination

$$\alpha = \sum_{\beta \in \Pi} A_{\alpha,\beta}\lambda_\beta \qquad (2.4)$$

where $A = [A_{\alpha,\beta}]_{\alpha,\beta\in\Pi}$ is the Cartan matrix of Σ.

Theorem 2.4.1. *$T(K)$ is the subgroup of $\widehat{T}(K)$ consisting of the diagonal automorphisms $h(\mu)$ where μ is the restriction to Λ_r of a K-character of Λ.*

This means that the quotient group $\widehat{T}(K)/T(K)$ is isomorphic to the group of K-characters of Λ/Λ_r. From (2.4) we see that the index of Λ_r in Λ is equal to the determinant of the Cartan matrix A and is given in Table 2.2. By investigating the Cartan integers further it is possible to show that Λ/Λ_r has the structure shown in Table 2.3. Then, in the case where K is the finite field $\mathbb{F}_q$ of q-elements, Table 2.4 gives the index d of $T(K)$ in $\widehat{T}(K)$.

Σ	A_n	B_n	C_n	D_n	E_6	E_7	E_8	G_2	F_4
$\det(A)$	$n+1$	2	2	4	3	2	1	1	1

Table 2.2: The determinants of the Cartan matrices

Σ	A_n	B_n	C_n	D_{2k+1}	D_{2k}	E_6	E_7	E_8	G_2	F_4
Λ/Λ_r	$\mathbb{Z}/(n+1)\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	1	1	1

Table 2.3: The structure of Λ/Λ_r

Σ	A_n	B_n	C_n	D_{2k+1}	D_{2k}	E_6	E_7	E_8	G_2	F_4
d	$(n+1,q-1)$	$(2,q-1)$	$(2,q-1)$	$(4,q-1)$	$(2,q-1)^2$	$(3,q-1)$	$(2,q-1)$	1	1	1

Table 2.4: The index $d=[\widehat{T}(\mathbb{F}_q):T(\mathbb{F}_q)]$

Lemma 2.4.2. *Let μ be a K-character of Λ_r, $\alpha\in\Sigma$, $r\in K$ and $w\in W$, then*

(i) $h(\mu)x_\alpha(r)h(\mu)^{-1}=x_\alpha(\mu(\alpha)r)$; and

(ii) $n_w h(\mu)n_w^{-1}=h(\mu')$

where μ' is the K-character of Λ_r given by $\mu'(\beta)=\mu(w^{-1}(\beta))$ for each $\beta\in\Sigma$.

This implies that $\widehat{T}(K)$ normalises each root subgroup $U_\alpha(K)$ and so therefore also $U(K)$. Thus, if we define $\widehat{B}(K) = \langle U(K), \widehat{T}(K)\rangle$ then we have $\widehat{B}(K) = \widehat{T}(K)U(K)$.

Definition 2.4.3. The **extended Chevalley group** $\widehat{G}(K)$ of type Σ is the subgroup of $\mathrm{Aut}(\mathfrak{L}_K)$ generated by $G(K)$ and $\widehat{T}(K)$.

Theorem 2.4.4. *$G(K)$ is a normal subgroup of $\widehat{G}(K)$ and $\widehat{G}(K)/G(K)$ is isomorphic to $\widehat{T}(K)/T(K)$.*

We conclude this chapter by identifying certain classical groups over K with either a Chevalley group or its extended version.

The **general linear group** $\mathrm{GL}_n(K)$ is the group of $n \times n$ invertible matrices with entries in K

$$\mathrm{GL}_n(K) = \{g \in M_n(K) : \det(g) \neq 0\}$$

and the **special linear group** $\mathrm{SL}_n(K)$ is the subgroup

$$\mathrm{SL}_n(K) = \{g \in \mathrm{GL}_n(K) : \det(g) = 1\}.$$

Further, the **projective general linear group** is $\mathrm{PGL}_n(K) = \mathrm{GL}_n(K)/Z(\mathrm{GL}_n(K))$ and the **projective special linear group** is $\mathrm{PSL}_n(K) = \mathrm{SL}_n(K)/Z(\mathrm{SL}_n(K))$ where $Z(\mathrm{GL}_n(K))$ and $Z(\mathrm{SL}_n(K))$ are the centres of $\mathrm{GL}_n(K)$ and $\mathrm{SL}_n(K)$ respectively.

Now, consider the symmetric bilinear form $\langle \cdot, \cdot \rangle$ on K^n defined by

$$\langle x, y\rangle = x_1 y_n + \cdots + x_n y_1$$

for each $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in K^n$. The **orthogonal group** $\mathrm{O}_n(K)$ is then the subgroup of $\mathrm{GL}_n(K)$ which preserves $\langle \cdot, \cdot \rangle$

$$\mathrm{O}_n(K) = \{g \in \mathrm{GL}_n(K) : \langle gx, gy\rangle = \langle x, y\rangle \text{ for each } x, y \in K^n\}$$

and $\mathrm{GO}_n(K)$ is the subgroup of $\mathrm{GL}_n(K)$ which preserves $\langle \cdot, \cdot \rangle$ up to a scalar

$$\mathrm{GO}_n(K) = \{g \in \mathrm{GL}_n(K) : \text{there is } a_g \in K^\times \text{ with } \langle gx, gy\rangle = a_g\langle x, y\rangle \text{ for each } x, y \in K^n\}.$$

Further, the **special orthogonal group** is $\mathrm{SO}_n(K) = \mathrm{O}_n(K) \cap \mathrm{SL}_n(K)$ and $\Omega_n(K)$ is the derived subgroup of $\mathrm{O}_n(K)$. Again, $\mathrm{PGO}_n(K) = \mathrm{GO}_n(K)/Z(\mathrm{GO}_n(K))$ and $\mathrm{P}\Omega_n(K) = \Omega_n(K)/Z(\Omega_n(K))$.

Similarly, if we consider the alternating bilinear form $\langle \cdot, \cdot \rangle'$ on K^{2n} given by

$$\langle x, y\rangle' = x_1 y_{2n} + \cdots + x_n y_{n+1} - x_{n+1} y_n - \cdots - x_{2n} y_1$$

for each $x = (x_1, \ldots, x_{2n}), y = (y_1, \ldots, y_{2n}) \in K^{2n}$, then the **symplectic group** $\mathrm{Sp}_{2n}(K)$ is the subgroup of $\mathrm{GL}_{2n}(K)$ which preserves $\langle \cdot, \cdot \rangle'$

$$\mathrm{Sp}_{2n} = \{g \in \mathrm{GL}_{2n}(K) : \langle gx, gy\rangle' = \langle x, y\rangle' \text{ for each } x, y \in K^{2n}\}$$

and $\mathrm{GSp}_{2n}(K)$ is the subgroup of $\mathrm{GL}_{2n}(K)$ which preserves $\langle \cdot, \cdot \rangle'$ up to a scalar

$$\mathrm{GSp}_n(K) = \{g \in \mathrm{GL}_{2n}(K) : \text{there is } a_g \in K^\times \text{ with } \langle gx, gy\rangle' = a_g \langle x, y\rangle' \text{ for each } x, y \in K^{2n}\}.$$

Finally, $\mathrm{PGSp}_n(K) = \mathrm{GSp}_n(K)/Z(\mathrm{GSp}_n(K))$ and $\mathrm{PSp}_n(K) = \mathrm{Sp}_n(K)/Z(\mathrm{Sp}_n(K))$.

Theorem 2.4.5. *Let $G(K)$ be the Chevalley group of type Σ over K and $\widehat{G}(K)$ its extension by the diagonal automorphisms.*

(i) If $\Sigma = A_n$, then $G(K) \simeq \mathrm{PSL}_{n+1}(K)$ and $\widehat{G}(K) \simeq \mathrm{PGL}_{n+1}(K)$.

(ii) If $\Sigma = B_n$, then $G(K) \simeq \Omega_{2n+1}(K)$ and $\widehat{G}(K) \simeq \mathrm{SO}_{2n+1}(K)$.

(iii) If $\Sigma = C_n$, then $G(K) \simeq \mathrm{PSp}_{2n}(K)$ and $\widehat{G}(K) \simeq \mathrm{PGSp}_{2n}(K)$.

(iv) If $\Sigma = D_n$, then $G(K) \simeq \mathrm{P}\Omega_{2n}(K)$ and $\widehat{G}(K)$ has index 2 in $\mathrm{PGO}_{2n}(K)$.

Chapter 3

Chevalley groups over rings

We now turn our attention to the finite ring $R = \mathfrak{o}/\mathfrak{p}^\ell$ obtained as the quotient of the ring of integers $\mathfrak{o}$ of a non-archimedean local field K by a power of its prime ideal $\mathfrak{p}$. The structure of the Chevalley groups over R is then inherited from $G(K)$ via the corresponding Chevalley groups over $\mathfrak{o}$. Additionally, certain normal subgroups arise from the ideal structure of R.

Proofs of the statements involving Chevalley groups over rings of integers can be found in [32]. The results for congruence subgroups are taken from [1] and a general reference for local fields is [26].

3.1 Local fields

Definition 3.1.1. A **non-archimedean absolute value** on a field K is a function $||\cdot|| : K \to \mathbb{R}$ such that for every $x, y \in K$

(i) $||x|| \geq 0$ with $||x|| = 0$ if and only if $x = 0$;

(ii) $||xy|| = ||x|| \cdot ||y||$; and

(iii) $||x + y|| \leq \max\{||x||, ||y||\}$.

K is then called a **non-archimedean local field** if it is complete with respect to this non-archimedean absolute value.

Examples 3.1.2. (i) Let p be a prime and consider the field $\mathbb{Q}$ of rational numbers. Each non-zero rational number $r \in \mathbb{Q}$ can be written as a quotient

$$r = \frac{x}{y}$$

for some non-zero integers $x, y \in \mathbb{Z}$. Further, there exist unique non-negative integers a and b so that

$$x = p^a x' \quad \text{and} \quad y = p^b y'$$

for some integers $x', y' \in \mathbb{Z}$ which are not divisible by p. Thus we may define the p-adic absolute value $|| \cdot ||_p : \mathbb{Q} \to \mathbb{R}$ on $\mathbb{Q}$ by $||0||_p = 0$ and for $r \neq 0$

$$||r||_p = p^{b-a}.$$

The non-archimedean local field $\mathbb{Q}_p$ of p-adic numbers is then the completion of $\mathbb{Q}$ with respect to the p-adic absolute value.

(ii) Again, let p be a prime but now consider the field $\mathbb{F}_p(t)$ of rational functions in t with coefficients in the field $\mathbb{F}_p$ of p elements. Each non-zero rational function $r(t) \in \mathbb{F}_p(t)$ can be written as a quotient

$$r(t) = \frac{x(t)}{y(t)}$$

for non-zero polynomials $x(t), y(t) \in \mathbb{F}_p[t]$. Further, there exist unique non-negative integers a and b so that

$$x(t) = t^a x'(t) \quad \text{and} \quad y(t) = t^b y'(t)$$

for some polynomials $x'(t), y'(t) \in \mathbb{F}_p[t]$ with non-zero constant terms. Thus we may again define a non-archimedean absolute value $|| \cdot ||_t : \mathbb{F}_p(t) \to \mathbb{R}$ on $\mathbb{F}_p(t)$

by $||0||_t = 0$ and for $r(t) \neq 0$

$$||r(t)||_t = p^{b-a}.$$

The non-archimedean local field $\mathbb{F}_p((t))$ of formal Laurent series in t over $\mathbb{F}_p$ is then the completion of $\mathbb{F}_p(t)$ with respect to this absolute value.

The **ring of integers** of a non-archimedean local field K with absolute value $||\cdot||$ is the subring

$$\mathfrak{o} = \{x \in K : ||x|| \leq 1\}.$$

The group of units in R is then clearly

$$\mathfrak{o}^\times = \{x \in K : ||x|| = 1\}$$

meaning that

$$\mathfrak{p} = \{x \in K : ||x|| < 1\}$$

is the unique maximal ideal of $\mathfrak{o}$. Thus, the quotient ring $\kappa = \mathfrak{o}/\mathfrak{p}$ is a field and is called the **residue class field** of K. Further, $\mathfrak{p}$ is a principal ideal and every ideal of $\mathfrak{o}$ is of the form $\mathfrak{p}^i$ for some $i \geq 0$.

Definition 3.1.3. Let $\mathfrak{o}$ be the ring of integers of a non-archimedean local field K with maximal ideal $\mathfrak{p}$ and finite residue class field κ of order q. Then, for a fixed integer $\ell > 1$ define R to be the quotient ring $R = \mathfrak{o}/\mathfrak{p}^\ell$.

R is clearly a commutative ring and, since $\mathfrak{p}$ is the unique maximal ideal of $\mathfrak{o}$, the unique maximal ideal of R is $\mathfrak{m} = \mathfrak{p}/\mathfrak{p}^\ell$. Thus, the group of units of R is $R^\times = \{r \in R : r \notin \mathfrak{m}\}$. Further, since $\mathfrak{p}$ is principal, $\mathfrak{m}$ must also be principal. Consequently, if π denotes the generator of $\mathfrak{m}$ then each ideal of R is of the form $\mathfrak{m}^i = \pi^i R$ for some $0 \leq i \leq \ell$ since each ideal of $\mathfrak{o}$ containing $\mathfrak{p}^\ell$ is of the form $\mathfrak{p}^i$ for some $0 \leq i \leq \ell$. In particular, $\mathfrak{m}^{\ell-1} = \pi^{\ell-1}R$ is the minimal non-zero ideal of R.

Lemma 3.1.4. *Let $0 \leq i \leq \ell$, then*

(i) $|R| = q^{\ell}$;

(ii) $|\mathfrak{m}^i| = q^{\ell - i}$; *and*

(iii) $|R^{\times}| = q^{\ell-1}(q-1)$.

Examples 3.1.5. (i) The ring $\mathfrak{o} = \mathbb{Z}_p$ of p-adic integers is the ring of integers of $\mathbb{Q}_p$. Further, the maximal ideal of $\mathfrak{o}$ is $\mathfrak{p} = p\mathbb{Z}_p$ and its residue class field $\kappa = \mathbb{Z}_p/p\mathbb{Z}_p$ is the field of p elements. Then, for $\ell > 1$ the ring $R = \mathbb{Z}_p/p^{\ell}\mathbb{Z}_p$ can be identified with the ring $\mathbb{Z}/p^{\ell}\mathbb{Z}$ of integers modulo p^{ℓ}.

(ii) The ring $\mathfrak{o} = \mathbb{F}_p[[t]]$ of formal power series in t is the ring of integers of $\mathbb{F}_p((t))$ and its maximal ideal is $\mathfrak{p} = t\mathbb{F}_p[[t]]$ so again the residue class field $\kappa = \mathbb{F}_p[[t]]/t\mathbb{F}_p[[t]]$ is the field of p elements. Thus, for $\ell > 1$ the ring $R = \mathbb{F}_p[[t]]/t^{\ell}\mathbb{F}_p[[t]]$ can be identified with the ring $\mathbb{F}_p[t]/t^{\ell}\mathbb{F}_p[t]$ of polynomials in t with coefficients in $\mathbb{F}_p$, modulo t^{ℓ}.

We also include a result about k-th roots in R which will be required later.

Lemma 3.1.6. *Let k be a positive integer which is not divisible by the characteristic of κ, then for every $r \in 1 + \mathfrak{m}$ there is some $s \in 1 + \mathfrak{m}$ with $r = s^k$.*

Proof. In fact, we will show that the only element $r \in 1+\mathfrak{m}$ with $r^k = 1$ is $r = 1$. This would then imply that the map from $1 + \mathfrak{m}$ to itself which sends r to r^k is injective. Thus, since $1 + \mathfrak{m}$ is finite, it must also be surjective and therefore any $r \in 1 + \mathfrak{m}$ is of the form $r = s^k$ for some $s \in 1 + \mathfrak{m}$.

Suppose that $r \in 1 + \mathfrak{m}$ is such that $r^k = 1$, but that $r \neq 1$. Then $r^k - 1 = 0$ implies that

$$(r^{k-1} + r^{k-2} + \cdots + r + 1)(r - 1) = 0$$

and therefore $r^{k-1} + r^{k-2} + \cdots + r + 1 \in \mathfrak{m}$ since $r - 1 \neq 0$. Consequently, if we consider the natural projection $\eta_1 : R \to \kappa$ which has $\mathfrak{m}$ as its kernel, then we see that $\eta_1(r) = 1$ and so

$$0 = \eta_1(r)^{k-1} + \eta_1(r)^{k-2} + \cdots + \eta_1(r) + \eta_1(1) = 1 + 1 + \cdots + 1 + 1 = k.$$

However, this implies that the characteristic of κ divides k, which is a contradiction. Hence we must have $r = 1$. □

3.2 Chevalley groups over finite local rings

Let $\mathfrak{o}$ be the ring of integers of a non-archimedean local field K and $\mathfrak{L}$ be a non-abelian simple Lie algebra over $\mathbb{C}$ with root system Σ and base Π. If $\mathfrak{L}_{\mathbb{Z}}$ again denotes the $\mathbb{Z}$-span of the Chevalley basis in $\mathfrak{L}$, then

$$\mathfrak{L}_{\mathfrak{o}} = \mathfrak{L}_{\mathbb{Z}} \otimes \mathfrak{o}$$

is a Lie ring where the Lie bracket is inherited from $\mathfrak{L}_{\mathbb{Z}}$. Further, since $\mathfrak{o}$ is a subring of K, we can consider $\mathfrak{L}_{\mathfrak{o}}$ to be a Lie subring of $\mathfrak{L}_K$. Thus $\mathrm{Aut}(\mathfrak{L}_{\mathfrak{o}})$ can be identified with the subgroup of $\mathrm{Aut}(\mathfrak{L}_K)$ consisting of the automorphisms of $\mathfrak{L}_K$ which preserve $\mathfrak{L}_{\mathfrak{o}}$. In particular, this means that for any $\alpha \in \Sigma$ and $r \in \mathfrak{o}$ the automorphism $x_\alpha(r)$ of $\mathfrak{L}_K$ can be considered as an automorphism of $\mathfrak{L}_{\mathfrak{o}}$.

Now, fix $\ell > 1$ and set $R = \mathfrak{o}/\mathfrak{p}^\ell$. Then

$$\mathfrak{L}_R = \mathfrak{L}_{\mathbb{Z}} \otimes R$$

is again a Lie ring with the Lie bracket inherited from $\mathfrak{L}_{\mathbb{Z}}$. Moreover, the natural projection $\eta : \mathfrak{o} \to R$ gives rise to a Lie ring homomorphism $\eta : \mathfrak{L}_{\mathfrak{o}} \to \mathfrak{L}_R$ and this in turn induces a homomorphism of automorphism groups

$$\eta : \mathrm{Aut}(\mathfrak{L}_{\mathfrak{o}}) \to \mathrm{Aut}(\mathfrak{L}_R).$$

Consequently, if for each $\alpha \in \Sigma$ and $r \in R$ we define

$$\overline{x}_\alpha(r) = \eta(x_\alpha(r'))$$

where $r' \in \mathfrak{o}$ has $\eta(r') = r$, then $\overline{x}_\alpha(r)$ is an automorphism of $\mathfrak{L}_R$.

Definition 3.2.1. The **Chevalley group** of type Σ over the ring R is the subgroup

$$G(R) = \langle \overline{x}_\alpha(r) : \alpha \in \Sigma, r \in R \rangle$$

of the automorphism group $\mathrm{Aut}(\mathfrak{L}_R)$.

Let $G(\mathfrak{o}) = G(K) \cap \mathrm{Aut}(\mathfrak{L}_\mathfrak{o})$. By [32, Theorem 18, Corollary 3]

$$G(\mathfrak{o}) = \langle x_\alpha(r) : \alpha \in \Sigma, r \in \mathfrak{o} \rangle$$

and so we have

$$\eta(G(\mathfrak{o})) = \langle \eta(x_\alpha(r)) : \alpha \in \Sigma, r \in \mathfrak{o} \rangle = \langle \overline{x}_\alpha(r) : \alpha \in \Sigma, r \in R \rangle = G(R).$$

With this in mind, for each subgroup $H(K)$ of $G(K)$, we let $H(\mathfrak{o})$ denote the subgroup $H(\mathfrak{o}) = H(K) \cap \mathrm{Aut}(\mathfrak{L}_\mathfrak{o})$ of $G(\mathfrak{o})$ and $H(R)$ its corresponding image $H(R) = \eta(H(\mathfrak{o}))$ in $G(R)$.

From Definition 2.3.1 it is clear that $x_\alpha(r) \in \mathrm{Aut}(\mathfrak{L}_\mathfrak{o})$ if and only if $r \in \mathfrak{o}$. Thus, $U_\alpha(\mathfrak{o}) = \{x_\alpha(r) : r \in \mathfrak{o}\}$ for each $\alpha \in \Sigma$ and so $U_\alpha(R) = \{\overline{x}_\alpha(r) : r \in R\}$. Further, for any $r, s \in R$ we see that if $r', s' \in \mathfrak{o}$ are such that $\eta(r') = r$ and $\eta(s') = s$, then

$$\overline{x}_\alpha(r)\overline{x}_\alpha(s) = \eta(x_\alpha(r'))\eta(x_\alpha(s')) = \eta(x_\alpha(r' + s')) = \overline{x}_\alpha(r + s).$$

Thus the root subgroup $U_\alpha(R)$ is isomorphic to the additive group R.

More generally, the Chevalley Commutator Formula for $G(K)$ gives the corresponding commutator formula for $G(R)$.

Theorem 3.2.2 (Chevalley Commutator Formula). *For any $\alpha, \beta \in \Sigma$ with $\alpha \neq \pm\beta$ and $r, s \in R$ we have*

$$[\overline{x}_\alpha(r), \overline{x}_\beta(s)] = \prod_{i,j>0} \overline{x}_{i\alpha+j\beta}(c_{i,j,\alpha,\beta}(-r)^i s^j)$$

where the constants $c_{i,j,\alpha,\beta}$ are considered as elements of R.

Proof. Let $r', s' \in \mathfrak{o}$ be such that $\eta(r') = r$ and $\eta(s') = s$, then we see that

$$[\overline{x}_\alpha(r), \overline{x}_\beta(s)] = [\eta(x_\alpha(r')), \eta(x_\beta(s'))] = \eta([x_\alpha(r'), x_\beta(s')])$$

and the result follows from Theorem 2.3.2. $\square$

In particular, this implies that if $rs = 0$, then $x_\alpha(r)$ and $x_\beta(s)$ commute.

By [32, Lemma 49(b)] we have $U(\mathfrak{o}) = \langle U_\alpha(\mathfrak{o}) : \alpha \in \Sigma^+ \rangle$ and so applying η we obtain

$$U(R) = \eta(U(\mathfrak{o})) = \langle \eta(U_\alpha(\mathfrak{o})) : \alpha \in \Sigma^+ \rangle = \langle U_\alpha(R) : \alpha \in \Sigma^+ \rangle.$$

Similarly, we have $U^-(R) = \langle U_\alpha(R) : \alpha \in \Sigma^- \rangle$. Thus, the commutator formula once again gives the following result.

Lemma 3.2.3. *(i) Each $g \in U(R)$ can be expressed as*

$$g = \prod_{\alpha \in \Sigma^+} \overline{x}_\alpha(r_\alpha)$$

for some $r_\alpha \in R$.

(ii) Each $g \in U^-(R)$ can be expressed as

$$g = \prod_{\alpha \in \Sigma^-} \overline{x}_\alpha(r_\alpha)$$

for some $r_\alpha \in R$.

For each $\alpha \in \Sigma$ the homomorphism ϕ_α in Proposition 2.3.4 restricts to a homomorphism $\phi_\alpha : \mathrm{SL}_2(\mathfrak{o}) \to G(\mathfrak{o})$ [32, Lemma 48] with

$$\phi_\alpha(\mathrm{SL}_2(\mathfrak{o})) = \phi_\alpha(\mathrm{SL}_2(K)) \cap \mathrm{Aut}(\mathfrak{L}_\mathfrak{o}). \qquad \text{[32, Lemma 49(d)]}$$

Thus, if we define a map $\overline{\phi}_\alpha : \mathrm{SL}_2(R) \to G(R)$ by setting for each $g \in \mathrm{SL}_2(R)$

$$\overline{\phi}_\alpha(g) = \eta(\phi_\alpha(g'))$$

where $g' \in \mathrm{SL}_2(\mathfrak{o})$ has $\eta(g') = g$, then this must also be a homomorphism.

Proposition 3.2.4. *For each $\alpha \in \Sigma$, the map $\overline{\phi}_\alpha : \mathrm{SL}_2(R) \to G(R)$ given by*

$$\overline{\phi}_\alpha\left(\begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix}\right) = x_\alpha(r) \qquad \textit{and} \qquad \overline{\phi}_\alpha\left(\begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix}\right) = x_{-\alpha}(s)$$

for every $r, s \in R$ is a homomorphism.

Consequently, if we again define

$$\overline{h}_\alpha(r) = \overline{\phi}_\alpha\left(\begin{bmatrix} r & 0 \\ 0 & r^{-1} \end{bmatrix}\right) \qquad \text{and} \qquad \overline{n}_\alpha(r) = \overline{\phi}_\alpha\left(\begin{bmatrix} 0 & r \\ -r^{-1} & 0 \end{bmatrix}\right)$$

for each $\alpha \in \Sigma$ and $r \in R^\times$, then we have $\overline{h}_\alpha(r) = \eta(h_\alpha(r'))$ and $\overline{n}_\alpha(r) = \eta(n_\alpha(r'))$ for some $r' \in \mathfrak{o}^\times$ with $\eta(r') = r$. Thus, we immediately obtain the corresponding versions of Lemmas 2.3.5 and 2.3.6.

Further, Proposition 3.2.4 allows us to easily calculate the following special case of the commutator $[x_\alpha(r), x_{-\alpha}(s)]$.

Lemma 3.2.5. *Let $r, s \in R$ with $r^2s^2 = 0$, then for any $\alpha \in \Sigma$*

$$[\overline{x}_\alpha(r), \overline{x}_{-\alpha}(s)] = \overline{h}_\alpha(1 + rs)\overline{x}_\alpha(-r^2s)\overline{x}_{-\alpha}(rs^2).$$

Proof. The result follows from Proposition 3.2.4 and the fact that

$$\begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix}\begin{bmatrix} 1 & -r \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -s & 1 \end{bmatrix} = \begin{bmatrix} 1+rs & 0 \\ 0 & 1-rs \end{bmatrix}\begin{bmatrix} 1 & -r^2s \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ rs^2 & 1 \end{bmatrix}$$

in $\mathrm{SL}_2(R)$. □

Now, $T(\mathfrak{o}) = \langle h_\alpha(r) : r \in \mathfrak{o}^\times, \alpha \in \Sigma \rangle$ [32, Lemma 49(a)] with $B(\mathfrak{o}) = T(\mathfrak{o})U(\mathfrak{o})$ [32, Lemma 49(a)]. Thus,

$$T(R) = \eta(T(\mathfrak{o})) = \langle \eta(\overline{h}_\alpha(r)) : r \in \mathfrak{o}^\times, \alpha \in \Sigma \rangle = \langle \overline{h}_\alpha(r) : r \in R^\times, \alpha \in \Sigma \rangle$$

and $B(R) = \eta(B(\mathfrak{o})) = \eta(T(\mathfrak{o})U(\mathfrak{o})) = \eta(T(\mathfrak{o}))\eta(U(\mathfrak{o})) = T(R)U(R)$. In this case, however, $B(R)$ and $N(R)$ do not in general form a BN-pair for $G(R)$. In particular, although for each $w \in W$ we have $n_w \in N(\mathfrak{o})$ and so may define

$$\overline{n}_w = \eta(n_w),$$

they do not form a complete set of $(B(R), B(R))$-double coset representatives in $G(R)$. Indeed, we will see later that in general the $(B(R), B(R))$-double coset structure of $G(R)$ is much more complicated than was the case for $G(K)$.

3.3 Extended Chevalley groups over finite local rings

Let μ be a K-character of Λ_r such that $h(\mu) \in \mathrm{Aut}(\mathfrak{L}_\mathfrak{o})$, then by (2.2) it is clear that we need $\mu(\alpha) \in \mathfrak{o}^\times$ for every $\alpha \in \Sigma$. Conversely, if μ is a K-character of Λ_r with $\mu(\alpha) \in \mathfrak{o}^\times$ for every $\alpha \in \Sigma$, then obviously $h(\mu) \in \mathrm{Aut}(\mathfrak{L}_\mathfrak{o})$. Thus, if we let $\widehat{T}(\mathfrak{o}) = \widehat{T}(K) \cap \mathrm{Aut}(\mathfrak{L}_\mathfrak{o})$ then we see that

$$\widehat{T}(\mathfrak{o}) = \{h(\mu) : \mu(\alpha) \in \mathfrak{o}^\times \text{ for every } \alpha \in \Sigma\}.$$

Now, any K-character μ of Λ_r with $\mu(\alpha) \in \mathfrak{o}^\times$ for each $\alpha \in \Sigma$ gives rise to an R-character $\eta(\mu)$ of Λ_r by setting for every $\alpha \in \Sigma$

$$\eta(\mu)(\alpha) = \eta(\mu(\alpha)).$$

Further, suppose that $\overline{\mu}$ is an R-character of Λ_r. If for each $\alpha \in \Pi$ we choose $s_\alpha \in \mathfrak{o}^\times$ with $\eta(s_\alpha) = \overline{\mu}(\alpha)$, then we may define a K-character μ of Λ_r by setting $\mu(\alpha) = s_\alpha$. Thus, we must have $\eta(\mu) = \overline{\mu}$ since $\eta(\mu(\alpha)) = \eta(s_\alpha) = \overline{\mu}(\alpha)$ for every $\alpha \in \Pi$.

As a consequence of this, if for each R-character $\overline{\mu}$ of Λ_r we define the diagonal automorphism $\overline{h}(\overline{\mu})$ of $\mathfrak{L}_R$ by

$$\overline{h}(\overline{\mu}) \cdot (h_\beta \otimes 1_R) = (h_\beta \otimes 1_R) \qquad \text{and} \qquad \overline{h}(\overline{\mu}) \cdot (e_\beta \otimes 1_R) = \overline{\mu}(\alpha)(e_\beta \otimes 1_R), \quad (3.1)$$

then $\overline{h}(\overline{\mu}) = \eta(h(\mu))$ for some $h(\mu) \in \widehat{T}(\mathfrak{o})$. In particular, this means the corresponding version of Lemma 2.4.2 must hold for $\overline{h}(\overline{\mu})$. Further, setting $\widehat{T}(R) = \eta(\widehat{T}(\mathfrak{o}))$, we obtain

$$\widehat{T}(R) = \{\overline{h}(\overline{\mu}) : \overline{\mu} \text{ is an } R\text{-character of } \Lambda_r\}.$$

Moreover, the proof of Proposition 2.4.1 from [2] is also valid in this situation.

Theorem 3.3.1. *$T(R)$ is the subgroup of $\widehat{T}(R)$ consisting of the diagonal automorphisms $\overline{h}(\overline{\mu})$ where $\overline{\mu}$ is the restriction to Λ_r of an R-character of Λ.*

For each $r \in R^\times$ and $\alpha \in \Pi$, let

$$\overline{y}_\alpha(r) = \overline{h}(\overline{\nu}_{\alpha,r})$$

where $\overline{\nu}_{\alpha,r} : \Lambda_r \to R^\times$ is the R-character of Λ_r given by

$$\overline{\nu}_{\alpha,r}(\beta) = \begin{cases} r & \text{if } \alpha = \beta; \\ 1 & \text{otherwise} \end{cases}$$

for each $\beta \in \Pi$. Then we see that any R-character $\overline{\mu}$ of Λ_r can be uniquely expressed as the sum

$$\overline{\mu} = \prod_{\alpha \in \Pi} \overline{\nu}_{\alpha,\overline{\mu}(\alpha)}$$

and so any diagonal automorphism $\overline{h}(\overline{\mu}) \in \widehat{T}(R)$ can be uniquely expressed as the product

$$\overline{h}(\overline{\mu}) = \prod_{\alpha \in \Pi} \overline{h}(\overline{\nu}_{\alpha,\overline{\mu}(\alpha)}).$$

In particular, this implies that $|\widehat{T}(R)| = q^{(\ell-1)n}(q-1)^n$.

Lemma 3.3.2. *For each $\alpha \in \Pi$, $r \in R^\times$ and $\beta \in \Sigma$, $s \in R$ we have*

$$y_\alpha(r)x_\beta(s)y_\alpha(r)^{-1} = x_\beta(r^{k_\alpha}s)$$

where k_α is as in (2.1).

Finally, if we set $\widehat{B}(\mathfrak{o}) = \widehat{B}(K) \cap \mathrm{Aut}(\mathfrak{L}_\mathfrak{o})$ and $\widehat{B}(R) = \eta(\widehat{B}(\mathfrak{o}))$, then by adapting the proof of [32, Lemma 49(a)] we have $\widehat{B}(\mathfrak{o}) = \widehat{T}(\mathfrak{o})U(\mathfrak{o})$ which implies that

$$\widehat{B}(R) = \eta(\widehat{B}(\mathfrak{o})) = \eta(\widehat{T}(\mathfrak{o}))(U(\mathfrak{o})) = \widehat{T}(R)U(R).$$

Definition 3.3.3. The **extended Chevalley group** of type Σ over R is the subgroup $\widehat{G}(R)$ of $\mathrm{Aut}(\mathfrak{L}_R)$ generated by $G(R)$ and $\widehat{T}(R)$.

Theorem 3.3.4. *$G(R)$ is a normal subgroup of $\widehat{G}(R)$ with $\widehat{G}(R)/G(R) \simeq \widehat{T}(R)/T(R)$.*

3.4 Congruence subgroups

In the last section of this chapter we consider certain normal subgroups of $G(R)$ and $\widehat{G}(R)$ which are obtained from the ideals of R. If for each $1 \leq i \leq \ell - 1$, we set

$$\mathfrak{L}_{\mathfrak{o}/\mathfrak{p}^i} = \mathfrak{L}_\mathbb{Z} \otimes \mathfrak{o}/\mathfrak{p}^i$$

then the natural projection $\eta_i : R \to \mathfrak{o}/\mathfrak{p}^i$ again gives rise to a homomorphism of automorphism groups

$$\eta_i : \mathrm{Aut}(\mathfrak{L}_R) \to \mathrm{Aut}(\mathfrak{L}_{\mathfrak{o}/\mathfrak{p}^i}).$$

The intersection of the kernel of this homomorphism with $G(R)$ is then the congruence subgroup $K_i = \ker(\eta_i) \cap G(R)$. Further, for each subgroup $H(R)$ of $G(R)$, let $H(\mathfrak{m}^i)$ denote the subgroup $H(\mathfrak{m}^i) = \ker(\eta_i) \cap H(R)$.

In particular, for each $\alpha \in \Sigma$ we see that $\overline{x}_\alpha(r) \in \ker(\eta_i)$ if and only if $r \in \mathfrak{m}^i$. Thus $U_\alpha(\mathfrak{m}^i) = \{\overline{x}_\alpha(r) : r \in \mathfrak{m}^i\}$ and is isomorphic to the additive group $\mathfrak{m}^i$.

Lemma 3.4.1. *(i) Each $g \in U(\mathfrak{m}^i)$ can be expressed as*

$$g = \prod_{\alpha \in \Sigma^+} \overline{x}_\alpha(r_\alpha)$$

for some $r_\alpha \in \mathfrak{m}^i$;

(ii) Each $g \in U^-(\mathfrak{m}^i)$ can be expressed as

$$g = \prod_{\alpha \in \Sigma^-} \overline{x}_\alpha(r_\alpha)$$

for some $r_\alpha \in \mathfrak{m}^i$; and

(iii) $K_i = U^-(\mathfrak{m}^i)T(\mathfrak{m}^i)U(\mathfrak{m}^i)$.

Similarly, consider the congruence subgroup $\widehat{K}_i = \ker(\eta_i) \cap \widehat{G}(R)$ of $\widehat{G}(R)$ and let $\widehat{H}(\mathfrak{m}^i) = \ker(\eta_i) \cap \widehat{H}(R)$ for any subgroup $\widehat{H}(R)$ of $G(R)$.

Lemma 3.4.2. *(i) $\widehat{T}(\mathfrak{m}^i) = \{\overline{h}(\overline{\mu}) : \overline{\mu}(\alpha) \in 1 + \mathfrak{m}^i$ for each $\alpha \in \Sigma\}$; and*

(ii) $\widehat{K}_i = U^-(\mathfrak{m}^i)\widehat{T}(\mathfrak{m}^i)U(\mathfrak{m}^i)$.

Proof. (i) Suppose that $\overline{\mu}$ is an R-character of Λ_r with $\overline{h}(\overline{\mu}) \in \ker(\eta_i)$. Then, by (3.1) we must have $\eta_i(\overline{\mu}(\alpha)) = 1$ and so $\overline{\mu}(\alpha) \in 1 + \mathfrak{m}^i$ for every $\alpha \in \Sigma$. Conversely, if $\overline{\mu}(\alpha) \in 1 + \mathfrak{m}^i$ for each $\alpha \in \Sigma$ then clearly $\overline{h}(\overline{\mu}) \in \ker(\eta_i)$.

(ii) This can be shown by adapting the proof of Lemma 3.4.1(iii) from [1]. $\square$

We conclude this chapter by recording some commutator calculations.

Lemma 3.4.3. *Let $r \in \mathfrak{m}$ and $s \in R$ be such that $rs \in \mathfrak{m}^{\ell-1}$.*

(i) If $\alpha \in \Sigma^-$ and $\beta \in \Sigma^+$ have $\mathrm{ht}(\alpha) + \mathrm{ht}(\beta) \geq 0$, then $[\overline{x}_\alpha(r), \overline{x}_\beta(s)] \in B(\mathfrak{m}^{\ell-1})$.

(ii) If $\alpha \in \Sigma^-$ and $\beta \in \Sigma^+$ are such that $\alpha + \beta \in \Sigma_{-1}$, then for some $v \in B(\mathfrak{m}^{\ell-1})$ we have $[\overline{x}_\alpha(r), \overline{x}_\beta(s)] = \overline{x}_{\alpha+\beta}(c_{1,1,\alpha,\beta}(-r)s)v$.

(iii) If $\alpha \in \Sigma^-$ and $\beta \in \Sigma^+$ have $\mathrm{ht}(\alpha) + \mathrm{ht}(\beta) = -1$, but $\alpha + \beta \notin \Sigma$ then $[\overline{x}_\alpha(r), \overline{x}_\beta(s)] \in B(\mathfrak{m}^{\ell-1})$.

Proof. Let $r \in \mathfrak{m}$ and $s \in R$ be such that $rs \in \mathfrak{m}^{\ell-1}$.

(i) Consider $\beta \in \Sigma^+$ with $\beta \neq -\alpha$. The commutator formula implies that

$$[\overline{x}_\alpha(r), \overline{x}_\beta(s)] = \prod_{i,j>0} \overline{x}_{i\alpha+j\beta}(c_{i,j,\alpha,\beta}(-r)^i s^j).$$

Now, since $r \in \mathfrak{m}$ and $s \in R$ with $rs \in \mathfrak{m}^{\ell-1}$ we have $(-r)^i s = 0$ for any $i > 1$. Thus

$$[\overline{x}_\alpha(r), \overline{x}_\beta(s)] = \prod_{j>0} \overline{x}_{\alpha+j\beta}(c_{1,j,\alpha,\beta}(-r)s^j).$$

Further, for any $j > 0$ with $\alpha + j\beta \in \Sigma$ we must have $\alpha + j\beta \in \Sigma^+$ since $\mathrm{ht}(\alpha+j\beta) = \mathrm{ht}(\alpha)+j\mathrm{ht}(\beta) \geq (j-1)\mathrm{ht}(\beta) \geq 0$. Hence $[\overline{x}_\alpha(r), \overline{x}_\beta(s)] \in B(\mathfrak{m}^{\ell-1})$.

If $\beta = -\alpha$, then by Lemma 3.2.5

$$[\overline{x}_\alpha(r), \overline{x}_{-\alpha}(s)] = \overline{h}_\alpha(1+rs)\overline{x}_{-\alpha}(rs^2) \in B(\mathfrak{m}^{\ell-1}).$$

(ii) Again, since $r \in \mathfrak{m}$ and $s \in R$ with $rs \in \mathfrak{m}^{\ell-1}$ we have

$$[\overline{x}_\alpha(r), \overline{x}_\beta(s)] = \prod_{j>0} \overline{x}_{\alpha+j\beta}(c_{1,j,\alpha,\beta}(-r)s^j).$$

However, on this occasion $\alpha + j\beta \in \Sigma$ implies $\alpha + j\beta \in \Sigma^+$ only for any $j > 1$. Consequently,

$$[\overline{x}_\alpha(r), \overline{x}_\beta(s)] = \overline{x}_{\alpha+\beta}(c_{1,1,\alpha,\beta}(-r)s) \prod_{j>1} \overline{x}_{\alpha+j\beta}(c_{1,j,\alpha,\beta}(-r)s^j)$$

where $v = \prod_{j>1} \overline{x}_{\alpha+j\beta}(c_{1,j,\alpha,\beta}(-r)s^j) \in B(\mathfrak{m}^{\ell-1})$.

(iii) This follows immediately from the proof of (ii) since $\alpha+\beta \notin \Sigma$ implies that the $j=1$ term does not appear and we obtain

$$[\overline{x}_\alpha(r), \overline{x}_\beta(s)] = \prod_{j>1} \overline{x}_{\alpha+j\beta}(c_{1,j,\alpha,\beta}(-r)s^j) \in B(\mathfrak{m}^{\ell-1}). \qquad \square$$

The above calculations will be used in conjunction with the following general result about commutators.

Lemma 3.4.4. *For any a, b, c we have the following:*

(i) $[a, bc] = [a, b]\left(b[a, c]b^{-1}\right)$*; and*

(ii) $[ab, c] = \left(a[b, c]a^{-1}\right)[a, c]$.

Proof.

(i) $[a, bc] = a(bc)a^{-1}(c^{-1}b^{-1}) = (aba^{-1}b^{-1})b(aca^{-1}c^{-1})b^{-1} = [a, b]\left(b[a, c]b^{-1}\right)$.

(ii) $[ab, c] = (ab)c(b^{-1}a^{-1})c^{-1} = a(bcb^{-1}c^{-1})a^{-1}(aca^{-1}c^{-1}) = \left(a[b, c]a^{-1}\right)[b, c]$. □

In particular, if b commutes with $[a, c]$ then $[a, bc] = [a, b][a, c]$. Similarly, if a commutes with $[b, c]$ then $[ab, c] = [b, c][a, c]$.

Chapter 4

Construction

It turns out to be easier to consider the analogue of the Steinberg character first for the extended Chevalley groups over R. As there is now no risk of confusion we remove the reference to R from the notation for the groups so that, for example, $\widehat{G}(R)$ is simply written as $\widehat{G}$. Similarly, we omit the bars from the notation for the elements of $\widehat{G}$ which means that, for example, $\overline{x}_\alpha(r)$ is denoted by $x_\alpha(r)$.

Additionally we will require that the residue class field κ has **good** characteristic (cf. [3]), i.e.

(i) char $\kappa \neq 2$ if $\Sigma = B_n$, C_n or D_n;

(ii) char $\kappa \neq 2$ or 3 if $\Sigma = F_4$, G_2, E_6 or E_7; and

(iii) char $\kappa \neq 2$, 3 or 5 if $\Sigma = E_8$.

This will ensure that certain combinations of the Chevalley commutator constants are invertible in R.

As both Steinberg's original construction [29] for the general linear group and Curtis' later definition [4] for finite groups with BN-pairs were in terms of permutation characters over parabolic subgroups, we begin by considering the concept of a parabolic subgroup in our situation.

4.1 Parabolic subgroups

Definition 4.1.1. A subgroup $\widehat{P}$ of $\widehat{G}$ is **parabolic** if it is of the form

$$\widehat{P} = \langle U_\alpha(\mathfrak{m}^{i_\alpha}), \widehat{B} : \alpha \in \Sigma^- \rangle \tag{4.1}$$

for some $0 \leq i_\alpha \leq \ell$.

Usually the parabolic subgroups of $\widehat{G}$ are taken to be all of the subgroups containing $\widehat{B}$. By [34], when char $\kappa \neq 2$ or 3 every subgroup containing $\widehat{B}$ must have the form in (4.1). However, when char $\kappa = 2$, and char $\kappa = 3$ for certain Σ, there exist subgroups of $\widehat{G}$ which contain $\widehat{B}$ but are not of this form. We will need that our parabolic subgroups can be constructed using the root subgroups of $\widehat{G}$.

Example 4.1.2. The parabolic subgroups of $\widehat{G} = \mathrm{PGL}_2(R)$ are of the form

$$\widehat{B}_i = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \widehat{G} : c \in \mathfrak{m}^i \right\}$$

for each $0 \leq i \leq \ell$.

Let $S = \{\alpha \in \Sigma : -\alpha \in \Pi\}$ and for each $\alpha \in S$ consider the parabolic subgroup

$$\widehat{H}_\alpha = \langle X_\alpha, \widehat{B} \rangle$$

where $X_\alpha = U_\alpha(\mathfrak{m}^{\ell-1})$.

Lemma 4.1.3. $\widehat{H}_\alpha = X_\alpha \widehat{B} = \widehat{B} X_\alpha$.

Proof. We need to show that $X_\alpha \widehat{B} \subseteq \widehat{B} X_\alpha$ since, by considering inverses, this would also imply that $\widehat{B} X_\alpha \subseteq X_\alpha \widehat{B}$. Thus, we would obtain $X_\alpha \widehat{B} = \widehat{B} X_\alpha$ which must therefore be equal to $\widehat{H}_\alpha$.

Fix $r \in \mathfrak{m}^{\ell-1}$ and let $\beta \in \Sigma^+$. Clearly $\mathrm{ht}(\alpha) + \mathrm{ht}(\beta) \geq 0$, and so Lemma 3.4.3(i) implies that $[x_\alpha(r), x_\beta(s)] \in \widehat{B}(\mathfrak{m}^{\ell-1})$ for any $s \in R$. Thus if $u \in U$ is expressed as

$u = \prod_{\beta \in \Sigma^+} x_\beta(s_\beta)$ for some $s_\beta \in R$, then by Lemma 3.4.4(i)

$$[x_\alpha(r), u] = [x_\alpha(r), \prod_{\beta \in \Sigma^+} x_\beta(s)] = \prod_{\beta \in \Sigma^+} [x_\alpha(r), x_\beta(s)] \in \widehat{B}(\mathfrak{m}^{\ell-1}).$$

Hence $x_\alpha(r)u = [x_\alpha(r), u]ux_\alpha(r) \in \widehat{B}X_\alpha$ for each $x_\alpha(r) \in X_\alpha$ and $u \in U$, implying that $X_\alpha U \subseteq \widehat{B}X_\alpha$.

Further, for each $h(\mu) \in \widehat{T}$

$$x_\alpha(r)h(\mu) = h(\mu)h(\mu)^{-1}x_\alpha(r)h(\mu) = h(\mu)x_\alpha(\mu(\alpha)^{-1}r).$$

Thus $x_\alpha(r)h(\mu) \in \widehat{B}X_\alpha$ for each $x_\alpha(r) \in X_\alpha$ and $h(\mu) \in \widehat{T}$, which gives $X_\alpha\widehat{T} \subseteq \widehat{B}X_\alpha$. Consequently,

$$X_\alpha\widehat{B} = X_\alpha(\widehat{T}U) = (X_\alpha\widehat{T})U \subseteq (\widehat{B}X_\alpha)U = \widehat{B}(X_\alpha U) \subseteq \widehat{B}(\widehat{B}X_\alpha) = \widehat{B}X_\alpha$$

as required. □

Proposition 4.1.4. *$\{\widehat{H}_\alpha : \alpha \in S\}$ are the minimal parabolic subgroups which strictly contain $\widehat{B}$.*

Proof. It is clear from Lemma 4.1.3 that each $\widehat{H}_\alpha$ is a minimal parabolic subgroup strictly containing $\widehat{B}$. Thus we need to show that if $\widehat{P}$ is a minimal parabolic subgroup which strictly contains $\widehat{B}$, then $\widehat{P} = \widehat{H}_\alpha$ for some α.

First note that there must be some $\alpha \in \Sigma^-$ and $0 \leq i < \ell$ with $U_\alpha(\mathfrak{m}^i) \leq \widehat{P}$ since $\widehat{P}$ is parabolic, but not equal to $\widehat{B}$. In particular this means that we have $U_\alpha(m^{\ell-1}) \leq \widehat{P}$. Thus, if $\alpha \in S$ then we must have $\widehat{H}_\alpha \leq \widehat{P}$ implying that $\widehat{P} = \widehat{H}_\alpha$ by minimality.

Suppose that $\alpha \notin S$. Then there is a root $\beta \in \Sigma^+$ with $\alpha + \beta = \gamma$ for some $\gamma \in S$. For any $r \in \mathfrak{m}^{\ell-1}$ and $s \in R$ we have $x_\alpha(r), x_\beta(s) \in \widehat{P}$ and so $[x_\alpha(r), x_\beta(s)] \in \widehat{P}$. However, by Lemma 3.4.3(ii) we have $[x_\alpha(r), x_\beta(s)] = x_\gamma(c_{1,1,\alpha,\beta}(-r)s)v$ for some $v \in \widehat{B}(\mathfrak{m}^{\ell-1})$. Thus, if for each $t \in \mathfrak{m}^{h-1}$ we choose $r \in \mathfrak{m}^{h-1}$ and $s \in R$ such that

$c_{1,1,\alpha+\beta}(-r)s = t$, we obtain $[x_\alpha(r), x_\beta(s)]v^{-1} = x_\gamma(t) \in \widehat{P}$ and therefore $X_\gamma \leq \widehat{P}$. Hence $\widehat{H}_\gamma \leq \widehat{P}$ implying that $\widehat{P} = \widehat{H}_\gamma$ by minimality. □

Now, for each non-empty $J = \{\alpha_{j_1}, \ldots, \alpha_{j_k}\} \subseteq S$ consider the parabolic subgroup

$$\widehat{H}_J = \langle \widehat{H}_{\alpha_{j_1}}, \ldots, \widehat{H}_{\alpha_{j_k}} \rangle.$$

Then, since for each $\alpha, \beta \in S$

$$\widehat{H}_\alpha \widehat{H}_\beta = \widehat{B} X_\alpha X_\beta \widehat{B} = \widehat{B} X_\beta X_\alpha \widehat{B} = \widehat{H}_\beta \widehat{H}_\alpha \tag{4.2}$$

we must have $\widehat{H}_J = \widehat{H}_{\alpha_{j_1}} \cdots \widehat{H}_{\alpha_{j_k}}$. In particular, this means that if we set

$$X_J = \{x_{\alpha_{j_1}}(r_1) \cdots x_{\alpha_{j_k}}(r_k) : r_1, \ldots, r_k \in \mathfrak{m}^{\ell-1}\},$$

then

$$\widehat{H}_J = \widehat{H}_{\alpha_{j_1}} \cdots \widehat{H}_{\alpha_{j_k}} = X_{\alpha_{j_1}} \widehat{B} \cdots X_{\alpha_{j_k}} \widehat{B} = X_{\alpha_{j_1}} \cdots X_{\alpha_{j_k}} \widehat{B} = X_J \widehat{B} \tag{4.3}$$

and similarly $\widehat{H}_J = \widehat{B} X_J$. Additionally, let $\widehat{H}_\emptyset = \widehat{B}$ and $X_\emptyset = \{1\}$ so that again $\widehat{H}_\emptyset = X_\emptyset \widehat{B} = \widehat{B} X_\emptyset$.

Lemma 4.1.5. *Let $I, J \subseteq S$, then*

(i) $|\widehat{H}_J| = q^{|J|} |\widehat{B}|$;

(ii) $\widehat{H}_I \widehat{H}_J = \widehat{H}_J \widehat{H}_I$;

(iii) $\widehat{H}_I \cap \widehat{H}_J = \widehat{H}_{I \cap J}$; *and*

(iv) $\langle \widehat{H}_I, \widehat{H}_J \rangle = \widehat{H}_{I \cup J}$.

Proof. (i) $|\widehat{H}_J| = |X_J \widehat{B}| = |X_J||\widehat{B}| = q^{|J|}|\widehat{B}| = |\widehat{H}_J|$ since $X_J \cap \widehat{B} = \{1\}$.

(ii) $\widehat{H}_I \widehat{H}_J = \widehat{B} X_I X_J \widehat{B} = \widehat{B} X_J X_I \widehat{B} = \widehat{H}_J \widehat{H}_I$.

(iii) $\widehat{H}_I \cap \widehat{H}_J = X_I \widehat{B} \cap X_J \widehat{B} = (X_I \cap X_J)\widehat{B} = X_{I \cap J} \widehat{B}$.

(iv) This is immediate from the definition. □

The analogue of the Steinberg character is then defined to be an alternating sum of permutation characters over the parabolic subgroups $\widehat{H}_J$ for $J \subseteq S$.

Definition 4.1.6. St_ℓ is the virtual character

$$\mathrm{St}_\ell = \sum_{J \subseteq S} (-1)^{|J|} (1_{\widehat{H}_J})^{\widehat{G}}.$$

Remarks 4.1.7. (i) If we apply the definition of the parabolic subgroup $\widehat{H}_\alpha$ in the case where $h = 1$, then we obtain the minimal standard parabolic subgroups of $\widehat{G}(\kappa)$ strictly containing $\widehat{B}(\kappa)$. Thus, the subgroups $\widehat{H}_J$ are exactly the standard parabolic subgroups and the expression for St_ℓ as an alternating sum of permutation characters reduces to the formula for the Steinberg character given by Curtis [4].

(ii) Further, for $\mathrm{PGL}_n(R)$ the definition of St_ℓ can be inflated to give a corresponding alternating sum of permutation characters in $\mathrm{GL}_n(R)$. The resulting expression is identical to the alternating sum obtained by Lees [21, Corollary 3.23] for the character afforded by the top homology space of the simplicial complex he described. Consequently, St_ℓ is the same as the analogue of the Steinberg character defined by Lees after reduction modulo the centre of $\mathrm{GL}_n(R)$.

4.2 Module affording St_ℓ

The definition of St_ℓ in the previous section was as a virtual character of $\widehat{G}$. Thus, we would like to show that it is actually a character of $\widehat{G}$ and to accomplish this we describe a method of constructing modules whose characters are alternating sums of permutation characters.

Let G be an arbitrary finite group. For each subgroup H of G define e_H to be the idempotent

$$e_H = \frac{1}{|H|} \sum_{h \in H} h$$

in the group ring $\mathbb{C}G$ and recall that the permutation module $\mathbb{C}Ge_H$ affords the permutation character $(1_H)^G$.

Further, if K is a subgroup of G which contains H, then $e_H e_K = e_K$ and so $\mathbb{C}Ge_K \subseteq \mathbb{C}Ge_H$. Indeed, $\mathbb{C}Ge_H$ can be expressed as $\mathbb{C}Ge_H = \mathbb{C}G(e_H - e_K) \oplus \mathbb{C}Ge_K$ which implies that $\mathbb{C}G(e_H - e_K)$ affords the character $(1_H)^G - (1_K)^G$.

Now, let H_1 and H_2 be two subgroups of G. Then it can be shown that the intersection of the corresponding permutation modules is $\mathbb{C}Ge_{H_1} \cap \mathbb{C}Ge_{H_2} = \mathbb{C}Ge_{H_{\{1,2\}}}$ where $H_{\{1,2\}} = \langle H_1, H_2 \rangle$. Consequently

$$\mathbb{C}Ge_{H_1} + \mathbb{C}Ge_{H_2} = \mathbb{C}G(e_{H_1} - e_{H_{\{1,2\}}}) \oplus \mathbb{C}G(e_{H_2} - e_{H_{\{1,2\}}}) \oplus \mathbb{C}Ge_{H_{\{1,2\}}}$$

and thus affords the character

$$(1_{H_1})^G - (1_{H_{\{1,2\}}})^G + (1_{H_2})^G - (1_{H_{\{1,2\}}})^G + (1_{H_{\{1,2\}}})^G = (1_{H_1})^G + (1_{H_2})^G - (1_{H_{\{1,2\}}})^G.$$

Hence, if we let $H_\emptyset$ denote the intersection $H_1 \cap H_2$, then any module M with

$$\mathbb{C}Ge_{H_\emptyset} = M \oplus (\mathbb{C}Ge_{H_1} + \mathbb{C}Ge_{H_2})$$

must afford the character

$$(1_{H_\emptyset})^G - (1_{H_1})^G - (1_{H_2})^G + (1_{H_{\{1,2\}}})^G.$$

To continue this approach with more than two subgroups we need to place certain restrictions on the choice of subgroup. Suppose that $H_1, \ldots, H_k$ are subgroups of G such that $H_i H_j = H_j H_i$ for each i, j. Again, set $H_\emptyset = H_1 \cap \cdots \cap H_k$ and for each non-empty $J = \{j_1, \ldots, j_l\} \subseteq S = \{1, \ldots, k\}$ let $H_J = \langle H_{j_1}, \ldots, H_{j_l} \rangle = H_{j_1} \cdots H_{j_l}$. In this situation the corresponding permutation modules satisfy the following distributive law.

Lemma 4.2.1. $(\mathbb{C}Ge_{H_1} + \cdots + \mathbb{C}Ge_{H_{k-1}}) \cap \mathbb{C}Ge_{H_k} = \mathbb{C}Ge_{H_{\{1,k\}}} + \cdots + \mathbb{C}Ge_{H_{\{k-1,k\}}}$.

Proof. For each i we see that $\mathbb{C}Ge_{H_{\{i,k\}}} \subseteq \mathbb{C}Ge_{H_k}$ and so therefore we must have $\mathbb{C}Ge_{H_{\{1,k\}}} + \cdots + \mathbb{C}Ge_{H_{\{k-1,k\}}} \subseteq (\mathbb{C}Ge_{H_1} + \cdots + \mathbb{C}Ge_{H_{k-1}}) \cap \mathbb{C}Ge_{H_k}$. Now consider an element $m \in (\mathbb{C}Ge_{H_1} + \cdots + \mathbb{C}Ge_{H_{k-1}}) \cap \mathbb{C}Ge_{H_k}$. Then $m = me_{H_k}$ and also $m = r_1 e_{H_1} + \cdots + r_{k-1} e_{H_{k-1}}$ for some $r_i \in \mathbb{C}G$. Thus, since $e_{H_i} e_{H_k} = e_{H_{\{i,k\}}}$ for each i, we see that

$$m = me_{H_k} = r_1 e_{H_1} e_{H_k} + \cdots + r_{k-1} e_{H_{k-1}} e_{H_k} = r_1 e_{H_{\{1,k\}}} + \cdots + r_{k-1} e_{H_{\{k-1,k\}}}.$$

Hence $(\mathbb{C}Ge_{H_1} + \cdots + \mathbb{C}Ge_{H_{k-1}}) \cap \mathbb{C}Ge_{H_k} \subseteq \mathbb{C}Ge_{H_{\{1,k\}}} + \cdots + \mathbb{C}Ge_{H_{\{k-1,k\}}}$. □

The sum of the permutation modules can then be shown to give a character which is expressible as an alternating sum of permutation characters.

Proposition 4.2.2. $\mathbb{C}Ge_{H_1} + \cdots + \mathbb{C}Ge_{H_k}$ *affords the character*

$$\sum_{\emptyset \neq J \subseteq S} (-1)^{|J|-1} (1_{H_J})^G.$$

Proof. We proceed by induction on k, noting that $k = 1$ is true since $\mathbb{C}Ge_{H_1}$ affords $(1_{H_1})^G$. Now, $\mathbb{C}Ge_{H_1} + \cdots + \mathbb{C}Ge_{H_k} = (\mathbb{C}Ge_{H_1} + \cdots + \mathbb{C}Ge_{H_{k-1}}) \oplus M$ where M is a submodule of $\mathbb{C}Ge_{H_k}$ such that $\mathbb{C}Ge_{H_k} = M \oplus (\mathbb{C}Ge_{H_1} + \cdots + \mathbb{C}Ge_{H_{k-1}}) \cap \mathbb{C}Ge_{H_k}$. By induction, $\mathbb{C}Ge_{H_1} + \cdots + \mathbb{C}Ge_{H_{k-1}}$ gives the character

$$\sum_{\emptyset \neq J \subseteq \{1,\ldots,k-1\}} (-1)^{|J|-1} (1_{H_J})^G. \tag{4.4}$$

Further, by Lemma 4.2.1, $(\mathbb{C}Ge_{H_1} + \cdots + \mathbb{C}Ge_{H_{k-1}}) \cap \mathbb{C}Ge_{H_k}$ produces

$$\sum_{\emptyset \neq J \subseteq \{1,\ldots,k-1\}} (-1)^{|J|-1} (1_{H_{J \cup \{k\}}})^G$$

and thus M must afford

$$(1_{H_k})^G - \sum_{\emptyset \neq J \subseteq \{1,\ldots,k-1\}} (-1)^{|J|-1} (1_{H_{J \cup \{k\}}})^G = \sum_{k \in J \subseteq S} (-1)^{|J|-1} (1_{H_J})^G. \tag{4.5}$$

Hence the character given by $\mathbb{C}Ge_{H_1} + \cdots + \mathbb{C}Ge_{H_k}$ is the sum of the characters (4.4) and (4.5), i.e. the character

$$\sum_{\emptyset \neq J \subseteq S} (-1)^{|J|-1} (1_{H_J})^G. \qquad \square$$

The alternating sum produced by Proposition 4.2.2 is not quite in the right form needed for St_ℓ. To obtain the correct form we consider the $\mathbb{C}G$-module $\mathbb{C}Ge$ where

$$e = \sum_{J \subseteq S} (-1)^{|J|} e_{H_J}.$$

Lemma 4.2.3. *e is an idempotent.*

Proof. By definition, $H_I H_J = H_{I \cup J}$ and so $e_{H_I} e_{H_J} = e_{H_{I\cup J}}$ for each $I, J \subseteq S$. Thus, $e_{H_I} e = \sum_{J \subseteq S} (-1)^{|J|} e_{H_I} e_{H_J} = \sum_{J \subseteq S} (-1)^{|J|} e_{H_{I \cup J}} = 0$ for each $\emptyset \neq I \subseteq S$ and $e_{H_\emptyset} e = \sum_{J \subseteq S} (-1)^{|J|} e_{H_\emptyset} e_{H_J} = \sum_{J \subseteq S} (-1)^{|J|} e_{H_J} = e$ for $I = \emptyset$. Consequently, $e^2 = \sum_{I \subseteq S} (-1)^{|I|} e_{H_I} e = e_{H_\emptyset} e = e$ as required. $\square$

Further, $\mathbb{C}Ge$ is the complement in the permutation module over $H_\emptyset$ of the sum of the permutation modules over the parabolic subgroups H_k.

Lemma 4.2.4. $\mathbb{C}Ge_{H_\emptyset} = \mathbb{C}Ge \oplus (\mathbb{C}Ge_{H_1} + \cdots + \mathbb{C}Ge_{H_k})$.

Proof. By the proof of Lemma 4.2.3 we have $e_{H_i} e = 0$ for each i and so therefore $\mathbb{C}Ge \cap (\mathbb{C}Ge_{H_1} + \cdots + \mathbb{C}Ge_{H_k}) = 0$. Further, $e_{H_\emptyset} = e + \sum_{\emptyset \neq J \subseteq S} (-1)^{|J|-1} e_{H_J}$ implies that $\mathbb{C}Ge_{H_\emptyset} = \mathbb{C}Ge + (\mathbb{C}Ge_{H_1} + \cdots + \mathbb{C}Ge_{H_k})$. $\square$

As a consequence the character afforded by $\mathbb{C}Ge$ is also an alternating sum of permutation characters.

Corollary 4.2.5. *$\mathbb{C}Ge$ affords the character*

$$\zeta = \sum_{J \subseteq S} (-1)^{|J|} (1_{H_J})^G.$$

Proof. This is immediate from Lemma 4.2.4 and Proposition 4.2.2. □

Finally, returning to the extended Chevalley group $\widehat{G}$ with its parabolic subgroups $\widehat{H}_J$ we obtain the desired result.

Theorem 4.2.6. St_ℓ *is the character afforded by the module* $\mathbb{C}\widehat{G}e$ *where* e *is the idempotent*

$$e = \sum_{J \subseteq S} (-1)^{|J|} e_{\widehat{H}_J}.$$

Proof. The character ζ from Corollary 4.2.5 is exactly the expression for St_ℓ given in Definition 4.1.6. □

4.3 Induction from a Steinberg character

If we let $\widehat{H} = \widehat{H}_S$ and define χ to be the character

$$\chi = \sum_{J \subseteq S} (-1)^{|J|} (1_{\widehat{H}_J})^{\widehat{H}}$$

afforded by the $\mathbb{C}H$-module $\mathbb{C}He$, then it is clear that St_ℓ is induced from χ. In fact, we will show that χ is essentially the Steinberg character of $\widehat{H}$.

For each $\alpha \in S$,

$$y_{-\alpha}(-1)x_\alpha(r)y_{-\alpha}(-1) = y_{-\alpha}(-1)x_\alpha(r)y_{-\alpha}(-1)^{-1} = x_\alpha(-r).$$

Thus if we set

$$\sigma_\alpha = x_\alpha(\pi^{\ell-1})y_{-\alpha}(-1)$$

then

$$\sigma_\alpha^2 = x_\alpha(\pi^{\ell-1})y_{-\alpha}(-1)x_\alpha(\pi^{\ell-1})y_{-\alpha}(-1) = x_\alpha(\pi^{\ell-1})x_\alpha(-\pi^{\ell-1}) = 1.$$

Further, for $\beta \in S$ with $\alpha \neq \beta$

$$y_{-\alpha}(-1)x_\beta(r) = y_{-\alpha}(-1)x_\beta(r)y_{-\alpha}(-1)^{-1}y_{-\alpha}(-1) = x_\beta(r)y_{-\alpha}(-1)$$

which implies that

$$\begin{aligned}
\sigma_\alpha\sigma_\beta &= x_\alpha(\pi^{\ell-1})y_{-\alpha}(-1)x_\beta(\pi^{\ell-1})y_{-\beta}(-1) \\
&= x_\alpha(\pi^{\ell-1})x_\beta(\pi^{\ell-1})y_{-\alpha}(-1)y_{-\beta}(-1) \\
&= x_\beta(\pi^{\ell-1})x_\alpha(\pi^{\ell-1})y_{-\beta}(-1)y_{-\alpha}(-1) \\
&= x_\beta(\pi^{\ell-1})y_{-\beta}(-1)x_\alpha(\pi^{\ell-1})y_{-\alpha}(-1) \\
&= \sigma_\beta\sigma_\alpha.
\end{aligned}$$

Consequently, if for each non-empty subset $J = \{\alpha_{j_1}, \ldots, \alpha_{j_k}\} \subseteq S$ we write

$$\sigma_J = \sigma_{\alpha_{j_1}} \cdots \sigma_{\alpha_{j_k}}$$

with $\sigma_\emptyset = 1$, then the group $\widehat{N}$ generated by $\{\sigma_\alpha : \alpha \in S\}$ is $\widehat{N} = \{\sigma_J : J \subseteq S\}$.

We will now show that $\widehat{B}$ and $\widehat{N}$ together form a BN-pair for $\widehat{H}$. The first property of BN-pairs we need to prove is that $\widehat{B}$ and $\widehat{N}$ generate $\widehat{H}$.

Lemma 4.3.1. $\widehat{H} = \langle \widehat{B}, \widehat{N} \rangle$.

Proof. Clearly $\widehat{B} \leq \widehat{H}$ and $\widehat{N} \leq \widehat{H}$ so therefore $\langle \widehat{B}, \widehat{N} \rangle \leq \widehat{H}$. Now, fix $\alpha \in S$. Let $r \in \pi^{\ell-1}R^\times$ and choose $s \in R^\times$ such that $r = \pi^{\ell-1}s$. Then

$$\begin{aligned}
y_{-\alpha}(s^{-1})\sigma_\alpha y_{-\alpha}(-s) &= y_{-\alpha}(s^{-1})x_\alpha(\pi^{\ell-1})y_{-\alpha}(-1)y_{-\alpha}(-s) \\
&= y_{-\alpha}(s^{-1})x_\alpha(\pi^{\ell-1})y_{-\alpha}(s) \\
&= x_\alpha(\pi^{\ell-1}s) \\
&= x_\alpha(r). \qquad (4.6)
\end{aligned}$$

Hence $x_\alpha(r) \in \langle \widehat{B}, \widehat{N} \rangle$ for each $r \in \mathfrak{m}^{\ell-1}$ and $\alpha \in S$, implying that $\widehat{H} \leq \langle \widehat{B}, \widehat{N} \rangle$. $\square$

Next, we prove two results regarding the $(\widehat{B}, \widehat{B})$-double coset structure of $\widehat{H}$.

Proposition 4.3.2. *For any $\alpha \in S$ and $J \subseteq S$*

$$\sigma_\alpha \widehat{B} \sigma_J \subseteq \widehat{B}\sigma_J\widehat{B} \cup \widehat{B}\sigma_\alpha\sigma_J\widehat{B}.$$

Proof. Let $b \in \widehat{B}$ and suppose that $\alpha \notin J$. Then, since $\sigma_\alpha b \in \widehat{H}_\alpha$ with $\sigma_\alpha b \notin \widehat{B}$, by Lemma 4.1.3 we may write $\sigma_\alpha b = b' x_\alpha(r)$ for some $r \in \pi^{\ell-1}R^\times$ and $b' \in \widehat{B}$. Let $s \in R^\times$ be such that $r = \pi^{\ell-1}s$, then $y_{-\alpha}(s^{-1})\sigma_J = \sigma_J y_{-\alpha}(s^{-1})$ and by (4.6)

$$\sigma_\alpha b \sigma_J = b' x_\alpha(r)\sigma_J = b' y_{-\alpha}(s^{-1})\sigma_\alpha y_{-\alpha}(-s)\sigma_J = b' y_{-\alpha}(s^{-1})\sigma_\alpha\sigma_J y_{-\alpha}(-s) \in \widehat{B}\sigma_\alpha\sigma_J\widehat{B}.$$

Now, suppose that $\alpha \in J$. In particular, this implies that $\sigma_J = \sigma_\alpha \sigma_{J-\{\alpha\}}$. Since $\sigma_\alpha b \sigma_\alpha \in \widehat{H}_\alpha$, by Lemma 4.1.3 we have $\sigma_\alpha b \sigma_\alpha = b' x_\alpha(r)$ for some $r \in \mathfrak{m}^{\ell-1}$ and $b' \in \widehat{B}$. If $r = 0$, then $\sigma_\alpha b \sigma_\alpha = b'$ and so

$$\sigma_\alpha b \sigma_J = \sigma_\alpha b \sigma_\alpha \sigma_{J-\{\alpha\}} = b' \sigma_{J-\{\alpha\}} = b' \sigma_\alpha \sigma_J \in \widehat{B}\sigma_\alpha\sigma_J\widehat{B}.$$

If $r \neq 0$, then (4.6) with $s \in R^\times$ such that $r = \pi^{\ell-1}s$ again gives

$$\begin{aligned}
\sigma_\alpha b \sigma_J &= b' x_\alpha(r)\sigma_{J-\{\alpha\}} \\
&= b' y_{-\alpha}(s^{-1})\sigma_\alpha y_{-\alpha}(-s)\sigma_{J-\{\alpha\}} \\
&= b' y_{-\alpha}(s^{-1})\sigma_\alpha \sigma_{J-\{\alpha\}} y_{-\alpha}(-s) \\
&= b' y_{-\alpha}(s^{-1})\sigma_J y_{-\alpha}(-s) \in \widehat{B}\sigma_J\widehat{B}.
\end{aligned}$$

Hence $\sigma_\alpha \widehat{B} \sigma_J \subseteq \widehat{B}\sigma_J\widehat{B} \cup \widehat{B}\sigma_\alpha\sigma_J\widehat{B}$. □

However, the second result only holds when the order q of the residue class field is greater than 2.

Lemma 4.3.3. *If $q \neq 2$, then $\sigma_\alpha \widehat{B} \sigma_\alpha \neq \widehat{B}$ for any $\alpha \in S$.*

Proof. Suppose that $q \neq 2$, then there is some $s \in R^\times$ with $1 - s \in R^\times$. Thus

$$\begin{aligned}
&\sigma_\alpha y_{-\alpha}(1-s)^{-1}\sigma_\alpha \\
&= x_\alpha(\pi^{\ell-1})y_{-\alpha}(-1)y_{-\alpha}(1-s)^{-1}x_\alpha(\pi^{\ell-1})y_{-\alpha}(-1) \\
&= x_\alpha(\pi^{\ell-1})y_{-\alpha}(-1)y_{-\alpha}(1-s)^{-1}x_\alpha(\pi^{\ell-1})y_{-\alpha}(1-s)y_{-\alpha}(1-s)^{-1}y_{-\alpha}(-1) \\
&= x_\alpha(\pi^{\ell-1})y_{-\alpha}(-1)x_\alpha(\pi^{\ell-1}(1-s))y_{-\alpha}(-1)y_{-\alpha}(1-s)^{-1} \\
&= x_\alpha(\pi^{\ell-1})x_\alpha(\pi^{\ell-1}(s-1))y_{-\alpha}(1-s)^{-1} \\
&= x_\alpha(\pi^{\ell-1}s)y_{-\alpha}(1-s)^{-1}
\end{aligned}$$

and so $\sigma_\alpha \widehat{B} \sigma_\alpha \neq \widehat{B}$. □

For $q \neq 2$ this is sufficient to show that $\widehat{B}$ forms part of a BN-pair for $\widehat{H}$. However, Lemma 4.3.3 does not hold when $q = 2$ and in this case we do not quite obtain a BN-pair.

Theorem 4.3.4. *(i) If $q \neq 2$, then $\widehat{B}$ and $\widehat{N}$ form a BN-pair for $\widehat{H}$.*

(ii) If $q = 2$, then $\widehat{B}$ is normal in $\widehat{H}$ and $\widehat{H}/\widehat{B} \simeq \widehat{N}$.

Proof. (i) Suppose that $q \neq 2$. By Lemma 4.3.1, $\widehat{B}$ and $\widehat{N}$ together generate $\widehat{H}$. Further, $\widehat{B} \cap \widehat{N} = \{1\}$ is trivially normal in $\widehat{H}$ and $\widehat{N}/(\widehat{B} \cap \widehat{N}) = \widehat{N}$ is generated by the set of involutions $\{\sigma_\alpha : \alpha \in S\}$. The result then follows from Proposition 4.3.2 and Lemma 4.3.3.

(ii) Now suppose that $q = 2$. Then, since $[\widehat{H}_\alpha : \widehat{B}] = 2$ for each α, we must have $\sigma_\alpha \widehat{B} \sigma_\alpha^{-1} = \widehat{B}$. Thus $\sigma_J \widehat{B} \sigma_J^{-1} = \widehat{B}$ for each $J \subseteq S$ and so $\widehat{B}$ must be normal in $\widehat{H}$, since $\widehat{B}$ and $\widehat{N}$ generate $\widehat{H}$ by Lemma 4.3.1. Finally, $\widehat{N}$ forms a complete set of left coset representatives for $\widehat{B}$ in $\widehat{H}$. □

When $q \neq 2$ the subgroups $\widehat{H}_J$ are exactly the parabolic subgroups of $\widehat{H}$ as a finite group with BN-pair. Thus the expression for St_ℓ as an alternating sum of permutation characters is identical to the formula for the Steinberg character of $\widehat{H}$ given by Curtis [4]. Similarly, when $q = 2$ the quotient groups $\widehat{H}_J/\widehat{B}$ are the parabolic subgroups of $\widehat{H}/\widehat{B}$ as a Coxeter group and so the expression for St_ℓ is Solomon's formula [27] for the sign character of $\widehat{H}/\widehat{B}$ inflated to $\widehat{H}$.

Corollary 4.3.5. *(i) If $q \neq 2$, then χ is the Steinberg character of $\widehat{H}$.*

(ii) If $q = 2$, then χ is the sign character of $\widehat{H}/\widehat{B}$ inflated to $\widehat{H}$.

Chapter 5

Homology

Before we show that St_ℓ is an irreducible character, we will prove that it is afforded by a homology space of a simplicial complex analogous to the combinatorial building in the finite field case. We use the approach to homology representations contained in [11].

5.1 Definitions

A **poset** is a set Ω together with a partial ordering $<$ of its elements. Each poset Ω defines a simplicial complex $\Delta(\Omega)$ whose vertices are the elements of Ω and k-simplices are the $(k+1)$-chains in Ω, i.e. subsets $\{\omega_0, \dots, \omega_k\}$ of Ω such that $\omega_0 < \cdots < \omega_k$.

For each k, let $C_k(\Omega)$ denote the $\mathbb{Z}$-space spanned by the k-simplices in $\Delta(\Omega)$. Further, for each k define the linear map $\partial_k : C_k(\Omega) \to C_{k-1}(\Omega)$ by

$$\partial_k(\gamma) = \sum_{i=0}^{k} (-1)^i \gamma_i$$

for each $\gamma = (\omega_0 < \cdots < \omega_k) \in C_k(\Omega)$ where

$$\gamma_i = (\omega_0 < \cdots < \omega_{i-1} < \widehat{\omega}_i < \omega_{i+1} < \cdots < \omega_k)$$

with $\widehat{\omega}_i$ meaning that the term ω_i has been omitted. Then $\partial_{k+1}\partial_k = 0$ for each k and so the graded vector space

$$C(\Omega) = \sum_{k=0}^{\infty} C_k(\Omega)$$

is a chain complex with boundary homomorphism $\partial = \{\partial_k\}_{k=0}^{\infty}$. Consequently, since $\text{im } \partial_{k+1} \subseteq \ker \partial_k$ we may consider the quotient space

$$H_k(\Omega) = \ker \partial_k / \text{im } \partial_{k+1}.$$

This is called the k**-th homology space** of Ω.

Now, suppose that G is an arbitrary finite group which acts on the poset Ω in such a way that the partial order is preserved, i.e. so that if $\omega < \omega'$ then $g\omega < g\omega'$ for each $g \in G$. This gives rise to an action on the simplicial complex $\Delta(\Omega)$ via

$$g(\omega_0 < \cdots < \omega_k) = g\omega_0 < \cdots < g\omega_k.$$

Clearly G sends k-simplices to k-simplices and so we also obtain a G-action on the $\mathbb{Z}$-space $C_k(\Omega)$ for each k. Moreover, for each k-simplex $\gamma = (\omega_0 < \cdots < \omega_k)$ we see that

$$\begin{aligned} g\gamma_i &= g(\omega_0 < \cdots < \omega_{i-1} < \widehat{\omega}_i < \omega_{i+1} < \cdots < \omega_k) \\ &= (g\omega_0 < \cdots < g\omega_{i-1} < g\widehat{\omega}_i < g\omega_{i+1} < \cdots < g\omega_k) \\ &= (g\gamma)_i \end{aligned}$$

implying that

$$g\partial_k(\gamma) = \sum_{i=0}^{k} (-1)^i g\gamma_i = \sum_{i=0}^{k} (-1)^i (g\gamma)_i = \partial_k(g\gamma).$$

Hence the G-action commutes with the boundary homomorphism ∂_k and so extends to an action on the k-th homology space $H_k(\Omega)$.

Finally, in order to calculate the homology spaces we also need to consider their reduced versions. These are obtained from the augmented chain complex $\widetilde{C}(\Omega)$ which has $\mathbb{Z}$-spaces

$$\widetilde{C}_k(\Omega) = \begin{cases} C_k(\Omega) & \text{if } k \neq -1; \\ \mathbb{Z} & \text{if } k = -1 \end{cases}$$

and boundary map $\widetilde{\partial} = \{\widetilde{\partial}_k\}_{k=-1}^{\infty}$ with $\widetilde{\partial}_k = \partial_k$ for $k \neq 0$ and $\widetilde{\partial}_0 : \widetilde{C}_0(\Omega) \to \widetilde{C}_{-1}(\Omega)$ given by $\widetilde{\partial}_0(\omega) = 1$ for each $\omega \in \Omega$. The **reduced k-th homology space** is then the quotient space

$$\widetilde{H}_k(\Omega) = \ker \widetilde{\partial}_k / \operatorname{im} \widetilde{\partial}_{k+1}$$

and is related to the k-th homology space [22] via

$$H_k(\Omega) = \begin{cases} \widetilde{H}_k(\Omega) \oplus \mathbb{Z} & \text{if } k = 0; \\ \widetilde{H}_k(\Omega) & \text{otherwise.} \end{cases} \tag{5.1}$$

5.2 Combinatorial building for $\widehat{G}$

We now return to the extended Chevalley group $\widehat{G}$ over R. Suppose that $\widehat{G}$ has rank $n > 1$ with $S = \{\alpha_1, \ldots, \alpha_n\}$.

Definition 5.2.1. For each $1 \leq i \leq n$, let $J_i = S - \{\alpha_i\}$ and define Ω to be the poset of left cosets

$$\Omega = \{g\widehat{H}_{J_i} : g \in \widehat{G}, 1 \leq i \leq n\}$$

where $g_i\widehat{H}_{J_i} < g_j\widehat{H}_{J_j}$ if $i < j$ and $g_i\widehat{H} = g_j\widehat{H}$. Then $\widehat{G}$ acts on Ω by permuting the cosets.

Lemma 5.2.2. *Every k-simplex γ in $\Delta(\Omega)$ is of the form*

$$\gamma = g(\widehat{H}_{J_{i_0}} < \cdots < \widehat{H}_{J_{i_k}})$$

for some $g \in \widehat{G}$ and $i_0 < \cdots < i_k$.

Proof. Suppose that $\gamma = (g_{i_0}\widehat{H}_{J_{i_0}} < \cdots < g_k\widehat{H}_{J_{i_k}})$. In particular this means that we must have $g_{i_0}\widehat{H} = g_{i_j}\widehat{H}$ and therefore $g_{i_0}^{-1}g_{i_j} \in \widehat{H}$ for each j. Consequently, we may choose an element $x_{i_j} \in X_{\alpha_{i_j}}$ such that $g_{i_0}^{-1} g_{i_j} \widehat{H}_{J_{i_j}} = x_{i_j} \widehat{H}_{J_{i_j}}$ which then implies that $g_0 x_{i_j} \widehat{H}_{J_{i_j}} = g_{i_j} \widehat{H}_{J_{i_j}}$.

Now, if we set $g = g_{i_0} x_{i_1} \cdots x_{i_k}$ then for each j

$$g\widehat{H}_{J_{i_j}} = g_{i_0} x_{i_1} \cdots x_{i_k} \widehat{H}_{J_{i_j}} = g_{i_0} x_{i_j} \widehat{H}_{J_{i_j}} = g_{i_j} \widehat{H}_{J_{i_j}}.$$

Hence $g(\widehat{H}_{J_{i_0}} < \cdots < \widehat{H}_{J_{i_k}}) = (g_{i_0} \widehat{H}_{J_{i_0}} < \cdots < g_{i_k} \widehat{H}_{J_{i_k}})$. □

In particular, this means that for any k-simplex $(g_{i_0} \widehat{H}_{J_{i_0}} < \cdots < g_{i_k} \widehat{H}_{J_{i_k}})$ in $\Delta(\Omega)$ we have

$$g_{i_0} \widehat{H}_{J_{i_0}} \cap \cdots \cap g_{i_k} \widehat{H}_{J_{i_k}} = g\widehat{H}_{J_{i_0}} \cap \cdots \cap g\widehat{H}_{J_{i_k}} = g(\widehat{H}_{J_{i_0}} \cap \cdots \cap \widehat{H}_{J_{i_k}}) = g\widehat{H}_{S - \{\alpha_{i_0}, \ldots, \alpha_{i_k}\}}$$

which is non-empty.

Conversely, given $g_{i_0} \widehat{H}_{J_{i_0}}, \ldots, g_{i_k} \widehat{H}_{J_{i_k}} \in \Omega$ with $i_0 < \cdots < i_k$ and non-empty intersection $g_{i_0} \widehat{H}_{J_{i_0}} \cap \cdots \cap g_{i_k} \widehat{H}_{J_{i_k}}$, consider any element $g \in g_{i_0} \widehat{H}_{J_{i_0}} \cap \cdots \cap g_{i_k} \widehat{H}_{J_{i_k}}$. Then for each j we have $g \in g_{i_j} \widehat{H}_{J_{i_j}}$ and so $g = g_{i_j} h_j$ for some $h_j \in \widehat{H}_{J_{i_j}}$. Thus $g\widehat{H} = g_{i_j} h_j \widehat{H} = g_{i_j} \widehat{H}$ implies that $(g_{i_0} \widehat{H}_{J_{i_0}} < \cdots < g_{i_k} \widehat{H}_{J_{i_k}})$ is a k-simplex in $\Delta(\Omega)$.

Hence, $\Delta(\Omega)$ can be identified with the simplicial complex which has vertices $\{g\widehat{H}_{J_i} : g \in \widehat{G}, 1 \le i \le n\}$ and where $(g_{i_0} \widehat{H}_{J_{i_0}}, \ldots, g_{i_k} \widehat{H}_{J_{i_k}})$ is a k-simplex if and only if $g_{i_0} \widehat{H}_{J_{i_0}} \cap \cdots \cap g_{i_k} \widehat{H}_{J_{i_k}}$ is non-empty.

Lemma 5.2.3. *Let γ be a k-simplex in $\Delta(\Omega)$, then* $\mathrm{Stab}_{\widehat{G}}(\gamma) = g\widehat{H}_J g^{-1}$ *for some $g \in \widehat{G}$ and $J \subseteq S$ with $|J| = n - k - 1$.*

Proof. By Lemma 5.2.2 we know that $\gamma = (g\widehat{H}_{J_{i_0}} < \cdots < g\widehat{H}_{J_{i_k}})$ for some $g \in \widehat{G}$ and $i_0 < \cdots < i_k$. Further, it is clear that $g' \in \mathrm{Stab}_{\widehat{G}}(\gamma)$ if and only if $g'g\widehat{H}_{J_{i_j}} = g\widehat{H}_{J_{i_j}}$ for every j. However, this holds exactly when $g' \in g\widehat{H}_{J_{i_j}} g^{-1}$ for every j. Hence,

$$\mathrm{Stab}_{\widehat{G}}(\gamma) = g\widehat{H}_{J_{i_1}} g^{-1} \cap \cdots \cap g\widehat{H}_{i_k} g^{-1} = g(\widehat{H}_{J_{i_1}} \cap \cdots \cap \widehat{H}_{J_{i_k}}) g^{-1} = g\widehat{H}_J g^{-1}$$

where $J = S - \{\alpha_{i_1}, \ldots, \alpha_{i_k}\}$. □

Now, consider the subset

$$\Omega_{\widehat{H}} = \{g\widehat{H}_{J_i} : g \in \widehat{H}, 1 \le i \le n\}$$

of Ω obtained by taking only the left cosets in $\widehat{H}$. Then $\Delta(\Omega)$ can be expressed as the disjoint union of left translates of $\Delta(\Omega_{\widehat{H}})$.

Lemma 5.2.4. *Let $\mathcal{T}$ be a left transversal for $\widehat{H}$ in $\widehat{G}$, then*

$$\Delta(\Omega) = \bigcup_{t \in \mathcal{T}} g \cdot \Delta(\Omega_{\widehat{H}})$$

where the union is disjoint.

Proof. Let γ be a k-simplex in $\Delta(\Omega)$, then by Lemma 5.2.2 we may assume that $\gamma = g(\widehat{H}_{J_{i_0}} < \cdots < g\widehat{H}_{J_{i_k}})$ for some $g \in \widehat{G}$ and $i_0 < \cdots < i_k$. If we write g as $g = th$ for some $t \in \mathcal{T}$ and $h \in \widehat{H}$, then we see that

$$\gamma = g(\widehat{H}_{J_{i_0}} < \cdots < \widehat{H}_{J_{i_k}}) = th(\widehat{H}_{J_{i_0}} < \cdots < \widehat{H}_{J_{i_k}})$$

where $h(\widehat{H}_{J_{i_0}} < \cdots < \widehat{H}_{J_{i_k}})$ is a k-simplex in $\Delta(\Omega_{\widehat{H}})$. Thus $\Delta(\Omega)$ can be written as the union

$$\Delta(\Omega) = \bigcup_{t \in \mathcal{T}} t \cdot \Delta(\Omega_{\widehat{H}}).$$

Further, if for some $t, t' \in \mathcal{T}$ and $h, h' \in \widehat{H}$ we have

$$th(\widehat{H}_{J_{i_0}} < \cdots < \widehat{H}_{J_{i_k}}) = t'h'(\widehat{H}_{J_{i_0}} < \cdots < \widehat{H}_{J_{i_k}}),$$

then $t\widehat{H} = t'\widehat{H}$ and so $t = t'$. Hence the union must be disjoint. □

As a consequence of this, the homology spaces for Ω over $\mathbb{C}$ are induced from the homology spaces for $\Omega_{\widehat{H}}$ over $\mathbb{C}$.

Theorem 5.2.5. $H_k(\Omega) \otimes \mathbb{C} = \mathrm{Ind}_{\widehat{H}}^{\widehat{G}} H_k(\Omega_{\widehat{H}}) \otimes \mathbb{C}$.

Proof. By Lemma 5.2.4 it is clear that

$$C_k(\Omega) = \bigoplus_{t \in \mathcal{T}} t \cdot C_k(\Omega_{\widehat{H}})$$

and so $C_k(\Omega) \otimes \mathbb{C} = \mathrm{Ind}_{\widehat{H}}^{\widehat{G}} C_k(\Omega_{\widehat{H}}) \otimes \mathbb{C}$ for each k.

Now let ∂ denote the boundary homomorphism for $C(\Omega)$ and ∂' the boundary homomorphism for $C(\Omega_{\widehat{H}})$. It is clear from the definition that the restriction of ∂_k to $C_k(\Omega_{\widehat{H}})$ is exactly ∂'_k. Thus, if for each $\gamma \in C_k(\Omega)$ we write $\gamma = \sum_{t \in \mathcal{T}} t\gamma_t$ for some $\gamma_t \in C_k(\Omega_{\widehat{H}})$, then

$$\partial_k(\gamma) = \sum_{t \in \mathcal{T}} \partial_k(t\gamma_t) = \sum_{t \in \mathcal{T}} t\partial_k(\gamma_t) = \sum_{t \in \mathcal{T}} t\partial'_k(\gamma_t).$$

Consequently, $\partial_k \otimes 1 : C_k(\Omega) \otimes \mathbb{C} \to C_{k-1}(\Omega) \otimes \mathbb{C}$ is the homomorphism induced from $\partial'_k \otimes 1 : C_k(\Omega_{\widehat{H}}) \otimes \mathbb{C} \to C_{k-1}(\Omega_{\widehat{H}}) \otimes \mathbb{C}$ for each k. Hence, we must have $H_k(\Omega) \otimes \mathbb{C} = \mathrm{Ind}_{\widehat{H}}^{\widehat{G}} H_k(\Omega_{\widehat{H}}) \otimes \mathbb{C}$. □

Remark 5.2.6. When $\widehat{G} = \mathrm{PGL}_n(R)$, the simplicial complex $\Delta(\Omega)$ is equivalent to the simplicial complex defined by Lees [20]. While Lees did not explicitly consider the subcomplex $\Delta(\Omega_{\widehat{H}})$, in determining the homology spaces of $\Delta(\Omega)$ he did use the fact that it could be expressed as the disjoint union of certain equivalent subcomplexes. Indeed, we will use his approach to calculate the homology spaces of $\Omega_{\widehat{H}}$ and so consequently the homology spaces of Ω.

5.3 Homology spaces of $\Omega_{\widehat{H}}$

The main idea in [20] used to find the homology spaces of $\Omega_{\widehat{H}}$ is to show that it arises from simpler posets using the following construction.

Definition 5.3.1. The **join** of two disjoint posets Ω and Ω' is defined to be the poset $\Omega * \Omega'$ with the underlying set $\Omega \cup \Omega'$ and where $\omega_1 < \omega_2$ if

(i) $\omega_1, \omega_2 \in \Omega$ and $\omega_1 < \omega_2$;

(ii) $\omega_1, \omega_2 \in \Omega'$ and $\omega_1 < \omega_2$; or

(iii) $\omega_1 \in \Omega$ and $\omega_2 \in \Omega'$.

The corresponding simplicial complex $\Delta(\Omega * \Omega')$ is then the topological join of the simplicial complexes $\Delta(\Omega)$ and $\Delta(\Omega')$. Consequently, by [24] we know that the reduced homology spaces of $\Omega * \Omega'$ are given by

$$\widetilde{H}_{k+1}(\Omega * \Omega') = \bigoplus_{i+j=k} \widetilde{H}_i(\Omega) \otimes H_j(\Omega'). \tag{5.2}$$

Now, if for each i we consider the subset

$$\Omega_i = \{g\widehat{H}_{J_i} : g \in \widehat{H}\},$$

then the simplicial complex $\Delta(\Omega_i)$ consists only of q distinct points. Therefore, the reduced homology spaces of Ω_i are

$$\widetilde{H}_k(\Omega_i) = \begin{cases} \mathbb{Z}^{(q-1)} & \text{if } k = 0; \\ 0 & \text{otherwise.} \end{cases} \tag{5.3}$$

Further, it is clear that $\Omega_{\widehat{H}}$ can be expressed as the join

$$\Omega_{\widehat{H}} = \Omega_1 * \cdots * \Omega_n.$$

Thus, by repeatedly applying (5.2) we see that

$$\widetilde{H}_{k+n-1}(\Omega_{\widehat{H}}) = \bigoplus_{i_1+\cdots+i_n=k} \widetilde{H}_{i_1}(\Omega_1) \otimes \cdots \otimes \widetilde{H}_{i_n}(\Omega_n)$$

and so from (5.3) we obtain

$$\widetilde{H}_k(\Omega_{\widehat{H}}) = \begin{cases} \mathbb{Z}^{(q-1)^n} & \text{if } k = n-1; \\ 0 & \text{otherwise.} \end{cases}$$

The homology spaces of $\Omega_{\widehat{H}}$ then follow from (5.1).

Proposition 5.3.2. *The homology spaces of $\Omega_{\widehat{H}}$ are*

$$H_k(\Omega_{\widehat{H}}) = \begin{cases} \mathbb{Z} & \text{if } k = 0; \\ \mathbb{Z}^{(q-1)^n} & \text{if } k = n-1; \\ 0 & \text{otherwise.} \end{cases}$$

We would like to show that χ is afforded by the top homology space of $\Omega_{\widehat{H}}$, but first we need to examine the bottom homology space.

Lemma 5.3.3. *$H_0(\Omega_{\widehat{H}}) \otimes \mathbb{C}$ affords the trivial character of $\widehat{H}$.*

Proof. For each $i > 1$ and $g \in \widehat{H}$, we see that $(\widehat{H}_{J_1}) = (g\widehat{H}_{J_i})$ in $H_0(\Omega_{\widehat{H}})$ since $(\widehat{H}_{J_1} < g\widehat{H}_{J_i})$ is a 1-simplex with $\partial_1(\widehat{H}_{J_1} < g\widehat{H}_{J_i}) = (\widehat{H}_{J_1}) - (g\widehat{H}_{J_i})$. Further, $(g\widehat{H}_{J_1} < g\widehat{H}_{J_i})$ is also a 1-simplex and so again $\partial_1(g\widehat{H}_{J_1} < g\widehat{H}_{J_i}) = (g\widehat{H}_{J_1}) - (g\widehat{H}_{J_i})$ implies that $(g\widehat{H}_{J_1}) = (g\widehat{H}_{J_i}) = (\widehat{H}_{J_1})$ in $H_0(\Omega_{\widehat{H}})$. Hence $H_0(\Omega_{\widehat{H}}) \otimes \mathbb{C} = \mathbb{C}(\widehat{H}_{J_1})$ with $g(\widehat{H}_{J_1}) = (g\widehat{H}_{J_1}) = (\widehat{H}_{J_1})$ for every $g \in \widehat{H}$. $\square$

Now, the Hopf Trace Formula states that

$$\sum_{k=1}^{n-1}(-1)^k \operatorname{tr}(g, H_k(\Omega_{\widehat{H}})) = \sum_{k=1}^{n-1}(-1)^k \operatorname{tr}(g, C_k(\Omega_{\widehat{H}})).$$

However, from Lemma 5.2.2 it is clear that the k-simplices in $\Delta(\Omega_{\widehat{H}})$ can be expressed as the disjoint union

$$\bigcup_{i_0<\cdots<i_k} \widehat{H} \cdot (\widehat{H}_{J_{i_0}} < \cdots < \widehat{H}_{J_{i_k}}).$$

Thus $C_k(\Omega_{\widehat{H}}) \otimes \mathbb{C}$ can be expressed as the direct sum

$$C_k(\Omega_{\widehat{H}}) \otimes \mathbb{C} = \bigoplus_{i_0<\cdots<i_k} \mathbb{C}\widehat{H}(\widehat{H}_{J_{i_0}} < \cdots < \widehat{H}_{J_{i_k}})$$

which, by Lemma 5.2.3, then affords the character

$$\sum_{|J|=n-k-1} (1_{\widehat{H}_J})^{\widehat{H}}.$$

Hence, if we denote by ζ the character afforded by $H_{n-1}(\Omega_{\widehat{H}})$ we see that

$$(-1)^{n-1}\zeta + 1_{\widehat{H}} = \sum_{k=1}^{n-1}(-1)^k \sum_{|J|=n-k-1} (1_{\widehat{H}_J})^{\widehat{H}} = \sum_{J\subset S}(-1)^{n-1-|J|}(1_{\widehat{H}_J})$$

and rearranging we obtain

$$\zeta = \sum_{J\subset S}(-1)^{|J|}(1_{\widehat{H}_J}) + (-1)^n(1_{\widehat{H}}) = \sum_{J\subseteq S}(-1)^{|J|}(1_{\widehat{H}_J})^{\widehat{H}}.$$

Hence, we have shown the following result.

Theorem 5.3.4. *$H_{n-1}(\Omega_{\widehat{H}}) \otimes \mathbb{C}$ affords χ.*

Finally, Theorem 5.2.5 implies that the analogue of the Steinberg character arises from a representation on the top homology space of Ω.

Corollary 5.3.5. *$H_{n-1}(\Omega) \otimes \mathbb{C}$ affords the character St_ℓ.*

Remark 5.3.6. If $q \neq 2$ then $\Omega_{\widehat{H}}$ is actually the combinatorial building for $\widehat{H}$ and if $q = 2$ then $\Omega_{\widehat{H}}$ is the Coxeter complex of $\widehat{H}/\widehat{B}$ on which $\widehat{B}$ acts trivially.

Chapter 6

Characterisation

We now wish to show that St_ℓ is an irreducible character of $\widehat{G}$ and to do that we need to prove that $(\mathrm{St}_\ell, \mathrm{St}_\ell) = 1$. However, since St_ℓ is given as an alternating sum of permutation characters over the parabolic subgroups $\widehat{H}_J$, we see that

$$(\mathrm{St}_\ell, \mathrm{St}_\ell) = \sum_{I,J\subseteq S} (-1)^{|I|+|J|} |\mathcal{D}_{\widehat{G}}(\widehat{H}_I, \widehat{H}_J)|$$

where $\mathcal{D}_{\widehat{G}}(\widehat{H}_I, \widehat{H}_J)$ denotes the set of $(\widehat{H}_I, \widehat{H}_J)$-double cosets in $\widehat{G}$. Thus, we need to examine the double coset structure of $\widehat{G}$.

In fact, using this approach we obtain a characterisation of St_ℓ in terms of permutation characters over parabolic subgroups which is similar to Curtis' characterisation [4] of the Steinberg character for the finite field case.

6.1 Example: $\mathrm{PGL}_2(R)$

We begin by considering the case where $\widehat{G} = \mathrm{PGL}_2(R)$.

Lemma 6.1.1. *For each* $0 \le k \le \ell$

$$\widehat{B} \begin{bmatrix} 1 & 0 \\ \pi^k & 1 \end{bmatrix} \widehat{B} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \widehat{G} : c \in \pi^k R^\times \right\}$$

Proof. Let $a, c, d', d' \in R^\times$ and $b, b' \in R$. Then

$$\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \pi^k & 1 \end{bmatrix} \begin{bmatrix} a' & b' \\ 0 & d' \end{bmatrix} = \begin{bmatrix} aa' + bd' + \pi^k a'b & ab' + bd' + \pi^k bb' \\ \pi^k a'd & dd' + \pi^k b'd \end{bmatrix}$$

where $\pi^k a'd \in \pi^k R^\times$. Thus

$$\widehat{B} \begin{bmatrix} 1 & 0 \\ \pi^k & 1 \end{bmatrix} \widehat{B} \subseteq \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \widehat{G} : c \in \pi^k R^\times \right\}.$$

To show the reverse inclusion, suppose that $a, b, c, d \in R$ are such that $ad-bc \in R^\times$ with $c = \pi^k r$ for some $r \in R^\times$. If $k > 0$ then we must have $a \in R^\times$ and so

$$\begin{bmatrix} 1 & 0 \\ 0 & a^{-1}r \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \pi^k & 1 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & adr^{-1} - \pi^k b \end{bmatrix} = \begin{bmatrix} a & b \\ \pi^k r & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

If $k = 0$ then $c \in R^\times$ and

$$\begin{bmatrix} 1 & ac^{-1} - 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c & b + d - ac^{-1}d \\ 0 & ac^{-1}d - b \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \tag{6.1}$$

Hence

$$\widehat{B} \begin{bmatrix} 1 & 0 \\ \pi^k & 1 \end{bmatrix} \widehat{B} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \widehat{G} : c \in \pi^k R^\times \right\}$$

as required. □

Lemma 6.1.2. *$\widehat{G}$ can be expressed as the disjoint union*

$$\widehat{G} = \bigcup_{k=0}^{\ell} \widehat{B} \begin{bmatrix} 1 & 0 \\ \pi^k & 1 \end{bmatrix} \widehat{B}.$$

Proof. It is clear that any element of $\widehat{G}$ must lie in exactly one of the double-cosets in Lemma 6.1.1. □

More generally, we need to describe the $(\widehat{B}_i, \widehat{B}_j)$-double cosets for any $0 \leq i, j \leq \ell$. However, these are closely related to the $(\widehat{B}, \widehat{B})$-double cosets.

Lemma 6.1.3. *Let $0 < i, j, \leq \ell$. Then for any $0 \leq k < \min(i, j)$*

$$\widehat{B}_i \begin{bmatrix} 1 & 0 \\ \pi^k & 1 \end{bmatrix} \widehat{B}_j = \widehat{B} \begin{bmatrix} 1 & 0 \\ \pi^k & 1 \end{bmatrix} \widehat{B}.$$

Proof. Suppose that we have

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \widehat{B}_i \qquad \text{and} \qquad \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \in \widehat{B}_j.$$

Then $c \in \mathfrak{m}^i$ and $c' \in \mathfrak{m}^j$ which implies that $a', d \in R^\times$ since $i, j > 0$. Consequently,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \pi^k & 1 \end{bmatrix} \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} = \begin{bmatrix} aa' + bc' + \pi^k a'b & ab' + bd' + \pi^k bb' \\ a'c + c'd + \pi^k a'd & b'c + dd' + \pi^k b'd \end{bmatrix}$$

where $a'c \in \mathfrak{m}^i$, $c'd \in \mathfrak{m}^j$ and $\pi^k a'd \in \pi^k R^\times$. However, since $k < i$ and $k < j$ this means that $a'c + c'd + \pi^k a'd \in \pi^k R^\times$ and so

$$\widehat{B}_i \begin{bmatrix} 1 & 0 \\ \pi^k & 1 \end{bmatrix} \widehat{B}_j \subseteq \widehat{B} \begin{bmatrix} 1 & 0 \\ \pi^k & 1 \end{bmatrix} \widehat{B}.$$

The reverse inclusion is clear. □

When $k = \min(i, j)$ then either $\widehat{B}_i = \widehat{B}_k$ or $\widehat{B}_j = \widehat{B}_k$ and so

$$\widehat{B}_i \begin{bmatrix} 1 & 0 \\ \pi^k & 1 \end{bmatrix} \widehat{B}_j = \widehat{B}_k = \bigcup_{k'=k}^{\ell} \widehat{B} \begin{bmatrix} 1 & 0 \\ \pi^{k'} & 1 \end{bmatrix} \widehat{B}.$$

Hence, Lemma 6.1.2 immediately gives the corresponding decomposition of $\widehat{G}$ into $(\widehat{B}_i, \widehat{B}_j)$-double cosets.

Lemma 6.1.4. *For each $0 \leq i, j \leq \ell$, we can express $\widehat{G}$ as the disjoint union*

$$\widehat{G} = \bigcup_{k=0}^{\min(i,j)} \widehat{B}_i \begin{bmatrix} 1 & 0 \\ \pi^k & 1 \end{bmatrix} \widehat{B}_j.$$

Consequently, by counting the number of double cosets we are able to calculate the inner product of St_ℓ with the permutation character $(1_{\widehat{B}_i})^{\widehat{G}}$.

Proposition 6.1.5. *For each i,*

$$(\mathrm{St}_\ell, (1_{\widehat{B}_i})^{\widehat{G}}) = \begin{cases} 1 & \text{if } i = \ell; \\ 0 & \text{if } i < \ell. \end{cases}$$

Proof. If $i = \ell$, then by Lemma 6.1.4

$$(\mathrm{St}_\ell, (1_{\widehat{B}_\ell})^{\widehat{G}}) = |\mathcal{D}_{\widehat{G}}(\widehat{B}_\ell, \widehat{B}_\ell)| - |\mathcal{D}_{\widehat{G}}(\widehat{B}_{\ell-1}, \widehat{B}_\ell)| = (\ell+1) - \ell = 1$$

while if $i < \ell$ then

$$(\mathrm{St}_\ell, (1_{\widehat{B}_i})^{\widehat{G}}) = |\mathcal{D}_{\widehat{G}}(\widehat{B}_\ell, \widehat{B}_i)| - |\mathcal{D}_{\widehat{G}}(\widehat{B}_{\ell-1}, \widehat{B}_i)| = (i+1) - (i+1) = 0. \qquad \square$$

Corollary 6.1.6. St_ℓ *is an irreducible constituent of* $(1_{\widehat{B}})^{\widehat{G}}$.

Proof. By Proposition 6.1.5 we have

$$(\mathrm{St}_\ell, \mathrm{St}_\ell) = (\mathrm{St}_\ell, (1_{\widehat{B}_\ell})^{\widehat{G}}) - (\mathrm{St}_\ell, (1_{\widehat{B}_{\ell-1}})^{\widehat{G}}) = (\mathrm{St}_\ell, (1_{\widehat{B}})^{\widehat{G}}) = 1. \qquad \square$$

6.2 Double coset structure of $\widehat{H}$

In general, there seems to be a natural distinction between the double cosets of $\widehat{G}$ that are contained in $\widehat{H}$ and those that are not. We begin by examining the double coset structure of $\widehat{H}$. From Section 4.3 we already know that $\{\sigma_J : J \subseteq S\}$ forms a complete set of $(\widehat{B}, \widehat{B})$-double coset representatives. However, we will need a more detailed description of the double cosets for later use.

In the previous section we saw that each $(\widehat{B}, \widehat{B})$-double coset of $\mathrm{PGL}_2(R)$ was of the form

$$\widehat{B} \begin{bmatrix} 1 & 0 \\ \pi^k & 1 \end{bmatrix} \widehat{B} = \widehat{B}_k - \widehat{B}_{k+1}.$$

We will show that each $(\widehat{B}, \widehat{B})$-double coset $\widehat{B}\sigma_J\widehat{B}$ of $\widehat{H}$ can similarly be expressed as

$$\widehat{B}\sigma_J\widehat{B} = \widehat{H}_J - \bigcup_{I \subset J} \widehat{H}_I.$$

Recall that X_J forms a set of left coset representatives of $\widehat{B}$ in $\widehat{H}_J$. For each $J \subseteq S$ define $\mathfrak{X}_J$ to be the set of left coset representatives

$$\mathfrak{X}_J = X_J - \bigcup_{I \subset J} X_I.$$

Then we see that for non-empty $J \subseteq S$ this gives

$$\mathfrak{X}_J = \left\{ \prod_{\alpha \in J} x_\alpha(r_\alpha) : r_\alpha \in \pi^{\ell-1} R^\times \text{ for each } \alpha \in J \right\}$$

while for $J = \emptyset$ we obtain $\mathfrak{X}_\emptyset = \{1\}$.

Lemma 6.2.1. *$\widehat{T}$ transitively permutes the elements in $\mathfrak{X}_J$ for each $J \subseteq S$.*

Proof. Suppose that $x = \prod_{\alpha \in J} x_\alpha(r_\alpha) \in \mathfrak{X}_J$. Then, by Lemma 2.4.2 we see that for each $h(\mu) \in \widehat{T}$

$$h(\mu)xh(\mu)^{-1} = \prod_{\alpha \in J} h(\mu)x_\alpha(r_\alpha)h(\mu)^{-1} = \prod_{\alpha \in J} x_\alpha(\mu(\alpha)r_\alpha).$$

Further, we have $\mu(\alpha)r_\alpha \in \pi^{\ell-1}R^\times$ for each $\alpha \in J$ and so $h(\mu)xh(\mu)^{-1} \in \mathfrak{X}_J$.

Now, for any $x' = \prod_{\alpha \in J} x_\alpha(r'_\alpha) \in \mathfrak{X}_J$ we must have $r'_\alpha = r_\alpha s_\alpha$ for some $s_\alpha \in R^\times$. Consequently, if we define an R-character μ of Λ_r by $\mu(\alpha) = s_\alpha$ for each $\alpha \in S$, then we see that $\mu(\alpha)r_\alpha = s_\alpha r_\alpha = r'_\alpha$. Thus

$$h(\mu)xh(\mu)^{-1} = \prod_{\alpha \in J} x_\alpha(\mu(\alpha)r_\alpha) = \prod_{\alpha \in J} x_\alpha(r'_\alpha) = x'.$$

Hence the conjugation action of $\widehat{T}$ on $\mathfrak{X}_J$ is transitive. $\square$

In particular, Lemma 6.2.1 implies that $\widehat{T}$ transitively permutes the left cosets $x\widehat{B}$ for $x \in \mathfrak{X}_J$. Now, consider the subgroup $V = \widehat{T}(\mathfrak{m})U$ of $\widehat{B}$.

Lemma 6.2.2. *V is normal in $\widehat{H}$.*

Proof. Since $\widehat{B} = \widehat{T}U$ and $\widehat{T}$ stabilises both $\widehat{T}(\mathfrak{m})$ and U it is clear that V is normal in $\widehat{B}$. Thus, since $\widehat{H} = X_S\widehat{B}$, we only need to show that $xvx^{-1} \in V$ for each

$x \in X_S$ and $v \in V$. However, since $V = \widehat{T}(\mathfrak{m})U$ it suffices to show that for each $\alpha \in S$ and $x_\alpha(r) \in X_\alpha$ we have $x_\alpha(r)h(\mu)x_\alpha(r)^{-1} \in V$ for every $h(\mu) \in \widehat{T}(\mathfrak{m})$ and $x_\alpha(r)x_\beta(s)x_\alpha(r)^{-1} \in V$ for every $x_\beta(s) \in U$.

Fix $\alpha \in S$ and $x_\alpha(r) \in X_\alpha$. For every $h(\mu) \in \widehat{T}(\mathfrak{m})$

$$x_\alpha(r)h(\mu)x_\alpha(r)^{-1} = x_\alpha(r)h(\mu)x_\alpha(r)^{-1}h(\mu)^{-1}h(\mu) = x_\alpha(r)x_\alpha(-\mu(\alpha)r)h(\mu) = h(\mu)$$

since $\mu(\alpha) \in 1+\mathfrak{m}$ implies that $\mu(\alpha)r = r$ for each $\alpha \in S$. Further, for any $x_\beta(s) \in U$ we have $\mathrm{ht}(\alpha)+\mathrm{ht}(\beta) \geq 0$. Thus, by Lemma 3.4.3(i) we know that $[x_\alpha(r), x_\beta(s)] \in V$ and so $x_\alpha(r)x_\beta(s)x_\alpha(r)^{-1} = [x_\alpha(r), x_\beta(s)]x_\beta(s) \in V$. □

Thus, V preserves each left coset $x\widehat{B}$ for $x \in X_J$. Hence, if for each non-empty subset $J \subseteq S$ we set

$$x_J = \prod_{\alpha \in J} x_\alpha(\pi^{\ell-1})$$

with $x_\emptyset = 1$, then we obtain the following decomposition of $\widehat{H}$.

Proposition 6.2.3. *$\widehat{H}$ can be expressed as the disjoint union*

$$\widehat{H} = \bigcup_{J \subseteq S} \widehat{B}x_J\widehat{B}$$

where $\widehat{B}x_J\widehat{B} = \mathfrak{X}_J\widehat{B}$ for each $J \subseteq S$.

Proof. Let $b \in \widehat{B}$, then we can write $b = tv$ for some $t \in \widehat{T}$ and $v \in V$. Thus

$$bx_J\widehat{B} = tvx_J\widehat{B} = tx_Jx_J^{-1}vx_J\widehat{B} = tx_J\widehat{B} = tx_Jt^{-1}\widehat{B}.$$

Since conjugation by $\widehat{T}$ preserves $\mathfrak{X}_J$ we have $tx_Jt^{-1} \in \mathfrak{X}_J$ and so $\widehat{B}x_J\widehat{B} \subseteq \mathfrak{X}_J\widehat{B}$. However, since conjugation by $\widehat{T}$ is transitive on $\mathfrak{X}_J$ each element in $\mathfrak{X}_J$ is of the form tx_Jt^{-1} for some $t \in \widehat{T}$ and therefore $\mathfrak{X}_J\widehat{B} \subseteq \widehat{B}x_J\widehat{B}$. Finally, the expression of $\widehat{H}$ as a disjoint union of the double cosets $\widehat{B}x_J\widehat{B}$ follows from the fact that

$$X_S = \bigcup_{J \subseteq S} \mathfrak{X}_J$$

where the union is disjoint. □

From the definition of σ_J it is clear that $\widehat{B}x_J\widehat{B} = \widehat{B}\sigma_J\widehat{B}$ for each $J \subseteq S$. Further, we can use Proposition 6.2.3 to describe the $(\widehat{H}_J, \widehat{H}_{J'})$-double coset structure of $\widehat{H}$ for any $J, J' \subseteq S$.

Proposition 6.2.4. *For each $J, J' \subseteq S$ we can express $\widehat{H}$ as the disjoint union*

$$\widehat{H} = \bigcup_{I \subseteq S-(J\cup J')} \widehat{H}_J x_I \widehat{H}_{J'}.$$

Proof. By Proposition 6.2.3 we know that $\widehat{H}$ can be written as the union of double cosets $\widehat{H}_J x_I \widehat{H}_{J'}$ for $I \subseteq S$. Fix $I \subseteq S$ and consider the subsets $I_1 = I \cap J$, $I_2 = I - (J \cap J')$ and $I_3 = I \cap (J' - J)$. This gives a decomposition of I into the disjoint union $I = I_1 \cup I_2 \cup I_3$ where $I_1 \subseteq J$ and $I_3 \subseteq J'$. Thus $x_I = x_{I_1}x_{I_2}x_{I_3}$ with $x_{I_1} \in \widehat{H}_J$ and $x_{I_3} \in \widehat{H}_{J'}$. Consequently, $\widehat{H}_J x_I \widehat{H}_{J'} = \widehat{H}_J x_{I_1}x_{I_2}x_{I_3}\widehat{H}_{J'} = \widehat{H}_J x_{I_2}\widehat{H}_{J'}$ where $I_2 \subseteq S - (J \cup J')$. Hence each $(\widehat{H}_J, \widehat{H}_{J'})$-double coset representative x_I can be chosen with $I \subseteq S - (J \cup J')$.

Now, suppose that $\widehat{H}_J x_I \widehat{H}_{J'} = \widehat{H}_J x_{I'} \widehat{H}_{J'}$ for some $I, I' \subseteq S - (J \cup J')$. Then we must have $x_I = h_J x_{I'} h_{J'}$ for some $h_J \in \widehat{H}_J$ and $h_{J'} \in \widehat{H}_{J'}$. In particular, this means that $x_I x_{I'}^{-1} = h_J h_{J'}$. However, $h_J h_{J'} \in \widehat{H}_{J\cup J'}$ and $x_I x_{I'}^{-1} \in \widehat{H}_{I\cup I'}$. Thus we must have $x_I x_{I'}^{-1} \in \widehat{H}_{I\cup I'} \cap \widehat{H}_{J\cup J'} = \widehat{H}_{(I\cup I')\cap(J\cup J')} = \widehat{H}_\emptyset = \widehat{B}$ and therefore $x_I = x_{I'}$. □

Again, from Section 4.3 we know that χ is irreducible. However, we will show this explicitly using the double coset structure of $\widehat{H}$. More specifically, we will count the number of double cosets in $\widehat{H}$ in a particular way.

For non-empty subsets $J, J' \subseteq S$ define $\mathcal{E}_{\widehat{H}}(\widehat{H}_J, \widehat{H}_{J'})$ to be

$$\mathcal{E}_{\widehat{H}}(\widehat{H}_J, \widehat{H}_{J'}) = \mathcal{D}_{\widehat{H}}(\widehat{H}_J, \widehat{H}_{J'}),$$

the set of $(\widehat{H}_J, \widehat{H}_{J'})$-double cosets in $\widehat{H}$. Further, let $\mathcal{E}_{\widehat{H}}(\widehat{B}, \widehat{B})$ be

$$\mathcal{E}_{\widehat{H}}(\widehat{B}, \widehat{B}) = \mathcal{D}_{\widehat{H}}(\widehat{B}, \widehat{B}) - \{\widehat{B}x_S\widehat{B}\},$$

the set of $(\widehat{B}, \widehat{B})$-double cosets excluding $\widehat{B}x_S\widehat{B}$.

Now, fix $J' \subseteq S$ and let $\mathcal{E}$ denote the set

$$\mathcal{E} = \bigcup_{J \subseteq S} \mathcal{E}_{\widehat{H}}(\widehat{H}_J, \widehat{H}_{J'}).$$

$\mathcal{E}$ can then be expressed as the disjoint union $\mathcal{E} = \mathcal{E}_0 \cup \mathcal{E}_1$ where

$$\mathcal{E}_0 = \bigcup_{|J| \text{ even}} \mathcal{E}_{\widehat{H}}(\widehat{H}_J, \widehat{H}_{J'}) \qquad \text{and} \qquad \mathcal{E}_1 = \bigcup_{|J| \text{ odd}} \mathcal{E}_{\widehat{H}}(\widehat{H}_J, \widehat{H}_{J'}).$$

We will show that $\mathcal{E}_0$ and $\mathcal{E}_1$ contain the same number of double cosets by constructing a bijection between them.

Proposition 6.2.5. $|\mathcal{E}_0| = |\mathcal{E}_1|$.

Proof. Fix an ordering of the roots in S. Let $\widehat{H}_J x_I \widehat{H}_{J'} \in \mathcal{E}$ with $I \subseteq S - (J \cup J')$. We can only have $I = S$ if both $J = \emptyset$ and $J' = \emptyset$. However, this would imply that $\widehat{H}_J x_I \widehat{H}_{J'} = \widehat{B} x_S \widehat{B}$ and we have excluded this double coset from $\mathcal{E}$. Thus $S - I$ is non-empty and we may choose a minimal root $\alpha \in S - I$. Consequently, we are able to define a map $\Phi : \mathcal{E} \to \mathcal{E}$ by

$$\Phi(\widehat{H}_J x_I \widehat{H}_{J'}) = \begin{cases} \widehat{H}_{J-\{\alpha\}} x_I \widehat{H}_{J'} & \text{if } \alpha \in J; \\ \widehat{H}_{J \cup \{\alpha\}} x_I \widehat{H}_{J'} & \text{if } \alpha \notin J. \end{cases}$$

This is well-defined since for each double coset we are using the distinguished double coset representatives from Proposition 6.2.4. Further, note that since $\alpha \notin I$ we see that I is contained in $S - (J - \{\alpha\} \cup J')$ if $\alpha \in J$ and $S - (J \cup \{\alpha\} \cup J')$ if $\alpha \notin J$.

Now, suppose that $\widehat{H}_J x_I \widehat{H}_{J'} \in \mathcal{E}$ and $\alpha \in S - I$ is minimal. If $\alpha \in J$ then

$$\Phi(\widehat{H}_{J-\{\alpha\}} x_I \widehat{H}_{J'}) = \widehat{H}_{J-\{\alpha\} \cup \{\alpha\}} x_I \widehat{H}_{J'} = \widehat{H}_J x_I \widehat{H}_{J'}$$

whereas if $\alpha \notin J$

$$\Phi(\widehat{H}_{J \cup \{\alpha\}} x_I \widehat{H}_{J'}) = \widehat{H}_{J \cup \{\alpha\} - \{\alpha\}} x_I \widehat{H}_{J'} = \widehat{H}_J x_I \widehat{H}_{J'}.$$

Thus Φ is surjective and therefore bijective since $\mathcal{E}$ is finite.

Finally, it is clear that $\Phi(\mathcal{E}_0) \subseteq \mathcal{E}_1$ and $\Phi(\mathcal{E}_1) \subseteq \mathcal{E}_0$. Moreover, we must have $\Phi(\mathcal{E}_0) = \mathcal{E}_1$ and $\Phi(\mathcal{E}_1) = \mathcal{E}_0$ since Φ is surjective and $\mathcal{E} = \mathcal{E}_0 \cup \mathcal{E}_1$. Hence Φ restricts to a bijection $\Phi : \mathcal{E}_0 \to \mathcal{E}_1$ and so $|\mathcal{E}_0| = |\mathcal{E}_1|$. □

Theorem 6.2.6. *For each $J \subseteq S$*

$$(\chi, (1_{\widehat{H}_J})^{\widehat{H}}) = \begin{cases} 1 & \text{if } J = \emptyset; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Using the expression of χ as an alternating sum of permutation characters we see that

$$\begin{aligned} (\chi, (1_{\widehat{H}_J})^{\widehat{H}}) &= \sum_{I \subseteq S} (-1)^{|I|} ((1_{\widehat{H}_I})^{\widehat{H}}, (1_{\widehat{H}_J})^{\widehat{H}}) \\ &= \sum_{I \subseteq S} (-1)^{|I|} |\mathcal{D}_{\widehat{H}}(\widehat{H}_I, \widehat{H}_J)| \\ &= \sum_{|I| \text{ even}} |\mathcal{D}_{\widehat{H}}(\widehat{H}_I, \widehat{H}_J)| - \sum_{|I| \text{ odd}} |\mathcal{D}_{\widehat{H}}(\widehat{H}_I, \widehat{H}_J)|. \end{aligned}$$

Now, if $J \neq \emptyset$ this gives

$$(\chi, (1_{\widehat{H}_J})^{\widehat{H}}) = |\mathcal{E}_0| - |\mathcal{E}_1| = 0$$

while if $J = \emptyset$ we obtain

$$(\chi, (1_{\widehat{B}})^{\widehat{H}}) = |\mathcal{E}_0| + 1 - |\mathcal{E}_1| = 1.$$

□

Corollary 6.2.7. *χ is an irreducible constituent of $(1_{\widehat{B}})^{\widehat{H}}$.*

Proof. By Theorem 6.2.6 we have

$$(\chi, \chi) = \sum_{J \subseteq S} (-1)^{|J|} (\chi, (1_{\widehat{H}_J})^{\widehat{H}}) = (\chi, (1_{\widehat{B}})^{\widehat{G}}) = 1.$$

□

6.3 Characterisation of St_ℓ

Unfortunately, it seems to be difficult to explicitly describe the double cosets that are not contained in H. However, this is not necessary as we only need to consider an alternating sum involving the number of double cosets.

The key to showing that St_ℓ was irreducible for $\mathrm{PGL}_2(R)$ was the fact that given any $k < \ell$ we had

$$\widehat{B}\begin{bmatrix} 1 & 0 \\ \pi^k & 1 \end{bmatrix}\widehat{B} = \widehat{B}_{\ell-1}\begin{bmatrix} 1 & 0 \\ \pi^k & 1 \end{bmatrix}\widehat{B}$$

and so the double cosets cancelled in the alternating sum. Indeed, it will be sufficient for our purposes to show a similar result for the $(\widehat{B}, \widehat{B})$-double cosets of $\widehat{G}$ which are not contained in $\widehat{H}$.

Theorem 6.3.1. *For each $(\widehat{B}, \widehat{B})$-double coset $\widehat{B}g\widehat{B}$ not contained in $\widehat{H}$ we have $\widehat{B}g\widehat{B} = \widehat{H}_\alpha g\widehat{B}$ for some $\alpha \in S$ which depends only on the $(\widehat{H}, \widehat{H})$-double coset $\widehat{H}g\widehat{H}$.*

Appropriately enough for such an important result, the proof of Theorem 6.3.1 is long and is consequently postponed until the next section. However, from Theorem 6.3.1 we immediately obtain a similar result concerning the $(\widehat{H}_I, \widehat{H}_J)$-double cosets.

Proposition 6.3.2. *Let $\widehat{H}_I g\widehat{H}_J$ be an $(\widehat{H}_I, \widehat{H}_J)$-double coset not contained in $\widehat{H}$, then there is an $\alpha \in S$, depending only on the $(\widehat{H}, \widehat{H})$-double coset $\widehat{H}g\widehat{H}$, with*

$$\widehat{H}_I g\widehat{H}_J = \begin{cases} \widehat{H}_{I-\{\alpha\}} g\widehat{H}_J & \text{if } \alpha \in I; \\ \widehat{H}_{I\cup\{\alpha\}} g\widehat{H}_J & \text{if } \alpha \notin I. \end{cases}$$

Proof. Since $\widehat{B}g\widehat{B}$ is a $(\widehat{B}, \widehat{B})$-double coset not contained in $\widehat{H}$, by Theorem 6.3.1 $\widehat{B}g\widehat{B} = \widehat{H}_\alpha g\widehat{B}$ for some $\alpha \in S$ depending only on $\widehat{H}g\widehat{H}$. Thus, if $\alpha \in I$

$$\widehat{H}_{I-\{\alpha\}} g\widehat{H}_J = \widehat{H}_{I-\{\alpha\}}\widehat{B}g\widehat{B}\widehat{H}_J = \widehat{H}_{I-\{\alpha\}}\widehat{H}_\alpha g\widehat{B}\widehat{H}_J = \widehat{H}_I g\widehat{H}_J$$

and if $\alpha \notin I$

$$\widehat{H}_I g \widehat{H}_J = \widehat{H}_I \widehat{B} g \widehat{B} \widehat{H}_J = \widehat{H}_I \widehat{H}_\alpha g \widehat{B} \widehat{H}_J = \widehat{H}_{I \cup \{\alpha\}} g \widehat{H}_J. \qquad \square$$

This then gives us a method of pairing up the double cosets in a way which allows us to repeat the approach from the previous section.

Fix $J \subseteq S$ and consider the set of double cosets

$$\mathcal{D} = \bigcup_{I \subseteq S} \mathcal{D}_{\widehat{G}-\widehat{H}}(\widehat{H}_I, \widehat{H}_J)$$

where $\mathcal{D}_{\widehat{G}-\widehat{H}}(\widehat{H}_I, \widehat{H}_J)$ denotes the set of $(\widehat{H}_I, \widehat{H}_J)$-double cosets contained in $\widehat{G} - \widehat{H}$. $\mathcal{D}$ then decomposes into the disjoint union $\mathcal{D} = \mathcal{D}_0 \cup \mathcal{D}_1$ of subsets

$$\mathcal{D}_0 = \bigcup_{|I| \text{ even}} \mathcal{D}_{\widehat{G}-\widehat{H}}(\widehat{H}_I, \widehat{H}_J) \qquad \text{and} \qquad \mathcal{D}_1 = \bigcup_{|I| \text{ odd}} \mathcal{D}_{\widehat{G}-\widehat{H}}(\widehat{H}_I, \widehat{H}_J).$$

Again, we can show that $\mathcal{D}_0$ and $\mathcal{D}_1$ contain the same number of double cosets by constructing a bijection between them.

Theorem 6.3.3. $|\mathcal{D}_0| = |\mathcal{D}_1|$.

Proof. Fix an ordering of the roots in S. For each $\widehat{H}_I g \widehat{H}_J \in \mathcal{D}$ choose a minimal root $\alpha \in S$ from Proposition 6.3.2. Then we may define a map $\Psi : \mathcal{D} \to \mathcal{D}$ by setting

$$\Psi(\widehat{H}_I g \widehat{H}_J) = \begin{cases} \widehat{H}_{I-\{\alpha\}} g \widehat{H}_J & \text{if } \alpha \in I; \\ \widehat{H}_{I \cup \{\alpha\}} g \widehat{H}_J & \text{if } \alpha \notin I. \end{cases}$$

This is well-defined since if $\widehat{H}_I g \widehat{H}_J = \widehat{H}_I g' \widehat{H}_J$ then the minimal $\alpha \in S$ is the same for both choices of representative and, by Proposition 6.3.2, if $\alpha \in I$

$$\Psi(\widehat{H}_I g \widehat{H}_J) = \widehat{H}_{I-\{\alpha\}} g \widehat{H}_J = \widehat{H}_I g \widehat{H}_J = \widehat{H}_I g' \widehat{H}_J = \widehat{H}_{I-\{\alpha\}} g' \widehat{H}_J = \Psi(\widehat{H}_I g' \widehat{H}_J)$$

whereas if $\alpha \notin I$

$$\Psi(\widehat{H}_I g \widehat{H}_J) = \widehat{H}_{I \cup \{\alpha\}} g \widehat{H}_J = \widehat{H}_I g \widehat{H}_J = \widehat{H}_I g' \widehat{H}_J = \widehat{H}_{I \cup \{\alpha\}} g' \widehat{H}_J = \Psi(\widehat{H}_I g' \widehat{H}_J).$$

Now, let $\widehat{H}_I g \widehat{H}_J \in \mathcal{D}$ and suppose that $\alpha \in S$ is minimal from Proposition 6.3.2. If $\alpha \in I$, then α is also the minimal choice for $\widehat{H}_{I-\{\alpha\}} g \widehat{H}_J \in \mathcal{D}$ and therefore

$$\Psi(\widehat{H}_{I-\{\alpha\}} g \widehat{H}_J) = \widehat{H}_{I-\{\alpha\}\cup\{\alpha\}} g \widehat{H}_J = \widehat{H}_I g \widehat{H}_J.$$

Similarly if $\alpha \notin I$, then α is the minimal choice for $\widehat{H}_{I\cup\{\alpha\}} g \widehat{H}_J$ and so

$$\Psi(\widehat{H}_{I\cup\{\alpha\}} g \widehat{H}_J) = \widehat{H}_{I\cup\{\alpha\}-\{\alpha\}} g \widehat{H}_J = \widehat{H}_I g \widehat{H}_J.$$

Hence Ψ is surjective and, since $\mathcal{D}$ is finite, therefore bijective.

Finally, it is clear from its definition that $\Psi(\mathcal{D}_0) \subseteq \mathcal{D}_1$ and $\Psi(\mathcal{D}_1) \subseteq \mathcal{D}_0$. Thus, since Ψ is surjective and $\mathcal{D} = \mathcal{D}_0 \cup \mathcal{D}_1$, we must have $\Psi(\mathcal{D}_0) = \mathcal{D}_1$ and $\Psi(\mathcal{D}_1) = \mathcal{D}_0$. Hence Ψ restricts to a bijection $\Psi : \mathcal{D}_0 \to \mathcal{D}_1$ and so $|\mathcal{D}_0| = |\mathcal{D}_1|$. □

Corollary 6.3.4. *For each $J \subseteq S$,*

$$(\mathrm{St}_\ell, (1_{\widehat{H}_J})^{\widehat{G}}) = (\chi, (1_{\widehat{H}_J})^{\widehat{H}}).$$

Proof. For each $J \subseteq S$,

$$\begin{aligned}
(\mathrm{St}_\ell, (1_{\widehat{H}_J})^{\widehat{G}}) &= \sum_{I\subseteq S} (-1)^{|J|}((1_{\widehat{H}_I})^{\widehat{G}}, (1_{\widehat{H}_J})^{\widehat{G}}) \\
&= \sum_{I\subseteq S} (-1)^{|J|} |\mathcal{D}_{\widehat{G}}(\widehat{H}_I, \widehat{H}_J)| \\
&= \sum_{I\subseteq S} (-1)^{|J|} |\mathcal{D}_{\widehat{H}}(\widehat{H}_I, \widehat{H}_J)| + \sum_{I\subseteq S} (-1)^{|J|} |\mathcal{D}_{\widehat{G}-\widehat{H}}(\widehat{H}_I, \widehat{H}_J)| \\
&= \sum_{I\subseteq S} (-1)^{|J|} |\mathcal{D}_{\widehat{H}}(\widehat{H}_I, \widehat{H}_J)| + |\mathcal{D}_0| - |\mathcal{D}_1| \\
&= \sum_{I\subseteq S} (-1)^{|J|} |\mathcal{D}_{\widehat{H}}(\widehat{H}_I, \widehat{H}_J)| \\
&= (\chi, (1_{\widehat{H}_J})^{\widehat{G}}),
\end{aligned}$$

as required. □

Thus, from Theorem 6.2.6 we immediately have the following result.

Corollary 6.3.5. *For any $J \subseteq S$,*

$$(\mathrm{St}_\ell, (1_{\widehat{H}_J})^{\widehat{G}}) = \begin{cases} 1 & \text{if } J = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, the characterisation of St_ℓ is a simple consequence of its expression as an alternating sum of permutation characters and Corollary 6.3.5.

Theorem 6.3.6. St_ℓ *is the unique irreducible constituent of $(1_{\widehat{B}})^{\widehat{G}}$ which is not a constituent of $(1_{\widehat{P}})^{\widehat{G}}$ for any parabolic subgroup $\widehat{P}$ strictly containing $\widehat{B}$.*

Proof. Using Corollary 6.3.5,

$$(\mathrm{St}_\ell, \mathrm{St}_\ell) = \sum_{J \subseteq S} (-1)^{|J|} (\mathrm{St}_\ell, (1_{\widehat{H}_J})^{\widehat{G}}) = (\mathrm{St}_\ell, (1_{\widehat{B}})^{\widehat{G}}) = 1$$

and so St_ℓ is an irreducible constituent of $(1_{\widehat{B}})^{\widehat{G}}$. Further, St_ℓ cannot be a constituent of $(1_{\widehat{P}})^{\widehat{G}}$ for any parabolic subgroup $\widehat{P}$ which strictly contains $\widehat{B}$ since, by Proposition 4.1.4, we have $\widehat{P} \geq \widehat{H}_\alpha$ for some α. This then implies that

$$0 \leq (\mathrm{St}_\ell, (1_{\widehat{P}})^{\widehat{G}}) \leq (\mathrm{St}_\ell, (1_{\widehat{H}_\alpha})^{\widehat{G}}) = 0$$

which gives $(\mathrm{St}_\ell, (1_{\widehat{P}})^{\widehat{G}}) = 0$.

Conversely, suppose that ζ is an irreducible constituent of $(1_{\widehat{B}})^{\widehat{G}}$ which is not a constituent of $(1_{\widehat{P}})^{\widehat{G}}$ for any parabolic subgroup $\widehat{P}$ strictly containing $\widehat{B}$. In particular, this means that $(\zeta, (1_{\widehat{B}})^{\widehat{G}}) > 0$ and $(\zeta, (1_{\widehat{P}})^{\widehat{G}}) = 0$ for any parabolic subgroup $\widehat{P}$ which strictly contains $\widehat{B}$. Hence,

$$(\zeta, \mathrm{St}_\ell) = \sum_{J \subseteq S} (-1)^{|J|} (\zeta, (1_{\widehat{H}_J})^{\widehat{G}}) = (\zeta, (1_{\widehat{B}})^{\widehat{G}}) > 0$$

and, since ζ and St_ℓ are both irreducible, we must have $\zeta = \mathrm{St}_\ell$. $\square$

6.4 Proof of Theorem 6.3.1

We are trying to show that $\widehat{B}g\widehat{B} = \widehat{H}_\alpha g\widehat{B}$ for some $\alpha \in S$. However,

$$\widehat{H}_\alpha g\widehat{B} = \widehat{B}X_\alpha g\widehat{B} = \bigcup_{r\in\mathfrak{m}^{\ell-1}} \widehat{B}x_\alpha(r)g\widehat{B}.$$

Thus we need to show that there is an $\alpha \in S$ so that for each $r \in \mathfrak{m}^{\ell-1}$ we have $x_\alpha(r)g = b'gb$ for some $b, b' \in \widehat{B}$ since then $\widehat{B}x_\alpha(r)g\widehat{B} = \widehat{B}g\widehat{B}$. In fact we will prove that we can find $v, v' \in V = \widehat{T}(\mathfrak{m})U$ such that $[v, g] = v'x_\alpha(r)$ as this then implies that we have $x_\alpha(r)g = (v')^{-1}v'x_\alpha(r)g = (v')^{-1}vgv^{-1}$.

Proposition 6.4.1. *Let $\widehat{B}g\widehat{B}$ be a $(\widehat{B}, \widehat{B})$-double coset not contained in $\widehat{H}$. Then there exists an $\alpha \in S$ so that for every $r \in \mathfrak{m}^{\ell-1}$ we have $[v, g] = v'x_\alpha(r)$ for some $v, v' \in V$. Moreover, α depends only on the $(\widehat{H}, \widehat{H})$-double coset $\widehat{H}g\widehat{H}$.*

Proof of Theorem 6.3.1. Suppose that $\widehat{B}g\widehat{B}$ is a double coset of $\widehat{G}$ which is not contained in $\widehat{H}$ and let $\alpha \in S$ be as in the conclusion of Proposition 6.4.1. Then, for each $r \in \mathfrak{m}^{\ell-1}$ we have $[v, g] = v'x_\alpha(r)$ for some $v, v' \in V$ and therefore

$$g = v^{-1}[v, g]g^{-1}v^{-1} = v^{-1}v'x_\alpha(r)gv^{-1}$$

which implies that $\widehat{B}g\widehat{B} = \widehat{B}x_\alpha(r)g\widehat{B}$. Hence

$$\widehat{H}_\alpha g\widehat{B} = \bigcup_{r\in\mathfrak{m}^{\ell-1}} \widehat{B}x_\alpha(r)g\widehat{B} = \widehat{B}g\widehat{B}$$

where, by Proposition 6.4.1, α depends only on $\widehat{H}g\widehat{H}$. $\square$

The remainder of this section will be concerned with the proof of Proposition 6.4.1. We begin by showing that the $(\widehat{B}, \widehat{B})$-double coset representatives can be chosen to have a particular form. This is a weaker version of [15, Proposition 2.6].

Lemma 6.4.2. *Each $(\widehat{B}, \widehat{B})$-double coset has a representative of the form $g = kn_w$ for some $k \in U^-(\mathfrak{m})$ and $w \in W$.*

Proof. The natural projection $\eta_1 : \widehat{G} \to \widehat{G}(\kappa)$ maps $(\widehat{B}, \widehat{B})$-double cosets of $\widehat{G}$ to $(\widehat{B}(\kappa), \widehat{B}(\kappa))$-double cosets of $\widehat{G}(\kappa)$. Thus we may assume that the image of the double coset representative under η_1 is n_w for some $w \in W$ and so the double coset representative itself can be chosen to be kn_w for some $k \in K_1$ and $w \in W$. However, $K_1 = \widehat{B}(\mathfrak{m})U^-(\mathfrak{m})$ and therefore we may take $k \in U^-(\mathfrak{m})$. □

Assuming that the $(\widehat{B}, \widehat{B})$-double coset representative g has this form we can prove Proposition 6.4.1 by considering four different cases for k and w.

Lemma 6.4.3 (Case 1). *If $w \neq 1$, then there is an $\alpha \in S$ so that for each $r \in \mathfrak{m}^{\ell-1}$ we have $[v, g] = v' x_\alpha(r)$ for some $v, v' \in V$.*

Proof. By Proposition 2.1.3(ii) there must be some $\alpha \in S$ with $w^{-1}(\alpha) \in \Sigma^+$. Thus for every $r \in \mathfrak{m}^{\ell-1}$ we have $n_w^{-1} x_\alpha(r) n_w = x_{w^{-1}(\alpha)}(\pm r) \in U$. Hence, fixing $r \in \mathfrak{m}^{\ell-1}$ we see that $v = n_w^{-1} x_\alpha(r) n_w \in V$ gives

$$[v, g] = vgvg^{-1} = vkn_w n_w^{-1} x_\alpha(r) n_w n_w^{-1} k^{-1} = vkx_\alpha(r)k^{-1} = vx_\alpha(r). \qquad \square$$

If $w = 1$, then $g \in U^-(\mathfrak{m})$ with $g \notin \widehat{H}$ and we may express g as

$$g = \prod_{\beta \in \Sigma^-} x_\beta(r_\beta)$$

for some $r_\beta \in \mathfrak{m}$. Further, suppose that $r_\beta \in \pi^{i_\beta} R^\times$ for some $1 \leq i_\beta \leq \ell$ and consider $i = \min\{i_\beta : \beta \in \Sigma^-\}$. In the case where the minimum occurs only for roots in S there is again a reasonably straightforward choice for v.

Lemma 6.4.4 (Case 2). *Suppose that $i_\alpha = i$ for some $\alpha \in S$ and that $i_\beta > i$ for any root $\beta \in \Sigma^-$ with $\beta \notin S$. Then for each $r \in \mathfrak{m}^{\ell-1}$ we have $[v, g] = v' x_\alpha(r)$ for some $v, v' \in V$.*

Proof. First note that since $g \notin \widehat{H}$ we have $i < \ell$. Thus $1 + s \in R^\times$ for every $s \in \mathfrak{m}^{\ell-i-1}$ and so $y_{-\alpha}(1+s)^{-1} \in V$. For each $\beta \in \Sigma^-$ with $\beta \notin S$ we have $r_\beta s = 0$

since $i_\beta > i$. Thus by Lemma 3.3.2

$$[y_{-\alpha}(1+s)^{-1}, x_\beta(r_\beta)] = 1.$$

Further, for $\beta \in S$ with $\beta \neq \alpha$ we have $k_\alpha = 0$ and so again obtain

$$[y_{-\alpha}(1+s)^{-1}, x_\beta(r_\beta)] = 1.$$

Finally, for $\beta = \alpha$

$$\begin{aligned}
[y_{-\alpha}(1+s)^{-1}, x_\alpha(r_\alpha)] &= y_{-\alpha}(1+s)^{-1}x_\alpha(r_\alpha)y_{-\alpha}(1+s)x_\alpha(r_\alpha)^{-1} \\
&= x_\alpha(r_\alpha(1+s))x_\alpha(-r_\alpha) \\
&= x_\alpha(r_\alpha s).
\end{aligned}$$

Hence, if we fix $r \in \mathfrak{m}^{\ell-1}$ and choose $s \in \mathfrak{m}^{\ell-i-1}$ with $r_\alpha s = r$, then setting $v = y_{-\alpha}(1+s)^{-1} \in V$ we see that, by Lemma 3.4.4(i),

$$[v, g] = [v, \prod_{\beta \in \Sigma^-} x_\beta(r_\beta)] = \prod_{\beta \in \Sigma^-} [v, x_\beta(r_\beta)] = x_\alpha(r_\alpha s) = x_\alpha(r). \qquad \square$$

Now, suppose that $i_\gamma = i$ for some $\gamma \in \Sigma^-$ with $\gamma \notin S$ and let a denote the minimal height of such a root γ.

Lemma 6.4.5. *Let $\beta \in \Sigma_{-a-1}$ and $s_\beta \in \mathfrak{m}^{\ell-i-1}$, then*

$$[x_\beta(s_\beta), g] = v_\beta \prod_\gamma x_{\beta+\gamma}(c_{1,1,\beta,\gamma}(-s_\beta)r_\gamma)$$

for some $v_\beta \in V$ where the product runs over all $\gamma \in \Sigma_a$ with $i_\gamma = i$ and $\beta + \gamma \in S$.

Proof. If $\mathrm{ht}(\gamma) < a$, then $i_\gamma > i$ by the minimality of a. Thus, for any $\beta \in \Sigma_{-a-1}$ we have $s_\beta r_\gamma = 0$ and so

$$v_{\beta,\gamma} = [x_\beta(s_\beta), x_\gamma(r_\gamma)] = 1.$$

Similarly, if $\mathrm{ht}(\gamma) = a$ and $i_\gamma > i$ we again have

$$v_{\beta,\gamma} = [x_\beta(s_\beta), x_\gamma(r_\gamma)] = 1.$$

Further, if $\mathrm{ht}(\gamma) > a$ then by Lemma 3.4.3(i) we see that

$$v_{\beta,\gamma} = [x_\beta(s_\beta), x_\gamma(r_\gamma)] \in \widehat{B}(\mathfrak{m}^{\ell-1}).$$

Finally, suppose that $\mathrm{ht}(\gamma) = a$ and $i_\gamma = i$. By Lemma 3.4.3(iii), if $\beta + \gamma \notin \Sigma$

$$v_{\beta,\gamma} = [x_\beta(s_\beta), x_\gamma(r_\gamma)] \in \widehat{B}(\mathfrak{m}^{\ell-1})$$

whereas if $\beta + \gamma \in \Sigma$, there is an element $v_{\beta,\gamma} \in \widehat{B}(\mathfrak{m}^{\ell-1})$ with

$$[x_\beta(s_\beta), x_\gamma(r_\gamma)] = x_{\beta+\gamma}(c_{1,1,\beta,\gamma}(-s_\beta)r_\gamma)v_{\beta,\gamma}.$$

Now, set $\Sigma^- = \{\gamma_1, \ldots, \gamma_k\}$ and for each i let $v_i = x_{\gamma_1}(r_{\gamma_1}) \cdots x_{\gamma_i}(r_{\gamma_i})$. Then, since $v_i \in U^-(\mathfrak{m})$, it commutes with any commutator $[x_\beta(s_\beta), x_{\gamma_j}(r_{\gamma_j})]$. Thus, by Lemma 3.4.4(i)

$$\begin{aligned}
[x_\beta(s_\beta), g] &= [x_\beta(s_\beta), x_{\gamma_1}(r_{\gamma_1}) \cdots x_{\gamma_k}(r_{\gamma_k})] \\
&= [x_\beta(s_\beta), x_{\gamma_1}(r_{\gamma_1})](v_1[x_\beta(s_\beta), x_{\gamma_1}(r_{\gamma_2})]v_1^{-1}) \\
&\quad \cdots (v_{k-2}[x_\beta(s_\beta), x_{\gamma_{k-1}}(r_{\gamma_{k-1}})]v_{k-2}^{-1})(v_{k-1}[x_\beta(s_\beta), x_{\gamma_k}(r_{\gamma_k})]v_{k-1}^{-1}) \\
&= [x_\beta(s_\beta), x_{\gamma_1}][x_\beta(s_\beta), x_{\gamma_1}(r_{\gamma_2})] \\
&\quad \cdots [x_\beta(s_\beta), x_{\gamma_{k-1}}(r_{\gamma_{k-1}})][x_\beta(s_\beta), x_{\gamma_k}(r_{\gamma_k})].
\end{aligned}$$

Hence, using the description of the commutators given above, and the fact that since each $v_{\beta,\gamma}$ lies in $\widehat{B}(\mathfrak{m}^{\ell-1})$ it must also commute with any of the commutators, we obtain

$$[x_\beta(s_\beta), g] = v_\beta \prod_\gamma x_{\beta+\gamma}(c_{1,1,\beta,\gamma}(-s_\beta)r_\gamma)$$

where the product runs over all roots $\gamma \in \Sigma_a$ with $i_\gamma = i$ and $\beta + \gamma \in \Sigma$, and $v_\beta = \prod_{\gamma\in\Sigma_-} v_{\beta,\gamma} \in V$. $\square$

In particular, if in Lemma 6.4.5 there is only one $\gamma \in \Sigma_a$ with $i_\gamma = i$ and $\beta+\gamma \in S$, then we may choose v to be $x_\beta(s)$ for some s.

Lemma 6.4.6 (Case 3). *Suppose that $\beta \in \Sigma_{-a-1}$ is such that $\beta+\gamma \in S$ for exactly one $\gamma \in \Sigma_a$ with $i_\gamma = i$. Then, for each $r \in \mathfrak{m}^{\ell-1}$ we have $[v,g] = v'x_{\beta+\gamma}(r)$ for some $v, v' \in V$.*

Proof. By Lemma 6.4.5 we know that for any $s \in \mathfrak{m}^{\ell-i-1}$ we have

$$[x_\beta(s), g] = v'x_{\beta+\gamma}(c_{1,1,\beta,\gamma}(-s)r_\gamma)$$

for some $v' \in V$. Thus, if we fix an element $r \in \mathfrak{m}^{\ell-1}$ and choose $s \in \mathfrak{m}^{\ell-i-1}$ so that $c_{1,1,\beta,\gamma}(-s)r_\gamma = r$, then setting $v = x_\beta(s)$ gives $[v,g] = v'x_{\beta+\gamma}(r)$. □

Now, if we let $\mathcal{S} = \{\gamma \in \Sigma_a : i_\gamma = i\}$ then the following Proposition shows that Lemma 6.4.6 suffices for all but a small number of exceptional cases. In particular, it suffices whenever $\Sigma = A_n$, B_n, C_n or G_2.

Proposition 6.4.7. *Let Σ be an irreducible root system and $\mathcal{S}$ be a non-empty subset of Σ_i for some $i < -1$, with the exception of the following cases:*

(i) $\Sigma = D_{2n}$ *and* $\mathcal{S} = \Sigma_{1-2n}$*;*

(ii) $\Sigma = E_6$ *and* $\mathcal{S} = \Sigma_{-4}$*;*

(iii) $\Sigma = E_7$ *and* $\mathcal{S} = \Sigma_{-9}$*;*

(iv) $\Sigma = E_8$ *and* $\mathcal{S} = \Sigma_{-6}$, Σ_{-10} *or* Σ_{-15}*; and*

(v) $\Sigma = F_4$ *and* $\mathcal{S} = \Sigma_{-4}$

together with the corresponding sets obtained when Σ contains a subsystem equivalent to D_{2n}, E_6 or E_7. Then there exists a $\beta \in \Sigma_{-i-1}$ such that $\beta + \gamma \in S$ for exactly one $\gamma \in \mathcal{S}$.

Proof. The details can be found in Appendix B. □

For the exceptional cases we consider the element $v \in V$ given by

$$v = \prod_{\beta \in \Sigma_{-a-1}} x_\beta(s_\beta)$$

where for each $\beta \in \Sigma_{-a-1}$ we fix $s_\beta \in \mathfrak{m}^{\ell-i-1}$.

Lemma 6.4.8. *Let v be as above, then*

$$[v, g] = v' \prod_{\alpha \in S} x_\alpha(t_\alpha)$$

for some $v' \in V$, where for each $\alpha \in S$

$$t_\alpha = \sum_{\beta+\gamma=\alpha} c_{1,1,\beta,\gamma}(-s_\beta) r_\gamma$$

with the sum running over all $\beta \in \Sigma_{-a-1}$ and $\gamma \in \mathcal{S}$ such that $\beta + \gamma = \alpha$.

Proof. Setting $\Sigma_{-a-1} = \{\beta_1, \ldots, \beta_k\}$ and $v_i = x_{\beta_1}(s_{\beta_1}) \cdots x_{\beta_j}(s_{\beta_j}) \in V$ for each j, by Lemma 3.4.4(ii) we obtain

$$\begin{aligned} [v, g] &= [x_{\beta_1}(s_{\beta_1}) \cdots x_{\beta_k}(s_{\beta_k}), g] \\ &= \left(v_{k-1}[x_{\beta_k}(s_{\beta_k}), g] v_{k-1}^{-1}\right) \left(v_{k-2}[x_{\beta_k}(s_{\beta_{k-1}}), g] v_{k-2}^{-1}\right) \\ &\quad \cdots \left(v_1[x_{\beta_k}(s_{\beta_2}), g] v_1^{-1}\right) [x_{\beta_1}(s_{\beta_1}), g]. \end{aligned}$$

However, since $[x_\beta(s_\beta), g] \in V$ for each β and V is normal in $\widehat{H}$, this can be rearranged to give

$$[v, g] = v' \prod_{\beta \in \Sigma_{-a-1}} [x_\beta(s_\beta), g] \tag{6.2}$$

for some $v' \in V$.

Further, by Lemma 6.4.5 we know that for each $\beta \in \Sigma_{-a-1}$ there is some $v_\beta \in V$ with

$$[x_\beta(s_\beta), g] = v_\beta \prod_\gamma x_{\beta+\gamma}(c_{1,1,\beta,\gamma}(-s_\beta) r_\gamma),$$

where the product runs over all $\gamma \in \mathcal{S}$ with $\beta + \gamma \in S$. Thus, from (6.2) we obtain

$$[v, g] = v'' \prod_{\beta \in \Sigma_{-a-1}} \prod_{\gamma} x_{\beta+\gamma}(c_{1,1,\beta,\gamma}(-s_\beta) r_\gamma) \tag{6.3}$$

for some $v'' \in V$ where once again the second product runs over all $\gamma \in \mathcal{S}$ with $\beta + \gamma \in S$. Hence, combining terms in (6.3) we obtain

$$[v, g] = v'' \prod_{\alpha \in S} x_\alpha(t_\alpha)$$

where for each $\alpha \in S$

$$t_\alpha = \sum_{\beta+\gamma=\alpha} c_{1,1,\beta,\gamma}(-s_\beta) r_\gamma$$

with the sum running over all $\beta \in \Sigma_{-a-1}$ and $\gamma \in \mathcal{S}$ such that $\beta + \gamma = \alpha$. $\square$

Lemma 6.4.9 (Case 4). *Suppose that $\mathcal{S}$ is one of the exceptional cases from Proposition 6.4.7. Then there is an $\alpha \in S$ so that for every $r \in \mathfrak{m}^{\ell-1}$ we have $[v, g] = v' x_\alpha(r)$ for some $v, v' \in V$.*

Proof. By Lemma 6.4.8 we only need to prove that there is an $\alpha \in S$ so that for any $r \in \mathfrak{m}^{\ell-1}$ we may choose the s_β in such a way that $t_\alpha = r$ and $t_{\alpha'} = 0$ for $\alpha' \neq \alpha$. For each case, this can be shown explicitly and the details are contained in Appendix C. $\square$

Finally, we are able to prove Proposition 6.4.1.

Proof of Proposition 6.4.1. We have shown in Lemmas 6.4.3, 6.4.4, 6.4.6 and 6.4.9 that for any $g \notin \widehat{H}$ of the form $g = kn_w$ for some $k \in U^-(\mathfrak{m})$ and $w \in W$, there is an $\alpha \in S$ so that for any $r \in \mathfrak{m}^{\ell-1}$ we have $[v, g] = x_\alpha(r)v'$ for some $v, v' \in V$. If we can show that this α depends only on $\widehat{H} g \widehat{H}$ then we will be done.

Suppose that $g_1, g_2 \notin \widehat{H}$ are such that $\widehat{H} g_1 \widehat{H} = \widehat{H} g_2 \widehat{H}$ and fix $r \in \mathfrak{m}^{\ell-1}$. Then $g_2 = h_1 g_1 h_2$ for some $h_1, h_2 \in \widehat{H}$. Further, since $\widehat{H} = X_S$ and $\widehat{B} = \widehat{T} V$, we can express h_1 as $h_1 = x_1 t_1 u_1$ where $x_1 \in X_S$, $t_1 \in \widehat{T}$ and $u_1 \in V$. Thus

$$h_1^{-1} x_\alpha(r) h_1 = u_1^{-1} t_1^{-1} x_1^{-1} x_\alpha(r) x_1 t_1 u_1 = u_1^{-1} t_1^{-1} x_\alpha(r) t_1 u_1 = u_1^{-1} x_\alpha(r') u_1 = u_1' x_\alpha(r')$$

with $r' \in \mathfrak{m}^{\ell-1}$ and $u_1' \in V$.

Now, suppose that $v_1, v_1' \in V$ are such that $[v_1, g_1] = v_1' x_\alpha(r')$. Then, since V is normal in $\widehat{H}$, we have $v_2 = h_2^{-1} v_1 h_2 \in V$. Finally,

$$\begin{aligned}
[v_2, g_2] &= v_2 g_2 v_2^{-1} g_2^{-1} \\
&= v_2 (h_1 g_1 h_2)(h_2^{-1} v_1^{-1} h_2)(h_2^{-1} g_1^{-1} h_1^{-1}) \\
&= v_2 h_1 g_1 v_1^{-1} g_1^{-1} h_1^{-1} \\
&= v_2 h_1 (v_1^{-1} v_1) g_1 v_1^{-1} g_1^{-1} h_1^{-1} \\
&= v_2 h_1 v_1 [v_1, g_1] h_1^{-1} \\
&= v_2 h_1 v_1 (v_1' x_\alpha(r')) h_1^{-1} \\
&= v_2 h_1 v_1 v_1' ((u_1')^{-1} u_1') x_\alpha(r') h_1^{-1} \\
&= v_2 h_1 v_1 v_1' (u_1')^{-1} (u_1' x_\alpha(r')) h_1^{-1} \\
&= v_2 h_1 v_1 v_1' (u_1')^{-1} (h_1^{-1} x_\alpha(r) h_1) h_1^{-1} \\
&= v_2' x_\alpha(r)
\end{aligned}$$

where $v_2' = v_2 h_1 (v_1 v_1' (u_1')^{-1}) h_1^{-1} \in V$. □

Chapter 7

Hecke algebras

An alternative construction of the analogue of the Steinberg character can be given in terms of the Hecke algebra $\mathcal{H}(\widehat{G}, \widehat{B})$ of $\widehat{G}$ over $\widehat{B}$. We are able to define a linear character ψ of $\mathcal{H}(G, B)$ by first considering a certain linear character ϕ of the Hecke algebra $\mathcal{H}(\widehat{H}, \widehat{B})$ of $\widehat{H}$ and then showing that ϕ extends uniquely to a linear character ψ of $\mathcal{H}(G, B)$. The analogue St_ℓ is then the unique irreducible constituent of the permutation character over B which corresponds to ψ.

7.1 Definitions and standard results

We begin with some standard definitions and results for an arbitrary finite group G with subgroup B. Proofs can be found in [6] or [10].

Definition 7.1.1. The **Hecke algebra** $\mathcal{H}(G, B)$ of G over B is the subalgebra

$$\mathcal{H}(G, B) = e_B \mathbb{C} G e_B$$

of $\mathbb{C}G$.

Proposition 7.1.2. *For each (B, B)-double coset BgB define*

$$\beta_g = \mathrm{ind}(g) e_B g e_B,$$

where

$$\operatorname{ind}(g) = [B : B \cap g^{-1}Bg].$$

Then $\{\beta_g : BgB \in \mathcal{D}_G(B,B)\}$ *forms a basis for* $\mathcal{H}(G,B)$ *which is independent of the choice of double coset representatives.*

Further, it can be shown that for any $BgB \in \mathcal{D}_G(B,B)$

$$\operatorname{ind}(g) = \frac{|BgB|}{|B|}$$

and the standard basis element β_g is given by

$$\beta_g = |B|^{-1} \sum_{h \in BgB} h.$$

In the same way as for groups, irreducible characters of the Hecke algebra arise from simple $\mathcal{H}(G,B)$-modules by taking the traces of the linear maps corresponding to multiplication by the elements of $\mathcal{H}(G,B)$. The connection between the irreducible characters of the Hecke algebra and of the group G is given by the following theorem.

Theorem 7.1.3. *Let* ζ *be an irreducible character of* G *with* $(\zeta, (1_B)^G) > 0$, *then the restriction of* ζ *to* $\mathcal{H}(G,B)$ *is an irreducible character of* $\mathcal{H}(G,B)$ *of degree* $(\zeta, (1_B)^G)$. *Conversely, each irreducible character* ψ *of* $\mathcal{H}(G,B)$ *is the restriction of a unique irreducible character of* G.

For linear characters of the Hecke algebra, i.e. characters which are non-zero homomorphisms from $\mathcal{H}(G,B)$ to $\mathbb{C}$, we are also able to give an idempotent which generates a module affording the corresponding irreducible character of G.

Theorem 7.1.4. *Let* ψ *be a linear character of* $\mathcal{H}(G,B)$, *then* ψ *is the restriction of a unique irreducible character* ζ *of* G *with* $(\psi, (1_B)^G) = 1$. *Moreover,*

$$e = \frac{\zeta(1)}{[G:B]} \sum_{BgB \in \mathcal{D}_G(B,B)} \frac{1}{\operatorname{ind}(g)} \phi(\beta_{g^{-1}})\beta_g$$

is a primitive idempotent in $\mathbb{C}G$ *such that* $\mathbb{C}Ge$ *affords* ζ.

7.2 Example: $\mathrm{PGL}_2(R)$

Now consider the Hecke algebra $\mathcal{H}(\widehat{G},\widehat{B})$ for $\widehat{G} = \mathrm{PGL}_2(R)$. For each $0 \le i \le \ell$, let β_i denote the basis element of $\mathcal{H}(\widehat{G},\widehat{B})$ corresponding to the $(\widehat{B},\widehat{B})$-double coset

$$\widehat{B}\begin{bmatrix} 1 & 0 \\ \pi^i & 1 \end{bmatrix}\widehat{B}.$$

Clearly, $\beta_\ell = e_{\widehat{B}}$. Further, for each $0 \le i \le \ell - 1$

$$\widehat{B}\begin{bmatrix} 1 & 0 \\ \pi^i & 0 \end{bmatrix}\widehat{B} = \widehat{B}_i - \widehat{B}_{i+1}$$

implies that

$$\beta_i = |\widehat{B}|^{-1}\sum_{g\in\widehat{B}_i-\widehat{B}_{i+1}} g = |\widehat{B}|^{-1}\sum_{g\in\widehat{B}_i} g - |\widehat{B}|^{-1}\sum_{g\in\widehat{B}_{i+1}} g = |\widehat{B}_i:\widehat{B}|e_{\widehat{B}_i} - |\widehat{B}_{i+1}:\widehat{B}|e_{\widehat{B}_{i+1}}.$$

For ease of notation, set $c_i = |\widehat{B}_i : \widehat{B}|$ for each $0 \le i \le \ell$ and $c_{\ell+1} = 0$ so that

$$\beta_i = c_i e_{\widehat{B}_i} - c_{i+1} e_{\widehat{B}_{i+1}}$$

for every $0 \le i \le \ell$.

Theorem 7.2.1. *(i) For each $0 \le i \le \ell$*

$$\beta_i^2 = (c_i - 2c_{i+1})\,\beta_i + (c_i - c_{i+1})\sum_{j=i+1}^{\ell}\beta_j. \tag{7.1}$$

(ii) $\beta_i\beta_j = \beta_j\beta_i$ for each $0 \le i,j \le \ell$.

(iii) For each $0 \le i < j \le \ell$

$$\beta_i\beta_j = (c_j - c_{j+1})\,\beta_i. \tag{7.2}$$

Proof. (i) For each i

$$\begin{aligned}
\beta_i^2 &= \left(c_i e_{\widehat{B}_i} - c_{i+1} e_{\widehat{B}_{i+1}}\right)^2 \\
&= c_i^2 e_{\widehat{B}_i}^2 - c_i c_{i+1} e_{\widehat{B}_i} e_{\widehat{B}_{i+1}} - c_{i+1} c_i e_{\widehat{B}_{i+1}} e_{\widehat{B}_i} + c_{i+1}^2 e_{\widehat{B}_{i+1}}^2 \\
&= c_i^2 e_{\widehat{B}_i} - 2c_i c_{i+1} e_{\widehat{B}_i} + c_{i+1}^2 e_{\widehat{B}_{i+1}} \\
&= (c_i - 2c_{i+1}) \left(c_i e_{\widehat{B}_i} - c_{i+1} e_{\widehat{B}_{i+1}}\right) + (c_i - c_{i+1}) \left(c_{i+1} e_{\widehat{B}_{i+1}}\right) \\
&= (c_i - 2c_{i+1}) \beta_i + (c_i - c_{i+1}) \sum_{j=i+1}^{\ell} \beta_j.
\end{aligned}$$

(ii) This follows immediately from the fact that $e_{\widehat{B}_i} e_{\widehat{B}_j} = e_{\widehat{B}_j} e_{\widehat{B}_i}$.

(iii) For each $i < j$ we see that

$$\begin{aligned}
\beta_i \beta_j &= \left(c_i e_{\widehat{B}_i} - c_{i+1} e_{\widehat{B}_{i+1}}\right) \left(c_j e_{\widehat{B}_j} - c_{j+1} e_{\widehat{B}_{j+1}}\right) \\
&= c_i c_j e_{\widehat{B}_i} e_{\widehat{B}_j} - c_i c_{j+1} e_{\widehat{B}_i} e_{\widehat{B}_{j+1}} - c_{i+1} c_j e_{\widehat{B}_{i+1}} e_{\widehat{B}_j} + c_{i+1} c_{j+1} e_{\widehat{B}_{i+1}} e_{\widehat{B}_{j+1}} \\
&= c_i c_j e_{\widehat{B}_i} - c_i c_{j+1} e_{\widehat{B}_i} - c_{i+1} c_j e_{\widehat{B}_{i+1}} + c_{i+1} c_{j+1} e_{\widehat{B}_{i+1}} \\
&= (c_j - c_{j+1}) \left(c_i e_{\widehat{B}_i} - c_{i+1} e_{\widehat{B}_{i+1}}\right) \\
&= (c_j - c_{j+1}) \beta_i
\end{aligned}$$

as required. □

Having described the multiplication of the basis elements in $\mathcal{H}(\widehat{G}, \widehat{B})$ we want to construct a linear character of $\mathcal{H}(\widehat{G}, \widehat{B})$ which will correspond to St_ℓ. However, it is easier to check that a linear map of $\mathcal{H}(\widehat{G}, \widehat{B})$ is a homomorphism by noting that (7.1) can be rewritten as

$$(\beta_i - (c_i - c_{i+1})\beta_\ell) \sum_{j=i}^{\ell} \beta_j = 0 \tag{7.3}$$

and (7.2) as

$$\beta_i (\beta_j - (c_j - c_{j+1})\beta_\ell) = 0. \tag{7.4}$$

Proposition 7.2.2. *The map $\psi : \mathcal{H}(\widehat{G}, \widehat{B}) \to \mathbb{C}$ given by*

$$\psi(\beta_i) = \begin{cases} 1 & \text{if } i = \ell; \\ -1 & \text{if } i = \ell - 1; \\ 0 & \text{if } i < \ell - 1 \end{cases}$$

is a homomorphism.

Proof. To show that ψ is a homomorphism it suffices to check that it respects (7.3) and (7.4). If $i = \ell$ then

$$\psi(\beta_\ell) - (c_\ell - c_{\ell+1})\psi(\beta_\ell) = 1 - (1 - 0)(1) = 0$$

whereas if $i < \ell$ then

$$\sum_{j=i}^{\ell} \psi(\beta_j) = 0 + \cdots + 0 + (-1) + 1 = 0.$$

Consequently, for every i we have

$$(\psi(\beta_i) - (c_i - c_{i+1})\psi(\beta_\ell)) \sum_{j=i}^{\ell} \psi(\beta_j) = 0.$$

Similarly, if $i < j$ and $i = \ell - 1$ then $j = \ell$ and so again

$$\psi(\beta_\ell) - (c_\ell - c_{\ell+1})\psi(\beta_\ell) = 0$$

while $i < \ell - 1$ implies $\psi(\beta_i) = 0$. Thus, for every $i < j$

$$\psi(\beta_i)(\psi(\beta_j) - (c_j - c_{j+1})\psi(\beta_\ell)) = 0.$$

Hence ψ is a homomorphism. □

The linear character ψ is then the restriction of a unique irreducible character of $\widehat{G}$ which appears as a constituent of $(1_{\widehat{B}})^{\widehat{G}}$ with multiplicity 1. This character is the analogue of the Steinberg character.

Lemma 7.2.3. *ψ is the restriction of* St_ℓ *to* $\mathcal{H}(\widehat{G}, \widehat{B})$.

Proof. From Proposition 6.1.5 we know that $(\mathrm{St}_\ell, (1_{\widehat{B}_\ell})^{\widehat{G}}) = 1$ and so we have

$$\mathrm{St}_\ell(\beta_\ell) = \mathrm{St}_\ell(e_{\widehat{B}_\ell}) = \frac{1}{|\widehat{B}_\ell|}\sum_{g\in\widehat{B}_\ell} \mathrm{St}_\ell(g) = (\mathrm{St}_\ell, (1_{\widehat{B}_\ell})^{\widehat{G}}) = 1.$$

Further, $(\mathrm{St}_\ell, (1_{\widehat{B}_i})^{\widehat{G}}) = 0$ for $i < \ell$ implying that

$$\mathrm{St}_\ell(e_{\widehat{B}_i}) = \frac{1}{|\widehat{B}_i|}\sum_{g\in\widehat{B}_i} \mathrm{St}_\ell(g) = (\mathrm{St}_\ell, (1_{\widehat{B}_i})^{\widehat{G}}) = 0.$$

Thus,

$$\mathrm{St}_\ell(\beta_{\ell-1}) = \mathrm{St}_\ell(c_{\ell-1}e_{\widehat{B}_{\ell-1}} - c_\ell e_{\widehat{B}_\ell}) = c_{\ell-1}\mathrm{St}_\ell(e_{\widehat{B}_{\ell-1}}) - c_\ell \mathrm{St}_\ell(e_{\widehat{B}_\ell}) = -1$$

and for $i < \ell - 1$

$$\mathrm{St}_\ell(\beta_i) = \mathrm{St}_\ell(c_i e_{\widehat{B}_i} - c_{i+1}e_{\widehat{B}_{i+1}}) = c_i\mathrm{St}_\ell(e_{\widehat{B}_i}) - c_{i+1}\mathrm{St}_\ell(e_{\widehat{B}_{i+1}}) = 0.$$

Hence $(\mathrm{St}_\ell)_{\mathcal{H}(\widehat{G},\widehat{B})} = \psi$. □

7.3 Hecke algebra of $\widehat{H}$

For each $\alpha \in S$ we have $\widehat{B}x_{\{\alpha\}}\widehat{B} = \widehat{H}_\alpha - \widehat{B}$ and so the corresponding basis element β_α of $\mathcal{H}(\widehat{H}, \widehat{B})$ can be expressed as

$$\beta_\alpha = qe_{\widehat{H}_\alpha} - e_{\widehat{B}}.$$

More generally, for each $J \subseteq S$

$$\widehat{B}x_J\widehat{B} = \widehat{H}_J - \bigcup_{I\subset J} \widehat{H}_I$$

and so the corresponding basis element β_J is given by

$$\beta_J = \sum_{I\subseteq J}(-1)^{|J|-|I|}q^{|I|}e_{\widehat{H}_I}.$$

This allows us to easily determine the multiplication of the basis elements.

Theorem 7.3.1. *Let $\alpha, \alpha' \in S$ and $J \subseteq S$, then*

(i) $\beta_\alpha^2 = (q-2)\beta_\alpha + (q-1)\beta_\emptyset$;

(ii) $\beta_\alpha\beta_{\alpha'} = \beta_{\alpha'}\beta_\alpha$;

(iii) $\beta_\alpha\beta_J = \beta_{\{\alpha\}\cup J}$ *if* $\alpha \notin J$; *and*

(iv) $\beta_\alpha\beta_J = (q-2)\beta_J + (q-1)\beta_{J-\{\alpha\}}$ *if* $\alpha \in J$.

Proof. (i) For each $\alpha \in S$ we see that

$$\begin{aligned}
\beta_\alpha^2 &= (qe_{\widehat{H}_\alpha} - e_{\widehat{B}})2 \\
&= q^2e^2_{\widehat{H}_\alpha} - qe_{\widehat{H}_\alpha}e_{\widehat{B}} - qe_{\widehat{B}}e_{\widehat{H}_\alpha} + e^2_{\widehat{B}} \\
&= q^2e_{\widehat{H}_\alpha} - 2qe_{\widehat{H}_\alpha} + e_{\widehat{B}} \\
&= (q-2)(qe_{\widehat{H}_\alpha} - e_{\widehat{B}}) + (q-1)e_{\widehat{B}} \\
&= (q-2)\beta_\alpha + (q-1)\beta_\emptyset
\end{aligned}$$

(ii) This follows immediately from the fact that $e_{\widehat{H}_\alpha}e_{\widehat{H}_{\alpha'}} = e_{\widehat{H}_{\alpha'}}e_{\widehat{H}_\alpha}$.

(iii) Suppose that $\alpha \notin J$, then

$$\begin{aligned}
\beta_\alpha\beta_J &= (qe_{\widehat{H}_\alpha} - e_{\widehat{B}})\sum_{I\subseteq J}(-1)^{|J|-|I|}q^{|I|}e_{\widehat{H}_I} \\
&= q\sum_{I\subseteq J}(-1)^{|J|-|I|}q^{|I|}e_{\widehat{H}_\alpha}e_{\widehat{H}_I} - \sum_{I\subseteq J}(-1)^{|J|-|I|}q^{|I|}e_{\widehat{H}_I} \\
&= \sum_{I\subseteq J}(-1)^{|J|-|I|}q^{|I|+1}e_{\widehat{H}_{\{\alpha\}\cup I}} + \sum_{I\subseteq J}(-1)^{|J|-|I|+1}q^{|I|}e_{\widehat{H}_I} \\
&= \sum_{\alpha\in I\subseteq\{\alpha\}\cup J}(-1)^{|\{\alpha\}\cup J|-|I|}q^{|I|}e_{\widehat{H}_I} + \sum_{\alpha\notin I\subseteq\{\alpha\}\cup J}(-1)^{|\{\alpha\}\cup J|-|I|}q^{|I|}e_{\widehat{H}_I} \\
&= \sum_{I\subseteq\{\alpha\}\cup J}(-1)^{|\{\alpha\}\cup J|-|I|}q^{|I|}e_{\widehat{H}_I} \\
&= \beta_{\{\alpha\}\cup J}.
\end{aligned}$$

(iv) Suppose that $\alpha \in J$, then by parts (i) and (iii)

$$\begin{aligned}
\beta_\alpha \beta_J &= \beta_\alpha^2 \beta_{J-\{\alpha\}} \\
&= ((q-2)\beta_\alpha - (q-1)\beta_\emptyset)\beta_{J-\{\alpha\}} \\
&= (q-2)\beta_\alpha\beta_{J-\{\alpha\}} - (q-1)\beta_\emptyset\beta_{J-\{\alpha\}} \\
&= (q-2)\beta_J - (q-1)\beta_{J-\{\alpha\}}
\end{aligned}$$

as required. □

Corollary 7.3.2. *$\mathcal{H}(\widehat{H}, \widehat{B})$ is generated by the basis elements $\{\beta_\alpha\}_{\alpha \in S}$ together with the quadratic relations*

$$(\beta_\alpha - (q-1)\beta_\emptyset)(\beta_\alpha + \beta_\emptyset) = 0 \tag{7.5}$$

and the homogeneous relations $\beta_\alpha \beta_{\alpha'} = \beta_{\alpha'} \beta_\alpha$ for each $\alpha, \alpha' \in S$.

Proof. As the proof shows, Theorem 7.3.1(iv) can be obtained from (iii) using (i). Thus, it is clear that $\mathcal{H}(\widehat{H}, \widehat{B})$ is generated by $\{\beta_\alpha\}_{\alpha \in S}$ together with Theorem 7.3.1 (i) and (ii). However,

$$\begin{aligned}
(\beta_\alpha - (q-1)\beta_\emptyset)(\beta_\alpha + \beta_\emptyset) &= \beta_\alpha\beta_\alpha - (q-1)\beta_\emptyset\beta_\alpha + \beta_\alpha\beta_\emptyset - (q-1)\beta_\emptyset\beta_\emptyset \\
&= \beta_\alpha^2 - (q-2)\beta_\alpha - (q-1)\beta_\emptyset
\end{aligned}$$

and so (i) is equivalent to (7.5). □

Proposition 7.3.3. *The linear map $\phi : \mathcal{H}(\widehat{H}, \widehat{B}) \to \mathbb{C}$ given by $\phi(\beta_J) = (-1)^{|J|}$ is a homomorphism.*

Proof. By Corollary 7.3.2 it suffices to show that ϕ respects (7.5), but $\phi(\beta_\alpha) = -1$ implies that $\phi(\beta_\alpha) + \phi(\beta_\emptyset) = 0$ for each $\alpha \in S$ and so

$$(\phi(\beta_\alpha) - (q-1)\phi(\beta_\emptyset))(\phi(\beta_\alpha) + \phi(\beta_\emptyset)) = 0.$$

Hence ϕ is a homomorphism. □

Lemma 7.3.4. *ϕ is the restriction of χ to $\mathcal{H}(\widehat{H}, \widehat{B})$.*

Proof. By Theorem 6.2.6, $(\chi, (1_{\widehat{B}})^{\widehat{G}}) = 1$ and so $\chi(e_{\widehat{B}}) = 1$. Thus χ restricts to a non-zero homomorphism of $\mathcal{H}(\widehat{H}, \widehat{B})$. Further, for each $\alpha \in S$ we see that $(\chi, (1_{\widehat{H}_\alpha})^{\widehat{G}}) = 0$ and thus

$$\chi(\beta_\alpha) = q\chi(e_{\widehat{H}_\alpha}) - \chi(e_{\widehat{B}}) = -1.$$

Hence, for each non-empty subset $J = \{\alpha_1, \ldots, \alpha_k\} \subseteq S$,

$$\chi(\beta_J) = \chi(\beta_{\alpha_1} \cdots \beta_{\alpha_k}) = \chi(\beta_{\alpha_1}) \cdots \chi(\beta_{\alpha_k}) = (-1)^k = (-1)^{|J|}$$

and $\chi_{\mathcal{H}(\widehat{H},\widehat{B})} = \phi$. □

7.4 Extending to the Hecke algebra of $\widehat{G}$

To obtain the linear character of $\mathcal{H}(\widehat{G}, \widehat{B})$ that gives the analogue St_ℓ we will show that ϕ extends uniquely to a linear character ψ of $\mathcal{H}(\widehat{G}, \widehat{B})$.

Let $\mathcal{K}$ denote the subspace of $\mathcal{H}(\widehat{G}, \widehat{B})$ spanned by the basis elements corresponding to $(\widehat{B}, \widehat{B})$-double cosets not contained in $\widehat{H}$. Then it is clear that we obtain the vector space decomposition

$$\mathcal{H}(\widehat{G}, \widehat{B}) = \mathcal{H}(\widehat{H}, \widehat{B}) \oplus \mathcal{K}.$$

Lemma 7.4.1. *$\mathcal{K}$ is a left and right $\mathcal{H}(\widehat{H}, \widehat{B})$-module.*

Proof. Let β be a basis element of $\mathcal{H}(\widehat{G}, \widehat{B})$ corresponding to a $(\widehat{B}, \widehat{B})$-double coset $\widehat{B}g\widehat{B}$ with $g \notin \widehat{H}$. Then for any $h \in \widehat{H}$ and $b, b' \in \widehat{B}$ we must also have $h(bgb') \notin \widehat{H}$. Consequently, if $\beta_\alpha \in \mathcal{H}(\widehat{H}, \widehat{B})$ we must have $\beta_\alpha \beta \in \mathcal{K}$. Hence, $\mathcal{K}$ is a left $\mathcal{H}(\widehat{H}, \widehat{B})$-module. The proof for multiplication by $\mathcal{H}(\widehat{H}, \widehat{B})$ on the right is similar. □

When we proved that ψ was a linear character of the Hecke algebra for $\mathrm{PGL}_2(R)$ in Proposition 7.2.2 we needed that it was 0 on the basis elements β_i with $i < \ell$. If we similarly define ψ to be 0 on $\mathcal{K}$ then we again obtain a linear character of $\mathcal{H}(\widehat{G}, \widehat{B})$.

Theorem 7.4.2. *ϕ extends uniquely to the homomorphism $\psi : \mathcal{H}(\widehat{G}, \widehat{B}) \to \mathbb{C}$ defined by $\psi(\beta) = 0$ for any $\beta \in \mathcal{K}$.*

Proof. We begin by showing that ψ is a homomorphism. As ϕ is already a homomorphism on $\mathcal{H}(\widehat{H}, \widehat{B})$, it suffices to show that $\psi(\beta\beta') = \psi(\beta)\psi(\beta')$ for basis elements $\beta, \beta' \in \mathcal{H}(\widehat{G}, \widehat{B})$ where at least one of β or β' lies in $\mathcal{K}$. However, this means that one of $\psi(\beta)$ or $\psi(\beta')$ is zero and so we need to show that $\psi(\beta\beta') = 0$.

Suppose that $\beta \in \mathcal{K}$ and that β corresponds to the double coset $\widehat{B}g\widehat{B}$ with $g \notin \widehat{H}$. If we express $\beta\beta'$ as $\beta\beta' = \gamma + \gamma'$ for some $\gamma \in \mathcal{H}(\widehat{H}, \widehat{B})$ and $\gamma \in \mathcal{K}$, then we see that

$$\psi(\beta\beta') = \psi(\gamma) + \psi(\gamma') = \phi(\gamma) + 0 = \phi(\gamma).$$

Thus we would like to show that $\phi(\gamma) = 0$. By Theorem 6.3.1 we know that $\widehat{B}g\widehat{B} = \widehat{H}_\alpha g\widehat{B}$ for some α and so $e_{\widehat{H}_\alpha}\beta = \beta$. Consequently,

$$\beta\beta' = e_{\widehat{H}_\alpha}\beta\beta' = e_{\widehat{H}_\alpha}\gamma + e_{\widehat{H}_\alpha}\gamma'$$

where $e_{\widehat{H}_\alpha}\gamma \in \mathcal{H}(\widehat{H}, \widehat{B})$ and $e_{\widehat{H}_\alpha}\gamma' \in \mathcal{K}$ by Lemma 7.4.1. In particular, this means that $e_{\widehat{H}_\alpha}\gamma = \gamma$ and $e_{\widehat{H}_\alpha}\gamma' = \gamma'$. Therefore

$$\beta_\alpha\gamma = (qe_{\widehat{H}_\alpha} - e_{\widehat{B}})\gamma = qe_{\widehat{H}_\alpha}\gamma - e_{\widehat{B}}\gamma = (q-1)\gamma$$

and so, since ϕ is a homomorphism on $\mathcal{H}(\widehat{H}, \widehat{B})$,

$$(q-1)\phi(\gamma) = \phi(\beta_\alpha\gamma) = \phi(\beta_\alpha)\phi(\gamma) = -\phi(\gamma).$$

Hence $\phi(\gamma) = 0$ and $\psi(\beta\beta') = \psi(\beta)\psi(\beta')$. The case where $\beta' \in \mathcal{K}$ is similar, using the fact that $\mathcal{K}$ is also a right $\mathcal{H}(\widehat{H}, \widehat{B})$-module.

Now, suppose that ψ' is any extension of ϕ to $\mathcal{H}(\widehat{G}, \widehat{B})$. Let $\beta \in \mathcal{K}$ be a basis element corresponding to the double coset $\widehat{B}g\widehat{B}$ with $g \notin \widehat{H}$. Again $\widehat{B}g\widehat{B} = \widehat{H}_\alpha g\widehat{B}$ for some $\alpha \in S$ and so $\beta_\alpha\beta = (q-1)\beta$. Hence

$$(q-1)\psi'(\beta) = \psi'(\beta_\alpha\beta) = \psi'(\beta_\alpha)\psi'(\beta) = -\psi'(\beta)$$

implies that $\psi'(\beta) = 0$ and $\psi' = \psi$. $\square$

Lemma 7.4.3. *ψ is the restriction of St_ℓ to $\mathcal{H}(\widehat{G}, \widehat{B})$.*

Proof. As in the proof of Lemma 7.3.4, $(\mathrm{St}_\ell, (1_{\widehat{B}})^{\widehat{G}}) = 1$ and $(\mathrm{St}_\ell, (1_{\widehat{H}_\alpha})^{\widehat{G}}) = 0$ imply that St_ℓ restricts to a homomorphism of $\mathcal{H}(\widehat{G}, \widehat{B})$ where $\mathrm{St}_\ell(\beta_J) = (-1)^{|J|}$ for each $J \subseteq S$. Hence the restriction of St_ℓ to $\mathcal{H}(\widehat{G}, \widehat{B})$ is an extension of ϕ and so, by uniqueness, must be ψ. □

Finally, we show that the idempotent e used to define the module affording St_ℓ in Theorem 4.2.6 is exactly the idempotent obtained from the linear character ψ of $\mathcal{H}(\widehat{G}, \widehat{B})$.

Proposition 7.4.4. *The primitive idempotent corresponding to ψ is*

$$e = \sum_{I \subseteq S} (-1)^{|I|} e_{\widehat{H}_I}.$$

Proof. By Theorem 7.1.4 we know that the idempotent is given by the formula

$$e = \frac{\mathrm{St}_\ell(1)}{[\widehat{G} : \widehat{B}]} \sum_{\widehat{B}g\widehat{B} \in \mathcal{D}_{\widehat{G}}(\widehat{B}, \widehat{B})} \frac{1}{\mathrm{ind}(g)} \psi(\beta_{g^{-1}}) \beta_g.$$

Since $\mathrm{St}_\ell = \chi^{\widehat{G}}$ we see that $\mathrm{St}_\ell(1) = \chi(1)[\widehat{G} : \widehat{H}]$ where

$$\chi(1) = \sum_{J \subseteq S} (-1)^{|J|} [\widehat{H} : \widehat{H}_J] = \sum_{J \subseteq S} (-1)^{|J|} q^{|S|-|J|} = (q-1)^{|S|}.$$

In particular, this means that

$$\frac{\mathrm{St}_\ell(1)}{[\widehat{G} : \widehat{B}]} = \frac{\chi(1)}{[\widehat{H} : \widehat{B}]} = \frac{(q-1)^{|S|}}{q^{|S|}}.$$

Further, if $\widehat{B}g\widehat{B}$ is not contained in $\widehat{H}$ then $\widehat{B}g^{-1}\widehat{B}$ also must not be contained in $\widehat{H}$. Thus the corresponding basis element $\beta_{g^{-1}}$ lies in $\mathcal{K}$ and $\psi(\beta_{g^{-1}}) = 0$. Consequently,

$$e = \frac{(q-1)^{|S|}}{q^{|S|}} \sum_{J \subseteq S} \frac{1}{\mathrm{ind}(x_J)} \phi(\beta_{x_J^{-1}}) \beta_J.$$

Now, for each $J \subseteq S$

$$\mathrm{ind}(x_J) = \frac{|\widehat{B}x_J\widehat{B}|}{|\widehat{B}|} = |\mathfrak{X}_J| = (q-1)^{|J|}.$$

Moreover, since $x_J^{-1} \in \widehat{B} x_J \widehat{B}$ we have $\beta_{x_J^{-1}} = \beta_J$ and thus $\phi(\beta_{x_J^{-1}}) = \phi(\beta_J) = (-1)^{|J|}$.

Hence

$$\begin{aligned} e &= \frac{(q-1)^{|S|}}{q^{|S|}} \sum_{J \subseteq S} \frac{1}{(q-1)^{|J|}} (-1)^{|J|} \beta_J \\ &= \sum_{J \subseteq S} (-1)^{|J|} (q-1)^{|S|-|J|} q^{-|S|} \sum_{I \subseteq J} (-1)^{|J|-|I|} q^{|I|} e_{\widehat{H}_I} \\ &= \sum_{I \subseteq S} (-1)^{|I|} q^{|I|-|S|} \sum_{I \subseteq J} (q-1)^{|S|-|J|} e_{\widehat{H}_I} \\ &= \sum_{I \subseteq S} (-1)^{|I|} e_{\widehat{H}_I} \end{aligned}$$

as required. □

Chapter 8

Restriction

Having described the analogue of the Steinberg character for the extended Chevalley group over R we now turn our attention to the Chevalley group G itself. Further, we strengthen the requirement on the residue class field κ and assume that its characteristic is **very good** (cf. [3]), i.e.

(i) char κ is good; and

(ii) char κ does not divide $n+1$ if $\Sigma = A_n$.

It is possible to repeat the approach in Chapter 4 to construct an analogue St'_ℓ for G. However, it turns out that the resulting character is merely the restriction of St_ℓ of G. Further, we will show that St'_ℓ is irreducible only when $G = \widehat{G}$.

8.1 Restriction to the Chevalley group

Definition 8.1.1. A subgroup P of G is **parabolic** if it is of the form

$$P = \langle U_\alpha(i_\alpha), B : \alpha \in \Sigma^- \rangle$$

for some $0 \leq i_\alpha \leq \ell$.

However, these are exactly the subgroups obtained by intersecting the parabolic subgroups of $\widehat{G}$ with G.

Lemma 8.1.2. *Each parabolic subgroup P of G is of the form $P = \widehat{P} \cap G$ for some parabolic subgroup $\widehat{P}$ of $\widehat{G}$.*

Proof. We may express $\widehat{B}$ as $\widehat{B} = \widehat{T}B$ where $\widehat{T}$ preserves each root subgroup $U_\alpha(\mathfrak{m}^{i_\alpha})$. Thus, if we consider the corresponding parabolic subgroup $\widehat{P} = \langle U_\alpha(i_\alpha), \widehat{B} : \alpha \in \Sigma^- \rangle$ of $\widehat{G}$, then we see that

$$\widehat{P} = \langle U_\alpha(i_\alpha), \widehat{T}B : \alpha \in \Sigma^- \rangle = \widehat{T}\langle U_\alpha(i_\alpha), B : \alpha \in \Sigma^- \rangle = \widehat{T}P.$$

Hence $\widehat{P} \cap G = (\widehat{T}P) \cap G = (\widehat{T} \cap G)P = TP = P$. □

In particular, this means that if we let $H_\alpha = \langle X_\alpha, B \rangle$ for each $\alpha \in S$, then $H_\alpha = \widehat{H}_\alpha \cap G$ and so $\{H_\alpha : \alpha \in S\}$ are again the minimal parabolic subgroups of G which strictly contain B.

Further, if for each non-empty subset $J = \{\alpha_{j_1}, \ldots, \alpha_{j_k}\} \subseteq S$ we define

$$H_J = \langle H_{\alpha_{j_1}}, \ldots, H_{\alpha_{j_k}} \rangle$$

with $H_\emptyset = B$ then $H_J = \widehat{H}_J \cap G$ for each $J \subseteq S$. Consequently, we obtain the following results from the corresponding results for the parabolic subgroups $\widehat{H}_J$ of $\widehat{G}$.

Lemma 8.1.3. *Let $I, J \subseteq S$, then*

(i) $H_J = X_J B = B X_J$;

(ii) $|H_J| = q^{|J|}B$;

(iii) $H_I H_J = H_J H_I$;

(iv) $H_I \cap H_J = H_{I \cap J}$; *and*

(v) $\langle H_I, H_J \rangle = H_{I \cup J}$.

More importantly, we see that if we define the analogue St'_ℓ of the Steinberg character for G to be the virtual character

$$\mathrm{St}'_\ell = \sum_{J \subseteq S} (-1)^{|J|} (1_{H_J})^G,$$

then $\mathrm{St}'_\ell = (\mathrm{St}_\ell)_G$.

Lemma 8.1.4. *St'_ℓ is the restriction of St_ℓ to G.*

Proof. Since $\widehat{H}_J G = \widehat{G}$ for each $J \subseteq S$, Mackey theory implies that

$$\begin{aligned} (\mathrm{St}_\ell)_G &= \sum_{J \subseteq S} (-1)^{|J|} ((1_{\widehat{H}_J})^{\widehat{G}})_G \\ &= \sum_{J \subseteq S} (-1)^{|J|} (1_{\widehat{H}_J \cap G})^G \\ &= \sum_{J \subseteq S} (-1)^{|J|} (1_{H_J})^G \\ &= \mathrm{St}'_\ell. \end{aligned}$$

as required. □

Similarly, if we let $H = H_S$ and consider the character

$$\chi' = \sum_{J \subseteq S} (-1)^{|J|} (1_{H_J})^H,$$

then again $\mathrm{St}'_\ell = (\chi')^G$, where χ' is the restriction of χ to H.

Finally, we note that the additional restriction on the characteristic of κ ensures that the congruence subgroup $\widehat{K}_1$ of $\widehat{G}$ is actually contained in G.

Proposition 8.1.5. *$\widehat{K}_1$ is a subgroup of G.*

Proof. We need to show that $\widehat{T}(\mathfrak{m}) \leq T(\mathfrak{m})$, since then we would have $\widehat{T}(\mathfrak{m}) = T(\mathfrak{m})$ and therefore $\widehat{K}_1 = U^-(\mathfrak{m})\widehat{T}(\mathfrak{m})U(\mathfrak{m}) = U^-(\mathfrak{m})T(\mathfrak{m})U(\mathfrak{m}) \leq G$. Let $h(\mu) \in \widehat{T}(\mathfrak{m})$ so that μ is an R-character of Λ_r with $\mu(\alpha) \in 1 + \mathfrak{m}$ for each $\alpha \in \Sigma$. To show that $h(\mu) \in T$ we need to prove that μ is the restriction of some R-character μ' of Λ.

Let $A = [A_{\alpha,\beta}]_{\alpha,\beta\in\Pi}$ denote the Cartan matrix of Σ and recall that for each $\alpha \in \Pi$,

$$\alpha = \sum_{\beta\in\Pi} A_{\alpha,\beta}\lambda_\beta.$$

Thus, if we consider the inverse matrix $A^{-1} = [A'_{\alpha,\beta}]_{\alpha,\beta\in\Pi}$, then for each $\beta \in \Pi$

$$\lambda_\beta = \sum_{\alpha\in\Pi} A'_{\alpha,\beta}\alpha.$$

Now, since char κ does not divide $\det(A)$, by Lemma 3.1.6 we may choose an element $s_\alpha \in 1 + \mathfrak{m}$ for each $\alpha \in \Pi$, so that $\mu(\alpha) = s_\alpha^{\det(A)}$. Thus, since $\det(A)A'_{\alpha,\beta}$ is an integer for each $\alpha, \beta \in \Pi$, we are able to define

$$r_\beta = \prod_{\alpha\in\Pi} s_\alpha^{\det(A)A'_{\alpha,\beta}}$$

for every $\beta \in \Pi$. Consequently, if we consider the R-character μ' of Λ given by $\mu'(\lambda_\beta) = r_\beta$ for every $\beta \in \Pi$, then we see that for each $\alpha \in \Pi$

$$\mu'(\alpha) = \mu'\left(\sum_{\beta\in\Pi} A_{\alpha,\beta}\lambda_\beta\right) = \prod_{\beta\in\Pi} \mu'(\lambda_\beta)^{A_{\alpha,\beta}} = \prod_{\beta\in\Pi} r_\beta^{A_{\alpha,\beta}} = s_\alpha^{\det(A)} = \mu(\alpha).$$

Hence, μ is the restriction of μ' and $h(\mu) \in T(\mathfrak{m})$. □

As a consequence of this, we see that

$$\widehat{G}/G \simeq (\widehat{G}/\widehat{K}_1)/(G/\widehat{K}_1) \simeq \widehat{G}(\kappa)/G(\kappa) \simeq \widehat{T}(\kappa)/T(\kappa)$$

and therefore $[\widehat{G} : G] = d$ where d is as in Table 2.4.

8.2 Example: $\mathrm{PSL}_2(R)$, char $\kappa \neq 2$

We now examine the situation where $G = \mathrm{PSL}_2(R)$ and the characteristic of κ is odd. Let $R_2^\times = \{r^2 : r \in R^\times\}$ denote the set of squares in $R^\times$. As char $\kappa \neq 2$, we know that $[R^\times : R_2^\times] = 2$ and so if we fix a non-square element $\epsilon \in R^\times$, then $R^\times$ can be expressed as the disjoint union

$$R^\times = R_2^\times \cup \epsilon R_2^\times.$$

Lemma 8.2.1. *For each* $1 \leq k \leq \ell - 1$ *we have*

$$\left(\widehat{B}\begin{bmatrix}1 & 0\\ \pi^k & 1\end{bmatrix}\widehat{B}\right)\bigcap G = B\begin{bmatrix}1 & 0\\ \pi^k & 1\end{bmatrix}B \bigcup B\begin{bmatrix}1 & 0\\ \pi^k\epsilon & 1\end{bmatrix}B$$

where the union is disjoint, while for $k = 0$

$$\left(\widehat{B}\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}\widehat{B}\right)\bigcap G = B\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}B. \tag{8.1}$$

Proof. Suppose that $1 \leq k \leq \ell - 1$. If we are able to show that

$$B\begin{bmatrix}1 & 0\\ \pi^k & 1\end{bmatrix}B = \left\{\begin{bmatrix}a & b\\ c & d\end{bmatrix} \in G : c \in \pi^k R^\times, ac \in \pi^k R_2^\times\right\} \tag{8.2}$$

and

$$B\begin{bmatrix}1 & 0\\ \pi^k\epsilon & 1\end{bmatrix}B = \left\{\begin{bmatrix}a & b\\ c & d\end{bmatrix} \in G : c \in \pi^k R^\times, ac \in \pi^k \epsilon R_2^\times\right\} \tag{8.3}$$

then the first result will follow from Lemma 6.1.1.

For any $a, c \in R^\times$ and $b, d \in R$,

$$\begin{bmatrix}a & b\\ 0 & a^{-1}\end{bmatrix}\begin{bmatrix}1 & 0\\ \pi^k & 1\end{bmatrix}\begin{bmatrix}c & d\\ 0 & c^{-1}\end{bmatrix} = \begin{bmatrix}ac + \pi^k bc & ad + bc^{-1} + \pi^k bd\\ \pi^k a^{-1}c & a^{-1}c^{-1} + \pi^k a^{-1}d\end{bmatrix}$$

where $(ac + \pi^k bc)(\pi^k a^{-1}c) = \pi^k c^2(1 + \pi^k a^{-1}b) \in \pi^k R_2^\times$. Thus

$$B\begin{bmatrix}1 & 0\\ \pi^k & 1\end{bmatrix}B \subseteq \left\{\begin{bmatrix}a & b\\ c & d\end{bmatrix} \in G : c \in \pi^k R^\times, ac \in \pi^k R_2^\times\right\}.$$

Further, suppose that $a, b, c, d \in R$ are such that $c \in \pi^k R^\times$, $ac \in \pi^k R_2^\times$ and $ad - bc = 1$. Then we must have $a \in R^\times$ and so $c = \pi^k a^{-1} r^2$ for some $r \in R^\times$. Thus $d = a^{-1}(1 + \pi^k a^{-1} b r^2)$ and we see that

$$\begin{bmatrix}ar^{-1} & 0\\ 0 & a^{-1}r\end{bmatrix}\begin{bmatrix}1 & 0\\ \pi^k & 1\end{bmatrix}\begin{bmatrix}r & a^{-1}br\\ 0 & r^{-1}\end{bmatrix} = \begin{bmatrix}a & b\\ c & d\end{bmatrix}.$$

Hence,

$$B\begin{bmatrix}1 & 0\\ \pi^k & 1\end{bmatrix}B = \left\{\begin{bmatrix}a & b\\ c & d\end{bmatrix} \in G : c \in \pi^k R^\times, ac \in \pi^k R_2^\times\right\}.$$

The proof of (8.3) is similar and (8.1) follows from the proof of Lemma 6.1.1 since if $ad - bc = 1$ then (6.1) involves only matrices from G. □

Consequently, this together with Lemma 6.1.2 gives the decomposition of G in terms of (B, B)-double cosets.

Lemma 8.2.2. *G can be expressed as the disjoint union*

$$G = B \bigcup_{k=1}^{\ell-1}\left(B\begin{bmatrix}1 & 0\\ \pi^k & 1\end{bmatrix}B \bigcup B\begin{bmatrix}1 & 0\\ \pi^k\epsilon & 1\end{bmatrix}B\right) \bigcup B\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}B.$$

For each $0 \le i \le \ell$, let B_i denote the parabolic subgroup

$$B_i = \left\{\begin{bmatrix}a & b\\ c & d\end{bmatrix} \in G : c \in \mathfrak{m}^i\right\}.$$

Again, the (B_i, B_j)-double cosets of G are related to the (B, B)-double cosets.

Lemma 8.2.3. *For any $1 \le i, j \le \ell$ and $k < \min(i, j)$,*

$$B_i\begin{bmatrix}1 & 0\\ \pi^k & 1\end{bmatrix}B_j = B\begin{bmatrix}1 & 0\\ \pi^k & 1\end{bmatrix}B$$

and

$$B_i\begin{bmatrix}1 & 0\\ \pi^k\epsilon & 1\end{bmatrix}B_j = B\begin{bmatrix}1 & 0\\ \pi^k\epsilon & 1\end{bmatrix}B. \tag{8.4}$$

Further, for $k = 0$

$$B_i\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}B_j = B\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}B. \tag{8.5}$$

Proof. Let

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in B_i \qquad \text{and} \qquad \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \in B_j.$$

Then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \pi^k & 1 \end{bmatrix} \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} = \begin{bmatrix} aa' + bc' + \pi^k a'b & ab' + bd' + \pi^k bb' \\ a'c + c'd + \pi^k a'd & b'c + dd' + b'd \end{bmatrix}.$$

Further, since $c \in \mathfrak{m}^i$ and $c' \in \mathfrak{m}^j$ with both $k < i$ and $k < j$ then we see that we may write $aa' + bc' + \pi^k a'b = aa' + \pi r$ and $a'c + c'd + \pi^k a'd = \pi^k(a'd + \pi s)$ for some $r, s \in R$. Thus

$$\begin{aligned} &(aa' + bc' + a'b)(a'c + c'd + \pi^k a'd) \\ &\quad = (aa' + \pi r)(\pi^k(a'd + \pi s)) \\ &\quad = \pi^k(a(a')^2 d + \pi(aa's + a'dr + \pi rs)) \\ &\quad = \pi^k(a')^2 ad(1 + \pi((a')^{-1}d^{-1}s + a^{-1}(a')^{-1}r + a^{-1}(a')^{-2}d^{-1}rs)) \in \pi^k R_2^\times \end{aligned}$$

since $ad = 1 + bc \in 1 + \mathfrak{m}$. Hence

$$B_i \begin{bmatrix} 1 & 0 \\ \pi^k & 1 \end{bmatrix} B_j \subseteq B \begin{bmatrix} 1 & 0 \\ \pi^k & 1 \end{bmatrix} B$$

and the reverse inclusion is clear.

The proof of (8.4) is similar while the proof of (8.5) follows immediately from the $k = 0$ case in Lemma 6.1.3. □

Thus, we obtain the corresponding decomposition of G into (B_i, B_j)-double cosets.

Lemma 8.2.4. *For each $0 \le i, j \le \ell$ if we set $k = \min(i, j)$ then G can be expressed as the disjoint union*

$$G = B_k \bigcup_{k'=1}^{k-1} \left(B_i \begin{bmatrix} 1 & 0 \\ \pi^{k'} & 1 \end{bmatrix} B_j \bigcup B_i \begin{bmatrix} 1 & 0 \\ \pi^{k'}\epsilon & 1 \end{bmatrix} B_j \right) \bigcup B_i \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} B_j.$$

Proposition 8.2.5. *For each* $0 \leq i \leq \ell$,

$$(\mathrm{St}'_\ell, (1_{B_i})^G) = \begin{cases} 2 & \text{if } i = \ell; \\ 0 & \text{if } i < \ell. \end{cases}$$

Proof. If $i = \ell$ then by Proposition 8.2.4

$$(\mathrm{St}'_\ell, (1_{B_\ell})^G) = |\mathcal{D}_G(B_\ell, B_\ell)| - |\mathcal{D}_G(B_{\ell-1}, B_\ell)| = 2\ell - 2(\ell - 1) = 2,$$

while for $1 \leq i \leq \ell - 1$

$$(\mathrm{St}'_\ell, (1_{B_i})^G) = |\mathcal{D}_G(B_\ell, B_i)| - |\mathcal{D}_G(B_{\ell-1}, B_i)| = 2i - 2i = 0$$

and finally, for $i = 0$

$$(\mathrm{St}'_\ell, (1_{B_0})^G) = |\mathcal{D}_G(B_\ell, B_0)| - |\mathcal{D}_G(B_{\ell-1}, B_0)| = 1 - 1 = 0. \qquad \square$$

Corollary 8.2.6. $(\mathrm{St}'_\ell, \mathrm{St}'_\ell) = 2$.

Proof. Using Proposition 8.2.5 we see that

$$(\mathrm{St}'_\ell, \mathrm{St}'_\ell) = (\mathrm{St}'_\ell, (1_{B_\ell})^G) - (\mathrm{St}'_\ell, (1_{B_{\ell-1}})^G) = (\mathrm{St}'_\ell, (1_B)^G) = 2. \qquad \square$$

8.3 Double coset structure of H

As can be seen from the previous section, the easiest method of finding the double cosets of H is by examining how the double cosets of $\widehat{H}$ decompose on intersection with H.

From Proposition 6.2.3 we know that $\widehat{B} x_J \widehat{B} = \mathfrak{X}_J \widehat{B}$ for each $J \subseteq S$. Thus,

$$(\widehat{B} x_J \widehat{B}) \cap H = (\mathfrak{X}_J \widehat{B}) \cap H = \mathfrak{X}_J B.$$

The action of $\widehat{T}$ on $\mathfrak{X}_J$ restricts to an action of T, but this is no longer transitive in general. Consequently, consider the decomposition

$$\mathfrak{X}_J = \bigcup_{i=1}^{d_J} \mathfrak{X}_J^{(i)}$$

of $\mathfrak{X}_J$ into its T-orbits $\mathfrak{X}_J^{(1)}, \dots, \mathfrak{X}_J^{(d_J)}$ and for each i choose a representative $x_J^{(i)} \in \mathfrak{X}_J^{(i)}$. As in the proof of Proposition 6.2.3, we see that

$$Bx_J^{(i)}B = \mathfrak{X}_J^{(i)}B$$

and therefore we have the disjoint union

$$(\widehat{B}x_J\widehat{B}) \cap H = \bigcup_{i=1}^{d_J} Bx_J^{(i)}B. \tag{8.6}$$

While it is difficult calculate d_J in general, for $J = S$ it turns out to be the index d of G in $\widehat{G}$.

Lemma 8.3.1. $d_S = d$.

Proof. Let $h(\mu) \in T$ and $x = \prod_{\alpha \in S} x_\alpha(r_\alpha) \in \mathfrak{X}_S$. Then

$$h(\mu)xh(\mu)^{-1} = \prod_{\alpha \in S} h(\mu)x_\alpha(r_\alpha)h(\mu)^{-1} = \prod_{\alpha \in S} x_\alpha(\mu(\alpha)r_\alpha).$$

Thus $h(\mu) \in \mathrm{Stab}_T(x)$ if and only if $\mu(\alpha)r_\alpha = r_\alpha$ for each $\alpha \in S$. However, since $r_\alpha \in \pi^{\ell-1}R^\times$, this is true exactly when $\mu(\alpha) \in 1+\mathfrak{m}^{\ell-1}$ for each $\alpha \in S$. Consequently, $\mathrm{Stab}_T(x) = T(\mathfrak{m})$.

In particular, this means that for each i

$$|\mathfrak{X}_S^{(i)}| = [T : \mathrm{Stab}_T(x_S^{(i)})] = [T : T(\mathfrak{m})] = \frac{[\widehat{T} : \widehat{T}(\mathfrak{m})]}{[\widehat{T} : T]} = \frac{(q-1)^n}{d}.$$

Finally, since

$$(q-1)^n = |\mathfrak{X}_S| = \sum_{i=1}^{d_S} |\mathfrak{X}_S^{(i)}| = d_S\frac{(q-1)^n}{d}$$

we must have $d_S = d$. □

Further, we see that for any $J, J' \subseteq S$ and $I \subseteq S - (J \cup J')$, the $(\widehat{H}_J, \widehat{H}_{J'})$-double coset $\widehat{H}_J x_I \widehat{H}_{J'}$ also decomposes into d_I distinct $(H_J, H_{J'})$-double cosets.

Lemma 8.3.2. *For each $J, J' \subseteq S$ and $I \subseteq S - (J \cup J')$,*

$$(\widehat{H}_J x_I \widehat{H}_{J'}) \cap H = \bigcup_{i=1}^{d_J} H_J x_I^{(i)} H_{J'}$$

where the union is disjoint.

Proof. From (8.6) it is clear that

$$(\widehat{H}_J x_I \widehat{H}_{J'}) \cap H = \bigcup_{i=1}^{d_J} H_J x_I^{(i)} H_{J'}$$

and so we need only show that this union is disjoint. Suppose that for some i, i' we have $H_J x_I^{(i)} H_{J'} = H_J x_I^{(i')} H_{J'}$. Then $x_I^{(i)} = g x_I^{(i')} g'$ for some $g \in H_J$ and $g' \in H_{J'}$. If we write $g = xb$ and $g = b'x'$ for some $b, b' \in B$, $x \in X_J$ and $x' \in X_{J'}$, then we see that $b x_I^{(i')} b' = x^{-1} x_I^{(i)} (x')^{-1} = x^{-1}(x')^{-1} x_I^{(i)}$ and so $x^{-1}(x')^{-1} x_I^{(i)} \in X_I$. However, since $x_I^{(i)} \in X_I$ this would imply that $x^{-1}(x')^{-1} \in X_I$. On the other hand, $x^{-1}(x')^{-1} \in X_{J \cup J'}$ and so $x^{-1}(x')^{-1} \in X_I \cap X_{J \cup J'} = X_\emptyset = \{1\}$. Hence, $b x_I^{(i')} b' = x_I^{(i)}$ which gives $B x_I^{(i)} B = B x_I^{(i')} B$ and so $i = i'$. $\square$

Lemma 8.3.3. *For each $J, J' \subseteq S$, G can be expressed as the disjoint union*

$$G = \bigcup_{I \subseteq S - (J \cup J')} \bigcup_{i=1}^{d_I} H_J x_I^{(i)} H_{J'}.$$

Finally, this allows us to adapt the approach in Section 6.2. For each non-empty subset $J, J' \subseteq S$ define $\mathcal{E}'_H(H_J, H_{J'})$ to be

$$\mathcal{E}'_H(H_J, H_{J'}) = \mathcal{D}_H(H_J, H_{J'}),$$

the set of $(H_J, H_{J'})$-double cosets in H. Further, let $\mathcal{E}'_H(B, B)$ denote

$$\mathcal{E}'_H(B, B) = \mathcal{D}_H(B, B) - \{B x_S^{(1)} B, \dots, B x_S^{(d)} B\},$$

the set of (B, B)-double cosets excluding $B x_S^{(1)} B, \dots, B x_S^{(d)} B$.

Fix $J' \subseteq S$ and set

$$\mathcal{E}' = \bigcup_{J \subseteq S} \mathcal{E}'_H(H_J, H_{J'}).$$

$\mathcal{E}'$ can then expressed as the disjoint union $\mathcal{E}' = \mathcal{E}'_0 \cup \mathcal{E}'_1$ where

$$\mathcal{E}'_0 = \bigcup_{|J| \text{ even}} \mathcal{E}'_H(H_J, H_{J'}) \quad \text{and} \quad \mathcal{E}'_1 = \bigcup_{|J| \text{ odd}} \mathcal{E}'_H(H_J, H_{J'}).$$

Again, we can show that $\mathcal{E}'_0$ and $\mathcal{E}'_1$ contain the same number of double cosets.

Lemma 8.3.4. $|\mathcal{E}'_0| = |\mathcal{E}'_1|$.

Proof. Fix an ordering of the roots in S. Let $H_J x_I^{(i)} H_{J'} \in \mathcal{E}'$ with $I \subseteq S - (J \cup J')$. Then, since we are excluding double cosets of the form $Bx_S^{(i)}B$, we have $I \neq S$ and so we may choose a minimal root $\alpha \in S - I$. Consequently, we can define a map $\Phi' : \mathcal{E}' \to \mathcal{E}'$ by

$$\Phi'(H_J x_I^{(i)} H_{J'}) = \begin{cases} H_{J-\{\alpha\}} x_I^{(i)} H_{J'} & \text{if } \alpha \in J; \\ H_{J \cup \{\alpha\}} x_I^{(i)} H_{J'} & \text{if } \alpha \notin J. \end{cases}$$

This is well defined since $x_I^{(i)}$ is a distinguished double coset representative from Lemma 8.3.3. Further, since $\alpha \notin I$ we see that if $\alpha \in J$ then $I \subseteq S - (J - \{\alpha\} \cup J')$ and so $x_I^{(i)}$ is again a distinguished $(H_{J-\{\alpha\}}, H_{J'})$-double coset representative. Similarly, if $\alpha \notin J$ then $I \subseteq S - (J \cup \{\alpha\} \cup J')$ and $x_I^{(i)}$ is a distinguished $(H_{J \cup \{\alpha\}}, H_{J'})$-double coset representative.

Now suppose that $H_J x_I^{(i)} H_{J'} \in \mathcal{E}'$ and $\alpha \in S - I$ is minimal. If $\alpha \in J$, then $\alpha \notin J - \{\alpha\}$ gives

$$\Phi'(H_{J-\{\alpha\}} x_I^{(i)} H_{J'}) = H_{J-\{\alpha\} \cup \{\alpha\}} x_I^{(i)} H_{J'} = H_J x_I^{(i)} H_{J'}$$

and if $\alpha \notin J$, then $\alpha \in J \cup \{\alpha\}$ implies that

$$\Phi'(H_{J \cup \{\alpha\}} x_I^{(i)} H_{J'}) = H_{J \cup \{\alpha\} - \{\alpha\}} x_I^{(i)} H_{J'} = H_J x_I^{(i)} H_{J'}.$$

Hence Φ' is surjective and so therefore bijective since $\mathcal{E}'$ is finite.

Finally, it is clear that $\Phi'(\mathcal{E}_0') \subseteq \mathcal{E}_1'$ and $\Phi'(\mathcal{E}_1') \subseteq \mathcal{E}_0'$. Thus, since Φ' is surjective and $\mathcal{E}' = \mathcal{E}_0' \cup \mathcal{E}_0'$, we must have $\Phi'(\mathcal{E}_0') = \mathcal{E}_1'$ and $\Phi'(\mathcal{E}_1') = \mathcal{E}_0'$. Consequently, Φ' restricts to a bijection $\Phi' : \mathcal{E}_0' \to \mathcal{E}_1'$ implying that $|\mathcal{E}_0'| = |\mathcal{E}_1'|$. $\square$

Theorem 8.3.5. *For each $J \subseteq S$*

$$(\chi', (1_{H_J})^H) = \begin{cases} d & \text{if } J = \emptyset; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Using the expression of χ' as an alternating sum of permutation characters we see that

$$\begin{aligned} (\chi', (1_{H_J})^H) &= \sum_{I \subseteq S} (-1)^{|I|} ((1_{H_I})^H, (1_{H_J})^H) \\ &= \sum_{I \subseteq S} (-1)^{|I|} |\mathcal{D}_H(H_I, H_J)| \\ &= \sum_{|I| \text{ even}} |\mathcal{D}_H(H_I, H_J)| - \sum_{|I| \text{ odd}} |\mathcal{D}_H(H_I, H_J)|. \end{aligned}$$

Now, if $J \neq \emptyset$ then this gives

$$(\chi, (1_{H_J})^H) = |\mathcal{E}_0'| - |\mathcal{E}_1'| = 0$$

whereas if $J = \emptyset$ then we obtain

$$(\chi', (1_B)^H) = (|\mathcal{E}_0'| + d) - |\mathcal{E}_1'| = d.$$

$\square$

Corollary 8.3.6. $(\chi', \chi') = d$.

Proof. From Theorem 8.3.5 we see that

$$(\chi', \chi') = \sum_{J \subseteq S} (-1)^{|J|} (\chi', (1_{H_J})^H) = (\chi', (1_B)^H) = d.$$

$\square$

8.4 Reducibility of St'_ℓ

The main result about the $(\widehat{B},\widehat{B})$-double coset structure from Chapter 6 was Theorem 6.3.1 and the key to its proof was Proposition 6.4.1.

Proposition 6.4.1 stated that given any $g \notin \widehat{H}$ there was an $\alpha \in S$, depending only on the $(\widehat{H},\widehat{H})$-double coset $\widehat{H}g\widehat{H}$, such that for any $r \in R$ we had $[v,g] = v'x_\alpha(r)$ for some $v, v' \in V$. In particular, if we take $g \in G$ with $g \notin H$, then we must also have $g \notin \widehat{H}$. Further, since $\widehat{T}(\mathfrak{m}) \leq G$ we also have $V \leq G$. Consequently, Proposition 6.4.1 immediately implies that we have the corresponding version of Theorem 6.3.1 for G.

Theorem 8.4.1. *For each (B,B)-double coset BgB not contained in H we have $BgB = H_\alpha gB$ for some $\alpha \in S$ which depends only on the (H,H)-double coset HgH.*

Further, the argument in Section 6.3 relies only on Theorem 6.3.1 and so remains valid in our current situation.

Proposition 8.4.2. *For each $J \subseteq S$, we have*

$$(\mathrm{St}'_\ell, (1_{H_J})^G) = (\chi', (1_{H_J})^H).$$

Thus, Theorem 8.3.5 implies the following result.

Corollary 8.4.3. *For each $J \subseteq S$,*

$$(\mathrm{St}'_\ell, (1_{H_J})^G) = \begin{cases} d & \text{if } J = \emptyset; \\ 0 & \text{otherwise.} \end{cases}$$

Finally, we are able to determine when St'_ℓ is irreducible.

Theorem 8.4.4. St'_ℓ *is irreducible if and only if $\widehat{G} = G$.*

Proof. By Corollary 8.4.3

$$(\mathrm{St}'_\ell, \mathrm{St}'_\ell) = \sum_{J \subseteq S} (-1)^{|J|} (\mathrm{St}'_\ell, (1_{H_J})^G) = (\mathrm{St}'_\ell, (1_B)^G) = d.$$

Hence $(\mathrm{St}'_\ell, \mathrm{St}'_\ell) = 1$ if and only if $d = [\widehat{G} : G] = 1$. □

Chapter 9

Decomposition

In Section 8.4 we saw that the analogue St'_ℓ of the Steinberg character for G was only irreducible when $G = \widehat{G}$. We will now describe how it decomposes into its irreducible constituents. Further, we will characterise the distinct irreducible constituents in terms of characters which can be regarded as analogues of the Gelfand-Graev character in the finite field case.

9.1 Example: $\mathrm{PSL}_2(R)$, char $\kappa \neq 2$

Consider the group $G = \mathrm{PSL}_2(R)$ when the characteristic of κ is odd. From Corollary 8.2.6 we know that $(\mathrm{St}'_\ell, \mathrm{St}'_\ell) = 2$ and so St'_ℓ must decompose into the sum of two distinct irreducible constituents. Let $\widetilde{V}$ denote the normal subgroup of $B_{\ell-1}$

$$\widetilde{V} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G : a, d \in 1 + \mathfrak{m},\ c \in \mathfrak{m}^{\ell-1} \right\}.$$

Fix a non-trivial linear character $\lambda' : \mathfrak{m}^{\ell-1} \to \mathbb{C}$ of the additive group $\mathfrak{m}^{\ell-1}$. Then, if for $r \in \mathfrak{m}^{\ell-1}$ we choose $r' \in R$ with $r = \pi^{\ell-1} r'$, we can define a map $\lambda_r : \widetilde{V} \to \mathbb{C}$ by

$$\lambda_r \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \lambda'(r'c).$$

Lemma 9.1.1. *λ_r is a linear character of $\widetilde{V}$ for each $r \in \mathfrak{m}^{\ell-1}$.*

Proof. Let

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \in \widetilde{V}.$$

Then, we have $a'c = c$ and $c'd = c'$ which gives

$$\begin{aligned} \lambda_r\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}\right) &= \lambda_r\left(\begin{bmatrix} aa'+bc' & ab'+bd' \\ c+c' & cb'+dd' \end{bmatrix}\right) \\ &= \lambda'(r'(c+c')) \\ &= \lambda'(r'c)\lambda'(r'c') \\ &= \lambda_r\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)\lambda_r\left(\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}\right) \end{aligned}$$

where $r' \in R$ is such that $r = \pi^{\ell-1}r'$. □

Lemma 9.1.2. *For each $r \in \pi^{\ell-1}R^\times$ the distinct conjugates of λ_r in $B_{\ell-1}$ are $\{\lambda_s : s \in rR_2^\times\}$.*

Proof. Let $s \in rR_2^\times$ so that $s = rt^2$ for some $t \in R^\times$. For each

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \widetilde{V}$$

we see that

$$\lambda_r\left(\begin{bmatrix} t^{-1} & 0 \\ 0 & t \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix}\right) = \lambda_r\left(\begin{bmatrix} a & t^{-2}b \\ t^2c & d \end{bmatrix}\right) = \lambda'(r't^2c) = \lambda_s\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right).$$

Thus λ_s is a conjugate of λ_r.

Now, suppose that

$$\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \in B_{\ell-1}.$$

Then $ac' = c'$ and $c'd = c'$ which means that

$$\lambda_r\left(\begin{bmatrix} d' & -b' \\ -c' & a' \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}\right) = \lambda_r\left(\begin{bmatrix} aa'd' - a'b'c + bc'd' - b'c' & ab'd' - (b')^2c + b(d')^2 - b'dd' \\ (a')^2c & a'b'c - b'c' + a'dd' - bc'd' \end{bmatrix}\right)$$
$$= \lambda'(r'(a')^2c)$$
$$= \lambda_{r(a')^2}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right).$$

Hence any conjugate of λ_r is of the form λ_s for some $s \in rR_2^\times$. □

Further, $[B_{\ell-1} : \widetilde{V}] = |R_2^\times|$ and so $\mathrm{Stab}_{B_{\ell-1}}(\lambda_r) = \widetilde{V}$ for each $r \in \mathfrak{m}^{\ell-1}$. Hence, Clifford Theory implies that $\lambda_r^{B_{\ell-1}}$ is irreducible and

$$(\lambda_r^{B_{\ell-1}})_{\widetilde{V}} = \sum_{s \in rR_2^\times} \lambda_s.$$

Now, for each $r \in \pi^{\ell-1}R^\times$ consider the induced character $\zeta_r = \lambda_r^G$ of G.

Lemma 9.1.3. *Let $r, s \in \pi^{\ell-1}R^\times$, then*

$$(\zeta_r, \zeta_s) = \begin{cases} 1 & \text{if } s \in rR_2^\times; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By the Intertwining Number Theorem

$$(\zeta_r, \zeta_s) = (\lambda_r^G, \lambda_s^G) = \sum_{g \in \mathfrak{D}_G(\widetilde{V}, \widetilde{V})} ((\lambda_r)_{g\widetilde{V}g^{-1} \cap V}, (\lambda_s^g)_{g\widetilde{V}g^{-1} \cap V}). \tag{9.1}$$

Consequently, suppose that

$$g = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \in G.$$

If $g \in B_{\ell-1}$, then we know from the proof of the previous Lemma that $\lambda_s^g = \lambda_{s(a')^2}$. Thus if $s \in rR_2^\times$ then there is exactly one $(\widetilde{V}, \widetilde{V})$-double coset representative g in $B_{\ell-1}$ for which $(\lambda_r, \lambda_s^g) = 1$, while if $s \notin rR_2^\times$ then $(\lambda_r, \lambda_s^g) = 0$ for all $(\widetilde{V}, \widetilde{V})$-double coset representatives.

Now, suppose that $c' \in \pi^i R^\times$ for some $1 \le i \le \ell - 1$. If we fix $a \in \mathfrak{m}^{\ell-1}$, then since $a' \in R^\times$ we can find an element $t \in \mathfrak{m}^{\ell-i-1}$ such that $ta'c' = a$. Further, since $1 + t \in 1 + \mathfrak{m}$ and char $\kappa \neq 2$, there must be some $c \in R^\times$ with $c^2 = 1 + t$. Thus we obtain

$$\begin{bmatrix} d' & -b' \\ -c' & a' \end{bmatrix} \begin{bmatrix} c^{-1} & 0 \\ 0 & c \end{bmatrix} \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} = \begin{bmatrix} a'c^{-1}d' - b'cc' & b'd'c^{-1} - b'cd' \\ a & a'cd' - b'c'^{-1} \end{bmatrix}. \tag{9.2}$$

and so

$$\lambda_r \left(\begin{bmatrix} c^{-1} & 0 \\ 0 & c \end{bmatrix} \right) = 1 \qquad \text{whereas} \qquad \lambda_s^g \left(\begin{bmatrix} c^{-1} & 0 \\ 0 & c \end{bmatrix} \right) = \lambda'(s'a).$$

Hence $(\lambda_r)_{g\tilde{V}g^{-1}\cap\tilde{V}} \neq (\lambda_s^g)_{g\tilde{V}g^{-1}\cap\tilde{V}}$ and $((\lambda_r)_{g\tilde{V}g^{-1}\cap\tilde{V}}, (\lambda_s^g)_{g\tilde{V}g^{-1}\cap\tilde{V}}) = 0$.

Finally, suppose that $c' \in R^\times$. Fix $a \in \mathfrak{m}^{\ell-1}$ and choose $b \in \mathfrak{m}^{\ell-1}$ such that $a = -b(c')^2$. Then

$$\begin{bmatrix} d' & -b' \\ -c' & a' \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} = \begin{bmatrix} 1 + bc'd' & b(d')^2 \\ a & 1 - bc'd' \end{bmatrix} \tag{9.3}$$

which means that

$$\lambda_r \left(\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \right) = 1 \qquad \text{while} \qquad \lambda_s^g \left(\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \right) = \lambda'(s'a).$$

Thus $(\lambda_r)_{g\tilde{V}g^{-1}\cap\tilde{V}} \neq (\lambda_s^g)_{g\tilde{V}g^{-1}\cap\tilde{V}}$ and $((\lambda_r)_{g\tilde{V}g^{-1}\cap\tilde{V}}, (\lambda_s^g)_{g\tilde{V}g^{-1}\cap\tilde{V}}) = 0$.

Hence from (9.1) we obtain

$$(\zeta_r, \zeta_s) = \begin{cases} 1 & \text{if } s \in rR_2^\times; \\ 0 & \text{otherwise.} \end{cases} \qquad \square$$

In particular, for $r \in \pi^{\ell-1}R_2^\times$ and $s \in \pi^{\ell-1}\epsilon R_2^\times$, this means that ζ_r and ζ_s are distinct irreducible characters of G.

Lemma 9.1.4. *For each $r \in \pi^{\ell-1}$ and $0 \le i \le \ell$*

$$(\zeta_r, (1_{B_i})^G) = \begin{cases} 1 & \textit{if } i = \ell; \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Let $r \in \pi^{\ell-1}R^{\times}$ and $0 \leq i \leq \ell$. Then, by the Intertwining Number Theorem

$$((1_{B_i})^G, \zeta_r) = ((1_{B_i})^G, \lambda_r^G) = \sum_{B_i g\widetilde{V} \in \mathfrak{D}_G(B_i, \widetilde{V})} (1_{g\widetilde{V}g^{-1} \cap B_i}, (\lambda_r^g)_{g\widetilde{V}g^{-1} \cap B_i}). \tag{9.4}$$

Consequently, $a', b', c', d' \in R$ with $a'd' - b'c' = 1$ and $c' \notin \mathfrak{m}^i$.

Suppose that $c' \in \pi^j R^{\times}$ with $1 \leq j \leq \ell - 1$. If we fix $a \in \mathfrak{m}^{\ell-1}$ and set $c \in R^{\times}$ as in (9.2), then again we see that

$$\lambda_r^g\left(\begin{bmatrix} c^{-1} & 0 \\ 0 & c \end{bmatrix}\right) = \lambda'(r'a).$$

Thus $(\lambda_r^g)_{g\widetilde{V}g^{-1} \cap B_i} \neq 1_{g\widetilde{V}g^{-1} \cap B_i}$ and so $(1_{g\widetilde{V}g^{-1} \cap B_i}, (\lambda_r^g)_{g\widetilde{V}g^{-1} \cap B_i}) = 0$.

Similarly, if $c \in R^{\times}$ then fixing $a \in \mathfrak{m}^{\ell-1}$ and setting $b \in \mathfrak{m}^{\ell-1}$ as in (9.3) we again obtain

$$\lambda_r^g\left(\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}\right) = \lambda'(r'a).$$

Therefore $(\lambda_r^g)_{g\widetilde{V}g^{-1} \cap B_i} \neq 1_{g\widetilde{V}g^{-1} \cap B_i}$ and so $(1_{g\widetilde{V}g^{-1} \cap B_i}, (\lambda_r^g)_{g\widetilde{V}g^{-1} \cap B_i}) = 0$.

Hence, the $g \neq 1$ terms disappear in (9.4) and we find that

$$((1_{B_i})^G, \zeta_r) = (1_{B_i \cap \widetilde{V}}, (\lambda_r)_{B_i \cap \widetilde{V}})$$

which is 1 if $i = \ell$ and 0 otherwise. □

Thus, for each $r \in \pi^{\ell-1}R^{\times}$

$$(\mathrm{St}'_\ell, \zeta_r) = ((1_{B_\ell})^G, \zeta_r) - ((1_{B_{\ell-1}})^G, \zeta_r) = 1.$$

Hence, if $r \in \pi^{\ell-1}R_2^{\times}$ and $s \in \pi^{\ell-1}\epsilon R_2^{\times}$, then ζ_r and ζ_s must be the distinct irreducible constituents of St'_ℓ.

Lemma 9.1.5. $\mathrm{St}'_\ell = \zeta_r + \zeta_s$ *for any* $r \in \pi^{\ell-1}R^{\times}$ *and* $s \in \pi^{\ell-1}\epsilon R^{\times}$.

9.2 Clifford theory for χ'

Our approach to determining the decomposition of St'_ℓ into its irreducible constituents is to first decompose χ'. This is achieved by examining its restriction to the subgroup $\widetilde{V} = X_S V$ of H. Indeed, since $H = T\widetilde{V}$ and T preserves both X_S and V, we see that $\widetilde{V}$ is a normal subgroup of H.

Consider the abelian group X_S. This is generated by the root subgroups X_α for $\alpha \in S$ and so any linear character λ of X_S is uniquely determined by its restriction to each X_α. Further, since V is normal in H, each linear character λ of X_S extends to a linear character of $\widetilde{V}$ which we will again denote by λ.

In particular, let $\mathfrak{X}$ denote the set of linear characters λ of $\widetilde{V}$ which are obtained from linear characters of X_S with $\lambda_{X_\alpha} \neq 1_{X_\alpha}$ for each $\alpha \in S$. The number of such characters is equal to the degree of χ'.

Lemma 9.2.1. $|\mathfrak{X}| = \chi'(1)$.

Proof. Each root subgroup X_α is abelian of order q. Thus, there are $q-1$ possible choices of non-trivial character λ_{X_α} for each $\alpha \in S$ and so a total of $(q-1)^n$ possibilities for λ. However, we also find that

$$\chi'(1) = \sum_{J \subseteq S} (-1)^{|J|} [H : H_J] = \sum_{J \subseteq S} (-1)^{|J|} q^{|S|-|J|} = (q-1)^n. \qquad \square$$

Indeed, the linear characters in $\mathfrak{X}$ completely describe the restriction of χ' to $\widetilde{V}$.

Lemma 9.2.2. $\chi'_{\widetilde{V}} = \sum_{\lambda \in \mathfrak{X}} \lambda$.

Proof. Let $\lambda \in \mathfrak{X}$, then since $H_J \widetilde{V} = H$ for each $J \subseteq S$, Frobenius reciprocity and Mackey theory imply that

$$(\lambda, \chi'_{\widetilde{V}}) = \sum_{J \subseteq S} (-1)^{|J|} (\lambda, ((1_{H_J})^H)_{\widetilde{V}}) = \sum_{J \subseteq S} (-1)^{|J|} (\lambda_{X_J V}, 1_{X_J V}). \qquad (9.5)$$

Now, if $J \neq \emptyset$ then for any $\alpha \in S$

$$0 \leq (\lambda_{X_J V}, 1_{X_J V}) \leq (\lambda_{X_\alpha}, 1_{X_\alpha}) = 0.$$

Consequently, the $J \neq \emptyset$ terms disappear in (9.5) and we obtain

$$(\lambda, \chi'_{\widetilde{V}}) = (\lambda_V, 1_V) = 1.$$

Thus each $\lambda \in \mathfrak{X}$ appears as a constituent of χ' with multiplicity 1 and the result follows from Lemma 9.2.1. □

As an immediate consequence of Lemma 9.2.2 we have the following result.

Lemma 9.2.3. *H permutes the characters in $\mathfrak{X}$.*

Proof. For each $g \in H$ we have $(\chi'_{\widetilde{V}})^g = \sum_{\lambda \in \mathfrak{X}} \lambda^g$. However, since $\chi'_{\widetilde{V}}$ is the restriction of a character from H we must have $(\chi'_{\widetilde{V}})^g = \chi_{\widetilde{V}}$ and so $(\chi'_{\widetilde{V}})^g = \chi'_{\widetilde{V}} = \sum_{\lambda \in \mathfrak{X}} \lambda$. Hence $\lambda^g \in \mathfrak{X}$ for each $\lambda \in \mathfrak{X}$. □

Further, the stabiliser of each character in $\mathfrak{X}$ is the group $\widetilde{V}$ itself.

Lemma 9.2.4. $\mathrm{Stab}_H(\lambda) = \widetilde{V}$ *for each* $\lambda \in \mathfrak{X}$.

Proof. Clearly $\widetilde{V} \leq \mathrm{Stab}_H(\lambda)$ so consider $g \in \mathrm{Stab}_H(\lambda)$. As $H = T\widetilde{V}$ we may express g as $g = tv$ for some $t \in T$ and $v \in \widetilde{V}$. In particular, if $g \in \mathrm{Stab}_H(\lambda)$ then since $\widetilde{V} \leq \mathrm{Stab}_H(\lambda)$ we must have $t \in \mathrm{Stab}_H(\lambda)$.

Now, suppose that $t = h(\mu)$ for some R-character of μ of Λ. Then for each $r \in \mathfrak{m}^{\ell-1}$ and $\alpha \in S$ we need that

$$\lambda(x_\alpha(r)) = \lambda^t(x_\alpha(r)) = \lambda(h(\mu)^{-1} x_\alpha(r) h(\mu)) = \lambda(x_\alpha(\mu(\alpha)^{-1} r)).$$

However, since $\lambda_{X_\alpha} \neq 1_{X_\alpha}$ this is only possible if $\mu(\alpha)^{-1} r = r$ for every $r \in \mathfrak{m}^{\ell-1}$. Hence, $\mu(\alpha) \in 1 + \mathfrak{m}$ for each $\alpha \in S$ implying that $t \in T(\mathfrak{m})$ and $g \in \widetilde{V}$. □

By Lemma 9.2.4, for each $\lambda \in \mathcal{X}$ we have

$$[H : \mathrm{Stab}_H(\lambda)] = [H : X_S V] = [T : T(\mathfrak{m})] = \frac{(q-1)^n}{d}.$$

Thus, the size of each H-orbit on $\mathcal{X}$ is $(q-1)^n/d$ and $\mathcal{X}$ decomposes into the disjoint union of d orbits, i.e. $\mathcal{X}$ can be written as the disjoint union

$$\mathcal{X} = \bigcup_{i=1}^{d} \mathcal{X}_i$$

where the action on $\mathcal{X}_i$ is transitive for each i. Therefore, we choose a representative λ_i from each orbit $\mathcal{X}_i$ and define $\chi_i = (\lambda_i)^H$.

Theorem 9.2.5.

$$\chi' = \sum_{i=1}^{d} \chi_i$$

where $\chi_1, \ldots, \chi_d$ are the distinct irreducible constituents of χ'.

Proof. Lemma 9.2.4 and Clifford Theory imply that each χ_i is irreducible with

$$(\chi_i)_{\widetilde{V}} = \sum_{\lambda \in \mathcal{X}_i} \lambda.$$

In particular, this means that the χ_i are distinct since the orbits $\mathcal{X}_i$ are disjoint. Further,

$$(\chi', \chi_i) = (\chi', (\lambda_i)^H) = (\chi'_{\widetilde{V}}, \lambda_i) = \sum_{\lambda \in \mathcal{X}} (\lambda, \lambda_i) = 1.$$

Thus each χ_i is a constituent of χ' with multiplicity 1 and the result then follows from the fact that $\sum_{i=1}^{d} \chi_i(1) = (q-1)^n = \chi'(1)$. □

Remark 9.2.6. (i) If we consider the extended Chevalley group $\widehat{G}$, then $\widehat{H}$ transitively permutes the linear characters in $\mathcal{X}$. Thus, for any $\lambda \in \mathcal{X}$ we have $\chi = \lambda^{\widehat{H}}$ which is therefore irreducible.

(ii) It is possible to parameterise the characters in $\mathcal{X}$ by the elements of $\mathfrak{X}_S$ in a manner similar to the definition of λ_r from the previous section. The H-orbits

$\mathcal{X}_i$ in $\mathcal{X}$ then correspond exactly with the T-orbits $\mathfrak{X}_S^{(i)}$ in $\mathfrak{X}_S$. Consequently, there is an explicit connection between the distinct irreducible constituents of χ' and the distinct (B,B)-double cosets contained in $(\widehat{B}x_S\widehat{B}) \cap H$.

9.3 Decomposition of St'_ℓ

For each i, if we consider $\zeta_i = (\chi_i)^G$ then it is clear from Theorem 9.2.5 that

$$\mathrm{St}'_\ell = \sum_{i=1}^{d} \zeta_i.$$

Thus, we would like to show that $\zeta_1, \ldots, \zeta_d$ are the distinct irreducible constituents of St'_ℓ.

Proposition 9.3.1. *For each* $1 \le i, j \le d$,

$$(\zeta_i, \zeta_j) = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{if } i \neq j. \end{cases}$$

Proof. By the Intertwining Number Theorem

$$\begin{aligned} (\zeta_i, \zeta_j) &= ((\chi_i)^G, (\chi_j)^G) \\ &= \sum_{HgH \in \mathcal{D}_G(H,H)} ((\chi_i)_{gHg^{-1}\cap H}, (\chi_j)^g_{gHg^{-1}\cap H}). \end{aligned} \tag{9.6}$$

Now, by Proposition 6.4.1 we know that if $HgH \neq H$ then there is an $\alpha \in S$ so that for every $r \in \mathfrak{m}^{\ell-1}$ we have $[v_r, g] = v'_r x_\alpha(r)$ for some $v_r, v'_r \in V$. Thus $gv_rg^{-1} = v''_r x_\alpha(r)$ for some $v_r, v''_r \in V$.

Consequently, if we let $Y = \langle gv_rg^{-1} : r \in \mathfrak{m}^{\ell-1}\rangle$, then $Y \le gHg^{-1} \cap H$. Further, for each $\lambda \in \mathcal{X}$ and $r \in \mathfrak{m}^{\ell-1}$ we see that

$$\lambda(gv_rg^{-1}) = \lambda(v''_r x_\alpha(r)) = \lambda(v''_r)\lambda(x_\alpha(r)) = \lambda(x_\alpha(r))$$

and so $\lambda_Y \neq 1_Y$. However, $\lambda^g_Y = 1_Y$ since for every $r \in \mathfrak{m}^{\ell-1}$

$$\lambda^g(gv_rg^{-1}) = \lambda(v_r) = 1.$$

Thus

$$\begin{aligned}
0 &\leq ((\chi_i)_{gHg^{-1}\cap H}, (\chi_j^g)_{gHg^{-1}\cap H}) \\
&\leq ((\chi_i)_Y, (\chi_j^g)_Y) \\
&= \sum_{\lambda\in\mathfrak{X}_i,\ \lambda'\in\mathfrak{X}_j} (\lambda_Y, (\lambda')^g_Y) \\
&= \sum_{\lambda\in\mathfrak{X}_i,\ \lambda'\in\mathfrak{X}_j} (\lambda_Y, 1_Y) \\
&= 0.
\end{aligned}$$

Hence the $g \neq 1$ terms in (9.6) disappear and we obtain

$$(\zeta_i, \zeta_j) = (\chi_i, \chi_j)$$

which, by Theorem 9.2.5, is 1 if $i = j$ and 0 otherwise. □

Proposition 9.3.1 implies that the ζ_i are distinct irreducible characters of G. Thus, from Theorem 9.2.5 we immediately obtain the decomposition of St'_ℓ into its irreducible constituents.

Theorem 9.3.2.

$$\mathrm{St}'_\ell = \sum_{i=1}^{d} \zeta_i$$

where $\zeta_1, \ldots, \zeta_d$ are the distinct irreducible constituents of St'_ℓ.

Remark 9.3.3. If we consider the extended Chevalley group $\widehat{G}$, then from the previous section we obtain $\zeta = \mathrm{St}_\ell$. Thus, Proposition 9.3.1 implies that St_ℓ is irreducible.

9.4 Gelfand-Graev characters for G

Finally, we characterise the irreducible constituents of St'_ℓ in terms of analogues of the Gelfand-Graev character for G. Our approach to the construction of the analogues follows [15].

Definition 9.4.1. A linear character θ of U is **non-degenerate** if $\theta_{U_\alpha(\mathfrak{m}^{\ell-1})}$ is non-trivial for each $\alpha \in \Pi$. Further, let Θ denote the set of non-degenerate linear characters of U.

Lemma 9.4.2. $|\Theta| = q^{n(\ell-1)}(q-1)^n$.

Proof. Any $\theta \in \Theta$ is completely determined by its values on the root subgroups U_α for $\alpha \in \Pi$. Each root subgroup U_α is abelian of order q^ℓ with $U_\alpha(\mathfrak{m}^{\ell-1})$ as a subgroup of index $q^{\ell-1}$. Thus there are q^ℓ linear characters of U_α of which exactly $q^{\ell-1}$ are trivial on $U_\alpha(\mathfrak{m}^{\ell-1})$. Hence for each $\alpha \in \Pi$ there are $q^{\ell-1}(q-1)$ possible choices for each linear character which is non-trivial on $U_\alpha(\mathfrak{m}^{\ell-1})$ and so a total of $q^{n(\ell-1)}(q-1)^n$ possibilities for θ. $\square$

Lemma 9.4.3. $\mathrm{Stab}_B(\theta) = U$ *for each* $\theta \in \Theta$.

Proof. Clearly $U \leq \mathrm{Stab}_B(\theta)$ so suppose that $g \in \mathrm{Stab}_B(\theta)$. Since $B = TU$ we may express g as $g = tu$ for some $t \in T$ and $u \in U$. Then we see that $g \in \mathrm{Stab}_B(\theta)$ implies $t \in \mathrm{Stab}_B(\theta)$ since $U \leq \mathrm{Stab}_B(\theta)$.

Now, let $t = h(\mu)$ for some R-character μ of Λ. For each $\alpha \in S$ and $r \in R$ we need that

$$\theta(x_\alpha(r)) = \theta^t(x_\alpha(r)) = \theta(h(\mu)^{-1}x_\alpha(r)h(\mu)) = \theta(x_\alpha(\mu(\alpha)^{-1}r)).$$

However, since $\theta_{U_\alpha(\mathfrak{m}^{\ell-1})} \neq 1_{U_\alpha(\mathfrak{m}^{\ell-1})}$, this means that we need $\mu(\alpha)^{-1}r = r$ for every $r \in R$ and so $\mu(\alpha) = 1$ for each $\alpha \in \Pi$. Hence $t = 1$ and $g \in U$. $\square$

In particular, for each $\theta \in \Theta$

$$[B : \mathrm{Stab}_B(\theta)] = [B : U] = |T| = \frac{q^{n(\ell-1)}(q-1)^n}{d}.$$

Thus Θ decomposes into the disjoint union of d orbits under the action of B, i.e.

$$\Theta = \bigcup_{i=1}^{d} \Theta_i$$

where B permutes the elements of Θ_i transitively for each i. Note that this is the same as the number of H-orbits in $\mathfrak{X}$. In fact, there is a connection between them.

First note that

$$n_{w_0} U n_{w_0}^{-1} \cap H = X_S.$$

Further, for each $\alpha \in S$ and any $x_\alpha(r) \in X_\alpha$ we see that

$$\theta^{n_{w_0}}(x_\alpha(r)) = \theta(n_{w_0}^{-1} x_\alpha(r) n_{w_0}) = \theta(x_{w_0^{-1}(\alpha)}(r))$$

where $w_0^{-1}(\alpha) \in \Pi$. Consequently, the non-degeneracy of θ implies that $\theta_{X_\alpha}^{n_{w_0}} \neq 1_{X_\alpha}$ for each $\alpha \in S$. Thus $\theta_{X_S}^{n_{w_0}} = \lambda_{X_S}$ for exactly one $\lambda \in \mathfrak{X}$ and we can then set

$$\Theta_i = \{\theta \in \Theta : \theta_{X_S}^{n_{w_0}} = \lambda_{X_S} \text{ for some } \lambda \in \mathfrak{X}_i\}.$$

Now, suppose that $\theta_{X_S}^{n_{w_0}} = \lambda_{X_S}$ for some $\lambda \in \mathfrak{X}_i$. Let $g \in B$ and express g as $g = tu$ for $t \in T$ and $u \in U$. Then we see that

$$(\theta^g)_{X_S}^{n_{w_0}} = (\theta^t)_{X_S}^{n_{w_0}} = (\theta_{X_S}^{n_{w_0}})^{n_{w_0}^{-1} t n_{w_0}} = \lambda_{X_S}^{n_{w_0}^{-1} t n_{w_0}}$$

where $n_{w_0}^{-1} t n_{w_0} \in T$. Thus, we also have $\lambda^{n_{w_0}^{-1} t n_{w_0}} \in \mathfrak{X}_i$ and so Θ_i is indeed a B-orbit in Θ.

Definition 9.4.4. For each $1 \leq i \leq d$, let θ_i be a representative from the orbit Θ_i. The analogue Γ_i of the Gelfand-Graev character is then the induced character θ_i^G.

We will now show that the irreducible constituents of St'_ℓ are given by the characters Γ_i. We begin with two lemmas involving the (U, B)-double coset structure of G. The proof of the first is similar to the proof of Lemma 6.4.2

Lemma 9.4.5. *Each (U, B)-double coset has a representative of the form $g = kn_w$ for some $k \in U^-(\mathfrak{m})$ and $w \in W$.*

In particular, there is a unique (U, B)-double coset corresponding to $w = w_0$.

Lemma 9.4.6. *$Ukn_{w_0}B = Un_{w_0}B$ for any $k \in U^-(\mathfrak{m})$.*

Proof. Let $k \in U^-(\mathfrak{m})$, then we have $n_{w_0}^{-1}kn_{w_0} \in U$ and so we see that

$$Ukn_{w_0}B = Un_{w_0}n_{w_0}^{-1}kn_{w_0}B = Un_{w_0}B. \qquad \square$$

Thus, we see that Γ_i and $(1_B)^G$ have a unique common constituent.

Proposition 9.4.7. *$(\Gamma_i, (1_B)^G) = 1$ for each i.*

Proof. By the Intertwining Number Theorem,

$$(\Gamma_i, (1_B)^G) = ((\theta_i)^G, (1_B)^G) = \sum_{UgB \in \mathfrak{D}_G(U,B)} ((\theta_i)_{gUg^{-1}\cap B}, 1_{gUg^{-1}\cap B}). \tag{9.7}$$

Suppose that $g = kn_w$ is a (U, B)-double coset representative with $w \neq w_0$. Then we must have $w(\alpha) = \alpha$ for some $\alpha \in \Pi$ and so $n_w x_\alpha(r) n_w^{-1} = x_\alpha(r)$ for every $x_\alpha(r) \in U_\alpha(\mathfrak{m}^{\ell-1})$. Further,

$$gx_\alpha(r)g^{-1} = kn_w x_\alpha(r) n_w^{-1} k^{-1} = kx_\alpha(r)k^{-1} = x_\alpha(r)$$

and $U_\alpha(\mathfrak{m}^{\ell-1}) \leq gUg^{-1} \cap B$. Therefore we see that

$$0 \leq ((\theta_i)_{gUg^{-1}\cap B}, 1_{gUg^{-1}\cap B}) \leq ((\theta_i)_{U_\alpha(\mathfrak{m}^{\ell-1})}, 1_{U_\alpha(\mathfrak{m}^{\ell-1})}) = 0$$

since θ is non-degenerate. Hence the $g \neq n_{w_0}$ terms disappear in (9.7) and we are left with

$$(\Gamma_i, (1_B)^G) = ((\theta_i)_{n_{w_0}Un_{w_0}^{-1}\cap B}, 1_{n_{w_0}Un_{w_0}^{-1}\cap B}) = 1$$

since $n_{w_0}Un_{w_0}^{-1} \cap B = \{1\}$. $\square$

Further, each ζ_i is a constituent of exactly one of the analogues of the Gelfand-Graev character.

Proposition 9.4.8. *For each i, j*

$$(\zeta_i, \Gamma_j) = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By the Intertwining Number Theorem,

$$(\zeta_i, \Gamma_j) = (\chi_i^G, \theta_j^G) = \sum_{g \in \mathfrak{D}_G(H,U)} ((\chi_i)_{gUg^{-1}\cap H}, (\theta_j)^g_{gUg^{-1}\cap H}). \tag{9.8}$$

Again, if we consider an (H, U)-double coset representative $g = kn_w$ with $w \neq w_0$, then g commutes with the elements of $U_\alpha(\mathfrak{m}^{\ell-1})$ for some $\alpha \in \Pi$. Thus, we see that $U_\alpha(\mathfrak{m}^{\ell-1}) \leq gHg^{-1} \cap U$ and $(\theta_j)^g_{U_\alpha(\mathfrak{m}^{\ell-1})} = \theta_{U_\alpha(\mathfrak{m}^{\ell-1})}$. Consequently,

$$\begin{aligned} 0 &\leq ((\chi_i)_{gUg^{-1}\cap H}, (\theta_j)^g_{gUg^{-1}\cap H}) \\ &\leq ((\chi_i)_{U_\alpha(\mathfrak{m}^{\ell-1})}, (\theta_j)^g_{U_\alpha(\mathfrak{m}^{\ell-1})}) \\ &= \sum_{\lambda \in \mathfrak{X}_i} (\lambda_{U_\alpha(\mathfrak{m}^{\ell-1})}, (\theta_j)_{U_\alpha(\mathfrak{m}^{\ell-1})}) \\ &= 0 \end{aligned}$$

since $\lambda_{U_\alpha(\mathfrak{m}^{\ell-1})} = 1_{U_\alpha(m^{\ell-1})}$ for each $\lambda \in \mathfrak{X}_i$ and θ_j is non-degenerate.

Hence the $g \neq n_{w_0}$ terms disappear in (9.8) and we obtain

$$(\zeta_i, \Gamma_i) = ((\chi_i)_{n_{w_0}Un_{w_0}^{-1}\cap H}, (\theta_j)^{n_{w_0}}_{n_{w_0}Un_{w_0}^{-1}\cap H}) = \sum_{\lambda \in \mathfrak{X}_i} (\lambda_{X_S}, (\theta_j)^{n_{w_0}}_{X_S})$$

which is 1 if $i = j$ and 0 otherwise. □

Consequently, Propositions 9.4.7 and 9.4.8 imply that each irreducible constituent of St'_ℓ is the unique common constituent of the permutation character over B and exactly one of the analogues of the Gelfand-Graev character.

Theorem 9.4.9. *ζ_i is the unique common constituent of Γ_i and $(1_B)^G$.*

Remark 9.4.10. In the case of the extended Chevalley group $\widehat{G}$, we see that $\widehat{B}$ transitively permutes the elements of Θ and so there is a unique Gelfand-Graev character $\Gamma_0 = \theta^{\widehat{G}}$. Again, Γ_0 has a unique common constituent with the permutation character $(1_{\widehat{B}})^{\widehat{G}}$, which must therefore be St_ℓ. In the case of $\mathrm{PGL}_n(R)$, this is exactly the construction given by Hill [15] for the analogue of the Steinberg character.

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Appendix A

Irreducible root systems

For use in Appendices B and C we need an explicit description of the irreducible root system of each type together with a base and the corresponding set of positive roots. These have been taken from [13] with a reordering of the simple roots in the root systems of types E_6, E_7 and E_8 so that they agree with the Dynkin diagrams in Table 2.1. Here $e_1, \ldots, e_n$ denote the standard basis vectors of $\mathbb{R}^n$ and

$$e_\epsilon = \frac{1}{2}\sum_{i=1}^{n} \epsilon_i e_i$$

for each $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in \mathcal{E}^n$ where $\mathcal{E} = \{\pm 1\}$. This notation is shortened further by recording only the signs involved. For example, the first simple root in E_6 is

$$e_{+------+} = \frac{1}{2}e_1 - \frac{1}{2}e_2 - \frac{1}{2}e_3 - \frac{1}{2}e_4 - \frac{1}{2}e_5 - \frac{1}{2}e_6 - \frac{1}{2}e_7 + \frac{1}{2}e_8.$$

$\mathbf{A}_n$ $(n \geq 1)$:

Let $\mathfrak{E}$ denote the subspace of $\mathbb{R}^{n+1}$ which is orthogonal to $e_1 + \cdots + e_{n+1}$. The root system of type A_n is then

$$\Sigma = \{\pm(e_i - e_j) : 1 \leq i < j \leq n+1\}.$$

A base for Σ is given by $\Pi = \{\alpha_i = e_i - e_{i+1} : 1 \le i \le n\}$ and the corresponding positive roots are

$$\Sigma^+ = \{e_i - e_j : 1 \le i < j \le n+1\}.$$

$\mathbf{B}_n$ $(n \ge 2)$:

Let $\mathfrak{E} = \mathbb{R}^n$, then the root system of type B_n is

$$\Sigma = \{\pm e_i \pm e_j : 1 \le i < j \le n\} \cup \{\pm e_i : 1 \le i \le n\}.$$

A base for Σ is given by $\Pi = \{\alpha_i = e_i - e_{i+1} : 1 \le i \le n-1\} \cup \{\alpha_n = e_n\}$ and the corresponding positive roots are

$$\Sigma^+ = \{e_i - e_j : 1 \le i < j \le n\} \cup \{e_i + e_j : 1 \le i < j \le n\} \cup \{e_i : 1 \le i \le n\}.$$

$\mathbf{C}_n$ $(n \ge 3)$:

Let $\mathfrak{E} = \mathbb{R}^n$, then the root system of type C_n is

$$\Sigma = \{\pm e_i \pm e_j : 1 \le i < j \le n\} \cup \{\pm 2e_i : 1 \le i \le n\}.$$

A base for Σ is given by $\Pi = \{\alpha_i = e_i - e_{i+1} : 1 \le i \le n-1\} \cup \{\alpha_n = 2e_n\}$ and the corresponding positive roots are

$$\Sigma^+ = \{e_i - e_j : 1 \le i < j \le n\} \cup \{e_i + e_j : 1 \le i < j \le n\} \cup \{2e_i : 1 \le i \le n\}.$$

$\mathbf{D}_n$ $(n \ge 4)$:

Let $\mathfrak{E} = \mathbb{R}^n$, then the root system of type D_n is

$$\Sigma = \{\pm e_i \pm e_j : 1 \le i < j \le n\}.$$

A base for Σ is given by $\Pi = \{\alpha_i = e_i - e_{i+1} : 1 \le i \le n-1\} \cup \{\alpha_n = e_{n-1} + e_n\}$ and the corresponding positive roots are

$$\Sigma^+ = \{e_i - e_j : 1 \le i < j \le n\} \cup \{e_i + e_j : 1 \le i < j \le n\}.$$

$\mathbf{E_6}$:

Let $\mathfrak{E}$ denote the subspace of $\mathbb{R}^8$ which is orthogonal to both e_1+e_2 and e_2-e_3. The root system of type E_6 is then

$$\Sigma = \{\pm e_i \pm e_j : 4 \leq i < j \leq 8\} \cup \{e_\epsilon : \epsilon \in \mathcal{E}^n, \prod_{i=1}^{8} \epsilon_i = 1, \epsilon_1 = -\epsilon_2 = -\epsilon_3\}.$$

A base for Σ is given by

$$\begin{aligned} \Pi &= \{\alpha_1 = e_{+-------+}, \alpha_2 = e_7 + e_8, \alpha_3 = e_7 - e_8, \alpha_4 = e_6 - e_7, \\ &\quad \alpha_5 = e_5 - e_6, \alpha_6 = e_4 - e_5\} \end{aligned}$$

and the corresponding positive roots are

$$\begin{aligned} \Sigma^+ &= \{e_i - e_j : 4 \leq i < j \leq 8\} \cup \{e_i + e_j : 4 \leq i < j \leq 8\} \\ &\quad \cup \{e_\epsilon : \epsilon \in \mathcal{E}^n, \prod_{i=1}^{8} \epsilon_i = 1, \epsilon_1 = -\epsilon_2 = -\epsilon_3 = 1\}. \end{aligned}$$

$\mathbf{E_7}$:

Let $\mathfrak{E}$ denote the subspace of $\mathbb{R}^8$ which is orthogonal to $e_1 + e_2$. The root system of type E_7 is then

$$\Sigma = \{\pm(e_1 - e_2)\} \cup \{\pm e_i \pm e_j : 3 \leq i < j \leq 8\} \cup \{e_\epsilon : \epsilon \in \mathcal{E}^n, \prod_{i=1}^{8} \epsilon_i = 1, \epsilon_1 = -\epsilon_2\}.$$

A base for Σ is given by

$$\begin{aligned} \Pi &= \{\alpha_1 = e_{+-------+}, \alpha_2 = e_7 + e_8, \alpha_3 = e_7 - e_8, \alpha_4 = e_6 - e_7, \\ &\quad \alpha_5 = e_5 - e_6, \alpha_6 = e_4 - e_5, \alpha_7 = e_3 - e_4\} \end{aligned}$$

and the corresponding positive roots are

$$\begin{aligned} \Sigma^+ &= \{e_1 - e_2\} \cup \{e_i - e_j : 3 \leq i < j \leq 8\} \cup \{e_i + e_j : 3 \leq i < j \leq 8\} \\ &\quad \cup \{e_\epsilon : \epsilon \in \mathcal{E}^n, \prod_{i=1}^{8} \epsilon_i = 1, \epsilon_1 = -\epsilon_2 = 1\}. \end{aligned}$$

$\mathbf{E}_8$:

Let $\mathfrak{E} = \mathbb{R}^8$, then the root system of type E_8 is

$$\Sigma = \{\pm e_i \pm e_j : 1 \leq i < j \leq 8\} \cup \{e_\epsilon : \epsilon \in \mathcal{E}^n, \prod_{i=1}^{8} \epsilon_i = 1\}.$$

A base for Σ is given by

$$\begin{aligned} \Pi &= \{\alpha_1 = e_{+------+}, \alpha_2 = e_7 + e_8, \alpha_3 = e_7 - e_8, \alpha_4 = e_6 - e_7, \\ &\qquad \alpha_5 = e_5 - e_6, \alpha_6 = e_4 - e_5, \alpha_7 = e_3 - e_4, \alpha_8 = e_2 - e_3\} \end{aligned}$$

and the corresponding positive roots are

$$\begin{aligned} \Sigma^+ &= \{e_i - e_j : 1 \leq i < j \leq 8\} \cup \{e_i + e_j : 1 \leq i < j \leq 8\} \\ &\qquad \cup \{e_\epsilon : \epsilon \in \mathcal{E}^n, \prod_{i=1}^{8} \epsilon_i = 1, \epsilon_1 = 1\}. \end{aligned}$$

$\mathbf{F}_4$:

Let $\mathfrak{E} = \mathbb{R}^4$, then the root system of type F_4 is

$$\Sigma = \{\pm e_i \pm e_j : 1 \leq i < j \leq 4\} \cup \{\pm e_i : 1 \leq i \leq 4\} \cup \{e_\epsilon : \epsilon \in \mathcal{E}^4\}.$$

A base for Σ is given by $\Pi = \{\alpha_1 = e_2 - e_3, \alpha_2 = e_3 - e_4, \alpha_3 = e_4, \alpha_4 = e_{+---}\}$ and the corresponding positive roots are

$$\Sigma^+ = \{e_i - e_j : 1 \leq i < j \leq 4\} \cup \{e_i + e_j : 1 \leq i < j \leq 4\} \cup \{e_i : 1 \leq i \leq 4\} \cup \{e_\epsilon : \epsilon \in \mathcal{E}^4, \epsilon_1 = 1\}.$$

$\mathbf{G}_2$:

Let $\mathfrak{E} = \mathbb{R}^2$, then the root system of type G_2 is

$$\Sigma = \{\pm\alpha, \pm\beta, \pm(\alpha + \beta), \pm(\alpha + 2\beta), \pm(\alpha + 3\beta), \pm(2\alpha + 3\beta)\}.$$

A base for Σ is given by $\Pi = \{\alpha, \beta\}$ and the corresponding positive roots are

$$\Sigma^+ = \{\alpha, \beta, \alpha + \beta, \alpha + 2\beta, \alpha + 3\beta, 2\alpha + 3\beta\}.$$

Appendix B

Proof of Proposition 6.4.7

To simplify notation, we replace the set $\mathcal{S}$ of negative roots with the corresponding set of positive roots and prove the following equivalent Proposition.

Proposition B.1. *Let Σ be an irreducible root system and $\mathcal{S}$ be a non-empty subset of Σ_i for some $i > 1$, with the exception of the following cases:*

(i) $\Sigma = D_{2n}$ *and* $\mathcal{S} = \Sigma_{2n-1}$*;*

(ii) $\Sigma = E_6$ *and* $\mathcal{S} = \Sigma_4$*;*

(iii) $\Sigma = E_7$ *and* $\mathcal{S} = \Sigma_9$*;*

(iv) $\Sigma = E_8$ *and* $\mathcal{S} = \Sigma_6$, Σ_{10} *or* Σ_{15}*;*

(v) $\Sigma = F_4$ *and* $\mathcal{S} = \Sigma_4$*;*

together with the corresponding sets obtained when Σ contains a subsystem equivalent to D_{2n}, E_6 or E_7. Then there exists a $\beta \in \Sigma_{i-1}$ such that $\gamma - \beta \in \Pi$ for exactly one $\gamma \in \mathcal{S}$.

We prove Proposition B.1 by considering each irreducible root system separately.

A_n $(n \geq 1)$:

Lemma B.2. *Let $\Sigma = A_n$ and $\mathcal{S}$ be a non-empty subset of Σ_i for some $i > 1$. Then there is a $\beta \in \Sigma_{i-1}$ such that $\gamma - \beta \in \Pi$ for exactly one $\gamma \in \mathcal{S}$.*

Proof. Suppose that $\gamma = e_i - e_j \in \mathcal{S}$ with i minimal. Then since $\gamma \notin \Pi$ we may set $\beta = e_i - e_{j-1} \in \Sigma_{i-1}$. Further, if $\alpha \in \Pi$ has $\alpha + \beta \in \Sigma$, then $\alpha = \alpha_{i-1}$ or α_j. However, $\alpha_{i-1} + \beta = e_{i-1} - e_{j-1} \notin \mathcal{S}$ and $\alpha_j + \beta = \gamma$. □

B_n $(n \geq 2)$:

Lemma B.3. *Let $\Sigma = B_n$ and $\mathcal{S}$ be a non-empty subset of Σ_i for some $i > 1$. Then there is a $\beta \in \Sigma_{i-1}$ such that $\gamma - \beta \in \Pi$ for exactly one $\gamma \in \mathcal{S}$.*

Proof. (i) *Suppose that $e_i - e_j \in \mathcal{S}$ for some $i < j$.* Choose $\gamma = e_i - e_j \in \mathcal{S}$ with i minimal. Again we may set $\beta = e_i - e_{j-1} \in \Sigma_{i-1}$ and we see that $\alpha \in \Pi$ with $\alpha + \beta \in \Sigma$ only for $\alpha = \alpha_{i-1}$ or α_j. However, $\alpha_{i-1} + \beta = e_{i-1} - e_{j-1} \notin \mathcal{S}$ and $\alpha_j + \beta = \gamma$.

(ii) *Suppose that there are no roots of type* (i), *but $e_i + e_j \in S$ for some $i < j \neq n$.* Choose $\gamma = e_i + e_j \in \mathcal{S}$ with i minimal. Then setting $\beta = e_i + e_{j+1} \in \Sigma_{i-1}$ then $\alpha \in \Pi$ with $\alpha + \beta \in \Sigma$ implies that $\alpha = \alpha_{i-1}$ or α_j, but we see that $\alpha_{i-1} + \beta = e_{i-1} + e_{j+1} \notin \mathcal{S}$ and $\alpha_j + \beta = \gamma$.

(iii) *Suppose that there are no roots of types* (i) *or* (ii), *but $e_i \in \mathcal{S}$ for some i.* Choose $\gamma = e_i \in \mathcal{S}$ with i minimal and note that since $\gamma \notin \Pi$ we have $i \neq n$. Then setting $\beta = e_i - e_n \in \Sigma_{i-1}$ we see that $\alpha \in \Pi$ with $\alpha + \beta \in \Sigma$ means that $\alpha = \alpha_{i-1}$ or α_n, but $\alpha_{i-1} + \beta = e_{i-1} - e_n \notin \mathcal{S}$ and $\alpha_n + \beta = \gamma$.

(iv) *Suppose that there are no roots of types* (i) - (iii). We must have $e_i - e_n \in \mathcal{S}$ for some i, so choose $\gamma = e_i - e_n$ with i minimal. Again, since $\gamma \notin \Pi$, we have

$i \neq n-1$ and so we may set $\beta = e_{i+1} - e_n \in \Sigma_{i-1}$. If $\alpha \in \Pi$ has $\alpha + \beta \in \Sigma$ then $\alpha = \alpha_i$ or α_n, but $\alpha_n + \beta = e_{i+1} \notin \mathcal{S}$ and $\alpha_i + \beta = \gamma$. $\square$

C_n $(n \geq 3)$:

Lemma B.4. *Let $\Sigma = C_n$ and $\mathcal{S}$ be a non-empty subset of Σ_i for some $i > 1$. Then there is a $\beta \in \Sigma_{i-1}$ such that $\gamma - \beta \in \Pi$ for exactly one $\gamma \in \mathcal{S}$.*

Proof. (i) *Suppose that $e_i - e_j \in \mathcal{S}$ for some $i < j$.* This is the same type (i) for B_n.

(ii) *Suppose that there are no roots of type* (i), *but $e_i + e_j \in \mathcal{S}$ for some $i < j \neq n$.* This is the same as type (ii) for B_n.

(iii) *Suppose that there are no roots of type* (i) *or* (ii), *but $e_i + e_n \in \mathcal{S}$ for some i.* Choose $\gamma = e_i + e_n \in \mathcal{S}$ with i minimal. Then if we set $\beta = e_i - e_n \in \Sigma_{i-1}$ we see that $\alpha \in \Pi$ with $\alpha + \beta \in \Sigma$ implies that $\alpha = \alpha_{i-1}$ or α_n. However, $\alpha_{i-1} + \beta = e_{i-1} + e_n \notin \mathcal{S}$ and $\alpha_n + \beta = \gamma$.

(iv) *Suppose that there are no roots of types* (i) - (iii). We must have $2e_i \in \mathcal{S}$ for some i, so choose $\gamma = 2e_i \in \mathcal{S}$ with i minimal and note that $i \neq n$ since $\gamma \notin \Pi$. Setting $\beta = e_i + e_{i+1} \in \Sigma_{i-1}$ if $\alpha \in \Pi$ has $\alpha + \beta \in \Sigma$ then $\alpha = \alpha_{i-1}$ or α_i, but $\alpha_{i-1} + \beta = e_{i-1} + e_{i+1} \notin \mathcal{S}$ and $\alpha_i + \beta = \gamma$. $\square$

G_2:

Lemma B.5. *Let $\Sigma = G_2$ and $\mathcal{S}$ be a non-empty subset of Σ_i for some $i > 1$. Then there is a $\beta \in \Sigma_{i-1}$ such that $\gamma - \beta \in \Pi$ for exactly one $\gamma \in \mathcal{S}$.*

Proof. There is at most one root of each height $i > 1$. $\square$

D_n $(n \geq 4)$:

Lemma B.6. *Let $\Sigma = D_{2n}$ and $\mathcal{S}$ be a non-empty subset of Σ_{2n-1} with the exception of $\mathcal{S} = \Sigma_{2n-1}$. Then there is a $\beta \in \Sigma_{i-1}$ such that $\gamma - \beta \in \Pi$ for exactly one $\gamma \in \mathcal{S}$.*

Proof. First note that $\mathrm{ht}(e_i - e_j) = j - i$ and $\mathrm{ht}(e_i + e_j) = 4n - i - j$. Thus

$$\Sigma_{2n-1} = \{\gamma_0, \gamma_1, \dots, \gamma_n\}$$

where $\gamma_0 = e_1 - e_{2n}$ and $\gamma_i = e_i + e_{2n-i+1}$ for $1 \leq i \leq n$. Similarly,

$$\Sigma_{2n-2} = \{\beta_0, \beta_1, \dots, \beta_n\}$$

where $\beta_0 = e_2 - e_{2n}$, $\beta_1 = e_1 - e_{n-1}$ and $\beta_i = e_i + e_{2n-i+2}$ for $2 \leq i \leq n$. Moreover, the possible ways of expressing each root in Σ_{2n-1} as the sum of a simple root and a root in Σ_{2n-2} are

$$\begin{array}{lllllll}
\gamma_0 & = & \alpha_1 + \beta_0 & = & \alpha_{2n-1} + \beta_1 & & \\
\gamma_1 & = & \alpha_1 + \beta_2 & = & \alpha_{2n} + \beta_1 & & \\
\gamma_2 & = & \alpha_2 + \beta_3 & = & \alpha_{2n-1} + \beta_2 & = & \alpha_{2n} + \beta_0 \\
\gamma_n & = & \alpha_{n+1} + \beta_n & & & &
\end{array}$$

with

$$\gamma_i = \alpha_i + \beta_{i+1} = \alpha_{2n-i+1} + \beta_i$$

for $3 \leq i \leq n-1$ in general. In particular, it is clear that for $i \geq 1$ we have $\gamma - \beta_i \in \Pi$ if and only if $\gamma = \gamma_i$ or γ_{i-1}. Finally, since there must be some $i \geq 1$ so that exactly one of γ_i or γ_{i-1} lies in $\mathcal{S}$, setting $\beta = \beta_i$ we have $\gamma - \beta \in \Pi$ for exactly one $\gamma \in \mathcal{S}$. $\square$

Lemma B.7. *Let $\Sigma = D_n$ and $\mathcal{S}$ be a non-empty subset of Σ_{2k}. Then there is a $\beta \in \Sigma_{2k-1}$ such that $\gamma - \beta \in \Pi$ for exactly one $\gamma \in \mathcal{S}$.*

Proof. (i) *Suppose that $e_i + e_j \in \mathcal{S}$ for some $i < j$.* Choose $\gamma = e_i + e_j \in \mathcal{S}$ with i maximal. Since $\mathcal{S} \subseteq \Sigma_{2k}$ we cannot have $j = i + 1$. Thus, if we choose

$\beta = e_{i+1} + e_j \in \Sigma_{2k-1}$ then we see that $\alpha + \beta \in \Sigma$ only for $\alpha = \alpha_i$ or α_j. However, $\alpha_j + \beta = e_{i+1} + e_{j+1} \notin \mathcal{S}$ and $\alpha_i + \beta = \gamma$.

(ii) *Suppose that $e_i + e_j \notin \mathcal{S}$ for any $i < j$.* Choose $\gamma = e_i - e_j \in \mathcal{S}$ with i minimal. Setting $\beta = e_i - e_{j-1} \in \Sigma_{2k-1}$ we see that $\alpha + \beta \in \Sigma$ only for $\alpha = \alpha_{i-1}$, α_k or α_n if $j = n$. However, $\alpha_{i-1} + \beta = e_{i-1} - e_{j-1} \notin \mathcal{S}$, if $j = n$ then $\alpha_n + \beta = e_i + e_n \notin \mathcal{S}$, and finally $\alpha_j + \beta = \gamma$. $\square$

Lemma B.8. *Let $\Sigma = \mathcal{D}_n$ and $\mathcal{S}$ be a non-empty subset of Σ_{2k-1}, with the exception of $\mathcal{S} = \{e_{2k-n-1} - e_n, e_{2k-n-1} + e_n, e_{2k-n} + e_{n+1}, \ldots, e_{n-k} + e_{n-k+1}\}$. Then there is a $\beta \in \Sigma_{2k-2}$ such that $\gamma - \beta \in \Pi$ for exactly one $\gamma \in \mathcal{S}$.*

Proof. (i) *Suppose that $e_i - e_j \in \mathcal{S}$ for some $i < j$ with $i < 2k - n - 1$.* Choose $\gamma = e_i - e_j \in \mathcal{S}$ with i minimal, so that $j < n$ by assumption. Then setting $\beta = e_i - e_{j-1} \in \Sigma_{2k-2}$ we see that $\alpha + \beta \in \Sigma$ only for $\alpha = \alpha_{i-1}$ or α_j. However, $\alpha_{i-1} + \beta = e_{i-1} - e_{j-1} \notin \mathcal{S}$ and $\alpha_j + \beta = \gamma$.

(ii) *Suppose that $e_i + e_j \in \mathcal{S}$ for some $i < j$ with $i < 2k - n - 1$.* Choose $\gamma = e_i + e_j \in \mathcal{S}$ with i minimal, so that $j < n$ by assumption. Then setting $\beta = e_i + e_{j-1} \in \Sigma_{2k-2}$ we see that $\alpha + \beta \in \Sigma$ only for $\alpha = \alpha_{i-1}$ or α_j. However, $\alpha_{i-1} + \beta = e_{i-1} + e_{j-1} \notin \mathcal{S}$ and $\alpha_j + \beta = \gamma$.

(iii) *No roots of type* (i) *or* (ii). In this case we see that $\mathcal{S}$ can be identified with a proper subset of Σ_{2k-1} for D_{2k}. The result then follows from Lemma B.6. $\square$

F_4:

Lemma B.9. *Let $\Sigma = F_4$ and $\mathcal{S}$ be a non-empty subset of Σ_i for some $i > 1$, with the exception of $\mathcal{S} = \Sigma_4$. Then there is a $\beta \in \Sigma_{i-1}$ such that $\gamma - \beta \in \Pi$ for exactly one $\gamma \in \mathcal{S}$.*

Proof. We order the roots in Σ_i so that for each $\gamma \in \Sigma_i$ we can find a $\beta \in \Sigma_{i-1}$ with $\gamma - \beta \in \Pi$, but $\gamma' - \beta \notin \Pi$ for any smaller $\gamma' \in \Sigma_i$. Thus, given a set $\mathcal{S} \subseteq \Sigma_i$ we need only choose $\beta \in \Sigma_{i-1}$ corresponding to the maximal root $\gamma \in \mathcal{S}$.

Table B.1 gives the roots in Σ_i in decreasing order and the choice of β in each case for $i = 2, 3, 5, 6$ and 7. Thus, for example if $\mathcal{S} = \{e_{+-+-}, e_3 + e_4\} \subset \Sigma_3$ then $\gamma = e_{+-+-}$ is the maximal element of $\mathcal{S}$ and so $\beta = e_{+--+}$ is the required root in Σ_2.

$\mathrm{ht}(\gamma)$	2		
γ	$e_2 - e_4$	e_3	e_{+--+}
β	$e_2 - e_3$	$e_3 - e_4$	e_{+---}

$\mathrm{ht}(\gamma)$	3		
γ	e_2	e_{+-+-}	$e_3 + e_4$
β	$e_2 - e_4$	e_{+--+}	e_3

$\mathrm{ht}(\gamma)$	5		
γ	e_{++-+}	$e_2 + e_3$	$e_1 - e_2$
β	e_{++--}	$e_2 + e_4$	e_{+-++}

$\mathrm{ht}(\gamma)$	6	
γ	$e_1 - e_3$	e_{+++-}
β	$e_1 - e_2$	e_{+-++}

$\mathrm{ht}(\gamma)$	7	
γ	$e_1 - e_4$	e_{++++}
β	$e_1 - e_3$	e_{+++-}

Table B.1: The roots Σ_2, Σ_3, Σ_5, Σ_6 and Σ_7 for $\Sigma = F_4$

Note that there is exactly one root of each height 8 to 11 and so the result is then trivially true. Thus, it remains only to show that if $\mathcal{S}$ is strictly contained in Σ_4 then there is a choice of β. Table B.2 again gives the roots in $\Sigma_4 - \{\gamma'\}$ in decreasing order for each $\gamma' \in \Sigma_4$ and the choice of β in each case. $\square$

γ'	e_{++--}	
γ	$e_2 + e_4$	e_{+-++}
β	e_2	e_{+-+-}

γ'	$e_2 + e_4$	
γ	e_{++--}	e_{+-++}
β	e_2	e_{+-+-}

γ'	e_{+-++}	
γ	e_{++--}	$e_2 + e_4$
β	e_{+-+-}	$e_3 + e_4$

Table B.2: The roots in $\Sigma_4 - \{\gamma'\}$ for $\Sigma = F_4$

E_7, E_6, E_8:

Lemma B.10. *Let $\Sigma = E_7$, E_6, or E_8 and $\mathcal{S}$ be a non-empty subset of Σ_i for some $i > 1$, with the exception of the following cases:*

(i) $\Sigma = E_6$, E_7 *or* E_8 *and* $\mathcal{S} = \{e_6 + e_7, e_5 - e_8, e_5 + e_8\} \subseteq \Sigma_3$;

(ii) $\Sigma = E_6$, E_7 *or* E_8 *and* $\mathcal{S} = \Sigma_4 \cap E_6$.

(iii) $\Sigma = E_7$, *or* E_8 *and* $\mathcal{S} = \{e_3 - e_8, e_4 + e_7, e_5 + e_6, e_3 + e_8\} \subseteq \Sigma_5$;

(iv) $\Sigma = E_8$ *and* $\mathcal{S} = \Sigma_6$;

(v) $\Sigma = E_7$ *or* E_8 *and* $\mathcal{S} = \Sigma_9 \cap E_7$.

(vi) $\Sigma = E_8$ *and* $\mathcal{S} = \Sigma_{10}$;

(vii) $\Sigma = E_8$ *and* $\mathcal{S} = \Sigma_{15}$.

Then there is a $\beta \in \Sigma_{i-1}$ such that $\gamma - \beta \in \Pi$ for exactly one $\gamma \in \mathcal{S}$.

Proof. This can be shown in the same way as the proof of Lemma B.9 by explicitly describing which β to choose in any given situation. The details are omitted. □

Note that if we let Σ' denote the subsystem spanned by $\{\alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ in $\Sigma = E_6$, E_7 or E_8 then $\mathcal{S} = \{e_6 + e_7, e_5 - e_8, e_5 + e_8\}$ is the set of roots of height 3 in Σ'. Further, Σ' is equivalent to D_4 under the identification $\alpha'_1 = \alpha_5$, $\alpha'_2 = \alpha_4$, $\alpha'_3 = \alpha_3$ and $\alpha'_4 = \alpha_2$.

Similarly, if Σ'' is the subset of $\Sigma = E_7$ or E_8 spanned by $\{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}$ then Σ'' is equivalent to D_6 with $\mathcal{S} = \{e_3 - e_8, e_4 + e_7, e_5 + e_6, e_3 + e_8\}$ as the set of roots of height 5.

Appendix C

Proof of Lemma 6.4.9

To simplify notation, we replace the set $\mathcal{S}$ of negative roots with the corresponding set of positive roots and then write $\mathcal{S} = \{\gamma_1, \ldots, \gamma_m\}$, $\Sigma_{-a-1} = \{\beta_1, \ldots, \beta_{m'}\}$ and $\Pi = \{\alpha_1, \ldots, \alpha_n\}$. Further, we choose $r'_j \in R^\times$ with $r_{\gamma_j} = \pi^i r_j$ and $s'_j \in R$ with $s_{\beta_j} = \pi^{\ell-i-1} s'_j$ for each j. Thus, if for each l we set

$$t'_l = \sum_{j,k} c_{1,1,\beta_j,-\gamma_k}(-s'_j) r_k$$

where the sum runs over all j and k with $\gamma_k - \beta_j = \alpha_l$, then $t_{-\alpha_l} = \pi^{\ell-1} t'_l$. Hence we wish to find an l so that for every $r \in R$ we can choose the s'_j in such a way that $t'_l = r$ and $t'_k = 0$ for $k \neq l$.

In particular, we need to know the constants $c_{1,1,\beta_j,-\gamma_k}$ for each j, k. From [2] we see that $c_{1,1,\beta,-\gamma} = N_{\beta,-\gamma}$ and

$$\frac{N_{\beta,-\gamma}}{(\alpha,\alpha)} = \frac{N_{\alpha,\beta}}{(\gamma,\gamma)}$$

where $\gamma - \beta = \alpha$. The structure constants $N_{\alpha,\beta}$ for $\Sigma = D_{2n}$ were then obtained using the explicit description of the simple Lie algebra of type D_{2n} contained in [2], for $\Sigma = F_4, E_6$ and E_7 they were taken from [14], and for $\Sigma = E_8$ they were calculated using the GAP computer algebra package [12].

$\mathcal{S} = \Sigma_{2n-1}$ in D_{2n}

As in the proof of Lemma B.6, set $\mathcal{S} = \Sigma_{2n-1} = \{\gamma_0, \gamma_1, \ldots, \gamma_n\}$ where

$$\gamma_0 = e_1 - e_{2n}, \quad \gamma_i = e_i + e_{2n-i+1} \quad \text{for } 1 \leq i \leq n$$

and $\Sigma_{2n-2} = \{\beta_0, \beta_1, \ldots, \beta_n\}$ with

$$\beta_0 = e_2 - e_{2n}, \quad \beta_1 = e_1 - e_{n-1}, \quad \beta_i = e_i + e_{2n-i+2} \quad \text{for } 2 \leq i \leq n.$$

The different ways of expressing each $\gamma \in \mathcal{S}$ as the sum of a simple root α and a root β of height $2n-2$ are given in the proof of Lemma B.6 and the corresponding non-zero structure constants are

$$\begin{aligned}
&N_{\alpha_1,\beta_0} = 1, \qquad N_{\alpha_1,\beta_2} = 1, \quad N_{\alpha_{2n-1},\beta_1} = -1,\\
&N_{\alpha_{2n-1},\beta_2} = 1, \quad N_{\alpha_{2n},\beta_0} = 1, \quad N_{\alpha_{2n},\beta_1} = -1,\\
&N_{\alpha_i,\beta_{i+1}} = 1 \quad \text{for } 2 \leq i \leq n-1,\\
&N_{\alpha_i,\beta_{2n-i+1}} = 1 \quad \text{for } n+1 \leq i \leq 2n-2.
\end{aligned}$$

Consequently, we see that

$$\begin{aligned}
&t'_1 = -r'_0 s'_0 - r'_1 s'_1, \quad t'_{2n-1} = -r'_2 s'_2 + r'_0 s'_1, \quad t'_{2n} = r'_1 s'_2 - r'_2 s'_0,\\
&t'_i = -r'_i s'_{i+1} \quad \text{for } 2 \leq i \leq n-1, \qquad t'_n = 0,\\
&t'_i = -r'_{2n-i+1} s'_{2n-i+1} \quad \text{for } n+1 \leq i \leq 2n-2.
\end{aligned}$$

Thus, if we set

$$s'_0 = -\tfrac{1}{2}(r'_0)^{-1} r', \quad s'_1 = -\tfrac{1}{2}(r'_0)^{-1}(r_1)^{-1} r'_2 r', \quad s'_2 = -\tfrac{1}{2}(r'_1)^{-1} r'$$

and $s'_i = 0$ for $3 \leq i \leq n$ then we obtain $t'_1 = r'$ and $t'_2 = t'_3 = \cdots = t'_{2n} = 0$.

$\mathcal{S} = \Sigma_4$ in F_4

Consider $\mathcal{S} = \{\gamma_1, \gamma_2, \gamma_3\} = \Sigma_4$ where

$$\gamma_1 = e_{+-++}, \quad \gamma_2 = e_{++--}, \quad \gamma_3 = e_2 + e_4.$$

Further, if we set $\Sigma_3 = \{\beta_1, \beta_2, \beta_3\}$ where

$$\beta_1 = e_2, \quad \beta_2 = e_{+-+-}, \quad \beta_3 = e_3 + e_4$$

then the decompositions of each $\gamma \in \mathcal{S}$ into the sum of a simple root $\alpha \in \Pi$ and a root $\beta \in \Sigma_3$ with the corresponding structure constants $N_{\alpha,\beta}$ are given in Table C.1.

$+$	β_1	β_2	β_3
α_1		γ_2	γ_3
α_2			
α_3	γ_3	γ_1	
α_4	γ_2		γ_1

N	β_1	β_2	β_3
α_1	0	1	1
α_2	0	0	0
α_3	-2	-1	0
α_4	-1	0	-1

Table C.1: The decompositions and structure constants for $\mathcal{S} = \Sigma_4$ in F_4

Consequently,

$$t_1' = -2r_2's_2' - r_3's_3' \quad t_2' = 0, \quad t_3' = r_3's_1' + r_1's_2', \quad t_4' = r_2's_1' + r_1's_3'$$

and so if we let

$$s_1' = \tfrac{1}{3}r_1'(r_2')^{-1}(r_3')^{-1}r', \quad s_2' = -\tfrac{1}{3}(r_2')^{-1}r', \quad s_3' = -\tfrac{1}{3}(r_3')^{-1}r'$$

then we obtain $t_1' = r'$ and $t_2' = t_3' = t_4' = 0$.

$\mathcal{S} = \Sigma_4$ in E_6

Consider $\mathcal{S} = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5\} = \Sigma_4$ where

$$\gamma_1 = e_{+----+++}, \quad \gamma_2 = e_{+---+---}, \quad \gamma_3 = e_5 + e_7, \quad \gamma_4 = e_4 + e_8, \quad \gamma_5 = e_4 - e_8.$$

Further, if we set

$$\beta_1 = e_{+----+--}, \quad \beta_2 = e_6 + e_7, \quad \beta_3 = e_5 + e_8, \quad \beta_4 = e_5 - e_8, \quad \beta_5 = e_4 - e_7$$

then the decompositions of each $\gamma \in \mathcal{S}$ into the sum of a simple root $\alpha \in \Pi$ and a root $\beta \in \Sigma_3$ with the corresponding structure constants $N_{\alpha,\beta}$ are given in Table C.2.

$+$	β_1	β_2	β_3	β_4	β_5
α_1		γ_1		γ_2	
α_2	γ_1			γ_3	γ_4
α_3			γ_3		γ_5
α_4					
α_5	γ_2	γ_3			
α_6			γ_4	γ_5	

N	β_1	β_2	β_3	β_4	β_5
α_1	0	1	0	1	0
α_2	1	0	0	1	1
α_3	0	0	1	0	1
α_4	0	0	0	0	0
α_5	-1	-1	0	0	0
α_6	0	0	-1	-1	0

Table C.2: The decompositions and structure constants for $\mathcal{S} = \Sigma_4$ in E_6

Consequently

$$t'_1 = -r'_1 s'_2 - r'_2 s'_4, \quad t'_2 = -r'_1 s'_1 - r'_3 s'_4 - r'_4 s'_5, \quad t'_3 = -r'_3 s'_3 - r'_5 s'_5,$$
$$t'_4 = 0, \qquad t'_5 = r'_2 s'_1 + r'_3 s'_2, \qquad t'_6 = r'_4 s'_3 + r'_5 s'_4$$

and so if we let

$$s'_1 = \tfrac{2}{3}(r'_1)^{-1}(r'_2)^{-1} r'_3 r', \quad s'_2 = -\tfrac{2}{3}(r'_1)^{-1} r', \qquad s'_3 = \tfrac{1}{3}(r'_2)^{-1}(r'_4)^{-1} r'_5 r',$$
$$s'_4 = -\tfrac{1}{3}(r'_2)^{-1} r', \qquad s'_5 = -\tfrac{1}{3}(r'_2)^{-1} r'_3 (r'_4)^{-1} r'$$

then we obtain $t'_1 = r'$ and $t'_2 = t'_3 = t'_4 = t'_5 = t'_6 = 0$.

$\mathcal{S} = \Sigma_9$ in E_7

Consider $\mathcal{S} = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\} = \Sigma_9$ where

$$\gamma_1 = e_{+-+--++-}, \quad \gamma_2 = e_{+-+-+--+}, \quad \gamma_3 = e_{+--++-+-}, \quad \gamma_4 = e_3 + e_4 \ .$$

If we set $\Sigma_8 = \{\beta_1, \beta_2, \beta_3, \beta_4\}$ where

$$\beta_1 = e_{+-+--+-+}, \quad \beta_2 = e_{+--+-++-}, \quad \beta_3 = e_3 + e_5, \quad \beta_4 = e_{+--++--+}$$

then the decompositions of each $\gamma \in \mathcal{S}$ into the sum of a simple root $\alpha \in \Pi$ and a root $\beta \in \Sigma_8$ with the corresponding structure constants $N_{\alpha,\beta}$ are given in Table C.3.

$+$	β_1	β_2	β_3	β_4
α_1			γ_2	
α_2				
α_3	γ_1			γ_3
α_4				
α_5	γ_2	γ_3		
α_6			γ_4	
α_7		γ_1		γ_2

N	β_1	β_2	β_3	β_4
α_1	0	0	1	0
α_2	0	0	0	0
α_3	1	0	0	1
α_4	0	0	0	0
α_5	1	1	0	0
α_6	0	0	1	0
α_7	0	-1	0	-1

Table C.3: The decompositions and structure constants for $\mathcal{S} = \Sigma_9$ in E_7

Consequently,

$$t'_1 = -r'_2 s'_3, \quad t'_2 = 0, \quad t'_3 = -r'_1 s'_1 - r'_3 s'_4, \quad t'_4 = 0,$$
$$t'_5 = -r'_2 s'_1 - r'_3 s'_2, \quad t'_6 = -r'_4 s'_3, \quad t'_7 = r'_1 s'_2 + r_2 s'_4$$

and so if we let

$$s'_1 = -\tfrac{1}{2}(r'_1)^{-1} r', \quad s'_2 = \tfrac{1}{2}(r'_1)^{-1} r'_2 (r'_3)^{-1} r', \quad s'_3 = 0, \quad s'_4 = -\tfrac{1}{2}(r'_3)^{-1} r'$$

then we obtain $t'_3 = r'$ and $t'_1 = t'_2 = t'_4 = t'_5 = t'_6 = t'_7 = 0$.

$\mathcal{S} = \Sigma_6$ in E_8

Consider $\mathcal{S} = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7\} = \Sigma_6$ where

$$\gamma_1 = e_{+---++-+}, \quad \gamma_2 = e_{+---++-+}, \quad \gamma_3 = e_{+-+-----}, \quad \gamma_4 = e_4 + e_6,$$
$$\gamma_5 = e_3 + e_7, \quad \gamma_6 = e_2 + e_8, \quad \gamma_7 = e_2 - e_8.$$

Further, if we set $\Sigma_5 = \{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6\}$ where

$$\beta_1 = e_{+----+-++}, \quad \beta_2 = e_{+--+----}, \quad \beta_3 = e_5 + e_6, \quad \beta_4 = e_4 + e_7,$$
$$\beta_5 = e_3 + e_8, \qquad \beta_6 = e_3 - e_8, \qquad \beta_7 = e_2 - e_7$$

then the decompositions of each $\gamma \in \mathcal{S}$ into the sum of a simple root $\alpha \in \Pi$ and a root $\beta \in \Sigma_5$ with the corresponding structure constants $N_{\alpha,\beta}$ are given in Table C.4.

$+$	β_1	β_2	β_3	β_4	β_5	β_6	β_7
α_1			γ_1	γ_2		γ_3	
α_2		γ_2				γ_5	γ_6
α_3					γ_5		γ_7
α_4	γ_1			γ_4			
α_5							
α_6	γ_2		γ_4				
α_7		γ_3		γ_5			
α_8					γ_6	γ_7	

N	β_1	β_2	β_3	β_4	β_5	β_6	β_7
α_1	0	0	-1	-1	0	-1	0
α_2	0	-1	0	0	0	-1	-1
α_3	0	0	0	0	-1	0	-1
α_4	1	0	0	1	0	0	0
α_5	0	0	0	0	0	0	0
α_6	1	0	1	0	0	0	0
α_7	0	1	0	1	0	0	0
α_8	0	0	0	0	1	1	0

Table C.4: The decompositions and structure constants for $\mathcal{S} = \Sigma_6$ in E_8

Consequently,

$$t'_1 = r'_1 s'_3 + r'_2 s'_4 + r'_3 s'_6, \quad t'_2 = r'_2 s'_2 + r'_5 s'_6 + r'_6 s'_7, \quad t'_3 = r'_5 s'_5 + r'_7 s'_7,$$
$$t'_4 = -r'_1 s'_1 - r'_4 s'_4, \qquad t'_5 = 0, \qquad t'_6 = -r'_2 s'_1 - r'_4 s'_3,$$
$$t'_7 = -r'_3 s'_2 - r'_5 s'_4, \qquad t'_8 = -r'_6 s'_5 - r'_7 s'_6$$

and so if we let

$$s'_1 = -\tfrac{2}{5}(r'_1)^{-1}(r'_2)^{-1}r'_4 r', \quad s'_2 = -\tfrac{2}{5}(r'_2)^{-1}(r'_3)^{-1}r'_5 r', \quad s'_3 = \tfrac{2}{5}(r'_1)^{-1}r',$$
$$s'_4 = \tfrac{2}{5}(r'_2)^{-1}r', \qquad s'_5 = -\tfrac{1}{5}(r'_3)^{-1}(r'_6)^{-1}r'_7 r', \quad s'_6 = \tfrac{1}{5}(r'_3)^{-1}r',$$
$$s'_7 = \tfrac{1}{5}(r'_3)^{-1}r'_5(r'_6)^{-1}r'$$

then we obtain $t'_1 = r'$ and $t'_2 = t'_3 = t'_4 = t'_5 = t'_6 = t'_7 = 0$.

$\mathcal{S} = \Sigma_{10}$ in E_8

Consider the set $\mathcal{S} = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6\} = \Sigma_6$ where

$$\gamma_1 = e_{+--+++--}, \quad \gamma_2 = e_{+-+-+-+-}, \quad \gamma_3 = e_{++---++-},$$
$$\gamma_4 = e_{+-++---+}, \quad \gamma_5 = e_{++--+--+} \text{ and } \gamma_6 = e_2 + e_4.$$

Further, if we set $\Sigma_9 = \{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6\}$ where

$$\beta_1 = e_{+--++-+-}, \quad \beta_2 = e_{+-+--++-}, \quad \beta_3 = e_{+-+-+--+},$$
$$\beta_4 = e_{++---+-+}, \quad \beta_5 = e_3 + e_4 \text{ and } \beta_6 = e_2 + e_5$$

then the decompositions of each $\gamma \in \mathcal{S}$ into the sum of a simple root $\alpha \in \Pi$ and a root $\beta \in \Sigma_9$ with the corresponding structure constants $N_{\alpha,\beta}$ are given in Table C.5.

$+$	β_1	β_2	β_3	β_4	β_5	β_6
α_1					γ_4	γ_5
α_2						
α_3			γ_2	γ_3		
α_4	γ_1					
α_5		γ_2		γ_5		
α_6			γ_4			γ_6
α_7	γ_2					
α_8		γ_3	γ_5		γ_6	

N	β_1	β_2	β_3	β_4	β_5	β_6
α_1	0	0	0	0	-1	-1
α_2	0	0	0	0	0	0
α_3	0	0	-1	-1	0	0
α_4	1	0	0	0	0	0
α_5	0	1	0	1	0	0
α_6	0	0	1	0	0	1
α_7	1	0	0	0	0	0
α_8	0	1	1	0	1	0

Table C.5: The decompositions and structure constants for $\mathcal{S} = \Sigma_{10}$ in E_8

Consequently,

$$t'_1 = r'_4 s'_5 + r'_5 s'_6, \quad t'_2 = 0, \quad t'_3 = r'_2 s'_3 + r'_3 s'_4,$$
$$t'_4 = -r'_1 s'_1, \quad t'_5 = -r'_2 s'_2 - r'_5 s'_4, \quad t'_6 = -r'_4 s'_3 - r'_6 s'_6,$$
$$t'_7 = -r'_2 s'_1, \quad t'_8 = -r'_3 s'_2 - r'_5 s'_3 - r'_6 s'_5$$

and so if we let

$$s'_1 = 0, \qquad s'_2 = -\tfrac{1}{3}(r'_3)^{-1}(r'_4)^{-1}r'_6 r',$$
$$s'_3 = -\tfrac{1}{3}(r'_4)^{-1}(r'_5)^{-1}r'_6 r', \quad s'_4 = \tfrac{1}{3}(r'_3)^{-1}r'_2(r'_4)^{-1}(r'_5)^{-1}r'_6 r',$$
$$s'_5 = \tfrac{2}{3}(r'_4)^{-1}r', \qquad s'_6 = \tfrac{1}{3}(r'_5)^{-1}r'$$

then we obtain $t'_1 = r'$ and $t'_2 = t'_3 = t'_4 = t'_5 = t'_6 = t'_7 = t'_8 = 0$.

$\mathcal{S} = \Sigma_{15}$ in E_8

Consider $\mathcal{S} = \{\gamma_1, \gamma_2, \gamma_3\} = \Sigma_{15}$ where

$$\gamma_1 = e_{+-++++-+}, \quad \gamma_2 = e_{++-++-++}, \quad \gamma_3 = e_{+++--+++}, \quad \gamma_4 = e_{+++-+---}.$$

Further, if we set $\Sigma_{14} = \{\beta_1, \beta_2, \beta_3, \beta_4\}$ where

$$\beta_1 = e_{+-+++-++}, \quad \beta_2 = e_{++-+-+++}, \quad \beta_3 = e_{++-++---}, \quad \beta_4 = e_{+++--+--}$$

then the decompositions of each $\gamma \in \mathcal{S}$ into the sum of a simple root $\alpha \in \Pi$ and a root $\beta \in \Sigma_{14}$ with the corresponding structure constants $N_{\alpha,\beta}$ are given in Table C.6.

$+$	β_1	β_2	β_3	β_4
α_1				
α_2			γ_2	γ_3
α_3				
α_4	γ_1			
α_5		γ_2		γ_4
α_6				
α_7		γ_3	γ_4	
α_8	γ_2			

N	β_1	β_2	β_3	β_4
α_1	0	0	0	0
α_2	0	0	1	1
α_3	0	0	0	0
α_4	1	0	0	0
α_5	0	1	0	1
α_6	0	0	0	0
α_7	0	1	1	0
α_8	1	0	0	0

Table C.6: The decompositions and structure constants for $\mathcal{S} = \Sigma_{15}$ in E_8

Consequently,

$$t_1' = 0, \qquad t_2' = -r_2's_3' - r_3's_4', \quad t_3' = 0, \qquad t_4' = -r_1's_1',$$
$$t_5' = -r_2's_2' - r_4's_4', \quad t_6' = 0, \qquad t_7' = -r_3's_2' - r_4's_3', \quad t_8' = -r_2's_1'$$

and so if we let

$$s_1' = 0, \quad s_2' = \tfrac{1}{2}(r_2')^{-1}(r_3')^{-1}r_4'r', \quad s_3' = -\tfrac{1}{2}(r_2')^{-1}r', \quad s_4' = -\tfrac{1}{2}(r_3')^{-1}r'$$

then we obtain $t_2' = r'$ and $t_1' = t_3' = t_4' = t_5' = t_6' = t_7' = t_8' = 0$.

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