Gravitational and electromagnetic field of static and ultrarelativistic objects in nonlocal theory

by

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Abstract

This thesis studies solutions of nonlocal modifications of Maxwell and linearized gravity equations. We focus on a wide class of Lorentz invariant theories known as ghost free models in which a form factor of nonlocality is chosen so that no new additional unphysical degrees of freedom are present. Such form factors introduce a length scale ℓ that determines the range in which the effects of nonlocality are important. Using the Green function method we obtain solutions in such theories for stationary fields created by point-like and extended objects. By performing the boost transformations of the obtained stationary solutions and taking the Penrose limit we obtain solutions of the nonlocal theory describing the electromagnetic and gravitational field of ultrarelativistic objects. The key role in this derivation is played by a property of the factorization of the Green functions in the Penrose limit. The properties of electromagnetic and gravitational fields of ultrarelativistic objects are discussed and concrete examples are presented.

Preface

This thesis is an original work by Jose Pinedo Soto. All results were obtained by using the provided references and nothing else.

Chapters 2 - 4 are based on two research papers [1, 2] which are the result of collaboration with Prof. Valeri Frolov and Dr. Jens Boos. The author of this thesis was substantially involved in every step of the research process. Chapters 1 and 5 present the reader with a general context to understand the relevance of the thesis and give an outlook to future venues of research. Appendixes A to F are complementary information to their corresponding section or chapter.

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Chapter 1

Introduction

The fundamental theories of theoretical physics such as General Relativity and Maxwell theories are local. In particular, this means that they can be derived from an action S which is of the form

$$S = \int dx L \tag{1.1}$$

were the Lagrangian density L is a function of the field and a finite number of its derivatives calculated at a given point x. In such theories a point-like particle "feels" the field with which it interacts only at the place where it is located. Another important feature of these theories is their local Lorentz invariance. As a result their action S as well as their Lagrangian density L are scalars under coordinate transformations.

However, there exist a long-standing problem with these theories. A static field of a point-like object in the Maxwell theory as well as in linearized gravity is divergent at the origin. This results in the infinity of self-energy of such sources. This classical singularity problem is a manifestation of similar divergences present in their quantum counterparts. In General Relativity this problem is "reincarnated" in a well known hassle: the generic and inevitable existence of singularities in cosmology and inside black holes. One says that the standard General Relativity is ultraviolet (UV) incomplete and it requires a modification in the regime where the spacetime curvature becomes very large.

There are a lot of different modifications of General Relativity which were proposed in order to solve the singularity problem and to “cure” this fundamental “disease” of General Relativity. One interesting and promising approach is using nonlocal modifications of gravity theory.

The idea of nonlocality in physics is quite old. It has been explored for quite some time [3–11]. More recently the interest in nonlocal theory greatly increased. This is mainly motivated by the development of string theory [12–15]. The main focus of this thesis is the study of the so-called infinite derivative theories. This is a proposed modification of the local equations that preserves the local Lorentz invariance of the theory. At the linearized level the standard $\Box$ operator is changed to $f(\Box)\Box$, where a nonlocal form factor $f(z)$ is chosen such that it does not vanish in the complex plane of z, and hence it has a unique inverse. As a result, no new unphysical degrees of freedom are present, at least at tree level. For this reason, such nonlocal theories are sometimes refereed to as “ghost-free” [16, 17]. Recently, there has been substantial activity devoted to study of nonlocal generalizations of General Relativity. The main motivation behind this study is an attempt to solve the long standing problems of General Relativity: cosmological and black hole singularities.

The nonlocal theories of gravity have appealing UV properties [18, 19]. Linearized solutions of the nonlocal ghost-free gravity equations for stationary objects were derived and discussed in many publications (see Refs. [20, 21] and references therein). It has been demonstrated that in the weak-field regime this class of theories regularizes the gravitational field of point-like sources [21–24] as well as thin brane-like extended objects [20, 25, 26]. Paper [27] contains a nice summary of these results. Discussion of the nonlocal gravity models in the strong-field regime in connection with black holes can be found in [28–32] and in references therein; for cosmological applications see [33, 34]. Nonlocal infinite-derivative form factors have also been explored in quantum theory [35, 36] as well as quantum field theory [37–43].

It should be emphasized that most of the publications devoted to study of the solutions of the nonlocal linearized gravity equations are focused on the four-dimensional theories with static sources. In our work [1] we obtained a far-going generalization of these results. Namely,

- We studied nonlocal linearized gravity equations in arbitrary number of space-time dimensions;
- We discussed wide class of so-called GF_N theories, where form factor $f(\Box)$ is the exponential of the N-th power of the $\Box$ operator;
- We obtained stationary solutions for sources which besides mass also have spin.

Chapter 3 of the thesis, which is based on the paper [1], contains discussion of these results.

It is interesting that while there are many publications devoted to the linearized nonlocal gravity, the subject of nonlocal modifications of the Maxwell equations remained in the "shadows". However, there exists a well known similarity between the Maxwell equations with those of linearized gravity. To fill this "gap" we studied a class of higher dimensional Lorentz invariant nonlocal generalizations of the Maxwell theory, which (at least at tree level) do not contain unphysical extra degrees of freedom and in this sense they are "ghost-free". Construction of such theories and their properties are discussed in our work [2]. Chapter 2 of this thesis is based on these results.

If one obtains a solution of the field equations for a stationary source in the frame of reference where this source is at rest (rest frame) one can easily find how these solutions look like in a frame which moves with a constant velocity with respect to the rest frame. This is possible whenever one is dealing with a Lorentz invariant theory. In the Maxwell theory the electric field of a point charge is spherically symmetric and its equipotential surfaces are spheres. If such a charge moves with a constant velocity with respect to an observer then its equipotential surfaces are squeezed and take on an elliptic form, which is a direct result of relativistic Lorentz contraction. In the limit when the velocity of the charge tends to the speed of light, this squeezing is so strong that the field of the charge is practically confined to a null plane and becomes similar to a plane wave (see Ref. [44] and references therein).

Similar effects have been studied in gravity and are well known, in fact, the study of the gravitational field of ultrarelativistic particles and beams of light is a very old subject. The first solution describing the gravitational field of beams of light ("pencils") was found by Tolman, Ehrenfest and Podolski in 1931 [45]. These authors used a linear approximation of the Einstein equations. One of their main conclusions was that the gravitational force acting on a massless particle moving in the same direction as the beam of light vanishes. Later, Bonnor [46] presented a solution for the gravitational field produced by a cylindrical beam of a null fluid. This model can be interpreted as a description of a high frequency light beam in the geometric optics approximation when diffraction effects are neglected.[1] The gravitational field of a spinning pencil of light was obtained by Bonnor in 1970 [48], see also Refs. [49, 50]. Higher-dimensional solutions describing the gravitational field of spinning ultrarelativistic objects and light beams were obtained in [51, 52]. The latter work introduced the name "gyraton" for such spinning ultrarelativistic objects, which is now used in the literature quite frequently. There exist different generalizations of standard gyraton solutions, such as solutions for charged gyratons [53], gyratons in asymptotically AdS spacetimes [54], in a generalized Melvin universe with cosmological constant [55], and string gyratons in supergravity [56]. Gyraton solutions of the Einstein equations belong to the wide class of so-called Kundt metrics [57]. A comprehensive discussion of gyratons in the Robinson–Trautman and Kundt classes of metrics can be found in [58–61].

There is another problem that has been widely discussed in the literature and which is closely related to gyratons. In 1970, Aichelburg and Sexl [62] constructed a

[1] More recent studies of light beams beyond the geometric optics approximation can be found in [47] and references therein.

metric of a massive ultrarelativistic particle. In its rest frame, the gravitational field of such a particle of mass m is described by the Schwarzschild metric. In order to obtain the metric when this particle moves with a very high velocity they applied a boost transformation and considered the limit where the velocity of the object tends to the speed of light, and hence the Lorentz factor γ diverges. They demonstrated that keeping the value of the energy $E = \gamma m$ fixed yields a limiting metric which is now called the Aichelburg–Sexl solution. For this solution the gravitational field of a particle is localized at the null plane tangent to the null vector of the particle's four-velocity. Later, Penrose [63] demonstrated that this is a generic property of any metric that is boosted to the speed of light, provided the corresponding energy is kept fixed, and this special limiting case has hence been dubbed as the "Penrose limit". Aichelburg–Sexl-type metrics have been widely used for the study of the gravitational interaction of two ultrarelativistic particles as well as black hole production via their collision. The area of the apparent horizon in this process just before the moment of collision was calculated in [64] and has been widely used for estimating black hole formation cross sections in the collision of ultrarelativistic particles (see e.g. [65–69] and references therein).

Since in the Penrose limit the initial mass m of the particle tends to zero, one can obtain the Aichelburg–Sexl metric by starting with a linearized, weak-field gravity solution for a point-like particle. By considering a superposition of such solutions it is easy to construct the gravitational field of extended objects in linearized gravity. In particular, one may consider first a line distribution of mass, and then boost the solution. Due to the Lorentz contraction in the direction of motion the visible size of the body in this direction shrinks. This means that in order to obtain a solution for the ultrarelativistic case featuring a finite energy distribution profile one needs not only to take the Penrose limit keeping γm constant, but also simultaneously keep the parameter L/γ fixed, where L is the size of the object in the direction of motion as measured on the rest frame. Such a procedure can be applied to a spinning object provided the rotation takes place within the plane orthogonal to the direction of motion. One can show that in such a procedure one reconstructs the linearized gravitational field of a gyraton. This method is described in details in chapter 5 of the book [70].

The main motivation for the study of the fields of ultrarelativistic sources in nonlocal Lorentz invariant gravity is the hope to understand how a small scale modification of gravity might become important for the process of mini black hole formation in the collision of ultrarelativistic particles. For example, it was shown that if the Einstein–Hilbert action is modified by the inclusion of higher-derivative as well as infinite-derivative terms, there exists a mass gap for black hole formation [71–74].

Electromagnetic and linearized gravitational fields of ultrarelativistic objects in the "ghost-free" higher dimensional nonlocal theories were studied in our works [1] and [2]. Solutions of the equations for such sources are discussed in chapter 4 of the thesis which is based on these publications.

1.1 Thesis overview

The thesis is organized as follows. In chapter 2 we start by constructing stationary solutions for point like particles and charged magnetized "pencils" in Maxwell theory. After this we introduce a higher dimensional nonlocal generalization of such a theory. Using the Green function method we construct solutions for the same type of sources in this modified theory. Chapter 3 follows the spirit of the previous chapter by presenting linearized Einstein theory and its generalization for an arbitrary number of dimensions. Solutions for stationary sources are discussed for this theory. We then present a nonlocal action for the linearized gravity and using the Green function method obtain stationary solutions in this theory. In chapter 4 we "boost" the obtained stationary solutions and after taking their Penrose limit we construct solutions describing the field of the ultrarelativistic sources of the nonlocal Maxwell and linearized gravity equations. Properties of these solutions are explored. The last chapter 5 contains summary and discussion of the results presented in the thesis. Several appendices included at the end of the thesis contain details of the calculations and additional useful information.

Chapter 2

Maxwell Equations and Their Nonlocal Modification

2.1 Maxwell theory

2.1.1 Equations

In this chapter we discuss nonlocal generalizations of the Maxwell equations and their solutions. These generalizations will be introduced in section 2.3. We shall use a formulation for such a theory in an arbitrary number of spacetime dimensions. However, it is instructive at first to discuss the case of the standard four-dimensional Maxwell theory.

We assume that the background metric is flat such that using Cartesian coordinates $X^\mu = (t, \boldsymbol{x})$ we can write it in the form

$$ds^2 = \eta_{\mu\nu} dX^\mu dX^\nu = -dt^2 + \sum_{i=1}^{3} (dx^i)^2 \,. \tag{2.1}$$

However, in what follows we shall use not only the flat (Cartesian) coordinates but curved coordinates as well. For this purpose it is convenient to write the Maxwell equations in a general covariant form. We denote the metric by $g_{\mu\nu}$ then one has

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \,. \tag{2.2}$$

We shall also use the covariant derivatives which we denote by ∇_μ or by $(\ldots)_{;\mu}$. In Cartesian coordinates one has

$$\nabla_\mu = \frac{\partial}{\partial X^\mu} \,. \tag{2.3}$$

The Maxwell equations are obtained by varying the following action

$$S_M[A_\mu] = -\int \left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - A_\mu j^\mu\right)\sqrt{-g}\; d^4x\,, \tag{2.4}$$

over A_μ. Here $F_{\mu\nu}$ is the electromagnetic field strength tensor

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu\,. \tag{2.5}$$

In these definitions we have used Heaviside units and put the speed of light equal to unity, $c \equiv 1$. The variation of the action 2.4 gives

$$\nabla_\nu F^{\nu\mu} = \frac{1}{\sqrt{-g}}\partial_\nu(\sqrt{-g}F^{\nu\mu}) = j^\mu\,. \tag{2.6}$$

In the Cartesian coordinates these equations reduce to

$$\frac{\partial}{\partial X^\nu}F^{\nu\mu} = j^\mu\,. \tag{2.7}$$

After substituting the expression for the field strength $\boldsymbol{F}$ in terms of the vector potential $\boldsymbol{A}$ into (2.7) one gets

$$\Box A^\nu - \nabla^\nu\nabla_\mu A^\mu = j^\nu\,. \tag{2.8}$$

Maxwell equations are invariant under the gauge transformation $A_\mu \to A_\mu + \lambda_\mu$, where λ is an arbitrary function of coordinates. Using this freedom one can impose the following gauge fixing condition (“Lorenz gauge”) $\nabla_\mu A^\mu = 0$ which implies

$$\Box A_\mu = j_\mu\,. \tag{2.9}$$

2.1.2 Stationary field of point-like objects

The Maxwell equations are linear. This means that it is sufficient to find a solution for a stationary point-like source. The field of an extended object can be obtained by integrating such a solution over the volume occupied by the object with a proper weight representing the charge and magnetic moment density distributions.

Let $X^\mu = (t, \boldsymbol{x})$ be Cartesian coordinates. Let us consider the following stationary conserved current

$$j^\mu = q\delta^\mu_t \delta^{(3)}(\boldsymbol{x}) + \delta^\mu_i M^{ik}\partial_k \delta^{(3)}(\boldsymbol{x})\,. \tag{2.10}$$

Here, q is the charge of the point particle and $M_{ik} = -M_{ki}$ is a constant, antisymmetric matrix that parameterizes the particle's intrinsic magnetic moment. This current obeys the conservation law $\partial_\mu j^\mu = 0$.

We write the electromagnetic potential in the form

$$A_\mu(\boldsymbol{x}) = \delta^t_\mu \varphi(\boldsymbol{x}) + \delta^i_\mu A_i(\boldsymbol{x})\,, \tag{2.11}$$

and impose the Lorentz gauge condition on it $\partial_\mu A^\mu = 0$. Then the Maxwell equations take the form

$$\triangle\varphi = -q\delta^{(3)}(\boldsymbol{x})\,, \tag{2.12}$$

$$\triangle A_i = M_i{}^k \partial_k \delta^{(3)}(\boldsymbol{x})\,. \tag{2.13}$$

Here $\triangle$ is a flat Laplacian

$$\triangle = \sum_{i=1}^{3} \partial^2_{x_i}\,. \tag{2.14}$$

Let us denote by $G(\boldsymbol{x})$ the Green function of the Laplace operator

$$\triangle G(\boldsymbol{x}) = \delta^{(3)}(\boldsymbol{x})\,. \tag{2.15}$$

One has

$$G(\boldsymbol{x}) = \frac{1}{4\pi|\boldsymbol{x}|}\,. \tag{2.16}$$

Then solutions of the field equations (2.12) and (2.13) are

$$\varphi(\boldsymbol{x}) = q\, G(\boldsymbol{x})\,, \quad A_i(\boldsymbol{x}) = -M_i{}^k \partial_k G(\boldsymbol{x})\,. \tag{2.17}$$

The potential 1-form $\boldsymbol{A} = A_\mu \mathrm{d}X^\mu$ then takes the form

$$\begin{aligned} A_\mu \mathrm{d}x^\mu &= \varphi \mathrm{d}t + A_i \mathrm{d}x^i \\ &= q\, G(r)\mathrm{d}t - M_i{}^k \partial_k G(\boldsymbol{x})\mathrm{d}x^i\,. \end{aligned} \tag{2.18}$$

These general expressions can be simplified. Let us note that the three-dimensional matrix M_{ij} which enters the expression for the potential A_i is antisymmetric and it has constant coefficients. By using rigid three-dimensional rotations such an object can be written in the following canonical form (see appendix B)

$$\boldsymbol{M} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & m \\ 0 & -m & 0 \end{pmatrix} . \tag{2.19}$$

We denote the corresponding three-dimensional orthogonal coordinates (Darboux basis) as $(\xi, \vec{x}_\perp)$. Two coordinates $\vec{x}_\perp$ span the two-dimensional plane Π orthogonal to ξ-axes.

Consider an electric current loop in the plane Π with the center at a point $\xi = \vec{x}_\perp = 0$. If I is the current in the loop and S is its area then the dipole magnetic moment of the current loop is $m = IS$. The current for the point-like source which enters (2.10) can be obtained as a limit in which the radius of the current loop becomes infinitely small while the value of the dipole magnetic moment is kept constant.

2.2 Stationary electromagnetic field of extended objects

Since the Maxwell equations are linear their solutions for extended charged and/or magnetized objects can be obtained by superimposing the described solutions for the corresponding point-like objects. In other words, they can be written as integrals containing the Green functions multiplied by the currents which enter the right-hand side of the Maxwell equations. In many cases when the current distributions possess sufficient symmetry these integrals can be calculated in an explicit form (see e.g. [75]). In this section we present the expressions for the electromagnetic field for two special cases which will be used in our further discussion of the field of ultrarelativistic objects. Namely, we consider electrically charged and magnetized pencil-like objects. In both cases the transverse size of the pencil is infinitely small.

Later we consider two reference frames: one is the frame where the object is at rest and the other one in which the object moves with constant velocity. In order to distinguish them we denote the rest frame by $\bar{S}$ and the moving frame by S. For this reason from now on we denote the coordinates (and other objects) in the rest frame with a bar over them, and omit the bar when we shall be dealing with the moving frame. Hence, we denote the length of the pencil as measured in the rest frame by $\bar{L}$. We specify these coordinates $\bar{X}^\mu$ in the rest frame such that one of the spatial axes is directed along the linear extension of the pencil, and we denote this coordinate by $\bar{\xi}$, while the coordinates in the directions orthogonal to the pencil are labeled $\vec{x}_\perp$. Thus we have

$$\bar{X}^\mu = (\bar{t}, \bar{\xi}, \vec{x}_\perp)\,. \tag{2.20}$$

We choose the origin of the coordinate system such that the end points of the pencil are located at $\bar{\xi} = \pm\bar{L}/2$. In what follows, we shall boost the pencil in the $\bar{\xi}$-direction.

We consider two types of pencils. One is a uniformly charged pencil with a total electric charge $\bar{q}$, and the second type corresponds to a uniformly magnetized pencil with a total magnetic moment $\bar{m}$. To distinguish these cases we refer to them as q-pencil and m-pencil, respectively.

2.2.1 Field of a q-pencil in its rest frame

We start with the case of a q-pencil and assume that its charge density distribution is uniform. If $\bar{q}$ is the electric charge and $\bar{L}$ the length of the pencil, then its charge density is

$$\bar{\lambda} = \frac{\bar{q}}{\bar{L}}\delta^{(2)}(\vec{x}_\perp)\Theta(\bar{\xi}| - \bar{L}/2, \bar{L}/2)\,, \tag{2.21}$$

and the 4-current $\bar{j}^\mu$ takes the form

$$\bar{j}^\mu = \delta^\mu_{\bar{t}}\bar{\lambda}\,. \tag{2.22}$$

Here $\Theta(x|x_-,x_+) = \theta(x-x_-)\theta(x_+-x)$ is a step function equal to 1 in the interval (x_-,x_+) and zero outside it. In the Coulomb gauge $\partial_j A^j = 0$ we may choose the vector potential to be of the form[1]

$$\boldsymbol{A} \equiv \bar{A}_\mu \mathrm{d}\bar{X}^\mu = \bar{\phi}\mathrm{d}\bar{t}\,. \tag{2.23}$$

Solving the field equation for the potential $\bar{\phi}$,

$$\bar{\triangle}\bar{\phi} = -\bar{\lambda}\,, \tag{2.24}$$

one finds

$$\bar{\phi}(\bar{\xi},\rho) = \frac{\bar{q}}{4\pi\bar{L}}\int\limits_{-\bar{L}/2}^{\bar{L}/2}\frac{\mathrm{d}\bar{\xi}'}{\sqrt{(\bar{\xi}-\bar{\xi}')^2+\rho^2}}\,, \tag{2.25}$$

where $\rho = |\vec{x}_\perp|$. Taking the integral one obtains

$$\bar{\phi}(\bar{\xi},\rho) = \frac{\bar{q}}{4\pi\bar{L}}\ln\left(\frac{\bar{\xi}_+ + \sqrt{\bar{\xi}_+^2+\rho^2}}{\bar{\xi}_- + \sqrt{\bar{\xi}_-^2+\rho^2}}\right)\,. \tag{2.26}$$

We defined $\bar{\xi}_\pm = \bar{\xi} \pm \bar{L}/2$ for convenience.

2.2.2 Field of an m-pencil in its rest frame

Let us denote by $\{\rho,\varphi\}$ polar coordinates in the plane orthogonal to the pencil. Then the Minkowski metric takes the form[2]

$$\mathrm{d}s^2 = -\mathrm{d}\bar{t}^2 + \mathrm{d}\bar{\xi}^2 + \mathrm{d}\rho^2 + \rho^2\mathrm{d}\varphi^2\,. \tag{2.27}$$

To obtain the field of the m-pencil let us consider first the magnetic field of a solenoid with current density

$$\begin{aligned}\boldsymbol{J} &= \bar{J}_\varphi \mathrm{d}\varphi\,,\\ \bar{J}_\varphi &= \frac{\bar{m}}{\pi\bar{L}R}\delta(\rho-R)\Theta(\bar{\xi}|-\bar{L}/2,\bar{L}/2)\,.\end{aligned} \tag{2.28}$$

[1] $\boldsymbol{A}$ as a differential form is invariant under Lorentz transformations. For this reason we omit the bar on any bold-faced objects here and in what follows.

[2] Recall that the angular φ-component do not refer to an orthonormal basis but rather the ∂_φ-vector with norm ρ. Care should be taken when comparing our results to the literature, where sometimes we find expressions evaluated in orthonormal frames with the unit basis vector $\hat{\varphi} = \partial_\varphi/\rho$.

Here R is the radius of the solenoid, $\bar{L}$ is its length measured in the frame $\bar{S}$, and $\bar{m}$ denotes the magnetic moment of the solenoid which is proportional to the magnetic flux inside of it. Since the magnetic field is static and axially symmetric one can put $\bar{\boldsymbol{A}} \equiv \bar{A}_\mu \mathrm{d}\bar{X}^\mu = \bar{A}_\varphi \mathrm{d}\varphi$, and the potential $\bar{A}_\varphi$ in the limit $R \to 0$ is

$$\bar{A}_\varphi = \frac{\bar{m}}{4\pi \bar{L}} \left(\frac{\bar{\xi}_+}{\sqrt{\bar{\xi}_+^2 + \rho^2}} - \frac{\bar{\xi}_-}{\sqrt{\bar{\xi}_-^2 + \rho^2}} \right) . \tag{2.29}$$

Here, as earlier, $\bar{\xi}_\pm = \bar{\xi} \pm \bar{L}/2$. For details of this calculation we refer to Appendix A. One can also check that the expression (2.29) coincides with the magnetic field of a monopole–anti-monopole pair located on the $\bar{\xi}$-axis at the points separated by distance $\bar{L}$.

2.3 Nonlocal Maxwell equations

2.3.1 Action and field equations

We present now a far going generalization of the results presented in the previous section. Namely, we consider a spacetime with arbitrary number of dimensions $D \geq 4$ and we do not assume that the electric charge and magnetic moment densities are constant. We shall also obtain results valid for both higher-dimensional Maxwell theory as well as for its nonlocal ghost-free generalization.

Consider D-dimensional flat spacetime with Cartesian coordinates $X^\mu = (t, \boldsymbol{x})$, with $\boldsymbol{x} = (x^i)$, $i = 1, \dots d$, and $d = D - 1$. The Minkowski metric is

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \sum_{i=1}^{d} (\mathrm{d}x^i)^2 = -\mathrm{d}t^2 + \mathrm{d}\boldsymbol{x}^2 . \tag{2.30}$$

Let us consider a vector field A_μ obeying linear equations. We assume that these equations can be derived from an action and impose the following conditions on it:

1. The action $S[A_\mu]$ is a scalar and it can be written in the form

$$S[A_\mu] = \int \mathrm{d}^D x \sqrt{g} L . \tag{2.31}$$

   where L is the Lagrangian density;

2. L is bilinear in A_μ

$$L = A_\mu \mathcal{O}^{\mu\nu} A_\nu \,, \tag{2.32}$$

where $\mathcal{O}^{\mu\nu}$ is a symmetric rank two tensor operator constructed from the metric $g_{\mu\nu}$ and covariant derivatives ∇_μ;

3. The action $S[A_\mu]$ is gauge invariant, that is it is invariant under transformations

$$A_\mu(x) \to \hat{A}_\mu(x) = A_\mu(x) + \lambda_{,\mu}(x) \,, \tag{2.33}$$

where $\lambda(x)$ is an arbitrary function.

Since in the flat spacetime $g_{\mu\nu}$ and $\nabla_\mu\nabla_\nu$ are the only symmetric rank two tensors that can be constructed form the metric and covariant derivatives the operator $\mathcal{O}^{\mu\nu}$ which enters relation (2.32) has the following form

$$\mathcal{O}^{\mu\nu} = h(\Box) g^{\mu\nu} + f(\Box)\nabla^\mu\nabla^\nu \,. \tag{2.34}$$

In a flat spacetime and in the Cartesian coordinates one has $g_{\mu\nu} = \eta_{\mu\nu}$ and $\nabla_\mu = \partial/\partial X^\mu$, but it is convenient to work with a covariant form of the action. One only needs to remember that in this case the operators ∇_μ commute and their action on the metric vanishes, $\nabla_\lambda g_{\mu\nu} = 0$. Let us also emphasize that we shall use the action solely for deriving the field equations for A_μ and for that reason one may integrate by parts without considering the contribution of the boundary terms.

We denote by $S[\hat{A}_\mu]$ the action (2.31) for the field $\hat{A}_\mu$ defined by (2.33). Then one has

$$\delta_\lambda S \equiv S[\hat{A}_\mu] - S[A_\mu] = \int \mathrm{d}^D x \sqrt{g}\, J \,, \tag{2.35}$$

$$J = -\lambda \left[h(\Box) + f(\Box)\Box\right] \nabla^\mu A_\mu + \frac{1}{2}\lambda\Box \left[h(\Box) + f(\Box)\Box\right] \lambda \,. \tag{2.36}$$

Let us consider first the second term in J which is quadratic in the gauge function λ. Since this function is arbitrary, the last term vanishes only if

$$h(\Box) = -f(\Box)\Box \,. \tag{2.37}$$

However, under this condition the first term on the right-hand side of (2.36) which is linear in λ vanishes as well. Hence the condition (2.37) guarantees that the action (2.31)-(2.34) is gauge invariant. As a result the action for nonlocal higher dimensional modification of the Maxwell theory takes the form

$$S[A_\mu] = -\frac{1}{2}\int \mathrm{d}^D x\sqrt{g}A_\mu f(\Box)[g^{\mu\nu}\Box - \nabla^\mu\nabla^\nu]A_\nu\,. \tag{2.38}$$

After integration by parts one can write the action in the form

$$\begin{aligned} S[A_\mu] &= -\int \mathrm{d}^D x\sqrt{g}\left[\frac{1}{4}F_{\mu\nu}f(\Box)F^{\mu\nu} - j^\mu A_\mu\right]\,, \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu\,. \end{aligned} \tag{2.39}$$

We added to this action a term describing an interaction the electromagnetic field with a conserved external current j^μ. The operator $f(\Box)$ in this action is called a *form factor*, and its precise form specifies a nonlocal model.

The D-dimensional version of the local Maxwell theory can be easily obtained as a special case of the action (2.39). It is sufficient to impose one more condition, namely to require that

- The field equations for the field A_μ obtained from the action (2.31) are not higher than the second order in derivatives.

This condition implies that $f(\Box)$ is in fact a constant. One can always put this constant to be equal to 1 by simply re-scaling the field variables A_μ. The action for the local Maxwell field in D-dimensional spacetime is

$$S[A_\mu] = -\int \mathrm{d}^D x\sqrt{g}\left[\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j^\mu A_\mu\right]\,. \tag{2.40}$$

The field equations obtained by varying the action (2.39) with respect to A_μ are

$$f(\Box)\nabla_\mu F^{\mu\nu} = j^\nu\,, \tag{2.41}$$

$$\partial_{[\rho}F_{\mu\nu]} = 0\,. \tag{2.42}$$

The first of these equations is obtained by the variation of the action (2.39). The second equation implies that locally $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Inserting this expression into the first equation one finds

$$f(\Box)\left(\Box A^\nu - \nabla^\nu \nabla_\mu A^\mu\right) = j^\nu\,. \tag{2.43}$$

We may now exploit the gauge invariance in A_μ to fix the gauge to the convenient choice ("Lorenz gauge") $\nabla_\mu A^\mu = 0$ which implies

$$f(\Box)\Box A_\mu = j_\mu\,. \tag{2.44}$$

From now on we shall work exclusively in the Lorenz gauge.

2.3.2 Form factors

Nontrivial generalizations of the Maxwell theory can be obtained if the form factor is not a constant but it is a nontrivial function of the box-operator. For discussion of such cases it is convenient to use the following procedure. Let us first consider an analytic function $f(z)$ of a complex variable z. Such a function can be written in the form

$$f(z) = \sum_{n=0}^{\infty} f_n z^n\,. \tag{2.45}$$

The operator $f(\Box)$ is obtained from $f(z)$ by substituting $z \to \Box$.

In the case when only a finite number q of terms is present in the series (2.45) the form-factor function is a polynomial of order q

$$P_q(z) = z^q + f_{q-1} z^{q-1} + \ldots f_1 z + f_0\,. \tag{2.46}$$

The corresponding field equations (2.44) are higher derivative modification of the Maxwell theory. In a general case such models have the following common problem. A polynomial of order q has q zeroes z_j, $j = 1, \ldots q$ in the complex plane of a variable z [3]. As a result the inverse function of $zP_q(z)$ which enters the equation (2.44) has

[3] Some of these zeroes can coincide. For simplicity, we do not consider such cases here.

$q+1$ poles so that one has

$$\frac{1}{zP_q(z)} = \frac{C_0}{z} + \sum_{j=1}^{q} \frac{C_j}{z - z_j}\,. \tag{2.47}$$

Each of such poles corresponds to a propagating degree of freedom of the field A_μ.

In a general case if one combines all the terms in the right-hand side one gets the following expression

$$\frac{C_0}{z} + \sum_{j=1}^{q} \frac{C_j}{z - z_j} = \frac{Q_q}{zP_q(z)}\,. \tag{2.48}$$

Here $Q_q(z)$ is a polynomial of order q. Relation (2.47) implies that in fact $Q_q(z) = 1$. This imposes $q+1$ conditions on the coefficients C_0 and C_j, which uniquely determine them. In particular, the condition that $Q_q(z)$ does not contain z^q implies that

$$C_0 + \sum_{j=1}^{q} C_j = 0\,. \tag{2.49}$$

For the local Maxwell theory $P_q(z) = 1$ so that $C_0 = 1$ and $C_{j\geq 1} = 0$. Let us keep the condition $C_0 \geq 0$ for a nontrivial polynomial $P_q(z)$. Then relation (2.49) shows that at least some of the residues $C_{j\geq 1}$ should be negative. The corresponding unphysical modes are known as ghosts.

Recently it was proposed that this fundamental problem of the higher-derivative models can be solved by considering special nonlocal modifications of the corresponding local theory [16, 17]. Namely, one can consider such form factors $f(z)$ which are regular and do not have zeroes in the total complex plane. Consider the function $f(z) = \exp[P(z)]$, where $P(z)$ is an entire function. Notice that this form factor has no zeroes since $P(z)$ is everywhere holomorphic and therefore the exponent is well defined in the whole complex plane. This means that the inverse function $zf(z)$ does not introduce new poles.

This class of theories is often referred to as "ghost-free" as they do not introduce new unphysicall degrees of freedom. To make our consideration more concrete we restrict ourselves by considering the form factors of the following form

$$f(\Box) = \exp\left[(-\ell^2\Box)^N\right]\,, \quad \ell > 0\,. \tag{2.50}$$

Here N is a positive integer number and $\ell > 0$ is a characteristic scale of nonlocality which is estimated to be $\ell < 10^{-20}$m [76]. We call nonlocal models with these form factors $\mathrm{GF_N}$ theories. These form factors are choosen so that in the limiting case of $\ell \to 0$ one recovers the local theory since $f(0) = 1$. The latter condition also guarantees that the residue of the pole of $zf(z)$ at $z = 1$ is 1, and such a theory correctly reproduces the properties of the corresponding local theory in the infrared regime.

2.4 Static Green functions

Our next step consists of finding solutions that describe the electromagnetic field of stationary sources. Namely we assume the current j^ν which enters the right-hand side of the equation (2.44) does not depend of time t. We also assume that there are no free propagating electromagnetic waves. Since the field does not depend on time one may substitute the $\Box$-operator by the Laplace operator

$$\triangle = \nabla^2 = \sum_{i=1}^{d} \partial_i^2 \,. \tag{2.51}$$

In both local and nonlocal cases the solution of the field equations for a stationary sources can be found by using the corresponding Green function. For the nonlocal theory such a Green function is a solution of the following differential equation:

$$f(\triangle)\triangle\mathcal{G}_d(\boldsymbol{x}' - \boldsymbol{x}) = -\delta^{(d)}(\boldsymbol{x}' - \boldsymbol{x}) \,. \tag{2.52}$$

Here and later we use the notation $\mathcal{G}_d(\boldsymbol{x})$ for the Green function of the nonlocal theory. For the local theory it coincides with the usual Green function $G_d(\boldsymbol{x})$. For a $\mathrm{GF_N}$ theory one has

$$f(\triangle) = \exp[(-\ell^2\triangle)^N] \,. \tag{2.53}$$

Using the Fourier transformation one can obtain a useful representation of a static Green function $\mathcal{G}_d$. In this subsection we describe its form following the paper [73].

We denote the d-dimensional vector of momentum by $\mathbf{k} = (k_1, \ldots, k_d)$ and write

$$\mathcal{G}_d(\boldsymbol{x}' - \boldsymbol{x}) = \int \frac{d^d\mathbf{k}}{(2\pi)^d} e^{i\mathbf{k}(\boldsymbol{x}-\boldsymbol{x}')} \bar{\mathcal{G}}_d(\mathbf{k}). \tag{2.54}$$

Here $\boldsymbol{kx} = \sum_{j=1}^{d} k_j x^j$. Substituting this expression into the equation (2.52) and using a similar Fourier transform representation for the delta function

$$\delta^d(\boldsymbol{x}' - \boldsymbol{x}) = \int \frac{d^d\mathbf{k}}{(2\pi)^d} e^{i\mathbf{k}(\boldsymbol{x}-\boldsymbol{x}')}\,, \tag{2.55}$$

one gets

$$\bar{\mathcal{G}}_d(\mathbf{k}) = \frac{1}{k^2 f(-k^2)}.$$

Let us denote the angle between vectors $\boldsymbol{k}$ and $\boldsymbol{x} - \boldsymbol{x}'$ by θ, then

$$\boldsymbol{kx} = kr\cos\theta, \quad k = |\boldsymbol{k}|, \quad r = |\boldsymbol{x} - \boldsymbol{x}'|\,, \tag{2.56}$$

and the Green function reads

$$\mathcal{G}_d(x - x') = A_{d-2} \int_0^\infty \frac{dk}{(2\pi)^d} \frac{k^{d-3}}{f(-k^2)} \int_0^\pi d\theta\, \sin^{d-2}\theta\, e^{ikr\cos(\theta)}\,. \tag{2.57}$$

Here

$$A_{d-2} = 2\frac{\pi^{(d-1)/2}}{\Gamma\left(\frac{d-1}{2}\right)}\,, \tag{2.58}$$

is the area of a unit sphere S^{d-2}.

Integration over θ gives the expression for the Green function $\mathcal{G}_d(x, x')$ in terms of an integral containing the Bessel function

$$\mathcal{G}_d(x, x') = \frac{1}{2\pi} \int_0^\infty \frac{dk}{kf(-k^2)} \left(\frac{k}{2\pi r}\right)^{\frac{d}{2}-1} J_{\frac{d}{2}-1}(kr)\,. \tag{2.59}$$

The Green function $\mathcal{G}_d(x, x')$ depends only on the distance r between points. After changing of the integration variable $z = kr$ one gets

$$\mathcal{G}_d(r) = \frac{1}{(2\pi)^{d/2}\, r^{d-2}} \int_0^\infty dz\, \frac{z^{\frac{d-4}{2}}}{f(-z^2/r^2)}\, J_{\frac{d}{2}-1}(z)\,. \tag{2.60}$$

Here $d \geq 3$. For the local Maxwell theory one has $f = 1$. In this case the integral in (2.60) is a constant. Calculating this integral one reproduces the standard expression for a local Green function

$$G_d(x, x') = \frac{\Gamma\left(\frac{d}{2} - 1\right)}{4\pi^{d/2} r^{d-2}}. \tag{2.61}$$

There is also a recursive formula relating the Green functions in the spaces of different dimensions

$$\mathcal{G}_{d+2}(r) = -\frac{1}{2\pi r} \frac{\partial \mathcal{G}_d(r)}{\partial r}. \tag{2.62}$$

This result and its derivation can be found in [73]. This means that if the Green function is known for $d = 3, 4$ then the higher-dimensional Green functions can be obtained by using Eq. (2.62). This property allows one to find the Green functions in an explicit form for some special nonlocal models. For example, in the simplest case of GF_1 theory one has

$$\mathcal{G}_3(r) = \frac{1}{4\pi r} \mathrm{erf}\left(\frac{r}{2\ell}\right), \tag{2.63}$$

$$\mathcal{G}_4(r) = \frac{1}{4\pi^2 r^2}\left[1 - e^{-r^2/(4\ell^2)}\right], \tag{2.64}$$

where $\mathrm{erf}(z)$ denotes the error function [77]. In the limit $\ell \to 0$ one recovers the well-known local expressions. Expressions for Green functions $\mathcal{G}_d^N(r)$ for some of GF_N theories can be found in [1]. We collect these results in the appendix F.

An important property of the Green functions in such nonlocal ghost-free theories is that they are regular at $r = 0$. This can be demonstrated by using the expansion of the Bessel function in (2.59) for small r. Additional information about nonlocal static Green functions can be found in [27].

2.5 Stationary solutions for point like sources

Let us consider a conserved stationary external current

$$j^\mu = \delta^\mu_t q \delta^{(d)}(\boldsymbol{x}) + \delta^\mu_i M^{ik} \partial_k \delta^{(d)}(\boldsymbol{x}). \tag{2.65}$$

Here, q is the charge of the point particle and $M_{ik} = -M_{ki}$ is a constant, antisymmetric matrix that parametrizes the particle's intrinsic magnetic moment.

Let us write $d = 2k + \sigma$, where $\sigma = 0$ if d is even and $\sigma = 1$ if d is odd. Any skew symmetric $d \times d$ matrix can be put in a block diagonal form by means of a rigid rotation in the corresponding d-dimensional space [78–80]. In this new Darboux coordinate basis the matrix $\boldsymbol{M}$ has the form (see appendix B)

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{m}_1 & 0 & \ldots & 0 & 0 \\ 0 & \boldsymbol{m}_2 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & \boldsymbol{m}_k & 0 \\ 0 & 0 & \ldots & 0 & 0 \end{bmatrix} . \tag{2.66}$$

Here $\boldsymbol{m}_a$ are 2×2 matrices

$$\boldsymbol{m}_a = \begin{bmatrix} 0 & m_a \\ -m_a & 0 \end{bmatrix} . \tag{2.67}$$

The last column and row in (2.66) which contain zeroes are present when $\sigma = 1$ and they are absent for $\sigma = 0$. One can identify k quantities m_a with independent components of the magnetic moment. In the four dimensional spacetime when $d = 3$ only one block is present and there is only one component of the magnetic moment.

We write the electromagnetic potential as

$$A_\mu(\boldsymbol{x}) = \delta^t_\mu \varphi(\boldsymbol{x}) + \delta^i_\mu A_i(\boldsymbol{x}) , \tag{2.68}$$

then the field equations take the form

$$f(\triangle)\triangle\varphi = -q\delta^{(d)}(\boldsymbol{x}) , \tag{2.69}$$

$$f(\triangle)\triangle A_i = M_i{}^k \partial_k \delta^{(d)}(\boldsymbol{x}) . \tag{2.70}$$

Solutions of these equations can be written in terms of the Green function and they are

$$\varphi(\boldsymbol{x}) = q\,\mathcal{G}_d(\boldsymbol{x}) , \quad A_i(\boldsymbol{x}) = -M_i{}^k \partial_k \mathcal{G}_d(\boldsymbol{x}) . \tag{2.71}$$

The potential 1-form $\boldsymbol{A} = A_\mu \mathrm{d}X^\mu$ then takes the form

$$\begin{aligned} A_\mu \mathrm{d}X^\mu &= \varphi \mathrm{d}t + A_i \mathrm{d}x^i \\ &= q\, \mathcal{G}_d^N(r)\mathrm{d}t - M_i{}^k \partial_k \mathcal{G}_d(\boldsymbol{x})\mathrm{d}x^i \,. \end{aligned} \tag{2.72}$$

Using the relation

$$\partial_i \mathcal{G}_d(\boldsymbol{x}) = \frac{x_i}{r}\partial_r \mathcal{G}_d(\boldsymbol{x}) = -2\pi x_i \mathcal{G}_{d+2}(\boldsymbol{x})\,, \tag{2.73}$$

we may also write

$$A_\mu \mathrm{d}X^\mu = q\, \mathcal{G}_d^N(r)\mathrm{d}t + 2\pi M_i{}^k x_k \mathcal{G}_{d+2}(\boldsymbol{x})\mathrm{d}x^i \,. \tag{2.74}$$

2.6 Pencil-like sources

Let us discuss now a special type of higher-dimensional extended charged and magnetized objects which are similar to the four-dimensional pencil-type sources. For this purpose we single out one spatial coordinate which we denote by $\bar{\xi}$. This will be a direction of the pencil. We denote the coordinates as $\bar{X}^\mu = (\bar{t}, \bar{\xi}, \boldsymbol{x}_\perp)$ and write the metric in the form

$$\mathrm{d}s^2 = -\mathrm{d}\bar{t}^2 + \mathrm{d}\bar{\xi}^2 + \mathrm{d}\boldsymbol{x}_\perp^2 \,. \tag{2.75}$$

As earlier, we denote quantities calculated in the source's rest frame $\bar{S}$ with bars. The space orthogonal to $\bar{\xi}$ is $(d-1)$-dimensional and we denote $d-1 = 2n+\epsilon$. Here $\epsilon = 0$ if d is odd and $\epsilon = 1$ if d is even. The metric of this space is

$$\mathrm{d}\boldsymbol{x}_\perp^2 = \sum_{j=1}^{d-1} (\mathrm{d}x_\perp^j)^2 \,. \tag{2.76}$$

In the transverse space with coordinates $x_\perp^j$, we choose n mutually orthogonal two-planes Π_a, $a = 1, \dots, n$. If d is odd and $\epsilon = 0$ these n two-planes span the complete transverse space. For d even one has $\epsilon = 1$ and in order to fully span the transverse space, besides n two-planes, there exist one more one-dimensional direction. We denote the corresponding coordinate by z; see Fig. 2.1 for a visualization of this decomposition.

In a general case certainly there is an ambiguity in the choice of a set of two-planes. We assume that for a given antisymmetric matrix $\boldsymbol{M}$ the planes Π_a coincide with eigen two-planes of this matrix. In particular, this means that the current (2.65) has non-vanishing components only in the directions transverse to $\bar{\xi}$. In four spacetime dimensions, where $n = 1$, this assumption implies that the vector of the magnetic moment generated by the current is directed along the $\bar{\xi}$-direction. The above condition imposed on M_{ij} plays a similar role in higher dimensions. It is convenient to use the Darboux coordinates associated with two-planes Π_a. Namely, we denote by $(y_a, \hat{y}_a)$ orthonormal coordinates in each of the two-planes Π_a, so that the metric takes the form

$$\mathrm{d}s^2 = -\mathrm{d}\bar{t}^2 + \mathrm{d}\bar{\xi}^2 + \sum_{a=1}^{n}(dy_a^2 + d\hat{y}_a^2) + \epsilon\, dz^2 \,. \tag{2.77}$$

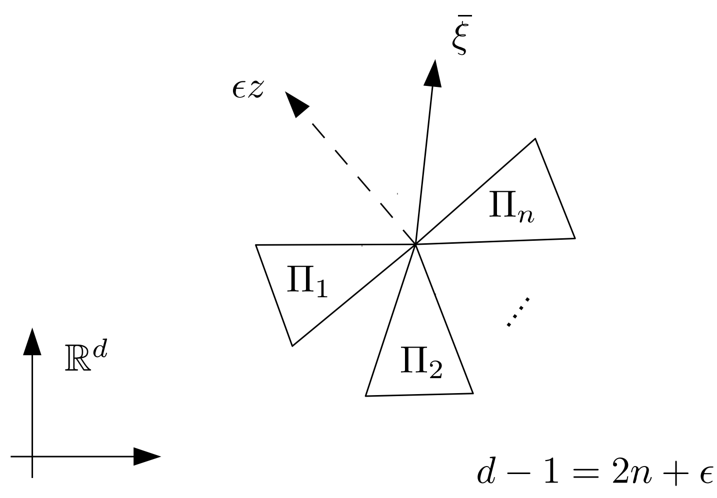

Figure 2.1: Darboux decomposition of d-dimensional space into n mutually orthogonal Darboux planes Π_a and a transverse z-direction if $\epsilon = 1$ [1]

In what follows it is also convenient to introduce polar coordinates $\{\rho_a, \varphi_a\}$ in each of the two-planes Π_a related to $(y_a, \hat{y}_a)$ as follows

$$y_a = \rho_a \cos\varphi_a, \qquad \hat{y}_a = \rho_a \sin\varphi_a \,. \tag{2.78}$$

In these coordinates the metric takes the form

$$\mathrm{d}s^2 = -\mathrm{d}\bar{t}^2 + \mathrm{d}\bar{\xi}^2 + \sum_{a=1}^{n} (\mathrm{d}\rho_a^2 + \rho_a^2 \mathrm{d}\varphi_a^2) + \epsilon \mathrm{d}z^2 \,, \tag{2.79}$$

and the field of the point-like source (2.74) can be expressed as

$$\begin{aligned} \bar{A}_\mu \mathrm{d}\bar{X}^\mu &= \bar{q}\, \mathcal{G}_d(\bar{r})\mathrm{d}\bar{t} - 2\pi \mathcal{G}_{d+2}(\bar{r}) \sum_{a=1}^{n} \bar{m}_a \rho_a^2 \mathrm{d}\varphi_a \,, \\ \bar{r}^2 &= \bar{\xi}^2 + \boldsymbol{x}_\perp^2 = \bar{\xi}^2 + \sum_{a=1}^{n} \rho_a^2 + \epsilon z^2 \,. \end{aligned} \tag{2.80}$$

Now we want to construct a solution for a stationary charged and magnetized higher dimensional extended object. For simplicity we limit our consideration to charged and/or magnetized pencils whose transverse charge and magnetic moment densities are δ-shaped, but we allow a density profile in the pre-boosted $\bar{\xi}$-direction to be arbitrary functions of $\bar{\xi}$. We denote these densities by $\bar{\lambda}(\bar{\xi})$ and $\bar{\mu}_a(\bar{\xi})$ for the charged and magnetized pencils, respectively. Then, the conserved external current takes the following form:

$$j^\mu = \delta^\mu_{\bar{t}} \lambda(\bar{\xi}) \delta^{(d-1)}(\boldsymbol{x}_\perp) + \delta^\mu_i \bar{\mu}^{ik}(\bar{\xi}) \partial_k \delta^{(d-1)}(\boldsymbol{x}) \,. \tag{2.81}$$

We shall make the following additional assumptions

- The antisymmetric matrix function $\bar{\mu}^{ik}(\bar{\xi})$ is orthogonal to the $\bar{\xi}$ direction, $\bar{\mu}^{i\bar{\xi}}(\bar{\xi}) = 0$;
- The two-dimensional eigen planes of the antisymmetric matrix function $\bar{\mu}^{ik}(\bar{\xi})$ are parallel propagated along $\bar{\xi}$ axis.

These assumptions allow one to use Darboux coordinates $(y_a, \hat{y}_a)$ which are also parallel propagating along $\bar{\xi}$. Consider a Darboux plane Π_a and denote by

$$\boldsymbol{e}^{(a)} = e^{(a)i}\partial_{x^i} = \partial_{y^a}, \quad \boldsymbol{e}^{(\hat{a})} = e^{(\hat{a})i}\partial_{x^i} = \partial_{\hat{y}^a} \,, \tag{2.82}$$

a pair of orthonormal vectors in it. The one-forms dual to these vectors are

$$\boldsymbol{\omega}^{(a)} = \omega_i^{(a)} dx^i = dy^a, \quad \boldsymbol{\omega}^{(\hat{a})} = \omega_i^{(\hat{a})} dx^i = d\hat{y}^a \,. \tag{2.83}$$

The volume element in the a-th Darboux frame is

$$\epsilon^{(a)} = \boldsymbol{\omega}^{(a)} \wedge \boldsymbol{\omega}^{(\hat{a})} \,. \tag{2.84}$$

Then one has

$$\bar{\mu}_{ik}(\bar{\xi}) = \sum_{a=1}^{n} \mu_a(\bar{\xi}) \epsilon_{ij}^{(a)} \,. \tag{2.85}$$

Under these assumptions the conserved current of the pencil takes the form

$$j^\mu = \left[\delta_t^\mu \lambda(\bar{\xi}) + \sum_{\alpha=1}^{n} \mu_\alpha(\bar{\xi}) \epsilon^{(a)\mu j} \partial_j \right] \delta^{(d-1)}(x_\perp) \,. \tag{2.86}$$

Here $\lambda(\bar{\xi})$ and $\mu_\alpha(\bar{\xi})$ are the charge density the magnetic moment density distributed along the pencil.

The total charge $\bar{q}$ and total magnetic moment $\bar{m}_a$ of the pencil are given by the line integrals

$$\bar{q} = \int_{-\infty}^{\infty} \mathrm{d}\bar{\xi}\, \bar{\lambda}(\bar{\xi}) \,, \quad \bar{m}_a = \int_{-\infty}^{\infty} \mathrm{d}\bar{\xi}\, \bar{\mu}_a(\bar{\xi}) \,. \tag{2.87}$$

If the pencil has a finite length the integrals should be taken over a finite interval of $\bar{\xi}$.

One can easily obtain the following expressions for the components of the vector potential $\boldsymbol{A}$ generated by the current (2.86)

$$A_t = \int_{-\infty}^{\infty} d\bar{\xi}' \lambda(\bar{\xi}') \mathcal{G}_d(r) \,, \tag{2.88}$$

$$A_\alpha = -2\pi \int_{-\infty}^{\infty} d\bar{\xi}' \mathcal{G}_{d+2}(r) \sum_{\alpha=1}^{n} \mu_\alpha(\bar{\xi}') \rho_\alpha^2 \,.$$

Here we have used the notations $r^2 = (\bar{\xi}' - \bar{\xi})^2 + \mathbf{x}_\perp^2$ and $\mathbf{x}_\perp^2 = \sum_{\alpha=1}^{n} \rho_\alpha^2 + \epsilon z^2$; where $\epsilon = 0$ for an odd number of d spatial dimensions and $\epsilon = 1$ for even number of spatial dimensions.

2.7 Summary of chapter 2

In this chapter we discussed models which generalize the Maxwell theory. This generalization can be performed in two directions. First, we assumed that the number of spacetime dimensions is greater than four. And second, which is less trivial, we constructed the nonlocal action for the higher dimensional electromagnetic field which preserves both Lorentz and gauge invariance. Such an action contains one arbitrary function of the $\Box$-operator, called a form factor. It can be chosen so that no new unphysical degrees of freedom arise. We use the Green function method to construct solutions for stationary charged and magnetized objects. A characteristic property of these solutions is that they are regular for point like sources. We also described stationary solutions for a special pencil-like type of extended objects. These solutions will be used in the chapter 4 in which we discuss the field of the ultrarelativistic objects.

Chapter 3

Linearized Gravity and its Ghost-Free Modification

3.1 Einstein gravity

In Einstein theory of gravity the gravitational filed is described by a metric $g_{\mu\nu}$

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \, . \tag{3.1}$$

This metric obeys the Einstein equations which can be obtained from the Einstein-Hilbert action

$$S_g[g] = \frac{1}{2\kappa} \int dx \sqrt{-g} \, R \, . \tag{3.2}$$

Here R is the Ricci scalar and

$$\kappa = 8\pi G \, , \tag{3.3}$$

where G is Newton's coupling constant.

Action (3.2) can be obtained by imposing the following conditions:

1. The action of the gravitational field depends only on the metric and its derivatives;

2. It can be written in the form

$$S_g[g] = \int dx \sqrt{-g} L(g, \partial g, \dots) \, . \tag{3.4}$$

3. The Lagrangian density L is a scalar under the coordinate transformations;

4. The field equations obtained from S do not contain higher than the second derivatives of the metric.

These conditions imply that $L = C_0 R + C_1$, where C_0 and C_1 are constants. The parameter C_1 is responsible for the cosmological constant. In what follows we shall study weak gravitational field on a flat spacetime background and put the cosmological constant equal to zero. In order to properly reproduce Newton gravity in the weak field aproximation the constant C_0 should be taken as follows $C_0 = \frac{1}{2\kappa}$.

In the presence of matter the action takes the form

$$S = S_g[g] + S_m[g, \ldots] \,, \tag{3.5}$$

where $S_m[g, \ldots]$ is the action for the matter in the gravitational field. The dots in this action stands for the dynamical variables responsible for the matter degrees of freedom. The stress-energy tensor of the matter is defined as follows

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S_{\mathrm{m}}}{\delta g_{\mu\nu}} \,. \tag{3.6}$$

The variation of the action (3.5) gives the following Einstein equations

$$\begin{aligned} G_{\mu\nu} &= \kappa T_{\mu\nu} \,, \\ G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \,. \end{aligned} \tag{3.7}$$

Here $R_{\mu\nu}$ is the Ricci tensor.

3.2 Linearized Einstein gravity

3.2.1 Action and field equations

To obtain linearized gravity equations we write the metric in the form

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \,, \tag{3.8}$$

where $\eta_{\mu\nu}$ is a flat metric and $h_{\mu\nu}$ is its perturbation. The equations for $h_{\mu\nu}$ can be obtained from the Einstein equations (3.7). One can proceed in two ways which give the same result.

As earlier, we denote by X^μ Cartesian coordinate in which $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$. To obtain linear equations for the perturbation $h_{\mu\nu}$ it is sufficient to keep the quantities of the second order in h in the action (3.5) expansion. One can show (see e.g. [73]) that

$$S_g = -\frac{1}{2\kappa}\int dX\left(-\frac{1}{2}h^{\mu\nu}\Box h_{\mu\nu} + h^{\mu\nu}\partial_\mu\partial_\alpha\, h^\alpha{}_\nu - h^{\mu\nu}\partial_\mu\partial_\nu h + \frac{1}{2}h\Box h\right). \tag{3.9}$$

The variation of this action (with matter term included) gives

$$\begin{aligned} G_{\mu\nu} \equiv \Box\, h_{\mu\nu} - \partial_\sigma\big(\partial_\nu\, h_\mu{}^\sigma + \partial_\mu h_\nu{}^\sigma\big) \\ + \eta_{\mu\nu}\big(\partial_\rho\partial_\sigma h^{\rho\sigma} - \Box h\big) + \partial_\mu\partial_\nu h = -2\kappa T_{\mu\nu}\,, \end{aligned} \tag{3.10}$$

where $h = \eta^{\alpha\beta}h_{\alpha\beta}$ denotes the trace of $h_{\mu\nu}$. The same linearized gravity equations can be obtained starting with Einstein equations (3.7). In this case it is sufficient to keep only linear terms in $h_{\mu\nu}$ in the Eintein tensor $G_{\mu\nu}$ expansion.

3.2.2 Gauge invariance

Suppose we have a perturbed metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. Or, written in contravariant form $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$. Let us make the coordinate transformation

$$X^\mu \to \bar{X}^\mu = X^\mu + \xi^\mu(X)\,, \tag{3.11}$$

where ξ^μ is small. Then the metric $\boldsymbol{g}$ in the new coordinates is

$$\begin{aligned} &\bar{g}^{\mu\nu} = \frac{\partial\bar{X}^\mu}{\partial X^\alpha}\frac{\partial\bar{X}^\nu}{\partial X^\beta}g^{\alpha\beta} = \\ &(\delta^\mu_\alpha + \xi^\mu_{,\alpha})(\delta^\nu_\beta + \xi^\nu_{,\beta})(\eta^{\alpha\beta} - h^{\alpha\beta}) = \eta^{\mu\nu} - \bar{h}^{\mu\nu} + \dots\,, \end{aligned} \tag{3.12}$$

where

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu}\,. \tag{3.13}$$

The dots in the above relation stand for the omitted higher order terms. The linearized gravity action and equations of motion are obtained starting with the covariant Einstein theory. This guarantees that the linearized action and equations are invariant under the gauge transformation (3.13). This property can also be shown directly (see appendix C).

3.2.3 Equations in de Donder gauge

Linearized equations (3.10) can be simplified. For this purpose we denote

$$\hat{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu} \, . \tag{3.14}$$

The inverse transformation is

$$h_{\mu\nu} = \hat{h}_{\mu\nu} - \frac{1}{2} \hat{h} \, \eta_{\mu\nu} \, . \tag{3.15}$$

We also impose the de Donder gauge conditions $\partial_\mu \hat{h}^{\mu\nu} = 0$. Then, Eq. (3.10) simplifies greatly and takes the form

$$\Box \hat{h}_{\mu\nu} = -2\kappa T_{\mu\nu} \, . \tag{3.16}$$

The conservation law $\partial_\mu T^{\mu\nu} = 0$ implies that the imposed gauge conditions are consistent.

3.3 Stationary field of point-like sources

As earlier we denote by

$$X^\mu = (t, \boldsymbol{x}), \qquad \boldsymbol{x} = (x^1, x^2, x^3) \, , \tag{3.17}$$

Cartesian coordinates in the flat spacetime. Let us consider now a case when the stress-energy tensor is stationary, that is it does not depend on time t. We also assume that there are no freely propagating gravitational waves. Then the linearized gravitational field $h_{\mu\nu}$ is also stationary and one can replace the box-operator in (3.16) by the three-dimensional Laplace operator $\triangle$. Thus one has

$$\triangle \hat{h}_{\mu\nu} = -2\kappa T_{\mu\nu} \,. \tag{3.18}$$

Since the field equations are linear it is sufficient to find the field created by a point-like source. A solution for an extended object can be obtained by integrating these solutions with functions describing the corresponding stress-energy tensor distribution. The stress-energy of a point-like spinning particle can be written in the form

$$T_{\mu\nu} = \delta^t_\mu \delta^t_\nu \, m\delta(\boldsymbol{x}) + \delta^t_{(\mu}\delta^i_{\nu)} \, j_i{}^j \frac{\partial}{\partial x^j}\delta(\boldsymbol{x}) \,. \tag{3.19}$$

Here $\delta(\boldsymbol{x})$ is a three-dimensional delta-function, m is the mass of the particle and j_{ij} is a constant antisymmetric matrix parameterizing its angular momentum. As for the electromagnetic field current one can use Darboux coordinates in which the antisymmetric 3×3 matrix $\boldsymbol{j}$ takes the form

$$\boldsymbol{j} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & j \\ 0 & -j & 0 \end{pmatrix} \,. \tag{3.20}$$

The structure of the stress-energy tensor (3.19) implies that only the following components of $\hat{\boldsymbol{h}}$ do not vanish

$$\phi = \hat{h}_{tt}, \qquad A_i = \hat{h}_{ti} \,. \tag{3.21}$$

A solution for such a source takes the form

$$\phi(r) = \frac{\kappa m}{4\pi r} \,, \tag{3.22}$$

$$A_i(\boldsymbol{x}) = -\frac{\kappa j_{ij} x^j}{4\pi r^3} \,. \tag{3.23}$$

Here $r = |\boldsymbol{x}|$ and $\kappa = 8\pi G$. Let us note that $\phi = -2\varphi$, where φ is Newton's potential of a point source of mass m. Using Darboux coordinates $(t, z, y, \hat{y})$ in which the matrix $\boldsymbol{j}$ is of the form (3.20) one can write the components of A_i as follows

$$\boldsymbol{A} = (0, A_y, A_{\hat{y}}) \,, \tag{3.24}$$

$$A_y = \frac{\kappa j}{4\pi r^3}\hat{y}, \quad A_{\hat{y}} = -\frac{\kappa j}{4\pi r^3} y \,. \tag{3.25}$$

One may write the metric perturbation $\hat{\boldsymbol{h}}$ in spherical coordinates (t, r, ϕ, θ) using the change of variables

$$\begin{aligned} y &= r\cos\phi\sin\theta \,, \\ \hat{y} &= r\sin\phi\sin\theta \,, \\ z &= r\cos\theta \,. \end{aligned} \tag{3.26}$$

Then, the perturbation of the metric becomes

$$\hat{\boldsymbol{h}} = \frac{\kappa m}{4\pi r}\mathrm{d}t^2 - \frac{\kappa j \sin^2\theta}{2\pi r}\mathrm{d}\phi\,\mathrm{d}t \,. \tag{3.27}$$

The full perturbed metric is

$$\begin{aligned} ds^2 = &-\left(1 - \frac{2m}{r}\right)\mathrm{d}t^2 - \frac{4j\sin^2\theta}{r}\mathrm{d}\phi\,\mathrm{d}t + \left(1 + \frac{2m}{r}\right) \\ &\times \left[\mathrm{d}r^2 + r^2(d\theta^2 + \sin^2\theta\mathrm{d}\phi)\right] \,. \end{aligned} \tag{3.28}$$

Here, we substituted the value of $\kappa = 8\pi G$ and after this we put $G = 1$. For studying the motion of nonrelativistic particles (in Newton's limit) the factor $\frac{2m}{r}$ in the spatial part of the metric can be omitted and one gets

$$ds^2 = -\left(1 - \frac{2m}{r}\right)\mathrm{d}t^2 - \frac{4j\sin^2\theta}{r}\mathrm{d}\phi\,\mathrm{d}t + \mathrm{d}r^2 + r^2(d\theta^2 + \sin^2\theta\mathrm{d}\phi) \,. \tag{3.29}$$

This metric has a resemblance to the vacuum solution for Einstein equations for a rotating black whole, known as the Kerr metric. In Boyer-Lindquist coordinates it has the form

$$ds^2 = -\left(1 - \frac{2mr}{\Sigma}\right)\mathrm{d}t^2 - \frac{4jr\sin^2\theta}{\Sigma}\mathrm{d}t\,\mathrm{d}\phi + \frac{A\sin^2\theta}{\Sigma}\mathrm{d}\phi^2 + \frac{\Sigma}{\Delta}\mathrm{d}r^2 + \Sigma\mathrm{d}\theta^2 \,. \tag{3.30}$$

Where

$$\begin{aligned} &\Sigma = r^2 + \frac{j^2}{m^2}\cos^2\theta \,, \quad \Delta = r^2 - 2mr + \frac{j^2}{m^2} \,, \\ &A = (r^2 + \frac{j^2}{m^2})^2 - \Delta\frac{j^2}{m^2}\sin^2\theta \,. \end{aligned} \tag{3.31}$$

Taking the limit $r \to \infty$ the metric (3.30) becomes

$$\mathrm{d}s^2 \simeq -\left(1 - \frac{2m}{r}\right)\mathrm{d}t^2 - \frac{4j\sin^2\theta}{r}\mathrm{d}\phi\mathrm{d}t + \mathrm{d}r^2 + r^2(d\theta^2 + \sin^2\theta\mathrm{d}\phi)\,. \tag{3.32}$$

Therefore, we see that our perturbed metric (3.29) has exactly the same form as (3.32), this is, the asymptotic form of the Kerr metric.

3.4 Linearized higher-dimensional Einstein gravity

Higher-dimensional generalization of the Einstein equations are now often discussed in the literature. Mainly this is motivated by string theory which is naturally formulated when the number of spacetime dimensions D is greater than four. For this purpose one can use the following generalization of the action (3.2)

$$S_g[g] = \frac{1}{2\kappa}\int d^Dx\sqrt{-g}\,R\,. \tag{3.33}$$

Here R is the Ricci scalar and

$$\kappa = 8\pi G^{(D)}\,, \tag{3.34}$$

where $G^{(D)}$ is the higher-dimensional Newton's coupling constant. The form of the higher-dimensional Einstein equations (3.7) and their linearized version (3.10) remains the same. As ealier it is convenient to introduce new variables $\hat{h}_{\mu\nu}$

$$\hat{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu}\,. \tag{3.35}$$

After imposing the de Donder gauge fixing condition

$$\partial_\mu\hat{h}^{\mu\nu} = 0\,, \tag{3.36}$$

Eq. (3.10) simplifies greatly and takes the form

$$\Box\hat{h}_{\mu\nu} = -2\kappa T_{\mu\nu}\,. \tag{3.37}$$

The conservation law $\partial_\mu T^{\mu\nu} = 0$ implies that the imposed gauge conditions are consistent. The only significant difference where the number of dimensions is important is the inverse transformation for $\boldsymbol{h}$ which now takes the form

$$h_{\mu\nu} = \hat{h}_{\mu\nu} - \frac{1}{d-1}\hat{h}\eta_{\mu\nu} \,. \tag{3.38}$$

Once again, we denote by $d = D - 1$ the number of spatial dimensions.

3.5 Stationary field of point-like sources in higher dimensions

Following a similar procedure as in the case of 4D linearized Einstein gravity, one can find stationary solutions for an arbitrary number of dimensions. As earlier we denote

$$X^\mu = (t, \boldsymbol{x}), \qquad \boldsymbol{x} = (x^1, x^2, \ldots, x^d) \,. \tag{3.39}$$

Again, we are interested in the case where the stress-energy tensor does not depend on time. Therefore our equation takes the same form as (3.18) only that this time the Laplace operator is acting on all d spatial dimensions. A point-like spinning particle in an arbitrary number of dimensions has the following stress-energy

$$T_{\mu\nu} = \delta^t_\mu \delta^t_\nu \, m\delta^{(d)}(\boldsymbol{x}) + \delta^t_{(\mu}\delta^i_{\nu)} \, j_i{}^j \frac{\partial}{\partial x^j}\delta^{(d)}(\boldsymbol{x}) \,. \tag{3.40}$$

Now $\delta^{(d)}(\boldsymbol{x})$ is the d-dimensional delta function and all the other variables remain the same as on the Einstein case. The general structure of $\boldsymbol{h}$ remains unchanged and therefore we have that the only non-vanishing components are

$$\phi = \hat{h}_{tt}, \qquad A_i = \hat{h}_{ti} \,, \tag{3.41}$$

where $i = 1, 2, \ldots, d$. The solution for the metric perturbation can be found using the d-dimensional Green functions. These solutions are

$$\begin{aligned} \phi(r) &= 2\kappa\frac{d-2}{d-1} m \, G_d(r) \,, \\ A_i(\boldsymbol{x}) &= -2\pi\kappa j_{ij} x^j \, G_{d+2}(r) \,. \end{aligned} \tag{3.42}$$

Using (2.61) one finds

$$\phi(r) = \frac{\Gamma\left(\frac{d}{2}\right)}{(d-1)\pi^{\frac{d}{2}}} \frac{\kappa m}{r^{d-2}}\,, \tag{3.43}$$

$$A_i(\boldsymbol{x}) = -\frac{\Gamma\left(\frac{d}{2}\right)}{2\pi^{\frac{d}{2}}} \frac{\kappa j_{ij} x^j}{r^d}\,. \tag{3.44}$$

We choose the sign of A_i such that in the three-dimensional case $d = 3$ one obtains the standard Lense–Thirring expression ($j_{xy} = j$ and $\kappa = 8\pi G$)

$$A_i(\boldsymbol{x})\mathrm{d}x^i \sim \frac{2Gj}{r^3}(x\mathrm{d}y - y\mathrm{d}x) = \frac{2Gj}{r}\sin^2\theta\,\mathrm{d}\varphi\,. \tag{3.45}$$

These solutions can be generalized for an extended pencil-like object as we will see later.

3.6 Linearized nonlocal gravity

Nonlocal ghost-free equations for linearized gravity were discussed in many publications starting with pioneer work by Mazumdar and collaborators [16, 81]. A higher dimensional generalization of these results was obtained in [73]. Let us remind here the main steps of the derivation of the corresponding nonlocal linearized action and present the final result of this analysis.

We consider the metric perturbation $h_{\mu\nu}$ over D-dimensional flat spacetime background. To obtain the most general action for the linearized nonlocal gravity one can proceed in the same way as it was done in chapter 2 for the nonlocal Maxwell theory. Namely, one can start with the most general action which is quadratic in the metric perturbation $\boldsymbol{h}$ and which contains an arbitrary number of derivatives. Using the assumption that the action and its Lagrangian are gauge invariant scalars and omitting the total derivatives one can arrive at the final version of the action which will be given later (see equation (3.49)).

However, this procedure requires quite lengthy calculations. The same answer can be obtained in much more "cheap and clever" way which was proposed by Mazumdar and collaborators [16, 81]. For this purpose it is convenient to use the "curved metric" language and to write the corresponding action in a covariant form. This guarantees that the resulting equations are covariant. Infinitely small coordinate transformation $x^\mu \to x^\mu + \xi^\mu$ results in the change $h_{\mu\nu} \to h_{\mu\nu} - \xi_{\mu;\nu} - \xi_{\nu;\mu}$. On the flat-space background this is nothing but the gauge transformation of the perturbation $\boldsymbol{h}$. If the field equations for $\boldsymbol{h}$ are derived from a scalar action then this guarantees the gauge invariance of these equations.

We start with the action

$$S = \frac{1}{2\kappa}\int d^D x \sqrt{g}\, L \,. \tag{3.46}$$

The most general form of the scalar Lagrangian L which is quadratic in curvature is

$$L = R + R_{\mu_1\nu_1\lambda_1\sigma_1} O^{\mu_1\nu_1\lambda_1\sigma_1}_{\mu_2\nu_2\lambda_2\sigma_2} R^{\mu_2\nu_2\lambda_2\sigma_2} \,. \tag{3.47}$$

Here $O^{\mu_1\nu_1\lambda_1\sigma_1}_{\mu_2\nu_2\lambda_2\sigma_2}$ is an operator constructed from ∇_μ and the metric $g_{\mu\nu}$. This Lagrangian will be used to obtain the linearized equations for the perturbation. Each of the curvature terms which enter (3.47) are already of the first order in $\boldsymbol{h}$. This means that the terms proportional to the third and higher order in curvature do not contribute to the field equations in the linear order.

One can simplify the action (3.46)-(3.47) by using the symmetry properties of the curvature tensor, Bianchi identities and by omitting the terms which are total derivatives. Moreover, since the goal is to get the equations on the flat background one can commute the covariant derivative. Really, the commutator of the covariant derivatives is proportional to the curvature and it vanishes on the flat background. As it was shown in [16, 81] after using these properties the action (3.46)-(3.47) greatly simplifies and it reduces to the form

$$S = \frac{1}{2\kappa_d}\int \mathrm{d}^D x \sqrt{-g}\left[R + RF_1(\Box)R + R_{\mu\nu}F_2(\Box)R^{\mu\nu} + R_{\mu\nu\lambda\sigma}F_3(\Box)R^{\mu\nu\lambda\sigma}\right].$$

This form of the action is also valid in the case where the number of spacetime dimensions is greater than 4 [73]. This form of the quadratic in curvature action can be further simplified using the following observation [82, 83]: the "Gauss-Bonnet structures" of the form ($k \geq 1$) obey the following relations

$$ {}^{*}R^{\alpha\beta\gamma\sigma}\,\Box^{k}\,{}^{*}R_{\alpha\beta\gamma\sigma} = R^{\alpha\beta\gamma\sigma}\Box^{k}R_{\alpha\beta\gamma\sigma} - 4R^{\alpha\beta}\Box^{k}R_{\alpha\beta} + R\,\Box^{k}R = O(R^{3}) + \mathrm{div}\,. \tag{3.48} $$

These relations are valid in an arbitrary number of spacetime dimensions. This property allows one to reduce the term with the form factor F_3 to the other ones with form factors F_1 and F_2 by omitting the terms which are either of the third and higher order in curvature or are total divergences. As a result, the general higher derivative action can be written in a form which contains only two arbitrary functions of the box operator [81]. The result is the following: The most general linearized action S in a Lorentz invariant theory with an arbitrary number of derivatives and quadratic in the perturbation $h_{\mu\nu}$ can be written in the form [16]

$$ \begin{aligned} S = \frac{1}{2\kappa}\int \mathrm{d}^{D}X\Big(&\frac{1}{2}h^{\mu\nu}\,f(\Box)\Box\,h_{\mu\nu} - h^{\mu\nu}\,f(\Box)\partial_{\mu}\partial_{\alpha}\,h^{\alpha}{}_{\nu} + h^{\mu\nu}\,\tilde{f}(\Box)\partial_{\mu}\partial_{\nu}h \\ &- \frac{1}{2}h\,\tilde{f}(\Box)\Box h + \frac{1}{2}h^{\mu\nu}\,\frac{f(\Box)-\tilde{f}(\Box)}{\Box}\partial_{\mu}\partial_{\nu}\partial_{\alpha}\partial_{\beta}\,h^{\alpha\beta}\Big)\,. \end{aligned} \tag{3.49} $$

We write this action in Cartesian coordinates and use partial derivatives instead of the covariant ones. $\Box$ is the d'Alembert operator of Minkowski space, $\Box = \eta^{\mu\nu}\partial_{\mu}\partial_{\nu}$. The functions $f(\Box)$ and $\tilde{f}(\Box)$ can be chosen freely to parameterize different Lorentz-invariant modifications of gravity, subject only to the constraint

$$ f(0) = \tilde{f}(0) = 1\,, \tag{3.50} $$

which guarantees the proper Newtonian limit; see also the related discussions in Refs. [16, 42]. In the case of $f(\Box) = \tilde{f}(\Box) = 1$ one recovers the linearized General Relativity.

The field equations corresponding to the action (3.49) are

$$ \begin{aligned} f(\Box)\big[\Box\,h_{\mu\nu} - \partial_{\sigma}\big(\partial_{\nu}\,h_{\mu}{}^{\sigma} + \partial_{\mu}h_{\nu}{}^{\sigma}\big)\big] + \tilde{f}(\Box)\big[\eta_{\mu\nu}\big(\partial_{\rho}\partial_{\sigma}h^{\rho\sigma} - \Box h\big) + \partial_{\mu}\partial_{\nu}h\big] \\ + \frac{f(\Box)-\tilde{f}(\Box)}{\Box}\partial_{\mu}\partial_{\nu}\partial_{\rho}\partial_{\sigma}h^{\rho\sigma} = -2\kappa T_{\mu\nu}\,, \end{aligned} \tag{3.51} $$

where $T_{\mu\nu}$ is the energy-momentum tensor of matter, and $h = \eta^{\alpha\beta} h_{\alpha\beta}$ denotes the trace of $h_{\mu\nu}$.

From now on we shall restrict ourselves to the case of

$$\tilde{f}(\Box) = f(\Box)\,. \tag{3.52}$$

This condition guarantees that no extra massive scalar modes are present in the theory and the only physical degrees of freedom are massless gravitons [16]. We denote

$$\hat{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}\,. \tag{3.53}$$

The inverse transformation is

$$h_{\mu\nu} = \hat{h}_{\mu\nu} - \frac{1}{d-1} \hat{h} \eta_{\mu\nu}\,. \tag{3.54}$$

We also impose the de Donder gauge conditions $\partial_\mu \hat{h}^{\mu\nu} = 0$. Then, Eq. (3.51) simplifies greatly and takes the form

$$f(\Box)\Box \hat{h}_{\mu\nu} = -2\kappa T_{\mu\nu}\,. \tag{3.55}$$

The conservation law $\partial_\mu T^{\mu\nu} = 0$ implies that the imposed gauge conditions are consistent.

3.7 Nonlocal stationary solutions

Equations (3.55) written in the Cartesian coordinates are similar to the equations for the electromagnetic potential in the nonlocal Maxwell theory which were discussed in the previous chapter, and one can solve them by using the same method as described earlier in chapter 2.

We assume that $T_{\mu\nu}$ does not depend on time. For a stationary metric generated by such a stress-energy tensor the $\Box$-operator reduces to the d-dimensional Laplace operator $\triangle = \delta^{ij}\partial_i\partial_j$. We denote

$$\mathcal{D} = f(\triangle)\triangle\,. \tag{3.56}$$

In what follows we also assume that the form factor f is of the form (2.50), that is, we consider GF_N nonlocal models.

Then, we can solve the field equations (3.55) by using the static Green function

$$\mathcal{D}\mathcal{G}_d(\boldsymbol{x},\boldsymbol{x'}) = -\delta^{(d)}(\boldsymbol{x}-\boldsymbol{x'})\,. \tag{3.57}$$

These Green functions are the same as we introduced earlier in the case of nonlocal Maxwell theory.

The stress-energy tensor for stationary distribution of spinning matter can be written in the form (see, e.g. [70])

$$T_{\mu\nu} = \rho(\boldsymbol{x})\delta^t_\mu\delta^t_\nu + \delta^t_{(\mu}\delta^i_{\nu)}\frac{\partial}{\partial x^j}j_i{}^j(\boldsymbol{x})\,. \tag{3.58}$$

Here $\rho(\boldsymbol{x})$ is the mass density and $j_i{}^j(\boldsymbol{x})$ is the angular momentum density. A solution $h_{\mu\nu}$ of the field equations (3.55) for this source can be written as follows:

$$\boldsymbol{h} = h_{\mu\nu}\mathrm{d}X^\mu\mathrm{d}X^\nu\,, \tag{3.59}$$

$$\boldsymbol{h} = \phi\left(\mathrm{d}t^2 + \frac{1}{d-2}\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j\right) + 2A_i\mathrm{d}x^i\mathrm{d}t\,,$$

$$\phi(\boldsymbol{x}) = 2\kappa\frac{d-2}{d-1}\int \mathrm{d}^d y\,\rho(\boldsymbol{y})\mathcal{G}_d(\boldsymbol{x}-\boldsymbol{y})\,, \tag{3.60}$$

$$A_i(\boldsymbol{x}) = \kappa\int \mathrm{d}^d y\,j_i{}^j(\boldsymbol{y})\frac{\partial\mathcal{G}_d(\boldsymbol{x}-\boldsymbol{y})}{\partial x^j}\,. \tag{3.61}$$

Due to the translational symmetry of Eq. (3.57), the Green function $\mathcal{G}_d(\boldsymbol{x},\boldsymbol{x'})$ is a function of $\boldsymbol{x}-\boldsymbol{x'}$, while due to the spherical symmetry it depends on the radius variable $r = |\boldsymbol{x}-\boldsymbol{x'}|$ alone. Thus one has

$$\mathcal{G}_d(\boldsymbol{x}-\boldsymbol{x'}) = \mathcal{G}_d(r)\,. \tag{3.62}$$

Using the property of the Green function (2.62) the expression for $\boldsymbol{A}$ can written in the form

$$A_i(\boldsymbol{x}) = -2\pi\kappa\int \mathrm{d}^d y\,j_{ij}(\boldsymbol{y})(x^j-y^j)\mathcal{G}_{d+2}(\boldsymbol{x}-\boldsymbol{y})\,. \tag{3.63}$$

3.7.1 Point-like particle

The stress-energy tensor for a point-like particle is

$$T_{\mu\nu} = \delta^t_\mu \delta^t_\nu \, m\delta^{(d)}(\boldsymbol{x}) + \delta^t_{(\mu}\delta^i_{\nu)} \, j_i{}^j \frac{\partial}{\partial x^j}\delta^{(d)}(\boldsymbol{x}) \,, \tag{3.64}$$

where m is the mass of the particle and j_{ij} is a constant antisymmetric matrix parametrizing its angular momentum. A solution for the perturbed metric (3.59)–(3.61) for such a source takes the form[1]

$$\begin{aligned} \phi(r) &= 2\kappa\frac{d-2}{d-1} m \, \mathcal{G}_d(r) \,, \\ A_i(\boldsymbol{x}) &= -2\pi\kappa j_{ij} x^j \, \mathcal{G}_{d+2}(r) \,. \end{aligned} \tag{3.65}$$

Since all corresponding static nonlocal Green functions in GF_N models are regular at $r = 0$, the same property is valid for the solutions (3.65).

At large distances one recovers the standard expressions known from linearized General Relativity [85]:

$$\phi(r) \sim \frac{\Gamma\left(\frac{d}{2}\right)}{(d-1)\pi^{\frac{d}{2}}} \frac{\kappa m}{r^{d-2}} \,, \tag{3.66}$$

$$A_i(\boldsymbol{x}) \sim -\frac{\Gamma\left(\frac{d}{2}\right)}{2\pi^{\frac{d}{2}}} \frac{\kappa j_{ij} x^j}{r^d} \,. \tag{3.67}$$

We choose a sign of A_i such that in the three-dimensional case $d = 3$ one obtains the standard Lense–Thirring expression ($j_{xy} = j$ and $\kappa = 8\pi G$)

$$A_i(\boldsymbol{x}) \mathrm{d}x^i \sim \frac{2Gj}{r^3}(x\mathrm{d}y - y\mathrm{d}x) = \frac{2Gj}{r}\sin^2\theta \, \mathrm{d}\varphi \,. \tag{3.68}$$

For illustration purposes let us explicitly write the expressions for the metric perturbations for $d = 3$ dimensions in the simplest GF_1 model

$$\begin{aligned} \phi(r) &= \frac{\kappa m}{4\pi r} \operatorname{erf}\left(\frac{r}{2\ell}\right) \,, \\ A_i(\boldsymbol{x}) &= -\frac{\kappa j_{ij} x^j}{4\pi r^3} \left[\operatorname{erf}\left(\frac{r}{2\ell}\right) - \frac{r}{\sqrt{\pi}\ell} e^{(r/2\ell)^2} \right] \,. \end{aligned} \tag{3.69}$$

[1] In four-dimensional spacetime, this solution can be used to obtain a metric for a spinning ring discussed in [84].

3.8 Pencil-like gravitational sources

As we did in the previous chapter, we now focus on a particular type of higher-dimensional objects with a finite extension in one spatial direction and a zero transverse size. We assume that this object is massive and it rotates around its own axis. We consider the field of such an object in the frame $\bar{S}$ where it is rest. As earlier, we denote by $\bar{t}$ the time coordinate in the rest frame and by $\bar{\xi}$ a coordinate along the pencil direction. We also write the flat metric in the form

$$\mathrm{d}s^2 = -\mathrm{d}\bar{t}^2 + \mathrm{d}\bar{\xi}^2 + \mathrm{d}\boldsymbol{x}_\perp^2\,. \tag{3.70}$$

To specify the stress-energy tensor (3.58) to the case of the pencil-like distribution of the spinning matter we proceed as follows. We assume that both the matter density ρ and the density of the angular momentum j_{ij} are concentrated on the line representing the pencil. We introduce the quantities $\bar{\lambda}(\bar{\xi})$ and $\bar{j}_{ij}(\bar{\xi})$ as the mass and angular momentum line densities, respectively. They describe the distribution of the mass and angular momentum along the pencil. We also impose the following restrictions on the tensor structure of j_{ij}:

- Matrix functions $j_{ij}(\bar{\xi})$ are orthogonal to $\bar{\xi}$ direction, $j_{i\bar{\xi}}(\bar{\xi}) = 0$;
- Eigen two-planes Π_a of $j_{ij}(\bar{\xi})$ are parallel propagated along the $\bar{\xi}$ axis.

The second property implies that by using the rigid spatial rotations the matrix functions $j_{ij}(\bar{\xi})$ can be presented in the form

$$\bar{\boldsymbol{j}} \hat{=} \begin{pmatrix} 0 & \bar{j}_1 & & & \dots & & & 0 \\ -\bar{j}_1 & 0 & & & & & & \\ & & 0 & \bar{j}_2 & & & & \\ & & -\bar{j}_2 & 0 & & & & \\ \vdots & & & & \ddots & & & \\ & & & & & 0 & \bar{j}_n & \\ & & & & & -\bar{j}_n & 0 & \\ 0 & & & & & & & 0 \end{pmatrix} . \tag{3.71}$$

Here $\bar{j}_a$ are functions of $\bar{\xi}$ alone.

Let us specify the $(d-1)$ coordinates $x_\perp^j$ orthogonal to the $\bar{\xi}$-direction further:

$$\begin{aligned} x_\perp^j &= (y^a, \hat{y}^a, \epsilon z) , \quad a = 1, \dots, n , \\ n &= \left\lfloor \frac{d-1}{2} \right\rfloor , \quad d = 2n + 1 + \epsilon . \end{aligned} \tag{3.72}$$

One can say that the $(d-1)$-dimensional "transverse space" orthogonal to the ξ-axis is spanned by n mutually orthogonal two-planes Π_a, and $(y^a, \hat{y}^a)$ are right-handed coordinates in these planes. We shall refer to these planes as *Darboux planes.* If the number of spacetime dimensions $d+1$ is odd one has $\epsilon = 1$ and besides these two-planes there exists an additional one-dimensional z-axis which is orthogonal to each of the planes as well as to ξ-axis. In even spacetime dimensions there is no such additional z coordinate: for example, in four spacetime dimensions there exists only one two-plane orthogonal to the ξ-direction. We denote by $\boldsymbol{e}^{(a)} = \partial_{y^a}$ and $\hat{\boldsymbol{e}}^{(a)} = \partial_{\hat{y}^a}$ unit vectors along the y^a-axis and $\hat{y}^a$-axis, respectively. The 1-forms dual to these vectors are $\boldsymbol{\omega}^{(a)} = \mathrm{d}y^a$ and $\hat{\boldsymbol{\omega}}^{(a)} = \mathrm{d}\hat{y}^a$ such that the volume 2-form for each Darboux plane Π_a is given by $\boldsymbol{\epsilon}^{(a)} = \boldsymbol{\omega}^{(a)} \wedge \hat{\boldsymbol{\omega}}^{(a)}$.

In these coordinates the stress-energy tensor of a thin spinning pencil is

$$T_{\mu\nu} = \left[\delta^{\bar{t}}_\mu \delta^{\bar{t}}_\nu \bar{\lambda}(\bar{\xi}) + \sum_{a=1}^{n} \left(\bar{j}_a(\bar{\xi}) \delta^{\bar{t}}_{(\mu} \delta^i{}_{\nu)} \epsilon_i^{(a)j} \partial_j \right) \right] \delta^{(d-1)}(\boldsymbol{x}_\perp) . \tag{3.73}$$

We assume that this object has a finite length in $\bar{\xi}$, such that both $\bar{\lambda}(\bar{\xi})$ and $j_a(\bar{\xi})$ vanish when $\bar{\xi}$ is outside some interval $(0,\bar{L})$. We call $\bar{L}$ the length of the pencil. The mass and the angular momentum of such a pencil are

$$\bar{m} = \int \mathrm{d}\bar{\xi}\,\bar{\lambda}(\bar{\xi})\,, \tag{3.74}$$

$$\bar{J}_{ij} = \int \mathrm{d}\bar{\xi}\,\bar{\jmath}_{ij}(\bar{\xi})\,, \tag{3.75}$$

$$\bar{\jmath}_{ij}(\bar{\xi}) = \sum_{a=1}^{n} \epsilon_{ij}^{(a)}\bar{\jmath}_a(\bar{\xi})\,. \tag{3.76}$$

The gravitational field $h_{\mu\nu}$ for the thin spinning pencil is

$$\boldsymbol{h} = \bar{\phi}\left[\mathrm{d}t^2 + \frac{1}{d-2}(\mathrm{d}\bar{\xi}^2 + \mathrm{d}\boldsymbol{x}_\perp^2)\right] + 2\bar{A}_i \mathrm{d}x_\perp^i \mathrm{d}t\,, \tag{3.77}$$

where

$$\begin{aligned} \bar{\phi}(\bar{\xi}, x_\perp^i) &= 2\kappa\frac{d-2}{d-1}\int \mathrm{d}\bar{\xi}'\bar{\lambda}(\bar{\xi}')\mathcal{G}_d(\bar{r})\,, \\ \bar{A}_i(\bar{\xi}, x_\perp^i) &= -2\pi\kappa\int \mathrm{d}\bar{\xi}'\bar{\jmath}_{ij}(\bar{\xi}')x_\perp^j\mathcal{G}_{d+2}(\bar{r})\,. \end{aligned} \tag{3.78}$$

The expression for $\bar{r}^2$ is given in (2.80).

3.9 Summary of chapter 3

In this chapter we discussed linearized gravity equations in four and higher dimensions. We started with a higher dimensional generalization of General Relativity, and after this we described the nonlocal action of the so called linearized ghost-free gravity theories. In both cases we constructed solutions for point-like and pencil-like spinning objects. For this purpose we used the Green function method. As a result, solutions of the nonlocal linearized gravity equations for stationary sources have a lot of similarities with those constructed in the previous chapter for nonlocal Maxwell equations. In particular, the field of a point-like spinning particle in the ghost-free gravity is regular at the origin. In the next chapter we use the obtained stationary solutions of nonlocal Maxwell and gravity equations to obtain solutions for the field of ultrarelativistic sources.

Chapter 4

Field of Ultrarelativistic Objects

4.1 Boost transformation and Penrose limit

In this chapter we discuss electromagnetic and gravitational field of ultrarelativistic pencils with a finite length. We demonstrate that the corresponding solutions can be obtained by boosting the obtained earlier solutions for stationary sources. For this purpose we first rewrite these solutions in an inertial frame S which moves with respect to a static frame $\bar{S}$ with velocity $\beta = v/c$. After this we take the limit $\beta \to 1$. In order to obtain a finite result in such a limit one should properly rescale the parameters which enter $\bar{S}$ solutions. This rescaling depends on the spin of the field and it is different for the electromagnetic and gravitational cases. In the gravitational theory instead of the mass of the object in the frame $\bar{S}$ one should keep the energy in the S frame fixed. Penrose demonstrated that such a rescaling is valid not only in the linearized gravity. In his famous paper [63] Penrose wrote:

> We envisage a succession of observers travelling in the spacetime M whose world lines approach the null geodesic γ more and more closely; so we picture these observers as travelling with greater and greater speeds, approaching that of light. As their speeds increase they must correspondingly recalibrate their clocks to run faster and faster (assuming that all spacetime measurements are referred to clock measurements in the standard way), so that in the limit the clocks measure the affine parame-

ter along γ. (Without clock recalibration a degenerate spacetime metric would result.) In the limit, the observers measure the spacetime to have the plane-wave structure.

The corresponding procedure for getting solutions for the field of ultrarelativistic objects is called the Penrose limit. We apply this procedure to the solutions of electromagnetic and linearized gravity equations both for local and nonlocal versions of these theories. We shall also consider the cases of four-dimensional and higher dimensional spacetimes.

Let us consider two frames. The first one is frame $\bar{S}$ where the matter creating the gravitational field is at rest. The second frame S moves with a constant velocity β with respect to $\bar{S}$. We adapt now the choice of the coordinates which is convenient for this situation. Let ξ be a coordinate along the vector of velocity of S and denote by $\boldsymbol{x}_\perp$ the $d-1$ coordinates orthogonal to the ξ-direction. To distinguish the rest frame coordinates from the coordinates in the boosted frame we use a bar for the rest frame coordinates and write

$$X^\mu = (t, \xi, x_\perp^i), \quad \bar{X}^\mu = (\bar{t}, \bar{\xi}, x_\perp^i)\,. \tag{4.1}$$

The index $i = 1, 2, \ldots, d-1$ enumerates the coordinates transverse to the direction of motion. We omit the bar for the coordinates $x_\perp^i$ since the Lorentz transformation for the motion in the ξ-direction does not affect their values. The background Minkowski metric is

$$\mathrm{d}s_0^2 = -\mathrm{d}\bar{t}^2 + \mathrm{d}\bar{\xi}^2 + \mathrm{d}\boldsymbol{x}_\perp^2 = -\mathrm{d}t^2 + \mathrm{d}\xi^2 + \mathrm{d}\boldsymbol{x}_\perp^2\,. \tag{4.2}$$

Here, $(\bar{t}, \bar{\xi})$ are coordinates in the rest frame $\bar{S}$ and (t, ξ) are the corresponding coordinates in the moving frame S. For the remainder of the chapter, we denote all quantities defined with respect to the rest frame $\bar{S}$ with a bar. For example, the radial distance from the origin to a point $(\bar{\xi}, x_\perp^i)$ is $\bar{r}^2 = \bar{\xi}^2 + \boldsymbol{x}_\perp^2$. We specify $(d-1)$ coordinates $x_\perp^j$ orthogonal to the $\bar{\xi}$-direction in the same way as it was done in section 2.6.

In what follows we apply the Penrose limit to the previously described solutions for electromagnetic and gravitational "pencils". We shall apply to these solutions the boost transformation in the $\bar{\xi}$-direction

$$\bar{t} = \gamma\,(t - \beta\xi)\ , \quad \bar{\xi} = \gamma\,(\xi - \beta t)\ . \tag{4.3}$$

Here $\beta = v/c$ is the boost parameter and v is the velocity of the frame S with respect to the rest frame $\bar{S}$. For fixed ξ, that is, for a fixed point in frame S one has $\bar{\xi} = -\gamma\beta t + \text{const}$. This means that the frame S moves in the negative ("left") direction of $\bar{\xi}$ with respect to the rest frame $\bar{S}$. In other words, a pencil which is at rest with respect to $\bar{S}$ moves with a positive velocity in S frame.

We introduce the retarded and advanced null coordinates in the S frame defined as follows:

$$u = \frac{t-\xi}{\sqrt{2}}\,, \quad v = \frac{t+\xi}{\sqrt{2}}\,. \tag{4.4}$$

Then (4.3) implies

$$\bar{t} = \frac{\gamma}{\sqrt{2}}[(1+\beta)u + (1-\beta)v]\,, \tag{4.5}$$

$$\bar{\xi} = \frac{\gamma}{\sqrt{2}}[-(1+\beta)u + (1-\beta)v]\,. \tag{4.6}$$

In the ultrarelativistic limit, $\beta \to 1$, one has

$$\bar{t} \to \sqrt{2}\gamma u\,, \quad \bar{\xi} \to -\sqrt{2}\gamma u\,. \tag{4.7}$$

This implies that if one considers the matter distribution of an ultrarelativistec pencil in the frame $\bar{S}$ such that it is located in the strip between $\bar{\xi} = 0$ and $\bar{\xi} = \bar{L}$, then by keeping the length $L = \bar{L}/\gamma$ fixed in the moving frame the pencil will be located between the region $u = -L/\sqrt{2}$ and $u = 0$ of spacetime after taking the Penrose limit; see Fig. 4.1.

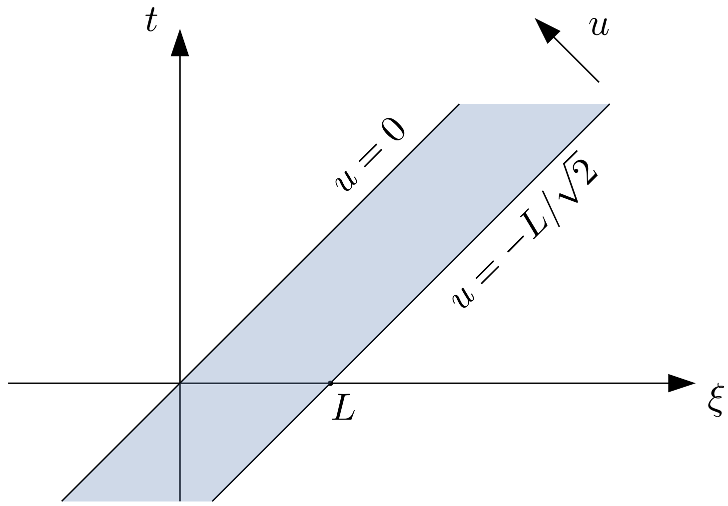

Figure 4.1: The pencil of length L moves within the two-dimensional (t, ξ)-section of Minkowski space in the frame S.

4.2 Green functions in the Penrose limit

As we have demonstrated in the previous chapters solutions for the stationary sources in the rest frame $\bar{S}$ can be found explicitly by using the Green functions of the equation

$$f(\triangle)\triangle\mathcal{G}_d(\boldsymbol{x}-\boldsymbol{x}') = -\delta^{(d)}(\boldsymbol{x}-\boldsymbol{x}')\,. \tag{4.8}$$

Here f is the form factor of the nonlocal theory and $d = D-1$ is the number of spatial dimensions. For $f = 1$ $\mathcal{G}_d(\boldsymbol{x}-\boldsymbol{x}') = G_d(\boldsymbol{x}-\boldsymbol{x}')$ where G_d is the Green function of d dimensional Laplace equation.

In order to obtain the Penrose limit of the Green function it is convenient to use the following trick. Let us write the following representation of the static Green function $\mathcal{G}_d(r)$

$$\mathcal{G}_d(r) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\frac{\mathrm{d}\eta}{f(-\eta\ell^2)\eta}\int\limits_{-\infty}^{\infty}\mathrm{d}\tau\, K_d(r|\tau)\, e^{i\eta\tau}\,. \tag{4.9}$$

Here

$$K_d(r|\tau) = \frac{1}{(4\pi i\tau)^{\frac{d}{2}}}e^{i\frac{r^2}{4\tau}}\,, \tag{4.10}$$

is the d-dimensional heat kernel in imaginary time $\tau = -it$ which satisfies the following equations

$$\triangle K_d(r|\tau) = -i\partial_\tau K_d(r|\tau)\,, \tag{4.11}$$

$$\lim_{\tau\to 0} K_d(r|\tau) = \delta^{(d)}(\boldsymbol{r})\,. \tag{4.12}$$

The derivation of the representation (4.9) for the static Green function is given in Appendix E.

The following property makes this representation very useful for the study of the Penrose limit of solutions: Relation (4.9) expresses the Green function $\mathcal{G}_d(\bar{r})$ as a double Fourier transform, wherein the radius $\bar{r}$ enters only quadratically via the exponential function $\sim \exp[i\bar{r}^2/(4\tau)]$. This observation allows us to perform the Penrose limit procedure in a very general and convenient form. In the representation (4.9) the only quantity which is "sensitive" to the boost is the heat kernel K_d. It factorizes such that the boost-sensitive factor is the exponent of the form $\sim \exp[i(\bar{\xi} - \bar{\xi}')^2/4\tau]$, which for large γ factors takes the form $\sim \exp[i\gamma(u-u')^2/2\tau]$. To take the Penrose limit we use the following relation (see also [86]):

$$\delta(u) = \lim_{\epsilon\to 0} \frac{1}{\sqrt{2\pi i\epsilon}} e^{i\frac{u^2}{2\epsilon}}\,. \tag{4.13}$$

Denote $\epsilon = \tau/\gamma^2$ and apply this relation to (4.9) to obtain

$$\lim_{\gamma\to\infty} \gamma\, \mathcal{G}_d(\bar{r}) = \frac{1}{\sqrt{2}} \mathcal{G}_{d-1}(r_\perp)\delta(u-u')\,, \tag{4.14}$$

where $r_\perp^2 = \delta_{ij} x_\perp^i x_\perp^j$.

This means that in the Penrose limit the Green function is factorized and becomes a product of the delta function and a similar Green function in a space with one spatial dimension less. Note that this property is universal in the following sense: It is valid in any number of dimensions and for an arbitrary choice of the form factor.

4.3 Electromagnetic field of ultrarelativistic charged and magnetized objects

The main tool for the study of the fields of ultrarelativistic objects both in the electromagnetic and gravitational theories is using the transformation law of the Green functions in the Penrose limit (4.14) described in the previous section. However, these two cases are technically slightly different. This difference is connected with the spin of the fields. For the electromagnetic field the spin is one and it is described by a vector potential $\boldsymbol{A}$. For gravity the spin is two and the linearized metric is a rank two tensor. As a result the behavior of the charge and magnetic moment under the boost transformation differs from the behavior of the energy and angular momentum of the gravity sources.

4.3.1 Electromagnetic field in the Penrose limit

We consider first the electromagnetic field of ultrarelativistic objects. For simplicity we limit our consideration to charged and magnetized pencils whose transverse charge and magnetic moment densities are δ-shaped, but we allow a density profile in the pre-boosted $\bar{\xi}$-direction to be arbitrary functions of $\bar{\xi}$. Let us denote these densities by $\bar{\lambda}(\bar{\xi})$ and $\bar{\mu}_a(\bar{\xi})$ for the charged and magnetized pencils, respectively. Results for objects with a transverse extension can be obtained by superposing these solutions.

We use the obtained earlier (in chaper 2) expressions for the potential $\bar{A}_\mu$ for a pencil like distribution of the electric charge and magnetic moment

$$\bar{A}_{\bar{t}} = \int_{-\infty}^{\infty} \mathrm{d}\bar{\xi}'\, \bar{\lambda}(\bar{\xi}')\mathcal{G}_d(\bar{r})\,, \tag{4.15}$$

$$\bar{A}_a = -2\pi \int_{-\infty}^{\infty} \mathrm{d}\bar{\xi}'\, \mathcal{G}_{d+2}(\bar{r}) \sum_{a=1}^{n} \bar{\mu}_a(\bar{\xi}')\rho_a^2\,, \tag{4.16}$$

$$\bar{r}^2 = (\bar{\xi}' - \bar{\xi})^2 + \boldsymbol{x}_\perp^2, \quad \boldsymbol{x}_\perp^2 = \sum_{a=1}^{n} \rho_a^2 + \epsilon z^2\,. \tag{4.17}$$

Both $\bar{\lambda}(\bar{\xi})$ and $\bar{\mu}_a(\bar{\xi})$ are one-dimensional line densities.

We boost these solutions by applying the Lorentz transformation (4.3) to it. After this we need to take the limit $\gamma \to \infty$. In order to obtain finite (and non-vanishing) expressions for the field in the Penrose limit one should keep the total charge q fixed, $q = \bar{q}$, while the magnetic moment should be rescaled as follows $m = \gamma \bar{m}$. In accordance with this scaling we define

$$\lambda(u) = \lim_{\gamma\to\infty} \sqrt{2}\gamma\bar{\lambda}(-\sqrt{2}\gamma u)\,, \tag{4.18}$$

$$\mu_a(u) = \lim_{\gamma\to\infty} \sqrt{2}\bar{\mu}_a(-\sqrt{2}\gamma u)\,. \tag{4.19}$$

For this reason, again making use of the relation (4.14), one finds the following expressions for the ultrarelativistic charged and magnetized pencils:

$$A_\mu \mathrm{d}X^\mu = \lambda(u)\mathcal{G}^N_{d-1}(r_\perp)\mathrm{d}u - \pi\mathcal{G}^N_{d+1}(r_\perp)\sum_{a=1}^{n}\mu_a(u)\rho_a^2\mathrm{d}\varphi_a\,. \tag{4.20}$$

The spacetime metric in (u, v) coordinates is

$$\mathrm{d}s^2 = -2\mathrm{d}u\mathrm{d}v + \sum_{a=1}^{n}(\mathrm{d}\rho_a^2 + \rho_a^2\mathrm{d}\varphi_a^2) + \epsilon\mathrm{d}z^2\,. \tag{4.21}$$

The potential (4.20) is regular as $r_\perp \to 0$, which is in stark contrast to the results one obtains in standard local Maxwell theory.

4.3.2 Properties of solutions

Let us now make some remarks concerning the properties of the obtained solutions within nonlocal Maxwell theory. To that end, the obtained solution (4.20) can be written in the form

$$\boldsymbol{A} = \sum_{a=0}^{n}\lambda_a(u)a_a(r_\perp)\boldsymbol{\zeta}_a\,, \tag{4.22}$$

where $\lambda_0(u) = \lambda(u)$ and $\lambda_{a\geq 1}(u) = \mu_a(u)$. We denoted by $\boldsymbol{\zeta}$ the following Killing vectors:

$$\boldsymbol{\zeta}_0 = \partial_v, \qquad \boldsymbol{\zeta}_a = \partial_{\varphi_a}\,. \tag{4.23}$$

It is easy to check that

$$\mathcal{L}_{\boldsymbol{\zeta}_a} \boldsymbol{A} = 0\,, \tag{4.24}$$

where $\mathcal{L}_{\boldsymbol{\zeta}}$ is the Lie derivative in $\boldsymbol{\zeta}$ direction. These relations show that the boosted solutions have the expected symmetries, that is, no dependence on advanced time v as well as rotational isometries in the φ_a-directions of Π_a two-planes.

Another observation is the following. In the absence of the magnetic moments, $\mu_a = 0$, both in the local and nonlocal case, the electromagnetic field $\boldsymbol{F}$ is null [1],

$$F_{\mu\alpha} F^{\alpha}{}_{\nu} = S u_{,\mu} u_{,\nu}\,, \quad F^2 = \frac{1}{2} F_{\mu\nu} F^{\mu\nu} = 0\,. \tag{4.25}$$

In general, the presence of the magnetic field violates this property. However, the case of electrodynamics in four-dimensional spacetime is an exception. To demonstrate this, let us write the potential 1-form $\boldsymbol{A}$ as

$$\boldsymbol{A} = b(u) c(\rho) \boldsymbol{\zeta}_0 + B(u) C(\rho) \boldsymbol{\zeta}_1\,, \tag{4.26}$$

where $\boldsymbol{\zeta}_0$ and $\boldsymbol{\zeta}_1$ are the 1-forms that are dual to their respective Killing vector. Then, calculations show that

$$F^2 = B^2 \left(\rho \frac{\mathrm{d}C}{\mathrm{d}\rho} + 2C \right)^2 . \tag{4.27}$$

Thus $F^2 = 0$ only when $C = C_0/\rho^2$. This is precisely the case for the field of the magnetized and charged ultrarelativistic pencil in four dimensions in the framework of the standard local Maxwell theory. For the theory in higher dimensions this property is violated. Let us emphasize that in the nonlocal theory $F^2 \neq 0$ for ultrarelativisic magnetized pencils not only in the higher dimensions, but in four spacetime dimensions as well.

[1] Let us note that for a static q-pencil of fixed length $\bar{L}$ the invariant F^2 does not vanish. However, when one takes the Penrose limit, the length $\bar{L}$ is not fixed but is multiplied by γ. As a result, F^2 decreases and becomes zero in the limit $\gamma \to \infty$.

4.4 Gravitational field of ultrarelativistic massive and spinning objects

We already mentioned that there exist a similarity between electromagnetic and linearized gravity equations. However, there is also a quite important difference. In the electromagnetism a charge can have both, positive and negative signs. For this reason one can consider neutral magnetized objects. In gravity the situation is quite different. The energy of physical objects is positive. One cannot put the energy of an object equal to zero and keep only the spin-induced component of the gravitational field. This means that in application of the results discussed in this section to a beam of spinning massless particles one should analyze the relation between the energy density and spin distribution in the beam. This can be done for example in the framework of the geometric optics approximation. This subject is beyond the scope of this thesis. In what follows we focus on the solutions for the gravitational field of ultrarelativistic sources in linearized gravity and its nonlocal ghost-free generalization. In this approach we keep the energy and angular momentum density parameters arbitrary.

4.4.1 Scaling properties

Let us first make a simple remark concerning the scaling properties of the pencil characteristics under a boost transformation (4.3). Let us assume that both mass and angular momentum are uniformly distributed along the pencil and their densities in the rest $\bar{S}$ frame, $\bar{\lambda} = \bar{m}/\bar{L}$ and $\bar{j} = \bar{J}/\bar{L}$ are constant. Because of the Lorentz contraction, the length of the same pencil, as measured in the moving frame S is $L = \bar{L}/\gamma$, while its energy is $E = m = \gamma\bar{m}$. As a result, the linear energy density of the pencil in the S frame is $\lambda = \gamma^2\bar{\lambda}$. In the Penrose limit the energy E is taken to be fixed. Thus the energy density λ grows to infinity as $\gamma \to \infty$. To keep it finite, one needs to rescale $\bar{L} \to \gamma\bar{L}$ in the boost process, such that the length L remains unchanged[2].

Because we keep the ratio $\bar{L}/\gamma$ constant during the Penrose limit, the linear density scales as follows:

$$\lambda(u) = \lim_{\gamma\to\infty} \sqrt{2}\gamma^2\,\overline{\lambda}(-\sqrt{2}\gamma u)\,. \tag{4.28}$$

This guarantees that in the Penrose limit the product $\overline{m}\gamma$ and the ratio $\bar{L}/\gamma$ remain constant,

$$E = \gamma\,\overline{m} = \gamma \int\limits_{-\infty}^{\infty} \mathrm{d}\bar{\xi}\,\overline{\lambda}(\bar{\xi}) = \int\limits_{-\infty}^{\infty} \mathrm{d}u\,\lambda(u) = \mathrm{const}\,. \tag{4.29}$$

The angular momentum line density $\bar{j}_{ij}(\bar{\xi})$ "lives" in transverse space and its tensorial structure is unaffected by the boost. Using this property we define the boosted linear density of the angular momentum in the S frame as follows:

$$j_{ij}(u) = \lim_{\gamma\to\infty} \sqrt{2}\gamma\,\bar{j}_{ij}(-\sqrt{2}\gamma u)\,, \tag{4.30}$$

$$j_{a}(u) = \lim_{\gamma\to\infty} \sqrt{2}\gamma\,\bar{j}_{a}(-\sqrt{2}\gamma u)\,. \tag{4.31}$$

The total angular momentum J of the boosted pencil remains unchanged, finite and has the form

$$J_{ij} = \int\limits_{-\infty}^{\infty} \mathrm{d}\bar{\xi}\,\bar{j}_{ij}(\bar{\xi}) = \int\limits_{-\infty}^{\infty} \mathrm{d}u j_{ij}(u)\,. \tag{4.32}$$

4.4.2 Metric

After the Penrose limit, as defined above, the resulting metric takes the form

$$\begin{aligned} \boldsymbol{g} &= (\eta_{\mu\nu} + h_{\mu\nu})\,\mathrm{d}X^{\mu}\mathrm{d}X^{\nu} \\ &= -2\mathrm{d}u\mathrm{d}v + \phi\mathrm{d}u^2 + 2A_i\mathrm{d}x_{\perp}^{i}\mathrm{d}u + \mathrm{d}\boldsymbol{x}_{\perp}^{2}\,, \end{aligned} \tag{4.33}$$

[2]Let us emphasize that in this thesis we do not discuss restrictions on the properties of the matter which creates the gravitational field. If for example one requires that the matter obeys the null energy condition then one can expect that $|\bar{J}| \leq R\bar{m}$, where R is the transverse size of the pencil. So that to keep $|\bar{J}|$ fixed while $\bar{m} \to 0$ without violation of the null energy condition one needs to consider "thick pencils". Let us note that beyond the linear approximation in higher dimensions the situation is quite different. Namely, Myers and Perry in [85] demonstrated that *"for $N > 5$ black holes with a fixed mass may have arbitrarily large angular momentum". (N is the umber of spatial dimensions, which we denoted by d).*

where we defined

$$\phi = \lim_{\gamma\to\infty} 2\gamma^2 \frac{d-1}{d-2}\bar{\phi}\,, \quad A_i = \lim_{\gamma\to\infty} \sqrt{2}\gamma \bar{A}_i\,. \tag{4.34}$$

Here, $\bar{\phi}$ and $\bar{A}_i$ are given by (3.78). The integrands in their representations contain the Green function $\mathcal{G}_d(\bar{r})$. Performing the limit $\gamma \to \infty$ in the relations (4.34) for the potential ϕ and the gravitomagnetic potential A_i using the factorization property of the Green functions in the Penrose limit one finally gets

$$\phi = 2\sqrt{2}\kappa\lambda(u)\mathcal{G}_{d-1}(r_\perp)\,, \tag{4.35}$$

$$A_i = -2\pi\kappa j_{ij}(u)x_\perp^j \mathcal{G}_{d+1}(r_\perp)\,. \tag{4.36}$$

Introducing polar coordinates $\{\rho_a, \varphi_a\}$ in each Darboux plane Π_a such that

$$y^a = \rho_a \cos\varphi_a\,, \quad \hat{y}^a = \rho_a \sin\varphi_a\,, \tag{4.37}$$

one may use the relation

$$j_{ij}\, x_\perp^i \mathrm{d}x_\perp^j = \sum_{a=1}^{n} j_a \rho_a^2 \mathrm{d}\varphi_a\,, \tag{4.38}$$

to rewrite the gravitomagnetic potential 1-form as

$$A_i(\boldsymbol{x}_\perp)\mathrm{d}x_\perp^i = 2\pi\kappa\mathcal{G}_{d+1}(r_\perp)\sum_{a=1}^{n} j_a(u)\rho_a^2 \mathrm{d}\varphi_a\,, \tag{4.39}$$

which makes the rotational symmetry in each Darboux plane manifest.

4.5 Gravitational field of ghost-free gyratons

Ultrarelativistic spinning objects creating the gravitational field described in the previous section are often called gyratons (see e.g. [70]). In this section we present and discuss gyraton-like solutions in General Relativity and in infinite-derivative nonlocal gravity.

In General Relativity, the form factor $f(\Box)$ is simply

$$f(\Box) = 1\,, \tag{4.40}$$

whereas in infinite-derivative GF_N gravity model this form factor as the form

$$f(\Box) = \exp\left[(-\Box\ell^2)^N\right] \,. \tag{4.41}$$

The static Green function (2.60) can be computed for a wide range of theories, but in the context of the present thesis we shall consider General Relativity as well as two infinite-derivative theories corresponding to the choices $N = 1$ and $N = 2$, which we shall hence refer to as GF_1 and GF_2. It is also possible to extend these studies to arbitrary number of spatial dimensions d.

4.5.1 Gyratons in d=3

Gyraton metrics in General Relativity

As a warm-up, let us consider the well-known gyraton solutions of (3+1)-dimensional General Relativity [44, 46, 51, 52]. The relevant two-dimensional and four-dimensional static Green functions are

$$G_2(r) = -\frac{1}{2\pi}\log(r)\,, \quad G_4(r) = \frac{1}{4\pi^2 r^2}\,. \tag{4.42}$$

Since in $d = 3$ the transverse space is two-dimensional we have $n = 1$ and $\epsilon = 0$. Therefore we may write $|\boldsymbol{x}_\perp| = \rho$, call the polar angle φ, and denote by $j(u)$ the linear density of the angular momentum in the S frame. Then, the gravitational potentials ϕ and $\boldsymbol{A} = A_i \mathrm{d}x^i$ are

$$\phi(u,\rho) = -\frac{\sqrt{2}\kappa\lambda(u)}{2\pi}\log(\rho)\,, \tag{4.43}$$

$$\boldsymbol{A}(u) = \frac{\kappa j(u)}{2\pi}\mathrm{d}\varphi\,. \tag{4.44}$$

This gravitomagnetic field is locally exact such that

$$\boldsymbol{F} = \mathrm{d}\boldsymbol{A} = 0\,. \tag{4.45}$$

Observe, however, that the gravitomagnetic charge does not vanish:

$$Q_0 = \int\limits_{\mathcal{A}} \boldsymbol{F} = \oint\limits_{\partial A} \boldsymbol{A} = \kappa j(u)\,. \tag{4.46}$$

Here, $\mathcal{A}$ denotes a surface in the Darboux plane. For later convenience we may assume $\mathcal{A}$ to be a circle of radius ρ. However, in a given null plane $u = \text{const}$ this charge does not depend on the choice of the contour $\partial\mathcal{A}$. As we shall see soon, this property is no longer valid in nonlocal gravity, and effectively the gravitomagnetic current is spread out of the $\rho = 0$ line in the direction transverse to the motion.

Gyraton metrics in ghost-free gravity

We consider now a similar gyraton solutions in the nonlocal theories GF_1 and GF_2. The two dimensional static Green function for GF_1 theory can be written as

$$\mathcal{G}_2(r) = -\frac{1}{4\pi}\mathrm{Ein}\left(\frac{r^2}{4\ell^2}\right)\,, \tag{4.47}$$

where $\mathrm{Ein}(x)$ denotes the complementary exponential integral and $E_1(x)$ is the exponential integral [77],

$$\mathrm{Ein}(x) = \int\limits_0^x \mathrm{d}z\frac{1-e^{-z}}{z} = E_1(x) + \ln x + \gamma\,, \tag{4.48}$$

$$E_1(x) = e^{-x}\int\limits_0^\infty \mathrm{d}z\frac{e^{-z}}{z+x} = -\mathrm{Ei}(-x)\,, \tag{4.49}$$

and $\gamma = 0.577\ldots$ is the Euler–Mascheroni constant. Then, the gravitational potentials ϕ and $\boldsymbol{A}$ take the form

$$\phi(u,\rho) = -\frac{\sqrt{2}\kappa\lambda(u)}{2\pi}\mathrm{Ein}\left(\frac{\rho^2}{4\ell^2}\right)\,, \tag{4.50}$$

$$\boldsymbol{A}(u,\boldsymbol{x}_\perp) = \frac{\kappa j(u)}{2\pi}\left[1-\exp\left(-\frac{r_\perp^2}{4\ell^2}\right)\right]\mathrm{d}\varphi\,. \tag{4.51}$$

This gravitomagnetic field is no longer exact and hence the gravitomagnetic charge depends on the radius,

$$Q_1(\rho) = \kappa j(u)\left[1-\exp\left(-\frac{\rho^2}{4\ell^2}\right)\right]\,. \tag{4.52}$$

At large distances, $\rho \gg \ell$, we recover the gyraton solution obtained in General Relativity. In GF_2 theory one has

$$\begin{aligned}\mathcal{G}_2(r) = \frac{y}{2\pi}\Big[& \sqrt{\pi}\,{}_1F_3\left(\tfrac{1}{2};\ 1, \tfrac{3}{2}, \tfrac{3}{2};\ y^2\right) \\ & - y\,{}_2F_4\left(1, 1;\ \tfrac{3}{2}, \tfrac{3}{2}, 2, 2;\ y^2\right)\Big]\,,\end{aligned} \tag{4.53}$$

where we defined $y = \rho^2/(16\ell^2)$. The gravitomagnetic charge now takes the form

$$\begin{aligned}Q_2(\rho) = -\kappa j(u)\Big[& 1 - {}_0F_2\left(\tfrac{1}{2}, \tfrac{1}{2}; y^2\right) \\ & - 2\sqrt{\pi}y\,{}_0F_2\left(1, \tfrac{3}{2}; y^2\right)\Big]\,.\end{aligned} \tag{4.54}$$

See Fig. 4.2 for a plot of these charges. Interestingly, the GF_1 charge is monotonic, whereas the GF_2 charge exhibits an oscillatory behavior.

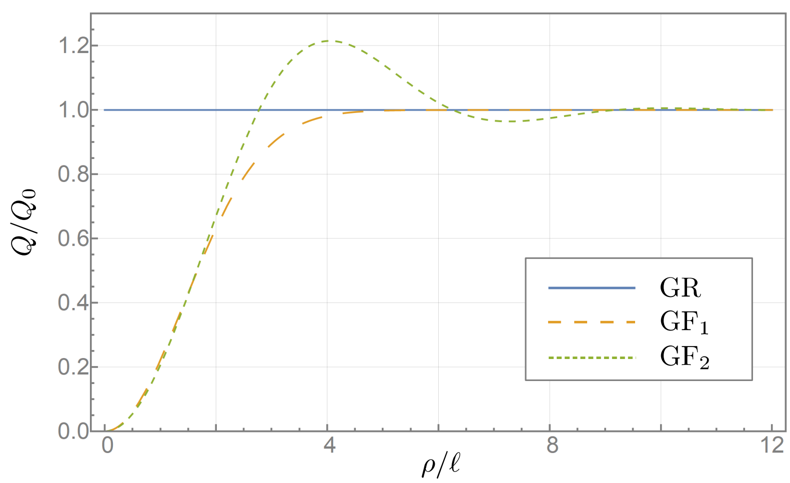

Figure 4.2: The gravitomagnetic charges on a plane $u = \text{const.}$ of the four-dimensional gyraton in linearized General Relativity as well as linearized GF_1 and GF_2 theory plotted as a function of ρ/ℓ. The charges are normalized to the value Q_0 encountered in General Relativity.

Curvature invariants

One may wonder about the geometric properties of the four-dimensional gyraton spacetime

$$\begin{aligned}\boldsymbol{g} = & -2\mathrm{d}u\mathrm{d}v + \phi(u, x, y)\mathrm{d}u^2 + \mathrm{d}x^2 + \mathrm{d}y^2 \\ & + 2\left[A_x(u, x, y)\mathrm{d}x + A_y(u, x, y)\mathrm{d}y\right]\mathrm{d}u\,.\end{aligned} \tag{4.55}$$

This spacetime is a pp-wave because it features a covariantly constant null Killing vector $\boldsymbol{k} = \partial_v$ [57],

$$\nabla_\nu k^\mu = 0\,. \tag{4.56}$$

This property remains valid for any choice of the functions ϕ, A_x and A_y, provided their functional dependence remains the same. Since pp-wave spacetimes have vanishing scalar polynomial curvature invariants one finds

$$R = R_{\mu\nu}R^{\mu\nu} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = 0\,. \tag{4.57}$$

For this reason they remain unchanged for solutions found in the context of linearized infinite-derivative gravity as compared to linearized General Relativity.

4.5.2 Gyratons in d $\geq$ 4 dimensions

$d = 4$ case

In five spacetime dimensions one has $d = 4$, which implies that $n = 1$ and $\epsilon = 1$. In this case there is only one Darboux plane orthogonal to ξ as well as one additional z-axis. Let us write the transverse distance as $r_\perp^2 = \rho^2 + z^2$, where ρ is the radial variable in the Darboux plane. Then, from Eqs. (4.35) as well as (4.39), one obtains

$$\phi = 2\sqrt{2}\kappa\lambda(u)\mathcal{G}_3(r_\perp)\,, \tag{4.58}$$

$$A_i \mathrm{d}x_\perp^i = -\frac{\kappa}{r_\perp}\frac{\mathrm{d}}{\mathrm{d}r_\perp}\mathcal{G}_3(r_\perp)j(u)\rho^2\mathrm{d}\varphi\,, \tag{4.59}$$

where φ is the polar angle in the Darboux plane, $j(u)$ is the angular momentum eigenfunction, and $\lambda(u)$ describes the density profile. The explicit expressions for the functions $\mathcal{G}_3$ in linearized General Relativity as well as in GF_1 and GF_2 theories are given in Appendix F.

Higher dimensions

In higher dimensions one can proceed analogously to find expressions for the gyraton metrics. Instead of repeating previous steps, we give here an algorithmic procedure of how to construct such solutions in an arbitrary number of higher dimensions.

First, given the number of spatial dimensions d, determine the number of Darboux planes n using (3.72). If d is even there will be an independent z-axis as well. Due to the rotational symmetry around the pre-boosted $\bar{\xi}$-direction it makes sense to introduce polar coordinates in each Darboux plane called $\{\rho_a, \varphi_a\}$ where a labels the Darboux planes. This construction is unique, provided one fixes the direction of the polar angles φ_a to be right-handed with respect to the original $\bar{\xi}$-direction.

Second, one introduces the perpendicular radius variable $r_\perp$ according to

$$r_\perp^2 = \sum_{a=1}^{n} \rho_a^2 + \epsilon z^2 \,. \tag{4.60}$$

Recall that $\epsilon = 1$ if d is even, and $\epsilon = 0$ if d is odd. Now one can insert this radius variable into (4.35) and (4.39). In order to determine the static Green function $\mathcal{G}_d(r_\perp)$ in higher dimensions one may utilize the recursion formula (2.62) as well as Appendix F.

Last, one may want to start with a known line energy density $\bar{\lambda}(\bar{\xi})$ as well as angular momentum line densities $\bar{j}_a(\bar{\xi})$ in the original rest frame. In that case, Eqs. (4.28) and (4.30) provide prescriptions as to how to retrieve the resulting functions $\lambda(u)$ and $j_a(u)$ in retarded time.

Realistic gyratons may also have a finite transverse thickness, but due to the linearity of the problem it is always possible to supplement a transverse density function in (3.73) and construct the gravitational field of a "thick gyraton" by superposition.

4.6 Summary of chapter 4

This chapter is devoted to study fields of objects moving with very large speed close to the speed of light. For this purpose we apply the Lorentz boost to the stationary solutions of the nonlocal modification of the Maxwell and linearized gravity equations. We demonstrated that for a properly chosen scaling of the parameters of the stationary solutions one can obtain a well defined Penrose limit. We then proved a remarkable property that the Green functions: In the Penrose limit they are dimensionally reduced and factorized. Namely, after taking the Penrose limit, the Green function takes the form of a product of a delta-function localized at the null plane and a similar Green function in a space with one dimension less. We study and describe properties of the obtained solutions.

Chapter 5

Summary and Discussion

Let us summarize the results included in this thesis. We study higher dimensional Maxwell and Einstein equations. General interest in such higher dimensional theories was motivated by string theory in which the existence of extra dimensions is important for the consistency of the theory. We considered nonlocal modifications of Maxwell and linearized gravity where the action and the field equations contain infinite number of derivatives. The assumptions that the theory possess Lorentz invariance and is invariant under corresponding gauge transformation greatly restrict the form of the actions. The next important assumption was that the modified theories contain the same number of physical degrees of freedom as its local counterpart. This assumption requires that the form factor of the nonlocal theory does not have zeroes on the complex plane of its argument. We focused on wide class of models GF_N with form factors of the form

$$f(\Box) = \exp\left[(-\ell^2\Box)^N\right] \, . \tag{5.1}$$

The chapters 2 and 3 of the thesis are devoted to the study of stationary solutions of point-like and extended sources in nonlocal versions of the Maxwell and linearized gravity equations. The main tool for this study is the method of Green function. Using static Green functions we obtained solutions for the higher-dimensional electromagnetic field of charged and magnetized sources in the nonlocal modification of the Maxwell theory (chapter 2). Similar results are also obtained for the gravitational field of massive spinning objects in the weak field approximation of the nonlocal gravity equations (chapter 3).

The analyses of the obtained solutions show that all of the solutions for the point-like sources in the ghost-free modifications GF_N of the electromagnetism and gravity are regular at their origin. This is a common and interesting property of these models. Earlier publications demonstrated similar results for static (non-spinning) massive particles in the ghost-free gravity. We generalized these results in the two following ways:

- We proved the regularity of the gravitational field for massive spinning point-like particles in the modified nonlocal gravity;
- We obtained a similar result for the electromagnetic field of a point-like particle with electric charge and magnetic moment in the nonlocal modification of Maxwell theory.

In the second part of the thesis we studied the electromagnetic and gravitational field of ultrarelativistic objects. In our analysis we assume that the number $D \geq 4$ of spacetime dimensions can be arbitrary. We also considered both local and nonlocal theories. To find a field of an ultrarelativistic object we first made a boost transformation of the stationary solutions obtained in chapters 2 and 3. After this, we considered a limit when the velocity of the boost tends to the speed of light. In order to obtain a finite physically meaningful result we accompanied this process by additional scaling transformations of the parameters of the initial stationary solutions. We demonstrated that these scaling transformations depend on the nature (spin) of the field. A very important technical point in obtaining these results is played by a special integral representation of the Green functions (see appendix E). This representation demonstrates that in the Penrose limit the d dimensional Green function is factorized and it can be presented as a product of the delta-function localized at the null plane and the corresponding $(d-1)$-dimensional Green function. This universal behavior of the Green functions in the Penrose limit greatly simplifies obtaining the required solutions. We studied the obtained solutions for the fields of the ultrarelativistic objects and demonstrated that in the class of GF_N theories they are regular at the source position.

The results obtained in this thesis might have several interesting applications. They can be used for studying the scattering of two ultrarelativistic particles. This would allow one to single out properties of the scattering amplitudes which might indicate the the existence of extra dimensions and nonlocality. In the latter case, the experimental results might restrict the value of the nonlocality scale ℓ. Another more complicated problem is to study how non-linearity of the nonlocal gravity equations modified the results obtained in the linear approximation. This is a very important question connected with the existence of black-hole and cosmological singularities. Finally, let us mention one more interesting problem. In Einstein gravity the Penrose limit of a weak field static solution coincides with the Aichelburg-Sexl metric which is an exact solution to the full non-linear Einstein equations. An open question is whether this property (under some reasonable conditions) is valid for the considered nonlocal theories. It would be interesting to answer these questions, which go far beyond the scope of this thesis.

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Appendix A: Field of a magnetized pencil

In this appendix we give an expression for the potential of an m-pencil in three spatial dimensions, which is used in the main body of this thesis. To that end, let us consider a thin cylinder of radius R of the length L along the $\bar{\xi}$-axis. Let us further assume that the cylinder has a total electric charge Q and is rotating with the angular velocity ω around its symmetry axis. If one wishes to consider an uncharged magnetized object, then—by linearity of the Maxwell equations—one may just add another cylinder with the opposite charge $-Q$. At any rate, the magnetic moment $\mathbf{m}$ of a system of charges q_a located at $\boldsymbol{r}_a$ and moving with velocity $\boldsymbol{v}_a$ is given by the following expression [87]:

$$\boldsymbol{m} = \frac{1}{2}\sum_a q_a[\boldsymbol{r}_a \times \boldsymbol{v}_a]\,. \tag{A.1}$$

In cylindrical coordinates $\{\rho, \varphi, \bar{\xi}\}$ the magnetic moment of the rotating charged cylinder takes the form

$$\boldsymbol{m} = (0, 0, \bar{m}), \qquad \bar{m} = \frac{1}{2}QR^2\omega\,. \tag{A.2}$$

The current density of a system of charged particles is

$$\bar{\boldsymbol{J}} = \sum_a q_a \boldsymbol{v}_a \delta^{(d)}(\boldsymbol{r} - \boldsymbol{r}_a)\,. \tag{A.3}$$

For the rotating cylinder one finds

$$\begin{aligned} \bar{\boldsymbol{J}} &= (0, \bar{J}^{\varphi}, 0)\,, \\ \bar{J}^{\varphi} &= \frac{Q\omega}{2\pi R\bar{L}}\delta(\rho - R)\Theta\left(\bar{\xi}| - \bar{L}/2, \bar{L}/2\right)\,. \end{aligned} \tag{A.4}$$

The field equations (2.6) give the following equation for the potential $\bar{A}_\varphi$:

$$\rho\partial_\rho\left(\frac{1}{\rho}\partial_\rho\bar{A}_\varphi\right)+\partial_{\bar{\xi}}^2\bar{A}_\varphi=\bar{J}_\varphi\,. \tag{A.5}$$

Let us denote $\partial_\rho\bar{A}_\varphi=\rho Z$, then (A.5) gives

$$\frac{1}{\rho}\partial_\rho\left(\rho\partial_\rho Z\right)+\partial_{\bar{\xi}}^2 Z=j\,, \tag{A.6}$$

where $j(\rho)=\frac{1}{\rho}\partial_\rho\bar{J}_\varphi$. The left-hand side of this equation is nothing but the flat three-dimensional Laplace operator in cylindrical coordinates applied to the function $Z(\rho,\bar{\xi})$. Using the Green function of this operator, expressed in cylindrical coordinates, one then obtains

$$Z(\rho,\bar{\xi})=\int\limits_{-\bar{L}/2}^{\bar{L}/2}\mathrm{d}\bar{\xi}'\int\limits_0^{2\pi}\rho'\mathrm{d}\varphi' P,\quad P=-\int\limits_0^{\infty}\mathrm{d}\rho'\frac{j(\rho')}{4\pi r}\,, \tag{A.7}$$

$$r=(\rho^2+\rho'^2-2\rho\rho'\cos\varphi'+z^2)^{1/2}\,. \tag{A.8}$$

Here we abbreviated $z=\bar{\xi}-\bar{\xi}'$. The integral for P can be evaluated with the following result

$$P=\frac{Q\omega R}{8\pi^2\bar{L}}\frac{\partial}{\partial R}\left(\frac{1}{r}\right)\,, \tag{A.9}$$

where r is given by (A.8) subject to the substitution $\rho'=R$ in this expression. Using definition (A.2) of the magnetic moment one can write

$$Z(\rho,\bar{\xi})=\frac{\bar{m}}{4\pi^2 R\bar{L}}\int\limits_{-\bar{L}/2}^{\bar{L}/2}\mathrm{d}\bar{\xi}'\frac{\partial I}{\partial R}\,. \tag{A.10}$$

Here we introduced the shorthand notation

$$I=\int\limits_0^{2\pi}\frac{\mathrm{d}\varphi}{r}=\frac{4K\left(\frac{2\sqrt{R\rho}}{\sqrt{(\rho+R)^2+z^2}}\right)}{\sqrt{(\rho+R)^2+z^2}}\,, \tag{A.11}$$

where K is the complete elliptic integral of second type. Using its expansion for small valuse of its argument one can find

$$S=\lim_{R\to 0}\frac{1}{R}\partial_R I=\pi\frac{\rho^2-2z^2}{(\rho^2+z^2)^{5/2}}\,. \tag{A.12}$$

Using these results and restoring $\bar{A}_\varphi$ one obtains

$$\bar{A}_\varphi = \frac{\bar{m}}{4\pi\bar{L}} \int\limits_{-\bar{L}/2}^{\bar{L}/2} \mathrm{d}\bar{\xi}' \int\limits_0^{\rho} \mathrm{d}\rho' \rho' \, S \,. \tag{A.13}$$

Performing the integration over ρ' and $\bar{\xi}'$ finally yields

$$\bar{A}_\varphi = \frac{\bar{m}}{4\pi\bar{L}} \left(\frac{\bar{\xi}_+}{\sqrt{\rho^2 + \bar{\xi}_+^2}} - \frac{\bar{\xi}_-}{\sqrt{\rho^2 + \bar{\xi}_-^2}} \right) , \tag{A.14}$$

where we abbreviated $\bar{\xi}_\pm = \bar{\xi} \pm \bar{L}/2$. Let us mention that this expression for the field of a magnetized infinitely thin pencil can be also obtained by using the expression for the potential of a magnetized solenoid of finite radius R, which can be found in Jackson's book [75].

Appendix B: Darboux basis

Consider a skew-symmetric matrix $A_{ij} = -A_{ji}$ in $d = 2n + \epsilon$ dimensional space. Let γ_{ij} be the Euclidean metric in it. We define $A^i{}_j = \gamma^{ik} A_{kj}$. Let $\boldsymbol{v}$ be a vector with components v^j. We denote $\boldsymbol{A}\boldsymbol{v} = \boldsymbol{u}$, where $\boldsymbol{u}$ is a vector with components $u^i = A^i{}_j v^j$. Since for any vector $\boldsymbol{u}$ one has

$$< \boldsymbol{u}, \boldsymbol{u} >= \gamma_{ij} u^i u^j \geq 0 \,, \tag{B.1}$$

then

$$0 \leq \gamma_{ij} A^i{}_k v^k A^j{}_m v^m = -v^k A_k{}^j A^j{}_m v^m = - < \boldsymbol{v}, \boldsymbol{A}^2 \boldsymbol{v} > \,. \tag{B.2}$$

This means that a symmetric matrix $\boldsymbol{A}^2$ is non-positive definite.

Consider eigenvectors of $\boldsymbol{A}^2$. Let $\boldsymbol{v}$ be its eigenvector

$$\boldsymbol{A}^2 \boldsymbol{v} = -\lambda^2 \boldsymbol{v} \,. \tag{B.3}$$

Denote $\boldsymbol{u}\boldsymbol{A} = \boldsymbol{v}$. Then

$$\boldsymbol{A}^2 \boldsymbol{u} = \boldsymbol{A}\boldsymbol{A}^2 \boldsymbol{v} = -\lambda^2 \boldsymbol{A}\boldsymbol{v} = -\lambda^2 \boldsymbol{u} \,. \tag{B.4}$$

In other words, if $\boldsymbol{v}$ is the eigenvector of $\boldsymbol{A}^2$ with the eigenvalue $-\lambda^2$, then the same is true for the vector $\boldsymbol{u}$. It is easy to see that these two vectors are orthogonal, $< \boldsymbol{u}, \boldsymbol{v} >= 0$. We chose the vector $\boldsymbol{v}$ to have unit norm, then the vector $\hat{v} = \boldsymbol{u}/\lambda$ is also a unit vector.

Let us enumerate the eigenvalues and eigenvectors by index a. Let us assume that the matrix $\boldsymbol{A}$ in non-degenerate, that is:

- It has maximal number of non-vanishing eigenvalues n, that is $a = 1, \ldots, n$;

- All these eigenvalues are different.

Then it is easy to check that eigenvectors with different eigenvalues are mutually orthogonal. Really

$$-\lambda_{a_2}^2 < \boldsymbol{v}_{a_1}, \boldsymbol{v}_{a_2} >=< \boldsymbol{v}_{a_1}, \boldsymbol{A}^2 \boldsymbol{v}_{a_2} >=< \boldsymbol{v}_{a_2}, \boldsymbol{A}^2 \boldsymbol{v}_{a_1} >= -\lambda_{a_1}^2 < \boldsymbol{v}_{a_1}, \boldsymbol{v}_{a_2} > . \tag{B.5}$$

Since $\lambda_{a_2} \neq \lambda_{a_1}$ then $< \boldsymbol{v}_{a_1}, \boldsymbol{v}_{a_2} >= 0$. Thus one has n mutually orthonormal pairs of vectors $(\boldsymbol{v}_a, \hat{\boldsymbol{v}}_a)$. Each such pair spans a two-plane Π_a. For even $d = 2n$ these two-planes span all the d dimensional space. For odd $d = 2n+1$ there exist one more spatial direction, orthogonal to all the two planes. The basis constructed of $\boldsymbol{v}_a, \hat{\boldsymbol{v}}_a$ vectors is called Darboux basis. The corresponding Cartesian coordinates are known as Darboux coordinates.

In this Darboux basis the matrix $\boldsymbol{A}$ has the form

$$\boldsymbol{A} = \mathrm{diag}(\Lambda_1, ..., \Lambda_n, 0) . \tag{B.6}$$

Where

$$\Lambda_a = \begin{bmatrix} 0 & \lambda_a \\ -\lambda_a & 0 \end{bmatrix} . \tag{B.7}$$

The last term 0 in (B.6) is present when d is odd, so that $\epsilon = 1$. One can show that a similar representation is valid also when the matrix $\boldsymbol{A}$ is degenerate. Further details can be found in [80].

Appendix C: Gauge invariance of the linearized gravity equations

Let us show that both the action and the field equations of the linearized gravity are invariant under gauge transformations. Consider the following transformation of the metric perturbation

$$h_{\mu\nu} \to h_{\mu\nu} - (\xi_{\mu,\nu} + \xi_{\nu,\mu}) \ . \tag{C.1}$$

Here $\xi_\mu(X)$ are arbitrary functions. Let us apply this transformation to the linearized Einstein action

$$S_g = -\frac{1}{2\kappa}\int dX \left(-\frac{1}{2}h^{\mu\nu}\Box h_{\mu\nu} + h^{\mu\nu}\partial_\mu\partial_\alpha\, h^\alpha{}_\nu - h^{\mu\nu}\partial_\mu\partial_\nu h + \frac{1}{2}h\Box h\right) . \tag{C.2}$$

We denote by $\delta_\xi S_g$ the change of the action under the gauge transformation and keep only the first order ξ_μ terms in it. It is convenient to consider separately the variation of each of the four terms in the integrand of (C.2) which we denote by δS_{gi}, $i = 1, 2, 3, 4$. For the first term inside the parenthesis we have

$$\begin{aligned}\delta_\xi S_{g1} &= -\frac{1}{2\kappa}\int \mathrm{d}X\left(\frac{1}{2}(\xi^{\mu,\nu} + \xi^{\nu,\mu})\Box h_{\mu\nu} + \frac{1}{2}h^{\mu\nu}\Box(\xi_{\mu,\nu} + \xi_{\nu,\mu})\right)\\ &= -\frac{1}{2\kappa}\int \mathrm{d}X\left(h^{\mu\nu}\Box(\xi_{\mu,\nu} + \xi_{\nu,\mu})\right) .\end{aligned} \tag{C.3}$$

The variation of the second term yields

$$\begin{aligned}\delta_\xi S_{g2} &= -\frac{1}{2\kappa}\int \mathrm{d}X\left(-h^{\mu\nu}\partial_\mu\partial_\alpha(\xi^\alpha{}_{,\nu} + \xi_{\nu,}{}^\alpha) - (\xi^{\mu,\nu} + \xi^{\nu,\mu})\partial_\mu\partial_\alpha h^\alpha{}_\nu\right)\\ &= -\frac{1}{2\kappa}\int \mathrm{d}X\left(-2h^{\mu\nu}\partial_\mu\partial_\alpha(\xi^\alpha{}_{,\nu} + \xi_{\nu,}{}^\alpha)\right) .\end{aligned} \tag{C.4}$$

The variation of third term gives

$$\begin{aligned}\delta_\xi S_{g3} &= -\frac{1}{2\kappa}\int \mathrm{d}X\,(h^{\mu\nu}\partial_\mu\partial_\nu(\xi^\alpha{}_{,\alpha}+\xi_{\alpha,}{}^\alpha)+(\xi^{\mu,\nu}+\xi^{\nu,\mu})\partial_\mu\partial_\nu h)\\ &= -\frac{1}{\kappa}\int \mathrm{d}X\,(h^{\mu\nu}\partial_\mu\partial_\nu\xi^\alpha{}_{,\alpha}+h\Box\xi^\alpha{}_{,\alpha})\,.\end{aligned}\tag{C.5}$$

Finally

$$\begin{aligned}\delta_\xi S_{g4} &= -\frac{1}{2\kappa}\int dX\left(-\frac{1}{2}h\Box(\xi^\mu{}_{,\mu}+\xi_{\mu,}{}^\mu)-\frac{1}{2}(\xi^\alpha{}_{,\alpha}+\xi_{\alpha,}{}^\alpha)\Box h\right)\\ &= \frac{1}{\kappa}\int dX h\Box\xi^\mu{}_{,\mu}\,.\end{aligned}\tag{C.6}$$

Combining these results we obtain

$$\begin{aligned}\delta_\xi S_g = -\frac{1}{\kappa}\int \mathrm{d}X\Big(&h^{\mu\nu}\Box\xi_{\mu,\nu}-h^{\mu\nu}\partial_\mu\partial_\alpha(\xi^\alpha{}_{,\nu}+\xi_{\nu,}{}^\alpha)\\ &+h^{\mu\nu}\partial_\mu\partial_\nu\xi^\alpha{}_{,\alpha}\Big)\,.\end{aligned}\tag{C.7}$$

The expression under the integral vanishes and we have

$$\delta_\xi S_g = 0\,.\tag{C.8}$$

This proves the gauge invariance of the linearized action (C.1).

Similarly, for the linearized field equations

$$\begin{aligned}G_{\mu\nu} \equiv \partial_\sigma\big(\partial_\nu\, h_\mu{}^\sigma+\partial_\mu h_\nu{}^\sigma\big)&+\eta_{\mu\nu}\big(\partial_\rho\partial_\sigma h^{\rho\sigma}-\Box h\big)\\ &+\partial_\mu\partial_\nu h = -2\kappa T_{\mu\nu}\,.\end{aligned}\tag{C.9}$$

We can perform the same gauge transformation (C.1) to obtain the variation. Using the same procedure as we did for the action, that is, going term by term one obtains

$$\delta_\xi G^1_{\mu\nu} = -\Box(\xi_{\mu,\nu}+\xi_{\nu,\mu})\,,\tag{C.10}$$

$$\delta_\xi G^2_{\mu\nu} = \partial_\sigma\left[(\partial_\nu(\xi_{\mu,}{}^\sigma+\xi^\sigma{}_{,\mu})+\partial_\mu(\xi^\sigma_{\nu,}+\xi^\sigma{}_{,\nu})\right]\,,\tag{C.11}$$

$$\delta_\xi G^3_{\mu\nu} = -\eta_{\mu\nu}\partial_\rho\partial_\sigma(\xi^{\rho,\sigma}+\xi^{\sigma,\rho})+\eta_{\mu\nu}\Box(\xi^\alpha{}_{,\alpha}+\xi_{\alpha,}{}^\alpha)\,,\tag{C.12}$$

$$\delta_\xi G_{\mu\nu}{}^4 = -\partial_\mu\partial_\nu(\xi^\alpha{}_{,\alpha}+\xi_{\alpha,}{}^\alpha)\,.\tag{C.13}$$

First, we check that

$$\delta_\xi G_{\mu\nu}{}^3 = \eta_{\mu\nu}g^{\rho\alpha}g^{\sigma\beta}\left(\partial_\rho\partial_\sigma\partial_\beta\xi_\alpha+\partial_\rho\partial_\sigma\partial_\alpha\xi_\beta-\partial_\beta\partial_\sigma\partial_\alpha\xi_\rho-\partial_\beta\partial_\sigma\partial_\rho\xi_\alpha\right)\,.\tag{C.14}$$

If one makes the change $\rho \leftrightarrow \alpha$ on the first term and $\alpha \leftrightarrow \beta$, $\rho \leftrightarrow \sigma$ on the second one all terms cancel

$$\delta_\xi G^3_{\mu\nu} = 0\,. \tag{C.15}$$

By combining the remaining terms we get

$$\begin{aligned}\delta_\xi G_{\mu\nu} = \delta_\xi G^1_{\mu\nu} + \delta_\xi G^2_{\mu\nu} + \delta_\xi G^4_{\mu\nu} = g^{\alpha\beta}\Big(& -\partial_\alpha\partial_\beta\partial_\nu\xi_\mu - \partial_\alpha\partial_\beta\partial_\mu\xi_\nu + \partial_\alpha\partial_\nu\partial_\beta\xi_\mu \\ & + \partial_\alpha\partial_\mu\partial_\beta\xi_\nu + \partial_\alpha\partial_\nu\partial_\mu\xi_\beta + \partial_\alpha\partial_\mu\partial_\nu\xi_\beta \\ & - \partial_\mu\partial_\nu\partial_\alpha\xi_\beta - \partial_\mu\partial_\nu\partial_\beta\xi_\alpha\Big) = 0\,.\end{aligned} \tag{C.16}$$

Therefore, we conclude that both the linearized action and the linearized field equations are invariant under the gauge transformation. Let us emphasize that in the above relations we do not use the property that the number of spacetime dimensions is four. In fact, this result does not depend on the number of dimensions and is valid in the higher dimensional case.

Appendix D: Mass and angular momentum of extended objects

We denote by $X^\mu = (t, x^\alpha)$ Cartesian coordinates in $d+1$ dimensional Minkowski spacetime and use indices $\alpha, \beta, \ldots = 1, 2, \ldots, d$ from the beginning of the Greek alphabet to label spatial coordinates. Let us consider distribution of matter described by the stress-energy of the form

$$T_{00} = \rho(\boldsymbol{x}), \quad T_{0\alpha} = \frac{1}{2}\frac{\partial}{\partial x^\beta} j_{\alpha\beta}(\boldsymbol{x}), \quad T_{\alpha\beta} = 0, \tag{D.1}$$

where $j_{\alpha\beta}(\boldsymbol{x})$ is an anti-symmetric tensor function. It is easy to check that this stress-energy tensor satisfies the required conservation law $\partial_\mu T^{\mu\nu} = 0$. Denote by $\boldsymbol{\xi}_{(\mu)}$ a generator of the spacetime translations, and by $\boldsymbol{\zeta}_{(\alpha\beta)}$ the generators of the rigid spatial rotations, then one has

$$\boldsymbol{\xi}_{(\mu)} = \xi^\nu_{(\mu)}\partial_\nu = \partial_\mu, \tag{D.2}$$

$$\boldsymbol{\zeta}_{(\alpha\beta)} = \boldsymbol{\zeta}^\nu_{(\alpha\beta)}\partial_\nu = x_\alpha\partial_\beta - x_\beta\partial_\alpha. \tag{D.3}$$

The conserved quantities related to these symmetries are

$$P_\mu = \int \mathrm{d}^d x\, T_{0\nu}\xi^\nu_{(\mu)}, \tag{D.4}$$

$$J_{\alpha\beta} = \int \mathrm{d}^d x\, T_{0\gamma}\boldsymbol{\zeta}^\gamma_{(\alpha\beta)}. \tag{D.5}$$

Or in an explicit form

$$M = P_0 = \int \mathrm{d}^d x\, T_{00}, \tag{D.6}$$

$$P_\alpha = \int \mathrm{d}^d x\, T_{0\alpha}, \tag{D.7}$$

$$J_{\alpha\beta} = \int \mathrm{d}^d x\, (x_\alpha T_{0\beta} - x_\beta T_{0\alpha}). \tag{D.8}$$

We assume that the stress-energy tensor (D.1) either vanishes outside some compact region, or it is sufficiently fast decreasing at far spatial distance, so that the surface terms arising as a result of integration by parts in (D.8) vanish. Simple calculations give

$$M = \int \mathrm{d}^d x \, T_{00}, \;\; P_\alpha = 0, \;\; J_{\alpha\beta} = \int \mathrm{d}^d x \, j_{\alpha\beta} \,. \tag{D.9}$$

The relation $P_\alpha = 0$ implies that the stress-energy tensor (D.1) is written in the center of mass frame.

Appendix E: “Heat kernel” representation

The static Green function $\mathcal{G}_d$ considered in this thesis satisfies the relation

$$f(\triangle)\triangle\mathcal{G}_d(\boldsymbol{x},\boldsymbol{x'}) = -\delta(\boldsymbol{x}-\boldsymbol{x'})\,. \tag{E.1}$$

Here $\triangle$ is a Laplace operator in d-dimensional space. We denote by $K_d(\boldsymbol{x}|\tau)$ the d-dimensional “heat kernel” of $\triangle$ under the Wick rotation $t = -i\tau$[1]. It is defined as a solution of the equation

$$\triangle K_d(\boldsymbol{x}|\tau) = -i\partial_\tau K_d(\boldsymbol{x}|\tau)\,, \tag{E.2}$$

obeying the boundary conditions

$$\begin{aligned} \lim_{\tau\to 0} K_d(\boldsymbol{x}|\tau) &= \delta(\boldsymbol{x})\,,\\ \lim_{\tau\to\pm\infty} K_d(\boldsymbol{x}|\tau) &= 0\,. \end{aligned} \tag{E.3}$$

It has the following explicit form:

$$K_d(\boldsymbol{x}|\tau) = \frac{1}{(4\pi i\tau)^{d/2}}\exp\left(\frac{i\boldsymbol{x}^2}{4\tau}\right)\,. \tag{E.4}$$

Let us define the object $\mathcal{K}_d(\boldsymbol{x}|\tau)$ as a solution of the equation

$$f(\triangle)\mathcal{K}_d(\boldsymbol{x}|\tau) = iK_d(\boldsymbol{x}|\tau)\,. \tag{E.5}$$

Then it is easy to check the required Green function $\mathcal{G}_d$ can be written in the form

$$\mathcal{G}_d(\boldsymbol{x},\boldsymbol{x'}) = \int\limits_0^\infty \mathrm{d}\tau\,\mathcal{K}_d(\boldsymbol{x}-\boldsymbol{x'}|\tau)\,. \tag{E.6}$$

[1]Strictly speaking the notion of the heat kernel is introduced for the parabolic equations. The equation (E.2) has the form of the Schrödinger equation. For brevity, we still use the name “heat kernel” with quotation marks.

We introduce now the Fourier transform of $\mathcal{K}_d$ and its inverse by means of the relations

$$\begin{aligned}\widetilde{\mathcal{K}}_d(\boldsymbol{x}|\omega) &= \int\limits_{-\infty}^{\infty} \mathrm{d}\tau\, e^{i\omega\tau}\mathcal{K}_d(\boldsymbol{x}|\tau)\,,\\ \mathcal{K}_d(\boldsymbol{x}|\tau) &= \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi}\, e^{-i\omega\tau}\widetilde{\mathcal{K}}_d(\boldsymbol{x}|\omega)\,.\end{aligned} \tag{E.7}$$

Then we may write

$$\mathcal{G}_d(\boldsymbol{x},\boldsymbol{x'}) = \int\limits_0^{\infty} \mathrm{d}\tau \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \int\limits_{-\infty}^{\infty} \mathrm{d}\tau' e^{-i\omega(\tau-\tau')}\mathcal{K}_d(\boldsymbol{x}-\boldsymbol{x'}|\tau') \tag{E.8}$$

$$= \int\limits_0^{\infty} \mathrm{d}\tau \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \int\limits_{-\infty}^{\infty} \mathrm{d}\tau' e^{-i\omega(\tau-\tau')}\frac{i}{f(\triangle)}K_d(\boldsymbol{x}-\boldsymbol{x'}|\tau') \tag{E.9}$$

$$= \int\limits_0^{\infty} \mathrm{d}\tau \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \int\limits_{-\infty}^{\infty} \mathrm{d}\tau' e^{-i\omega(\tau-\tau')}\frac{i}{f(-i\partial_\tau)}K_d(\boldsymbol{x}-\boldsymbol{x'}|\tau') \tag{E.10}$$

$$= \int\limits_0^{\infty} \mathrm{d}\tau \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \int\limits_{-\infty}^{\infty} \mathrm{d}\tau' e^{-i\omega(\tau-\tau')}\frac{i}{f(-\omega)}K_d(\boldsymbol{x}-\boldsymbol{x'}|\tau')\,. \tag{E.11}$$

In the first equality we have used (E.5), then used the properties of the "heat kernel" via Eq. (E.4), and finally integrated by parts where the boundary terms vanish due to (E.3). The integral over τ can be easily calculated assuming that one takes care about its asymptotic behavior and uses the standard regularization. By using the relation

$$\int\limits_0^{\infty} \mathrm{d}\tau e^{-i\omega\tau} \equiv \lim_{\epsilon\to 0}\int\limits_0^{\infty} \mathrm{d}\tau e^{-i(\omega-i\epsilon)\tau} = \frac{-i}{\omega}\,, \tag{E.12}$$

one obtains

$$\mathcal{G}_d(\boldsymbol{x},\boldsymbol{x'}) = \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \int\limits_{-\infty}^{\infty} \mathrm{d}\tau' e^{i\omega\tau'}\frac{1}{\omega f(-\omega)}K_d(\boldsymbol{x}-\boldsymbol{x'}|\tau')\,, \tag{E.13}$$

which is the double Fourier representation for the Green function $\mathcal{G}_d$ used in the main body of the thesis.

Appendix F: Static infinite-derivative ghost-free Green functions

Let us consider theories with the form factor $f(\triangle)$ of the form $f^N(\triangle) = \exp\left[(-\triangle\ell^2)^N\right]$, where N is a positive integer number. We refer to such a theory as ghost-free gravity and use the abbreviation $\mathrm{GF_N}$ for such a theory. For $N = 0$, $f^0(\triangle) = 1$ and the corresponding theory is nothing but linearized General Relativity. Let us write $\mathcal{D}_N = f^N(\triangle)\triangle$ and denote by $\mathcal{G}_d^N$ a static Green function for $\mathrm{GF_N}$ theory in a space with d dimensions. Such a Green function obeys the equation

$$\mathcal{D}_N \mathcal{G}_d^N(r) = -\delta^{(d)}(\boldsymbol{r})\,. \tag{F.1}$$

For $N = 0$, that is, in General Relativity, we also use the notation $G_d(r) = \mathcal{G}_d^0(r)$. The static Green functions can be found by using Eqs. (2.62) and (2.60). In this appendix we collect exact expressions for these Green functions for General Relativity as well as $\mathrm{GF_1}$ and $\mathrm{GF_2}$ theory for the number of spatial dimensions $d = 1, 2, 3, 4$. Using the recursive relation (2.62) one can obtain their expression for $d \geq 5$. In what follows we will use the abbreviation $y = (r/4\ell)^2$.

$$G_1(r) = -\frac{r}{2}\,, \tag{F.2}$$

$$\mathcal{G}_1^1(r) = -\frac{r}{2}\mathrm{erf}\left(\frac{r}{2\ell}\right) - \ell\frac{\exp\left[-r^2/(4\ell^2)\right] - 1}{\sqrt{\pi}}\,, \tag{F.3}$$

$$\mathcal{G}_1^2(r) = -\frac{\ell}{\pi}\Big\{\; 2\Gamma(\tfrac{1}{4})y\,{}_1F_3\left(\tfrac{1}{4};\; \tfrac{3}{4},\tfrac{5}{4},\tfrac{3}{2};\; y^2\right) + \Gamma(\tfrac{3}{4})\Big[{}_1F_3\left(-\tfrac{1}{4};\; \tfrac{1}{4},\tfrac{1}{2},\tfrac{3}{4};\; y^2\right) - 1\Big]\Big\}\,, \tag{F.4}$$

$$G_2(r) = -\frac{1}{2\pi}\log\left(\frac{r}{r_0}\right)\,, \tag{F.5}$$

$$\mathcal{G}_2^1(r) = -\frac{1}{4\pi}\mathrm{Ein}\left(\frac{r^2}{4\ell^2}\right)\,, \tag{F.6}$$

$$\mathcal{G}_2^2(r) = -\frac{y}{2\pi}\Big[\;\sqrt{\pi}\,{}_1F_3\left(\tfrac{1}{2};\; 1,\tfrac{3}{2},\tfrac{3}{2};\; y^2\right) - y\,{}_2F_4\left(1,1;\; \tfrac{3}{2},\tfrac{3}{2},2,2;\; y^2\right)\Big]\,, \tag{F.7}$$

$$G_3(r) = \frac{1}{4\pi r}\,, \tag{F.8}$$

$$\mathcal{G}_3^1(r) = \frac{\mathrm{erf}[r/(2\ell)]}{4\pi r}\,, \tag{F.9}$$

$$\mathcal{G}_3^2(r) = \frac{1}{6\pi^2\ell}\Big[3\Gamma\left(\tfrac{5}{4}\right){}_1F_3\left(\tfrac{1}{4};\; \tfrac{1}{2},\tfrac{3}{4},\tfrac{5}{4};\; y^2\right) - 2y\Gamma\left(\tfrac{3}{4}\right){}_1F_3\left(\tfrac{3}{4};\; \tfrac{5}{4},\tfrac{3}{2},\tfrac{7}{4};\; y^2\right)\Big]\,, \tag{F.10}$$

$$G_4(r) = \frac{1}{4\pi^2 r^2}\,, \tag{F.11}$$

$$\mathcal{G}_4^1(r) = \frac{1 - \exp\left[-r^2/(4\ell^2)\right]}{4\pi^2 r^2}\,, \tag{F.12}$$

$$\mathcal{G}_4^2(r) = \frac{1}{64\pi^2 y\ell^2}\Big[1 - {}_0F_2\left(\tfrac{1}{2},\tfrac{1}{2};\; y^2\right) + 2\sqrt{\pi}y\,{}_0F_2\left(1,\tfrac{3}{2};\; y^2\right)\Big]\,. \tag{F.13}$$

Here we use the standard notation ${}_aF_b$ for the hypergeometric function [77].