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Irreducible Projective Representations of Finite Groups

by

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Abstract

Any free presentation for a finite group G may be used to construct an infinite group F having G as quotient modulo a central subgroup, having finite commutator subgroup F' determined up to isomorphism by G, and having the projective lifting property for G over all fields. This thesis is concerned with the study of those irreducible representations of F which arise as lifts of irreducible projective representations of G over fields of characteristic zero.

If k is such a field, we obtain a bijective correspondence between the set of primitive central idempotents of the group algebra kF and the set of F-orbits of irreducible k-characters of F'. In the case where k is algebraically closed, this correspondence extends to the set of projective equivalence classes of irreducible projective k-representations of G.

In general the group algebra kF embeds in a completely reducible ring KF having dimension $|G|\,|H^2(G, \mathbb{C}^\times)|$ over a purely transcendental field extension K of k. Analysis of the simple components of KF yields information on the general structure of certain simple k-algebras which appear as homomorphic images of kF, and on possible values of their Schur index and degree. These algebras determine irreducible projective representations of G over k, since they also appear as simple components of twisted group rings of G over k.

The problem of realizability of projective representations over small fields is considered in the light of the close connection between the equivalence classes of irreducible projective $\mathbb{C}$-representations of G and the F-orbits of absolutely irreducible characters of F'. In particular it is shown that if the field $k \subseteq \mathbb{C}$ is an ordinary splitting field for the finite group F', then every complex projective representation of G is projectively realizable in k.

Finally a detailed discussion of the irreducible projective representations of finite metacyclic groups over subfields of the field of complex numbers is included.

University of Alberta

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled **Irreducible Projective Representations of Finite Groups** submitted by **Rachel Quinlan** in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics.

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Contents

Chapter 1

Introduction

The projective representation theory of finite groups was introduced by Schur in 1904, and has received considerable attention since, particularly in the case of representations over algebraically closed fields. The subject is a natural but not entirely straightforward generalization of the theory of linear representations - a projective representation of a group G over a field k basically consists of two components : a homomorphism of G into a projective (not general) linear group over k, and a cocycle, which is a function of $G \times G$ into $k^{\times}$. It is the appearance of this cocycle which leads to limitations on any far-reaching general analogies between projective and linear representations.

Throughout this thesis we will assume that all fields under consideration have characteristic zero, although for many (though not all) of the results, the hypothesis that the characteristic of the field should not divide the order of the finite group under discussion would suffice. Also, G will always denote a finite group. The layout of the thesis is as follows : Chapter 1 is introductory, and consists mainly of standard definitions which are central to the subject. In Chapter 2 we introduce the idea of a generic central extension for a finite group, an object whose linear representation theory will be of fundamental importance. Chapters 3 and 4 concern the structure of the group algebras of these generic central extensions, and in Chapter 5 we consider the finite dimensional irreducible representations of such group algebras. In Chapters 6 and 7 we reach some conclusions about projective representations of finite groups over fields. These conclusions, and the general theory discussed earlier, are applied in Chapter 8 to the specific case of metacyclic groups.

1.1 Projective Representations and Twisted Group Rings

Let G be a (finite) group and let k be a field. A *linear representation* of G over k is of course a group homomorphism of G into a general linear group over k. A *projective representation* of G over k is a mapping $T : G \longrightarrow GL(n, k)$, which is not necessarily a group homomorphism, but which sends 1_G to $I_{GL(n,k)}$, and for which

$$\pi \circ T : G \longrightarrow PGL(n, k)$$

is a group homomorphism, where π is the usual projection of $GL(n, k)$ on $PGL(n, k)$. The positive integer n is called the *degree* of T. The *kernel* of T is the kernel of the homomorphism $\pi \circ T$, and T is said to be *faithful* if $\ker T$ is trivial.

Of course the fact that $\pi \circ T$ is a group homomorphism means that

$$T(xy) \in k^{\times}T(x)T(y), \ \forall \ x, y \in G.$$

Thus implicit in the definition of T is a function $f : G \times G \longrightarrow k^{\times}$ defined for $x, y \in G$ by

$$T(xy) = f(x, y)T(x)T(y). \tag{1.1}$$

If $x, y, z \in G$, we can use 1.1 to write $T(xyz)$ in two ways :-

$$\begin{aligned} T(xyz) &= f(xy, z)T(xy)T(z) \\ T(xyz) &= f(x, yz)T(x)T(yz) \end{aligned}$$

Further expansion of the right hand sides of these equations leads to the following condition on f :-

$$f(x, y)f(xy, z) = f(x, yz)f(y, z), \quad \forall x, y, z \in G. \tag{1.2}$$

Also, from the requirement that $T(1_G) = 1_{GL(n,k)}$ (which is a simplifying convention and imposes no real restrictions), combined with 1.1 we obtain

$$f(1_G, x) = f(x, 1_G) = 1, \quad \forall x \in G. \tag{1.3}$$

Any function $f : G \times G \longrightarrow k^\times$ satisfying 1.2 and 1.3 is called a *cocycle* of G in k. The set of all such cocycles is denoted $Z^2(G, k^\times)$ and forms a group under multiplication which is defined pointwise:-

$$f_1 f_2(x, y) = f_1(x, y) f_2(x, y), \quad \text{for } f_1, f_2 \in Z^2(G, k^\times), \quad x, y \in G.$$

The identity element of $Z^2(G, k^\times)$ is of course the trivial cocycle - the one which sends every element of $G \times G$ to 1. We note that if $T : G \longrightarrow GL(n, k)$ is a projective representation of G with cocycle $f \in Z^2(G, k^\times)$, then T is in fact a linear representation if and only if f is the trivial cocycle. If $f \in Z^2(G, k^\times)$ is the cocycle associated to a projective representation T of G by equation 1.1, T is often referred to as an f-representation.

Now let μ be any function taking G into the set of nonzero elements of the field k, and let $T : G \longrightarrow GL(n, k)$ be as above. Then we may define a function $T' : G \longrightarrow GL(n, k)$ by

$$T'(x) = \mu(x) T(x), \quad \text{for } x \in G.$$

That T' is again a projective representation of G is clear, since $\pi \circ T' = \pi \circ T$. For $x, y \in G$, we have

$$\begin{aligned} T'(xy) &= \mu(xy) T(xy) \\ &= \mu(xy) f(x, y) T(x) T(y) \\ &= \mu(xy) f(x, y) \mu(x)^{-1} T'(x) \mu(y)^{-1} T'(y) \\ &= \mu(xy) \mu(x)^{-1} \mu(y)^{-1} f(x, y) T'(x) T'(y). \end{aligned}$$

Thus $T' : G \longrightarrow GL(n, k)$ is a linear representation of G if and only if

$$f(x, y) = \mu(x) \mu(y) \mu(xy)^{-1} \quad \forall\, x, y \in G. \tag{1.4}$$

A cocycle f which satisfies 1.4 for all x, y in G is known as a *coboundary* of G in k. The subset of $Z^2(G, k^\times)$ consisting of the coboundaries is denoted by $B^2(G, k^\times)$ and forms a subgroup under multiplication. This leads to the definition of $H^2(G, k^\times)$, the second cohomology group of G with coefficients in k, as the quotient group :-

$$H^2(G, k^\times) := Z^2(G, k^\times) / B^2(G, k^\times).$$

The name $\delta\mu$ is generally given to the coboundary determined by the function $\mu : G \longrightarrow k^\times$ i.e.

$$\delta\mu(x, y) = \mu(x) \mu(y) \mu(xy)^{-1}, \quad \forall\, x, y \in G.$$

If T' and T are projective representations of G defined as above, then the cocycle f' of T' is $f\delta\mu$, and in particular f and f' belong to the same class in $H^2(G, k^\times)$. For an arbitrary choice of k the group $H^2(G, k^\times)$ may be infinite, but it is finite in the case where k is algebraically closed. This is a consequence of the divisibility of the multiplicative group of an algebraically closed k, which guarantees that every coset of $B^2(G, k^\times)$ in $Z^2(G, l^\times)$ includes a representative which takes values in the (finite) group of $|G|$th roots of unity in $k^\times$. The finite abelian group $H^2(G, \mathbb{C}^\times)$ is called the *Schur multiplier* of G and denoted by $M(G)$.

We now give a module-theoretic description of projective representations, which is directly analogous to the familiar interpretation of linear representations of groups as modules over their group rings. A projective representation of a group G over a field k is a module, not over the ordinary group ring kG, but over a slightly more general object. A *twisted group ring* of G over k is a k-algebra R having basis $\mathcal{E} = \{e_g, g \in G\}$ as a k-vector space, and in which the multiplication of the basis elements does not exactly replicate the multiplication in G (as in the case of ordinary group rings), but in which for $x, y \in G$ we have

$$e_x e_y \in k^\times e_{xy}.$$

Thus there exists a function $f : G \times G \longrightarrow k^\times$ defined by

$$e_x e_y = f(x, y) e_{xy}, \quad \forall\ x, y \in G.$$

We may extend the multiplication on $\mathcal{E}$ by k-linearity to a multiplication on R. Then the stipulation that multiplication in R should be associative leads to the requirement that f must satisfy the relation given by 1.2 on G. That f also satisfies 1.3 follows if we require that the identity element of R should be $1_k e_{1_G}$. Thus $f \in Z^2(G, k^\times)$. Finally addition in R is defined in the obvious way :-

$$\sum a_g e_g + \sum b_g e_g = \sum (a_g + b_g) e_g, \text{ for } a_g, b_g \in k,\ g \in G.$$

The ring R defined by these conditions is called the twisted group ring of G over k determined by f, and is usually denoted by $k^f G$. Suppose the cocycles f' and f represent the same class in $H^2(G, K^\times)$, so $f' = f\,\delta\mu$ for some function $\mu : G \longrightarrow k^\times$. Let $\mathcal{E}' = \{e'_g, g \in G\}$ be a basis for the twisted group ring $k^{f'} G$ for which

$$e'_x e'_y = f'(x, y) e'_{xy}, \quad \forall\ x, y \in G.$$

Then it is easily checked that the map $\phi : k^f G \longrightarrow k^{f'} G$ defined on $\mathcal{E}$ by $\phi(e_g) = e'_g$ and extended by k-linearity to $k^f G$ is an isomorphism of k-algebras. In particular, $k^f G$ is isomorphic to the ordinary group algebra kG if f is a coboundary.

Now suppose for some $f \in Z^2(G, k^\times)$ that T is a projective f-representation of G of degree n. Then we may regard T as a mapping from G into $GL(V)$, where V is a vector space of dimension n over k. Then the relation 1.1 defines the structure of a $k^f G$-module on V. On the other hand the choice of a k-basis for any $k^f G$-module defines a mapping of G into some general linear group over k, which is a projective f-representation of G. Thus we have an alternative characterization of projective representations of G in terms of modules over twisted group rings.

1.2 Irreducible Projective Representations and Projective Equivalence

Notions such as irreducibility and equivalence of projective representations are defined by direct analogy with the linear theory. We give these definitions in this section, and also try to indicate some of the limitations of this analogy, in particular why some of the most elementary results on completely reducible linear representations do not really translate smoothly into the projective setting, and why the definition of projective equivalence is inherently somewhat problematic.

Let $T : G \longrightarrow GL(n, k)$ be a projective f- representation of the finite group G over the field k. Thus T defines the structure of a $k^f G$-module on a vector space V of dimension n over k. Then T is said to be *irreducible* as a projective representation of G if V contains no proper $k^f G$-submodule.

If E is a field extension of k, we may define a projective representation T^E of G over E by composing T with the inclusion of $GL(n, k)$ in $GL(n, E)$. If T^E remains irreducible for all choices of E, T is said to be *absolutely irreducible*.

In the case where $\operatorname{char} k = 0$ or $\operatorname{char} k$ does not divide the order of G, every $k^f G$-module can be written as a direct sum of irreducible $k^f G$-modules. This is a consequence of Maschke's theorem applied to twisted group rings.

Theorem 1.2.1 (Maschke) *Let G be a finite group, and let k be a field for which $\operatorname{char} k = 0$ or $\operatorname{char} k$ does not divide $|G|$. Then if $f \in Z^2(G, k^\times)$, the twisted group ring $k^f G$ is completely* *reducible.* □

A proof of Maschke's theorem for twisted group rings can be found in [15].

It is worth mentioning that while twisted group rings share the property of complete reducibility with ordinary group rings (under the hypothesis of Maschke's theorem) it is possible for a twisted group ring to be simple though this is not possible for any ordinary group ring of a nontrivial finite group over a field. Let k and G be as in the statement of Theorem 1.2.1, and assume that G is not trivial. It is easy to see that the ordinary group ring kG contains at least two simple components, for

$$e = \frac{1}{|G|}\left(\sum_{g \in G} g\right)$$

is a central idempotent of kG which is equal to neither 0 nor 1.

However, examples of twisted group rings which are simple are easily found. If x generates a cyclic group C of order 2, let $f \in Z^2(C, \mathbb{Q}^\times)$ be the cocycle given by

$$f(1,1) = f(1,x) = f(x,1) = 1; \quad f(x,x) = 2.$$

Then the twisted group ring $\mathbb{Q}^f C$ is isomorphic to the quadratic field extension $\mathbb{Q}(\sqrt{2})$ of $\mathbb{Q}$. There also exist examples of twisted group rings which are central simple over their ground fields. This means that it is possible for the regular representation of a twisted group ring to be irreducible, even over a field which is algebraically closed.

For example, let $G \cong C_2 \times C_2$ and let a and b be generators for G. Let f be the cocycle in $Z^2(G, \mathbb{C})$ defined by the table

f	1	a	b	ab
1	1	1	1	1
a	1	-1	1	-1
b	1	-1	-1	1
ab	1	1	-1	-1

Then $\mathbb{C}^f G \cong \left(\frac{-1,-1}{\mathbb{C},-1}\right) \cong M_2(\mathbb{C})$, and the mapping $T : G \longrightarrow GL(2,\mathbb{C})$ defined by

$$a \longrightarrow \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad b \longrightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad ab \longrightarrow \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix},$$

is an irreducible projective representation of G over $\mathbb{C}$. with cocycle f. We observe that f as defined above cannot be a coboundary in $Z^2(G, \mathbb{C}^\times)$ since the twisted group ring $\mathbb{C}^f G$ is noncommutative and cannot therefore be isomorphic to $\mathbb{C}G$. Since $\mathbb{C}$ is algebraically closed, T is an absolutely irreducible representation of G. Thus $C_2 \times C_2$ has faithful absolutely irreducible projective representations, although all of its absolutely irreducible linear representations have degree 1 and are certainly not faithful.

If T_1 and T_2 are projective representations of G of degree n over the field k, they are said to be *linearly equivalent* if for some $A \in GL(n, k)$ we have

$$T_2(g) = A^{-1}T_1(g)A, \quad \forall\, g \in G.$$

It is easily observed that the same cocycle $f \in Z^2(G, k^\times)$ is associated to both T_1 and T_2 if they are linearly equivalent, and it is a consequence of Theorem 1.2.1 that every projective f-representation of G is linearly equivalent to one which can be written as a sum of irreducible f-representations. The irreducible constituents which appear in such a decomposition are unique up to linear equivalence.

Our original definition of projective representations, essentially as homomorphisms into projective general linear groups, suggests that "equivalence classes" of projective representations should perhaps be more inclusive than those determined by the above definition of linear equivalence. As above, suppose T_1 and T_2 are projective representations of degree n of G over the field k. We would like to declare T_1 and T_2 to be "equivalent" if for some $A \in GL(n, k)$ the representation T_1^A defined for $g \in G$ by

$$T_1^A(g) = A^{-1}T_1(g)A$$

satisfies $\pi \circ T_1^A = \pi \circ T_2$, where π as before denotes the usual projection of $GL(n, k)$ on $PGL(n, k)$.

The representations $T_1 : G \longrightarrow GL(n, k)$ and $T_2 : G \longrightarrow GL(n, k)$ are *projectively equivalent* over k if there exists a matrix $A \in GL(n, k)$ and a function $\mu : G \longrightarrow k^\times$ for which

$$\mu(g)A^{-1}T_1(g)A = T_2(g), \quad \forall\, g \in G. \tag{1.5}$$

It is easily seen that if T_1 and T_2 are as above, their cocycles need not be equal, but differ by the coboundary $\delta\mu$. We remark for later reference that in the special case where μ is a group homomorphism the same cocycle is associated to both T_1 and T_2. Even in this case however, T_1

and T_2 need not be linearly equivalent. Thus in general the cocycles associated to projectively equivalent representations belong to the same class in $H^2(G, k^\times)$. This is consistent with our comments on isomorphism of twisted group rings at the end of Section 1.1 : projectively equivalent representations correspond to modules over isomorphic twisted group rings. Since $H^2(G, k^\times)$ is typically infinite for an arbitrary choice of G and k, a finite group may have infinitely many projective equivalence classes of projective representations over a given field.

Projective equivalence is the analogue in projective representation theory of the concept of linear equivalence in linear representation theory. This correspondence is fairly tenuous in some respects however. Great caution is required in drawing any conclusions based on regarding projectively equivalent representations as "the same". For example, as we shall see in Chapter 6, the Schur index over a given field of an absolutely irreducible projective representation, which is defined exactly as in the linear setting, is not invariant under projective equivalence.

Another initially surprising and somewhat unsatisfactory fact is that a projective representation is not determined up to projective equivalence by the projective equivalence classes of its irreducible constituents. For let $G = \langle x \rangle$ be a cyclic group of order n, and let ξ be an nth root of unity in $\mathbb{C}$. We may define for $i = 1 \ldots n$ a (linear) representation R_i of G by $R_i(x) = \xi^i$. Each R_i is of course trivial when regarded as a projective representation of G, since it sends G into $\mathbb{C}^\times$. Moreover, the same cocycle in $Z^2(G, \mathbb{C}^\times)$, namely the trivial one, is associated to each R_i. Now let R be the linear representation of degree n of G defined by

$$R(x) = \mathrm{diag}(1, \xi, \xi^2, \ldots, \xi^{n-1}).$$

As a projective representation of G over $\mathbb{C}$, R is not only nontrivial but faithful, although each of its irreducible constituents is projectively trivial. This situation is caused by the general difficulty that if T_1 and T_2 are projectively equivalent irreducible representations of G which determine the same cocycle f in $Z^2(G, k^\times)$, the irreducible $k^f G$-modules determined by T_1 and T_2 need not be isomorphic.

Another significant difference between the projective and linear representation theories is that the sum of two projective representations need not be a projective representation, unless the same cocycle is associated to both summands. In addition, projectively equivalent representations generally do not have the same character, which means that one of the most powerful and beautiful aspects of linear representation theory, namely character theory, loses much of its scope when carried into the projective situation. Although there is a well-established and cohesive theory of projective characters (see [10], for example), it can necessarily apply only to one cocycle at a time. Also, the product of two projective characters is not in general a projective character, even if the same cocycle is associated to both of the corresponding representations.

Throughout the remainder of this thesis, if two projective representations of a group or algebra are described simply as "equivalent", we shall understand that they are projectively (and not necessarily linearly) equivalent.

Chapter 2

Covering Groups and Generic Central Extensions

Let $R : G \longrightarrow GL(n, k)$ be a linear representation of a finite group G over a field k. Then $\{R(g), g \in G\}$ generates a finite subgroup of $GL(n, k)$ which is isomorphic to $G/\ker(R)$.

Now suppose $T : G \longrightarrow GL(n, k)$ is a projective representation of G, and consider the group G^T generated in $GL(n, k)$ by $\{T(g), g \in G\}$. The order of G^T need not be finite; however if A denotes the intersection of G^T with $\mathcal{Z}(GL(n, k)) \cong k^\times$, then $\bar{G} := G^T/A$ is isomorphic to the image of G in $PGL(n, k)$ under the homomorphism $\pi \circ T$, where π is the usual surjection of $GL(n, k)$ on $PGL(n, k)$. Now G^T is of course an extension of its central subgroup A by the homomorphic image $\bar{G}$ of G; the abelian group A is not in general finite but is certainly finitely generated since its index in G^T is finite. We remark that if $k^\times$ is identified with $\mathcal{Z}(GL(n, k))$ then all values assumed by the cocycle $f \in Z^2(G, k^\times)$ associated to T appear in A; this follows from the fact that $T(g_1)T(g_2) = f(g_1, g_2)T(g_1g_2)$ for all g_1, g_2 in G.

Definition A *central extension* for the finite group G is a triple (A, B, ϕ) where B is a group having A as a finitely generated subgroup of its centre, and $\phi : B \longrightarrow G$ is a surjective group homomorphism with kernel A. Associated to the central extension (A, B) is the short exact sequence

$$1 \longrightarrow A \longrightarrow B \longrightarrow G \longrightarrow 1.$$

It will be convenient sometimes to refer to this sequence (instead of to (A, B, ϕ)) as a central extension for G. We will also sometimes avoid explicit mention of ϕ and refer to a central extension for G simply as (A, B).

From the comments preceding the above definitions it is apparent that every projective representation of G can be related to a linear representation of some central extension for the image of G under the homomorphism $\pi \circ T$ which sends G into a projective general linear group. Thus one approach to the study of projective representations of finite groups is to investigate the linear representations of their central extensions. This approach has been particularly fruitful in the case of projective representations over algebraically closed fields (see, for example [4]). One reason for this success is the fact that if $\bar{k}$ is an algebraically closed field, then every cocycle in $Z^2(G, \bar{k}^\times)$ is cohomologous to one which takes values in the group of $|G|$th roots of unity in $\bar{k}^\times$: this is a consequence of the divisibility of the multiplicative group of $\bar{k}$. Thus if $T : G \longrightarrow GL(n, \bar{k})$ is a projective representation, T is projectively equivalent, over $\bar{k}$, to a representation T_1 for which $G^{T_1} := \langle T_1(g), g \in G \rangle$ is a *finite* subgroup of $GL(n, \bar{k})$. In fact the situation is somewhat better than this, as we shall see in the next section.

Returning to the case where the field k is arbitrary, we cannot necessarily arrange for the group $G^T = \langle T(g), g \in G \rangle$ to be finite for every projective k-representation T, but G^T will always contain a subgroup of finite index which is free abelian of finite rank and central not only in G^T but in $GL(n, k)$.

2.1 Lifts and Finite Covering Groups

In this section we state without proof some fundamental results from the foundations of the theory of projective representations. All of these results are due to Schur, who introduced and extensively developed the subject in the early years of the twentieth century. We begin with an important definition.

Definition Let G be a finite group and let $T : G \longrightarrow GL(n, k)$ be a projective representation of G over a field k. Then if H is a group having G as a homomorphic image under the mapping ϕ, T is said to *lift* to H if there exists a *linear* representation $\tilde{T} : H \longrightarrow GL(n, k)$ for which the following diagram of group homomorphisms commutes :-

$$\begin{array}{ccc} H & \xrightarrow{\tilde{T}} & GL(n,k) \\ {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle \pi} \\ G & \xrightarrow[\pi \circ T]{} & PGL(n,k) \end{array}$$

In this situation we will refer to $\tilde{T}$ as a *lift* of T to H.

On the other hand, given a linear k-representation $\tilde{T}$ of H which sends $\ker \varphi$ into $k^\times$, we can obtain a projective representation T of G, by choosing a section η for G in H and defining $T(g) = \tilde{T}(\eta(g))$, for $g \in G$. Of course T then depends on the choice of section η, but only up to cohomology in $Z^2(G, k^\times)$.

In the case where $\tilde{T}$ is a lift to H of some projective k-representation T of G, it is clear that $\tilde{T}$ is an irreducible representation of H if and only if T is an irreducible projective representation of G. This follows for instance from the fact that the images of $\tilde{T}$ and T generate the same k-subalgebra of $GL(n,k)$, if n is the degree of T.

Now let G^* be a group having G as a homomorphic image. We will say that G^* has the *projective lifting property* for G over the field k if every projective k-representation of G is equivalent (over k) to one which can be lifted to G^*. The following result of Schur states that every finite group G has a finite central extension having the projective lifting property for G over $\mathbb{C}$. A proof can be found in Chapter 2 of [11].

Theorem 2.1.1 (Schur) *Let G be a finite group. Then there exists a central extension $\hat{G}$ of a finite abelian group A by G for which the following conditions hold :-*

i) $A \cong \mathrm{M}(G)$

ii) $A \subseteq \mathcal{Z}(\hat{G}) \cap \hat{G}'$

iii) $\hat{G}$ has the projective lifting property for G over $\mathbb{C}$. □

A group $\hat{G}$ having properties i), ii) and iii) of Theorem 2.1.1 will be called a *covering group* for G.

It is easily observed that if two projective representations of a finite group G over a field k have linearly equivalent lifts to some central extension H for G, then the original representations are projectively equivalent over k. For this reason there is no hope of proving a version of Theorem 2.1.1 which would apply without restriction on the field. The group $H^2(G, k^\times)$ is typically infinite for a given finite group G and field k, and thus G may have infinitely many inequivalent irreducible projective k-representations, which cannot be described by the finitely many irreducible linear representations of any proposed finite covering group. However, for every finite group G there exists a group F having G as a quotient by an infinite central subgroup, and having the projective lifting property for G over all fields. The main theme of this thesis is the investigation of central simple algebras arising from finite dimensional irreducible linear representations of these "infinite covering groups".

2.2 Generic Central Extensions

As usual let G be a finite group, and let $\tilde{F}$ be a free group of finite rank for which $\phi : \tilde{F} \longrightarrow G$ is a surjective group homomorphism with kernel $\tilde{R}$. Let

$$1 \longrightarrow A \longrightarrow H \xrightarrow{\phi_1} G \longrightarrow 1$$

be a central extension for G. Since $\tilde{F}$ is a free group, we can find a homomorphism $\alpha : \tilde{F} \longrightarrow H$ for which $\phi_1 \circ \alpha = \phi$. Then $\alpha(\tilde{R}) \subseteq A$ since $\phi_1 \circ \alpha(\tilde{R}) = 1$. Therefore α maps $\tilde{R}$ into $\mathcal{Z}(H)$ and so $[\tilde{F}, \tilde{R}] \subseteq \ker \alpha$. Thus the map $\phi : \tilde{F} \longrightarrow G$ induces a group surjection

$$\phi' : \tilde{F}/[\tilde{F}, \tilde{R}] \longrightarrow G; \quad \ker \phi' = \tilde{R}/[\tilde{F}, \tilde{R}]$$

This leads to the following lemma, after we define

$$F = \tilde{F}/[\tilde{F}, \tilde{R}]; \quad R = \tilde{R}/[\tilde{F}, \tilde{R}].$$

Lemma 2.2.1 *Let (A, B, φ) be a central extension for the finite group G. Then if F and R are defined as above for G, there exists a group homomorphism $\theta : F \longrightarrow B$ for which the following diagram commutes :-*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & R & \longrightarrow & F & \xrightarrow{\phi'} & G & \longrightarrow & 1 \\ & & \downarrow \theta|_R & & \downarrow \theta & & \downarrow id & & \\ 1 & \longrightarrow & A & \longrightarrow & B & \xrightarrow[\varphi]{} & G & \longrightarrow & 1 \end{array}$$

Note that R is central in F and so (R, F) is a central extension having the universal property described in Lemma 2.2.1 amongst all central extensions for G. For this reason we shall refer to a central extension (R, F) (or just F) obtained as above from a free presentation for G as a *generic central extension* for G.

We now show that any generic central extension F for G has the projective lifting property for G over all fields. To do this we need only show that every projective representation of G lifts (over the field in which it is realized) to *some* central extension for G. This well-known fact is the content of the next lemma.

Lemma 2.2.2 *Let $T : G \longrightarrow GL(n, k)$ be a projective representation of a finite group G over a field k, and let $\alpha \in Z^2(G, k^\times)$ be the cocycle associated to T. Define a group G_α by*

$$G_\alpha = \{(a, g) | a \in k^\times, g \in G\},$$

with multiplication given by

$$(a, g)\,(b, h) = (ab\,\alpha(g, h), gh)\,, \quad \text{for } a, b \in k^\times, \text{and } g, h \in G.$$

Then

i) G_α is a central extension of $k^\times$ by G.

ii) The map $\bar{T} : G_\alpha \longrightarrow GL(n, k)$ defined by $\bar{T}(a, g) = aT(g)$ is a linear k-representation of G_α and is a lift to G_α of T.

□

Both conclusions of Lemma 2.2.2 follow immediately from the various definitions. The group G_α is sometimes called an "α-covering group" for G over k. Lemmas 2.2.1 and 2.2.2 have the following important consequence.

Theorem 2.2.1 *Let G be a finite group, and let (R, F) be a generic central extension for G. Let $T : G \longrightarrow GL(n, k)$ be a projective representation of G over a field k. Then there exists a lift $\bar{T} : F \longrightarrow GL(n, k)$ of T to F.*

Proof Let $\alpha \in Z^2(G, k^\times)$ be the cocyle associated to T, and let G_α and $\bar{T}$ be defined as in Lemma 2.2.2. Then by Lemma 2.2.1 we can find a group homomorphism $\theta : F \longrightarrow G_\alpha$ for which the following diagram commutes :-

$$\begin{array}{ccccccccc}
1 & \longrightarrow & R & \longrightarrow & F & \longrightarrow & G & \longrightarrow & 1 \\
 & & \big\downarrow{\scriptstyle \theta|_R} & & \big\downarrow{\scriptstyle \theta} & & \big\downarrow{\scriptstyle id} & & \\
1 & \longrightarrow & k^\times & \longrightarrow & G_\alpha & \longrightarrow & G & \longrightarrow & 1 \\
 & & & & \big\downarrow{\scriptstyle \bar{T}} & & \big\downarrow{\scriptstyle T\circ\pi} & & \\
 & & & & GL(n,k) & \xrightarrow{\pi} & PGL(n,k) & &
\end{array}$$

Then $\tilde{T} := \bar{T} \circ \theta$ is a lift of T to F. □

Theorem 2.2.1 is the motivation for much of the work in this thesis : if (R, F) is a generic central extension for G, then every projective k-representation of G can be described in terms of a linear representation of F which sends R into $k^\times$, where the field k is entirely arbitrary. Rephrasing this statement in the language of simple rings, we see that every k-algebra arising as a simple component of a twisted group algebra of G over k can be realized as an image of the ordinary group algebra kF under a k-algebra homomorphism which sends kR into k. In the next section we discuss some properties of generic central extensions which will lead to conclusions about the structure of their group algebras and the nature of their finite dimensional representations.

2.3 Properties of Generic Central Extensions

Throughout this section let (R, F) be a fixed generic central extension for G. The group F is not determined by G up to isomorphism; its isomorphism type depends on a choice of presentation for G. We now describe some important and useful properties which are however shared by all generic central extensions.

We begin with the statement of a celebrated result of Schur. A proof can be found in Section 2.4 of [11].

Theorem 2.3.1 *The central subgroup $F' \cap R$ of F is isomorphic to $\mathrm{M}(G)$, the Schur multiplier of G.* □

The isomorphism mentioned in Theorem 2.3.1 will be of great use later, but for now we need only the fact the $F' \cap R$ is finite. This will enable us to prove the next lemma, which is also due to Schur.

Lemma 2.3.1 *Let* $\mathrm{t}(F)$ *denote the subset of* F *consisting of all the torsion elements. Then* $\mathrm{t}(F) = F'$.

Proof Let $x \in F'$. Then, since R has finite index in F, $x^n \in F' \cap R$ for some positive integer n. Then x has finite order by Theorem 2.3.1.

On the other hand the group F/F' is free abelian, since

$$F/F' = \frac{\tilde{F}/[\tilde{F}, \tilde{R}]}{\tilde{F}'/[\tilde{F}, \tilde{R}]} \cong \tilde{F}/\tilde{F}'.$$

Then any element of finite order in F must belong to F'. $\square$

Let (A_1, B_1) and (A_2, B_2) be generic central extensions for G. Then by Lemma 2.2.1, there exists a mapping $\theta : B_1 \longrightarrow B_2$ which takes A_1 into A_2 and whose kernel is contained in A_1. So the image of (A_1, B_1) in (A_2, B_2) is again a generic central extension for G. The homomorphisms defined in this way between different generic central extensions are not in general isomorphisms. However they restrict to isomorphisms on the commutator subgroups.

Lemma 2.3.2 *Let* (A, B) *and* (R, F) *be generic central extensions for* G. *Then* $B' \cong F'$.

Proof By Lemma 2.2.1, there exist group homomorphisms $\psi : B \longrightarrow F$ and $\theta : F \longrightarrow B$, for which the following diagram commutes :-

$$\begin{array}{ccccccccc}
1 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & G & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \psi|_A} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle id} & & \\
1 & \longrightarrow & R & \longrightarrow & F & \longrightarrow & G & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \theta|_R} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle id} & & \\
1 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & G & \longrightarrow & 1
\end{array}$$

Define a map $\eta : B \longrightarrow A$ by $\eta(x) = x^{\psi\theta}x^{-1}$, for $x \in B$. That $\eta(x) \in A$ is clear from the commutativity of the above diagram. Now let x, y be elements of B. Then

$$\begin{aligned}\eta(xy) &= (xy)^{\psi\theta}y^{-1}x^{-1} \\ &= x^{\psi\theta}y^{\psi\theta}y^{-1}x^{-1} \\ &= x^{\psi\theta}x^{-1}y^{\psi\theta}y^{-1}, \text{ since } y^{\psi\theta}y^{-1} \in A \subseteq \mathcal{Z}(B) \\ &= \eta(x)\eta(y)\end{aligned}$$

Thus η is a group homomorphism of B into the abelian group A, and so $B' \subseteq \ker(\eta)$. It then follows from the definition of η that $\theta \circ \psi : B \longrightarrow B$ restricts to the identity mapping on B'. Similarly $\psi \circ \theta : F \longrightarrow F$ restricts on F' to the identity mapping. Of course $\psi(B') \subseteq F'$ and $\theta(F') \subseteq B'$, so we conclude $B' \cong F'$. □

The following is another result of Schur, which relates generic central extensions to the finite covering groups of Section 2.1. A proof can be found in [11].

Theorem 2.3.2 *Let (R, F) be an essential generic central extension for G, and let S be a torsion free complement in R for its torsion subgroup $F' \cap R$. Then*

i) F/S is a covering group for G.

ii) If $\hat{G}$ is any covering group for G, then $\hat{G} \cong F/S$ for some choice of complement S for $F' \cap R$ in R. □

It is a consequence of Theorem 2.3.2 that F', and hence the commutator subgroup of any generic central extension for G, is isomorphic to $\hat{G}'$, where $\hat{G}$ is any covering group for G.

Chapter 3

Group Algebras of Generic Central Extensions

Let (R, F, ϕ) be a generic central extension for the finite group G, and let k be a field. We are interested in finite dimensional irreducible k-representations of F which send R into $k^\times$ and thus arise as lifts of irreducible projective representations of G : every such representation defines a finite dimensional simple k-algebra, which is the image of the group ring kF under the k-linear extension of the representation.

The group ring kF has infinite dimension over k and is not completely reducible. In this chapter however we shall see that kF embeds in a natural way in a ring which is completely reducible and has finite rank as a module over its centre, and in which we have recourse to all the results and methods from the theory of finite dimensional central simple algebras, yet from which we will later find we can recover all the relevant information about kF itself.

3.1 Extending the Centre

The torsion subgroup of R is $F' \cap R$ by Lemma 2.3.1; let S be a torsion free complement for $F' \cap R$ in R. Then S is a free abelian group of rank r=rank $\tilde{F}$, and kF contains the central subring kS, which is isomorphic to a ring of Laurent polynomials in r commuting variables.

Then kS is an integral domain, and moreover no element of kS can be a zerodivisor in kF : this follows from the centrality of kS in kF, and the fact that any transversal for S in F forms a basis for kF as a right module over kS. Thus we can form from kF a ring of quotients $(kS)^{-1}kF$, in which every nonzero element of kS is invertible. We will denote this ring of quotients by KF, where K denotes the field of quotients of kS. Then K is a purely transcendental field extension of k of transcendence degree r. Any basis for the free abelian group S forms an algebraically independent generating set for K over k.

Any transversal for S in F is a K-basis for KF, so KF is a finite dimensional K-algebra. Furthermore, KF is completely reducible. This is a consequence of the following lemma, of which a more detailed proof can be found in Section 1.2 of [14].

Lemma 3.1.1 *KF is isomorphic to a twisted group ring of the finite group $\hat{G} = F/S$ over K, and is completely reducible.*

Proof Choose a section μ for $\hat{G}$ in F : i.e. for each $x \in \hat{G}$, choose a preimage $\mu(x) \in F$ for x. Then $T = \{\mu(x), x \in \hat{G}\}$ is a transversal for S in F, and thus forms a K-basis for KF. We can define a map $f : \hat{G} \times \hat{G} \longrightarrow K^{\times}$ by $f(x,y) = \mu(x)\mu(y)\mu(xy)^{-1}$, for $x, y \in \hat{G}$. Then $f \in Z^2(\hat{G}, K^{\times})$ and the bijective correspondence between $\hat{G}$ and T defined by $x \longleftrightarrow \mu(x)$ establishes a K-algebra isomorphism between KF and $K^f\hat{G}$. The complete reducibility of KF is now immediate from Maschke's theorem, if char k does not divide the order of $\hat{G}$. □

3.2 Primitive Idempotents of KF

Since KF is a completely reducible ring, it can be written as a direct sum of simple K-algebras, and its identity element is a sum of nonzero primitive central idempotents. These idempotents are the projections of 1 on the various simple components of KF, and they are pairwise orthogonal (i.e. the product of any pair of them in KF is 0). In this section we show that the primitive central idempotents of KF belong not only to kF but to the finite dimensional completely reducible k-algebra kF'.

We begin by considering the central idempotents of kF. The following result is extremely useful, particularly in the light of Lemma 2.3.1. A proof can be found in [14], Section 4.3.

Lemma 3.2.1 *Let $\mathcal{G}$ be any group, and let $\mathcal{F}$ be any field. Then the support of any central idempotent of the ordinary group algebra $\mathcal{FG}$ generates a finite normal subgroup of $\mathcal{G}$.* □

Since the set of torsion elements of F is equal to its commutator subgroup F', it is an immediate consequence of Lemma 3.2.1 that every central idempotent of kF belongs to kF'. Let I denote the set of primitive central idempotents of kF'. Of course a central element of kF' need not be central in kF; F acts on I by conjugation. For each $f \in I$, the sum in kF of the F-conjugates of f is a central idempotent in kF. Let $\mathcal{I}$ denote the set of elements of this type :-

$$\mathcal{I} = \left\{ \sum_{x \in T_f} f^x, \ f \in I \right\},$$

where for each $f \in I$, T_f is a transversal in F for $C_F(f)$. We will show that $\mathcal{I}$ is the full set of primitive central idempotents of KF.

Theorem 3.2.1 *Let e be the sum in kF of an F-orbit of primitive central idempotents of kF'. Then $A := KFe$ is a simple ring.*

Proof :

1. Let Z denote the centre of A. Then Z is a direct sum of fields as A, being a two-sided ideal of KF, is a completely reducible ring. To show that A is simple, it suffices to show that Z is a field.

Suppose the contrary, so that $e = e' + e''$, where e' and e'' are nonzero central idempotents of KF for which $e'e'' = 0$. The ring KF is obtained from kF by adjoining (to the centre) the field of quotients of kS, where S is a torsion free complement for $F' \cap R$ in R. Then we can find central elements a' and a'' of kF (of kS in fact), for which $a'e'$ and $a''e''$ are central elements of kF. Then

$$a'e'a''e'' = a'a''e'e'' = 0.$$

To complete the proof of Theorem 3.2.1 then, it is enough to show that the centre of kFe contains no zerodivisors. The remainder of this section will be devoted to a proof of this fact.

2. Let e_1 be a primitive central idempotent of kF' for which $ee_1 = e_1$ (so $e_1 = e$ if e is primitive in kF'). Then e is the sum in kF of the distinct F-conjugates of e_1, and if we define

$F_1 = C_F(e_1)$ we have

$$kFe \cong \mathrm{M}_s\left(kF_1e_1\right). \tag{3.1}$$

where $s = [F : F_1]$ (see [14], Section 6.1). Certainly $F_1 \supseteq \mathcal{Z}(F)$, so F_1 has finite index in F; also $F_1 \supseteq F'$ since e_1 is central in kF'. The set of torsion elements of F_1 is F', so all central idempotents of the ring kF_1 have support in F'. Hence e_1 is a primitive central idempotent of kF_1.

3. We now establish some notation. Let $A_1^k = kF_1e_1$, $B_1 = kF'e_1$. Then B_1 is a simple component of kF', so $B_1 = \mathrm{M}_n(D_1)$ where D_1 is a finite dimensional division algebra over k. Let $\mathcal{E} = \{\varepsilon_{ij}\}_{1\le i,j\le n}$ be a system of matrix units in B_1. Then $D_1 \cong C_{B_1}(\mathcal{E})$ and $A_1^k = \mathrm{M}_n(\Delta^k)$, where $\Delta^k = C_{A_1^k}(\mathcal{E})$; $D_1 \subseteq \Delta^k$ obviously.

4. Let T be a transversal for F' in F_1. Then T generates $A_1^k = kF_1e_1$ as a right module over $B_1 = kF'e_1$ (of course e_1 commutes with each element of T, by definition of F_1). Furthermore, since $e_1 \in kF'$, T is right independent over B_1.

Then A_1^k is a crossed product over B_1 by the group F_1/F'. Since F_1/F' is a subgroup of finite index in F/F', it is a free abelian group of finite rank (equal to the free rank of $\bar{F}$). We will use this crossed product structure of A_1^k over B_1 to describe Δ^k as a crossed product over D_1, again by a free abelian group, and to conclude that Δ^k contains no zerodivisors.

$B_1 = kF'e_1$ is invariant under conjugation by elements of F_1, and for each $t \in T$ the set

$$\mathcal{E}^t = \left\{t^{-1}\varepsilon_{ij}t,\ \varepsilon_{ij} \in \mathcal{E}\right\}$$

is a system of matrix units in B_1. Then $\mathcal{E}$ and $\mathcal{E}^t$ are conjugate in B_1 (see [6], theorem 2.13) : that is, we can find an element $b(t)$ of $\mathcal{U}(B_1)$ for which

$$t^{-1}\varepsilon_{ij}t = b(t)^{-1}\varepsilon_{ij}b(t),\quad \forall \varepsilon_{ij} \in \mathcal{E}.$$

Then $c(t) := b(t)^{-1}t$ centralizes $\mathcal{E}$. Thus each $t \in T$ can be written in the form

$$t = b(t)c(t).$$

where $b(t) \in \mathcal{U}(B_1)$, $c(t) \in \mathcal{U}(\Delta^k)$.

5. Let $S = \{c(t), t \in T\}$. Then

i) S is right independent over D_1:-

Suppose that

$$\sum_{i=1}^{n} d_i c(t_i) = 0$$

for $d_1, \ldots, d_n \in D_1^{\times}$, $t_1, \ldots, t_n \in T$. Then

$$\sum_{i=1}^{n} d_i b(t_i)^{-1} b(t_i) c(t_i) = \sum_{i=1}^{n} d_i b(t_i)^{-1} t_i = 0.$$

This contradicts the right independence of T over B_1, since $d_i b(t_i)^{-1} \in B_1$ for each i.

ii) S generates Δ^k as a right D_1-module:-

$D_1[S] \subseteq \Delta^k$ clearly. On the other hand, suppose $\alpha \in \Delta^k$. Then, since $\alpha \in A_1^k$, α can be written uniquely in the form

$$\alpha = \sum_{i=1}^{n} b_i t_i,$$

where $b_i \in B_1$, $t_i \in T$ for $i = 1 \ldots n$. Then since $\alpha \in \Delta^k$, for each $\varepsilon \in \mathcal{E}$ we have

$$\begin{aligned} \varepsilon\alpha &= \alpha\varepsilon, \\ \Longrightarrow \sum_{i=1}^{n} \varepsilon b_i t_i &= \sum_{i=1}^{n} b_i t_i \varepsilon = \sum_{i=1}^{n} b_i \varepsilon^{t_i^{-1}} t_i. \\ \Longrightarrow \varepsilon b_i &= b_i t_i \varepsilon t_i^{-1}. \\ \Longrightarrow \varepsilon b_i t_i &= b_i t_i \varepsilon, \text{ for } i = 1 \ldots n. \end{aligned}$$

Then $b_i t_i \in \Delta^k$ for each i. Now $b_i t_i = b_i b(t_i) c(t_i)$, and since $c(t_i) \in \mathcal{U}(\Delta^k)$, we have $b_i b(t_i) \in \Delta^k$ also. Then

$$\beta_i := b_i b(t_i) \in \Delta^k \cap B_1 = D_1,$$

and $\alpha = \sum_{i=1}^{n} \beta_i c(t_i)$, where $\beta_i \in D$, $c(t_i) \in S$. Hence $\Delta^k = D_1[S]$.

iii) Suppose $t_1, t_2 \in T$ and $t_1 t_2 \in F't$, $t \in T$. Then $c(t_1)c(t_2) \in D_1^{\times} c(t)$:-

We require to show $c(t_1)c(t_2)c(t)^{-1} \in D_1$. By definition of $c(t_i)$ we have

$$\begin{aligned} c(t_1)c(t_2)c(t)^{-1} &= b(t_1)^{-1} t_1 b(t_2)^{-1} t_2 t^{-1} b(t)^{-1} \\ &= b(t_1)^{-1} \left(t_1 b(t_2)^{-1} t_1^{-1}\right) t_1 t_2 t^{-1} b(t)^{-1}. \end{aligned}$$

Since $b(t_2)^{-1} \in B_1 = kF'e_1$, and $t_1 \in F_1$, certainly $t_1 b(t_2)^{-1} t_1^{-1} \in B_1$. Also, $t_1 t_2 t^{-1} \in F' \subset B_1$, so $c(t_1)c(t_2)c(t)^{-1} \in B_1 \cap \Delta^k = D_1$. Then $c(t_1)c(t_2) \in D_1 c(t)$, as required.

6. By 5. above, Δ^k is a crossed product over D_1 by a group isomorphic to F_1/F'. Then, by the following lemma (see [19]), Δ^k is a domain.

Lemma (Higman) *Suppose $S \subseteq R$ are rings for which R is a crossed product over S by a group H having the property that every finitely generated subgroup has an infinite cyclic image. Then R is a domain whenever S is a domain.*

Since Δ^k is a crossed product over a division algebra by a free abelian group, the lemma applies and we conclude that Δ^k is a domain. Now

$$kFe \cong \mathrm{M}_s(kF_1e_1) \cong \mathrm{M}_{sn}(\Delta^k),$$

and $\mathcal{Z}(kFe) \cong \mathcal{Z}(\Delta^k)$. so kFe contains no central zerodivisors. The centre of $A = KFe$ is then also a domain, hence it is a field since KFe is a completely reducible ring. This completes the proof of Theorem 3.2.1. □

We conclude Chapter 3 with some observations on the proof of Theorem 3.2.1.

Lemma 3.2.2 *In the context and notation of Theorem 3.2.1, suppose that F_0 is a subgroup of F_1, for which $\langle R, F' \rangle \subseteq F_0$. Then $A_0 := KF_0e_1$ is a simple K-algebra.*

Proof Once we observe that kF_0e_1 is a crossed product over $kF'e_1$ by the free abelian group F_0/F', we may apply steps 4-6 of the proof of Theorem 3.2.1 to conclude that A_0 is simple. □

It is worth remarking also that the proof of Theorem 3.2.1 reveals more about the structure of the simple components of KF than simply its statement. The following theorem follows directly from steps 4-6 of this proof, and from Lemma 3.2.2.

Theorem 3.2.2 *Let e be a primitive central idempotent of KF, and let e_1 be a primitive central idempotent of kF' for which $ee_1 = e_1$, so $kF'e_1 \cong M_n(D_1)$ where D_1 is a finite dimensional division algebra over k. Let F_0 be a subgroup of F_1 for which $F' \subseteq F_0$. Then $KF_0e_1 \cong M_n(\Delta_0)$, where $\Delta_0 \supseteq D_1$ is a finite dimensional division algebra over K. In particular any set of n^2 matrix units in $kF'e_1$ is a full set of matrix units for KF_0e_1, contained in kF'.* □

Chapter 4

Structure of the Simple Components of KF

Let KF be the completely reducible ring defined in Section 3.1. Throughout this chapter we fix a primitive central idempotent e of KF, and let A denote the simple algebra KFe. If e is not primitive in kF', let e_1 be a primitive central idempotent of kF' for which $ee_1 = e_1$. Then we have seen that if e_1 has s distinct conjugates under the action of F, then A is isomorphic to a ring of $s \times s$ matrices over a simple K-algebra isomorphic to KF_1e_1, where $F_1 = C_F(e_1)$.

We now establish some notation which will be used throughout the remainder of this work, and define some objects which will be central to our discussion of the structure of A.

4.1 Notation and Background

Let $A_1 = KF_1e_1$; A_1 is a simple subring of A by Lemma 3.2.2. and A_1 contains the finite dimensional simple k-algebra $B_1 = kF'e_1$. If $kF'e_1 \cong M_n(D_1)$ for some division ring D_1 then $A_1 \cong M_n(D)$ where D is a division ring containing D_1. In particular any set of n^2 matrix units in $kF'e_1$ is a full set of matrix units for A_1.

Let Z denote the centre of A_1, and let X denote the group F_1e_1. Of course $X \cong F_1/F^{e_1}$, where F^{e_1} is the subgroup of F' defined by

$$F^{e_1} = \{x \in F : xe_1 = e_1\}.$$

That F^{e_1} is normal in F_1 is clear since F_1 centralizes e_1. Let T_X denote the torsion subgroup of X; $T_X = F'e_1$. Clearly $X' \subseteq T_X$, but this is not in general an equality since F_1' need not be equal to F'.

Let E denote the centre of $B_1 = kF'e_1 = k[T_X]$. Then E is a finite field extension of k. Let $F_0 = C_{F_1}(E)$. $X_0 = C_X(E)$, and $A_0 = K[X_0]$. It is easily checked that $X_0 = F_0e_1$. Then A_0 is a simple ring by Lemma 3.2.2 and the centre L of A_0 contains E.

4.2 Normal Field Extensions in $\mathcal{Z}(A_0)$

We will show that A_0 is precisely the centralizer in A of E; hence $Z \subseteq A_0$. Also X/X_0 acts as the full Galois group of the finite field extension $\mathcal{Z}(A_0)/Z$.

Lemma 4.2.1 $C_A(E) = A_0$

Proof We will show that $C_{KF_1}(E) = KF_0$. Then $C_{A_1}(E) = KF_0 \cap A_1 = KF_0e = A_0$.

That $KF_0 \subseteq C_{KF_1}(E)$ is clear. To prove the other inclusion we use the fact that kF_1 is a crossed product over kF' by the free abelian group F_1/F'. Let $\mathcal{T}$ be a transversal for F' in F_1, and let $\alpha \in C_{KF_1}(E)$. Multiplying by a suitable element of K if necessary, we can assume $\alpha \in kF_1$. Then α can be written uniquely in the form

$$\alpha = \sum_{t\in\mathcal{T}} \alpha_t t.$$

where $\alpha_t \in kF'$, $\alpha_t = 0$ for all but finitely many t. Now let $\theta \in E$. Then

$$\theta\alpha = \alpha\theta \implies \sum_{t\in\mathcal{T}} \theta\alpha_t t = \sum_{t\in\mathcal{T}} \alpha_t t\theta = \sum_{t\in\mathcal{T}} \alpha_t\theta^t t$$

$$\implies \sum_{t\in\mathcal{T}} (\theta - \theta^t)\alpha_t t = 0.$$

Then $(\theta - \theta^t)\alpha_t = 0$ for each t, and since $\theta - \theta^t \in E$ (a field) we have either

$$\theta = \theta^t, \forall\theta \in E, \text{ and } t \in F_0$$

or

$$\alpha_t = 0$$

Hence $\text{supp}(\alpha) \subset F_0$, $\forall \alpha \in C_{KF_1}(E)$, and $C_{KF_1}(E) = KF_0$, $C_{A_1}(E) = A_0$. □

It follows from Lemma 4.2.1 that A_0 contains Z, the centre of A_1. Let L denote the algebra generated by Z and E. Then L is a field, since it is contained in the centre of the simple algebra A_0. In fact L is precisely the centre of A_0, for

$$A_0 = C_A(E) = C_A(ZE) = C_A(L).$$

By the double centralizer theorem, $C_A(C_A(L)) = L$, since L is a simple Z-subalgebra of the central simple Z-algebra A_1. Then

$$C_A(A_0) = C_A(C_A(L)) = L$$

and $\mathcal{Z}(A_0) = L$ since $L \subseteq A_0$.

Thus L is a finite field extension of Z, since $L = ZE$ and $E = \mathcal{Z}(kF'e)$ has finite dimension over k.

Lemma 4.2.2 *L/Z is a normal extension of fields, with Galois group isomorphic to X/X_0 ($\cong F_1/F_0$).*

Proof That the extension is normal is easy to see, since L is generated over Z (as E is over k) by sums in B_1 of conjugacy classes from T_X. These are central in B_1 but not necessarily in A_1. If $\mathcal{C}$ is such a class sum, then the polynomial $\prod_x (t - \mathcal{C}^x)$, where x runs through a transversal for $C_F(\mathcal{C})$ in F, has coefficients in Z and splits in L, hence the normality of L over Z.

Now X acts by conjugation on L and the kernel of this action is $C_X(L) = X_0$. The fixed field of L under the action of X/X_0 is just $\mathcal{Z}(A) = Z$, hence $\text{Gal}(L/Z) \cong X/X_0$. □

4.3 Tensor Product Structure of A_0

Let B denote the subalgebra of A_0 generated over Z by $T_X = F'e_1$, and let $C = C_{A_0}(B)$. Then we will see that B is a central simple L-algebra, whence $A_0 = B \otimes_L C$. Furthermore, since

B_1, B, A_0 and A_1 all have the same set of matrix units by Theorem 3.2.2, C is a division algebra.

To show that B is central simple over L. we make use of the following lemma (Proposition 12.4a in [16]) :-

Lemma 4.3.1 *Let $\mathcal{A}$ be a finite dimensional algebra over a field $\mathcal{F}$. Let $\mathcal{B}$ and $\mathcal{C}$ be $\mathcal{F}$-subalgebras of $\mathcal{A}$ for which*

i) $\mathcal{B}$ is central simple over $\mathcal{F}$.

ii) $\mathcal{C}$ centralizes $\mathcal{B}$.

iii) $\mathcal{A} = \mathcal{B}\mathcal{C}$.

Then $\mathcal{A} \cong \mathcal{B} \otimes_{\mathcal{F}} \mathcal{C}$. □

Theorem 4.3.1 *$B = Z[T_X]$ is a central simple L-algebra.*

Proof : First we show that B is semisimple. Suppose not, and let I be a nonzero nilpotent ideal in B. Let $\bar{I}$ denote the two-sided ideal generated by I in A_0. A typical element of $(\bar{I})^t$ is a K-linear combination of elements of the form $x_1\alpha_1x_2\alpha_2\ldots\alpha_px_{p+1}$, where $\alpha_1,\ldots\alpha_p$ are elements of I. $x_1,\ldots x_{p+1}$ are elements of X_0, and $p \geq t$. Suppose q is the nilpotency class of I. We now show that $\bar{I}$ is also nilpotent, of class at most q.

Since every element of X_0 is invertible, the expression

$$a = x_1\alpha_1\ldots x_q\alpha_qx_{q+1} \qquad x_i \in X_0, \alpha_i \in I, \tag{4.1}$$

can be written in the form

$$a = x\alpha'_1\ldots\alpha'_q, \tag{4.2}$$

where $x \in X_0$ and for $i = 1\ldots q$ α'_i is of the form $x'^{-1}\alpha_ix'$ for some $x' \in X_0$. Since $T_X \trianglelefteq X_0$ and X_0 centralizes E. conjugation by x' induces a central automorphism of $kF'e_1$ and hence of B, since B is generated by T_X over the centre of A_0. Furthermore, since $kF'e_1$ is simple, this automorphism is inner by the Noether-Skolem theorem (see [7], Section 4.6). Thus for $i = 1 \ldots q$, $\alpha_i' = c_i^{-1}\alpha_i c_i$ for some $c_i \in \mathcal{U}(kF'e_1)$, whence $\alpha_i' \in I$. It then follows that $a = 0$ in 4.2, since $(I)^q = 0$. Thus $\bar{I}$ is nilpotent of class at most q, which contradicts the simplicity of A_0. We conclude that B is semisimple. Since L is a field, it now suffices to show that $\mathcal{Z}(B) = L$.

The subalgebra of B generated by K and E is a field, since it is contained in the centre of A_0. Also

$$KE \cong K \otimes_k E,$$

since K is purely transcendental over k, and E is a finite extension of k. The algebra B is finite dimensional over KE, since KF has finite dimension over K.

It is apparent that any E-basis of $B_1 = kF'e_1$ remains independent over the field KE, whence $\dim_E(B_1) = \dim_{KE}(K[T_X])$, and $K[T_X] \cong KE \otimes_E B_1$ (see [16], Proposition 9.2c). Then $K[T_X]$ is a central simple KE-algebra, since KE is simple and B_1 is central simple over E (see Lemma 12.4b of [16]). Then

Now $K[T_X] \cong K \otimes_k B_1$ is a simple subalgebra of B, and its centre is $K \otimes_k E = KE$. Also, B is generated by L and $K[T_X]$, and L centralizes $K[T_X]$. Then we can apply Lemma 4.3.1 to conclude

$$B = L \otimes_{KE} K[T_X].$$

Since $K[T_X]$ is a central simple KE-algebra and L is simple, B is then a simple ring and its centre is $L \otimes_{KE} KE = L$. □

Thus B is a central simple subalgebra of the finite-dimensional simple L-algebra A_0, hence so also is the division algebra $C = C_{A_0}(B)$, and we reach the following conclusion (see Theorem 4.7 in [7]):-

Theorem 4.3.2 $A_0 = B \otimes_L C$. □

The simple ring B_1 is a ring of $n \times n$ matrices over a division algebra D_1, and by Theorem 3.2.2, any set $\mathcal{E}$ of n^2 matrix units for B_1 is a full set of matrix units for A_0 and hence for B, since $B_1 \subseteq B \subseteq A_0$. Thus $B \cong M_n(D)$, where D is a division ring containing a copy of D_1.

Also, since $B = L \otimes_{KE} K[T_X]$, we have

$$\begin{aligned} \dim_{KE}(B) &= \dim_{KE}(L)\dim_{KE}(K[T_X]) \\ \Longrightarrow \dim_L(B) &= \dim_{KE} K[T_X]. \end{aligned}$$

Furthermore, since K/k is a purely transcendental field extension, any E-basis for B_1 is a KE-basis for $K[T_X]$, whence $\dim_{KE} K[T_X] = \dim_E(kF'e_1)$. Then $\dim_L(B) = \dim_E(kF'e_1)$, and the degrees of the simple algebras B and $kF'e_1$ coincide. Since these algebras also have the same set of matrix units, their Schur indices also coincide. Then the degree and Schur index of B depend only on the simple component $\langle e_1\rangle$ of kF'.

We now define $T_X{}^+ = X \cap B$. The notation, and the idea of studying this group are both suggested by [18]. Then $T_X{}^+$ contains T_X, and $T_X{}^+$ is a subgroup of $X_0 = C_X(L)$, since B centralizes L. For $x \in F_1$, let $\hat{C}_x$ denote the sum in kF of the (finitely many) F_1-conjugates of x. Define

$$\mathcal{P} = \left\{ x \in F_1 : \hat{C}_x e_1 \neq 0 \right\}$$

We note that $\mathcal{P} \subseteq F_0$. For suppose $x \in \mathcal{P}$: then $0 \neq \hat{C}_x e_1 \in Z = \mathcal{Z}(A_1)$. Since $e_1 \in kF'$, we have $\hat{C}_x e_1 = xc$, where $c \in kF'e_1 = B_1$. Then x must centralize $E = \mathcal{Z}(B_1)$, so $x \in F_0$.

Lemma 4.3.2 *$T_X{}^+$ consists precisely of elements of X_0 of the form cxe_1, where $c \in F'$ and $x \in \mathcal{P}$.*

Proof : Certainly $T_X \subseteq T_X{}^+$. Suppose $x \in \mathcal{P}$. Then $\hat{C}_x = x\theta_x$ where $\theta_x \in kF'$. Since $\hat{C}_x \in Z^\times$, $\theta_x e_1$ is a unit in $\mathcal{U}(kF'e_1)$. Then $\theta_x e_1 \in \mathcal{U}(B)$, and $xe_1 \in B$, as $x\theta_x e_1 \in Z$.

On the other hand, suppose $t \in F_0$ satisfies $te_1 \in B$. Then, since $B = Z\langle T_X\rangle$, we can write te_1 in the form

$$te_1 = \sum_{x \in \mathcal{P}_1} \alpha_x \hat{C}_x e_1,$$

where $0 \neq \alpha_x \in KF'$ for $x \in \mathcal{P}_1$, $\mathcal{P}_1 \subseteq \mathcal{P}$.

If S is a free abelian subgroup of R for which K is the field of quotients of kS, we can find an element $a \neq 0$ of kS for which

$$ate_1 = \sum_{x \in \mathcal{P}_1} a\alpha_x \hat{C}_x e_1,$$

and $a\alpha_x \in kF'[S]$ for each $x \in \mathcal{P}_1$. Also, for $x \in \mathcal{P}_1$, $\hat{C}_x e_1 = x\theta_x$, where $\theta_x \in \mathcal{U}(kF'e_1)$. Then

$$ate_1 = \sum_{x \in \mathcal{P}_1} a\alpha_x x\theta_x e_1.$$

We now regard each of these expressions as an element of the group ring kF. Let $y \in \mathrm{supp}(ate_1)$. Then since $a \in kS$ and $e_1 \in kF'$, $y = sct$ for some $s \in S$ and $c \in F'$. Then sct must appear in the support of $\sum_{x \in \mathcal{P}_1} a\alpha_x x\theta_x e_1$, where $\alpha_x\theta_x \in kF'$ for each x. Then, since $e_1 \in kF'$ also, we must have

$$sct = s'c'x,$$

where $s' \in S$, $c' \in F'$, and $x \in \mathcal{P}$. This completes the proof of the lemma since $x \in \mathcal{P} \implies s'x \in \mathcal{P}$, as $s' \in \mathcal{Z}(F)$. □

We remark that if $x \in F_0$ satisfies $\dot{C}_x e_1 \neq 0$ where $\dot{C}_x$ denotes the sum in kF_0 of the F_0-conjugates of x, then $xe_1 \in {T_X}^+$. This follows from the fact that B contains the centre of A_0.

We will denote the preimage of ${T_X}^+$ in F by F'^+, i.e.

$$F'^+ = \{x \in F : xe_1 \in B\}, \tag{4.3}$$

and we will denote the image of F'^+ in G by G^+. Thus G^+ is a subgroup of $G_0 = \phi(F_0)$, and G^+ contains G'.

Fix a transversal $\mathbf{T}$ for T_X in X_0, with the property $\mathbf{T} = \mathcal{T}\mathcal{S}$, where $\mathcal{T}$ and $\mathcal{S}$ are transversals for T_X in ${T_X}^+$ and ${T_X}^+$ in X_0 respectively. Now let $\alpha_1 \in C$. We can multiply α_1 by a nonzero element a of K if necessary, to obtain $\alpha = a\alpha_1 \in kF$, and using the crossed product structure of kF over kF' we can write α uniquely in the form

$$\alpha = \sum_{t \in \mathbf{T}} \alpha_t t,$$

where each α_t belongs to kF'. Of course $C_{A_0}(B) = C_{A_0}(T_X)$, so let $c \in T_X$. Then

$$\begin{aligned} c\alpha = \alpha c \quad &\implies \quad \sum_{t \in \mathbf{T}} c\alpha_t t = \sum_{t \in \mathbf{T}} \alpha_t c^{t^{-1}} t \\ &\implies \quad c\alpha_t = \alpha_t tct^{-1}, \ \forall t \in \mathbf{T} \\ &\implies \quad c\alpha_t t = \alpha_t tc, \ \forall t \in \mathbf{T} \end{aligned}$$

Then for each $t \in \mathbf{T}$, $\alpha_t t$ centralizes X', so $\alpha_t t \in C$.

Now B_1 is a simple ring, and X_0 centralizes $E = \mathcal{Z}(B_1)$, so conjugation by any element of X_0 induces an inner automorphism of B_1, by the Noether-Skolem theorem. Then for each $t \in \mathbf{T}$ we can choose an element β_t of $\mathcal{U}(B_1)$ for which $\beta_t t \in C_{A_0}(T_X) = C$. Also β_t is determined by t up to multiplication by elements of $E^\times$. From now on we fix for each $t \in \mathbf{T}$ an element γ_t of C for which $\gamma_t = \beta_t t$, $\beta_t \in \mathcal{U}(kF'e_1)$. We remark that since $B_1 \subseteq kF'$, γ_t belongs not only to KF_0 but to kF_0. It is clear from the above discussion that C is generated over L (over K, in fact) by $\mathcal{B} = \{\gamma_t\}_{t\in\mathbf{T}}$. This set is linearly independent over E but not in general over L. However a certain subset of $\mathcal{B}$ will constitute an L-basis of C.

Theorem 4.3.3 *Let $t \in \mathbf{T}$. Then $\gamma_t \in L$ if and only if $t \in {T_X}^+$.*

Proof ($\Longrightarrow$) Suppose $\gamma_t \in L$. Then $\gamma_t = \beta_t t$. and $\beta_t \in B_1$. Since $t \in X_0 = F_0 e_1$, we have $\gamma_t \in \mathcal{Z}(kF_0 e_1)$; in particular γ_t belongs to the centre of the group ring kF_0. Then

$$\gamma_t = \sum_{x\in\mathcal{X}} a_x \hat{C}_x e_1, \quad a_x \in k^\times,$$

where $\mathcal{X}$ is some subset of F_0 and $\hat{C}_x$ denotes the sum in A_0 of the distinct F_0 -conjugates of x; $\hat{C}_x e_1 \neq 0$ for $x \in \mathcal{X}$. Then $\hat{C}_x = b_x x$ where $b_x \in kF'e_1$ (b_x is a sum of simple commutators), and we can write

$$\gamma_t = \sum_{x\in\mathcal{X}} \alpha_x x e_1,$$

where $\alpha_x \in kF'$. Finally each $x \in \mathcal{X}$ can be written as $x = c_x t_x$ where $c_x \in T_X$, $t_x \in \mathbf{T}$. Then

$$\gamma_t = \beta_t t = \sum_{x\in\mathcal{X}} \alpha'_x t_x,$$

where $\alpha'_x \in kF'$ for each x. Then, since the elements of $\mathbf{T}$ are independent over kF', we must have $t_x = t$ for each $x \in \mathcal{X}$. Then $x \in T_X t$, $\forall x \in \mathcal{X}$, and in particular there exists an element c of F' for which $\hat{C}_{ct} e_1 \neq 0$, whence $t \in {T_X}^+$ by the remark following the proof of Lemma 4.3.2.

($\Longleftarrow$) Suppose $t \in {T_X}^+$. Then, by Lemma 4.3.2 $t = cx$, where $c \in T_X$ and $x \in F_0$ satisfies $\hat{C}_x e_1 \neq 0$. Now $\hat{C}_x e_1 = \theta_x x$. where $\theta_x \in \mathcal{U}(kF'e_1)$.

$$\hat{C}_x e_1 = \theta_x c^{-1} c x = \theta_x c^{-1} t,$$

where $\theta_x c^{-1} \in \mathcal{U}(kF'e_1)$. Then $\beta_t t \in E^\times \theta_x c^{-1} t$, since $\theta_x c^{-1} t \in L$ and in particular $\theta_x c^{-1} t$ centralizes B. Hence $\gamma_t = \beta_t t \in L$. □

Recall that $\mathbf{T} = \mathcal{T}\mathcal{S}$, where $\mathcal{S}$ is a transversal for $T_X{}^+$ in X_0, and $\mathcal{T}$ a transversal for T_X in $T_X{}^+$. The elements of $\mathcal{B}$ possess the following important properties :-

Lemma 4.3.3 *i) Suppose $t_1, t_2 \in \mathbf{T}$ and let $t \in \mathbf{T}$ represent the coset $T_X t_1 t_2$. Then $\gamma_{t_1}\gamma_{t_2} \in E^\times \gamma_t$.*

ii) Suppose $s_1, s_2 \in \mathcal{S}$ and let $ts \in \mathbf{T}$ represent the coset $T_X s_1 s_2$, where $t \in \mathcal{T}$, $s \in \mathcal{S}$. Then $\gamma_{s_1}\gamma_{s_2} \in L^\times \gamma_s$.

Proof : **i)**

$$\begin{aligned}
\gamma_{t_1}\gamma_{t_2} &= \beta_{t_1} t_1 \beta_{t_2} t_2 = \beta_{t_1} \beta_{t_2}^{t_1^{-1}} t_1 t_2 \\
&= \beta_{t_1} \beta_{t_2}^{t_1^{-1}} ct, \text{ where } c \in T_X.
\end{aligned}$$

Then $\beta_{t_1}\beta_{t_2}^{t_1^{-1}} ct \in C$, and $\beta_{t_1}\beta_{t_2}^{t_1^{-1}} c \in \mathcal{U}(kF'e_1)$, and

$$\beta_{t_1}\beta_{t_2}^{t_1^{-1}} c \in E^\times \beta_t \Longrightarrow \beta_{t_1}\beta_{t_2}^{t_1^{-1}} ct \in E^\times \beta_t t$$

$$\gamma_{t_1}\gamma_{t_2} \in E^\times \gamma_t.$$

ii) By i), $\gamma_{s_1}\gamma_{s_2} \in E^\times \gamma_{ts}$. Also by i), $\gamma_t \gamma_s \in E^\times \gamma_t \gamma_s$. Then

$$\gamma_{s_1}\gamma_{s_2} \in E^\times \gamma_t \gamma_s.$$

However $t \in T_X{}^+ \Longrightarrow \gamma_t \in L^\times$, hence $\gamma_{s_1}\gamma_{s_2} \in L^\times \gamma_s$. □

4.4 The Centre of A_0

The field $L = \mathcal{Z}(A_0)$ is generated as a vector space over KE by $\{\gamma_t\}_{t\in\mathcal{T}}$, for suppose $\lambda \in L^\times$, and choose $a \in K$ for which $a\lambda \in kF \cap L$. Then $a\lambda$ can be written uniquely in the form

$$a\lambda = \sum_{t\in\mathbf{T}} \lambda_t t,$$

where $\lambda_t \in kF'$, $\lambda_t = 0$ for all but finitely many $t \in \mathcal{T}$. Then it follows easily from the centrality of $a\lambda$ in A_0 that $\lambda_t t \in L$ for each t, i.e. $\lambda_t = 0$ or $t \in T_X{}^+$ and $\lambda_t \in E^\times \gamma_t$. Then

$$\lambda = \sum_{t\in\mathcal{T}_1} a^{-1}\lambda_t{}'\gamma_t,$$

where $\mathcal{T}_1$ is the subset of $\mathcal{T}$ upon which $\lambda_t \neq 0$, and $\lambda_t' = \lambda_t \beta_t^{-1} \in E^\times$, for $t \in \mathcal{T}_1$.

Next we determine a transcendence basis for L over E.

$T_X{}^+/T_X$ is a free abelian group, of which $\langle Re_1, T_X\rangle/T_X$ is a subgroup of finite index (since R has finite index in F). Then both are free abelian groups of the same rank. Since

$$\langle Re_1, T_X\rangle/T_X \cong Re_1/Re_1 \cap T_X \cong R/F' \cap R,$$

this rank is r, which is the rank of the finite abelian group G/G' and is equal to the transcendence degree of K over k. Now we can find a basis $\{\bar{t}_1, \ldots, \bar{t}_r\}$ of $T_X{}^+/T_X$, for which $\{\bar{t}_1^{j_1}, \ldots, \bar{t}_r^{j_r}\}$ is a basis of $\langle Re_1, T_X\rangle/T_X$. Here $\bar{t}_i$ denotes the coset $t_i T_X$, and $t_i \in \mathcal{T}$ for $i = 1 \ldots r$.

Theorem 4.4.1 *L/E is a purely transcendental field extension with transcendence basis*

$$\Gamma = \{\gamma_{t_1}, \ldots, \gamma_{t_r}\}.$$

Proof : First we show that K is contained in $E(\Gamma)$, the algebra generated by Γ over E. For this it suffices to show that $E(\Gamma)$ contains Re_1, since K is the field of quotients of a subring of kR. Note that the torsion subgroup $Re_1 \cap T_X$ of Re_1 is contained in $E^\times$.

For $i = 1 \ldots r$ we have $t_i^{j_i} \in Re_1 T_X$, so $t_i^{j_i} = s_i c_i$, where $s_i \in Re_1$, $c_i \in T_X$. Here s_i and c_i are determined uniquely up to multiplication by elements of $Re_1 \cap T_X$. Then

$$\langle Re_1, T_X\rangle = \langle s_i c_i, T_X\rangle_{i=1\ldots r} = \langle s_i, T_X\rangle_{i=1\ldots r}.$$

Then since T_X is finite and r is the rank of the free abelian group

$$R/F' \cap R \cong Re_1/Re_1 \cap T_X,$$

$\langle s_1, \ldots, s_r\rangle$ must be a torsion-free complement for $Re_1 \cap T_X$ in Re_1.

Now for $i = 1 \ldots r$, it follows from Lemma 4.3.3 that $(\gamma_{t_i})^{j_i} = s_i \theta_i$, where $\theta_i \in L \cap kF'e_1 = E$. Hence $s_i \in E(\Gamma)$ for $i = 1 \ldots r$, $E(\Gamma)$ contains Re_1, and $E(\Gamma)$ contains K. In fact KE is generated as an E-algebra by

$$\Gamma_1 := \left\{(\gamma_{t_1})^{j_1}, \ldots, (\gamma_{t_r})^{j_r}\right\}. \tag{4.4}$$

Now E/k is an algebraic field extension, and K/k is purely transcendental of transcendence degree r. Then the transcendence degree of KE/E is also r, and so Γ_1 is a transcendence basis for KE/E. In particular Γ_1 is an algebraically independent set over E, and so also is Γ.

That $L = E(\Gamma)$ now follows from Lemma 4.3.3 and the fact that $\{\gamma_t\}_{t\in\mathcal{T}}$ generates L as a vector space over KE. This completes the proof of Theorem 4.4.1 : L/E is a purely transcendental field extension of transcendence degree r, and Γ is a transcendence basis of L/E for which $L = E(\Gamma)$. □

4.5 The Division Algebra C

Let $\mathbf{B} = \{\gamma_s\}_{s\in\mathcal{S}}$. It is apparent now that $\mathbf{B}$ is an L-basis for C. It is immediate from Lemma 4.3.3 that $C = L[\mathbf{B}]$, since $C = K[\mathcal{B}]$, and $\gamma_t \in L$ whenever $t \in \mathcal{T}$. That $\mathbf{B}$ is independent over L follows from the independence of $\mathbf{T}$ over $kF'e_1$. For suppose we have $\{l_s\} \subset L$ for which $\sum_{s\in\mathcal{S}} l_s\gamma_s = 0$. Multiplying by a suitable element of K if necessary, we can suppose that $l_s \in \mathcal{Z}(kF_0e_1)$ for each $s \in \mathcal{S}$. Then each l_s can be written in the form

$$l_s = \sum_{t\in\mathcal{T}} a_{ts}\gamma_t,$$

where $a_{ts} \in E$, and $a_{ts} = 0$ for all but finitely many t. Then

$$\sum_{s\in\mathcal{S}}\sum_{t\in\mathcal{T}} a_{ts}\gamma_t\gamma_s = 0.$$

Now $\gamma_t\gamma_s = a'_{ts}ts$ for some $a'_{ts} \in \mathcal{U}(kF'e_1)$, by Lemma 4.3.3, and so

$$\sum_{s,t} b_{ts}ts = 0$$

where $b_{ts} = a'_{ts}a_{ts} \in kF'e_1$. Then $b_{ts} = 0\ \forall t, s$ since $\mathbf{T}$ is independent over $kF'e_1$, and so $a_{ts} = 0\ \forall\ t, s$. Then $l_s = 0$, $\forall s \in \mathcal{S}$, and $\mathbf{B}$ is independent over L.

The centre of C consists only of L, since $A_0 = B \otimes_L C$ and $L = \mathcal{Z}(A_0) = \mathcal{Z}(B)$. Also, $\dim_L(C) = [X_0 : T_X{}^+]$ since $\mathbf{B}$ is an L-basis for C. Then of course $[X_0 : T_X{}^+]$ is a square. Now $T_X{}^+ \trianglelefteq X_0$, and the quotient $X_0/T_X{}^+$ is abelian since $T_X{}^+ \supseteq T_X \supseteq X'$, and finite since $T_X{}^+$ contains $\mathcal{Z}(X)$ which has finite index in X, hence in X_0.

For $s \in \mathcal{S}$, let $\bar{s}$ denote the element $sT_X{}^+$ of $X_0/T_X{}^+$. Then we can find elements $s_1, \ldots, s_k$ of $\mathcal{S}$ for which

$$X_0/T_X{}^+ = \langle\bar{s}_1\rangle \times \cdots \times \langle\bar{s}_k\rangle,$$

where $\bar{s}_i$ has order d_i in $X_0/T_X{}^+$, $d_k \mid d_{k-1} \cdots \mid d_1$ for $i = 1 \ldots k$. Let $\mathcal{S}_1$ denote the subset of $\mathcal{S}$ consisting of the elements $s_1, \ldots, s_k$.

Theorem 4.5.1 *C is a twisted group ring of the finite abelian group $X_0/T_X{}^+$ over L.*

Proof : Let H denote the subgroup of $C^\times$ generated by $\{\gamma_s\}_{s\in\mathcal{S}}$, and let $\bar{H} = H/H\cap L^\times$. Define a map $\phi : X_0/T_X{}^+ \longrightarrow \bar{H}$ on $\mathcal{S}_1$ by

$$\phi(\bar{s}_i) = \bar{\gamma}_{s_i}.$$

Then it follows easily from Lemma 4.3.3 that ϕ extends to an isomorphism of groups.

Now the assignment

$$\bar{s}_1^{r_1}\bar{s}_2^{r_2}\ldots\bar{s}_k^{r_k} \longrightarrow \gamma_{s_1}^{r_1}\gamma_{s_2}^{r_2}\ldots\gamma_{s_k}^{r_k}$$

where $0 \leq r_i \leq d_i$ for $i = 1\ldots k$, defines the structure of a twisted group ring on C. It is immediate from Lemma 4.3.3 that for each choice of $r_1,\ldots,r_k$,

$$\gamma_{s_1}^{r_1}\gamma_{s_2}^{r_2}\ldots\gamma_{s_k}^{r_k} \in L^\times\gamma_{s_1^{r_1}\ldots s_k^{r_k}}.$$

and so C is generated over L by elements of the form $\gamma_{s_1}^{r_1}\gamma_{s_2}^{r_2}\ldots\gamma_{s_k}^{r_k}$. That these elements are independent over L for different choices of $r_1,\ldots,r_k$ is clear, since **B** is independent over L.

Then $C \cong L^f\left(X_0/T_X{}^+\right)$, where the cocycle $f \in Z^2\left(X_0/T_X{}^+, L^\times\right)$ is defined by

$$f(\bar{s}_1^{r_1}\ldots\bar{s}_k^{r_k},\ \bar{s}_1^{q_1}\ldots\bar{s}_k^{q_k}) = \gamma_{s_1}^{r_1}\ldots\gamma_{s_k}^{r_k}\gamma_{s_1}^{q_1}\ldots\gamma_{s_k}^{q_k}\left(\gamma_{s_1}^{r_1+q_1}\ldots\gamma_{s_k}^{r_k+q_k}\right)^{-1}.$$

Of course a different choice for $\mathcal{S}_1$ will yield a cocycle which differs from f by a coboundary in $Z^2\left(X_0/T_X{}^+, L^\times\right)$. □

It is well known (see [21]), that if a finite abelian group $\mathcal{A}$ has a central simple twisted group algebra over a field $\mathcal{F}$, then $\mathcal{A}$ must be a group of symmetric type (i.e. the direct product of two isomorphic abelian groups), and $\mathcal{F}$ must contain a root of unity of order equal to the exponent of $\mathcal{A}$. For clarity we include a proof of these facts; in the process we obtain a fairly explicit description of twisted group rings of this type as tensor products of symbol algebras. This (applied to C) will be useful later in determining possible values of the Schur index and degree of irreducible projective representations of G over various fields.

Lemma 4.5.1 *Let $\mathcal{A}$ be a finite abelian group, let $\mathcal{F}$ be a field, and let $f \in Z^2(\mathcal{A},\mathcal{F}^\times)$. Then the map*

$$\phi : \mathcal{A}\times\mathcal{A} \longrightarrow \mathcal{F}^\times$$

defined for $a, b \in \mathcal{A}$ by

$$\phi(a,b) = \frac{f(a,b)}{f(b,a)}$$

is an antisymmetric pairing on $\mathcal{A}$.

Proof : We require to show for $a, b, c \in \mathcal{A}$ that

$$\phi(ab,c) = \phi(a.c)\phi(b,c), \quad \text{or} \quad \frac{f(ab,c)}{f(c,ab)} = \frac{f(a,c)}{f(c,a)}\frac{f(b,c)}{f(c,b)}.$$

This follows easily from the usual cocycle law : if x, y, z are elements of a group G, and $\alpha \in \mathrm{H}^2(G, A)$ for any abelian group A, we have

$$\alpha(x.y)\alpha(xy,z) = \alpha(x,yz)\alpha(y,z).$$

Note

$$\frac{f(ab,c)}{f(c,ab)} = \frac{f(a,bc)f(b,c)}{f(ca.b)f(c.a)}$$

so we need only show $f(a,bc)f(c,b) = f(ca,b)f(a.c)$. Since $\mathcal{A}$ is abelian we have

$$f(a,bc)f(c.b) = f(a,cb)f(c,b) = f(a,c)f(ac,b) = f(a,c)f(ca,b).$$

Then ϕ is a pairing; that ϕ is antisymmetric is clear. □

Theorem 4.5.2 *Let $\mathcal{A}$ be a finite abelian group of exponent d, let $\mathcal{F}$ be a field of characteristic zero, and suppose that the twisted group algebra $\mathcal{F}^f\mathcal{A}$ is a central simple $\mathcal{F}$-algebra for some $f \in \mathrm{H}^2(\mathcal{A}, \mathcal{F}^\times)$. Then*

i) $\mathcal{F}$ contains a root of unity of order d.

ii) $\mathcal{A}$ is of symmetric type.

iii If $\mathcal{A} \cong (C_{d_1} \times C_{d_1}) \times (C_{d_2} \times C_{d_2}) \times \cdots \times (C_{d_n} \times C_{d_n})$, where $d_n \mid d_{n-1} \mid \cdots \mid d_2 \mid d_1 = d$, then

$$\mathcal{F}^f(\mathcal{A}) \cong \left(\frac{A_1, B_1}{\xi_1, k}\right) \otimes_k \left(\frac{A_2, B_2}{\xi_2, k}\right) \otimes_k \cdots \otimes_k \left(\frac{A_n, B_n}{\xi_n, k}\right),$$

where for $i = 1 \ldots n$, $A_i, B_i \in \mathcal{F}^\times$ and ξ_i is a root of unity of order d_i in $\mathcal{F}$.

Proof : For $x \in \mathcal{A}$ let $\bar{x}$ denote the basis element of $\mathcal{F}^f\mathcal{A}$ corresponding to x. Consider the antisymmetric pairing ϕ on $\mathcal{A}$ defined as in Lemma 4.5.1. For $a, b \in \mathcal{A}$, $\phi(a, b)$ has order in $\mathcal{F}^\times$ equal to the least common multiple of the orders of a and b in $\mathcal{A}$. Also the restriction of ϕ to $C \times C$ is trivial for any cyclic subgroup C of $\mathcal{A}$, since

$$\phi(x^i, x^j) = (\phi(x, x))^{ij} = 1, \ \forall x \in \mathcal{A}.$$

That $\mathcal{F}^f\mathcal{A}$ is a central simple over $\mathcal{F}$ means that ϕ is nondegenerate, for suppose for some $a \in \mathcal{A}$ that $\phi(a, x) = 1, \ \forall x \in \mathcal{A}$. Then $f(a, x) = f(x, a) \ \forall x \in \mathcal{A}$, and $\bar{a}$ belongs to the centre of $\mathcal{F}^f\mathcal{A}$.

Now choose an element a_1 of order d in $\mathcal{A}$. There exists an element b_1 of $\mathcal{A}$ (also of order d) for which $\phi(a_1, b_1)$ has order d in $\mathcal{F}^\times$, otherwise some $(\bar{a})^{d'}$ with $d' < d$ would be central in $\mathcal{F}^f\mathcal{A}$. (This proves i)). Finally $\langle a_1, b_1 \rangle \cong C_d \times C_d$, and

$$\mathcal{F}(\bar{a}_1, \bar{b}_1) \cong \left(\frac{(\bar{a}_1)^d, (\bar{b}_1)^d}{\xi_1, \mathcal{F}} \right).$$

Now let $\mathcal{O}_1$ denote the orthogonal complement of $\langle \bar{a}_1, \bar{b}_1 \rangle$ in $\mathcal{A}$, with respect to ϕ :-

$$\mathcal{O}_1 = \{ x \in \mathcal{A} \ : \ \phi(a_1, x) = \phi(b_1, x) = 1 \}.$$

Certainly $\mathcal{O}_1$ is a group, and since $\phi(a, b)$ is a power of $\phi(a_1, b_1)$ for all $a, b \in \mathcal{A}$, it is easily checked that $\mathcal{A} = \langle a_1, b_1, \mathcal{O}_1 \rangle$. Also $\langle a_1, b_1 \rangle \cap \mathcal{O}_1 = \{1\}$, so $\mathcal{A} = \langle a_1, b_1 \rangle \times \mathcal{O}_1$. Let

$$\mathcal{C} = C_{\mathcal{F}^f\mathcal{A}}\left(\mathcal{F}(\bar{a}_1, \bar{b}_1)\right).$$

Then $\mathcal{C}$ is generated as a vector space over $\mathcal{F}$ by $\{\bar{x} : x \in \mathcal{O}_1\}$, for suppose

$$\alpha = \sum_{a_i \in \mathcal{A}} s_i \bar{a}_i \in \mathcal{C}.$$

Then

$$\bar{a}_1 \alpha = \sum_{a_i \in \mathcal{A}} s_i f(a_1, a_i) \overline{a_1 a_i} = \alpha \bar{a}_1 = \sum_{a_i \in \mathcal{A}} s_i f(a_i, a_1) \overline{a_1 a_i}.$$

Then for each i, either $s_i = 0$ or

$$f(a_1, a_i) = f(a_i, a_1), \text{ and } \phi(a_1, a_i) = 1.$$

Similarly for b_1; hence $\alpha \in \mathcal{F}(\mathcal{O})$, and by Lemma 4.3.1

$$\mathcal{F}^f\mathcal{A} = \mathcal{F}(\bar{a}_1, \bar{b}_1) \otimes_{\mathcal{F}} \mathcal{F}(\mathcal{O}).$$

$\mathcal{F}(\mathcal{O})$ is again a central simple $\mathcal{F}$-algebra, and we can repeat the argument to complete the proof :-

$$\mathcal{F}^f\mathcal{A} \cong \left(\frac{A_1, B_1}{\xi_1, \mathcal{F}}\right) \otimes_k \left(\frac{A_2, B_2}{\xi_2, \mathcal{F}}\right) \otimes_k \cdots \otimes_k \left(\frac{A_n, B_n}{\xi_n, \mathcal{F}}\right)$$

where

$$\mathcal{A} = \langle a_1\rangle \times \langle b_1\rangle \times \langle a_2\rangle \times \langle b_2\rangle \times \ldots \langle a_n\rangle \times \langle b_n\rangle.$$

$\langle a_i\rangle \times \langle b_i\rangle \cong C_{d_i} \times C_{d_i}$. □

We have proved the following :-

Corollary 4.5.1 *We can find a subset*

$$\mathcal{S}_0 = \{r_1, s_1, \ldots, r_k, s_k\}$$

of $\mathcal{S}$ for which

i) $X_0/T_X{}^+ = \langle\bar{r}_1\rangle \times \langle\bar{s}_1\rangle \times \ldots \langle\bar{r}_k\rangle \times \langle\bar{s}_k\rangle$,
where $\bar{r}_i = r_iT_X{}^+$, $\bar{s}_i = s_iT_X{}^+$.

ii) $\mathrm{ord}(\bar{r}_i) = \mathrm{ord}(\bar{s}_i) = d_i;\ d_k \mid d_{k-1} \mid \cdots \mid d_1$.

iii) $C = \left(\frac{R_1, S_1}{\xi_1, L}\right) \otimes_L \left(\frac{R_2, S_2}{\xi_2, L}\right) \otimes_L \cdots \otimes_L \left(\frac{R_k, S_k}{\xi_k, L}\right)$

where $R_i = (\gamma_{r_i})^{d_i} \in L^\times$, $S_i = (\gamma_{s_i})^{d_i} \in L^\times$ *and* ξ_i *is a root of unity of order* d_i *in* E.

Since C is a central division algebra over L, each symbol algebra appearing in iii) above is a central division algebra over L. The algebra $\left(\frac{R_i, S_i}{\xi_i, L}\right)$ has index d_i and exponent d_i over L. The index of C itself is

$$d = d_1 d_2 \ldots d_k = \sqrt{|X_0/T_X{}^+|},$$

and its exponent is $d_1 = \exp(X_0/T_X{}^+)$.

4.6 Cyclic Division Algebra Extensions

Now $A_0 = B \otimes_L C$, where $B \cong \mathrm{M}_n(D)$ for some division algebra D. Then $A_0 \cong \mathrm{M}_n(\Delta_0)$, where $\Delta_0 \cong D \otimes_L C$ is a division algebra, and

$$\mathrm{ind}(A_0) = \mathrm{ind}(B)\mathrm{ind}(C) = \mathrm{ind}(B)\sqrt{[X_0 : T_X{}^+]}.$$

As before, let

$$\mathcal{E} = \{\varepsilon_{ij} : 1 \le i, j \le n\}$$

be a system of matrix units for B. (So $\mathcal{E} \subseteq B_1$, and by Theorem 3.2.2 $\mathcal{E}$ is also a system of matrix units for each of the rings $kF'e_1$, A_0, and A_1.)

$A_1 = KF_1e_1 \cong \mathrm{M}_n(\Delta)$, where $\Delta = C_{A_1}(\mathcal{E})$, and A_1 is obtained from A_0 by adjoining the elements of a transversal for X_0 in X. Since X/X_0 is a finite abelian group, we can find elements $x_1, \ldots x_l$ of X for which

$$X/X_0 = \langle \bar{x}_1 \rangle \times \cdots \times \langle \bar{x}_l \rangle,$$

where $\bar{x}_i = x_i X_0$ in X/X_0, and the order of $\bar{x}_i$ is m_i: $m_l \mid m_{l-1} \mid \cdots \mid m_1$.

By Lemma 4.2.2, the conjugation action of X on L induces an isomorphism between X/X_0 and $\mathrm{Gal}(L/Z)$. For $i = 1 \ldots l$, let ϕ_i denote the automorphism of B defined by

$$\phi_i(\alpha) = x_i^{-1}\alpha x_i, \quad \text{for } \alpha \in B.$$

Of course ϕ_i restricts to a Z-automorphism of L, which we also denote by ϕ_i. Now $\mathcal{E}^{\phi_i}$ is another system of matrix units for B, and is therefore conjugate in B to $\mathcal{E}$ (see [6], theorem 2.13). Then we can find a unit b_{x_i} of B_1 for which $\theta_{x_i} := b_{x_i}x_i \in C_{A_1}(\mathcal{E}) = \Delta$. Now $B = Z\langle T_X \rangle$, and $\mathcal{U}(B)$ is invariant under the action of X, and so for $r > 0$ we have $(\theta_{x_i})^r = bx_i^r \in \Delta$, where $b \in \mathcal{U}(B)$; also $\theta_{x_i}\theta_{x_j} = bx_ix_j \in \Delta$; again $b \in \mathcal{U}(B)$. It then follows, if H denotes the subgroup of $\Delta^\times$ generated by $\theta_{x_1}, \ldots, \theta_{x_l}$, that

$$X/X_0 \cong H/H \cap \Delta_0^\times.$$

Also, since $A_0 \supseteq B$, A_1 is generated as an A_0-module by

$$\mathcal{B} = \left\{\theta_{x_1}^{t_1} \ldots \theta_{x_l}^{t_l} : 0 \le t_i < m_i\right\}.$$

Also, since $\mathcal{B} \subseteq \Delta^\times$, Δ is generated by $\mathcal{B}$ as a module over Δ_0.

Consider the algebra $\Delta_1 := \Delta_0(\theta_{x_1})$. Since $\theta_{x_1} = b_{x_1}x_1$ and $b_{x_1} \in \mathcal{U}(B)$, b_{x_1} centralizes L and conjugation by θ_{x_1} induces the Z-automorphism ϕ_1 of L. Let L_1 denote the fixed field of ϕ_1. Then, since L/Z is a Galois extension with Galois group X/X_0, L/L_1 is a cyclic field extension of degree m_1, with Galois group $\langle\phi_1\rangle$. Hence Δ_1 is a generalized cyclic extension (see [8]) of Δ_0, and Δ_1 is a central division algebra over L_1. Similarly $\Delta_2 := \Delta_1(\theta_{x_2})$ is a generalized cyclic algebra over Δ_1, etc: we can build Δ up from Δ_0 by a series of cyclic algebra extensions.

Now $A_1 \cong \mathrm{M}_n(\Delta)$. Finally, since $A \cong M_s(A_1)$, $s = [F : C_F(e_1)]$, we have $A \cong \mathrm{M}_{sn}(\Delta)$; A is a central simple Z-algebra of degree $sn\,\mathrm{ind}(\Delta)$.

Theorem 4.6.1 *The Schur Index of A divides the order of G.*

Proof : $A = M_{sn}(\Delta)$ and $\mathrm{ind}(A) = \mathrm{ind}(\Delta)$. Certainly $\mathrm{ind}(\Delta) = [X : X_0]\,\mathrm{ind}(A_0)$, and

$$\mathrm{ind}(A) = [X : X_0]\sqrt{[X_0 : T_X{}^+]}\,\mathrm{ind}(B).$$

Now let G_1, G_0 and G^+ denote the images of F_1, F_0 and F'^+ respectively in G. Note that $G' \subseteq G^+ \subseteq G_0 \subseteq G_1$. Also, since $Re_1 \subseteq T_X{}^+ \subseteq X_0 \subseteq X_1$, we have

$$[X : X_0] = [G_1 : G_0] \text{ and } [X_0 : T_X{}^+] = [G_0 : G^+].$$

Then $[X : X_0]\sqrt{[X_0 : T_X{}^+]}$ divides $[G : G^+]$. Furthermore, $\mathrm{ind}(B)$ divides $| G' |$. To see this, let θ denote the irreducible k-character of F' determined by the component $kF'e_1$ of kF', and let χ be an absolutely irreducible character of F' appearing in θ. Then $\mathrm{ind}(B) = m_k(\chi)$, the Schur Index of χ over k, and so $\mathrm{ind}(B)$ divides the degree $\chi(1)$ of χ. Now $\chi(1)$ divides $[F' : A]$, where A is any abelian normal subgroup of F'. In particular then, since $F' \cap R \cong M(G)$ is central in F', we can conclude that $\mathrm{ind}(B)$ divides $[F' : F' \cap R] =| G' |$. Then $\mathrm{ind}(A)$ divides $|G|$ since $[X : X_0]\sqrt{[X_0 : T_X{}^+]}$ divides $[G : G^+]$, and $G^+ \supseteq G'$. □

Chapter 5

Irreducible Projective k-Representations of G

In this chapter we consider the finite dimensional simple k-algebras which arise as images of the group ring kF under k-linear extensions of lifts to F of irreducible projective k-representations of G. Much of the structure of the simple components of the completely reducible ring KF, as described in Chapter 4, is reproduced in these algebras. This is not really surprising, for let A be a simple component of KF. Then A is isomorphic to a ring of matrices over its simple subring A_1, and by Relation 3.1, kF contains a system of matrix units for this extension of rings. Furthermore, by the discussion in section 4.6, A_1 is obtained from its subring A_0 by a series of generalized cyclic extensions, each of which entails the adjunction of an element of kF. The algebra A_0 is the tensor product over its centre L of the central simple L-algebras B and C: here C is a division algebra which by Corollary 4.5.1 can be written as a tensor product of symbol algebras of the form

$$\left(\frac{(\gamma_r)^d, (\gamma_s)^d}{L, \zeta_d}\right),$$

where γ_r, $\gamma_s \in kF$, and ζ_d is a dth root of unity in $\mathcal{Z}(kF')$. The L-algebra B is generated over L by a subring of kF', and L itself is a purely transcendental field extension of a field $E \subseteq \mathcal{Z}(kF')$ generated over E by a transcendence basis contained in kF (Theorem 4.4.1). The central point here is that A can be described largely in terms of the behaviour of various elements of kF. Many of the properties of these elements (not including algebraic independence over k of course) survive under finite dimensional k-representations of kF, leading to a useful resemblance between the simple components of KF described in Chapter 4 and the finite dimensional simple images of kF.

5.1 Ordinary k-Characters of F'

Let $\mathrm{Irr}_k(F')$ denote the set of irreducible (ordinary) k-characters of F'. There is a natural bijective correspondence between $\mathrm{Irr}_k(F')$ and the set I of primitive central idempotents of the group ring kF' : for each $\theta_i \in \mathrm{Irr}_k(F')$, let R_i be an irreducible representation of kF' affording the character θ_i on F'. and let e_i be the (unique) primitive central idempotent of kF' which is not annihilated by R_i.

There is a well known relationship between the coefficients from k appearing in the elements of I, and the values assumed by the characters in $\mathrm{Irr}_k(F')$. We give a brief description of this relationship below; for the details see Section 14.1 of [14].

Throughout the following let $\bar{k}$ be a finite field extension of k which is a splitting field for F'. We can assume that $\bar{k}$ is a cyclotomic extension of k, hence Galois. Let the set of primitive central idempotents of $\bar{k}F'$ be $\bar{I} = \{f_1, \ldots, f_t\}$. Then $\bar{I}$ is in bijective correspondence with the set $\mathrm{Irr}(F')$ of absolutely irreducible characters of F' (all of which are afforded by $\bar{k}$-representations). For $i = 1 \ldots t$, let $\chi_i \in \mathrm{Irr}(F')$ denote the character of that irreducible representation of $\bar{k}F'$ which does not annihilate f_i. Then the coefficients appearing in f_i are related to the values assumed by χ_i according to the following formula :-

$$f_i = \sum_{x \in C} \frac{\chi_i(1)}{|F'|} \chi_i(x^{-1}) \hat{x}, \tag{5.1}$$

where C is a set of representatives for the conjugacy classes of F', and for $x \in F'$, $\hat{x}$ denotes the sum in kF' of the distinct F'-conjugates of x.

Now the Galois group $\mathcal{G}$ of $\bar{k}$ over k acts on $\bar{k}F'$ by

$$\left(\sum_{i=1}^{r} a_i x_i\right)^{\sigma} = \sum_{i=1}^{r} a_i^{\sigma} x_i,$$

for $a_1, \ldots, a_r \in \bar{k}$, $x_1, \ldots, x_r \in F'$. and $\sigma \in \mathcal{G}$. Every element of the subring kF' of $\bar{k}F'$ is of course fixed by this action. and every primitive central idempotent of kF' is the sum in kF of the distinct $\mathcal{G}$-conjugates of some $f_i \in \bar{I}$.

We also have an action of $\mathcal{G}$ on $\mathrm{Irr}(F')$ defined for $x \in F'$, $\chi \in \mathrm{Irr}(F')$, and $\sigma \in \mathcal{G}$ by

$$\chi^{\sigma}(x) = (\chi(x))^{\sigma}.$$

The sum of the distinct $\mathcal{G}$-conjugates of $\chi \in \mathrm{Irr}(F')$ is an irreducible k-character of F', and every element of $\mathrm{Irr}_k(F')$ is such a sum of absolutely irreducible characters. Let $f_1 \in \bar{I}$ be the primitive central idempotent of $\bar{k}F'$ corresponding to $\chi_1 \in \mathrm{Irr}(F')$. Then it follows from 5.1 that $f^{\sigma} \in \bar{I}$ corresponds to $\chi_1^{\sigma} \in \mathrm{Irr}(F')$, for $\sigma \in \mathcal{G}$. Consequently, if $e_1 \in I$ is the sum of the distinct $\mathcal{G}$-conjugates of f_1, and $\theta_1 \in \mathrm{Irr}_k(F')$ is the sum of the distinct $\mathcal{G}$-conjugates of χ_1, we have

$$e_1 = \sum_{x \in C} \frac{\theta_1(1)}{|F'|} \theta_1(x^{-1})\hat{x}, \tag{5.2}$$

where C and $\hat{x}$ are defined as in 5.1.

In this situation we have the following result concerning the centre of the simple k-algebra $B_1 = kF'e_1$.

Theorem 5.1.1 *If θ_1 is an irreducible k-character of F' corresponding to the primitive central idempotent e_1 of kF', and χ_1 is an absolutely irreducible constituent of θ_1, then $\mathcal{Z}(kF'e_1)$ is isomorphic to the field $k(\chi_1)$ obtained by adjoining to k all values assumed by χ_1 on F'.*

Proof Suppose f_1 is the primitive central idempotent of $\bar{k}F'$ corresponding to χ_1. Then the projection ϕ of the simple ring $kF'e_1$ on $kF'e_1f_1 = kF'f_1 \subseteq \bar{k}F'f_1$ restricts to an embedding of the centre. This centre is generated over k by elements of the form $\hat{x}e_1$, where $\hat{x}$ is the sum in kF' of the distinct F'-conjugates of $x \in F'$. The formula

$$\hat{x} = \sum_{i=1}^{t} \frac{[F' : C_{F'}(x)]}{\chi_i(1)} \chi_i(x) f_i, \tag{5.3}$$

which is related to 5.1, expresses $\hat{x}$ as a $\bar{k}$-linear combination of primitive central idempotents of $\bar{k}F'$, with coefficients involving absolutely irreducible character values of F' (see Section 14.1 of [14]). Then $\phi(\hat{x})$ has the form $a\chi_i(x)f_i$, where $a \in k^{\times}$. This completes the proof since the field $\mathcal{Z}(kF'e_1) = \mathcal{Z}(kF')e_1$ is isomorphic to its image under ϕ. □

Now for $e_i \in I$, let θ_i denote the irreducible k-character of F' corresponding to e_i. We have an action of F on the set of k-characters of F' defined for $x \in F'$, $y \in F$, and a character θ by

$$\theta^{y}(x) = \theta(yxy^{-1}).$$

This action restricts to an action of F on $\mathrm{Irr}_k(F')$, under which $\theta_i^y = \theta_j$ if and only if $e_i^y = e_j$. Of course F also acts on I by conjugation and, as mentioned in Chapter 3, the primitive central idempotents of kF are the sums in kF of the F-orbits of elements of I. If $e_1 \in I$, let $\mathcal{T}$ be a right transversal in F for $C_F(e_1)$. Then $e = \sum_{y\in\mathcal{T}} e_1^y$ is a primitive central idempotent of kF, and

$$\begin{aligned} e &= \sum_{y\in\mathcal{T}}\sum_{x\in\mathcal{C}}\left(\frac{\theta_1(1)}{|F'|}\theta_1(x^{-1})\hat{x}\right)^y \\ &= \sum_{y\in\mathcal{T}}\sum_{x\in\mathcal{C}}\left(\frac{\theta_1(1)}{|F'|}\theta_1(x^{-1})\hat{x^y}\right) \\ &= \sum_{x\in\mathcal{C}}\sum_{y\in\mathcal{T}}\frac{\theta_1(1)}{|F'|}\theta_1^{y^{-1}}(y^{-1}x^{-1}y)\hat{x^y} \end{aligned}$$

As x runs through the set $\mathcal{C}$ of representatives for the conjugacy classes of F', so also does x^y, and so we have

$$e = \sum_{x\in\mathcal{C}}\sum_{y\in\mathcal{T}}\frac{\theta_1(1)}{|F'|}\theta_1^{y^{-1}}(x^{-1})\hat{x} \tag{5.4}$$

$$= \sum_{x\in\mathcal{C}}\frac{\theta(1)}{[F:C_F(e_1)]|F'|}\theta(x^{-1})\hat{x}. \tag{5.5}$$

where $\theta = \sum_{y\in\mathcal{T}}\theta_1^{y^{-1}}$ is a k-character of F' which is invariant under the action of F and irreducible with respect to this property. Thus we obtain a bijective correspondence between the set $\mathcal{I}$ of primitive central idempotents of kF and the set of irreducible F-invariant k-characters of F'.

For $e \in \mathcal{I}$, it is clear that $\mathcal{Z}(kFe)\cap kF'$ is a field, since it is contained in the field $Z = \mathcal{Z}(KFe)$; in fact $\mathcal{Z}(kFe)\cap kF'$ is isomorphic to a particular character field for F'. The following result can be established by an argument similar to the one given in the proof of Theorem 5.1.1 where the primitive central idempotent f_1 of $\bar{k}F'$ is replaced essentially by the sum of its distinct F-conjugates - by a primitive central idempotent f of $\bar{k}F$ for which $ef = f$, where $\bar{k}$ is as before a Galois extension of k which is a splitting field for F'.

Theorem 5.1.2 *If θ is the irreducible F-invariant character of F' corresponding to $e \in \mathcal{I}$, let χ be the sum in $\mathrm{Irr}(F')$ of the distinct F-conjugates of some absolutely irreducible character χ_1 of F' which is a constituent of θ. Then $\mathcal{Z}(kFe)\cap kF'$ is isomorphic to $k(\chi)$, the field obtained from k by adjoining all values assumed by χ on F'.* □

Finally, in the situation where $e_1 \in I$ corresponds to the character $\theta_1 \in \mathrm{Irr}_k(F')$, we remark that the inertia subgroup (i.e. the stabilizer) of θ_1 in F is precisely the centralizer F_1 of e_1, which certainly has finite index in F. The following version of Clifford's theorem ([1]) then asserts that every finite dimensional irreducible k-representation R of F, for which θ_1 is a constituent of the character of $R|_{F'}$, is induced from an irreducible k-representation of F_1.

Theorem 5.1.3 *Let τ be the character of a finite dimensional irreducible k-representation of F, and let $\theta \in \mathrm{Irr}_k(F')$ be an irreducible constituent of the k-character $\mathrm{Res}^F_{F'}\tau$ of F'. Then*

$$\tau = \mathrm{Ind}^F_{F_1}\Theta,$$

where Θ is an irreducible k-character of the inertia group F_1 of θ in F, whose restriction to F' is an integer multiple of θ. □

For a proof of Theorem 5.1.3 see [2], Chapters 49-50.

5.2 Lifting of Projective Representations

Let $T : G \longrightarrow GL(d,k)$ be an irreducible projective representation of G over k, and let (R, F, ϕ) be a generic central extension for G. By Theorem 2.2.1 we can find an ordinary irreducible representation $\tilde{T} : F \longrightarrow GL(d,k)$, for which the following diagram commutes :-

$$\begin{array}{ccccc} F & & \xrightarrow{\quad\tilde{T}\quad} & & GL(d,k) \\ \downarrow{\scriptstyle\phi} & & & & \downarrow{\scriptstyle\pi} \\ G & \xrightarrow[T]{} & GL(d,k) & \xrightarrow[\pi]{} & PGL(d,k) \end{array}$$

Here π is the usual projection of $GL(d,k)$ on $PGL(d,k)$, so $\pi \circ T : G \longrightarrow PGL(d,k)$ is a homomorphism of groups.

Of course $\tilde{T}$ extends by k-linearity to a ring homomorphism of kF into $\mathrm{M}_d(k)$, which we also denote by $\tilde{T}$. The subring A^T of $M_n(k)$ generated as an algebra over k either by $\{T(g), g \in G\}$ or by $\{\tilde{T}(x), x \in F\}$ is simple; it is isomorphic as a k-algebra to some simple component of the twisted group ring $k^f G$, if $f \in Z^2(G, k^\times)$ is the cocycle associated to T. Thus $\tilde{T}$ annihilates all but one of the primitive central idempotents of kF and $A^T = \tilde{T}(kFe_T)$, for some $e_T \in \mathcal{I}$. To justify this notation we need the following lemma.

Lemma 5.2.1 *e_T depends only on T and not on the choice of lift $\tilde{T}$.*

Proof : Suppose $\tilde{T}$ and $\tilde{T}'$ are lifts of T to F. Then the map $\psi : F \longrightarrow k^\times$ defined for $x \in F$ by

$$\psi(x) = \tilde{T}(x)\tilde{T}'(x^{-1})$$

is a group homomorphism. for let $x, y \in F$. Then

$$\begin{aligned} \psi(xy) &= \tilde{T}(xy)\tilde{T}'(xy^{-1}) \\ &= \tilde{T}(x)\tilde{T}(y)\tilde{T}'(y^{-1})\tilde{T}'(x^{-1}) \\ &= \tilde{T}(x)\tilde{T}'(x^{-1})\tilde{T}(y)\tilde{T}'(y^{-1}) \\ &= \psi(x)\psi(y) \end{aligned}$$

Since ψ is a homomorphism from F into the abelian group $k^\times$, $\psi|_{F'}$ is trivial. Then $\tilde{T}$ and $\tilde{T}'$ have the same restriction to F', which contains the support of every central idempotent of kF. It then follows that $\tilde{T}$ and $\tilde{T}'$ determine the same primitive idempotent of kF. □

We say that the irreducible projective k-representation T of G *belongs to* the component kFe_T of kF (or to the idempotent e_T) if $\tilde{T}(e_T) = 1$ in $M_n(k)$ for any lift $\tilde{T}$ of T to F.

If a study of the group ring kF and its components is to be successful in determining information about the projective representations of G over k, we might at least hope that (projectively) equivalent irreducible representations of G should belong to the same component of kF. It is easily checked that this is indeed the case.

Lemma 5.2.2 *Let T_1 and T_2 be projectively equivalent irreducible projective representations of G over k, of degree d. Then T_1 and T_2 belong to the same component of kF.*

Proof: For some $A \in GL(d, k)$ and for some function $\mu : G \longrightarrow k^\times$, we have

$$T_2(g) = \mu(g)A^{-1}T_1(g)A, \quad \forall g \in G.$$

Define $T_2' : G \longrightarrow GL(d, k)$ for $g \in G$ by

$$T_2'(g) = \mu(g)T_1(g).$$

Then T_2' is another projective k-representation of G, equivalent to both T_1 and T_2. Let $\tilde{T}_2 : F \longrightarrow GL(d, k)$ be a lift of T_2 to F, and for $x \in F$ define

$$\tilde{T}_2'(x) = A\tilde{T}_2(x)A^{-1}.$$

Then $\tilde{T}_2'$ is a lift of T_2' to F. Clearly $\tilde{T}_2$ and $\tilde{T}_2'$ are linearly equivalent ordinary representations of F, and their restrictions to F' are linearly equivalent. Then $\tilde{T}_2$ and $\tilde{T}_2'$ determine the same component of kF. Finally, since $T_2'(g) = \mu(g)T_1(g)$ for all $g \in G$, any lift of T_2' to F is also a lift of T_1. In particular if $\tilde{T}_1$ is a lift of T_1 to F, then $\tilde{T}_1|_{F'} = \tilde{T}_2'|_{F'}$ by Lemma 5.2.1. This completes the proof. □

From now on we fix a primitive central idempotent e of kF, an irreducible projective k-representation T of G of degree d belonging to e, and a lift $\tilde{T}$ of T to F. The simple k-subalgebra of $M_d(k)$ generated by the image of G under T will be denoted by A^T.

If e is not primitive as a central idempotent of kF', let $e_1 \in I$ satisfy $ee_1 = e_1$. By Theorem 5.1.3, $\tilde{T}$ is induced from an irreducible representation $\tilde{T}_1$ of $F_1 = C_F(e_1)$ of degree $d_1 = d/s$ (where $s = [F : F_1]$). The image A_1^T of kFe_1 under $\tilde{T}_1$ is a simple k-subalgebra of $M_{d_1}(k)$.

Thus we may confine our attention to the subring $A_1^k = kF_1e_1$ of $A^k = kFe$ and its simple images. If K is the usual transcendental extension of k, defined as the field of quotients of a central subring of kF, our knowledge from Chapter 4 of the simple K-algebra $A_1 = KF_1e_1$ will lead to some conclusions concerning possible values of the Schur index and degree of irreducible projective representations of G over k, at least in terms of the corresponding invariants for irreducible linear representations of F'.

In the following discussion involving the rings A_1^k and A_1, we use much of the notation established in Chapter 4, some of which we recall here for convenience. Thus $B_1 = kF'e_1$, $E = \mathcal{Z}(B_1)$, and $F_0 = C_F(E)$; A_0 denotes the simple K-algebra KF_0e_1, and A_0^k the subring kF_0e_1 of A_1^k. The centre of A_0 is denoted L. $L = ZE$ where Z is the centre of A_1, and A_0 is the tensor product over L of $B = Z\langle F'\rangle e_1$ and the division algebra $C = C_{A_0}(B)$. We define E_1 to be the field $E \cap Z$.

We will denote by A_1^T and A_0^T respectively the images of kF_1e_1 and kF_0e_1 under $\tilde{T}_1$. So A_0^T is a k-subalgebra of A_1^T.

5.3 The Simple k-Algebra A_1^T

Let $\tilde{T}_{\div}$ denote the restriction of $\tilde{T}_1$ to the subgroup F'^{+} of F_0 determined by e_1 as in 4.3. Then $\tilde{T}_{\div}$ may not be irreducible, but it is certainly completely reducible since it is a lift of a projective k-representation of G^+. Let $A_{\div}^T$ denote the k-subalgebra of A_0^T generated over k by the image of F'^{+} under $\tilde{T}_{\div}$. Of course $\tilde{T}_1$ (hence $\tilde{T}_{\div}$) restricts to an embedding of the simple ring B_1 in A_1^T (or $A_{\div}^T$). We let B_1^T denote the image of B_1 in A_1^T, and by abuse of notation we identify the fields E and E_1 with their images under $\tilde{T}_1$. It is immediate from Lemma 4.2.2 and the injectivity of $\tilde{T}_1|_E$ that the conjugation action of $F_1^T := \tilde{T}_1(F_1)$ on E in A_1^T induces an isomorphism of F_1^T/F_0^T and $\mathrm{Gal}(E/E_1)$ (where $F_0^T = \tilde{T}_1(F_0)$).

The purpose of the next series of results is to describe the centre of the completely reducible algebra $A_{\div}^T$ as the tensor product over E_1 of the fields $E = \mathcal{Z}(B_1^T)$ and $Z^T = \mathcal{Z}(A_1^T)$. Some of the methods used are suggested by arguments appearing in [9].

Lemma 5.3.1 $Z^T \subseteq A_0^T$.

Proof: First we show that the image in A_1^T of any transversal for F_0 in F_1 is right independent over A_0^T. If not, let m be the least positive integer for which there exists a transversal $\tau = \{x_1, \ldots, x_l\}$ for F_0 in F_1, such that for some nonzero elements $\alpha_1, \ldots, \alpha_m$ of A_0^T we have

$$\alpha_1 x_{i_1}^T + \cdots + \alpha_m x_{i_m}^T = 0 \text{ in } A_1^T. \tag{5.6}$$

Here x_j^T denotes the image under $\tilde{T}_1$ of $x_j \in \tau$. We may assume that $x_{i_1} = 1$, since $\tau' = \{x_1 x_{i_1}^{-1}, \ldots, x_l x_{i_1}^{-1}\}$ is again a transversal for F_0 in F_1, and

$$\alpha_1 + \alpha_2 (x_{i_2}^T)^{-1} + \cdots + \alpha_m x_{i_m}^T (x_{i_1}^T)^{-1} = 0.$$

Since the E_1-automorphisms of E defined as conjugation by the elements $x_1^T, \ldots, x_l^T$ generate the full Galois group of E over E_1, we can find an element a of E which does not commute with all of $x_{i_2}^T, \ldots, x_{i_m}^T$. Then

$$a\alpha_1 + a\alpha_2 x_{i_2}^T + \cdots + a\alpha_m x_{i_m}^T = 0 \tag{5.7}$$

$$a\alpha_1 + \alpha_2' x_{i_2}^T + \cdots + a\alpha_m' x_{i_m}^T = 0, \tag{5.8}$$

where for $j = 2 \ldots m$, $\alpha_j' = \alpha_j x_{i_j}^T a (x_{i_j}^T)^{-1} \in A_0^T$. Since $\alpha_j' \neq \alpha_j$ for at least one j, subtracting 5.8 from 5.7 leads to a contradiction to the minimality of m. This proves the right independence over A_0^T of the image of τ.

That $Z^T \subseteq A_0^T$ is then an immediate consequence of the remark preceding the statement of Lemma 5.3.1 : every element x of A_1^T can be written in the form

$$x = \sum_{i=1}^{l} \alpha_i x_i^T,$$

for some $\alpha_1, \ldots, \alpha_l$ in A_0^T. Since the automorphisms of E defined by conjugation by $x_1^T, \ldots, x_l^T$ generate the Galois group of E over E_1, x centralizes E if and only if $\alpha_i = 0$ for all those i for which $x_i \notin F_0$. □

Lemma 5.3.2 $\mathcal{Z}(A_0^T) \subseteq A_{\div}^T$

Proof : We begin as with Lemma 5.3.1 by showing that the image in A_0^T of any transversal for F'^+ in F_0 is right independent over $A_{\div}^T$. If not, again let m be minimal for which there exists such a transversal $\mathcal{S}$ with the property that for some nonzero elements $\alpha_1, \ldots, \alpha_m$ of $A_{\div}^T$,

$$\alpha_1 y_{i_1}^T + \cdots + \alpha_m y_{i_m}^T = 0 \text{ in } A_0^T. \tag{5.9}$$

Here $y_{i_j}^T$ denotes the image under $\tilde{T}_1$ of $y_{i_j} \in \mathcal{S}$.

Let $S_0 = \{r_1, s_1, \ldots, r_k, s_k\}$ be as in Corollary 4.5.1. We recall that

$$F_0/F'^+ \cong \langle \bar{r}_1 \rangle \times \langle \bar{s}_1 \rangle \times \cdots \times \langle \bar{r}_k \rangle \times \langle \bar{s}_k \rangle,$$

and $\langle \bar{r}_i \rangle \cong \langle \bar{s}_i \rangle \cong C_{d_i}$. Also, if the units γ_{r_i} and γ_{s_i} are defined as in Section 4.3, then $[\gamma_{r_i}, \gamma_{s_i}]$ is a d_ith root of unity in E, and γ_{r_i} and γ_{s_i} commute with those γ_{r_j} and γ_{s_j} for which $j \neq i$. Furthermore these commutator relations survive in A_0^T since $\tilde{T}_1$ embeds E in A_0^T.

It follows from Lemma 4.3.3 and Corollary 4.5.1 that each $y_i \in \mathcal{S}$ can be written in the form

$$y_i = \delta_i (\gamma_{r_1})^{l_{r_1}(i)} (\gamma_{s_1})^{l_{s_1}(i)} \ldots (\gamma_{r_k})^{l_{r_k}(i)} (\gamma_{s_k})^{l_{s_k}(i)}, \tag{5.10}$$

where δ_i is an invertible element of kF_0e_1 (δ_i is the product of an element of the group F'^+ and a unit of B_1). Thus the expression 5.9 may be written in the form

$$\sum_{j=1}^{m} \alpha'_j \sigma_{i_j}^T = 0, \tag{5.11}$$

where $\alpha'_j = \alpha_j \delta_{i_j} \in kF_0$ and

$$\sigma_{i_j}^T = (\gamma_{r_1}^T)^{l_{r_1}(i_j)} (\gamma_{s_1}^T)^{l_{s_1}(i_j)} \ldots (\gamma_{r_k}^T)^{l_{r_k}(i_j)} (\gamma_{s_k}^T)^{l_{s_k}(i_j)}.$$

As usual $\gamma_{r_t}^T$ and $\gamma_{s_t}^T$ denote respectively the images of γ_{r_t} and γ_{s_t} under $\tilde{T}_1$.

There is no loss of generality in assuming that for some i either γ_{r_t} or γ_{s_t} appears in some but not all of the $\sigma_{i_j}^T$ in 5.9. Certainly $m \geq 2$ and so some $\gamma_{r_t}^T$ (or $\gamma_{s_t}^T$ - say $\gamma_{r_t}^T$) appears with different exponents in two different $\sigma_{i_j}^T$. Then we may eliminate $\gamma_{r_t}^T$ from some, but not all, of the $\sigma_{i_j}^T$ by multiplying the expression 5.11 on the right by a suitable power of $\gamma_{r_t}^T$. What we obtain still has the general form of 5.9 for *some* transversal for F'^+ in F_0, since $(\gamma_{r_t}^T)^j$ is the product of $\gamma_{r_t}^T$ with a unit from B_1^T.

Now some but not all of the $\sigma_{i_j}^T$ commute with the unit $\gamma_{s_t}^T$ of A_0^T. For each $\sigma_{i_j}^T$ we have

$$(\gamma_{s_t}^T)^{-1}\sigma_{i_j}^T\gamma_{s_t}^T = \xi\sigma_{i_j}^T.$$

Here ξ is a root of unity in E, which is equal to 1 for some but not all i_j. Then comparing the expression 5.11 to its conjugate by $\gamma_{s_t}^T$ will (as in the proof of Lemma 5.3.1) lead to a contradiction to the choice of m. This establishes the right independence over $A_\oplus^T$ of the image under $\tilde{T}_1$ of a transversal for F'^+ in F_0.

Finally A_0^T is generated as a right module over $A_\oplus^T$ by the image of any such transversal. The result then follows from the commutator relations among the elements $\gamma_{r_t}^T$ and $\gamma_{s_t}^T$, since any central element of A_0^T must centralize every $\gamma_{r_t}^T$ and $\gamma_{s_t}^T$. □

We will make further use of Lemma 5.3.2 and its proof shortly, in a discussion of the structure of the simple components of $A_\oplus^T$. First however we investigate the Z^T-dimension of $A_\oplus^T$.

Lemma 5.3.3 *The dimension over E_1 of $A_\oplus^T$ is equal to* $\dim_{E_1}(Z^T)\dim_{E_1}(B_1)$.

Proof : It follows from the original definition of F'^+ that $A_\oplus^T$ is generated over Z^T by B_1^T; hence $\dim_{E_1}(A_\oplus^T) \leq \dim_{E_1}(Z^T)\dim_{E_1}(B_1^T)$.

The simple ring B_1 is a ring of $n \times n$ matrices over a central E-division algebra D_1, and if $\mathcal{E}$ is a system of n^2 matrix units in B_1, then it is easily checked that the image $\mathcal{E}^T$ of $\mathcal{E}$ under the k-linear extension of $\tilde{T}_1$ to kF_1e_1 is again a set of n^2 distinct elements satisfying the identities of matrix units. Thus

$$A_\oplus^T \cong M_n\left(C_{A_\oplus^T}(\mathcal{E}^T)\right);$$

(see [14], Lemma 6.1.5).

Certainly $C_{A^T_\oplus}(\mathcal{E}^T)$ contains $D_1^T = \tilde{T}_1(D_1)$ and Z^T. Let $\mathcal{B}$ be a basis for Z^T over E_1 in $A^T_\oplus$. We now show that $\mathcal{B}$ is right independent over D_1^T. Suppose not, and let m be minimal for which we can find elements $b_{i_1}, \ldots, b_{i_m}$ in $\mathcal{B}$ and nonzero $\alpha_1, \ldots, \alpha_m$ in D_1^T such that :-

$$\sum_{j=1}^{m} \alpha_j b_{i_j} = 0. \tag{5.12}$$

Since D_1^T is a division algebra, we may multiply 5.12 on the left by $(\alpha_1)^{-1}$ - thus there is no loss of generality in assuming that $\alpha_1 = 1$. Since $\mathcal{B}$ is linearly independent over E_1, not all of the α_j belong to E_1. If not all of them belong to E, we can find an element d^T of D_1^T which commutes with α_1 but not with all of $\alpha_2, \ldots, \alpha_m$. Then

$$\alpha_1 b_{i_1} + \alpha_2 b_{i_2} + \cdots + \alpha_m b_{i_m} = 0$$

$$\alpha_1 b_{i_1} + (d^T)^{-1}\alpha_2 d^T b_{i_2} + \cdots + (d^T)^{-1}\alpha_m d^T b_{i_m} = 0$$

Thus $\sum_{j=2}^{m} \alpha_j' b_{i_j} = 0$, where $\alpha_j' = \alpha_j - (d^T)^{-1}\alpha_j d^T$. This contradicts the choice of m since each α_j' belongs to D_1^T but not all of them are equal to zero.

In the case where every α_j belongs to E, we may apply the same argument, but using a suitably chosen element x^T of F_1^T in the place of d^T. Certainly $E = \mathcal{Z}(D_1^T)$ is stabilized under conjugation by elements of F_1^T, and the fact that the automorphisms defined by such conjugations generate all of $\mathrm{Gal}(E/E_1)$ guarantees the existence of a suitable x^T.

Thus $\mathcal{B}$ is right independent over D_1^T, and the dimension over E_1 of $C_{A^T_\oplus}(\mathcal{E}^T)$ is at least equal to $\dim_{E_1}(D_1^T)[Z^T : E_1]$. Then

$$\dim_{E_1}(A^T_\oplus) \geq n^2 \dim_{E_1}(D_1^T)[Z^T : E_1] = \dim_{E_1}(B_1^T)\dim_{E_1}(Z^T).$$

This completes the proof. □

The following corollary is an immediate consequence of Lemma 5.3.3 (see [16], Proposition 9.2c).

Corollary 5.3.1 *$A^T_\oplus$ is the tensor product over E_1 of Z^T and B_1^T.* □

In particular then, the centre of $A^T_\oplus$ is isomorphic to the tensor product over E_1 of Z^T and E, which is a direct sum of field composita of E and Z^T. Thus $A^T_\oplus$ is simple if and only if E and Z^T are linearly disjoint. Otherwise $\mathcal{Z}(A^T_\oplus)$ is a direct sum of isomorphic fields, and its centrally primitive idempotents are conjugate under the action of $\mathrm{Gal}(E/E_1)$. Hence the components of $\mathcal{Z}(A_{\oplus}^T)$ are all centralized by F_0^T and are permuted transitively by F_1^T. The transitivity of this action of course follows from the isomorphism $F_1^T/F_0^T \cong \mathrm{Gal}(E/E_1)$.

Let A_+^T be a simple component of $A_{\oplus}^T$ and let $\tilde{T}_+$ be the irreducible representation of F'^+ (or kF'^+) defined as the composition of $\tilde{T}_{\oplus}$ with the projection of $A_{\oplus}^T$ on A_+^T. Then by Clifford's theorem

$$\tilde{T}_1 = \mathrm{Ind}_{I^+}^{F_1}\left(\tilde{T}_{I^+}\right).$$

where I^+ is the inertia subgroup of $\tilde{T}_+$ in F_1 and $\tilde{T}_{I^+}$ is an irreducible representation of I^+ whose restriction to F'^+ is a sum of irreducible constituents each equivalent to $\tilde{T}_+$. From the fact that F_0 centralizes E, and hence every primitive central idempotent of $A_{\oplus}^T$, it follows that $I^+ \supseteq F_0$.

Before investigating the irreducible representation $\tilde{T}_{I^+}$, we digress briefly to consider the field Z^T and how it arises as a finite extension of k. By Lemmas 5.3.1 and 5.3.2, the algebras A_0^T and $A_{\oplus}^T$ have the same centre EZ^T : from the proof of Lemma 5.3.1 we know that A_0^T is obtained from $A_{\oplus}^T$ by the adjunction of elements which are centralized by $A_{\oplus}^T$ but do not centralize each other. It follows from Lemma 5.3.3 that the $\tilde{T}_1$-image $\mathcal{B}^T$ of any E_1-basis for B_1 is right independent over Z^T; a dimension count ensures that if $\mathcal{B}'$ is an E-basis for B_1, then the image of $\mathcal{B}'$ under $\tilde{T}_1$ is independent over $EZ^T = \mathcal{Z}(A_0^T)$. Recall from the remarks following Theorem 4.3.1 that

$$\dim_L(B) = \dim_E(B_1),$$

where B and L are defined as in Section 4.3; i.e. K is the purely transcendental extension of k of Section 3.1, B is the simple ring generated over $Z = \mathcal{Z}(KF_1e_1)$ by F', and $L = \mathcal{Z}(B) = ZE$. It follows that any E-basis for B_1 is an L-basis for B and hence a basis for kF'^+e_1 as a right module over its centre. We conclude that Z^TE is precisely the image under $\tilde{T}_1$ of the centre $L \cap kF_0e_1$ of kF_0e_1. By analogy with the notation of Chapter 4, we will denote this algebra by L^T. Recall from Theorem 4.4.1 that $\mathcal{Z}(kF_0e_1)$ is generated as an E-algebra by

$$\Gamma = \{\gamma_{t_1}, \ldots, \gamma_{t_r}\} \subseteq A_0^k.$$

where γ_{t_i} has the form $\gamma_{t_i} = \beta_{t_i}t_i$ for some $t_i \in F'^+e_1$ and $\beta_{t_i} \in \mathcal{U}(B_1)$. Thus L^T is generated as an E-algebra by $\Gamma^T := \tilde{T}_1(\Gamma)$.

Now let L_+^T be the centre of A_+^T : then L_+^T is the image of L^T under the projection of $A_{\oplus}^T$ on A_+^T, and is of course a field compositum of Z^T and E.

We have from 4.4 a sequence $j_1, \ldots, j_r$ of positive integers for which

$$\Gamma_1 = \left\{ \gamma_{t_i}^{j_i} \right\}_{i=1\ldots r}$$

is a transcendence basis for KE/E, and $KE = E(\Gamma_1)$. Moreover, for each i, $\gamma_{t_i}^{j_i} = s_i \alpha_{t_i}$, where $s_i \in Re_1$, $\alpha_{t_i} \in E^\times$, and $\langle s_1, \ldots, s_r \rangle$ is a torsion-free complement for $(F' \cap R)e_1$ in Re_1. In fact $KE = E(s_1, \ldots, s_r)$ and $\{s_1, \ldots, s_r\}$ is another transcendence basis for KE over E.

Since $s_i \in Re_1$ for $i = 1 \ldots r$, and $\tilde{T}$ is a lift to F of a projective k-representation of G, we have $\tilde{T}_+(s_i) \in k^\times$ for each i. Let $\tilde{T}_+(s_i) = s_i^T \in k^\times$: since $\{s_1, \ldots, s_r\}$ is an algebraically independent set over E, we are free to choose each $s_i^T \in k^\times$ completely arbitrarily. This amounts to a choice of group homomorphism $\tilde{T}|_R : R \longrightarrow k^\times$. We will see later that the choice of $\tilde{T}|_R$ does not fully determine the irreducible projective representation T of G, but that it does determine (up to a coboundary) the cocycle in $Z^2(G, k^\times)$ associated to T. Furthermore it is the choice of $\{s_i^T\}_{i=1\ldots r}$ which determines the field L_+^T, which is a finite field extension of E, obtained by adjoining for $i = 1 \ldots r$ a root of a polynomial of the form

$$p_i(X) = X^{j_i} - \alpha_{t_i} s_i^T \in E[X].$$

This may not necessarily determine L_+^T up to isomorphism: for a given i, adjoining roots of different irreducible factors of $p_i(X)$ in $E[X]$ may not lead to the same field extension. However we can say that the degree of the field extension L_+^T/E is at most equal to

$$j_1 \ldots j_r = \left[F'^+ : RF' \right] = \left[G^+ : G' \right]. \tag{5.13}$$

It is easily seen that the action of I^+ on L_+^T (via its image under $\tilde{T}_{I^+}$) has kernel F_0 and leads to the isomorphism $I^+/F_0 \cong \mathrm{Gal}(L_+^T/Z_+^T)$.

The restriction $\tilde{T}_{0^+}$ to F_0 of the irreducible representation $\tilde{T}_{I^+}$ is also irreducible : this follows from the fact that the completely reducible algebras A_0^T and $A_{\pm}^T$ have the same centre and therefore have the same set of centrally primitive idempotents. Let A_{0+}^T denote that simple component of A_0^T which contains A_+^T as a central simple subalgebra. The role of A_{0+}^T in the following description of the structure of A_1^T is similar to that of A_0 in the description of A_1 in Chapter 4.

In fact the structure of A_{0+}^T as an extension of A_+^T is fairly easily discerned in the light of Lemma 5.3.2 and the results of Section 4.5. The projection of $A_{\pm}^T$ on A_+^T certainly restricts to an embedding of E, and the argument given in the proof of Lemma 5.3.2 can be reproduced to show that the image under $\tilde{T}_{0^+}$ of any transversal for F'^+ in F_0 is right independent over A_+^T. Let C_+^T denote the image under $\tilde{T}_{0^+}$ of the subalgebra of kF_1e_1 generated over $E(\Gamma)$ by the elements $\gamma_{r_1}, \gamma_{s_1}, \ldots, \gamma_{r_k}, \gamma_{s_k}$ of Corollary 4.5.1 and Lemma 5.3.2.

In A_0 the algebra C generated over L by $\{\gamma_{r_1}, \gamma_{s_1}, \ldots, \gamma_{r_k}, \gamma_{s_k}\}$ is precisely equal to the centralizer of F' in A_0, and by Corollary 4.5.1 it decomposes as a tensor product of symbol algebras :-

$$C = \left(\frac{\gamma_{r_1}^{d_1}, \gamma_{s_1}^{d_1}}{\xi_1, L}\right) \otimes_L \left(\frac{\gamma_{r_2}^{d_2}, \gamma_{s_2}^{d_2}}{\xi_2, L}\right) \otimes_L \cdots \otimes_L \left(\frac{\gamma_{r_k}^{d_k}, \gamma_{s_k}^{d_k}}{\xi_k, L}\right),$$

where:-

i) Each $(\gamma_{r_i})^{d_i}$ and $(\gamma_{s_i})^{d_i}$ belongs to $E(\Gamma)$.

ii) For $i = 1 \ldots k$, ξ_i is a root of unity of order d_i in E. Also $d_k | d_{k-1} | \ldots | d_1$.

If for each i we now define $R_i = (\gamma_{r_i})^{d_i}$ and $S_i = (\gamma_{s_i})^{d_i}$, then R_i and S_i belong to $E(\Gamma)$ and their images R_i^{T+} and S_i^{T+} respectively in $A_{0^+}^T$ have been determined by the choice of $\tilde{T}|_R : R \longrightarrow k^\times$. Now the images $\gamma_{r_i}^{T+}$ and $\gamma_{s_i}^{T+}$ of γ_{r_i} and γ_{s_i} are roots of the polynomials $X^{d_i} - R_i^{T+}$ and $X^{d_i} - S_i^{T+}$ respectively in $L^T[X]$. Since E contains a root of unity of order d_1 and $d_i | d_1$ for each i, this determines the image of the ring $E(\Gamma, \gamma_{r_i}, \gamma_{s_i})$ up to isomorphism. Since ξ_i is a root of unity in E, we have

$$\tilde{T}_{0^+}\left(E(\Gamma, \gamma_{r_i}, \gamma_{s_i})\right) = \left(\frac{R_i^{T+}, S_i^{T+}}{\xi_i, L^T}\right),$$

and C_+^T is a tensor product over L_+^T of symbol algebras :-

$$C_+^T = \left(\frac{R_1^T, S_1^T}{\xi_1, L^T}\right) \otimes_{L^T} \left(\frac{R_2^T, S_2^T}{\xi_2, L^T}\right) \otimes_{L^T} \cdots \otimes_{L^T} \left(\frac{R_k^T, S_k^T}{\xi_k, L^T}\right). \tag{5.14}$$

This tensor product decomposition of C_+^T is a consequence of Lemma 4.3.1. In particular C_+^T is a central simple L_+^T algebra, and since $\tilde{T}_+$ embeds B_1 in $A_{0^+}^T$, it is easily seen that C_+^T is precisely the centralizer in $A_{0^+}^T$ of $B_{1^+}^T = \tilde{T}_+(B_1)$. Then by Lemma 4.3.1, $A_{0^+}^T$ has a tensor product decomposition similar to that of A_0 described in Section 4.3. Here B_+^T denotes the algebra generated over L_+^T by $B_{1^+}^T$:-

$$A_{0^+}^T = B_+^T \otimes_{L_+^T} C_+^T. \tag{5.15}$$

The central simple Z_+^T-algebra $A_{r+}^T = \tilde{T}_{r+}(kI^+)$ may now be built up from A_{o+}^T by means of a series of cyclic extensions. Unlike the corresponding subalgebra C of A_0, the ring C_+^T may not be a division algebra : its index depends on the values of $R_1^T, S_1^T, \ldots, R_k^T, S_k^T$ in L_+^T. In any case A_{o+}^T is a simple ring, and for some division algebra D_+^T and some $t \geq 1$ we have

$$A_{o+}^T \cong \mathrm{M}_t(D_+^T).$$

Let $\mathcal{E}^T$ be a system of t^2 matrix units in A_{o+}^T, so D_+^T is the centralizer in A_0^T of $\mathcal{E}^T$. The group I^+/F_0 is abelian: suppose $I^+/F_0 = \langle \bar{y}_1 \rangle \times \cdots \times \langle \bar{y}_p \rangle$, where $\bar{y}_i = y_i F_0$ and the order of $\bar{y}_i$ is l_i; $l_p | l_{p-1} | \ldots | l_1$. For $i = 1 \ldots p$, let $y_i^{T_r}$ denote the image of y_i under $\tilde{T}_{r+}$: the conjugation action of $\langle y_1 \rangle$ on kF_0e_1 defines an action of $\langle y_1^{T_r} \rangle$ on A_{o+}^T by

$$(y_1^{T_r})^{-1} \tilde{T}_{o+}(\alpha) y_1^{T_r} = \tilde{T}_{o+}(y_1^{-1} \alpha y_1), \text{ for } \alpha \in kF_0e_1.$$

This action is well-defined since $\tilde{T}_{o+}$ maps kF_0e_1 *onto* A_{o+}^T. Let ϕ_1 denote the automorphism of A_0^T defined as conjugation by $y_1^{T_r}$ (note ϕ_1 is not an L_+^T-algebra automorphism, its restriction to L_+^T is a Z_+^T-automorphism of order l_1). Then $(\mathcal{E}^T)^{\phi_1}$ is another system of matrix units for A_{o+}^T, and so by Theorem 2.13 in [6] $(\mathcal{E}^T)^{\phi_1}$ is the image of $\mathcal{E}^T$ under an inner automorphism of A_{o+}^T: i.e. there exists a unit b_1^T of A_{o+}^T for which

$$e_{ij}^{\phi_1} = e_{ij}^{b_1^{-1}}$$

for each $e_{ij} \in \mathcal{E}^T$, so $\theta_{y_1} := y_1^{T_r} b_1$ centralizes $\mathcal{E}^T$. Now

$$(y_1)^{l_1} \in F_0 \Longrightarrow (y_1^{T_r})^{l_1} \in A_{o+}^T,$$

and $(\theta_{y_1})^{l_1} = \tilde{T}_{o+}\left((y_1)^{l_1}\right) a$, for some $a \in \mathcal{U}(A_{o+}^T)$. Then

$$(\theta_{y_1})^{l_1} \in C_{A_{o+}^T}(\mathcal{E}^T) = D_+^T.$$

So $(\theta_{y_1})^{l_1}$ centralizes L_+^T, and since b_1 centralizes L_+^T, the automorphism of L_+^T defined as conjugation by θ_{y_1} is the restriction to L_+^T of ϕ_1. Since $I^+/F_0 \cong \mathrm{Gal}\left(L_+^T/Z_+^T\right)$, the order of this automorphism is l_1. Let $L_1^T \subseteq L_+^T$ denote the fixed field of L_+^T under ϕ_1. Then L_+^T/L_1^T is a cyclic field extension of degree l_1, and the algebra generated over D_+^T by θ_{y_1} is a cyclic algebra extension of D_+^T, of degree $l_1 \mathrm{ind}(D_+^T)$ over its centre L_1^T. Then

$$D_+^T(\theta_{y_1}) \cong \mathrm{M}_{t_1}(\Delta_1^T)$$

where Δ_1^T is a central L_1^T-division algebra and

$$\begin{aligned} t_1 \mathrm{ind}(\Delta_1^T) &= l_1 \mathrm{ind}(D_+^T), \\ A_{o+}^T(y_1^{T_r}) = A_{o+}^T(\theta_{y_1}) &\cong M_{tt_1}(\Delta_1^T) = \mathrm{M}_{t_1'}(\Delta_1^T). \end{aligned}$$

Similarly, $A^T_{0+}(y_1^{T_I}, y_2^{T_I}) \cong \mathrm{M}_{t'_2}(\Delta_2^T)$, where $t'_2\mathrm{ind}(\Delta_2^T) = l_2 t'_1 \mathrm{ind}(\Delta_1^T)$ etc., and we may build A^T_{I+} up from A^T_{0+} by adjoining the $y_i^{T_I}$ one by one for $i = 1 \ldots l$. If after $i-1$ steps we have

$$A^T_{0+}(y_1^{T_I}, \ldots, y_{i-1}^{T_I}) \cong \mathrm{M}_{t'_{i-1}}(\Delta^T_{i-1})$$

for a division algebra Δ^T_{i-1} with centre L^T_{i-1} for which L^T_+/L^T_{i-1} is an abelian extension of degree $l_1 l_2 \ldots l_{i-1}$ then

$$A^T_{0+}(y_1^{T_I}, \ldots, y_i^{T_I}) \cong \mathrm{M}_{t'_{i-1}t_i}(\Delta_i^T). \tag{5.16}$$

where :-

i) $\mathrm{M}_{t_i}(\Delta_i^T)$ is a cyclic extension of Δ^T_{i-1}.

ii) $t_i\mathrm{ind}(\Delta_i^T) = l_i\mathrm{ind}(\Delta^T_{i-1})$.

iii) If $\mathcal{Z}(\Delta_i^T) = L_i^T$, then L^T_{i-1}/L^T_i is a cyclic field extension of degree l_i.

Hence $A^T_{I+} = A^T_{0+}(y_1^{T_I}, \ldots, y_p^{T_I}) = \tilde{T}_{I+}(kF_0)$ has the form

$$A^T_{I+} \cong M_{t'}(\Delta^T).$$

where Δ^T is a Z^T_+-central division algebra and

$$t'\mathrm{ind}(\Delta^T) = lt\,\mathrm{ind}(D^T_+).$$

where $l = l_1 l_2 \ldots l_p = [I^+ : F_0]$.

The irreducible representation $\tilde{T}_1$ of F_1 is induced from $\tilde{T}_{I+}$ and A_1^T is isomorphic to a ring of $m \times m$ matrices over A^T_{0+}, where $m = [F_1 : I^+]$. Finally A^T, the image of kF under $\tilde{T}$, is isomorphic to a ring of $ms \times ms$ matrices over A^T_{0+}. We conclude this chapter with some observations on the degree and Schur index of A^T.

Lemma 5.3.4 *The Schur index of A^T divides* $\mathrm{ind}(B_1)[I^+ : F_0]\sqrt{[F_0 : F'^+]}$.

Proof : Since A^T is a ring of matrices over A^T_{I+}, which is itself a ring of matrices over the division ring Δ^T, $\mathrm{ind}(A^T) = \mathrm{ind}(\Delta^T)$. The simple ring A^T_{I+} is obtained from A^T_{0+} through a series of cyclic extensions, and so its index is of the form $l'\,\mathrm{ind}(A^T_+)$ where $l'|l$ (for a detailed discussion of cyclic extensions of division algebras see [8], Section 1.4). By 5.15, the index of A^T_{o+} divides $\text{ind}(B_1)\,d$, where $d = d_1d_2\ldots d_k = \sqrt{[F_0 : F'^+]}$. Of course $\text{ind}(B_1)$ is the Schur index over k of any absolutely irreducible constituent of the irreducible k-representation of F' determined by the centrally primitive idempotent e_1 of kF'. □

For later reference we now gather together some information on the degree of the irreducible k-representation $\tilde{T}$ of F.

Theorem 5.3.1 *The degree of the irreducible k-representation $\tilde{T}$ is given by*

$$\deg \tilde{T} = [G : G_0]\sqrt{[G_0 : G^+]}\,\deg(B_1)[Z^T_+ : k]\,\text{ind}(A^T_{o+}).$$

Proof : The degree of $\tilde{T}$ is equal to $\deg(A^T)\text{ind}(A^T)[Z^T : k]$, where $\deg(A^T)$ denotes the *degree* of the central simple Z^T-algebra A^T i.e. $\deg(A^T) = \sqrt{\dim_{Z^T}(A^T)}$. Since A^T is isomorphic to $M_{ms}(A^T_{I+})$ where $ms = [F : I^+]$, we have $\deg(A^T) = ms\deg(A^T_{I+})$. The central simple Z^T_+-algebra A^T_{I+} is an extension of A^T_{o+} of degree $[I^+ : F_0]$, and by 5.15 $\deg(A^T_{o+}) = \deg(B_1)\deg(C^T_+)$. Thus since $\deg(C^T_+) = \sqrt{[F_0 : F'^+]}$,

$$\deg(\tilde{T}) = [F : I^+][I^+ : F_0]\sqrt{[F_0 : F'^+]}\,\text{ind}(A^T_{o+})[Z^T_+ : k],$$

whence the result. □

In Chapter 7 we will consider the case in which the k-representation $\tilde{T}$ of F is absolutely irreducible, which will lead to considerable simplification of the situation described here in Section 5.3 - in particular the centre of A^T will be precisely k, so that the centre of A^T_0 will be simply $k \otimes_k E \cong E$, and $\tilde{T}_1$ will restrict to an irreducible representation of F_0.

Chapter 6

Projective Equivalence and Projective Schur Index

Suppose T_1 and T_2 are projectively equivalent irreducible projective representations of G over a field k. Then by Lemma 5.2.2, T_1 and T_2 belong to the same component of kF, where F is a generic central extension for G. However since kF has only a finite number of components, and G generally has infinitely many mutually inequivalent irreducible projective representations over k, at least in a case where $H^2(G, k^\times)$ is infinite. we cannot hope that components of kF should always distinguish projectively inequivalent irreducible projective k-representations of G. In this chapter we show that in the case where k is algebraically closed, the components of kF correspond precisely to projective equivalence classes of irreducible projective representations of G over k.

For an arbitrary field k, the equivalence relation on the set of irreducible projective k-representations of G defined by belonging to a particular component of kF is obviously something weaker than projective equivalence over k. However we shall see that this relation does have an interpretation in terms of equivalence over extensions of k. We should bear in mind that, as mentioned in Chapter 1, two representations which are projectively inequivalent over a given field may become equivalent over some of its extensions.

6.1 Cocycles

Let T be an irreducible projective representation of G over a field k, and let $f \in \mathcal{Z}^2(G, k^\times)$ denote the cocycle associated to T, so for $x, y \in G$,

$$T(xy) = f(x, y)T(x)T(y).$$

Let $\bar{f}$ denote the class of f in $\mathrm{H}^2(G, k^\times)$. If (R, F, ϕ) is a generic central extension for G and $\tilde{T}$ is a lift of T to F, then $\tilde{T}$ sends every element of R to a scalar matrix in $GL(n, k)$, so $\tilde{T}|_R \in \mathrm{Hom}(R, k^\times)$.

Associated to the central extension

$$1 \longrightarrow R \longrightarrow F \longrightarrow G \longrightarrow 1$$

we have the transgression map $\mathrm{tra} : \mathrm{Hom}(R, k^\times) \longrightarrow \mathrm{H}^2(G, k^\times)$ defined as follows for $\eta \in \mathrm{Hom}(R, k^\times)$ and $x, y \in G$:-

First define $\eta' \in \mathcal{Z}^2(G, k^\times)$ for any section μ of G in F by

$$\eta'(x, y) = \eta\left(\mu(x)\mu(y)\mu(xy)^{-1}\right)$$

Then $\mathrm{tra}\,\eta$ is the class of η' in $\mathrm{H}^2(G, k^\times)$, and is independent of the choice of section.

We also have the *Hochschild-Serre exact sequence* (see [11]):-

$$\cdots \longrightarrow Hom(G, k^\times) \xrightarrow{\mathrm{inf}} Hom(F, k^\times) \xrightarrow{\mathrm{res}} Hom(R, k^\times) \xrightarrow{\mathrm{tra}} H^2(G, k^\times) \longrightarrow \cdots$$

Here inf is the inflation map : if $\theta \in \mathrm{Hom}(G, k^\times)$ then $\inf\theta \in \mathrm{Hom}(F, k^\times)$ is defined by $\inf\theta = \theta \circ \phi$. The mapping denoted by res is the usual restriction mapping.

Lemma 6.1.1 *Let $\tilde{T}$ be any lift of T to F, let $\bar{f} \in \mathrm{H}^2(G, k^\times)$ denote the cohomology class corresponding to T, and let $\eta \in \mathrm{Hom}(R, k^\times)$ denote the restriction to R of $\tilde{T}$. Then $\bar{f} = \mathrm{tra}\,\eta$.*

Proof : Let $\mu : G \longrightarrow F$ be a section of G in F, and use it to define $\eta' \in \mathcal{Z}^2(G, k^\times)$ as above. We will show $\eta' \sim f$ in $\mathrm{H}^2(G, k^\times)$, where f is the cocycle corresponding to the projective representation T. For $g, h \in G$ we have

$$\begin{aligned}
\eta'(g,h) &= \eta\left(\mu(g)\mu(h)\mu(gh)^{-1}\right) \\
&= \tilde{T}\left(\mu(g)\mu(h)\mu(gh)^{-1}\right) \\
&= \tilde{T}\left(\mu(g)\right)\tilde{T}\left(\mu(h)\right)\tilde{T}\left(\mu(gh)^{-1}\right). \\
f(g,h) &= T(g)T(h)\left(T(gh)\right)^{-1}
\end{aligned}$$

Define a map $\psi : G \longrightarrow k^{\times}$ by

$$\psi(g) = T(g)^{-1}\tilde{T}(\mu(g)) \quad \left(= \tilde{T}(\mu(g))T(g)^{-1}\right)$$

for $g \in G$. Then

$$\begin{aligned}
f^{-1}(g,h)\eta'(g,h) &= T(gh)T(h)^{-1}T(g)^{-1}\tilde{T}(\mu(g))\tilde{T}(\mu(h))\tilde{T}(\mu(gh))^{-1} \\
&= \psi(g)\psi(h)\psi(gh)^{-1}.
\end{aligned}$$

So $f^{-1}\eta'$ is a coboundary in $\mathcal{Z}^2(G, k^{\times})$, and $\bar{f} = \text{tra}\,\eta$ in $\text{H}^2(G, k^{\times})$ □

Note : $\text{tra}\,\eta$ is independent of the choice of $\tilde{T}$, for suppose $\tilde{T}_1$ and $\tilde{T}_2$ are both lifts of T to F. Then we have already seen that the map ψ of F into $k^{\times}$ defined for $x \in F$ by $\psi(x) = \tilde{T}_1(x)\tilde{T}_2(x)^{-1}$ is a group homomorphism. Then if η_1 and η_2 denote respectively the restrictions of $\tilde{T}_1$ and $\tilde{T}_2$ to R, we have $\eta_1\eta_2^{-1} = \psi|_R$. So $\eta_1\eta_2^{-1}$ is the restriction to R of an element of $\text{Hom}(F, k^{\times})$; i.e. $\eta_1\eta_2^{-1}$ is in the kernel of the transgression map and $\text{tra}\,\eta_1 = \text{tra}\,\eta_2$, by exactness of the Hochschild-Serre sequence.

6.2 Algebraically Closed Fields

From now on we replace the field k by $\mathbb{C}$, the field of complex numbers (or by any algebraically closed field of characteristic zero). In this section we will prove the following theorem :-

Theorem 6.2.1 *Let T_1 and T_2 be irreducible projective $\mathbb{C}$-representations of a finite group G. Then T_1 and T_2 are projectively equivalent over $\mathbb{C}$ if and only if they belong to the same component of $\mathbb{C}F$.*

We require to show that if T_1 and T_2 belong to the same component of $\mathbb{C}F$, then they are projectively equivalent (we already know the other direction). First we show that the cocycles associated to T_1 and T_2 belong to the same cohomology class in $M(G) = H^2(G, \mathbb{C}^\times)$.

Let T be any irreducible projective $\mathbb{C}$-representation of G, and as before let η denote the restriction to R of a lift $\tilde{T}$ of T to F. We will show that $\operatorname{tra}\eta$ depends only on $\eta|_{F'\cap R}$, which is a linear $\mathbb{C}$-character of $F' \cap R$ and is independent of the choice of $\tilde{T}$. First suppose that $\eta|_{F'\cap R}$ is trivial. Then η factors through $F' \cap R$ and can be regarded as a homomorphism of $R/F' \cap R$ into $\mathbb{C}^\times$. But $R/F' \cap R \cong RF'/F'$, so we have a map η' of RF'/F' into $\mathbb{C}^\times$ for which $\eta'(rF') = \eta(r)$. $\forall r \in R$. Finally, since $\mathbb{C}^\times$ is divisible, η' extends to a homomorphism of F/F' into $\mathbb{C}^\times$, which can be inflated to F. Hence η is the restriction to R of an element of $\operatorname{Hom}(F, \mathbb{C}^\times)$, and $\operatorname{tra}\eta = 1$ in $M(G)$.

Suppose that T_1 and T_2 are complex projective representations of G having lifts $\tilde{T}_1$ and $\tilde{T}_2$ respectively to F, for which $\eta_1|_{F'\cap R} = \eta_2|_{F'\cap R}$, where $\eta_i = \tilde{T}_i|_R$ for $i = 1, 2$. Then $\eta_1\eta_2^{-1}$ has trivial restriction to $F' \cap R$ and so $\eta_1\eta_2^{-1} \in \ker(\operatorname{tra})$, whence $\operatorname{tra}\eta_1 = \operatorname{tra}\eta_2$. Thus $\operatorname{tra}\eta$ in general depends only on the restriction of η to $F' \cap R$. Moreover, if $\eta|_{F'\cap R}$ is not trivial, then η is certainly not in the kernel of the transgression map, since no homomorphism of F into the abelian group $k^\times$ can have nontrivial restriction to F'.

Now if T_1 and T_2 are irreducible projective $\mathbb{C}$-representations of G belonging to the same component of $\mathbb{C}F$ and having lifts $\tilde{T}_1$ and $\tilde{T}_2$ respectively to F, then the ordinary representations $\tilde{T}_1|_{F'}$ and $\tilde{T}_2|_{F'}$ afford the same complex character of F'. Hence

$$\tilde{T}_1|_{F'\cap R} : F' \cap R \longrightarrow \mathbb{C}^\times \quad = \quad \tilde{T}_2|_{F'\cap R} : F' \cap R \longrightarrow \mathbb{C}^\times$$

and the cocycles associated to T_1 and T_2 represent the same class in $M(G)$.

We now fix some notation. If $\alpha \in M(G)$, let $\eta \in \operatorname{Hom}(R, \mathbb{C}^\times)$ satisfy $\alpha = \operatorname{tra}\eta$, and define $\theta_\alpha = \eta|_{F'\cap R}$. We have seen that θ_α determines α and does not depend on the choice of ϕ. Also, since $M(G) \cong F' \cap R \cong \operatorname{Hom}(F' \cap R, \mathbb{C}^\times)$ (as $F' \cap R$ is a finite abelian group and hence isomorphic to its dual), we have

$$\{\theta_\alpha, \alpha \in M(G)\} = \operatorname{Hom}(F' \cap R, \mathbb{C}^\times).$$

If T is an irreducible projective $\mathbb{C}$-representation of G whose cocycle belongs to the class $\alpha \in M(G)$, and $\tilde{T}$ is a lift of T to F, then it follows from Lemma 6.1.1 that

$$\tilde{T}|_{F'\cap R} = \theta_\alpha \in \operatorname{Hom}(F' \cap R, \mathbb{C}^\times).$$

We will show by a counting argument that the number of centrally primitive idempotents of $\mathbb{C}F$ is equal to the number of pairwise inequivalent irreducible projective representations of G over $\mathbb{C}$. We need some definitions and background.

Definition : Let $f \in Z^2(G, \mathbb{C}^\times)$. Then an element x of G is called *f-regular* if

$$f(x, h) = f(h, x), \ \forall h \in C_G(x).$$

Thus $x \in G$ is f-regular if whenever x and h commute in G, the basis elements corresponding to x and h commute in the twisted group ring $\mathbb{C}^f G$. It is easily checked that if $x \in G$ is f-regular, it is also f'-regular whenever $f' \in Z^2(G, \mathbb{C}^\times)$ is cohomologous to f, and that any G-conjugate of x is f-regular. Thus for $\alpha \in M(G)$ we can define an α-regular conjugacy class of G. We will make use of the following (see [20]) :

Theorem 6.2.2 (Tappe, 1977) *If $\alpha \in M(G)$, the number n_α of mutually inequivalent irreducible α-representations of G over $\mathbb{C}$ is equal to the number of α-regular conjugacy classes of G contained in G'.*

Let $\mathcal{I}$ denote the set of centrally primitive idempotents of $\mathbb{C}F$, and let $\mathcal{S}_F$ and $\mathcal{S}_G$ denote respectively the set of conjugacy classes of F contained in F' and the set of conjugacy classes of G contained in G'. We require to show that

$$|\mathcal{I}| = \sum_{\alpha \in M(G)} n_\alpha.$$

For each $\mathcal{C} \in \mathcal{S}_F$, let $\hat{\mathcal{C}} = \sum_{x \in \mathcal{C}} x$ in $\mathbb{C}F$. Then $|\mathcal{I}| = |\mathcal{S}_F|$, since $\mathcal{I}$ and $\{\hat{\mathcal{C}}, \mathcal{C} \in \mathcal{S}_F\}$ are bases for the same vector space over $\mathbb{C}$, namely $\mathcal{Z}(\mathbb{C}F) \cap \mathbb{C}F'$.

Let $C \in \mathcal{S}_G$ and let $\mathcal{C} \in \mathcal{S}_F$ satisfy $\phi(\mathcal{C}) = C$. In this situation we will say that $\mathcal{C}$ *lies over* C. Choose some $X \in \mathcal{C}$ and define

$$\mathcal{Z}_\mathcal{C} := \{Z \in F' \cap R \, : \, ZX \in \mathcal{C}\}.$$

$\mathcal{Z}_\mathcal{C}$ is a subgroup of $F' \cap R$, for suppose $Z_1, Z_2 \in \mathcal{Z}_\mathcal{C}$; say $X^a = Z_1 X$, $X^b = Z_2 X$, for some $a, b \in F$. Then $X^{ab} = Z_1 X^b = Z_1 Z_2 X$, and $Z_1 Z_2 \in \mathcal{Z}_\mathcal{C}$, since $Z_1 \in F' \cap R \subseteq \mathcal{Z}(F)$. It is easily checked that $\mathcal{Z}_\mathcal{C}$ does not depend on the choice of $X \in \mathcal{C}$. It does not depend on the choice of $\mathcal{C}$ either; for suppose $\mathcal{C}' \in \mathcal{S}_F$ is another conjugacy class lying over C. If $X \in \mathcal{C}$ then some element X' of $\mathcal{C}'$ has the same image in G as X. Then $X' = rX$, where $r \in F' \cap R$. Since $F' \cap R \subseteq \mathcal{Z}(F)$, $X'^a = rX^a$, $\forall a \in F$, and $r\mathcal{C} \subseteq \mathcal{C}'$. Similarly $r^{-1}\mathcal{C}' \subseteq \mathcal{C}$, and $\mathcal{C}' = r\mathcal{C}$. Then if $ZX \in \mathcal{C}$ for some $Z \in F' \cap R$, we have $ZrX \in \mathcal{C}'$, and so the subgroup Z_C of $F' \cap R$ is well-defined for each $C \in \mathcal{S}_G$.

Furthermore, if $Z \in F' \cap R$ and $\mathcal{C} \in \mathcal{S}_F$ lies over $C \in \mathcal{S}_G$, then either $Z\mathcal{C} = \mathcal{C}$ and $Z \in \mathcal{Z}_C$, or $Z\mathcal{C}$ is another element of $\mathcal{S}_F$ lying over C. Thus the number of conjugacy classes of F which are contained in F' and lie over $C \in \mathcal{S}_G$ is $[F' \cap R : \mathcal{Z}_C]$. Hence

$$|\mathcal{S}_F| = \sum_{C \in \mathcal{S}_G} [F' \cap R : \mathcal{Z}_C]. \tag{6.1}$$

For $\alpha \in \mathrm{M}(G)$, we define the subgroup I_α of $F' \cap R$ as the kernel of the homomorphism $\theta_\alpha : F' \cap R \longrightarrow \mathbb{C}^\times$.

Lemma 6.2.1 *Let $C \in \mathcal{S}_G$, and let $\alpha \in M(G)$. Then C is α-regular if and only if $\mathcal{Z}_C \subseteq I_\alpha$.*

Proof : An element x of G is α-regular if and only if $f(x, y) = f(y, x)$ whenever $f \in \mathrm{Z}^2(G, \mathbb{C}^\times)$ is a cocycle representing α, and $y \in C_G(x)$. This can be restated as follows : $x \in G$ is α-regular if and only if $T(x)T(y) = T(y)T(x)$ for all $y \in C_G(x)$ and all irreducible α-representations $T : G \longrightarrow GL(n\mathbb{C})$. Let T be such a representation, let $y \in C_G(x)$, and let X and Y be preimages in F for x and y respectively. Let $\tilde{T}$ be a lift to F of T. Then $\tilde{T}(X) = c_X T(X)$ and $\tilde{T}(Y) = c_Y T(Y)$ for some c_X and c_Y in $\mathbb{C}^\times$: so

$$\begin{aligned} \tilde{T}(YXY^{-1}X^{-1}) &= c_Y T(y) c_X T(x) T(y)^{-1} c_Y^{-1} T(x)^{-1} c_X^{-1} \\ &= T(y)T(x)T(y)^{-1}T(x)^{-1} \end{aligned}$$

So we have the following characterization of α-regularity : $x \in G$ is α-regular if and only if for each $X \in \phi^{-1}(x)$ we have $\tilde{T}(YXY^{-1}X^{-1}) = 1$, $\forall Y \in \phi^{-1}(C_G(x))$, whenever $\tilde{T}$ is a lift to F of an irreducible α-representation $T : G \longrightarrow GL(n, \mathbb{C})$ of G.

Now suppose $x \in G'$ and let $C \in \mathcal{S}_G$ denote the conjugacy class of x in G. Note that

$$\begin{aligned} Y \in \phi^{-1}(C_G(x)) &\iff YXY^{-1}X^{-1} \in F' \cap R \\ &\iff YXY^{-1} \in (F' \cap R)X \\ &\iff YXY^{-1}X^{-1} \in \mathcal{Z}_C. \end{aligned}$$

Hence $\mathcal{Z}_C = \{YXY^{-1}X^{-1} | Y \in \phi^{-1}(C_G(x))\}$, and x is α-regular if and only if $\mathcal{Z}_C \subseteq \ker(\tilde{T})$. However, by Lemma 6.1.1, the restriction to $F' \cap R$ of $\tilde{T}$, regarded as a homomorphism of $F' \cap R$ into $\mathbb{C}^\times$, is simply θ_α. This proves the lemma; C is α-regular if and only if $\mathcal{Z}_C \subseteq \ker(\theta_\alpha) = I_\alpha$.
□

Proof of Theorem 6.2.1: The number of components of $\mathbb{C}F$ is

$$|\mathcal{S}_F| = \text{no. of conjugacy classes of } F \text{ in } F'$$

Let $\alpha \in M(G)$. By Theorem 6.2.2 the number n_α of inequivalent irreducible α-representations of G over $\mathbb{C}$ is equal to the number of α-regular conjugacy classes of G contained in G'. For each $C \in \mathcal{S}_G$, let

$$M_C = \{\alpha \in M(G) : C \text{ is } \alpha\text{-regular}\}$$

Then $\sum_{C \in \mathcal{S}_G} |M_C| = \sum_{\alpha \in M(G)} n_\alpha$, and

$$\begin{aligned} M_C &= \{\alpha \in M(G) : \mathcal{Z}_C \subseteq I_\alpha\} \\ &= \{\alpha \in M(G) : \theta_\alpha \text{ factors through } \mathcal{Z}_C\}. \end{aligned}$$

Then (since $\{\theta_\alpha\}_{\alpha \in M(G)} = \text{Hom}(F' \cap R, \mathbb{C}^\times)$), we have

$$|M_C| = |\text{Hom}\left(\frac{F' \cap R}{\mathcal{Z}_C}, \mathbb{C}^\times\right)| = [F' \cap R : \mathcal{Z}_C].$$

and by 6.1

$$\sum_{C \in \mathcal{S}_G} |M_C| = \sum_{C \in \mathcal{S}_G} [F' \cap R : \mathcal{Z}_C] = |\mathcal{S}_F| = \sum_{\alpha \in M} n_\alpha.$$

Hence the number of primitive central idempotents of $\mathbb{C}F$ is equal to the number of inequivalent irreducible projective $\mathbb{C}$-representations of G. □

It is interesting to note that the finite covering groups of Section 2.1 do not share the property of generic central extensions described in Theorem 6.2.1. Let $\hat{G}$ be a finite covering group for G. Then $\hat{G}$ is an extension by G of a subgroup A of $\hat{G}' \cap \mathcal{Z}(\hat{G})$, for which $A \cong M(G)$, and every complex projective representation of G lifts to a complex linear representation of $\hat{G}$. However, projective equivalence of irreducible projective representations of G does not imply linear equivalence of their lifts to $\hat{G}$; thus an absolutely irreducible projective representation of G does not "belong to" a unique component of the group ring $\mathbb{C}\hat{G}$. For example, in the case where G is cyclic of order d, $M(G)$ is trivial and G is its own covering group. In this case the d distinct absolutely irreducible linear characters of G all correspond to the same projective equivalence class of irreducible projective representations of G (namely the trivial one). In this case if F is a generic central extension for G (arising from a one-generator presentation), then F is infinite cyclic and $\mathbb{C}F$ has only one component.

In general let ρ be an absolutely irreducible character of $\hat{G}$ afforded by the representation $R : \hat{G} \longrightarrow GL(n, \mathbb{C})$. Then R sends A into $\mathbb{C}^\times$ and the choice of a section μ for G in $\hat{G}$ defines an irreducible projective representation R_P of G by

$$R_P(g) = R(\mu(g)), \text{ for } g \in G.$$

Suppose now that $\psi \in \mathrm{Hom}(\hat{G}, \mathbb{C}^\times)$. Then $R^\psi : G \longrightarrow GL(n, \mathbb{C})$ defined for $g \in G$ by $R^\psi(g) = R(g)\psi(g)$ is another irreducible representation of G which is linearly equivalent to R if and only if $\psi(g) = 1$ whenever $\rho(g) \neq 0$ on $\hat{G}$. However since $\pi \circ R = \pi \circ R^\psi$, where $\pi : GL(n, \mathbb{C}) \longrightarrow PGL(n, \mathbb{C})$ is the usual projection, the projective representations of G determined by R and R^ψ are projectively equivalent. The number of additional absolutely irreducible characters of $\hat{G}$ (or of simple components of $\mathbb{C}\hat{G}$) which determine the same projective equivalence class of irreducible projective representations of G as ρ is at least equal to the number of homomorphisms of $\hat{G}$ into $\mathbb{C}$ whose restriction to $\hat{G}^\rho$ is not the identity, where $\hat{G}^\rho$ is the subgroup of $\hat{G}$ generated by those conjugacy classes upon which ρ is nonzero. Since $\mathbb{C}^\times$ is divisible every element of $\mathrm{Hom}(\hat{G}^\rho, \mathbb{C}^\times)$ which restricts to the identity on $\hat{G}' \cap \hat{G}^\rho$ extends to an element of $\mathrm{Hom}(\hat{G}, \mathbb{C}^\times)$, hence

$$1 + \left|\{\psi \in \mathrm{Hom}(\hat{G}, \mathbb{C}^\times) : \psi(\hat{G}^\rho) \neq 1\}\right| = \left|\mathrm{Hom}(\hat{G}^\rho / \hat{G}^\rho \cap \hat{G}', \mathbb{C}^\times)\right| = [\hat{G}^\rho : \hat{G}^\rho \cap \hat{G}'].$$

6.3 Projective Schur Index and Projective Characters

In this section we use Theorem 6.2.1 and the results of Section 5.3 to reach some conclusions concerning possible values of the Schur index of absolutely irreducible projective representations. We begin with some general background information and terminology. Throughout the following discussion, if R is a (linear or projective) representation of a group G, and E is a field containing all entries appearing in the matrices $T(g)$, $g \in G$, we will denote by R_E the E-representation of G defined by

$$R_E(g) = R(g), \ \forall g \in G.$$

Now let T be an absolutely irreducible complex projective representation of degree d of G, and let $\tau : G \longrightarrow \mathbb{C}$ be the character of T. If E is a subfield of $\mathbb{C}$, then T is said to be *(linearly) realizable* over E if there exists a matrix $A \in GL(d, \mathbb{C})$ for which the projective representation T' of G defined for $g \in G$ by

$$T'(g) := A^{-1}T(g)A$$

sends every element of G into $GL(d, E)$.

In this case it is clear that the representations T and T' have the same character, and also that the same cocycle $f \in Z^2(G, \mathbb{C}^\times)$ is associated to each of them. Thus any field $E \subseteq \mathbb{C}$ over which T is realizable must contain all values assumed by the character τ, and all values assumed by the cocycle f.

Now let k be a subfield of $\mathbb{C}$. The *projective Schur index* of T over k is defined as

$$m_k(T) = \min(E : k(\tau, f)),$$

where $k(\tau, f)$ is the field obtained from k by adjoining all character and cocycle values of T, and the minimum is taken over all extensions E of k over which T is realizable. It is clear from the definition that $m_k(T) = m_{k(\tau,f)}(T)$.

There is another characterization of the projective Schur index, in terms of the index of division rings associated to irreducible representations. This is described in the following theorem, of which a proof can be found in [10], Section 8.3.

Theorem 6.3.1 *Let k be a subfield of $\mathbb{C}$, and let P be an irreducible projective k-representation of G, with cocycle $h \in Z^2(G, k^\times)$. The image of k^fG under the k-linear extension of P is a simple k-algebra; let m be its index. Finally let T be an irreducible constituent of the complex projective representation $P_{\mathbb{C}}$ of G. Then $m_k(T) = m$.* □

We observe that in Theorem 6.3.1, there is no loss of generality in assuming at the outset that k contains the character and cocycle values of every absolutely irreducible constituent of T. This is equivalent to the assumption that the centre of the simple algebra $T(k^fG)$ is just k.

The next theorem establishes a connection between absolutely irreducible projective characters of G and a particular subgroup of G which was encountered in Chapter 4. Recall that if e is the primitive central idempotent of $\mathbb{C}F$ to which the irreducible projective representation T of

G belongs, then the choice of a primitive central idempotent e_1 of $\mathbb{C}F'e$ determines subgroups F_1, F_0, and F'^{+} of F. Here $F_1 = C_F(e_1)$, $F_0 = C_F(\mathcal{Z}(kF'e_1))$, and F'^{+} is the subgroup of F consisting of those $x \in F$ for which xe_1 belongs to the ring generated by F' over the centre of kF_0e_1. Let G^+ denote the image of F'^{+} in G. We will arrive shortly at an interpretation of G^+ in terms of the character of T. We begin with a pair of lemmas, both of which follow from Lemma 4.3.2.

Lemma 6.3.1 *G^+ is independent of the choice of e_1.*

Proof Suppose e_1 and e_2 are primitive central idempotents of $kF'e$, and let

$$F_1^1 = C_F(e_1), \quad F_0^1 = C_F(\mathcal{Z}(kF_1^1e_1)), \quad F'^{+}_1 = \{x \in F_1^1 : xe_1 \in \mathcal{Z}(kFe_1)[F']\}$$
$$F_1^2 = C_F(e_2), \quad F_0^2 = C_F(\mathcal{Z}(kF'e_2)), \quad F'^{+}_2 = \{x \in F_1^2 : xe_2 \in \mathcal{Z}(kF_1^2e_2)[F']\}$$

We will show that $F'^{+}_1 = F'^{+}_2$. By Lemma 4.3.2, F_1^+ is generated by F' and the set

$$\mathcal{P}_1 = \{x \in F_1^1 : \hat{C}_x^1 e_1 \neq 0\},$$

where $\hat{C}_x^1$ denotes the sum in kF of the F_1^1- conjugates of x. Choose $y \in F$ for which $e_2 = y^{-1}e_1y$. Then it is clear that $F_1^2 = y^{-1}F_1^1y$. Furthermore, if we define

$$\mathcal{P}_2 = \{x \in F_2^1 : \hat{C}_x^2 e_2 \neq 0\},$$

where $\hat{C}_x^2$ is the sum in kF of the F_1^2-conjugates of x, then $\mathcal{P}_2 = y^{-1}\mathcal{P}_1y$. In particular $\mathcal{P}_2 \subseteq \langle \mathcal{P}_1, F' \rangle$. Similarly $\mathcal{P}_1 \subseteq \langle \mathcal{P}_2, F' \rangle$, hence $F'^{+}_1 = F'^{+}_2$, and the result. □

Lemma 6.3.2 *Let T be an irreducible complex projective representation of G belonging to the primitive central idempotent e of $\mathbb{C}F$, and let k be a subfield of $\mathbb{C}$ for which $e \in kF$. Let f be a primitive central idempotent of $kF'e$, and let e_1 be a primitive central idempotent of $\mathbb{C}F'f$. Define*

$$F_1 = C_F(e_1) \qquad F_1^k = C_F(f)$$
$$F_0 = C_F(\mathcal{Z}(\mathbb{C}F'e_1)) \qquad F_0^k = C_F(\mathcal{Z}(kF'f))$$
$$F'^{+}_{\mathbb{C}} = \{x \in F : xe_1 \in \mathcal{Z}(\mathbb{C}F_0e_1)[F']\} \quad F'^{+}_k = \{x \in F : xf \in \mathcal{Z}(kF_0^kf)[F']\}$$

Then

i) $F_1 \subseteq F_1^k$

ii) $F_0 = F_0^k$

iii) $F'^+_{\mathbb{C}} = F'^+_k$

Proof Let k_1 be the field obtained from k by adjoining all coefficients of e_1. Then by Section 5.1, f is the sum of the distinct Galois conjugates of e_1 under the action of Gal(k_1/k). Any element of F which centralizes e_1 also centralizes each of these conjugates, hence i).

Now suppose $x \in F_0^k$, and for $c \in F'$ let $\hat{c}$ denote the sum in kF of the F'-conjugates of c. Then x centralizes $\hat{c}f$ for all $c \in F'$. Since e_1 is central in $\mathbb{C}F'$, it is a $\mathbb{C}$-linear combination of elements of the form $\hat{c}$, $c \in F'$; say $e_1 = \sum a_i\hat{c}_i$, where each a_i is a nonzero complex number and each c_i is an element of F'. Then $e_1 = e_1 f = \sum a_i\hat{c}_i f$, and e_1 is a $\mathbb{C}$-linear combination of central elements of $kF'f$. Then x centralizes e_1, and $F_0^k \subseteq F_1$. However $F_1 = F_0$ since the centre of $\mathbb{C}F'e_1$ is just $\mathbb{C}$.

To complete the proof of ii), let $x \in F_0$. So x centralizes $\hat{c}e_1$ for all $c \in F'$. Then x also centralizes $\hat{c}f$ which is the sum of the distinct conjugates of $\hat{c}e_1$ under the action of Gal(k_1/k). Thus $x \in F_0^k$ and $F_0 = F_0^k$.

To prove iii), let $x \in F_0$, and let $\hat{C}_x$ and $\hat{C}_x^k$ denote respectively the sum in kF of the distinct F_1- conjugates of x and the sum in kF of the distinct F_1^k- conjugates of x. First suppose $\hat{C}_x^k f \neq 0$. Then $\hat{C}_x f \neq 0$, since $\hat{C}_x^k f$ is a sum of F_1^k-conjugates of $\hat{C}_x f$, as f is centralized by F_1^k. Then $\hat{C}_x e_1 \neq 0$, since $\hat{C}_x f$ is a sum of Gal(k_1/k)-conjugates of $\hat{C}_x e_1$.

On the other hand suppose $\hat{C}_x e_1 \neq 0$. Then $\hat{C}_x f \neq 0$ since $\hat{C}_x e_1 = \hat{C}_x f e_1$. By ii) then, the sum of the F_0-conjugates of x has nonzero projection on $kF_0 f$, hence $x \in F_k^+$.

That $F'^+_{\mathbb{C}} = F'^+_k$ is now immediate from Lemma 4.3.2. □

Thus the subgroup F'^+ of F and its image G^+ in G depend only on the choice of a primitive central idempotent e of $\mathbb{C}F$ and not on the subsequent choice of simple component of $\mathbb{C}Fe$, or on the choice of underlying field k (provided that $e \in kF$). We now return to the absolutely irreducible complex projective representation T of G belonging to e, and show that G^+ can be related to the character τ of T. First we briefly discuss some general properties of projective characters.

Projective characters differ greatly from linear characters, even in their most fundamental properties. For example a projective character is generally not a class function, since a projective representation is generally not a homomorphism into a general linear group. Also, projectively equivalent representations normally do not have the same character, since they are not merely conjugate but may also differ by any function of the group into the set of nonzero field elements. However, if ρ is the character of a projective representation P of G, we can define a function $\rho^* : G \longrightarrow \{0, 1\}$ by

$$\rho^*(g) = \begin{cases} 0 & \text{if} \quad \rho(g) = 0 \\ 1 & \text{if} \quad \rho(g) \neq 0 \end{cases} \quad . \quad \text{for } g \in G.$$

It is easily seen that ρ^* is a class function on G, and that if ρ_1 is the character of a projective representation P_1 of P which is projectively equivalent to P, then $\rho_1^* = \rho^*$. We also remark that if $\tilde{P}$ is a lift of P to a generic central extension (R, F, ϕ) of G, and if $\tilde{\rho}$ denotes the character of $\tilde{P}$, we may define a function $\tilde{\rho}^* : F \longrightarrow \{0, 1\}$ by

$$\tilde{\rho}^*(x) = \begin{cases} 0 & \text{if} \quad \tilde{\rho}(x) = 0 \\ 1 & \text{if} \quad \tilde{\rho}(x) \neq 0 \end{cases} \quad . \quad \text{for } x \in F.$$

Then $\tilde{\rho}^* = \rho^* \circ \phi$.

Theorem 6.3.2 *Let T be an absolutely irreducible projective representation of G belonging to the primitive central idempotent e of $\mathbb{C}F$, and let τ be the character of T. Let G^+ be the image in G of the subgroup F'^+ of F defined by e as in Section 4.3. Then $G^+ \supseteq \langle G', (\tau^*)^{-1}(1)\rangle$.*

Proof For $x \in F$, let $\hat{C}_x$ denote the sum in $\mathbb{C}F$ of all the F-conjugates of x. Let $\tilde{T}$ denote both a lift of T to F and its extension to $\mathbb{C}F$, and let $\tilde{\tau}$ be the character of $\tilde{T}$. Then

$$(\tilde{\tau}^*)^{-1}(1) = \{x \in F : \hat{C}_x e \neq 0\}.$$

This follows from the fact that $\tilde{T}$ maps $\mathbb{C}F$ onto a full matrix ring over $\mathbb{C}$. For suppose $\hat{C}_x e \neq 0$. Then $\tilde{T}(\hat{C}_x)$ is a nonzero scalar matrix, hence $\tilde{\tau}(x) \neq 0$ since $\tilde{\tau}$ is a class function on F. Similarly $\tilde{\tau}(x) = 0$ if $\hat{C}_x e = 0$.

Let e_1 be a primitive central idempotent of $\mathbb{C}F'e$, and as usual let $F_1 = C_F(e_1)$. Then $F_1 \trianglelefteq F$; this follows from the fact that F' centralizes e_1. Thus $(\tilde{\tau}^*)^{-1}(1) \subseteq F_1$ since by Theorem 5.1.3, $\tilde{T}$ is induced from an irreducible representation $\tilde{T}_1$ of F_1.

For $x \in F_1$, let $\hat{C}_x$ denote the sum in $\mathbb{C}F$ of the F_1-conjugates of x, and let

$$\mathcal{P} = \{x \in F_1 : \hat{C}_x e_1 \neq 0\}.$$

We now show that $\langle \mathcal{P}, F' \rangle \supseteq \langle (\tilde{\tau}^*)^{-1}(1), F' \rangle$.

Let $x \in (\tau^*)^{-1}(1)$. Then $\hat{C}_x e \neq 0$, and $\hat{C}_x e_1 \neq 0$, since $\hat{C}_x$ is centralized by F and e is a sum of F-conjugates of e_1. Let $\mathcal{S}$ be a transversal for F_1 in F. Then

$$\hat{C}_x = \sum_{s \in \mathcal{S}} s^{-1} \hat{C}_x s,$$

and

$$\sum_{s \in \mathcal{S}} s^{-1} \hat{C}_x s e_1 \neq 0 \Longrightarrow \exists s_1 \in \mathcal{S} \text{ for which } s_1^{-1} \hat{C}_x s_1 e_1 \neq 0.$$

Finally $s_1^{-1} \hat{C}_x s_1 = \hat{C}_{s_1^{-1} x s_1}$; hence $s_1^{-1} x s_1 = [s_1, x^{-1}]x \in \mathcal{P}$, and $x \in \langle \mathcal{P}, F' \rangle$. Thus $x \in F'^{+}$, by Lemma 4.3.2. □

The following theorem is the result of combining Lemma 5.2.1, Lemma 5.3.4, Theorem 6.3.2, and Theorem 6.3.1.

Theorem 6.3.3 *Let T be an irreducible complex projective representation of G with character τ, and let $\tilde{T}$ be a lift of T to a generic central extension F for G. Let k be a subfield of $\mathbb{C}$, and let $m_k(\tilde{T}')$ be the Schur index over k of some irreducible constituent $\tilde{T}'$ of the representation $\tilde{T}|_{F'}$ of F'. Then the Schur index $m_k(T)$ of T over k has the form*

$$m_k(T) = m'd,$$

where m' divides $m_k(\tilde{T}')$ and d divides $[G : \langle G', (\tau^*)^{-1}(1) \rangle]$. □

If f is the primitive central idempotent of $\mathbb{C}F$ to which T belongs, and e is the primitive central idempotent of kF for which $ef = f$, then e defines the usual subgroups F_1, F_0 and F'^{+} of F. Then the factor d which appears in the statement of Theorem 6.3.3 has the form $d = d_1 d_2$ where d_1 divides $[F_1 : F_0]$ and d_2^2 divides $[F_0 : F'^{+}]$.

It is well known that the Schur index over a given field of an absolutely irreducible projective representation of G is not invariant under projective equivalence. The reasons for this somewhat unsatisfactory situation are explained by Theorems 6.2.1 and 6.3.1. To see this let T be an irreducible complex projective representation of G, and let (R, F) be a generic central extension for G. If e is the primitive central idempotent of $\mathbb{C}F$ to which T belongs, let k be a subfield of $\mathbb{C}$ for which $e \in kF$. From the discussion in Section 5.3 it is apparent that there may be many possibilities for the value of the index of a simple k-algebra arising as an image of kFe under an irreducible representation sending R into $k^\times$. However, by Theorem 6.3.1, each of these is the Schur index over k of some absolutely irreducible representation T_1 of G belonging to e. Finally T_1 is projectively equivalent to T by Theorem 6.2.1.

A theorem of Fein [3] states that every irreducible complex projective representation of G is projectively equivalent to one having Schur index 1 over $\mathbb{Q}$. This representation is then linearly realizable, not over $\mathbb{Q}$, but over an extension of $\mathbb{Q}$ obtained by adjoining the relevant cocycle and character values. Fein's theorem is related to the fact that every cocycle in $Z^2(G, \mathbb{C}^\times)$ is cohomologous to one taking values in the group of $|G|$th roots of unity in $\mathbb{C}^\times$, and to questions concerning realizibility over cyclotomic fields, which will be discussed in Chapter 7.

Chapter 7

Projective Splitting Fields

In this chapter we investigate some questions on the subject of realizability of projective representations over different fields. If $\bar{k}$ is an algebraically closed field with a subfield k_0, and $T : G \longrightarrow GL(n, \bar{k})$ is a projective representation of a finite group G over $\bar{k}$, then we say that T is *projectively realizable* over k_0 if T is projectively equivalent over $\bar{k}$ to a representation sending every element of G to a matrix in $GL(n, k_0)$. We say that k_0 is a *projective splitting field* for G if every projective $\bar{k}$-representation of G is projectively realizable over k_0.

The problem of determining projective splitting fields for a given finite group G bears some resemblance to the corresponding problem in the theory of linear representations, but the two are far from being entirely analogous. The differences arise mainly from the differing notions of projective and linear equivalence of representations, as outlined in Chapter 1.

If T_1 and T_2 are projectively inequivalent projective representations of G over the field k_0, they may become projectively equivalent over an extension k of k_0. Examples are extremely easy to find, even for cyclic groups of very small order. For instance, let $G = \langle \alpha \rangle$ be cyclic of order 2, and consider the rational projective representations T_1 and T_2 of G defined by

$$T_1(\alpha) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad T_2(\alpha) = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$$

Of course both T_1 and T_2 must send the identity element of G to the identity matrix in $GL(2, \mathbb{Q})$. The representation T_1 is linear so its cocycle is the identity element of $Z^2(G, \mathbb{Q}^\times)$, and the cocycle

$f \in Z^2(G, \mathbb{Q}^\times)$ corresponding to T_2 is given by

$$f(\alpha, \alpha) = 2; \qquad f(\alpha, 1) = f(1, \alpha) = f(1, 1) = 1.$$

The twisted group ring $\mathbb{Q}^f G$ of course has dimension 2 over $\mathbb{Q}$ and is isomorphic to the field $\mathbb{Q}(\sqrt{2})$. The representations T_1 and T_2 are certainly not equivalent over $\mathbb{Q}$, since T_2 is irreducible as a projective $\mathbb{Q}$-representation of G and T_1 is not. However, over the field $\mathbb{Q}(\sqrt{2})$ we have

$$T_2'(\alpha) = A^{-1}T_2(\alpha)A = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & -\sqrt{2} \end{pmatrix}; \text{ where } A = \begin{pmatrix} \sqrt{2} & -\sqrt{2} \\ 1 & 1 \end{pmatrix} \text{ in } GL\left(2, \mathbb{Q}(\sqrt{2})\right).$$

Thus T_2 is equivalent over $\mathbb{Q}(\sqrt{2})$ to the representation T_2' defined above. It is apparent that T_2' is projectively equivalent over $\mathbb{Q}(\sqrt{2})$ to T_1. The cocycle f is a coboundary in $Z^2(G, \mathbb{Q}(\sqrt{2})^\times)$ but not in $Z^2(G, \mathbb{Q}^\times)$. In fact for $g, h \in G$ we have

$$f(g, h) = \frac{\mu(g)\mu(h)}{\mu(gh)},$$

where $\mu : G \longrightarrow \mathbb{Q}(\sqrt{2})$ is defined by $\mu(1) = 1$, $\mu(\alpha) = \sqrt{2}$. Although the example $G = C_2$ is a particularly simple one, it demonstrates the general point : to discuss projective equivalence of representations, we need to specify a field over which to work.

The same is not true in the linear case, for suppose now that R_1 and R_2 are ordinary representations of degree n of a finite group G over a field k_0. Then if k is an extension of k_0, R_1 and R_2 are equivalent over k if and only if there exists a matrix $A \in GL(n, k)$ for which

$$R_2(g) = A^{-1}R_1(g)A, \ \forall \ g \in G.$$

It turns out that if such a matrix A exists in $GL(n, k)$, one also exists in $GL(n, k_0)$. In the case where R_1 and R_2 are irreducible, this is a consequence of the Noether-Skolem theorem, and the general case follows. Thus R_1 and R_2 are linearly equivalent over an extension of k_0 if and only if they are linearly equivalent over k_0 itself.

Let $\bar{k}$ denote the algebraic closure of k_0. Then k_0 is an ordinary splitting field for G if and only if every $\bar{k}$-representation of G is linearly equivalent to a k_0-representation of G. This means that every absolutely irreducible representation of G can be realized over k_0, and that every irreducible k_0-representaion of G is absolutely irreducible (i.e. remains irreducible when regarded as a representation over any extension of k_0). This last remark follows from the fact that every ordinary character of G can be afforded by a k_0-representation, and that a linear representation of G is determined up to linear equivalence by its character. If k_0 is an ordinary splitting field for G, then the group ring k_0G is a direct sum of matrix rings *over* k_0, and the representations of G over $\bar{k}$ are essentially identical to its representations over k_0.

In the search for projective splitting fields, the situation is not quite so rigid. Again let $\bar{k}$ be the algebraic closure of a field k_0, and now suppose that k_0 is a projective splitting field for G. This means that every projective $\bar{k}$-representation of G is projectively equivalent over $\bar{k}$ to one which sends every element of G to a matrix having entries in k_0. However this does not require that every irreducible projective k_0-representation of G remain irreducible over extensions of k_0, or that all simple components of twisted group rings of G over k_0 have Schur index 1. The $\mathbb{Q}$-representation T_2 of C_2 mentioned earlier provides an example : $\mathbb{Q}$ is certainly a projective splitting field for C_2, since all absolutely irreducible projective representations of cyclic groups are trivial; however T_2 is a non-trivial irreducible projective representation of C_2 over $\mathbb{Q}$.

7.1 Necessary Conditions for Projective Splitting Fields

Suppose $\bar{k}$ is an algebraically closed field, containing k as a subfield. Theorem 6.2.1, which establishes the one-to-one correspondence between (projective) equivalence classes of absolutely irreducible projective representations of G and primitive central idempotents of $\bar{k}F$, will be extremely useful in determining conditions under which k may be a projective splitting field for G.

Let T be an irreducible projective representation of G over $\bar{k}$, belonging to the component $\langle e \rangle$ of $\bar{k}F$. Suppose T is projectively equivalent (over $\bar{k}$) to the k-representation T' of G. Then T' is absolutely irreducible, and the central idempotent of kF to which it belongs must remain primitive in $\bar{k}F$. Then $e \in kF$, and if k is a projective splitting field for G, kF and $\bar{k}F$ must have the same set of primitive central idempotents. The following lemma is now an immediate consequence of Theorem 5.1.2.

Lemma 7.1.1 *Suppose k is a projective splitting field for a finite group G. Then if F is a generic central extension for G, k must contain all F-invariant character values of F'.* □

If k is a projective splitting field for G, $Z^2(G, k^\times)$ must of course contain a representative for every element of $M(G)$. We remark that this is true of any field k which contains all F-invariant character values of F' : this follows from the centrality of $F' \cap R$ in F. If θ is a character of $F' \cap R$, then some integer multiple of θ arises as the restriction to $F' \cap R$ of an F-invariant character of F'. To see this note that θ can certainly be extended to a character θ' of R, since $F' \cap R$ is a direct factor of R. Then the induced character $\mathrm{Ind}_R^F \theta'$ of F restricts on F' to an F-invariant character, each of whose irreducible constituents restricts to θ on $F' \cap R$. That θ itself is F-invariant is obvious since $F' \cap R \subseteq \mathcal{Z}(F)$. Then any field k which satisfies the condition of Lemma 7.1.1 must contain all character values of $F' \cap R$ and must in particular contain a root of unity of order $\exp(M(G))$. Then since every element of $Z^2(G, \bar{k}^\times)$ is cohomologous to one taking values in the group of $\exp(M(G))^{\text{th}}$ roots of unity, $Z^2(G, k^\times)$ contains a representative from every cohomology class in $Z^2(G, \bar{k}^\times)$.

As we might guess by looking at the corresponding situation in the theory of linear representations, the condition of Lemma 7.1.1 is not sufficient to guarantee that k will be a projective splitting field for G. Suppose that k is any field, and that T is an irreducible projective representation of G over k, belonging to the component $\langle e \rangle$ of kF; as usual let $\tilde{T}$ be a lift of T to F. Then by Theorem 5.1.2, we have

$$\mathcal{Z}(kFe) \cap kF'e \cong k(\chi),$$

where χ is the sum of an F-orbit of absolutely irreducible characters of F' appearing in $\tilde{T}|_{F'}$. Thus k and $k(\chi)$ must be equal if T is absolutely irreducible.

The analogous result in the the theory of linear representations of finite groups says that if R is an irreducible linear representation of G over a field k, then the centre of the simple algebra generated over k by $\{R(g), g \in G\}$ is isomorphic to the character field $k(\theta)$, where θ is the character of any absolutely irreducible constituent of R. Thus R can be absolutely irreducible only if $k = k(\theta)$ for all such θ. Of course a field k which contains all character values of G need not be a splitting field; the group ring kG may still have simple components of Schur index exceeding 1.

The following example shows that the same situation arises in the projective case : even over a field k which contains all F-invariant character values of F', where F is a generic central extension for the finite group G, division algebras apearing in the simple components of kF' can sometimes (though not always) obstruct the realizability over k of certain projective representations of G.

Example : Let S_4 be the group of permutations of the set $\{a, b, c, d\}$. The Schur multiplier of S_4 has order 2, and the group $\tilde{S_4}$ defined by

$$\tilde{S_4} = \langle z, t_1, t_2, t_3 | z^2 = 1, t_i^2 = z, [t_i, z] = 1, (t_i t_{i+1})^3 = z, [t_i, t_j] = z \text{ for } |i - j| > 1\rangle$$

is a covering group for S_4 (see [4]). The map ϕ defined on generators by

$$\phi(z) = 1, \ \phi(t_1) = (a\ b), \ \phi(t_2) = (b\ c), \ \phi(t_3) = (c\ d),$$

extends to a surjection of $\tilde{S_4}$ on S_4. The subgroup $\tilde{A_4} = \phi^{-1}(A_4)$ of $\tilde{S_4}$ is a covering group for A_4, and it includes the element z in its commutator subgroup, for it follows easily from the above presentation for $\tilde{S_4}$ that

$$z = [\alpha, \beta] \in \tilde{A_4}',$$

where $\alpha = t_1 t_3, \beta = t_1 t_2 t_1 t_2 t_3 t_2$, $\alpha, \beta \in \tilde{A_4}$. Then $\tilde{A_4}'$ is a central extension of C_2 by $C_2 \times C_2$ (which is isomorphic to the commutator subgroup of A_4. Furthermore $\tilde{A_4}'$ is not abelian, for the elements $\phi(\alpha)$ and $\phi(\beta)$ are pairs of disjoint transpositions in S_4 and therefore belong to A_4', hence $z \in (\tilde{A_4}')'$. Then $\tilde{A_4}'$ is isomorphic to either to the dihedral group D_8 of order 8 or the quaternion group Q_8 of order 8, and is generated by α and β. It is easily checked that both α and β have order 4 in $\tilde{A_4}$; hence $\tilde{A_4}' \cong Q_8$ as it is generated by two non-commuting elements of order 4.

Now let F be a generic central extension for A_4. Then $F' \cong Q_8$, and $\mathbb{Q}$ contains all character values of F'. However $\mathbb{Q}$ is not an ordinary splitting field for F' since the group ring $\mathbb{Q}F'$ has a quaternion division algebra as one of its simple components. Let e be the primitive central idempotent of $\mathbb{Q}F'$ corresponding to this component; of course $\mathbb{Q}F'$ and $\mathbb{C}F'$ have the same set of primitive central idempotents. The idempotent e remains central in $\mathbb{Q}F$, since the character of F' corresponding to e is non-zero only on the subgroup $\langle z\rangle$, which is central in F. We now show, by considering the structure of the simple rings $\mathbb{Q}Fe$ and $\mathbb{C}Fe$ in the context and notation of Chapter 4, that no absolutely irreducible complex projective representation of A_4 belonging to e can be realized over $\mathbb{Q}$.

Certainly $C_F(e) = F$ and so $F_1 = F$ in $\mathbb{Q}Fe$, and in $\mathbb{C}Fe$. Also, since $\mathbb{Q}F'e$ and $\mathbb{C}F'e$ are central simple algebras over $\mathbb{Q}$ and $\mathbb{C}$ respectively, $F_0 = F$ for each of these rings. Moreover, in each case $F'^{+} = F$ also, since $[F_0 : F'^{+}]$ must be a square dividing $[A_4 : A_4'] = 3$.

By Theorem 5.3.1, the degree of the absolutely irreducible representation T is 2, since the minimal F-invariant character of F' determined by e has degree 2, and $[F : F'^{+}] = 1$. Then

T cannot be realized over $\mathbb{Q}$, since $\mathbb{Q}F'e$ is a quaternion division algebra over $\mathbb{Q}$, and the $\mathbb{Q}$-representation of F' corresponding to e has degree 4.

This example demonstrates the type of problem that can arise in attempts at realizing representations over various fields. In general however the existence of matrix rings over non-commutative division algebras as simple components of kF' need not always preclude the realizability of projective representations of G over k. If as usual (R, F) is a generic central extension for a finite group G, let χ_1 be an absolutely irreducible character of F' having Schur index m over a field k which contains all F-invariant character values of F'. Let χ denote the sum of the F-conjugates of χ_1, so χ is a minimal F-invariant character of F'. Then χ determines a primitive central idempotent e of kF, and e remains primitive in $\bar{k}F$ for all field extensions $\bar{k}$ of k. Each simple component of the ring $kF'e$ is a matrix ring over a division algebra of index m, having a field isomorphic to $k(\chi_1)$ as the centre.

Now let T be an (absolutely) irreducible projective representation of G belonging to e, and let $\bar{k} \supseteq k$ be an algebraically closed field containing all entries appearing in all matrices $T(g)$, $g \in G$. Let f be a primitive central idempotent in the completely reducible ring $\bar{k}F'e$. Then if $\bar{F}_1 = C_F(f)$, let $\bar{Z}$ denote the centre of the ring $\bar{k}\bar{F}_1 f$, and define

$$\bar{F}'^{+} = \{x \in \bar{F}_1 | xf \in \bar{Z}\langle F' \rangle\}.$$

Then by Theorem 5.3.1, the degree of the representation T is given by

$$\deg(T) = [F : \bar{F}_1] \sqrt{[\bar{F}_1 : \bar{F}'^{+}]}\, \chi_1(1).$$

Now let e_1 be the sum of the conjugates of f under the action of $\mathrm{Gal}(\bar{k}/k)$. Then e_1 is a primitive central idempotent of $kF'e$. Let $F_1 = C_F(e_1)$, $F_0 = C_F(\mathcal{Z}(kF'e_1))$, and $Z = \mathcal{Z}(kF_0e_1)$. Define the subgroup F'^{+} of F_0 as in 4.3 :

$$F'^{+} = \{x \in F | xe_1 \in Z\langle F' \rangle\}.$$

Then $F_0 = \bar{F}_1$ and $F'^{+} = \bar{F}'^{+}$ by Lemma 6.3.2. Let T_2 be an irreducible projective k-representation of G belonging to e, having a lift $\tilde{T}_2$ to F. If T_2 is absolutely irreducible as a projective representation of G, then $\tilde{T}_2$ must map kFe onto a matrix ring over k. By Theorem 5.3.1, the degree of T_2 is given by

$$\deg(T_2) = [F : F_0] d_1 \sqrt{[F_0 : F'^{+}]}\, d_2 \chi_1(1),$$

where d_1 is a divisor of $[F_1 : F_0]$ and d_2 is the Schur index of the simple $k(\chi_1)$-algebra $A_0^{T_2} = \tilde{T}_2(kF_0)$. Thus T_2 is absolutely irreducible (and projectively equivalent over $\bar{k}$ to T) if and only if $d_1 = d_2 = 1$, in which case

$$\deg(T_2) = [F : F_0]\sqrt{[F_0 : F'^+]}\deg(\chi_1) = [F : \bar{F}_1]\sqrt{[\bar{F}_1 : \bar{F}'^+]}\deg(\chi_1) = \deg(T).$$

This requires (at least) that $A_0^{T_2}$ be a ring of matrices of degree $\sqrt{[F_0 : F'^+]}\deg(\chi_1)$ over a field isomorphic to $k(\chi_1)$. Since the character of $\tilde{T}_2|_{F'}$ is a multiple of χ, the Schur index m of χ over k must divide $\sqrt{[F_0 : F'^+]}$, if T is realizable over k.

We obtain the following necessary (but generally insufficient) conditions for a field k to be a projective splitting field for G :-

Theorem 7.1.1 *Let G be a finite group with generic central extension (R, F) and suppose k is a projective splitting field for G. Then*

i) k contains all F-invariant character values of F'.

ii) If m is the Schur index over k of some absolutely irreducible (ordinary) character of F', then some subgroup of G/G' has a homomorphic image of symmetric type, whose order is divisible by m^2. □

7.2 A Sufficient Condition for Projective Splitting Fields

Suppose k is a field satisfying the conditions of Theorem 7.1.1 for the finite group G with generic central extension F. Then k is a projective splitting field for G if and only if every (projective) equivalence class of absolutely irreducible projective representations of G includes a representative T lifting to an ordinary representation $\tilde{T}$ of F which maps kF onto a full matrix ring over k. Of course this is equivalent to the statement that $T(g)$ should have entries in k for all $g \in G$, but our approach to the realizability problem will be to use the results from Chapters 4 and 5 to investigate the structure of simple k-algebras which arise as images of kF under k-linear extensions of ordinary representations of F which send R into $k^\times$.

Let e be a centrally primitive idempotent of kF, and let e_1, and subsequently F_0, F_1, A_0 and A_1 be defined as in Section 4.1 for the component $\langle e\rangle$ of kF. Then $L = \mathcal{Z}(A_0)$, and it follows from Theorem 4.4.1 and the discussion in Section 5.3 that there exists a transcendence basis $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_r\}$ for L over $\mathcal{Z}(kF'e_1)$, for which irreducible projective k-representations of G are determined up to equivalence by the choice of images for $\gamma_1, \ldots, \gamma_r$ in their lifts to F. These images must always be algebraic over k, and in cases where the associated projective representations of G are to be absolutely irreducible, they must be inside $k^\times$.

If now T is an absolutely irreducible projective representation of G belonging to e (which is a centrally primitive idempotent of $\bar{k}$ for all extensions $\bar{k}$ of k), then T is projectively realizable over k if and only if some choice of $\gamma_1^T, \ldots, \gamma_r^T$ in $k^\times$ defines an irreducible linear representation of F which sends e to 1, R into $k^\times$, and under which the image of kF_1e_1 is a matrix ring over k.

There is one situation in which we can guarantee the existence of such choices for $\gamma_1^T, \ldots, \gamma_r^T$: namely when k is an ordinary splitting field for F'. In this case the problem simplifies in two ways. Firstly, if k is a splitting field for F', then every simple component of kF' is a matrix ring over k, and so there is no danger of difficulties arising from division algebras appearing at the level of kF', as in the example of A_4 over $\mathbb{Q}$. Secondly, in the case where k is a splitting field for F' the centre of every simple component of kF' is just k, whence $F_1 = F_0$ for each component of kF. Thus we need only show that a suitable choice of $\gamma_1^T, \ldots, \gamma_r^T$ will ensure that every symbol algebra appearing in the tensor product description of A_0^T (see 5.14) is a matrix ring over k. The proof of the following theorem indicates how such a splitting can always be arranged.

Theorem 7.2.1 *Let G be a finite group with generic central extension F, and let k be an algebraic number field contained in $\mathbb{C}$. Then if k is an ordinary splitting field for F', k is a projective splitting field for G.*

Proof Choose a centrally primitive idempotent e of $\mathbb{C}F$. Then $e \in kF$ of course, since k contains all character values of F', hence all coefficients appearing in central idempotents of $\mathbb{C}F$. The rings kFe and $\mathbb{C}Fe$ resemble each other closely; this is a consequence of the fact that the field k splits F'. In particular, if e_1 is a centrally primitive idempotent of $\mathbb{C}F'$ for which $e_1e = e$, then $e_1 \in kF'$ also, and the subgroup F_0 of F defined as in Chapter 4 is the same for the rings kFe and $\mathbb{C}Fe$ - in each case this is just $C_F(e_1)$. Also, by Lemma 6.3.2, the rings kFe_1 and $\mathbb{C}Fe_1$ define the same subgroup F'^+ of F (F'^+ is the intersection of F with the ring generated by F' over the centre of kF_1e_1 or $\mathbb{C}F_1e_1$).

Now let $s = [F : F_0]$ and let d be the degree of the absolutely irreducible linear character of F' determined by e_1. Let $K_\mathbb{C}$ denote the usual purely transcendental field extension of $\mathbb{C}$, obtained by adjoining to $\mathbb{C}$ all quotients from $\mathbb{C}[S]$, where S is a torsion-free complement for $F' \cap R$ in R. Let K_k denote the corresponding purely transcendental extension of k, (i.e. K_k is the field of quotients of $k[S]$); and let $Z_\mathbb{C}$ and Z_k denote the centres of the simple rings $K_\mathbb{C}F_1e_1$ and $K_kF_1e_1$ respectively. These simple rings are similar in structure : by the results of Chapter 4, each is a ring of $d \times d$ matrices over a division algebra of index m, where $m^2 = [F_0 : F'^+]$.

By Theorem 6.2.1. all absolutely irreducible projective representations of G belonging to e are projectively equivalent (over $\mathbb{C}$). Let $T : G \longrightarrow GL(n, \mathbb{C})$ be one of these. Then $n = sdm$ and any lift $\tilde{T}$ of T to F maps $\mathbb{C}F$ onto a ring of $n \times n$ matrices over $\mathbb{C}$ (of course all symbol algebras over $\mathbb{C}$ are split).

On the other hand, any irreducible linear k-representation $\tilde{T}'$ of F belonging to $\langle e \rangle$, sending R into $k^\times$ and having degree n, defines (by restriction to a section for G in F) a realization of T over k. In what follows we show how to construct an absolutely irreducible representation $\tilde{T}_1$ of F_0 for which we may define such a $\tilde{T}'$ by

$$\tilde{T}' = \mathrm{Ind}_{F_0}^{F}(\tilde{T}_1).$$

Since k is a splitting field for F', we may assume that $\tilde{T}_1|_{F'}$ is a k-representation of F' of degree md, which we need to suitably extend to F_0. The free abelian group F'^+/F' has finite index in F_0/F', and the quotient F_0/F'^+ is of symmetric type, by Theorems 4.5.1 and 4.5.2. Then we can invoke the fundamental theorem of finitely generated abelian groups to find a basis

$$\mathcal{B} = \{\bar{a}_1, \bar{b}_1, \ldots, \bar{a}_q, \bar{b}_q, \bar{c}_1, \ldots, \bar{c}_s\},$$

of F_0/F', for which

$$\mathcal{B}' = \{\bar{a}_1^{d_1}, \bar{b}_1^{d_1}, \ldots, \bar{a}_q^{d_q}, \bar{b}_q^{d_q}, \bar{c}_1, \ldots, \bar{c}_s\}$$

is a basis for F'^+/F', and $d_q|d_{q-1}|\ldots|d_1$. Here $2q + s = r$ is the rank of the free group $\bar{F}$, and we have

$$F_0/F'^+ \cong C_{d_1} \times C_{d_1} \times \ldots C_{d_q} \times C_{d_q}.$$

For $i = 1 \ldots q$ and $j = 1 \ldots s$, choose representatives a_i, b_i and c_j for the F'-cosets $\bar{a}_i, \bar{b}_i$ and $\bar{c}_j$ respectively in F_0. Then as in Section 4.3 we can find units α_i, β_i and δ_j in $kF'e_1$ for which every

$$\gamma_{a_i} = a_i\alpha_i,\ \gamma_{b_i} = b_i\beta_i, \text{ and } \gamma_{c_j} = c_j\delta_j$$

centralizes F' in $kF_0\epsilon_1$. Then by Theorem 4.4.1

$$\Gamma := \left\{(\gamma_{a_1})^{d_1}, (\gamma_{b_1})^{d_1}, \ldots, (\gamma_{a_q})^{d_q}, (\gamma_{b_q}^{d_q}), \gamma_{c_1}, \ldots, \gamma_{c_s}\right\}$$

is a transcendence basis for Z_k over k. Furthermore, after applying the procedure of Theorem 4.5.1 if necessary, we can assume that

$$K_k F_0 \epsilon_1 \cong M_d\left[\left(\frac{(\gamma_{a_1})^{d_1}, (\gamma_{b_1})^{d_1}}{Z_k, \zeta_{d_1}}\right) \otimes \cdots \otimes \left(\frac{(\gamma_{a_q})^{d_q}, (\gamma_{b_q})^{d_q}}{Z_k, \zeta_{d_q}}\right)\right],$$

where for $i = 1 \ldots q$, ζ_{d_i} is a root of unity of order d_i in Z_k (hence in k, since Z_k is purely transcendental over k).

We can now extend $\tilde{T}_1$ to F_0 by choosing images $A_1, B_1, \ldots, A_q, B_q, C_1, \ldots C_s$ in $k^\times$ for the elements $(\gamma_{a_1})^{d_1}, (\gamma_{b_1})^{d_1}, \ldots, (\gamma_{a_q})^{d_q}, (\gamma_{b_q})^{d_q}, \gamma_{c_1}, \ldots, \gamma_{c_s}$ of Γ, as in Section 5.3. The image $A_0^{\tilde{T}_1}$ of $kF_0\epsilon_1$ under $\tilde{T}_1$ is a ring of $d \times d$ matrices over a tensor product of symbol algebras :-

$$A_0^{\tilde{T}_1} = M_d\left[\left(\frac{A_1, B_1}{k, \zeta_{d_1}}\right) \otimes \cdots \otimes \left(\frac{A_q, B_q}{k, \zeta_{d_q}}\right)\right]$$

Suitable choices of $A_1, B_1, \ldots, A_q, B_q$ will guarantee that each of these symbol algebras splits over k : for instance we may choose each B_i from the group of d_ith powers in $k^\times$ to ensure for $i = 1 \ldots q$ that

$$B_i \in N_{k\sqrt[d_i]{A_i}/k}\left(k(\sqrt[d_i]{A_i})^\times\right), \text{ and } \left(\frac{A_i, B_i}{k, \zeta_{d_i}}\right) \cong M_{d_i}(k).$$

Under such a choice, $\tilde{T}_1$ sends kF_0e_1 onto a simple ring which is isomorphic to $M_{md}(k)$.

Finally, $\tilde{T}' := \mathrm{Ind}_{F_0}^{F}\tilde{T}_1$ is a linear representation of F whose restriction to any section for G in F defines as in Section 2.1 an irreducible representation of G which is realizable over k and which belongs to e and is therefore projectively equivalent to the original T by Theorem 6.2.1. This completes the proof of Theorem 7.2.1: given an irreducible complex projective representation T of G belonging to the component $\langle e\rangle$ of $\mathbb{C}F$, the assumption that kF' is a direct sum of matrix rings over k for a field $k \subseteq \mathbb{C}$ is enough to guarantee the existence of an absolutely irreducible projective k-representation T_1 of G belonging to the component $\langle e\rangle$ of kF. □

The following result, due to H. Opolka (see [13]), is an easy consequence of Theorem 7.2.1, in view of the fact that a field $\mathcal{F}$ which contains a root of unity of order equal to the exponent of the finite group $\mathcal{G}$ is an ordinary splitting field for $\mathcal{G}$. This well-known result is originally due to Brauer, who obtained it as a consequence of his celebrated theorem on induced characters. Details can be found in [5].

Corollary 7.2.1 *Suppose G is a finite group, and k is a field containing a root of unity of order* $\exp(G)\exp(M(G))$. *Then k is a projective splitting field for G.*

Proof Let F be a generic central extension for G. Then since F' is a central extension of $M(G)$ by G', its exponent is a divisor of $\exp(G)\exp(M(G))$. Then k contains a root of unity of order $\exp(F')$ and is therefore an ordinary splitting field for F'. Then k is a projective splitting field for G by Theorem 7.2.1. □

Chapter 8

Metacyclic Groups

In this chapter we apply the methods developed so far to the case where G is a finite metacyclic group. In this case it is possible to describe a generic central extension (R, F) for G quite explicitly, mainly due to the fact that F' is cyclic and kF' is a direct sum of cyclotomic field extensions of k. We obtain a detailed description of the irreducible projective k-representations of G, where k is a subfield of $\mathbb{C}$. The main results are :-

i) Determination of minimal projective splitting fields for metacyclic groups, and

ii) Determination of those metacyclic groups which have faithful irreducible projective representations over $\mathbb{C}$. This result is originally due to Ng (see [12]). We give an alternative proof.

8.1 Generic Central Extensions of Metacyclic Groups

Throughout this chapter we let G denote the metacyclic group defined by:-

$$G = \langle x, y | x^m = 1, y^s = x^t, y^{-1}xy = x^r \rangle. \tag{8.1}$$

Here $\gcd(r, m) = 1$, $m | t(r-1)$, and $r^s \equiv 1 \bmod m$. Also, we may assume (by suitable choice of the generator x) that $t | m$.

Let $\tilde{F}$ be a free group of rank 2, with generators $\tilde{X}$ and $\tilde{Y}$, and let $\tilde{R}$ be the kernel of the homomorphism of $\tilde{F}$ onto G defined by

$$\tilde{X} \longrightarrow x, \quad \tilde{Y} \longrightarrow y.$$

Then $F := \tilde{F}/[\tilde{F}, \tilde{R}]$ is a generic central extension of $R := \tilde{R}/[\tilde{F}, \tilde{R}]$ by G. Let X and Y denote the images of $\tilde{X}$ and $\tilde{Y}$ respectively in F. Then $R \subseteq \mathcal{Z}(F)$, and

$$R = \langle X^m, Y^s X^{-t}, Y^{-1}XYX^{-r} \rangle.$$

Moreover, $R = S \times (F' \cap R)$ where S is a free abelian group of rank 2, and $F' \cap R$, the torsion subgroup of R, is isomorphic to the Schur multiplier of G.

We will use the following notation in the description of F:-

$$\begin{aligned} \alpha(i) &= 1 + r + r^2 + \ldots r^{i-1} = \frac{r^i - 1}{r - 1}, \quad \text{for } i > 0 \\ j &= \gcd(m, r-1) \\ n &= \gcd(\alpha(s), t) \end{aligned}$$

Let c denote the element $[Y^{-1}, X] = Y^{-1}XYX^{-1}$ of F. Then $c = aX^{r-1}$ where $a = Y^{-1}XYX^{-r} \in R$. In particular then $[X, c] = 1$ in F. Also, since $X^Y = cX$, we have

$$c^Y = (aX^{r-1})^Y = a(cX)^{r-1} = ac^{r-1}X^{r-1} = c^r.$$

Thus $\langle c \rangle \trianglelefteq F$. Also, $X^{Y^i} = c^{\alpha(i)}X$, for $i \in \mathbb{Z}_{>0}$.

Lemma 8.1.1 $\langle c \rangle = F'$

Proof : Since $YX = XYc$ we can write every element of F in the form $X^i Y^j c^k$ for some integers i, j and k. Then we need only check that

$$[X^{i_1}Y^{j_1}c^{k_1}, X^{i_2}Y^{j_2}c^{k_2}] \in \langle c \rangle,$$

for all choices of i_1, j_1, k_1 and i_2, j_2, k_2. In fact by the normality of $\langle c \rangle$ in F, it is enough to check

$$[X^{i_1}Y^{j_1}, X^{i_2}Y^{j_2}] \in \langle c \rangle, \quad \forall i_1, j_1, i_2, j_2.$$

For any $i \in \mathbb{Z}$, $(X^i)^Y = c^i X^i$, and since Y normalizes $\langle c \rangle$ we have $(X^i)^{Y^j} \in \langle c \rangle X^i$, $\forall i, j$. Thus

$$[X^{i_1}Y^{j_1}, X^{i_2}Y^{j_2}] \in X^{i_1}X^{i_2}X^{-i_1}X^{-i_2}\langle c \rangle.$$

i.e. $[X^{i_1}Y^{j_1}, X^{i_2}Y^{j_2}] \in \langle c \rangle$, proving the lemma. □

The commutator subgroup $G' = \langle x^{r-1} \rangle$ of G has order m/j, and so $F' \cap R = \langle c^{m/j} \rangle$. Also, since $[X, c] = 1$ in F, we have

$$\begin{aligned} c^2 &= Y^{-1}XYX^{-1}Y^{-1}XYX^{-1} \\ &= X^{-1}Y^{-1}XY(Y^{-1}XYX^{-1}) \\ &= X^{-1}Y^{-1}X^2YX^{-1} \\ &= Y^{-1}X^2YX^{-2} \\ &= [Y^{-1}, X^2] \end{aligned}$$

Similarly we find that

$$c^i = [Y^{-1}, X^i] \tag{8.2}$$

in general. Then

$$c^t = [Y^{-1}, X^t] = [Y^{-1}, X^tY^{-s}] = 1,$$

since $X^tY^{-s} \in R$. Also, since $X^{Y^i} = c^{\alpha(i)}X$ for all positive integers i, we find that

$$c^{\alpha(s)} = [Y^{-s}, X] = [X^tY^{-s}, X] = 1.$$

Then the order of c in F divides $n = \gcd(\alpha(s), t)$. In fact this order is exactly n, since by theorem 2.3.1 $F' \cap R$ is isomorphic to the Schur multiplier of G, which is cyclic of order nj/m. In general if G is the metacyclic group of 8.1, then $M(G)$ is cyclic of order $\dfrac{\gcd(\alpha(s), t)\gcd(m, r-1)}{m}$. For a proof of this fact see [12] (for example).

Finally we remark that it is not difficult to find a pair of generators for a free abelian complement S for $F' \cap R$ in R. We have

$$R = \langle X^m, Y^sX^{-t}, Y^{-1}XYX^{-r} \rangle$$

Also,

$$R \cong (R/F' \cap R) \times (F' \cap R); \qquad R/F' \cap R \cong RF'/F'.$$

This latter group is of course free abelian of rank 2, since it has finite index in F/F'. Under the usual surjection of $R/F' \cap R$ on RF'/F', we obtain

$$X^m \longrightarrow \bar{X}^m, \quad Y^sX^{-t} \longrightarrow \bar{Y}^s\bar{X}^{-t}, \quad Y^{-1}XYX^{-r} \longrightarrow \bar{X}^{(-r+1)}$$

Then $RF'/F' = \langle \bar{X}^j \rangle \times \langle \bar{Y}^s \bar{X}^{-t} \rangle$, where $j = \gcd(m, r-1)$. If $j = s_1 m - s_2(r-1)$, the elements

$$a_1 := Y^s X^{-t} \text{ and } a_2 := (X^m)^{s_1} (Y^{-1} X Y X^{-r})^{s_2}$$

generate a free abelian complement S for $F' \cap R$ in R.

8.2 Primitive Idempotents for Metacyclic G

As usual let k be a field of characterisitic zero; we will assume $k \subseteq \mathbb{C}$, and let G be the metacyclic group with the presentation of 8.1. Then the group ring kF contains the central subring kS. Let K denote the field of quotients of kS: $K = k(a_1, a_2)$ is a purely transcendental field extension of k of transcendence degree 2. The ring KF is completely reducible.

Now let ξ denote a primitive nth root of unity in $\mathbb{C}$, and for each $d|n$ let ξ_d denote the primitive dth root of unity $\xi^{n/d}$. Of course

$$\mathbb{Q}C_n \cong \bigoplus_{d|n} \mathbb{Q}(\xi_d).$$

and

$$\begin{aligned} kF' \cong kC_n &\cong k \otimes_{\mathbb{Q}} \mathbb{Q}C_n \\ &\cong \bigoplus_{d|n} k \otimes_{\mathbb{Q}} \mathbb{Q}(\xi_d) \\ k \otimes_{\mathbb{Q}} \mathbb{Q}(\xi_d) &\cong [k \cap \mathbb{Q}(\xi_d) : \mathbb{Q}] k(\xi_d) \\ kF' &\cong \bigoplus_{d|n} [k \cap \mathbb{Q}(\xi_d) : \mathbb{Q}] k(\xi_d) \end{aligned}$$

The group ring kF' is a direct sum of cyclotomic field extensions of k by nth roots of unity.

It is possible to fully describe the primitive central idempotents of kF', and hence of kF. For $i = 0, 1, \ldots, n-1$, let $\widehat{\xi^i c}$ denote the element $\sum_{j=1}^{n} (\xi^i c)^j$ of $\mathbb{C}F'$, and let $f_i = \frac{1}{n}\widehat{\xi^i c}$. It is routine to check that $\mathcal{F} = \{f_0, \ldots, f_n\}$ is the full set of primitive idempotents of $\mathbb{C}F'$, which is of course isomorphic to the direct sum of n copies of $\mathbb{C}$. In the component $\langle f_i \rangle$ of $\mathbb{C}F'$, c is identified with ξ^{-i}, since $\xi^{-i} f_i = c f_i$. Given a field automorphism τ of $\mathbb{C}$, we can define a $\mathbb{Q}$-algebra automorphism of τ' of $\mathbb{C}F'$ by

$$\left(\sum_{i=0}^{n-1} a_i c^i \right)^{\tau'} = \sum_{i=0}^{n-1} a_i^{\tau} c^i, \quad a_i \in \mathbb{C}.$$

In particular this defines a faithful action of Gal($\mathbb{Q}(\xi)/\mathbb{Q}$) on $\mathcal{F}$. For $d|n$, let $\mathcal{F}_d$ denote the subset of $\mathcal{F}$ consisting of those f_i for which $\gcd(i,n) = n/d$, so ξ^i has order d. Then the subsets $\mathcal{F}_d$ are the orbits of $\mathcal{F}$ under the action of Gal($\mathbb{Q}(\xi)/\mathbb{Q}$). This action restricts to an action of $\mathcal{G} := \text{Gal}(\mathbb{Q}(\xi)/k \cap \mathbb{Q}(\xi))$, under which $\mathcal{F}_d$ splits into further orbits. Since each element of $\mathcal{F}_d$ has the form $\frac{1}{n}\widehat{\xi^i c}$, where ξ^i is a primitive dth root of unity in $\mathbb{C}$, each of these orbits has $[\mathbb{Q}(\xi_d) : \mathbb{Q}(\xi_d) \cap k]$ elements, and the number of them is

$$\frac{[\mathbb{Q}(\xi_d) : \mathbb{Q}]}{[\mathbb{Q}(\xi_d) : \mathbb{Q}(\xi_d) \cap k]} = [\mathbb{Q}(\xi_d) \cap k : \mathbb{Q}].$$

Note that this is also the number of copies of $\mathbb{Q}(\xi_d)$ which appear as simple components of kF'. Each primitive idempotent of kF' is the sum in $\mathbb{C}F'$ of an orbit of $\mathcal{F}$ under the action of $\mathcal{G}$.

Let I denote the set of primitive central idempotents of kF', and for each $d|n$ let I_d denote the subset of I comprising those e' for which ce' is a root of unity of order d in the field $kF'e$.

Now each element of I_d is the sum in kF' of $\mathcal{G}$-conjugates of some $f_i \in \mathcal{F}_d$; ξ^i is a primitive dth root of unity. Observe that every such f_i has the same coefficient set - namely the set of dth roots of unity in $\mathbb{C}$:-

$$f_i = \frac{1}{n}\sum_{j=0}^{n-1} \left((\xi^i c)\right)^j .$$

Then every element of I_d also has the same coefficient set, consisting of some rational multiples of elements of k of the form

$$Tr_{\mathbb{Q}(\xi)/\mathbb{Q}(\xi_d)\cap k}(\xi_d^j), \quad j = 1 \dots d. \tag{8.3}$$

where $\xi_d \in \mathbb{C}$ has order d.

It follows from the primitive element theorem that the set of coefficients appearing in any $e' \in I_d$ generates $\mathbb{Q}(\xi_d) \cap k$ over $\mathbb{Q}$, for let α be a primitive element for this extension. Then α can be written in the form

$$\alpha = a_{\phi(d)-1}\xi_d^{\phi(d)-1} + \dots + a_1\xi_d + a_0, \quad a_0, \dots, a_{\phi(d)-1} \in \mathbb{Q}.$$

Then $\sum_{\sigma\in\mathcal{G}} \alpha^\sigma$ is an integer multiple of α and clearly belongs to the field generated over $\mathbb{Q}$ by elements of the type of 8.3.

In general I is not a central subset of kF; conjugation by elements of F induces F-actions on $\mathcal{F}$ and I, under which the subsets $\mathcal{F}_d$ and I_d are stabilized for all divisors d of n. The action of F on $\mathcal{F}$ can be described in terms of the Galois group of $\mathbb{Q}(\xi)$ over $\mathbb{Q}$ in the following sense : X of course centralizes F' and hence $\mathcal{F}$, so every F-conjugate of $f \in \mathcal{F}$ has the form f^{Y^a} for some $a \in \mathbb{Z}$. Now $c^Y = c^r$ and so for $i = 1 \ldots n-1$ we have

$$f_i^Y = \frac{1}{n}(\widehat{\xi^i c})^Y = \frac{1}{n}(\widehat{\xi^i c^r}).$$

Certainly c^r has order n in F since $\gcd(r, n) = 1$. Let r' be an inverse for r in $\mathbb{Z}_n$, so $c^{rr'} = c$. Then the coefficient of c in f_i^Y is $\frac{1}{n}\xi^{ir'}$, and

$$f_i^Y = \frac{1}{n}(\widehat{\xi^{ir'} c}) = f_{ir'}.$$

Since $\gcd(r', n) = 1$, $\xi^{ir'}$ has the same order as ξ^i.

Let ρ denote the automorphism of $\mathbb{Q}(\xi)$ defined by $\xi^\rho = \xi^{r'}$. Then ρ extends in the usual way to a $\mathbb{Q}$-algebra automorphism of $\mathbb{Q}(\xi)F'$, under which

$$f_i^Y = f_i^\rho, \quad \forall f_i \in \mathcal{F}.$$

The same applies to elements of I, since each of them is a sum of elements of $\mathcal{F}$. It is easily seen that the subsets $\mathcal{F}_d$ and I_d of $\mathcal{F}$ and I respectively are stabilized by $\langle\rho\rangle$, for each divisor d of n.

Each primitive central idempotent of kF is the sum of all elements of a $\langle\rho\rangle$-orbit of I. Fix $d|n$. Then all elements of I_d have the same coefficient set, and so all have the same number of $\langle\rho\rangle$-conjugates. Since this coefficient set generates $\mathbb{Q}(\xi_d) \cap k$ as a field over $\mathbb{Q}$, the number of conjugates of $e' \in I_d$ is the order of the restriction of ρ to $\mathbb{Q}(\xi_d) \cap k$. For example I_d is central in kF if and only if the fixed field of ρ contains $\mathbb{Q}(\xi_d) \cap k$. Let E denote this fixed field :-

$$E = \{x \in \mathbb{Q}(\xi) : x^\rho = x\}.$$

In general the number of elements in a $\langle\rho\rangle$-orbit of I_d is $[\mathbb{Q}(\xi_d) \cap k : \mathbb{Q}(\xi_d) \cap k \cap E]$. The primitive central idempotents of kF corresponding to components upon which the projection of F' has order d are of course the sums of the elements of these orbits. The number of simple components of KF of this type is

$$\frac{[\mathbb{Q}(\xi_d) \cap k : \mathbb{Q}]}{[\mathbb{Q}(\xi_d) \cap k : \mathbb{Q}(\xi_d) \cap k \cap E]} = [\mathbb{Q}(\xi_d) \cap k \cap E : \mathbb{Q}],$$

obviously a divisor of $\phi(d)$. In the case where $k = \mathbb{Q}$ this is 1; in the case where $\xi_d \in k$ it is $[\mathbb{Q}(\xi_d) \cap E : \mathbb{Q}]$. We will denote by $\mathcal{I}$ the full set of primitive central idempotents of kF, and by $\mathcal{I}_d$ the subset of $\mathcal{I}$ consisting of those elements e for which $F'e$ has order d, $d|n$. The coefficient set of any element of $\mathcal{I}_d$ of course generates $\mathbb{Q}(\xi_d) \cap E$ as a field extension of $\mathbb{Q}$.

8.3 Cyclic Algebras in KF

Throughout this section let $e_d \in \mathcal{I}_d$. We will investigate the simple algebra KFe_d in the context of the discussion in Chapter 4. If $kF'e_d$ is not simple, let e_{1d} be a centrally primitive idempotent of kF' for which $e_{1d}e_d = e_{1d}$. Then

$$KFe_d \cong M_{l_d}(KF_1^d e_{1d}),$$

where $F_1^d = C_F(e_{1d})$ and $[F : F_1] = l_d$ is the number of conjugates of e_{1d} under the action of F. From the description in Section 8.2 of the primitive idempotents of kF we have

$$l_d = [\mathbb{Q}(\xi_d) \cap k : \mathbb{Q}(\xi_d) \cap k \cap E].$$

Since X centralizes c and hence kF', we observe that $F_1^d = \langle X, Y^{l_d}, F' \rangle$. For the remainder of this section we assume that e_d behaves as the multiplicative identity element, and by reference to an element, subset, or subgroup of F_1^d we shall understand its projection on the simple ring $A_1^d := KF_1^d e_d$.

Now $kF'e_{1d}$ is a field isomorphic to $k(\xi_d)$ and so $F_0^d := C_F(\mathcal{Z}(kF'))$ is just the centralizer in F_1^d of c. The element Y^i centralizes c in A_1^d if and only if

$$Y^{-i}cY^i = c^{r^i} = c,$$

i.e. if and only if $d | r^i - 1$. Let $b = \mathrm{ord}_d(r)$. Then $F_0^d = \langle X, Y^b, c \rangle$, and F_1^d / F_0^d, which is cyclic of order b/l_d, is isomorphic to the Galois group of the field extension $k(\xi_d)/k(\xi_d) \cap E$, since

$$\mathcal{Z}(A_1^d) \cap kF'e_{1d} \cong k(\xi_d) \cap E.$$

Let A_0^d be the centralizer in A_1^d of $kF'e_d$. By Theorem 4.3.2 we know that

$$A_0^d = B \otimes_L C,$$

where B is the simple algebra generated by F' over the centre L of A_0^d, and $C = C_{A_0^d}(B)$. In this case $B = L$, since F' is central in A_0^d. Then $F'^+ := F_0 \cap B$ is just the centre of F_0. It follows from the centrality of c in F_0 that F'^+ is generated by c, $\langle X \rangle \cap \mathcal{Z}(F_0)$, and $\langle Y^b \rangle \cap \mathcal{Z}(F_0)$. We need to find generators for each of the latter cyclic groups : suppose X^i generates $\langle X \rangle \cap \mathcal{Z}(F_0)$. Then

$$Y^{-b}X^iY^b = c^{i\alpha(b)}X^i = X^i \Longrightarrow d | i\alpha(b).$$

On the other hand, suppose $(Y^b)^i \in \mathcal{Z}(F_0)$. Then Y^{bi} centralizes X, and

$$y^{-bi}XY^{bi} = c^{\alpha(ib)}X = X \Longrightarrow d|\alpha(ib).$$

Claim 8.3.1 *$d|\alpha(ib)$ if and only if $d|i\alpha(b)$.*

Proof Let $g = d/\gcd(d, \alpha(b))$. Note that $b = \mathrm{ord}_d(r)$ so $d|r^b - 1$, $d|\alpha(b)(r-1)$. Then $g|r-1$, and $r \equiv 1 \bmod g$. Also note that

$$\alpha(ib) = (1 + r^b + r^{2b} + \cdots + r^{(i-1)b})\alpha(b)$$

- this is immediate from the definition of $\alpha(ib)$.

$$\begin{aligned} d|\alpha(ib) &\iff d|(1 + r^b + r^{2b} + \cdots + r^{(i-1)b})\alpha(b) \\ &\iff g|1 + r^b + \cdots + r^{(i-1)b} \\ &\iff g|i. \end{aligned}$$

This proves the claim, since g divides i if and only if $g\gcd(d, \alpha(b))$ divides $i\gcd(d, \alpha(b))$, i.e. if and only if d divides $i\alpha(b)$. □

Thus $\langle X\rangle \cap B = \langle X^g\rangle$, and $\langle Y^b\rangle = \langle (Y^b)^g\rangle$, where $g = d/\gcd(d, \alpha(b))$. We observe that $gb = \mathrm{ord}_{dj_d}(r)$ where $j_d = \gcd(d, r-1)$, since $d|\alpha(gb) \iff dj_d|(r^{gb} - 1)$.

Then X^g and Y^{gb} generate L as a purely transcendental field extension of $kF'e_{1d}$, which is isomorphic to $k(\xi_d)$. Now $B = L = \mathcal{Z}(A_0)$ and so $C = C_{A_0}(B) = A_0$. Then A_0 is a symbol algebra of degree g over L :-

$$A_0 = \left(\frac{X^g, (Y^b)^g}{\zeta, L}\right).$$

Here $\zeta = [Y^b, X]$ is a primitive root of unity of order g in L.

Let Z denote the centre of A_1, a subfield of L. Then, by Lemma 4.2.2, the Galois group of L/Z is cyclic of order b/l_d, generated by the automorphism σ defined as the restriction to L of

$$\tilde{\sigma} : A_1 \longrightarrow A_1$$

$$\theta^{\tilde{\sigma}} := \theta^{Y^{l_d}} \tag{8.4}$$

That $\tilde{\sigma}$ restricts to the identity mapping on Z is clear, since $Y^{l_d} \in A_1$. The central simple Z-algebra A_1 is cyclic of degree $d' = gb/l_d$.

$$A_1 = \left(L(X)/Z, \sigma, (Y^{l_d})^{d'}\right)$$

A_1 is a symbol algebra if and only if the dth roots of unity in $kF'e_{1d}$ are centralized by F_1. In this case $F_1 = F_0$ and $A_1 = A_0$. The field Z is purely transcendental of transcendence degree 2 over

$$Z \cap kF'e_{1d} \cong k(\xi_d) \cap E.$$

Now $L = kF'(X^g, Y^{bg})$, and $Y^{bg} \in Z$ since it is centralized by X. By Lemma 4.3.2, there exists some $c^i \in F'$ for which the sum of F_1-conjugates of $X^g c^i$ has nonzero projection on $\langle e_{1d}\rangle$; thus we obtain an element $C_{X^g} = X^g c'$ of $Z : c' \in kF'$. Finally, Z is generated as a field over $Z \cap kF'$ by the set $\{C_{X^g}, Y^{gb}\}$. To see this note that L is generated over $(Z \cap kF')\langle C_{X^g}, Y^{gb}\rangle$ by c, and that c is a root of a polynomial of degree b/l_d over $Z \cap kF'$.

8.4 Irreducible k-Representations

The construction of an irreducible projective k-representation T of G belonging to e_d now entails the determination of images under a lift $\tilde{T}$ of T to F for the elements C_{X^g} and Y^{gb}, which generate Z over $kF' \cap Z$ in KF_1e_1. The images of C_{X_g} and Y^{gb} need not belong to k, but are certainly algebraic over k; for example since $\langle Y\rangle \cap R = \langle Y^s\rangle$, $gb|s$ and the image of Y^{gb} under $\tilde{T}$ satisfies $\left(\tilde{T}(Y^{gb})\right)^{s/gb} \in k^\times$. Similarly $\left(\tilde{T}(X^g)\right)^{m/g} \in k^\times$, and $C_{X^g} = X^g c'$ where $c' \in kF'$ and the image of c' is determined (up to a choice of basis) by e_{1d}.

Let A^T be the k-algebra generated by $\{T(g), g \in G\}$, or $\{\tilde{T}(x), x \in F\}$: then A^T is a central simple algebra over a field Z^T which is a finite extension of k, and by Theorem 5.1.2 $Z^T \cap \tilde{T}(kF') \cong k(\chi)$ where χ is a sum of the F-conjugates of any absolutely irreducible (linear) character of F' appearing in $\tilde{T}|_{F'}$. From the results of Section 5.3 we know that $A_0^T := \tilde{T}(kF_0)$ is a direct sum of simple components each of which is a symbol algebra of degree g over a field which is generated by Z^T and a copy of $kF'e_{1d}$. The number of such components depends upon the field Z^T and in particular on the tensor product $Z^T \otimes_{Z^T \cap \tilde{T}(kF')} \tilde{T}(kF')$. If $I^T \subseteq \tilde{T}(F_1)$ is the stabilizer of the simple component A_{0+}^T of A_0^T under the conjugation action of $\tilde{T}(F_1)$ on A_0^T, then the subalgebra A_I^T of A^T generated by I over A_{0+}^T is simple, and is a symbol algebra of degree gb/l_d over a field Z_+^T which is isomorphic to Z^T. In this situation A^T is isomorphic to a ring of matrices of degree $[F : I]$ over A_I^T, where $I = \tilde{T}^{-1}(I^T)$.

We are interested in particular in *absolutely* irreducible projective representations of G, i.e. representations which remain irreducible when regarded as maps into general linear groups over

$\mathbb{C}$. The representation T described above is absolutely irreducible if and only if A^T is a full matrix ring over k: this requires firstly that $Z^T = k$, so the images P and Q respectively of C_{X^g} and Y^{gb} are elements of $k^\times$. Also, by Theorem 5.1.2, we must have $k = k(\chi)$, whenever χ is the sum of the F-conjugates of an absolutely irreducible character of F' appearing in $\tilde{T}|_{F'}$. Finally, in order for T to be absolutely irreducible we require that $\text{ind}(A^T) = \text{ind}(A_1^T) = 1$.

The stipulation that $Z^T = k$ of course means that A_0^T is a simple ring and much of the complexity of Section 5.3 is avoided. In this case

$$A_0^T \cong \left(\frac{P(c'^T)^{-1}, Q}{\zeta, L^T}\right),$$

where $c'^T = \tilde{T}(c')$, and $L^T = \tilde{T}(kF')$. Furthermore if $(Y^{l_d})^T$ denotes the image under $\tilde{T}$ of Y^{l_d}, then

$$A_1^T = A_0^T\left((Y^{l_d})^T\right) \cong (k(X^T)/k, \sigma^T, Q)$$

is a cyclic algebra of degree $d' = gb/l_d$ over k. Here $X^T = \tilde{T}(X)$ and σ^T is the automorphism induced in A_1^T by the automorphism σ of kF defined in 8.4 as conjugation by Y^{l_d}. Now $A_1^T \cong M_{d'}(k)$ if and only if $Q = N_{k(X^T)/k}(\alpha)$ for some $\alpha \in k(X^T)$. This can easily be arranged by the choice of P and Q in $k^\times$: for example we may choose $Q \in (k^\times)^{d'}$. Then $A^T \cong M_{gb}(k)$ and T is an absolutely irreducible projective representation of G.

Theorem 8.4.1 *Let G be the metacyclic group of 8.1. and let $n = |G'||M(G)|$. Let k be a subfield of the field $\mathbb{C}$ of complex numbers, and let $\xi \in \mathbb{C}$ be a primitive nth root of unity. Then k is a projective splitting field for G if and only if k contains the fixed field of $\mathbb{Q}(\xi)$ under the automorphism σ which sends ξ to ξ^r.*

Proof : Suppose that $k \subseteq \mathbb{C}$ is a projective splitting field for G. Then by Theorem 7.1.1 k contains $\mathbb{Q}(\xi)^\sigma$, since this is precisely the field obtained from $\mathbb{Q}$ by adjoining all F-invariant character values of F', where F as usual is a generic central extension for G.

On the other hand, suppose $k \supseteq \mathbb{Q}(\xi)^\sigma$. Then kF and $\mathbb{C}F$ have the same set $\mathcal{I}$ of primitive central idempotents. Let $e \in \mathcal{I}$, and let χ denote the F-invariant character of F' corresponding to e. Then $k = k(\chi)$, and as above we may construct k-representations of F which behave as lifts of absolutely ireducible projective representations of G belonging to e. The result is then a consequence of Theorem 6.2.1. □

Example : Let G, k and σ be as above. Then $\mathbb{Q}$ is a projective splitting field for G if and only if σ generates the full Galois group of $\mathbb{Q}(\xi)/\mathbb{Q}$. This Galois group, which is of course isomorphic to $\mathcal{U}(\mathbb{Z}_n)$, is cyclic if and only if $n = p^a$ or $2p^a$ for an odd prime p, or if $n = 2$ or 4. The metacyclic groups for which $\mathbb{Q}$ is a projective splitting field are described by the following theorem.

Theorem 8.4.2 *Suppose $\mathbb{Q}$ is a projective splitting field for the metacyclic group G of 8.1. Then one of the following holds :-*

i) $n = p^a$ or $2p^a$, where p is an odd prime, $a > 0$, and $\text{ord}_r(n) = \phi(n)$.

ii) n=4 and $|G'| = 4$, $M(G)$ trivial, $r \equiv 3 \bmod 4$.

iii) n=4 and $|G'| = 2$, $M(G) \cong C_2$, $r \equiv 3 \bmod 4$.

iv) n=2 and $|G'| = 2$, $M(G)$ trivial.

v) n=2 and $|G'| = 1$, $M(G) \cong C_2$: $G \cong C_2 \times C_l$, $2|l$.

vi) G is cyclic.

8.5 Faithful Projective Representations

Let $T : G \longrightarrow GL(n, k)$ be a projective representation of the finite metacyclic group G over a field k. We recall from section 1.1 that the *kernel* of T is defined as the kernel of the group homomorphism

$$\bar{T} = \pi \circ T : G \longrightarrow PGL(n, k).$$

i.e. $\ker(T) = \{g \in G : T(g) \in k^\times\}$. The representation T is said to be *faithful* if $\ker(T) = \{1\}$; in this case T embeds G in the projective general linear group over k. We will determine the metacyclic groups which have faithful absolutely irreducible projective representations, and the smallest fields over which these representations can be realized. A related question asks which metacyclic groups have central simple twisted group rings over a given field k. We will give an answer to this question also.

Lemma 8.5.1 *Suppose the metacyclic group of 8.1 has a faithful absolutely irreducible representation T over the field $k \subseteq \mathbb{C}$. Then $\gcd(t, \alpha(s)) = m$.*

Proof : Let $\tilde{T}$ be a lift of T to F, extending to a surjective ring homomorphism $\tilde{T} : kF \longrightarrow M_l(k)$. Since T is absolutely irreducible, $\tilde{T}$ sends the centre of kF into k. Let $\langle e \rangle$ be the component of kF to which T belongs (e as usual being a primitive central idempotent of kF) and let d be the order of the group $F'e$. Then $X^d \in \mathcal{Z}(kFe)$ since $Y^{-1}X^dY = c^dX^d$. Thus $\tilde{T}(X^d) \in k^\times$ and $T(x^d) \in k^\times$, so $x^d \in \ker T$. Certainly $d|m$ since $\gcd(t, \alpha(s)) = |F'|$ divides m (as $t|m$). Then d must be equal to m since T is faithful. □

We observe that the condition $m|t$ in Lemma 8.5.1 implies immediately that $y^s = 1$, i.e. G is a semidirect product of $\langle x \rangle$ by $\langle y \rangle$.

Now if $A^T = \tilde{T}(kF)$ where $\tilde{T}$ is a lift to F of the representation T of Lemma 8.5.1, then from Section 5.3 we know that $\tilde{T}(kFe) \cong M_l(k)$ is isomorphic to a ring of $l_m \times l_m$ matrices over the cyclic algebra

$$A_1^T = \left(k(X^T)/k, \sigma, (Y^T)^{\mathrm{ord}_{mj}(r)}\right),$$

where X^T and Y^T denote respectively the images of X and Y under $\tilde{T}$. The degree of A_1^T is $\mathrm{ord}_{mj}(r)/l_m$, and since

$$\langle Y \rangle \cap \mathcal{Z}(F) = \langle Y^{\mathrm{ord}_{mj}(r)} \rangle$$

it follows that

$$\langle y \rangle \cap \ker(T) \supseteq \langle y^{\mathrm{ord}_{mj}}(r) \rangle.$$

The reverse inclusion also holds, since $\tilde{T}$ embeds F' in $M_l(k)$ as $|F'e| = |F'|$. Then if $[Y^i, X] \neq 1$ for some i, $\tilde{T}(Y^i) \notin k$. Thus $\tilde{T}$ sends no element of $\langle Y \rangle$ which is not central in F into k, and

$$\langle y \rangle \cap \ker(T) = \langle y^{\mathrm{ord}_{mj}}(r) \rangle.$$

Since T is faithful, we conclude that $s = \mathrm{ord}_{mj}(r)$.

Certainly $\mathrm{ord}_{mj}(r)|s$ as

$$\begin{aligned} m = \gcd(\alpha(s), m) &\implies m|\alpha(s) \\ &\implies mj|\alpha(s)(r-1) \\ &\implies mj|r^s - 1 \\ &\implies \mathrm{ord}_{mj}(r)|s. \end{aligned}$$

Of course this is not true for all metacyclic groups; it uses the condition $m = n = |G'||\mathrm{M}(G)|$. We require that s be minimal with the property that $m|\alpha(s)$ in order to ensure that the projective representation T of G be faithful. This condition is sufficient, for suppose now that

$$\tilde{T}(X^iY^l) \in k^\times.$$

Then, since the action of $\langle Y \rangle$ on X survives under $\tilde{T}$, Y must centralize X^i in Fe. Then $m|i$ and $\tilde{T}(X^i) \in k^\times$, hence $\tilde{T}(Y^l) \in k^\times$ also, so $s|l$, and $x^iy^l = 1$ in G.

Then T is a faithful projective representation of G, of degree $s = \text{ord}_{mj}(r)$, where $j = \gcd(m, r-1)$. We summarize these results in the following theorem :

Theorem 8.5.1 *The metacyclic group G of 8.1 has faithful absolutely irreducible representations if and only if the following conditions hold :-*

i) $m|t$; $G = \langle x \rangle \rtimes \langle y \rangle$.

ii) $m|\alpha(s)$, and s is minimal with this property. □

This theorem is originally due to H.N. Ng -see [12]. The next corollary is an immediate consequence of Theorem 8.5.1, for suppose that for some $f \in Z^2(G, \mathbb{C}^\times)$ the twisted group ring $\mathbb{C}^f(G)$ is central simple of degree s over $\mathbb{C}$. Then $|G| = \dim_{\mathbb{C}}(\mathbb{C}^f(G)) = s^2$ and the isomorphism

$$\mathbb{C}^f(G) \xrightarrow{\cong} M_s(\mathbb{C})$$

restricts to a faithful absolutely irreducible f-representation of G of degree s.

Corollary 8.5.1 *The metacyclic group G of 8.1 has a central simple twisted group algebra over $\mathbb{C}$ if and only if G satisfies both conditions of Theorem 8.5.1 and in addition $m = s$.* □

If G is a metacyclic group satisfying the conditions of Corollary 8.5.1, we have $m = s$, $j = \gcd(s, r-1)$, and $\text{ord}_{sj}(r) = s$, whence $s|\phi(sj)$. The order of G' is s/j, and the order of $M(G)$ is j.

Now let G be the metacylic group with presentation

$$\langle x, y | x^s = 1, y^s = 1, y^{-1}xy = x^r \rangle,$$

and suppose k is a subfield of $\mathbb{C}$ over which G has a central simple twisted group ring. Let T be the associated faithful irreducible projective k-representation of G, and let $f \in Z^2(G, k^\times)$

be the associated cocycle. The degree of T is $s' = s_T s$, where s_T denotes the Schur index of T, which is equal to the Schur index of the simple algebra $k^f G$. Let $\tilde{T} : F \longrightarrow GL(s', k)$ be a lift of T to F, extending to a ring homomorphism $\tilde{T} : kF \longrightarrow M_{s'}(k)$. Then, by Theorem 5.1.2, the centre of $\tilde{T}(kF)$ contains the values assumed by the sum of the F-conjugates of any absolutely irreducible character of F' appearing in $\tilde{T}|_{F'}$. Then the idempotent e of kF to which T belongs remains primitive in $\mathbb{C}F$.

Now if ξ_s is a primitive sth root of unity in $\mathbb{C}$, k contains the fixed field of $\mathbb{Q}(\xi_s)$ under the automorphism sending ξ_s to ξ_s^r, which is generated over $\mathbb{Q}$ by the coefficients appearing in e. Then k is a projective splitting field for G by Theorem 8.4.1. We obtain the following result :-

Theorem 8.5.2 *If G is a metacyclic group possessing central simple twisted group algebras over a field k, then k is a projective splitting field for G.* □

In the above setting KFe is a cyclic algebra of degree $s = \mathrm{ord}_{sj}(r)$ over its centre Z. Moreover, KFe is a ring of matrices over the central simple Z-algebra A_1 generated over Z by X and Y^{l_s}, where

$$l_s = [\mathbb{Q}(\xi_s) \cap k : \mathbb{Q}(\xi_s)^{\sigma}],$$

A_1 is a cyclic algebra of degree s/l_s given by

$$A_1 = (Z(X)/Z, \sigma, Y^s) :$$

σ is as usual defined as conjugation by Y. Now if $\tilde{T}$ is a lift to kF of an irreducible projective k-representation T of G belonging to e, we have

$$\tilde{T}(kF) \cong M_{l_s}\left(\underbrace{\left(k(\sqrt[s]{P})/k, \sigma^T, Q\right)}_{A_1^T}\right),$$

where $P, Q \in k^\times$ satisfy $\tilde{T}(X^s) = P$, $\tilde{T}(Y^s) = \tilde{T}((Y^l)^{s/l_s}) = Q$. The degree of the k-algebra $\tilde{T}(kF)$ is s; its index is $\mathrm{ind}(A_1^T)$, a divisor of s/l_s. By different choices of P and Q we can arrange for $\mathrm{ind}(\tilde{T}(kF))$ to be *any* divisor of s/l_s. In particular choosing

$$Q \in \mathrm{N}_{k(\sqrt[s]{P})/k}\left(k(\sqrt[s]{A})^\times\right)$$

will yield an irreducible k-representation T_1 of G for which $\{T_1(g), g \in G\}$ generates the ring of $s \times s$ matrices over k. The representation T_1 of G is of course absolutely irreducible and corresponds to a twisted group algebra of G over k which is isomorphic to $M_s(k)$. Also, if d divides s/l, G has a central simple twisted group algebra over k which is isomorphic to $M_{s/d}(D)$ where D is some central k-division algebra of degree d. We have proved the following theorem.

Theorem 8.5.3 *Let G be a metacyclic group and let k be a field of characteristic 0. Then if G has a central simple twisted k-group algebra of index s, it also has central simple twisted k-group algebras of index d, for any divisor d of s.* □

Chapter 9

Conclusion

We conclude with some general remarks, and by mentioning some possibilities for further work, arising from or suggested by the material included in this thesis.

The study in Chapter 8 of the projective representations of metacyclic groups was obviously expedited by the fact that a generic central extension of a metacyclic group has cyclic commutator subgroup. In particular the fact that kF' is a direct sum of fields when G is a metacyclic group facilitates the search for absolutely irreducible representations and splitting fields, since no complications arise from a requirement to split division algebras appearing in kF'. It is likely that the approach of Chapter 8 may be extended to yield specific information on the projective representations of a broader class of groups, perhaps some or all finite groups having metabelian generic central extensions. This class does not include all metabelian groups; however it does include all groups which are nilpotent of class 2. If F is a generic central extension for a group G which is nilpotent of class 2, then $\gamma_4(F)$ is trivial, whence F' is abelian (see [17]). The class also includes all abelian groups, whose generic central extensions are nilpotent of class 2, and whose projective representations have been extensively studied.

Although there is very little explicit reference to cocycles, and cocycle computations are avoided completely in the approach taken here to the study of projective representations, it is perhaps worth mentioning the implicit role of a particular cocycle, namely the one which is defined by Lemma 3.1.1. The simple K-algebras which are investigated in Chapter 4 arise as simple components of twisted group rings of (finite) covering groups for G, not over k but over purely transcendental extensions K of k. These projective K-representations of $\hat{G} = F/S$ might be described as "generic" projective k-representations of G, since all projective k-representations of G arise from "specializing" elements of certain transcendence bases for K/k to values in k. Over algebraically closed fields, Theorem 2.1.1 relates *linear* representations of $\hat{G}$ to projective representations of G; over more general fields, Lemma 3.1.1 relates certain *projective* representations of $\hat{G}$ to projective representations of G. It is easily seen that these representations are in general genuinely projective - the cocycle in $Z^2(\hat{G}, K^\times)$ determined by Lemma 3.1.1 can be a coboundary only in a case where G is perfect.

It may be possible to improve some of the results of Chapter 7 on projective splitting fields. For example it would be of interest to know under what conditions on the group G a field k satisfying the first conclusion of Theorem 7.1.1 is a projective splitting field for G. This is certainly not always the case : however in constructing k-representations of F arising as lifts of projective k-representations of G, and belonging to a particular component of kF, we have some freedom in choosing images for certain central elements of kF which are transcendental over k. It may be possible, perhaps under some assumptions on G, to investigate the existence of choices which might split not only the symbol algebras appearing in the image of kF'^+, but also any division algebras appearing in kF' as well as the further cyclic extensions which arise from the action of F_1/F_0 as a Galois group, as described in Chapter 5. It would also be of interest to know for which groups the converse of Theorem 7.2.1 is true.

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