University of Alberta

Coupled Modeling of Multiphase Flow in Reservoir and Horizontal Well

by

Ping Gui ©

A thesis submitted to the Faculty of Graduate Studies and Research
in partial fulfillment of the requirements for the degree of

Master of Science
in
Petroleum Engineering

Department of Civil and Environmental Engineering

Edmonton, Alberta
Spring 2008

Your file Votre référence
ISBN: 978-0-494-45918-8
Our file Notre référence
ISBN: 978-0-494-45918-8

ABSTRACT

Simulation of oil and gas flow in horizontal wells is much more complex than that for vertical wells. Traditional reservoir simulation usually treats horizontal wells as the simplified source/sink term, which could lead to erroneous performance predictions. Therefore, horizontal wellbore hydraulics and radial influx through perforations should be modeled as part of the reservoir simulation.

This research developed a 3-D, multiphase, fully implicit reservoir/wellbore coupled model with hybrid grid technique. Using this coupled model, the flow behavior in both reservoir and horizontal wellbore can be simulated simultaneously to reflect the flow characteristics and interactions between the reservoir and the horizontal wellbore.

The coupled model is used to study the transient flow behavior in the horizontal wellbore. Simulation results reveal finite conductivity and non-uniform influx distribution features in horizontal wellbore. Sensitivity analyses of reservoir permeability, initial gas saturation and perforation distribution are conducted using the developed reservoir and wellbore coupled model.

ACKNOWLEDGEMENT

I would like to express my gratitude to my supervisors Dr. Luciane B. Cunha and Dr. Jose Carlos (JC) Cunha for their valuable guidance during my study at the University of Alberta. This thesis would not have been possible without their constant support and encouragement throughout my study and research.

I would like to acknowledge the Natural Sciences and Engineering Research Council of Canada and Department of Civil and Environmental Engineering for providing financial support for this research and my study at the University of Alberta.

Acknowledgement is also due to Dr. Ronaldo Vicente for his valuable discussions in this work. I also wish to extend my gratitude to my professors and colleagues from the School of Mining and Petroleum Engineering at the Department of Civil and Environmental Engineering, University of Alberta. I want to thank them for all their helpful teaching and discussions.

Especially, I would like to give my special thanks to my wife, daughter and parents, for their love, encouragement and understanding throughout the years.

CONTENTS

LIST OF TABLES

LIST OF FIGURES

NOMENCLATURE

NOTATION

Symbol	Designates
A_x, A_y, A_z	Section area, m^2
B	Formation volume factor, m^3/std.m^3
C_r	Rock compressibility, 1/Pa
d	Inner diameter of the wellbore, m
$\vec{F}$	Residual function vector
f	Moody's friction factor
H_o	Oil hold-up in wellbore
J	Jacobian matrix
K	Absolute permeability, millidarcy (md)
k_{rg}	Gas relative permeability
k_{ro}	Oil relative permeability
nx, ny, nz	Number of gridblocks in x, y and z direction
P_{ref}	Reference pressure at a reference depth (Z_{ref}), Pa
$P_{c_{go}}$	Capillary pressure between oil and gas, Pa
q_{gsc}, q_{osc}	Oil and gas influx from reservoir, m^3/day at standard condition
$r_{i,j,k}$	Radius of the grid block center, m
$r_{i\pm1/2,j,k}$	Radius of the grid block interfaces, m
R_s	Solution gas/oil ratio, std m^3/std m^3
S_w, S_g, S_o	Water, gas and oil saturation, %
t	Time, seconds
T	Transmissibility, m^3/(Pa • s)
$\upsilon_r, \upsilon_\theta, \upsilon_z$	Volumetric velocities through perforations, m^3/s
V	Control volume of the element, m^3
v	Velocity of the fluid flow in wellbore, m/s
$\vec{X}$	Vector of unknowns
$\Delta x, \Delta y, \Delta z$	Grid block length in x, y and z direction, m
x_e, y_e, z_e	Reservoir limit in x, y and z direction, m
z_{base}	Depth of the reservoir base, m
z_{ref}	Reference depth, m

NOTATION (Continued…)

Symbol	Designates
Δzz	Length of the wellbore segment, m
Ze	Formation thickness, m
ε	Tolerance for computation convergence
μ	Viscosity, Pa • s
ρ_{osc}, ρ_{gsc}	Oil and gas densities at standard condition, kg/m^3
ϕ	Porosity, %
$\bar{\gamma}$	Specific gravity

SUBSCRIPTS

Symbol	Designates
o, g, w	Oil, gas and water
m	Fluid mixture
x, y ,z	Cartesian directions
r, θ, zz	Radial directions
Sc.	Standard conditions
Sp.	Specified

SUPERSCRIPTS

Symbol	Designates
n	Current time step
n+1	Next time step
v	Current iteration
v+1	Next iteration

CHAPTER 1: INTRODUCTION

Horizontal wells are widely used in many new oil and gas fields as well as in further development of mature fields. With horizontal drilling oil companies can achieve better economic production and increase recovery from old producing fields. Additionally, horizontal wells have been used to speed up recovery and improve the rate of return, which is very important to the project's economic feasibility. Not surprisingly horizontal drilling is becoming more and more popular in the world. Horizontal wells can extend to thousands of feet, making a lengthy exposure of the reservoir accordingly. (See Table 1.1 for the top 10 extended-reach drilling wells in the world)

Table 1.1 - Top 10 extended reach drilling wells in the world*

Rank	Horizontal Displacement (meter)	Measured Depth (meter)	TVD (meter)	Operator	Well	Field	Area
1	10,728	11,278	1,637	BP	M-16Z	Wytch Farm	UK Land
2	10,585	11,184	1,657	TotalFinaElf	CN-1	Ara	Argentina
3	10,114	10,658	1,605	BP	M-11Y	Wytch Farm	UK Land
4	10,089	11,134	2,600	ExxonMobil	Z2 (EM)	Chayvo	Sakhalin
5	10,082	10,917	2,600	ExxonMobil	Z7 (EM)	Chayvo	Sakhalin
6	9,771	10,994	2,600	ExxonMobil	Z1 (EM)	Chayvo	Sakhalin
7	9,736	10,675	2,600	ExxonMobil	Z3 (EM)	Chayvo	Sakhalin
8	9,533	10,522	2,538	ExxonMobil	ZG2 (EM)	Chayvo	Sakhalin
9	9,509	10,536	2,537	ExxonMobil	ZG1 (EM)	Chayvo	Sakhalin
10	9,243	10,183	2,612	ExxonMobil	Z4 (EM)	Chayvo	Sakhalin

**Data from Schlumberger, Status: Sept. 2006*

The success of horizontal wells can be attributed to their advantage over vertical wells, which includes increased well productivity, reduction of water coning and improved overall cost-effectiveness of reservoir development (Babu and Odeh, 1989). Horizontal wells can increase the productivity by enhancing reservoir contact area particularly for reservoirs with low permeability, high anisotropy, thin layers and naturally fractured. Because of the smaller drawdown, horizontal wells are also advantageous for reservoirs with potential water or gas coning problems.

Because of the aforementioned facts, oil industry and research institutes put a considerable effort on related research, including developing simulation tools to study the flow behavior and the production performance in the horizontal wellbore, as well as its influence on the reservoir (Pedrosa and Aziz, 1985; Ertekin et al., 2001). However, these models either assumed simplified source/sink term for wellbore or treated the wellbore flow as a regular pipe flow assuming no mass transfer through the perforation, which usually led to erroneous predictions of performance behavior. The horizontal wellbore hydraulics and radial influx through perforations should be modeled as part of the reservoir simulation. Only the use of a coupled model (reservoir fluid flow and horizontal wellbore hydraulics) can describe the flow interaction between the reservoir and wellbore and hence reveal the actual flow behavior in the reservoir and wellbore.

Proper and accurate simulation of the performance of horizontal wells is much more complex than that for vertical wells (Dikken, 1990; Ozkan et al., 1992 and 1993; Novy, 1995). Depending on the completion method used in a horizontal well, the radial influx could change the flow behavior along the wellbore and complicate the modeling of the problem (Vicente et al., 2001a and 2001b). For gas-liquid wellbore flow, the mass transfer between reservoir and wellbore alters gas and liquid flow rates along the wellbore. Fluid flow in the horizontal wellbore is different from the flow in a pipe because of the mass transfer between the wellbore and reservoir (Ouyang and Aziz, 1998). The influx amount of fluid transfer between the reservoir and the wellbore will introduce different flow patterns in the horizontal wellbore. In return, the pressure drop and liquid fraction along the wellbore affect the reservoir pressure distribution and consequently the mass transfer between the reservoir and wellbore. Neglecting the interaction between the reservoir and wellbore can lead to incorrect predictions of performance behavior, especially for high-permeability and high production field cases (Seines et al., 1990; Gui and Cunha, 2006). Potential problems include, among others, overestimation of well productivity and breakthrough times. Unless the wellbore pressure drop is very small and can be neglected, the reservoir model and the wellbore model must be solved simultaneously to precisely predict production behavior of horizontal wells (Gui et al., 2007).

To understand the flow behavior in horizontal wellbore and accurately predict its performance and interaction with the reservoir, this study developed a fully implicit, three-dimensional reservoir simulator coupled with multiphase flow in the horizontal wellbore. The built-in wellbore model is used to analyze the flow pattern, flow dynamics and the influence from reservoir. To take advantage of the radial nature of flow around wellbore, the developed coupled model considers a hybrid grid system (local grid refinement) with Cartesian grids for the reservoir region and cylindrical grids for the wellbore region (Pedrosa and Aziz, 1985). The coupled system needs to be solved simultaneously to ensure the interaction effect between horizontal wellbore and reservoir, and to meet certain computational stability requirements as well.

This coupled model has been validated or compared with different commercial models and research results. Based on an effective and rigorous coupled model, different numerical applications were conducted to study the influence of horizontal wellbore hydraulics and its interaction with the reservoir. These investigation and applications include finite conductivity analysis using the coupled model, transient pressure analysis and sensitivity analysis of perforation distribution, etc.

In this thesis, Chapter 2 reviews recent articles on reservoir simulation development that considered wellbore hydraulics. Chapter 3 presents the methodology for development of the coupled model. The flow model in reservoir domain is also developed in this chapter. In Chapter 4, a multiphase flow wellbore model is developed to determine pressure drop, liquid holdup and flow pattern along the wellbore. Chapter 5 describes the coupling process of integrating the reservoir model and wellbore model. Typical cases are studied to validate or compare the developed coupled model.

In Chapter 6, several applications have been studied using the reservoir/wellbore coupled model. The coupled model is first used to study the effect of finite conductivity and non-uniform influx distribution. Comparisons including influx distribution, flow rate distribution, drawdown and pressure distribution along the wellbore are made between the proposed coupled model and a non-coupled traditional model. Chapter 6 also presents the application of coupled model in transient pressure analysis. Sensitivity analysis have been conducted to study the effect of reservoir permeability and initial gas saturation on the transient flow behavior. The effect of perforation distribution on wellbore flow behavior is studied using the proposed coupled model at the end of Chapter 6.

Chapter 7 concludes this research work and further discusses possible extension work. To make this thesis report more readable, several appendices are incorporated at the end, including the detailed derivation of the reservoir and wellbore models.

CHAPTER 2: LITERATURE REVIEW

As mentioned in the introduction, researchers have been working on reservoir simulation and wellbore hydraulics simulation for many years. Recently, the interaction between reservoir fluid flow and wellbore fluid flow gradually attracted researchers' attention, due to the currently extensive application of horizontal wells. This chapter summarizes recent studies on reservoir/wellbore interaction and coupled flow models, which are reviewed and organized in chronological order.

Well test analysis is a typical application that involves both reservoir flow modeling and wellbore flow modeling. Winterfeld (1986) studied the pressure buildup process in a multiphase wellbore/reservoir system (vertical well). Wellbore flow is described by mass and momentum conservation, while reservoir inflow is explained by mass balance and Darcy's law. A fully implicit finite-difference scheme is used to discretize the wellbore flow and reservoir inflow equations. A single-phase example illustrates wellbore storage and wellbore fluid inertia and two-phase examples illustrate phase redistribution shortly after the shut-in. Almehaide et al. (1989) proposed a black-oil simulator for vertical injectors to study the interaction between wellbore multiphase flow and the effect of gravity segregation in the wellbore. The authors also used this coupled model to investigate the impact of wellbore phase segregation on pressure buildup response and the influence of multiphase flow during well testing processes.

Stone et al. (1989) developed a fully implicit, three-dimensional, thermal numerical model to simulate the fluid flow in the reservoir and the wellbore. The authors introduced a simple multiphase flow model for the wellbore with different flow regimes. This model had stability problems if the flow rate in the wellbore was very high. In addition, the flow regime also had stability problem during transition periods from stratified flow to bubble, slug or annular flow. In such cases, the well dynamics either took place on a much smaller time scale, or some time steps may have to be cut. Dikken (1990) developed a model to couple the well flow and reservoir flow, both of which were single phase turbulent flow and assumed steady state flow. The author pointed out that it was an approximate method because it treated reservoir flow in non-communicating sheets perpendicular to the horizontal section. The author used a parameter α to treat the effect of perpendicular inflow from reservoir to wellbore, instead of using the conventional pipe equations. The author defined inflow from the reservoir using the productivity index (PI) and assumed that PI was constant along the well. Dikken pointed out that turbulent horizontal wellbore flow may result in reduction in drawdown at positions far away from the wellbore toe. Additionally, the reduced drawdown may result in a decrease of the total production as a function of well length. The total production may become virtually constant for lengths exceeding a certain value.

Pressure loss in wellbore is one of the key points in the reservoir/wellbore coupling. Among various pressure losses in wellbore, the friction loss is the most apparent and has been widely studied. Seines et al. (1990) outlined a method for implementing friction loss in the horizontal well for simulation studies. The authors pointed out that friction pressure loss can be important for highly productive reservoirs with limited pressure drawdown availability. The study showed that proper length of the horizontal wellbore should be determined by considering the total well life and expected flow rates. A non-slip homogeneous model is employed to determine the frictional pressure drop when more than one free phase is present in the wellbore. As mentioned in the introduction chapter, flow in the horizontal wellbore is different from the flow in a pipe because of the mass transfer through the perforation. Therefore, pressure drop calculation should take the perforation influx into consideration. Islam and Chakma (1990) studied both experimental and numerical modeling of horizontal wells and the effects of perforations. The authors concluded that the perforations can increase the pressure drop in the wellbore. In this study, hybrid technique was adopted and the well model was coupled with a three-phase simulator.

One of the difficulties of reservoir/wellbore coupled model development involves the simultaneous solution of two different systems: the reservoir partial differential equation and the wellbore partial differential equation (PDE). Flow in the reservoir is described as a parabolic type PDE, while that in the horizontal wellbore is a hyperbolic type PDE.

Simultaneous solution of both parabolic PDE and hyperbolic PDE in the same matrix equation brings out computational and stability problems. To fix this problem, Collins et al. (1991) came up with a special "dual-porosity" method to couple the wellbore flow and reservoir flow with hybrid grids. The wellbore was represented by a row of cylindrical wellbore grid blocks where well "permeability" and "relative permeability" were adjusted to yield the pressure drop and phase slippage predicted by multiphase flow correlations. With this approach, wellbore equations were transformed into a form similar to reservoir flow equations, with the same primary variables. However, the shortcoming of this methodology is that many parameters have to be oversimplified to adjust wellbore equations to reservoir flow equation forms. Some of these parameters have strong non-linearity characteristics, closely related to the pressure and saturation change during the calculation. Therefore the results may not reflect the interaction effect because of the oversimplification.

Obviously, there are many parameters that can have important influence on wellbore pressure drop, from wellbore configuration to production scheme. Folefac et al. (1991) pointed out that well length, well diameter and perforated interval have significant effect on the pressure drop in the wellbore. Meanwhile, two-phase flow conditions in the wellbore can increase the pressure drop compared to single-phase flow. The proposed wellbore model is based on a one-dimensional mixture momentum balance and a drift flux expression for the slip velocity between the phases. Ihara and Shimizu(1993) studied flow dynamics for horizontal wellbore by considering acceleration pressure drop. An experiment was conducted to generate relevant data of flow in the horizontal wellbore and interactions with the reservoir as well. The authors discussed several factors in the sensitivity analysis for field application, including reservoir permeability, length of horizontal well section, drainage area and producing gas/oil ratio. Novy (1995) proposed a criterion to identify when friction becomes important and should be considered in a particular wellbore/reservoir system. The equations are solved numerically for both single-phase oil and single-phase gas flow, with the assumption of steady state flow and constant well production index in the system. Novy pointed out that Dikken's results overstate the effects of friction in tubes with rough walls. Novy suggested a criterion to analyze the wellbore friction, which is the ratio of wellbore pressure drop to draw-down at the producing end.

Traditional reservoir simulation models usually assume constant pressure along the horizontal wellbore and treat the horizontal well as an infinite conductivity medium (Ertekin, 2001). However, it is well known in the industry that a horizontal well can have a finite conductivity effect, especially when the flow rate is high. A general, semi-analytical model that couples wellbore and reservoir hydraulics was presented by Ozkan et al. (1992). Dimensionless groups were defined for general applicability. The influx and pressure distributions along the length of the well were studied. The authors concluded that when the pressure drop in the wellbore becomes significant compared to the drawdown, a larger portion of the fluid will enter the wellbore near the heel of the well. Penmatcha et al. (1997) also developed a semi-analytical well-model for both single phase oil and two-phase oil and gas. The model is flexible to incorporate any friction factor correlation in the wellbore part. The ratio of wellbore pressure drop to reservoir drawdown is an indication of the frictional effects on well productivity. The authors further pointed out that the breakthrough occurs first at the heel of the well due to the pressure drop in the well. This research studied the optimum well length for a horizontal well using a proposed model. Penmatcha and Aziz (1998) later developed a comprehensive, 3-D reservoir/wellbore coupling model to compare the infinite conductivity and finite-conductivity wellbore flow behavior. The semi-analytical finite-conductivity well model considers both frictional and acceleration pressure drops in the wellbore. Liu and Jiang (1998) also studied finite conductivity effect on well productivity and used a coupled model for the optimization of horizontal wellbore length.

Wellbore pressure drop consists of wall friction pressure loss, acceleration pressure loss and pressure drop caused by gravity. For a horizontal well, gravity-caused wellbore pressure drop has little effect on inflow distribution. Ouyang and Huang (1998) developed a wellbore/reservoir flow coupling model and conducted a series of sensitivity studies on the wellbore pressure drop. The authors observed that wellbore pressure drop caused by wall friction and acceleration affects inflow distribution. The infinite conductivity assumption can still be applied for an inclined well provided that both frictional and acceleration pressure drops are negligible in the well. The coupled model can be applied to determine well index, wellbore pressure profile and wellbore inflow/outflow distribution during the well's production or injection life.

Vicente (2000) developed a reservoir and horizontal wellbore coupled model. The wellbore model assumes homogenous flow. The coupled model has been applied to horizontal well design, finite conductivity analysis and transient pressure analysis. A reservoir/wellbore coupled model is also a useful tool to balance the inflow/injection profile, perforation and completion design and production optimization (Vicente et al., 2001a and 2001b). Baba and Tiab (2001) analyzed the effect of finite conductivity horizontal well on transient-pressure behavior using a semi-analytical model. The author pointed that a completion scheme should be optimized on the basis of stabilized flux distributions and additional pressure drop along the horizontal wellbore. Ouyang and Huang (2005) evaluated the well completion impacts on the performance of horizontal and multilateral wells. The study can be applied to design completion under various flow and reservoir scenarios.

CHAPTER 3: DEVELOPMENT OF THE RESERVOIR FLOW MODEL

As summarized in the literature review, researchers have realized the influence of wellbore pressure loss on reservoir flow and the effect of finite conductivity of the horizontal wellbore. Some of them developed semi-analytical well-models focusing on the wellbore flow characteristics. Recently, development of the reservoir and wellbore coupled model became a source of interest to researchers. Basically, there are two major types of methodologies to develop a coupled model in both reservoir and wellbore domains: semi-analytical simulation and numerical simulation. The advantage of semi-analytical methods is the better understanding of the interaction between wellbore and reservoir flow. But a semi-analytical method is based on some simplified assumptions such as a homogeneous reservoir and single phase flow. Although a numerical simulation method is more complex to implement, this method can better represent actual heterogeneous reservoirs and wellbore conditions. Numerical simulation methods are also more robust for the coupling process with multiphase flow. This research proposes the use of numerical simulation methods to conduct the coupling simulation of the reservoir/wellbore system.

The coupled model comprises two main domains: the reservoir and the wellbore. The basic principle behind this coupled model is the pressure continuity and mass balance at the sandface. The fluid flow in the reservoir is described as a parabolic type partial differential equation, while the fluid flow in the horizontal wellbore is a hyperbolic type partial differential equation. Both systems are integrated into one equation matrix for the simultaneous solution. Therefore, variables like reservoir pressure, saturation, and wellbore pressure, velocity and holdup can be obtained at the same time. All those variables interplay with each other in the coupled system and reflect the interaction between reservoir and wellbore.

Chapter 3 and Chapter 4 focus on the development of the reservoir flow model and wellbore flow model, respectively. The following several sections in this chapter present the development process of a 3-D, fully implicit black-oil reservoir simulation model. To make the report more readable, detailed derivations in the development process are incorporated at the end of this thesis as an appendix.

3.1 Hybrid Local Grid Refinement

To better simulate the flow near the horizontal wellbore and save computational cost, the reservoir domain can be further divided into near-wellbore region and reservoir region far away from the wellbore. The near-wellbore flow in the porous medium has typical radial streamlines. Therefore, the use of cylindrical coordinates is advantageous in the near-wellbore region. Based on this, a hybrid grid system is developed to locally refine the grid system and follow the actual flow geometry. The hybrid system is composed of two main elementary meshes: a cylindrical mesh in the wellbore vicinity and Cartesian mesh in the reservoir far away from the wellbore (Pedrosa and Aziz, 1985), as shown in the figure 3.1 and figure 3.2.

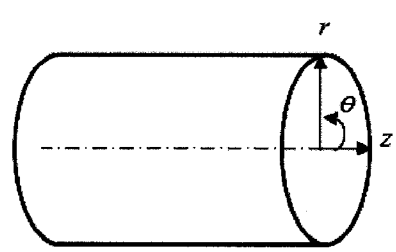

Figure 3.1 - Cylindrical coordinates

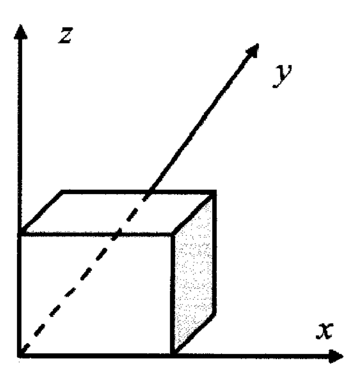

Figure 3.2 - Cartesian coordinates

The hybrid system includes both Cartesian and Cylindrical coordinates, as shown in figure 3.3. For the reservoir regions far away from the wellbore, the flow streamlines can be taken as linear, a rectangular grid system is good enough to simulate the fluid flow. For the wellbore vicinity, the cylindrical grid can locally follow the actual flow radial streamlines and generates results of more accuracy around the drainage areas. From the coupling point of view, a hybrid grid system provides a better platform for the purpose of precisely simulating the interaction between flow in the horizontal wellbore and the reservoir.

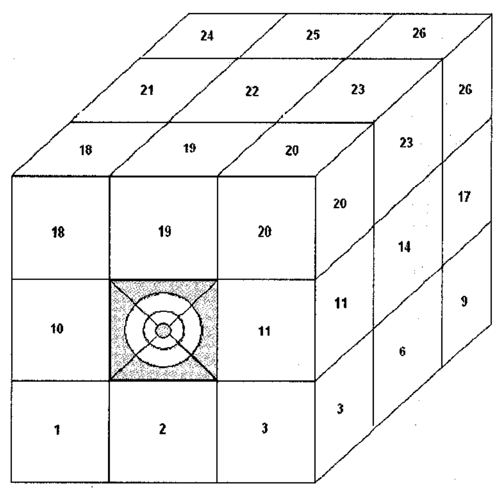

Figure 3.3 - The hybrid system and grid ordering of the Cartesian system

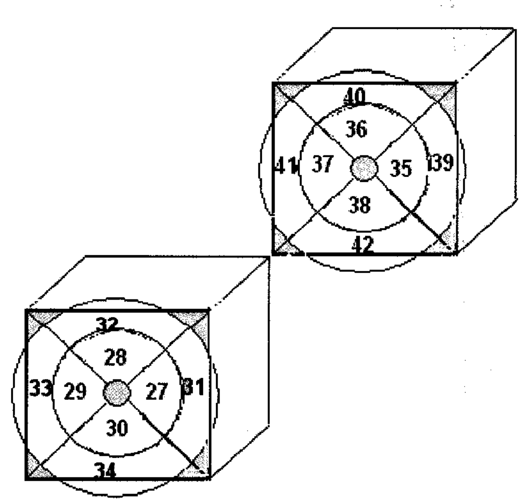

Figure 3.4 - Grid ordering of the cylindrical system

To make the grid as structured as possible, an apparent radial mesh and an apparent Cartesian mesh are applied to the mixed area (irregular shape) between wellbore region and reservoir regions, see figure 3.5. The apparent transmissibility is calculated according to the particular grid system. To get the apparent transmissibility for the cylindrical system, it is necessary to assume an apparent radial block with the same volume as the original irregular block and then calculate the external radius of the apparent radial block. Similarly for the Cartesian system, an apparent Cartesian block with same volume of the irregular zone can be assumed to calculate the Cartesian transmissibility. Different weight factors are also calculated to update the properties in the irregular regions for both apparent radial grids and apparent Cartesian grids.

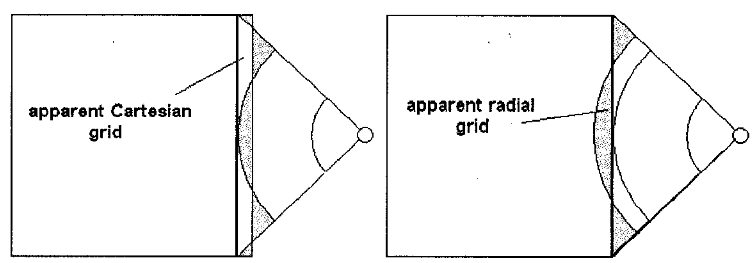

Figure 3.5 - Treatment of the mixed irregular area using apparent mesh

The Cartesian and cylindrical hybrid structure gives the calculation equation system a unique structure too. In this study, the Newton-Raphson method is used to solve the partial differential equations of reservoir flow model, see sections 3.2 and 3.9. The Newton-Raphson method involves calculating the Jacobian matrix and the right hand side vector. The Jacobian matrix is built up by calculating all the derivatives of the residual function with respect to each unknown for each iteration. Because of the Cartesian and cylindrical hybrid system, the Jacobian matrix also has a unique structure as shown in figure 3.6. It is divided into four zones: Cartesian system, cylindrical system, and two side-wings hybrid grid zones.

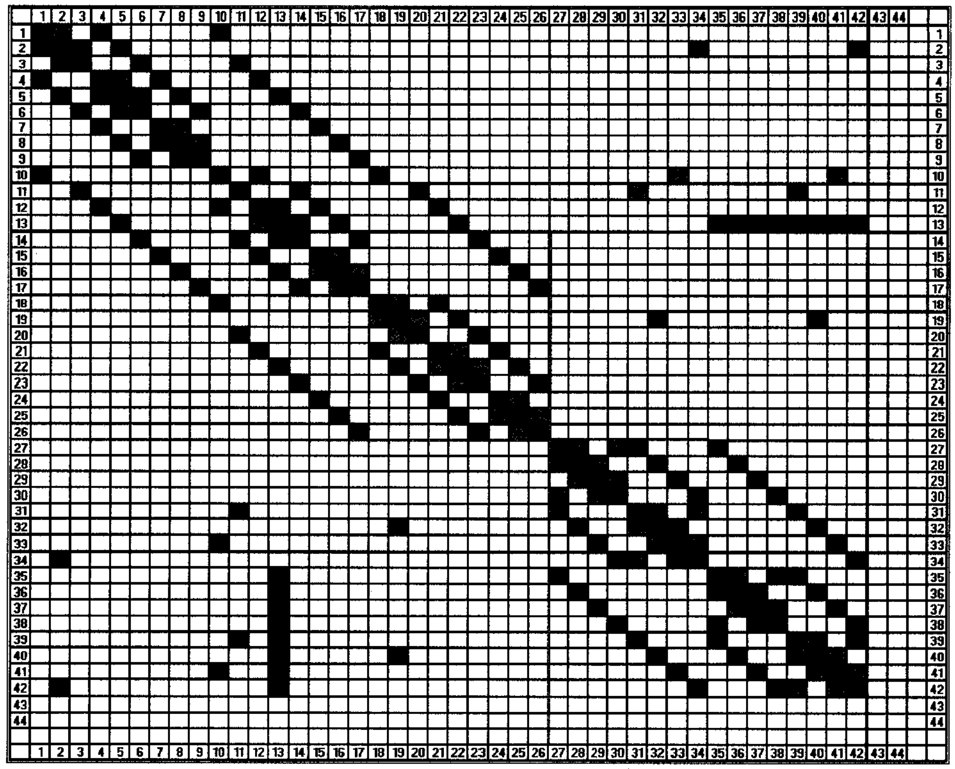

Figure 3.6 - Ordering scheme and matrix structure for a sample case

The mathematical models for oil and gas flow in reservoir, as for both Cartesian and cylindrical regions, will be developed in the following sections. Since the development of reservoir model under Cartesian and cylindrical coordinates are similar in many respects, sections 3.2 to 3.9 only illustrate the development of the reservoir model in Cartesian coordinates. Details of the model development in cylindrical coordinates can be referred to in Appendix 1.

3.2 Partial Differential Equations for the Reservoir Flow Model

Reservoir flow equations can be derived based on a control volume of the element, which is $V = \Delta x \cdot \Delta y \cdot \Delta z$ in Cartesian coordinates and $V = \Delta r \cdot (r\Delta\theta) \cdot \Delta z$ in cylindrical coordinates. The area perpendicular to flow is: $A_x = \Delta y \cdot \Delta z$; $A_y = \Delta x \cdot \Delta z$; $A_z = \Delta y \cdot \Delta x$

in Cartesian coordinates, and $A_r = (r\Delta\theta) \cdot \Delta z$; $A_\theta = \Delta r \cdot \Delta z$; $A_z = (r\Delta\theta) \cdot \Delta r$ in cylindrical coordinates.

Considering the flow continuity through the control volume, the continuity equation can be expressed as the following (without external source/sink for oil and gas phases):

(Mass in) – (Mass out) = (Mass Accumulation)

Therefore, the continuity equation in both Cartesian and cylindrical coordinates can be expressed in the following way:

$$-\frac{\partial(\rho \upsilon_x A_x)}{\partial x}\Delta x - \frac{\partial(\rho \upsilon_y A_y)}{\partial y}\Delta y - \frac{\partial(\rho \upsilon_z A_z)}{\partial z}\Delta z = V\frac{\partial(\phi\rho)}{\partial t}$$

$$-\frac{\partial(\rho \upsilon_r A_r)}{\partial r}\Delta r - \frac{\partial(\rho \upsilon_\theta A_\theta)}{\partial \theta}\Delta \theta - \frac{\partial(\rho \upsilon_z A_z)}{\partial z}\Delta z = V\frac{\partial(\phi\rho)}{\partial t}$$

(3.1)

For the Cartesian coordinates (same throughout this chapter if not specified, derivation and development in cylindrical coordinates for near-wellbore region can be found in Appendix 1), υ_x, υ_y and υ_z are volumetric velocities in *x*, *y* and *z* direction separately and can be obtained with Darcy's Law, (for phase *l*):

$$\upsilon_x = -\frac{k_x k_{rl}}{\mu_l}\left(\frac{\partial P}{\partial x} - \gamma\frac{\partial h}{\partial x}\right) \quad (3.1.a)$$

$$\upsilon_y = -\frac{k_y k_{rl}}{\mu_l}\left(\frac{\partial P}{\partial y} - \gamma\frac{\partial h}{\partial y}\right) \quad (3.1.b)$$

$$\upsilon_z = -\frac{k_z k_{rl}}{\mu_l}\left(\frac{\partial P}{\partial z} - \gamma\frac{\partial h}{\partial z}\right) \quad (3.1.c)$$

Substitute Equations (3.1.a) ~ (3.1.c) to (3.1):

$$-\frac{\partial[-\frac{k_x k_{rl}}{\mu_l}(\frac{\partial P}{\partial x}-\gamma\frac{\partial h}{\partial x})\rho_l A_x]}{\partial x}\Delta x-\frac{\partial[-\frac{k_y k_{rl}}{\mu_l}(\frac{\partial P}{\partial y}-\gamma\frac{\partial h}{\partial y})\rho_l A_y]}{\partial y}\Delta y$$
$$-\frac{\partial[-\frac{k_z k_{rl}}{\mu_l}(\frac{\partial P}{\partial z}-\gamma\frac{\partial h}{\partial z})\rho_l A_z]}{\partial z}\Delta z=V\frac{\partial(\phi\rho_l S_l)}{\partial t} \quad (3.2)$$

Using $B_l=\frac{\rho_{sc}}{\rho_l}$, $\frac{\partial(\rho_l)}{\partial t}=\frac{\partial(\frac{1}{B_l})}{\partial t}$, the above equation becomes:

$$\frac{\partial[A_x\frac{k_x k_{rl}}{B_l\mu_l}(\frac{\partial P}{\partial x}-\gamma\frac{\partial h}{\partial x})]}{\partial x}\Delta x+\frac{\partial[A_y\frac{k_y k_{rl}}{B_l\mu_l}(\frac{\partial P}{\partial y}-\gamma\frac{\partial h}{\partial y})]}{\partial y}\Delta y$$
$$+\frac{\partial[A_z\frac{k_z k_{rl}}{B_l\mu_l}(\frac{\partial P}{\partial z}-\gamma\frac{\partial h}{\partial z})]}{\partial z}\Delta z=V\frac{\partial(\phi\frac{S_l}{B_l})}{\partial t} \quad (3.3)$$

To summarize, the reservoir flow equations in the Cartesian coordinates are:

For the gas phase:

$$\frac{\partial}{\partial x}[\frac{A_x k_x k_{rg}}{B_g\mu_g}(\frac{\partial P_g}{\partial x}-\gamma_g\frac{\partial h}{\partial x})+\frac{A_x R_s k_x k_{ro}}{B_o\mu_o}(\frac{\partial P_o}{\partial x}-\gamma_o\frac{\partial h}{\partial x})]\Delta x$$
$$+\frac{\partial}{\partial y}[\frac{A_y k_y k_{rg}}{B_g\mu_g}(\frac{\partial P_g}{\partial y}-\gamma_g\frac{\partial h}{\partial y})+\frac{A_y R_s k_y k_{ro}}{B_o\mu_o}(\frac{\partial P_o}{\partial y}-\gamma_o\frac{\partial h}{\partial y})]\Delta y$$
$$+\frac{\partial}{\partial z}[\frac{A_z k_z k_{rg}}{B_g\mu_g}(\frac{\partial P_g}{\partial z}-\gamma_g\frac{\partial h}{\partial z})+\frac{A_z R_s k_z k_{ro}}{B_o\mu_o}(\frac{\partial P_o}{\partial z}-\gamma_o\frac{\partial h}{\partial z})]\Delta z$$
$$=V\frac{\partial}{\partial t}(\phi\frac{S_g}{B_g}+\phi R_s\frac{S_o}{B_o}) \quad (3.4)$$

For the oil phase:

$$\frac{\partial}{\partial x}[\frac{A_x k_x k_{ro}}{B_o\mu_o}(\frac{\partial P_o}{\partial x}-\gamma_o\frac{\partial h}{\partial x})]\Delta x+\frac{\partial}{\partial y}[\frac{A_y k_y k_{ro}}{B_o\mu_o}(\frac{\partial P_o}{\partial y}-\gamma_o\frac{\partial h}{\partial y})]\Delta y$$
$$+\frac{\partial}{\partial z}[\frac{A_z k_z k_{ro}}{B_o\mu_o}(\frac{\partial P_o}{\partial z}-\gamma_o\frac{\partial h}{\partial z})]\Delta z=V\frac{\partial}{\partial t}(\phi\frac{S_o}{B_o}) \quad (3.5)$$

Considering the oil and gas production as external source/sink term in the equations, we have:

For the gas phase:

$$\frac{\partial}{\partial x}\left[\frac{A_x k_x k_{rg}}{B_g \mu_g}\left(\frac{\partial P_g}{\partial x}-\gamma_g\frac{\partial h}{\partial x}\right)+\frac{A_x R_s k_x k_{ro}}{B_o \mu_o}\left(\frac{\partial P_o}{\partial x}-\gamma_o\frac{\partial h}{\partial x}\right)\right]\Delta x$$
$$+\frac{\partial}{\partial y}\left[\frac{A_y k_y k_{rg}}{B_g \mu_g}\left(\frac{\partial P_g}{\partial y}-\gamma_g\frac{\partial h}{\partial y}\right)+\frac{A_y R_s k_y k_{ro}}{B_o \mu_o}\left(\frac{\partial P_o}{\partial y}-\gamma_o\frac{\partial h}{\partial y}\right)\right]\Delta y$$
$$+\frac{\partial}{\partial z}\left[\frac{A_z k_z k_{rg}}{B_g \mu_g}\left(\frac{\partial P_g}{\partial z}-\gamma_g\frac{\partial h}{\partial z}\right)+\frac{A_z R_s k_z k_{ro}}{B_o \mu_o}\left(\frac{\partial P_o}{\partial z}-\gamma_o\frac{\partial h}{\partial z}\right)\right]\Delta z \qquad (3.6)$$
$$=V\frac{\partial}{\partial t}\left(\phi\frac{S_g}{B_g}+\phi R_s\frac{S_o}{B_o}\right)+R_s q_o V+q_{fg}V$$

For the oil phase:

$$\frac{\partial}{\partial x}\left[\frac{A_x k_x k_{ro}}{B_o \mu_o}\left(\frac{\partial P_o}{\partial x}-\gamma_o\frac{\partial h}{\partial x}\right)\right]\Delta x+\frac{\partial}{\partial y}\left[\frac{A_y k_y k_{ro}}{B_o \mu_o}\left(\frac{\partial P_o}{\partial y}-\gamma_o\frac{\partial h}{\partial y}\right)\right]\Delta y$$
$$+\frac{\partial}{\partial z}\left[\frac{A_z k_z k_{ro}}{B_o \mu_o}\left(\frac{\partial P_o}{\partial z}-\gamma_o\frac{\partial h}{\partial z}\right)\right]\Delta z=V\frac{\partial}{\partial t}\left(\phi\frac{S_o}{B_o}\right)+q_o V \qquad (3.7)$$

For water:

$$\frac{\partial}{\partial t}\left(\frac{S_w}{B_w}\right)=0 \qquad (3.8)$$

In above equations:

q_o: Produced oil rate per unit of rock volume;

q_{fg}: Produced free gas rate per unit of rock volume;

z: Depth with respect to a reference plane;

$\gamma_f=\frac{g}{g_c}\rho_f$: Gravity of gas, oil or water, which is related to the pressure

$k_{rg}=k_{rg}(S_g)$: Gas relative permeability (a function of gas saturation)

$k_{ro}=k_{ro}(S_w,S_g)$: Oil relative permeability (a function of water and gas saturation)

To solve the above equations, the following additional relationships should be considered:

$S_o + S_g + S_w = 1$ and $S_w = S_{wi}$

So oil saturation $S_o = 1 - S_{wi} - S_g$ (3.9)

$$P_{c_{go}}(S_{liquid} = 1 - S_g) = P_g - P_o,\ P_g = P_o + P_{c_{go}}(S_g) \quad (3.10)$$

Where, S_w, S_g, S_o are water, gas and oil saturation, and $P_{c_{go}}$ is the capillary pressure between oil and gas.

The physical system in analysis here presents the following initial and boundary conditions:

- For Pressure: the reservoir is initially in equilibrium, thus:

$$P(x, y, z, 0) = P_i \quad (3.11)$$

- For Saturation: the reservoir is initially under-saturated and with water saturation S_{wi}, thus:

$$S_w(x, y, z, 0) = S_{wi}$$
$$S_o(x, y, z, 0) = 1 - S_{wi} \quad (3.12)$$
$$S_g(x, y, z, 0) = 0$$

3.3 Pre-Processing of the Reservoir Flow Model

To build residual function and Jacobian matrix, several important pre-processing steps should be taken before the discretization of the partial differential equations. The following sections (3.3.1 to 3.3.5) discuss on the calculation of geometric factors at the interface, determination of initial pressure distribution, calculation of porous volume of grid blocks, calculation of different properties at the interfaces and determination of the transmissibility terms.

3.3.1 Geometric Factors at the Interface

To simplify the derivation of transmissibility and residual function in the next step, geometric factors (*GF*) are calculated at the interfaces of each cell. The first step of geometric factor calculation is to determine the grid block position. Here, block centered scheme is adopted in the discretization of the flow equations. In all three directions *x*, *y*, *z*, the grids are equally distributed and the increments are constants, which can be calculated by:

$$\Delta x = \frac{x_e}{nx},\ \Delta y = \frac{y_e}{ny},\ \Delta z = \frac{z_e}{nz} \tag{3.13}$$

Where,

nx is the number of gridblocks in the direction x;

ny is the number of gridblocks in the direction y;

nz is the number of gridblocks in the direction z;

x_e is reservoir limit in x direction;

y_e is reservoir limit in y direction;

z_e is reservoir limit in z direction;

The position of the grid nodes in each cell occurs in the geometric point in which the average pressure $\overline{P}$ of the cell is applied (here is also the geometric center).

The grid point position of each cell can be calculated as below:

$$x_{i,j,k} = x_{i-1/2,j,k} + \frac{\Delta x}{2} \tag{3.14}$$

$$y_{i,j,k} = y_{i,j-1/2,k} + \frac{\Delta y}{2} \tag{3.15}$$

$$z_{i,j,k} = z_{i,j,k-1/2} + \frac{\Delta z}{2} \tag{3.16}$$

Where i=1,2,...nx; j=1,2,... ny and k=1,2,...nz.

Taking z direction as an example, assuming the depth of the medium point of cell (i,j,k) in layer k=1 is $Z_{i,j,1}$:

$$Z_{i,j,1} = Z_{base} + \frac{\Delta z}{2} \tag{3.17}$$

Where, $\Delta z = \frac{Z_e}{nz}$

For layers k = 2 to nz the following expression is valid:

$$Z_{i,j,k} = Z_{i,j,1} + (k-1)\Delta z$$

Where, i = 1 to nx, j = 1 to ny and k = 2 to nz;

nz: Number of grid points in direction z;

Z_{base}:Depth of the reservoir base;

Z_e: Formation thickness.

The computation of the geometric factor is simply a harmonic mean of the absolute permeabilities close to the interface, weighted with respect to the distances between the grid-points and the interface.

For direction x:

$$GF_{x_{i-1/2,j,k}} = \frac{\Delta y_j \cdot \Delta z_k}{\frac{x_{i-1/2} - x_{i-1}}{Kx_{i-1,j,k}} + \frac{x_i - x_{i-1/2}}{Kx_{i,j,k}}} \tag{3.18}$$

And

$$GF_{x_{i+1/2,j,k}} = \frac{\Delta y_j \cdot \Delta z_k}{\frac{x_{i+1/2} - x_i}{Kx_{i,j,k}} + \frac{x_{i+1} - x_{i+1/2}}{Kx_{i+1,j,k}}} \tag{3.19}$$

Same here for direction y and direction z

$$GF_{y_{i,j-1/2,k}} = \frac{\Delta x_i \cdot \Delta z_k}{\frac{y_{j-1/2} - y_{j-1}}{Ky_{i,j-1,k}} + \frac{y_j - y_{j-1/2}}{Ky_{i,j,k}}} \qquad GF_{y_{i,j+1/2,k}} = \frac{\Delta x_i \cdot \Delta z_k}{\frac{y_{j+1/2} - y_j}{Ky_{i,j,k}} + \frac{y_{j+1} - y_{j+1/2}}{Ky_{i,j+1,k}}}$$

$$GF_{z_{i,j,k-1/2}} = \frac{\Delta x_i \cdot \Delta y_j}{\frac{z_{k-1/2} - z_{k-1}}{Kz_{i,j,k-1}} + \frac{z_k - z_{k-1/2}}{Kz_{i,j,k}}} \qquad GF_{z_{i,j,k+1/2}} = \frac{\Delta x_i \cdot \Delta y_j}{\frac{z_{k+1/2} - z_k}{Kz_{i,j,k}} + \frac{z_{k+1} - z_{k+1/2}}{Kz_{i,j,k+1}}}$$

Where,

$x_{i\pm1/2,j,k}$: Grid block interfaces in x direction;

$y_{i,j\pm1/2,k}$: Grid block interfaces in y direction;

$z_{i,j,k\pm1/2}$: Grid block interfaces in z direction;

$x_{i,j,k}$: Grid points center in x direction;

$y_{i,j,k}$: Grid points center in y direction;

$z_{i,j,k}$: Grid points center in z direction;

Δx : Length of each grid block in x direction;

Δz : Cell thickness;

Δy : Length of each grid block in y direction;

$Kx_{i,j,k}$: Absolute permeability in direction x of blocks (i,j,k)

3.3.2 Initial Pressure Distribution

The initial pressure distribution is related to the depth of the grid points ($Z_{i,j,k}$). After calculating the depth of the grid points (see section 3.3: direction z), the initial pressure of each block can be determined through an iterative process based on the following equations:

$$P_{i,j,k} = P_{ref} + \bar{\gamma}(Z_{ref} - Z_{i,j,k}) \tag{3.20}$$

Where,

$P_{i,j,k}$: Initial pressure of block (i,j,k);

P_{ref} : Reference pressure at a reference depth (z_{ref});

z_{ref} : Reference depth;

$z_{i,j,k}$: The depth of the medium point of cell (i,j,k);

$\bar{\gamma}$: Specific gravity;

$$\bar{\gamma} = f(\bar{P}) \text{ And } \bar{P} = \frac{P_{i,j,k} + P_{ref}}{2} \rightarrow P_{i,j,k} = 2\bar{P} - P_{ref}$$

$$2\bar{P} - P_{ref} = P_{i,j,k} = P_{ref} + \bar{\gamma}(Z_{ref} - Z_{i,j,k})$$

Where, $P_{i,j,k}$ is needed to calculate $\bar{P}$ and $\bar{\gamma}$, but $\bar{\gamma}$ is needed to calculate $\bar{P}$ and $P_{i,j,k}$. So an iteration method (Newton-Raphson method) should be used to solve the problem:

Define a residual function F_R and its derivative with respect to $\bar{P}$:

$$\begin{aligned} F_R &= 2\bar{P} - P_{ref} - P_{i,j,k} = 2\bar{P} - P_{ref} - [P_{ref} + \bar{\gamma}(Z_{ref} - Z_{i,j,k})] \\ &= 2(\bar{P} - P_{ref}) - \bar{\gamma}(Z_{ref} - Z_{i,j,k}) = 0 \end{aligned} \quad (3.21)$$

Derivative of F_R with respect to $\bar{P}$:

$$\frac{\partial F_R}{\partial \bar{P}} = 2 - (Z_{ref} - Z_{i,j,k})\frac{d\bar{\gamma}}{d\bar{P}} \quad (3.22)$$

Resulting Newton-Raphson system:

$$\frac{\partial F_R}{\partial \bar{P}} \cdot \Delta\bar{P} = -F_R \quad (3.23)$$

So based on (3.21), (3.22) and (3.23) and considering a tolerance ε for convergence, the $P_{i,j,k}$ can be determined for each grid through the following iterative technique:

1. Given P_{ref}, Z_{ref}, $Z_{i,j,k}$, calculate $\bar{\gamma}(P_{ref})$ by interpolating at oil PVT table;
2. Calculate $P_{i,j,k}$ with the expression: $P_{i,j,k} = P_{ref} + \bar{\gamma}(Z_{ref} - Z_{i,j,k})$;
3. Determine $\bar{P}$ using $\bar{P} = \frac{P_{i,j,k} + P_{ref}}{2}$;
4. Given $\bar{P}$, obtain $\bar{\gamma}(\bar{P})$ by interpolating at oil PVT table;
5. Knowing $\bar{\gamma}(\bar{P})$ and $\bar{P}$, calculate the residual function value:

$$F_R = 2(\overline{P} - P_{ref}) - \overline{\gamma}(Z_{ref} - Z_{i,j,k})$$

6. Calculate the derivative of $\overline{\gamma}$ with respect to $\overline{P}$, $(\frac{d\overline{\gamma}}{d\overline{P}})$ from oil PVT table and $\overline{P}$;

7. Calculate the derivative of F_R:

$$\frac{\partial F_R}{\partial \overline{P}} = 2 - (Z_{ref} - Z_{i,j,k})\frac{d\overline{\gamma}}{d\overline{P}}$$

8. Calculate $\overline{P}$ with $\Delta\overline{P} = -\frac{F_R}{(\partial F_R / \partial \overline{P})}$

9. If $\Delta\overline{P} > \varepsilon$, $\overline{P} = \overline{P_{previous}} + \Delta\overline{P_{calculated}}$, then go to 4.

   If $\Delta\overline{P} \le \varepsilon$, $P_{i,j,k} = 2\overline{P} - P_{ref}$, $P_{ref} = P_{i,j,k}$ and $Z_{ref} = Z_{i,j,k}$, then go to next loop.

3.3.3 Pore Volume of Grid Blocks

The porous volume of a cell (*i,j,k*) can be calculated with:

$$V_{P_{i,j,k}} = \Delta x \Delta y \Delta z \phi_{inicial\ k} \qquad (3.24)$$

Where,

Δz : Cell thickness;

Δy : Length of each grid block in *y* direction;

Δx : Length of each grid block in *x* direction;

$\phi_{inicial\ k}$ is the porosity of each cell at initial condition (the porosity is defined for each layer *k*)

$$\phi_{inicial\ k} = \phi_k^{\ 1}(1 - C_r P_{initial_{i,j,k}}) \qquad (3.25)$$

Where, $\phi_k^{\ 1}$ is the porosity at atmospheric pressure and $P_{initial_{i,j,k}}$ is the initial pressure in each cell, C_r is the rock compressibility;

The pore volume of each cell is computed prior to the simulation process with the expressions above and it is corrected at each stage of the iterative process by:

$$V_{P_{i,j,k}} = V_{P_{i,j,k}}[1 - C_r(P^n_{oi,j,k} - P^{\nu+1}_{oi,j,k})] \tag{3.26}$$

Where,

$P^n_{oi,j,k}$ is the oil pressure at cell (i,j,k) at the time-step n;

$P^{\nu+1}_{oi,j,k}$ is the oil pressure at cell (i,j,k) with iteration $\nu+1$.

3.3.4 Properties at the Interfaces

There are three steps to determine the properties at the interface. First, the weighting factors should be calculated for each Cartesian direction, (w_x, w_y, w_z). Then properties values at the interfaces can be determined by either Property weighting method or Pressure weighting method. The derivatives of the properties at the interface should also be calculated for later transmissibility computation and equation matrix terms.

For the x direction:

$$w_x = \frac{x_{i-1/2,j,k} - x_{i-1,j,k}}{x_{i,j,k} - x_{i-1,j,k}} \tag{3.27}$$

Where,

$$x_{i-1/2,j,k} - x_{i-1,j,k} = \frac{\Delta x}{2}$$

$$x_{i,j,k} - x_{i-1,j,k} = \Delta x$$

So $w_x = 0.5$. As for Cartesian grid, the weight factor in y and z direction can be derived in the same way as in x direction.

The computation of the properties at the interface can be approached in either Property weighting method or Pressure weighting method.

- Property Weighting

x direction: $f_{i-1/2,j,k} = w_x f_{i,j,k} + (1-w_x) f_{i-1,j,k}$ and $f_{i+1/2,j,k} = w_x f_{i+1,j,k} + (1-w_x) f_{i,j,k}$

y direction: $f_{i,j-1/2,k} = w_y f_{i,j,k} + (1-w_y) f_{i,j-1,k}$ and $f_{i,j+1/2,k} = w_y f_{i,j+1,k} + (1-w_y) f_{i,j,k}$

z direction: $f_{i,j,k-1/2} = w_z f_{i,j,k} + (1-w_z) f_{i,j,k-1}$ and $f_{i,j,k+1/2} = w_z f_{i,j,k+1} + (1-w_z) f_{i,j,k}$

Where, f represents a given oil property.

(3.28)

- Pressure Weighting

Calculate the "average pressure" at the interface ($P_{o_{i\pm1/2,j,k}}$, $P_{o_{i,j\pm1/2,k}}$ and $P_{o_{i,j,k\pm1/2}}$) through a weighted average based on:

$P_{o_{i-1,j,k}}$ and $P_{o_{i,j,k}}$ (for $i-\frac{1}{2},j,k$), $P_{o_{i,j,k}}$ and $P_{o_{i+1,j,k}}$ (for $i+\frac{1}{2},j,k$)

$P_{o_{i,j-1,k}}$ and $P_{o_{i,j,k}}$ (for $i,j-\frac{1}{2},k$), $P_{o_{i,j,k}}$ and $P_{o_{i,j+1,k}}$ (for $i,j+\frac{1}{2},k$)

$P_{o_{i,j,k-1}}$ and $P_{o_{i,j,k}}$ (for $i,j,k-\frac{1}{2}$), $P_{o_{i,j,k}}$ and $P_{o_{i,j,k+1}}$ (for $i,j,k+\frac{1}{2}$)

The interface average pressure will be:

x direction:

$$P_{o_{i-1/2,j,k}} = w_x P_{o_{i,j,k}} + (1-w_x) P_{o_{i-1,j,k}}, \quad P_{o_{i+1/2,j,k}} = w_x P_{o_{i+1,j,k}} + (1-w_x) P_{o_{i,j,k}}$$

y direction:

$$P_{o_{i,j-1/2,k}} = w_y P_{o_{i,j,k}} + (1-w_y) P_{o_{i,j-1,k}}, \quad P_{o_{i,j+1/2,k}} = w_y P_{o_{i,j+1,k}} + (1-w_y) P_{o_{i,j,k}}$$

z direction:

$$P_{o_{i,j,k-1/2}} = w_z P_{o_{i,j,k}} + (1-w_z) P_{o_{i,j,k-1}}, \quad P_{o_{i,j,k+1/2}} = w_z P_{o_{i,j,k+1}} + (1-w_z) P_{o_{i,j,k}}$$

(3.29)

Based on $P_{o_{i\pm1/2,j,k}}$, $P_{o_{i,j\pm1/2,k}}$ and $P_{o_{i,j,k\pm1/2}}$, the properties $f_{i\pm1/2,j,k}$, $f_{i,j\pm1/2,k}$ and $f_{i,j,k\pm1/2}$ can be obtained through interpolation using PVT tables, thus:

x direction: $f_{i\pm1/2,j,k}$ is a function of $P_{o_{i\pm1/2,j,k}}$

y direction: $f_{i,j\pm1/2,k}$ is a function of $P_{o_{i,j\pm1/2,k}}$

z direction: $f_{i,j,k\pm1/2}$ is a function of $P_{o_{i,j,k\pm1/2}}$

Note that all the computations described here consider the pressure in the oil phase. This is correct to the computation of the variables: $\gamma_o, \mu_o, b_o, R_s$, which are functions of the pressure in the oil phase. However, for μ_g, b_g and γ_g, it is necessary to consider the pressure in the gas phase:

$$Pg_{i\pm1/2,j,k}, Pg_{i,j\pm1/2,k}, Pg_{i,j,k\pm1/2}$$

Where, $P_g = P_o + P_{cgo}$

Because of the non-linear characteristics of the above properties with respect to the reservoir pressure at the interface, it is necessary to obtain the derivatives of each property (μ_g, b_g, b_o, R_s, γ_g, γ_o) with respect to the reservoir pressure. This will be later used to calculate the transmissibility and other matrix terms.

The derivatives of each property to reservoir pressure can be expressed as the following:

$$\frac{df_{i\pm1/2,j,k}}{dP_{o_{i-1,j,k}}}, \frac{df_{i\pm1/2,j,k}}{dP_{o_{i,j,k}}}, \frac{df_{i\pm1/2,j,k}}{dP_{o_{i+1,j,k}}};$$

$$\frac{df_{i,j\pm1/2,k}}{dP_{o_{i,j-1,k}}}, \frac{df_{i,j\pm1/2,k}}{dP_{o_{i,j,k}}}, \frac{df_{i,j\pm1/2,k}}{dP_{o_{i,j+1,k}}};$$

$$\frac{df_{i,j,k\pm1/2}}{dP_{o_{i,j,k-1}}}, \frac{df_{i,j,k\pm1/2}}{dP_{o_{i,j,k}}}, \frac{df_{i,j,k\pm1/2}}{dP_{o_{i,j,k+1}}}$$

Where f stands for μ_o, μ_g, b_g, b_o, R_s, γ_g, γ_o

Thus for x direction:

$$f_{i-1/2,j,k} = w_x f_{i,j,k} + (1-w_x) f_{i-1,j,k} \text{ and } f_{i+1/2,j,k} = w_x f_{i+1,j,k} + (1-w_x) f_{i,j,k} \quad (3.30)$$

$$\frac{df_{i-1/2,j,k}}{dP_{o_{i-1,j,k}}} = \frac{d[w_x f_{i,j,k} + (1-w_x) f_{i-1,j,k}]}{dPo_{i-1,j,k}} = (1-w_x)\frac{df_{i-1,j,k}}{dP_{o_{i-1,j,k}}}$$

$$\frac{df_{i-1/2,j,k}}{dP_{o_{i,j,k}}} = \frac{d[w_x f_{i,j,k} + (1-w_x) f_{i-1,j,k}]}{dPo_{i,j,k}} = w_x \frac{df_{i,j,k}}{dP_{o_{i,j,k}}}$$

$$\frac{df_{i-1/2,j,k}}{dP_{o_{i+1,j,k}}} = 0$$

$$\frac{df_{i+1/2,j,k}}{dP_{o_{i-1,j,k}}} = 0$$

$$\frac{df_{i+1/2,j,k}}{dP_{o_{i,j,k}}} = \frac{d[w_x f_{i+1,j,k} + (1-w_x) f_{i,j,k}]}{dPo_{i,j,k}} = (1-w_x)\frac{df_{i,j,k}}{dP_{o_{i,j,k}}}$$

$$\frac{df_{i+1/2,j,k}}{dP_{o_{i+1,j,k}}} = \frac{d[w_x f_{i+1,j,k} + (1-w_x) f_{i,j,k}]}{dPo_{i+1,j,k}} = w_x \frac{df_{i+1,j,k}}{dP_{o_{i+1,j,k}}}$$

(3.31)

For y direction:

$$f_{i,j-1/2,k} = w_y f_{i,j,k} + (1-w_y) f_{i,j-1,k} \text{ and } f_{i,j+1/2,k} = w_y f_{i,j+1,k} + (1-w_y) f_{i,j,k}$$

$$\frac{df_{i,j-1/2,k}}{dP_{o_{i,j-1,k}}} = \frac{d[w_y f_{i,j,k} + (1-w_y) f_{i,j-1,k}]}{dPo_{i,j-1,k}} = (1-w_y)\frac{df_{i,j-1,k}}{dP_{o_{i,j-1,k}}}$$

$$\frac{df_{i,j-1/2,k}}{dP_{o_{i,j,k}}} = \frac{d[w_y f_{i,j,k} + (1-w_y) f_{i,j-1,k}]}{dPo_{i,j,k}} = w_y \frac{df_{i,j,k}}{dP_{o_{i,j,k}}}$$

$$\frac{df_{i,j-1/2,k}}{dP_{o_{i,j+1,k}}} = 0$$

$$\frac{df_{i,j+1/2,k}}{dP_{o_{i,j-1,k}}} = 0$$

$$\frac{df_{i,j+1/2,k}}{dP_{o_{i,j,k}}} = \frac{d[w_y f_{i,j+1,k} + (1-w_y) f_{i,j,k}]}{dPo_{i,j,k}} = (1-w_y)\frac{df_{i,j,k}}{dP_{o_{i,j,k}}}$$

$$\frac{df_{i,j+1/2,k}}{dP_{o_{i,j+1,k}}} = \frac{d[w_y f_{i,j+1,k} + (1-w_y) f_{i,j,k}]}{dPo_{i,j+1,k}} = w_y \frac{df_{i,j+1,k}}{dP_{o_{i,j+1,k}}}$$

(3.32)

For z direction:

$$f_{i,j,k-1/2} = w_z f_{i,j,k} + (1-w_z) f_{i,j,k-1} \text{ and } f_{i,j,k+1/2} = w_z f_{i,j,k+1} + (1-w_z) f_{i,j,k}$$

$$\frac{df_{i,j,k-1/2}}{dP_{o_{i,j,k-1}}} = \frac{d[w_z f_{i,j,k} + (1-w_z) f_{i,j,k-1}]}{dPo_{i,j,k-1}} = (1-w_z) \frac{df_{i,j,k-1}}{dP_{o_{i,j,k-1}}}$$

$$\frac{df_{i,j,k-1/2}}{dP_{o_{i,j,k}}} = \frac{d[w_z f_{i,j,k} + (1-w_z) f_{i,j,k-1}]}{dPo_{i,j,k}} = w_z \frac{df_{i,j,k}}{dP_{o_{i,j,k}}}$$

$$\frac{df_{i,j,k-1/2}}{dP_{o_{i,j,k+1}}} = 0$$

$$\frac{df_{i,j,k+1/2}}{dP_{o_{i,j,k-1}}} = 0$$

$$\frac{df_{i,j,k+1/2}}{dP_{o_{i,j,k}}} = \frac{d[w_z f_{i,j,k+1} + (1-w_z) f_{i,j,k}]}{dPo_{i,j,k}} = (1-w_z) \frac{df_{i,j,k}}{dP_{o_{i,j,k}}}$$

$$\frac{df_{i,j,k+1/2}}{dP_{o_{i,j,k+1}}} = \frac{d[w_z f_{i,j,k+1} + (1-w_z) f_{i,j,k}]}{dPo_{i,j,k+1}} = w_z \frac{df_{i,j,k+1}}{dP_{o_{i,j,k+1}}}$$

(3.33)

For the boundaries:

$i = 1, j = 1$ to ny, $k = 1$ to nz: $f_{1/2,j,k} = f(P_{o_{1/2,j,k}})$; $P_{o_{1/2,j,k}} = P_{o_{1,j,k}}$

$j = 1, i = 1$ to nx, $k = 1$ to nz: $f_{i,1/2,k} = f(P_{o_{i,1/2,k}})$; $P_{o_{i,1/2,k}} = P_{o_{i,1,k}}$

$k = 1, i = 1$ to nx, $j = 1$ to ny: $f_{i,j,1/2} = f(P_{o_{i,j,1/2}})$; $P_{o_{i,j,1/2}} = P_{o_{i,j,1}} + \gamma_{i,j,1} \frac{\Delta z}{2}$

$i = nx, j = 1$ to ny, $k = 1$ to nz: $f_{nx+1/2,j,k} = f(P_{o_{nx+1/2,j,k}})$; $P_{o_{nx+1/2,j,k}} = P_{o_{nx,j,k}}$

$i = 1$ to nx, $j = ny$, $k = 1$ to nz: $f_{i,ny+1/2,k} = f(P_{o_{i,ny+1/2,k}})$; $P_{o_{i,ny+1/2,k}} = P_{o_{i,ny,k}}$

$i = 1$ to $nx, j = 1$ to $ny, k = nz$: $f_{i,j,nz+1/2} = f(P_{o\,i,j,nz+1/2})$;

$$P_{o\,i,j,nz+1/2} = P_{o\,i,j,nz} - \gamma_{i,j,nz}\frac{\Delta z}{2}$$

3.3.5 Transmissibility Terms (Upstream Weighting)

In the treatment of the transmissibility and its derivatives, an upstream weighting scheme is adopted.

The concept of fluid potential is given by

$$\Phi = \int_{P^0}^{P}\frac{dP}{\rho} + g(z - z^0) \qquad (3.34)$$

Where, the coordinate system with z direction upward;

z^0 : Reference depth;

P^0 : Reference pressure at a reference depth z^0.

The potential difference is $\Delta\Phi_{1,2}$:

$$\Phi_1 - \Phi_2 = \int_{P^0}^{P_1}\frac{dP}{\rho} + g(z_1 - z^0) - \int_{P^0}^{P_2}\frac{dP}{\rho} + g(z_2 - z^0)$$
$$\Delta\Phi_{1,2} = \int_{P_2}^{P_1}\frac{dP}{\rho} + g(z_1 - z_2) \qquad (3.35)$$

Assuming that P and density are continuous function inside the domain (P_1, P_2), potential can be calculated as:

$$\Delta\Phi_{1,2} = \frac{(P_1 - P_2)}{\overline{\rho}} + \gamma(z_1 - z_2)$$

$$\Delta\Phi_{1,2} = (P_1 - P_2) + \overline{\rho}g(z_1 - z_2) \qquad (3.36)$$

Where, $\overline{\rho}g = \overline{\gamma}$, and $\overline{\gamma}$ is the specific gravity between points 1 and 2.

Thus, the verification of the flow direction of the gas and oil phases in directions x, y and z between two cells will be made in the following way:

For direction x:

For the gas phase:

$$POT_{gx1} = (P_o^{\,v}{}_{i,j,k} - P_o^{\,v}{}_{i-1,j,k}) - \gamma^{\,v}_{g_{i-1/2,j,k}}(h_{x_{i-1,j,k}} - h_{x_{i,j,k}}) + P_c^{\,v}{}_{i,j,k} - P_c^{\,v}{}_{i-1,j,k}$$

$$POT_{gx2} = (P_o^{\,v}{}_{i+1,j,k} - P_o^{\,v}{}_{i,j,k}) - \gamma^{\,v}_{g_{i+1/2,j,k}}(h_{x_{i,j,k}} - h_{x_{i+1,j,k}}) + P_c^{\,v}{}_{i+1,j,k} - P_c^{\,v}{}_{i,j,k}$$

If $POT_{gx} \geq 0$ then $\delta_{gx} = 1$

If $POT_{gx} < 0$ then $\delta_{gx} = 0$

(3.37)

For the oil phase:

$$POT_{ox1} = (P_o^{\,v}{}_{i,j,k} - P_o^{\,v}{}_{i-1,j,k}) - \gamma^{\,v}_{o_{i-1/2,j,k}}(h_{x_{i-1,j,k}} - h_{x_{i,j,k}})$$

$$POT_{ox2} = (P_o^{\,v}{}_{i+1,j,k} - P_o^{\,v}{}_{i,j,k}) - \gamma^{\,v}_{o_{i+1/2,j,k}}(h_{x_{i,j,k}} - h_{x_{i+1,j,k}})$$

If $POT_{ox} \geq 0$ then $\delta_{ox} = 1$

If $POT_{ox} < 0$ then $\delta_{ox} = 0$

(3.38)

For direction y:

For the gas phase:

$$POT_{gy1} = (P_o^{\,v}{}_{i,j,k} - P_o^{\,v}{}_{i,j-1,k}) - \gamma^{\,v}_{g_{i,j-1/2,k}}(h_{y_{i,j-1,k}} - h_{y_{i,j,k}}) + P_c^{\,v}{}_{i,j,k} - P_c^{\,v}{}_{i,j-1,k}$$

$$POT_{gy2} = (P_o^{\,v}{}_{i,j+1,k} - P_o^{\,v}{}_{i,j,k}) - \gamma^{\,v}_{g_{i,j+1/2,k}}(h_{y_{i,j,k}} - h_{y_{i,j+1,k}}) + P_c^{\,v}{}_{i,j+1,k} - P_c^{\,v}{}_{i,j,k}$$

If $POT_{gy} \geq 0$ then $\delta_{gy} = 1$

If $POT_{gy} < 0$ then $\delta_{gy} = 0$

(3.39)

For the oil phase:

$$POT_{oy1} = (P_o^{\,v}{}_{i,j,k} - P_o^{\,v}{}_{i,j-1,k}) - \gamma^{\,v}_{o_{i,j-1/2,k}}(h_{y_{i,j-1,k}} - h_{y_{i,j,k}})$$

$$POT_{oy2} = (P_o^{\,v}{}_{i,j+1,k} - P_o^{\,v}{}_{i,j,k}) - \gamma^{\,v}_{o_{i,j+1/2,k}}(h_{y_{i,j,k}} - h_{y_{i,j+1,k}})$$

If $POT_{oy} \geq 0$ then $\delta_{oy} = 1$

If $POT_{oy} < 0$ then $\delta_{oy} = 0$ (3.40)

For direction z:

For the gas phase:

$$POT_{gz1} = (P_o^{\,v}{}_{i,j,k} - P_o^{\,v}{}_{i,j,k-1}) - \gamma^{\,v}_{g_{i,j,k-1/2}}(z_{i,j,k-1} - z_{i,j,k}) + P_c^{\,v}{}_{i,j,k} - P_c^{\,v}{}_{i,j,k-1}$$

$$POT_{gz2} = (P_o^{\,v}{}_{i,j,k+1} - P_o^{\,v}{}_{i,j,k}) - \gamma^{\,v}_{g_{i,j,k+1/2}}(z_{i,j,k} - z_{i,j,k+1}) + P_c^{\,v}{}_{i,j,k+1} - P_c^{\,v}{}_{i,j,k}$$

If $POT_{gz} \geq 0$ then $\delta_{gz} = 1$

If $POT_{gz} < 0$ then $\delta_{gz} = 0$

(3.41)

For the oil phase:

$$POT_{oz1} = (P_o^{\,v}{}_{i,j,k} - P_o^{\,v}{}_{i,j,k-1}) - \gamma^{\,v}_{o_{i,j,k-1/2}}(z_{i,j,k-1} - z_{i,j,k})$$

$$POT_{oz2} = (P_o^{\,v}{}_{i,j,k+1} - P_o^{\,v}{}_{i,j,k}) - \gamma^{\,v}_{o_{i,j,k+1/2}}(z_{i,j,k} - z_{i,j,k+1})$$

If $POT_{oz} \geq 0$ then $\delta_{oz} = 1$

If $POT_{oz} < 0$ then $\delta_{oz} = 0$

(3.42)

The following is a general form of transmissibility expression used and its derivatives with respect to oil pressure and saturation in x direction.

$$T_{xl_{i-1/2,j,k}} = \frac{GF_{x_{i-1/2,j,k}}\, b_{l_{i-1/2,j,k}}}{\mu_{l_{i-1/2,j,k}}}[\delta_{lx} k_{rl_{i,j,k}} + (1-\delta_{lx}) k_{rl_{i-1,j,k}}]$$

$$T_{xl_{i+1/2,j,k}} = \frac{GF_{x_{i+1/2,j,k}}\, b_{l_{i+1/2,j,k}}}{\mu_{l_{i+1/2,j,k}}}[\delta_{lx} k_{rl_{i+1,j,k}} + (1-\delta_{lx}) k_{rl_{i,j,k}}]$$

(3.43)

General Form transmissibility and its derivatives with respect to oil saturation:

$$\frac{\partial T_{xl_{i-1/2,j,k}}}{\partial S_{o_{i-1,j,k}}} = (1-\delta_{lx})\frac{GF_{x_{i-1/2,j,k}}\, b_{l_{i-1/2,j,k}}}{\mu_{l_{i-1/2,j,k}}}\frac{dk_{rl_{i-1,j,k}}}{dS_{o_{i-1,j,k}}}$$

$$\frac{\partial T_{xl_{i-1/2,j,k}}}{\partial S_{o_{i,j,k}}} = \delta_{lx}\frac{GF_{x_{i-1/2,j,k}}\, b_{l_{i-1/2,j,k}}}{\mu_{l_{i-1/2,j,k}}}\frac{dk_{rl_{i,j,k}}}{dS_{o_{i,j,k}}}$$

$$\frac{\partial T_{xl_{i-1/2,j,k}}}{\partial S_{o_{i+1,j,k}}} = 0$$

$$\frac{\partial T_{xl_{i+1/2,j,k}}}{\partial S_{o_{i-1,j,k}}} = 0$$

$$\frac{\partial T_{xl_{i+1/2,j,k}}}{\partial S_{o_{i,j,k}}} = (1-\delta_{lx})\frac{GF_{x_{i+1/2,j,k}} b_{l_{i+1/2,j,k}}}{\mu_{l_{i+1/2,j,k}}}\frac{dk_{rl_{i,j,k}}}{dS_{o_{i,j,k}}}$$

$$\frac{\partial T_{xl_{i+1/2,j,k}}}{\partial S_{o_{i+1,j,k}}} = \delta_{lx}\frac{GF_{x_{i+1/2,j,k}} b_{l_{i+1/2,j,k}}}{\mu_{l_{i+1/2,j,k}}}\frac{dk_{rl_{i+1,j,k}}}{dS_{o_{i+1,j,k}}}$$

(3.44)

General Form transmissibility and its derivatives with respect to oil pressure:

$$\frac{\partial T_{xl_{i+1/2,j,k}}}{\partial P_{o_{i-1,j,k}}} = 0$$

$$\frac{\partial T_{xl_{i+1/2,j,k}}}{\partial P_{o_{i,j,k}}} = (w_x - 1)\cdot T_{xl_{i+1/2,j,k}}\left(\frac{1}{\mu_{l_{i+1/2,j,k}}}\frac{d\mu_{l_{i,j,k}}}{dP_{o_{i,j,k}}} - \frac{1}{b_{l_{i+1/2,j,k}}}\frac{db_{l_{i,j,k}}}{dP_{o_{i,j,k}}}\right)$$

$$\frac{\partial T_{xl_{i+1/2,j,k}}}{\partial P_{o_{i+1,j,k}}} = -w_x \cdot T_{xl_{i+1/2,j,k}}\left(\frac{1}{\mu_{l_{i+1/2,j,k}}}\frac{d\mu_{l_{i+1,j,k}}}{dP_{o_{i+1,j,k}}} - \frac{1}{b_{l_{i+1/2,j,k}}}\frac{db_{l_{i+1,j,k}}}{dP_{o_{i+1,j,k}}}\right)$$

$$\frac{\partial T_{xl_{i-1/2,j,k}}}{\partial P_{o_{i-1,j,k}}} = (w_x - 1)\cdot T_{xl_{i-1/2,j,k}}\left(\frac{1}{\mu_{l_{i-1/2,j,k}}}\frac{d\mu_{l_{i-1,j,k}}}{dP_{o_{i-1,j,k}}} - \frac{1}{b_{l_{i-1/2,j,k}}}\frac{db_{l_{i-1,j,k}}}{dP_{o_{i-1,j,k}}}\right)$$

$$\frac{\partial T_{xl_{i-1/2,j,k}}}{\partial P_{o_{i,j,k}}} = -w_x \cdot T_{xl_{i-1/2,j,k}}\left(\frac{1}{\mu_{l_{i-1/2,j,k}}}\frac{d\mu_{l_{i,j,k}}}{dP_{o_{i,j,k}}} - \frac{1}{b_{l_{i-1/2,j,k}}}\frac{db_{l_{i,j,k}}}{dP_{o_{i,j,k}}}\right)$$

$$\frac{\partial T_{xl_{i-1/2,j,k}}}{\partial P_{o_{i+1,j,k}}} = 0$$

(3.45)

Note that for closed boundaries, the transmissibility and derivatives will be zero. Similar derivatives can be obtained in the y and z directions, please see Appendix 2 for the computation of the transmissibility and its derivatives with respect to oil pressure and saturation.

3.4 Finite Difference Equations

To obtain the numerical solutions, the partial differential equations for reservoir flow have to be discretized to be finite difference equations. In this study, block centered scheme are adopted for the spatial discretization. Since the coupling model has strong non-linear characteristics, to ensure the stability and accuracy of the model, a fully implicit method is used in the time discretization for the reservoir partial differential equations.

For the gas phase:

$$\begin{aligned}
&\frac{\partial}{\partial x}[\frac{A_x k_x k_{rg}}{B_g \mu_g}(\frac{\partial P_g}{\partial x}-\gamma_g \frac{\partial h_x}{\partial x})+\frac{A_x R_s k_x k_{ro}}{B_o \mu_o}(\frac{\partial P_o}{\partial x}-\gamma_o \frac{\partial h_x}{\partial x})]\Delta x \\
&+\frac{\partial}{\partial y}[\frac{A_y k_y k_{rg}}{B_g \mu_g}(\frac{\partial P_g}{\partial y}-\gamma_g \frac{\partial h_y}{\partial y})+\frac{A_y R_s k_y k_{ro}}{B_o \mu_o}(\frac{\partial P_o}{\partial y}-\gamma_o \frac{\partial h_y}{\partial y})]\Delta y \\
&+\frac{\partial}{\partial z}[\frac{A_z k_z k_{rg}}{B_g \mu_g}(\frac{\partial P_g}{\partial z}-\gamma_g \frac{\partial h_z}{\partial z})+\frac{A_z R_s k_z k_{ro}}{B_o \mu_o}(\frac{\partial P_o}{\partial z}-\gamma_o \frac{\partial h_z}{\partial z})]\Delta z \\
&=V\frac{\partial}{\partial t}(\phi\frac{S_g}{B_g}+\phi R_s \frac{S_o}{B_o})+R_s q_o V+q_{fg}V
\end{aligned}$$

For the oil phase:

$$\begin{aligned}
&\frac{\partial}{\partial x}[\frac{A_x k_x k_{ro}}{B_o \mu_o}(\frac{\partial P_o}{\partial x}-\gamma_o \frac{\partial h_x}{\partial x})]\Delta x+\frac{\partial}{\partial y}[\frac{A_y k_y k_{ro}}{B_o \mu_o}(\frac{\partial P_o}{\partial y}-\gamma_o \frac{\partial h_y}{\partial y})]\Delta y \\
&+\frac{\partial}{\partial z}[\frac{A_z k_z k_{ro}}{B_o \mu_o}(\frac{\partial P_o}{\partial z}-\gamma_o \frac{\partial h_z}{\partial z})]\Delta z=V\frac{\partial}{\partial t}(\phi\frac{S_o}{B_o})+q_o V
\end{aligned}$$

The detailed discretization procedure, including derivation of the flow terms and accumulation terms in the oil and gas flow equations, is attached in the Appendix 2. The final finite difference equations for gas and oil are as the followings:

For the gas phase:

$$
\begin{aligned}
& T_{xo_{i+1/2,j,k}} \cdot R_s[(P^{n+1}_{o_{i+1,j,k}} - P^{n+1}_{o_{i,j,k}}) - \gamma^{n+1}_{o_{i+1/2,j,k}}(h_{xi+1} - h_{xi})_{j,k}] \\
& - T_{xo_{i-1/2,j,k}} \cdot R_s[(P^{n+1}_{o_{i,j,k}} - P^{n+1}_{o_{i-1,j,k}}) - \gamma^{n+1}_{o_{i-1/2,j,k}}(h_{xi} - h_{xi-1})_{j,k}] \\
& + T_{yo_{i,j+1/2,k}} \cdot R_s[(P^{n+1}_{o_{i,j+1,k}} - P^{n+1}_{o_{i,j,k}}) - \gamma^{n+1}_{o_{i,j+1/2,k}}(h_{y_{j+1}} - h_{y_j})_{i,k}] \\
& - T_{yo_{i,j-1/2,k}} \cdot R_s[(P^{n+1}_{o_{i,j,k}} - P^{n+1}_{o_{i,j-1,k}}) - \gamma^{n+1}_{o_{i,j-1/2,k}}(h_{y_j} - h_{y_{j-1}})_{i,k}] \\
& + T_{zo_{i,j,k+1/2}} \cdot R_s[(P^{n+1}_{o_{i,j,k+1}} - P^{n+1}_{o_{i,j,k}}) - \gamma^{n+1}_{o_{i,j,k+1/2}}(z_{k+1} - z_k)_{i,j}] \\
& - T_{zo_{i,j,k-1/2}} \cdot R_s[(P^{n+1}_{o_{i,j,k}} - P^{n+1}_{o_{i,j,k-1}}) - \gamma^{n+1}_{o_{i,j,k-1/2}}(z_k - z_{k-1})_{i,j}] \\
& + T_{xg_{i+1/2,j,k}} \cdot [(P^{n+1}_{o_{i+1,j,k}} - P^{n+1}_{o_{i,j,k}}) - \gamma^{n+1}_{g_{i+1/2,j,k}}(h_{xi+1} - h_{xi})_{j,k}] \\
& - T_{xg_{i-1/2,j,k}} \cdot [(P^{n+1}_{o_{i,j,k}} - P^{n+1}_{o_{i-1,j,k}}) - \gamma^{n+1}_{g_{i-1/2,j,k}}(h_{xi} - h_{xi-1})_{j,k}] \\
& + T_{yg_{i,j+1/2,k}} \cdot [(P^{n+1}_{o_{i,j+1,k}} - P^{n+1}_{o_{i,j,k}}) - \gamma^{n+1}_{g_{i,j+1/2,k}}(h_{y_{j+1}} - h_{y_j})_{i,k}] \\
& - T_{yg_{i,j-1/2,k}} \cdot [(P^{n+1}_{o_{i,j,k}} - P^{n+1}_{o_{i,j-1,k}}) - \gamma^{n+1}_{g_{i,j-1/2,k}}(h_{y_j} - h_{y_{j-1}})_{i,k}] \\
& + T_{zg_{i,j,k+1/2}} \cdot [(P^{n+1}_{o_{i,j,k+1}} - P^{n+1}_{o_{i,j,k}}) - \gamma^{n+1}_{g_{i,j,k+1/2}}(z_{k+1} - z_k)_{i,j}] \\
& - T_{zg_{i,j,k-1/2}} \cdot [(P^{n+1}_{o_{i,j,k}} - P^{n+1}_{o_{i,j,k-1}}) - \gamma^{n+1}_{g_{i,j,k-1/2}}(z_k - z_{k-1})_{i,j}] \\
& + T^{n+1}_{xg_{i+1/2,j,k}} \cdot (P^{n+1}_{c_{i+1,j,k}} - P^{n+1}_{c_{i,j,k}}) - T^{n+1}_{xg_{i-1/2}} \cdot (P^{n+1}_{c_{i,j,k}} - P^{n+1}_{c_{i-1,j,k}}) \\
& + T^{n+1}_{yg_{i,j+1/2,k}} \cdot (P^{n+1}_{c_{i,j+1,k}} - P^{n+1}_{c_{i,j,k}}) - T^{n+1}_{yg_{j-1/2}} \cdot (P^{n+1}_{c_{i,j,k}} - P^{n+1}_{c_{i,j-1,k}}) \\
& + T^{n+1}_{zg_{i,j,k+1/2}} \cdot (P^{n+1}_{c_{i,j,k+1}} - P^{n+1}_{c_{i,j,k}}) - T^{n+1}_{zg_{k-1/2}} \cdot (P^{n+1}_{c_{i,j,k}} - P^{n+1}_{c_{i,j,k-1}}) \\
& = \frac{V^n_{p_{i,j,k}}}{\Delta t}\{(1 + C_r \Delta_t P_o) \cdot [b_g{}^{n+1}(1 + S_w{}^n (C_r + C_w)\Delta_t P_o - S_w{}^n - S_o{}^{n+1}) + (R_s S_o b_o)^{n+1}] \\
& - b_g{}^n (1 - S_w{}^n - S_o{}^n) - (R_s S_o b_o)^n\}_{i,j,k} + R^{n+1}_{so\, i,j} Q^{n+1}_{o\, i,j,k} + Q^{n+1}_{fg\, i,j,k}
\end{aligned}
\tag{3.46}
$$

For the oil phase:

$$
\begin{aligned}
& T_{xo_{i+1/2,j,k}}[(P^{n+1}_{o_{i+1,j,k}} - P^{n+1}_{o_{i,j,k}}) - \gamma^{n+1}_{o_{i+1/2,j,k}}(h_{xi+1} - h_{xi})_{j,k}] \\
& - T_{xo_{i-1/2,j,k}}[(P^{n+1}_{o_{i,j,k}} - P^{n+1}_{o_{i-1,j,k}}) - \gamma^{n+1}_{o_{i-1/2,j,k}}(h_{xi} - h_{xi-1})_{j,k}] \\
& + T_{yo_{i,j+1/2,k}}[(P^{n+1}_{o_{i,j+1,k}} - P^{n+1}_{o_{i,j,k}}) - \gamma^{n+1}_{o_{i,j+1/2,k}}(h_{y_{j+1}} - h_{y_j})_{i,k}] \\
& - T_{yo_{i,j-1/2,k}}[(P^{n+1}_{o_{i,j,k}} - P^{n+1}_{o_{i,j-1,k}}) - \gamma^{n+1}_{o_{i,j-1/2,k}}(h_{y_j} - h_{y_{j-1}})_{i,k}] \\
& + T_{zo_{i,j,k+1/2}}[(P^{n+1}_{o_{i,j,k+1}} - P^{n+1}_{o_{i,j,k}}) - \gamma^{n+1}_{o_{i,j,k+1/2}}(z_{k+1} - z_k)_{i,j}] \\
& - T_{zo_{i,j,k-1/2}}[(P^{n+1}_{o_{i,j,k}} - P^{n+1}_{o_{i,j,k-1}}) - \gamma^{n+1}_{o_{i,j,k-1/2}}(z_k - z_{k-1})_{i,j}] \\
& = \frac{V^n_{p_{i,j,k}}}{\Delta t}[(1 + C_r \Delta_t P_o) \cdot (b_o S_o)^{n+1} - (S_o b_o)^n]_{i,j,k} + Q^{n+1}_{o_{i,j,k}}
\end{aligned}
\tag{3.47}
$$

3.5 Residual Function and Jacobian Matrix

After the discretization of the partial differential equations, the Newton-Raphson method is used to solve the system. The Newton-Raphson method involves calculating the Jacobian matrix and the right hand side vector. The Jacobian matrix is built by calculating all the derivatives of the residual function with respect to each unknown for each iteration. The right hand side vector is also updated with the residual function, which is expressed in terms of transmissibility, potential and source/sink term. The resulted system of equations obtained from the residual functions and associated derivatives can be expressed as:

$$\underset{\approx}{J}^{(\nu)}_{n+1} \cdot \delta \vec{X}^{(\nu+1)}_{n+1} = -\vec{F}^{(\nu)}_{n+1} \tag{3.48}$$

The Jacobian matrix structure is shown in figure 3.6, and contains four zones: Reservoir-Reservoir region, Reservoir-Wellbore region, Wellbore-Reservoir region, and Wellbore-Wellbore region. The Reservoir-Reservoir region is sub-divided as Cartesian system, cylindrical system, and hybrid system at side-wings zone.

The associated unknowns in the finite difference equations include reservoir pressure and saturation for each block, and wellbore pressure, liquid holdup and velocity for each segment. Because all the variables for both reservoir and wellbore domains should be solved simultaneously, the time-step control and convergence check are critical. The initial time-step is tested to ensure the convergence. To capture the early transient flow behaviour and to ensure the stability of the non-linear model system, the time-step is usually set to be small at the beginning and becomes automatic adaptive with increasing time.

The previous finite difference equations describe the flow of gas and oil in the reservoir, with associated unknowns:

$$P^{n+1}_{o_{i,j,k}}, P^{n+1}_{o_{i+1,j,k}}, P^{n+1}_{o_{i-1,j,k}}, P^{n+1}_{o_{i,j+1,k}}, P^{n+1}_{o_{i,j-1,k}}, P^{n+1}_{o_{i,j,k+1}}, P^{n+1}_{o_{i,j,k-1}}$$

$$S^{n+1}_{o_{i,j,k}}, S^{n+1}_{o_{i+1,j,k}}, S^{n+1}_{o_{i-1,j,k}}, S^{n+1}_{o_{i,j+1,k}}, S^{n+1}_{o_{i,j-1,k}}, S^{n+1}_{o_{i,j,k+1}}, S^{n+1}_{o_{i,j,k-1}}$$

These equations comprise a non-linear system of algebraic equations which will be solved with the Newton-Raphson Method. Therefore, the next step is to determine the residual function F_g and F_o that are associated with the gas and oil finite difference equations. The construction of the Jacobian matrix is made by calculating the derivatives of F_g and F_o with respect to $P^{n+1}_{o_{i,j,k}}, P^{n+1}_{o_{i+1,j,k}}, P^{n+1}_{o_{i-1,j,k}}, P^{n+1}_{o_{i,j+1,k}}, P^{n+1}_{o_{i,j-1,k}}, P^{n+1}_{o_{i,j,k+1}}, P^{n+1}_{o_{i,j,k-1}}$ and $S^{n+1}_{o_{i,j,k}}, S^{n+1}_{o_{i+1,j,k}}, S^{n+1}_{o_{i-1,j,k}}, S^{n+1}_{o_{i,j+1,k}}, S^{n+1}_{o_{i,j-1,k}}, S^{n+1}_{o_{i,j,k+1}}, S^{n+1}_{o_{i,j,k-1}}$.

The residual function F_g and F_o can be determined in the following formula.

For the gas phase:

$$
\begin{aligned}
F_g =\; & T_{xo_{i+1/2,j,k}} \cdot R_s[(P^{n+1}_{o_{i+1,j,k}} - P^{n+1}_{o_{i,j,k}}) - \gamma^{n+1}_{o_{i+1/2,j,k}}(h_{x\,i+1} - h_{x\,i})_{j,k}] \\
& - T_{xo_{i-1/2,j,k}} \cdot R_s[(P^{n+1}_{o_{i,j,k}} - P^{n+1}_{o_{i-1,j,k}}) - \gamma^{n+1}_{o_{i-1/2,j,k}}(h_{x\,i} - h_{x\,i-1})_{j,k}] \\
& + T_{yo_{i,j+1/2,k}} \cdot R_s[(P^{n+1}_{o_{i,j+1,k}} - P^{n+1}_{o_{i,j,k}}) - \gamma^{n+1}_{o_{i,j+1/2,k}}(h_{y\,j+1} - h_{y\,j})_{i,k}] \\
& - T_{yo_{i,j-1/2,k}} \cdot R_s[(P^{n+1}_{o_{i,j,k}} - P^{n+1}_{o_{i,j-1,k}}) - \gamma^{n+1}_{o_{i,j-1/2,k}}(h_{y\,j} - h_{y\,j-1})_{i,k}] \\
& + T_{zo_{i,j,k+1/2}} \cdot R_s[(P^{n+1}_{o_{i,j,k+1}} - P^{n+1}_{o_{i,j,k}}) - \gamma^{n+1}_{o_{i,j,k+1/2}}(z_{k+1} - z_k)_{i,j}] \\
& - T_{zo_{i,j,k-1/2}} \cdot R_s[(P^{n+1}_{o_{i,j,k}} - P^{n+1}_{o_{i,j,k-1}}) - \gamma^{n+1}_{o_{i,j,k-1/2}}(z_k - z_{k-1})_{i,j}] \\
& + T_{xg_{i+1/2,j,k}} \cdot [(P^{n+1}_{o_{i+1,j,k}} - P^{n+1}_{o_{i,j,k}}) - \gamma^{n+1}_{g_{i+1/2,j,k}}(h_{x\,i+1} - h_{x\,i})_{j,k}] \\
& - T_{xg_{i-1/2,j,k}} \cdot [(P^{n+1}_{o_{i,j,k}} - P^{n+1}_{o_{i-1,j,k}}) - \gamma^{n+1}_{g_{i-1/2,j,k}}(h_{x\,i} - h_{x\,i-1})_{j,k}] \\
& + T_{yg_{i,j+1/2,k}} \cdot [(P^{n+1}_{o_{i,j+1,k}} - P^{n+1}_{o_{i,j,k}}) - \gamma^{n+1}_{g_{i,j+1/2,k}}(h_{y\,j+1} - h_{y\,j})_{i,k}] \\
& - T_{yg_{i,j-1/2,k}} \cdot [(P^{n+1}_{o_{i,j,k}} - P^{n+1}_{o_{i,j-1,k}}) - \gamma^{n+1}_{g_{i,j-1/2,k}}(h_{y\,j} - h_{y\,j-1})_{i,k}] \\
& + T_{zg_{i,j,k+1/2}} \cdot [(P^{n+1}_{o_{i,j,k+1}} - P^{n+1}_{o_{i,j,k}}) - \gamma^{n+1}_{g_{i,j,k+1/2}}(z_{k+1} - z_k)_{i,j}] \\
& - T_{zg_{i,j,k-1/2}} \cdot [(P^{n+1}_{o_{i,j,k}} - P^{n+1}_{o_{i,j,k-1}}) - \gamma^{n+1}_{g_{i,j,k-1/2}}(z_k - z_{k-1})_{i,j}] \\
& + T^{n+1}_{xg_{i+1/2,j,k}} \cdot (P^{n+1}_{c_{i+1,j,k}} - P^{n+1}_{c_{i,j,k}}) - T^{n+1}_{xg_{i-1/2}} \cdot (P^{n+1}_{c_{i,j,k}} - P^{n+1}_{c_{i-1,j,k}}) \\
& + T^{n+1}_{yg_{i,j+1/2,k}} \cdot (P^{n+1}_{c_{i,j+1,k}} - P^{n+1}_{c_{i,j,k}}) - T^{n+1}_{yg_{j-1/2}} \cdot (P^{n+1}_{c_{i,j,k}} - P^{n+1}_{c_{i,j-1,k}}) \\
& + T^{n+1}_{zg_{i,j,k+1/2}} \cdot (P^{n+1}_{c_{i,j,k+1}} - P^{n+1}_{c_{i,j,k}}) - T^{n+1}_{zg_{k-1/2}} \cdot (P^{n+1}_{c_{i,j,k}} - P^{n+1}_{c_{i,j,k-1}}) \\
& - \frac{V^n_{p_{i,j,k}}}{\Delta t}\{(1 + C_r \Delta_t P_o) \cdot [b_g{}^{n+1}(1 + S_w{}^n (C_r + C_w)\Delta_t P_o - S_w{}^n - S_o{}^{n+1}) + (R_s S_o b_o)^{n+1}] \\
& - b_g{}^n (1 - S_w{}^n - S_o{}^n) - (R_s S_o b_o)^n\}_{i,j,k} - R^{n+1}_{so\,i,j} Q^{n+1}_{o\,i,j,k} - Q^{n+1}_{fg\,i,j,k}
\end{aligned}
\tag{3.49}
$$

For the oil phase:

$$
\begin{aligned}
F_o = & T_{xo_{i+1/2,j,k}}[(P^{n+1}_{o_{i+1,j,k}} - P^{n+1}_{o_{i,j,k}}) - \gamma^{n+1}_{o_{i+1/2,j,k}}(h_{x i+1} - h_{x i})_{j,k}] \\
& - T_{xo_{i-1/2,j,k}}[(P^{n+1}_{o_{i,j,k}} - P^{n+1}_{o_{i-1,j,k}}) - \gamma^{n+1}_{o_{i-1/2,j,k}}(h_{x i} - h_{x i-1})_{j,k}] \\
& + T_{yo_{i,j+1/2,k}}[(P^{n+1}_{o_{i,j+1,k}} - P^{n+1}_{o_{i,j,k}}) - \gamma^{n+1}_{o_{i,j+1/2,k}}(h_{y j+1} - h_{y j})_{i,k}] \\
& - T_{yo_{i,j-1/2,k}}[(P^{n+1}_{o_{i,j,k}} - P^{n+1}_{o_{i,j-1,k}}) - \gamma^{n+1}_{o_{i,j-1/2,k}}(h_{y j} - h_{y j-1})_{i,k}] \\
& + T_{zo_{i,j,k+1/2}}[(P^{n+1}_{o_{i,j,k+1}} - P^{n+1}_{o_{i,j,k}}) - \gamma^{n+1}_{o_{i,j,k+1/2}}(z_{k+1} - z_{k})_{i,j}] \\
& - T_{zo_{i,j,k-1/2}}[(P^{n+1}_{o_{i,j,k}} - P^{n+1}_{o_{i,j,k-1}}) - \gamma^{n+1}_{o_{i,j,k-1/2}}(z_{k} - z_{k-1})_{i,j}] \\
& - \frac{V^n_{p_{i,j,k}}}{\Delta t}[(1 + C_r \Delta_t P_o)\cdot(b_o S_o)^{n+1} - (S_o b_o)^n]_{i,j,k} - Q^{n+1}_{o_{i,j,k}}
\end{aligned}
\tag{3.50}
$$

Above residual functions F_g and F_o can be expressed with transmissibility and potential. Transmissibility terms have been developed in section 3.7. The potential can be defined by the following expressions:

$$
\begin{aligned}
POT_{gx1} &= (P^v_{oi,j,k} - P^v_{oi-1,j,k}) - \gamma^v_{g\,i-1/2,j,k}(h_{x\,i-1,j,k} - h_{x\,i,j,k}) + (P^v_{ci,j,k} - P^v_{ci-1,j,k}) \\
POT_{gx2} &= (P^v_{oi+1,j,k} - P^v_{oi,j,k}) - \gamma^v_{g\,i+1/2,j,k}(h_{x\,i,j,k} - h_{x\,i+1,j,k}) + (P^v_{ci+1,j,k} - P^v_{ci,j,k}) \\
POT_{gy1} &= (P^v_{oi,j,k} - P^v_{oi,j-1,k}) - \gamma^v_{g\,i,j-1/2,k}(h_{y\,i,j-1,k} - h_{y\,i,j,k}) + (P^v_{ci,j,k} - P^v_{ci,j-1,k}) \\
POT_{gy2} &= (P^v_{oi,j+1,k} - P^v_{oi,j,k}) - \gamma^v_{g\,i,j+1/2,k}(h_{y\,i,j,k} - h_{y\,i,j+1,k}) + (P^v_{ci,j+1,k} - P^v_{ci,j,k}) \\
POT_{gz1} &= (P^v_{oi,j,k} - P^v_{oi,j,k-1}) - \gamma^v_{g\,i,j,k-1/2}(z_{i,j,k-1} - z_{i,j,k}) + (P^v_{ci,j,k} - P^v_{ci,j,k-1}) \\
POT_{gz2} &= (P^v_{oi,j,k+1} - P^v_{oi,j,k}) - \gamma^v_{g\,i,j,k+1/2}(z_{i,j,k} - z_{i,j,k+1}) + (P^v_{ci,j,k+1} - P^v_{ci,j,k})
\end{aligned}
$$

$$
\begin{aligned}
POT_{ox1} &= (P^v_{oi,j,k} - P^v_{oi-1,j,k}) - \gamma^v_{o\,i-1/2,j,k}(h_{x\,i-1,j,k} - h_{x\,i,j,k}) \\
POT_{ox2} &= (P^v_{oi+1,j,k} - P^v_{oi,j,k}) - \gamma^v_{o\,i+1/2,j,k}(h_{x\,i,j,k} - h_{x\,i+1,j,k}) \\
POT_{oy1} &= (P^v_{oi,j,k} - P^v_{oi,j-1,k}) - \gamma^v_{o\,i,j-1/2,k}(h_{y\,i,j-1,k} - h_{y\,i,j,k}) \\
POT_{oy2} &= (P^v_{oi,j+1,k} - P^v_{oi,j,k}) - \gamma^v_{o\,i,j+1/2,k}(h_{y\,i,j,k} - h_{y\,i,j+1,k}) \\
POT_{oz1} &= (P^v_{oi,j,k} - P^v_{oi,j,k-1}) - \gamma^v_{o\,i,j,k-1/2}(z_{i,j,k-1} - z_{i,j,k}) \\
POT_{oz2} &= (P^v_{oi,j,k+1} - P^v_{oi,j,k}) - \gamma^v_{o\,i,j,k+1/2}(z_{i,j,k} - z_{i,j,k+1})
\end{aligned}
$$

(3.51)

Where,
$$(z_{i,j,k-1} - z_{i,j,k}) = (z_{i,j,k} - z_{i,j,k+1}) = -\Delta z = -\frac{z_e}{nz}$$

Then the gas and oil residual functions can be formulated as:

For the gas phase:

$$\begin{aligned}
F_g &= \\
& T_{xo_{i+1/2,j,k}} \cdot R_{s_{i+1/2,j,k}} \cdot POT_{ox2} - T_{xo_{i-1/2,j,k}} \cdot R_{s_{i-1/2,j,k}} \cdot POT_{ox1} \\
& + T_{yo_{i,j+1/2,k}} \cdot R_{s_{i,j+1/2,k}} \cdot POT_{oy2} - T_{yo_{i,j-1/2,k}} \cdot R_{s_{i,j-1/2,k}} \cdot POT_{oy1} \\
& + T_{zo_{i,j,k+1/2}} \cdot R_{s_{i,j,k+1/2}} \cdot POT_{oz2} - T_{zo_{i,j,k-1/2}} \cdot R_{s_{i,j,k-1/2}} \cdot POT_{oz1} \\
& + T_{xg_{i+1/2,j,k}} \cdot POT_{gx2} - T_{xg_{i-1/2,j,k}} \cdot POT_{gx1} \\
& + T_{yg_{i,j+1/2,k}} \cdot POT_{gy2} - T_{yg_{i,j-1/2,k}} \cdot POT_{gy1} \\
& + T_{zg_{i,j,k+1/2}} \cdot POT_{gz2} - T_{zg_{i,j,k-1/2}} \cdot POT_{gz1} \\
& - \frac{V^n_{p_{i,j,k}}}{\Delta t} \{(1 + C_r \Delta_t P_o) \cdot [b_g{}^{n+1}(1 + S_w{}^n (C_r + C_w)\Delta_t P_o - S_w{}^n - S_o{}^{n+1}) + (R_s S_o b_o)^{n+1}] \\
& - b_g{}^n (1 - S_w{}^n - S_o{}^n) - (R_s S_o b_o)^n\}_{i,j,k} - R^{n+1}_{so\ i,j} Q^{n+1}_{o\ i,j,k} - Q^{n+1}_{fg\ i,j,k}
\end{aligned} \tag{3.52}$$

For the oil phase:

$$\begin{aligned}
F_o &= \\
& T_{xo_{i+1/2,j,k}} \cdot POT_{ox2} - T_{xo_{i-1/2,j,k}} \cdot POT_{ox1} \\
& + T_{yo_{i,j+1/2,k}} \cdot POT_{oy2} - T_{yo_{i,j-1/2,k}} \cdot POT_{oy1} \\
& + T_{zo_{i,j,k+1/2}} \cdot POT_{oz2} - T_{zo_{i,j,k-1/2}} \cdot POT_{oz1} \\
& - \frac{V^n_{p_{i,j,k}}}{\Delta t} [(1 + C_r \Delta_t P_o) \cdot (b_o S_o)^{n+1} - (S_o b_o)^n]_{i,j,k} - Q^{n+1}_{o_{i,j,k}}
\end{aligned} \tag{3.53}$$

The residual functions, $F^{n+1}_{g_{i,j,k}}$ and $F^{n+1}_{o_{i,j,k}}$ can be approximated in an iterative way, $F^{\stackrel{(v+1)}{n+1}}_{g_{i,j,k}}$ and $F^{\stackrel{(v+1)}{n+1}}_{o_{i,j,k}}$ with a truncated Taylor Series expansion centered at current iteration:

$$F^{\stackrel{(v+1)}{n+1}}_{i,j,k} = F^{\stackrel{(v)}{n+1}}_{i,j,k} + \sum_{\forall x} \left(\frac{\partial F_{i,j,k}}{\partial x}\right)^{\stackrel{(v)}{n+1}} \delta x^{\stackrel{(v+1)}{n+1}}_{i,j,k} = 0 \tag{3.54}$$

Where,

$F = F_g$ and F_o

$i = 1$ to nx, $j = 1$ to ny, $k = 1$ to nz

$$x = P^{n+1}_{o_{i,j,k}}, P^{n+1}_{o_{i+1,j,k}}, P^{n+1}_{o_{i-1,j,k}}, P^{n+1}_{o_{i,j+1,k}}, P^{n+1}_{o_{i,j-1,k}}, P^{n+1}_{o_{i,j,k+1}}, P^{n+1}_{o_{i,j,k-1}}$$

$$S^{n+1}_{o_{i,j,k}}, S^{n+1}_{o_{i+1,j,k}}, S^{n+1}_{o_{i-1,j,k}}, S^{n+1}_{o_{i,j+1,k}}, S^{n+1}_{o_{i,j-1,k}}, S^{n+1}_{o_{i,j,k+1}}, S^{n+1}_{o_{i,j,k-1}}$$

$$\delta x^{\overset{(v+1)}{n+1}}_{i,j,k} = x^{\overset{(v+1)}{n+1}}_{i,j,k} - x^{\overset{(v)}{n+1}}_{i,j,k}$$

At $n+1$ step, for the gas phase:

$$\begin{aligned}
(F_{g_{i,j,k}})^{(v+1)} = (F_{g_{i,j,k}})^{(v)} &+ \left(\frac{\partial F_{g_{i,j,k}}}{\partial P_{o_{i,j,k}}}\right)^{(v)} \delta P^{(v+1)}_{oi,j,k} + \left(\frac{\partial F_{g_{i,j,k}}}{\partial P_{o_{i+1,j,k}}}\right)^{(v)} \delta P^{(v+1)}_{oi+1,j,k} + \left(\frac{\partial F_{g_{i,j,k}}}{\partial P_{o_{i-1,j,k}}}\right)^{(v)} \delta P^{(v+1)}_{oi-1,j,k} + \\
&\left(\frac{\partial F_{g_{i,j,k}}}{\partial P_{o_{i,j+1,k}}}\right)^{(v)} \delta P^{(v+1)}_{oi,j+1,k} + \left(\frac{\partial F_{g_{i,j,k}}}{\partial P_{o_{i,j-1,k}}}\right)^{(v)} \delta P^{(v+1)}_{oi,j-1,k} + \left(\frac{\partial F_{g_{i,j,k}}}{\partial P_{o_{i,j,k+1}}}\right)^{(v)} \delta P^{(v+1)}_{oi,j,k+1} + \left(\frac{\partial F_{g_{i,j,k}}}{\partial P_{o_{i,j,k-1}}}\right)^{(v)} \delta P^{(v+1)}_{oi,j,k-1} \\
&+ \left(\frac{\partial F_{g_{i,j,k}}}{\partial S_{o_{i,j,k}}}\right)^{(v)} \delta S^{(v+1)}_{oi,j,k} + \left(\frac{\partial F_{g_{i,j,k}}}{\partial S_{o_{i+1,j,k}}}\right)^{(v)} \delta S^{(v+1)}_{oi+1,j,k} + \left(\frac{\partial F_{g_{i,j,k}}}{\partial S_{o_{i-1,j,k}}}\right)^{(v)} \delta S^{(v+1)}_{oi-1,j,k} + \left(\frac{\partial F_{g_{i,j,k}}}{\partial S_{o_{i,j+1,k}}}\right)^{(v)} \delta S^{(v+1)}_{oi,j+1,k} \\
&+ \left(\frac{\partial F_{g_{i,j,k}}}{\partial S_{o_{i,j-1,k}}}\right)^{(v)} \delta S^{(v+1)}_{oi,j-1,k} + \left(\frac{\partial F_{g_{i,j,k}}}{\partial S_{o_{i,j,k+1}}}\right)^{(v)} \delta S^{(v+1)}_{oi,j,k+1} + \left(\frac{\partial F_{g_{i,j,k}}}{\partial S_{o_{i,j,k-1}}}\right)^{(v)} \delta S^{(v+1)}_{oi,j,k-1} + \left(\frac{\partial F_{g_{i,j,k}}}{\partial P_{wf}}\right)^{(v)} \delta P_{wf} = 0
\end{aligned} \tag{3.55}$$

Thus,

$$\begin{aligned}
&\left(\frac{\partial F_{g_{i,j,k}}}{\partial P_{o_{i,j,k}}}\right)^{(v)} \delta P^{(v+1)}_{oi,j,k} + \left(\frac{\partial F_{g_{i,j,k}}}{\partial P_{o_{i+1,j,k}}}\right)^{(v)} \delta P^{(v+1)}_{oi+1,j,k} + \left(\frac{\partial F_{g_{i,j,k}}}{\partial P_{o_{i-1,j,k}}}\right)^{(v)} \delta P^{(v+1)}_{oi-1,j,k} + \\
&\left(\frac{\partial F_{g_{i,j,k}}}{\partial P_{o_{i,j+1,k}}}\right)^{(v)} \delta P^{(v+1)}_{oi,j+1,k} + \left(\frac{\partial F_{g_{i,j,k}}}{\partial P_{o_{i,j-1,k}}}\right)^{(v)} \delta P^{(v+1)}_{oi,j-1,k} + \left(\frac{\partial F_{g_{i,j,k}}}{\partial P_{o_{i,j,k+1}}}\right)^{(v)} \delta P^{(v+1)}_{oi,j,k+1} + \left(\frac{\partial F_{g_{i,j,k}}}{\partial P_{o_{i,j,k-1}}}\right)^{(v)} \delta P^{(v+1)}_{oi,j,k-1} \\
&+ \left(\frac{\partial F_{g_{i,j,k}}}{\partial S_{o_{i,j,k}}}\right)^{(v)} \delta S^{(v+1)}_{oi,j,k} + \left(\frac{\partial F_{g_{i,j,k}}}{\partial S_{o_{i+1,j,k}}}\right)^{(v)} \delta S^{(v+1)}_{oi+1,j,k} + \left(\frac{\partial F_{g_{i,j,k}}}{\partial S_{o_{i-1,j,k}}}\right)^{(v)} \delta S^{(v+1)}_{oi-1,j,k} + \\
&\left(\frac{\partial F_{g_{i,j,k}}}{\partial S_{o_{i,j+1,k}}}\right)^{(v)} \delta S^{(v+1)}_{oi,j+1,k} + \left(\frac{\partial F_{g_{i,j,k}}}{\partial S_{o_{i,j-1,k}}}\right)^{(v)} \delta S^{(v+1)}_{oi,j-1,k} + \left(\frac{\partial F_{g_{i,j,k}}}{\partial S_{o_{i,j,k+1}}}\right)^{(v)} \delta S^{(v+1)}_{oi,j,k+1} + \left(\frac{\partial F_{g_{i,j,k}}}{\partial S_{o_{i,j,k-1}}}\right)^{(v)} \delta S^{(v+1)}_{oi,j,k-1} \\
&+ \left(\frac{\partial F_{g_{i,j,k}}}{\partial P_{wf}}\right)^{(v)} \delta P_{wf} = -(F_{g_{i,j,k}})^{(v)}
\end{aligned} \tag{3.56}$$

Analogous, for the oil phase:

$$(F_{o_{i,j,k}})^{(v+1)} = (F_{o_{i,j,k}})^{(v)} + (\frac{\partial F_{o_{i,j,k}}}{\partial P_{o_{i,j,k}}})^{(v)} \delta P_{oi,j,k}^{(v+1)} + (\frac{\partial F_{o_{i,j,k}}}{\partial P_{o_{i+1,j,k}}})^{(v)} \delta P_{oi+1,j,k}^{(v+1)} + (\frac{\partial F_{o_{i,j,k}}}{\partial P_{o_{i-1,j,k}}})^{(v)} \delta P_{oi-1,j,k}^{(v+1)} +$$
$$(\frac{\partial F_{o_{i,j,k}}}{\partial P_{o_{i,j+1,k}}})^{(v)} \delta P_{oi,j+1,k}^{(v+1)} + (\frac{\partial F_{o_{i,j,k}}}{\partial P_{o_{i,j-1,k}}})^{(v)} \delta P_{oi,j-1,k}^{(v+1)} + (\frac{\partial F_{o_{i,j,k}}}{\partial P_{o_{i,j,k+1}}})^{(v)} \delta P_{oi,j,k+1}^{(v+1)} + (\frac{\partial F_{o_{i,j,k}}}{\partial P_{o_{i,j,k-1}}})^{(v)} \delta P_{oi,j,k-1}^{(v+1)}$$
$$+ (\frac{\partial F_{o_{i,j,k}}}{\partial S_{o_{i,j,k}}})^{(v)} \delta S_{oi,j,k}^{(v+1)} + (\frac{\partial F_{o_{i,j,k}}}{\partial S_{o_{i+1,j,k}}})^{(v)} \delta S_{oi+1,j,k}^{(v+1)} + (\frac{\partial F_{o_{i,j,k}}}{\partial S_{o_{i-1,j,k}}})^{(v)} \delta S_{oi-1,j,k}^{(v+1)} + (\frac{\partial F_{o_{i,j,k}}}{\partial S_{o_{i,j+1,k}}})^{(v)} \delta S_{oi,j+1,k}^{(v+1)}$$
$$+ (\frac{\partial F_{o_{i,j,k}}}{\partial S_{o_{i,j-1,k}}})^{(v)} \delta S_{oi,j-1,k}^{(v+1)} + (\frac{\partial F_{o_{i,j,k}}}{\partial S_{o_{i,j,k+1}}})^{(v)} \delta S_{oi,j,k+1}^{(v+1)} + (\frac{\partial F_{o_{i,j,k}}}{\partial S_{o_{i,j,k-1}}})^{(v)} \delta S_{oi,j,k-1}^{(v+1)} + (\frac{\partial F_{o_{i,j,k}}}{\partial P_{wf}})^{(v)} \delta P_{wf} = 0$$

(3.57)

Thus,

$$(\frac{\partial F_{o_{i,j,k}}}{\partial P_{o_{i,j,k}}})^{(v)} \delta P_{oi,j,k}^{(v+1)} + (\frac{\partial F_{o_{i,j,k}}}{\partial P_{o_{i+1,j,k}}})^{(v)} \delta P_{oi+1,j,k}^{(v+1)} + (\frac{\partial F_{o_{i,j,k}}}{\partial P_{o_{i-1,j,k}}})^{(v)} \delta P_{oi-1,j,k}^{(v+1)} +$$
$$+ (\frac{\partial F_{o_{i,j,k}}}{\partial P_{o_{i,j+1,k}}})^{(v)} \delta P_{oi,j+1,k}^{(v+1)} + (\frac{\partial F_{o_{i,j,k}}}{\partial P_{o_{i,j-1,k}}})^{(v)} \delta P_{oi,j-1,k}^{(v+1)} + (\frac{\partial F_{o_{i,j,k}}}{\partial P_{o_{i,j,k+1}}})^{(v)} \delta P_{oi,j,k+1}^{(v+1)} + (\frac{\partial F_{o_{i,j,k}}}{\partial P_{o_{i,j,k-1}}})^{(v)} \delta P_{oi,j,k-1}^{(v+1)}$$
$$+ (\frac{\partial F_{o_{i,j,k}}}{\partial S_{o_{i,j,k}}})^{(v)} \delta S_{oi,j,k}^{(v+1)} + (\frac{\partial F_{o_{i,j,k}}}{\partial S_{o_{i+1,j,k}}})^{(v)} \delta S_{oi+1,j,k}^{(v+1)} + (\frac{\partial F_{o_{i,j,k}}}{\partial S_{o_{i-1,j,k}}})^{(v)} \delta S_{oi-1,j,k}^{(v+1)} +$$
$$+ (\frac{\partial F_{o_{i,j,k}}}{\partial S_{o_{i,j+1,k}}})^{(v)} \delta S_{oi,j+1,k}^{(v+1)} + (\frac{\partial F_{o_{i,j,k}}}{\partial S_{o_{i,j-1,k}}})^{(v)} \delta S_{oi,j-1,k}^{(v+1)} + (\frac{\partial F_{o_{i,j,k}}}{\partial S_{o_{i,j,k+1}}})^{(v)} \delta S_{oi,j,k+1}^{(v+1)} + (\frac{\partial F_{o_{i,j,k}}}{\partial S_{o_{i,j,k-1}}})^{(v)} \delta S_{oi,j,k-1}^{(v+1)}$$
$$+ (\frac{\partial F_{o_{i,j,k}}}{\partial P_{wf}})^{(v)} \delta P_{wf} = -(F_{o_{i,j,k}})^{(v)}$$

(3.58)

To solve this problem with the support of the Newton-Raphson method, the following iterative process should be used:

$$\underset{\approx}{J}{}_{n+1}^{(v)} \cdot \delta \vec{X}_{n+1}^{(v+1)} = -\vec{F}_{n+1}^{(v)}$$

Where,

$\underset{\approx}{J}{}_{n+1}^{(v)}$ is the Jacobian Matrix;

$\vec{F}_{n+1}^{(v)}$ is the Residual function vector: $F_{g_{i,j,k}}$, $F_{o_{i,j,k}}$;

$\overset{(v+1)}{\vec{X}^{n+1}}$ is the vector of unknowns for future iteration (variables $P_{o_{i,j,k}}$, $S_{o_{i,j,k}}$, $i = 1$ to $nx, j = 1$ to ny, $k = 1$ to nz); $\overset{(v+1)}{\vec{X}^{n+1}} = \overset{(v)}{\vec{X}^{n+1}} + \delta \overset{(v+1)}{\vec{X}^{n+1}}$

$\overset{(v)}{\vec{X}^{n+1}}$ is the vector of unknowns for current iteration (variables $P_{o_{i,j,k}}, S_{o_{i,j,k}}$, $i = 1$ to $nx, j = 1$ to ny, $k = 1$ to nz);

$\delta \overset{(v+1)}{\vec{X}^{n+1}}$ is the vector with variations from current to future iteration (variables $P_{o_{i,j,k}}, S_{o_{i,j,k}}$, $i = 1$ to $nx, j = 1$ to ny, $k = 1$ to nz);

The iterative process is repeated until,

$$\left| \delta \overset{(v+1)}{\vec{X}} \right|_{\forall x} \leq tolerance \begin{cases} saturation\ tolerance\ if\ X = S_o \\ pressure\ tolerance\ if\ X = P_o \end{cases} \tag{3.59}$$

The associated unknowns in the finite difference equations of reservoir flow model are:

$$P^{n+1}_{o_{i,j,k}}, P^{n+1}_{o_{i+1,j,k}}, P^{n+1}_{o_{i-1,j,k}}, P^{n+1}_{o_{i,j+1,k}}, P^{n+1}_{o_{i,j-1,k}}, P^{n+1}_{o_{i,j,k+1}}, P^{n+1}_{o_{i,j,k-1}}$$

$$S^{n+1}_{o_{i,j,k}}, S^{n+1}_{o_{i+1,j,k}}, S^{n+1}_{o_{i-1,j,k}}, S^{n+1}_{o_{i,j+1,k}}, S^{n+1}_{o_{i,j-1,k}}, S^{n+1}_{o_{i,j,k+1}}, S^{n+1}_{o_{i,j,k-1}}$$

As mentioned before, it is necessary to obtain all the derivatives of F_g and F_o with respect to P_o and S_o for each grid location, including the derivatives of potentials and accumulation terms with respect to P_o and S_o. All the details about derivation can be found in Appendix 2.

In the above sections, the partial differential equations and residual functions include the source / sink terms for both oil and gas phases, as a traditional reservoir simulator does. To compare traditional uncoupled flow model to the proposed coupled model, this section briefly introduces the development of the source/sink terms in the traditional reservoir flow model. For the source/sink terms, the oil flow rate can be obtained by applying the Taylor series expansion during each iteration.

$$\overset{v+1}{q}{}^{n+1}_{ok} = \overset{v}{q}{}^{n+1}_{ok} + \left.\frac{\partial q_{ok}}{\partial S_o}\right|^{(v)}_{n+1} (\overset{v+1}{S}{}^{n+1}_{ok} - \overset{v}{S}{}^{n+1}_{ok}) + \left.\frac{\partial q_{ok}}{\partial P_o}\right|^{(v)}_{n+1} (\overset{v+1}{P}{}^{n+1}_{ok} - \overset{v}{P}{}^{n+1}_{ok}) + \left.\frac{\partial q_{ok}}{\partial P_{wf}}\right|^{(v)}_{n+1} (\overset{v+1}{P}{}^{n+1}_{wfref} - \overset{v}{P}{}^{n+1}_{wfref}) \tag{3.60}$$

The wellbore flow pressure at reference elevation can be calculated as:

$$\overset{v+1}{P}{}^{n+1}_{wfref} = \overset{v}{P}{}^{n+1}_{wfref} - \frac{\sum_k [(\overset{v+1}{q}{}^{n+1}_{ok} - \overset{v}{q}{}^{n+1}_{ok}) - \left.\frac{\partial q_{ok}}{\partial S_o}\right|^{(v)}_{n+1} (\overset{v+1}{S}{}^{n+1}_{ok} - \overset{v}{S}{}^{n+1}_{ok}) - \left.\frac{\partial q_{ok}}{\partial P_o}\right|^{(v)}_{n+1} (\overset{v+1}{P}{}^{n+1}_{ok} - \overset{v}{P}{}^{n+1}_{ok})]}{\sum_k [\left.\frac{\partial q_{ok}}{\partial P_{wf}}\right|^{(v)}_{n+1}]} \tag{3.61}$$

Therefore, wellbore pressure can be calculated as:

$$P_{wfk} = P_{wfref} + \bar{\gamma}\Delta H_k \tag{3.62}$$

Where in (3.60) and (3.61),

$$\left.\frac{\partial q_{ok}}{\partial S_o}\right|^{(v)}_{n+1} = WI \cdot \frac{\partial \lambda_{ok}}{\partial S_{ok}} (P_{ok} - P_{wfk})$$

$$\left.\frac{\partial q_{ok}}{\partial P_o}\right|^{(v)}_{n+1} = WI \cdot \frac{\partial \lambda_{ok}}{\partial P_{ok}} (P_{ok} - P_{wfk}) + WI \cdot \lambda_{ok}$$

$$\left.\frac{\partial q_{ok}}{\partial P_{wf}}\right|^{(v)}_{n+1} = -WI \cdot \lambda_{ok} \cdot \frac{\partial P_{wfk}}{\partial P_{wfref}} = -WI \cdot \lambda_{ok} \cdot [1 + \frac{\partial \bar{\gamma}_{wb}}{\partial P_{wfref}} (H_k - H_{ref})] \approx -WI \cdot \lambda_{ok}$$

$$\lambda_o = \frac{K_H K_{ro}}{B_o \mu_o} = \frac{b_o K_H K_{ro}}{\mu_o}$$

K_H is the geometric average of the absolute permeability in horizontal direction:

$$K_H = (K_x K_y)^{0.5}$$

Well index WI can be expressed as $WI = \dfrac{2\pi\Delta z}{\ln\dfrac{r_{eq}}{r_w} + S}$ and S=0

r_{eq} is calculated with Peaceman's model for non-square well block

$r_{eq} \approx 0.208\Delta x$ or in more general form:

$$r_{eq} = 0.28\frac{[(k_y/k_x)^{1/2}(\Delta x)^2 + (k_x/k_y)^{1/2}(\Delta y)^2]^{1/2}}{(k_x/k_y)^{1/4} + (k_y/k_x)^{1/4}}.$$

The gas flow can be determined by using the mobility ratio method based on oil flow rate or by applying the Taylor series expansion as well.

$$Q_g = (\frac{\lambda_{g_{i,j,k}}}{\lambda_{o_{i,j,k}}} + R_{s_{i,j,k}}) \cdot Q_o \quad \rightarrow \quad Q_g = [(\frac{k_{rg}\mu_o b_g}{k_{ro}\mu_g b_o})_{i,j,k} + R_{s_{i,j,k}}] \cdot Q_o$$

Or

$$\overset{v+1}{q_{gk}^{n+1}} = \overset{v}{q_{gk}^{n+1}} + \left.\frac{\partial q_{gk}}{\partial S_o}\right|_{n+1}^{(v)} (\overset{v+1}{S_{ok}^{n+1}} - \overset{v}{S_{ok}^{n+1}}) + \left.\frac{\partial q_{gk}}{\partial P_o}\right|_{n+1}^{(v)} (\overset{v+1}{P_{ok}^{n+1}} - \overset{v}{P_{ok}^{n+1}}) + \left.\frac{\partial q_{gk}}{\partial P_{wf}}\right|_{n+1}^{(v)} (\overset{v+1}{P_{wfref}^{n+1}} - \overset{v}{P_{wfref}^{n+1}}) \tag{3.62}$$

Where,

$$\left.\frac{\partial q_{gk}}{\partial P_o}\right|_{n+1}^{(v)} = \left.\frac{\partial (q_{ok} \cdot Rs + q_{fgk})}{\partial P_o}\right|_{n+1}^{(v)} == [Rs \cdot \frac{\partial q_{ok}}{\partial P_o} + q_{ok}\frac{\partial Rs}{\partial P_o}] + [WI \cdot \frac{\partial \lambda_{fgk}}{\partial P_o}(P_{ok} - P_{wfk}) + WI \cdot \lambda_{fgk}]$$

$$\left.\frac{\partial q_{gk}}{\partial S_o}\right|_{n+1}^{(v)} = \left.\frac{\partial (q_{ok} \cdot Rs + q_{fgk})}{\partial S_o}\right|_{n+1}^{(v)} = Rs \cdot \frac{\partial q_{ok}}{\partial S_o} + WI \cdot \frac{\partial \lambda_{gk}}{\partial S_{ok}}(P_{ok} - P_{wfk})$$

$$\left.\frac{\partial q_{gk}}{\partial P_{wf}}\right|_{n+1}^{(v)} = \left.\frac{\partial (q_{ok} \cdot Rs + q_{fgk})}{\partial P_{wf}}\right|_{n+1}^{(v)} = Rs \cdot \frac{\partial q_{ok}}{\partial P_{wf}} - WI \cdot \lambda_{gfk}$$

However, a reservoir flow model using simplified source/sink term cannot fully reflect the interaction between wellbore flow and reservoir flow. Therefore the reservoir/wellbore coupled model will not use source sink terms to express production or injection rate. Instead, a two phase wellbore flow model is developed separately to reflect the interaction between reservoir and wellbore. Chapter 4 presents the development of two phase wellbore flow model.

CHAPTER 4: DEVELOPMENT OF THE WELLBORE FLOW MODEL

Since the simplified source/sink term cannot fully reflect the interaction between wellbore flow and reservoir flow, the coupled model needs to integrate a two phase wellbore flow model to the reservoir model in order to determine pressure drop, liquid holdup and flow pattern along the wellbore. This chapter will present the development of this two phase wellbore flow model.

The horizontal wellbore geometry is different than a regular pipeline because of the influx from the reservoir through the perforations. To study the interaction caused by the influx and pressure drop in the wellbore, the horizontal wellbore is divided into segments (multi-segmented well). Then the wellbore flow equations can be derived accordingly based on the control volume in each wellbore segment. The two phase wellbore model is based on a homogeneous flow pattern (dispersed bubble flow) and consists of mass conservation equations and the momentum equation. The radial volumetric influx from reservoir to the wellbore is calculated with mass conservation equations to capture the interaction between the wellbore and the reservoir. The influx along the wellbore makes the flow behavior and wellbore model more complex (Ouyang and Aziz, 1998). For gas-liquid wellbore flow, the influx can alter gas and liquid flow rates along the wellbore. As a result, gas and liquid superficial velocities and slip velocity will change along the wellbore. Additionally, the influx amount of fluid transfer between the reservoir and the wellbore will introduce different flow patterns in the horizontal wellbore. The momentum equation in the multi-segmented wellbore model deals with different types of pressure losses, such as segment pressure difference, wellbore wall friction and acceleration components. The variables in the wellbore equations include fluid velocity, wellbore pressure and reservoir pressure of wellbore block. For a multi-phase case, variables also include saturation distribution in reservoir and liquid holdup in the wellbore.

When the pressure drop in the wellbore and the influx from the reservoir are considered simultaneously with conventional reservoir simulation, the results can be reasonably accurate, especially when high flow rate and consequently high wellbore pressure drops are present. In addition, flowing properties are updated with real-time local flowing conditions. Therefore, the reservoir/wellbore coupled model can reveal the fluid flow in the reservoir near the wellbore and also the actual characteristics in the horizontal wellbore. The following sections elaborate on the development of a wellbore flow model.

4.1 Partial Differential Equations

The horizontal wellbore can be seen as a regular pipeline, but with influx from the reservoir through the perforations. To develop the wellbore flow model, the horizontal wellbore is divided into segments (multi-segmented well), as shown in figure 4.1. The block centered scheme is adopted to discretize the wellbore equations in the horizontal direction.

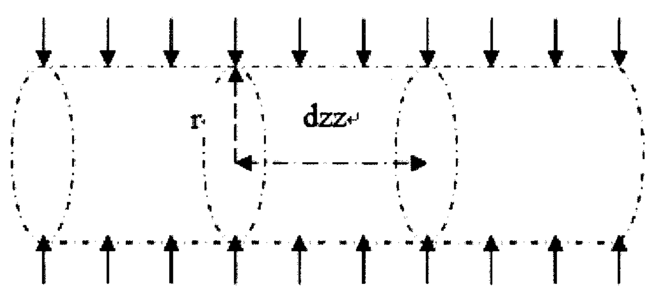

Figure 4.1 - Horizontal wellbore segments

In the horizontal direction *zz* we have a grid naturally regular and the increments *dzz* (constants) can be calculated by: $dzz = \frac{L}{nzz}$

Where,

nzz is the number of segments in the horizontal wellbore;

L is the total length of the horizontal wellbore;

dzz is the length of each segment of the wellbore;

Therefore, the grid block interfaces positions $(i \pm \frac{1}{2})$ can be expressed as:

$$x_i = x_{i-1/2} + \frac{dzz}{2} \tag{4.1}$$

In the wellbore system, the horizontal wellbore flow equations can be derived based on a control volume in each wellbore section, which is $V = dzz \cdot (\pi r^2)$. The position of the grid nodes in each cell occurs in the geometric point where the average pressure $\overline{P}$ of the cell is applied. Taking into consideration the one-dimension flow continuity and momentum balance through the control volume, we have the continuity equation and oil and gas mixture momentum equation as the following (*dzz* is expressed as Δx in the formula):

Oil Continuity Equation:

$$\frac{\partial}{\partial x}(A_x \frac{H_o}{B_o} v)\Delta x + q_{osc} = V_b \frac{\partial}{\partial t}(\frac{H_o}{B_o}) \tag{4.1}$$

Extended with partial derivatives

$$A_x[H_o v \frac{\partial b_o}{\partial x} + b_o v \frac{\partial H_o}{\partial x} + b_o H_o \frac{\partial v}{\partial x}]\Delta x + q_{osc} = V_b \frac{\partial}{\partial t}(b_o H_o) \tag{4.2}$$

Gas Continuity Equations:

$$\frac{\partial}{\partial x}(A_x \frac{H_g}{B_g} v + A_x R_s \frac{H_o}{B_o} v)\Delta x + q_{gsc-free} + R_s q_{osc} = V_b \frac{\partial}{\partial t}(\frac{H_g}{B_g} + R_s \frac{H_o}{B_o}) \tag{4.3}$$

Extended with partial derivatives:

$$\begin{aligned} &A_x[H_g v \frac{\partial b_g}{\partial x}\Delta x + b_g v \frac{\partial H_g}{\partial x}\Delta x + b_g H_g \frac{\partial v}{\partial x}\Delta x] + \\ &A_x[R_s H_o v \frac{\partial b_o}{\partial x}\Delta x + R_s b_o v \frac{\partial H_o}{\partial x}\Delta x + R_s b_o H_o \frac{\partial v}{\partial x}\Delta x + b_o H_o v \frac{\partial R_s}{\partial x}\Delta x] + \\ &q_{gsc-free} + R_s q_{osc} = V_b[\frac{\partial(b_g H_g)}{\partial t} + \frac{\partial(R_s b_o H_o)}{\partial t}] \end{aligned} \tag{4.4}$$

Mixture Momentum Equation

$$(\frac{\partial P}{\partial x}) + \rho v(\frac{\partial v}{\partial x}) - \frac{v}{V_b}[\rho_{osc} q_{osc} + \rho_{gsc}(q_{gsc-free} + R_s q_{osc})] - (\frac{\rho f v|v|}{2d}) = \rho \frac{\partial v}{\partial t} \tag{4.5}$$

Where,

A_x is the section area πr^2;

H_o is the oil hold-up, and $H_g = 1 - H_o$;

B_o, B_g are the formation volume factors for oil and gas;

v is the velocity of the mixture flow in wellbore, assuming homogenous flow;

R_s is the gas/oil solution ratio;

Δx is the length of the wellbore section =dzz;

q_{gsc}, q_{osc} are oil and gas influx from reservoir per unit of wellbore section;

V_b is the control volume $V = \Delta zz \cdot (\pi r^2)$;

P is the pressure in wellbore;

ρ_{osc}, ρ_{gsc} are oil and gas densities at standard condition;

ρ is the mixture density in the wellbore;

f is the Moody's friction factor;

d is the inner diameter of the wellbore.

To solve the above equations, we must also consider the following additional relationships:

- The influx from reservoir:

$$Q_{osc} = \frac{\partial}{\partial r}[k_r A_r \frac{k_{ro}}{\mu_o b_o}(\frac{\partial P_o}{\partial r} - \gamma_o \frac{\partial G}{\partial r})] \cdot \Delta r \tag{4.6}$$

$$Q_{gsc} = \frac{\partial}{\partial r}[k_r A_r \frac{k_{rg}}{\mu_g b_g}(\frac{\partial P_g}{\partial r} - \gamma_g \frac{\partial G}{\partial r}) + k_r A_r \frac{R_s k_{ro}}{\mu_o b_o}(\frac{\partial P_o}{\partial r} - \gamma_o \frac{\partial G}{\partial r})] \cdot \Delta r \tag{4.7}$$

- The density of mixture in the wellbore:

$\rho_l = \frac{\rho_{lsc}}{B_l}$, where l represent oil or gas

$$\rho = H_o(\frac{\rho_{osc} + \rho_{gsc} R_s}{B_o}) + H_g \frac{\rho_{gsc}}{B_g} \tag{4.8}$$

$$\frac{\partial \rho}{\partial P_{wb\,i}} = H_o[\frac{\rho_{gsc}}{B_o}\frac{\partial R_s}{\partial P_{wb\,i}} + (\rho_{osc} + \rho_{gsc} R_s)\frac{\partial b_o}{\partial P_{wb\,i}}] + H_g \rho_{gsc} \frac{\partial b_g}{\partial P_{wb\,i}} \tag{4.9}$$

- The viscosity of mixture in the wellbore:

$$\mu = H_o\mu_o + H_g\mu_g \tag{4.10}$$

Determination of other properties and derivatives at the wellbore block interface can be approached using either Property weighting method or Pressure weighting method, similar to the interface properties calculation in Chapter 3, please refer to section 3.3.4 for more details.

4.2 Finite Difference Equations

To discretize the wellbore partial differential equations (4.1 to 4.5), a block centered scheme is used for the spatial discretization and a fully implicit method for the time discretization (backward discretization).

For the oil continuity equation:

$\frac{\partial}{\partial x}(A_x \frac{H_o}{B_o} v)\Delta x + q_{osc} = V_b \frac{\partial}{\partial t}(\frac{H_o}{B_o})$, using b_o (=1/B_o) instead of B_o, there is:

$$A_x \frac{\partial}{\partial x}(b_o H_o v)\Delta x + q_{osc} = V_b \frac{\partial}{\partial t}(b_o H_o)$$

$$A_x[H_o v \frac{\partial b_o}{\partial x}\Delta x + b_o v \frac{\partial H_o}{\partial x}\Delta x + b_o H_o \frac{\partial v}{\partial x}\Delta x] + q_{osc} = V_b \frac{\partial}{\partial t}(b_o H_o)$$

$$A_x[H_o v \frac{\partial b_o}{\partial x} + b_o v \frac{\partial H_o}{\partial x} + b_o H_o \frac{\partial v}{\partial x}]\Delta x + q_{osc} = V_b \frac{\partial}{\partial t}(b_o H_o)$$

So after finite difference treatment:

$$A_x[H_o v \frac{(b_o)_{i+1/2}^{n+1} - (b_o)_{i-1/2}^{n+1}}{\Delta x} + b_o v \frac{(H_o)_{i+1/2}^{n+1} - (H_o)_{i-1/2}^{n+1}}{\Delta x} + b_o H_o \frac{(v)_{i+1/2}^{n+1} - (v)_{i-1/2}^{n+1}}{\Delta x}]\Delta x + q_{osc}$$
$$= V_b[\frac{(b_o H_o)_i^{n+1} - (b_o H_o)_i^n}{\Delta t}]$$

$$A_x\{H_o v \cdot [(b_o)_{i+1/2}^{n+1} - (b_o)_{i-1/2}^{n+1}] + b_o v \cdot [(H_o)_{i+1/2}^{n+1} - (H_o)_{i-1/2}^{n+1}] + b_o H_o \cdot [(v)_{i+1/2}^{n+1} - (v)_{i-1/2}^{n+1}]\} + q_{osc}$$
$$= V_b[\frac{(b_o H_o)_i^{n+1} - (b_o H_o)_i^n}{\Delta t}]$$

(4.11)

For the wellbore equation, the subscript i can be changed to k for the direction along wellbore in the wellbore coordinates system. So the above finite difference equation becomes:

$$A_x H_o v\cdot[(b_o)_{k+1/2}^{n+1}-(b_o)_{k-1/2}^{n+1}]+A_x b_o v\cdot[(H_o)_{k+1/2}^{n+1}-(H_o)_{k-1/2}^{n+1}]+A_x b_o H_o\cdot[(v)_{k+1/2}^{n+1}-(v)_{k-1/2}^{n+1}]$$
$$+\sum_{j=1}^{N_\theta}\{[\frac{A_r K_r K_{ro}}{\mu_o B_o \Delta r}]_{1/2,j,k}^{n+1}(p_{o1,j,k}^{n+1}-p_{o-wb-k}^{n+1})-[\frac{A_r K_r K_{ro}\gamma_o}{\mu_o B_o \Delta r}]_{1/2,j,k}^{n+1}(G_{1,j,k}-G_{wb-k})]\}$$
$$=V_b[\frac{(b_o H_o)_k^{n+1}-(b_o H_o)_k^n}{\Delta t}]$$

(4.12)

For the gas continuity equation:

$$A_x\frac{\partial}{\partial x}(\frac{H_g}{B_g}v+R_s\frac{H_o}{B_o}v)\Delta x+q_{gsc-free}+R_s q_{osc}=V_b\frac{\partial}{\partial t}(\frac{H_g}{B_g}+R_s\frac{H_o}{B_o})$$

$$A_x\frac{\partial}{\partial x}(\frac{H_g}{B_g}v)+A_x\frac{\partial}{\partial x}(R_s\frac{H_o}{B_o}v)\Delta x+q_{gsc-free}+R_s q_{osc}=V_b\frac{\partial}{\partial t}(\frac{H_g}{B_g})+V_b\frac{\partial}{\partial t}(R_s\frac{H_o}{B_o})$$

Reorganizing the above equation, we have:

$$A_x[H_g v\frac{\partial b_g}{\partial x}\Delta x+b_g v\frac{\partial H_g}{\partial x}\Delta x+b_g H_g\frac{\partial v}{\partial x}\Delta x]$$
$$A_x[R_s H_o v\frac{\partial b_o}{\partial x}\Delta x+R_s b_o v\frac{\partial H_o}{\partial x}\Delta x+R_s b_o H_o\frac{\partial v}{\partial x}\Delta x+b_o H_o v\frac{\partial R_s}{\partial x}\Delta x]$$
$$+q_{gsc-free}+R_s q_{osc}$$
$$=V_b[\frac{\partial(b_g H_g)}{\partial t}+\frac{\partial(R_s b_o H_o)}{\partial t}]$$

So, after a finite difference treatment:

$$[A_x v H_g]_i^{n+1}[(b_g)_{i+1/2}^{n+1}-(b_g)_{i-1/2}^{n+1}]+[A_x H_g b_g]_i^{n+1}[(v)_{i+1/2}^{n+1}-(v)_{i-1/2}^{n+1}]+[A_x v b_g]_i^{n+1}[(H_g)_{i+1/2}^{n+1}-(H_g)_{i-1/2}^{n+1}]$$
$$+R_{s,i}^{n+1}X_o+[A_x v H_o b_o]_i^{n+1}[(R_{s i+1/2}^{n+1})-(R_{s i-1/2}^{n+1})]$$
$$+\sum_{j=1}^{N\theta}\{[\frac{A_r K_r K_{rg}}{\mu_g B_g \Delta r}]_{i+1/2,j,k}^{n+1}(p_{gi+1,j,k}^{n+1}-p_{g-wb-i}^{n+1})-[\frac{A_r K_r K_{rg}\gamma_g}{\mu_g B_g \Delta r}]_{i+1/2,j,k}^{n+1}(G_{i+1,j,k}-G_{wb-i})]\}+R_{s,i}^{n+1}q_{osc,i}^{n+1}$$
$$=[\frac{V_b}{\Delta t}]_i^{n+1}\{[(b_g H_g)_i^{n+1}-(b_g H_g)_i^n]+[(R_s b_o H_o)_i^{n+1}-(R_s b_o H_o)_i^n]\}$$

(4.13)

Similar to the oil equation, the gas continuity finite difference equation in the wellbore coordinates system becomes:

$$[A_x vH_g]_k^{n+1}[(b_g)_{k+1/2}^{n+1}-(b_g)_{k-1/2}^{n+1}]+[A_x H_g b_g]_k^{n+1}[(v)_{k+1/2}^{n+1}-(v)_{k-1/2}^{n+1}]+[A_x vb_g]_k^{n+1}[(H_g)_{k+1/2}^{n+1}-(H_g)_{k-1/2}^{n+1}]$$
$$+R_{s,k}^{n+1}X_o+[A_x vH_o b_o]_k^{n+1}[(R_{sk+1/2}^{n+1})-(R_{sk-1/2}^{n+1})]$$
$$+\sum_{j=1}^{N\theta}\{[\frac{A_r K_r K_{rg}}{\mu_g B_g \Delta r}]_{1/2,j,k}^{n+1}(p_{g1,j,k}^{n+1}-p_{g-wb-k}^{n+1})-[\frac{A_r K_r K_{rg}\gamma_g}{\mu_g B_g \Delta r}]_{1/2,j,k}^{n+1}(G_{1,j,k}-G_{wb-k})]\}+R_{s,k}^{n+1}q_{osc,k}^{n+1}$$
$$=[\frac{V_b}{\Delta t}]_k^{n+1}\{[(b_g H_g)_k^{n+1}-(b_g H_g)_k^n]+[(R_s b_o H_o)_k^{n+1}-(R_s b_o H_o)_k^n]\}$$

(4.14)

Where,

$$X_o=A_x H_o v\cdot[(b_o)_{k+1/2}^{n+1}-(b_o)_{k-1/2}^{n+1}]+A_x b_o v\cdot[(H_o)_{k+1/2}^{n+1}-(H_o)_{k-1/2}^{n+1}]+A_x b_o H_o\cdot[(v)_{k+1/2}^{n+1}-(v)_{k-1/2}^{n+1}]$$

$$q_{osc}=\sum_{j=1}^{N_\theta}\{[\frac{A_r K_r K_{ro}}{\mu_o B_o \Delta r}]_{1/2,j,k}^{n+1}(p_{o1,j,k}^{n+1}-p_{o-wb-k}^{n+1})-[\frac{A_r K_r K_{ro}\gamma_o}{\mu_o B_o \Delta r}]_{1/2,j,k}^{n+1}(G_{1,j,k}-G_{wb-k})]\}$$

For the mixture momentum equation:

$$(\frac{\partial P}{\partial x})+\rho v(\frac{\partial v}{\partial x})-\frac{v}{V_b}[\rho_{osc}q_{osc}+\rho_{gsc}(q_{gsc-free}+R_s q_{osc})]-(\frac{\rho f v|v|}{2d})=\rho\frac{\partial v}{\partial t}$$

After finite difference treatment, there is:

$$\frac{(P)_{k+1/2}^{n+1}-(P)_{k-1/2}^{n+1}}{\Delta x}+\rho v(\frac{(v)_{k+1/2}^{n+1}-(v)_{k-1/2}^{n+1}}{\Delta x})-\frac{v}{V_b}[\rho_{osc}q_{osc}+\rho_{gsc}(q_{gsc-free}+R_s q_{osc})]-(\frac{\rho f v|v|}{2d})$$
$$=\rho\frac{(v)_k^{n+1}-(v)_k^n}{\Delta t}$$

$$[A_x]_k^{n+1}[P_{k+1/2}^{n+1}-P_{k-1/2}^{n+1}]+[A_x\rho v]_k^{n+1}(v_{k+1/2}^{n+1}-v_{k-1/2}^{n+1})-v_k^{n+1}[\rho_{osc}q_{osc}+\rho_{gsc}(q_{gsc-free}+R_s q_{osc})]$$
$$-(\frac{V_b\rho f v|v|}{2d})_k^{n+1}=[\frac{\rho V_b}{\Delta t}]_k^{n+1}[(v)_k^{n+1}-(v)_k^n]$$

(4.15)

4.3 Residual Function and Jacobian Matrix

The above finite difference equations (4.12 to 4.15) have the following unknowns in both the wellbore and reservoir system:

$$P^{n+1}_{wb,k-1}, P^{n+1}_{wb,k}, P^{n+1}_{wb,k+1}, H^{n+1}_{o,k-1}, H^{n+1}_{o,k}, H^{n+1}_{o,k+1}, v^{n+1}_{m,k-1}, v^{n+1}_{m,k}, v^{n+1}_{m,k+1}, P^{n+1}_{o,(1,j,k)}, S^{n+1}_{o,(1,j,k)}$$

Similar to the methodology of reservoir flow model, the above wellbore finite difference equations also comprise a non-linear system of algebraic equations which can be solved by using the Newton-Raphson Method. Thus, we need to determine the residual function associated with the wellbore finite difference equations, F_{wg}, F_{wo} and F_{wm}.

Residual Function for the oil continuity equation:

$$F^{n+1}{}_{wo} = A_x H_o v \cdot [(b_o)^{n+1}_{k+1/2} - (b_o)^{n+1}_{k-1/2}] + A_x b_o v \cdot [(H_o)^{n+1}_{k+1/2} - (H_o)^{n+1}_{k-1/2}] + A_x b_o H_o \cdot [(v)^{n+1}_{k+1/2} - (v)^{n+1}_{k-1/2}] + \\ + \sum_{j=1}^{N\theta} \{ [\frac{A_r K_r K_{ro}}{\mu_o B_o \Delta r}]^{n+1}_{1/2,j,k} (p^{n+1}_{o1,j,k} - p^{n+1}_{o-wb-k}) - [\frac{A_r K_r K_{ro} \gamma_o}{\mu_o B_o \Delta r}]^{n+1}_{1/2,j,k} (G_{1,j,k} - G_{wb-k})] \} \\ - V_b [\frac{(b_o H_o)^{n+1}_k - (b_o H_o)^n_k}{\Delta t}] \tag{4.16}$$

Residual Function for the gas continuity equation:

$$F^{n+1}{}_{wg} = [A_x v H_g]^{n+1}_k [(b_g)^{n+1}_{k+1/2} - (b_g)^{n+1}_{k-1/2}] + [A_x H_g b_g]^{n+1}_k [(v)^{n+1}_{k+1/2} - (v)^{n+1}_{k-1/2}] + [A_x v b_g]^{n+1}_k [(H_g)^{n+1}_{k+1/2} - (H_g)^{n+1}_{k-1/2}] \\ + R^{n+1}_{s,k} X_o + [A_x v H_o b_o]^{n+1}_k [(R^{n+1}_{s k+1/2}) - (R^{n+1}_{s k-1/2})] \\ + \sum_{j=1}^{N\theta} \{ [\frac{A_r K_r K_{rg}}{\mu_g B_g \Delta r}]^{n+1}_{1/2,j,k} (p^{n+1}_{g1,j,k} - p^{n+1}_{g-wb-k}) - [\frac{A_r K_r K_{rg} \gamma_g}{\mu_g B_g \Delta r}]^{n+1}_{1/2,j,k} (G_{1,j,k} - G_{wb-k})] \} + R^{n+1}_{s,k} q^{n+1}_{osc,k} \\ - [\frac{V_b}{\Delta t}]^{n+1}_k \{ [(b_g H_g)^{n+1}_k - (b_g H_g)^n_k] + [(R_s b_o H_o)^{n+1}_k - (R_s b_o H_o)^n_k] \} \tag{4.17}$$

Residual Function for the mixture momentum equation:

$$F^{n+1}_{wm} = [A_x]^{n+1}_k [P^{n+1}_{k+1/2} - P^{n+1}_{k-1/2}] + [A_x \rho v]^{n+1}_k (v^{n+1}_{k+1/2} - v^{n+1}_{k-1/2}) - v^{n+1}_k [\rho_{osc} q_{osc} + \rho_{gsc} (q_{gsc-free} + R_s q_{osc})]^{n+1}_k \\ - (\frac{V_b \rho f v |v|}{2d})^{n+1}_k - [\frac{\rho V_b}{\Delta t}]^{n+1}_k [(v)^{n+1}_k - (v)^n_k] \tag{4.18}$$

The residual functions $F^{n+1}_{wg_k}, F^{n+1}_{wo_k}, F^{n+1}_{wm_k}$ can be approximated in an iterative way, $\overset{(v+1)}{F^{n+1}_{wg_k}}, \overset{(v+1)}{F^{n+1}_{wo_k}}, \overset{(v+1)}{F^{n+1}_{wm_k}}$ with a truncated Taylor Series expansion centered at current iteration:

$$\overset{(v+1)}{F_k^{n+1}} = \overset{(v)}{F_k^{n+1}} + \sum_{\forall x} (\frac{\partial F_k}{\partial x})^{\overset{(v)}{n+1}} \overset{(v+1)}{\delta x_k^{n+1}} = 0$$

Where,

$F = F_{wg}$, F_{wo} and F_{wm} , $k = 1$ to *nzz*

$$x = P^{n+1}_{wb,k-1}, P^{n+1}_{wb,k}, P^{n+1}_{wb,k+1}, H^{n+1}_{o,k-1}, H^{n+1}_{o,k}, H^{n+1}_{o,k+1}, v^{n+1}_{m,k-1}, v^{n+1}_{m,k}, v^{n+1}_{m,k+1}, P^{n+1}_{o,(1,j,k)}, S^{n+1}_{o,(1,j,k)}$$

$$\delta x_i^{(v+1)\,n+1} = x_i^{(v+1)\,n+1} - x_i^{(v)\,n+1}$$

At n+1 step, we have:

For the gas phase:

$$\begin{aligned}
&(F_{wgk})^{(v+1)} = (F_{wgk})^{(v)} + \\
&+ (\frac{\partial F_{wgk}}{\partial P_{wb,k}})^{(v)} \partial P^{(v+1)}_{wb,k} + (\frac{\partial F_{wgk}}{\partial P_{wb,k-1}})^{(v)} \partial P^{(v+1)}_{wb,k-1} + (\frac{\partial F_{wgk}}{\partial P_{wb,k+1}})^{(v)} \partial P^{(v+1)}_{wb,k+1} + \\
&+ (\frac{\partial F_{wgk}}{\partial H_{o,k}})^{(v)} \partial H^{(v+1)}_{o,k} + (\frac{\partial F_{wgk}}{\partial H_{o,k-1}})^{(v)} \partial H^{(v+1)}_{o,k-1} + (\frac{\partial F_{wgk}}{\partial H_{o,k+1}})^{(v)} \partial H^{(v+1)}_{o,k+1} + \\
&+ (\frac{\partial F_{wgk}}{\partial v_{m,k}})^{(v)} \partial v^{(v+1)}_{m,k} + (\frac{\partial F_{wgk}}{\partial v_{m,k-1}})^{(v)} \partial v^{(v+1)}_{m,k-1} + (\frac{\partial F_{wgk}}{\partial v_{m,k+1}})^{(v)} \partial v^{(v+1)}_{m,k+1} \\
&+ (\frac{\partial F_{wgk}}{\partial P_{o,(1,j,k)}})^{(v)} \partial P^{(v+1)}_{o,(1,j,k)} + (\frac{\partial F_{wgk}}{\partial S_{o,(1,j,k)}})^{(v)} \partial S^{(v+1)}_{o,(1,j,k)} \\
&= 0
\end{aligned} \tag{4.19}$$

Thus,

$$\begin{aligned}
&(\frac{\partial F_{wgk}}{\partial P_{wb,k}})^{(v)} \partial P^{(v+1)}_{wb,k} + (\frac{\partial F_{wgk}}{\partial P_{wb,k-1}})^{(v)} \partial P^{(v+1)}_{wb,k-1} + (\frac{\partial F_{wgk}}{\partial P_{wb,k+1}})^{(v)} \partial P^{(v+1)}_{wb,k+1} + \\
&+ (\frac{\partial F_{wgk}}{\partial H_{o,k}})^{(v)} \partial H^{(v+1)}_{o,k} + (\frac{\partial F_{wgk}}{\partial H_{o,k-1}})^{(v)} \partial H^{(v+1)}_{o,k-1} + (\frac{\partial F_{wgk}}{\partial H_{o,k+1}})^{(v)} \partial H^{(v+1)}_{o,k+1} + \\
&+ (\frac{\partial F_{wgk}}{\partial v_{m,k}})^{(v)} \partial v^{(v+1)}_{m,k} + (\frac{\partial F_{wgk}}{\partial v_{m,k-1}})^{(v)} \partial v^{(v+1)}_{m,k-1} + (\frac{\partial F_{wgk}}{\partial v_{m,k+1}})^{(v)} \partial v^{(v+1)}_{m,k+1} \\
&+ (\frac{\partial F_{wgk}}{\partial P_{o,(1,j,k)}})^{(v)} \partial P^{(v+1)}_{o,(1,j,k)} + (\frac{\partial F_{wgk}}{\partial S_{o,(1,j,k)}})^{(v)} \partial S^{(v+1)}_{o,(1,j,k)} \\
&= -(F_{wgk})^{(v)}
\end{aligned} \tag{4.20}$$

Analogous, for the oil phase:

$$
\begin{aligned}
&(F_{wok})^{(\nu+1)} = (F_{wok})^{(\nu)} + \\
&+ (\frac{\partial F_{wok}}{\partial P_{wb,k}})^{(\nu)} \partial P_{wb,k}^{(\nu+1)} + (\frac{\partial F_{wok}}{\partial P_{wb,k-1}})^{(\nu)} \partial P_{wb,k-1}^{(\nu+1)} + (\frac{\partial F_{wok}}{\partial P_{wb,k+1}})^{(\nu)} \partial P_{wb,k+1}^{(\nu+1)} + \\
&+ (\frac{\partial F_{wok}}{\partial H_{o,k}})^{(\nu)} \partial H_{o,k}^{(\nu+1)} + (\frac{\partial F_{wok}}{\partial H_{o,k-1}})^{(\nu)} \partial H_{o,k-1}^{(\nu+1)} + (\frac{\partial F_{wok}}{\partial H_{o,k+1}})^{(\nu)} \partial H_{o,k+1}^{(\nu+1)} + \\
&+ (\frac{\partial F_{wok}}{\partial v_{m,k}})^{(\nu)} \partial v_{m,k}^{(\nu+1)} + (\frac{\partial F_{wok}}{\partial v_{m,k-1}})^{(\nu)} \partial v_{m,k-1}^{(\nu+1)} + (\frac{\partial F_{wok}}{\partial v_{m,k+1}})^{(\nu)} \partial v_{m,k+1}^{(\nu+1)} \\
&+ (\frac{\partial F_{wok}}{\partial P_{o,(1,j,k)}})^{(\nu)} \partial P_{o,(1,j,k)}^{(\nu+1)} + (\frac{\partial F_{wok}}{\partial S_{o,(1,j,k)}})^{(\nu)} \partial S_{o,(1,j,k)}^{(\nu+1)} \\
&= 0
\end{aligned}
\tag{4.21}
$$

$$
\begin{aligned}
&(\frac{\partial F_{wok}}{\partial P_{wb,k}})^{(\nu)} \partial P_{wb,k}^{(\nu+1)} + (\frac{\partial F_{wok}}{\partial P_{wb,k-1}})^{(\nu)} \partial P_{wb,k-1}^{(\nu+1)} + (\frac{\partial F_{wok}}{\partial P_{wb,k+1}})^{(\nu)} \partial P_{wb,k+1}^{(\nu+1)} + \\
&+ (\frac{\partial F_{wok}}{\partial H_{o,k}})^{(\nu)} \partial H_{o,k}^{(\nu+1)} + (\frac{\partial F_{wok}}{\partial H_{o,k-1}})^{(\nu)} \partial H_{o,k-1}^{(\nu+1)} + (\frac{\partial F_{wok}}{\partial H_{o,k+1}})^{(\nu)} \partial H_{o,k+1}^{(\nu+1)} + \\
&+ (\frac{\partial F_{wok}}{\partial v_{m,k}})^{(\nu)} \partial v_{m,k}^{(\nu+1)} + (\frac{\partial F_{wok}}{\partial v_{m,k-1}})^{(\nu)} \partial v_{m,k-1}^{(\nu+1)} + (\frac{\partial F_{wok}}{\partial v_{m,k+1}})^{(\nu)} \partial v_{m,k+1}^{(\nu+1)} \\
&+ (\frac{\partial F_{wok}}{\partial P_{o,(1,j,k)}})^{(\nu)} \partial P_{o,(1,j,k)}^{(\nu+1)} + (\frac{\partial F_{wok}}{\partial S_{o,(1,j,k)}})^{(\nu)} \partial S_{o,(1,j,k)}^{(\nu+1)} \\
&= -(F_{wok})^{(\nu)}
\end{aligned}
\tag{4.22}
$$

For the momentum equation:

$$
\begin{aligned}
&(F_{wmk})^{(\nu+1)} = (F_{wmk})^{(\nu)} + \\
&+ (\frac{\partial F_{wmk}}{\partial P_{wb,k}})^{(\nu)} \partial P_{wb,k}^{(\nu+1)} + (\frac{\partial F_{wmk}}{\partial P_{wb,k-1}})^{(\nu)} \partial P_{wb,k-1}^{(\nu+1)} + (\frac{\partial F_{wmk}}{\partial P_{wb,k+1}})^{(\nu)} \partial P_{wb,k+1}^{(\nu+1)} + \\
&+ (\frac{\partial F_{wmk}}{\partial H_{o,k}})^{(\nu)} \partial H_{o,k}^{(\nu+1)} + (\frac{\partial F_{wmk}}{\partial H_{o,k-1}})^{(\nu)} \partial H_{o,k-1}^{(\nu+1)} + (\frac{\partial F_{wmk}}{\partial H_{o,k+1}})^{(\nu)} \partial H_{o,k+1}^{(\nu+1)} + \\
&+ (\frac{\partial F_{wmk}}{\partial v_{m,k}})^{(\nu)} \partial v_{m,k}^{(\nu+1)} + (\frac{\partial F_{wmk}}{\partial v_{m,k-1}})^{(\nu)} \partial v_{m,k-1}^{(\nu+1)} + (\frac{\partial F_{wmk}}{\partial v_{m,k+1}})^{(\nu)} \partial v_{m,k+1}^{(\nu+1)} \\
&+ (\frac{\partial F_{wmk}}{\partial P_{o,(1,j,k)}})^{(\nu)} \partial P_{o,(1,j,k)}^{(\nu+1)} + (\frac{\partial F_{wmk}}{\partial S_{o,(1,j,k)}})^{(\nu)} \partial S_{o,(1,j,k)}^{(\nu+1)} \\
&= 0
\end{aligned}
\tag{4.23}
$$

Thus,

$$\left(\frac{\partial F_{wmk}}{\partial P_{wb,k}}\right)^{(v)} \partial P_{wb,k}^{(v+1)} + \left(\frac{\partial F_{wmk}}{\partial P_{wb,k-1}}\right)^{(v)} \partial P_{wb,k-1}^{(v+1)} + \left(\frac{\partial F_{wmk}}{\partial P_{wb,k+1}}\right)^{(v)} \partial P_{wb,k+1}^{(v+1)} +$$
$$+ \left(\frac{\partial F_{wmk}}{\partial H_{o,k}}\right)^{(v)} \partial H_{o,k}^{(v+1)} + \left(\frac{\partial F_{wmk}}{\partial H_{o,k-1}}\right)^{(v)} \partial H_{o,k-1}^{(v+1)} + \left(\frac{\partial F_{wmk}}{\partial H_{o,k+1}}\right)^{(v)} \partial H_{o,k+1}^{(v+1)} +$$
$$+ \left(\frac{\partial F_{wmk}}{\partial v_{m,k}}\right)^{(v)} \partial v_{m,k}^{(v+1)} + \left(\frac{\partial F_{wmk}}{\partial v_{m,k-1}}\right)^{(v)} \partial v_{m,k-1}^{(v+1)} + \left(\frac{\partial F_{wmk}}{\partial v_{m,k+1}}\right)^{(v)} \partial v_{m,k+1}^{(v+1)} \tag{4.24}$$
$$+ \left(\frac{\partial F_{wmk}}{\partial P_{o,(1,j,k)}}\right)^{(v)} \partial P_{o,(1,j,k)}^{(v+1)} + \left(\frac{\partial F_{wmk}}{\partial S_{o,(1,j,k)}}\right)^{(v)} \partial S_{o,(1,j,k)}^{(v+1)}$$
$$= -(F_{wmk})^{(v)}$$

The system of equations obtained from the oil/gas continuity and mixture momentary residual functions can be expressed as follows:

$$\underset{\approx}{J}{}_{n+1}^{(v)} \cdot \delta \vec{X}_{n+1}^{(v+1)} = -\vec{F}_{n+1}^{(v)}$$

Where,

$$\vec{X}_{n+1}^{(v+1)} = \vec{X}_{n+1}^{(v)} + \delta \vec{X}_{n+1}^{(v+1)}$$

$\underset{\approx}{J}{}_{n+1}^{(v)}$ is the Jacobian Matrix

$\vec{F}_{n+1}^{(v)}$ is the Residual function vector F_{wg} , F_{wo} and F_{wm};

$\vec{X}_{n+1}^{(v+1)}$ is the vector of unknowns for future iteration;

$\vec{X}_{n+1}^{(v)}$ is the vector of unknowns for current iteration;

$\delta \vec{X}_{n+1}^{(v+1)}$ is the vector from current to future iteration;

Newton-Raphson method is used to solve this problem. To determine the unknowns at step $n+1$, it is necessary to know the previous P_{wb}, H_o, v_m (k =1 to nzz). Accordingly an iterative analysis should be conducted with an initial guess: $P_{wb\,k}^{(0)}, H_{o\,k}^{(0)}, v_k$ (k=1 to nzz).

With $\overrightarrow{X}^{(0)}_{n+1} = \overrightarrow{X}^{(0)}_{n}$, it is possible to calculate the Jacobian matrix $\underset{\approx}{J}^{(0)}$ as well as the vector $\overrightarrow{F}^{(0)}$.

Once $\underset{\approx}{J}^{(0)}$ and $\overrightarrow{F}^{(0)}$ are available, $\delta \overrightarrow{X}^{(1)}$ and consequently $\overrightarrow{X}^{(1)}$ can be obtained through the following expression:

$$\overrightarrow{X}^{(1)} = \overrightarrow{X}^{(0)} + \delta \overrightarrow{X}^{(1)}$$

This process can go on from $\overrightarrow{X}^{(1)}$ to the determination of $\overrightarrow{X}^{(2)}$ and so on until the following criterion is met.

$$\left| \delta \overrightarrow{X}^{(v+1)} \right| \underset{\forall x}{\leq} tolerance \ (for\ P_{wb}, H_o, v)$$

Similar to the reservoir flow model, to construct the Jacobian matrix, it is necessary to obtain all the derivatives of F_{wg} , F_{wo} and F_{wm} with respect to wellbore variables, including $P^{n+1}_{wb,k-1}, P^{n+1}_{wb,k}, P^{n+1}_{wb,k+1}, H^{n+1}_{o,k-1}, H^{n+1}_{o,k}, H^{n+1}_{o,k+1}, v^{n+1}_{m,k-1}, v^{n+1}_{m,k}, v^{n+1}_{m,k+1}, P^{n+1}_{o,(1,j,k)}, S^{n+1}_{o,(1,j,k)}$. The derivatives of F_{wg}, F_{wo}, F_{wm} regarding to reservoir variables $P_{o,(1,j,k)}, S_{o,(1,j,k)}$ are also necessary because they are relevant to the influx in the residual function, which reveals the connection between reservoir and wellbore on the sandface. All the details about the derivation can be found in Appendix 3.

CHAPTER 5: COUPLED MODEL DEVELOPMENT AND VALIDATION

Chapter 3 and Chapter 4 present the development of the reservoir flow model and the wellbore flow model, respectively. To couple both reservoir and horizontal wellbore flow, the above two flow models should be integrated into one solution system. This chapter presents the coupling process of reservoir/wellbore model. The developed coupled model is compared and validated with other research studies or commercial simulators.

5.1 Coupling Process of Reservoir/Wellbore Model

Flow in the reservoir is described as a parabolic partial differential equation, while flow in the horizontal wellbore is represented by a hyperbolic partial differential equation, but the coupled model has to solve their partial differential equations simultaneously. After the discretization of the reservoir and wellbore partial differential equations, a Newton-Raphson method is used to solve the system. The Newton-Raphson method involves calculating the Jacobian matrix and the right hand side vector. The Jacobian matrix is built up by calculating all the derivatives of residual function with respect to each unknown during each iteration. The right hand side vector is updated with the residual function. Only in this way, the coupled model can cover all three regions of study interest, namely, the reservoir part I (far away from the wellbore), the reservoir part II (wellbore vicinity) and the wellbore region.

The associated unknowns in the finite difference equations include reservoir pressure and saturation for each block, and wellbore pressure, liquid holdup and velocity for each segment. Because all the variables for both reservoir and wellbore domains should be solved simultaneously, the time-step control and convergence check are critical to ensure the convergence. Since the wellbore equation is more sensitive, the time-step is usually set to be very small at the beginning and become automatic adaptive later.

Figure 5.1 shows the Jacobian matrix structure of a sample case. The matrix has four zones: reservoir to reservoir region, reservoir to wellbore region, wellbore to reservoir region, and wellbore to wellbore region. The reservoir-reservoir region is sub-divided as Cartesian system, cylindrical system, and hybrid zones.

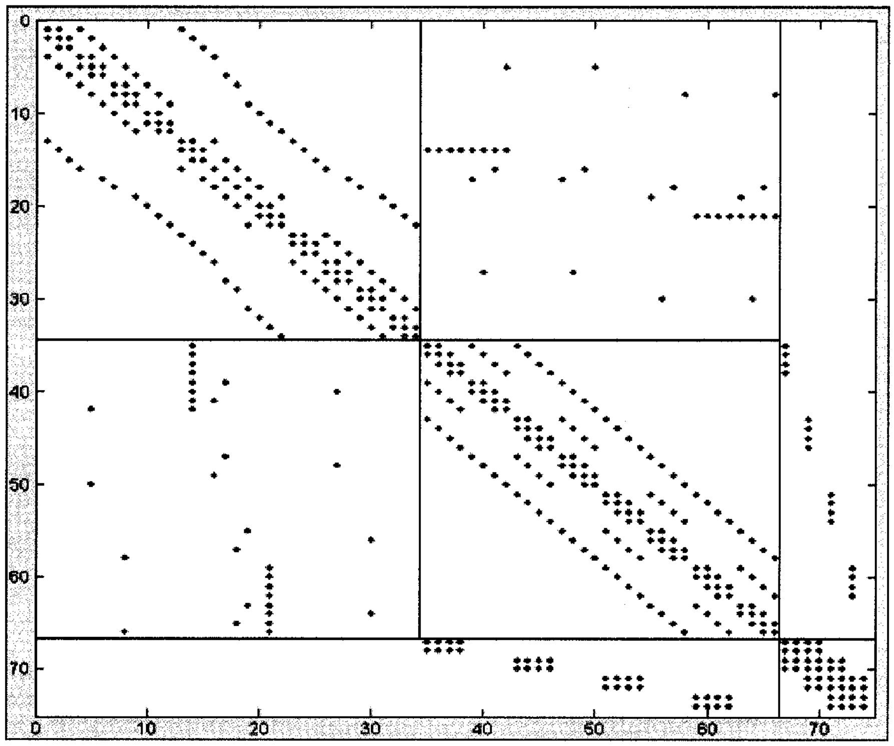

Figure 5.1 - Jacobian matrix structure for a sample case

The coupled model is implemented using a FORTRAN90 code. A selection of subroutines and files used in the source code are listed and explained in Table 5.1. Figure 5.2 shows the flowchart of the source code.

Table 5.1 - Selected subroutines and files used in the source code

Name	Function
MAIN	Main program to control the model development process
MODFILE	File to define variable modules
READRE	Subroutine to read reservoir and simulation input data
VLOCAT	Subroutine to allocate variables dimensions
READTB	Subroutine to read PVT and rock tables
INIPAR	Subroutine to initialize parameters
GRID	Subroutine to define grid
FATG	Subroutine to the computation of geometric factors
GRDDEP	Subroutine to compute grid points depth
DISTIP	Subroutine to compute initial pressure distribution
ATUP1	Subroutine to update the pressure dependent properties at n time-step
ATUP2F	Subroutine to update saturation and pressure dependent properties at each iteration
VPVTF	Subroutine for the interpolation of PVT properties in tables
TRANSO	Subroutine to compute transmissibility
FRGO	Subroutine to compute gas/oil residual function and derivatives in the Jacobian matrix
INTERP	Interpolation subroutine
BCGRAD	Solver subroutine using bi-conjugate gradient method
ENGINE*	MATLAB matrix solver engine
PRPLOT	Subroutine to print simulation results
Input.txt	Reservoir and simulation input data
Result(n).txt	Simulation output file. "*n*" depends on different simulation output.
*. Note: ENGINE part is located in the MAIN program file, which includes certain MATLAB variables' preparation, open, transfer and calculation process for Fortran code (Mix-languages programming).	

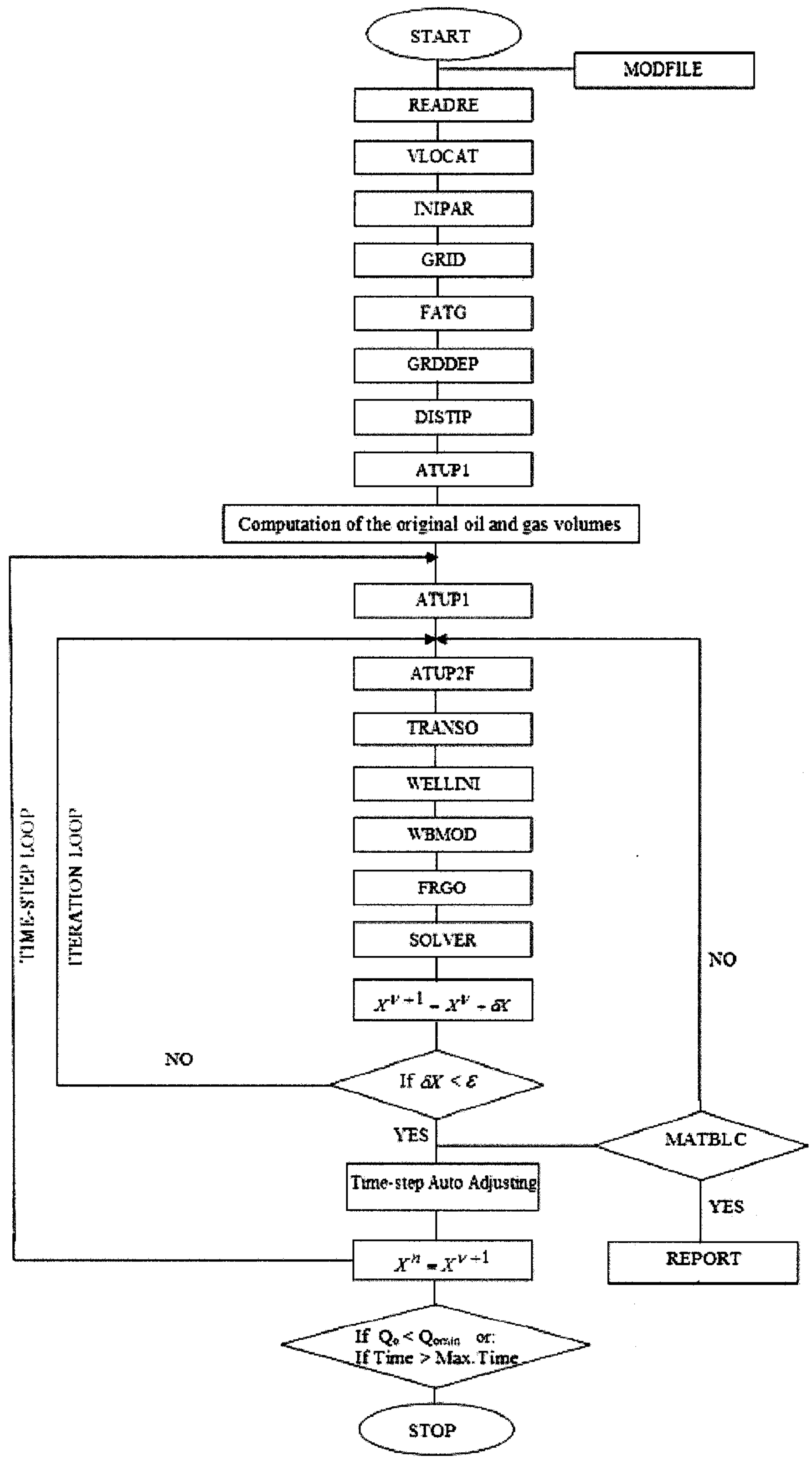

Figure 5.2 - Flow chart of the source code

5.2 Comparison and Validation of the Simulator Developed

Results using the coupled model are compared and validated with either commercial reservoir simulators or other research studies during various development stages, including hybrid local grid refinement, uncoupled model, and coupled model.

5.2.1 Validation of the Hybrid Local Grid Refinement (Uncoupled Model)

The reservoir simulator developed in this research uses hybrid local grid refinement technique (both Cartesian and cylindrical coordinates in the same system). Figure 5.3 shows the structure of hybrid grids and the horizontal well location.

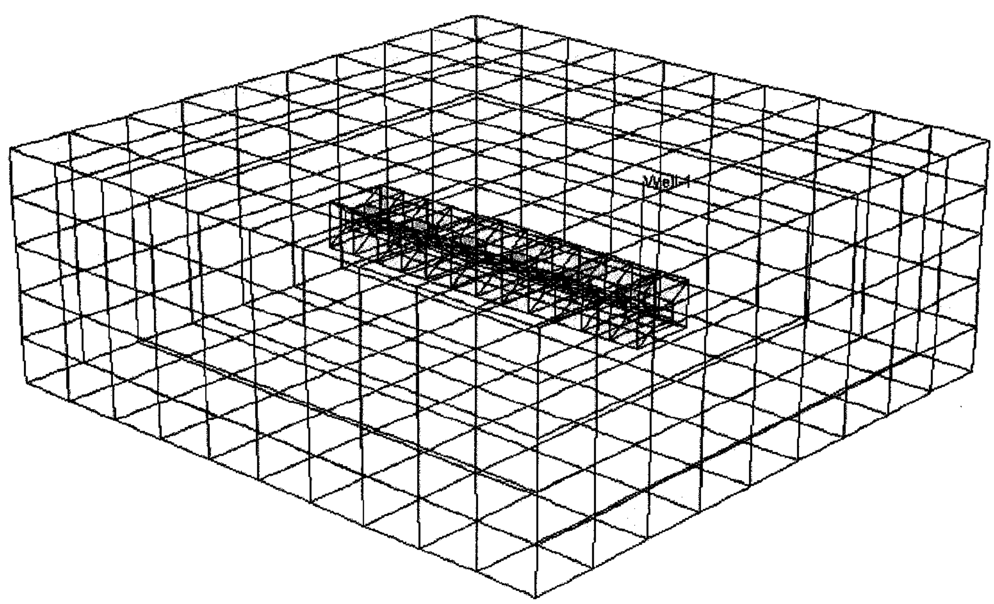

Figure 5.3 - Structure of hybrid grids and the horizontal well location

A sample case has been run to compare the pressure behavior between the developed model and a commercial reservoir simulator, both using the hybrid local refinement technique. The production scheme for this case starts as a constant flow rate, until pressure drops to a certain level, then changes to a constant pressure scheme. As illustrated in figure

5.4, a good agreement is reached on the pressure behavior calculated from the model developed with the commercial reservoir simulator.

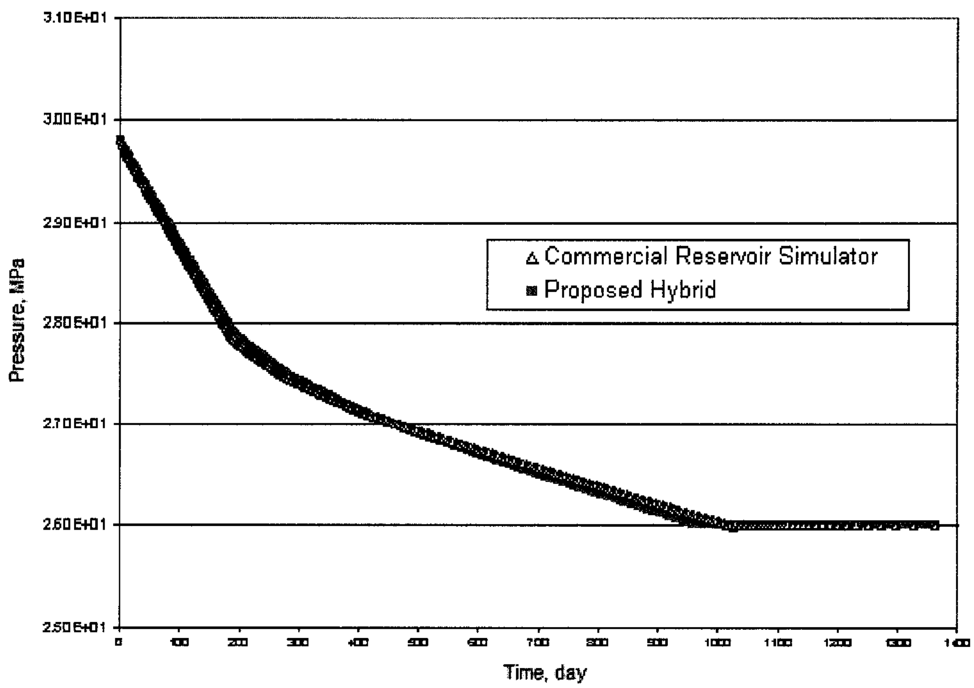

Figure 5.4 - Comparison of the pressure drop between the developed model and a commercial reservoir simulator

5.2.2 Uncoupled Model Validation

The model developed in this research has an option to turn off the coupling feature. This section shows the validation of this uncoupled model (infinite conductivity at the wellbore).

This case studies a single phase slightly compressible fluid flow (oil). Homogenous reservoir properties are assumed with permeability of 0.25 md and porosity of 0.15. Initial reservoir pressure 2500 psi is imposed at the middle layer of the reservoir. Production scheme is specified bottom-hole pressure. Detailed reservoir parameter and properties for this case study can be found in the work of Gökta and Ertekin (1999).

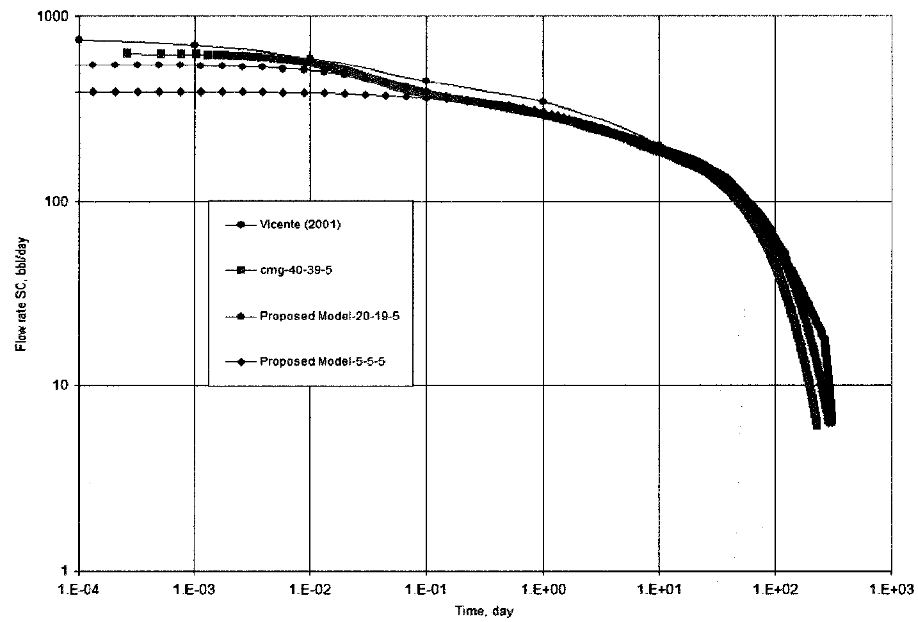

Figure 5.5 - Oil flow rate comparisons among the proposed simulator, other research and a commercial simulator

From figure 5.5, it is obvious that the difference among several simulators happens mainly on the early-time period (before 0.1 day). This is partially because of different time-step control methods used during simulation. Also, early-time difference comes from different grid refinement degrees (numbers in the legend represents the different grid refinement levels in different directions). The simulation result using coarse grid (curve 5-5-5) has a lower oil production in the early stage. With more refined grids (curve 20-19-5, with horizontal wellbore sitting on the same position), the simulation results can have a better agreement with the commercial simulator's result and results from Vicente's et al. model (2001a and 2001b).

Alternatively, hybrid grids can be used to refine the near-wellbore region and capture the early transient flow characteristics, as shown in figure 5.6. Because of the hybrid refinement, even a relatively coarse grid (5-5-5) case can have a good agreement with other curves. Steady-state assumption will not be valid for the transient behavior, even with a refined Cartesian grid.

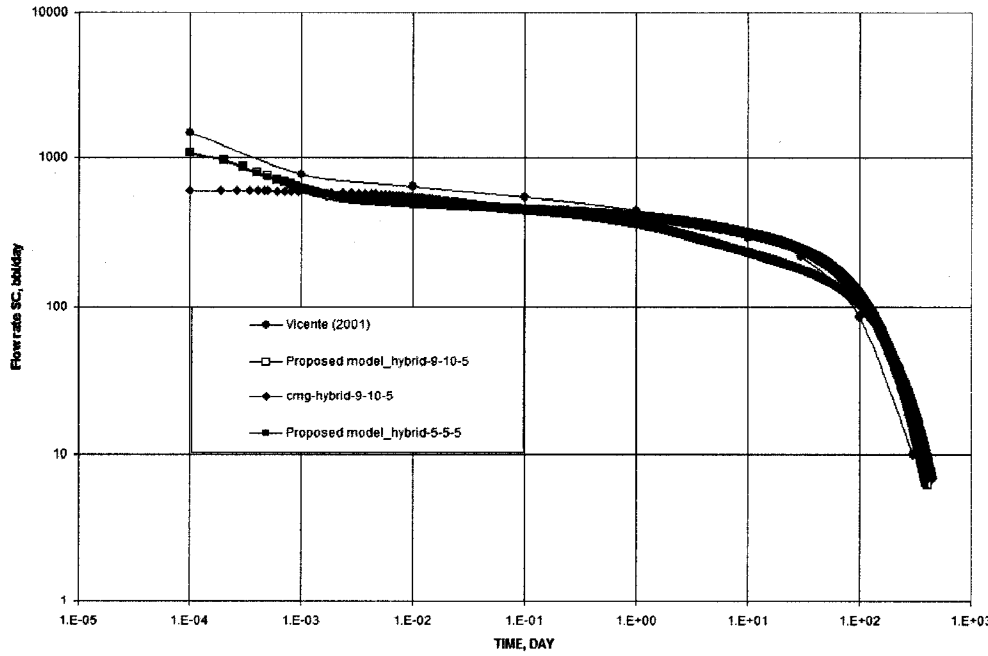

Figure 5.6 - Oil flow rate comparisons (using hybrid grid)

5.2.3 Coupled Model Validation (Single Phase)

The developed coupled model was first used to simulate a single phase fluid flow case (slightly compressible). Finite conductivity effect and non-uniform influx distribution in the horizontal wellbore is analyzed in this simulation. Results from the proposed model are compared with those of a recent research.

In this validation case, the reservoir has sealed off at the reservoir top and bottom boundary, allowing water support coming from the reservoir outside boundaries. Production scheme is constant oil flow rate at the heel. Some reservoir and fluid properties are listed in table 5.2.

Table 5.2 - Reservoir and fluid properties for validation of the coupled model

Properties	**Value**	**Unit**
Initial pressure	2300	psi
Reservoir size	4613 x 5249	ft
Formation thickness	72	ft
Horizontal permeability	8500	md
Vertical permeability	1500	md
Oil viscosity	1.43	cp
Oil density	55	lb/ft^3
Formation volume factor	1.16	bbl/stb
Total compressibility	6.9×10^{-6}	1/psi
Porosity	0.25	-
Vertical location of the well (from bottom)	11.5	ft
Wellbore radius	0.25	ft
Horizontal well length	2625	ft
Relative well surface roughness	10^{-4}	-

Figure 5.7 shows the oil influx distribution along the wellbore. Because of the water support around the reservoir boundary, the influx has a basic U shape influx distribution. The curve is asymmetric because of the finite conductivity of multi-segmented wellbore. The influx is lower at the toe side, but the pressure drop along the wellbore makes the influx gradually goes up toward the heel. As shown in figure 5.7, the two curves, one from the current model and the other from recent research (Vicente et al., 2000), have a good match in the influx distribution.

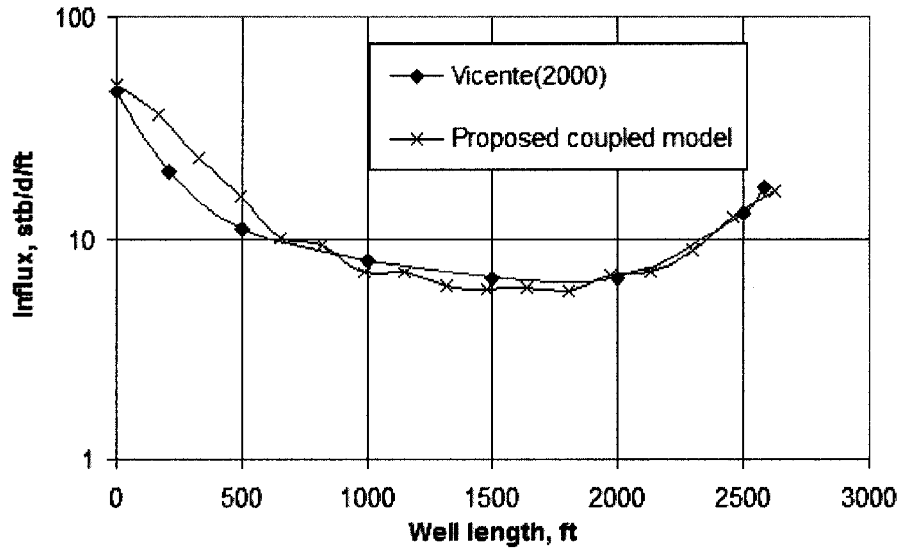

Figure 5.7 - Influx distribution along the wellbore

5.2.4 Two Phase Coupled Model Validation and Transient Flow Analysis

The following validation case investigates the transient two phase flow behavior in the reservoir and horizontal wellbore. The proposed model is used to simulate a field scenario. The oil field is located in a highly permeable North Sea reservoir. The horizontal permeability is 1 Darcy and the anisotropy ratio equals 0.1. The initial average reservoir pressure is 17.24 MPa and initial gas saturation 0.1. It is assumed that the horizontal well extends through the reservoir without sloping from the heel to the toe. The fluid influx to the wellbore is assumed to be continuous from the toe to the heel. The reservoir has impermeable top, bottom and edge boundaries. Production scheme is considered as a constant flow rate at the heel. Reservoir and fluid properties are listed in table 5.3.

Table 5.3 - Reservoir and fluid properties for two phase coupled model validation and transient flow analysis

Properties	**Value**	**Unit**
Initial pressure	17.24	MPa
Reservoir size	91.44, 304.8	m
Formation thickness	80	m
Horizontal permeability	1000	md
Vertical permeability	100	md
Horizontal well length	304.8	m
Initial gas saturation	0.1	fraction
Maximum oil flow rate	158.99	m^3/day

By using a hybrid local grid refinement technique and a small time-step (auto-adaptive time step adjustment) in the early stage of the production, the simulation results can reveal the early-time transient flow characteristics in the reservoir and horizontal wellbore. Figure 5.8 draws comparisons of the transient pressure drop on a log-log coordinate between proposed model and a commercial reservoir simulator (with Multi-Segment Well technique). Basically the two curves have three stages: unloading at the very beginning, initial radial influx of oil and gas, and the starting of reservoir depletion. Overall the two curves have an agreement during all the three stages discussed above.

At the beginning of the unloading period, oil expansion dominates in the wellbore and the pressure drop is characterized by a unit slope line. Corresponding to the pressure behavior, figure 5.9 and 5.10 show that radial influx coming from well vicinities keeps increasing and maintains itself at a certain level. The specified oil flow rate is offset by the expansion of the fluid in the wellbore during this period. However, uncoupled model can not capture this early transient behavior and the oil flow rate equals to the specified rate from the beginning of the simulation, as shown in figure 5.9.

As the storage effect fades away, the initial radial influx of oil and gas from the vicinity of wellbore becomes more and more dominant, which directly influence the pressure drop behavior. With the increasing influx support coming from the wellbore vicinities, the pressure drop at the heel decreases to a certain level until the pressure transient propagates from the wellbore vicinities and reaches the reservoir domain. As pressure spreads through the reservoir domain, the pressure drop rate starts increasing and shows the effect of reservoir depletion.

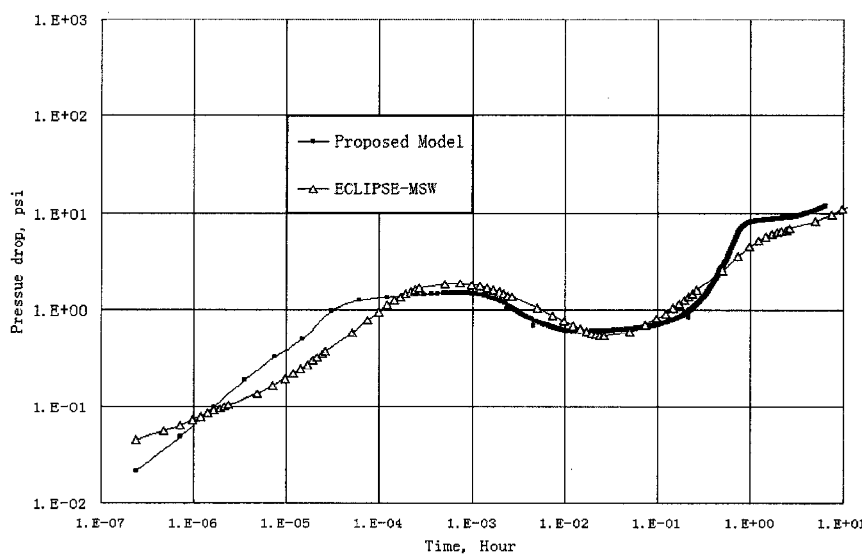

Figure 5.8 - Validation of transient pressure drop behavior

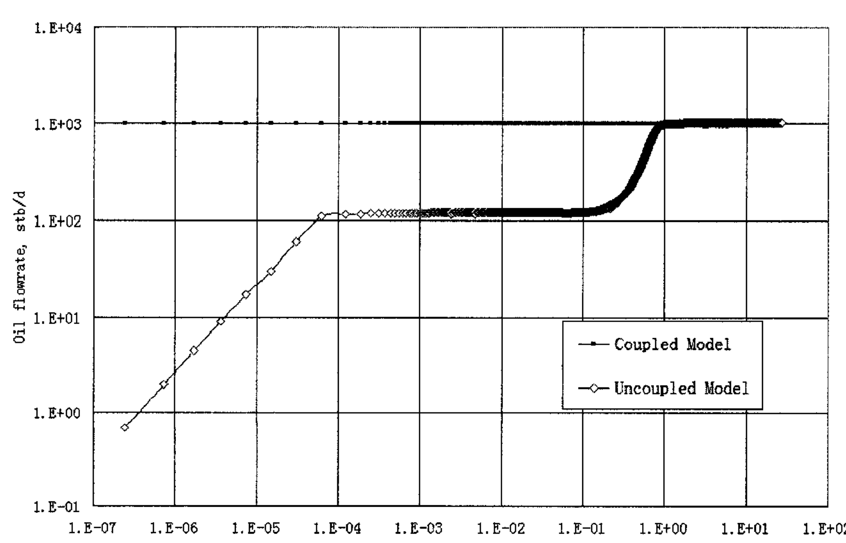

Figure 5.9 - Comparison of the oil flow rate between the developed coupled model and the uncoupled model

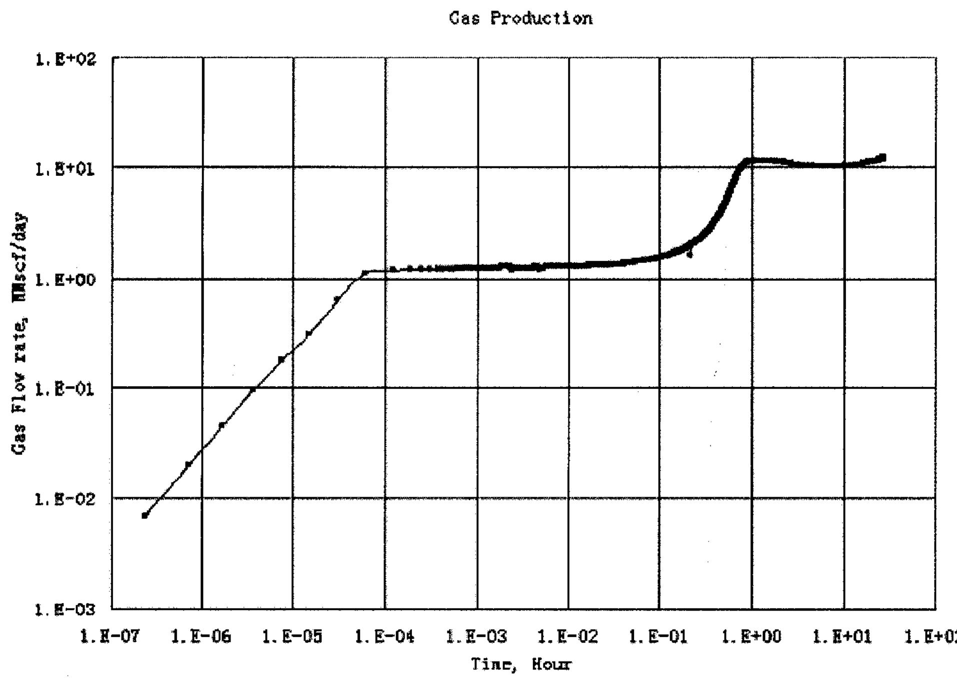

Figure 5.10 - Gas cumulative production from all wellbore segments

Figure 5.11 shows gas void fraction and liquid holdup changes in the wellbore. During the storage period, liquid holdup is very close to the initial oil saturation. Dissolved gas comes out from the oil due to pressure drop, but the amount is relatively small with a minuscule pressure drop. With the increasing radial influx of oil and gas from the vicinity of wellbore, more free gas enters the wellbore and liquid holdup starts dropping quickly. The gas phase keeps increasing and the oil phase decreasing. As mentioned before, after the unloading and initial radial influx period, reservoir depletion effect begins. Volumetrically a large amount of free gas enters into wellbore and occupies a major portion of the wellbore volume. Reflected in the holdup figure 5.11, gas void fraction and liquid holdup intersect at 0.5 point and gas void fraction quickly increases to a higher level.

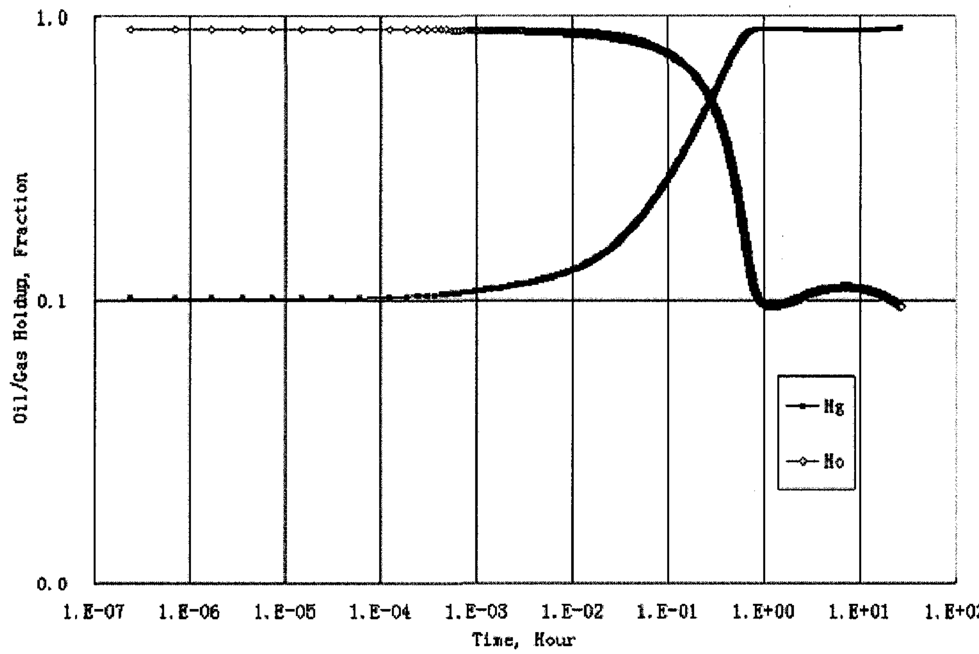

Figure 5.11 - Gas void fraction and liquid holdup

CHAPTER 6: APPLICATIONS OF THE DEVELOPED COUPLED MODEL

In this chapter, several applications are conducted using the developed reservoir/wellbore coupled model. First, the coupled model is used to study the effect of finite conductivity and non-uniform influx distribution in the horizontal wellbore. Comparisons are made between the coupled model and a non-coupled traditional model, including influx distribution, flow rate distribution, drawdown and pressure distribution along the wellbore. The second part of this chapter discusses about the application of the coupled model on transient flow. Sensitivity analyses are conducted to study the effect of reservoir permeability and initial gas saturation on the transient flow behavior. The effect of perforation distribution on wellbore flow behavior is investigated to in the third part.

6.1 Finite Conductivity and Non-Uniform Influx Distribution (Single Phase)

Since the coupled model integrates the reservoir model with a wellbore model, it follows naturally that the wellbore behavior will present a finite conductivity and non-uniform influx distribution instead of an infinite conductivity and uniform influx distribution. The following case studies the finite conductivity and non-uniform influx distribution effect in the horizontal wellbore. The oil field is located in a highly permeable North Sea reservoir. Table 6.1 lists the related reservoir data and other properties data. Some of the data are simplified with constant values. It is assumed that the horizontal well has no sloping from the heel to the toe. The horizontal well extends 800 meters in length and fluid influx to the wellbore is assumed to be continuous from the toe to the heel. The reservoir has an impermeable top and bottom boundaries and the edge boundaries have constant water support. Production scheme considers a constant flow rate at the heel.

Table 6.1 - Reservoir and fluid properties for finite conductivity and non-uniform influx distribution studies

Properties	**Value**	**Unit**
Initial pressure	15858.0	kPa
Reservoir size	1400.0 x 1600.0	m
Formation thickness	22.0	m
Horizontal permeability	8500	md
Vertical permeability	1500	md
Oil viscosity	1.43	cp
Oil density	881	kg/m^3
Formation volume factor	1.16	$m^3/sc\ m^3$
Total compressibility	0.001	1/MPa
Porosity	0.25	-
Vertical location of the well	3.5 from bottom	m
Wellbore radius	0.0762	m
Horizontal well length	800.0	m
Relative well surface roughness	0.0001	-

Comparisons about influx distribution, flow rate distribution, drawdown and pressure distribution along the wellbore, etc. are made between the proposed coupled model and a non-coupled traditional model.

Figure 6.1 shows the influx distribution along the wellbore. Because of the water support coming around the reservoir boundary, both models show a basic U shape influx distribution. The curve is symmetrical in the case of a traditional model because the wellbore model is considered as an infinite conductivity and the drawdown along the wellbore is only related to the reservoir pressure (at the wellbore block). For the coupled model, however, the curve is asymmetric due to the finite conductivity of the multi-segmented wellbore. At the toe side, the influx is lower than for the traditional model. Toward the heel, influx per unit length gradually increases and eventually becomes higher than the uncoupled model. This is because the wellbore pressure is changing along the wellbore and more influx will come from the heel since the drawdown is higher in that region. Attention is required especially if the flow rate is high and the reservoir has potential water or gas coning problem. It should also be noted that the prediction on coning may be underestimated by just using a traditional model.

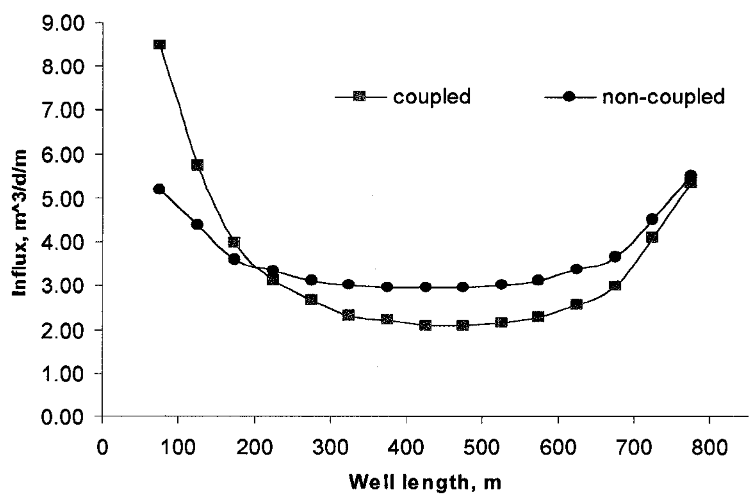

Figure 6.1 - Influx distribution along the wellbore

The analysis above can be further supported by drawing the wellbore pressure drop along with the well length, as shown in figure 6.2. The coupled wellbore model considers both frictional and accelerational pressure drops. With the increase of fluid velocity in the wellbore (see figure 6.3), the pressure drops goes up with the well length. However, the uncoupled model uses the infinite conductivity assumption and ignores both frictional and accelerational pressure drops. This could result in poor or erroneous predictions of well productivity, because the wellbore pressure distribution caused by wall friction and acceleration can affect the influx distribution and vice versa.

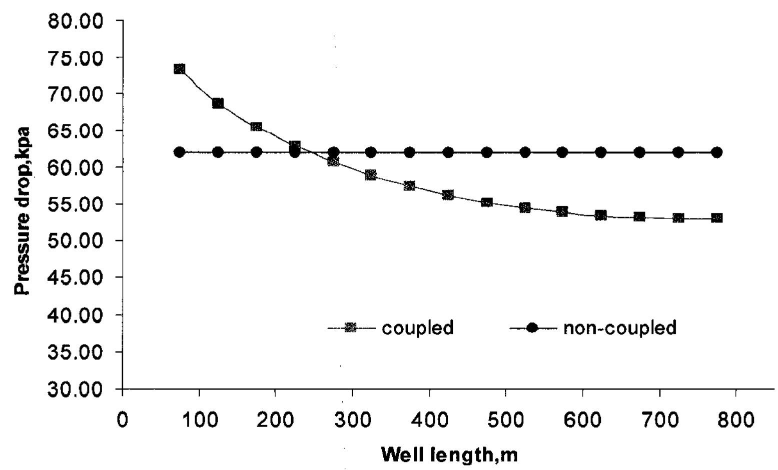

Figure 6.2 - Pressure drop from initial pressure along the wellbore

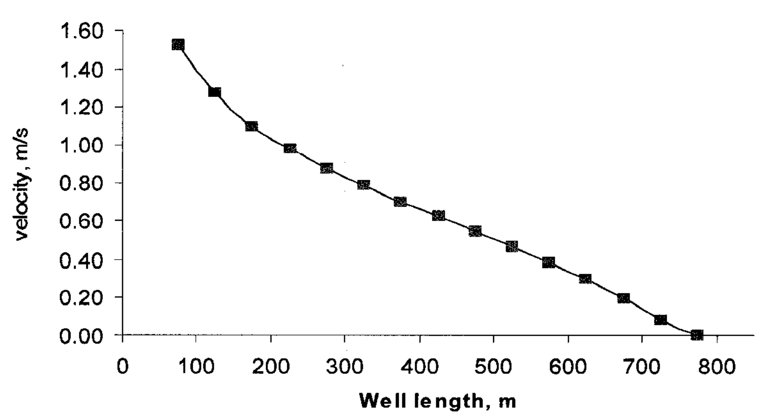

Figure 6.3 - Fluid velocity distribution along the wellbore

Figures 6.4 and 6.5 show the reservoir pressure distribution at the radial part of the hybrid grid, which represents wellbore vicinity. Compare figures 6.4 and 6.5 for both coupled and non-coupled models. It is obvious that the reservoir pressure is also affected by the non-uniform distributed influx. Figure 6.6 shows the drawdown difference between the coupled model and non-coupled model. It is necessary to clarify that although figure 6.6 (drawdown curve) and figure 6.1 (influx curve) has the same trend, the position of the crossing point is different. This is because the production scheme is constant flow rate and the total flow rate is distributed according to the mobility-potential method. In other words, the influx is determined by the ratio of local mobility-potential and the total flow rate of all wellbore segments, not the absolute local value of drawdown.

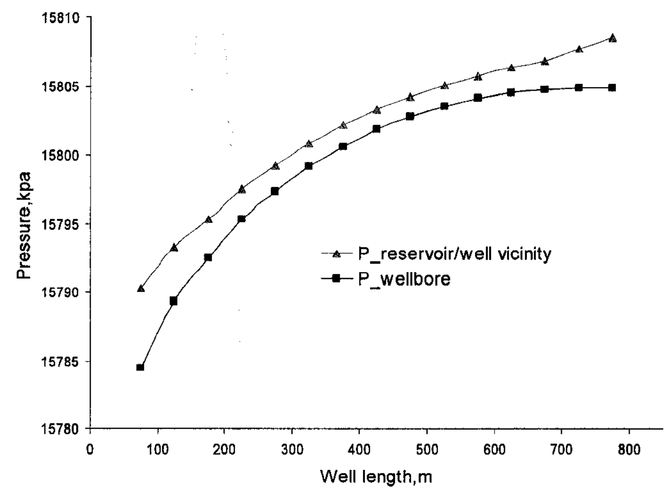

Figure 6.4 - Reservoir pressure (well vicinity) and wellbore pressure along the well (Coupled model)

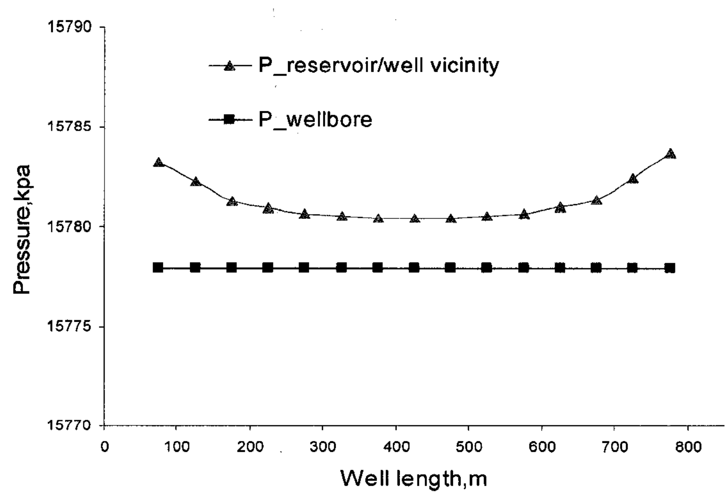

Figure 6.5 - Reservoir pressure (well vicinity) and wellbore pressure along the well (Non-Coupled model)

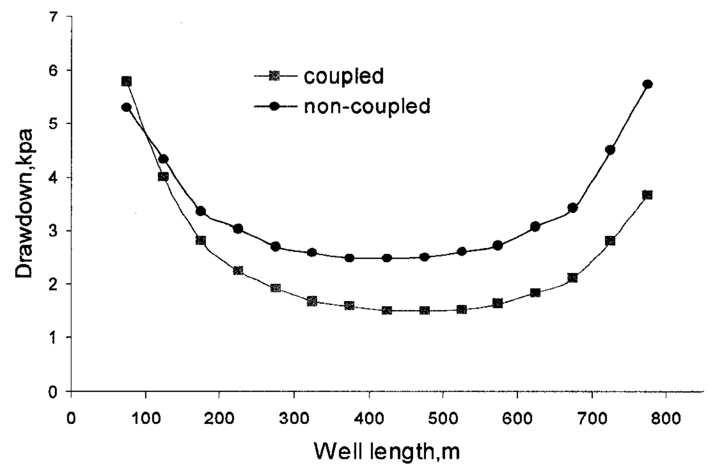

Figure 6.6 - Drawdown distribution along the wellbore

According to figure 6.7, the flow rate calculated from the traditional model is more linearly distributed than that of coupled model. From another point of view, the derivative curve of figure 6.7 actually reflects the influx performance (see figure 6.1).

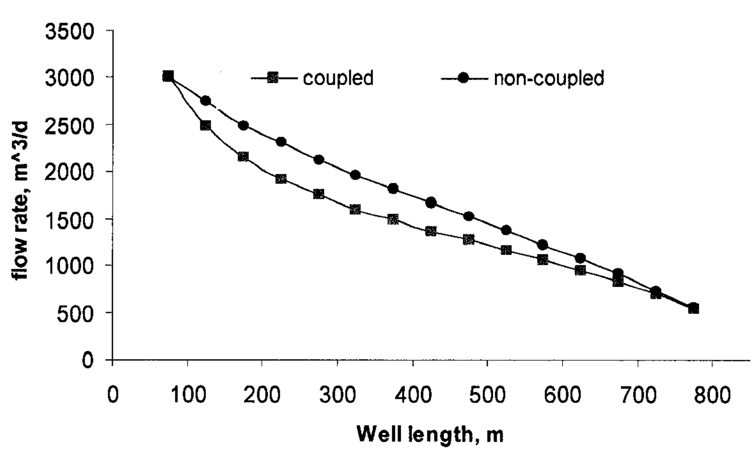

Figure 6.7 - Cumulative flow rate distribution along the wellbore

As mentioned before, the coupled model should be applied especially when the production rate is high. Figures 6.8 to 6.10 compare the influx and wellbore pressure under different flow rates. As flow rate goes down, the influx curve tends to be more symmetric and the wellbore pressure tends to be constant along the wellbore length. As a result, the flow rate distribution is more linear when the total production rate is relatively low. In other words, the wellbore conductivity tends to be infinite for the coupled model.

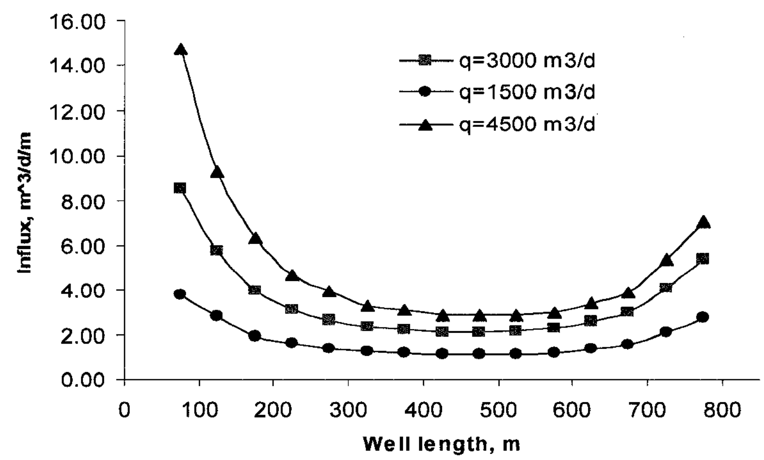

Figure 6.8 - Influx comparison for different flow rates

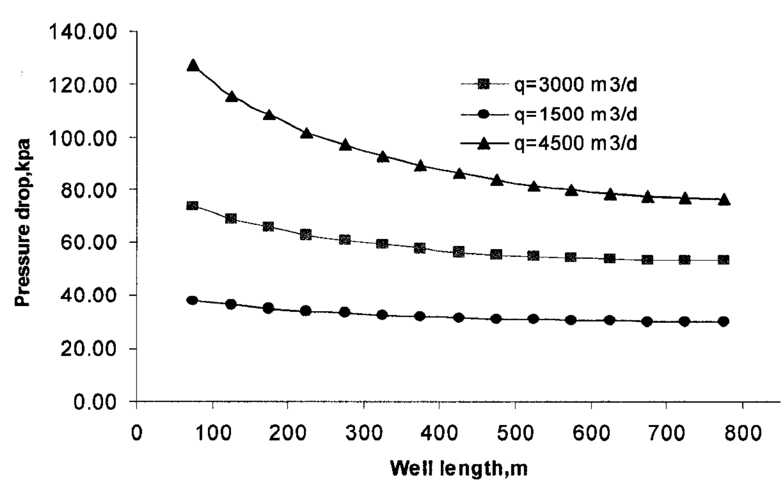

Figure 6.9 - Pressure drop comparison for different flow rates

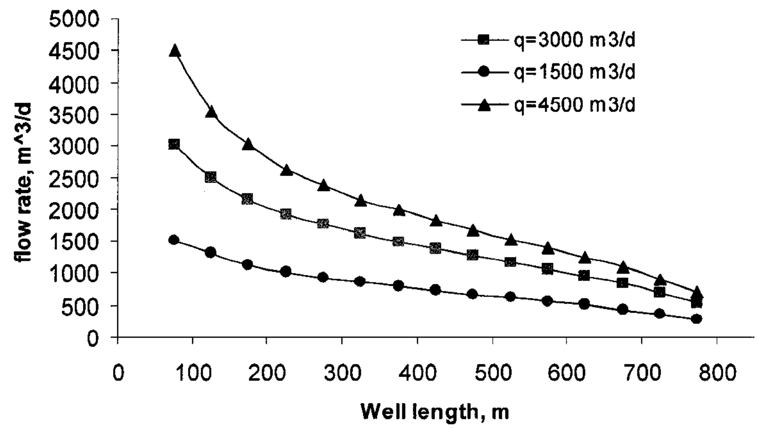

Figure 6.10 - Flow rate distribution comparison for different flow rates at the well heel

6.2 Finite Conductivity and Non-Uniform Influx Distribution (Two Phase)

Finite conductivity and non-uniform influx distribution is also studied on the same oil and gas two phase flow case, presented in Chapter 5, section 5.2.4. Figures 6.11 to 6.14 compare the pressure drop, flow rate and holdup behavior at different locations in the wellbore.

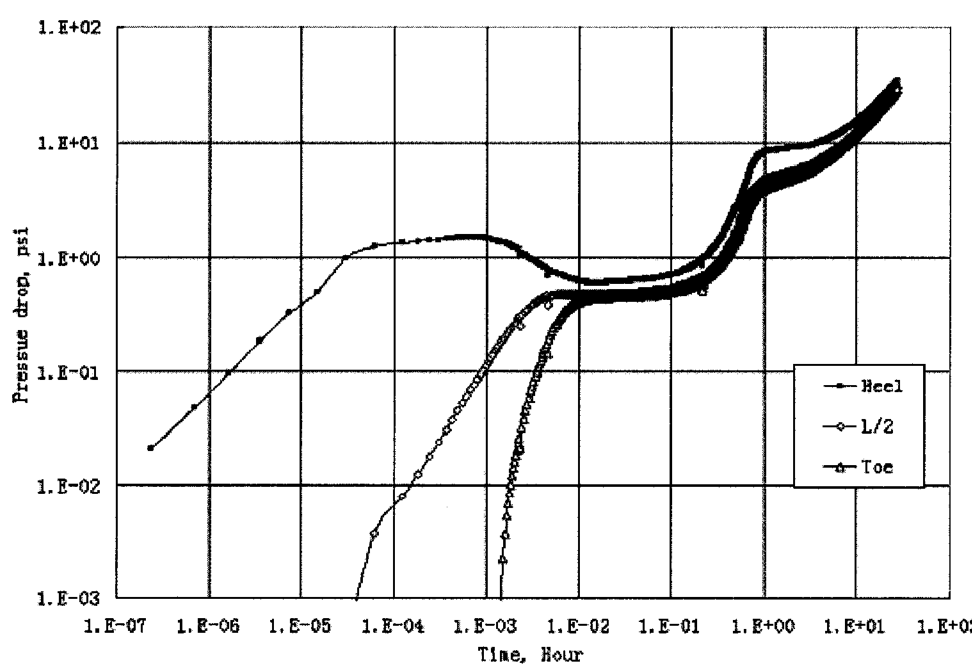

Figure 6.11 - Pressure drop at different well blocks

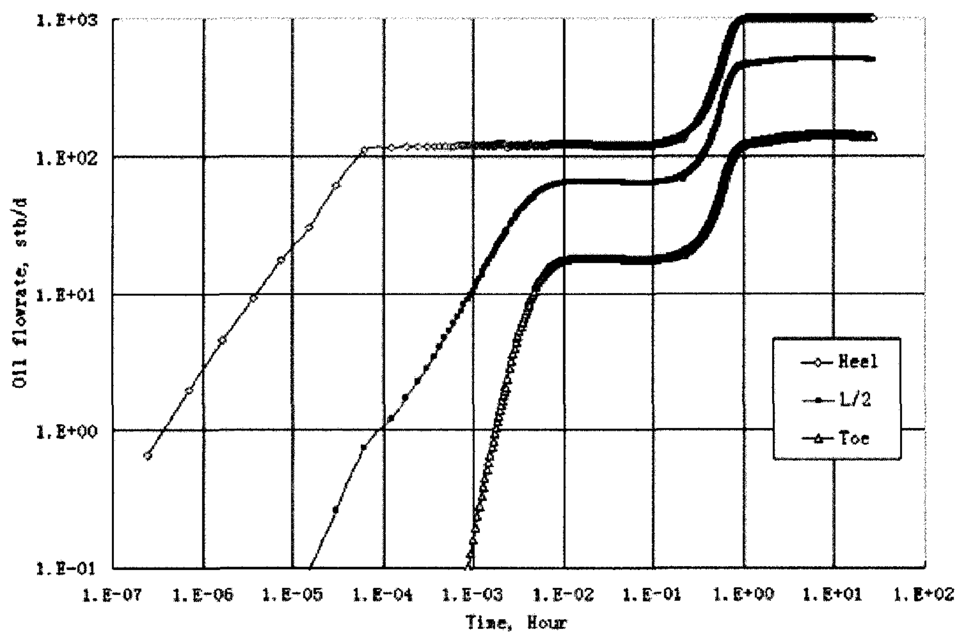

Figure 6.12 - Cumulative oil flow rate at different locations

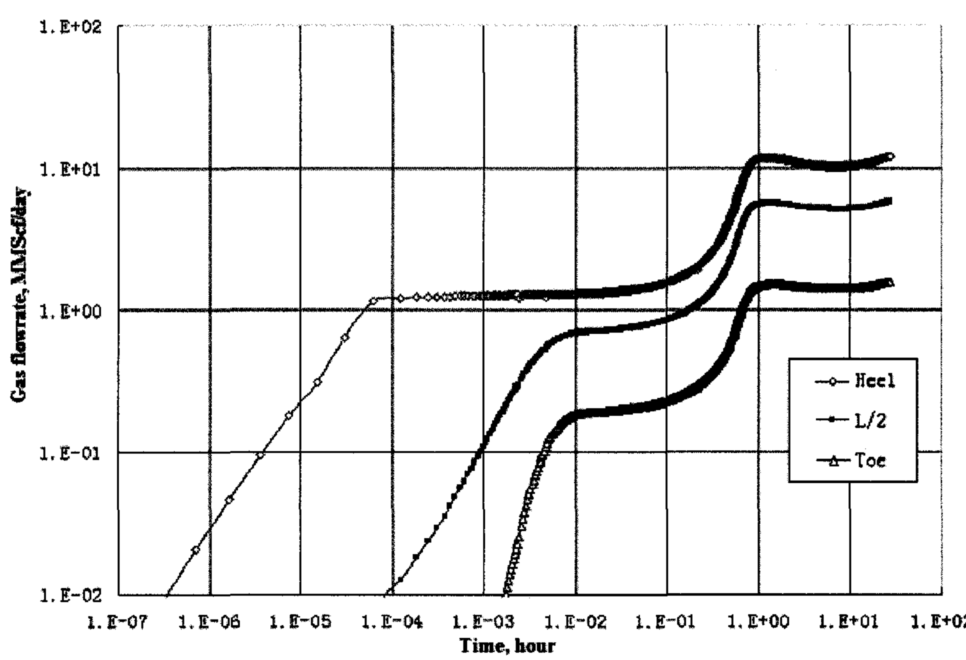

Figure 6.13 - Cumulative gas flow rate at different locations

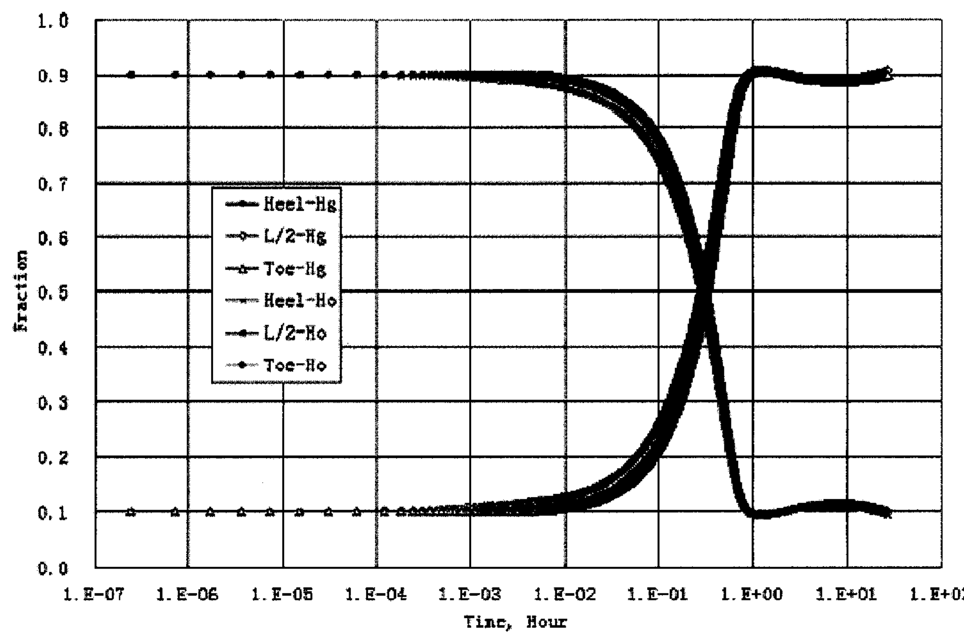

Figure 6.14 - Holdup at different locations

As shown in figure 6.11, three different pressure drop curves are compared to each other for pressure drop at the heel, the middle, and the toe. During the storage period, the difference is obvious because the fluid expansion effect propagates from the heel to the toe. The pressure drop at the well heel is higher than that on the toe. Even after the storage period, pressure drop at heel still remains the highest along the wellbore. This also reflects the necessary pressure difference from the toe to the heel to meet the flow condition, even though pressure drops at different locations tend to converge shortly after reservoir depletion begins.

As shown in figure 6.12 and 6.13, the cumulative oil and gas flow rate at different locations have the same trend. Because the influx is collected along the wellbore, the difference mainly reflects the accumulation of fluid through perforations. However, it should be noted that pressure distribution along the wellbore can influence the influx in return, so the influx is not necessarily uniformly distributed along the wellbore. Therefore the cumulative oil and gas flow rate is not proportional to the well segments length. Regarding to the gas void fraction and liquid holdup, the difference on various locations is not obvious because that oil and gas flow rate at the wellbore condition varies at almost the same scale.

Figure 6.15 shows the oil influx distribution along the wellbore. The wellbore unloading wave propagates along the wellbore at early times. The oil influx distribution evolves into a smooth profile after one minute.

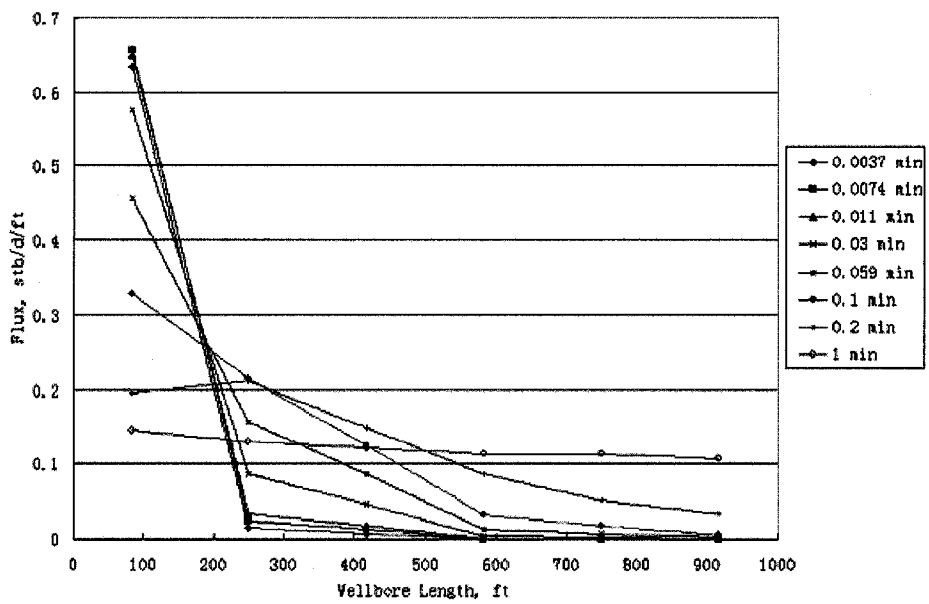

Figure 6.15 - Oil influx distribution at early time

6.3 Sensitivity Analysis on Permeability and Initial Gas Saturation

The transient flow behavior in the wellbore and its interaction with the reservoir are closely related to the reservoir properties and initial fluid phase existence. Thus a sensitivity analysis is conducted in this section to study the effect of reservoir permeability and initial gas saturation on the flow behavior.

Effect of Reservoir Permeability

Figures 6.16 to 6.18 compare the pressure drop and oil/gas flow rate for different reservoir permeabilities (k_h). The anisotropy ratio equals 0.1. Two different cases are compared to each other, the first one having k_h equal to 100 md and the second one with the

k_h equal to 10 md. The remaining data in both cases are the same as those presented in Chapter 5, section 5.2.4.

As shown in figure 6.16, during the storage period, reservoir permeability has no impact on the pressure drop behavior. This is expected to happen since during the wellbore storage period the whole process is mainly controlled by the wellbore geometry and fluid characteristics. Fluid expansion inside wellbore dominates during this period. With the fading of the fluid expansion effect, the importance of reservoir influx increases and the radial flow regime in reservoir starts to dominate the process. Higher reservoir permeability results in a lower pressure drop for the specified production rate.

Regarding to the oil and gas flow rate, the difference between the two cases is distinctive only during the storage period and initial radial influx period. Since the pressure drop is the same for both cases for the early storage period, the oil and gas flow rate for tighter reservoir is lower than that of the reservoir with a higher permeability, as shown in figure 6.17 and figure 6.18. Because the oil flow rate has been specified, the oil and gas flow rate for both cases tend to be the same after the storage period and initial radial influx period. For tighter reservoir (10 md case), this period will last longer than that for the 100 md reservoir.

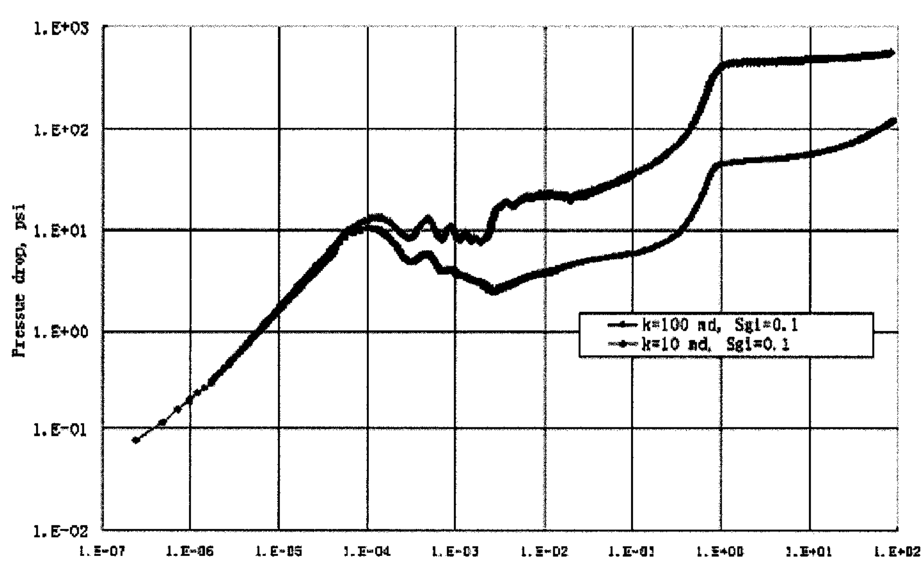

Figure 6.16 - Effect of reservoir permeability on pressure drop

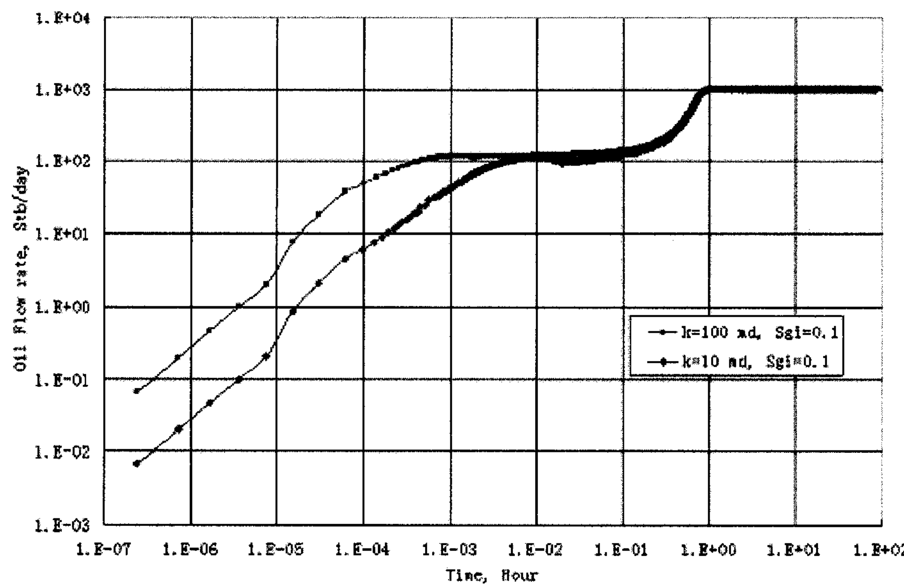

Figure 6.17 - Effect of reservoir permeability on oil flow rate

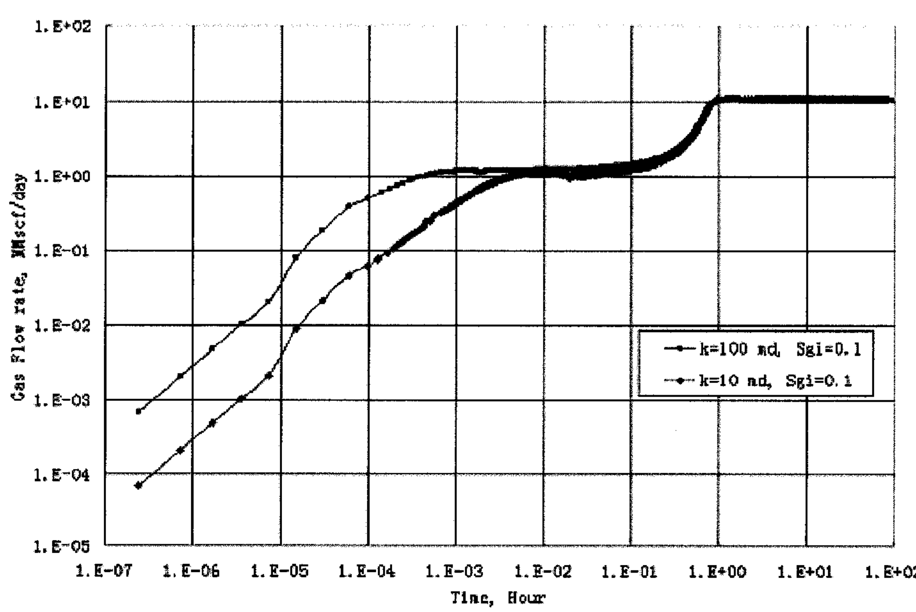

Figure 6.18 - Effect of reservoir permeability on gas flow rate

Effect of Initial Gas Saturation

Figures 6.19 to 6.21 analyze the pressure drop and oil/gas flow rate for different initial gas saturations. The horizontal reservoir permeability equals 100 md with the anisotropy ratio being 0.1 for both cases/scenarios.

As shown in figure 6.19, the pressure drop in the early storage period is almost the same for different initial gas saturations. However, during the radial influx period, the pressure drop is higher when the initial gas saturation is smaller. This can be explained from two aspects. First, as in figure 6.20 and 6.21, to maintain a specified oil flow rate, the reservoir with a lower initial gas saturation takes less time to reach the specified flow rate, implying a higher pressure drop during that period. Secondly, during this period the pressure drop mainly comes from viscous friction loss instead of reservoir depletion. When the initial gas saturation is lower, the fluid mixture in the wellbore is more viscous and the friction pressure loss is higher than that with a higher initial gas saturation.

When the reservoir depletion effect starts one hour later, the trend reverses with a higher pressure drop for the case/scenario of higher initial gas saturation. During this period, all cases reach to a specified oil flow rate and the depletion pressure drop dominates the total pressure drop instead of the friction. Higher initial gas saturation means lower initial oil saturation. And a reservoir with higher initial gas saturation depletes faster than that with lower initial gas saturation. This is why the case with higher initial gas saturation eventually gets higher pressure drop one hour after (figure 6.19). From these figures, it should be noted that different initial gas saturation results significantly different pressure drop. For example, lower Sgi (0.001) stands for slightly compressible or even single phase reservoir. Therefore, misuse of single phase flow model to predict two phase flow model would under-estimate the pressure drop, which could result in overoptimistic coning prediction.

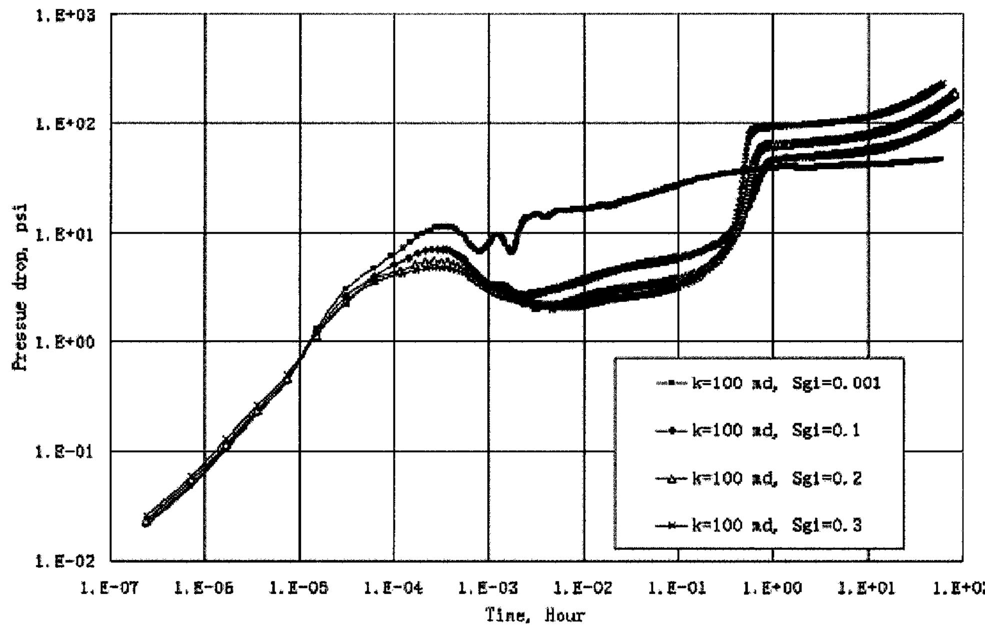

Figure 6.19 - Effect of initial gas saturation on pressure drop

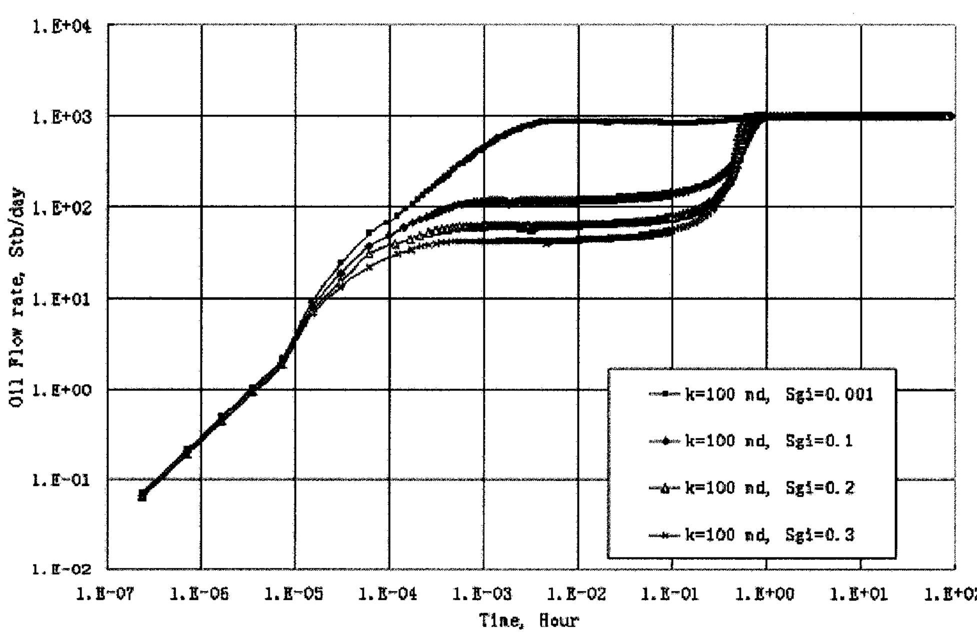

Figure 6.20 - Effect of initial gas saturation on oil production

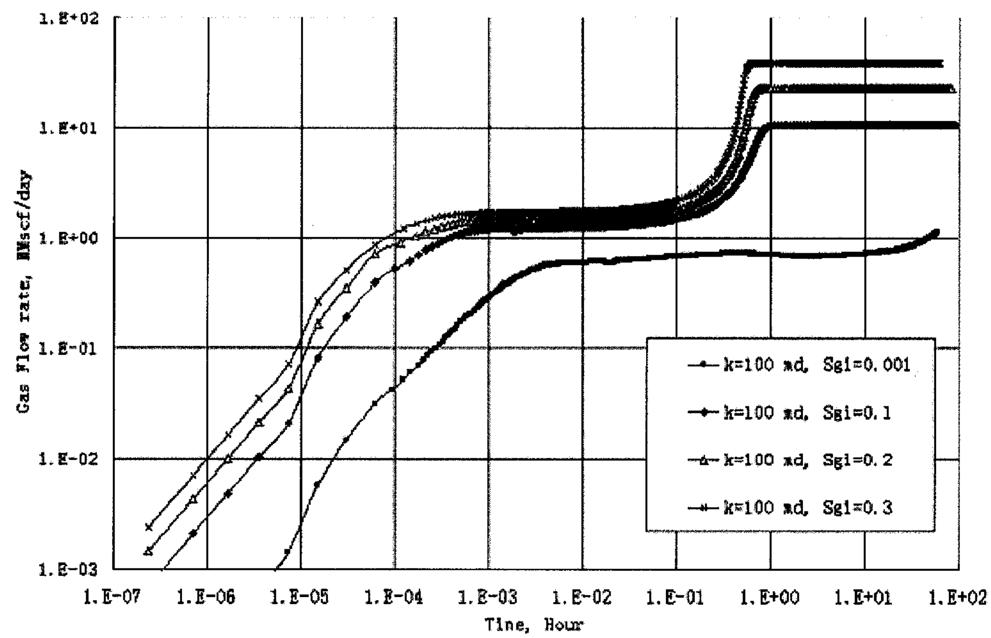

Figure 6.21 - Effect of initial gas saturation on gas production

6.4 Perforation Distribution along Horizontal Wellbore

Perforated completion is one of the most important methods for horizontal well completion. In this case, horizontal well deliverability cannot be calculated based on the whole horizontal wellbore length. Perforation distribution should be taken into consideration in the calculation. Perforation distribution profile along horizontal wellbore can have significant influence on the behavior of transient flow.

In this research, perforation can be controlled with the geometric factor in each horizontal well segment. The following experiments are conducted to study the influence of perforation distribution on the transient flow behavior of horizontal well. It is an oil and gas two phase flow on the assumption of homogenous flow pattern in the horizontal wellbore. The model is run under various scenarios so as to compare the perforation effects to each other, i.e. closing wellbore segments around the heel side, closing wellbore segments around the middle part, closing wellbore segments around the toe side and open hole. The remaining data in the above cases are the same as those presented in section 5.2.4. Comparisons are made for oil and gas flow rate at the heel, middle, and toe respectively, as well as near-wellbore reservoir pressure at the heel. In figures 6.22 to 6.30, legend '123' represents no perforations around the heel part, legend '456' represents no perforations around the middle part and legend '8910' represents no perforations around the toe part.

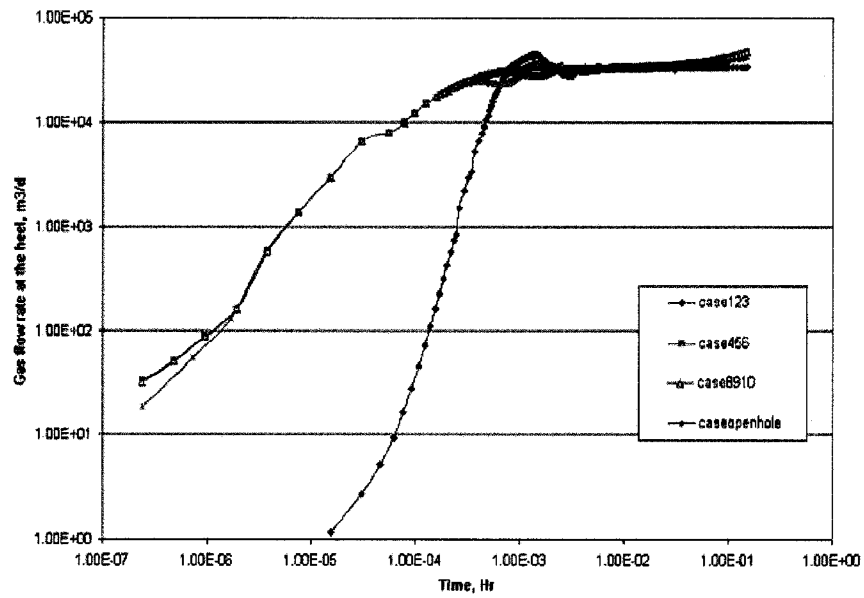

Figure 6.22 - Gas flow rate at the heel

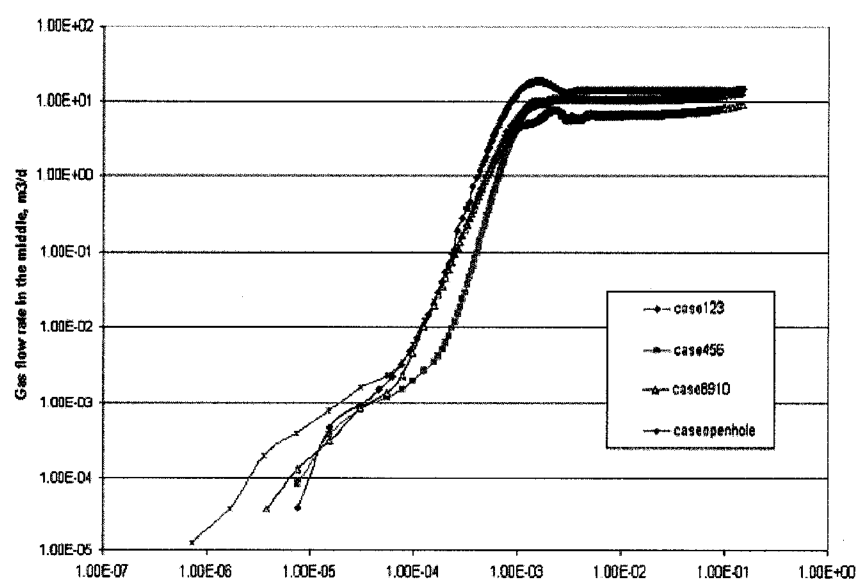

Figure 6.23 - Gas flow rate in the middle

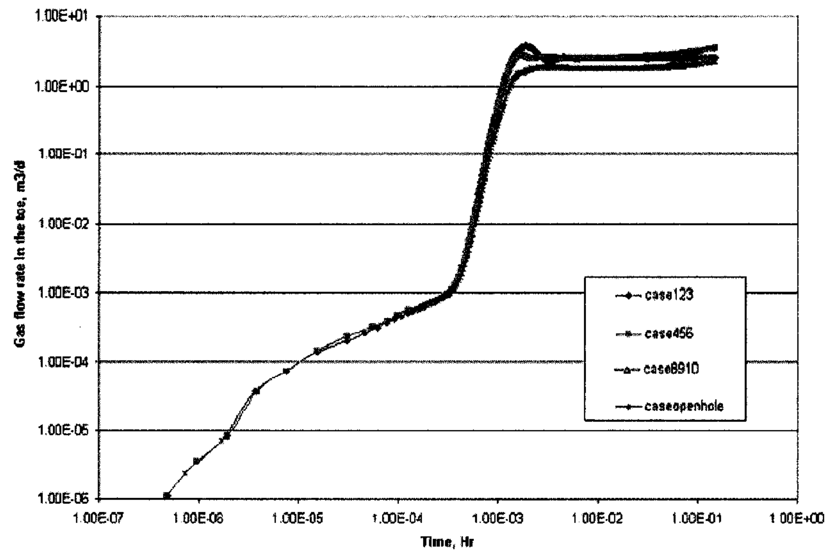

Figure 6.24 - Gas flow rate at the toe

Figures 6.22 to 6.24 show the gas flow rates at the heel, in the middle, and at the toe, respectively. The difference is obvious at the heel in the early stage. During this period, perforation distribution is significantly related to the initial radial influx, indicating a direct influence on the flow rate curves. Transient flow behavior in the wellbore propagates from the heel to the toe. And closing or reducing perforation density around the heel has significant effect on the transient flow behavior. There is not much difference in the toe part during the transient period. Oil flow rates have the same trend as that of gas flow rates in the heel, middle and toe part, as shown in figures 6.25 to 6.27.

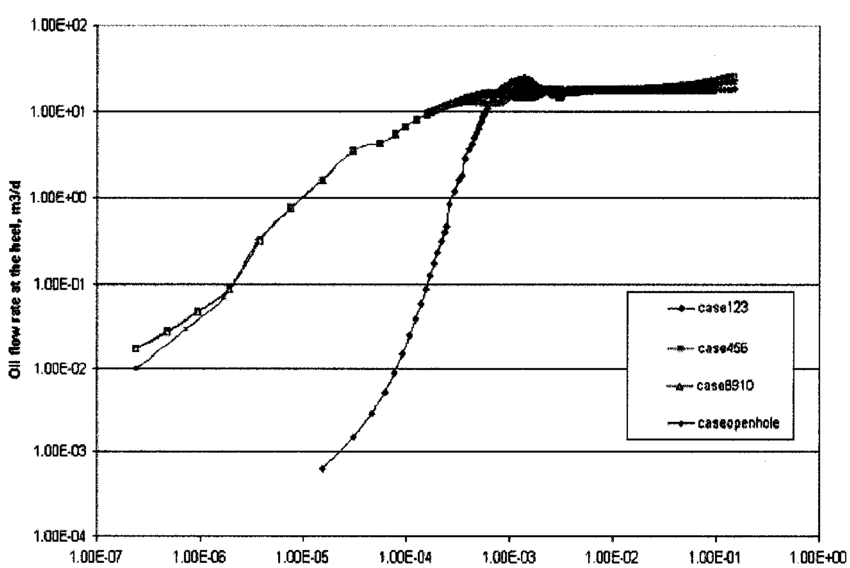

Figure 6.25 - Oil flow rate at the heel

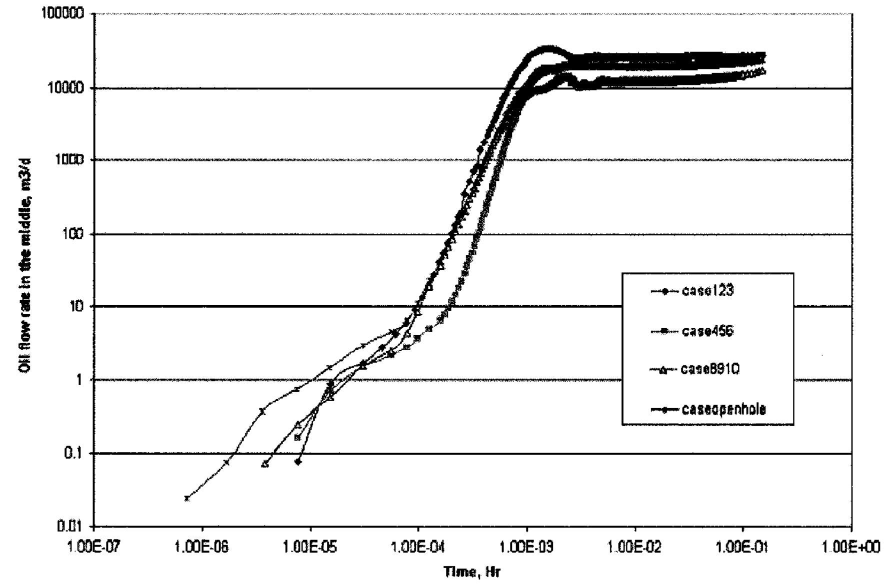

Figure 6.26 - Oil flow rate in the middle

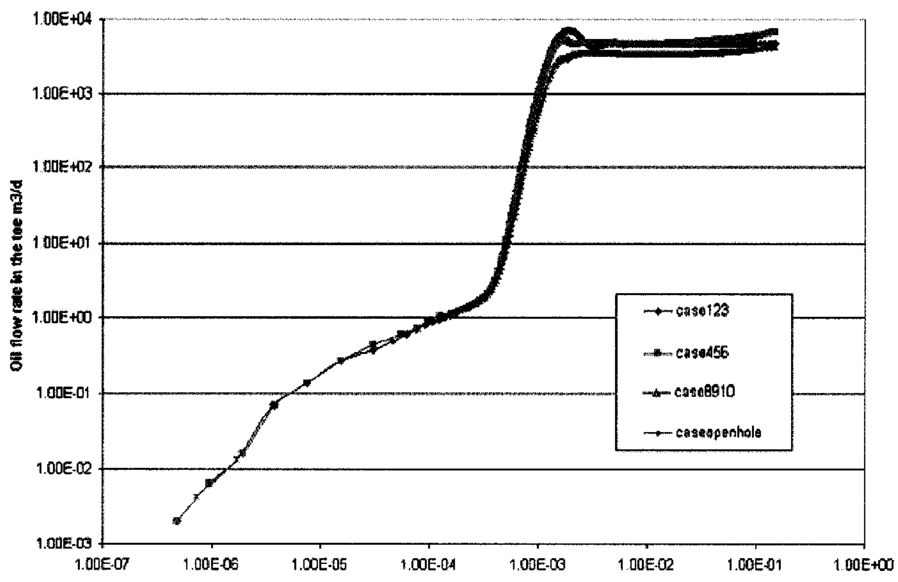

Figure 6.27 - Oil flow rate at the toe

Correspondingly, figure 6.28 presents the near-wellbore reservoir pressure change in the early stage. At the very beginning (wellbore storage period), when the reservoir pressure is still not that sensitive to the changes of influx, few distinctions are shown for different scenarios. When the wellbore storage period ends, perforation distribution then determines influx distribution from reservoir to wellbore. Hence reservoir pressure starts to show difference under various scenarios.

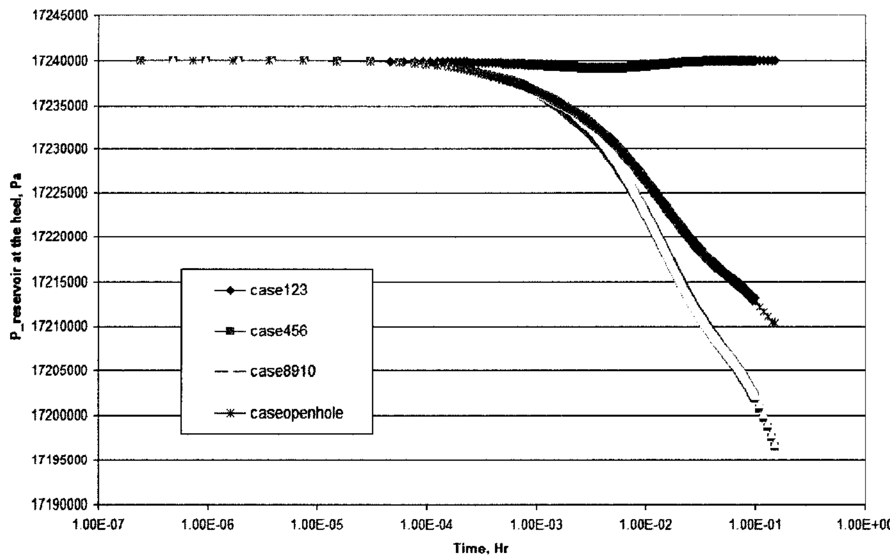

Figure 6.28 - Near-wellbore reservoir transient pressure at the heel

Besides the perforation distribution, permeability heterogeneity also influences the transient flow behaviour of horizontal well. Figures 6.29 to 6.30 analyze the perforation profile and different permeability heterogeneity influence on the transient flow behaviour. It is not hard to understand that reservoir pressure and fluid influx are strongly related to the permeability after the wellbore storage period. Therefore, lower permeability case has lower initial radial influx. And the influx profile, especially during the initial radial influx period, also depends on the perforation distribution.

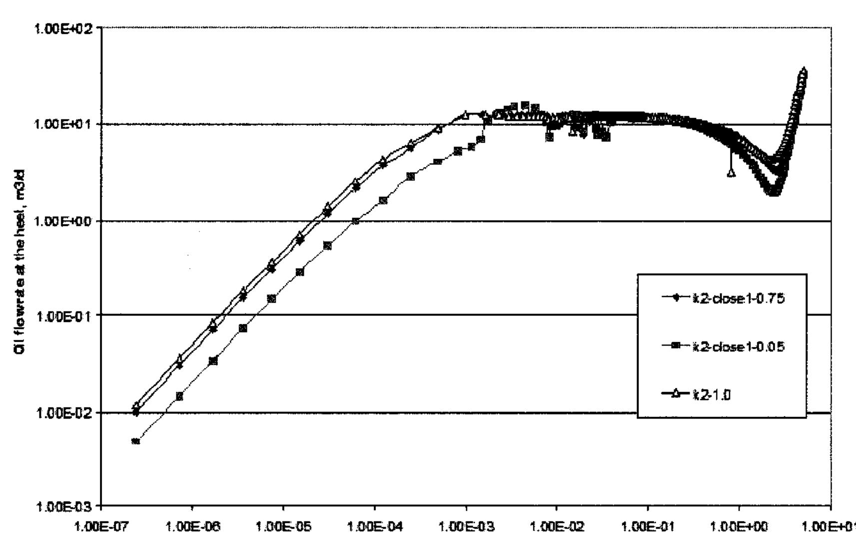

Figure 6.29 - Oil flow rate with different permeability (partially perforated wellbore)

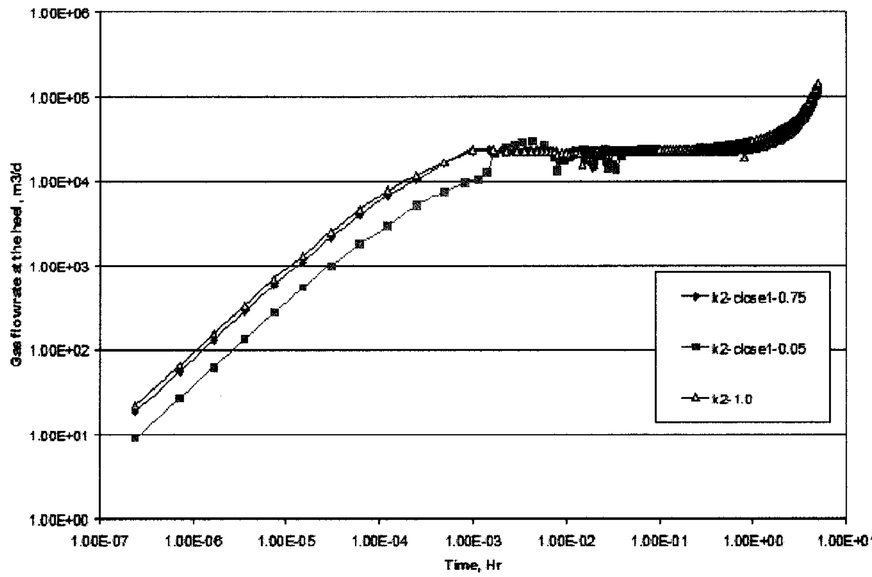

Figure 6.30 - Gas flow rate with different permeability (partially perforated wellbore)

CHAPTER 7: CONCLUSIONS AND RECOMMENDATIONS

This chapter summarizes the development and applications of reservoir and horizontal wellbore coupled model. Additionally, several recommendations are presented for future study in this area.

7.1 Conclusions

1. Flow behavior in both reservoir and horizontal wellbore domains can be simulated simultaneously by using the developed two phase coupled model. Since all the variables are simulated simultaneously, the results reflect the flow characteristics and interactions between reservoir and wellbore.

2. A 3-D fully implicit two-phase reservoir simulator is developed in this thesis. Hybrid local grid refinement technique is implemented in the simulator development. The hybrid grid system can accurately represent the radial flow around wellbore, therefore better serving the purpose of coupling the horizontal wellbore and the reservoir.

3. The multi-segmented horizontal wellbore model is built in the coupled model and can reveal the finite conductivity features in the horizontal wellbore. The influx and the effect accordingly are non-uniformly distributed in both the horizontal well and the reservoir.

4. A few applications using the developed coupled model analyze on the finite conductivity of horizontal well. Comparisons of influx distribution, flow rate distribution, drawdown and pressure distribution along the wellbore are made between the proposed coupled model and a non-coupled traditional model. Pressure and flow behavior at different locations in the wellbore are also studied upon to better understand the characteristics of the finite conductivity in horizontal well.

5. The coupled model predicts more influx coming from the heel. More attention is required if the flow rate is high and the reservoir has potential water or gas coning problem. It should be noted that the predictions on coning may be underestimated by just using a traditional model.

6. The frictional and accelerational pressure drop in the wellbore can not be ignored for high production rate wells. The wellbore pressure distribution caused by wall friction and acceleration can affect the distribution of the influx. The distribution of the reservoir pressure is also affected by the not-uniformly-distributed influx.

7. The coupled model should be used especially when the production rate is high. With a lower flow rate, the influx and wellbore pressure behavior tends to be similar to that of the non-coupled model. The wellbore conductivity tends to be infinite as that of the uncoupled model.

8. The coupled model is also used to study the two-phase flow behavior. Simulation results reflect the characteristics of the early-time transient flow in reservoir and horizontal wellbore. Pressure drop, oil/gas flow rate, and holdup behavior are studied the two-phase flow using the developed model.

9. This thesis analyzes the effect of reservoir permeability and initial gas saturation on the flow behavior. The results show that misuse of single phase flow model to predict two phase flow model could under-estimate the pressure drop, causing overoptimistic coning prediction.

10. Numerical experiments also show that perforation distribution profile along horizontal wellbore and permeability heterogeneity in reservoir have significant influence on the transient flow behavior.

7.2 Recommendations

1. Homogeneous flow pattern is assumed in this multi-segmented horizontal wellbore model. It is recommended to develop more flexible flow patterns in the wellbore model, e.g. stratified flow.

2. MATLAB™ solver is integrated into the FORTRAN™ source code to solve the Jacobian matrix equation. More efficient solvers, especially aiming at solving irregular sparse matrix from nonlinear hyperbolic / parabolic PDEs, should be developed /applied to enhance the capability of the developed simulator.

3. Well length optimization and perforation distribution optimization should be comprehensively studied with the coupled model. To achieve this, it is recommended to integrate the vertical part of the well into the coupling system.

BIBLIOGRAPHY

Almehaideb, R. A., Aziz, K., and Pedrosa Jr. O. A. "A Reservoir/Wellbore Model for Multiphase Injection and Pressure Transient Analysis". The SPE Middle East Meeting, Manama, Bahrain, 1989. Paper SPE 17941.

Baba, A., Tiab D. "Effect of Finite Conductivity Horizontal Well on Transient-Pressure Behavior". The SPE Permian Basin Oil and Gas Recovery Conference, Midland, Texas, 15-17 May 2001. Paper SPE 70013.

Babu, D. K. and Odeh, A. S. "Productivity of a Horizontal Well". The SPE Reservoir Engineering, 4(4): 417-421, 1989.

Collins, D. A., Farouq Ali, S. M., Sharma, R., Li, Y., and Jha, K.. "Field-Scale Simulation of Horizontal Wells with Hybrid Grid", The SPE Symposium on Reservoir Simulation, Anaheim, CA, 1991. Paper SPE 21218.

Dikken, B. J.: "Pressure Drop in Horizontal Wells and its Effects on Production Performance", Journal of Petroleum Technology, pp 1426-1433, November 1990.

Ertekin, T., Abou-Kassem, J.H. and King, G.R. Basic Applied Reservoir Simulation. Vol.7. 2001.

Folefac, A. N., Archer, J. S., and Issa, R. I. "Effect of Pressure Drop along Horizontal Wellbore on Well Performance". The Offshore Europe Conference, Aberdeen, 1991. Paper SPE 23094.

Gökta, B. and Ertekin, T. "Implementation of a Local Grid Refinement Technique in Modeling Slanted, Undulating Horizontal and Multi-Lateral Wells". The SPE Annual Technical Conference and Exhibition, Houston, Texas, 3-6 October 1999.

Gui, P. and Cunha, L.B., "A Coupled Model to Simulate the Fluid Flow in the Reservoir and Horizontal Wellbore". The 7th Canadian International Petroleum Conference, Calgary, Canada, June 2006.

Gui, P., Cunha, L.B., and Cunha, J.C., "A Numerical Two-Phase Flow Model Coupling Reservoir and Multi-Segment Horizontal Well". The 2007 SPE Rocky Mountain Oil & Gas Technology Symposium, Denver, USA, April 2007. Paper SPE 107989.

Ihara, M. and Shimizu, N. "Effect of Accelerational Pressure Drop in a Horizontal Wellbore". The SPE Annual Technical Conference and Exhibition, Houston, Texas, 1993. Paper SPE 26519.

Islam, M. R., and Chakma, A.: "Comprehensive Physical and Numerical Modeling of a Horizontal Well". The SPE Annual Technical Conference and Exhibition, New Orleans, Louisiana (1990). Paper SPE 20627.

Liu, X., and Jiang, Z. "The Model Coupling Fluid Flow in Reservoir with Flow in Horizontal Wellbore". The SPE International Conference on Horizontal Well Technology, Calgary, Canada, 1 - 4 November 1998. Paper SPE 50432.

Novy, R. A.. "Pressure Drops in Horizontal Wells: When Can They Be Ignored?" The SPE Reservoir Engineering, 10(1):29~35, 1995. Paper SPE 24941.

Ouyang L-B., and Aziz K., "A Simplified Approach to Couple Wellbore Flow and Reservoir Inflow for Arbitrary Well Configurations" The SPE Annual Technical Conference and Exhibition held in New Orleans, Louisiana, U.S.A., 27-30 September 1998. Paper SPE 48936.

Ouyang, L.-B. and Huang, B. "An Evaluation of Well Completion Impacts on the Performance of Horizontal and Multilateral Wells". The SPE Annual Technical Conference and Exhibition, Dallas, Texas, 9-12 October 2005. Paper SPE 96530.

Ozkan, E., Sarica, C., Haciislamoglu, M., and Raghavan, R.. "Effect of Conductivity on Horizontal Well Pressure Behavior". The SPE Annual Technical Conference and Exhibition, Washington, D. C., 1992. Paper SPE 24683.

Ozkan, E., Sarica, C., Haciislamoglu, M., and Raghavan, R. "The Influence of Pressure Drop along the Wellbore on Horizontal Well Productivity". The SPE Production Operations Symposium, Oklahoma. 1993. Paper SPE 25502.

Pedrosa Jr, O. A., Aziz, K., "Use of a Hybrid Grid in Reservoir Simulation", The Symposium on Reservoir Simulation, Dallas, 10-13 February 1985. Paper SPE 13507.

Penmatcha, V.R., Arbabi, S. and Aziz, K, “Effects of Pressure Drop in Horizontal Wells and Optimum Well Length”. The SPE Production Operations Symposium in Oklahoma City, Oklahoma, 9–11 March 1997. Paper SPE 37494.

Penmatcha, V.R., and Aziz, K. “A Comprehensive Reservoir/Wellbore Model for Horizontal Wells”. The SPE India Oil and Gas Conference and Exhibition in New Delhi. India, 17-19 February 1998. Paper SPE 39521.

Seines, K., Aavatsmark, I., Lien, S. C. and Rushworth, P. “Important Reservoir Considerations for Horizontal Wells: A Field Example”. The SPE Latin American Petroleum Engineering Conference, Rio de Janerio, Brazil, 1990. Paper SPE 21124.

Stone, T. W., Edmunds, N. R., and Kristoff, B. J. “A Comprehensive Wellbore/Reservoir Simulator”. The SPE Symposium on Reservoir Simulation, Houston, TX, 1989. Paper SPE 18419.

Vicente, R., Sarica, C. and Ertekin, T. “A Numerical Model Coupling Reservoir and Horizontal Well-Flow Dynamics: Transient Behavior of Single-Phase Liquid and Gas Flow”. The SPE/Petroleum Society of CIM International Conference on Horizontal Well Technology, Calgary, 2000. Paper SPE 65508.

Vicente, R., “A Numerical Model Coupling Reservoir and Horizontal Well Flow Dynamics”. PhD dissertation, The Pennsylvania State University, December 2000

Vicente R., Sarica C. and Ertekin T., “A Two-Phase Model Coupling Reservoir and Horizontal Well Flow Dynamics”. The SPE Latin American and Caribbean Petroleum Engineering Conference in Buenos Aires, Argentina, 25–28 March 2001. Paper SPE 69570.

Vicente, R., Sarica, C. and Ertekin, T., “An Investigation of Horizontal Well Completions Using a Two-Phase Model Coupling Reservoir and Horizontal Well Flow Dynamics”. The SPE Annual Technical Conference and Exhibition held in New Orleans, Louisiana, 30 September–3 October 2001. Paper SPE 71601.

Winterfeld, P. H. “Simulation of Pressure Buildup in a Multiphase Wellbore-Reservoir System". The SPE Annual Technical Conference and Exhibition, New Orleans, LA, 1986. Paper SPE 15534.

APPENDIX 1: RESERVOIR MODEL (PDE) IN CYLINDRICAL COORDINATES

This appendix presents the development of the partial differential equation for the reservoir model in cylindrical coordinates. Additionally, some of the grid and parameters treatment for reservoir model in the cylindrical coordinates are also introduced in this appendix.

A1.1 Partial Differential Equations

In the radial system, the reservoir flow equations can be derived based on a control volume of a radial element, which is $V = \Delta r \cdot (r\Delta\theta) \cdot \Delta z$, see figure A1.1.

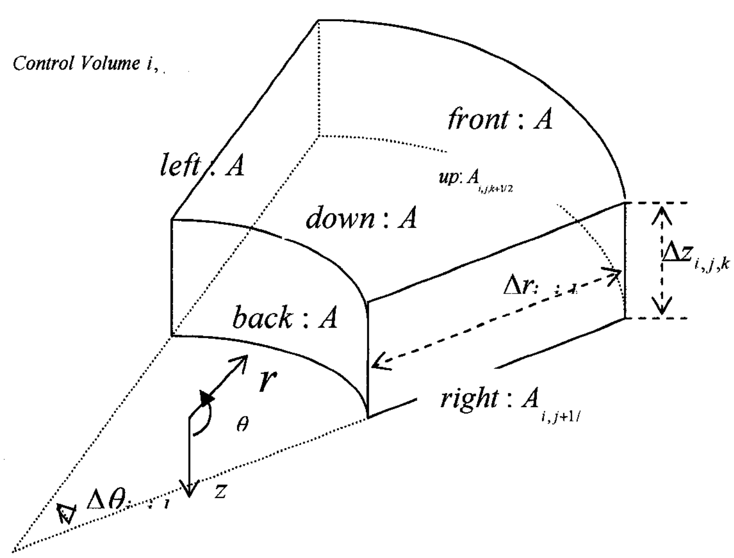

Figure A1.1 - Control volume in the cylindrical coordinates

The area perpendicular to flow is:

$$A_r = (r\Delta\theta) \cdot \Delta z \qquad A_\theta = \Delta r \cdot \Delta z \qquad A_z = (r\Delta\theta) \cdot \Delta r$$

Considering the flow continuity through the control volume, the mass conservation equation should be expressed based on the following (without external source/sink for oil and gas phases): (Mass in) – (Mass out) = (Mass Accumulation)

$$-\frac{\partial(\rho\upsilon_r A_r)}{\partial r}\Delta r-\frac{\partial(\rho\upsilon_\theta A_\theta)}{\partial\theta}\Delta\theta-\frac{\partial(\rho\upsilon_z A_z)}{\partial z}\Delta z=V\frac{\partial(\phi\rho)}{\partial t}\qquad\text{(A1.1)}$$

Where, $\upsilon_r, \upsilon_\theta$ *and* υ_z are volumetric velocities in $r-\theta-z$ direction, respectively, obtained with Darcy's Law.

$$\upsilon_r=-\frac{k_r k_{rl}}{\mu_l}\left(\frac{\partial P}{\partial r}-\gamma\frac{\partial h}{\partial r}\right)\qquad\text{(A1.1.a)}$$

$$\upsilon_\theta=-\frac{k_\theta k_{rl}}{\mu_l}\frac{1}{r}\left(\frac{\partial P}{\partial\theta}-\gamma\frac{\partial h}{\partial\theta}\right)\qquad\text{(A1.1.b)}$$

$$\upsilon_z=-\frac{k_z k_{rl}}{\mu_l}\left(\frac{\partial P}{\partial z}-\gamma\frac{\partial h}{\partial z}\right)\qquad\text{(A1.1.c)}$$

Substituting Eq. (1.1.a) ~ (1.1.c) to (1.1), there is:

$$-\frac{\partial\left[-\frac{k_r k_{rl}}{\mu_l}\left(\frac{\partial P}{\partial r}-\gamma\frac{\partial h}{\partial r}\right)\rho_l A_r\right]}{\partial r}\Delta r-\frac{\partial\left[-\frac{k_\theta k_{rl}}{\mu_l}\frac{1}{r}\left(\frac{\partial P}{\partial\theta}-\gamma\frac{\partial h}{\partial\theta}\right)\rho_l A_\theta\right]}{\partial\theta}\Delta\theta$$
$$-\frac{\partial\left[-\frac{k_z k_{rl}}{\mu_l}\left(\frac{\partial P}{\partial z}-\gamma\frac{\partial h}{\partial z}\right)\rho_l A_z\right]}{\partial z}\Delta z=V\frac{\partial(\phi\rho_l S_l)}{\partial t}\qquad\text{(A1.2)}$$

Using $B_l=\frac{\rho_{sc}}{\rho_l}$, $\frac{\partial(\rho_l)}{\partial t}=\frac{\partial(\frac{1}{B_l})}{\partial t}$, the above equation becomes:

$$\frac{\partial\left[A_r\frac{k_r k_{rl}}{B_l\mu_l}\left(\frac{\partial P}{\partial r}-\gamma\frac{\partial h}{\partial r}\right)\right]}{\partial r}\Delta r+\frac{1}{r}\frac{\partial\left[A_\theta\frac{k_\theta k_{rl}}{B_l\mu_l}\left(\frac{\partial P}{\partial\theta}-\gamma\frac{\partial h}{\partial\theta}\right)\right]}{\partial\theta}\Delta\theta$$
$$+\frac{\partial\left[A_z\frac{k_z k_{rl}}{B_l\mu_l}\left(\frac{\partial P}{\partial z}-\gamma\frac{\partial h}{\partial z}\right)\right]}{\partial z}\Delta z=V\frac{\partial(\phi\frac{S_l}{B_l})}{\partial t}\qquad\text{(A1.3)}$$

Adding the source sink terms, the reservoir flow equations in the cylindrical coordinates are expressed in the following formula:

For the gas phase:

$$\frac{\partial}{\partial r}[\frac{A_r k_r k_{rg}}{B_g \mu_g}(\frac{\partial P_g}{\partial r}-\gamma_g\frac{\partial h}{\partial r})+\frac{A_r R_s k_r k_{ro}}{B_o \mu_o}(\frac{\partial P_o}{\partial r}-\gamma_o\frac{\partial h}{\partial r})]\Delta r$$
$$+\frac{1}{r}\frac{\partial}{\partial \theta}[\frac{A_\theta k_\theta k_{rg}}{B_g \mu_g}(\frac{\partial P_g}{\partial \theta}-\gamma_g\frac{\partial h}{\partial \theta})+\frac{A_\theta R_s k_\theta k_{ro}}{B_o \mu_o}(\frac{\partial P_o}{\partial \theta}-\gamma_o\frac{\partial h}{\partial \theta})]\Delta \theta$$
$$+\quad\frac{\partial}{\partial z}[\frac{A_z k_z k_{rg}}{B_g \mu_g}(\frac{\partial P_g}{\partial z}-\gamma_g\frac{\partial h}{\partial z})+\frac{A_z R_s k_z k_{ro}}{B_o \mu_o}(\frac{\partial P_o}{\partial z}-\gamma_o\frac{\partial h}{\partial z})]\Delta z$$
$$=V\frac{\partial}{\partial t}(\phi\frac{S_g}{B_g}+\phi R_s\frac{S_o}{B_o})+R_s q_o V+q_{fg}V \tag{A1.4}$$

For the oil phase:

$$\frac{\partial}{\partial r}[\frac{A_r k_r k_{ro}}{B_o \mu_o}(\frac{\partial P_o}{\partial r}-\gamma_o\frac{\partial h}{\partial r})]\Delta r+\frac{1}{r}\frac{\partial}{\partial \theta}[\frac{A_\theta k_\theta k_{ro}}{B_o \mu_o}(\frac{\partial P_o}{\partial \theta}-\gamma_o\frac{\partial h}{\partial \theta})]\Delta \theta$$
$$+\frac{\partial}{\partial z}[\frac{A_z k_z k_{ro}}{B_o \mu_o}(\frac{\partial P_o}{\partial z}-\gamma_o\frac{\partial h}{\partial z})]\Delta z=V\frac{\partial}{\partial t}(\phi\frac{S_o}{B_o})+q_o V \tag{A1.5}$$

For water:

$$\frac{\partial}{\partial t}(\frac{S_w}{B_w})=0 \tag{A1.6}$$

Similar nomenclature and boundary conditions for above equations can be referred to Chapter 3. As described previously, the flow equations proposed here are linear equations related to 3-D flow in a $r-\theta-z$ cylindrical coordinates. The conversion from the linear flow geometry to a radial one is obtained through geometric factors presented in Chapter 3. In this way it is possible to refine the grid using hybrid technique around the wellbore, resulting in more accurate prediction. To achieve this, it is also necessary to compute the position of the grid point in a compatible way as presented in Chapter 3.

A1.2 Grid Definition and Ordering

Block centered method is adopted to discretize the flow equations. Two ordering schemes are shown here:

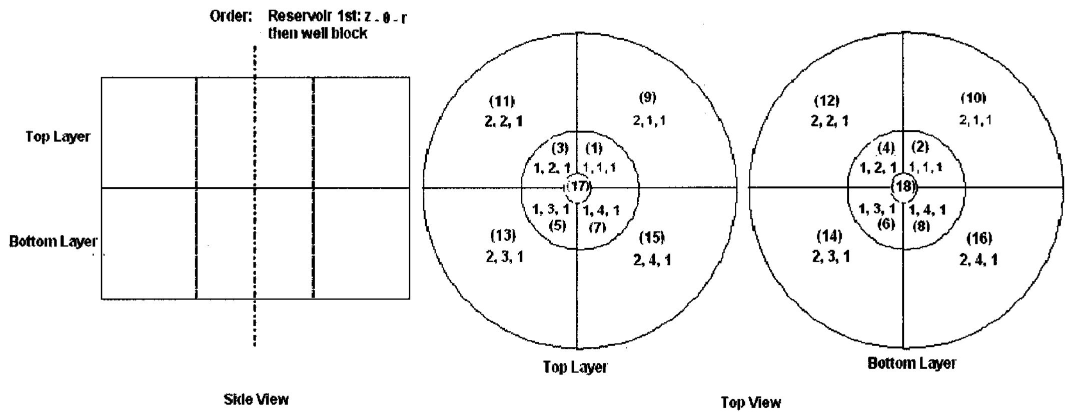

Figure A1.2 - Cylindrical grid ordering scheme 1

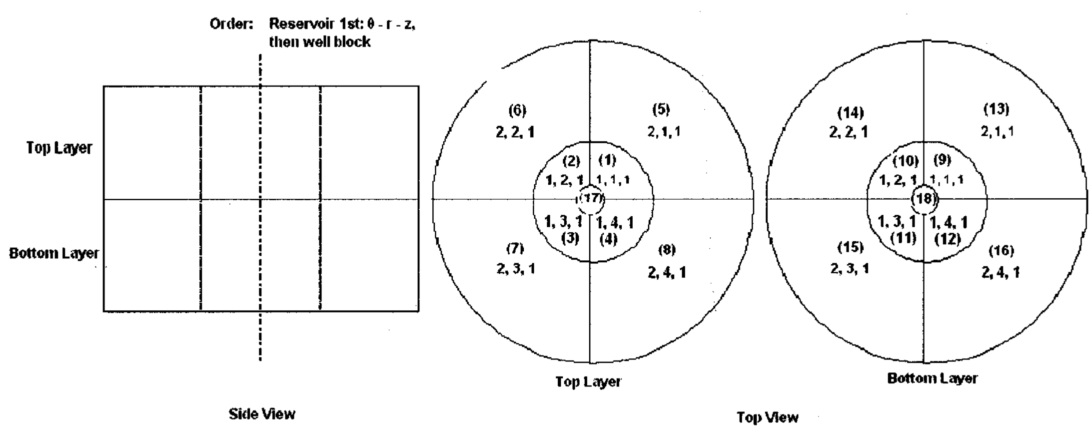

Figure A1.3 - Cylindrical grid ordering scheme 2

To reduce zero items in Jacobian matrix and keep it structured, it is reasonable to order the grid starting from smaller dimension (dimension z). Therefore, scheme 1 in figure A1.2 is adopted in this study.

A1.3 Computation of the Grid Points Position in Each Cell

The position of the grid node in each cell is the geometric point where the average pressure $\overline{p}$ of the cell is applied. The grid point position can be calculated in the following way for each direction:

a) Direction r

Grid point location in r direction can be obtained from the following expressions (Pedrosa, 1985).

$$r_{i,j,k} = r_{i-1/2,j,k} \exp(\frac{\alpha^2}{\alpha^2 - 1} \ln \alpha - 1/2) \qquad \text{Where,} \quad \alpha = \frac{r_{i+1/2,j,k}}{r_{i-1/2,j,k}} \qquad \text{(A1.7)}$$

Note that

$$r_{i-1/2,j,k} = r_w \exp[(i-1)\Delta\rho], \qquad r_{i+1/2,j,k} = r_w \exp[i\Delta\rho] \qquad \text{(A1.8)}$$

$i=1,2,\ldots nr;\ j=1,2,\ldots n\theta$; and $k=1,2,\ldots nz$

$$r_{i\pm1/2,j,1} = r_{i\pm1/2,j,2} = r_{i\pm1/2,j,3} = \ldots = r_{i\pm1/2,j,nz} \quad ,i = 1 \sim nr, j = 1 \sim n\theta$$

Thus,

$$\alpha = \frac{r_w \exp[i\Delta\rho]}{r_w \exp[(i-1)\Delta\rho]} = \frac{\exp[i\Delta\rho]\exp(\Delta\rho)}{\exp[i\Delta\rho]} = \exp(\Delta\rho), \ \textit{is CONSTANT} \qquad \text{(A1.9)}$$

So, $(\frac{\alpha^2}{\alpha^2 - 1} \ln \alpha - 1/2)$ is constant for a given r_e and r_w and nr.

And,

$$r_{i,j,k} = r_{i-1/2,j,k} \exp(\frac{\alpha^2}{\alpha^2 - 1} \ln \alpha - 1/2) = r_{i-1/2,j,k} \cdot Const. = Const. \cdot\ r_w \exp[(i-1)\Delta\rho]$$

(A1.10)

b) Direction θ

Grid point in the θ direction is simply located in the center of each cell in the θ direction:

$$\theta_{i,j,k} = \theta_{i,j-1/2,k} + \frac{\Delta\theta}{2} \qquad \text{(A1.11)}$$

Where i =1,2,…nr; j =1,2,… $n\theta$; and k =1,2,…nz ;

c) Direction z

Similar like the θ direction, grid point is also located in the centre of each cell in the vertical direction, i.e., with respect to the direction z

$$z_{i,j,k} = z_{i,j,k-1/2} + \frac{\Delta z}{2} \tag{A1.12}$$

Where i =1, 2,…nr; j=1,2,… $n\theta$; and k=1,2,…nzz ;

A1.4 Determination of Geometric Factors at the Interface

The geometric factors (GF) are calculated at the interfaces of each cell through the following expressions in the cylindrical system.

a) Direction r

$$GF_{r_{i-1/2,j,k}} = \frac{\Delta\theta_j \cdot \Delta z_k}{\frac{1}{Kr_{i-1,j,k}} \ln(\frac{r_{i-1/2,j,k}}{r_{i-1,j,k}}) + \frac{1}{Kr_{i,j,k}} \ln(\frac{r_{i,j,k}}{r_{i-1/2,j,k}})}$$

$$GF_{r_{i+1/2,j,k}} = \frac{\Delta\theta_j \cdot \Delta z_k}{\frac{1}{Kr_{i,j,k}} \ln(\frac{r_{i+1/2,j,k}}{r_{i,j,k}}) + \frac{1}{Kr_{i+1,j,k}} \ln(\frac{r_{i+1,j,k}}{r_{i+1/2,j,k}})}$$

(A1.13)

Where,

$r_{i\pm1/2,j,k}$: Radius of the grid block interfaces;

$r_{i,j,k}$: Radius of the grid points;

Δz : Cell thickness;

$\Delta\theta$: Regular angle of the grid block in θ direction;

$Kr_{i,j,k}$, $Kr_{i-1,j,k}$, $Kr_{i+1,j,k}$: Absolute permeability in direction r.

b) Direction θ

$$GF_{\theta_{i,j+1/2,k}} = \frac{\ln(r_{i+1/2,j,k} / r_{i-1/2,j,k}) \cdot \Delta z_k}{\dfrac{(\theta)_{i,j+1/2,k} - (\theta)_{i,j,k}}{K_{\theta_{i,j,k}}} + \dfrac{(\theta)_{i,j+1,k} - (\theta)_{i,j+1/2,k}}{K_{\theta_{i,j+1,k}}}}$$

(A1.14)

Where

$r_{i\pm 1/2,j,k}$: Radius of the grid block interfaces;

$(\theta)_{i,j+1/2,k} - (\theta)_{i,j,k} = \frac{\Delta\theta}{2}$: Regular angle of the grid block in θ direction (equally distributed);

$(\theta)_{i,j+1,k} - (\theta)_{i,j+1/2,k} = \frac{\Delta\theta}{2}$: Regular angle of the grid block in θ direction (equally distributed);

$K_{\theta_{i,j,k}}, K_{\theta_{i,j+1,k}}$: Absolute permeabilities in θ direction.

c) Direction z

$$GF_{z_{i,j,k+1/2}} = \frac{\dfrac{\Delta\theta_j}{2} \cdot (r^2_{i+1/2,j,k} - r^2_{i-1/2,j,k})}{\dfrac{z_{i,j,k+1/2} - z_{i,j,k}}{Kz_{i,j,k}} + \dfrac{z_{i,j,k+1} - z_{i,j,k+1/2}}{Kz_{i,j,k+1}}}$$

(A1.15)

Where

$r_{i\pm 1/2,j,k}$: Radius of the grid block interfaces;

$z_{i,j,k+1/2} - z_{i,j,k} = \frac{\Delta z}{2}$: Regular grid in z direction (equally distributed);

$z_{i,j,k+1} - z_{i,j,k+1/2} = \frac{\Delta z}{2}$: Regular grid in z direction (equally distributed);

$\Delta\theta_j (r^2_{i+1/2,j,k} - r^2_{i-1/2,j,k})$: Area $A_{i,j,k}$ in radial direction of the grid cell;

$Kz_{i,j,k}, Kz_{i,j,k+1}$: Absolute permeability in direction z.

A1.5 Determination of Weight Factors and Interfaces Properties

In r direction, the pressure profile is assumed to be linear in a logarithmic scale with respect to distance, as shown schematically as the following.

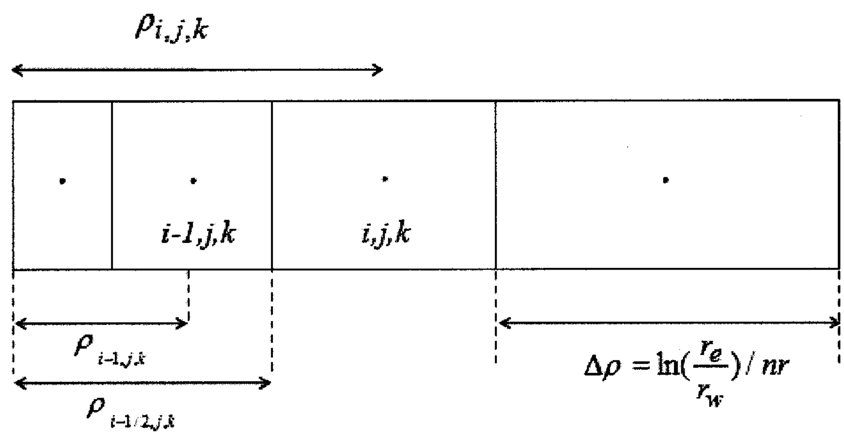

Figure A1.4 - Logarithmic scale linear distribution profile

Weight factor w_r is defined as

$$w_r = \frac{\rho_{i-1/2,j,k} - \rho_{i-1,j,k}}{\rho_{i,j,k} - \rho_{i-1,j,k}} \tag{A1.16}$$

Where, $\rho = \ln(\frac{r}{r_w})$, thus

$$w_r = \frac{\ln(\frac{r_{i-1/2,j,k}}{r_w}) - \ln(\frac{r_{i-1,j,k}}{r_w})}{\ln(\frac{r_{i,j,k}}{r_w}) - \ln(\frac{r_{i-1,j,k}}{r_w})} = \frac{\ln(\frac{r_{i-1/2,j,k}}{r_{i-1,j,k}})}{\ln(\frac{r_{i,j,k}}{r_{i-1,j,k}})} \tag{A1.17}$$

And because:

$$r_{i-1/2,j,k} = r_w \exp[(i-1)\Delta\rho]$$

$$r_{i-1,j,k} = r_{i-3/2,j,k} \exp(\frac{\alpha^2}{\alpha^2 - 1}\ln\alpha - 1/2) = r_{i-3/2,j,k} \cdot C = C \cdot r_w \exp[(i-2)\Delta\rho]$$

$$r_{i,j,k} = r_{i-1/2,j,k} \exp(\frac{\alpha^2}{\alpha^2 - 1}\ln\alpha - 1/2) = r_{i-1/2,j,k} \cdot C = C \cdot r_w \exp[(i-1)\Delta\rho]$$

So:

$$w_r = \frac{\ln\{\frac{r_w \exp[(i-1)\Delta\rho]}{C \cdot r_w \exp[(i-2)\Delta\rho]}\}}{\ln\{\frac{C \cdot r_w \exp[(i-1)\Delta\rho]}{C \cdot r_w \exp[(i-2)\Delta\rho]}\}} = \frac{\ln\{\frac{\exp[(i-1)\Delta\rho]}{C \cdot \exp[(i-2)\Delta\rho]}\}}{\ln\{\frac{\exp[(i-1)\Delta\rho]}{\exp[(i-2)\Delta\rho]}\}} = \frac{\ln(\frac{\exp\Delta\rho}{C})}{\ln(\exp\Delta\rho)} = \frac{\Delta\rho - \ln C}{\Delta\rho}$$

(A1.18)

Thus, $w_r = \frac{\Delta\rho - \ln C}{\Delta\rho}$ is constant with given r_w, r_e and nr

In θ direction, w_θ is simply 0.5 since the grid point is located in the center in the θ direction, shown in the following figure A1.5.

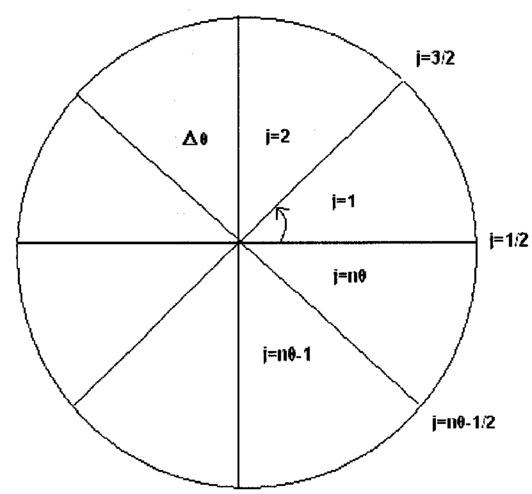

Figure A1.5 - Weight factor in angle direction

$$w_\theta = \frac{\theta_{i,j-1/2,k} - \theta_{i,j-1,k}}{\theta_{i,j,k} - \theta_{i,j-1,k}}, \text{ where } \theta_{i,j-1/2,k} - \theta_{i,j-1,k} = \frac{\Delta\theta}{2} \text{ and } \theta_{i,j,k} - \theta_{i,j,k-1} = \Delta\theta$$

$$So: w_\theta = 0.5$$

Similar for z direction, the weight factor w_z is defined as:

$$w_z = \frac{z_{i,j,k-1/2} - z_{i,j,k-1}}{z_{i,j,k} - z_{i,j,k-1}}$$

$$z_{i,j,k-1/2} - z_{i,j,k-1} = \frac{\Delta z}{2};\ z_{i,j,k} - z_{i,j,k-1} = \Delta z. \qquad So: w_z = 0.5$$

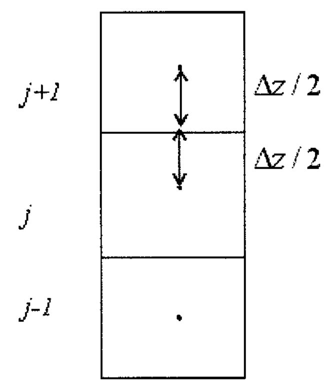

Figure A1.6 - Weight factor in *z* direction

The process of determining interface properties is similar like that in Cartesian coordinates.

APPENDIX 2: DERIVATION OF TRANSMISSIBILITY TERM, FINITE DIFFERENCE EQUATION AND RESIDUAL FUNCTION

This appendix presents the detailed derivation of the transmissibility terms and the finite difference equations based on partial differential equations for the reservoir model. It also includes the detail derivation of the residual function that is used for Jacobian matrix.

A2.1 Transmissibility Terms (Upstream Weighting)

The following part gives detailed computation process of the transmissibility terms. Derivatives of the transmissibility with respect to saturation and pressure are also presented in this section.

A2.1.1 Transmissibility Terms and Derivatives with Respect to Saturation

Transmissibility terms are defined as:

$$T_{xo_{i\pm1/2,j,k}} = \frac{A_{x_{i\pm1/2,j,k}} \lambda_{o_{i\pm1/2,j,k}}}{\Delta x_{i\pm1/2,j,k}}, \text{ where, } \frac{A_{x_{i\pm1/2,j,k}} \cdot k_x}{\Delta x_{i\pm1/2,j,k}} = GF_{x_{i\pm1/2,j,k}} \text{ and } A_{x_{i\pm1/2,j,k}} = \Delta y_{i,j,k} \cdot \Delta z_{i,j,k}$$

$$T_{yo_{i,j\pm1/2,k}} = \frac{A_{y_{i,j\pm1/2,k}} \lambda_{o_{i,j\pm1/2,k}}}{\Delta y_{i,j\pm1/2,k}}, \text{ where, } \frac{A_{y_{i,j\pm1/2,k}} \cdot k_y}{\Delta y_{i,j\pm1/2,k}} = GF_{y_{i,j\pm1/2,k}} \text{ and } A_{y_{i,j\pm1/2,k}} = \Delta x_{i,j,k} \cdot \Delta z_{i,j,k}$$

$$T_{zo_{i,j,k\pm1/2}} = \frac{A_{z_{i,j,k\pm1/2}} \lambda_{o_{i,j,k\pm1/2}}}{\Delta z_{i,j,k\pm1/2}}, \text{ where, } \frac{A_{z_{i,j,k\pm1/2}} \cdot k_z}{\Delta z_{i,j,k\pm1/2}} = GF_{z_{i,j,k\pm1/2}} \text{ and } A_{z_{i,j\pm1/2,k}} = \Delta y_{i,j,k} \Delta x_{i,j,k}$$

$$T_{xg_{i\pm1/2,j,k}} = \frac{A_{x_{i\pm1/2,j,k}} \lambda_{g_{i\pm1/2,j,k}}}{\Delta x_{i\pm1/2,j,k}}, \text{ where, } \frac{A_{x_{i\pm1/2,j,k}} \cdot k_x}{\Delta x_{i\pm1/2,j,k}} = GF_{x_{i\pm1/2,j,k}} \text{ and } A_{x_{i\pm1/2,j,k}} = \Delta y_{i,j,k} \cdot \Delta z_{i,j,k}$$

$$T_{yg_{i,j\pm1/2,k}} = \frac{A_{y_{i,j\pm1/2,k}} \lambda_{g_{i,j\pm1/2,k}}}{\Delta y_{i,j\pm1/2,k}}, \text{ where, } \frac{A_{y_{i,j\pm1/2,k}} \cdot k_y}{\Delta y_{i,j\pm1/2,k}} = GF_{y_{i,j\pm1/2,k}} \text{ and } A_{y_{i,j\pm1/2,k}} = \Delta x_{i,j,k} \cdot \Delta z_{i,j,k}$$

$$T_{zg_{i,j,k\pm1/2}} = \frac{A_{z_{i,j,k\pm1/2}} \lambda_{g_{i,j,k\pm1/2}}}{\Delta z_{i,j,k\pm1/2}}, \text{ where, } \frac{A_{z_{i,j,k\pm1/2}} \cdot k_z}{\Delta z_{i,j,k\pm1/2}} = GF_{z_{i,j,k\pm1/2}} \text{ and } A_{z_{i,j\pm1/2,k}} = \Delta y_{i,j,k} \Delta x_{i,j,k}$$

(A2.1)

And $\lambda_f = \frac{kk_{rf}}{\mu_f} b_f$

a) Direction x

Adopting the one point upstream scheme for the flow from $i+1$ to i for any j,k, there is:

$$T_{xl_{i-1/2,j,k}} = \frac{GF_{x\,i-1/2,j,k}\, b_{l_{i-1/2,j,k}}}{\mu_{l_{i-1/2,j,k}}} k_{rl_{i,j,k}}$$

$$T_{xl_{i+1/2,j,k}} = \frac{GF_{x\,i+1/2,j,k}\, b_{l_{i+1/2,j,k}}}{\mu_{l_{i+1/2,j,k}}} k_{rl_{i+1,j,k}}$$

(A2.2)

Thus,

$$\frac{\partial T_{xl_{i-1/2,j,k}}}{\partial S_{o_{i-1,j,k}}} = 0$$

$$\frac{\partial T_{xl_{i-1/2,j,k}}}{\partial S_{o_{i,j,k}}} = \frac{GF_{x\,i-1/2,j,k}\, b_{l_{i-1/2,j,k}}}{\mu_{l_{i-1/2,j,k}}} \frac{dk_{rl_{i,j,k}}}{dS_{o_{i,j,k}}}$$

$$\frac{\partial T_{xl_{i-1/2,j,k}}}{\partial S_{o_{i+1,j,k}}} = 0$$

$$\frac{\partial T_{xl_{i+1/2,j,k}}}{\partial S_{o_{i-1,j,k}}} = 0$$

$$\frac{\partial T_{xl_{i+1/2,j,k}}}{\partial S_{o_{i,j,k}}} = 0$$

$$\frac{\partial T_{xl_{i+1/2,j,k}}}{\partial S_{o_{i+1,j,k}}} = \frac{GF_{x\,i+1/2,j,k}\, b_{l_{i+1/2,j,k}}}{\mu_{l_{i+1/2,j,k}}} \frac{dk_{rl_{i+1,j,k}}}{dS_{o_{i+1,j,k}}}$$

(A2.3)

Flow from i to $i+1$ for any j, k,

$$T_{xl_{i-1/2,j,k}} = \frac{GF_{x\,i-1/2,j,k}\, b_{l_{i-1/2,j,k}}}{\mu_{l_{i-1/2,j,k}}} k_{rl_{i-1,j,k}}$$

$$T_{xl_{i+1/2,j,k}} = \frac{GF_{x\,i+1/2,j,k}\, b_{l_{i+1/2,j,k}}}{\mu_{l_{i+1/2,j,k}}} k_{rl_{i,j,k}}$$

(A2.4)

Thus,

$$\frac{\partial T_{xl_{i-1/2,j,k}}}{\partial S_{o_{i-1,j,k}}} = \frac{GF_{x\,i-1/2,j,k}\, b_{l_{i-1/2,j,k}}}{\mu_{l_{i-1/2,j,k}}} \frac{dk_{rl_{i-1,j,k}}}{dS_{o_{i-1,j,k}}}$$

$$\frac{\partial T_{xl_{i-1/2,j,k}}}{\partial S_{o_{i,j,k}}} = 0$$

$$\frac{\partial T_{xl_{i-1/2,j,k}}}{\partial S_{o_{i+1,j,k}}} = 0$$

$$\frac{\partial T_{xl_{i+1/2,j,k}}}{\partial S_{o_{i-1,j,k}}} = 0$$

$$\frac{\partial T_{xl_{i+1/2,j,k}}}{\partial S_{o_{i,j,k}}} = \frac{GF_{x\,i+1/2,j}\, b_{l_{i+1/2,j,k}}}{\mu_{l_{i+1/2,j,k}}} \frac{dk_{rl_{i,j,k}}}{dS_{o_{i,j,k}}}$$

$$\frac{\partial T_{xl_{i+1/2,j,k}}}{\partial S_{o_{i+1,j,k}}} = 0$$

(A2.5)

The above two situations can be generalized with a control parameter (δ_{lx}, $l = o, g$), the general form of transmissibility and its derivatives with respect to saturation can be expressed in the following forms for x direction:

$$\overleftarrow{\delta_{lx}} = 1, \quad \overrightarrow{\delta_{lx}} = 0$$

$$T_{xl_{i-1/2,j,k}} = \frac{GF_{x\,i-1/2,j,k}\, b_{l_{i-1/2,j,k}}}{\mu_{l_{i-1/2,j,k}}} [\delta_{lx} k_{rl_{i,j,k}} + (1-\delta_{lx}) k_{rl_{i-1,j,k}}]$$

$$T_{xl_{i+1/2,j,k}} = \frac{GF_{x\,i+1/2,j,k}\, b_{l_{i+1/2,j,k}}}{\mu_{l_{i+1/2,j,k}}} [\delta_{lx} k_{rl_{i+1,j,k}} + (1-\delta_{lx}) k_{rl_{i,j,k}}]$$

$$\frac{\partial T_{xl_{i-1/2,j,k}}}{\partial S_{o_{i-1,j,k}}} = (1-\delta_{lx})\frac{GF_{x\,i-1/2,j,k}\,b_{l_{i-1/2,j,k}}}{\mu_{l_{i-1/2,j,k}}}\frac{dk_{rl_{i-1,j,k}}}{dS_{o_{i-1,j,k}}}$$

$$\frac{\partial T_{xl_{i-1/2,j,k}}}{\partial S_{o_{i,j,k}}} = \delta_{lx}\frac{GF_{x\,i-1/2,j,k}\,b_{l_{i-1/2,j,k}}}{\mu_{l_{i-1/2,j,k}}}\frac{dk_{rl_{i,j,k}}}{dS_{o_{i,j,k}}}$$

$$\frac{\partial T_{xl_{i-1/2,j,k}}}{\partial S_{o_{i+1,j,k}}} = 0$$

$$\frac{\partial T_{xl_{i+1/2,j,k}}}{\partial S_{o_{i-1,j,k}}} = 0$$

$$\frac{\partial T_{xl_{i+1/2,j,k}}}{\partial S_{o_{i,j,k}}} = (1-\delta_{lx})\frac{GF_{x\,i+1/2,j,k}\,b_{l_{i+1/2,j,k}}}{\mu_{l_{i+1/2,j,k}}}\frac{dk_{rl_{i,j,k}}}{dS_{o_{i,j,k}}}$$

$$\frac{\partial T_{xl_{i+1/2,j,k}}}{\partial S_{o_{i+1,j,k}}} = \delta_{lx}\frac{GF_{x\,i+1/2,j,k}\,b_{l_{i+1/2,j,k}}}{\mu_{l_{i+1/2,j,k}}}\frac{dk_{rl_{i+1,j,k}}}{dS_{o_{i+1,j,k}}}$$

(A2.6)

Note that:

When $i=1$ and $j=1$ to ny; $k=1$ to nz, $T_{rl_{i-1/2,j,k}}$ =0 and $\frac{\partial T_{rl_{i-1/2,j,k}}}{\partial S_{o_{i-1,j,k}}} = 0$

When $i=nx$ and $j=1$ to ny; $k=1$ to nz, $T_{rl_{i+1/2,j,k}}$ =0 and $\frac{\partial T_{rl_{i+1/2,j,k}}}{\partial S_{o_{i,j,k}}} = 0$

b) Direction *y*

Similar to x direction, the transmissibility for flow from $j+1$ to j ($\delta_{ly}=1$) for any i,k :

$$T_{yl_{i,j-1/2,k}} = \frac{GF_{y\,i,j-1/2,k}\,b_{l_{i,j-1/2,k}}}{\mu_{l_{i,j-1/2,k}}}k_{rl_{i,j,k}}$$

$$T_{yl_{i,j+1/2,k}} = \frac{GF_{y\,i,j+1/2,k}\,b_{l_{i,j+1/2,k}}}{\mu_{l_{i,j+1/2,k}}}k_{rl_{i,j+1,k}}$$

(A2.7)

Thus,

$$\frac{\partial T_{yl_{i,j-1/2,k}}}{\partial S_{o_{i,j-1,k}}} = 0$$

$$\frac{\partial T_{yl_{i,j-1/2,k}}}{\partial S_{o_{i,j,k}}} = \frac{GF_{y_{i,j-1/2,k}} b_{l_{i,j-1/2,k}}}{\mu_{l_{i,j-1/2,k}}} \frac{dk_{rl_{i,j,k}}}{dS_{o_{i,j,k}}}$$

$$\frac{\partial T_{yl_{i,j-1/2,k}}}{\partial S_{o_{i,j+1,k}}} = 0$$

$$\frac{\partial T_{yl_{i,j+1/2,k}}}{\partial S_{o_{i,j-1,k}}} = 0$$

$$\frac{\partial T_{yl_{i,j+1/2,k}}}{\partial S_{o_{i,j,k}}} = 0$$

$$\frac{\partial T_{yl_{i,j+1/2,k}}}{\partial S_{o_{i,j+1,k}}} = \frac{GF_{y_{i,j+1/2,k}} b_{l_{i,j+1/2,k}}}{\mu_{l_{i,j+1/2,k}}} \frac{dk_{rl_{i,j+1,k}}}{dS_{o_{i,j+1,k}}}$$

(A2.8)

Flow from j to $j+1$: ($\delta_{ly} = 0$) for any i,k, there is

$$T_{yl_{i,j-1/2,k}} = \frac{GF_{y_{i,j-1/2,k}} b_{l_{i,j-1/2,k}}}{\mu_{l_{i,j-1/2,k}}} k_{rl_{i,j-1,k}}$$

$$T_{yl_{i,j+1/2,k}} = \frac{GF_{y_{i,j+1/2,k}} b_{l_{i,j+1/2,k}}}{\mu_{l_{i,j+1/2,k}}} k_{rl_{i,j,k}}$$

(A2.9)

Thus,

$$\frac{\partial T_{yl_{i,j-1/2,k}}}{\partial S_{o_{i,j-1,k}}} = \frac{GF_{y_{i,j-1/2,k}} b_{l_{i,j-1/2,k}}}{\mu_{l_{i,j-1/2,k}}} \frac{dk_{rl_{i,j-1,k}}}{dS_{o_{i,j-1,k}}}$$

$$\frac{\partial T_{yl_{i,j-1/2,k}}}{\partial S_{o_{i,j,k}}} = 0$$

$$\frac{\partial T_{yl_{i,j-1/2,k}}}{\partial S_{o_{i,j+1,k}}} = 0$$

$$\frac{\partial T_{yl_{i,j+1/2,k}}}{\partial S_{o_{i,j-1,k}}} = 0$$

$$\frac{\partial T_{yl_{i,j+1/2,k}}}{\partial S_{o_{i,j,k}}} = \frac{GF_{y_{i,j+1/2,k}} b_{l_{i,j+1/2,k}}}{\mu_{l_{i,j+1/2,k}}} \frac{dk_{rl_{i,j,k}}}{dS_{o_{i,j,k}}}$$

$$\frac{\partial T_{yl_{i,j+1/2,k}}}{\partial S_{o_{i,j+1,k}}} = 0$$

(A2.10)

General Form:

$$\delta_{ly} = 0 \;\; (j \to j+1), \quad \delta_{ly} = 1 \;\; (j+1 \to j)$$

$$T_{yl_{i,j-1/2,k}} = \frac{GF_{y_{i,j-1/2,k}} b_{l_{i,j-1/2,k}}}{\mu_{l_{i,j-1/2,k}}} [\delta_{ly} k_{rl_{i,j,k}} + (1-\delta_{ly}) k_{rl_{i,j-1,k}}]$$

$$T_{yl_{i,j+1/2,k}} = \frac{GF_{y_{i,j+1/2,k}} b_{l_{i,j+1/2,k}}}{\mu_{l_{i,j+1/2,k}}} [\delta_{ly} k_{rl_{i,j+1,k}} + (1-\delta_{ly}) k_{rl_{i,j,k}}]$$

(A2.11)

$$\frac{\partial T_{yl_{i,j-1/2,k}}}{\partial S_{o_{i,j-1,k}}} = (1-\delta_{ly}) \frac{GF_{y_{i,j-1/2,k}} b_{l_{i,j-1/2,k}}}{\mu_{l_{i,j-1/2,k}}} \frac{dk_{rl_{i,j-1,k}}}{dS_{o_{i,j-1,k}}}$$

$$\frac{\partial T_{yl_{i,j-1/2,k}}}{\partial S_{o_{i,j,k}}} = \delta_{ly} \frac{GF_{y_{i,j-1/2,k}} b_{l_{i,j-1/2,k}}}{\mu_{l_{i,j-1/2,k}}} \frac{dk_{rl_{i,j,k}}}{dS_{o_{i,j,k}}}$$

$$\frac{\partial T_{yl_{i,j-1/2,k}}}{\partial S_{o_{i,j+1,k}}} = 0$$

$$\frac{\partial T_{yl_{i,j+1/2,k}}}{\partial S_{o_{i,j-1,k}}} = 0$$

$$\frac{\partial T_{yl_{i,j+1/2,k}}}{\partial S_{o_{i,j,k}}} = (1-\delta_{ly})\frac{GF_{y_{i,j+1/2,k}} b_{l_{i,j+1/2,k}}}{\mu_{l_{i,j+1/2,k}}}\frac{dk_{rl_{i,j,k}}}{dS_{o_{i,j,k}}}$$

$$\frac{\partial T_{yl_{i,j+1/2,k}}}{\partial S_{o_{i,j+1,k}}} = \delta_{ly}\frac{GF_{y_{i,j+1/2,k}} b_{l_{i,j+1/2,k}}}{\mu_{l_{i,j+1/2,k}}}\frac{dk_{rl_{i,j+1,k}}}{dS_{o_{i,j+1,k}}}$$

(A2.12)

Note that,

When (j=1 and i=1 to nx; k=1 to nz), $T_{yl_{i,j-1/2,k}}$ =0 and $\frac{\partial T_{yl_{i,j-1/2,k}}}{\partial S_{o_{i-1,j,k}}} = 0$

When (j=ny and i=1 to nx; k=1 to nz), $T_{yl_{i,j+1/2,k}}$ =0 and $\frac{\partial T_{yl_{i,j+1/2,k}}}{\partial S_{o_{i,j,k}}} = 0$

c) Direction *z*

Similar for z direction, flow from k+1 to k: ($\delta_{l_z} = 1$) for any i, j, there is

$$T_{zl_{i,j,k-1/2}} = \frac{GF_{z_{i,j,k-1/2}} b_{l_{i,j,k-1/2}}}{\mu_{l_{i,j,k-1/2}}} k_{rl_{i,j,k}}$$

$$T_{zl_{i,j,k+1/2}} = \frac{GF_{z_{i,j,k+1/2}} b_{l_{i,j,k+1/2}}}{\mu_{l_{i,j,k+1/2}}} k_{rl_{i,j,k+1}}$$

(A2.13)

Thus,

$$\frac{\partial T_{zl_{i,j,k-1/2}}}{\partial S_{o_{i,j,k-1}}} = 0$$

$$\frac{\partial T_{zl_{i,j,k-1/2}}}{\partial S_{o_{i,j,k}}} = \frac{GF_{z_{i,j,k-1/2}} b_{l_{i,j,k-1/2}}}{\mu_{l_{i,j,k-1/2}}}\frac{dk_{rl_{i,j,k}}}{dS_{o_{i,j,k}}}$$

$$\frac{\partial T_{zl_{i,j,k-1/2}}}{\partial S_{o_{i,j,k+1}}} = 0$$

$$\frac{\partial T_{zl_{i,j,k+1/2}}}{\partial S_{o_{i,j,k-1}}} = 0$$

$$\frac{\partial T_{zl_{i,j,k+1/2}}}{\partial S_{o_{i,j,k}}} = 0$$

$$\frac{\partial T_{zl_{i,j,k+1/2}}}{\partial S_{o_{i,j,k+1}}} = \frac{GF_{z_{i,j,k+1/2}} b_{l_{i,j,k+1/2}}}{\mu_{l_{i,j,k+1/2}}} \frac{dk_{rl_{i,j,k+1}}}{dS_{o_{i,j,k+1}}}$$

(A2.14)

Flow from k to k+1: ($\delta_{l_z} = 0$) for any i,j, there is

$$T_{zl_{i,j,k-1/2}} = \frac{GF_{z_{i,j,k-1/2}} b_{l_{i,j,k-1/2}}}{\mu_{l_{i,j,k-1/2}}} k_{rl_{i,j,k-1}}$$

$$T_{zl_{i,j,k+1/2}} = \frac{GF_{z_{i,j,k+1/2}} b_{l_{i,j,k+1/2}}}{\mu_{l_{i,j,k+1/2}}} k_{rl_{i,j,k}}$$

(A2.15)

Thus,

$$\frac{\partial T_{zl_{i,j,k-1/2}}}{\partial S_{o_{i,j,k-1}}} = \frac{GF_{z_{i,j,k-1/2}} b_{l_{i,j,k-1/2}}}{\mu_{l_{i,j,k-1/2}}} \frac{dk_{rl_{i,j,k-1}}}{dS_{o_{i,j,k-1}}}$$

$$\frac{\partial T_{zl_{i,j,k-1/2}}}{\partial S_{o_{i,j,k}}} = 0$$

$$\frac{\partial T_{zl_{i,j,k-1/2}}}{\partial S_{o_{i,j,k+1}}} = 0$$

$$\frac{\partial T_{zl_{i,j,k+1/2}}}{\partial S_{o_{i,j,k-1}}} = 0$$

$$\frac{\partial T_{zl_{i,j,k+1/2}}}{\partial S_{o_{i,j,k}}} = \frac{GF_{z_{i,j,k+1/2}} b_{l_{i,j,k+1/2}}}{\mu_{l_{i,j,k+1/2}}} \frac{dk_{rl_{i,j,k}}}{dS_{o_{i,j,k}}}$$

$$\frac{\partial T_{zl_{i,j,k+1/2}}}{\partial S_{o_{i,j,k+1}}} = 0$$

(A2.16)

General Form:

$$\uparrow \ \delta_{l_z} = 0, \quad \delta_{l_z} = 1 \ \downarrow$$

$$T_{zl_{i,j,k-1/2}} = \frac{GF_{z\,i,j,k-1/2}\, b_{l_{i,j,k-1/2}}}{\mu_{l_{i,j,k-1/2}}} [\delta_{l_z} k_{rl_{i,j,k}} + (1-\delta_{l_z}) k_{rl_{i,j,k-1}}]$$

$$T_{zl_{i,j,k+1/2}} = \frac{GF_{z\,i,j,k+1/2}\, b_{l_{i,j,k+1/2}}}{\mu_{l_{i,j,k+1/2}}} [\delta_{l_z} k_{rl_{i,j,k+1}} + (1-\delta_{l_z}) k_{rl_{i,j,k}}]$$

(A2.17)

$$\frac{\partial T_{zl_{i,j,k-1/2}}}{\partial S_{o_{i,j,k-1}}} = (1-\delta_{l_z}) \frac{GF_{z\,i,j,k-1/2}\, b_{l_{i,j,k-1/2}}}{\mu_{l_{i,j,k-1/2}}} \frac{dk_{rl_{i,j,k-1}}}{dS_{o_{i,j,k-1}}}$$

$$\frac{\partial T_{zl_{i,j,k-1/2}}}{\partial S_{o_{i,j,k}}} = \delta_{l_z} \frac{GF_{z\,i,j,k-1/2}\, b_{l_{i,j,k-1/2}}}{\mu_{l_{i,j,k-1/2}}} \frac{dk_{rl_{i,j,k}}}{dS_{o_{i,j,k}}}$$

$$\frac{\partial T_{zl_{i,j,k-1/2}}}{\partial S_{o_{i,j,k+1}}} = 0$$

$$\frac{\partial T_{zl_{i,j,k+1/2}}}{\partial S_{o_{i,j,k-1}}} = 0$$

$$\frac{\partial T_{zl_{i,j,k+1/2}}}{\partial S_{o_{i,j,k}}} = (1-\delta_{l_z}) \frac{GF_{z\,i,j,k+1/2}\, b_{l_{i,j,k+1/2}}}{\mu_{l_{i,j,k+1/2}}} \frac{dk_{rl_{i,j,k}}}{dS_{o_{i,j,k}}}$$

$$\frac{\partial T_{zl_{i,j,k+1/2}}}{\partial S_{o_{i,j,k+1}}} = \delta_{l_z} \frac{GF_{z\,i,j,k+1/2}\, b_{l_{i,j,k+1/2}}}{\mu_{l_{i,j,k+1/2}}} \frac{dk_{rl_{i,j,k+1}}}{dS_{o_{i,j,k+1}}}$$

(A2.18)

Note that:

When (k=1 and i=1 to nx, j=1 to ny) $T_{zl_{i,j,k-1/2}}$ =0 and $\dfrac{\partial T_{zl_{i,j,k-1/2}}}{\partial S_{o_{i,j,k-1}}} = 0$

When (k=nz and i=1 to nx, j=1 to ny) $T_{zl_{i,j,k+1/2}}$ =0 and $\dfrac{\partial T_{zl_{i,j,k+1/2}}}{\partial S_{o_{i,j,k}}} = 0$

A2.1.2 Transmissibility Derivatives with Respect to Pressure

The following part is for the transmissibility derivatives with respect to all unknown pressure. The derivatives of the properties were presented in section 3.3.4 - Properties at the Interface, in Chapter 3)

a) Direction x:

$$\frac{\partial T_{xl_{i+1/2,j,k}}}{\partial P_{o_{i-1,j,k}}} = 0$$

$$\frac{\partial T_{xl_{i+1/2,j,k}}}{\partial P_{o_{i,j,k}}} = (w_x - 1) \cdot T_{xl_{i+1/2,j,k}} \left(\frac{1}{\mu_{l_{i+1/2,j,k}}} \frac{d\mu_{l_{i,j,k}}}{dP_{o_{i,j,k}}} - \frac{1}{b_{l_{i+1/2,j,k}}} \frac{db_{l_{i,j,k}}}{dP_{o_{i,j,k}}}\right)$$

$$\frac{\partial T_{xl_{i+1/2,j,k}}}{\partial P_{o_{i+1,j,k}}} = -w_x \cdot T_{xl_{i+1/2,j,k}} \left(\frac{1}{\mu_{l_{i+1/2,j,k}}} \frac{d\mu_{l_{i+1,j,k}}}{dP_{o_{i+1,j,k}}} - \frac{1}{b_{l_{i+1/2,j,k}}} \frac{db_{l_{i+1,j,k}}}{dP_{o_{i+1,j,k}}}\right)$$

$$\frac{\partial T_{xl_{i-1/2,j,k}}}{\partial P_{o_{i-1,j,k}}} = (w_x - 1) \cdot T_{xl_{i-1/2,j,k}} \left(\frac{1}{\mu_{l_{i-1/2,j,k}}} \frac{d\mu_{l_{i-1,j,k}}}{dP_{o_{i-1,j,k}}} - \frac{1}{b_{l_{i-1/2,j,k}}} \frac{db_{l_{i-1,j,k}}}{dP_{o_{i-1,j,k}}}\right)$$

$$\frac{\partial T_{xl_{i-1/2,j,k}}}{\partial P_{o_{i,j,k}}} = -w_x \cdot T_{xl_{i-1/2,j,k}} \left(\frac{1}{\mu_{l_{i-1/2,j,k}}} \frac{d\mu_{l_{i,j,k}}}{dP_{o_{i,j,k}}} - \frac{1}{b_{l_{i-1/2,j,k}}} \frac{db_{l_{i,j,k}}}{dP_{o_{i,j,k}}}\right)$$

$$\frac{\partial T_{xl_{i-1/2,j,k}}}{\partial P_{o_{i+1,j,k}}} = 0$$

(A2.19)

b) Direction *y*:

$$\frac{\partial T_{yl_{i,j+1/2,k}}}{\partial P_{o_{i,j-1,k}}} = 0$$

$$\frac{\partial T_{yl_{i,j+1/2,k}}}{\partial P_{o_{i,j,k}}} = (w_y - 1)\cdot T_{yl_{i,j+1/2,k}} \left(\frac{1}{\mu_{l_{i,j+1/2,k}}}\frac{d\mu_{l_{i,j,k}}}{dP_{o_{i,j,k}}} - \frac{1}{b_{l_{i,j+1/2,k}}}\frac{db_{l_{i,j,k}}}{dP_{o_{i,j,k}}}\right)$$

$$\frac{\partial T_{yl_{i,j+1/2,k}}}{\partial P_{o_{i,j+1,k}}} = -w_y \cdot T_{yl_{i,j+1/2,k}} \left(\frac{1}{\mu_{l_{i,j+1/2,k}}}\frac{d\mu_{l_{i,j+1,k}}}{dP_{o_{i,j+1,k}}} - \frac{1}{b_{l_{i,j+1/2,k}}}\frac{db_{l_{i,j+1,k}}}{dP_{o_{i,j+1,k}}}\right)$$

$$\frac{\partial T_{yl_{i,j-1/2,k}}}{\partial P_{o_{i,j-1,k}}} = (w_y - 1)\cdot T_{yl_{i,j-1/2,k}} \left(\frac{1}{\mu_{l_{i,j-1/2,k}}}\frac{d\mu_{l_{i,j-1,k}}}{dP_{o_{i,j-1,k}}} - \frac{1}{b_{l_{i,j-1/2,k}}}\frac{db_{l_{i,j-1,k}}}{dP_{o_{i,j-1,k}}}\right)$$

$$\frac{\partial T_{yl_{i,j-1/2,k}}}{\partial P_{o_{i,j,k}}} = -w_y \cdot T_{yl_{i,j-1/2,k}} \left(\frac{1}{\mu_{l_{i,j-1/2,k}}}\frac{d\mu_{l_{i,j,k}}}{dP_{o_{i,j,k}}} - \frac{1}{b_{l_{i,j-1/2,k}}}\frac{db_{l_{i,j,k}}}{dP_{o_{i,j,k}}}\right)$$

$$\frac{\partial T_{yl_{i,j-1/2,k}}}{\partial P_{o_{i,j+1,k}}} = 0$$

(A2.20)

c) Direction z:

$$\frac{\partial T_{zl_{i,j,k+1/2}}}{\partial P_{o_{i,j,k-1}}} = 0$$

$$\frac{\partial T_{zl_{i,j,k+1/2}}}{\partial P_{o_{i,j,k}}} = (w_z - 1)\cdot T_{zl_{i,j,k+1/2}} \left(\frac{1}{\mu_{l_{i,j,k+1/2}}}\frac{d\mu_{l_{i,j,k}}}{dP_{o_{i,j,k}}} - \frac{1}{b_{l_{i,j,k+1/2}}}\frac{db_{l_{i,j,k}}}{dP_{o_{i,j,k}}}\right)$$

$$\frac{\partial T_{zl_{i,j,k+1/2}}}{\partial P_{o_{i,j,k+1}}} = -w_z \cdot T_{zl_{i,j,k+1/2}} \left(\frac{1}{\mu_{l_{i,j,k+1/2}}}\frac{d\mu_{l_{i,j,k+1}}}{dP_{o_{i,j,k+1}}} - \frac{1}{b_{l_{i,j,k+1/2}}}\frac{db_{l_{i,j,k+1}}}{dP_{o_{i,j,k+1}}}\right)$$

$$\frac{\partial T_{zl_{i,j,k-1/2}}}{\partial P_{o_{i,j,k-1}}} = (w_z - 1)\cdot T_{zl_{i,j,k-1/2}} \left(\frac{1}{\mu_{l_{i,j,k-1/2}}}\frac{d\mu_{l_{i,j,k-1}}}{dP_{o_{i,j,k-1}}} - \frac{1}{b_{l_{i,j,k-1/2}}}\frac{db_{l_{i,j,k-1}}}{dP_{o_{i,j,k-1}}}\right)$$

$$\frac{\partial T_{zl_{i,j,k-1/2}}}{\partial P_{o_{i,j,k}}} = -w_z \cdot T_{zl_{i,j,k-1/2}} \left(\frac{1}{\mu_{l_{i,j,k-1/2}}}\frac{d\mu_{l_{i,j,k}}}{dP_{o_{i,j,k}}} - \frac{1}{b_{l_{i,j,k-1/2}}}\frac{db_{l_{i,j,k}}}{dP_{o_{i,j,k}}}\right)$$

$$\frac{\partial T_{zl_{i,j,k-1/2}}}{\partial P_{o_{i,j,k+1}}} = 0 \qquad \text{(A2.21)}$$

A2.2 Derivation of the Finite Difference Equations for Reservoir Model

A2.2.1 Discretization of the Reservoir Flow Equations

Block centered scheme for the spatial discretization and fully implicit method for the time discretization are used for the partial differential equations of the reservoir flow model.

For the gas phase:

$$
\begin{aligned}
&\frac{\partial}{\partial x}[\frac{A_x k_x k_{rg}}{B_g \mu_g}(\frac{\partial P_g}{\partial x}-\gamma_g \frac{\partial h_x}{\partial x})+\frac{A_x R_s k_x k_{ro}}{B_o \mu_o}(\frac{\partial P_o}{\partial x}-\gamma_o \frac{\partial h_x}{\partial x})]\Delta x \\
&+\frac{\partial}{\partial y}[\frac{A_y k_y k_{rg}}{B_g \mu_g}(\frac{\partial P_g}{\partial y}-\gamma_g \frac{\partial h_y}{\partial y})+\frac{A_y R_s k_y k_{ro}}{B_o \mu_o}(\frac{\partial P_o}{\partial y}-\gamma_o \frac{\partial h_y}{\partial y})]\Delta y \\
&+\frac{\partial}{\partial z}[\frac{A_z k_z k_{rg}}{B_g \mu_g}(\frac{\partial P_g}{\partial z}-\gamma_g \frac{\partial h_z}{\partial z})+\frac{A_z R_s k_z k_{ro}}{B_o \mu_o}(\frac{\partial P_o}{\partial z}-\gamma_o \frac{\partial h_z}{\partial z})]\Delta z \\
&=V\frac{\partial}{\partial t}(\phi\frac{S_g}{B_g}+\phi R_s\frac{S_o}{B_o})+R_s q_o V+q_{fg}V
\end{aligned}
\tag{A2.22}
$$

For the oil phase:

$$
\begin{aligned}
&\frac{\partial}{\partial x}[\frac{A_x k_x k_{ro}}{B_o \mu_o}(\frac{\partial P_o}{\partial x}-\gamma_o \frac{\partial h_x}{\partial x})]\Delta x+\frac{\partial}{\partial y}[\frac{A_y k_y k_{ro}}{B_o \mu_o}(\frac{\partial P_o}{\partial y}-\gamma_o \frac{\partial h_y}{\partial y})]\Delta y \\
&+\frac{\partial}{\partial z}[\frac{A_z k_z k_{ro}}{B_o \mu_o}(\frac{\partial P_o}{\partial z}-\gamma_o \frac{\partial h_z}{\partial z})]\Delta z=V\frac{\partial}{\partial t}(\phi\frac{S_o}{B_o})+q_o V
\end{aligned}
\tag{A2.23}
$$

Substituting $A_x=\Delta y\cdot\Delta z$, $A_y=\Delta x\cdot\Delta z$ and $A_z=\Delta y\cdot\Delta x$ into the above equations, we have:

For the gas phase:

$$
\begin{aligned}
&\frac{\partial}{\partial x}[\frac{\Delta y\Delta z\cdot k_x k_{rg}}{B_g \mu_g}(\frac{\partial P_g}{\partial x}-\gamma_g \frac{\partial h_x}{\partial x})+\frac{\Delta y\Delta z\cdot R_s k_x k_{ro}}{B_o \mu_o}(\frac{\partial P_o}{\partial x}-\gamma_o \frac{\partial h_x}{\partial x})]\Delta x \\
&+\frac{\partial}{\partial y}[\frac{\Delta x\Delta z\cdot k_y k_{rg}}{B_g \mu_g}(\frac{\partial P_g}{\partial y}-\gamma_g \frac{\partial h_y}{\partial y})+\frac{\Delta x\Delta z\cdot R_s k_y k_{ro}}{B_o \mu_o}(\frac{\partial P_o}{\partial y}-\gamma_o \frac{\partial h_y}{\partial y})]\Delta y \\
&+\frac{\partial}{\partial z}[\frac{\Delta y\Delta x\cdot k_z k_{rg}}{B_g \mu_g}(\frac{\partial P_g}{\partial z}-\gamma_g \frac{\partial h_z}{\partial z})+\frac{\Delta y\Delta x\cdot R_s k_z k_{ro}}{B_o \mu_o}(\frac{\partial P_o}{\partial z}-\gamma_o \frac{\partial h_z}{\partial z})]\Delta z \\
&=V\frac{\partial}{\partial t}(\phi\frac{S_g}{B_g}+\phi R_s\frac{S_o}{B_o})+R_s q_o V+q_{fg}V
\end{aligned}
\tag{A2.24}
$$

For the oil phase:

$$\frac{\partial}{\partial x}[\frac{\Delta y\Delta z\cdot k_x k_{ro}}{B_o\mu_o}(\frac{\partial P_o}{\partial x}-\gamma_o\frac{\partial h_x}{\partial x})]\Delta x+\frac{\partial}{\partial y}[\frac{\Delta x\Delta z\cdot k_y k_{ro}}{B_o\mu_o}(\frac{\partial P_o}{\partial y}-\gamma_o\frac{\partial h_y}{\partial y})]\Delta y$$
$$+\frac{\partial}{\partial z}[\frac{\Delta y\Delta x\cdot k_z k_{ro}}{B_o\mu_o}(\frac{\partial P_o}{\partial z}-\gamma_o\frac{\partial h_z}{\partial z})]\Delta z=V\frac{\partial}{\partial t}(\phi\frac{S_o}{B_o})+q_oV$$

(A2.25)

Discretization for the gas phase:

$$\frac{1}{\Delta x}[(\frac{\Delta y\Delta z\cdot k_x k_{rg}}{\Delta x\cdot B_g\mu_g})_{i+1/2,j,k}(\Delta P_g-\gamma_{g_{i+1/2}}\Delta h_x)_{j,k}-(\frac{\Delta y\Delta z\cdot k_x k_{rg}}{\Delta x\cdot B_g\mu_g})_{i-1/2,j,k}(\Delta P_g-\gamma_{g_{i-1/2}}\Delta h_x)_{j,k}$$
$$+(\frac{\Delta y\Delta z\cdot k_x k_{ro}\cdot R_s}{\Delta x\cdot B_o\mu_o})_{i+1/2,j,k}(\Delta P_o-\gamma_{o_{i+1/2}}\Delta h_x)_{j,k}-(\frac{\Delta y\Delta z\cdot k_x k_{ro}\cdot R_s}{\Delta x\cdot B_o\mu_o})_{i-1/2,j,k}(\Delta P_o-\gamma_{o_{i-1/2}}\Delta h_x)_{j,k}]\Delta x$$
$$+\frac{1}{\Delta y}[(\frac{\Delta x\Delta z\cdot k_y k_{rg}}{\Delta y\cdot B_g\mu_g})_{i,j+1/2,k}(\Delta P_g-\gamma_{g_{j+1/2}}\Delta h_y)_{i,k}-(\frac{\Delta x\Delta z\cdot k_y k_{rg}}{\Delta y\cdot B_g\mu_g})_{i,j-1/2,k}(\Delta P_g-\gamma_{g_{j-1/2}}\Delta h_y)_{i,k}$$
$$+(\frac{\Delta x\Delta z\cdot k_y k_{ro}\cdot R_s}{\Delta y\cdot B_o\mu_o})_{i,j+1/2,k}(\Delta P_o-\gamma_{o_{j+1/2}}\Delta h_y)_{i,k}-(\frac{\Delta x\Delta z\cdot k_y k_{ro}\cdot R_s}{\Delta y\cdot B_o\mu_o})_{i,j-1/2,k}(\Delta P_o-\gamma_{o_{j-1/2}}\Delta h_y)_{i,k}]\Delta y$$
$$+\frac{1}{\Delta z}[(\frac{\Delta y\Delta x\cdot k_z k_{rg}}{\Delta z\cdot B_g\mu_g})_{i,j,k+1/2}(\Delta P_g-\gamma_{g_{k+1/2}}\Delta z)_{i,j}-(\frac{\Delta y\Delta x\cdot k_z k_{rg}}{\Delta z\cdot B_g\mu_g})_{i,j,k-1/2}(\Delta P_g-\gamma_{g_{k-1/2}}\Delta z)_{i,j}$$
$$+(\frac{\Delta y\Delta x\cdot k_z k_{ro}\cdot R_s}{\Delta z\cdot B_o\mu_o})_{i,j,k+1/2}(\Delta P_o-\gamma_{o_{k+1/2}}\Delta z)_{i,j}-(\frac{\Delta y\Delta x\cdot k_z k_{ro}\cdot R_s}{\Delta z\cdot B_o\mu_o})_{i,j,k-1/2}(\Delta P_o-\gamma_{o_{k-1/2}}\Delta z)_{i,j}]\Delta z$$
$$=V\frac{1}{\Delta t}\cdot\Delta_t(\phi\frac{S_g}{B_g}+\phi R_s\frac{S_o}{B_o})_{i,j,k}+(R_s q_o V+q_{fg}V)_{i,j,k}$$

(A2.26)

For the oil phase:

$$\frac{1}{\Delta x}[(\frac{\Delta y\Delta z\cdot k_x k_{ro}}{\Delta x\cdot B_o\mu_o})_{i+1/2,j,k}(\Delta P_o-\gamma_{o_{i+1/2}}\Delta h_x)_{j,k}-(\frac{\Delta y\Delta z\cdot k_x k_{ro}}{\Delta x\cdot B_o\mu_o})_{i-1/2,j,k}(\Delta P_o-\gamma_{o_{i-1/2}}\Delta h_x)_{j,k}]\Delta x$$
$$+\frac{1}{\Delta y}[(\frac{\Delta x\Delta z\cdot k_y k_{ro}}{\Delta y\cdot B_o\mu_o})_{i,j+1/2,k}(\Delta P_o-\gamma_{o_{j+1/2}}\Delta h_y)_{i,k}-(\frac{\Delta x\Delta z\cdot k_y k_{ro}}{\Delta y\cdot B_o\mu_o})_{i,j-1/2,k}(\Delta P_o-\gamma_{o_{j-1/2}}\Delta h_y)_{i,k}]\Delta y$$
$$+\frac{1}{\Delta z}[(\frac{\Delta y\Delta x\cdot k_z k_{ro}}{\Delta z\cdot B_o\mu_o})_{i,j,k+1/2}(\Delta P_o-\gamma_{o_{k+1/2}}\Delta z)_{i,j}-(\frac{\Delta y\Delta x\cdot k_z k_{ro}}{\Delta z\cdot B_o\mu_o})_{i,j,k-1/2}(\Delta P_o-\gamma_{o_{k-1/2}}\Delta z)_{i,j}]\Delta z$$
$$=V\frac{1}{\Delta t}\cdot\Delta_t(\phi\frac{S_o}{B_o})_{i,j,k}+(q_o V)_{i,j,k}$$

(A2.27)

Thus for the gas phase:

$$
\begin{aligned}
&[(\frac{\Delta y\Delta z\cdot k_x k_{rg}}{\Delta x\cdot B_g\mu_g})_{i+1/2,j,k}(\Delta P_g-\gamma_{g_{i+1/2}}\Delta h_x)_{j,k}-(\frac{\Delta y\Delta z\cdot k_x k_{rg}}{\Delta x\cdot B_g\mu_g})_{i-1/2,j,k}(\Delta P_g-\gamma_{g_{i-1/2}}\Delta h_x)_{j,k}\\
&+(\frac{\Delta y\Delta z\cdot k_x k_{ro}\cdot R_s}{\Delta x\cdot B_o\mu_o})_{i+1/2,j,k}(\Delta P_o-\gamma_{o_{i+1/2}}\Delta h_x)_{j,k}-(\frac{\Delta y\Delta z\cdot k_x k_{ro}\cdot R_s}{\Delta x\cdot B_o\mu_o})_{i-1/2,j,k}(\Delta P_o-\gamma_{o_{i-1/2}}\Delta h_x)_{j,k}]\\
&+[(\frac{\Delta x\Delta z\cdot k_y k_{rg}}{\Delta y\cdot B_g\mu_g})_{i,j+1/2,k}(\Delta P_g-\gamma_{g_{j+1/2}}\Delta h_y)_{i,k}-(\frac{\Delta x\Delta z\cdot k_y k_{rg}}{\Delta y\cdot B_g\mu_g})_{i,j-1/2,k}(\Delta P_g-\gamma_{g_{j-1/2}}\Delta h_y)_{i,k}\\
&+(\frac{\Delta x\Delta z\cdot k_y k_{ro}\cdot R_s}{\Delta y\cdot B_o\mu_o})_{i,j+1/2,k}(\Delta P_o-\gamma_{o_{j+1/2}}\Delta h_y)_{i,k}-(\frac{\Delta x\Delta z\cdot k_y k_{ro}\cdot R_s}{\Delta y\cdot B_o\mu_o})_{i,j-1/2,k}(\Delta P_o-\gamma_{o_{j-1/2}}\Delta h_y)_{i,k}]\\
&+[(\frac{\Delta y\Delta x\cdot k_z k_{rg}}{\Delta z\cdot B_g\mu_g})_{i,j,k+1/2}(\Delta P_g-\gamma_{g_{k+1/2}}\Delta z)_{i,j}-(\frac{\Delta y\Delta x\cdot k_z k_{rg}}{\Delta z\cdot B_g\mu_g})_{i,j,k-1/2}(\Delta P_g-\gamma_{g_{k-1/2}}\Delta z)_{i,j}\\
&+(\frac{\Delta y\Delta x\cdot k_z k_{ro}\cdot R_s}{\Delta z\cdot B_o\mu_o})_{i,j,k+1/2}(\Delta P_o-\gamma_{o_{k+1/2}}\Delta z)_{i,j}-(\frac{\Delta y\Delta x\cdot k_z k_{ro}\cdot R_s}{\Delta z\cdot B_o\mu_o})_{i,j,k-1/2}(\Delta P_o-\gamma_{o_{k-1/2}}\Delta z)_{i,j}]\\
&=V\frac{1}{\Delta t}\cdot\Delta_t(\phi\frac{S_g}{B_g}+\phi R_s\frac{S_o}{B_o})_{i,j,k}+(R_s q_o V+q_{fg}V)_{i,j,k}
\end{aligned}
$$

(A2.28)

For the oil phase:

$$
\begin{aligned}
&[(\frac{\Delta y\Delta z\cdot k_x k_{ro}}{\Delta x\cdot B_o\mu_o})_{i+1/2,j,k}(\Delta P_o-\gamma_{o_{i+1/2}}\Delta h_x)_{j,k}-(\frac{\Delta y\Delta z\cdot k_x k_{ro}}{\Delta x\cdot B_o\mu_o})_{i-1/2,j,k}(\Delta P_o-\gamma_{o_{i-1/2}}\Delta h_x)_{j,k}]\\
&+[(\frac{\Delta x\Delta z\cdot k_y k_{ro}}{\Delta y\cdot B_o\mu_o})_{i,j+1/2,k}(\Delta P_o-\gamma_{o_{j+1/2}}\Delta h_y)_{i,k}-(\frac{\Delta x\Delta z\cdot k_y k_{ro}}{\Delta y\cdot B_o\mu_o})_{i,j-1/2,k}(\Delta P_o-\gamma_{o_{j-1/2}}\Delta h_y)_{i,k}]\\
&+[(\frac{\Delta y\Delta x\cdot k_z k_{ro}}{\Delta z\cdot B_o\mu_o})_{i,j,k+1/2}(\Delta P_o-\gamma_{o_{k+1/2}}\Delta z)_{i,j}-(\frac{\Delta y\Delta x\cdot k_z k_{ro}}{\Delta z\cdot B_o\mu_o})_{i,j,k-1/2}(\Delta P_o-\gamma_{o_{k-1/2}}\Delta z)_{i,j}]\\
&=V\frac{1}{\Delta t}\cdot\Delta_t(\phi\frac{S_o}{B_o})_{i,j,k}+(q_o V)_{i,j,k}
\end{aligned}
$$

(A2.29)

$$\frac{1}{\Delta t}\cdot\Delta_t(x)=\frac{x^{n+1}-x^n}{\Delta t};$$

$$(\Delta x)_{j,k}=\begin{cases}x_{i+1,j,k}-x_{i,j,k} & for\ i+1/2,j,k\\ x_{i,j,k}-x_{i-1,j,k} & for\ i-1/2,j,k\end{cases}$$

$$(\Delta y)_{i,k} = \begin{cases} y_{i,j+1,k} - y_{i,j,k} & for\ i, j+1/2, k \\ y_{i,j,k} - y_{i,j-1,k} & for\ i, j-1/2, k \end{cases}$$

$$(\Delta z)_{i,j} = \begin{cases} z_{i,j,k+1} - z_{i,j,k} & for\ i, j, k+1/2 \\ z_{i,j,k} - z_{i,j,k-1} & for\ i, j, k-1/2 \end{cases}$$

$$(\Delta P_o)_{j,k} = \begin{cases} P_{o_{i+1,j,k}} - P_{o_{i,j,k}} & for\ i+1/2, j, k \\ P_{o_{i,j,k}} - P_{o_{i-1,j,k}} & for\ i-1/2, j, k \end{cases};$$

$$(\Delta P_g)_{j,k} = \begin{cases} P_{g_{i+1,j,k}} - P_{g_{i,j,k}} & for\ i+1/2, j, k \\ P_{g_{i,j,k}} - P_{g_{i-1,j,k}} & for\ i-1/2, j, k \end{cases}$$

$$(\Delta P_o)_{i,k} = \begin{cases} P_{o_{i,j+1,k}} - P_{o_{i,j,k}} & for\ i, j+1/2, k \\ P_{o_{i,j,k}} - P_{o_{i,j-1,k}} & for\ i, j-1/2, k \end{cases};$$

$$(\Delta P_g)_{i,k} = \begin{cases} P_{g_{i,j+1,k}} - P_{g_{i,j,k}} & for\ i, j+1/2, k \\ P_{g_{i,j,k}} - P_{g_{i,j-1,k}} & for\ i, j-1/2, k \end{cases}$$

$$(\Delta P_o)_{i,j} = \begin{cases} P_{o_{i,j,k+1}} - P_{o_{i,j,k}} & for\ i, j, k+1/2 \\ P_{o_{i,j,k}} - P_{o_{i,j,k-1}} & for\ i, j, k-1/2 \end{cases};$$

$$(\Delta P_g)_{i,j} = \begin{cases} P_{g_{i,j,k+1}} - P_{g_{i,j,k}} & for\ i, j, k+1/2 \\ P_{g_{i,j,k}} - P_{g_{i,j,k-1}} & for\ i, j, k-1/2 \end{cases}$$

(A2.30)

$$T_{xo_{i\pm1/2,j,k}} = \frac{A_{x_{i\pm1/2,j,k}} \lambda_{o_{i\pm1/2,j,k}}}{\Delta x_{i\pm1/2,j,k}},\ where\ \frac{A_{x_{i\pm1/2,j,k}} \cdot k_x}{\Delta x_{i\pm1/2,j,k}} = GF_{x_{i\pm1/2,j,k}},\ A_{x_{i\pm1/2,j,k}} = \Delta y_{i,j,k} \cdot \Delta z_{i,j,k}$$

$$T_{yo_{i,j\pm1/2,k}} = \frac{A_{y_{i,j\pm1/2,k}} \lambda_{o_{i,j\pm1/2,k}}}{\Delta y_{i,j\pm1/2,k}}, \quad where \quad \frac{A_{y_{i,j\pm1/2,k}} \cdot k_y}{\Delta y_{i,j\pm1/2,k}} = GF_{y_{i,j\pm1/2,k}}, \quad A_{y_{i,j\pm1/2,k}} = \Delta x_{i,j,k} \cdot \Delta z_{i,j,k}$$

$$T_{zo_{i,j,k\pm1/2}} = \frac{A_{z_{i,j,k\pm1/2}} \lambda_{o_{i,j,k\pm1/2}}}{\Delta z_{i,j,k\pm1/2}}, \quad where \quad \frac{A_{z_{i,j,k\pm1/2}} \cdot k_z}{\Delta z_{i,j,k\pm1/2}} = GF_{z_{i,j,k\pm1/2}}, \quad A_{z_{i,j\pm1/2,k}} = \Delta y_{i,j,k} \Delta x_{i,j,k}$$

$$T_{xg_{i\pm1/2,j,k}} = \frac{A_{x_{i\pm1/2,j,k}} \lambda_{g_{i\pm1/2,j,k}}}{\Delta x_{i\pm1/2,j,k}}, \quad where \quad \frac{A_{x_{i\pm1/2,j,k}} \cdot k_x}{\Delta x_{i\pm1/2,j,k}} = GF_{x_{i\pm1/2,j,k}}, A_{x_{i\pm1/2,j,k}} = \Delta y_{i,j,k} \cdot \Delta z_{i,j,k}$$

$$T_{yg_{i,j\pm1/2,k}} = \frac{A_{y_{i,j\pm1/2,k}} \lambda_{g_{i,j\pm1/2,k}}}{\Delta y_{i,j\pm1/2,k}}, \quad where \quad \frac{A_{y_{i,j\pm1/2,k}} \cdot k_y}{\Delta y_{i,j\pm1/2,k}} = GF_{y_{i,j\pm1/2,k}}, \quad A_{y_{i,j\pm1/2,k}} = \Delta x_{i,j,k} \cdot \Delta z_{i,j,k}$$

$$T_{zg_{i,j,k\pm1/2}} = \frac{A_{z_{i,j,k\pm1/2}} \lambda_{g_{i,j,k\pm1/2}}}{\Delta z_{i,j,k\pm1/2}}, \quad where \quad \frac{A_{z_{i,j,k\pm1/2}} \cdot k_z}{\Delta z_{i,j,k\pm1/2}} = GF_{z_{i,j,k\pm1/2}}, \quad A_{z_{i,j\pm1/2,k}} = \Delta y_{i,j,k} \Delta x_{i,j,k}$$

(A2.31)

$$V_{i,j,k} = (\Delta x \cdot \Delta\theta \cdot \Delta z)_{i,j,k} \qquad Q_{o_{i,j,k}} = V_{i,j,k} q_{o_{i,j,k}} \qquad Q_{fg_{i,j,k}} = V_{i,j,k} q_{fg_{i,j,k}}$$

(A2.32)

The oil and gas reservoir flow equations can be written in the following forms:

For the gas phase:

$$\begin{aligned}
&[T_{xg_{i+1/2,j,k}}(\Delta P_g - \gamma_{g_{i+1/2}} \Delta h_x)_{j,k} - T_{xg_{i-1/2,j,k}}(\Delta P_g - \gamma_{g_{i-1/2}} \Delta h_x)_{j,k} \\
&+ T_{xo_{i+1/2,j,k}} \cdot R_s(\Delta P_o - \gamma_{o_{i+1/2}} \Delta h_x)_{j,k} - T_{xo_{i-1/2,j,k}} \cdot R_s(\Delta P_o - \gamma_{o_{i-1/2}} \Delta h_x)_{j,k}] \\
&+ [T_{yg_{i,j+1/2,k}}(\Delta P_g - \gamma_{g_{j+1/2}} \Delta h_y)_{i,k} - T_{yg_{i,j-1/2,k}}(\Delta P_g - \gamma_{g_{j-1/2}} \Delta h_y)_{i,k} \\
&+ T_{yo_{i,j+1/2,k}} \cdot R_s(\Delta P_o - \gamma_{o_{j+1/2}} \Delta h_y)_{i,k} - T_{yo_{i,j-1/2,k}} \cdot R_s(\Delta P_o - \gamma_{o_{j-1/2}} \Delta h_y)_{i,k}] \\
&+ [T_{zg_{i,j,k+1/2}}(\Delta P_g - \gamma_{g_{k+1/2}} \Delta z)_{i,j} - T_{zg_{i,j,k-1/2}}(\Delta P_g - \gamma_{g_{k-1/2}} \Delta z)_{i,j} \\
&+ T_{zo_{i,j,k+1/2}} \cdot R_s(\Delta P_o - \gamma_{o_{k+1/2}} \Delta z)_{i,j} - T_{zo_{i,j,k-1/2}} \cdot R_s(\Delta P_o - \gamma_{o_{k-1/2}} \Delta z)_{i,j}] \\
&= V \frac{1}{\Delta t} \cdot \Delta_t (\phi \frac{S_g}{B_g} + \phi R_s \frac{S_o}{B_o})_{i,j,k} + (R_s q_o V + q_{fg} V)_{i,j,k}
\end{aligned}$$

(A2.33)

Where, $P_{g_{i,j,k}} = P_{o_{i,j,k}} + P_{cgo_{i,j,k}}$

For the oil phase:

$$\begin{aligned}
&[T_{xo_{i+1/2,j,k}} \cdot (\Delta P_o - \gamma_{o_{i+1/2}} \Delta h_x)_{j,k} - T_{xo_{i-1/2,j,k}} (\Delta P_o - \gamma_{o_{i-1/2}} \Delta h_x)_{j,k}] \\
&+[T_{yo_{i,j+1/2,k}} (\Delta P_o - \gamma_{o_{j+1/2}} \Delta h_y)_{i,k} - T_{yo_{i,j-1/2,k}} (\Delta P_o - \gamma_{o_{j-1/2}} \Delta h_y)_{i,k}] \\
&+[T_{zo_{i,j,k+1/2}} (\Delta P_o - \gamma_{o_{k+1/2}} \Delta z)_{i,j} - T_{zo_{i,j,k-1/2}} (\Delta P_o - \gamma_{o_{k-1/2}} \Delta z)_{i,j}] \\
&= V \frac{1}{\Delta t} \cdot \Delta_t (\phi \frac{S_o}{B_o})_{i,j,k} + (q_o V)_{i,j,k}
\end{aligned} \qquad \text{(A2.34)}$$

A2.2.2 Development of the Flow Terms of the Oil and Gas Flow Equations

Developing the left hand side of above equations (where $P_{g_{i,j,k}} = P_{o_{i,j,k}} + P_{cgo_{i,j,k}}$), there is:

For the gas phase:

$$\begin{aligned}
LHS &= T_{xo_{i+1/2,j,k}} \cdot R_s[(P^{n+1}_{o_{i+1,j,k}} - P^{n+1}_{o_{i,j,k}}) - \gamma^{n+1}_{o_{i+1/2,j,k}} (h_{x\,i+1} - h_{x\,i})_{j,k}] \\
&- T_{xo_{i-1/2,j,k}} \cdot R_s[(P^{n+1}_{o_{i,j,k}} - P^{n+1}_{o_{i-1,j,k}}) - \gamma^{n+1}_{o_{i-1/2,j,k}} (h_{x\,i} - h_{x\,i-1})_{j,k}] \\
&+ T_{yo_{i,j+1/2,k}} \cdot R_s[(P^{n+1}_{o_{i,j+1,k}} - P^{n+1}_{o_{i,j,k}}) - \gamma^{n+1}_{o_{i,j+1/2,k}} (h_{y\,j+1} - h_{y\,j})_{i,k}] \\
&- T_{yo_{i,j-1/2,k}} \cdot R_s[(P^{n+1}_{o_{i,j,k}} - P^{n+1}_{o_{i,j-1,k}}) - \gamma^{n+1}_{o_{i,j-1/2,k}} (h_{y\,j} - h_{y\,j-1})_{i,k}] \\
&+ T_{zo_{i,j,k+1/2}} \cdot R_s[(P^{n+1}_{o_{i,j,k+1}} - P^{n+1}_{o_{i,j,k}}) - \gamma^{n+1}_{o_{i,j,k+1/2}} (z_{k+1} - z_k)_{i,j}] \\
&- T_{zo_{i,j,k-1/2}} \cdot R_s[(P^{n+1}_{o_{i,j,k}} - P^{n+1}_{o_{i,j,k-1}}) - \gamma^{n+1}_{o_{i,j,k-1/2}} (z_k - z_{k-1})_{i,j}] \\
&+ T_{xg_{i+1/2,j,k}} \cdot [(P^{n+1}_{o_{i+1,j,k}} - P^{n+1}_{o_{i,j,k}}) - \gamma^{n+1}_{g_{i+1/2,j,k}} (h_{x\,i+1} - h_{x\,i})_{j,k}] \\
&- T_{xg_{i-1/2,j,k}} \cdot [(P^{n+1}_{o_{i,j,k}} - P^{n+1}_{o_{i-1,j,k}}) - \gamma^{n+1}_{g_{i-1/2,j,k}} (h_{x\,i} - h_{x\,i-1})_{j,k}] \\
&+ T_{yg_{i,j+1/2,k}} \cdot [(P^{n+1}_{o_{i,j+1,k}} - P^{n+1}_{o_{i,j,k}}) - \gamma^{n+1}_{g_{i,j+1/2,k}} (h_{y\,j+1} - h_{y\,j})_{i,k}] \\
&- T_{yg_{i,j-1/2,k}} \cdot [(P^{n+1}_{o_{i,j,k}} - P^{n+1}_{o_{i,j-1,k}}) - \gamma^{n+1}_{g_{i,j-1/2,k}} (h_{y\,j} - h_{y\,j-1})_{i,k}] \\
&+ T_{zg_{i,j,k+1/2}} \cdot [(P^{n+1}_{o_{i,j,k+1}} - P^{n+1}_{o_{i,j,k}}) - \gamma^{n+1}_{g_{i,j,k+1/2}} (z_{k+1} - z_k)_{i,j}] \\
&- T_{zg_{i,j,k-1/2}} \cdot [(P^{n+1}_{o_{i,j,k}} - P^{n+1}_{o_{i,j,k-1}}) - \gamma^{n+1}_{g_{i,j,k-1/2}} (z_k - z_{k-1})_{i,j}] \\
&+ T^{n+1}_{xg_{i+1/2,j,k}} \cdot (P^{n+1}_{c_{i+1,j,k}} - P^{n+1}_{c_{i,j,k}}) - T^{n+1}_{xg_{i-1/2,j,k}} \cdot (P^{n+1}_{c_{i,j,k}} - P^{n+1}_{c_{i-1,j,k}}) \\
&+ T^{n+1}_{yg_{i,j+1/2,k}} \cdot (P^{n+1}_{c_{i,j+1,k}} - P^{n+1}_{c_{i,j,k}}) - T^{n+1}_{yg_{i,j-1/2,k}} \cdot (P^{n+1}_{c_{i,j,k}} - P^{n+1}_{c_{i,j-1,k}}) \\
&+ T^{n+1}_{zg_{i,j,k+1/2}} \cdot (P^{n+1}_{c_{i,j,k+1}} - P^{n+1}_{c_{i,j,k}}) - T^{n+1}_{zg_{i,j,k-1/2}} \cdot (P^{n+1}_{c_{i,j,k}} - P^{n+1}_{c_{i,j,k-1}}) \\
&= RHS = ACC_g + SST
\end{aligned} \qquad \text{(A2.35)}$$

For the oil phase:

$$
\begin{aligned}
&LHS = \\
&T_{xo_{i+1/2,j,k}}[(P^{n+1}_{o_{i+1,j,k}} - P^{n+1}_{o_{i,j,k}}) - \gamma^{n+1}_{o_{i+1/2,j,k}}(h_{x_{i+1}} - h_{x_i})_{j,k}] \\
&- T_{xo_{i-1/2,j,k}}[(P^{n+1}_{o_{i,j,k}} - P^{n+1}_{o_{i-1,j,k}}) - \gamma^{n+1}_{o_{i-1/2,j,k}}(h_{x_i} - h_{x_{i-1}})_{j,k}] \\
&+ T_{yo_{i,j+1/2,k}}[(P^{n+1}_{o_{i,j+1,k}} - P^{n+1}_{o_{i,j,k}}) - \gamma^{n+1}_{o_{i,j+1/2,k}}(h_{y_{j+1}} - h_{y_j})_{i,k}] \\
&- T_{yo_{i,j-1/2,k}}[(P^{n+1}_{o_{i,j,k}} - P^{n+1}_{o_{i,j-1,k}}) - \gamma^{n+1}_{o_{i,j-1/2,k}}(h_{y_j} - h_{y_{j-1}})_{i,k}] \\
&+ T_{zo_{i,j,k+1/2}}[(P^{n+1}_{o_{i,j,k+1}} - P^{n+1}_{o_{i,j,k}}) - \gamma^{n+1}_{o_{i,j,k+1/2}}(z_{k+1} - z_k)_{i,j}] \\
&- T_{zo_{i,j,k-1/2}}[(P^{n+1}_{o_{i,j,k}} - P^{n+1}_{o_{i,j,k-1}}) - \gamma^{n+1}_{o_{i,j,k-1/2}}(z_k - z_{k-1})_{i,j}] \\
&= RHS = ACC_o + SST
\end{aligned}
$$

(A2.36)

A2.2.3 Development of the Accumulation Terms of the Oil and Gas Flow Equations

a) For the gas phase:

$$\text{Flow Terms} = V_{i,j,k} \frac{1}{\Delta t} \cdot \Delta_t \left(\phi \frac{S_g}{B_g} + \phi R_s \frac{S_o}{B_o}\right)_{i,j,k} + Source\,/\,Sink\ \ Terms$$

Where $\frac{1}{B_l} = b_l$, so:

$$\text{Flow Terms} = V_{i,j,k} \frac{1}{\Delta t} \cdot \Delta_t (\phi S_g b_g + \phi R_s S_o b_o)_{i,j,k} + Source\,/\,Sink\ \ Terms$$

Considering $(1 - S_w - S_o) = S_g$, thus:

$$\text{Flow Terms} = V_{i,j,k} \frac{1}{\Delta t} \cdot \Delta_t [\phi(1 - S_w - S_o) b_g + \phi R_s S_o b_o]_{i,j,k} + Source\,/\,Sink\ \ Terms,$$

(A2.37)

Developing the accumulation part in above equation:

$$V_{i,j,k} \frac{1}{\Delta t} \cdot \Delta_t [\phi(1 - S_w - S_o) b_g + \phi R_s S_o b_o]_{i,j,k} =$$

$$= \frac{V_{i,j,k}[(\phi R_s S_o b_o)^{n+1} - (\phi R_s S_o b_o)^n + (\phi b_g)^{n+1} - (\phi b_g)^n - \Delta_t(\phi S_w b_g) - (\phi S_o b_g)^{n+1} + (\phi S_o b_g)^n]_{i,j,k}}{\Delta t}$$

(A2.38)

Where, the term $\Delta_t(\phi S_w b_g)$ in above equation can be developed as:

$$\Delta_t(\phi S_w b_g) = \frac{\partial(\phi S_w b_g)}{\partial t} = (S_w)^n \Delta_t(\phi b_g) + (\phi b_g)^{n+1} \Delta_t S_w$$

(A2.39)

Developing the water equation $\frac{\partial(\phi S_w b_w)}{\partial t} = 0$ as:

$$\Delta_t(\phi S_w b_w) = \frac{\partial(\phi S_w b_w)}{\partial t} = S_w{}^n \frac{\partial(\phi b_w)}{\partial t} + (\phi b_w)^{n+1} \frac{\partial(S_w)}{\partial t} = 0$$

(A2.40)

Thus:

$$\Delta_t S_w = \frac{\partial(S_w)}{\partial t} = \frac{-S_w{}^n}{(\phi b_w)^{n+1}} \frac{\partial(\phi b_w)}{\partial t} = \frac{-S_w{}^n}{(\phi b_w)^{n+1}} (\phi' b_w + \phi b'_w) \Delta_t P_o$$

$$= -S_w{}^n \left(\frac{1}{\phi} \frac{d\phi}{dP_o} + \frac{1}{b_w} \frac{db_w}{dP_o}\right) \Delta_t P_o = -S_w{}^n (C_r + C_w) \Delta_t P_o$$

(A2.41)

Where, $\phi' = \frac{d\phi}{dP_o}$; $b'_w = \frac{db_w}{dP_o}$ $\Delta_t P_o = \frac{dP_o}{dt}$, or $\Delta_t P_o = P_o^{n+1} - P_o^n$

Thus,

$$\Delta_t(\phi S_w b_g) = (S_w)^n \Delta_t(\phi b_g) + (\phi b_g)^{n+1} \Delta_t S_w$$

$$= (S_w)^n [(\phi b_g)^{n+1} - (\phi b_g)^n] - (\phi b_g)^{n+1} S_w{}^n (C_r + C_w) \Delta_t P_o$$

(A2.42)

Substituting $\Delta_t(\phi S_w b_g)$ into above main gas equation, there is,

$$\mathit{Flow\ Terms} = \frac{V_{i,j,k}}{\Delta t} [(\phi R_s S_o b_o)^{n+1} - (\phi R_s S_o b_o)^n + (\phi b_g)^{n+1} - (\phi b_g)^n$$

$$- (S_w)^n [(\phi b_g)^{n+1} - (\phi b_g)^n] + (\phi b_g)^{n+1} S_w{}^n (C_r + C_w) \Delta_t P_o - (\phi S_o b_g)^{n+1} + (\phi S_o b_g)^n]_{i,j,k} + SST$$

(A2.43)

Using $V^n_{p_{i,j,k}} = V_{i,j,k} \cdot \phi_k{}^n$ and $\phi_k{}^{n+1} = \phi_k{}^n (1 + Cr \Delta_t P_o)$, ($\phi_k$ is the porosity is defined for each layer *k*), flow terms becomes:

$$\frac{V_{p_{i,j,k}}^{n}}{\Delta t}\{(1+Cr\Delta_t P_o)\cdot[b_g^{\,n+1}(1+S_w^{\,n}(C_r+C_w)\Delta_t P_o - S_w^{\,n} - S_o^{\,n+1}) + (R_s S_o b_o)^{n+1}] - b_g^{\,n}(1-S_w^{\,n}-S_o^{\,n}) - (R_s S_o b_o)^{n}\}_{i,j,k} + SST \tag{A2.44}$$

b) For the Oil Equation:

$$\text{Flow Terms} = V\frac{1}{\Delta t}\cdot\Delta_t(\phi S_o b_o)_{i,j,k} + Source\,/\,Sink\ Terms$$

Developing the accumulation part in the above equation:

$$\frac{V_{i,j,k}\Delta_t[\phi S_o b_o]_{i,j,k}}{\Delta t} = \frac{V_{i,j,k}}{\Delta t}[(\phi S_o b_o)^{n+1} - (\phi S_o b_o)^{n}]_{i,j,k} \tag{A2.45}$$

Because $V_{p_{i,j,k}}^{n} = V_{i,j,k}\cdot\phi_k^{\,n}$ and $\phi_k^{\,n+1} = \phi_k^{\,n}(1+Cr\Delta_t P_o)$, above equation becomes:

$$\frac{V_{i,j,k}}{\Delta t}[(\phi S_o b_o)^{n+1} - (\phi S_o b_o)^{n}]_{i,j,k} = \frac{V_{p_{i,j,k}}^{n}}{\Delta t}[(1+Cr\Delta_t P_o)\cdot(b_o S_o)^{n+1} - (S_o b_o)^{n}]_{i,j,k} \tag{A2.46}$$

c) For the Water Equation

The water saturation can be calculated as:

$$\frac{\partial(S_w)}{\partial t} = -S_w^{\,n}(C_r+C_w)\Delta_t P_o$$

Where $\Delta_t P_o = P_{o_{i,j,k}}^{\;n+1} - P_{o_{i,j,k}}^{\;n}$ (A2.47)

$$S_{w_{i,j,k}}^{\;n+1} - S_{w_{i,j,k}}^{\;n} = -S_w^{\,n}(C_r+C_w)\Delta_t P_o$$

$$S_{w_{i,j,k}}^{\;\overset{(v+1)}{n+1}} = S_{w_{i,j,k}}^{\;n}[1-(C_r+C_w)\Delta_t P_o] \tag{A2.48}$$

A2.2.4 Main equations

Combining the flow terms and accumulation terms, the main equations for oil and gas can be obtained from the following expressions:

For the gas phase:

$$
\begin{aligned}
&T_{xo_{i+1/2,j,k}} \cdot R_s[(P^{n+1}_{o_{i+1,j,k}} - P^{n+1}_{o_{i,j,k}}) - \gamma^{n+1}_{o_{i+1/2,j,k}}(h_{x\,i+1} - h_{x\,i})_{j,k}] \\
&- T_{xo_{i-1/2,j,k}} \cdot R_s[(P^{n+1}_{o_{i,j,k}} - P^{n+1}_{o_{i-1,j,k}}) - \gamma^{n+1}_{o_{i-1/2,j,k}}(h_{x\,i} - h_{x\,i-1})_{j,k}] \\
&+ T_{yo_{i,j+1/2,k}} \cdot R_s[(P^{n+1}_{o_{i,j+1,k}} - P^{n+1}_{o_{i,j,k}}) - \gamma^{n+1}_{o_{i,j+1/2,k}}(h_{y\,j+1} - h_{y\,j})_{i,k}] \\
&- T_{yo_{i,j-1/2,k}} \cdot R_s[(P^{n+1}_{o_{i,j,k}} - P^{n+1}_{o_{i,j-1,k}}) - \gamma^{n+1}_{o_{i,j-1/2,k}}(h_{y\,j} - h_{y\,j-1})_{i,k}] \\
&+ T_{zo_{i,j,k+1/2}} \cdot R_s[(P^{n+1}_{o_{i,j,k+1}} - P^{n+1}_{o_{i,j,k}}) - \gamma^{n+1}_{o_{i,j,k+1/2}}(z_{k+1} - z_k)_{i,j}] \\
&- T_{zo_{i,j,k-1/2}} \cdot R_s[(P^{n+1}_{o_{i,j,k}} - P^{n+1}_{o_{i,j,k-1}}) - \gamma^{n+1}_{o_{i,j,k-1/2}}(z_k - z_{k-1})_{i,j}] \\
&+ T_{xg_{i+1/2,j,k}} \cdot [(P^{n+1}_{o_{i+1,j,k}} - P^{n+1}_{o_{i,j,k}}) - \gamma^{n+1}_{g_{i+1/2,j,k}}(h_{x\,i+1} - h_{x\,i})_{j,k}] \\
&- T_{xg_{i-1/2,j,k}} \cdot [(P^{n+1}_{o_{i,j,k}} - P^{n+1}_{o_{i-1,j,k}}) - \gamma^{n+1}_{g_{i-1/2,j,k}}(h_{x\,i} - h_{x\,i-1})_{j,k}] \\
&+ T_{yg_{i,j+1/2,k}} \cdot [(P^{n+1}_{o_{i,j+1,k}} - P^{n+1}_{o_{i,j,k}}) - \gamma^{n+1}_{g_{i,j+1/2,k}}(h_{y\,j+1} - h_{y\,j})_{i,k}] \\
&- T_{yg_{i,j-1/2,k}} \cdot [(P^{n+1}_{o_{i,j,k}} - P^{n+1}_{o_{i,j-1,k}}) - \gamma^{n+1}_{g_{i,j-1/2,k}}(h_{y\,j} - h_{y\,j-1})_{i,k}] \\
&+ T_{zg_{i,j,k+1/2}} \cdot [(P^{n+1}_{o_{i,j,k+1}} - P^{n+1}_{o_{i,j,k}}) - \gamma^{n+1}_{g_{i,j,k+1/2}}(z_{k+1} - z_k)_{i,j}] \\
&- T_{zg_{i,j,k-1/2}} \cdot [(P^{n+1}_{o_{i,j,k}} - P^{n+1}_{o_{i,j,k-1}}) - \gamma^{n+1}_{g_{i,j,k-1/2}}(z_k - z_{k-1})_{i,j}] \\
&+ T^{n+1}_{xg_{i+1/2,j,k}} \cdot (P^{n+1}_{c_{i+1,j,k}} - P^{n+1}_{c_{i,j,k}}) - T^{n+1}_{xg_{i-1/2,j,k}} \cdot (P^{n+1}_{c_{i,j,k}} - P^{n+1}_{c_{i-1,j,k}}) \\
&+ T^{n+1}_{yg_{i,j+1/2,k}} \cdot (P^{n+1}_{c_{i,j+1,k}} - P^{n+1}_{c_{i,j,k}}) - T^{n+1}_{yg_{i,j-1/2,k}} \cdot (P^{n+1}_{c_{i,j,k}} - P^{n+1}_{c_{i,j-1,k}}) \\
&+ T^{n+1}_{zg_{i,j,k+1/2}} \cdot (P^{n+1}_{c_{i,j,k+1}} - P^{n+1}_{c_{i,j,k}}) - T^{n+1}_{zg_{i,j,k-1/2}} \cdot (P^{n+1}_{c_{i,j,k}} - P^{n+1}_{c_{i,j,k-1}}) \\
&= \frac{V^n_{p_{i,j,k}}}{\Delta t}\{(1 + Cr\Delta_t P_o) \cdot [b_g^{\,n+1}(1 + S_w^{\,n}(C_r + C_w)\Delta_t P_o - S_w^{\,n} - S_o^{\,n+1}) + (R_s S_o b_o)^{n+1}] \\
&- b_g^{\,n}(1 - S_w^{\,n} - S_o^{\,n}) - (R_s S_o b_o)^n\}_{i,j,k} + R^{n+1}_{so_{i,j}} Q^{n+1}_{o_{i,j,k}} + Q^{n+1}_{fg_{i,j,k}}
\end{aligned}
\tag{A2.49}
$$

For the oil phase:

$$
\begin{aligned}
&T_{xo_{i+1/2,j,k}}[(P^{n+1}_{o_{i+1,j,k}} - P^{n+1}_{o_{i,j,k}}) - \gamma^{n+1}_{o_{i+1/2,j,k}}(h_{x\,i+1} - h_{x\,i})_{j,k}] \\
&- T_{xo_{i-1/2,j,k}}[(P^{n+1}_{o_{i,j,k}} - P^{n+1}_{o_{i-1,j,k}}) - \gamma^{n+1}_{o_{i-1/2,j,k}}(h_{x\,i} - h_{x\,i-1})_{j,k}] \\
&+ T_{yo_{i,j+1/2,k}}[(P^{n+1}_{o_{i,j+1,k}} - P^{n+1}_{o_{i,j,k}}) - \gamma^{n+1}_{o_{i,j+1/2,k}}(h_{y\,j+1} - h_{y\,j})_{i,k}] \\
&- T_{yo_{i,j-1/2,k}}[(P^{n+1}_{o_{i,j,k}} - P^{n+1}_{o_{i,j-1,k}}) - \gamma^{n+1}_{o_{i,j-1/2,k}}(h_{y\,j} - h_{y\,j-1})_{i,k}] \\
&+ T_{zo_{i,j,k+1/2}}[(P^{n+1}_{o_{i,j,k+1}} - P^{n+1}_{o_{i,j,k}}) - \gamma^{n+1}_{o_{i,j,k+1/2}}(z_{k+1} - z_k)_{i,j}] \\
&- T_{zo_{i,j,k-1/2}}[(P^{n+1}_{o_{i,j,k}} - P^{n+1}_{o_{i,j,k-1}}) - \gamma^{n+1}_{o_{i,j,k-1/2}}(z_k - z_{k-1})_{i,j}] \\
&= \frac{V^n_{p_{i,j,k}}}{\Delta t}[(1 + Cr\Delta_t P_o) \cdot (b_o S_o)^{n+1} - (S_o b_o)^n]_{i,j,k} + Q^{n+1}_{o_{i,j,k}}
\end{aligned}
\tag{A2.50}
$$

A2.3 Residual Function for Jacobian Matrix

The associated unknowns in above finite difference equations include:

$$P^{n+1}_{o_{i,j,k}}, P^{n+1}_{o_{i+1,j,k}}, P^{n+1}_{o_{i-1,j,k}}, P^{n+1}_{o_{i,j+1,k}}, P^{n+1}_{o_{i,j-1,k}}, P^{n+1}_{o_{i,j,k+1}}, P^{n+1}_{o_{i,j,k-1}} \text{ and}$$

$$S^{n+1}_{o_{i,j,k}}, S^{n+1}_{o_{i+1,j,k}}, S^{n+1}_{o_{i-1,j,k}}, S^{n+1}_{o_{i,j+1,k}}, S^{n+1}_{o_{i,j-1,k}}, S^{n+1}_{o_{i,j,k+1}}, S^{n+1}_{o_{i,j,k-1}}$$

These equations comprise a non-linear system of algebraic equations which will be solved by using the Newton-Raphson Method. Therefore, there is a need to determine the residual function associated with the gas and oil equations, F_g and F_o.

A2.3.1 Determine Oil and Gas Residual Function

For all the unknowns in the reservoir domain, there is

$$\delta x^{(v+1)}_{i,j,k\ n+1} = x^{(v+1)}_{i,j,k\ n+1} - x^{(v)}_{i,j,k\ n+1}$$

Thus at n+1, for the gas phase:

$$\begin{aligned}
&\left(\frac{\partial F_{g_{i,j,k}}}{\partial P_{o_{i,j,k}}}\right)^{(v)} \delta P^{(v+1)}_{oi,j,k} + \left(\frac{\partial F_{g_{i,j,k}}}{\partial P_{o_{i+1,j,k}}}\right)^{(v)} \delta P^{(v+1)}_{oi+1,j,k} + \left(\frac{\partial F_{g_{i,j,k}}}{\partial P_{o_{i-1,j,k}}}\right)^{(v)} \delta P^{(v+1)}_{oi-1,j,k} + \\
&\left(\frac{\partial F_{g_{i,j,k}}}{\partial P_{o_{i,j+1,k}}}\right)^{(v)} \delta P^{(v+1)}_{oi,j+1,k} + \left(\frac{\partial F_{g_{i,j,k}}}{\partial P_{o_{i,j-1,k}}}\right)^{(v)} \delta P^{(v+1)}_{oi,j-1,k} + \left(\frac{\partial F_{g_{i,j,k}}}{\partial P_{o_{i,j,k+1}}}\right)^{(v)} \delta P^{(v+1)}_{oi,j,k+1} + \left(\frac{\partial F_{g_{i,j,k}}}{\partial P_{o_{i,j,k-1}}}\right)^{(v)} \delta P^{(v+1)}_{oi,j,k-1} \\
&+\left(\frac{\partial F_{g_{i,j,k}}}{\partial S_{o_{i,j,k}}}\right)^{(v)} \delta S^{(v+1)}_{oi,j,k} + \left(\frac{\partial F_{g_{i,j,k}}}{\partial S_{o_{i+1,j,k}}}\right)^{(v)} \delta S^{(v+1)}_{oi+1,j,k} + \left(\frac{\partial F_{g_{i,j,k}}}{\partial S_{o_{i-1,j,k}}}\right)^{(v)} \delta S^{(v+1)}_{oi-1,j,k} + \\
&\left(\frac{\partial F_{g_{i,j,k}}}{\partial S_{o_{i,j+1,k}}}\right)^{(v)} \delta S^{(v+1)}_{oi,j+1,k} + \left(\frac{\partial F_{g_{i,j,k}}}{\partial S_{o_{i,j-1,k}}}\right)^{(v)} \delta S^{(v+1)}_{oi,j-1,k} + \left(\frac{\partial F_{g_{i,j,k}}}{\partial S_{o_{i,j,k+1}}}\right)^{(v)} \delta S^{(v+1)}_{oi,j,k+1} + \left(\frac{\partial F_{g_{i,j,k}}}{\partial S_{o_{i,j,k-1}}}\right)^{(v)} \delta S^{(v+1)}_{oi,j,k-1} \\
&+\left(\frac{\partial F_{g_{i,j,k}}}{\partial P_{wf}}\right)^{(v)} \delta P_{wf} = -F_{g_{i,j,k}}^{(v)}
\end{aligned}$$

(A2.51)

For the oil phase:

$$(\frac{\partial F_{o_{i,j,k}}}{\partial P_{o_{i,j,k}}})^{(v)} \delta P_{oi,j,k}^{(v+1)} + (\frac{\partial F_{o_{i,j,k}}}{\partial P_{o_{i+1,j,k}}})^{(v)} \delta P_{oi+1,j,k}^{(v+1)} + (\frac{\partial F_{o_{i,j,k}}}{\partial P_{o_{i-1,j,k}}})^{(v)} \delta P_{oi-1,j,k}^{(v+1)} +$$
$$+(\frac{\partial F_{o_{i,j,k}}}{\partial P_{o_{i,j+1,k}}})^{(v)} \delta P_{oi,j+1,k}^{(v+1)} + (\frac{\partial F_{o_{i,j,k}}}{\partial P_{o_{i,j-1,k}}})^{(v)} \delta P_{oi,j-1,k}^{(v+1)} + (\frac{\partial F_{o_{i,j,k}}}{\partial P_{o_{i,j,k+1}}})^{(v)} \delta P_{oi,j,k+1}^{(v+1)} + (\frac{\partial F_{o_{i,j,k}}}{\partial P_{o_{i,j,k-1}}})^{(v)} \delta P_{oi,j,k-1}^{(v+1)}$$
$$+(\frac{\partial F_{o_{i,j,k}}}{\partial S_{o_{i,j,k}}})^{(v)} \delta S_{oi,j,k}^{(v+1)} + (\frac{\partial F_{o_{i,j,k}}}{\partial S_{o_{i+1,j,k}}})^{(v)} \delta S_{oi+1,j,k}^{(v+1)} + (\frac{\partial F_{o_{i,j,k}}}{\partial S_{o_{i-1,j,k}}})^{(v)} \delta S_{oi-1,j,k}^{(v+1)} +$$
$$+(\frac{\partial F_{o_{i,j,k}}}{\partial S_{o_{i,j+1,k}}})^{(v)} \delta S_{oi,j+1,k}^{(v+1)} + (\frac{\partial F_{o_{i,j,k}}}{\partial S_{o_{i,j-1,k}}})^{(v)} \delta S_{oi,j-1,k}^{(v+1)} + (\frac{\partial F_{o_{i,j,k}}}{\partial S_{o_{i,j,k+1}}})^{(v)} \delta S_{oi,j,k+1}^{(v+1)} + (\frac{\partial F_{o_{i,j,k}}}{\partial S_{o_{i,j,k-1}}})^{(v)} \delta S_{oi,j,k-1}^{(v+1)}$$
$$+(\frac{\partial F_{o_{i,j,k}}}{\partial P_{wf}})^{(v)} \delta P_{wf} = -F_{o_{i,j,k}}^{(v)}$$

(A2.52)

For the gas phase:

$$F_g = T_{xo_{i+1/2}} \cdot R_{s_{i+1/2}} \cdot POT_{ox2} - T_{xo_{i-1/2}} \cdot R_{s_{i-1/2}} \cdot POT_{ox1}$$
$$+ T_{yo_{j+1/2}} \cdot R_{s_{j+1/2}} \cdot POT_{oy2} - T_{yo_{j-1/2}} \cdot R_{s_{j-1/2}} \cdot POT_{oy1}$$
$$+ T_{zo_{k+1/2}} \cdot R_{s_{k+1/2}} \cdot POT_{oz2} - T_{zo_{k-1/2}} \cdot R_{s_{k-1/2}} \cdot POT_{oz1}$$
$$+ T_{xg_{i+1/2}} \cdot POT_{gx2} - T_{xg_{i-1/2}} \cdot POT_{gx1}$$
$$+ T_{yg_{j+1/2}} \cdot POT_{gy2} - T_{yg_{j-1/2}} \cdot POT_{gy1}$$
$$+ T_{zg_{k+1/2}} \cdot POT_{gz2} - T_{zg_{k-1/2}} \cdot POT_{gz1}$$
$$- \frac{V_{p_{i,j,k}}^n}{\Delta t} \{(1 + Cr\Delta_t P_o) \cdot [b_g^{n+1}(1 + S_w^n (C_r + C_w)\Delta_t P_o - S_w^n - S_o^{n+1}) + (R_s S_o b_o)^{n+1}]$$
$$- b_g^n (1 - S_w^n - S_o^n) - (R_s S_o b_o)^n\}_{i,j,k} - R_{so_{i,j}}^{n+1} Q_{o_{i,j,k}}^{n+1} - Q_{fg_{i,j,k}}^{n+1}$$

(A2.53)

For the oil phase:

$$F_o = T_{xo_{i+1/2}} \cdot POT_{ox2} - T_{xo_{i-1/2}} \cdot POT_{ox1}$$
$$+ T_{yo_{j+1/2}} \cdot POT_{oy2} - T_{yo_{j-1/2}} \cdot POT_{oy1}$$
$$+ T_{zo_{k+1/2}} \cdot POT_{oz2} - T_{zo_{k-1/2}} \cdot POT_{oz1} \qquad \text{(A2.54)}$$
$$- \frac{V_{p_{i,j,k}}^n}{\Delta t} [(1 + Cr\Delta_t P_o) \cdot (b_o S_o)^{n+1} - (S_o b_o)^n]_{i,j,k} - Q_{o_{i,j,k}}^{n+1}$$

A2.3.2 Derivatives of Gas Residual Function with Respect to P_o and S_o

The derivatives of F_g with respect to $P^{n+1}_{o_{i,j,k}}, P^{n+1}_{o_{i+1,j,k}}, P^{n+1}_{o_{i-1,j,k}}, P^{n+1}_{o_{i,j+1,k}}, P^{n+1}_{o_{i,j-1,k}}, P^{n+1}_{o_{i,j,k+1}}, P^{n+1}_{o_{i,j,k-1}}$ and $S^{n+1}_{o_{i,j,k}}, S^{n+1}_{o_{i+1,j,k}}, S^{n+1}_{o_{i-1,j,k}}, S^{n+1}_{o_{i,j+1,k}}, S^{n+1}_{o_{i,j-1,k}}, S^{n+1}_{o_{i,j,k+1}}, S^{n+1}_{o_{i,j,k-1}}$ (all items are in current iteration) are calculated as the following:

a) Derivatives of F_g with respect to P_o

Derivatives of F_g with respect to $P_{o_{i,j,k}}$: $(\frac{\partial F_{g_{i,j,k}}}{\partial P_{o_{i,j,k}}})^{(v)}$

$$
\begin{aligned}
\frac{\partial F_g}{\partial P_{o_{i,j,k}}} &= POT_{ox2}\left(\frac{\partial T_{xo_{i+1/2,j,k}}}{\partial P_{o_{i,j,k}}} R_{s_{i+1/2,j,k}} + T_{xo_{i+1/2,j,k}} \frac{dR_{s_{i+1/2,j,k}}}{\partial P_{o_{i,j,k}}}\right) + T_{xo_{i+1/2,j,k}} R_{s_{i+1/2,j,k}} \frac{d(POT_{ox2})}{\partial P_{o_{i,j,k}}} \\
&- POT_{ox1}\left(\frac{\partial T_{xo_{i-1/2,j,k}}}{\partial P_{o_{i,j,k}}} R_{s_{i-1/2,j,k}} + T_{xo_{i-1/2,j,k}} \frac{dR_{s_{i-1/2,j,k}}}{\partial P_{o_{i,j,k}}}\right) - T_{xo_{i-1/2,j,k}} R_{s_{i-1/2,j,k}} \frac{d(POT_{ox1})}{\partial P_{o_{i,j,k}}} \\
&+ POT_{gx2} \frac{\partial T_{xg_{i+1/2,j,k}}}{\partial P_{o_{i,j,k}}} + T_{xg_{i+1/2,j,k}} \frac{d(POT_{gx2})}{\partial P_{o_{i,j,k}}} - POT_{gx1} \frac{\partial T_{xg_{i-1/2,j,k}}}{\partial P_{o_{i,j,k}}} - T_{xg_{i-1/2,j,k}} \frac{d(POT_{gx1})}{\partial P_{o_{i,j,k}}} \\
& POT_{oy2}\left(\frac{\partial T_{yo_{i,j+1/2,k}}}{\partial P_{o_{i,j,k}}} R_{s_{i,j+1/2,k}} + T_{yo_{i,j+1/2,k}} \frac{dR_{s_{i,j+1/2,k}}}{\partial P_{o_{i,j,k}}}\right) + T_{yo_{i,j+1/2,k}} R_{s_{i,j+1/2,k}} \frac{d(POT_{oy2})}{\partial P_{o_{i,j,k}}} \\
&- POT_{oy1}\left(\frac{\partial T_{yo_{i,j-1/2,k}}}{\partial P_{o_{i,j,k}}} R_{s_{i,j-1/2,k}} + T_{yo_{i,j-1/2,k}} \frac{dR_{s_{i,j-1/2,k}}}{\partial P_{o_{i,j,k}}}\right) - T_{yo_{i,j-1/2,k}} R_{s_{i,j-1/2,k}} \frac{d(POT_{oy1})}{\partial P_{o_{i,j,k}}} \\
&+ POT_{gy2} \frac{\partial T_{yg_{i,j+1/2,k}}}{\partial P_{o_{i,j,k}}} + T_{yg_{i,j+1/2,k}} \frac{d(POT_{gy2})}{\partial P_{o_{i,j,k}}} - POT_{gy1} \frac{\partial T_{yg_{i,j-1/2,k}}}{\partial P_{o_{i,j,k}}} - T_{yg_{i,j-1/2,k}} \frac{d(POT_{gy1})}{\partial P_{o_{i,j,k}}} \\
& POT_{oz2}\left(\frac{\partial T_{zo_{i,j,k+1/2}}}{\partial P_{o_{i,j,k}}} R_{s_{i,j,k+1/2}} + T_{zo_{i,j,k+1/2}} \frac{dR_{s_{i,j,k+1/2}}}{\partial P_{o_{i,j,k}}}\right) + T_{zo_{i,j,k+1/2}} R_{s_{i,j,k+1/2}} \frac{d(POT_{oz2})}{\partial P_{o_{i,j,k}}} \\
&- POT_{oz1}\left(\frac{\partial T_{zo_{i,j,k-1/2}}}{\partial P_{o_{i,j,k}}} R_{s_{i,j,k-1/2}} + T_{zo_{i,j,k-1/2}} \frac{dR_{s_{i,j,k-1/2}}}{\partial P_{o_{i,j,k}}}\right) - T_{zo_{i,j,k-1/2}} R_{s_{i,j,k-1/2}} \frac{d(POT_{oz1})}{\partial P_{o_{i,j,k}}} \\
&+ POT_{gz2} \frac{\partial T_{zg_{i,j,k+1/2}}}{\partial P_{o_{i,j,k}}} + T_{zg_{i,j,k+1/2}} \frac{d(POT_{gz2})}{\partial P_{o_{i,j,k}}} - POT_{gz1} \frac{\partial T_{zg_{i,j,k-1/2}}}{\partial P_{o_{i,j,k}}} - T_{zg_{i,j,k-1/2}} \frac{d(POT_{gz1})}{\partial P_{o_{i,j,k}}} \\
&- \frac{\partial ACC_g}{\partial P_{o_{i,j,k}}} - \frac{\partial (R_s Q_o)_{i,j,k}}{\partial P_{o_{i,j,k}}} - \frac{\partial (Q_{fg})_{i,j,k}}{\partial P_{o_{i,j,k}}} \qquad where, -\frac{\partial (R_s Q_o)_{i,j,k}}{\partial P_{o_{i,j,k}}} - \frac{\partial (Q_{fg})_{i,j,k}}{\partial P_{o_{i,j,k}}} = -\frac{\partial (Q_g)_{i,j,k}}{\partial P_{o_{i,j,k}}}
\end{aligned}
\tag{A2.55}
$$

Derivatives of F_g with respect to $P_{o_{i+1,j,k}}$: $(\frac{\partial F_{g_{i,j,k}}}{\partial P_{o_{i+1,j,k}}})^{(v)}$

$$\frac{\partial F_g}{\partial P_{o_{i+1,j,k}}} = POT_{ox2}(\frac{\partial T_{xo_{i+1/2,j,k}}}{\partial P_{o_{i+1,j,k}}} R_{s_{i+1/2,j,k}} + T_{xo_{i+1/2,j,k}} \frac{dR_{s_{i+1/2,j,k}}}{\partial P_{o_{i+1,j,k}}}) + T_{xo_{i+1/2,j,k}} R_{s_{i+1/2,j,k}} \frac{d(POT_{ox2})}{\partial P_{o_{i+1,j,k}}}$$
$$+ POT_{gx2} \frac{\partial T_{xg_{i+1/2,j,k}}}{\partial P_{o_{i+1,j,k}}} + T_{xg_{i+1/2,j,k}} \frac{d(POT_{gx2})}{\partial P_{o_{i+1,j,k}}} \tag{A2.56}$$

Derivatives of F_g with respect to $P_{o_{i-1,j,k}}$: $(\frac{\partial F_{g_{i,j,k}}}{\partial P_{o_{i-1,j,k}}})^{(v)}$

$$\frac{\partial F_g}{\partial P_{o_{i-1,j,k}}} = -POT_{ox1}(\frac{\partial T_{xo_{i-1/2,j,k}}}{\partial P_{o_{i-1,j,k}}} R_{s_{i-1/2,j,k}} + T_{xo_{i-1/2,j,k}} \frac{dR_{s_{i-1/2,j,k}}}{\partial P_{o_{i-1,j,k}}}) - T_{xo_{i-1/2,j,k}} R_{s_{i-1/2,j,k}} \frac{d(POT_{ox1})}{\partial P_{o_{i-1,j,k}}}$$
$$- POT_{gx1} \frac{\partial T_{xg_{i-1/2,j,k}}}{\partial P_{o_{i-1,j,k}}} - T_{xg_{i-1/2,j,k}} \frac{d(POT_{gx1})}{\partial P_{o_{i-1,j,k}}} \tag{A2.57}$$

Derivatives of F_g with respect to $P_{o_{i,j+1,k}}$: $(\frac{\partial F_{g_{i,j,k}}}{\partial P_{o_{i,j+1,k}}})^{(v)}$

$$\frac{\partial F_g}{\partial P_{o_{i,j+1,k}}} = POT_{oy2}(\frac{\partial T_{yo_{i,j+1/2,k}}}{\partial P_{o_{i,j+1,k}}} R_{s_{i,j+1/2,k}} + T_{yo_{i,j+1/2,k}} \frac{dR_{s_{i,j+1/2,k}}}{\partial P_{o_{i,j+1,k}}}) + T_{yo_{i,j+1/2,k}} R_{s_{i,j+1/2,k}} \frac{d(POT_{oy2})}{\partial P_{o_{i,j+1,k}}}$$
$$+ POT_{gy2} \frac{\partial T_{yg_{i,j+1/2,k}}}{\partial P_{o_{i,j+1,k}}} + T_{yg_{i,j+1/2,k}} \frac{d(POT_{gy2})}{\partial P_{o_{i,j+1,k}}} \tag{A2.58}$$

Derivatives of F_g with respect to $P_{o_{i,j-1,k}}$: $(\frac{\partial F_{g_{i,j,k}}}{\partial P_{o_{i,j-1,k}}})^{(v)}$

$$\frac{\partial F_g}{\partial P_{o_{i,j-1,k}}} = -POT_{oy1}(\frac{\partial T_{yo_{i,j-1/2,k}}}{\partial P_{o_{i,j-1,k}}} R_{s_{i,j-1/2,k}} + T_{yo_{i,j-1/2,k}} \frac{dR_{s_{i,j-1/2,k}}}{\partial P_{o_{i,j-1,k}}}) - T_{yo_{i,j-1/2,k}} R_{s_{i,j-1/2,k}} \frac{d(POT_{oy1})}{\partial P_{o_{i,j-1,k}}}$$
$$- POT_{gy1} \frac{\partial T_{yg_{i,j-1/2,k}}}{\partial P_{o_{i,j-1,k}}} - T_{yg_{i,j-1/2,k}} \frac{d(POT_{gy1})}{\partial P_{o_{i,j-1,k}}} \tag{A2.59}$$

Derivatives of F_g with respect to $P_{o_{i,j,k+1}}$: $(\frac{\partial F_{g_{i,j,k}}}{\partial P_{o_{i,j,k+1}}})^{(v)}$

$$\frac{\partial F_g}{\partial P_{o_{i,j,k+1}}} = POT_{oz2}(\frac{\partial T_{zo_{i,j,k+1/2}}}{\partial P_{o_{i,j,k+1}}} R_{s_{i,j,k+1/2}} + T_{zo_{i,j,k+1/2}} \frac{dR_{s_{i,j,k+1/2}}}{\partial P_{o_{i,j,k+1}}}) + T_{zo_{i,j,k+1/2}} R_{s_{i,j,k+1/2}} \frac{d(POT_{oz2})}{\partial P_{o_{i,j,k+1}}}$$
$$+ POT_{gz2} \frac{\partial T_{zg_{i,j,k+1/2}}}{\partial P_{o_{i,j,k+1}}} + T_{zg_{i,j,k+1/2}} \frac{d(POT_{gz2})}{\partial P_{o_{i,j,k+1}}}$$

(A2.60)

Derivatives of F_g with respect to $P_{o_{i,j,k-1}}$: $(\frac{\partial F_{g_{i,j,k}}}{\partial P_{o_{i,j,k-1}}})^{(v)}$

$$\frac{\partial F_g}{\partial P_{o_{i,j,k-1}}} = -POT_{oz1}(\frac{\partial T_{zo_{i,j,k-1/2}}}{\partial P_{o_{i,j,k-1}}} R_{s_{i,j,k-1/2}} + T_{zo_{i,j,k-1/2}} \frac{dR_{s_{i,j,k-1/2}}}{\partial P_{o_{i,j,k-1}}}) - T_{zo_{i,j,k-1/2}} R_{s_{i,j,k-1/2}} \frac{d(POT_{oz1})}{\partial P_{o_{i,j,k-1}}}$$
$$- POT_{gz1} \frac{\partial T_{zg_{i,j,k-1/2}}}{\partial P_{o_{i,j,k-1}}} - T_{zg_{i,j,k-1/2}} \frac{d(POT_{gz1})}{\partial P_{o_{i,j,k-1}}}$$

(A2.61)

b) Derivatives of F_g with respect to S_o

Derivatives of F_g with respect to $S_{o_{i,j,k}}$: $(\frac{\partial F_{g_{i,j,k}}}{\partial S_{o_{i,j,k}}})^{(v)}$

$$\frac{\partial F_g}{\partial S_{o_{i,j,k}}} = POT_{ox2}(\frac{\partial T_{xo_{i+1/2,j,k}}}{\partial S_{o_{i,j,k}}} R_{s_{i+1/2,j,k}}) - POT_{ox1}(\frac{\partial T_{xo_{i-1/2,j,k}}}{\partial S_{o_{i,j,k}}} R_{s_{i-1/2,j,k}})$$
$$+ POT_{gx2} \frac{\partial T_{xg_{i+1/2,j,k}}}{\partial S_{o_{i,j,k}}} + T_{xg_{i+1/2,j,k}} \frac{d(POT_{gx2})}{\partial S_{o_{i,j,k}}} - POT_{gx1} \frac{\partial T_{xg_{i-1/2,j,k}}}{\partial S_{o_{i,j,k}}} - T_{xg_{i-1/2,j,k}} \frac{d(POT_{gx1})}{\partial S_{o_{i,j,k}}}$$
$$POT_{oy2}(\frac{\partial T_{yo_{i,j+1/2,k}}}{\partial S_{o_{i,j,k}}} R_{s_{i,j+1/2,k}}) - POT_{oy1}(\frac{\partial T_{yo_{i,j-1/2,k}}}{\partial S_{o_{i,j,k}}} R_{s_{i,j-1/2,k}})$$
$$+ POT_{gy2} \frac{\partial T_{yg_{i,j+1/2,k}}}{\partial S_{o_{i,j,k}}} + T_{yg_{i,j+1/2,k}} \frac{d(POT_{gy2})}{\partial S_{o_{i,j,k}}} - POT_{gy1} \frac{\partial T_{yg_{i,j-1/2,k}}}{\partial S_{o_{i,j,k}}} - T_{yg_{i,j-1/2,k}} \frac{d(POT_{gy1})}{\partial S_{o_{i,j,k}}}$$
$$POT_{oz2}(\frac{\partial T_{zo_{i,j,k+1/2}}}{\partial S_{o_{i,j,k}}} R_{s_{i,j,k+1/2}}) - POT_{oz1}(\frac{\partial T_{zo_{i,j,k-1/2}}}{\partial S_{o_{i,j,k}}} R_{s_{i,j,k-1/2}})$$
$$+ POT_{gz2} \frac{\partial T_{zg_{i,j,k+1/2}}}{\partial S_{o_{i,j,k}}} + T_{zg_{i,j,k+1/2}} \frac{d(POT_{gz2})}{\partial S_{o_{i,j,k}}} - POT_{gz1} \frac{\partial T_{zg_{i,j,k-1/2}}}{\partial S_{o_{i,j,k}}} - T_{zg_{i,j,k-1/2}} \frac{d(POT_{gz1})}{\partial S_{o_{i,j,k}}}$$
$$- \frac{\partial ACC_g}{\partial S_{o_{i,j,k}}} - \frac{\partial (R_s Q_o)_{i,j,k}}{\partial S_{o_{i,j,k}}} - \frac{\partial (Q_{fg})_{i,j,k}}{\partial S_{o_{i,j,k}}} \quad where\ , - \frac{\partial (R_s Q_o)_{i,j,k}}{\partial S_{o_{i,j,k}}} - \frac{\partial (Q_{fg})_{i,j,k}}{\partial S_{o_{i,j,k}}} = - \frac{\partial (Q_g)_{i,j,k}}{\partial S_{o_{i,j,k}}}$$

(A2.62)

Derivatives of F_g with respect to $S_{o_{i+1,j,k}}$: $(\frac{\partial F_{g_{i,j,k}}}{\partial S_{o_{i+1,j,k}}})^{(\nu)}$

$$\frac{\partial F_g}{\partial S_{o_{i+1,j,k}}} = POT_{ox2}(\frac{\partial T_{xo_{i+1/2,j,k}}}{\partial S_{o_{i+1,j,k}}} R_{s_{i+1/2,j,k}}) + POT_{gx2}\frac{\partial T_{xg_{i+1/2,j,k}}}{\partial S_{o_{i+1,j,k}}} + T_{xg_{i+1/2,j,k}}\frac{d(POT_{gx2})}{\partial S_{o_{i+1,j,k}}}$$

(A2.63)

Derivatives of F_g with respect to $S_{o_{i-1,j,k}}$: $(\frac{\partial F_{g_{i,j,k}}}{\partial S_{o_{i-1,j,k}}})^{(\nu)}$

$$\frac{\partial F_g}{\partial S_{o_{i-1,j,k}}} = -POT_{ox1}(\frac{\partial T_{xo_{i-1/2,j,k}}}{\partial S_{o_{i-1,j,k}}} R_{s_{i-1/2,j,k}}) - POT_{gx1}\frac{\partial T_{xg_{i-1/2,j,k}}}{\partial S_{o_{i-1,j,k}}} - T_{xg_{i-1/2,j,k}}\frac{d(POT_{gx1})}{\partial S_{o_{i-1,j,k}}}$$

(A2.64)

Derivatives of F_g with respect to $S_{o_{i,j+1,k}}$: $(\frac{\partial F_{g_{i,j,k}}}{\partial S_{o_{i,j+1,k}}})^{(\nu)}$

$$\frac{\partial F_g}{\partial S_{o_{i,j+1,k}}} = POT_{oy2}(\frac{\partial T_{yo_{i,j+1/2,k}}}{\partial S_{o_{i,j+1,k}}} R_{s_{i,j+1/2,k}}) + POT_{gy2}\frac{\partial T_{yg_{i,j+1/2,k}}}{\partial S_{o_{i,j+1,k}}} + T_{yg_{i,j+1/2,k}}\frac{d(POT_{gy2})}{\partial S_{o_{i,j+1,k}}}$$

(A2.65)

Derivatives of F_g with respect to $S_{o_{i,j-1,k}}$: $(\frac{\partial F_{g_{i,j,k}}}{\partial S_{o_{i,j-1,k}}})^{(\nu)}$

$$\frac{\partial F_g}{\partial S_{o_{i,j-1,k}}} = -POT_{oy1}(\frac{\partial T_{yo_{i,j-1/2,k}}}{\partial S_{o_{i,j-1,k}}} R_{s_{i,j-1/2,k}}) - POT_{gy1}\frac{\partial T_{yg_{i,j-1/2,k}}}{\partial S_{o_{i,j-1,k}}} - T_{yg_{i,j-1/2,k}}\frac{d(POT_{gy1})}{\partial S_{o_{i,j-1,k}}}$$

(A2.66)

Derivatives of F_g with respect to $S_{o_{i,j,k+1}}$: $(\frac{\partial F_{g_{i,j,k}}}{\partial S_{o_{i,j,k+1}}})^{(\nu)}$

$$\frac{\partial F_g}{\partial S_{o_{i,j,k+1}}} = POT_{oz2}(\frac{\partial T_{zo_{i,j,k+1/2}}}{\partial S_{o_{i,j,k+1}}} R_{s_{i,j,k+1/2}}) + POT_{gz2}\frac{\partial T_{zg_{i,j,k+1/2}}}{\partial S_{o_{i,j,k+1}}} + T_{zg_{i,j,k+1/2}}\frac{d(POT_{gz2})}{\partial S_{o_{i,j,k+1}}}$$

(A2.67)

Derivatives of F_g with respect to $S_{o_{i,j,k-1}}$: $(\frac{\partial F_{g_{i,j,k}}}{\partial S_{o_{i,j,k-1}}})^{(\nu)}$

$$\frac{\partial F_g}{\partial S_{o_{i,j,k-1}}} = -POT_{oz}1(\frac{\partial T_{zo_{i,j,k-1/2}}}{\partial S_{o_{i,j,k-1}}} R_{s_{i,j,k-1/2}}) - POT_{gz1}\frac{\partial T_{zg_{i,j,k-1/2}}}{\partial S_{o_{i,j,k-1}}} - T_{zg_{i,j,k-1/2}}\frac{d(POT_{gz1})}{\partial S_{o_{i,j,k-1}}}$$

(A2.68)

A2.3.3 Derivatives of Oil Residual Function with Respect to P_o and S_o

The derivatives of F_o with respect to $P^{n+1}_{o_{i,j,k}}, P^{n+1}_{o_{i+1,j,k}}, P^{n+1}_{o_{i-1,j,k}}, P^{n+1}_{o_{i,j+1,k}}, P^{n+1}_{o_{i,j-1,k}}, P^{n+1}_{o_{i,j,k+1}}, P^{n+1}_{o_{i,j,k-1}}$ and $S^{n+1}_{o_{i,j,k}}, S^{n+1}_{o_{i+1,j,k}}, S^{n+1}_{o_{i-1,j,k}}, S^{n+1}_{o_{i,j+1,k}}, S^{n+1}_{o_{i,j-1,k}}, S^{n+1}_{o_{i,j,k+1}}, S^{n+1}_{o_{i,j,k-1}}$ (all items are in current iteration) are calculated as the following:

a) Derivatives of F_o with respect to P_o

Derivatives of F_o with respect to $P_{o_{i,j,k}}$: $(\frac{\partial F_{o_{i,j,k}}}{\partial P_{o_{i,j,k}}})^{(v)}$

$$\begin{aligned}\frac{\partial F_o}{\partial P_{o_{i,j,k}}} &= POT_{ox}2(\frac{\partial T_{xo_{i+1/2,j,k}}}{\partial P_{o_{i,j,k}}}) + T_{xo_{i+1/2,j,k}}\frac{d(POT_{ox}2)}{\partial P_{o_{i,j,k}}} \\ &- POT_{ox}1(\frac{\partial T_{xo_{i-1/2,j,k}}}{\partial P_{o_{i,j,k}}}) - T_{xo_{i-1/2,j,k}}\frac{d(POT_{ox}1)}{\partial P_{o_{i,j,k}}} \\ &POT_{oy}2(\frac{\partial T_{yo_{i,j+1/2,k}}}{\partial P_{o_{i,j,k}}}) + T_{yo_{i,j+1/2,k}}\frac{d(POT_{oy}2)}{\partial P_{o_{i,j,k}}} \\ &- POT_{oy}1(\frac{\partial T_{yo_{i,j-1/2,k}}}{\partial P_{o_{i,j,k}}}) - T_{yo_{i,j-1/2,k}}\frac{d(POT_{oy}1)}{\partial P_{o_{i,j,k}}} \\ &POT_{oz}2(\frac{\partial T_{zo_{i,j,k+1/2}}}{\partial P_{o_{i,j,k}}}) + T_{zo_{i,j,k+1/2}}\frac{d(POT_{oz}2)}{\partial P_{o_{i,j,k}}} \\ &- POT_{oz}1(\frac{\partial T_{zo_{i,j,k-1/2}}}{\partial P_{o_{i,j,k}}}) - T_{zo_{i,j,k-1/2}}\frac{d(POT_{oz}1)}{\partial P_{o_{i,j,k}}} - \frac{\partial ACC_o}{\partial P_{o_{i,j,k}}} - \frac{\partial (Q_o)_{i,j,k}}{\partial P_{o_{i,j,k}}}\end{aligned}$$

(A2.69)

Derivatives of F_o with respect to $P_{o_{i+1,j,k}}$: $(\frac{\partial F_{o_{i,j,k}}}{\partial P_{o_{i+1,j,k}}})^{(v)}$

$$\frac{\partial F_o}{\partial P_{o_{i+1,j,k}}} = POT_{ox}2(\frac{\partial T_{xo_{i+1/2,j,k}}}{\partial P_{o_{i+1,j,k}}}) + T_{xo_{i+1/2,j,k}}\frac{d(POT_{ox}2)}{\partial P_{o_{i+1,j,k}}}$$

(A2.70)

Derivatives of F_o with respect to $P_{o_{i-1,j,k}}$: $(\frac{\partial F_{o_{i,j,k}}}{\partial P_{o_{i-1,j,k}}})^{(v)}$

$$\frac{\partial F_o}{\partial P_{o_{i-1,j,k}}} = -POT_{ox1}(\frac{\partial T_{xo_{i-1/2,j,k}}}{\partial P_{o_{i-1,j,k}}}) - T_{xo_{i-1/2,j,k}} \frac{d(POT_{ox1})}{\partial P_{o_{i-1,j,k}}} \quad \text{(A2.71)}$$

Derivatives of F_o with respect to $P_{o_{i,j+1,k}}$: $(\frac{\partial F_{o_{i,j,k}}}{\partial P_{o_{i,j+1,k}}})^{(v)}$

$$\frac{\partial F_g}{\partial P_{o_{i,j+1,k}}} = POT_{oy2}(\frac{\partial T_{yo_{i,j+1/2,k}}}{\partial P_{o_{i,j+1,k}}}) + T_{yo_{i,j+1/2,k}} \frac{d(POT_{oy2})}{\partial P_{o_{i,j+1,k}}} \quad \text{(A2.72)}$$

Derivatives of F_o with respect to $P_{o_{i,j-1,k}}$: $(\frac{\partial F_{o_{i,j,k}}}{\partial P_{o_{i,j-1,k}}})^{(v)}$

$$\frac{\partial F_g}{\partial P_{o_{i,j-1,k}}} = -POT_{oy1}(\frac{\partial T_{yo_{i,j-1/2,k}}}{\partial P_{o_{i,j-1,k}}}) - T_{yo_{i,j-1/2,k}} \frac{d(POT_{oy1})}{\partial P_{o_{i,j-1,k}}} \quad \text{(A2.73)}$$

Derivatives of F_o with respect to $P_{o_{i,j,k+1}}$: $(\frac{\partial F_{o_{i,j,k}}}{\partial P_{o_{i,j,k+1}}})^{(v)}$

$$\frac{\partial F_o}{\partial P_{o_{i,j,k+1}}} = POT_{oz2}(\frac{\partial T_{zo_{i,j,k+1/2}}}{\partial P_{o_{i,j,k+1}}}) + T_{zo_{i,j,k+1/2}} \frac{d(POT_{oz2})}{\partial P_{o_{i,j,k+1}}} \quad \text{(A2.74)}$$

Derivatives of F_o with respect to $P_{o_{i,j,k-1}}$: $(\frac{\partial F_{o_{i,j,k}}}{\partial P_{o_{i,j,k-1}}})^{(v)}$

(A2.75)

$$\frac{\partial F_o}{\partial P_{o_{i,j,k-1}}} = -POT_{oz1}(\frac{\partial T_{zo_{i,j,k-1/2}}}{\partial P_{o_{i,j,k-1}}}) - T_{zo_{i,j,k-1/2}} \frac{d(POT_{oz1})}{\partial P_{o_{i,j,k-1}}}$$

b) Derivatives of F_o with respect to S_o

Derivatives of F_o with respect to $S_{o_{i,j,k}}$: $(\frac{\partial F_{o_{i,j,k}}}{\partial S_{o_{i,j,k}}})^{(\nu)}$

$$\frac{\partial F_o}{\partial S_{o_{i,j,k}}} = POT_{ox2}(\frac{\partial T_{xo_{i+1/2,j,k}}}{\partial S_{o_{i,j,k}}}) - POT_{ox1}(\frac{\partial T_{xo_{i-1/2,j,k}}}{\partial S_{o_{i,j,k}}}) + POT_{oy2}(\frac{\partial T_{yo_{i,j+1/2,k}}}{\partial S_{o_{i,j,k}}}) - POT_{oy1}(\frac{\partial T_{yo_{i,j-1/2,k}}}{\partial S_{o_{i,j,k}}}) + POT_{oz2}(\frac{\partial T_{zo_{i,j,k+1/2}}}{\partial S_{o_{i,j,k}}}) - POT_{oz1}(\frac{\partial T_{zo_{i,j,k-1/2}}}{\partial S_{o_{i,j,k}}}) - \frac{\partial ACC_o}{\partial S_{o_{i,j,k}}} - \frac{\partial (Q_o)_{i,j,k}}{\partial S_{o_{i,j,k}}} \tag{A2.76}$$

Derivatives of F_o with respect to $S_{o_{i+1,j,k}}$: $(\frac{\partial F_{o_{i,j,k}}}{\partial S_{o_{i+1,j,k}}})^{(\nu)}$

$$\frac{\partial F_o}{\partial S_{o_{i+1,j,k}}} = POT_{ox2}(\frac{\partial T_{xo_{i+1/2,j,k}}}{\partial S_{o_{i+1,j,k}}}) \tag{A2.77}$$

Derivatives of F_o with respect to $S_{o_{i-1,j,k}}$: $(\frac{\partial F_{o_{i,j,k}}}{\partial S_{o_{i-1,j,k}}})^{(\nu)}$

$$\frac{\partial F_o}{\partial S_{o_{i-1,j,k}}} = -POT_{ox1}(\frac{\partial T_{xo_{i-1/2,j,k}}}{\partial S_{o_{i-1,j,k}}}) \tag{A2.78}$$

Derivatives of F_o with respect to $S_{o_{i,j+1,k}}$: $(\frac{\partial F_{o_{i,j,k}}}{\partial S_{o_{i,j+1,k}}})^{(\nu)}$

$$\frac{\partial F_o}{\partial S_{o_{i,j+1,k}}} = POT_{oy2}(\frac{\partial T_{yo_{i,j+1/2,k}}}{\partial S_{o_{i,j+1,k}}}) \tag{A2.79}$$

Derivatives of F_o with respect to $S_{o_{i,j-1,k}}$: $(\frac{\partial F_{o_{i,j,k}}}{\partial S_{o_{i,j-1,k}}})^{(\nu)}$

$$\frac{\partial F_o}{\partial S_{o_{i,j-1,k}}} = -POT_{oy1}(\frac{\partial T_{yo_{i,j-1/2,k}}}{\partial S_{o_{i,j-1,k}}}) \tag{A2.80}$$

Derivatives of F_o with respect to $S_{o_{i,j,k+1}}$: $(\frac{\partial F_{o_{i,j,k}}}{\partial S_{o_{i,j,k+1}}})^{(\nu)}$

$$\frac{\partial F_o}{\partial S_{o_{i,j,k+1}}} = POT_{oz2}(\frac{\partial T_{zo_{i,j,k+1/2}}}{\partial S_{o_{i,j,k+1}}}) \tag{A2.81}$$

Derivatives of F_o with respect to $S_{o_{i,j,k-1}}$: $(\frac{\partial F_{o_{i,j,k}}}{\partial S_{o_{i,j,k-1}}})^{(v)}$

$$\frac{\partial F_o}{\partial S_{o_{i,j,k-1}}} = -POT_{oz1}(\frac{\partial T_{zo_{i,j,k-1/2}}}{\partial S_{o_{i,j,k-1}}}) \quad \text{(A2.82)}$$

A2.3.4 Derivatives of Potentials to P_o and S_o

a) Derivatives of terms $POT_{gx1}, POT_{gx2}, POT_{gy1}, POT_{gy2}, POT_{gz1}, POT_{gz2}$ are:

$$\frac{d(POT_{gx1})}{dP_{o_{i-1,j,k}}} = -1 - \frac{d\gamma_{g_{i-1/2,j,k}}}{dP_{o_{i-1,j,k}}}(h_{x_{i-1,j,k}} - h_{x_{i,j,k}}) = -1 - (1-w_x)\frac{d\gamma_{g_{i-1,j,k}}}{dP_{o_{i-1,j,k}}}(h_{x_{i-1,j,k}} - h_{x_{i,j,k}})$$

$$\frac{d(POT_{gx2})}{dP_{o_{i-1,j,k}}} = 0$$

$$\frac{d(POT_{gx1})}{dP_{o_{i,j,k}}} = 1 - \frac{d\gamma_{g_{i-1/2,j,k}}}{dP_{o_{i,j,k}}}(h_{x_{i-1,j,k}} - h_{x_{i,j,k}}) = 1 - w_x\frac{d\gamma_{g_{i,j,k}}}{dP_{o_{i,j,k}}}(h_{x_{i-1,j,k}} - h_{x_{i,j,k}})$$

$$\frac{d(POT_{gx2})}{dP_{o_{i,j,k}}} = -1 - \frac{d\gamma_{g_{i+1/2,j,k}}}{dP_{o_{i,j,k}}}(h_{x_{i,j,k}} - h_{x_{i+1,j,k}}) = -1 - (1-w_x)\frac{d\gamma_{g_{i,j,k}}}{dP_{o_{i,j,k}}}(h_{x_{i,j,k}} - h_{x_{i+1,j,k}})$$

$$\frac{d(POT_{gx1})}{dP_{o_{i+1,j,k}}} = 0$$

$$\frac{d(POT_{gx2})}{dP_{o_{i+1,j,k}}} = 1 - \frac{d\gamma_{g_{i+1/2,j,k}}}{dP_{o_{i+1,j,k}}}(h_{x_{i,j,k}} - h_{x_{i+1,j,k}}) = 1 - w_x\frac{d\gamma_{g_{i+1,j,k}}}{dP_{o_{i+1,j,k}}}(h_{x_{i,j,k}} - h_{x_{i+1,j,k}})$$

(A2.83)

$$\frac{d(POT_{gy1})}{dP_{o_{i,j-1,k}}} = -1 - \frac{d\gamma_{g_{i,j-1/2,k}}}{dP_{o_{i,j-1,k}}}(h_{y_{i,j-1,k}} - h_{y_{i,j,k}}) = -1 - (1-w_y)\frac{d\gamma_{g_{i,j-1,k}}}{dP_{o_{i,j-1,k}}}(h_{y_{i,j-1,k}} - h_{y_{i,j,k}})$$

$$\frac{d(POT_{gy2})}{dP_{o_{i,j-1,k}}} = 0$$

$$\frac{d(POT_{gy1})}{dP_{o_{i,j,k}}} = 1 - \frac{d\gamma_{g_{i,j-1/2,k}}}{dP_{o_{i,j,k}}}(h_{y_{i,j-1,k}} - h_{y_{i,j,k}}) = 1 - w_y\frac{d\gamma_{g_{i,j,k}}}{dP_{o_{i,j,k}}}(h_{y_{i,j-1,k}} - h_{y_{i,j,k}})$$

$$\frac{d(POT_{gy2})}{dP_{o_{i,j,k}}} = -1 - \frac{d\gamma_{g_{i,j+1/2,k}}}{dP_{o_{i,j,k}}}(h_{y_{i,j,k}} - h_{y_{i,j+1,k}}) = -1 - (1-w_y)\frac{d\gamma_{g_{i,j,k}}}{dP_{o_{i,j,k}}}(h_{y_{i,j,k}} - h_{y_{i,j+1,k}})$$

$$\frac{d(POT_{gy1})}{dP_{o_{i,j+1,k}}} = 0$$

$$\frac{d(POT_{gy2})}{dP_{o_{i,j+1,k}}} = 1 - \frac{d\gamma_{g_{i,j+1/2,k}}}{dP_{o_{i,j+1,k}}}(h_{y_{i,j,k}} - h_{y_{i,j+1,k}}) = 1 - w_y\frac{d\gamma_{g_{i,j+1,k}}}{dP_{o_{i,j+1,k}}}(h_{y_{i,j,k}} - h_{y_{i,j+1,k}})$$

(A2.84)

$$\frac{d(POT_{gz1})}{dP_{o_{i,j,k-1}}} = -1 + \frac{d\gamma_{g_{i,j,k-1/2}}}{dP_{o_{i,j,k-1}}}\Delta z = -1 + (1-w_z)\frac{d\gamma_{g_{i,j,k-1}}}{dP_{o_{i,j,k-1}}}\Delta z$$

$$\frac{d(POT_{gz2})}{dP_{o_{i,j,k-1}}} = 0$$

$$\frac{d(POT_{gz1})}{dP_{o_{i,j,k}}} = 1 + \frac{d\gamma_{g_{i,j,k-1/2}}}{dP_{o_{i,j,k}}}\Delta z = 1 + w_z\frac{d\gamma_{g_{i,j,k}}}{dP_{o_{i,j,k}}}\Delta z$$

$$\frac{d(POT_{gz2})}{dP_{o_{i,j,k}}} = -1 + \frac{d\gamma_{g_{i,j,k+1/2}}}{dP_{o_{i,j,k}}}\Delta z = -1 + (1-w_z)\frac{d\gamma_{g_{i,j,k}}}{dP_{o_{i,j,k}}}\Delta z$$

$$\frac{d(POT_{gz1})}{dP_{o_{i,j,k+1}}} = 0$$

$$\frac{d(POT_{gz2})}{dP_{o_{i,j,k+1}}} = 1 + \frac{d\gamma_{g_{i,j,k+1/2}}}{dP_{o_{i,j,k+1}}}\Delta z = 1 + w_z\frac{d\gamma_{g_{i,j,k+1}}}{dP_{o_{i,j,k+1}}}\Delta z$$

(A2.85)

b) Derivatives of terms $POT_{ox1}, POT_{ox2}, POT_{oy1}, POT_{oy2}, POT_{oz1}, POT_{oz2}$ are:

$$\frac{d(POT_{ox1})}{dP_{o_{i-1,j,k}}} = -1 - \frac{d\gamma_{o_{i-1/2,j,k}}}{dP_{o_{i-1,j,k}}}(h_{x_{i-1,j,k}} - h_{x_{i,j,k}}) = -1 - (1-w_x)\frac{d\gamma_{o_{i-1,j,k}}}{dP_{o_{i-1,j,k}}}(h_{x_{i-1,j,k}} - h_{x_{i,j,k}})$$

$$\frac{d(POT_{ox2})}{dP_{o_{i-1,j,k}}} = 0$$

$$\frac{d(POT_{ox1})}{dP_{o_{i,j,k}}} = 1 - \frac{d\gamma_{o_{i-1/2,j,k}}}{dP_{o_{i,j,k}}}(h_{x_{i-1,j,k}} - h_{x_{i,j,k}}) = 1 - w_x \frac{d\gamma_{o_{i,j,k}}}{dP_{o_{i,j,k}}}(h_{x_{i-1,j,k}} - h_{x_{i,j,k}})$$

$$\frac{d(POT_{ox2})}{dP_{o_{i,j,k}}} = -1 - \frac{d\gamma_{o_{i+1/2,j,k}}}{dP_{o_{i,j,k}}}(h_{x_{i,j,k}} - h_{x_{i+1,j,k}}) = -1 - (1 - w_x)\frac{d\gamma_{o_{i,j,k}}}{dP_{o_{i,j,k}}}(h_{x_{i,j,k}} - h_{x_{i+1,j,k}})$$

$$\frac{d(POT_{ox1})}{dP_{o_{i+1,j,k}}} = 0$$

$$\frac{d(POT_{ox2})}{dP_{o_{i+1,j,k}}} = 1 - \frac{d\gamma_{o_{i+1/2,j,k}}}{dP_{o_{i+1,j,k}}}(h_{x_{i,j,k}} - h_{x_{i+1,j,k}}) = 1 - w_x \frac{d\gamma_{o_{i+1,j,k}}}{dP_{o_{i+1,j,k}}}(h_{x_{i,j,k}} - h_{x_{i+1,j,k}})$$

(A2.86)

$$\frac{d(POT_{oy1})}{dP_{o_{i,j-1,k}}} = -1 - \frac{d\gamma_{o_{i,j-1/2,k}}}{dP_{o_{i,j-1,k}}}(h_{y_{i,j-1,k}} - h_{y_{i,j,k}}) = -1 - (1 - w_y)\frac{d\gamma_{o_{i,j-1,k}}}{dP_{o_{i,j-1,k}}}(h_{y_{i,j-1,k}} - h_{y_{i,j,k}})$$

$$\frac{d(POT_{oy2})}{dP_{o_{i,j-1,k}}} = 0$$

$$\frac{d(POT_{oy1})}{dP_{o_{i,j,k}}} = 1 - \frac{d\gamma_{o_{i,j-1/2,k}}}{dP_{o_{i,j,k}}}(h_{y_{i,j-1,k}} - h_{y_{i,j,k}}) = 1 - w_y \frac{d\gamma_{o_{i,j,k}}}{dP_{o_{i,j,k}}}(h_{y_{i,j-1,k}} - h_{y_{i,j,k}})$$

$$\frac{d(POT_{oy2})}{dP_{o_{i,j,k}}} = -1 - \frac{d\gamma_{o_{i,j+1/2,k}}}{dP_{o_{i,j,k}}}(h_{y_{i,j,k}} - h_{y_{i,j+1,k}}) = -1 - (1 - w_y)\frac{d\gamma_{o_{i,j,k}}}{dP_{o_{i,j,k}}}(h_{y_{i,j,k}} - h_{y_{i,j+1,k}})$$

$$\frac{d(POT_{oy1})}{dP_{o_{i,j+1,k}}} = 0$$

$$\frac{d(POT_{oy2})}{dP_{o_{i,j+1,k}}} = 1 - \frac{d\gamma_{o_{i,j+1/2,k}}}{dP_{o_{i,j+1,k}}}(h_{y_{i,j,k}} - h_{y_{i,j+1,k}}) = 1 - w_y \frac{d\gamma_{o_{i,j+1,k}}}{dP_{o_{i,j+1,k}}}(h_{y_{i,j,k}} - h_{y_{i,j+1,k}})$$

(A2.87)

$$\frac{d(POT_{oz1})}{dP_{o_{i,j,k-1}}} = -1 + \frac{d\gamma_{o_{i,j,k-1/2}}}{dP_{o_{i,j,k-1}}}\Delta z = -1 + (1 - w_z)\frac{d\gamma_{o_{i,j,k-1}}}{dP_{o_{i,j,k-1}}}\Delta z$$

$$\frac{d(POT_{oz}2)}{dP_{o_{i,j,k-1}}} = 0$$

$$\frac{d(POT_{oz}1)}{dP_{o_{i,j,k}}} = 1 + \frac{d\gamma_{o_{i,j,k-1/2}}}{dP_{o_{i,j,k}}}\Delta z = 1 + w_z \frac{d\gamma_{o_{i,j,k}}}{dP_{o_{i,j,k}}}\Delta z$$

$$\frac{d(POT_{oz}2)}{dP_{o_{i,j,k}}} = -1 + \frac{d\gamma_{o_{i,j,k+1/2}}}{dP_{o_{i,j,k}}}\Delta z = -1 + (1 - w_z)\frac{d\gamma_{o_{i,j,k}}}{dP_{o_{i,j,k}}}\Delta z$$

$$\frac{d(POT_{oz}1)}{dP_{o_{i,j,k+1}}} = 0$$

$$\frac{d(POT_{oz}2)}{dP_{o_{i,j,k+1}}} = 1 + \frac{d\gamma_{o_{i,j,k+1/2}}}{dP_{o_{i,j,k+1}}}\Delta z = 1 + w_z \frac{d\gamma_{o_{i,j,k+1}}}{dP_{o_{i,j,k+1}}}\Delta z$$

(A2.88)

Note:

$$\frac{dPOT_{gx1}}{dS_{o_{i-1,j,k}}} = \frac{dPOT_{gx1}}{dS_{o_{i,j,k}}} = \frac{dPOT_{gx1}}{dS_{o_{i+1,j,k}}} = \frac{dPOT_{gx2}}{dS_{o_{i-1,j,k}}} = \frac{dPOT_{gx2}}{dS_{o_{i,j,k}}} = \frac{dPOT_{gx2}}{dS_{o_{i+1,j,k}}} = 0$$

$$\frac{dPOT_{gy1}}{dS_{o_{i,j-1,k}}} = \frac{dPOT_{gy1}}{dS_{o_{i,j,k}}} = \frac{dPOT_{gy1}}{dS_{o_{i,j+1,k}}} = \frac{dPOT_{gy2}}{dS_{o_{i,j-1,k}}} = \frac{dPOT_{gy2}}{dS_{o_{i,j,k}}} = \frac{dPOT_{gy2}}{dS_{o_{i,j+1,k}}} = 0 \rightarrow \frac{dP_c}{dS_o}$$

$$\frac{dPOT_{gz1}}{dS_{o_{i,j,k-1}}} = \frac{dPOT_{gz1}}{dS_{o_{i,j,k}}} = \frac{dPOT_{gz1}}{dS_{o_{i,j,k+1}}} = \frac{dPOT_{gz2}}{dS_{o_{i,j,k-1}}} = \frac{dPOT_{gz2}}{dS_{o_{i,j,k}}} = \frac{dPOT_{gz2}}{dS_{o_{i,j,k+1}}} = 0$$

(A2.89)

$$\frac{dPOT_{ox1}}{dS_{o_{i-1,j,k}}} = \frac{dPOT_{ox1}}{dS_{o_{i,j,k}}} = \frac{dPOT_{ox1}}{dS_{o_{i+1,j,k}}} = \frac{dPOT_{ox2}}{dS_{o_{i-1,j,k}}} = \frac{dPOT_{ox2}}{dS_{o_{i,j,k}}} = \frac{dPOT_{ox2}}{dS_{o_{i+1,j,k}}} = 0$$

$$\frac{dPOT_{oy1}}{dS_{o_{i,j-1,k}}} = \frac{dPOT_{oy1}}{dS_{o_{i,j,k}}} = \frac{dPOT_{oy1}}{dS_{o_{i,j+1,k}}} = \frac{dPOT_{oy2}}{dS_{o_{i,j-1,k}}} = \frac{dPOT_{oy2}}{dS_{o_{i,j,k}}} = \frac{dPOT_{oy2}}{dS_{o_{i,j+1,k}}} = 0$$

$$\frac{dPOT_{oz1}}{dS_{o_{i,j,k-1}}} = \frac{dPOT_{oz1}}{dS_{o_{i,j,k}}} = \frac{dPOT_{oz1}}{dS_{o_{i,j,k+1}}} = \frac{dPOT_{oz2}}{dS_{o_{i,j,k-1}}} = \frac{dPOT_{oz2}}{dS_{o_{i,j,k}}} = \frac{dPOT_{oz2}}{dS_{o_{i,j,k+1}}} = 0$$

(A2.90)

A2.3.5 Derivatives of Accumulation Term with P_o and S_o

Derivative of the gas accumulation term with respect to $P_{o_{i,j,k}}$: $\frac{\partial ACC_g}{\partial P_{o_{i,j,k}}}$

$$\frac{\partial ACC_g}{\partial P_{oi,j,k}} = \frac{V^n_{pi,j,k}}{\Delta t}\{Cr\cdot[b_{gi,j,k}{}^{v}(1+S_{wi,j,k}{}^{n}(C_r+C^v{}_{wi,j,k})(P^v{}_{oi,j,k}-P^n{}_{oi,j,k})-S_{wi,j,k}{}^{n}-S_{oi,j,k}{}^{v})+(R_sS_ob_o)^v_{i,j,k}]+$$

$$+(1+Cr(P^v{}_{oi,j,k}-P^n{}_{oi,j,k}))\cdot[\frac{db_{gi,j,k}{}^{v}}{dP_{oi,j,k}}(1+S_{wi,j,k}{}^{n}(C_r+C^v{}_{wi,j,k})(P^v{}_{oi,j,k}-P^n{}_{oi,j,k})-S_{wi,j,k}{}^{n}-S_{oi,j,k}{}^{v})+$$

$$+S_{wi,j,k}{}^{n}b_{gi,j,k}{}^{v}(C_r+C^v{}_{wi,j,k})+(S_ob_o)_{i,j,k}{}^{v}\frac{dR_{si,j,k}{}^{v}}{dP_{oi,j,k}}+(S_oR_s)_{i,j,k}{}^{v}\frac{db_{oi,j,k}{}^{v}}{dP_{oi,j,k}}]\}$$

(A2.91)

Derivative of the accumulation term with respect to $S_{o_{i,j,k}}$: $\frac{\partial ACC_g}{\partial S_{o_{i,j,k}}}$

$$\frac{\partial ACC_g}{\partial S_{oi,j,k}} = \frac{V^n_{pi,j,k}}{\Delta t}\{[1+C_r(P^v{}_{oi,j,k}-P^n{}_{oi,j,k})](R_{si,j,k}{}^{v}b_{oi,j,k}{}^{v}-b_{gi,j,k}{}^{v})\}$$

(A2.92)

Derivative of the oil accumulation term with respect to $P_{o_{i,j,k}}$: $\frac{\partial ACC_o}{\partial P_{o_{i,j,k}}}$

$$\frac{\partial ACC_o}{\partial P_{oi,j,k}} = \frac{V^n_{pi,j,k}}{\Delta t}\{C_r(S_{oi,j,k}b_{oi,j,k})^v+[1+C_r(P^v{}_{oi,j,k}-P^n{}_{oi,j,k})]\cdot S_{oi,j,k}{}^{v}\frac{db_{oi,j,k}{}^{v}}{dP_{oi,j,k}}\}$$

(A2.93)

Derivative of the accumulation term with respect to $S_{o_{i,j,k}}$: $\frac{\partial ACC_o}{\partial S_{o_{i,j,k}}}$

$$\frac{\partial ACC_o}{\partial S_{oi,j,k}} = \frac{V^n_{pi,j,k}}{\Delta t}[1+C_r(P^v{}_{oi,j,k}-P^n{}_{oi,j,k})]b_{oi,j,k}{}^{v}$$

(A2.94)

Note:

$$\frac{dACC_g}{dP_{o_{i-1,j,k}}} = \frac{dACC_g}{dP_{o_{i+1,j,k}}} = \frac{dACC_g}{dP_{o_{i,j-1,k}}} = \frac{dACC_g}{dP_{o_{i,j+1,k}}} = \frac{dACC_g}{dP_{o_{i,j,k-1}}} = \frac{dACC_g}{dP_{o_{i,j,k+1}}} = 0$$

$$\frac{dACC_g}{dS_{o_{i-1,j,k}}} = \frac{dACC_g}{dS_{o_{i+1,j,k}}} = \frac{dACC_g}{dS_{o_{i,j-1,k}}} = \frac{dACC_g}{dS_{o_{i,j+1,k}}} = \frac{dACC_g}{dS_{o_{i,j,k-1}}} = \frac{dACC_g}{dS_{o_{i,j,k+1}}} = 0$$

$$\frac{dACC_o}{dP_{o_{i-1,j,k}}} = \frac{dACC_o}{dP_{o_{i+1,j,k}}} = \frac{dACC_o}{dP_{o_{i,j-1,k}}} = \frac{dACC_o}{dP_{o_{i,j+1,k}}} = \frac{dACC_o}{dP_{o_{i,j,k-1}}} = \frac{dACC_o}{dP_{o_{i,j,k+1}}} = 0$$

$$\frac{dACC_o}{dS_{o_{i-1,j,k}}} = \frac{dACC_o}{dS_{o_{i+1,j,k}}} = \frac{dACC_o}{dS_{o_{i,j-1,k}}} = \frac{dACC_o}{dS_{o_{i,j+1,k}}} = \frac{dACC_o}{dS_{o_{i,j,k-1}}} = \frac{dACC_o}{dS_{o_{i,j,k+1}}} = 0$$

(A2.95)

APPENDIX 3: DERIVATIVES OF WELLBORE RESIDUAL FUNCTION

Chapter 4 presents the partial differential equation and finite difference equations for wellbore domain, but with only some general equations. This appendix summarizes the detailed derivation of wellbore residual functions, and corresponding derivatives with respect to wellbore and reservoir unknowns.

A3.1 Derivatives of Mass Conservation Residual Function (Oil) with Respect to Wellbore Unknowns

The formula of the oil mass conservation residual function is

$$F^{n+1}{}_{wo} = A_x H_o v\cdot[(b_o)^{n+1}_{k+1/2} - (b_o)^{n+1}_{k-1/2}] + A_x b_o v\cdot[(H_o)^{n+1}_{k+1/2} - (H_o)^{n+1}_{k-1/2}] + A_x b_o H_o\cdot[(v)^{n+1}_{k+1/2} - (v)^{n+1}_{k-1/2}] +$$
$$+\sum_{j=1}^{N\theta}\{[\frac{A_r K_r K_{ro}}{\mu_o B_o \Delta r}]^{n+1}_{1/2,j,k}(p^{n+1}_{o1,j,k} - p^{n+1}_{o-wb-k}) - [\frac{A_r K_r K_{ro}\gamma_o}{\mu_o B_o \Delta r}]^{n+1}_{1/2,j,k}(G_{1,j,k} - G_{wb-k})]\}$$
$$-V_b[\frac{(b_o H_o)^{n+1}_k - (b_o H_o)^n_k}{\Delta t}]$$

The derivatives of the oil mass conservation residual function with respect to wellbore unknowns, $P^{n+1}_{wb,k-1}, P^{n+1}_{wb,k}, P^{n+1}_{wb,k+1}, H^{n+1}_{o,k-1}, H^{n+1}_{o,k}, H^{n+1}_{o,k+1}, v^{n+1}_{m,k-1}, v^{n+1}_{m,k}, v^{n+1}_{m,k+1}$, is listed in the following section to construct the wellbore part in the Jacobian matrix.

a) Derivatives of F_{wo} with respect to P_{wb}

Derivatives of F_{wo} with respect to $P_{wb\,k}$: $(\frac{\partial F_{wok}}{\partial P_{wb\ k}})^{(v)}$

$$(\frac{\partial F_{wok}}{\partial P_{wb\ k}})^{(v)} = [A_x H_o]^{n+1}_k[\frac{(v\ )^{n+1}_{k+1} - (v\ )^{n+1}_{k-1}}{2}](\frac{\partial b_{ok}}{\partial P_{wb\ k}}) + [A_x v\ ]^{n+1}_k[\frac{(H_o)^{n+1}_{k+1} - (H_o)^{n+1}_{k-1}}{2}](\frac{\partial b_{ok}}{\partial P_{wb\ k}})$$
$$+\sum_{j=1}^{N\theta}\{-(\frac{A_r K_r K_{ro}}{\mu_o B_o \Delta r})^{n+1}_{1/2,j,k}\} + \sum_{j=1}^{N\theta}\{\frac{\partial T_{o,1/2,j,k}}{\partial P_{wb,k}}(p^{n+1}_{o1,j,k} - p^{n+1}_{o-wb-k} - G_{1,j,k} + G_{wb-k})\}$$
$$-[\frac{V_b}{\Delta t}]^{n+1}_k (H_o)^{n+1}_k(\frac{\partial b_{ok}}{\partial P_{wb\ k}})$$

(A3.1)

Where,

$$\frac{\partial T_{1/2,j,k}}{\partial P_{wb,k}} = (w_r - 1)\cdot T_{1/2,j,k}\left(\frac{1}{\mu_{l1/2,j,k}}\frac{d\mu_{0,j,k}}{\partial P_{wb,k}} - \frac{1}{b_{l1/2,j,k}}\frac{db_{0,j,k}}{\partial P_{wb,k}}\right)$$

$$= \left(\frac{r_w}{r_1} - 1\right)\cdot T_{1/2,j,k}\left(\frac{1}{\mu_{l1/2,j,k}}\frac{d\mu_{wb,k}}{\partial P_{wb,k}} - \frac{1}{b_{1/2,j,k}}\frac{db_{wb,k}}{\partial P_{wb,k}}\right)$$

$$w_r = \frac{r_{1/2,j,k} - r_{0,j,k}}{r_{1,j,k} - r_{0,j,k}} = \frac{r_w - 0}{r_{1,j,k} - 0} = \frac{r_w}{r_1}$$

(A3.2)

Derivatives of F_{wo} with respect to $P_{wb\ k+1}$: $\left(\frac{\partial F_{wok}}{\partial P_{wb\ \ k+1}}\right)^{(v)}$

$$\left(\frac{\partial F_{wok}}{\partial P_{wb\ \ k+1}}\right)^{(v)} = (0.5)[A_x v\ H_o]_k^{(v)}\left(\frac{\partial b_{ok+1}}{\partial P_{wb\ \ k+1}}\right)$$

(A3.3)

Derivatives of F_{wo} with respect to $P_{wb\ k-1}$: $\left(\frac{\partial F_{wok}}{\partial P_{wb\ \ k-1}}\right)^{(v)}$

$$\left(\frac{\partial F_{wok}}{\partial P_{wb\ \ k-1}}\right)^{(v)} = -(0.5)[A_x v\ H_o]_k^{(v)}\left(\frac{\partial b_{ok-1}}{\partial P_{wb\ \ k-1}}\right)$$

(A3.4)

b) Derivatives of F_{wo} with respect to H_o

The following subscript *i* represents the wellbore direction:

Derivatives of F_{wo} with respect to $H_{o\,i}$: $\left(\frac{\partial F_{woi}}{\partial H_{o\ i}}\right)^{(v)}$

$$\left(\frac{\partial F_{woi}}{\partial H_{o\ i}}\right)^{(v)} = [A_x v\ ]_i^{(v)}[(b_o)_{i+1/2}^{v} - (b_o)_{i-1/2}^{v}] + [A_x b_o]_i^{v}[(v\ )_{i+1/2}^{v} - (v\ )_{i-1/2}^{v}] - [\frac{V_b}{\Delta t}]_i^{v}(b_o)_i^{v}$$

$$= [A_x v\ ]_i^{(v)}\left[\frac{(b_o)_{i+1}^{v} - (b_o)_{i-1}^{v}}{2}\right] + [A_x b_o]_i^{v}\left[\frac{(v\ )_{i+1}^{v} - (v\ )_{i-1}^{v}}{2}\right] - [\frac{V_b}{\Delta t}]_i^{v}(b_o)_i^{v}$$

(A3.5)

Derivatives of F_{wo} with respect to $H_{o\ i+1}$: $(\frac{\partial F_{wo_i}}{\partial H_{o\ i+1}})^{(v)}$

$$(\frac{\partial F_{wo_i}}{\partial H_{o\ i+1}})^{(v)} = 0.5[A_x v\ b_o]_i^v \tag{A3.6}$$

Derivatives of F_{wo} with respect to $H_{o\ i-1}$: $(\frac{\partial F_{wo_i}}{\partial H_{o\ i-1}})^{(v)}$

$$(\frac{\partial F_{wo_i}}{\partial H_{o\ i-1}})^{(v)} = -0.5[A_x v\ b_o]_i^v \tag{A3.7}$$

c) Derivatives of F_{wo} with respect to v

Derivatives of F_{wo} with respect to v_i : $(\frac{\partial F_{wo_i}}{\partial v_i})^{(v)}$

$$(\frac{\partial F_{wo_i}}{\partial v_i})^{(v)} = [A_x H_o]_i^v [\frac{(b_o)_{i+1}^v - (b_o)_{i-1}^v}{2}] + [A_x b_o]_i^v [\frac{(H_o)_{i+1}^v - (H_o)_{i-1}^v}{2}] \tag{A3.8}$$

Derivatives of F_{wo} with respect to v_{i+1}: $(\frac{\partial F_{wo_i}}{\partial v_{i+1}})^{(v)}$

$$(\frac{\partial F_{wo_i}}{\partial v_{i+1}})^{(v)} = 0.5[A_x H_o b_o]_i^v \tag{A3.9}$$

Derivatives of F_{wo} with respect to v_{i-1}: $(\frac{\partial F_{wo_i}}{\partial v_{i-1}})^{(v)}$

$$(\frac{\partial F_{wo_i}}{\partial v_{i-1}})^{(v)} = -0.5[A_x H_o b_o]_i^v \tag{A3.10}$$

A3.2 Derivatives of Mass Conservation Residual Function (Gas) with respect to Wellbore Unknowns

The mass conservation residual function for the gas is shown in formula (4.17) in Chapter 4

$$F^{n+1}{}_{wg} = [A_x v H_g]_k^{n+1}[(b_g)_{k+1/2}^{n+1} - (b_g)_{k-1/2}^{n+1}] + [A_x H_g b_g]_k^{n+1}[(v)_{k+1/2}^{n+1} - (v)_{k-1/2}^{n+1}] + [A_x v b_g]_k^{n+1}[(H_g)_{k+1/2}^{n+1} - (H_g)_{k-1/2}^{n+1}]$$
$$+ R_{s,k}^{n+1} X_o + [A_x v H_o b_o]_k^{n+1}[(R_{s\,k+1/2}^{n+1}) - (R_{s\,k-1/2}^{n+1})]$$
$$+ \sum_{j=1}^{N\theta}\{[\frac{A_r K_r K_{rg}}{\mu_g B_g \Delta r}]_{1/2,j,k}^{n+1}(p_{g1,j,k}^{n+1} - p_{g-wb-k}^{n+1}) - [\frac{A_r K_r K_{rg}\gamma_g}{\mu_g B_g \Delta r}]_{1/2,j,k}^{n+1}(G_{1,j,k} - G_{wb-k})]\} + R_{s,k}^{n+1} q_{osc,k}^{n+1}$$
$$- [\frac{V_b}{\Delta t}]_k^{n+1}\{[(b_g H_g)_k^{n+1} - (b_g H_g)_k^{n}] + [(R_s b_o H_o)_k^{n+1} - (R_s b_o H_o)_k^{n}]\}$$

The derivatives of the above residual function with respect to wellbore unknowns, $P_{wb,k-1}^{n+1}, P_{wb,k}^{n+1}, P_{wb,k+1}^{n+1}, H_{o,k-1}^{n+1}, H_{o,k}^{n+1}, H_{o,k+1}^{n+1}, v_{m,k-1}^{n+1}, v_{m,k}^{n+1}, v_{m,k+1}^{n+1}$, are calculated in the following section to construct the wellbore part in the Jacobian matrix.

a) Derivatives of F_{wg} with respect to P_{wb}

Derivatives of F_{wg} with respect to $P_{wb\,i}$: $(\frac{\partial F_{wgi}}{\partial P_{wb\ i}})^{(v)}$

$$(\frac{\partial F_{wgi}}{\partial P_{wb\ i}})^{(v)} = [A_x(1.0 - H_o)]_i^v[\frac{(v\ )_{i+1}^v - (v\ )_{i-1}^v}{2}](\frac{\partial b_{gi}}{\partial P_{wb\ i}}) + [A_x v\ ]_i^v[\frac{(H_o)_{i-1}^v - (H_o)_{i+1}^v}{2}](\frac{\partial b_{gi}}{\partial P_{wb\ i}})$$
$$+ (\frac{\partial R_{si}}{\partial P_{wb\ i}})X_o + R_{si}\{[A_x H_o]_i^v[\frac{(v\ )_{i+1}^v - (v\ )_{i-1}^v}{2}](\frac{\partial b_{oi}}{\partial P_{wb\ i}}) + [A_x v\ ]_i^v[\frac{(H_o)_{i+1}^v - (H_o)_{i-1}^v}{2}](\frac{\partial b_{oi}}{\partial P_{wb\ i}})\}$$
$$+ [A_x v\ H_o]_i^v[\frac{(R_{s,i+1}^v) - (R_{s,i-1}^v)}{2}](\frac{\partial b_{oi}}{\partial P_{wb\ i}})$$
$$+ \sum_{j=1}^{N\theta}[-(\frac{A_r K_r K_{rg}}{\mu_g B_g \Delta r})_{i+1/2,j,k}^v] + \sum_{j=1}^{N\theta}[\frac{\partial T_{g,1/2,j,k}}{\partial P_{wb,k}}(p_{g1,j,k}^{n+1} - p_{o-wb-k}^{n+1} - G_{1,j,k} + G_{wb-k})]$$
$$+ R_{si}\sum_{j=1}^{N\theta}\{-[\frac{A_r K_r K_{ro}}{\mu_o B_o \Delta r}]_{i+1/2,j,k}^v\} + \sum_{j=1}^{N\theta}\{[\frac{A_r K_r K_{ro}}{\mu_o B_o \Delta r}]_{i+1/2,j,k}^v(p_{oi+1,j,k}^v - p_{o-wb-i}^v - G_{i+1,j,k} + G_{wb-i})]\}(\frac{\partial R_{si}}{\partial P_{wb\ i}})$$
$$- [\frac{V_b}{\Delta t}]_i^v\{(1.0 - H_o)_i^v(\frac{\partial b_{gi}}{\partial P_{wb\ i}}) + R_{si}H_{oi}(\frac{\partial b_{oi}}{\partial P_{wb\ i}}) + H_{oi}b_{oi}(\frac{\partial R_{si}}{\partial P_{wb\ i}})\}$$

(A3.11)

Derivatives of F_{wg} with respect to $P_{wb\,i+1}$: $(\frac{\partial F_{wgi}}{\partial P_{wb\ i+1}})^{(v)}$

$$(\frac{\partial F_{wgi}}{\partial P_{wb\ i+1}})^{(v)} = 0.5[A_x v\ (1.0 - H_o)]_i^v (\frac{\partial b_{gi+1}}{\partial P_{wb\ i+1}}) + 0.5R_{s,i}^v [A_x v\ H_o]_i^v (\frac{\partial b_{oi+1}}{\partial P_{wb\ i+1}}) + 0.5[A_x v\ H_o b_o]_i^v (\frac{\partial R_{si+1}}{\partial P_{wb\ i+1}})$$

(A3.12)

Derivatives of F_{wg} with respect to $P_{wb\,i-1}$: $(\frac{\partial F_{wgi}}{\partial P_{wb\ i-1}})^{(v)}$

$$(\frac{\partial F_{wgi}}{\partial P_{wb\ i-1}})^{(v)} = -[0.5A_x v\ (1.0 - H_o)]_i^v (\frac{\partial b_{gi-1}}{\partial P_{wb\ i-1}}) - [0.5R_{s,i}^v A_x v\ H_o]_i^v (\frac{\partial b_{oi-1}}{\partial P_{wb\ i-1}}) - [0.5A_x v\ H_o b_o]_i^v (\frac{\partial R_{si-1}}{\partial P_{wb\ i-1}})$$

(A3.13)

b) Derivatives of F_{wg} with respect to H_o

Derivatives of F_{wg} with respect to $H_{o\,i}$: $(\frac{\partial F_{wgi}}{\partial H_{o\ i}})^{(v)}$

$$(\frac{\partial F_{wgi}}{\partial H_{o\ i}})^{(v)} = [-A_x v\ ]_i^v [\frac{(b_g)_{i+1}^v - (b_g)_{i-1}^v}{2}] + [-A_x b_g]_i^{n+1} [\frac{(v\ )_{i+1}^v - (v\ )_{i-1}^v}{2}]$$
$$+ R_{s,i}^{n+1} [A_x v\ ]_i^v [\frac{(b_o)_{i+1}^v - (b_o)_{i-1}^v}{2}] + [A_x b_o]_i^v [\frac{(v\ )_{i+1}^v - (v\ )_{i-1}^v}{2}] + [A_x b_o v\ ]_i^v [\frac{(R_{s,i+1}^v) - (R_{s,i-1}^v)}{2}]$$
$$- [\frac{V_b}{\Delta t}]_i^v [(-b_g + R_s b_o)_i^v]$$

(A3.14)

Derivatives of F_{wg} with respect to $H_{o\,i+1}$: $(\frac{\partial F_{wgi}}{\partial H_{o\ i+1}})^{(v)}$

$$(\frac{\partial F_{wgi}}{\partial H_{o\ i+1}})^{(v)} = 0.5[-A_x v\ b_g]_i^v + 0.5R_{s,i}^{n+1} [A_x v\ b_o]_i^v \qquad \text{(A3.15)}$$

Derivatives of F_{wg} with respect to $H_{o\,i-1}$: $(\frac{\partial F_{wgi}}{\partial H_{o\ i-1}})^{(v)}$

$$(\frac{\partial F_{wgi}}{\partial H_{o\ i-1}})^{(v)} = 0.5[A_x v\ b_g]_i^v + 0.5R_{s,i}^{n+1} [-A_x v\ b_o]_i^v \qquad \text{(A3.16)}$$

c) Derivatives of F_{wg} with respect to v

Derivatives of F_{wg} with respect to v_i : $(\frac{\partial F_{wgi}}{\partial v_i})^{(v)}$

$$(\frac{\partial F_{wgi}}{\partial v_i})^{(v)} = [A_x H_g]_i^v [\frac{(b_g)_{i+1}^v - (b_g)_{i-1}^v}{2}] + [A_x b_g]_i^v [\frac{(H_g)_{i+1}^v - (H_g)_{i-1}^v}{2}]$$
$$+ R_{s,i}^v \{[A_x H_o]_i^v [\frac{(b_o)_{i+1}^v - (b_o)_{i-1}^v}{2}] + [A_x b_o]_i^v [\frac{(H_o)_{i+1}^v - (H_o)_{i-1}^v}{2}]\} + [A_x H_o b_o]_i^v [\frac{(R_s)_{i+1}^v - (R_s)_{i-1}^v}{2}]$$

(A3.17)

Derivatives of F_{wg} with respect to v_{i+1}: $(\frac{\partial F_{wgi}}{\partial v_{i+1}})^{(v)}$

$$(\frac{\partial F_{wgi}}{\partial v_{i+1}})^{(v)} = 0.5[A_x H_g b_g]_i^v + 0.5 R_{si} [A_x H_o b_o]_i^v \quad \text{(A3.18)}$$

Derivatives of F_{wg} with respect to v_{i-1}: $(\frac{\partial F_{wgi}}{\partial v_{i-1}})^{(v)}$

$$(\frac{\partial F_{wgi}}{\partial v_{i-1}})^{(v)} = -0.5[A_x H_g b_g]_i^v - 0.5 R_{si} [A_x H_o b_o]_i^v \quad \text{(A3.19)}$$

A3.3 Derivatives of Momentum Residual Function (Mixture) with Respect to Wellbore Unknowns

The mixture momentum residual function is the formula (4.18) in Chapter 4

$$F_{wm}^{n+1} = [A_x]_k^{n+1}[P_{k+1/2}^{n+1} - P_{k-1/2}^{n+1}] + [A_x \rho v]_k^{n+1}(v_{k+1/2}^{n+1} - v_{k-1/2}^{n+1}) - v_k^{n+1}[\rho_{osc} q_{osc} + \rho_{gsc}(q_{gsc-free} + R_s q_{osc})]_k^{n+1}$$
$$-(\frac{V_b \rho f v|v|}{2d})_k^{n+1} - [\frac{\rho V_b}{\Delta t}]_k^{n+1}[(v)_k^{n+1} - (v)_k^n]$$

(4.18)

The derivatives of the mixture momentum residual function with respect to wellbore unknowns, $P_{wb,k-1}^{n+1}, P_{wb,k}^{n+1}, P_{wb,k+1}^{n+1}, H_{o,k-1}^{n+1}, H_{o,k}^{n+1}, H_{o,k+1}^{n+1}, v_{m,k-1}^{n+1}, v_{m,k}^{n+1}, v_{m,k+1}^{n+1}$, are calculated in the following section to construct the wellbore part in the Jacobian matrix.

a) Derivatives of F_{wm} with respect to P_{wb}

Derivatives of F_{wm} with respect to $P_{wb\,i}$: $(\frac{\partial F_{wm_i}}{\partial P_{wb\ i}})^{(v)}$

$$(\frac{\partial F_{wm_i}}{\partial P_{wb\ i}})^{(v)} = [A_x v]_i^{n+1} \frac{(v_{i+1}^v - v_{i-1}^v)}{2}(\frac{\partial \rho}{\partial P_{wb\ i}}) - v_i^v \{\rho_{osc} \sum_{j=1}^{N\theta}[-(\frac{A_r K_r K_{ro}}{\mu_o B_o \Delta r})_{1/2,j,k}^v] + \rho_{gsc} \sum_{j=1}^{N\theta}[-(\frac{A_r K_r K_{rg}}{\mu_g B_g \Delta r})_{1/2,j,k}^v] + \rho_{gsc} R_s \sum_{i=1}^{N\theta}[-(\frac{A_r K_r K_{ro}}{\mu_o B_o \Delta r})_{1/2,j,k}^v] + \rho_{gsc} q_{osc} (\frac{\partial R_s}{\partial P_{wb\ i}})\} - (\frac{V_b f\,|v|\,v}{2d})_i^v (\frac{\partial \rho}{\partial P_{wb\ i}}) - (\frac{V_b}{\Delta t})_i^v (v_i^v - v_i^n)(\frac{\partial \rho}{\partial P_{wb\ i}})$$

(A3.20)

Derivatives of F_{wm} with respect to $P_{wb\,i+1}$: $(\frac{\partial F_{wm_i}}{\partial P_{wb\ i+1}})^{(v)}$

$$(\frac{\partial F_{wm_i}}{\partial P_{wb\ i+1}})^{(v)} = 0.5 A_x \quad \text{(A3.21)}$$

Derivatives of F_{wm} with respect to $P_{wb\,i-1}$: $(\frac{\partial F_{wm_i}}{\partial P_{wb\ i-1}})^{(v)}$

$$(\frac{\partial F_{wm_i}}{\partial P_{wb\ i-1}})^{(v)} = -0.5 A_x \quad \text{(A3.22)}$$

b) Derivatives of F_{wm} with respect to H_o

$$(\frac{\partial F_{wm_i}}{\partial H_{o\ i}})^{(v)} = (\frac{\partial F_{wm_i}}{\partial H_{o\ i+1}})^{(v)} = (\frac{\partial F_{wm_i}}{\partial H_{o\ i-1}})^{(v)} = 0 \quad \text{(A3.23)}$$

c) Derivatives of F_{wm} with respect to v

Derivatives of F_{wm} with respect to v_i : $(\frac{\partial F_{wm_i}}{\partial v_i})^{(v)}$

$$(\frac{\partial F_{wm_i}}{\partial v_i})^{(v)} = [A_x \rho]_i^v \frac{(v_{i+1}^v - v_{i-1}^v)}{2} - [\rho_{osc} q_{osc} + \rho_{gsc}(q_{gsc-free} + R_s q_{osc})]_i^v - (\frac{V_b \rho f v}{d})_i^v - (\frac{\rho V_b}{\Delta t})_i^v$$

(A3.24)

Derivatives of F_{wm} with respect to v_{i+1} : $(\frac{\partial F_{wm_i}}{\partial v_{i+1}})^{(v)}$

$$(\frac{\partial F_{wm_i}}{\partial v_{i+1}})^{(v)} = 0.5[A_x \rho v]_i^v \tag{A3.25}$$

Derivatives of F_{wm} with respect to v_{i-1} : $(\frac{\partial F_{wm_i}}{\partial v_{i-1}})^{(v)}$

$$(\frac{\partial F_{wm_i}}{\partial v_{i-1}})^{(v)} = [-0.5A_x \rho v]_i^v \tag{A3.26}$$

A3.4 Derivatives of Wellbore Residual Function with respect to Reservoir Unknowns

The derivatives of F_{wg}, F_{wo}, F_{wm} with respect to reservoir unknowns $P_{o,1,j,k}, S_{o,1,j,k}$ are also derived because of the connection between the reservoir and the wellbore at the sandface. The derivatives are actually related to the influx terms in the residual function.

a) Mass Conservation Residual Function (Oil)

Specified oil flow rate only means the total production flow rate is constant, but for each wellbore section, the oil flow rate is not constant. Therefore,

$$\frac{\partial Q_o}{\partial P_{o1,j,1}}, \frac{\partial Q_o}{\partial S_{o1,j,1}} \neq 0 \tag{A3.27}$$

Same for the bottom-hole pressure (BHP), constant BHP production scheme only refers to constant heel point pressure of the horizontal wellbore. Along the wellbore, pressure should be calculated together with flow rate by the coupled model.

$$\frac{\partial F_{wo}}{\partial P_{o1,j,k}} = \sum_{j=1}^{N_Q}[\frac{\partial Q_{oj}}{\partial P_{o1,j,k}}], \text{ and } \frac{\partial Q_{oj}}{\partial P_{o1,j,k}} = POT_{o1}\frac{\partial T_{o1/2,j,k}}{\partial P_{o1,j,k}} + T_{o1/2,j,k}\frac{\partial POT_{o1}}{\partial P_{o1,j,k}} \tag{A3.28}$$

$$\frac{\partial Q_{oj}}{\partial P_{o1,j,k}} = -POT_{o1} \cdot w_r \cdot T_{o1/2,j,k} (\frac{1}{\mu_{o1/2,j,k}} \frac{d\mu_{o1,j,k}}{dP_{o1,j,k}} - \frac{1}{b_{o1/2,j,k}} \frac{db_{o1,j,k}}{dP_{o1,j,k}}) + T_{o1/2,j,k} [1 - w_r \frac{d\gamma_{o1,j,k}}{dP_{o1,j,k}} (Z_c - Z_{1,j,k})]$$

(A3.29)

Where, $POT_{o1} = P_{o1,j,k} - P_{wf} - \gamma_{o1/2,j,1} (Z_c - Z_{1,j,k})$

$$w_r = \frac{r_{1/2,j,k} - r_{0,j,k}}{r_{1,j,k} - r_{0,j,k}} = \frac{r_w - 0}{r_{1,j,k} - 0} = \frac{r_w}{r_1}$$

$$\frac{\partial F_{wo}}{\partial S_{o1,j,i}} = \sum_{j=1}^{N\theta} [\frac{\partial Q_{oj}}{\partial S_{o1,j,i}}]$$

$$\frac{\partial Q_{oj}}{\partial S_{o1,j,k}} = POT_{o1} \frac{\partial T_{o1/2,j,k}}{\partial S_{o1,j,k}}$$

b) Mass Conservation Residual Function (Gas)

$$Q_g = (\frac{\lambda_{g1,j,k}}{\lambda_{o1,j,k}} + R_{s1,j,k}) \cdot Q_o$$

$$Q_g = [(\frac{k_{rg}\mu_o b_g}{k_{ro}\mu_g b_o})_{1,j,k} + R_{s1,j,k}] \cdot Q_o \quad (j=1 \text{ to } n\theta)$$

$$\frac{\partial F_{wg}}{\partial P_{o1,j,k}} = \sum_{j=1}^{N\theta} [\frac{\partial Q_{gj}}{\partial P_{o1,j,k}}]$$ Thus:

$$\frac{\partial Q_g}{\partial P_{o1,j,k}} = Q_o \cdot [\frac{k_{rg}\mu_o b_g}{k_{ro}\mu_g b_o} (\frac{1}{b_g} \frac{db_g}{dP_{o1,j,k}} + \frac{1}{\mu_o} \frac{d\mu_o}{dP_{o1,j,k}} - \frac{1}{\mu_g} \frac{d\mu_g}{dP_{o1,j,k}} - \frac{1}{b_o} \frac{db_o}{dP_{o1,j,k}}) + \frac{dR_s}{dP_{o1,j,k}}]_{1,j,k}$$
$$+ (\frac{k_{rg}\mu_o b_g}{k_{ro}\mu_g b_o} + R_s)_{1,j,k} \frac{\partial Q_o}{\partial P_{o1,j,k}}$$

(A3.30)

$$\frac{\partial F_{wg}}{\partial S_{o1,j,i}} = \sum_{j=1}^{N\theta} [\frac{\partial Q_{gj}}{\partial S_{o1,j,i}}]$$ Thus:

$$\frac{\partial Q_g}{\partial S_{o1,j,k}} = Q_o \cdot \frac{\mu_o b_g}{k_{ro}\mu_g b_o} (\frac{dk_{rg}}{dS_{o1,j,k}} - \frac{k_{rg}}{k_{ro}} \frac{dk_{ro}}{dS_{o1,j,k}})_{1,j,k} + (\frac{k_{rg}}{k_{ro}} \frac{\mu_o}{\mu_g} \frac{b_g}{b_o} + R_s)_{1,j,k} \frac{\partial Q_o}{\partial S_{o1,j,k}}$$

(A3.31)

c) For the Mixture Momentum Equation

$$\frac{\partial F_{wm}}{\partial P_{o1,j,k}} = -v_i [\rho_{osc} \Sigma \frac{\partial Q_{oj}}{\partial P_{o1,j,k}} + \rho_{gsc} \Sigma \frac{\partial Q_{gj}}{\partial P_{o1,j,k}}]$$

(A3.32)

$$\frac{\partial F_{wm}}{\partial S_{o1,j,k}} = -v_i [\rho_{osc} \Sigma \frac{\partial Q_{oj}}{\partial S_{o1,j,k}} + \rho_{gsc} \Sigma \frac{\partial Q_{gj}}{\partial S_{o1,j,k}}]$$

(A3.33)

Where, $\frac{\partial Q_{oa}}{\partial P_{o1,j,k}}$, $\frac{\partial Q_{ga}}{\partial P_{o1,j,k}}$, $\frac{\partial Q_{oa}}{\partial S_{o1,j,k}}$, $\frac{\partial Q_{ga}}{\partial S_{o1,j,k}}$

APPENDIX 4: SOURCE CODE

The FORTRAN™ and MATLAB™ source code, as well as the input and output files for a case study, can be found in the attached CD.