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EXTENDED STRUCTURES IN QUANTUM FIELD THEORY

by

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ABSTRACT

A detailed study of extended structures in quantum systems is presented. This is first done by using perturbation theory where the quantal Hilbert space of such a system is constructed as the direct product of a quantum mechanical realization of the quantum coordinate and its canonical conjugate, the total momentum, in the Schrödinger picture and the Fock space of the physical particle fields. A systematic scheme for perturbative computation and renormalization is developed.

Then more general aspects of such systems are investigated in the context of the asymptotic condition. The algebra of the Poincaré group generators, together with the asymptotic condition, lead to the form of the Hamiltonian. It is shown that the appearance of the quantum coordinate, the asymptotic condition and the form of the Hamiltonian are consistent with the requirement that the Poincaré group generators generate Poincaré group transformations in the asymptotic region.

Finally, an application of the boson method to quantum electrodynamics in solids is presented. There, the overall macroscopic properties of the extended structure dominate observable phenomena. Macroscopic equations which govern the macroscopic properties of the system are derived. The relation to linear response theory is explored and the classical Maxwell equations are derived.

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CHAPTER 1

INTRODUCTION

In the past decade a substantial effort has been devoted to the extension of the methodology of quantum field theory to the analysis of systems with extended structure. Systems of this kind possess spatially inhomogeneous ground states and are said to contain extended objects. These extended objects are embedded in the quantum system and interact with the quanta. Their description in the context of quantum field theory is an important problem in modern physics.

A chief ingredient in the study of extended objects is the fact that many classical nonlinear field equations possess soliton solutions.[77] These are either dynamically or topologically stable nontrivial solutions of the nonlinear field equations. They are stable under small time-dependent perturbations. It is of considerable interest to study the role of these solutions in the corresponding quantum field theory.[76,78-81]

In the study of elementary particles, the development of dual models[68-70] led to attempts to incorporate extended objects into quantum field theory. These efforts were fueled by the discovery of the string,[71] monopole[72] and instanton[73] solutions in classical gauge theories. However, the role played by extended objects at the level of elementary particles is as yet not well understood.

In the study of condensed matter, quantum field theoretic methods are directed at the analysis of systems consisting of large numbers of interacting particles. The homogeneous ground states of many of these systems exhibit a high degree of order. In quantum field theory these ordered states find a natural description through the dynamical rearrangement of symmetry which accompanies spontaneous symmetry breakdown.[74,75] Besides the homogeneous phases, many extended objects may appear.[76] Point defects, grain boundaries and dislocations in crystals and vortex lines in superconductors are examples. Furthermore, for given external fields, the macroscopic electromagnetic fields and currents in a solid may be regarded as extended structures. These are all examples of extended systems which are embedded in and coexist with quantum systems. The study of systems such as these requires the development of a consistent methodology for the analysis of extended objects in quantum field theory.

The macroscopic properties of a quantum system are governed by the dynamics of extended structures within the system. In general, these dynamics involve the details of the interaction between extended objects and the quanta of the system.

One significant feature of these systems is the appearance of certain quantities of microscopic origin as basic observable parameters which govern the behavior of macroscopic extended systems. An example is the conductivity of a metal which is defined as the ratio of macroscopic electric current to macroscopic electric field but is also a quantity which is determined microscopically, through the current-current correlation functions. This type of relationship is known in linear response theory.[27]

On a finer scale, the position and overall shape of an extended object may actually be quantum mechanical quantities; that is, the fact that the extended object is constructed from a large condensation of quanta and is embedded in a quantum system means that the object itself is a quantum mechanical quantity. When an extended object is small, the quantum fluctuation of its position may be observable. When the extended object is large, to a good approximation, it may behave classically.

In the quantum field theoretic analysis of extended systems, there are three important steps. The first is the derivation of classical field equations which the quantities which characterize the extended structure must satisfy. Then, there is the solution of these equations with given boundary conditions. Finally, it is necessary to study the interaction between the extended system and the quanta.

In the second step, the term boundary condition is used in a general sense. It may mean the specification of the classical field configurations at large space-or time-like distances from extended objects in order to distinguish the topological class of the solution. In the study of condensed matter, it may also mean that, given certain external fields, the free energy of the system must be minimized.

Typically, in the case when elementary boson fields are present, a classical solution of the classical Euler equations corresponding to the Heisenberg field equations is regarded as a first approximation (the tree approximation) to the vacuum matrix elements of the boson fields in an expansion in Planck's constant, $\hbar$ (a loop expansion). Quantum corrections to these quantities are to be added perturbatively. However, perturbative calculation immediately encounters a technical problem, the so-called "zero mode" problem. The Heisenberg field equations for the boson fields, when linearized about the classical solutions, possess zero frequency eigensolutions, the "zero modes".[8]

Since they have zero eigenvalues, the parts of the Green functions corresponding to these solutions are not well defined and it is therefore difficult to integrate the inhomogeneous differential equations of perturbation theory. This reflects the fact that these zero modes are not observed as excitation modes. The zero modes arise when the classical solution is noninvariant under some symmetry transformations which leave the Heisenberg field equations invariant. Though zero modes are not observed as excitations, the quantum fluctuations associated with them can be observable. Therefore, each independent zero mode which appears indicates a quantum mechanical degree of freedom of the extended object.

One method which has been proposed for dealing with the zero mode problem is the collective coordinate method.[21-24,82-87] This method anticipates the appearance of the zero modes by performing a canonical transformation which elevates the symmetry parameters which give rise to the zero modes to the status of dynamical variables. Thus, the quantum mechanical degrees of freedom of the extended object appear explicitly in an effective Lagrangian. The introduction of the parameters as dynamical degrees of freedom requires the imposition of certain constraints on the field operators. These constraints can be arranged in such a way as to eliminate the components of the fields which are proportional to the zero mode wavefunctions. Perturbative calculation is then possible. This method has been used for the computation of quantum corrections to the ground state energy and soliton-soliton scattering in some model quantum field theories.[91-94]

However, the collective coordinate method, while providing a useful computational tool, has a major shortcoming in that it leaves the properties of the quantum states obscure. There are many phenomena, such as physical particle-extended object scattering and particle-particle reactions in the presence of an extended object, whose analysis requires knowledge of the physical Hilbert space. In order to construct the physical Hilbert space it is necessary to work in the operator formalism of quantum field theory.

A systematic method for describing extended systems has been investigated within the orthodox formalism of quantum field theory. In this method (the boson method),[28,37,88,89] extended objects are created by a boson condensation process. The Heisenberg equation is first solved for the case of a spatially homogeneous ground state. Then certain extended objects are created in this system by means of the boson transformation which is the mathematical expression of a boson condensation. It has been shown that, in the tree approximation, the boson transformation leads to the corresponding classical field equation and their soliton solutions.[7]

Consider a quantum field theory consisting of a set of Heisenberg fields $\{\psi_i\}$ satisfying the Heisenberg field equations

$$\Lambda_i(\partial)\psi_i(x) = F_i[\psi] \quad . \tag{1.1}$$

Suppose that this quantum field theory is consistently realized in a Fock space of the physical fields $\{\rho^0_\alpha(x),\ \varphi^0_\kappa(x)\}$ which satisfy the equations

$$\Lambda_\alpha(\partial)\rho^0_\kappa(x) = 0 \quad , \qquad \Lambda_\kappa(\partial)\varphi^0_\kappa(x) = 0 \tag{1.2}$$

and ρ^0_α are boson fields and φ^0_κ are fermions. Furthermore, it is assumed that the field equations in (1.1) and (1.2) exhibit space and time-translational invariance; that is, that this realization represents a homogeneous phase of the quantum system. A solution of this quantum field theory is given when all matrix elements of the Heisenberg fields in the Fock space of the physical fields are given. This information is expressed compactly in an expression known as the dynamical map:

$$\psi_i(x) = \psi_i[x;\ \rho_\alpha^o,\ \varphi_\kappa^o] \quad . \tag{1.3}$$

The boson transformation theorem[89] states that the Heisenberg fields

$$\psi_i^f(x) = \psi_i[x;\ \rho_\alpha^o + f_\alpha,\ \varphi_\kappa^o] \tag{1.4}$$

satisfies the field equation

$$\Lambda_i(\partial)\psi_i^f(x) = F_i[\psi^f(x)] \tag{1.5}$$

when $f_\alpha(x)$ are c-number functions satisfying

$$\lambda_\alpha(\partial)f_\alpha(x) = 0 \quad . \tag{1.6}$$

The operator translation

$$\rho_\alpha^o(x) \rightarrow \rho_\alpha^o(x) + f_\alpha(x) \tag{1.7}$$

is called the boson transformation. It corresponds to a condensation of the boson, $\rho_\alpha^o(x)$. The Heisenberg fields, $\psi_i^f(x)$, describe the quantum system with extended structure.

The vacuum expectation values of $\psi_i^f(x)$ in the Fock space of $\{\rho_\alpha^o(x),\ \varphi_\kappa^o(x)\}$ are called the order parameters. They are sums of all possible connections, through the many-point Green functions, of the Heisenberg fields $\psi_i(x)$ with the classical functions, $f_\alpha(x)$. In the approximation where only the quantum tree graphs contribute to the Green functions, this classical field obeys the classical Euler equation. Thus, different choices of $f_\alpha(x)$ lead, in the tree approximation, to different solutions of the classical field equations.

In general the function $f_\alpha(x)$ need not be Fourier transformable but may carry certain singularities.[90,91] In fact for a relativistic theory, if $f_\alpha(x)$ are static, they must necessarily have either topological or divergent singularities (for further reference see reference 91).

The physical fields $\rho^0_\alpha(x)$ and $\varphi^0_\kappa(x)$ are modified by their interaction with the extended objects. In the presence of extended objects, the physical fields are, in fact, complicated functionals of the boson transformation functions $f_\alpha(x)$. They consist of infinite summations of the fields $\rho^0_\alpha(x)$ and $\varphi^0_\kappa(x)$ interacting through the many-point Green functions with the classical fields $f_\alpha(x)$. In general, besides scattering states, there appear bound states of the particles to the extended object and also quantum mechanical modes associated with the translation of the system. The latter mode is known as the quantum coordinate. Its appearance is a natural result of the canonical commutation relations of the Heisenberg fields.[3] Quantum coordinates are the quantum mechanical degrees of freedom corresponding to the zero modes.

The quantal Hilbert space of the system with extended structure is therefore different from the Fock space of the fields $\{\rho^0_\alpha(x), \varphi^0_\kappa(x)\}$. An exploration of the structure of the physical Hilbert space and some features of the solution of the quantum field theory within this Hilbert space is the subject of the next three chapters. For simplicity, these chapters will examine a relativistic one-component boson model in 1+1-dimensions. In this model there is only one quantum coordinate, that corresponding to translations.

In chapter II, perturbation theory is used to calculate the first few terms of the dynamical map in the tree approximation. The quantum coordinate always appears in the combination x-Q with the spatial coordinate x. The total momentum acts as the canonical conjugate of the quantum coordinate. The dynamical map is used to calculate the Hamiltonian and total momentum of the system. Then the canonical commutation relations are used in order to determine the commutation relations between the physical particle fields and the quantum coordinate. It is found that the quantum coordinate can be specified without interference from the physical particles at one particular time. This necessitates the use of the Schrödinger picture in its quantization. The Hilbert space of the system is constructed as the direct product of the quantum mechanical realization of the quantum coordinate and its canonical conjugate, the total momentum, in the Schrödinger picture and the Fock-like representation of the physical particle fields.

In chapter III, this scheme is generalized to include quantum corrections. The computational scheme is outlined and an example of one-loop renormalization is given.

In chapter IV, an outline of the formulation of quantum field theory in terms of the asymptotic condition is given. The existence of the quantum coordinate is an important ingredient in the definition of the asymptotic region. The appearance of the total momentum and the quantum coordinate in the generators of the Poincaré group is analyzed. This leads to the form of the asymptotic Hamiltonian. Then it is shown that the asymptotic condition, the asymptotic Hamiltonian and the

Poincaré algebra are consistent. This completes the detailed study of the dynamics of extended objects. Eventually the methods developed here must be generalized to more complicated and more realistic systems.

In chapter V, the boson method is used to study the macroscopic electromagnetic properties of solids. First a detailed picture of non-relativistic quantum electrodynamics for such systems is presented. Then macroscopic equations which govern the "classical" behavior of extended systems and macroscopic Maxwell equations are derived. In this analysis, interactions between extended structures and quanta are ignored. Various observable parameters, such as the dielectric constant and electric conductivity are identified.

CHAPTER II

A PERTURBATIVE LOOK AT EXTENDED OBJECT DYNAMICS I: THE TREE APPROXIMATION

In order to get an intuitive feeling for the structure of a quantum field theory with an extended object it is useful to explore the theory by perturbative calculation. Of particular importance is a careful evaluation of the role of quantum coordinates. In this chapter, the consideration is limited to an extended object which is assumed to be a topological soliton in a 1+1-dimensional space-time. There, the only quantum coordinate which appears is Q, corresponding to spatial translations. The goal of this work is to derive certain basic general properties of the quantum system with an extended object which should survive in other more general theories.

At first, only the tree approximation is considered. The problem of inserting the quantum corrections will be discussed once the structure of the tree approximation is well understood.

At the outset, an important question is whether the quantum coordinate Q(t) and the free physical fields in the presence of an extended object can be chosen as independent dynamical variables at all times. The quantal Hilbert space of the system must be chosen as a direct product of the Fock-like space of the physical particles and some quantum-mechanical realization of the quantum coordinate and its canonical conjugate, the canonical momentum in a representation where the two subspaces are independent.

An analysis of the structure of quantum field theory with extended objects using the Poincaré group[2] has indicated that the quantum coordinate, Q(t), is indeed not independent of the physical particle creation and annihilation operators at all times.

Briefly speaking, the commutation relations and the Heisenberg equations of the quantum field theory lead to the conservation of the generators of space-time transformations as well as their algebra. In the presence of an extended object, Poincaré symmetry is not manifest. It must be recovered through the presence of the quantum coordinate, Q, and its time derivatives. This fact provides much information on the way in which the quantum coordinate must appear in the Heisenberg fields. In this context, the concept of "c-q transmutation" was introduced: the dynamics together with the canonical formalism require the presence of certain combinations of the quantum mechanical operators and the space-time coordinates.

For the case of a relativistic scalar field in a 1+1-dimensional space-time, reference 2 arrived at the following form the the Hamiltonian

$$H = \frac{m}{\sqrt{1-\dot{Q}^2}} + \sqrt{1-\dot{Q}^2}\;\tilde{H}_0 + \ldots \tag{2.1}$$

where $\tilde{H}_0$ is the free Hamiltonian of the physical fields and $\dot{Q}$ is the time derivative of Q:

$$[Q, H] = i\dot{Q} \tag{2.2}$$

This result indicated that the operator Q(t) was not independent of the physical fields at all times.

The root of this complication lies in the fact that the wave-functions of the physical particles take into account the dynamics of the interaction between the physical particles and the extended object through the self-consistent potential[3] but the possibility that the extended object may recoil is not considered (the position of the self-consistent potential is fixed). However, if the underlying dynamics have translational symmetry, the recoil of the object under scattering by a quantum particle must be recovered. The mechanism for this recovery of the kinematics is the appearance of the quantum coordinate.

The free Hamiltonian of the physical particles, $\tilde{H}_0$, generates the time evolution due to interactions between the particles and the extended object assuming that the extended object does not recoil. As the time evolution proceeds, however, momentum is transferred to the extended object. This requires terms in the full Hamiltonian which mix $\dot{Q}$ and $\tilde{H}_0$. The fact that this effect is kinematical is the reason for its appearance through an analysis of the Poincaré symmetry.

In the following, the method of perturbation theory is used to explore this structure. Much of what is said here can be found in reference 4.

The coupled differential equations of the perturbative scheme have been derived in reference 1. They must be integrated in a way which respects certain boundary conditions and then solved by iteration.

However this procedure immediately encounters a technical complication known as the "zero mode" problem.[8] Because of the overall translational invariance of the theory the homogeneous equation has a discrete solution with zero frequency which corresponds to translations of the extended object. This solution is called the "zero mode" or "translation mode" and the quantum mechanical operator corresponding to it is the quantum coordinate. The part of the Green function corresponding to the translation mode is not well defined and the perturbative equations can only be integrated directly for components orthogonal to this solution. It will be shown that the components of the perturbative equations which are proportional to the translation mode can be determined to within the addition of solutions of the homogeneous equation. Any solutions of the homogeneous equation which may be added when integrating the perturbative equations must be determined from the boundary conditions.

It is, however, not obvious from the outset what boundary conditions are compatible with the canonical commutation relations. It was shown in reference 3 that the presence of the quantum coordinate is required by the commutation relations. It is shown in appendix A that, if $\dot{Q} = 0$, iteration of the perturbative equations leads to Q appearing everywhere in the combination $x - Q$ with the explicit spatial variable x. Therefore, it seems reasonable to impose boundary conditions which lead to Q always appearing in the combination $x - Q$ even when $\dot{Q} \neq 0$.

This specifies the way in which Q and $\dot{Q}$ must appear. All other boundary conditions are then determined by requiring consistency with the canonical commutation relation.

1. THE DYNAMICAL MAP

Consider a one-component boson field in (1+1)-dimensions satisfying the Heisenberg field equation

$$(\partial^2 + m^2)\psi(x) = F[\psi(x)] \quad . \tag{2.3}$$

It is assumed that this equation, together with the equal-time commutation relation

$$[\psi(x), \dot{\psi}(y)]_{x^o=y^o} = i\,\delta(x-y) \qquad (2.4)$$

can be realized in the Fock space of a single free boson field $\rho_o(x)$ which satisfies the field equation

$$(\partial^2 + m^2)\rho_o(x) = 0 \quad . \qquad (2.5)$$

Then the boson transformation $\rho_o(x) \to \rho_o(x) + f(x)$ leads to the field equation

$$(\partial^2 + m^2)\psi^f(x) = F[\psi^f(x)] \quad . \qquad (2.6)$$

The following is a study of the solutions of this equation in the tree approximation.

The vacuum expectation value of $\psi^f(x)$ in the Fock space of $\rho_o(x)$ is called the order parameter. It is the sum of all connections through the many-point Green's functions of the Heisenberg field $\psi(x)$ with the classical function f(x). For purposes which will be clear later, the order parameter will be denoted by $\psi_{-1}(x)$. One can obtain the n-particle term in the dynamical map of $\psi^f(x)$ by removing n of the functions f(x) from the order parameter and replacing them with a normal product of n of the physical fields $\rho_o(x)$:

$$\psi_{n-1}(x) = :\delta_f^n \psi_{-1}(x): \qquad (2.7)$$

where

$$\delta_f = \int d^2y\ \rho_o(y)\,\frac{\delta}{\delta f(y)} \quad . \qquad (2.8)$$

It is useful to consider a power counting parameter λ. Then

$$\psi_\lambda^f(x) = \sum_{n=-1}^{\infty} \lambda^n\ \psi_n(x) \quad . \qquad (2.9)$$

In the tree approximation, the order parameter satisfies the equation

$$(\partial^2 + m^2)\psi_{-1}(x) = F[\psi_{-1}(x)] \quad . \tag{2.10}$$

Equation (2.6) leads to

$$(\partial^2 + m^2)\psi_n(x) = \delta_f^{n+1} \; F[\psi_{-1}(x)] \tag{2.11}$$

or

$$(\partial^2 + m^2)\psi_n(x) = \sum \frac{1}{\ell!} F_\ell[\psi_{-1}(x)]\psi_{\alpha_1}(x)\ldots\psi_{\alpha_\ell}(x) \tag{2.12}$$

where $\alpha_1+\ldots+\alpha_\ell + \ell = n + 1$; $\alpha_1,\ldots,\alpha_\ell \geq 0$ and

$$F_\ell[\psi_{-1}(x)] = \frac{\partial^\ell \, F[\psi_{-1}(x)]}{\partial \, \psi_{-1}(x)^\ell} \quad . \tag{2.13}$$

The right hand side of equation (2.12) contains a term which is linear in $\psi_n(x)$. When this term is subtracted from each side of the equation, the right hand side contains components of order strictly less than n.

$$\{\partial^2 + m^2 - F_1[\psi_{-1}(x)]\} \; \psi_n(x) = \sum \frac{1}{\ell!} F_\ell[\psi_{-1}(x)]\psi_{\alpha_1}(x)\ldots\psi_{\alpha_\ell}(x) \tag{2.14}$$

where $\ell \geq 2$, $n \geq 0$. Equation (2.14) is supplemented by the classical field equation (2.10).

It is assumed that $\psi_{-1}(x)$ behaves as $x \to \pm\infty$ in such a way that $F_1[\psi_{-1}(x)] \to 0$ and $\psi_{-1}'(x) \to 0$ faster than any polynomial. We also assume that $\psi_{-1}(x)$ is static.

The quantum field $\psi_0(x)$ satisfies the linear homogeneous equation

$$\{\partial^2 + m^2 - F_1[\psi_{-1}(x)]\}\psi_0(x) = 0 \tag{2.15}$$

possessing the well-known solution

$$\psi_0(x) = - Q\, \psi'_{-1}(x) + \tilde{\psi}_0(x) \qquad (2.16)$$

where the prime denotes differentiation by the spatial coordinate. The operator Q is the quantum coordinate and $\tilde{\psi}_0(x)$ is written as

$$\tilde{\psi}_0(x) = \sum_i \frac{1}{\sqrt{2\omega_i}} \{u_i(x) e^{-i\omega_i t} \alpha_i + u_i^*(x) e^{i\omega_i t} \alpha_i^\dagger\}$$

$$+ \frac{1}{\sqrt{2\pi}} \int \frac{dk}{\sqrt{2\omega_k}} \{u_k(x) e^{-i\omega_k t} \alpha_k + u_k^*(x) e^{i\omega_k t} \alpha_k^\dagger\} \qquad (2.17)$$

where $u_i(x)$, ω_i and $u_k(x)$, ω_k are the bound state and scattering state wavefunctions and eigenfrequencies respectively of equation (2.15). Note that Q depends on time, t. The function $\psi'_{-1}(x)$ is commonly referred to as the zero mode or translation mode wavefunction.

<u>Definition</u>: $\tilde{\psi}(x) = \psi(x)$ when $Q = \dot{Q} = 0$.

The field $\tilde{\psi}(x)$ obviously satisfies the same field equation as $\psi(x)$. In appendix A, it is shown that when $\dot{Q}$ and all higher time derivatives are zero,

$$\psi(x, t) = \tilde{\psi}(x - \lambda Q, t) \qquad (2.18)$$

Since the translation mode wavefunction is orthogonal to the wavefunctions of the quantum states associated with $\tilde{\psi}_0(x)$, it follows that

$$Q = - \frac{1}{M} \int dx\, \psi'_{-1}(x)\, \psi_0(x) \qquad (2.19)$$

where

$$M = \int dx\, \psi'_{-1}(x)\, \psi'_{-1}(x) \qquad (2.20)$$

Since $\psi_{-1}(x)$ is time-dependent, equation (2.16) leads to

$$\ddot{Q} = -\frac{1}{M}\int dx\ \psi'_{-1}(x)\ \ddot{\psi}_0(x)$$

which, combined with equation (2.9) gives

$$\ddot{Q} = -\frac{1}{M}\int dx\ \psi'_{-1}(x)\{\ \frac{\partial^2}{\partial x^2} - m^2 + F_1[\psi_{-1}(x)]\}\ \psi_0(x)$$

$$= -\frac{1}{M}\int dx\{\psi'_{-1}(x)\psi''_0(x) - \psi'''_{-1}(x)\ \psi_0(x)\}$$

$$\ddot{Q} = 0 \qquad (2.21)$$

The remaining task is to determine how the presence of $\dot{Q}$ modifies equation (2.18). A solution of this problem has already been given in references 1 and 2. There it was shown that the effect of $\dot{Q}$ is taken into account by the replacements

$$t \rightarrow T = \sqrt{1-\dot{Q}^2}\ t - \frac{\dot{Q}}{\sqrt{1-\dot{Q}^2}}\ (x - Q)$$

$$x \rightarrow X = \frac{1}{\sqrt{1-\dot{Q}^2}}\ (x - Q) \qquad (2.22)$$

or

$$\psi(x,\ t) = \tilde{\psi}(X,\ T) \qquad (2.23)$$

Here, X and T are called the generalized coordinates.

The following is devoted to a further exploration of the perturbative development of equation (2.23).

Consider the field equations corresponding to the first few orders of equation (2.14):

$$\{\partial^2 + m^2 - F_1[\psi_{-1}(x)]\}\ \psi_1(x) = F_2[\psi_{-1}(x)]\ \psi_0^2(x)/2! \qquad (2.24)$$

$$\{\partial^2 + m^2 - F_1[\psi_{-1}(x)]\}\psi_2(x) = F_3[\psi_{-1}(x)]\ \psi_0^3(x)/3!$$

$$+ F_2[\ \psi_{-1}(x)]\psi_0(x)\ \psi_1(x) \quad . \qquad (2.25)$$

The solution of these equations must be consistent with the replacement $x \to x - Q$. Substitution of equation (2.16) into (2.24) leads to

$$\{\partial^2 + m^2 - F_1[\psi_{-1}(x)]\}\psi_1(x) = \frac{\dot{Q}^2}{2!} F_2[\psi_{-1}(x)]\ \psi_{-1}'(x)^2$$

$$- \dot{Q}\ F_2[\psi_{-1}(x)]\ \psi_{-1}'(x)\tilde{\psi}_0(x) + F_2[\psi_{-1}(x)]\ \tilde{\psi}_0^2(x)/2!$$

$$= \frac{Q^2}{2!}\ \{F[\psi_{-1}(x)]'' - F_1[\psi_{-1}(x)]\ \psi_{-1}''(x)\} - Q\{(F_1[\psi_{-1}(x)]\ \tilde{\psi}_0(x))'$$

$$- F_1[\psi_{-1}(x)]\ \tilde{\psi}_0'(x)\} + \{\partial^2 + m^2 - F_1[\psi_{-1}(x)]\}\ \tilde{\psi}_1(x)$$

$$= \frac{Q^2}{2!}\ \{\partial^2 + m^2 - F_1[\psi_{-1}(x)]\}\ \psi_{-1}''(x) - Q\ \{\partial^2 + m^2 - F_1[\psi_{-1}(x)]\}\ \tilde{\psi}_0'(x)$$

$$+ \{\partial^2 + m^2 - F_1[\psi_{-1}(x)]\}\ \tilde{\psi}_1(x)$$

$$= \{\partial^2 + m^2 - F_1[\psi_{-1}(x)]\}\ \{\ \frac{Q^2}{2!}\ \psi_{-1}''(x) - Q\ \tilde{\psi}_0'(x) + \tilde{\psi}_1(x)\}$$

$$- \dot{Q}^2\ \psi_{-1}''(x) + 2\dot{Q}\ \dot{\tilde{\psi}}_0'(x) \quad . \qquad (2.26)$$

Using the relations

$$\{\partial^2 + m^2 - F_1[\psi_{-1}(x)]\}\ x\psi_{-1}'(x) = -\ 2\ \psi_{-1}''(x) \qquad (2.27)$$

and

$$\{\partial^2 + m^2 - F_1[\psi_{-1}(x)]\}x\dot{\tilde{\psi}}_0(x) = -\ 2\ \dot{\tilde{\psi}}_0'(x) \qquad (2.28)$$

equation (2.26) can be written as

$$\{\partial^2 + m^2 - F_1[\psi_{-1}(x)]\}\ \psi_1(x) = \{\partial^2 + m^2 - F_1[\psi_{-1}(x)]\}\ \{\frac{Q^2}{2!}\ \psi''_{-1}(x) + \frac{\dot{Q}^2}{2!}\ x\ \psi'_{-1}(x) - Q\ \tilde{\psi}'_0(x) - \dot{Q}x\ \dot{\tilde{\psi}}_0(x) + \tilde{\psi}_1(x)\}$$

and therefore

$$\psi_1(x) = \frac{Q^2}{2!}\ \psi''_{-1}(x) + \frac{\dot{Q}^2}{2!}\ x\ \psi'_{-1}(x) - Q\ \tilde{\psi}'_0(x) - \dot{Q}x\ \dot{\tilde{\psi}}_0(x) + \tilde{\psi}_1(x) \qquad (2.29)$$

where the possible contribution of a solution of the linear homogeneous equation is ignored.

Now, consider (2.25). A series of manipulations similar to those used in deriving equation (2.26) leads to

$$\{\partial^2 + m^2 - F_1[\psi_{-1}(x)]\}\ \psi_2(x) = \{\partial^2 + m^2 - F_1[\psi_{-1}(x)]\}\ \{- \frac{Q^3}{3!}\ \psi'''_{-1}(x) - Q\ \frac{\dot{Q}^2}{2!}\ x\ \psi''_{-1}(x) + \frac{Q^2}{2!}\ \tilde{\psi}''_0(x) + Q\dot{Q}x\ \dot{\tilde{\psi}}'_0(x) + \frac{\dot{Q}^2}{2!}\ x\ \tilde{\psi}'_0(x) - Q\ \tilde{\psi}'_1(x) - \dot{Q}x\ \dot{\tilde{\psi}}_1(x) + \tilde{\psi}_2(x)\} - 2\dot{Q}^2 x\ \ddot{\tilde{\psi}}'_0(x) \qquad (2.30)$$

Use of the identities

$$\{\partial^2 + m^2 - F_1[\psi_{-1}(x)]\}\ x^2\ \ddot{\tilde{\psi}}_0(x) = 4x\ \ddot{\tilde{\psi}}'_0(x) - 2\ \ddot{\tilde{\psi}}_0(x) \qquad (2.31)$$

and

$$\{\partial^2 + m^2 - F_1[\psi_{-1}(x)]\}\ t\ \dot{\tilde{\psi}}_0(x) = 2\ \ddot{\tilde{\psi}}_0(x)$$

leads to

$$\psi_2(x) = -\frac{Q^3}{3!}\psi'''_{-1}(x) - Q\frac{\dot{Q}^2}{2!}x\,\psi''_{-1}(x) + \frac{Q^2}{2!}\tilde{\psi}''_0(x)$$

$$+ Q\dot{Q}x\,\dot{\tilde{\psi}}'_0(x) + \frac{\dot{Q}^2}{2!}x\,\tilde{\psi}'_0(x) + \frac{\dot{Q}^2}{2!}x^2\,\ddot{\tilde{\psi}}_0(x)$$

$$+ \frac{\dot{Q}^2}{2!}t\,\dot{\tilde{\psi}}_0(x) - \dot{Q}x\,\dot{\tilde{\psi}}_1(x) - Q\,\tilde{\psi}'_1(x) + \tilde{\psi}_2(x) \quad , \qquad (2.33)$$

where the contribution from a solution of the linear homogeneous equation is ignored. However, equation (2.33) does not satisfy

$$\frac{\partial}{\partial x}\psi_1(x) = -\frac{\partial}{\partial Q}\psi_2(x) \qquad (2.34)$$

as is required by the replacement $x \to x - \lambda Q$. The difference appears through the explicit x-dependence of $\psi_1(x)$:

$$\psi'_1(x) = -\frac{\partial}{\partial Q}\psi_2(x) + \frac{\dot{Q}^2}{2!}\psi'_{-1}(x) - \dot{Q}\,\dot{\tilde{\psi}}_0(x) \quad . \qquad (2.35)$$

This difference must be remedied by introducing into $\psi_2(x)$ appropriate terms which satisfy the linear homogeneous field equation. The first term on the right-hand side of equation (2.35) can be compensated by adding $-Q\,\dot{Q}^2/2!\,\psi'_{-1}(x)$ to $\psi_2(x)$. The second term can be compensated by adding $-(Q-\dot{Q}t)\dot{Q}\,\dot{\tilde{\psi}}_0(x)$.

Then equation (2.33) becomes

$$\psi_2(x) = -\frac{Q^3}{3!}\psi'''_{-1}(x) - Q\frac{\dot{Q}^2}{2!}x\,\psi''_{-1}(x) - Q\frac{\dot{Q}^2}{2!}\psi'_{-1}(x)$$

$$+ \frac{Q^2}{2!}\tilde{\psi}''_0(x) + Q\dot{Q}x\,\dot{\tilde{\psi}}'_0(x) + Q\dot{Q}\,\dot{\tilde{\psi}}_0(x) + \frac{\dot{Q}^2}{2!}x\,\tilde{\psi}'_0(x)$$

$$+ \frac{\dot{Q}^2}{2!}x^2\,\ddot{\tilde{\psi}}_0(x) - \frac{\dot{Q}^2}{2!}t\,\dot{\tilde{\psi}}_0(x) - \dot{Q}x\,\dot{\tilde{\psi}}_1(x) - Q\,\tilde{\psi}'_1(x)$$

$$+ \tilde{\psi}_2(x) \quad . \qquad (2.36)$$

The dynamical map is

$$\psi^f(x) = \psi_{-1}(x) - Q\psi'_{-1}(x) + \tilde{\psi}_0(x) + \psi_1(x) + \psi_2(x) + \ldots . \quad (2.37)$$

The solution determined here is consistent with a power series expansion of equations (2.22) and (2.23): $\psi^f(x, t) = \tilde{\psi}^f(X, T)$.

2. THE CANONICAL MOMENTUM

Consider the quantity

$$P_\lambda = -\int dx\ \dot{\psi}^f_\lambda(x)\ \psi^{f'}_\lambda(x) \quad . \tag{2.38}$$

The field equation (2.6) leads to the conservation law

$$\dot{P}_\lambda = 0 \tag{2.39}$$

and the equal-time commutation relation (2.4) leads to

$$[P_\lambda, \psi^f_\lambda(x)] = i\ \psi^{f'}_\lambda(x) \tag{2.40}$$

and

$$[P_\lambda, \dot{\psi}^f_\lambda(x)] = i\ \dot{\psi}^{f'}_\lambda(x) \quad . \tag{2.41}$$

Thus, P_λ generates spatial translations and can be identified with the canonical momentum. Since x and Q appear only in the combination $x - Q$ a condition which is sufficient to yield equations (2.40) and (2.41) is that

$$[Q, P_\lambda] = i/\lambda \tag{2.42}$$

and

$$[\tilde{\psi}_0(x), P_\lambda] = [\dot{\tilde{\psi}}_0(x), P_\lambda] = 0 \quad . \tag{2.43}$$

Combining equations (2.9) and (2.38) leads to

$$P_n = -\int dx\ \{\sum \dot{\psi}_k(x)\ \psi'_\ell(x)\} \tag{2.44}$$

where $k + \ell = n$; $k \geq 0$; $\ell \geq -1$. The first few orders of equation (2.44) are

$$P_{-1} = -\int dx\, \dot{\psi}_o(z)\psi'_{-1}(x) \tag{2.45}$$

$$P_o = -\int dz\, \{\dot{\psi}_1(x)\psi'_{-1}(x) + \dot{\psi}_o(x)\psi'_o(x)\} \tag{2.46}$$

$$P_1 = -\int dx\, \{\dot{\psi}_2(x)\psi'_{-1}(x) + \dot{\psi}_1(x)\psi'_o(x) + \dot{\psi}_o(x)\psi'_1(x)\} \quad . \tag{2.47}$$

Direct substitution of equations (2.16), (2.29) and (2.36) into equations (2.45), (2.46) and (2.47) leads to (see reference 4)

$$P_{-1} = M\dot{Q} \tag{2.48}$$

$$P_o = \tilde{P}_o \tag{2.49}$$

$$P_1 = \frac{M}{2}\dot{Q}^3 + \frac{Q}{2}\tilde{H}_o + \tilde{P}_1 \quad , \tag{2.50}$$

where

$$\tilde{P}_n = P_n \quad \text{with} \quad Q = \dot{Q} = 0 \quad , \tag{2.51}$$

and

$$\tilde{H}_o = \frac{1}{2}\int dx\, \{\dot{\tilde{\psi}}_o(x)\dot{\tilde{\psi}}_o(x) - \tilde{\psi}_o(x)\ddot{\tilde{\psi}}_o(x)\} \quad . \tag{2.52}$$

The quantities $\tilde{P}_n$ are intimately related to the fields $\tilde{\psi}_n(x)$ which obey the field equations

$$\{\partial^2 + m^2 - F_1[\psi_{-1}(x)]\}\, \tilde{\psi}_n(x)$$

$$= \sum \frac{1}{\ell!} F_\ell[\psi_{-1}(x)]\, \tilde{\psi}_{\alpha_1}(x)\ldots\tilde{\psi}_{\alpha_\ell}(x) \tag{2.53}$$

where $\alpha_1+\ldots+\alpha_\ell + \ell = n + 1$, $n \geq 0$, $\alpha_1,\ldots,\alpha_\ell \geq 0$. Because the operator on the left hand side of the equation (2.53) has a zero eigenvalue corresponding to the eigenfunction $\psi'_{-1}(x)$, this equation cannot be integrated directly. However, if

$$\tilde{\psi}_n(x) = \alpha_{n-1}(t)\psi'_{-1}(x) + \hat{\tilde{\psi}}_n(x) \tag{2.54}$$

where

$$\int dx\ \psi'_{-1}(x)\ \hat{\tilde{\psi}}_n(x) = 0 \tag{2.55}$$

and

$$\alpha_{n-1}(t) = \frac{1}{M}\int dx\ \psi'_{-1}(x)\ \tilde{\psi}_n(x) \tag{2.56}$$

equation (2.53) can be integrated for $\hat{\tilde{\psi}}_n(x)$ using the Green function

$$\tilde{g}(x, y) = \int \frac{d}{2\pi} e^{i\omega(x^0-y^0)} \left\{ \sum_i \frac{u_i(x)u_i^*(y)}{\omega_i^2 - \omega^2} + \int\frac{dk}{2\pi}\ \frac{u_k(x)u_k^*(y)}{\omega_k^2 - \omega^2} \right\} \tag{2.57}$$

Then, equations (2.53) and (2.56) lead to

$$\ddot{\alpha}_{n-1}(t) = \frac{1}{M}\int dx\ \psi'_{-1}(x) \sum \frac{1}{\ell!} F_\ell[\psi_{-1}(x)]\ \tilde{\psi}_{\alpha_1}(x)\ldots\tilde{\psi}_{\alpha_\ell}(x) \tag{2.58}$$

$$= -\frac{1}{M}\int dx \sum \frac{1}{(\ell-1)!} F_{\ell-1}[\psi_{-1}(x)]\ \tilde{\psi}_{\alpha_1}(x)\ldots\tilde{\psi}_{\alpha_{\ell-1}}(x)\tilde{\psi}'_{\alpha_\ell}(x) \tag{2.59}$$

where $\alpha_1+\ldots+\alpha_\ell + \ell = n + 1$; $\alpha_1,\ldots,\alpha_\ell \geq 0$; $\ell \geq 2$. Using equation (2.12) in equation (2.59) leads to

$$\ddot{\alpha}_{n-1}(t) = -\frac{1}{M}\frac{\partial}{\partial t}\int dx \sum \dot{\tilde{\psi}}_k(x)\tilde{\psi}'_\ell(x) \tag{2.60}$$

or

$$\dot{\alpha}_{n-1}(t) = -\frac{1}{M}\int dx \sum \dot{\tilde{\psi}}_k(x)\tilde{\psi}'_\ell(x) + \dot{\beta}_{n-1} \tag{2.61}$$

where $k + \ell = n - 1$, $k, \ell \geq 0$ and

$$\ddot{\beta}_n = 0 \tag{2.62}$$

Now, consider the cases n = 1 and n = 2 or equation (2.61).

$$\dot{\alpha}_o(t) = -\frac{1}{M}\int dx\, \dot{\tilde{\psi}}_o(x)\tilde{\psi}_o'(x) + \dot{\beta}_o \qquad (2.63)$$

$$\dot{\alpha}_1(t) = -\frac{1}{M}\int dx\, \{\dot{\tilde{\psi}}_o(x)\tilde{\psi}_1'(x) + \dot{\tilde{\psi}}_1(x)\tilde{\psi}_o'(x)\} + \dot{\beta}_1 \quad . \qquad (2.64)$$

Using the identity

$$\{\partial^2 + m^2 - F_1[\psi_{-1}(x)]\}\, x\tilde{\psi}_o(x) = -\, 2\tilde{\psi}_o'(x) \quad , \qquad (2.65)$$

equation (2.63) can be written as

$$\dot{\alpha}_o(t) = \frac{\partial}{\partial t}\frac{1}{2M}\int dxx\, \{\dot{\tilde{\psi}}_o(x)\dot{\tilde{\psi}}_o(x) - \tilde{\psi}_o(x)\ddot{\tilde{\psi}}_o(x)\} + \dot{\beta}_o \qquad (2.66)$$

or

$$\alpha_o(t) = \frac{1}{2M}\int dxx\, \{\dot{\tilde{\psi}}_o(x)\dot{\tilde{\psi}}_o(x) - \tilde{\psi}_o(x)\ddot{\tilde{\psi}}_o(x)\} + \beta_o \quad . \qquad (2.67)$$

Similarly, using the identity

$$\{\partial^2 + m^2 - F_1[\psi_{-1}(x)]\}\, x\, \tilde{\psi}_1(x) = x\, \frac{1}{2!} F_2[\psi_{-1}(x)]\, \tilde{\psi}_o^2(x) - 2\, \tilde{\psi}_1'(x) \quad , \qquad (2.68)$$

and equation (2.65), equation (2.64) can be written as

$$\alpha_1(t) = \frac{1}{M}\int dx\{\, x\, \{\dot{\tilde{\psi}}_o(x)\dot{\tilde{\psi}}_1(x) - \frac{2}{3}\ddot{\tilde{\psi}}_o(x)\tilde{\psi}_1(x) - \frac{1}{3}\tilde{\psi}_o(x)\ddot{\tilde{\psi}}_1(x)\} - \frac{1}{3}\tilde{\psi}_o'(x)\tilde{\psi}_1(x)\} + \beta_1 \quad . \qquad (2.69)$$

The quantities β_n must be constructed from the operators $\{\alpha_o,\ \alpha_i^\dagger,\ \alpha_k,\ \alpha_k^\dagger\}$ defined in equation (2.17). The choice of β_n depends on the choice of boundary conditions. Equation (2.62) leads to

$$\{\partial^2 + m^2 - F_1[\psi_{-1}(x)]\}\beta_n\, \psi_{-1}'(x) = 0 \qquad (2.70)$$

that is, $\beta_n \psi'_{-1}(x)$ is a solution of the homogeneous free field equation. Therefore, $\beta_n \psi'_{-1}(x)$ appears in $\tilde{\psi}_n(x)$ in the same way as any solution of the homogeneous free field equation which may be added when integrating equation (2.53). In this way, all ambiguities arising from the presence of the so-called "zero mode" have been reduced to the choice of boundary conditions. These boundary conditions must be chosen in a way which is consistent with the solution that we are seeking. This solution corresponds to an expression for the Heisenberg field $\psi^f(x)$ in terms of the physical fields $\tilde{\psi}_o(x)$ and the operators Q and P such that the commutation relations between members of the set $\{Q, P, \tilde{\psi}_o(x), \dot{\tilde{\psi}}_o(x)\}$ lead to the canonical commutation relation (2.4).

The only commutation relations between members of the set $\{Q, P, \tilde{\psi}_o(x), \dot{\tilde{\psi}}_o(x)\}$ which have so far been specified are those in equations (2.42) and (2.43). Physical considerations dictate one more requirement. The states of the system corresponding to $\tilde{\psi}_o(x)$ should be particle-like. That is, it is required that

$$[\alpha_i, \alpha_j^\dagger] = \delta_{ij} \quad , \quad [\alpha_k, \alpha_\ell^\dagger] = \delta(k - \ell)$$

and (2.71)

$$[\alpha_i, \alpha_j] = 0 \quad , \quad [\alpha_k, \alpha_\ell] = 0 \quad .$$

Equations (2.71), (2.17) and the completeness of the set of functions $\{\psi'_{-1}(x), u_i(x), u_k(x)\}$ lead to

$$[\tilde{\psi}_o(x), \dot{\tilde{\psi}}_o(y)]_{x^o=y^o} = i \,\{\delta(x-y) - \frac{1}{M}\psi'_{-1}(x)\,\psi'_{-1}(y)\} \qquad (2.72)$$

and

$$[\tilde{\psi}_0(x), \tilde{\psi}_0(y)]_{x^0=y^0} = [\dot{\tilde{\psi}}_0(x), \dot{\tilde{\psi}}_0(y)]_{x^0=y^0} = 0 \quad . \tag{2.73}$$

Using the linear homogeneous field equation (2.15) it is possible to show that

$$\dot{\tilde{H}}_0 = 0 \tag{2.74}$$

where $\tilde{H}_0$ is given in equation (2.52). This fact, together with equations (2.72) and (2.73) leads to

$$[\tilde{\psi}_0(x), \tilde{H}_0] = i\,\dot{\tilde{\psi}}_0(x) \tag{2.75}$$

and

$$[\dot{\tilde{\psi}}_0(x), \tilde{H}_0] = i\,\ddot{\tilde{\psi}}_0(x) \quad . \tag{2.76}$$

Thus, $\tilde{H}_0$ generates the time translations of the fields $\tilde{\psi}_0(x)$ and $\dot{\tilde{\psi}}_0(x)$.

Equation (2.51) leads to

$$\tilde{P}_n = -\int dx\{\sum_{\substack{k+m=n \\ k,m\;0}} \dot{\tilde{\psi}}_k(x)\,\tilde{\psi}'_m(x) + \dot{\tilde{\psi}}_{n+1}(x)\,\psi'_{-1}(x)\} \tag{2.77}$$

which, using equations (2.56) and (2.61) reduces to

$$\tilde{P}_n = -M\,\dot{\beta}_n \quad . \tag{2.78}$$

This shows that the appearance of $\tilde{P}_n$ is directly related to the boundary conditions.

We have now determined the canonical momentum to first order as

$$P_\lambda = \frac{1}{\lambda}\,M\dot{Q} - M\,\dot{\beta}_0 + \lambda\,\frac{M}{2}\,Q^3 + \lambda\,\dot{Q}\,\tilde{H}_0 - \lambda\,M\dot{\beta}_1 + \dots \quad . \tag{2.79}$$

3. THE HAMILTONIAN

Now, consider the quantity

$$H = \int dx \, \{ \tfrac{1}{2} \dot{\psi}^f(x)^2 + \tfrac{1}{2} \psi^{f'}(x)^2 + \tfrac{1}{2} m^2 \psi^f(x)^2$$

$$+ V[\psi^f(x)] \} \quad , \qquad (2.80)$$

where $V[\psi^f(x)]$ is a local function of $\psi^f(x)$ such that

$$V_1[\psi^f(x)] = - F[\psi^f(x)] \qquad (2.81)$$

and $V[\psi_{-1}(x)] \to 0$ as $x \to \pm \infty$. Using the Heisenberg field equation (2.3) it is possible to show that

$$\dot{H} = 0 \quad . \qquad (2.82)$$

Then, using the commutation relation (2.4) leads to

$$[\psi^f(x), H] = i \, \dot{\psi}^f(x) \qquad (2.83)$$

$$[\dot{\psi}^f(x), H] = i \, \ddot{\psi}^f(x) \qquad (2.84)$$

and H is the canonical Hamiltonian.

When the power counting parameter is included, equation (2.80) becomes

$$H_\lambda = \int dx \, \{ \tfrac{1}{2} \dot{\psi}^f_\lambda(x)^2 + \tfrac{1}{2} \psi^{f'}_\lambda(x)^2 + \tfrac{1}{2} m^2 \, \psi^f_\lambda(x)^2$$

$$+ \lambda^{-2} \, V[\lambda \psi_f(x)] \} \quad . \qquad (2.85)$$

The interaction term, $V[\lambda\psi_f(x)]$, can be expanded about the classical field, $\psi_{-1}(x)$, to get

$$H_n = \int dx \{ \sum [\frac{1}{2} \dot{\psi}_k(x)\dot{\psi}_m(x) + \frac{1}{2} \psi_k'(x)\psi_m'(x) + \frac{1}{2} m^2 \psi_k(x)\psi_m(x)]$$

$$+ \frac{1}{m!} V_m[\psi_{-1}(x)] \psi_{\alpha_1}(x)\ldots\psi_{\alpha_m}(x)\} \quad (2.86)$$

where $k + m = n$ in the first summation and $\alpha_1+\ldots+\alpha_m + m = n + 1$; $\alpha_1,\ldots\alpha_m \geq 0$ in second summation. Equation (2.86) together with equations (2.10), (2.15), (2.20), (2.24) and (2.25) lead to

$$H_{-2} = M \quad (2.87)$$

$$H_{-1} = 0 \quad (2.88)$$

$$H_o = \int dx \{\frac{1}{2} \dot{\psi}_o(x)\dot{\psi}_o(x) - \frac{1}{2} \psi_o(x)\ddot{\psi}_o(x)\} \quad (2.89)$$

$$H_1 = \int dx \{ \dot{\psi}_o(x)\dot{\psi}_1(x) - \frac{1}{3}\psi_o(x)\ddot{\psi}_1(x) - \frac{2}{3} \ddot{\psi}_o(x)\psi_1(x)\} \quad (2.90)$$

$$H_2 = \int dx \{ \dot{\psi}_2(x)\dot{\psi}_o(x) - \frac{1}{4} \ddot{\psi}_2(x)\psi_o(x) - \frac{3}{4} \psi_2(x)\ddot{\psi}_o(x) + \frac{1}{2} \dot{\psi}_1(x)\dot{\psi}_1(x) - \frac{1}{2} \psi_1(x)\ddot{\psi}_1(x)\} \quad . \quad (2.91)$$

These quantities may be calculated by direct substitution of equations (2.16), (2.29) and (2.38) (see reference 4). The result is

$$H_\lambda = \frac{1}{\lambda^2} + \frac{1}{2} M\dot{Q}^2 + \tilde{H}_o - \lambda\dot{Q}\,\dot{\beta}_o + \lambda^2\{\frac{3}{8} M\dot{Q}^4 + \frac{1}{2} \dot{Q}^2 \tilde{H}_o - \dot{Q}\,\dot{\beta}_1 + \tilde{H}_2\} + \ldots \quad (2.92)$$

When Q and P are chosen as independent dynamical variables, $\dot{Q}$ is a dependent variable and may be determined from inversion of equation

(2.79).

$$\dot{Q} = \frac{\hat{P}}{M} + \lambda\dot{\beta}_o - \lambda^2 \{\frac{1}{2}\frac{\hat{P}^3}{M^3} + \frac{\hat{P}}{M^2}\tilde{H}_o - \dot{\beta}_1\} + \ldots \tag{2.93}$$

where $\hat{P} = \lambda P$.

Combining equations (2.92) and (2.93) leads to

$$H_\lambda = \frac{M}{\lambda^2}\{1 + \lambda^2 \frac{\hat{P}^2}{2M^2} + \frac{1}{8}\lambda^4 \frac{\hat{P}^4}{M^4}\} + \{1 - \frac{1}{2}\lambda^2 \frac{\hat{P}^2}{M^2}\}\tilde{H}_o$$
$$+ \lambda^2\{\tilde{H}_2 - M\frac{\dot{\beta}_o^2}{2}\} + \ldots \tag{2.94}$$

By substituting equations (2.93) into equations (2.16), (2.29) and (2.36) equation (2.37) may be written as

$$\psi^f_\lambda(x) = \frac{1}{\lambda}\psi_{-1}(x) + \{-Q\psi'_{-1}(x) + \tilde{\psi}_o(x)\}$$
$$+ \lambda\{\frac{Q^2}{2!}\psi''_{-1}(x) + \frac{1}{2!}\frac{\hat{P}^2}{M^2}x\psi'_{-1}(x) - Q\tilde{\psi}'_o(x) - \frac{\hat{P}}{M}x\dot{\tilde{\psi}}_o(x)$$
$$+ \tilde{\psi}_1(x)\} + \lambda^2\{-\frac{Q^3}{3!}\psi'''_{-1}(x) - \frac{1}{2!}Q\frac{\hat{P}^2}{M^2}x\psi''_{-1}(x) - \frac{1}{2!}Q\frac{\hat{P}^2}{M^2}\psi'_{-1}(x)$$
$$+ \frac{\hat{P}}{M}\dot{\beta}_o x\psi'_{-1}(x) + \frac{Q^2}{2!}\tilde{\psi}''_o(x) + \frac{Q\hat{P}}{M}x\dot{\tilde{\psi}}'_o(x) + \frac{Q\hat{P}}{M}\dot{\tilde{\psi}}_o(x)$$
$$+ \frac{1}{2}\frac{\hat{P}^2}{M^2}x\tilde{\psi}'_o(x) + \frac{1}{2}x^2\frac{\hat{P}^2}{M^2}\tilde{\psi}''_o(x) - \frac{1}{2}t\frac{\hat{P}^2}{M^2}\dot{\tilde{\psi}}_o(x) - x\dot{\beta}_o\dot{\tilde{\psi}}_o(x)$$
$$- x\frac{\hat{P}}{M}\dot{\tilde{\psi}}_1(x) - Q\tilde{\psi}'_1(x) + \tilde{\psi}_2(x)\} + \ldots \tag{2.95}$$

and

$$\pi^f_\lambda(x) = \{-\frac{\hat{P}}{M}\tilde{\psi}'_{-1}(x) + \dot{\tilde{\psi}}_o(x)\} + \lambda\{\frac{Q\hat{P}}{M}\psi''_{-1}(x)$$

$$- \dot{\beta}_o \psi'_{-1}(x) - \frac{\hat{P}}{M} \tilde{\psi}'_o(x) - Q \dot{\tilde{\psi}}'_o(x) - x \frac{\hat{P}}{M} \ddot{\tilde{\psi}}_o(x)$$

$$+ \dot{\tilde{\psi}}_1(x)\} + \lambda^2 \{ - \frac{1}{2} Q^2 \frac{\hat{P}}{M} \psi'''_{-1}(x) - \frac{1}{2} x \frac{\hat{P}^3}{M^3} \psi''_{-1}(x)$$

$$+ \frac{\hat{P}}{M^2} \tilde{H}_o \psi'_{-1}(x) - \dot{\beta}_1 \psi'_{-1}(x) + Q \dot{\beta}_o \psi''_{-1}(x) + \frac{Q^2}{2!} \dot{\tilde{\psi}}''_o(x)$$

$$+ \frac{Q\hat{P}}{M} \tilde{\psi}''_o(x) + \frac{\hat{P}^2}{M^2} x \dot{\tilde{\psi}}'_o(x) + \frac{Q\hat{P}}{M} x \ddot{\tilde{\psi}}'_o(x)$$

$$+ \frac{\hat{P}^2}{M^2} \dot{\tilde{\psi}}_o(x) + \frac{Q\hat{P}}{M} \ddot{\tilde{\psi}}_o(x) + \frac{1}{2} x \frac{\hat{P}^2}{M^2} \dot{\tilde{\psi}}'_o(x)$$

$$+ \frac{1}{2} x^2 \frac{\hat{P}^2}{M^2} \dddot{\tilde{\psi}}_o(x) - \frac{1}{2} \frac{\hat{P}^2}{M^2} \dot{\tilde{\psi}}_o(x) - \frac{1}{2} t \frac{\hat{P}^2}{M^2} \ddot{\tilde{\psi}}_o(x)$$

$$- \dot{\beta}_o \tilde{\psi}'_o(x) - x \dot{\beta}_o \ddot{\tilde{\psi}}_o(x) - x \frac{\hat{P}}{M} \ddot{\tilde{\psi}}_1(x)$$

$$- \frac{\hat{P}}{M} \tilde{\psi}'_1(x) - Q \dot{\tilde{\psi}}'_1(x) + \dot{\tilde{\psi}}_2(x)\} + \ldots \qquad (2.96)$$

Where $\pi^f_\lambda(x) = \dot{\psi}^f_\lambda(x)$.

Equations (2.94), (2.95) and (2.96) together must satisfy

$$i\, \pi^f_\lambda(x) = [\psi^f_\lambda(x), H_\lambda] \qquad . \qquad (2.97)$$

Equation (2.97) leads to the conditions

$$\dot{\beta}_o = \dot{\beta}_1 = \tilde{H}_2 = 0 \qquad (2.98)$$

and

$$[Q, \tilde{H}_o] = 0 \qquad . \qquad (2.99)$$

Combining equations (2.93) and (2.94) leads to

$$i\dot{Q} = [Q, H] \tag{2.100}$$

The time derivative of Q is generated by the full Hamiltonian.

The operators given in equations (2.95) and (2.96) must satisfy the equal time commutation relations. In the next section the commutation relations will be calculated perturbatively in order to arrive at some further conditions on commutation relations within the operator set $\{Q, P, \tilde{\psi}_0(x), \dot{\tilde{\psi}}_0(x)\}$ and also to determine β_0 and β_1.

4. THE CANONICAL COMMUTATION RELATIONS

The canonical equal time commutation relation

$$[\psi^f(x), \tilde{\pi}^f(y)]_{x^0=y^0} = i\,\delta(x - y) \tag{2.101}$$

will now be used to derive more information about the commutation relations between members of the set $\{Q, P, \tilde{\psi}_0(x), \dot{\tilde{\psi}}_0(x)\}$. If $\psi^f(x)$ and $\pi^f(x)$ are written according to equations (2.95) and (2.96) as

$$\psi^f(x) = \frac{1}{\lambda}\psi_{-1}(x) + \psi_0(x) + \lambda\psi_1(x) + \lambda^2\psi_2(x) + \ldots \tag{2.102}$$

and

$$\pi^f(x) = \pi_0(x) + \lambda\pi_1(x) + \lambda^2\pi_2(x) + \ldots \tag{2.103}$$

equation (2.101) leads to

$$[\psi_0(x), \pi_0(y)]_0 = i\,\delta(x - y) \tag{2.104}$$

$$[\psi_0(x), \pi_1(y)]_0 + [\psi_1(x), \pi_0(y)]_0 + [\psi_0(x), \pi_0(y)]_1 = 0 \tag{2.105}$$

$$[\psi_0(x), \pi_2(y)]_0 + [\psi_1(x), \pi_1(y)]_0 + [\psi_2(x), \pi_0(y)]_0$$

$$+ [\psi_0(x)\ \pi_1(y)]_1 + [\psi_1(x), \pi_0(y)]_1 + [\psi_0(x), \pi_0(y)]_2$$

$$= 0 \qquad (2.106)$$

where all commutators are at equal times.

The subscripts on the commutators in equations (2.104-106) denote the order in λ of the commutator which must be considered. Equations (2.42), (2.43), (2.72) and (2.104) lead to

$$[Q, \dot{\tilde{\psi}}_0(x)]_0 = 0 \qquad (2.107)$$

Equation (2.105) then requires the conditions (see reference 5)

$$[\psi_0(x), \pi_0(y)]_1 = 0 \qquad (2.108)$$

and

$$[\beta_0, \tilde{\psi}_0(x)] = [\beta_0, \dot{\tilde{\psi}}_0(x)] = 0 \quad . \qquad (2.109)$$

Equation (2.109) together with the fact that β_0 must be constructed from operators of the set $\{\alpha_i, \alpha_i^\dagger, \alpha_k, \alpha_k^\dagger\}$ means that β_0 is a c-number which we choose as zero. Equation (2.108) means that

$$[\psi_0(x), \pi_1(y)]_1 + [\psi_1(x), \pi_0(y)]_1 = 0 \qquad (2.110)$$

in equation (2.106). The reason for this is that $\psi_1(x)$ and $\pi_1(y)$ are constructed from operators of the set $\{Q, P, \tilde{\psi}_0(x), \dot{\tilde{\psi}}_0(x)\}$ which have no commutators of order λ. Then equation (2.106) leads to (see reference 5)

$$[\beta_1, \tilde{\psi}_0(x)] = [\beta_1, \dot{\tilde{\psi}}_0(x)] = 0 \qquad (2.111)$$

and

$$[Q, \dot{\tilde{\psi}}_0(x)] = i\lambda^2 t \frac{\hat{P}}{M^2} \ddot{\tilde{\psi}}_0(x) + \ldots \qquad (2.112)$$

By the same argument as that following equation (2.109), equation (2.111) implies that

$$\beta_1 = 0 \qquad (2.113)$$

Equations (2.75) and (2.93) lead to

$$[\dot{Q}, \tilde{\psi}_0(x)] = i\lambda^2 \frac{\hat{P}^2}{M^2} \dot{\tilde{\psi}}_0(x) + \ldots \qquad (2.114)$$

which, together with equation (2.112) and the fact that $\tilde{\psi}_0(x)$ satisfies the homogeneous field equation and is orthogonal to $\psi'_{-1}(x)$ leads to

$$[Q, \tilde{\psi}_0(x)] = i\lambda^2 t \frac{\hat{P}}{M^2} \dot{\tilde{\psi}}_0(x) + \ldots \qquad (2.115)$$

From the above results, the set of physical operators $\{P, Q, \tilde{\psi}_0(x), \dot{\tilde{\psi}}_0(x)\}$ obey the algebra

$$[Q, \hat{P}] = i \qquad (2.116)$$

$$[\tilde{\psi}_0(x), \dot{\tilde{\psi}}_0(y)]_{x^0=y^0} = i\,\{\delta(x-y) - \frac{\psi'_{-1}(x)\psi'_{-1}(y)}{M}\} \qquad (2.117)$$

$$[Q(t), \tilde{\psi}_0(x)]_{t=x^0} = i\lambda^2 t \frac{\hat{P}}{M^2} \dot{\tilde{\psi}}_0(x) + \ldots \qquad (2.118)$$

$$[Q(t), \dot{\tilde{\psi}}_0(x)]_{t=x^0} = i\lambda^2 t \frac{\hat{P}}{M^2} \ddot{\tilde{\psi}}_0(x) + \ldots , \qquad (2.119)$$

where all other equal time commutation relations vanish.

If the Schrodinger picture operator q is defined by

$$Q(t) = e^{iHt}\, q\, e^{-iHt} = q + \dot{Q}t \qquad (2.120)$$

equations (2.114) and (2.118) lead to

$$[q, \tilde{\psi}_0(x)] = [q, \dot{\tilde{\psi}}_0(x)] = 0 \tag{2.121}$$

for all times. Also, since $[P, H] = 0$,

$$[q, P] = i \quad . \tag{2.122}$$

Thus the operators of the set $\{q, P\}$ commute with those of the set $\{\tilde{\psi}_0(x), \dot{\tilde{\psi}}_0(x)\}$. The Hilbert space of the system is now constructed in the following way: take the direct product of the Schrödinger picture realization of the quantum mechanical operators, $\{q, P\}$, with the Fock-like representation of the particle like excitations, $\{\tilde{\psi}_0(x), \dot{\tilde{\psi}}_0(x)\}$. The vacuum in this representation is denoted by $|0_F\rangle$ and satisfies the condition $\alpha_i|0_F\rangle = \alpha_k|0_F\rangle = 0$.

It has been shown here that the position of the extended object can be chosen as an independent dynamical variable at one particular time. This necessitates the use of the Schrödinger representation for the operators $\{q, P\}$. In the solution chosen, q always appears in the combination $x - q$ with x and is the canonical conjugate of the total momentum, P.

The Hamiltonian has the form

$$H = M \left\{1 + \frac{P^2}{2M^2} - \frac{P^4}{4M^4}\right\} + \left\{1 - \frac{P^2}{2M^2}\right\} \tilde{H}_0 + \ldots \quad . \tag{2.123}$$

It contains the kinetic term of the extended object, the free Hamiltonian of the physical particles and a term which mixes these. It will be shown in Chapter IV that the full Hamiltonian is given by

$$H = \sqrt{P^2 - (M + \tilde{H}_0)^2} \quad . \tag{2.124}$$

In the next chapter, the information gained from the computations of this chapter will be used to calculate some quantum corrections of the lower order terms in the dynamical map and the Hamiltonian.

CHAPTER III

A PERTURBATIVE LOOK AT EXTENDED OBJECT DYNAMICS II: RENORMALIZATION AND QUANTUM CORRECTIONS

In the previous chapter, the dynamical maps of the Heisenberg fields were calculated perturbatively by considering the quantum tree graphs only. Contractions of physical fields as well as operator ordering were ignored. It is now necessary to address the problem of inserting the quantum corrections.

For a given quantum field theory, the central role of renormalization theory is the determination of the parameters which appear in the field equation. Once these parameters are determined in terms of some other experimentally determinable parameters, the quantum field theory is internally complete and can in principle be used to make predictions about observable phenomena.

Conventional renormalization theory is heavily dependent on manifest translation invariance. The combinatorics of perturbation theory are usually formulated in terms of the Feynman rules in momentum space[9-14] where they have a simple form. In theories where locality is manifest, counterterms arise in a natural way[12]. The dimensions of the counterterms are restricted by the power counting apparatus of renormalization theory. In a renormalizable theory the counterterms have the same dimensions as parameters in the classical Lagrangian and can be amalgated with them. They are then determined by the comparison of certain physical predictions with experiment. This constitutes a definition of the parameters of the theory.

In the case where an extended object is present, translational invariance is not manifest. It is recovered through the appearance of the quantum coordinate. This fact not only complicates the combinatorics of perturbation theory, but it also makes difficult the task of setting criteria by which the field theory is renormalized.

The boson theory provides a vehicle by which the quantum field theory with an extended object is understood in terms of the same field theory without an extended object. In the latter case, all of the parameters of the theory can be determined by standard renormalization theory. Consistency with the boson theory requires that the same parameters be used when an extended object is present. The theory with an extended object is then internally complete once the boundary conditions are specified and the space-dependent order parameter is calculated. Implicit in this notion is the assumption that both phases can exist. Which phase does appear in a particular case depends on the boundary conditions and not on the local structure of the theory.

This choice of parameters is also motivated by physical considerations. If the extended object is sufficiently localized, then, asymptotically in some spatial direction, the dynamics of the quantum field theory become free of its influence. This means that the physical particle wavefunctions are like plane waves which propagate with the physical mass of the particles. It is reasonable to define the observable particle masses and vertices in this region as those of the theory with no extended object. The parameters of the theory are determined by this condition as those of the theory when no extended object is present. The theory with no extended object is called the vacuum sector.

The boson theory also provides a strong statement about renormalizability. If the quantum field theory is renormalizable in the vacuum sector, all matrix elements of the renormalized Heisenberg fields can be made finite be an appropriate choice of counterterms. The boson theory leads to expressions for the matrix elements of the renormalized Heisenberg fields in the presence of an extended object in terms of series of matrix elements of the vacuum sector Heisenberg fields convoluted with the boson functions. This series then contains no ultraviolet divergences in the sense that all ultraviolet divergences should cancel in perturbative calculations. The series itself may, however, be asymptotic and may require the canonical commutation relations to define its sum.[3]

In this chapter it will be seen that the inclusion of quantum corrections entails two major modifications of the procedure of the previous chapter where the tree approximation was considered. These are (i) the careful treatment of operator ordering and (ii) the inclusion of the counterterms. The counterterms which are included are those which are calculated for the same theory in the vacuum sector. With these considerations, the theory is internally complete. When the particle-like fields in the dynamical map are put in normal ordered form, the contractions of these physical fields combined with the counterterms constitute the quantum corrections.

In this chapter, the dynamical map computed in chapter II will be calculated including the quantum corrections. The condition that the quantum coordinate, q, appear in the combination $x - Q$ with the explicit spatial coordinate x everywhere in the dynamical map will be retained. This also means that the canonical conjugate of q must be the canonical momentum, P, when all quantum corrections are considered. As an illustrative example of the computational procedure, the one-loop quantum corrections of the order parameter and the ground state energy will be explicitly calculated for the $\lambda\phi^4$ model in 1+1 - dimensions.

1. THE DYNAMICAL MAP

Consider the one component boson field in 1+1-dimensions which was considered in chapter II. The Heisenberg equation is

$$(\partial^2 + m^2)\, \psi^f(x) = F[\psi^f(x)] \quad . \tag{3.1}$$

As in equation (2.12) the right hand side of equation (3.1) can be expanded as

$$\lambda^{-1} F[\lambda\psi^f(x)] = \sum \lambda^{\ell+\alpha_1+\ldots+\alpha_\ell-1}\, \frac{1}{\ell!}\, F_\ell[\psi_{-1}(x)]\, \psi_{\alpha_1}(x)\ldots\psi_{\alpha_\ell}(x) \tag{3.2}$$

where the fields $\psi_\alpha(x)$ are defined in equation (2.9). However, each function $F_\ell[\psi_{-1}(x)]$ must contain counterterms. These must be of higher orders in λ^2.

$$F_\ell[\psi_{-1}(x)] = \sum \lambda^m\, F_\ell^m\, [\psi_{-1}(x)] \quad . \tag{3.3}$$

The quantum corrections arise from contractions of pairs of physical fields. For this reason, m must always be even in equation (3.3). The terms with $m = 0$ correspond to the tree approximation. Considering equations (3.2) and (3.3), equation (3.1) becomes

$$(\partial^2 + m^2)\, \psi_n(x) = \sum \frac{1}{\ell!}\, F_\ell^m\, [\psi_{-1}(x)]\, \psi_{\alpha_1}(x)\ldots\psi_{\alpha_\ell}(x) \quad . \tag{3.4}$$

Where $\ell + m + \alpha_1 + \ldots + \alpha_\ell = n + 1$ and $\alpha_1, \ldots, \alpha_\ell \geq 0$. The first few orders of equation (3.4) are

$$\{\partial^2 + m^2 - F_1^0[\psi_{-1}(x)]\}\ \psi_0(x) = 0 \tag{3.5}$$

$$\{\partial^2 + m^2 - F_1^0[\psi_{-1}(x)]\}\ \psi_1(x) = F_0^2[\psi_{-1}(x)] + F_2^0[\psi_{-1}(x)]\psi_0^2(x)/2! \tag{3.6}$$

$$\{\partial^2 + m^2 - F_1^0[\psi_{-1}(x)]\}\ \psi_2(x) = F_1^2[\psi_{-1}(x)]\ \psi_0(x)$$

$$+ F_3^0[\psi_{-1}(x)]\ \psi_0^3(x)^3/3!$$

$+ F_2^0[\psi_{-1}(x)]\ (\psi_0(x)\psi$ [illegible] $+ \psi_1(x)\psi_0(x))\ /2!$ (3.7)

In chapter II it was shown that the solution of equation (3.5) must be taken as

$$\psi_0(x) = -\left(q + \frac{p}{m}t\right)\ \psi'_{-1}(x) + \tilde{\psi}_0(x) \quad . \tag{3.8}$$

The computational technique can be summarized in the following steps: (i) Having computed the dynamical map to some given order, use the field equation to calculate the dynamical map to the next order, including the counterterms and paying careful attention to operator ordering.

(ii) Add symmetrized (Hermitian) solutions of the homogeneous equation until the following relation is satisfied:

$$-\frac{\partial}{\partial q}\psi_{n+1}(x) = \frac{\partial}{\partial x}\psi_n(x) \quad . \tag{3.9}$$

(iii) Add solutions of the homogeneous equation until the canonical momentum calculated using the equation*

$$P_{n+1} = -\frac{1}{2}\int dx \sum_k (\dot{\psi}_k(x)\psi'_{n-k}(x) + \psi'_{n-k}(x)\dot{\psi}_k(x)) \qquad (3.10)$$

satisfies

$$P_o = P \quad , \quad P_{n>0} = 0 \quad . \qquad (3.11)$$

(iv) Add solutions of the homogeneous equation until the canonical commutation relations are satisfied.

(v) Normal order the terms of the dynamical map to find the quantum corrections.

It is easy to see that to zeroth order the solution given in equation (3.8) satisfies the condition (3.9), (3.11) and the equal-time commutation relation to zeroth order

$$[\psi_o(x), \dot{\psi}_o(x)]_{x^o=y^o} = i\,\delta(x - y) \qquad (3.12)$$

when the operators $\{q, P, \tilde{\psi}_o(x), \dot{\tilde{\psi}}_o(x)\}$ obey the algebra given in equations (2.117), (2.121) and (2.122). Then, according to chapter II, the last term in equation (3.6) has the solution

$$\psi_1(x) = \frac{1}{2!}(q + \frac{P}{M}t)^2\psi''_{-1}(x) + \frac{1}{2!}\frac{P^2}{M^2}x\,\psi'_{-1}(x) - (q + \frac{P}{M}t)\,\tilde{\psi}'_o(x)$$
$$- \frac{P}{M}x\,\dot{\tilde{\psi}}_o(x) + \tilde{\psi}_1(x) \qquad (3.13)$$

where

$$\{\partial^2 + m^2 - F^o_1[\psi_{-1}(x)]\}\,\tilde{\psi}_1(x) = F^2_o[\psi_{-1}(x)] + F^o_2[\psi_{-1}(x)]\,\tilde{\psi}^2_o(x)/2! \quad . \qquad (3.14)$$

---

* Note that P in equation (3.10) is the same as $\hat{P}$ in chapter II.

Equation (3.13) immediately leads to

$$-\frac{\partial}{\partial q}\psi_1(x) = \psi_0'(x) \qquad (3.15)$$

Furthermore, it can be shown that

$$P_1 = -\frac{1}{2}\int dx\, \{\dot{\psi}_1(x)\psi_{-1}'(x) + \psi_{-1}'(x)\dot{\psi}_1(x) + \psi_0'(x)\dot{\psi}_0(x) + \dot{\psi}_0(x)\psi_0'(x)\}$$

$$= 0 \qquad (3.16)$$

when the condition

$$\dot{\alpha}_0(t) = \frac{1}{M}\int dx\, \psi_{-1}'(x)\dot{\tilde{\psi}}_1(x) = -\frac{1}{2M}\int dx\, \{\dot{\tilde{\psi}}_0(x)\tilde{\psi}_0'(x) + \tilde{\psi}_0'(x)\dot{\tilde{\psi}}_0(x)\} \qquad (3.17)$$

is satisfied. Also, as was shown in chapter II (see reference 4), the commutation relation

$$[\psi_0(x), \dot{\psi}_1(y)] + [\psi_1(x), \dot{\psi}_0(y)] = 0 \qquad (3.18)$$

follows directly.* Thus the solution for $\psi_1(x)$ is the one given in equation (3.13). Equation (3.14) can be rewritten as

$$\{\partial^2 + m^2 - F_1^0[\psi_{-1}(x)]\}\,\tilde{\psi}_1(x) = F_0^2[\psi_{-1}(x)] + F_2^0[\psi_{-1}(x)]\langle 0_F|\tilde{\psi}_0(x)^2|0_F\rangle/2!$$

$$+ F_2^0[\psi_{-1}(x)]\,\frac{1}{2!}\,:\tilde{\psi}_0(x)^2: \qquad (3.19)$$

or

$$\tilde{\psi}_1(x) = \bar{\psi}_1(x) + \frac{1}{M}\psi_{-1}'(x)\int dy\, y:\dot{\tilde{\psi}}_0(y)\dot{\tilde{\psi}}_0(y) - \tilde{\psi}_0(y)\ddot{\tilde{\psi}}_0(y):$$

$$+ \int d^2y\, \tilde{g}(x, y)\, F_2^0[\psi_{-1}(y)]\, :\tilde{\psi}_0^2(y):/2! \qquad (3.20)$$

---

* Note that $\beta_0 = 0$.

where

$$\{\partial^2 + m^2 - F_1^o[\psi_{-1}(x)]\}\ \bar{\psi}_1(x) = F_o^2[\psi_{-1}(x)]$$

$$+ F_2^o[\psi_{-1}(x)]\ \langle 0_F|\tilde{\psi}_o^2(x)|0_F\rangle/2! \qquad (3.21)$$

and $\bar{\psi}_1(x)$ is the one-loop quantum correction of the order parameter.

Now consider equation (3.7). According to chapter II (equation (2.36)) the solution is

$$\psi_2(x) = -\frac{1}{3!}(q + \frac{P}{M}t)^3\psi_{-1}'''(x) - \frac{1}{4M^2}[(q + \frac{P}{M}t)P^2 + P^2(q + \frac{P}{M}t)]$$

$$\times [x\,\psi_{-1}''(x) + \psi_{-1}'(x)]$$

$$+ \frac{1}{2!}(q + \frac{P}{M}t)^2\tilde{\psi}_o''(x) + \frac{1}{2M}[(q + \frac{P}{M}t)P + P(q + \frac{P}{M}t)]$$

$$\times [x\,\dot{\tilde{\psi}}_o'(x) + \dot{\tilde{\psi}}_o(x)]$$

$$+ \frac{1}{2!}\frac{P^2}{M^2}[x\tilde{\psi}_o'(x) + x^2\ddot{\tilde{\psi}}_o(x)] - \frac{P}{M}x\dot{\tilde{\psi}}_1(x) - (q + \frac{P}{M}t)\ \tilde{\psi}_1'(x) + \tilde{\psi}_2(x)\ , \qquad (3.22)$$

where

$$\{\partial^2 + m^2 - F_1^o[\psi_{-1}(x)]\}\ \tilde{\psi}_2(x) = F_1^2[\psi_{-1}(x)]\ \tilde{\psi}_o(x) + F_3^o[\psi_{-1}(x)]\ \frac{\tilde{\psi}_o^3(x)}{3!}$$

$$+ F_2^o[\psi_{-1}(x)]\ \frac{1}{2!}[\tilde{\psi}_o(x)\tilde{\psi}_1(x) + \tilde{\psi}_1(x)\tilde{\psi}_o(x)]\ . \qquad (3.23)$$

Equation (3.22) satisfies the condition

$$-\frac{\partial}{\partial q}\psi_2(x) = \psi_1'(x) \qquad . \qquad (3.24)$$

A straightforward calculation of the canonical momentum

$$P_2 = - \int dx \{ \psi'_{-1}(x)\dot{\psi}_2(x) + \frac{1}{2}(\dot{\psi}_0(x)\psi'_1(x) + \psi'_1(x)\dot{\psi}_0(x))$$
$$+ \frac{1}{2}(\psi'_0(x)\dot{\psi}_1(x) + \dot{\psi}_1(x)\psi'_0(x))\} \quad , \qquad (3.25)$$

leads to

$$P_2 = \frac{P^3}{2M^2} + \frac{P}{M}\tilde{H}_0 \quad , \qquad (3.26)$$

where

$$\tilde{H}_0 = \int dx \{ V_0^2[\psi_{-1}(x)] + \frac{1}{2}\dot{\tilde{\psi}}_0(x)\dot{\tilde{\psi}}_0(x) - \frac{1}{2}\tilde{\psi}_0(x)\ddot{\tilde{\psi}}_0(x) \} \qquad (3.27)$$

where $V_0^2[\psi_{-1}(x)]$ arises from the counterterms in the "potential" term of the Hamiltonian, $V[\psi(x)]$, defined in equations (2.80) and (2.81).

In order to satisfy equation (3.11) it is necessary to add the terms

$$\frac{P^3}{2M^3}\psi'_{-1}(x) + \frac{P}{M^2}\tilde{H}_0\,\psi'_{-1}(x)$$

to $\dot{\psi}_2(x)$ which means that it is necessary to add

$$\frac{P^3}{2M^3}\,t\,\psi'_{-1}(x) + \frac{P}{M^2}\tilde{H}_0\,t\,\psi'_{-1}(x)$$

to $\psi_2(x)$.

It is then possible, using the same procedure as that in chapter II (see reference 4), to show that with no further modification $\psi_2(x)$ satisfies the canonical commutation relation

$$[\psi_2(x), \dot{\psi}_0(y)] + [\psi_0(x), \dot{\psi}_2(y)] + [\psi_1(x), \dot{\psi}_1(y)] = 0 \ . (3.28)$$

Thus

$$\psi_2(x) = -\frac{1}{3!}(q+\frac{P}{M}t)^3\psi_{-1}'''(x) - \frac{1}{4M^2}[(q+\frac{P}{M}t)P^2 + P^2(q+\frac{P}{M}t)]$$

$$\times (x\,\psi_{-1}''(x) + \psi_{-1}'(x))$$

$$+ \frac{P^3}{2M^3}\,t\,\psi_{-1}'(x) + \frac{P}{M^2}\tilde{H}_o\,t\,\psi_{-1}'(x) + \frac{1}{2}(q+\frac{P}{M}t)^2\,\tilde{\psi}_o''(x)$$

$$+ \frac{1}{2M}[(q+\frac{P}{M}t)P + P(q+\frac{P}{M}t)][x\,\dot{\tilde{\psi}}_o'(x) + \dot{\tilde{\psi}}_o(x)]$$

$$+ \frac{1}{2!}\frac{P^2}{M^2}[x\,\tilde{\psi}_o'(x) + x^2\,\ddot{\tilde{\psi}}_o(x) - t\,\dot{\tilde{\psi}}_o(x)] - \frac{P}{M}x\,\dot{\tilde{\psi}}_1(x)$$

$$- (q+\frac{P}{M}t)\,\tilde{\psi}_1'(x) + \tilde{\psi}_2(x) \qquad (3.29)$$

The operator $\tilde{\psi}_2(x)$ is a solution of equation (3.23). When this solution is put in normal ordered form the part which is linear in $\tilde{\psi}_o(x)$ is the one-loop quantum correction of the physical particle wavefunction.

Notice that in both the first and second orders, once the dynamical map is adjusted so that the canonical momentum has the correct form (step (ii)), the equal-time commutation relation is automatically satisfied. Whether this situation persists in higher orders is at present unknown. If this is the case, step (iv) of the computation scheme would be unnecessary.

The quantity $\tilde{H}_o$ given in equation (3.27) is the free Hamiltonian of the physical field $\tilde{\psi}_o(x)$. It contains a counterterm the origin of which will be examined in the next section.

2. THE HAMILTONIAN

The counterterms appearing in the renormalized Hamiltonian also produce additional terms in the "potential" which was defined in equation (2.81). These are of higher orders in λ^2:

$$V[\psi^f(x)] = \sum \frac{1}{k!} V_k^m[\psi_{-1}(x)]\ \psi_{\alpha_1}(x)\ldots\psi_{\alpha_k}(x) \tag{3.30}$$

where $k + m + \alpha_1 \ldots + \alpha_k = n + 2$.

Equation (2.86) for the nth order Hamiltonian becomes

$$H_n = \int dx\ \{ \sum [\frac{1}{2} \dot{\psi}_k(x)\dot{\psi}_{n-k}(x) + \frac{1}{2} \psi_k'(x)\psi_{n-k}'(x) + \frac{1}{2} m^2\ \psi_k(x)\psi_{n-k}(x)]$$

$$+ \sum \frac{1}{k!} V_k^n[\psi_{-1}(x)]\ \psi_{\alpha_1}(x)\ldots\psi_{\alpha_k}(x) \quad . \tag{3.31}$$

The first few orders of equation (3.31) are

$$H_{-2} = M \tag{3.32}$$

$$H_{-1} = 0 \tag{3.33}$$

$$H_0 = \frac{P^2}{2M} + \tilde{H}_c \tag{3.34}$$

where $\tilde{H}_0$ is the quantity given in equation (3.27). This means that the term containing $\tilde{H}_0$ in $\psi_2(x)$ is finite when the renormalized Hamiltonian is finite. The quantity $V_0^2[\psi_{-1}(x)]$ is defined by

$$V_0^2[\psi_{-1}(x)]' = - F_0^2[\psi_{-1}(x)] \tag{3.35}$$

to within the addition of a constant which must be adjusted so that the Hamiltonian is finite. Since this constant must in general be divergent it is not defined uniquely by this condition. It is usually defined by physical considerations (see reference 6) as the one-loop ground state energy of the vacuum sector. The vacuum matrix element of the Hamiltonian when an extended object is present is then defined in reference to the ground state energy of the vacuum sector. Equation (3.34) can be written as

$$\tilde{H}_0 = \frac{P^2}{2M} + \frac{1}{2}\int dx \, : \dot{\tilde{\psi}}_0(x)\dot{\tilde{\psi}}_0(x) - \tilde{\psi}_0(x)\ddot{\tilde{\psi}}_0(x) :$$

$$+ \int dx \, \{\frac{1}{2} \langle 0_F|\dot{\tilde{\psi}}_0(x)\dot{\tilde{\psi}}_0(x) - \tilde{\psi}_0(x)\ddot{\tilde{\psi}}_0(x)|0_F\rangle + V_0^{2}[\psi_{-1}(x)]\}. \quad (3.36)$$

The last term in equation (3.36) is the one-loop correction of the ground state energy (the mass of the extended object).

In the next section, the computational scheme outlined here will be applied to a particular model - the $\lambda\phi^4$ model in 1+1-dimensions. This model is known to have a topologically nontrivial solution of the classical field equation. In the quantum field theory, this solution is the tree approximation of the space-dependent order parameter.

3. THE $\lambda\phi^4$ MODEL

Consider now a 1+1-dimensional boson model with Lagrangian density

$$\mathcal{L}(x) = \frac{1}{2}\,\partial_\mu\phi(x)\partial^\mu\phi(x) + \frac{1}{2}\,\mu^2\phi^2(x) - \frac{1}{4}\,\lambda\phi^4(x) \quad . \quad (3.37)$$

Initially, the solution of interest exhibits a constant non-vanishing vacuum expectation value. The renormalized field, $\psi(x)$, and the renormalized vacuum expectation value, v, are defined by

$$\phi(x) = Z^{\frac{1}{2}}(v + \psi(x)) \quad , \qquad \langle 0|\psi(x)|0\rangle = 0 \quad (3.38)$$

where Z is the wave function renormalization factor. The vertex and mass renormalization factors are defined by

$$\lambda = \frac{g^2}{2} Z_1 Z^{-1} \tag{3.39}$$

$$\mu = \frac{m^2}{2} Z_2 Z^{-1} \tag{3.40}$$

The parameters Z, Z_1 and Z_2 are determined by the following conditions:

$$\Delta^{-1}(p) \Big|_{p^2=m^2} = 0 \tag{3.41}$$

$$\frac{\partial}{\partial p^2} \Delta^{-1}(p) \Big|_{p^2=m^2} = 1 \tag{3.42}$$

$$\Gamma(p, q, r) \Big|_{p^2=q^2=r^2=m^2} = 3mg \tag{3.43}$$

Here $\Delta(p)$ is the propagator and $\Gamma(p, q, r)$ is the three point vertex function of the ψ field:

$$<0|T\psi(x)\psi(y)|0> = \frac{i}{(2\pi)^2} \int d^2p \; \Delta(p) e^{ip(x-y)} \tag{3.44}$$

$$<0|T\psi(x)\psi(y)\psi(z)|0> = [\frac{i}{(2\pi)^2}]^2 \int d^2p \; d^2q \; d^2r \; \delta(p+q+r)$$

$$e^{i(px+qy+rz)} \Delta(p)\Delta(q)\Delta(r) \; \Gamma(p, q, r) \tag{3.45}$$

The determination of the parameters in the one-loop approximation was presented in detail in reference 6 using the standard techniques of vacuum sector quantum field theory. The resulting field equation is

$$(\partial^2 + m^2)\psi(x) = \frac{3mg}{2} \Delta_o + \frac{g^2}{2} \{ 3\Delta_o - \frac{\sqrt{3}}{2} \} \psi(x)$$

$$- \frac{g^3}{m} \{ \frac{13\sqrt{3}}{16} + \frac{9}{8\pi} \} \psi^2(x) - \frac{g^4}{m^2} \{ \frac{5}{12}\sqrt{3} + \frac{3}{4\pi} \} \psi^3(x)$$

$$- \frac{3mg}{2} \psi^2(x) - \frac{g^2}{2} \psi^3(x) \quad , \tag{3.46}$$

where

$$\Delta_o = \frac{1}{2\pi} \int dk \; \frac{1}{2\sqrt{k^2 + m^2}} \quad . \tag{3.47}$$

Equation (3.44) leads to

$$F_o^o[\psi_{-1}(x)] = - \frac{3m}{2} \psi_{-1}^2(x) - \frac{1}{2} \psi_{-1}^3(x) \tag{3.48}$$

$$F_o^2[\psi_{-1}(x)] = \frac{3m}{2} \Delta_o + \left(\frac{3\Delta_o}{2} - \frac{\sqrt{3}}{4} \right) \psi_{-1}(x) - \frac{1}{m} \left(\frac{13\sqrt{3}}{16} + \frac{9}{8\pi}\right) \psi_{-1}^2(x)$$

$$- \frac{1}{m^2} \left\{\frac{5\sqrt{3}}{12} + \frac{3}{4\pi}\right\} \psi_{-1}^3(x) \tag{3.49}$$

where g has taken the role of the power counting parameter. The tree approximation to the space-dependent order parameter is a quantity which satisfies the equation

$$(\partial^2 + m^2) \psi_{-1}(x) = F_o^o[\psi_{-1}(x)] \tag{3.50}$$

where $F_o^o[\psi_{-1}(x)]$ is given in equation (3.48). A solution of this equation is

$$\psi_{-1}(x) = m (\zeta(x) - 1) \tag{3.51}$$

where

$$\zeta(x) = \tanh \frac{m}{2} (x - a) \tag{3.52}$$

The self-consistent potential is

$$F_1^o[\psi_{-1}(x)] = -\ 3m\ \psi_{-1}(x) - \frac{3}{2}\,\psi_{-1}^2(x) \tag{3.53}$$

$$= \frac{3m}{2}\,(1 - \zeta(x)^2) \tag{3.54}$$

and equation (3.5) becomes

$$\{\partial^2 - \frac{m^2}{2}\,(1 - 3\zeta(x)^2)\}\ \psi_o(x) = 0 \quad . \tag{3.55}$$

Equation (3.51) has solutions and eigen-frequencies

$$\psi'_{-1}(x) = \frac{m}{2}\,(1 - \zeta^2(x)) \tag{3.56}$$

$$u_1(x) = \sqrt{\frac{3m}{4}}\ \zeta(x)\ (1 - \zeta^2(x)) \quad , \quad \omega_1 = \sqrt{\frac{3}{4}}\ m \quad , \tag{3.57}$$

$$u(x,\ m\ u) = [1 - \frac{3\zeta^2(x)(1 - \zeta^2(x))}{1 + 4\ u^2} - \frac{3(1 - \zeta^2(x))^2}{4(1 + u^2)}]^{\frac{1}{2}}$$

$$\times\ \exp\{i\ \delta(mu,\ x) + ikx\} \tag{3.58}$$

where

$$\delta(mu,\ x) = \tan^{-1}\left[\frac{-12u(1+u^2)(1-\zeta(x)) + 4u(1-\zeta(x))^2}{2(1+4u^2)(1+u^2) - 6(1+u^2)(1-\zeta(x)) + 3(1-2u^2)(1-\zeta(x))^2}\right] \tag{3.59}$$

and

$$\omega_k = \sqrt{k^2 + m^2} \tag{3.60}$$

The wave functions given in equations (3.56), (3.57), and (3.59) form a complete set.

$$\frac{\psi'_{-1}(x)\psi'_{-1}(y)}{M} + u_1(x)u_1(y) + \int\frac{dk}{2\pi}\,u(x,\ k)u^*(y,\ k) = \delta(x - y) \tag{3.61}$$

The physical field $\tilde{\psi}_o(x)$ has the form

$$\tilde{\psi}_0(x) = \frac{1}{\sqrt{2\omega_1}} u_1(x) \left(\alpha_1 e^{-i\omega_1 t} + \alpha_1 e^{i\omega_1 t}\right)$$

$$+ \int \frac{dk}{\sqrt{4\pi\omega_k}} [u(x, k)\alpha_k e^{-i\omega_k t} + u^*(x, k)\alpha_k^\dagger e^{i\omega_k t}] \qquad (3.62)$$

Using the fact that

$$\frac{\partial}{\partial x} = \frac{\partial\zeta}{\partial x}\frac{\partial}{\partial\zeta} = \frac{m}{2}(1 - \zeta(x)^2)\frac{\partial}{\partial\zeta} \qquad (3.63)$$

equation (3.21) for the one-loop correction to the order parameter can be written as

$$\{\frac{\partial^2}{\partial t^2} - \frac{m^2}{4}(1-\zeta^2)\frac{\partial}{\partial\zeta}(1-\zeta^2)\frac{\partial}{\partial\zeta} - \frac{m^2}{2}(1-3\zeta^2)\}\{\bar{\psi}_1(\zeta) - \frac{1}{m}\left(\frac{7}{8\sqrt{3}} + \frac{3}{4\pi}\right)\}$$

$$= m\{\frac{\sqrt{3}}{8} + \frac{9}{16\pi}\}\zeta + m\{\frac{-13\sqrt{3}}{24} - \frac{15}{8\pi}\}\zeta^3 + m\{\frac{3\sqrt{3}}{24} + \frac{9}{16\pi}\}\zeta^5 . \qquad (3.64)$$

where the quantity

$$\langle 0_F|\tilde{\psi}_0^2(x)|0_F\rangle = \Delta_0 + \frac{\sqrt{3}}{12}\zeta^2(1-\zeta^2) - \frac{3}{8\pi}(1-\zeta^2)^2 \qquad (3.65)$$

was computed in reference 6.

This equation can be integrated using the boundary condition that $\bar{\psi}_1(x) \to 0$ as $x \to \infty$ and assuming that the order parameter is static to get

$$\bar{\psi}_1(x) = -\frac{1}{m}\left(\frac{7\sqrt{3}}{24} + \frac{3}{4\pi}\right)(\zeta(x) - 1) + \frac{1}{m}\left(\frac{\sqrt{3}}{12} + \frac{3}{8\pi}\right)\zeta(x)(1 - \zeta^2(x)) \qquad (3.66)$$

which is the same as the result of reference 6.

Equation (3.66) satisfies

$$\bar{\psi}_1(x) \to -2\left[-\frac{1}{m}\ \frac{7\sqrt{3}}{24} + \frac{3}{4\pi}\right] \quad \text{as } x \to \infty \quad , \tag{3.67}$$

where the quantity in square brackets is the one-loop quantum correction of the order parameter, v, in the vacuum sector.

Now, consider the Hamiltonian. From equation (3.56),

$$M = \int \psi'_{-1}(x)\psi'_{-1}(x)\,dx = 2m^3/3 \quad . \tag{3.68}$$

From equation (3.35) and (3.49),

$$V_0^2[\psi_{-1}(x)] = -\frac{3m}{2}\Delta_0\,\psi_{-1}(x) - \left(\frac{3\Delta_0}{2} - \frac{\sqrt{3}}{4}\right)\ \frac{1}{2}\psi_{-1}^2(x)$$

$$+\frac{1}{m}\left(\frac{13\sqrt{3}}{16} + \frac{9}{8\pi}\right)\ \frac{1}{3}\psi_{-1}^3(x) + C \quad . \tag{3.69}$$

The constant is to be chosen as the one-loop ground state energy in the vacuum sector. This is calculated in reference 6 as

$$C = -\frac{m^2}{8}\left[6\,\Delta_0 - \frac{\sqrt{3}}{4} - \frac{9}{4\pi}\right] - \int\frac{dk}{4\pi}\ \omega_k \quad . \tag{3.70}$$

The relation

$$\frac{1}{2}\langle 0_F|\dot{\tilde{\psi}}_0(x)\dot{\tilde{\psi}}_0(x) - \tilde{\psi}_0(x)\ddot{\tilde{\psi}}_0(x)|0_F\rangle = \frac{\sqrt{3}}{16}m^2\zeta^2(x)\ (1-\zeta^2(x))$$

$$-\frac{3}{4}m^2\ (1-\zeta^2(x))\ \Delta_0 + \int\frac{dk}{4\pi}\,\omega_k \tag{3.71}$$

is also derived in reference 6.

Since the last terms of equation (3.70) and (3.71) are quadratically divergent, their sum must be treated carefully. To investigate this situation, it is convenient to split the points in the operator products (in the left hand side of equation (3.71)) by a space-like distance ε. Then the sum of the last terms in equations (3.70) and (3.71) is

$$E_0(x) = \int \frac{dk}{4\pi} \; [\omega_k \; e^{i(k+\delta'(k,x))\varepsilon} - \omega_k e^{ik\varepsilon}] \tag{3.72}$$

The splitting has no effect on the other terms in equations (3.70) and (3.71). The quantity $E_0(x)$ is evaluated by requiring that the energies be compared between states with the same wave numbers for small ε. The result is

$$\begin{aligned} E_0(x) &= \lim_{\varepsilon \to 0} \int \frac{dk}{4\pi} [\omega_k - (1 + \frac{d}{dk}\delta'(k, x)\omega_k + \delta'(k, x)]e^{ik} \\ &= - \int \frac{dk}{4\pi} \frac{d}{dk} [\sum_{n=0}^{\infty} \frac{d^n k}{dk^n} \frac{(\delta'(k, x))^{n+1}}{(n+1)!}] \\ &= - \frac{1}{4\pi} \omega_k \; \delta'(k, x) \Big|_{-\infty}^{\infty} \\ &= - \frac{3m^2}{8\pi} (1 - \zeta^2(x)) \quad . \end{aligned} \tag{3.73}$$

Combining equations (3.69), (3.70), (3.71) and (3.73) leads to

$$\tilde{H}_0 \doteq \frac{1}{2} \int dx \; :\dot{\tilde{\psi}}_0(x)\dot{\tilde{\psi}}_0(x) - \tilde{\psi}_0(x)\ddot{\tilde{\psi}}_0(x): - \left(\frac{1}{2\sqrt{3}} + \frac{3}{\pi}\right) m \tag{3.74}$$

The energy of the soliton in the one-loop approximation is given by equation (3.68) plus the last term in equation (3.74) as

$$E = \frac{2m^3}{3g^2} - \left(\frac{1}{2\sqrt{3}} + \frac{3}{\pi} \right) m \tag{3.75}$$

4. SOME GENERAL CONSIDERATIONS

In section 1 it was found that, to the order considered, the counterterms always appear in the dynamical map implicitly through the operators $\tilde{H}_0$ and $\tilde{\psi}_n(x)$. Whether this happens in higher orders is not known. This does, however, support the idea that the appearance of q and P in the dynamical map can be summarized by the replacement of the space time coordinates (t, x) by some generalized coordinates (T, X) which are quantum mechanical operators.[1,2,5] It is evident that the generalized coordinates must contain the operator $\tilde{H}_0$. This means that they do not commute with the fields $\tilde{\psi}_n(x)$. Thus the operator ordering must be carefully specified when (t, x) are replaced by (T, X). This will be discussed in chapter IV.

Consider, now, the operators $\tilde{\psi}_n(x)$. They have so far been determined as functionals of the free physical fields $\tilde{\psi}_0(x)$; with ordinary product ordering leading to the dynamical map

$$\tilde{\psi}(x) = \sum \int dy_1 \ldots dy_n \, c_0(x, y_1, \ldots, y_n) \tilde{\psi}_0(y_1) \ldots \tilde{\psi}_0(y_n) \quad . \qquad (3.76)$$

The functions $c_0(x, y_1 \ldots, y_n)$ contain counterterms. The process of putting the operator product on the left hand side of equation (3.76), into normal ordered form produces contractions which, when combined with the counterterms are the quantum corrections.

A given operator can be put in normal ordered form by using the following combinatorial formula

$$\tilde{0}(x) = \sum_{n=0}^{\infty} \frac{1}{m!} \int dy_1 \ldots dy_m \, :\tilde{\psi}_0(y_1) \ldots \tilde{\psi}_0(y_m): \times \langle 0_F | \frac{\delta}{\delta\tilde{\psi}_0(y_1)} \ldots \frac{\delta}{\delta\tilde{\psi}_0(y_n)} \tilde{0}(x) | 0_F \rangle \quad . \qquad (3.77)$$

Equation (3.76) can be rewritten as

$$\tilde{\psi}(x) = \sum_m \int dy_1 \ldots dy_m \, c(x, y_1, \ldots, y_m) : \tilde{\psi}_0(y_1) \ldots \tilde{\psi}_0(y_m) : \qquad (3.78)$$

where the functions $c(x, y_1,\ldots,y_n)$ can be determined in terms of $c_0(x, y_1,\ldots,y_n)$ by using equation (3.77).

$$c_0(x, y_1,\ldots,y_n) = \sum_{n\geq k} \int dz_1\ldots dz_n \; c_0(x, z_1,\ldots,z_n)$$

$$\times \sum_{\text{perm}} \delta(y_1 - z_{j_i})\ldots\delta(y_k - z_{j}) \; \langle 0_F|\tilde{\psi}_0(z_{j_{k+1}})\ldots\tilde{\psi}_0(z_{j_n})|0_F\rangle \tag{3.79}$$

where the sum over permutations includes all permutations of the values of the indices $(j_i,\ldots j_k)$ over $(1,\ldots,n)$ with the remaining variables $(z_{j_{k+1}},\ldots,z_{j_i})$ inside the vacuum expectation value retaining their consecutive ordering.

The quantity $c(x)$ is the full space-dependent order parameter, while $c(x, y)$ is a kernel which changes the tree approximation of the physical particle wavefunction into the full wavefunction.

So far, considerations have been limited to field theories in 1+1-dimensions with no internal degrees of freedom. The purpose of this simplification is to focus attention on the role of the quantum coordinate for position. In higher space-time dimensions and in more complicated field theories there may be many more quantum coordinates associated with extended objects.

For example, in 1+3-dimensions, a nonspherical object would induce a nonspherical self-consistent potential. This nonspherical self-consistent potential may lead to angular momentum transfer when physical particles scatter from the extended object. Thus, angular momentum conservation is not manifest and must be recovered through the appearance of a quantum coordinate which corresponds to the intrinsic spin of the extended object. This is shown quantitatively in reference 2.

There may also be quantum coordinates corresponding to internal degrees of freedom representing, say, the intrinsic isospin of an extended object.[15,16,17,18,19] Since these coordinates may not commute with each other or with the physical particle fields, the structure of the Hilbert space for such a theory is very complicated. The theory for extended objects as developed here is only, so far, applicable when the quantum coordinates for translation are the only ones which appear.

There is also some difficulty in treating an extended object which would arise from an explicitly time dependent order parameter. Such objects would induce a time-dependent self-consistent potential and physical fields of well-defined frequency would not exist. This problem may, however, be only a limitation of the perturbative methods developed here. Soliton-soliton scattering in the Sine-Gordon theory has been studied using the collective coordinate method.[20,21,22] There, quantum corrections of the soliton-soliton scattering phase shift were calculated. However, the structure of the physical Hilbert space is unknown.

In the next chapter an intuitive development of the formalism of quantum field theory with a static extended object in terms of the asymptotic condition will be outlined.

CHAPTER IV

THE ASYMPTOTIC CONDITION AND HAMILTONIAN

In the previous two chapters, the tool of perturbation theory was used to illuminate the basic features of a quantum field theory with an extended object. One significant feature is the appearance of the quantum coordinates. It has been shown quite generally that the quantum coordinates appear whenever there is an extended object.[2,3] The existence of q means that the position of the extended object is a quantum mechanical quantity which can be taken to be q itself. This means that q appears in the combination $x - q$ in all quantities where reference is made to the position of the extended object.

Furthermore, it is reasonable to define the positions of all quantities of the theory with reference to the position of the extended object. Then q appears as $x - q$ everywhere that x appears explicitly. It is assumed that a solution of this type can be found self-consistently.

The canonical momentum, P, must generate spatial translations. A minimal condition which is sufficient for this requirement is

$$[q, P] = i \qquad (4.1)$$

and that P commutes with the physical particle creation and annihilation operators, that is, that P is the canonical conjugate of q. This is another condition which it is assumed can be imposed on the solution.

The positions of the physical particles are defined with respect to the position of the extended object. The physical particle wave-functions are modified by their interaction with the extended object through the self-consistent potential. If the self-consistent potential is finite and sufficiently localized, the physical particles are practically free at large distances from the object. This consideration has already been used in the discussion of renormalization and quantum corrections in chapter III. In this region, the physical particle wavefunctions obey the translation invariant field equation

$$(\partial^2 + m^2)\chi^{in}(x) \;\text{[illegible]} \tag{4.2}$$

where

$$\chi^{in}(x) = \lim_{\substack{<|x-q|> \to \infty \\ t \to -\infty}} [\psi(x,\ t) - <0_F|\psi(x,\ t)|0_F>] \tag{4.3}$$

is the asymptotic field. Here, $<|x-q|>$ is the average of $|x-q|$ in a wavepacket state. The limit in equation (4.3) is to be understood as a limit of matrix elements of each side for wavepacket states. The infinite time limit eliminates the particle-particle interactions. The asymptotic region must be defined with respect to the position of the extended object which is the quantum mechanical quantity, q.

It is for this reason that the appearance of q as $x - q$ is an essential feature of the formulation of quantum field theory with an extended object in terms of the asymptotic condition.

The Heisenberg field as expressed by the dynamical map is a functional of the creation and annihilation operators α_k and $\alpha_k^\dagger$ as well as the quantum coordinate and its canonical conjugate, q and P. The asymptotic field is also a functional of these operators.

In this chapter the consideration will again be confined to 1+1-dimensional systems.

1. THE POINCARÉ ALGEBRA AND THE POSITION OPERATOR

It is assumed that the operators q and P and the creation and annihilation operators obey the algebra

$$[q, P] = i \quad , \tag{4.4}$$

$$[\alpha_k, \alpha_\ell^\dagger] = \delta(k - \ell) \quad , \tag{4.5}$$

with all other commutators vanishing. Thus, P does not contain any creation or annihilation operators. It is appropriate to introduce the other generators of the Poincaré group, the Hamiltonian, H, and the boost generator, M_{01}. Together with P, they satisfy the algebra

$$[P, H] = 0 \quad , \tag{4.6}$$

$$[P, M_{01}] = iH \quad , \tag{4.7}$$

$$[H, M_{01}] = iP \quad . \tag{4.8}$$

Considering the problem at hand, it is useful to introduce the position operator

$$Q = -\frac{1}{2}(M_{01} H^{-1} + H^{-1} M_{01}) \tag{4.9}$$

This operator represents the center of mass of a quantum system in configuration space. It obeys the commutation relations

$$[Q, P] = i \tag{4.10}$$

$$[Q, H] = i\, PH^{-1} \quad . \tag{4.11}$$

Through the dynamical map, the Heisenberg fields are expressed in terms of the operators $\{q, P, \alpha_k, \alpha_k^\dagger\}$:

$$\psi^f(x) = \psi^f(x - q,\ t,\ P,\ \alpha_k^\dagger,\ \alpha_k)$$

Then the operators Q, P, H, M_{01} are also expressed in terms of these operators. From equation (4.6), H cannot contain q. From equation (4.6) and (4.10),

$$Q = q + \delta Q(P,\ \alpha_k^\dagger,\ \alpha_k) \tag{4.12}$$

and

$$[\delta Q,\ P] = 0$$

Using equations (4.11) and (4.12) leads to

$$\frac{i\partial H}{\partial P} + [\delta Q,\ H] = i\ PH^{-1} \tag{4.13}$$

It is now necessary to proceed with the further assumption that H contains the creation and annihilation operators α_k and $\alpha_k^\dagger$ only in the combination $\alpha_k^\dagger \alpha_k = n_k$. Then equation (4.13) leads to the fact that δQ also has this property and, therefore, that

$$[\delta Q,\ H] = 0 \tag{4.14}$$

This means that

$$\frac{\partial H}{\partial P} = PH^{-1} \tag{4.15}$$

Equation (4.7) then leads to the fact that M_{01} also contains α_k and $\alpha_k^\dagger$ only in the combination $\alpha_k^\dagger \alpha_k$. Equation (4.15) can be integrated to get

$$H = \sqrt{P^2 + M^2} \tag{4.16}$$

where M depends only on $\alpha_k^\dagger \alpha_k$.

2. THE POSITION OPERATOR AND THE QUANTUM COORDINATE

The dynamical maps of all Heisenberg fields are expressed in terms of the operator set $\{q, P, \alpha_k^\dagger, \alpha_k\}$. Consider the unitary transformation

$$\psi(x) \to e^{iN} \psi(x) e^{-iN} \tag{4.17}$$

where

$$-\frac{\partial}{\partial P} N(P, \alpha^\dagger \alpha) = \delta Q(P, \alpha^\dagger \alpha) \tag{4.18}$$

Under this transformation

$$\alpha_k \to e^{iN} \alpha_k e^{-iN} = \alpha_k'(P, \alpha_k) \tag{4.19}$$

$$P \to P \tag{4.20}$$

$$q \to e^{iN} q e^{-iN} = q' \tag{4.21}$$

$$Q \to e^{iN} Q e^{-iN} = q \tag{4.22}$$

$$\begin{aligned} \psi(x - q, t, P, \alpha_k, \alpha_k^\dagger) &\to \psi(x - q', t, P, \alpha_k', \alpha_k^{\dagger\prime}) \\ &= \psi'(x - q, t, P, \alpha_k, \alpha_k^\dagger) \end{aligned} \tag{4.23}$$

and

$$Q[\psi'] = q \tag{4.24}$$

Since N commutes with H and P, $\psi'(x)$ satisfies the Heisenberg equation. It is thus possible to choose a representation where $q = Q$. In the following it is assumed that this representation is chosen.

Then, the operator q obeys the algebra of equations (4.10) and (4.11). Equation (4.11) leads to

$$\dot{Q} = PH^{-1} \tag{4.25}$$

and

$$\ddot{Q} = 0 \tag{4.26}$$

where the Heisenberg representation operator $Q(t)$ is defined by

$$Q(t) = e^{iHt}\, q\, e^{-iHt} = q + \dot{Q}t \tag{4.27}$$

Equations (4.10), (4.11) and (4.25) lead to

$$[q, \dot{Q}] = i(1 - \dot{Q}^2)\, H^{-1} \tag{4.28}$$

The Lorentz boost generator M_{01} is given by

$$M_{01} = -\frac{1}{2}(qH + Hq) \tag{4.29}$$

which, combined with equation (4.28) leads to

$$i[\dot{Q}, M_{0k}] = -(1 - \dot{Q}^2) \tag{4.30}$$

and

$$i[q, M_{01}] = \frac{1}{2}(q\dot{Q} + \dot{Q}q) \tag{4.31}$$

which means that

$$i[Q(t), M_{01}] = -t + \frac{1}{2}\{Q(t)\dot{Q} + \dot{Q}Q(t)\} \tag{4.32}$$

3. THE MASS OPERATOR

In equation (4.16), the Hamiltonian has the form $H = \sqrt{P^2 + M^2}$ where $M(n_k)$ is the mass operator. It is a Poincaré invariant quantity which does not contain P. If a state is chosen where $\langle p^2 \rangle$ is very close to zero, then, far away from the extended object in the infinite past, the energy of the physical state is determined by counting the individual particle energies:

$$H = \int dk\, \omega_k\, \alpha_k^\dagger \alpha_k + \mu = M(n_k) \qquad (4.33)$$

With this consideration, the states which are bound to the extended object must be included.

$$M = \sum_i \omega_i\, \alpha_i^\dagger \alpha_i + \int dk\, \omega_k\, \alpha_k^\dagger \alpha_k + \mu \qquad (4.34)$$

where

$$\omega_k^2 = k^2 + m^2 \qquad (4.35)$$

and μ is the mass of the extended object.

The form of the Hamiltonian in equation (4.16) means that the observable energy levels of the quantum system are those of the physical particles which are broadened by the quantum fluctuation of the momentum of the extended object.

This form of the Hamiltonian agrees with the expansion computed in chapter II (equation (2.123)).

4. THE TRANSFORMATION PROPERTIES

This section will discuss how the Lorentz transformation of the

Heisenberg fields is induced through operator transformations of the physical operators.

Defining $\dot{Q}(\theta)$ by

$$\dot{Q}(\theta) = e^{-iM_{01}\theta}\,\dot{Q}\,e^{iM_{01}\theta} \tag{4.36}$$

leads to

$$\frac{\partial}{\partial\theta}\dot{Q}(\theta) = -(1 - \dot{Q}^2(\theta)) \tag{4.37}$$

which can be integrated to get

$$\dot{Q}(\theta) = \tanh(A - \theta) \tag{4.38}$$

where

$$\dot{Q} = \tanh A \tag{4.39}$$

Then, $q(\theta)$, which is defined by

$$q(\theta) = e^{-iM_{01}\theta}\,q\,e^{iM_{01}\theta} \tag{4.40}$$

can be found using equation (4.31) to be (see reference 5)

$$q(\theta) = \frac{1}{2}\{B\cosh(A-\theta)^{-1} + \cosh(A-\theta)^{-1}B^{\dagger}\} , \tag{4.41}$$

where

$$B = q\cosh A \tag{4.42}$$

The mass operator $M(n_k)$ and $(B+B^{\dagger})/2$ are invariant under Poincaré group transformations. Consider the generalized spatial coordinate

$$X(x, t) = \frac{1}{2}\left\{\frac{1}{\sqrt{1-\dot{Q}^2}}(x - Q(t)) + (x - Q(t))\frac{1}{\sqrt{1-\dot{Q}^2}}\right\} . \tag{4.43}$$

Using equations (4.9), (4.11) and (4.27), equation (4.43) can be rewritten in terms of A and B as

$$X(x, t) = x \cosh A - t \sinh A - \frac{1}{2}(B + B^{\dagger}) \quad . \tag{4.44}$$

Then using equations (4.38), (4.39) and (4.42) leads to

$$\begin{aligned} X(x, t, \theta) &= e^{-iM_{01}\theta} X(x, t)\, e^{iM_{01}\theta} \\ &= x \cosh(A - \theta) - t \sinh(A - \theta) - \frac{1}{2}(B + B^{\dagger}) \\ &= x' \cosh A - t' \sinh A - \frac{1}{2}(B + B^{\dagger}) \end{aligned} \tag{4.45}$$

where

$$\begin{aligned} x' &= x \cosh\theta + t \sinh\theta \\ t' &= x \cosh\theta + t \sinh\theta \end{aligned} \tag{4.46}$$

or

$$X(x, t, \theta) = X(x', t') \tag{4.47}$$

It can be proven that $X(x, t)$ is the only function which is linear in x and which satisfies the condition (4.47) together with the condition $x = X$ when $Q = \dot{Q} = 0$. This is called the "q-c transmutation": when the coordinates (x, t) appear through the form $X(x, t)$, the Lorentz transformation and spacetime translations of (x, t) are induced by operator transformations of q and P. Since H and M_{01} contain only the operators P and n_k, in general they induce a p-dependent change of phase of the creation and annihilation operators. Where the explicit spacetime coordinates in the dynamical map do not appear in the phases of creation and annihilation operators, ~~their Poincaré~~ transformations must be induced through the qc transmutation, in particular, the quantity $\varphi(x, t)$ defined by

$$\langle 0_F|\psi(x, t)|0_F\rangle = \varphi(x, t)\big|_{n_k=0} \qquad (4.48)$$

must be a function of (x, t) through X(x, t).

$$\varphi(x, t) = \varphi(X(x, t)) \quad . \qquad (4.49)$$

The Heisenberg field has the form

$$\psi(x, t) = \varphi(X(x, t)) + \psi^{in}(x, t) + \ldots \qquad (4.50)$$

where $\psi^{in}(x, t)$ is the physical field in the presence of the extended object. It satisfies a linear field equation which contains the self-consistent potential induced by the extended object. As in equation (4.3), in the limit as $\langle|x - q|\rangle \to \infty$, $\psi^{in}(x, t) \to \chi^{in}(x, t)$, when $\chi^{in}(x, t)$ satisfies the free field equation

$$\left(-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} - m^2\right) \chi^{in}(x, t) = 0 \quad . \qquad (4.51)$$

In this asymptotic region, the Poincaré generators must still induce the Poincaré transformations. The way in which q and P appear in these generators has already been determined in equations (4.16) and (4.29). The asymptotic field, $\chi^{in}(x - q, t)$, contains P, q, and n_k as well as α_k and $\alpha_k^\dagger$. It must be shown that χ^{in} contains P and n_k in such a way that the Poincaré transformations are consistent with equation (4.51) and the form of the mass operator in equation (4.34). This was done in reference 5.

There, it was shown that, when $\chi^{in}(x, t)$ is expanded as

$$\chi^{in}(x-q,\ t) = \frac{1}{\sqrt{2\pi}}\int\frac{dk}{\sqrt{2\omega_k}}\,[A_k(p,\ \alpha,\ \alpha^{\dagger})e^{ik(x-q)-i\omega_k t} + c.c.] \quad (4.52)$$

the condition

$$\chi^{in}(x,\ t) = e^{iHt}\,\chi^{in}(x,\ 0)\,e^{-iHt} \quad , \qquad (4.53)$$

together with the form of the Hamiltonian and mass operators in equations (4.16) and (4.34) and the fact that χ^{in} must obey the field equation (4.51),

$$\omega_k^2 = k^2 + m^2 \qquad (4.54)$$

lead to the form

$$\chi^{in}(x,\ t) = \frac{1}{\sqrt{2\pi}}\int d\ell\ f^{\ell}\ P_q\{\exp(ik_\ell(x-q) - i(\omega_{k_\ell} t)\alpha_\ell\} + c.c., \quad (4.55)$$

where

$$\omega_{k_\ell} = m\cosh(\theta_\ell + A) \qquad (4.56)$$

$$k_\ell = m\sinh(\theta_\ell + A) \qquad (4.57)$$

and

$$m\cosh\theta_\ell = \omega_\ell + \ell^2/2m \quad . \qquad (4.58)$$

In equation (4.55), the ordering symbol P_q means that q should be put on the right hand side of all other operators.

The coefficient f_ℓ is determined by the condition

$$\chi^{in}(x',\ t') = e^{-iM_{01}\theta}\,\chi^{in}(x,\ t)\,e^{iM_{01}\theta} \qquad (4.59)$$

as

$$f_\ell = \sqrt{\frac{M(n_k)\cosh A + m\cosh(\theta_\ell + A)}{M(n_k)\cosh A}}\; f_o^\ell \; . \tag{4.60}$$

Equation (4.55) can be written as

$$\chi^{in}(x,\ t) = \frac{1}{\sqrt{2\pi}}\int d\ell\ f^\ell\ P_q\ \exp\{i\,\tilde{\ell}\,X(x,\ t) - i\tilde{\omega}_\ell T(x,\ t)\}\ \alpha_\ell + \text{c.c.} \tag{4.61}$$

where

$$\tilde{\omega}_\ell = m\cosh\theta_\ell \tag{4.62}$$

$$\tilde{\ell} = m\sinh\theta_\ell \tag{4.63}$$

and $X(x,\ t)$ and $T(x,\ t)$ are the generalized coordinates given in equation (2.22).

It has thus been shown that the form of the Hamiltonian given by equations (4.16) and (4.34) is consistent with the asymptotic condition of equations (4.3) and (4.51).

5. SUMMARY

In this chapter the intuitive picture of a quantum field theory with an extended object which was gained through the perturbative computations of the previous two chapters was used to derive some general properties of such a system. An important aspect of this picture, the appearance of the quantum coordinate, has been regarded as essential in the definition of the asymptotic region. Furthermore, since all distances are measured with reference to the position of the extended object, the total momentum plays the role of the canonical conjugate of the quantum coordinate and is independent of the physical particle-like fields. The further reasonable assumption that the Hamiltonian is diagonal in the physical Hilbert space is substantiated by the perturbative calculations of chapters II and III.

Given this framework and the algebra of the generators of the Poincaré group the fact that these generators must generate Poincaré transformations in the asymptotic region is used to demonstrate the consistency of the derived form of the Hamiltonian and the appearance of the quantum coordinate with the asymptotic condition. A further result is the form of the asymptotic field itself.

These considerations have been limited to 1+1-dimensions. In reference 5 similar considerations were made for 3+1-dimensions where there appear no quantum coordinates besides those for position. There, the results are similar to those outlined here. However, the situation where quantum coordinates corresponding to internal degrees of freedom or spin appear has not been explored in this formalism.

CHAPTER V

QUANTUM ELECTRODYNAMICS IN SOLIDS

In this chapter an application of the boson method will be described. Most of the theory developed in the previous chapters is as yet useful only for simple models. The boson method itself has, however, been used to study "classical" aspects of extended systems. These are situations where the size of the extended objects is very large. In many experimental situations, observable effects are dominated by the overall classical shape of the extended structure and the microscopic structure which must be described by the full quantum theory is relatively less important.

This is the case in many aspects of the quantum field theoretic analysis in condensed matter physics. In systems such as crystals, where the microscopic behavior is described by phonons, many extended structures such as dislocations, point defects and grain boundaries may be present. These are large objects and, to a good approximation they can be treated as classical objects. The quantum coordinates and quantum fluctuations of the objects shape can, at least at the outset, be ignored. Many interesting physical phenomena are within the range of this type of analysis.[43,44]

The classical macroscopic behavior is governed by a set of equations which are obtained from the microscopic Heisenberg field equations. These macroscopic equations retain some of the information contained in the Heisenberg equations. Thus, some microscopic quantities are of direct relevance to the macroscopic properties of a system. This idea is familiar from linear response theory.[27] The classical macroscopic structure of a system is determined by solving these macroscopic equations with given boundary conditions.

This chapter contains a discussion at the fundamental level of the electromagnetic properties of solids. In quantum electrodynamics, the local gauge invariance is an important symmetry which imposes certain restrictions on particle interactions. These restrictions appear in the form of certain relations among the Green functions, known as Ward-Takahashi relations.[25,26] These relations also provide information which helps to determine the quasi-particle structure of the theory. Once the physical particle picture is established, the boson method is used to derive a set of macroscopic equations which, when solved with certain boundary conditions determine the macroscopic structure of the system. This program has been followed for the theory of superconductivity.[28,29,30] The work reviewed here[31] concentrates on the normal phase and is intended to be complementary to the previously developed theories of superconductivity.

The electromagnetic properties of solids are a widely discussed subject in solid state physics.[32,33,34] A common derivation of the plasma modes consists of looking for the collective oscillation due to the Coulomb interactions in an electron gas.[32] From the quantum field theoretic point of view this oscillation appears as the quasi-photon or the plasmon which acquires a mass through the coulomb interactions. Therefore, an important task in the quantum field theoretic analysis is the calculation of the proper self-energy of the photon.

In the quantum theoretical treatment of the collective mode which was presented in reference 32, the gauge condition for the vector potential is very complicated. There, certain auxiliary degrees of freedom are introduced and are then eliminated by a subsidiary condition which serves to recover gauge invariance. This procedure has become known as the collective field method.[45] In this method, however, the properties of the quantum states are obscure.

An important prerequisite for the applicability of the boson method to a given theory is the establishment of the physical particle picture. This means that the theory must have a well defined particle interpretation in its homogeneous phase. It is well known that the local gauge invariance of quantum electrodynamics makes the physical particle picture difficult to establish. In relativistic quantum electrodynamics, the second quantization of the electromagnetic field requires the specification of the gauge condition.[35,36] In the Gupta-Bleuler formalism the Hilbert space is divided into physical and unphysical spaces and the unphysical Hilbert space is composed of particle states which do not interact with physical particles. There, the redundant components of the vector potential do not appear in the physical matrix elements of observable quantities becuase of the Gupta-Bleuler gauge condition.

It is also known that the choice of gauge condition in quantum electrodynamics is not arbitrary when consistency with the physical particle picture is required.[37,38] The physical or quasi-particle picture means that the Hilbert space considered is the linear space of wave-packet states of physical or quasi-particles.

The gauge condition can be formulated so that it becomes one of the field equations which is obtained from the Lagrangian.[39] This method will be used in the following in order to study quantum electrodynamics in solids. In this study, first a particle interpretation will be derived, then the macroscopic equations will be found. The model treated will be that of an electron gas in a uniform positive background charge.

The macroscopic equations determine the "classical" electromagnetic fields and the electric current in a solid in the presence of given external fields and currents. These equations can be linearized. In this approximation they become the classical Maxwell equations.

1. THE CANONICAL FORMALISM, DYNAMICAL MAPS AND GAUGE SYMMETRY

In the electron gas model, the electrons interact through the electromagnetic interaction as well as other interactions. The dynamics of the system are governed by a Lagrangian density

$$\mathcal{L}(x) = \mathcal{L}[A_\mu(x), \psi(x)] \tag{5.1}$$

where $A_\mu(x)$ is the electromagnetic vector potential and $\psi(x)$ is the electron Heisenberg field. The Lagrangian density is invariant under the local gauge transformation

$$A_\mu(x) \to A_\mu(x) + \partial_\mu\lambda(x) \tag{5.2}$$

$$\psi(x) \to e^{-ie\lambda(x)} \psi(x) \quad , \tag{5.3}$$

$$\mathcal{L}[A_\mu(x) + \partial_\mu\lambda(x), e^{-ie\lambda(x)}\psi(x)] = \mathcal{L}[A_\mu(x), \psi(x)] \quad . \tag{5.4}$$

In this chapter the metric with signature (-,+,+,+) will be used. In the summation convention used, Greek indices are summed over 0,1,2,3 and Roman indices over 1,2,3. The system of units employed is where

$\hbar = c = 1$ and $e^2/4\pi\hbar c = 1/137$ (Heaviside-Lorentz rationalized units).
The Lagrangian has the form

$$\mathcal{L}(x) = -\frac{1}{4} F_{\mu\nu}(x)\, F^{\mu\nu}(x)$$
$$- \mathcal{L}_e[\psi(x),\ (\partial_\mu + ieA_\mu(x))\psi(x),\ F_{\mu\nu}(x)] \tag{5.5}$$

where

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) \tag{5.6}$$

In the parabolic band model which will be studied in the following sections $\mathcal{L}_e(x)$ is given by

$$\mathcal{L}_e(x) = i\psi^\dagger(x)(\partial^0 + ieA^0(x))\psi(x) - \frac{1}{2m}[(\vec{\nabla} - ieA(x))\psi^\dagger(x)\cdot(\vec{\nabla} + ie\vec{A}(x))\psi(x)$$
$$- K_F^2\psi^\dagger(x)\psi(x)] - en_e A_0(x) - V[\psi^\dagger,\ \psi,\ F_{ij}] \ . \tag{5.7}$$

In the V-term, ψ and $\psi^\dagger$ carry no derivatives. The constant $-en_e$ is the positive ion charge density. The electromagnetic interaction in the first two terms is called the minimal interaction and the electromagnetic interaction in the V-term is called the magnetic interaction.

In order to remedy the complication that the Lagrangian density in equation (5.5) does not contain a canonical conjugate of A_0 it is necessary to introduce a supplementary field B(x). This is accomplished by adding to the Lagrangian density the terms

$$\mathcal{L}_{GF} = B(x)D_\mu(\partial)A^\mu(x) - \frac{1}{2} B(x)\alpha(-i\vec{\nabla})B(x) \tag{5.8}$$

where

$$D^\mu(\partial) = \left(\frac{\partial}{\partial t}\ ,\ v^2(-i\vec{\nabla})\ \vec{\nabla}\right) \ . \tag{5.9}$$

Now, B(x) becomes the canonical conjugate of $A_0(x)$.

The field equations obtained from the Lagrangian are

$$i(\partial^0 + ieA^0(x))\psi(x) = -\frac{1}{2m}\{(\vec{\nabla} + ie\vec{A}(x))^2 - K_F^2\}\,\psi(x) + \frac{\partial V}{\partial\psi^\dagger(x)} \quad (5.10)$$

$$-\partial^\nu F_{\nu\mu}(x) = j_\mu(x) - D_\mu(\partial)B(x) \quad (5.11)$$

$$D^\mu(\partial)A_\mu(x) = \alpha(-i\vec{\nabla})B(x) \ , \quad (5.12)$$

where

$$j_\mu(x) = g_\mu^o\,(-e(\psi^\dagger(x)\psi(x) - n_e)) + g_\mu^i\{\frac{ie}{2m}(\psi^\dagger(x)\nabla_i\psi(x) - \nabla_i\psi^\dagger(x)\psi(x)) - \frac{e^2}{m}A_i(x)\psi^\dagger(x)\psi(x)\} \ . \quad (5.13)$$

Equation (5.12) is the gauge condition

Overall charge neutrality requires that

$$\int_\Omega d^3x\, \langle 0|j_o(x)|0\rangle = 0 \quad (5.14)$$

where Ω is a unit cell of the lattice. When the ion charge distribution is assumed to be uniform, equation (5.14) leads to

$$\langle 0|\psi^\dagger(x)\psi(x)|0\rangle = n_e \ . \quad (5.15)$$

The current given in equation (5.13) is conserved,

$$\partial^\mu j_\mu(x) = 0 \quad (5.16)$$

This together with the field equation (5.11) leads to

$$\partial^\mu D_\mu(\partial)B(x) = 0 \ , \quad (5.17)$$

that is, $B(x)$ is a free field. This implies that $B(x)$ does not interact with the other fields. The requirement that all matrix elements of observables between physically observable states should satisfy the usual Maxwell equation leads to the condition

$$\langle a|D_\mu(\partial)B(x)|b\rangle = 0 \qquad (5.18)$$

for the physical states $|a\rangle$ and $|b\rangle$. This condition splits the Hilbert space into physical and unphysical subspaces.

Since $B(x)$ does not interact with the other fields, these two subspaces are not mixed by the dynamics. Equation (5.18) is called the physical state condition.

The canonical momenta of the fields are obtained from the Lagrangian as

$$\pi_i(x) = -F_0^i(x) \qquad (5.19)$$

$$\pi_0(x) = B(x) \qquad (5.20)$$

$$\pi_\psi(x) = -i\psi^\dagger(x) \qquad (5.21)$$

Using these, the following equal-time canonical commutation relations can be derived

$$\{\psi(x), \psi^\dagger(y)\} = \delta(\vec{x} - \vec{y}) \qquad (5.22)$$

$$[A_i(x), A_j(y)] = [A_i(x), A_0(y)] = [A_i(x), B(y)]$$

$$= [A_0(x), A_0(y)] = [B(x), B(y)] = 0 \qquad (5.23)$$

$$[A_0(x), B(y)] = i\,\delta(\vec{x} - \vec{y}) \qquad (5.24)$$

$$[A_i(x),\ \partial^0 A_j(y)] = i\,\delta_{ij}\,\delta(\vec{x} - \vec{y}) \tag{5.25}$$

$$[A^0(x),\ \partial^0 A_i(y)] = 0 \tag{5.26}$$

$$[B(x),\ \partial^0 A_i(y)] = i\,\partial_i^y\,\delta(\vec{x} - \vec{y}) \tag{5.27}$$

$$[A_i(x),\ \partial^0 B(y)] = i\,\partial_i^x\,\delta(\vec{x} - \vec{y}) \tag{5.28}$$

$$[A_0(x),\ \partial^0 B(y)] = 0 \tag{5.29}$$

$$[B(x),\ \partial^0 B(y)] = 0 \tag{5.30}$$

$$[A_i(x),\ \partial^0 A_0(y)] = [B(x),\ \partial^0 A_0(y)] = 0 \tag{5.31}$$

$$[A^0(x),\ \partial^0 A_0(y)] = i\,\alpha(-i\vec{\nabla}_y)\,\delta(\vec{x} - \vec{y}) \tag{5.32}$$

Equations (5.17), (5.24) and (5.29) lead to (see reference 31)

$$\partial^\mu D_\mu(\partial_x)\ \langle 0|T\ B(x)\ A_\nu(y)|0\rangle = -\,i\partial_\nu^x\,\delta^{(4)}(x - y) \tag{5.33}$$

and

$$\partial^\mu D_\mu(\partial_y)\ \langle 0|T\ j_\nu(x)\ B(y)|0\rangle = 0 \tag{5.34}$$

Equations (5.33) and (5.34) lead to

$$\langle 0|T\ B(x)\ A_\nu(y)|0\rangle = -\,i\partial_\nu^x\,\frac{1}{D(\partial)}\,\delta^{(4)}(x - y) \tag{5.35}$$

$$\langle 0|T\ j_\nu(x)\ B(y)|0\rangle = 0 \tag{5.36}$$

where

$$D(\partial) = \partial^\mu D_\mu(\partial) \tag{5.37}$$

Equations (5.17), (5.23) and (5.30) imply that

$$[B(x), B(y)] = 0 \tag{5.38}$$

for any x and y, that is, that B has zero norm. Thus, B must be a combination of positive and negative norm particles

$$B(x) = Z_b^{-\frac{1}{2}}(-i\vec{\nabla})\,(b^o(x) - \chi^o(x)) \tag{5.39}$$

where b^o and χ^o are free fields satisfying

$$D(\partial)\, b^o(x) = D(\partial)\, \chi^o(x) = 0 \tag{5.40}$$

and

$$[\chi^o(x), \partial^o\chi^o(y)]_{x^o=y^o} = -\,[b^o(x), \partial^o b^o(y)]_{x^o=y^o} = i\,\delta(\vec{x}-\vec{y}), \tag{5.41}$$

It will be shown later that a vector quasi-particle, the plasmon, $U_\mu^o(x)$ appears in the dynamical map of $A_\mu(x)$. Its field equation can be put in the following form[29,30]

$$\{-\partial^2 g_\mu{}^\lambda + \partial_\mu\partial^\lambda + m^2{}_\mu{}^\lambda(-i\vec{\nabla})\}\, U_\lambda^o(x) = 0 \tag{5.42}$$

Then equations (5.33), (5.34) and (5.39) lead to the following dynamical maps (see reference 31)

$$\begin{aligned} A_\mu(x) = {} & Z^{\frac{1}{2}}{}_\mu{}^\lambda(-i\vec{\nabla})U_\lambda^o(x) + [\tilde{\alpha}(-i\vec{\nabla})\partial_\mu \\ & + \tilde{\alpha}_s(-i\vec{\nabla})\nabla_\mu]\, Z_o^{-\frac{1}{2}}(-i\vec{\nabla})\,(b^o(x) - \chi^o(x)) \\ & - Z_b^{\frac{1}{2}}(-i\vec{\nabla})\partial_\mu b^o(x) + :\tilde{A}_\mu(x;\, U_\nu^o,\, b^o - \chi^o,\, \varphi^o): \end{aligned} \tag{5.43}$$

$$\begin{aligned} j_\mu(x) = {} & -\,Z^{\frac{1}{2}}{}_\mu{}^\lambda(-i\vec{\nabla})m^2{}_\mu{}^\sigma(-i\vec{\nabla})\; U_\sigma^o(x) \\ & + (1 + \tilde{\alpha}_s(-i\vec{\nabla})\vec{\nabla}^2)Z_b^{-\frac{1}{2}}(-i\vec{\nabla})D_\mu(\partial)(b^o(x) - \chi^o(x)) \end{aligned}$$

$$: j_\mu(x;\, U^o_\nu,\, b^o - \chi^o(x),\, \varphi^o) : \tag{5.44}$$

$$B(x) = Z^{-\frac{1}{2}}(-i\vec{\nabla})\,(b^o(x) - \chi^o(x)) \tag{5.45}$$

$$\psi(x) = Z_\psi^{\frac{1}{2}}(-i\vec{\nabla})\,\varphi^o(x) + \ldots \tag{5.46}$$

where

$$\alpha_\mu(\partial) = \tilde{\alpha}(-i\vec{\nabla})\partial_\mu + \tilde{\alpha}_s(-i\vec{\nabla})\nabla_\mu \ ,$$
$$\nabla_\mu = (0, \vec{\nabla}) \tag{5.47}$$

and equation (5.12) leads to

$$\alpha(-i\vec{\nabla}) = v^2(-i\vec{\nabla})\,\tilde{\alpha}_s(-i\vec{\nabla})\,\vec{\nabla}^2 \ . \tag{5.48}$$

In equation (5.46), $\varphi^o(x)$ is the quasi-electron. In equations (5.43) and (5.44), $\tilde{A}_\mu$ and $\tilde{j}_\mu$ contain nonlinear terms in U^o_ν, $b^o - \chi^o$ and φ^o.

Consider the local gauge transformation,

$$e^{-iQ(t,\lambda)}\,\psi(x)\,e^{iQ(t,\lambda)} = e^{-ie\lambda(x)}\,\psi(x) \tag{5.49}$$

$$e^{-iQ(t,\lambda)}\,A_\mu(x)\,e^{iQ(t,\lambda)} = A_\mu(x) + \partial_\mu\lambda(x) \ . \tag{5.50}$$

The generator, $Q(t, \lambda)$, is given by

$$Q(t, \lambda) = \int d^3x\,[-\pi_i(x)\nabla_i\lambda(x) - \pi_o(x)\partial_o\lambda(x)$$
$$- e(\psi^\dagger(x)\psi(x) - n_e)\lambda(x)] \ . \tag{5.51}$$

When $\lambda(x)$ is smooth and damps rapidly enough as $|\vec{x}| \to \infty$, integration by parts and the field equation (5.11) lead to

$$Q(t, \lambda) = \int d^3x\; B(x)\,\overleftrightarrow{\partial_o}\,\lambda(x) \ . \tag{5.52}$$

When

$$\partial^\mu D_\mu(\partial)\lambda(x) = 0 \qquad (5.53)$$

the generator is time independent

$$\dot{Q}(t, \lambda) = 0$$

and

$$e^{-iQ(t,\lambda)}\chi^o(x)\, e^{iQ(t,\lambda)} = \chi^o(x) - Z_o^{-\frac{1}{2}}(-i\vec{\nabla})\lambda(x) \qquad (5.54)$$

$$e^{-iQ(t,\lambda)}b^o(x)\, e^{iQ(t,\lambda)} = b^o(x) - Z_b^{-\frac{1}{2}}(-i\vec{\nabla})\lambda(x) \quad . \qquad (5.55)$$

This implies that the local gauge transformations of the Heisenberg fields is induced by the field translation of the quasi-particle fields χ^o and b^o. Therefore, the dynamical maps take the form

$$\psi(x) = :\exp[ieZ_b^{\frac{1}{2}}(-i\vec{\nabla})\chi^o(x)]\ \Psi(x;\, U_\mu^o,\, b^o - \chi^o,\, \varphi^o): \qquad (5.56)$$

$$A_\mu(x) = :\, - Z_b^{\frac{1}{2}}(-i\vec{\nabla})\partial_\mu b^o(x) + \bar{A}_\mu(x;\, U_\mu^o,\, b^o - \chi^o,\, \varphi^o): \, . \qquad (5.57)$$

Then, in equations (5.56) and (5.57), the field translation in equations (5.54) and (5.55) induce the gauge transformation of the Heisenberg fields even when $D(\partial)\lambda(x) \neq 0$. The gauge symmetry is rearranged into the translational symmetry of χ^o and b^o. The dynamical rearrangement of the local gauge symmetry here is the same as that in the superconducting phase.[28] Consider the phase symmetry,

$$e^{-i\theta Q}\psi(x)\, e^{i\theta Q} = e^{-ie\theta}\psi(x) \qquad (5.58)$$

The generator is given by

$$Q = -\int d^3x\ j_o(x) \quad , \tag{5.59}$$

which is the total charge. Since the current is conserved, Q is time-independent. Therefore, the dynamical map of Q must consist of terms which are linear and bilinear in the physical fields. Furthermore, the linear terms should be gapless bosons. Since b^o, χ^o and U_μ^o have no charge, bilinear terms of these fields do not appear in Q. Therefore, Q must have the form

$$Q = \int d^3x\ \{C_B\ \partial_o(b^o(x) - \chi^o(x)) - e_r\ \varphi^{o\dagger}(x)\ \varphi^o(x)\} \tag{5.60}$$

Then Q generates the transformations

$$e^{-i\theta Q}\ \varphi^o(x)\ e^{i\theta Q} = e^{-ie_r\theta}\ \varphi^o(x) \tag{5.61}$$

$$e^{-i\theta Q}\ \chi^o(x)\ e^{i\theta Q} = \chi^o(x) + C_B\theta \tag{5.62}$$

$$e^{-i\theta Q}\ b^o(x)\ e^{i\theta Q} = b^o(x) + C_B\theta \quad . \tag{5.63}$$

Equation (5.56) shows that $\psi(x)$ carries the operator factor $\exp[ieZ_b^{\frac{1}{2}}(-i\vec{\nabla})\chi^o(x)]$. Thus the transformation in equation (5.62) creates the phase factor $\exp[iCZ_b^{\frac{1}{2}}(0)\theta]$. Since there is no spontaneous breakdown of the phase transformation, $\psi(x)$ should be proportional to $\varphi^o(x)$. Thus, the total phase created by the transformations of equations (5.61) and (5.62) is $\exp[i(C_B Z_b^{\frac{1}{2}}(0) - e_r)\theta]$ which should equal $\exp[-ie\theta]$. Therefore

$$e_r - C_B Z_b^{\frac{1}{2}}(0) = e \tag{5.64}$$

If $|\varphi^o\rangle$ is a state containing one physical electron, then from equation (5.60)

$$<\varphi^o|Q|\varphi^o> = -e_r <\varphi^o|\varphi^o> \qquad (5.65)$$

and e_r is the renormalized charge. Then

$$C_B = Z_b^{-1/2}(0)(Z_0(0) - 1)e \qquad (5.66)$$

where $Z_0(0)$ is the charge renormalization factor (see reference 31, appendix B). From equation (5.44)

$$Z_0(p) = \alpha_s(p)\,\vec{p}^{\,2} \qquad (5.67)$$

The phase transformation here is thus induced by a combination [illegible]eld translation of the b^o and χ^o fields and a phase transform-[illegible]n of φ^o. The quasi-electron field transforms with the renormalized charge and the field translation of b^o and χ^o compensates the difference between the renormalized and bare charges.

This is in contrast to the theory of superconductivity where, because of the spontaneous breakdown of phase symmetry, the entire phase transformation is induced by the field translation of b^o and χ^o.[28,29,30]

The physical state condition (5.18) can be expressed as

$$(b^o(x) - \chi^o(x))^{(-)}\,|\,\mathrm{phys}> = 0 \qquad (5.68)$$

where the symbol (-) denotes the parts of b^o and χ^o containing annihilation operators. Gauge invariant operators have the form

$$O_H(x) = O(x;\, U_\mu^o,\, b^o - \chi^o,\, \varphi^o) \qquad (5.69)$$

2. THE PHOTON SELF-ENERGY

In this section the Heisenberg field equations and the canonical commutation relations will be used to analyze the properties of the quasi-photon which is called the plasmon.

Consider the following definitions of the momentum space Green functions

$$<0|T\, A_\mu(x)A_\nu(y)|0> = \frac{i}{(2\pi)^4}\int d^4p\; e^{ip(x-y)}\Delta_{\mu\nu}(p) \tag{5.70}$$

$$<0|T\, j_\mu(x)A_\nu(y)|0> = \frac{i}{(2\pi)^4}\int d^4p\; e^{ip(x-y)}\; J^A_{\mu\nu}(p) \tag{5.71}$$

$$<0|T\, j_\mu(x)j_\nu(y)|0> = \frac{i}{(2\pi)^4}\int d^4p\; e^{ip(x-y)}\; J_{\mu\nu}(p)\quad . \tag{5.72}$$

Use of the field equations (5.10 - 5.12) and the equal-time commutation relations (5.19 - 5.32) (see reference 31) leads to the following set of relations:

$$(p^2 g_\mu{}^\lambda - p_\mu p^\lambda)\Delta_{\lambda\nu}(p) = - g_{\mu\nu} + \frac{D_\mu(p)p_\nu}{D(p)} + J^A_{\mu\nu}(p) \tag{5.73}$$

$$J^A_{\mu\nu}(p)(p^2 g^\nu{}_\lambda - p^\nu p_\lambda) = \tilde{J}_{\mu\lambda}(p) \tag{5.74}$$

$$D^\lambda(p)\Delta_{\lambda\nu}(p) = \alpha(\vec{p})p_\nu \frac{1}{D(p)} \tag{5.75}$$

$$J^A_{\mu\nu}(p)D^\nu(p) = 0 \tag{5.76}$$

$$p^\mu J^A_{\mu\nu}(p) = 0 \tag{5.77}$$

where

$$\tilde{J}_{\mu\nu}(p) = J_{\mu\nu}(p) - g^i_\mu g^j_\nu\, R_{ij}(p) \tag{5.78}$$

and

$$R_{ij}(x)\delta^{(4)}(x-y) = \langle 0|[j_i(x), \partial^0 A_j(y)]|0\rangle \delta(x^0 - y^0) \quad . \qquad (5.79)$$

The proper self-energy function of the photon is defined by

$$J^A_{\mu\nu}(p) = -\sum_\mu{}^\lambda(p)\,\Delta_{\lambda\nu}(p) \quad . \qquad (5.80)$$

With the definition (5.80), equations (5.73 - 5.77) can be solved as (see reference 31)

$$\Delta_{\mu\nu}(p) = -\frac{Z_T(p)}{p^2 + m_T^2(\vec{p})}\,T_{\mu\nu}(p) - \frac{v^2(\vec{p})Z(p)}{D(p)+m^2(\vec{p})}\left[K_{\mu\nu}(p) + \frac{p_\mu p_\nu}{m^2(\vec{p})}\right]$$

$$+ \frac{v^2(\vec{p})Z(p)}{D(p)}\left[K_{\mu\nu}(p) + \frac{p_\mu p_\nu}{m^2(\vec{p})}\right] + (\alpha(\vec{p}) - v^2(\vec{p})Z(p))\,\frac{p_\mu p_\nu}{D(p)^2} \qquad (5.81)$$

$$J^A_{\mu\nu}(p) = \frac{Z_T(p)(m_T^2(\vec{p}) + (p^2 + m_T^2(\vec{p}))(Z_T^{-1}(p) - 1))}{p^2 + m_T^2(\vec{p})}\,T_{\mu\nu}(p)$$

$$+ \frac{D(p) + m^2(\vec{p}) + Z(p)m^2(\vec{p})}{D(p) + m^2(\vec{p})}\left[L_{\mu\nu}(p) - \frac{p_\mu(p)p_\nu}{D(p)}\right] \qquad (5.82)$$

$$J_{\mu\nu}(p) = \frac{p^2 Z_T(p)(m_T^2(\vec{p}) + (p^2 + m_T^2(\vec{p}))(Z_T^{-1}(p) - 1)}{p^2 + m_T^2(\vec{p})}\,T_{\mu\nu}(p)$$

$$+ \frac{D(p) + m^2(\vec{p}) + Z(p)m^2(\vec{p})}{D(p) + m^2(\vec{p})}\,(p^2 L_{\mu\nu}(p) - p_\mu p_\nu) \qquad (5.83)$$

where

$$T_{\mu\nu}(p) = \begin{pmatrix} 0 & 0 \\ 0 & \delta_{ij} - p_i p_j/\vec{p}^2 \end{pmatrix} \qquad (5.84)$$

$$L_{\mu\nu}(p) = \begin{pmatrix} -1 & 0 \\ 0 & \frac{p_i p_j}{p^2} \end{pmatrix} \qquad (5.85)$$

$$K_{\mu\nu}(p) = \begin{pmatrix} -1 & 0 \\ 0 & v^2(\vec{p})\,\frac{p_i p_j}{\vec{p}^2} \end{pmatrix}$$

and, where

$$\Sigma_{\mu\nu}(p) = \Sigma_T(p)T_{\mu\nu}(p) + \Sigma_0(p)(p^2 L_{\mu\nu}(p) - p_\mu p_\nu) \qquad (5.87)$$

$$1 + \Sigma_0(p) = -\frac{D(p) + m^2(\vec{p})}{m^2(\vec{p})}\, Z^{-1}(p) \qquad (5.88)$$

$$p^2 + \Sigma_T(p) = (p^2 + m_T^2(\vec{p}))Z_T^{-1}(p) \qquad (5.89)$$

The quantity $\alpha(\vec{p})$ is determined in such a way that there is no double pole in $\Delta_{\mu\nu}(p)$ in equation (5.81). Thus,

$$\alpha(\vec{p}) = v^2(\vec{p})Z_0(\vec{p}) \qquad (5.90)$$

where

$$Z_0(\vec{p}) = Z(p)\Big|_{D(p) = 0} \qquad (5.91)$$

The pole-terms of the propagators $\Delta_{\mu\nu}$, $J^A_{\mu\nu}$ and $J_{\mu\nu}$ can also be derived from the dynamical maps of $A_\mu(x)$ and $J_\mu(x)$ given in equations (5.43) and (5.44). Comparisons of these with equations (5.81 - 5.83) leads to (see reference 31)

$$\tilde{\alpha}_s(\vec{p}) = -Z_0(\vec{p})/\vec{p}^2 \qquad (5.92)$$

$$Z_b(\vec{p}) - 2\tilde{\alpha}(\vec{p}) = -\frac{Z_o(\vec{p})}{\vec{p}^2} + \frac{v^2(\vec{p})Z_o(\vec{p})}{m^2(\vec{p})} - v^2(\vec{p})\tilde{Z}_o(\vec{p}) \quad , \qquad (5.93)$$

where

$$Z_o(\vec{p}) = Z(p)\Big|_{D(p) = 0} \qquad (5.94)$$

$$\tilde{Z}_o(\vec{p}) = \tilde{Z}(p)\Big|_{D(p) = 0} \qquad (5.95)$$

where

$$Z(p) = Z_o(\vec{p}) + D(p)\,\tilde{Z}(p) \qquad (5.96)$$

Thus, once the proper self-energy of the photon is calculated, equations (5.88) and (5.89) can be used to compute all of the parameters which appear in the Green functions in equations (5.31), (5.82) and (5.83). Equations (5.90), (5.92) and (5.93) can be used to compute terms in the dynamical maps in equations (5.43) and (5.44).

The quantities $\tilde{\alpha}(\vec{p})$, $Z_b(\vec{p})$ and $v^2(\vec{p})$ cannot be determined uniquely. However, because of the physical state condition (5.68) and the form of the gauge invariant operators (5.69), the massless particles b^o and χ^o are unobservable and $Z_b(\vec{p})$ and $\tilde{\alpha}(\vec{p})$ do not appear in any physical quantities. The only conditions required are

$$Z_b(\vec{p}) \neq 0 \quad , \quad v^2(\vec{p}) \neq 0 \quad , \qquad (5.97)$$

since their inverses must be well defined.

In summary, in this and the previous sections, the canonical formalism has been used to establish the physical particle picture for quantum electrodynamics in solids. The quasi-particles are the plasmon U_μ^0, the massless scalar fields b^0 and χ^0 and the quasi-electron field φ^0. In the next section, the boson method will be used to derive equations which the macroscopic electromagnetic fields and electric current must obey.

3. THE MACROSCOPIC EQUATIONS

The dynamical maps of the Heisenberg fields are given in equations (5.43), (5.44) and (5.56). Consider the boson transformation

$$Z_b^{\frac{1}{2}}(-i\vec{\nabla})\chi^0(x) \rightarrow Z_b^{\frac{1}{2}}(-i\vec{\nabla})\chi^0(x) + f(x) \tag{5.98}$$

$$Z_b^{\frac{1}{2}}(-i\vec{\nabla})b^0(x) \rightarrow Z_b^{\frac{1}{2}}(-i\vec{\nabla})b^0(x) + f(x) \tag{5.99}$$

$$Z^{\frac{1}{2}}(-i\vec{\nabla})_\mu{}^\nu U_\nu^0(x) \rightarrow Z^{\frac{1}{2}}(-i\vec{\nabla})_\mu{}^\nu U_\nu^0(x) + u_\mu(x) \tag{5.100}$$

The boson transformations of b^0 and χ^0 in equations (5.98) and (5.99) are required to be the same so that the physical state condition (5.68) is preserved.

It was shown in equations (5.54) and (5.55) that when $f(x)$ in equations (5.98) and (5.99) is a regular function which satisfies the free field equation

$$\partial_\mu D^\mu(\partial)\; f(x) = 0 \tag{5.101}$$

the boson transformation is simply a gauge transformation. Therefore, observable physical phenomena can arise from $f(x)$ only when it is in some way singular. This singularity can be a topological singularity defined by

$$[\partial_\mu, \partial_\nu]f(x) = G^{\dagger}_{\mu\nu}(x) \neq 0 \quad \text{for some } \mu, \nu, x \quad . \tag{5.102}$$

In this case, when the dynamical maps are constructed, terms which arise from the noncommutavity of derivatives must be included.

With the notation

$$f^0(x) = e\, Z_b^{\frac{1}{2}}(-i\vec{\nabla})\chi^0(x) \tag{5.103}$$

$$a^0_\mu(x) = Z^{\frac{1}{2}}(-i\vec{\nabla})_\mu{}^\lambda U^0_\lambda(x) + \alpha_\mu(\partial) Z_b^{-\frac{1}{2}}(-i\vec{\nabla})(b^0(x) - \chi^0(x)) - Z_b^{\frac{1}{2}}(-i\vec{\nabla})\partial_\mu b^0(x) \tag{5.104}$$

the dynamical maps (5.43) and (5.56) have the form

$$A_\mu(x) = a^0_\mu(x) + :\bar{A}_\mu(x): \tag{5.105}$$

$$\psi(x) = :e^{if^0(x)} \{Z_\psi^{\frac{1}{2}}(-i\vec{\nabla})\varphi^0(x) + \bar{\psi}(x)\}: \tag{5.106}$$

Since the field equations for $A_\mu(x)$ and $\psi(x)$ are of the form

$$\lambda(\partial_\mu + ieA_\mu)\psi = j_\psi[\psi, (\partial_\lambda + ieA_\lambda)\psi, F_{ij}] \tag{5.107}$$

$$-\partial^\nu F_{\nu\mu} = j_\mu[\psi, (\partial_\lambda + ieA_\lambda)\psi, F_{ij}] - D_\mu B \tag{5.108}$$

$f^0(x)$ and $a^0_\mu(x)$ appear in the field equations through the combinations $a^0_\mu(x) - \frac{1}{e}\partial_\mu f^0(x)$, $f^0_{\mu\nu}(x) = \partial_\mu a^0_\nu(x) - \partial_\nu a^0_\mu(x)$ and $B(x)$. Therefore, after the noncommutavity of derivatives is considered, the dynamical maps are given by

$$\psi(x) = :e^{if^0(x)}\, \Psi[x; a^0_\mu - \frac{1}{e}\partial_\mu f^0, f^0_{\mu\nu}, B, \varphi^0]: \tag{5.109}$$

$$A_\mu(x) = :a^0_\mu(x) + \tilde{A}_\mu[x; a^0_\lambda - \frac{1}{e}\partial_\lambda f^0, f^0_{\mu\nu}, B, \varphi^0]: \tag{5.110}$$

$$j_\mu(x) = :-\Sigma_\mu^{(1)\lambda}(\partial)[a_\lambda^o(x) - \frac{1}{e}\partial_\lambda f^o(x)] - \Sigma_\mu^{(2)\lambda}(\partial)a_\lambda^o(x)$$
$$+ \tilde{j}_\mu(x;\, a_\lambda^o - \frac{1}{e}\partial_\lambda f^o,\, f_{\mu\nu}^o,\, B,\, \rho^o):$$

Here, $\Sigma_\mu^{(1)\lambda}(\partial)$ is the photon self-energy which connects to an external photon line through the minimal interaction and $\Sigma_\mu^{(2)\lambda}(\partial)$ is connected to an external photon line through the magnetic interaction (see reference 31).

The boson transformation in equations (5.98), (5.99) and (5.100) can be rewritten as

$$f^o(x) \to f^o(x) + f(x) \tag{5.112}$$

$$a_\mu^o(x) \to a_\mu^o(x) + a_\mu(x) \tag{5.113}$$

$$B(x) \to B(x) \tag{5.114}$$

After the boson transformation, the classical fields

$$\mathcal{A}_\mu(x) = \langle 0|A_\mu(x)|0\rangle = a_\mu(x) + \tilde{A}_\mu(x;\, a_\lambda - \frac{1}{e}\partial_\lambda f,\, f_{\lambda\sigma},\, 0,\, 0) \tag{5.115}$$

$$\mathcal{J}_\mu(x) = \langle 0|j_\mu(x)|0\rangle = -\Sigma_\mu^{(1)\lambda}(\partial)(a_\lambda(x) - \frac{1}{e}\partial_\lambda f(x))$$
$$- \Sigma_\mu^{(2)\lambda}(\partial)a_\lambda(x) + \tilde{j}_\mu(x;\, a_\lambda - \frac{1}{e}\partial_\lambda f,\, f_{\lambda\sigma},\, 0,\, 0)\ , \tag{5.116}$$

satisfy the equations

$$-\partial^\mu \mathcal{F}_{\mu\nu}(x) = \mathcal{J}_\nu(x) \tag{5.117}$$

$$D^\mu(\partial)\mathcal{A}_\mu(x) = 0\ , \tag{5.118}$$

where

$$\mathcal{F}_{\mu\nu}(x) = \partial_\mu \mathcal{A}_\nu(x) - \partial_\nu \mathcal{A}_\mu(x) \qquad (5.119)$$

The quantities $\mathcal{A}_\mu$ and $\mathcal{J}_\mu$ are the classical macroscopic fields and currents which are created by the boson transformation. Equations (5.118) and (5.119) must be solved under given boundary conditions to determine these fields. In the next section the linear approximation to equations (5.118) and (5.119) will be examined.

4. LINEAR RESPONSE THEORY AND THE CLASSICAL MAXWELL EQUATIONS

Consider the equations (assuming no magnetic interactions):

$$-\partial^\lambda f_{\lambda\mu}(x) = -\Sigma^\lambda_\mu(\partial)(a_\lambda(x) - \frac{1}{e}\partial_\lambda f(x)) + j^{ext}_\mu(x) \quad , \qquad (5.120)$$

$$f_{\mu\nu}(x) = \partial_\mu a_\nu(x) - \partial_\nu a_\mu(x) \quad , \qquad (5.121)$$

$$D^\mu(\partial) a_\mu(x) = 0 \quad , \qquad (5.122)$$

and

$$D(\partial) f(x) = 0 \quad . \qquad (5.123)$$

Equation (5.120) is obtained by linearizing equation (5.117). The quantity $j^{ext}_\mu(x)$ is an external current which may be supplied to the system, for example, by an electron beam incident on the sample.

The magnetic induction, $\vec{B}(x)$, and the electric field, $\vec{E}(x)$, are given by

$$B_i(x) = \frac{1}{2}\varepsilon_{ijk} f_{jk}(x) \qquad (5.124)$$

$$E_i(x) = f_{oi}(x) \quad . \qquad (5.125)$$

Then, equations (5.87), (5.120), (5.124) and (5.125) lead to

$$\partial_0 \vec{B}(x) - \vec{\nabla} \times \vec{E}(x) = 0 \qquad (5.126)$$

$$\vec{\nabla} \cdot \vec{B}(x) = 0 \qquad (5.127)$$

$$\vec{\nabla} \cdot \vec{E}(x) = - \Sigma_0(\partial)\vec{\nabla} \cdot \vec{E}(x) - \frac{1}{e} \Sigma_0(\partial)\vec{\nabla} \cdot [\vec{\nabla}, \partial_0]f(x) + j_0^{ext}(x) \qquad (5.128)$$

$$\vec{\nabla} \cdot \vec{B}(x) - \partial^0\vec{E}(x) = (\Sigma_T(\partial) + \partial^2\Sigma_0(\partial)) \frac{1}{\vec{\nabla}^2} (\vec{\nabla} \times \vec{B}(x)) - \Sigma_0(\partial)(\vec{\nabla} \times \vec{B}(x) - \partial^0\vec{E}(x)) + \frac{1}{e} (\Sigma_T(\partial) + \partial^2\Sigma_0(\partial)) \times \frac{1}{\vec{\nabla}^2} \nabla_k[\nabla_k, \vec{\nabla}]f(x) - \frac{1}{e} \Sigma_0(\partial)\partial^\lambda[\partial_\lambda, \vec{\nabla}]f(x) + \vec{j}^{ext}(x) \qquad (5.129)$$

Using these equations, and given boundary conditions, the fields $\vec{E}(x)$ and $\vec{B}(x)$ can be determined from $f(x)$, which contributes only when it has a topological singularity.

When an external field is applied to the system (with $j_\mu^{ext}(x)=0$), the state of thermal equilibrium of the system is determined by specifying $f(x)$ in such a way that the Gibbs free energy is minimized.

When the function $f(x)$ is not singular, equations (5.128) and (5.129) lead to the following relations between the test current and the electromagnetic fields:

$$j_0^{ext}(x) = (1 + \Sigma_0(\partial))\vec{\nabla} \cdot \vec{E}(x) \qquad (5.130)$$

$$\vec{j}^{ext}(x) = \vec{\nabla} \times \vec{B}(x) - \vec{E}(x) - \frac{1}{\vec{\nabla}^2} (\Sigma_T(\partial) - \partial_0^2\Sigma_0(\partial))\vec{\nabla} \times \vec{B}(x) - \Sigma_0(\partial)\dot{\vec{E}}(x) \qquad (5.131)$$

The complex dielectric tensor is defined as[40,41,42]

$$j_i^{ind}(p) = ip_o[1 - \hat{\varepsilon}(p)]_{ij}E_j(p) \tag{5.132}$$

where $\vec{j}(p)$ and $\vec{E}(p)$ are the Fourier transforms of $\vec{j}(x)$ and $\vec{E}(x)$.

Since the induced current, $j_\mu^{ind}(x)$ is given by

$$j_\mu^{ind}(x) = -\partial^\lambda f_{\lambda\mu}(x) - j_\mu^{ext}(x) \tag{5.133}$$

and $\vec{B}(x)$ is related to $\vec{E}(x)$ by equation (5.126), equations (5.131) and (5.132) lead to

$$j_i^{ind}(p) = \frac{i}{p_o}\{\Sigma_T(p)\delta_{ij} - (\Sigma_T(p) + p_o^2\Sigma_o(p))\frac{p_ip_j}{\vec{p}^2}\}E_j(p) \tag{5.134}$$

$$j_o^{ind}(p) = i(1 + \Sigma_o(p))\vec{p}\cdot\vec{E}(p) \tag{5.135}$$

comparing equations (5.134) and (5.132) leads to

$$\hat{\varepsilon}_{ij}(p) = \delta_{ij} - \frac{1}{p_o^2}\{\Sigma_T(p)\delta_{ij} - (\Sigma_T(p) + p_o^2\Sigma_o(p))\frac{p_ip_j}{\vec{p}^2}\} \tag{5.136}$$

$$\hat{\varepsilon}_{ij}(p) = \delta_{ij} - \frac{1}{p_o^2}\Sigma_{ij}(p) . \tag{5.137}$$

The displacement field $\vec{D}(x)$, defined by

$$\vec{D}(x) = \hat{\varepsilon}(\partial)\vec{E}(x) \tag{5.138}$$

satisfies

$$\vec{\nabla}\cdot\vec{D}(x) = j_o^{ext}(x) \tag{5.139}$$

Therefore, the complex dielectric constant $\varepsilon(q)$, which is defined by $\vec{\nabla}\cdot\vec{D}(x) = \varepsilon(\partial)\vec{\nabla}\cdot\vec{E}(x)$ is given by

$$\varepsilon(q) = 1 + \Sigma_0(q) \quad . \tag{5.140}$$

The complex conductivity tensor $\hat{\sigma}_{ij}(p)$ defined by

$$j_i^{ind}(p) = \hat{\sigma}_{ij}(p)\, E_j(p) \tag{5.141}$$

is related to $\hat{\varepsilon}_{ij}(p)$ by

$$\hat{\sigma}_{ij}(p) = i\, p_0[1 - \hat{\varepsilon}(p)]_{ij} \tag{5.142}$$

which, considering equation (5.137), leads to

$$\hat{\sigma}_{ij}(p) = i\, \frac{1}{p_0}\, \Sigma_{ij}(p) \quad . \tag{5.143}$$

Note that in equation (5.87), the combination $\Sigma_T(p) + p_0^2\Sigma_0(p)$ must vanish at $\vec{p} = 0$ in order that $\Sigma_{\mu\nu}(p)$ be well defined there. Then $\hat{\varepsilon}_{ij}(p)$ and $\hat{\sigma}_{ij}(p)$ become diagonal in the limit $\vec{p} \to 0$:

$$\hat{\varepsilon}_{ij}(p_0, 0) = (1 + \Sigma_0(p_0, 0))\delta_{ij} \tag{5.144}$$

$$\hat{\sigma}_{ij}(p_0, 0) = -\, i\, p_0\Sigma_0(p_0, 0)\delta_{ij} \tag{5.145}$$

The dielectric constant, ε, and the conductivity, σ, for the static limit are given by the real parts of $\hat{\varepsilon}(p)$ and $\hat{\sigma}(p)$ at $p_0 = 0$, respectively:

$$\varepsilon = \lim_{p_0 \to 0} \mathrm{Re}\,(1 + \Sigma_0(p_0, 0)) \tag{5.146}$$

$$\sigma = \lim_{p_0 \to 0} p_0\ \mathrm{Im}\,\Sigma_0(p_0, 0)\ . \tag{5.147}$$

Thus, in this section the classical Maxwell equations (5.126 - 5.129) have been found. This led to the linear response functions in equations (5.137), (5.140) and (5.143) and in their static limits to the dielectric constant and conductivity, ϵ and σ, in equations (5.146) and (5.147).

5. APPLICATIONS

In addition to the normal phase which has been considered here, this type of analysis has been applied to the superconducting phase.[29,30] The electromagnetic interactions have been found to be of considerable importance in the analysis of the unusual magnetic and superconducting behavior of some ternary compounds, for example $RERh_4B_4$, $REMo_6S_8$, $REMo_6Se_8$, and $RERh_xSn_y$, where RE stands for a rare earth element. These compounds exhibit various unusual properties; the reentrant phase transition to a ferromagnetic normal state at low temperatures in $ErRh_4B_4$[50] and $HoMo_6S_8$[51], the coexistence of superconducting and antiferromagnetic orders in $GdMo_6S_8$, $TbMo_6S_8$, $DyMo_6S_8$, etc.[52-56], a periodic structure near the reentrant phase transition temperatures in $ErRh_4B_4$[57,58] and $HoMo_6S_8$ and the anomalous temperature dependence of the upper critical field near the magnetic phase transition temperature.[52,56,59-63]

In these compounds, the electrons which are responsible for superconductivity are mainly the 4d-electrons of Rh and Mo. The rare earth ions have large magnetic moments which are coupled to each other by a weak exchange interaction. The exchange interaction between the conduction electrons and the rare earth ions has been shown by band theoretical calculations to be extremely weak.[46-49] The conduction electrons interact with each other through the phonon mediated BCS interaction. Since the exchange interaction between the conduction electrons and the rare earth ions is very weak, the electromagnetic interaction is important and the macroscopic equations for the electromagnetic fields in a magnetic superconductor are useful.[64,65]

It was once believed, however, that no matter how weak this exchange interaction, the singularity in the magnetic susceptibility would destroy the superconducting phase.[65] A more precise analysis has shown that this is not the case.[67] A careful treatment of the infrared singularity shows that an infrared divergence is avoided. The only effect of the exchange interaction is to lower the superconducting transition temperature, T_c, slightly. Based on this fact the model which is commonly used in the analysis of magnetic superconductors[64,65] ignores the exchange interaction between the conduction electrons and the rare earth ions. The best present day model[64,65] obtained from these methods and its success indicates that the neglect of this exchange interaction is justified.

CHAPTER VI

DISCUSSION

In the previous chapters some aspects of extended structure in quantum field theory were discussed. The structure of the theory was first discussed in the context of perturbation theory. The tree approximation was used to gain an understanding of the physical Hilbert space of such a system. It was then shown how quantum corrections could be included. Finally, some general consequences of the presence of an extended object were explored with reference to the asymptotic condition.

This work was confined to the study of (1+1)-dimensional models with no continuous internal symmetries. The purpose of this simplification was to isolate the quantum coordinate for position. In higher dimensional models with internal symmetries, there may be many more quantum coordinates. Furthermore, the structure of the generators of the Poincaré group and the internal symmetry groups is very complicated. It has been shown that, in (3+1)-dimensional models, there may be quantum coordinates corresponding to the "spin" of extended objects which have a nonspherical shape. There may also be quantum coordinates corresponding to internal symmetries which are broken by the extended object. All of these complications must be overcome in the study of more realistic physical systems. The appearance of the quantum coordinate and its role in the recovery of Poincaré invariance are features which should survive in many more general situations.

The work in chapter II led to the construction of the physical Hilbert space. Knowledge of the structure of the Hilbert space, together with a systematic method of including quantum corrections, facilitate the perturbative calculation of observable physical quantities such as scattering cross-sections.

The calculations in chapter II did not lead to the gauge condition used in the collective coordinate method. It is possible that this choice of gauge condition results from a different choice of boundary conditions from those chosen here. However, this comparison requires further study.

Finally, the study of quantum electrodynamics in solids has many applications. It would also be interesting to analyze the role of quantum coordinates and the structure of the Hilbert space for this system.

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APPENDIX A

THE QUANTUM COORDINATE

In this appendix, it is proven that the quantum coordinate, Q, appears in the combination $x - Q$ with the spatial coordinate, x, when $\dot{Q} = 0$. Consider the field equation

$$\Lambda(\partial)\psi_\lambda^f(x) = \lambda^{-1}F[\lambda\psi_\lambda^f(x)] \quad . \tag{A.1}$$

With the power series expansion

$$\psi_\lambda^f(x) = \sum_{n=-1}^{\infty} \lambda^n \psi_n(x) \tag{A.2}$$

equation (A.2) can be written as

$$\Lambda(\partial)\psi_n(x) = \sum \frac{1}{\ell!} F_\ell[\psi_{-1}(x)]\psi_{\alpha_1}(x)\ldots\psi_{\alpha_\ell}(x) \tag{A.3}$$

where $n+1 = \ell + \alpha_1 + \ldots + \alpha_\ell$ and $\alpha_1, \ldots, \alpha_\ell \geq 0$. When $n = -1$,

$$\Lambda(\partial)\psi_{-1}(x) = F[\psi_{-1}(x)] \tag{A.4}$$

and, when $n = 0$,

$$\{\Lambda(\partial) - F_1[\psi_{-1}(x)]\}\psi_0(x) = 0 \tag{A.5}$$

which has the solution

$$\psi_0(x) = -Q \cdot \nabla\psi_{-1}(x) + \tilde{\psi}_0(x) \quad . \tag{A.6}$$

Using the definition $\tilde{\psi}_n(x) = \psi_n(x)$ when $Q = \dot{Q} = 0$, and the fact that equation (A.3) must be satisfied even when $Q = 0$ leads to

$$\Lambda(\partial)\tilde{\psi}_n(x) = \sum \frac{1}{\ell!} F_\ell[\psi_{-1}(x)]\tilde{\psi}_{\alpha_1}(x)\ldots\tilde{\psi}_{k_\ell}(x) \quad . \tag{A.7}$$

Theorem: When $\dot{Q} = 0$, the following relation holds:

$$\psi_n(x) = \sum_{k=0}^{n+1} \frac{1}{k!} (-Q \cdot \nabla)^k \tilde{\psi}_{n-k}(x) \quad . \tag{A.8}$$

Proof: Assume that equation (A.8) holds for $\psi_m(x)$ where $m < n$. It has already been seen that this is the case for $\psi_o(x)$. Equations (A.3) and (A.8) lead to

$$\{\Lambda(\partial) - F_1[\psi_{-1}(x)]\} \psi_n(x) = \sum \frac{1}{\ell!} F_\ell[\psi_{-1}(x)]$$

$$\times \frac{(-Q\cdot\nabla)^{k_1}}{k_1!} \tilde{\psi}_{\alpha_1-k_1}(x)\ldots \frac{(-Q\cdot\nabla)^{k_\ell}}{k_\ell!} \tilde{\psi}_{\alpha_\ell-k_\ell}(x) \tag{A.9}$$

where $\alpha_1+\ldots+\alpha_\ell + \ell = n + 1$; $\alpha_1,\ldots,\alpha_\ell \geq 0$; $0 \leq k_1,\ldots,k_\ell \leq \alpha_1 + 1,\ldots, \alpha_\ell + 1$. Equation (A.9) can be rearranged as

$$\{\Lambda(\partial) - F_1[\psi_{-1}(x)]\} \psi_n(x) = \sum \frac{1}{j!(\ell-j)!} F_\ell[\psi_{-1}(x)] \frac{(-Q\cdot\nabla)^{\alpha_1+1}}{(\alpha_1+1)!} \psi_{-1}(x)$$

$$\ldots \frac{(-Q\cdot\nabla)^{\alpha_j+1}}{(\alpha_j+1)!} \psi_{-1}(x) \frac{(-Q\cdot\nabla)^{k_{j+1}}}{k_{j+1}!} \tilde{\psi}_{\alpha_{j+1}-k_{j+1}}(x) \ldots$$

$$\times \frac{(-Q\cdot\nabla)^{k_\ell}}{k_\ell!} \tilde{\psi}_{\alpha_\ell-k_\ell}(x) \tag{A.10}$$

where $0 \leq j \leq \ell$. Equation (A.10) can be further rearranged as

$$\{\Lambda(\partial) - F_1[\psi_{-1}(x)]\} \psi_n(x) = \sum \frac{1}{(\ell-j)!} \frac{1}{\beta_1!\ldots\beta_p!} F[\psi_{-1}(x)]$$

$$\times [\frac{1}{1!} (-Q\cdot\nabla)\psi_{-1}(x)]^{\beta_1}$$

$$\ldots\left[\frac{1}{p!}\,(-Q\cdot\nabla)^{p}\psi_{-1}(x)\right]^{\beta_p}\frac{(-Q\cdot\nabla)^{k_{j+1}}}{k_{j+1}!}\,\tilde{\psi}_{\alpha_{j+1}-k_{j+1}}(x)$$

$$\times\ \ldots\frac{(-Q\cdot\nabla)^{k_\ell}}{k_\ell!}\,\tilde{\psi}_{\alpha_\ell-k_\ell}(x) \qquad \text{(A.11)}$$

where $1\cdot\beta_1 + 2\cdot\beta_2 + \ldots + p\cdot\beta_p = n + 1 - \ell + j - \alpha_{j+1} - \ldots - \alpha_\ell$, $\beta_1 + \ldots + \beta_p = j$. On the right hand side of equation (A.11), when $j = \ell$, the term $\ell = 1$ is missing. If this term is added and subtracted, the derivatives of $\psi_{-1}(x)$ can be combined into

$$\{\Lambda(\partial) - F_1[\psi_{-1}(x)]\}\ \psi_n(x) = \sum \frac{1}{(\ell-j)!\,\eta!}\ (-Q\cdot\nabla)^{\eta}\ F_{\ell-j}[\psi_{-1}(x)]$$

$$\times\ \frac{(-Q\cdot\nabla)^{k_{j+1}}}{k_{j+1}!}\ \tilde{\psi}_{\alpha_{j+1}-k_{j+1}}(x)$$

$$\ldots\ \frac{(-Q\cdot\nabla)^{k_\ell}}{k_\ell!}\ \tilde{\psi}_{\alpha_\ell-k_\ell}(x) - F_1[\psi_{-1}(x)]\,\frac{(-Q\cdot\nabla)^{n+1}}{(n+1)!}\ \psi_{-1}(x)\ , \qquad \text{(A.12)}$$

where $\eta + \ell - j + \alpha_{j+1} + \ldots + \alpha_\ell = n + 1$; or

$$\{\Lambda(\partial) - F_1[\psi_{-1}(x)]\}\ \psi_n(x) = \sum \frac{1}{\xi!\,\eta!}\ (-Q\cdot\nabla)^{\eta}\ F_\xi[\psi_{-1}(x)]$$

$$\times\ \frac{(-Q\cdot\nabla)^{k_1}}{k_1!}\ \tilde{\psi}_{\alpha_1-k_1}(x)$$

$$\ldots\ \frac{(-Q\cdot\nabla)^{k_\xi}}{k_\xi!}\ \tilde{\psi}_{\alpha_1-k_1}(x) - F_1[\psi_{-1}(x)]\ \frac{(-Q\cdot\nabla)^{n+1}}{(n+1)!}\ \psi_{-1}(x) \qquad \text{(A.13)}$$

where $\eta + \xi\ \alpha_1 + \ldots + \alpha_\xi = n + 1$. The right hand side of equation (A.13) can be further rearranged as

$$\{\Lambda(\partial) - F_1[\psi_{-1}(x)]\}\ \psi_n(x) = \sum \frac{1}{\xi!\eta!} (-Q \cdot \nabla)^\eta\ F_\xi[\psi_{-1}(x)]$$

$$\times \frac{(-Q \cdot \nabla)^{k_1}}{k_1!}\ \psi_{\ell_1}(x)$$

$$\therefore \frac{(-Q \cdot \nabla)^{k_\xi}}{k_\xi!}\ \tilde{\psi}_{\ell_\xi}(x) - F_1[\psi_{-1}(x)] \frac{(-Q \cdot \nabla)^{n+1}}{(n+1)!}\ \psi_{-1}(x) \quad , \qquad \text{(A.14)}$$

where $\eta + \xi + k_1 + \ldots + k_\xi + \ell_1 + \ldots + \ell_\xi = n + 1$; $k_1, \ldots, k_\xi \geq 0$; $\ell_1 \ldots, \ell_\xi \geq 0$; which leads to

$$\{\Lambda(\partial) - F_1[\psi_{-1}(x)]\}\ \psi_n(x) = \sum \frac{1}{\xi!\eta!\zeta!} (-Q \cdot \nabla)^\eta\ F_\xi[\psi_{-1}(x)](-Q \cdot \nabla)^\zeta$$

$$\{\tilde{\psi}_{\ell_1}(x) \ldots \tilde{\psi}_{\ell_\xi}(x)\} - F_1[\psi_{-1}(x)] \frac{(-Q \cdot \nabla)^{n+1}}{(n+1)!}\ \psi_{-1}(x) \quad , \qquad \text{(A.15)}$$

where $\eta + \xi + \zeta + \ell_1 + \ldots + \ell_\xi = n + 1$; $n \geq 0$; $\ell_1, \ldots, \ell_\xi \geq 0$; $\eta + \xi \geq 2$. A term on the right hand side of equation (A.15) with $\xi = 1$ and $\eta = 0$ can satisfy the first condition of the summation but not the fourth, that $\xi + \eta \geq 2$. The fourth condition can be released by adding and subtracting this term on the right hand side (the terms with $\xi = 0$, $\eta = 0, 1$ cannot satisfy the first condition):

$$\{\Lambda(\partial) - F_1[\psi_{-1}(x)]\}\ \psi_n(x) = \sum' \frac{1}{\xi!\eta!\zeta!} (-Q \cdot \nabla)^\eta\ F_\zeta[\psi_{-1}(x)]$$

$$\times\ (-Q \cdot \nabla)^\zeta\ \{\psi_{\ell_1}(x) \ldots \psi_{\ell_\xi}(x)\} - \sum F_1[\psi_{-1}(x)] \frac{(-Q \cdot \nabla)^\zeta}{\zeta!}\ \psi_{n-\zeta}(x)$$

$$- F_1[\psi_{-1}(x)] \frac{(-Q \cdot \nabla)^{n+1}}{(n+1)!}\ \psi_{-1}(x) \quad , \qquad \text{(A.16)}$$

where $0 \le \zeta \le \eta$ in the second sum. Equation (A.16) can be written as

$$\{\Lambda(\partial) - F_1[\psi_{-1}(x)]\}\ \psi_n(x) = \sum \frac{(-Q\cdot\nabla)^\eta}{\eta!}\ \{F_\xi[\psi_{-1}(x)]\tilde{\psi}_{\ell_1}(x)\ldots\tilde{\psi}_{\ell_\xi}(x)$$

$$- \sum_{\zeta=0}^{n+1} F_1[\psi_{-1}(x)]\ \frac{(-Q\cdot\nabla)^\zeta}{\zeta!}\ \tilde{\psi}_{\eta-\zeta}(x)\ \} \qquad \text{(A.17)}$$

which, using equation (A.7) leads to

$$\{\Lambda(\partial) - F_1[\psi_{-1}(x)]\}\ \psi_n(x) = \sum_{\eta=0}^{n+1} \{\ \frac{(-Q\cdot\nabla)^\eta}{\eta!}\ \ \Lambda(\partial)$$

$$- F_1[\psi_{-1}(x)]\ \frac{(-Q\cdot\nabla)^\eta}{\eta!}\}\ \tilde{\psi}_{n-\eta}(x) \qquad \text{(A.18)}$$

when it is assumed that the time derivatives of Q are zero, that is, that $(-Q\cdot\nabla)$ commutes with $\Lambda(\partial)$, equation (A.18) can be rewritten as

$$\{\Lambda(\partial) - F_1[\psi_{-1}(x)]\}\ \psi_n(x)$$

$$= \{\Lambda(\partial) - F_1[\psi_{-1}(x)]\}\ \sum_{\eta=0}^{n+1} \frac{(-Q\cdot\nabla)^\eta}{\eta!}\ \tilde{\psi}_{n-\eta}(x) \qquad \text{(A.19)}$$

or

$$\psi_n(x) = \sum_{\eta=0}^{n+1} \frac{(-Q\cdot\nabla)^\eta}{\eta!}\ \tilde{\psi}_{n-\eta}(x) \quad . \qquad \text{(A.20)}$$

This completes the proof of the theorem. It is now easy to prove the following theorem.

<u>Theorem</u>:

$$\psi_\lambda(x) = \sum_{n=-1}^{\infty} \lambda^n \tilde{\psi}_n(x - \lambda Q,\ t) \quad . \qquad \text{(A.21)}$$

<u>Proof</u>: A combination of equations (A.2) and (A.20) leads to

$$\psi_\lambda(x) = \sum_{n=-1}^{\infty} \sum_{\eta=0}^{n+1} \frac{1}{\eta!} (-Q \cdot \nabla)^\eta \, \psi_{n-\eta}(x) \, \lambda^n$$

$$= \sum_{n=-1}^{\infty} \sum_{\eta=0}^{n+1} \frac{\lambda^\eta}{\eta!} (-\dot{Q} \cdot \nabla)^\eta \, \psi_{n-\eta}(x) \, \lambda^{n-\eta}$$

$$= \sum_{n=-1}^{\infty} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} (-Q \cdot \nabla)^k \, \tilde{\psi}_n(x) \, \lambda^n$$

$$= \sum_{n=-1}^{\infty} \lambda^n \, \tilde{\psi}_n(x - \lambda Q, \, t) \quad .$$

Thus, the theorem is proved and, when $\dot{Q} = 0$,

$$\psi_\lambda(x, \, t) = \tilde{\psi}_\lambda(x - \lambda Q, \, t) \; . \tag{A.22}$$