The Exponential Habit Forming Utility: Explicit Forms and Graphical Illustrations

by

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A thesis submitted in partial fulfillment of the requirements for the degree of

Master of Science

in

Mathematical Finance

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Abstract

This thesis focuses on characterizing the optimal consumption and investment strategy for an investor, in a simple financial market, when his consumption habit is considered in the utility formulation. We consider a continuous-time market model for which we maximize the overall utility within an infinite horizon.

Using the Bellman's Principle, we derive the associated Hamilton-Jacobi-Bellman equation (called HJB hereafter). For the case of HARA utility (exponential, power and logarithmic), the solution to the corresponding HJB is explicitly described. Furthermore, for the HARA case, the optimal consumption, consumption habit and wealth processes are described by a stochastic differential equation (called SDE hereafter). We pay particular attention to the case of exponential utility, where the obtained SDE is solved explicitly.

By applying graphing method, we explain the relationships between the optimal consumption/wealth/habit and the system's parameters. This result is meaningful, since it implies the potential influence to investors when the system's parameters are changing.

Acknowledgements

First of all, I would like to express my sincerest gratitude to my supervisor Dr. Tahir Choulli. During my study at the University of Alberta, he provided tremendous help and patient guidance. In my view, he is a responsible supervisor as he went through almost every small detail in my thesis, and even he did some troublesome calculation for me. It would be much more difficult in finishing this thesis without his support.

I also wish to thank Dr. Christoph Frei, Dr. Abel Cadenillas and Dr. Peter Minev, for accepting to act as members of my defense committee.

It has been a truly enjoyable experience to work with the staff in the Department of Mathematical and Statistical Science, whose professional services are gratefully acknowledged.

Thanks to all my friends at Edmonton. During the two-year life at the University of Alberta, many friends have shown their encouragement to me, advised me, and shared their own experience with me. I will never forget the big time with you guys, and may our friendship be forever.

Finally, special thanks to my family, for your long-lasting understanding and support. I dedicate this thesis to them with love and gratitude.

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Chapter 1

Introduction

Mathematical finance is an area where researchers develop quantitative methods and analysis for financial problems. It is a multidisciplinary field, which involves financial and economic theory, the engineering designs and methods, mathematical and statistical tools, and computer skills. As the pace of financial innovation increases, commercial banks, investment banks, hedge funds, insurance companies, corporate treasuries and other financial institutions apply the methods of financial mathematics to problems such as derivative securities valuation, portfolio structuring, risk management, and scenario simulation.

Among the most popular topics in this field, the topic of investment optimization draws attention of many financial mathematicians. Under the modern portfolio theory, we assume that investors try to minimize risk while striving for the highest return possible. The theory states that investors will act rationally, always making decisions aimed at maximizing their return for their acceptable level of risk. The pioneer of this field is Harry Markowitz, who stated in 1952 that it is possible for different portfolios to have varying levels of risk and return. Each investor must decide how much risk he or she can tolerate, and allocate their portfolio according to the efficient frontier which shows a set of optimal portfolios that offers the highest expected return for a defined level of risk.

Another popular mathematical finance topic is Asset Pricing Theory, which has attracted many researchers since the middle of the last century. In finance, the agents can use a variety of pricing strategies when selling a product or service. Finding the right pricing rules is an important element for a successful business. In 1973, Fisher Black, Robert Merton and Myron Scholes developed the Black–Scholes Model, which is one of the most important concept in financial theory. It is widely used in determining fair prices of options, and many empirical tests have shown that the Black–Scholes price is close to the observed prices, although this model is often applied with adjustments and corrections. In addition, CAPM, another capital asset pricing model, describes the relationship between risk and expected return. CAPM says that the expected return of a security or a portfolio equals the rate on a risk-free security plus a risk premium.

1.1 Optimal Portfolio and Optimal Consumption

This optimization topic is an important area of financial economics. For different optimization goals, researchers have developed the optimal strategies in various directions. The research on this subject was initiated by Yaari (see [20]) in 1965, who raised the problem of consumption optimization for an individual who might be expected to react to his lifetime uncertainty. In his model, Yaari concentrated on lifetime uncertainty and ignored other uncertainties by building a deterministic investment environment. Merton (see [11]) investigated the optimum consumption and portfolio rules in a continuous-time model. His work extended these optimal rules for more general cases, and addressed an optimization problem without any bequest motive. For different specialized utility formation, he derived the optimal consumption in the feedback form. Combining the optimality theory with insurance application, Hakansson (see [7]) put forward an optimization problem allowing the individual to hold insurance in case of risk, and he discussed the corresponding optimal strategy for specialized models. As a specialized case, Bommier (see [2]) explored how survival uncertainty may affect individual preference under the assumption of risk aversion with respect to the length of life. In the same spirit, Young (see [22]) also focused on the reaction to lifetime uncertainty. In this work, the optimization target is to minimize the probability of lifetime ruin, and the author derived the optimal investment strategy under the assumption of constant consumption rate. Cuoco and Cvitanić (see [3]) examined the effect of that "large" investor whose portfolio choices may affect the price process of traded asset. By martingale and duality techniques, they proved the existence of optimal policies for large investors. Under the assumption that long-term macroeconomic conditions will influence investor's behavior. Sotomayor and Cadenillas (see [16]) added another stochastic process to represent the regime switching, then solve the optimization problem for specific HARA utility function.

Generally speaking, for some certain targets, the corresponding optimality conditions were obtained in previous works. In this thesis, I will examine an optimization problem with the target of maximizing expected overall utility.

1.2 Habit Formulation Utilities

In the last two decades, many researchers believe that the consumption habit formation plays an important role in optimization theory and utility formulation, and use time-separable utility function to represent the agent's preference. This idea started as early as 1930, when I. Fisher carefully examined the measurability of the utility function, and emphasized the importance of nonseparable utility formation. In the literature on habit formation, Pollak (see [15]) built a model binding a specification of linear habit formation. He proved the existence of long-run utility functions which depend on the parameters of short-run utility function and those of habit formation. Detemple and Zapatero (see [4]) discussed the asset prices in an general exchange economy where the preferences of the agents is formed by previous consumption. They derived closed-form solutions for the interest and the value at risk. Merton (see [11]) gave a brief discussion on an optimization problem for two-asset discrete market, and then passed to the case of continuous time within an infinite horizon. Sundaresan (see [17]) constructed a model in which consumer's utility depends on the consumption history. Applying the Hamilton-Jacobi-Bellman equation, he gave a feedback form consumption in a simple example. With simulation method, the consumption paths generated from this model is formed to be less fluctuating compared with the case of separable utility function. Hindy, Huang and Zhu (see [8]) explored the interaction between the durability of consumption goods and habit formation over consumption flow. Applying numerical techniques, they solved a free-boundary singular control problem.

1.3 Summary

In this thesis, we study the problem of optimal consumption and investment rules in a simple financial market within an infinite lifetime horizon for a investor with habit formation.

The thesis is organized as follows. In Chapter 2, we introduce the financial concept and the mathematical tools which are applied throughout the thesis. In Chapter 3, we specify the market model on which the remaining part of the thesis is based on. Then, we describe the optimal value function as a solution to a partial differential equation (PDE). Afterwards, we solve explicitly this obtained PDE for the three cases of HARA utility (Exponential, power and logarithmic). In Chapter 4, we focus on the global solution for optimal consumption, optimal consumption habit, and optimal wealth processes. Especially, we discuss the exponential habit formation in details. In Chapter 5, we apply graphing method to describe the effect of the model's parameters on the optimal investment, consumption rate, consumption habit or wealth. Finally, in Chapter 6, we conclude this thesis by summarizing the main finding and contribution of the thesis.

Chapter 2

Preliminaries on Stochastic and Financial Concepts

In this chapter, we introduce some financial concepts, and discuss the mathematical tools used throughout the rest of the thesis.

2.1 Market Structure

We start by describing and introducing the financial market. A financial market is a market in which people can trade financial derivatives, commodities, and other fungible items of value. In the market, everything for trade has a corresponding price. In ordinary usage, price is the quantity of payment or compensation given by one party to another in return for goods or services. The law of the markets determines that a suitable price is the one which can keep a balance between supply and demand. Usually, there exists low transaction costs when trading activities happen in real life.

However, to make the research concise, we need to simplify the market structure in this thesis. We will focus on a single small investor, which means that his transaction will not influence the market equilibrium. We also consider this single-agent economy with frictionless markets and no taxes. It is only stocks and bonds that are tradeable in our model. Bond is a riskless asset, and its rate of return is a positive number $r(t)$. $r(t)$ is also called interest rate at time t. The price process of the bond is denoted by $P_0(t)$ $(t \geqslant 0)$, and follows

$$dP_0(t) = r(t)P_0(t)dt. \tag{2.1}$$

Equivalently, given the initial bond price P_0, for every $t \in [0, \infty)$

$$P_0(t) = P_0 \exp\Big(\int_0^t r(s)ds\Big),\ \ t \geqslant 0. \tag{2.2}$$

Different from bonds, usually many kinds of stocks exist in the financial market. We assume that the market consists of m stocks. We denote by $P_i(t)$ the price of the i^{th} stock at time t $(i = 1, 2..., m)$. The dynamic of the stock price process is given by:

$$dP_i(t) = b_i(t)P_i(t)dt + P_i(t)\sum_{j=1}^{d}\sigma_{ij}(s)dW_t^j. \tag{2.3}$$

Equivalently, given the initial stock price P_i, for every $t \in [0, \infty)$, we have

$$P_i(t) = P_i \exp\left\{\int_0^t \left[b_i(s) - \frac{1}{2}\sum_{j=1}^{d}\sigma_{ij}^2(s)\right] ds + \sum_{j=1}^{d}\int_0^t \sigma_{ij}(s)dW_s^j\right\}. \tag{2.4}$$

In the equations above, the interest rate $r(t)$, the stock appreciation rate $b(t) \triangleq (b_1(t), ..., b_m(t))$, the volatility matrix $\sigma(t) = \{\sigma_{ij}(t)\}_{1\leqslant i\leqslant m, 1\leqslant j\leqslant d}$ are the coefficients/parameters of the model. Precisely, $r(t) : [0, \infty) \times \Omega \to \mathbb{R}$ and

$b_i(t) : [0, \infty) \times \Omega \to \mathbb{R}$ are positive scalars, while the volatility of i^{th} stock $\sigma_i(t) : [0, \infty] \times \Omega \to \mathbb{R}^m$ describes the price dispersion rate. All these processes are assumed to be $\mathcal{F}_t$-adapted.

The process $W_t \triangleq \left\{W_t^1, W_t^2, ..., W_t^d\right\}$ is a d-dimensional standard Wiener process. It is assumed that $m \leqslant d$. If $m = d$ and the volatility matrix is nonsingular, those stocks create a complete market. A financial market where every payoff can be replicated is called complete. Otherwise, the market is incomplete such as the case $m < d$, where an infinite number of risk neutral probability measures exist.

We further assume that the investors can buy stocks and bonds with their capital. The investment activity is characterized by portfolio $\pi(t) \triangleq (\pi_0(t), \pi_1(t), ..., \pi_m(t))$, where $\pi_i(t) \triangleq N_i(t)P_i(t)$. $N_0(t)$ represents the amount of bond, and $N_i(t)$ represents the amount of i^{th} stock at time t, $i = 1, ..., m$. In our model, short-selling is allowed, which means that N_i can be any real number for $i = 0, ..., m$.

2.2 Habit Utility Formation

In this section, we define some mathematical variables that describe the state of agent. At first, it is assumed that the agents have to make a continuous-time expense flow in order to live, so we use $c(t)$ to represent the consumption rate of the agent at time t. Consequently, the agents will gain happiness from their consumption, and we call this effect as utility. In economics, utility is a description of preferences over some set of goods and services. In mathematics, utility is a function $U : (0, \infty) \to \mathbb{R}$ that is increasing and concave. It is a single variable function with respect to consumption rate. In macroeconomics, the utility function must satisfy the following condition.

Assumption 1: *$U(c)$ is continuously differentiable and satisfies*

$$\frac{\partial U(c)}{\partial c} > 0, \quad \frac{\partial^2 U(c)}{\partial c^2} < 0, \quad \lim_{c \to +\infty} \frac{\partial U(c)}{\partial c} = 0. \tag{2.5}$$

This assumption means that the utility function has to be strictly increasing and strictly concave. From a financial view, the marginal utility is strictly decreasing, and it goes to zero as consumption rate approaches positive infinity.

In our model, we study the problem of optimal consumption and investment rules for an agent with habit formation. Therefore, we expand the original utility $U(c)$ to a multi-variable function $U(c, z)$ with respect to consumption rate $c(t)$ and consumption habit level $z(t)$. The habit index process $z(t)$ is given by

$$dz(t) = \beta(c(t) - z(t))dt, \quad z(0) = z_0, \tag{2.6}$$

Equivalently, given the initial consumption habit z_0,

$$z(t) = z_0 e^{-\beta t} + \int_0^t \beta e^{\beta(s-t)} c(s) ds. \tag{2.7}$$

In this formulation, z_0 is the initial consumption preference level. β is called habit formulation factor, and it represents the weight of nearby consumption in the formulation of habit. As time passes, the preference places less weight on historical consumption at a given past date. From the differential form, we can see that the consumption habit will increase if the momentary consumption rate exceeds the consumption habit. The higher β is, the fast $z(t)$ is adjusted to current consumption rate. If $\beta = 0$, the preference index is a constant and stays at z_0.

Similarly, the habit-related utility function has to satisfy the following condition.

Assumption 2: *$U(c,z)$ is continuously differentiable and satisfies:*

1. $\frac{\partial U(c,z)}{\partial c} > 0$. For fixed historical consumption rate, an increase in current consumption will increase utility.

2. $\frac{\partial U(c,z)}{\partial z} < 0$. For fixed current consumption, an increase in historical consumption rate will decrease utility.

3. $\frac{\partial^2 U(c,z)}{\partial c^2} < 0$. Marginal utility will decreases as current consumption increases. It indicates that utility function $U(c,z)$ is concave down for c.

4. $\lim\limits_{c \to +\infty} \frac{\partial U(c,z)}{\partial c} = 0$. Marginal utility approaches 0 as consumption rate goes to infinity.

In order to simplify calculations, various formations have been assumed in utility functions. The utility function can be specialized as follows: (just examples, not limited to those cases)

1. Exponential utility function: $u(c,z) = -\frac{1}{\Phi_1} e^{-\Phi_1 c + \Phi_2 z}$, where $\Phi_1 > 0, \Phi_2 \geqslant 0$. The parameter Φ_2 describes the strength of intertemporal dependence.

2. Power utility function: $u(c,z) = \frac{\{c-z\}^A}{A}, \quad A < 1$. This utility formation has the property that as c approaches z, the marginal utility goes to infinity. Therefore, the agent would never allow his consumption level to be lower than his consumption habit.

3. Logarithmic utility function: $u(c,z) = \log\{c-z\}$. Same as power utility function, as $c \to z$, the marginal utility goes to infinity. Therefore, the consumption habit determines the lower limit of consumption rate.

4. Generalized power utility function: $u(c,z) = \frac{1-\gamma}{\gamma}\left(\frac{mc-nz}{1-\gamma} + \eta\right)^{\gamma}$. The

last three kinds of utilities usually referred as HARA utilities. The word HARA is an abbreviation for "hyperbolic absolute risk aversion". A utility function is said to exhibit hyperbolic absolute risk aversion if and only if the level of risk tolerance is a linear function of consumption. HARA function is often applied in model to characterize the property of risk aversion mathematically.

In chapters 3 and 4, we examine the property of exponential utility function in depth, and make a brief discussion for the generalized power utility function.

2.3 The Consumer Optimization Problem

In our economy, the agent starts with an initial capital x_0, and no endowment will be added at any time $t \in (0, \infty)$. We use $x(t)$ to represent the wealth of the agent at time t. At any time t, the consumer must decide his consumption rate $c(t)$ and investment strategy $\pi(t)$. Then, the wealth process is determined by

$$\begin{cases} dx(t) = \left[r(t)x(t) + \sum_{i=1}^{m} \left(b_i(t) - r(t)\right) \pi_i(t) - c(t) \right] dt + \sum_{j=1}^{d} \sum_{i=1}^{m} \pi_i(t) \sigma_{ij}(t) dW_t^j, \\ x(0) = x_0. \end{cases} \tag{2.8}$$

Equivalently, for every $t \in [0, \infty)$,

$$x(t) = \exp\left[\int_0^t r(s)ds\right]\left\{x_0 + \int_0^t \exp\left[-\int_0^s r(u)du\right]\left[\pi(s)(b(s) - r(s)\cdot 1)^\top\right.\right.$$
$$\left.\left. - \; c_s\right] ds + \sum_{j=1}^d \int_0^t \exp\left[-\int_0^s r(u)du\right]\pi(s)\sigma_{\cdot j}(s)dW_s^j\right\}. \tag{2.9}$$

For a given utility function U, a given initial capital x_0, and a given initial consumption preference z_0, we consider the following target function

$$J(x_0, z_0; c(\cdot), \pi(\cdot)) = E_0\left\{\int_0^\infty e^{-\delta t}U(c(t), z(t))dt\right\}. \tag{2.10}$$

$J(x_0, z_0; c(\cdot), \pi(\cdot))$ represents the overall expected utility within an infinite horizon. Here, E_0 is the expectation operator, and δ is the subjective discount rate, which describes the level of investor's impatience.

The initial condition includes the initial wealth x_0 and initial consumption habit z_0, and the variables for control are consumption flow $c(\cdot)$ and investment flow $\pi(\cdot)$. Our objective is to maximize the overall expected utility $J(x_0, z_0; c(\cdot), \pi(\cdot))$ over the set of pairs $\langle c(\cdot), \pi(\cdot)\rangle$. Mathematically, the problem can be expressed as follows,

$$\max_{\{c(\cdot),\pi(\cdot)\}} J(x_0, z_0; c(\cdot), \pi(\cdot)) = \max_{\{c(\cdot),\pi(\cdot)\}} E_0\left\{\int_0^\infty e^{-\delta t}U(c(t), z(t))dt\right\}. \tag{2.11}$$

This stochastic control problem is the main focus of this thesis.

2.4 Itô's Formula

In this section, we introduce Itô's Formula for an n-dimensional Itô process having the form of

$$X_i(t) = X_i(0) + \int_0^t K_i(s)ds + \sum_{j=1}^{m} \int_0^t H_{ij}(s)dW_j(s).$$

Theorem 1: *(Itô's Formula) Let $f : [0, \infty) \times \mathbb{R}^n \mapsto \mathbb{R}$ be a $C^{1,2}$-function. That is, f is continuous, continuously differentiable with respect to the first variable (time), and twice continuously differentiable with respect to the last n variables (space).*

Then, for every $t \geqslant 0$,

$$\begin{aligned}
&f(t, X_1(t), ..., X_n(t)) \\
=&f(0, X_1(0), ..., X_n(0)) \\
&+ \int_0^t f_t(s, X_1(s), ..., X_n(s))ds + \sum_{i=1}^{n} \int_0^t f_{x_i}(s, X_1(s), ..., X_n(s))dX_i(s) \\
&+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \int_0^t f_{x_i x_j}(s, X_1(s), ..., X_n(s))d\langle X_i, X_j \rangle_s.
\end{aligned}$$

2.5 Optimization Tools

This section provides mathematical results and arguments useful for solving our main target defined in (2.11). To this end, we consider $(\Omega, \mathbb{F}, \mathbb{F}_{t \geqslant 0}, \mathbf{P})$ as a given filtered probability space which satisfies the usual condition, on which is defined an m-dimensional standard Brownian motion $W(t)$. We set up the following stochastic framework for dynamic programming:

$$\begin{cases} dx(t) = b\left(t, x(t), z(t), c(t), \pi(t)\right) dt + \sigma\left(t, x(t), z(t), c(t), \pi(t)\right) dW(t), \\ dz(t) = \beta(t, x(t), z(t), c(t), \pi(t))dt, \\ x(s) = x, z(s) = z. \end{cases} \tag{2.12}$$

along with a target function

$$J(s, x, z; c(\cdot), \pi(\cdot)) = E\left\{\int_s^\infty e^{-\delta t} u(c(t), z(t))dt\right\}. \tag{2.13}$$

We emphasize that in (2.12) the initial state x, z is a deterministic (almost surely) variable under $(\Omega, \mathbb{F}, \mathbf{P})$, and the mathematical expectation E is with respect to the probability $\mathbf{P}$.

Under the framework (2.12), we maximize $J(s, x, z; c(\cdot), \pi(\cdot))$, and find the optimal strategy $\langle c^*(\cdot), \pi^*(\cdot)\rangle$ such that

$$J(s, x, z; c^*(\cdot), \pi^*(\cdot)) = \max_{\{c(\cdot), \pi(\cdot)\}} J(s, x, z; c(\cdot), \pi(\cdot)). \tag{2.14}$$

Then, we define the following functions

$$V(s, x, z) \triangleq \max_{\{c(\cdot), \pi(\cdot)\}} J(s, x, z; c(\cdot), \pi(\cdot)), \tag{2.15}$$

$$J(x, z; c(\cdot), \pi(\cdot)) \triangleq J(0, x, z; c(\cdot), \pi(\cdot)), \tag{2.16}$$

and

$$V(x, z) \triangleq \max_{\{c(\cdot), \pi(\cdot)\}} J(x, z; c(\cdot), \pi(\cdot)). \tag{2.17}$$

If we put $\tilde{c}(s) = c(t+s)$, $\tilde{\pi}(s) = \pi(t+s)$, for $s \geq 0$, then we have

$$\begin{aligned} J(t,x,z;c(\cdot),\pi(\cdot)) &= E\left\{\int_t^\infty e^{-\delta s}u(c(s),z(s))ds\right\} \\ &= e^{-\delta t}E\left\{\int_0^\infty e^{-\delta s}u(\tilde{c}(s),\tilde{z}(s))ds\right\} \\ &= e^{-\delta t}J(0,x,z;\tilde{c}(\cdot),\tilde{\pi}(\cdot)). \end{aligned} \tag{2.18}$$

As a result, we obtain

$$V(t,x,z) = e^{-\delta t}V(x,z), \tag{2.19}$$

and get

$$\begin{aligned} V_t(t,x,z) &= -\delta e^{-\delta t}V(x,z), \\ V_{xx}(t,x,z) &= e^{-\delta t}V_{xx}(x,z), \\ V_x(t,x,z) &= e^{-\delta t}V_x(x,z), \\ V_z(t,x,z) &= e^{-\delta t}V_z(x,z). \end{aligned} \tag{2.20}$$

The following theorem is the stochastic version of Bellman's Principle of Optimality.

Theorem 2: *For any $(s,x,z) \in [0,\infty) \times \mathbb{R} \times \mathbb{R}$, the function V defined in (2.15) satisfies the following equation:*

$$V(s,x,z) = \max_{\{c(\cdot),\pi(\cdot)\}} E\left\{\int_s^{\hat{s}} e^{-\delta t}u(c(t),z(t))dt + V(\hat{s},x(\hat{s}),z(\hat{s}))\right\},$$
$$0 \leq s < \hat{s} < \infty$$

Based on this equation, we can derive the following HJB equation associated to the control problem (2.17).

Proposition 1: *If the function $V(x,z)$ defined in (2.17) is continuously twice* *differentiable, then it is the solution of following HJB equation:*

$$\delta V(x,z) + \inf_{\{c(\cdot),\pi(\cdot)\}} G(t,x,z,c,\pi,V_x,V_z,V_{xx}) = 0. \tag{2.21}$$

Here

$$\begin{aligned} G(t,x,z,c,\pi,V_x,V_z,V_{xx}) = &- V_x(x,z)\cdot b\,(t,x,z,c,\pi) - V_z(x,z)\cdot \beta(t,x,z,c,\pi) \\ &- \frac{1}{2}V_{xx}(x,z)\cdot[\sigma\,(t,x,z,c,\pi)]^2 - u(c,z), \end{aligned}$$

and

$$V_x(x,z) \triangleq \frac{\partial V(x,z)}{\partial x},\ V_z(x,z) \triangleq \frac{\partial V(x,z)}{\partial z},\ V_{xx}(x,z) \triangleq \frac{\partial^2 V(x,z)}{\partial x^2}.$$

Proof. Fix $(s,x,z) \in [0,\infty)\times\mathbb{R}^n\times\mathbb{R}^n$ and $\{c(\cdot),\pi(\cdot)\}$. Then assume $\hat{s}$ which satisfies $0 \le s < \hat{s} < \infty$. By Bellman's Principle (see Theorem 2), we have

$$V(s,x,z) = \max_{\{c(\cdot),\pi(\cdot)\}} E\left\{\int_s^{\hat{s}} e^{-\delta t}u(c(t),z(t))dt + V(\hat{s},x(\hat{s}),z(\hat{s}))\right\}.$$

This implies that

$$V(s,x,z) \ge E\left\{\int_s^{\hat{s}} e^{-\delta t}u(c(t),z(t))dt + V(\hat{s},x(\hat{s}),z(\hat{s}))\right\}.$$

After simplification, we write

$$0 \le -\frac{E\left\{V(\hat{s},x(\hat{s}),z(\hat{s})) - V(s,x,z)\right\}}{\hat{s}-s} - \frac{E\int_s^{\hat{s}} e^{-\delta t}u(c(t),z(t))dt}{\hat{s}-s}.$$

Thus, using Itô's Formula and taking $\hat{s} \downarrow s$ afterwards, we obtain

$$\begin{aligned} 0 \leq & - V_t(s,x,z) - V_x(s,x,z) \cdot b\,(s,x,z,c,\pi) - V_z(s,x,z) \cdot \beta(s,x,z,c,\pi) \\ & - \frac{1}{2} V_{xx}(s,x,z) \cdot [\sigma\,(s,x,z,c,\pi)]^2 - e^{-\delta s} u(c,z). \end{aligned} \tag{2.22}$$

Then, by plugging (2.20) in the above equality, we get

$$\begin{aligned} 0 \leq & - (-\delta V(x,z)) - V_x(x,z) \cdot b\,(s,x,z,c,\pi) - V_z(x,z) \cdot \beta(s,x,z,c,\pi) \\ & - \frac{1}{2} V_{xx}(x,z) \cdot [\sigma\,(s,x,z,c,\pi)]^2 - u(c,z). \end{aligned}$$

Further simplifications imply that, for any c, π

$$\begin{aligned} 0 \leq & \delta V(x,z) + \{ -V_x(x,z) \cdot b\,(s,x,z,c,\pi) - V_z(x,z) \cdot \beta(s,x,z,c,\pi) \\ & - \frac{1}{2} V_{xx}(x,z) \cdot [\sigma\,(s,x,z,c,\pi)]^2 - u(c,z) \Big\}. \end{aligned}$$

Therefore, we deduce that

$$0 \leq \delta V(x,z) + \inf_{\{c(\cdot),\pi(\cdot)\}} G(s,x,z,c,\pi,V_x,V_z,V_{xx}). \tag{2.23}$$

Then, for any $\varepsilon > 0, 0 \leq s < \hat{s} < \infty$ with $\hat{s} - s > 0$ small enough, there exists a pair $\langle c(\cdot), \pi(\cdot) \rangle$ such that

$$V(s,x,z) - \varepsilon(\hat{s} - s) \leq E\left\{ \int_s^{\hat{s}} e^{-\delta t} u(c(t), z(t)) dt + V(\hat{s}, x(\hat{s}), z(\hat{s})) \right\}$$

After a simple transformation, we get

$$\varepsilon \geq -\frac{E\left\{ V(\hat{s}, x(\hat{s}), z(\hat{s})) - V(s,x,z) \right\}}{\hat{s} - s} - \frac{E \int_s^{\hat{s}} e^{-\delta t} u(c(t), z(t)) dt}{\hat{s} - s}$$

Again, using Ito's formula and let $\hat{s} \downarrow s$, we obtain

$$\begin{aligned} e^{\delta s}\varepsilon &\geq \delta V(x,z) + G(s,x,z,c,\pi,V_x,V_z,V_{xx}) \\ &\geq \delta V(x,z) + \inf_{\{c(\cdot),\pi(\cdot)\}} G(s,x,z,c,\pi,V_x,V_z,V_{xx}). \end{aligned}$$

Therefore, we conclude that

$$0 \geq \delta V(x,z) + \inf_{\{c(\cdot),\pi(\cdot)\}} G(s,x,z,c,\pi,V_x,V_z,V_{xx}). \qquad (2.24)$$

By combining (2.23) and (2.24), our conclusion (2.21) follows immediately. This ends the proof of the proposition. □

Applying the HJB equation, we may get a partial differential equation for V. In most cases, the explicit form of V is difficult to find. However, if we the function V is smooth enough, then we can apply the following theorem to test whether a given admissible pair of $\langle c(\cdot),\pi(\cdot)\rangle$ is optimal. It is also called Verification Theorem.

Theorem 3: *(Verification Theorem) Let v be a solution of the HJB equation (2.21). For any $(s,x,z) \in [0,\infty) \times \mathbb{R} \times \mathbb{R}$, the function v satisfies the following equation:*

$$v(s,x,z) \geqslant J(s,x,z;c(\cdot),\pi(\cdot))$$

Furthermore, a given admissible pair $\langle x^*(\cdot),z^*(\cdot),c^*(\cdot),\pi^*(\cdot)\rangle$ is optimal for *the control problem given in (2.11) if and only if*

$$\begin{aligned}
&v_t(t, x^*(t), z^*(t)) \\
=&H(t, x^*(t), z^*(t), c^*(t), \pi^*(t), v_x(t, x^*(t), z^*(t)), v_z(t, x^*(t), z^*(t)), v_{xx}(t, x^*(t), z^*(t)))
\end{aligned}$$

where

$$\begin{aligned}
&H(t, x, z, c, \pi, v_x(t, x, z), v_z(t, x, z), v_{xx}(t, x, z)) \\
\triangleq& -v_x(t, x, z) \cdot b\,(t, x, z, c, \pi) - v_z(t, x, z) \cdot \beta(t, x, z, c, \pi) \\
&- \frac{1}{2} v_{xx}(t, x, z) \cdot [\sigma\,(t, x, z, c, \pi)]^2 - e^{-\delta t} u(c(t), z(t))
\end{aligned}$$

Proof. By Itô formula, we get

$$\begin{aligned}
&\frac{d}{dt} v(t, x, z) \\
=&v_t(t, x, z) + \langle v_x(t, x, z), b\,(t, x, z, c, \pi)\rangle + \langle v_z(t, x, z), \beta\,(t, x, z, c, \pi)\rangle \\
=&v_t(t, x, z) + v_x(t, x, z) \cdot b\,(t, x, z, c, \pi) + v_z(t, x, z) \cdot \beta(t, x, z, c, \pi) \\
&+ \frac{1}{2} v_{xx}(t, x, z) \cdot [\sigma\,(t, x, z, c, \pi)]^2 \\
=&- e^{-\delta t} u(c(t), z(t)) \\
&+ \{v_t(t, x, z) - H(t, x, z, c, \pi, v_x(t, x, z), v_z(t, x, z), v_{xx}(t, x, z))\} \\
\leqslant&- e^{-\delta t} u(c(t), z(t)),
\end{aligned} \tag{2.25}$$

where the last inequality has been proved in equation (2.22). Integrating both sides from s to ∞, we get

$$\begin{aligned}
v(s, x, z) &\geqslant v(\infty, x, z) + J(s, x, z; c(\cdot), \pi(\cdot)) \\
&= J(s, x, z; c(\cdot), \pi(\cdot))
\end{aligned}$$

Next, applying the second equality in (2.25) to $\langle c^*(\cdot), \pi^*(\cdot)\rangle$, and integrating from s to ∞:

$$\begin{aligned} v(s,x,z) =& J(s,x,z;c^*(\cdot),\pi^*(\cdot)) - \int_s^\infty \{v_t(t,x^*(t),z^*(t)) - H(t,x^*(t),z^*(t), \\ & c^*(t),\pi^*(t),v_x(t,x^*(t),z^*(t)),v_z(t,x^*(t),z^*(t)),v_{xx}(t,x^*(t),z^*(t)))\}dt \end{aligned}$$

Combined with (2.22), we get:

$$\begin{aligned} & v_t(t,x^*(t),z^*(t)) \\ =& H(t,x^*(t),z^*(t),c^*(t),\pi^*(t),v_x(t,x^*(t),z^*(t)),v_z(t,x^*(t),z^*(t)),v_{xx}(t,x^*(t),z^*(t))) \end{aligned}$$

This ends the proof of the theorem. □

In conclusion, thanks to the HJB equation, we can deduce a partial differential equation in the form of (2.21). Then, we can apply dynamic programming principle to find the optimal controls $c^*(\cdot)$ and $\pi^*(\cdot)$.

In mathematics, dynamic programming is a method for solving complex multi-variable problems by breaking then down into simpler subproblems. Many problems reflect a need to choose among multiple alternatives. We now generalize the following techniques.

Theorem 4: *If $f(x_1, x_2, ..., x_n)$ is differentiable with respect to each of its arguments and reaches a maximum or a minimum at the stationary point $(x_1^*, x_2^*, ..., x_n^*)$, then each of the partial derivatives evaluated at that point equals zero, i,e.*

$$f_1(x_1^*, x_2^*, ..., x_n^*) = 0$$

$$...$$

$$...$$

$$...$$

$$f_n(x_1^*, x_2^*, ..., x_n^*) = 0$$

For a continuously differentiable multi-variable real function, a point P is critical if all of the partial derivatives of the function are zero at P. In calculus, the second derivative test is a criterion for determining whether a given critical point of a real function is a local maximum or a local minimum using the value of the second derivative at the point. For a function of more than one variable, the second derivative test generalizes to a test based on the eigenvalues of the function's Hessian matrix at the critical point.

The Hessian matrix is a square matrix of second-order partial derivatives of a function. It describes the local curvature of a function of many variables. Given a real function

$$f(x_1, x_2, ..., x_n),$$

If all second partial derivatives of f exist and are continuous over the domain of the function, then the Hessian matrix of f is defined by

$$H(f)_{ij}(\mathbf{x}) = D_i D_j f(\mathbf{x}),$$

where $\mathbf{x} = (x_1, x_2, ..., x_n)$ and D_i is the differential operator with respect to the ith argument. Thus, the Hessian matrix of f can be written as

$$H(f) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}.$$

A non-degenerate critical point is a local maximum if and only if the Hessian matrix is negative definite; it is a local minimum if and only if the Hessian matrix is positive definite; otherwise, it is a saddle point.

2.6 Second Order Linear Differential Equations

In this subsection, we introduce the method in solving the second order non-homogeneous linear differential equations, which can be expressed as:

$$y'' + py' + qy = f(x), \tag{2.26}$$

where p and q are two constants and $f(x)$ is any continuous function.

Theorem 5: *let λ_1 and λ_2 are the complex roots of $\lambda^2 + p\lambda + q = 0$. Then, the particular solution to (2.26) is given as follows.*

1. *If λ_1 and λ_2 are distinct real roots, then*

$$y(x) = \frac{1}{\lambda_2 - \lambda_1}\left[e^{\lambda_2 x}\int f(x)e^{-\lambda_2 x}dx - e^{\lambda_1 x}\int f(x)e^{-\lambda_1 x}dx\right].$$

2. *If $\lambda_1 = \lambda_2$ are equal roots, then*

$$y(x) = e^{\lambda_1 x}\left[x\int f(x)e^{-\lambda_1 x}dx - \int xf(x)e^{-\lambda_1 x}dx\right].$$

3. If λ_1 and λ_2 are two complex conjugate roots (i.e. $\lambda_1 = \mu + \nu i$ and $\lambda_2 = \mu - \nu i$, $\nu > 0$), then

$$y(x) = \frac{1}{\nu}e^{\mu x}\left[\sin(\nu x)\int e^{-\mu x}f(x)\cos(\nu x)dx - \cos(\nu x)\int e^{-\mu x}f(x)\sin(\nu x)dx\right].$$

Theorem 6: *Using the same notations of Theorem 5, The general solution to (2.26) is given as follows:*

1. If λ_1 and λ_2 are distinct real roots, then

$$y(x) = C_1e^{\lambda_1 x} + C_2e^{\lambda_2 x} + \frac{1}{\lambda_2 - \lambda_1}\left[e^{\lambda_2 x}\int f(x)e^{-\lambda_2 x}dx - e^{\lambda_1 x}\int f(x)e^{-\lambda_1 x}dx\right].$$

2. If λ_1 and λ_2 are equal roots, then

$$y(x) = (C_1 + C_2x)e^{\lambda_1 x} + e^{\lambda_1 x}\left[x\int f(x)e^{-\lambda_1 x}dx - \int xf(x)e^{-\lambda_1 x}dx\right].$$

3. If λ_1 and λ_2 are two complex conjugate roots, then

$$\begin{aligned} y(x) =& e^{\mu x}[C_1\cos(\nu x) + C_2\sin(\nu x)] \\ &+ \frac{1}{\nu}e^{\mu x}\left[\sin(\nu x)\int e^{-(\mu x)}f(x)\cos(\nu x)dx - \cos(\nu x)\int e^{-\mu x}f(x)\sin(\nu x)dx\right]. \end{aligned}$$

2.7 Gaussian White Noise Process

In mathematics, Gaussian white noise is a real-valued process $\dot{W}(t)$ such that $\dot{W}(t) \sim N(0, \sigma^2)$, for all $t > 0$, and $\dot{W}(t)$ is independent of $\sigma(\dot{W}(u), u < t)$.

The relationship between Wiener process (or Brownian Motion) and Gaussian white noise Process is

$$W_t - W_s = \int_s^t \dot{W}(u)du, \text{ for } t > s.$$

In this thesis, by white noise we mean the generalized Gaussian process which is informally given by the time derivative of the Wiener process. We will apply this concept as a key idea in solving the obtained stochastic differential equation.

Chapter 3

HARA Habit Utilities: Description of the Optimal Value Function

In this chapter, we solve an optimization problem of an agent who begins with an initial endowment, and optimally consume and invest in a standard simple market. The objective of this agent is to maximize the overall utility of consumption in an infinite horizon. For this problem, we calculate the optimal value function, and give the feedback-form solutions for the optimal controls.

3.1 Introduction of Optimization Problem

At first, we describe mathematically the simple financial market model that will be investigated. We consider a continuous-time economy on an infinite time span. The agents in this market are assumed to be "small investor", which means his actions have no influence on the market prices. In addition, the transaction is smooth, which indicates all transaction costs are ignored.

This market consists of a riskless bond and one stock. The stock price is driven by a one dimension Brownian motion $W(t)$ $(m = d = 1)$, which indicates the completeness of the market. The interest rate $r(t)$, the expected return of stock $b(t)$ and the volatility matrix $\sigma(t)$ are assumed to be constants, i.e.

$$r(t) = r, \quad b(t) = b, \quad \sigma(t) = \sigma. \tag{3.1}$$

Thus, the bond price at time t is denoted by $p_0(t)$ and follows

$$dp_0(t) = rp_0(t)dt, \qquad p_0(0) = p_0. \tag{3.2}$$

Equivalently, for every $t \in [0, \infty)$

$$p_0(t) = p_0 \exp(rt). \tag{3.3}$$

The stock price at time t is denoted by $p_1(t)$ and satisfies

$$dp_1(t) = b(t)p_1(t)dt + \sigma p_1(t)dW_t, \qquad p_1(0) = p_1. \tag{3.4}$$

Equivalently, for every $t \in [0, \infty)$

$$p_1(t) = p_1 \exp\left\{(b - \frac{1}{2}\sigma^2)t + \sigma W_t\right\}. \tag{3.5}$$

Since in this economy $m = d = 1$, then σ is a real number that we assume to be positive.

Now, we consider the wealth process $x(t)$ of an investor. We assume that the investor starts with an initial wealth x_0 and no endowment will be added to his asset. For any $t \in [0, \infty)$, the investor consumes and/or invests only throughout the investment period. The consumption activities are characterized by its consumption rate $c(t)$, $t \geqslant 0$. Since there is only one stock in this market, then the portfolio $\pi(t) \triangleq \pi_1(t) = N_1(t)p_1(t)$ is a real number. Thus, the wealth process can be rewritten as

$$dx(t) = [rx(t) + (b - r)\,\pi(t) - c(t)]\,dt + \pi(t)\sigma dW_t, \quad x(0) = x_0. \tag{3.6}$$

Equivalently,

$$x(t) = e^{rt}x_0 + \int_0^t e^{-r(s-t)}\,[\pi(s)(b-r) - c(s)]\,ds + \int_0^t e^{-r(s-t)}\pi(s)\sigma dW_s. \tag{3.7}$$

To define the utility maximization problem faced by the investor, we first introduce the utility function. In economics, utility is a representation of preferences over some set of goods and services, while mathematically speaking, the utility is a function satisfying some conditions. For details, we refers the reader to Assumption 2. The agent in our model has a utility function $U(c)$ corresponding to his consumption rate $U : \mathbb{R}^2 \to \mathbb{R}$.

For a given utility function U, a given initial capital x_0, and a given initial consumption preference z_0, we consider the following target function.

$$J(x_0, z_0; c(\cdot), \pi(\cdot)) = E_0 \left\{ \int_0^\infty e^{-\delta t} U(c(t), z(t)) dt \right\}. \tag{3.8}$$

$J(x_0, z_0; c(\cdot), \pi(\cdot))$ represents the overall expected utility within a infinite horizon. Here, E_0 is the expectation operator, and δ is the subjective discount rate, which describes the level of investor's impatience. Our objective is to maximize the overall expected utility $J(x_0, z_0; c(\cdot), \pi(\cdot))$ under $c(\cdot)$ and $\pi(\cdot)$.

It can be mathematically expressed as

$$V(x_0, z_0) := \max_{\{c(\cdot),\pi(\cdot)\}} J(x_0, z_0; c(\cdot), \pi(\cdot)) = \max_{\{c(\cdot),\pi(\cdot)\}} E_0 \left\{ \int_0^\infty e^{-\delta t} U(c(t), z(t)) dt \right\}. \tag{3.9}$$

This function V is called the optimal value function that we will describe in the forthcoming section.

3.2 The Optimal Value Function

In this section, we characterize the optimal value function.

Theorem 7: *For the optimization problem constructed in Section 3.1, for any $x > 0, z > 0$, the optimal value function $V(x, z)$ defined in equation (3.9) is the solution for the following partial differential equation.*

$$\begin{aligned} \delta V =& U(I(V_x - \beta V_z, z), z) + [rx - I(V_x - \beta V_z, z)]V_x \\ &+ [\beta(I(V_x - \beta V_z, z) - z)]V_z - \frac{(b-r)^2 V_x^2}{2\sigma^2 V_{xx}}, \end{aligned} \tag{3.10}$$

where $I(\cdot, z) = U_c(\cdot, z)^{-1}$. To be concise, from here we use V, V_x, V_z, V_{xx} instead of $V(x, z)$, $V_x(x, z)$, $V_z(x, z)$, $V_{xx}(x, z)$.

Proof. Thanks to (2.21), the corresponding HJB equation for the optimization problem in (3.9) is given by

$$\delta V = \max_{\{c,\pi\}} \left\{ u(c, z) + [rx + (b-r)\pi - c] V_x + \beta(c - z)V_z + \frac{1}{2}\pi^2\sigma^2 V_{xx} \right\}. \tag{3.11}$$

In order to calculate the optimal controls c^* and π^* in the above equation, we calculate the corresponding first derivatives and put them equal to zero.

The derivative with respect to c leads to

$$\frac{\partial u(c,z)}{\partial c} - V_x(x,z) + \beta V_z(x,z) = 0, \tag{3.12}$$

while the derivative with respect to π implies that:

$$(b-r)V_x(x,z) + \pi\sigma^2 V_{xx}(x,z) = 0. \tag{3.13}$$

On the one hand, by solving both equations above, the optimal controls c^* and π^* are given by

$$\begin{aligned} c^* :=& c^*(x,z) = I(V_x(x,z) - \beta V_z(x,z), z), \\ \pi^* :=& \pi^*(x,z) = -\frac{(b-r)V_x(x,z)}{\sigma^2 V_{xx}(x,z)}. \end{aligned} \tag{3.14}$$

On the other hand, (3.11) becomes

$$\delta V = u(c^*, z) + (rx + (b-r)\pi^* - c^*)V_x + \beta(c^* - z)V_z + \frac{1}{2}(\pi^*)^2\sigma^2 V_{xx}. \tag{3.15}$$

Hence by inserting (3.14) into (3.15), (3.10) follows immediately. This ends the proof of the theorem. □

3.3 Exponential Habit Utility Formation

In the previous sections, the utility function considered is general. Here, we focus on the interesting and particular case of exponential utility function.

Precisely, we consider the case where $U(c, z)$ is given by

$$u(c, z) = -\frac{1}{\phi_1} e^{-\phi_1 c + \phi_2 z}. \tag{3.16}$$

In economics, the parameter ϕ_2 represents the level of intertemporal dependence. According to Assumption 2, to make this utility formation reasonable, we assume that

$$\phi_1 > 0 \quad \text{and} \quad \phi_2 \geqslant 0. \tag{3.17}$$

For this case, we will calculate the function V explicitly.

Proposition 2: *For the utility $U(c, z)$ defined in (3.16), the solution to (3.10) is given by:*

$$V(x, z) := -b_1 e^{-b_2 x + b_3 z}, \tag{3.18}$$

where

$$\begin{cases} b_1 := \frac{1}{\phi_1 r} \exp\left(1 - \frac{2\delta\sigma^2 + (b-r)^2}{2\sigma^2 r}\right) > 0, \\ b_2 := r\phi_1 - \frac{r\beta\phi_2}{r+\beta} = \frac{r\phi_1\beta}{r+\beta}\left(\frac{r+\beta}{\beta} - \frac{\phi_2}{\phi_1}\right), \\ b_3 := \frac{r\phi_2}{r+\beta} \geqslant 0. \end{cases} \tag{3.19}$$

Proof. Firstly, we assume that the solution to (3.10), denoted by V, takes the form of

$$V(x, z) = -b_1 e^{-b_2 x + b_3 z}. \tag{3.20}$$

Then, thanks to (3.14), we calculate c^* and π^* in this case as follows.

$$\begin{aligned} c^*(x, z) &= \frac{\phi_2}{\phi_1} z - \frac{1}{\phi_1} \ln(V_x(x, z) - \beta V_z(x, z)) \\ &= \frac{b_2}{\phi_1} x + \frac{\phi_2 - b_3}{\phi_1} z - \frac{1}{\phi_1} \ln(b_1 b_2 + b_1 \beta b_3), \end{aligned} \tag{3.21}$$

and

$$\pi^*(x,z) = -\frac{(b-r)V_x(x,z)}{\sigma^2 V_{xx}(x,z)} = \frac{b-r}{b_2\sigma^2}. \tag{3.22}$$

Hence $\pi^*(x,z)$ is constant in (x,z) and as a result the optimal portfolio does not depend on time nor on randomness.

A combination of (3.10) and (3.16) leads to

$$\delta V = (V_x - \beta V_z)\left(-\frac{1}{\phi_1} - \frac{\phi_2}{\phi_1}z + \frac{\ln(V_x - \beta V_z)}{\phi_1}\right) - \beta z V_z - \frac{(b-r)^2 V_x^2}{2\sigma^2 V_{xx}} + rxV_x.$$

Using (3.20), we calculate V_x, V_z and plugging them afterwards in the above equation, this latter becomes

$$\begin{aligned}\delta &= x\left[\frac{1}{\phi_1}b_2(b_2+\beta b_3) - b_2 r\right] \\ &\quad + z\left[\frac{1}{\phi_1}(-b_2-\beta b_3)(b_3-\phi_2) - \beta b_3\right] \\ &\quad + \left[\frac{-b_2-\beta b_3}{\phi_1}[\ln(b_1b_2+\beta b_1 b_3) - 1] - \frac{(b-r)^2}{2\sigma^2}\right].\end{aligned}$$

Since x and z are arbitrary elements of $(0,\infty)$, this equation is equivalent to

$$\begin{cases} \frac{1}{\phi_1}b_2(b_2+\beta b_3) - b_2 r = 0, \\ \frac{1}{\phi_1}(-b_2-\beta b_3)(b_3-\phi_2) - \beta b_3 = 0, \\ \frac{-b_2-\beta b_3}{\phi_1}[\ln(b_1b_2+\beta b_1b_3) - 1] - \frac{(b-r)^2}{2\sigma^2} = \delta. \end{cases}$$

Then, (3.19) follows from solving the above system. This ends the proof of the proposition. □

3.4 General Power and Log Habit Utility

In this section, we examine another form of HARA habit utility formation given by

$$u(c,z) = \frac{1-\gamma}{\gamma}\left(\frac{mc-nz}{1-\gamma}+\eta\right)^{\gamma},\ c \geqslant z, \tag{3.23}$$

with restrictions on parameters

$$m \geqslant n \geqslant 0, \quad \gamma < 1 \ \text{ and } \ \eta \geqslant 0,$$

For this case, we calculate explicitly the optimal value function V.

Proposition 3: *For the utility $U(c,z)$ defined in (3.23), under the assumption $x + pz + q > 0$. the solution to (3.10) is given by:*

$$V(x,z) := \Omega\{x+pz+q\}^{\gamma}, \tag{3.24}$$

where

$$\begin{cases} \Omega := \frac{m}{\gamma-\gamma p\beta}\left[\frac{\left(\delta+\frac{\gamma(b-r)^2}{2(\gamma-1)\sigma^2}-\gamma r\right)m}{(1-\gamma)^2(1-p\beta)}\right]^{\gamma-1}, \\ p := -\frac{n}{mr+\gamma m-\gamma n}, \\ q := \frac{\eta(1-\gamma)}{rm}(1-p\beta). \end{cases} \tag{3.25}$$

Proof. Firstly, we assume that the solution to (3.10), denoted by V, takes the form of

$$V(x,z) = \Omega\{x+pz+q\}^{\gamma}. \tag{3.26}$$

Then, thanks to (3.14), we calculate c^* and π^* for this case as follows.

$$\begin{aligned}\pi^*(x,z) =& -\frac{(x+pz+q)(b-r)}{(\gamma-1)\sigma^2},\\ c^*(x,z) =& \frac{1}{m}\left\{(1-\gamma)[(\frac{\Omega\gamma-\Omega\gamma p\beta}{m})^{\frac{1}{\gamma-1}}(x+pz+q)-\eta]+nz\right\}.\end{aligned} \tag{3.27}$$

By calculating V_x, V_z, and V_{xx} from (3.26) and inserting them into (3.10) afterwards, we obtain

$$\begin{aligned}\delta\Omega(x+pz+q)^\gamma =& \frac{1-\gamma}{\gamma}\left(\frac{mc-nz}{1-\gamma}+\eta\right)^\eta + \Omega\gamma(x+pz+q)^{\gamma-1}(rx-c^*)\\ &+\Omega pr(x+pz+q)^{\gamma-1}\beta(c^*-z) - \frac{\Omega\gamma(b-r)^2}{2(\gamma-1)\sigma^2}(x+pz+q)^\gamma.\end{aligned} \tag{3.28}$$

Thus, a combination of (3.27) and (3.28) implies that

$$\begin{aligned}&(x+pz+q)\left\{\delta\Omega+\frac{\Omega\gamma(b-r)^2}{2(\gamma-1)\sigma^2}+\left(\frac{\Omega\gamma-\Omega\gamma p\beta}{m}\right)^{\frac{1}{\gamma-1}}\left[\frac{-\Omega(1-\gamma)^2(1-p\beta)}{m}\right]\right\}\\ =&\Omega\gamma rx+\frac{\Omega\gamma}{m}(-n+p\gamma(n-m))z+\frac{\eta\Omega\gamma(1-\gamma)(1-p\beta)}{m}.\end{aligned} \tag{3.29}$$

This is equivalent to

$$\Omega\gamma r = \frac{\frac{\Omega\gamma}{m}(-n+p\gamma(n-m))}{p} = \frac{\frac{\eta\Omega\gamma(1-\gamma)}{m}(1-p\beta)}{q}. \tag{3.30}$$

Then, we derive that

$$p = -\frac{n}{mr+\gamma m-\gamma n} \quad \text{and} \quad q = \frac{\eta(1-\gamma)}{rm}(1-p\beta). \tag{3.31}$$

In order to get Ω, we plug p and q back to (3.29) and obtain

$$\Omega = \frac{m}{\gamma - \gamma p\beta} \left[\frac{\left(\delta + \frac{\gamma(b-r)^2}{2(\gamma-1)\sigma^2} - \gamma r\right) m}{(1-\gamma)^2(1-p\beta)} \right]^{\gamma-1}. \tag{3.32}$$

This ends the proof of the proposition. $\square$

As a special case, which is frequently investigated, we consider the following utility function.

$$u(c,z) = \frac{\{c-z\}^A}{A}, \quad c-z>0 \quad \text{and} \quad A<1, \tag{3.33}$$

This case can be obtained from (3.23) by putting $m = n = 1$, $\eta = 0$ and $\gamma = A$. The corresponding optimal controls are given by the following proposition.

Proposition 4: *For the case of utility formation defined in (3.33), under the assumption that* $\delta - Ar + \frac{A(b-r)^2}{2(A-1)\sigma^2} > 0$, *the optimal value function* V *can be expressed as*

$$V(x,z) = \Omega\{rx - z\}^A,$$

where

$$\Omega = \frac{\left[\delta - Ar + \frac{A(b-r)^2}{2(A-1)\sigma^2}\right]^{A-1}}{(1-A)^{A-1}A(r+\beta)^A}.$$

And the optimal controls are

$$\begin{aligned} c^*(x,z) =& z + \frac{\delta - Ar + \frac{A(b-r)^2}{2(A-1)\sigma^2}}{(1-A)(r+\beta)}(rx-z), \\ \pi^*(x,z) =& -\frac{(rx-z)(b-r)}{(A-1)\sigma^2 r}. \end{aligned}$$

Another special case of utility formulation is logarithmic utility function, which can be expressed as

$$u(c,z) = \log\{c - z\}, \quad c > z \tag{3.34}$$

This case can be obtained from (3.23) by putting $m = n = 1$, $\eta = 0$, and let γ goes to 0.

Similarly, we can get the optimal control by the following proposition.

Proposition 5: *For the case of utility formation defined in (3.34), the optimal value function V is given by*

$$V(x,z) = \frac{1}{\delta}\log\{rx - z\} + M,$$

where

$$M = \frac{1}{\delta}\left(\log(\frac{\delta}{r+\beta}) + \frac{r}{\delta} - 1 + \frac{(b-r)^2}{2\delta\sigma^2}\right).$$

Furthermore, the optimal controls are given by

$$\pi^*(x,z) = \frac{(rx-z)(b-r)}{r\sigma^2}, \quad c^*(x,z) = z + \frac{\delta(rx-z)}{r+\beta}.$$

Chapter 4

The Optimal Portfolio and Consumption Processes

In chapter 3, we examined several particular cases of habit utility formation. For each case, we derived explicitly the optimal value function and the optimal controls in feedback forms. Herein, we continue investigating the optimal consumption. Throughout this chapter, $c^*(t)$, $z^*(t)$ and $x^*(t)$ denote the optimal consumption process, the optimal consumption habit process and the optimal wealth process respectively. Thanks to the instantaneous optimal portfolio and consumption obtained in chapter 3, we give the explicit description for $c^*(t)$, $x^*(t)$ and $z^*(t)$ in this chapter.

4.1 Exponential Habit Utility Function

For the case of exponential utility formation defined in (3.16), the dynamic of the process c^* is described in the following.

Theorem 8: *The process $L_t := e^{\beta t}c^*(t)$ satisfies the following stochastic dif-*

ferential equation

$$dX_t = \left[\alpha_1 X_t + \alpha_2 \int_0^t X_s ds\right] dt + dK_t, \tag{4.1}$$

with the initial condition

$$X_0 = \frac{r x_0[\phi_1(r+\beta) - \phi_2\beta]}{(r+\beta)\phi_1} + \frac{\beta\phi_2}{\phi_1(r+\beta)} z_0 - \frac{2(r-\delta)\sigma^2 - (b-r)^2}{2\sigma^2 r\phi_1}. \tag{4.2}$$

Here

$$\begin{aligned}
dK_t &:= \left(\gamma_1 e^{\beta t} + \gamma_2 z_0\right) dt + \gamma_3 e^{\beta t} dW_t, \\
\alpha_1 &:= \beta + r - \frac{b_2}{\phi_1} + \beta\frac{\phi_2 - b_3}{\phi_1} = \frac{\beta(\phi_1+\phi_2)}{\phi_1}, \\
\alpha_2 &:= -\frac{\phi_2 - b_3}{\phi_1}\beta(r+\beta) = -\beta^2\frac{\phi_2}{\phi_1}, \\
\gamma_1 &:= \frac{2\sigma^2(r-\delta) + (b-r)^2}{2\phi_1\sigma^2}, \\
\gamma_2 &:= -\frac{\phi_2 - b_3}{\phi_1}(r+\beta), \\
\gamma_3 &:= \frac{b-r}{\phi_1\sigma}.
\end{aligned} \tag{4.3}$$

Proof. Put

$$\tilde{c}_t^* := e^{-rt}c^*(t), \quad \tilde{z}_t^* := e^{-rt}z^*(t), \quad \text{and} \quad \tilde{x}_t^* := e^{-rt}x^*(t), \quad t \geqslant 0. \tag{4.4}$$

Then, by using Itô's Formula, (2.6) and (2.7), we derive

$$\begin{aligned}
d\tilde{z}_t^* &= -re^{-rt}z^*(t) + e^{-rt}dz^*(t) \\
&= -(r+\beta)e^{-rt}z^*(t)dt + \beta e^{-rt}c^*(t)dt \\
&= -(r+\beta)e^{-rt}[z_0 e^{-\beta t} + \int_0^t \beta e^{\beta(s-t)}c^*(s)ds]dt + \beta e^{-rt}c^*(t)dt.
\end{aligned} \tag{4.5}$$

Similarly, we use Itô's Formula and (3.6) and get

$$\begin{aligned} d\tilde{x}_t^* &= -re^{-rt}x^*(t) + e^{-rt}dx^*(t) \\ &= e^{-rt}\left[\frac{(b-r)^2}{b_2\sigma^2} - c^*(t)\right]dt + \frac{b-r}{b_2\sigma}e^{-rt}dW_t. \end{aligned} \tag{4.6}$$

Due to (3.21), we have

$$\begin{aligned} \tilde{c}_t^* =& \frac{b_2}{\phi_1}\tilde{x}_t^* + \frac{\phi_2 - b_3}{\phi_1}\tilde{z}_t^* - \frac{1}{\phi_1}\ln(b_1b_2 + b_1\beta b_3)e^{-rt} \\ =& \frac{b_2}{\phi_1}\tilde{x}_t^* + \frac{\phi_2 - b_3}{\phi_1}\tilde{z}_t^* - \frac{2\sigma^2(r-\delta) - (b-r)^2}{2\phi_1\sigma^2 r}e^{-rt}. \end{aligned} \tag{4.7}$$

Then, by differentiating the above equation and inserting (4.5) and (4.6) afterwards, we obtain

$$\begin{aligned} d\tilde{c}_t^* =& \frac{b_2}{\phi_1}d\tilde{x}_t^* + \frac{\phi_2 - b_3}{\phi_1}d\tilde{z}_t^* + \frac{2\sigma^2(r-\delta) - (b-r)^2}{2\phi_1\sigma^2}e^{-rt}dt \\ =& \frac{b_2}{\phi_1}\left\{e^{-rt}\left[\frac{(b-r)^2}{b_2\sigma^2} - c^*(t)\right]dt + \frac{b-r}{b_2\sigma}e^{-rt}dW_t\right\} \\ &+ \frac{\phi_2 - b_3}{\phi_1}\left[-(r+\beta)e^{-rt}[z_0e^{-\beta t} + \int_0^t \beta e^{\beta(s-t)}c^*(s)ds]dt + \beta e^{-rt}c^*(t)dt\right] \\ &+ \frac{2\sigma^2(r-\delta) - (b-r)^2}{2\phi_1\sigma^2}e^{-rt}dt \\ =& \left[-\frac{\phi_2 - b_3}{\phi_1}(r+\beta)e^{-(r+\beta)t}z_0 + \frac{2\sigma^2(r-\delta) + (b-r)^2}{2\phi_1\sigma^2}e^{-rt}\right]dt \\ &+ \left[-\frac{b_2}{\phi_1} + \beta\frac{\phi_2 - b_3}{\phi_1}\right]\tilde{c}_t^* dt - \frac{\phi_2 - b_3}{\phi_1}\beta(r+\beta)e^{-(r+\beta)t}\left(\int_0^t e^{(\beta+r)s}\tilde{c}_s^* ds\right)dt \\ &+ \frac{b-r}{\phi_1\sigma}e^{-rt}dW_t. \end{aligned} \tag{4.8}$$

Consider the process

$$L_t := e^{(\beta+r)t}\tilde{c}_t^* = e^{\beta t}c^*(t), \quad t \geqslant 0, \tag{4.9}$$

whose dynamic will be derived below using (4.8).

$$\begin{aligned}
dL_t =&(\beta+r)e^{(\beta+r)t}\tilde{c}_t^* dt + e^{(\beta+r)t} d\tilde{c}_t^* \\
=&\left[-\frac{\phi_2-b_3}{\phi_1}(r+\beta)z_0 + \frac{2\sigma^2(r-\delta)+(b-r)^2}{2\phi_1\sigma^2}e^{\beta t}\right]dt \\
&+\left[\beta+r-\frac{b_2}{\phi_1}+\beta\frac{\phi_2-b_3}{\phi_1}\right]L_t dt - \frac{\phi_2-b_3}{\phi_1}\beta(r+\beta)\left(\int_0^t L_s ds\right)dt \\
&+\frac{b-r}{\phi_1\sigma}e^{\beta t}dW_t \\
=&\alpha_1 L_t dt + \alpha_2\left(\int_0^t L_s ds\right)dt + dK_t,
\end{aligned} \tag{4.10}$$

where K_t, α_1 and α_2 are defined in (4.3). Furthermore, the initial value of the process L can be calculated as follows.

$$\begin{aligned}
L_0 = c^*(0) = &\frac{b_2}{\phi_1}x_0 + \frac{\phi_2-b_3}{\phi_1}z_0 - \frac{1}{\phi_1}ln[b_1b_2+b_1\beta b_3] \\
= &\frac{rx_0[\phi_1(r+\beta)-\phi_2\beta]}{(r+\beta)\phi_1} + \frac{\beta\phi_2}{\phi_1(r+\beta)}z_0 - \frac{2(r-\delta)\sigma^2-(b-r)^2}{2\sigma^2 r\phi_1}.
\end{aligned}$$

This ends the proof of the theorem. □

Below, we will prove the uniqueness of the solution to the stochastic differential equation (4.1)–(4.2).

Proposition 6: *The solution to the stochastic differential equation (4.1)–(4.2), when it exists, is unique.*

Proof. Suppose that there are two solutions L^1 and L^2 to (4.1)–(4.2), and put

$D_t := L_t^1 - L_t^2$. Thus, we calculate

$$\begin{cases} dD_t = \alpha_1 D_t dt + \alpha_2 \left(\int_0^t D_s ds\right) dt, \\ D_0 = 0. \end{cases} \tag{4.11}$$

By putting $\xi_t := \int_0^t D_s ds$, we obtain

$$\begin{cases} \xi_t'' = \alpha_1 \xi_t' + \alpha_2 \xi_t, \\ \xi_0' = 0, \\ \xi_0 = 0. \end{cases} \tag{4.12}$$

Using Theorem 6, we deduce the following cases.

Case 1. If $\Delta := \alpha_1^2 - 4\alpha_2 > 0$, then $\xi_t = C_1 e^{r_1 t} + C_2 e^{r_2 t}$ where r_1 and r_2 are the distinct real roots to the equation $x^2 - \alpha_1 x - \alpha_2 = 0$ and C_1, C_2 are constants. From the initial conditions, we obtain

$$\begin{cases} C_1 + C_2 = 0, \\ r_1 C_1 + r_2 C_2 = 0. \end{cases} \tag{4.13}$$

Thus, $C_1 = C_2 = 0$, which means $\xi_t \equiv 0$, $t \geqslant 0$. Hence $L^1 \equiv L^2$.

Case 2. If $\Delta := \alpha_1^2 - 4\alpha_2 = 0$, then $\xi_t = (C_1 + C_2 t) e^{r_1 t}$ where r_1 is the unique real root to the equation $x^2 - \alpha_1 x - \alpha_2 = 0$, and C_1 and C_2 are constants. The initial conditions lead to

$$\begin{cases} C_1 = 0, \\ C_1 r_1 + C_2 = 0. \end{cases} \tag{4.14}$$

As a result, we get $C_1 = C_2 = 0$, or equivalently $\xi \equiv 0$.

Case 3. If $\Delta := \alpha_1^2 - 4\alpha_2 < 0$, then $\xi_t = [C_1 \cos(\nu t) + C_2 \sin(\nu t)]e^{\mu t}$, where $\mu \pm \nu i$ ($\nu > 0$) are the distinct complex roots to $x^2 - \alpha_1 x - \alpha_2 = 0$. The initial conditions imply $C_1 = C_2 = 0$, which means $\xi_t \equiv 0$.

In conclusion, in all cases, we obtained $L^1 = L^2$. This proves the uniqueness of solution to (4.1)–(4.2). □

Proposition 7: *Put $I_t := \int_0^t L_s ds$. Then, I is a solution to the non-homogeneous linear differential equation:*

$$I_t'' - \alpha_1 I_t' - \alpha_2 I_t = G_t, \tag{4.15}$$

with the initial conditions

$$\begin{aligned} I_0 &= 0, \\ I_0' &= \frac{r x_0[\phi_1(r+\beta) - \phi_2\beta]}{(r+\beta)\phi_1} + \frac{\beta\phi_2}{\phi_1(r+\beta)} z_0 - \frac{2(r-\delta)\sigma^2 - (b-r)^2}{2\sigma^2 r\phi_1}. \end{aligned}$$

Here G_t is the following process

$$G_t = -\frac{\phi_2 - b_3}{\phi_1}(r+\beta)z_0 + \frac{2\sigma^2(r-\delta) + (b-r)^2}{2\phi_1\sigma^2} e^{\beta t} + \frac{b-r}{\phi_1\sigma} e^{\beta t}\dot{W}_t, \tag{4.16}$$

where $\dot{W}$ is the white noise.

Proof. Thanks to Theorem 8, $I_t = \int_0^t L(s)ds$ satisfies

$$I_t'' - \alpha_1 I_t' - \alpha_2 I_t = \frac{dK_t}{dt} = G_t. \tag{4.17}$$

Finally, we put $t = 0$ to get the initial conditions

$$I_0 = 0, \tag{4.18}$$

$$\begin{aligned} I_0' &= c^*(0) \\ &= \frac{rx_0[\phi_1(r+\beta) - \phi_2\beta]}{(r+\beta)\phi_1} + \frac{\beta\phi_2}{\phi_1(r+\beta)}z_0 - \frac{2(r-\delta)\sigma^2 - (b-r)^2}{2\sigma^2 r\phi_1}. \end{aligned} \tag{4.19}$$

This ends the proof of Proposition 7. □

According to Theorem 6, in order to solve (4.15), we need to discuss its characteristic equation:

$$\lambda^2 - \frac{\beta(\phi_1 + \phi_2)}{\phi_1}\lambda + \beta^2\frac{\phi_2}{\phi_1} = 0. \tag{4.20}$$

The discriminant of the above equation is:

$$\Delta := \beta^2(\frac{\phi_2}{\phi_1} - 1)^2 \geqslant 0. \tag{4.21}$$

Thus, we obtain two cases whether $\phi_1 \neq \phi_2$ or $\phi_1 = \phi_2$. These two cases will be discussed separately in two subsections.

4.1.1 The case of $\phi_1 \neq \phi_2$

For this case, $\lambda^2 - \alpha_1\lambda - \alpha_2 = 0$ has two solutions given by

$$\lambda_1 := \frac{\beta\left(1 + \frac{\phi_2}{\phi_1} + \left|1 - \frac{\phi_2}{\phi_1}\right|\right)}{2} \quad \text{and} \quad \lambda_2 := \frac{\beta\left(1 + \frac{\phi_2}{\phi_1} - \left|1 - \frac{\phi_2}{\phi_1}\right|\right)}{2}. \tag{4.22}$$

Theorem 9: *Consider λ_1 and λ_2 given in (4.22), and γ_1, γ_2 and γ_3 given in (4.3). Then, the following assertions hold.*

1. *The optimal consumption rate process $c^*(t)$ is given by*

$$c^*(t) = c^*(0)\kappa_1(0,t)+\int_0^t \kappa_1(s,t)\left(\gamma_1 e^{\beta s}+\gamma_2 z_0\right)dt+\gamma_3\int_0^t \kappa_1(s,t)e^{\beta s}dW_s,\ t\geqslant 0, \tag{4.23}$$

where $\kappa_1(s,t)$ is defined by

$$\kappa_1(s,t) := \frac{e^{-\beta t}}{\sqrt{\Delta}}\left[\lambda_1 e^{\lambda_1(t-s)}-\lambda_2 e^{\lambda_2(t-s)}\right],\ t\geqslant s\geqslant 0.$$

2. *The optimal consumption habit $z^*(t)$ is given by*

$$\begin{aligned} z^*(t) =& z_0 e^{-\beta t}+c^*(0)\kappa_2(0,t)+\int_0^t \kappa_2(s,t)\left(\gamma_1 e^{\beta s}+\gamma_2 z_0\right)dt \\ &+\gamma_3\int_0^t \kappa_2(s,t)e^{\beta s}dW_s,\ t\geqslant 0, \end{aligned} \tag{4.24}$$

where $\kappa_2(s,t)$ is defined by

$$\kappa_2(s,t) := \frac{\beta e^{-\beta t}}{\sqrt{\Delta}}\left[e^{\lambda_1(t-s)}-e^{\lambda_2(t-s)}\right],\ t\geqslant s\geqslant 0.$$

3. *The optimal wealth process $x^*(t)$ is given by*

$$\begin{aligned} x^*(t) =& e^{rt}x_0+\frac{(b-r)^2}{rb_2\sigma^2}(e^{rt}-1)+\frac{b-r}{b_2\sigma}e^{rt}\int_0^t e^{-rs}dW_s-c^*(0)\kappa_3(0,t) \\ &-\int_0^t \kappa_3(s,t)\left(\gamma_1 e^{\beta s}+\gamma_2 z_0\right)dt-\gamma_3\int_0^t \kappa_3(s,t)e^{\beta s}dW_s. \end{aligned} \tag{4.25}$$

where $\kappa_3(s,t)$ is defined by

$$\kappa_3(s,t) := \frac{\lambda_1\left[e^{(\lambda_1-\beta)t-\lambda_1 s}-e^{rt-(r+\beta)s}\right]}{\sqrt{\Delta}(\lambda_1-r-\beta)}-\frac{\lambda_2\left[e^{(\lambda_2-\beta)t-\lambda_2 s}-e^{rt-(r+\beta)s}\right]}{\sqrt{\Delta}(\lambda_2-r-\beta)},\ t\geqslant s\geqslant 0. \tag{4.26}$$

Proof. 1) Consider the process

$$\bar{L}_t := c^*(0)e^{\beta t}\kappa_1(0,t) + e^{\beta t}\int_0^t \kappa_1(s,t)dK_s, \quad t \geqslant 0. \tag{4.27}$$

Then, thanks to Theorem 8 and Proposition 7, it is enough to prove $\bar{L}_t$ fulfills the SDE (4.1)–(4.2). To this end, we write

$$\begin{aligned}\bar{L}_t =& \frac{c^*(0)\lambda_1}{\sqrt{\Delta}}e^{\lambda_1 t} - \frac{c^*(0)\lambda_2}{\sqrt{\Delta}}e^{\lambda_2 t} \\ &+ \frac{\lambda_1}{\sqrt{\Delta}}\int_0^t e^{\lambda_1(t-s)}dK_s - \frac{\lambda_2}{\sqrt{\Delta}}\int_0^t e^{\lambda_2(t-s)}dK_s.\end{aligned} \tag{4.28}$$

Then, the dynamic of $d\bar{L}$ is given by

$$\begin{aligned}d\bar{L}_t = \Bigg[&\frac{c^*(0)\lambda_1^2}{\sqrt{\Delta}}e^{\lambda_1 t} - \frac{c^*(0)\lambda_2^2}{\sqrt{\Delta}}e^{\lambda_2 t} + \frac{\lambda_1^2}{\sqrt{\Delta}}\int_0^t e^{\lambda_1(t-s)}dK_s \\ &-\frac{\lambda_2^2}{\sqrt{\Delta}}\int_0^t e^{\lambda_2(t-s)}dK_s\Bigg]dt + dK_t,\end{aligned} \tag{4.29}$$

and $\int_0^t \bar{L}_s ds$ is given by

$$\begin{aligned}\int_0^t \bar{L}_s ds =& \frac{c^*(0)}{\sqrt{\Delta}}(e^{\lambda_1 t}-1) - \frac{c^*(0)}{\sqrt{\Delta}}(e^{\lambda_2 t}-1) \\ &+ \frac{1}{\sqrt{\Delta}}\int_0^t\left[\lambda_1\int_0^s e^{\lambda_1(s-u)}dK_u - \lambda_2\int_0^s e^{\lambda_2(s-u)}dK_u\right]ds.\end{aligned} \tag{4.30}$$

Since

$$\left(\lambda_i\int_0^s e^{\lambda_i(s-u)}dK_u\right)ds = d\left(\int_0^s e^{\lambda_i(s-u)}dK_u\right) - dK_s,\ i=1,2, \tag{4.31}$$

the equation (4.30) becomes

$$\int_0^t \bar{L}_s ds = \frac{c^*(0)}{\sqrt{\Delta}} e^{\lambda_1 t} - \frac{c^*(0)}{\sqrt{\Delta}} e^{\lambda_2 t} + \frac{1}{\sqrt{\Delta}} \int_0^t e^{\lambda_1 (t-s)} dK_s - \frac{1}{\sqrt{\Delta}} \int_0^t e^{\lambda_2 (t-s)} dK_s. \tag{4.32}$$

Then, by combining (4.28), (4.32) and

$$\lambda_i^2 = \alpha_1 \lambda_i + \alpha_2, \quad i = 1, 2,$$

we obtain

$$\begin{aligned} \alpha_1 \bar{L}_t + \alpha_2 \int_0^t \bar{L}_s ds =& \frac{c^*(0)\lambda_1^2}{\sqrt{\Delta}} e^{\lambda_1 t} - \frac{c^*(0)\lambda_2^2}{\sqrt{\Delta}} e^{\lambda_2 t} + \frac{\lambda_1^2}{\sqrt{\Delta}} \int_0^t e^{\lambda_1 (t-s)} dK_s \\ & - \frac{\lambda_2^2}{\sqrt{\Delta}} \int_0^t e^{\lambda_2 (t-s)} dK_s. \end{aligned} \tag{4.33}$$

Therefore, by inserting (4.33) in (4.29), we get

$$d\bar{L}_t = \left[\alpha_1 \bar{L}_t + \alpha_2 \int_0^t \bar{L}_s ds \right] dt + dK_t. \tag{4.34}$$

This proves that $\bar{L}$ is a solution to the SDE (4.1)–(4.2). Since this SDE has a unique solution (see Proposition 6), and $L(t) := e^{\beta t} c^*(t)$ is also a solution to this SDE, we get

$$\bar{L}_t = e^{\beta t} c^*(t), \quad t \geqslant 0.$$

Thus, we obtain

$$c^*(t) = c^*(0)\kappa_1(0,t) + \int_0^t \kappa_1(s,t) dK_s, \ t \geqslant 0, \tag{4.35}$$

The equality (4.23) follows directly from the equation (4.35). This ends the proof of the first assertion.

2) By inserting (4.35) into (2.7), we get

$$\begin{aligned} z^*(t) =& z_0 e^{-\beta t} + \int_0^t \beta e^{\beta(s-t)} c^*(s) ds \\ =& z_0 e^{-\beta t} + \int_0^t e^{\beta(s-t)} \left[c^*(0)\kappa_1(0,s) + \int_0^s \kappa_1(u,s) dK_u \right] ds. \end{aligned} \tag{4.36}$$

Since

$$\begin{aligned} \int_0^t e^{\beta(s-t)} c^*(0)\kappa_1(0,s) ds =& \frac{e^{-\beta t} c^*(0)}{\sqrt{\Delta}} \int_0^t \left(\lambda_1 e^{\lambda_1 s} - \lambda_2 e^{\lambda_2 s} \right) ds \\ =& \frac{e^{-\beta t} c^*(0)}{\sqrt{\Delta}} \left(e^{\lambda_1 t} - e^{\lambda_2 t} \right) \end{aligned} \tag{4.37}$$

and

$$\begin{aligned} & \int_0^t e^{\beta(s-t)} \left(\int_0^s \kappa_1(u,s) dK_u \right) ds \\ =& \frac{e^{-\beta t}}{\sqrt{\Delta}} \int_0^t \left(\int_0^s \left[\lambda_1 e^{\lambda_1(s-u)} - \lambda_2 e^{\lambda_2(s-u)} \right] dK_u \right) ds \\ =& \frac{e^{-\beta t}}{\sqrt{\Delta}} \left(\int_0^t e^{\lambda_1(t-s)} dK_s - \int_0^t e^{\lambda_2(t-s)} dK_s \right), \end{aligned} \tag{4.38}$$

the next equality (4.39) follows directly from the equation (4.36).

$$z^*(t) = z_0 e^{-\beta t} + c^*(0)\kappa_2(0,t) + \int_0^t \kappa_2(s,t) dK_s, \ t \geqslant 0. \tag{4.39}$$

Thus, by inserting the expression of dK_t given in (4.3) into (4.39) and simplification, we get (4.24), and this ends the proof of the second assertion.

3) Here, we will prove (4.25). By inserting (3.22) and (4.35) into (3.7), we

get

$$
\begin{aligned}
x^*(t) =& e^{rt}x_0 + \int_0^t e^{-r(s-t)} \left[\pi^*(s)(b-r) - c^*(s)\right] ds + \int_0^t e^{-r(s-t)}\pi^*(s)\sigma dW_s \\
=& e^{rt}x_0 + \frac{(b-r)^2}{rb_2\sigma^2}(e^{rt}-1) + \frac{b-r}{b_2\sigma}e^{rt}\int_0^t e^{-rs}dW_s \\
&- e^{rt}\int_0^t e^{-rs}\left[c^*(0)\kappa_1(0,s) + \int_0^s \kappa_1(u,s)dK_u\right] ds.
\end{aligned}
\tag{4.40}
$$

Now, since $\frac{\phi_2}{\phi_1} < \frac{r+\beta}{\beta}$ due to $b_2 > 0$, by using the notation in (4.26), we calculate

$$
\begin{aligned}
& e^{rt}\int_0^t e^{-rs}c^*(0)\kappa_1(0,s)ds \\
=& e^{rt}\int_0^t e^{-rs}c^*(0)\frac{e^{-\beta s}}{\sqrt{\Delta}}\left[\lambda_1 e^{\lambda_1 s} - \lambda_2 e^{\lambda_2 s}\right] ds \\
=& \frac{c^*(0)e^{rt}}{\sqrt{\Delta}}\left[\frac{\lambda_1}{\lambda_1 - r - \beta}\left(e^{(\lambda_1 - r - \beta)t} - 1\right) - \frac{\lambda_2}{\lambda_2 - r - \beta}\left(e^{(\lambda_2 - r - \beta)t} - 1\right)\right] \\
=& c^*(0)\kappa_3(0,t),
\end{aligned}
\tag{4.41}
$$

and

$$
\begin{aligned}
& e^{rt}\int_0^t e^{-rs}\int_0^s \kappa_1(u,s)dK_u ds \\
=& e^{rt}\int_0^t e^{-rs}\int_0^s \frac{e^{-\beta s}}{\sqrt{\Delta}}\left[\lambda_1 e^{\lambda_1(s-u)} - \lambda_2 e^{\lambda_2(s-u)}\right] dK_u ds \\
=& \frac{e^{rt}}{\sqrt{\Delta}}\left\{\frac{\lambda_1}{\lambda_1 - \beta - r}\left[e^{(\lambda_1 - \beta - r)t}\int_0^t e^{-\lambda_1 s}dK_s - \int_0^t e^{-(\beta + r)s}dK_s\right]\right. \\
& \left. - \frac{\lambda_2}{\lambda_2 - \beta - r}\left[e^{(\lambda_2 - \beta - r)t}\int_0^t e^{-\lambda_2 s}dK_s - \int_0^t e^{-(\beta + r)s}dK_s\right]\right\} \qquad (4.42) \\
=& \int_0^t \kappa_3(s,t)dK_s \qquad (4.43)
\end{aligned}
$$

A combination of (4.40)–(4.42) leads to

$$\begin{aligned} x^*(t) =& e^{rt}x_0 + \frac{(b-r)^2}{rb_2\sigma^2}(e^{rt}-1) + \frac{b-r}{b_2\sigma}e^{rt}\int_0^t e^{-rs}dW_s \\ & - c^*(0)\kappa_3(0,t) - \int_0^t \kappa_3(s,t)dK_s. \end{aligned} \tag{4.44}$$

By inserting the expression of dK_t given in (4.3) into the above equation, we get (4.25), and this ends the proof of the third assertion. In conclusion, the proof for the theorem is completed. □

4.1.2 The case of $\phi_1 = \phi_2$

For this case, the characteristic equation (4.20) has one single root $\lambda_1 := \beta$. Hence, the SDE (4.1)–(4.2) becomes

$$\begin{cases} dX_t = \left[2\beta X_t - \beta^2 \int_0^t X_s ds\right] dt + dK_t, \\ X_0 = \frac{r^2}{r+\beta}x_0 + \frac{\beta}{r+\beta}z_0 - \frac{2(r-\delta)\sigma^2-(b-r)^2}{2\sigma^2 r\phi_1}. \end{cases} \tag{4.45}$$

In this case, dK_t is given by

$$dK_t := \left[-\beta z_0 + \frac{2\sigma^2(r-\delta)+(b-r)^2}{2\phi_1\sigma^2}e^{\beta t}\right] dt + \frac{b-r}{\phi_1\sigma}e^{\beta t}dW_t. \tag{4.46}$$

Theorem 10: *Consider γ_1 and γ_3 given in (4.3), for the case of $\phi_1 = \phi_2$, the following properties hold.*

1. *The optimal consumption rate process $c^*(t)$ is given by*

$$c^*(t) = c^*(0)(1+\beta t) + \frac{\gamma_1\beta}{2}t^2 + \gamma_1 t - \beta z_0 t + \int_0^t \gamma_3(\beta t - \beta s + 1)dW_s, \quad t \geqslant 0. \tag{4.47}$$

2. The optimal consumption habit $z^*(t)$ is given by

$$z^*(t) = z_0 + \beta c^*(0)t - \beta z_0 t + \frac{\gamma_1\beta}{2}t^2 + \beta\gamma_3 \int_0^t (t-s)dW_s, \quad t \geqslant 0. \quad (4.48)$$

3. The optimal wealth process $x^*(t)$ is given by

$$\begin{aligned} x^*(t) =& x_0 + \left[\frac{r\beta}{r+\beta}x_0 - \frac{\beta}{r+\beta}z_0 + \frac{2r\sigma^2(r-\delta)+(r+2\beta)(b-r)^2}{2\phi_1\sigma^2 r^2}\right]t \\ &+ \frac{\beta}{2r}\gamma_1 t^2 + \gamma_3 \int_0^t \left[\frac{\beta+r}{r^2} + \frac{\beta}{r}(t-s)\right] dW_s, \quad t \geqslant 0. \end{aligned} \quad (4.49)$$

Proof. 1) Here, we prove (4.47). Consider the process

$$\bar{L}_t = c^*(0)(1+\beta t)e^{\beta t} + \int_0^t (\beta t - \beta s + 1)e^{\beta(t-s)}dK_s, \quad t \geqslant 0. \quad (4.50)$$

Then, thanks to Theorem 8 and Proposition 7, it is enough to prove that the process $\bar{L}_t$ satisfies the SDE (4.45)–(4.46). To this end, we calculate the dynamic of $d\bar{L}$ as follows.

$$\begin{aligned} d\bar{L}_t = \Big[& c^*(0)(2\beta+\beta^2 t)e^{\beta t} + (2\beta+\beta^2 t)e^{\beta t}\int_0^t e^{-\beta s}dK_s \\ & -\beta^2 e^{\beta t}\int_0^t s e^{-\beta s}dK_s\Big] dt + dK_t. \end{aligned} \quad (4.51)$$

Similarly, the dynamic of $\int_0^t \bar{L}_s ds$ is derived as

$$\int_0^t \bar{L}_s ds = c^*(0)te^{\beta t} + \int_0^t \left[(\beta s+1)e^{\beta s}\int_0^s e^{-\beta u}dK_u - \beta e^{\beta s}\int_0^s u e^{-\beta u}dK_u\right] ds. \quad (4.52)$$

By inserting

$$\left((\beta s+1)e^{\beta s}\int_0^s e^{-\beta u}dK_u\right)ds = d\left(s\int_0^s e^{\beta(s-u)}dK_u\right) - sdK_s,$$

and

$$\left(\beta e^{\beta s}\int_0^s ue^{-\beta u}dK_u\right)ds = d\left(\int_0^s ue^{\beta(s-u)}dK_u\right) - sdK_s,$$

into (4.52), we get

$$\int_0^t \bar{L}_s ds = c^*(0)te^{\beta t} + t\int_0^t e^{\beta(t-s)}dK_s - \int_0^t se^{\beta(t-s)}dK_s. \tag{4.53}$$

Then, by combining (4.50) and (4.53), we obtain

$$\begin{aligned} 2\beta\bar{L}_t - \beta^2\int_0^t \bar{L}_s ds =& c^*(0)(2\beta+\beta^2 t)e^{\beta t} + (2\beta+\beta^2 t)e^{\beta t}\int_0^t e^{-\beta s}dK_s \\ &- \beta^2 e^{\beta t}\int_0^t se^{-\beta s}dK_s. \end{aligned} \tag{4.54}$$

Now, we insert (4.54) into (4.51) and get

$$d\bar{L}_t = \left[\alpha_1\bar{L}_t + \alpha_2\int_0^t \bar{L}_s ds\right]dt + dK_t. \tag{4.55}$$

This proves that $\bar{L}$ is a solution to the SDE (4.45)–(4.46). Since this SDE has a unique solution (see Proposition 6), and $L(t) := e^{\beta t}c^*(t)$ is already a solution to this SDE, we deduce that

$$\bar{L}_t = e^{\beta t}c^*(t), \quad t \geqslant 0.$$

Thus, we get

$$c^*(t) = c^*(0)(1+\beta t) + \int_0^t e^{-\beta s}(\beta t - \beta s + 1)dK_s, \quad t \geqslant 0. \tag{4.56}$$

By plugging the expression of dK_t given in (4.46) into the above equation, we obtain (4.47) after simplification. This ends the proof of the first assertion.

2) Here, we will prove the equality (4.48). By inserting (4.56) into (2.7), we get

$$\begin{aligned} z^*(t) =& z_0 e^{-\beta t} + \int_0^t \beta e^{\beta(s-t)} c^*(s) ds \\ =& z_0 e^{-\beta t} + \int_0^t \beta e^{\beta(s-t)} \left[c^*(0)(1+\beta s) + \int_0^s e^{-\beta u}(\beta s - \beta u + 1) dK_u \right] ds. \end{aligned} \tag{4.57}$$

Since

$$\int_0^t \beta e^{\beta(s-t)} c^*(0)(1+\beta s) ds = \beta c^*(0) t,$$

and

$$\begin{aligned} & \int_0^t \beta e^{\beta(s-t)} \int_0^s e^{-\beta u}(\beta s - \beta u + 1) dK_u ds \\ =& \beta e^{-\beta t} \int_0^t \left[(\beta s + 1) e^{\beta s} \int_0^s e^{-\beta u} dK_u - \beta e^{\beta s} \int_0^s u e^{-\beta u} dK_u \right] ds \\ =& \beta e^{-\beta t} \left[t e^{\beta t} \int_0^t e^{-\beta s} dK_s - e^{\beta t} \int_0^t s e^{-\beta s} dK_s \right] \\ =& \beta \int_0^t (t-s) e^{-\beta s} dK_s, \end{aligned}$$

the equation (4.57) becomes

$$z^*(t) = z_0 e^{-\beta t} + \beta c^*(0) t + \beta \int_0^t (t-s) e^{-\beta s} dK_s, \quad t \geqslant 0.$$

By inserting the expression of dK_t given in (4.46) into this equation, we get (4.48) immediately, and this completes the proof of the second assertion.

3) Here, we will prove (4.49). By inserting (3.22) into (3.7), we get

$$
\begin{aligned}
x^*(t) =& e^{rt}x_0 + \int_0^t e^{-r(s-t)} \left[\pi^*(s)(b-r) - c^*(s)\right] ds + \int_0^t e^{-r(s-t)}\pi^*(s)\sigma dW_s \\
=& e^{rt}x_0 + \frac{(b-r)^2}{rb_2\sigma^2}(e^{rt}-1) + \frac{b-r}{b_2\sigma}e^{rt}\int_0^t e^{-rs}dW_s - e^{rt}\int_0^t e^{-rs}c^*(s)ds.
\end{aligned}
\tag{4.58}
$$

Here, we denote

$$
\psi(s,t) := e^{-\beta s}\left[\left(\frac{1}{r}+\frac{\beta}{r^2}\right)\left(e^{r(t-s)}-1\right) - \frac{\beta}{r}(t-s)\right].
$$

Similar as we did in calculating $z^*(t)$, we obtain that

$$
\begin{aligned}
& e^{rt}\int_0^t e^{-rs}c^*(s)ds \\
=& e^{rt}\int_0^t e^{-rs}\left[c^*(0)(1+\beta s) + \int_0^s e^{-\beta u}(\beta s - \beta u + 1)dK_u\right] ds \\
=& c^*(0)\left[-\frac{1}{r}(1-e^{rt}) - \frac{\beta}{r}t - \frac{\beta}{r^2}(1-e^{rt})\right] + \int_0^t e^{-\beta s}\left[(\frac{1}{r}+\frac{\beta}{r^2})(e^{r(t-s)}-1) \right. \\
& \left. -\frac{\beta}{r}(t-s)\right] dK_s \\
=& c^*(0)\psi(0,t) + \int_0^t \psi(s,t)dK_s.
\end{aligned}
$$

By inserting this equation into (4.58), we derive that

$$
\begin{aligned}
x^*(t) =& e^{rt}x_0 + \frac{(b-r)^2}{rb_2\sigma^2}(e^{rt}-1) + \frac{b-r}{b_2\sigma}e^{rt}\int_0^t e^{-rs}dW_s \\
& - c^*(0)\psi(0,t) - \int_0^t \psi(s,t)dK_s, \quad t \geqslant 0.
\end{aligned}
\tag{4.59}
$$

By inserting the expression of dK_t given in (4.46) into this equation, we get (4.49).

This ends the proof of the theorem. □

4.2 General Power Habit Utility Function

For the class of utilities defined in (3.23), we will describe the dynamic of the process c^* as follows.

Theorem 11: *The process $L_t := e^{\beta t} c^*(t)$ satisfies the following stochastic differential equation*

$$\begin{aligned} dX_t = &\left[(\beta + h_1)X_t + h_2\beta \int_0^t X_s ds + h_2 z_0 + h_3 e^{\beta t}\right] dt \\ &+ \left[h_4 X_t + h_5 \beta \int_0^t X_s ds + h_5 z_0 + h_6 e^{\beta t}\right] dW_t, \end{aligned} \tag{4.60}$$

with the initial condition

$$X_0 = n_1 x_0 + n_2 z_0 + n_3. \tag{4.61}$$

Here,

$$\begin{cases} h_1 = r + m_1(b-r) - n_1 + n_2\beta, \\ h_2 = -n_2(r + m_1(b-r)) + (b-r)m_2n_1 - n_2\beta, \\ h_3 = -n_3(r + m_1(b-r)) + (b-r)m_3n_1\beta, \\ h_4 = \sigma m_1, \\ h_5 = -\sigma n_2 m_1 + \sigma n_1 m_2, \\ h_6 = -\sigma n_3 m_1 + \sigma n_1 m_3. \end{cases} \tag{4.62}$$

and

$$\begin{cases} m_1 = \frac{b-r}{(1-\gamma)\sigma^2}, \\ m_2 = \frac{(b-r)p}{(1-\gamma)\sigma^2}, \\ m_3 = \frac{(b-r)q}{(1-\gamma)\sigma^2}, \\ n_1 = \frac{1-\gamma}{m}\left(\frac{\Omega\gamma-\Omega\gamma p\beta}{m}\right)^{\frac{1}{\gamma-1}}, \\ n_2 = \frac{1-\gamma}{m}\left(\frac{\Omega\gamma-\Omega\gamma p\beta}{m}\right)^{\frac{1}{\gamma-1}} p + \frac{n}{m}, \\ n_3 = \frac{1-\gamma}{m}\left(\frac{\Omega\gamma-\Omega\gamma p\beta}{m}\right)^{\frac{1}{\gamma-1}} q - \frac{(1-\gamma)\eta}{m}. \end{cases} \tag{4.63}$$

Proof. For the HARA utility defined in (3.23), we use the feedback form of the optimal controls in (3.27), and conclude that

$$\pi^*(t) = m_1 x^*(t) + m_2 z^*(t) + m_3 \quad \text{and} \quad c^*(t) = n_1 x^*(t) + n_2 z^*(t) + n_3. \tag{4.64}$$

Thus, the dynamic of c^* follows

$$
\begin{aligned}
dc^*(t) =& n_1 dx^*(t) + n_2 dz^*(t) \\
=& n_1 \left\{ [rx^*(t) - c^*(t) + (b-r)(m_1 x^*(t) + m_2 z^*(t) + m_3)] \, dt \right. \\
& \left. + (m_1 x^*(t) + m_2 z^*(t) + m_3) \sigma dW_t \right\} + n_2 \beta (c^*(t) - z^*(t)) dt.
\end{aligned} \tag{4.65}
$$

Since $x^*(t) = \frac{c^*(t) - n_2 z^*(t) - n_3}{n_1}$, the above equation can be simplified as

$$
dc^*(t) = (h_1 c^*(t) + h_2 z^*(t) + h_3) dt + (h_4 c^*(t) + h_5 z^*(t) + h_6) dW_t, \tag{4.66}
$$

Now we put $L_t = e^{\beta t} c^*(t)$, and derive

$$
\begin{aligned}
dL_t =& \beta L_t dt + e^{\beta t} dc^*(t) \\
=& \left[(\beta + h_1) L_t + h_2 \beta \int_0^t L_s ds + h_2 z_0 + h_3 e^{\beta t} \right] dt \\
& + \left[h_4 L_t + h_5 \beta \int_0^t L_s ds + h_5 z_0 + h_6 e^{\beta t} \right] dW_t.
\end{aligned} \tag{4.67}
$$

This ends the proof of the theorem. □

Chapter 5

Graphs and Financial Interpretation

In the previous chapters, the optimal consumption and investment strategies are derived in closed-form expressions. However, since most of the parameters are general scalars, the inner relationship between those parameters and the optimal strategies is not obvious enough. In this section, we apply graphing methods to investigate the properties of a particular case of habit utility.

For the exponential utility function defined in (3.16), there are two parameters ϕ_1 and ϕ_2 that we assume to be equal to one. In other words, we consider the following utility.

$$u(c, z) = -e^{-c+z}. \tag{5.1}$$

The interpretation of this formation lies in the exponential relationship between the utility and the value that the consumption rate exceeds the historical average discounted consumption level.

For this setting, we have $\Delta := \alpha_1^2 - 4\alpha_2 = 4\beta^2 - 4\beta^2 = 0$, where α_1 and α_2

are defined in (4.3). Hence, the parameters become

$$\begin{aligned}
&b_1 = \frac{1}{r}\exp(1 - \frac{2\delta\sigma^2 + (b-r)^2}{2\sigma^2 r}), \quad b_2 = \frac{r^2}{\beta + r}, \quad b_3 = \frac{r}{\beta + r}, \\
&\alpha_1 = 2\beta, \quad \alpha_2 = -\beta^2, \\
&c^*(0) = \frac{r^2}{r+\beta}x_0 + \frac{\beta}{r+\beta}z_0 - \frac{2\sigma^2(r-\delta) - (b-r)^2}{2r\sigma^2}, \\
&dK_t = \left[-\beta z_0 + \frac{2\sigma^2(r-\delta) + (b-r)^2}{2\sigma^2}e^{\beta t}\right]dt + \frac{b-r}{\sigma}e^{\beta t}dW_t.
\end{aligned}$$

In the rest of this chapter, we investigate the expected value of $\pi^*(t)$, $c^*(t)$, $z^*(t)$ and $x^*(t)$.

5.1 The Investment Strategy

We recall that the optimal portfolio π^* is given by

$$\pi^*(t) = \frac{b-r}{b_2\sigma^2} = \frac{(r+\beta)(b-r)}{r^2\sigma^2}. \tag{5.2}$$

We can obtain several conclusions from this expression. Firstly, It is obvious that the optimal portfolio is a positive constant (i.e. it does not depend on time nor on randomness). Secondly, since the agent is risk-averse according to his utility function, a higher stock return b and a lower risk σ will lead to a higher stock investment, and this expression also supports this conclusion. Moreover, π^* is also constant in x_0 and z_0, which indicates a possible situation that $\pi^*(t) > x^*(t)$. In that case, the investor will get loan from the bank with interest rate r. Finally, π^* is a linear function of habit factor β with positive slope $\frac{b}{r\sigma^2}$.

5.2 The Expected Consumption Rate

Proposition 8: *Under the assumption $\phi_1 = \phi_2 = 1$, the expected value of the optimal consumption rate $c^*(t)$ can be simplified as*

$$E[c^*(t)] = c^*(0)(1 + \beta t) - \beta z_0 t + \frac{\alpha_3 \beta}{2} t^2 + \alpha_3 t, \tag{5.3}$$

where α_3 is defined as

$$\alpha_3 = \frac{2\sigma^2(r - \delta) + (b - r)^2}{2\sigma^2}.$$

The expression (5.3) indicates that the curve should be a parabola. Naturally, the optimal consumption rate will start from $c^*(0) = \frac{r^2}{r+\beta} x_0 + \frac{\beta}{r+\beta} z_0 - \frac{2\sigma^2(r-\delta)-(b-r)^2}{2r\sigma^2}$. Under the assumption that $\alpha_3 > 0$, as t approaches ∞, $c^*(t)$ also approaches ∞.

Then, in order to describe directly the relationship between $E[c^*(t)]$ and the other parameters of the model (i.e. $x_0, z_0, b, \sigma, \delta, r, \beta$), we plot graphs that explain these relationships. In addition, We also focus on its first derivative with respect to some of those parameters. This part is necessary since we may find the numerical foundation of those graphs.

5.2.1 The Effect of Initial Wealth x_0

The first derivative with respect to x_0 is

$$\frac{\partial E[c^*(t)]}{\partial x_0} = \frac{r^2(1 + \beta t)}{r + \beta}. \tag{5.4}$$

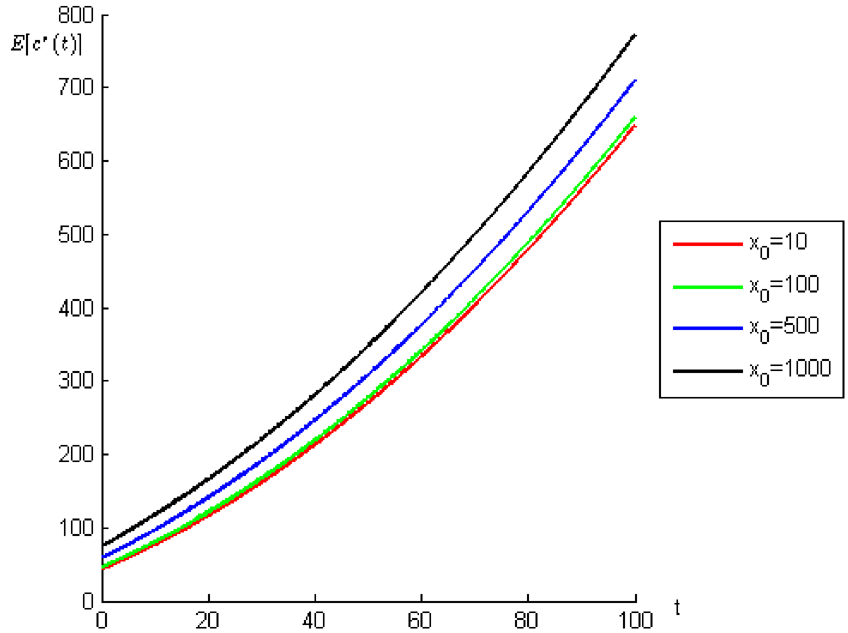

Figure 5.1: Expected consumption rate for different x_0

At time 0, $\frac{\partial E[c^*(t)]}{\partial x_0}\Big|_{t=0} = \frac{r^2}{r+\beta}$. As t approaches ∞, $\frac{\partial E[c^*(t)]}{\partial x_0}$ is positive and approaches ∞ as well.

Now we put $r = 0.05$, $\beta = 0.03$, $x_0 = 10, 100, 500, 1000$, $z_0 = 10$, $b = 0.07$, $\sigma = 0.01$, $\delta = 0.04$, the graph of $E[c^*(t)]$ is shown in figure 5.1.

5.2.2 The Effect of Initial Consumption Habit z_0

The first derivative with respect to z_0 is

$$\frac{\partial E[c^*(t)]}{\partial z_0} = \frac{\beta(1-rt)}{r+\beta}. \tag{5.5}$$

At time 0, we have $\frac{\partial E[c^*(t)]}{\partial z_0}\Big|_{t=0} = \frac{\beta}{r+\beta}$. Before the moment $t_0 = \frac{1}{r}$, $\frac{\partial E[c^*(t)]}{\partial z_0}$

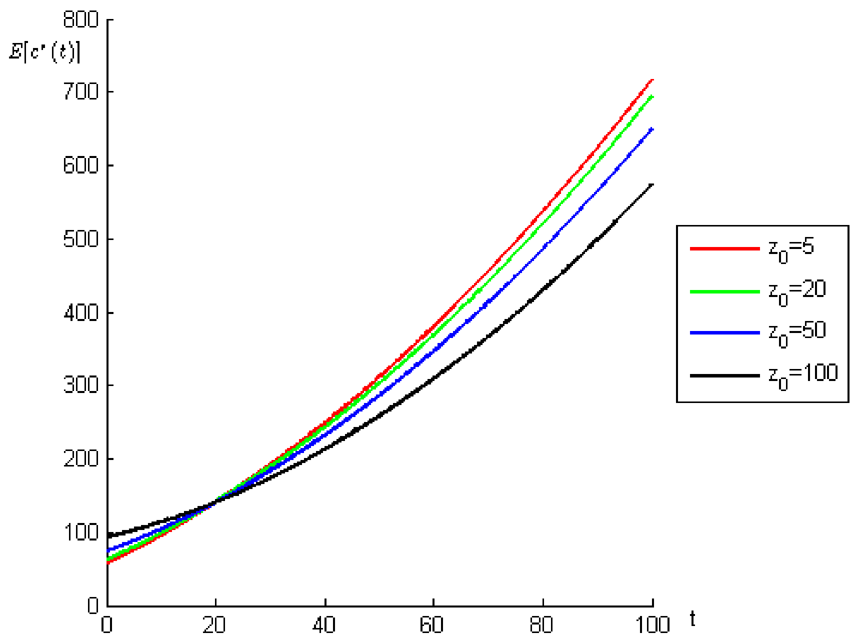

Figure 5.2: Expected consumption rate for different z_0

is positive, and becomes negative after that moment. As t goes to infinity, $\frac{\partial E[c^*(t)]}{\partial z_0}$ is approaching $-\infty$. This phenomenon shows that, even though a lower initial consumption habit decreases the initial consumption level, it will contribute to a higher consumption rate after time t_0.

Now we put $r = 0.05$, $\beta = 0.03$, $x_0 = 500$, $z_0 = 5, 20, 50, 100$, $b = 0.07$, $\sigma = 0.01$, $\delta = 0.04$, the graph of $E[c^*(t)]$ is shown in figure 5.2.

5.2.3 The Effect of Stock Return b

The first derivative with respect to b is

$$\frac{\partial E[c^*(t)]}{\partial b} = \frac{b-r}{\sigma^2}\left[\frac{1}{r} + (1+\frac{\beta}{r})t + \frac{\beta}{2}t^2\right]. \tag{5.6}$$

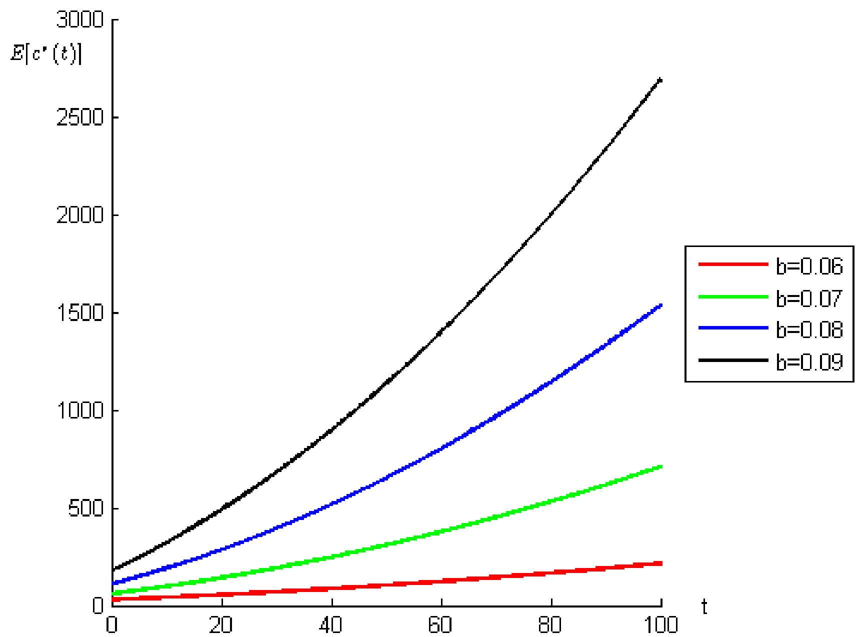

Figure 5.3: Expected consumption rate for different b

At time 0, we have $\frac{\partial E[c^*(t)]}{\partial b}\Big|_{t=0} = \frac{b-r}{r\sigma^2}$. This expression implies that $\frac{\partial E[c^*(t)]}{\partial b}$ is positive, and increases to ∞ as t goes to ∞. This can be explained by the fact that a higher stock return can boost the consumption activity of consumers. The increase indicates that this effect becomes more significant as time passes.

Now we put $r = 0.05$, $\beta = 0.03$, $x_0 = 500$, $z_0 = 10$, $b = 0.06, 0.07, 0.08, 0.09$, $\sigma = 0.01$, $\delta = 0.04$, the graph of $E[c^*(t)]$ is shown in figure 5.3.

5.2.4 The Effect of Stock Volatility σ

The first derivative with respect to σ is

$$\frac{\partial E[c^*(t)]}{\partial \sigma} = -\frac{(b-r)^2}{\sigma^3}\left[\frac{1}{r} + (1+\frac{\beta}{r})t + \frac{\beta}{2}t^2\right]. \tag{5.7}$$

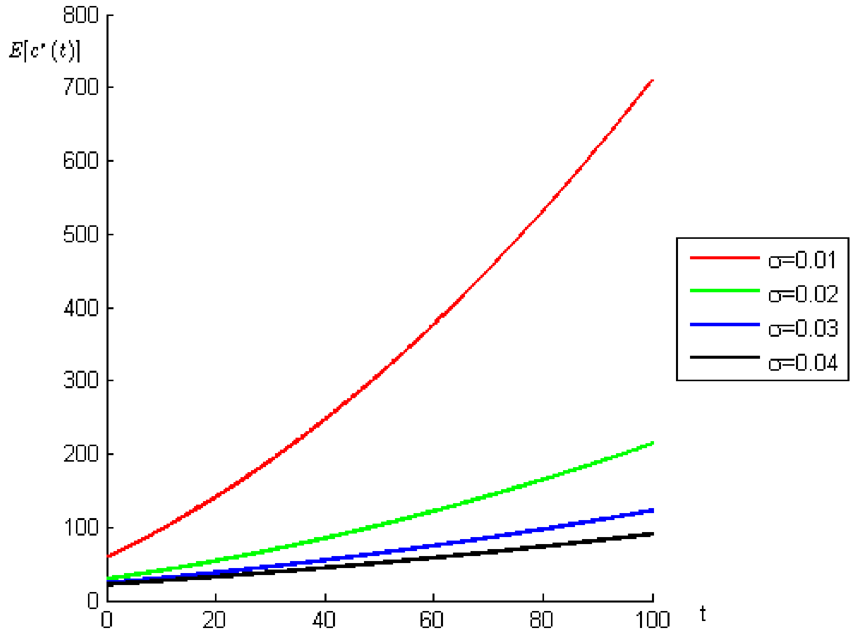

Figure 5.4: Expected consumption rate for different σ

At time 0, we have $\frac{\partial E[c^*(t)]}{\partial \sigma}\Big|_{t=0} = -\frac{(b-r)^2}{r\sigma^3}$. If we assume that σ is nonnegative, then it is obvious that this derivative is always negative, and approaches $-\infty$ when t goes to ∞. This result shows that higher volatility of the stock price will push down the consumption level. In economics, this fact means that people would be cautious in consumption when the potential risk becomes significant.

Now we put $r = 0.05$, $\beta = 0.03$, $x_0 = 500$, $z_0 = 10$, $b = 0.07$, $\sigma = 0.01, 0.02, 0.03, 0.04$, $\delta = 0.04$, the graph of $E[c^*(t)]$ is shown in figure 5.4.

5.2.5 The Effect of Subjective Discount Rate δ

The first derivative with respect to δ is

$$\frac{\partial E[c^*(t)]}{\partial \delta} = \frac{1}{r}(1+\beta t) - (\frac{\beta}{2}t^2 + t). \tag{5.8}$$

At time 0, we get $\frac{\partial E[c^*(t)]}{\partial \delta}\Big|_{t=0} = \frac{1}{r}$. Before the moment $t_1 = -\frac{1}{\beta} + \frac{1}{r} + \frac{1}{\beta}\sqrt{\frac{\beta^2}{r^2}+1}$, $\frac{\partial E[c^*(t)]}{\partial \delta}$ is positive, and becomes negative after t_1. As t goes to ∞, $\frac{\partial E[c^*(t)]}{\partial b}$ approaches $-\infty$. Since δ represents the extent of impatience, a higher δ causes a higher consumption rate at the beginning. However, it has negative impact on the consumption in the long run. Thus, a more patient investor will consume more in the long run.

Now we put $r = 0.05$, $\beta = 0.03$, $x_0 = 500$, $z_0 = 10$, $b = 0.07$, $\sigma = 0.01$, $\delta = 0.01, 0.1, 0.2, 0.5$, the graph of $E[c^*(t)]$ is shown in figure 5.5.

5.2.6 The Effect of Interest Rate r

The first derivative with respect to r is

$$\frac{\partial E[c^*(t)]}{\partial r} = \alpha_4(1+\beta t) + (1 + \frac{r-b}{\sigma^2})(\frac{\beta t^2}{2} + t), \tag{5.9}$$

where $\alpha_4 = \frac{r^2+2r\beta}{(r+\beta)^2}x_0 - \frac{\beta}{(r+\beta)^2}z_0 - \frac{\delta}{r^2} - \frac{b^2}{2r^2\sigma^2} + \frac{1}{2\sigma^2}$.

At $t = 0$, we have $\frac{\partial E[c^*(t)]}{\partial r}\Big|_{t=0} = \alpha_4$, and its sign depends on several parameters. However, this expression indicates that, in the long run, $\frac{\partial E[c^*(t)]}{\partial r}$ approaches ∞.

Now we put $r = 0.03, 0.04, 0.05, 0.06$, $\beta = 0.03$, $x_0 = 500$, $z_0 = 10$, $b = 0.07$, $\sigma = 0.01$, $\delta = 0.04$, the graph of $E[c^*(t)]$ is shown in figure 5.6.

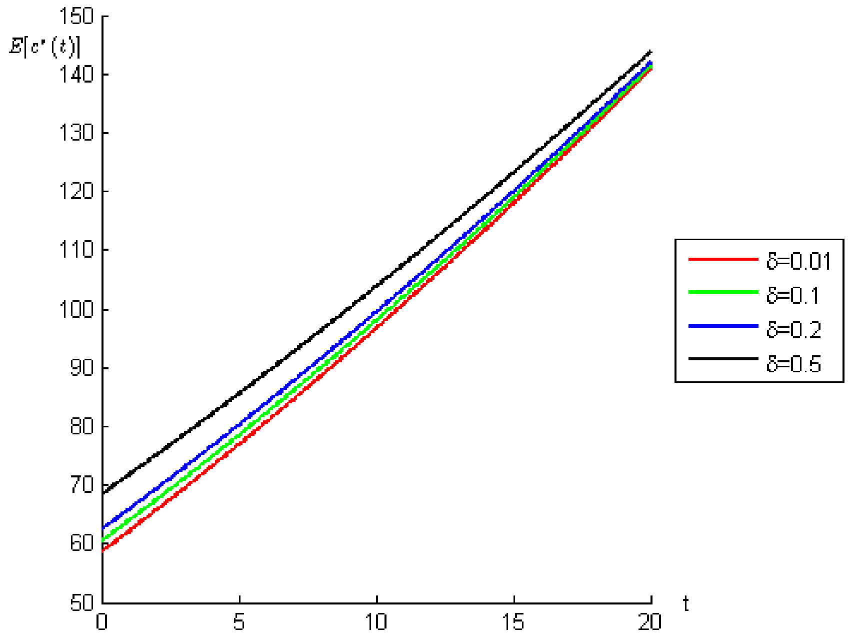

Figure 5.5: Expected consumption rate for different δ

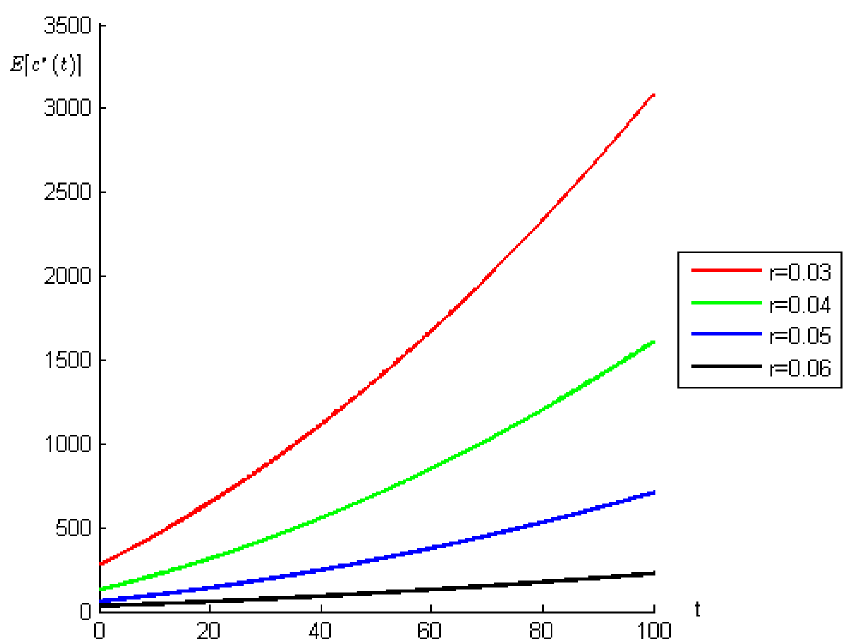

Figure 5.6: Expected consumption rate for different r

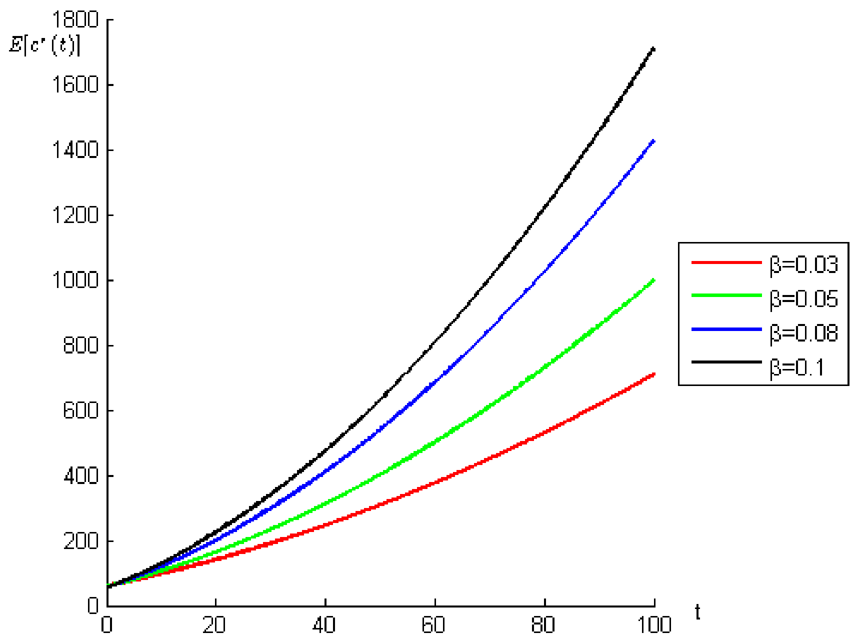

Figure 5.7: Expected consumption rate for different β

5.2.7 The Effect of Habit Formulation Factor β

The first derivative with respect to β is

$$\frac{\partial E[c^*(t)]}{\partial \beta} = \alpha_5(1+\beta t) + c^*(0)t - z_0 t + \frac{\alpha_3}{2}t^2, \tag{5.10}$$

where $\alpha_5 = \frac{-r^2 x_0 + r z_0}{(r+\beta)^2}$.

Obviously, the value of $\frac{\partial E[c^*(t)]}{\partial \beta}$ at time 0 coincides with α_5, and its sign depends on the value of $(-rx_0 + z_0)$. However, $\frac{\partial E[c^*(t)]}{\partial \beta}$ goes to ∞ as t increases to ∞. This result indicates that the sensitive agents have a high level of optimal consumption in the long run.

Now we put $r = 0.05$, $\beta = 0.03, 0.05, 0.08, 0.1$, $x_0 = 500$, $z_0 = 10$, $b = 0.07$, $\sigma = 0.01$, $\delta = 0.04$, the graph of $E[c^*(t)]$ is shown in figure 5.7.

5.3 The Expected Consumption Habit

Proposition 9: *Under the assumption $\phi_1 = \phi_2 = 1$, the expected value of the optimal consumption habit can be expressed as*

$$E[z^*(t)] = z_0(1 - \beta t) + \beta c^*(0)t + \frac{\beta\alpha_3}{2}t^2. \tag{5.11}$$

Since the optimal consumption rate $c(t)$ goes to infinity as $t \to 0$, it is clear that the consumption habit $z(t)$ approaches the same limit as well.

Then we consider its first derivative with respect to certain parameter, and the relationship between $E[z^*(t)]$ and those parameters will be shown in the forthcoming graphs:

5.3.1 The Effect of Initial Wealth x_0

The first derivative with respect to x_0 is

$$\frac{\partial E[z^*(t)]}{\partial x_0} = \frac{r^2\beta t}{r + \beta}. \tag{5.12}$$

At time 0, we have $\frac{\partial E[z^*(t)]}{\partial x_0} = 0\Big|_{t=0}$. As t goes to infinity, $\frac{\partial E[z^*(t)]}{\partial x_0}$ is positive and it approaches infinity as well.

We put $r = 0.05$, $\beta = 0.03$, $x_0 = 10, 100, 500, 1000$, $z_0 = 10$, $b = 0.07$, $\sigma = 0.01$, $\delta = 0.04$, the graph of $E[z^*(t)]$ is shown in figure 5.8.

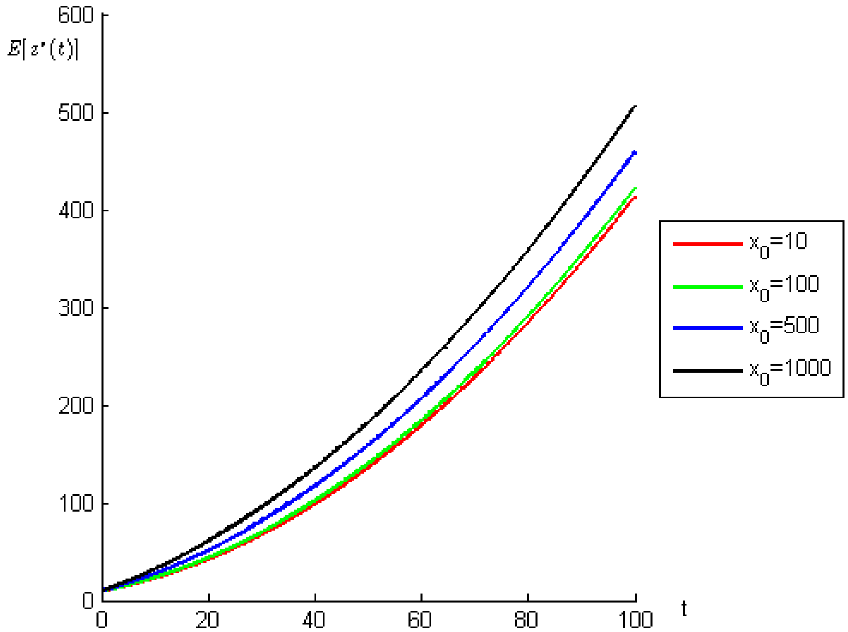

Figure 5.8: Expected consumption habit for different x_0

5.3.2 The Effect of Initial Consumption Habit z_0

The first derivative with respect to z_0 is

$$\frac{\partial E[z^*(t)]}{\partial z_0} = 1 - \frac{r\beta t}{r+\beta}. \tag{5.13}$$

At time 0, we have $\frac{\partial E[z^*(t)]}{\partial z_0}\Big|_{t=0} = 1$, and it is positive before a certain moment $t_2 = \frac{r+\beta}{r\beta}$. However, it becomes negative after that point, and approaches $-\infty$ as t goes to ∞.

Now, we put $r = 0.05$, $\beta = 0.03$, $x_0 = 500$, $z_0 = 5, 20, 50, 100$, $b = 0.07$, $\sigma = 0.01$, $\delta = 0.04$, the graph of $E[z^*(t)]$ is shown in figure 5.9.

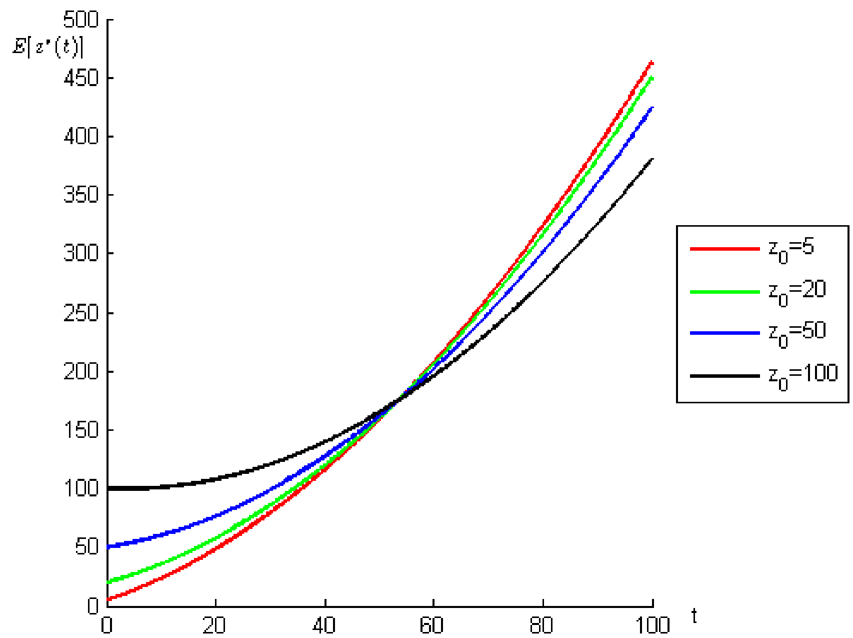

Figure 5.9: Expected consumption habit for different z_0

5.3.3 The Effect of Stock Return b

The first derivative with respect to b is

$$\frac{\partial E[z^*(t)]}{\partial b} = \frac{\beta(b-r)}{r\sigma^2}\left(t + \frac{r}{2}t^2\right). \tag{5.14}$$

At time 0, we get $\frac{\partial E[z^*(t)]}{\partial b}\Big|_{t=0} = 0$, and it is positive for any $t > 0$. As t approaches ∞, $\frac{\partial E[z^*(t)]}{\partial b}$ becomes infinite as well.

We put $r = 0.05$, $\beta = 0.03$, $x_0 = 500$, $z_0 = 10$, $b = 0.06, 0.07, 0.08, 0.09$, $\sigma = 0.01$, $\delta = 0.04$, the graph of $E[z^*(t)]$ is shown in figure 5.10.

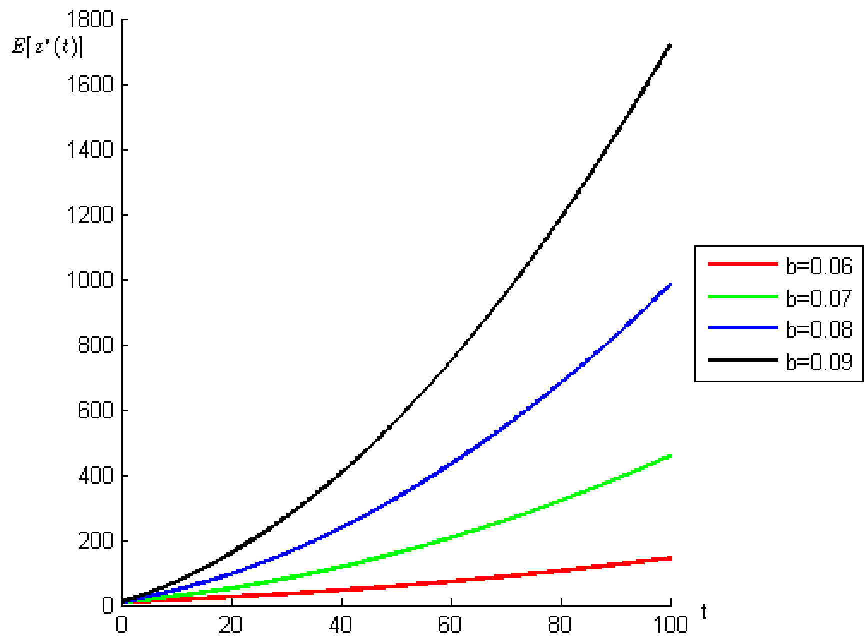

Figure 5.10: Expected consumption habit for different b

5.3.4 The Effect of Stock Volatility σ

The first derivative with respect to σ is

$$\frac{\partial E[z^*(t)]}{\partial \sigma} = -\frac{\beta(b-r)^2}{r\sigma^3}\left(t + \frac{r}{2}t^2\right) \tag{5.15}$$

At time 0, we get $\frac{\partial E[z^*(t)]}{\partial \sigma}\big|_{t=0} = 0$. For any $t \in (0, \infty)$, it is negative for positive σ. Moreover, it decreases to $-\infty$ as t approaches ∞.

Now, we put $r = 0.05$, $\beta = 0.03$, $x_0 = 500$, $z_0 = 10$, $b = 0.07$, $\sigma = 0.01, 0.02, 0.03, 0.04$, $\delta = 0.04$, the graph of $E[z^*(t)]$ is shown in figure 5.11.

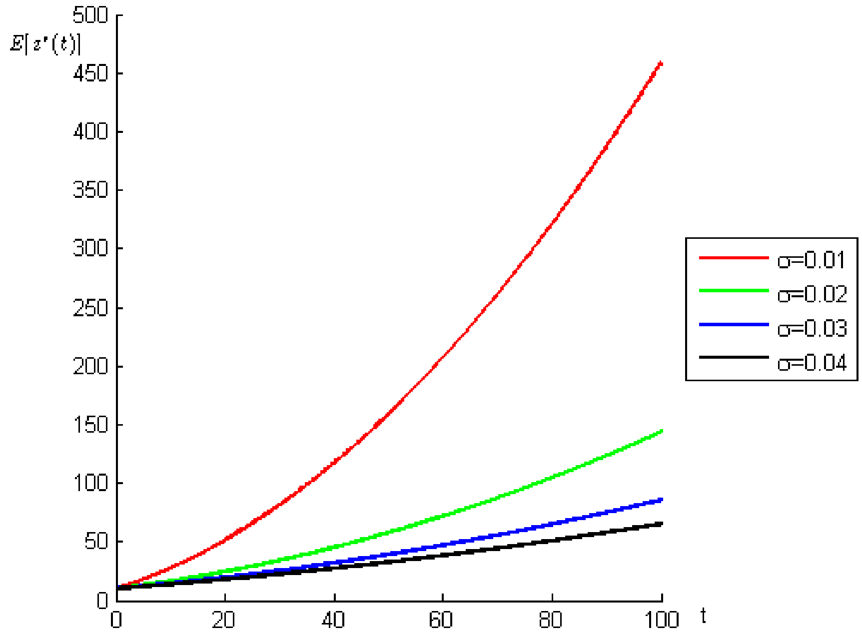

Figure 5.11: Expected consumption habit for different σ

5.3.5 The Effect of Subjective Discount Rate δ

The first derivative with respect to δ is

$$\frac{\partial E[z^*(t)]}{\partial \delta} = \frac{\beta t}{r} - \frac{\beta}{2} t^2. \tag{5.16}$$

For $t = 0$, we have $\frac{\partial E[z^*(t)]}{\partial \delta}\Big|_{t=0} = 0$, and it is positive before $t_3 := \frac{2}{r}$. However, after t_0, $\frac{\partial E[z^*(t)]}{\partial \delta}$ becomes negative, and approaches $-\infty$ as t increases to ∞. This result implies that the impatience leads to higher consumption habit at the beginning and lower consumption preference in the long run.

Now, we put $r = 0.05$, $\beta = 0.03$, $x_0 = 500$, $z_0 = 10$, $b = 0.07$, $\sigma = 0.01$, $\delta = 0.01, 0.1, 0.2, 0.5$, the graph of $E[z^*(t)]$ is shown in figure 5.12.

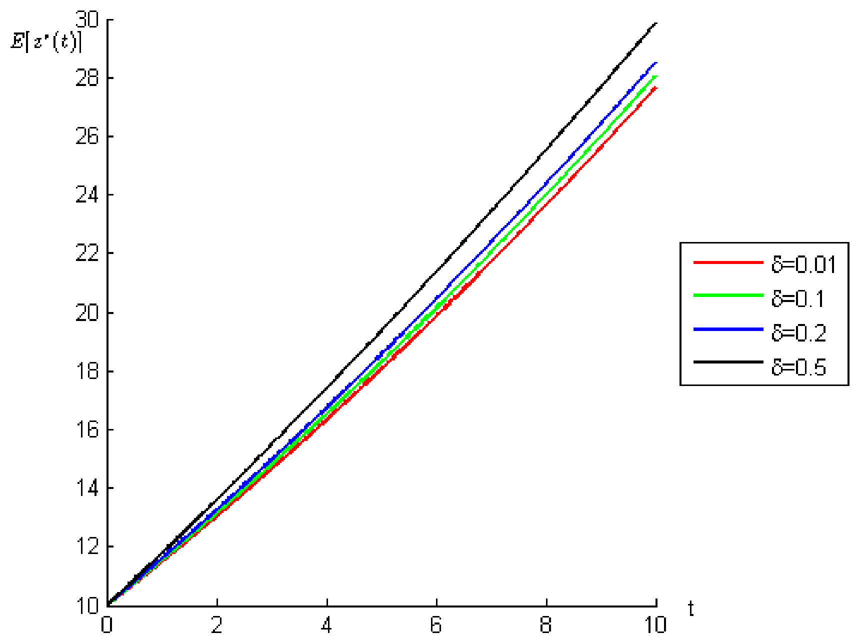

Figure 5.12: Expected consumption habit for different δ

5.3.6 The Effect of Interest Rate r

The first derivative with respect to r is

$$\frac{\partial E[z^*(t)]}{\partial r} = \alpha_4 \beta t + (1 + \frac{r-b}{\sigma^2})\frac{\beta t^2}{2}, \tag{5.17}$$

where $\alpha_4 = \frac{r^2+2r\beta}{(r+\beta)^2}x_0 - \frac{\beta}{(r+\beta)^2}z_0 - \frac{\delta}{r^2} - \frac{b^2}{2r^2\sigma^2} + \frac{1}{2\sigma^2}$.

At $t = 0$, we have $\frac{\partial E[z^*(t)]}{\partial r}\Big|_{t=0} = 0$, and it is strictly positive for any time $t > 0$. Then, $\frac{\partial E[z^*(t)]}{\partial r}$ goes to ∞ as time passes. This result implies that the higher interest rate leads to a higher consumption preference in the long run.

Now we put $r = 0.03, 0.04, 0.05, 0.06$, $\beta = 0.03$, $x_0 = 500$, $z_0 = 10$, $b = 0.07$, $\sigma = 0.01$, $\delta = 0.04$, the graph of $E[c^*(t)]$ is shown in figure 5.13.

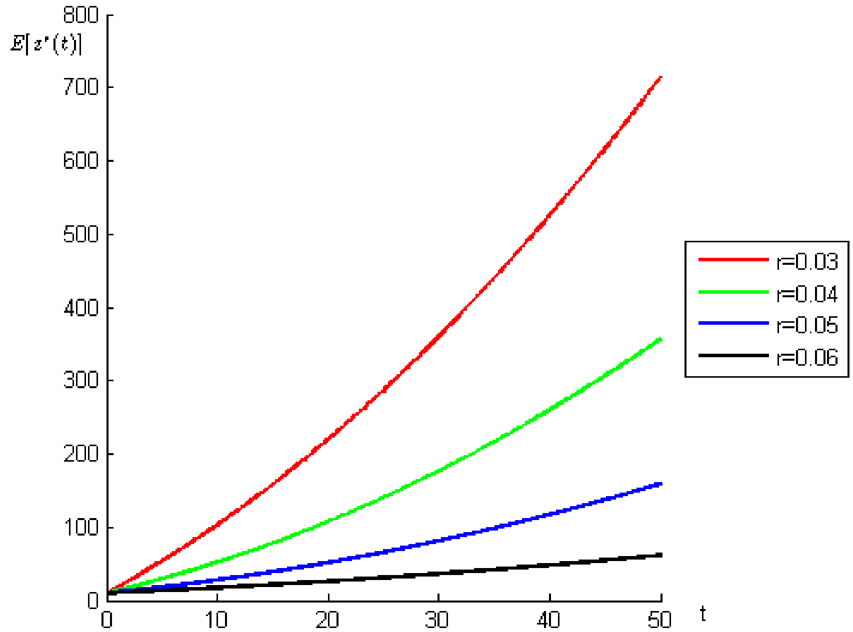

Figure 5.13: Expected consumption habit for different r

5.3.7 The Effect of Habit Formulation Rate β

The first derivative with respect to β is

$$\frac{\partial E[z^*(t)]}{\partial \beta} = \alpha_5 \beta t - z_0 t + \frac{\alpha_3}{2} t^2. \tag{5.18}$$

Obviously, the value of $\frac{\partial E[z^*(t)]}{\partial \beta}$ vanishes at time 0, and approaches ∞ as t increases to ∞. This result indicates that the sensitive agents have a high level of optimal consumption habit in the long run.

Now we put $r = 0.05$, $\beta = 0.03, 0.05, 0.08, 0.1$, $x_0 = 500$, $z_0 = 10$, $b = 0.07$, $\sigma = 0.01$, $\delta = 0.04$, the graph of $E[c^*(t)]$ is shown in figure 5.14.

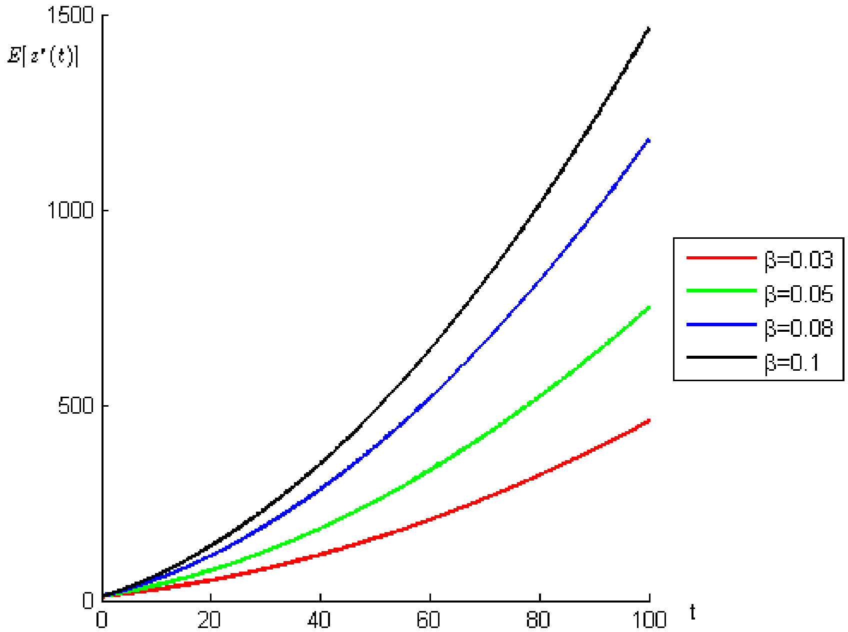

Figure 5.14: Expected consumption habit for different β

5.4 The Expected Wealth Value

Proposition 10: *Under the assumption $\phi_1 = \phi_2 = 1$, the expected value of the optimal wealth $x^*(t)$ can be simplified as*

$$\begin{aligned} E[x^*(t)] =& x_0 + \left(\frac{r\beta}{r+\beta} x_0 - \frac{\beta}{r+\beta} z_0 + \frac{\beta(b-r)^2}{r^2\sigma^2} + \frac{2\sigma^2(r-\delta)+(b-r)^2}{2r\sigma^2} \right) t \\ &+ \frac{2\beta\sigma^2(r-\delta)+\beta(b-r)^2}{4r\sigma^2} t^2. \end{aligned} \tag{5.19}$$

In the following, we consider its first derivative with respect to certain parameters. Then, the relationship between $E[x^*(t)]$ and those parameters will be shown in graphs.

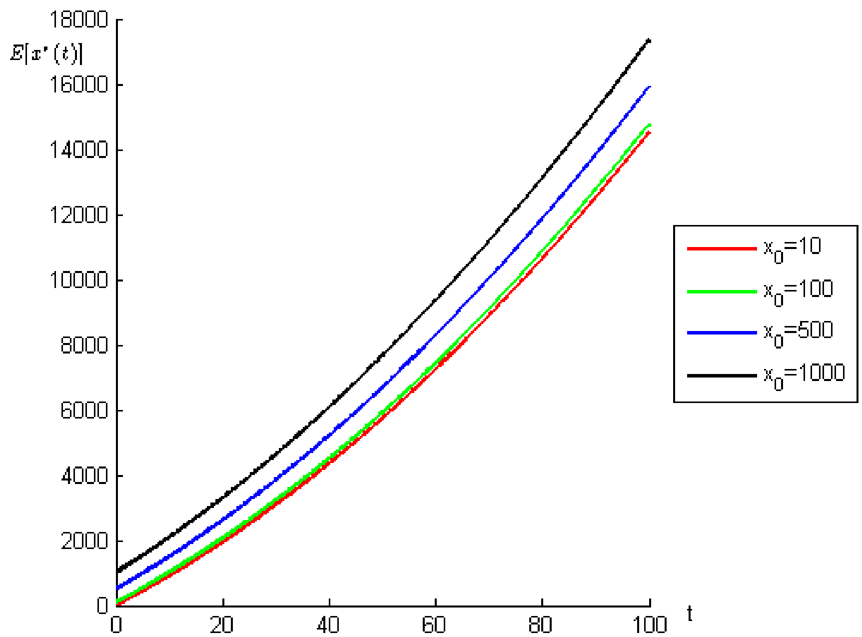

Figure 5.15: Expected wealth for different x_0

5.4.1 The Effect of Initial Wealth x_0

The first derivative with respect to x_0 is

$$\frac{\partial E[x^*(t)]}{\partial x_0} = 1 + \frac{r\beta}{r+\beta}t. \tag{5.20}$$

At time 0, we get $\frac{\partial E[x^*(t)]}{\partial x_0}\Big|_{t=0}$=1. As t approaches ∞, $\frac{\partial E[x^*(t)]}{\partial x_0}$ is approaching ∞, and it is positive. This result indicates that the impact of a small difference in initial wealth on future wealth will becomes greater as t increases.

Now, we put $r = 0.05$, $\beta = 0.03$, $x_0 = 10, 100, 500, 1000$, $z_0 = 10$, $b = 0.07$, $\sigma = 0.01$, $\delta = 0.04$, the graph of $E[x^*(t)]$ is shown in figure 5.15.

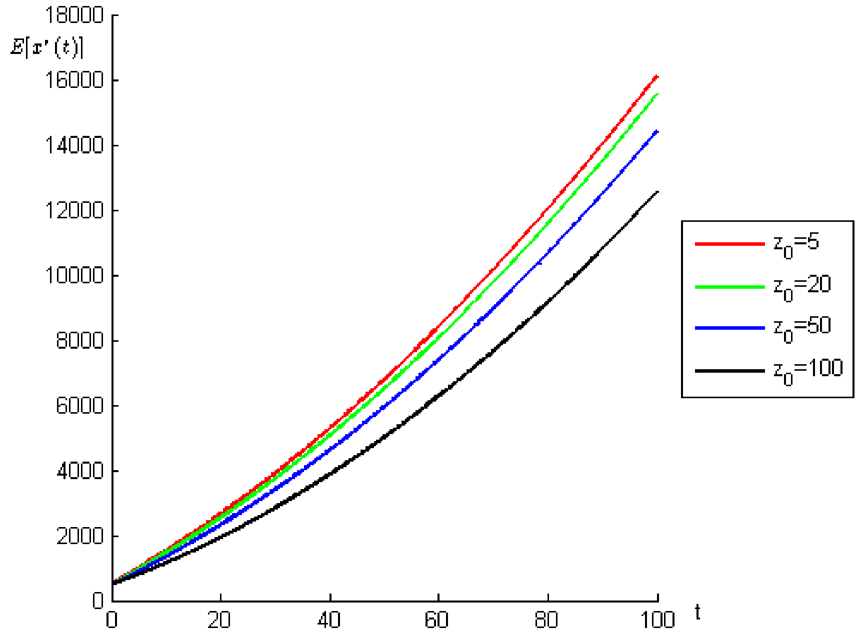

Figure 5.16: Expected wealth for different z_0

5.4.2 The Effect of Initial Consumption Habit z_0

The first derivative with respect to z_0 is

$$\frac{\partial E[x^*(t)]}{\partial z_0} = -\frac{\beta}{r+\beta}t. \tag{5.21}$$

At time 0, we have $\frac{\partial E[x^*(t)]}{\partial z_0}\Big|_{t=0} = 0$. As time passes, $\frac{\partial E[x^*(t)]}{\partial z_0}$ is always negative, and it approaches $-\infty$ as t goes to ∞. This result shows that the optimal wealth value is negatively related to the initial consumption habit for any $t > 0$.

Now, we put $r = 0.05$, $\beta = 0.03$, $x_0 = 500$, $z_0 = 5, 20, 50, 100$, $b = 0.07$, $\sigma = 0.01$, $\delta = 0.04$, the graph of $E[x^*(t)]$ is shown in figure 5.16.

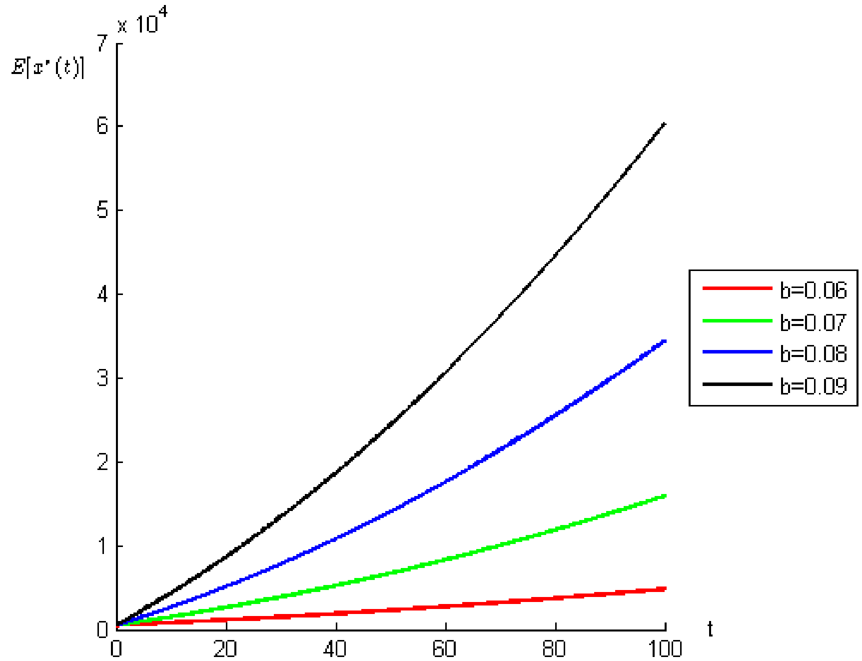

Figure 5.17: Expected wealth for different b

5.4.3 The Effect of Stock Return b

The first derivative with respect to b is

$$\frac{\partial E[x^*(t)]}{\partial b} = \frac{(2\beta + r)(b-r)}{r\sigma^2}t + \frac{(b-r)\beta}{2r\sigma^2}t^2. \tag{5.22}$$

At time 0, we have $\frac{\partial E[x^*(t)]}{\partial b}\Big|_{t=0} = 0$. Since stock return is always higher than interest rate, $\frac{\partial E[x^*(t)]}{\partial b}$ is positive as time passes. It is clear that a higher stock return leads to a higher level of wealth all the time.

Now, we put $r = 0.05$, $\beta = 0.03$, $x_0 = 500$, $z_0 = 10$, $b = 0.06, 0.07, 0.08, 0.09$, $\sigma = 0.01$, $\delta = 0.04$, the graph of $E[x^*(t)]$ is shown in figure 5.17.

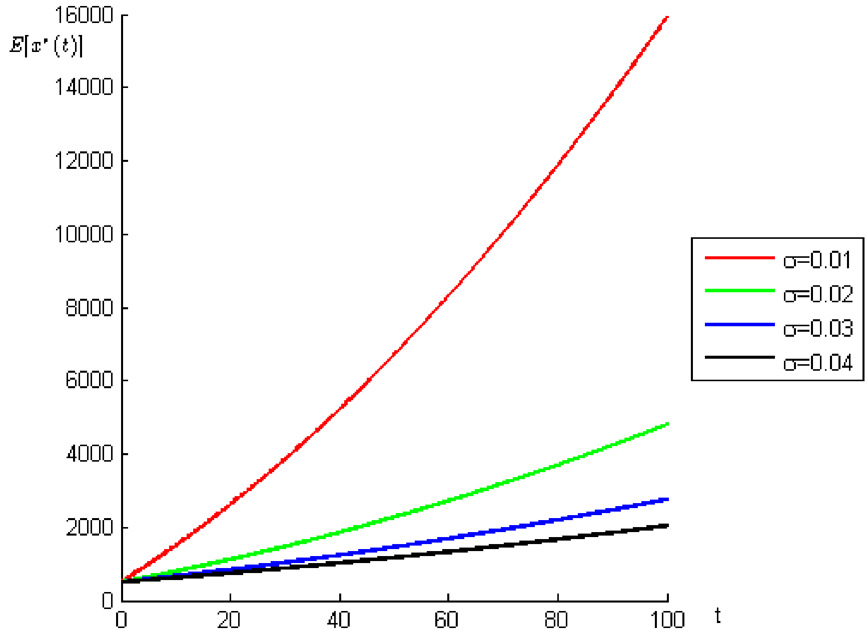

Figure 5.18: Expected wealth for different σ

5.4.4 The Effect of Stock Volatility σ

The first derivative with respect to σ is

$$\frac{\partial E[x^*(t)]}{\partial \sigma} = -\frac{(2\beta + r)(b-r)^2}{r^2\sigma^3}t - \frac{\beta(b-r)^2}{2r\sigma^3}t^2. \tag{5.23}$$

At time 0, we get $\frac{\partial E[x^*(t)]}{\partial \sigma}\big|_{t=0} = 0$. As t increases, $\frac{\partial E[x^*(t)]}{\partial \sigma}$ is negative and approaches $-\infty$. This result proves that higher volatility of stock price has negative influence on optimal wealth.

Now, we put $r = 0.05$, $\beta = 0.03$, $x_0 = 500$, $z_0 = 10$, $b = 0.07$, $\sigma = 0.01, 0.02, 0.03, 0.04$, $\delta = 0.04$, the graph of $E[x^*(t)]$ is shown in figure 5.18.

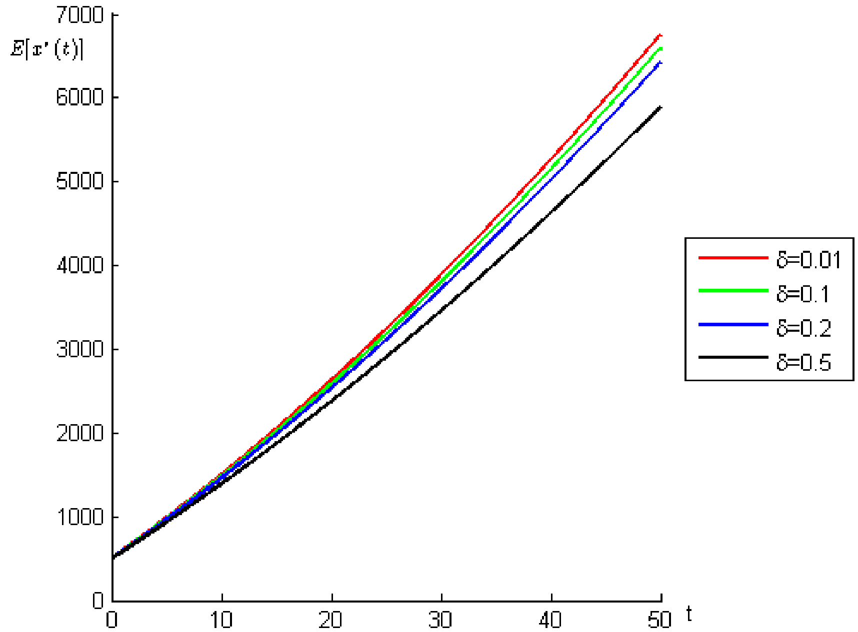

Figure 5.19: Expected wealth for different δ

5.4.5 The Effect of Subjective Discount Rate δ

The first derivative with respect to δ is

$$\frac{\partial E[x^*(t)]}{\partial \delta} = -\frac{1}{r}t - \frac{\beta}{2r}t^2. \tag{5.24}$$

At time 0, we get $\frac{\partial E[x^*(t)]}{\partial \delta} = 0$. As t goes to ∞, $\frac{\partial E[x^*(t)]}{\partial \delta}$ is negative and approaches $-\infty$. This result indicates that the impatient agents have to tolerate a low level of wealth compared with the patient agents.

Now, we put $r = 0.05$, $\beta = 0.03$, $x_0 = 500$, $z_0 = 10$, $b = 0.07$, $\sigma = 0.01$, $\delta = 0.01, 0.1, 0.2, 0.5$, the graph of $E[x^*(t)]$ is shown in figure 5.19.

5.4.6 The Effect of Interest Rate r

The first derivative with respect to r is

$$\begin{aligned}\frac{\partial E[x^*(t)]}{\partial r} =& t\left(\frac{\beta^2 x_0}{(r+\beta)^2}+\frac{\beta z_0}{(r+\beta)^2}-\frac{2\beta b^2}{r^3\sigma^2}+\frac{2b}{r^2\sigma^2}+\frac{\delta}{r^2}-\frac{b^2}{2r^2\sigma^2}+\frac{1}{2\sigma^2}\right)\\ &+t^2\left(\frac{\beta\sigma}{2r^2}-\frac{\beta b^2}{4r^2\sigma^2}+\frac{\beta}{4\sigma^2}\right).\end{aligned}$$

At time 0, we get $\frac{\partial E[x^*(t)]}{\partial r}\Big|_{t=0}=0$. The effect of r on the optimal wealth can be explained in two aspects. On one hand, higher interest rate causes a higher return from the bond investment. However, on the other hand, it may increase the cost the investors pay for the loan. This equality tells us that the effect of interest rate is very complex since it depends on many other parameters.

Now we put $r=0.03, 0.04, 0.05, 0.06$, $\beta=0.03$, $x_0=500$, $z_0=10$, $b=0.07$, $\sigma=0.01$, $\delta=0.04$, the graph of $E[c^*(t)]$ is shown in figure 5.20.

5.4.7 The Effect of Habit Formulation Factor β

The first derivative with respect to β is

$$\frac{\partial E[x^*(t)]}{\partial \beta}=\left(\frac{r^2}{(r+\beta)^2}x_0-\frac{r}{(r+\beta)^2}z_0+\frac{(b-r)^2}{r^2\sigma^2}\right)t+\frac{2\sigma^2(r-\delta)+(b-r)^2}{4r\sigma^2}t^2.$$

Similarly, we get $\frac{\partial E[x^*(t)]}{\partial \beta}\Big|_{t=0}=0$. Under the assumption that $\alpha_3>0$, $\frac{\partial E[x^*(t)]}{\partial \beta}$ approaches ∞ as time goes to ∞, which suggests that the sensitive investors can enjoy a high level of wealth in the long run.

Now we put $r=0.05$, $\beta=0.03, 0.05, 0.08, 0.1$, $x_0=500$, $z_0=10$, $b=0.07$, $\sigma=0.01$, $\delta=0.04$, the graph of $E[c^*(t)]$ is shown in figure 5.21.

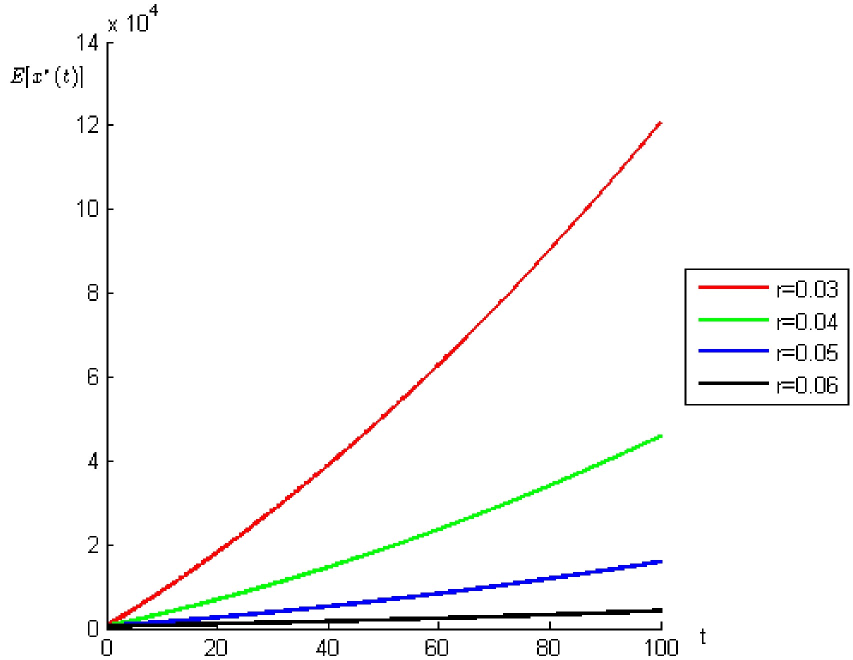

Figure 5.20: Expected consumption rate for different r

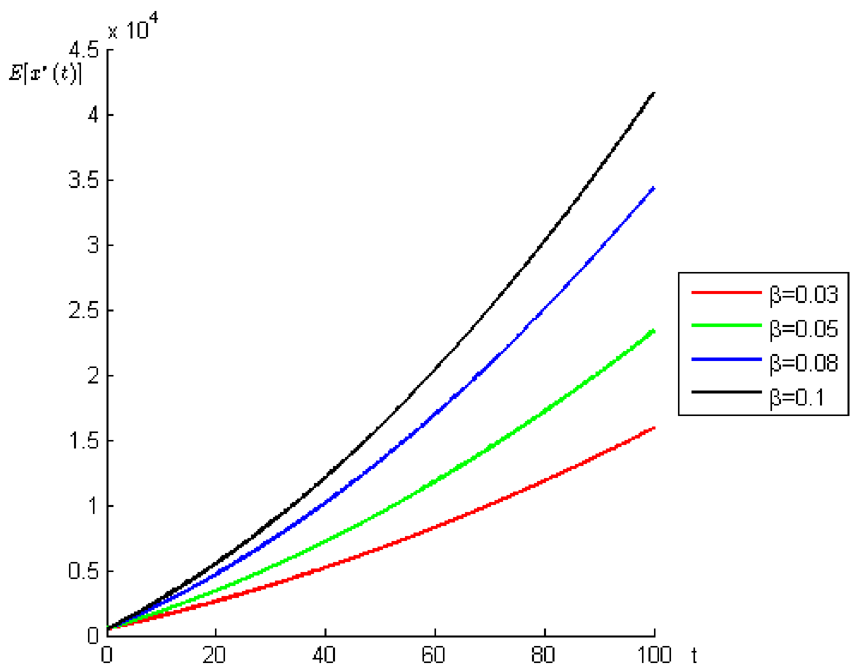

Figure 5.21: Expected wealth for different β

Chapter 6

Conclusions

In this thesis, we analysed a habit utility optimization problem for a "small" investor in a complete financial market. Our optimization target consists of maximizing the expected overall utility of an agent over his consumption and investment activities. This leads to a stochastic control problem for which we derived the corresponding Hamilton-Jacobi-Bellman equation (HJB equation). We focused more on the HARA case and calculated explicitly the optimal value function that is the solution of the obtained HJB equation.

In addition to the optimal value function, we calculated the optimal controls in feedback forms. Our results at this stage correct a mistake in Sundaresan's paper (see [17]). More importantly, for the case of exponential habit, we obtained a stochastic differential equation (SDE) that the optimal consumption rate process satisfies. This SDE has the form of

$$dX_t = \left[\alpha_1 X_t + \alpha_2 \int_0^t X_s ds\right] dt + dK_t,$$

where α_1, α_2 are general coefficients, and K_t is an Itô Process.

Using the white noise interpretation, we solved explicitly this SDE. Furthermore, we calculated explicitly the optimal consumption/habit process and the optimal wealth process. All these constitute our major theoretical contribution in this thesis.

Furthermore, we applied graphing method to describe the inner relationships between the optimal strategies and the model's parameters for an example of exponential habit utility (see (5.1)). From the obtained graphs, we gave financial interpretations to these relationships. Below, we outline some of these financial/economic conclusions.

- The optimal portfolio is positive and is constant in time and randomness.
- Lower initial consumption habit leads to lower consumption at the beginning. However, after a certain moment that we calculated, the level of life becomes higher.
- Higher stock volatility has negative influence on both optimal consumption rate and optimal wealth process. This fact indicates that the exponential utility agents are risk-averse.
- Patient consumers can enjoy higher level of life in the long run.
- Sensitive consumers have high level of consumption in the long run.

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