University of Alberta

ON THE EXTREMAL DISTANCE BETWEEN CONVEX BODIES

by

Carlos Hugo Jimenez ©

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in

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ABSTRACT

Let K and L be two convex bodies in $\mathbb{R}^n$ with non-empty interiors. Define the following (multiplicative) distance between them as follows

$$d(K,L) = \inf\left\{|\lambda| \,\middle|\, \lambda \in \mathbb{R},\ K-a \subset T(L-b) \subset \lambda(K-a)\right\},$$

where the infimum is taken over all $a, b \in \mathbb{R}^n$ and all linear bijections T from $\mathbb{R}^n$ into $\mathbb{R}^n$. It was recently proved, using John's decomposition of the identity ($I_n = \sum_{i=1}^m a_i u_i \otimes v_i$), that for any two convex bodies K and L the distance between them $d(K,L) \leq n$. It was also observed that if L is a non-degenerate simplex and K is centrally symmetric then we have $d(K,L) = n$. This gives rise to a natural question about the extremal case when $d(K,L) = n$ and none of these additional assumptions are made on K and L. In this thesis we investigate this question. We show that u_i, v_i from John's decomposition can be split in two hyperplanes. We also show that if one of the u_i's (or one of the v_i's) is in the convex hull of the others then one can reduce the number of elements in John's decomposition.

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To my Parents: Felix and Epifania.

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Chapter 1

Introduction

The study of n-dimensional normed spaces and their behavior as the dimension n grows to infinity has attracted great interest and has achieved great progress. This area, called Asymptotic Theory of finite-dimensional normed spaces, was originated in functional analysis and many of its results were later successfully used to solve problems in many different branches of mathematics. In convex geometry for instance, this area helps in the study of aspects of the convex bodies in high dimensional spaces. Indeed, with the one to one correspondence between norms on $\mathbb{R}^n$ and symmetric convex bodies in $\mathbb{R}^n$, the study of these two objects can be done as the study of one.

This theory mainly studies numerical invariants on the set of convex bodies. Such invariants usually depend on the dimension n. In most of this thesis, our attention focuses on the study of one of these invariants: a Banach-Mazur type distance. For this kind of distance, a set of natural questions always arises, such as: how big can the distance be?, what are the extremal cases? Special consideration has also been given to the study of similar questions not for convex bodies themselves but for their projections and sections.

Many answers about bounds of distances were found with a celebrated theorem by F. John [7], which gives as consequences not only an upper bound of $\sqrt{n}$ for the Banach-Mazur distance between any symmetric convex body and the Euclidian ball but also a characterization of the ellipsoid of maximal vol-

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ume inside any convex body using the contact points between the boundaries of the body and the ellipsoid. As an immediate consequence of this theorem we obtain an upper bound of n for the Banach-Mazur distance between any two symmetric convex bodies.

Over the years, several extensions of John's Theorem have been obtained. Lewis [9] and Milman (Chapter 14 in [14]) extended it to symmetric convex bodies. Years later Giannopoulos-Perissinaki-Tsolomitis in [4] and Bastero-Romance in [2] obtained other different extensions.

In this thesis, we use a more general extension of John's Theorem due to Gordon, Litvak, Meyer, and Pajor [6] to explore the scenario where a Banach-Mazur-type distance between two convex bodies is maximal. Theorem 3.8 in [6] extends John's Theorem to arbitrary convex bodies K and L in $\mathbb{R}^n$. It shows that if K is in a position of maximal volume inside L then (after some shift) we have the following decomposition of the identity

$$I_n = \sum_{i=1}^{m} a_i u_i \otimes v_i,$$

where $a_i \in \mathbb{R}$, $u_i \in \partial K \cap \partial L$, $v_i \in \partial K^0 \cap \partial L^0$ $(i \leq m)$ such that

$$\sum_{i=1}^{m} a_i u_i = \sum_{i=1}^{m} a_i v_i = 0 \text{ and } \langle u_i, v_i \rangle = 1 \text{ for every i}$$

(see Theorem 3.2 below for a more precise formulation). We prove here that if the distance between K and L is maximal then there exist two hyperplanes P_A and P_B such that $u_i \in P_A$ for $i \in A$, $v_i \in P_B$ for $i \in B$, where $A, B \subset \{1, ..., m\}$, and $A \cup B = \{1, ..., m\}$. We additionally discuss how to decrease the number of summands in John's decomposition. Namely, in Theorem 3.5 we show that if either $u_j \in Conv(u_i)_{i \neq j}$ or $v_j \in Conv(v_i)_{i \neq j}$ then one can find another John's decomposition of the identity with smaller number of points.

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Chapter 2

Preliminaries and Notation

In the present section we briefly introduce some notation and recall some concepts from functional analysis and convex geometry that will be used in the following chapters. For more details we refer to [1], [3], [10], [12], [14], and references therein.

Let us consider $\mathbb{R}^n$ equipped with the standard inner product $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and the canonical basis $\{e_1, ..., e_n\}$. We denote by α, β, λ, and μ real numbers, by x, y, z, w, u, and v vectors in $\mathbb{R}^n$.

We will denote the set of linear bijections from $\mathbb{R}^n$ into $\mathbb{R}^n$ by GL_n. The map $I_n \in GL_n$ denotes the Identity map on $\mathbb{R}^n$.

For any vectors $x = (x_1, ..., x_n), y = (y_1, ..., y_n) \in \mathbb{R}^n$, $x \otimes y$ denotes the linear operator in $\mathbb{R}^n$ defined by $x \otimes y(z) = \langle x, z \rangle y$ for every $z \in \mathbb{R}^n$. In matrix form we can write this operator as

$$x \otimes y = \begin{pmatrix} x_1 y_1 & \cdots & x_n y_1 \\ \vdots & \ddots & \vdots \\ x_1 y_n & \cdots & x_n y_n \end{pmatrix}.$$

Note that if $\langle x, y \rangle = 1$, then

$$Tr(x \otimes y) = 1,$$

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where $Tr(x \otimes y)$ denotes the trace of the operator $x \otimes y$.

Furthermore, every operator can be presented using such notation. Indeed, it is easy to see that $T = \sum_{i=1}^{n} e_i \otimes Te_i$. We also notice that if $T = \sum_{i=1}^{n} x_i \otimes y_i$, then the trace of the operator T can be computed as

$$Tr(T) = \sum_{i=1}^{n} \langle x_i, y_i \rangle. \tag{2.1}$$

For a set $C \subset \mathbb{R}^n$, its convex hull can be defined as

$$Conv(C) = \left\{ \sum_{i=1}^{m} \lambda_i x_i \mid x_i \in C \ \forall\, 1 \le i \le m \quad \text{and} \quad \sum_{i=1}^{m} \lambda_i = 1,\ \lambda_i \ge 0 \right\}.$$

We call to any convex compact subset of $\mathbb{R}^n$ with non-empty interior a **convex body**. We will use the letters K, K', and L to denote convex bodies in the following sections. We say that a convex body is centrally symmetric (with respect to the origin) if $K = -K$. That is, $x \in K$ implies that $-x \in K$. For a convex body K, its **polar** is defined as the set $K^0 = \{x \mid \langle x, y \rangle \le 1 \text{ for every } y \in K\}$.

A convex body K' is said to be an affine image of K if there exist $a \in \mathbb{R}^n$ and $T \in GL_n$ such that $K' = TK + a$.

Along this work we denote volume of a convex body K by $vol(K)$. We say that K is in a position of maximal volume inside L if $K \subset L$ and for every affine image K' of K such that $K' \subset L$ one has $vol(K) \ge vol(K')$.

A function $\|\cdot\| : \mathbb{R}^n \to [0, \infty)$ is said to be a norm if it satisfies the following properties:

(a) $\|x\| \ge 0$ and $\|x\| = 0$ iff $x = 0$,

(b) $\|x + y\| \le \|x\| + \|y\|$ for all $x, y \in X$, and

(c) $\|\alpha x\| = |\alpha| \|x\|$ for all $\alpha \in \mathbb{R}$ and $x \in X$.

We call the pair $X = (\mathbb{R}^n, \|\cdot\|)$ a normed space. The unit ball of X with respect to $\|\cdot\|$ is $B_X = \{y \mid \|y\| \le 1\}$.

Given a convex body K we denote its interior by $int(K)$ and its boundary by ∂K.

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Let K be a convex body with $0 \in int(K)$, the **Minkowski functional of K** or **Gauge of K** is defined as $\|x\|_K = \inf\{\lambda > 0 \mid x \in \lambda K\}$. The functional $\|\cdot\|_K$ defined in this way is sublinear and positively homogeneous. If in addition K is centrally symmetric then $\|x\|_K = \|-x\|_K$ and, therefore, $\|\cdot\|_K$ defines a norm.

Given a norm on $\mathbb{R}^n$, for $X = (\mathbb{R}^n, \|\cdot\|)$ we note that its unit ball B_X is convex, compact, centrally-symmetric, and has non-empty interior. Thus, we have the immediate bijection between norms on $\mathbb{R}^n$ and the set of centrally symmetric convex bodies. To simplify the notation we will use $(\mathbb{R}^n, B_X)$ for $X = (\mathbb{R}^n, \|\cdot\|)$. Furthermore, if X^* denotes the dual of X then X^* can be identified with $(\mathbb{R}^n, B_{X^*})$, where $B_{X^*} = (B_X)^0$, via $f(x) = \sum_{i=1}^n x_i f_i = \langle x, f\rangle$, for every $x \in X = (\mathbb{R}^n, B_X)$ and $f \in X^* = (\mathbb{R}^n, B_{X^*})$.

If $X = (\mathbb{R}^n, B_X)$ and $Y = (\mathbb{R}^n, B_Y)$ are two n-dimensional normed spaces, the **Banach-Mazur** distance between them is defined by

$$d_{BM}(X,Y) := \inf_{T \in GL_n} \|T : X \to Y\| \cdot \|T^{-1} : Y \to X\|. \tag{2.2}$$

This distance satisfies $d_{BM}(X,Y) \geq 1$ with equality if and only if the space X is isometric to Y. If $d_{BM}(K,L) \leq \lambda$ then we say that X and Y are λ-isomorphic. In geometrical terms it means that there exists a linear transformation ϕ such that

$$B_X \subseteq \phi(B_Y) \subseteq d_{BM}(K,L) B_X.$$

In light of the bijection between norms in $\mathbb{R}^n$ and centrally-symmetric convex bodies, we define an analogous notion of Banach-Mazur distance for convex bodies. However, the restriction to centrally symmetric convex bodies is not natural within a geometrical context. The following definition generalizes the concept of Banach-Mazur distance to arbitrary convex bodies.

For convex bodies K and L in $\mathbb{R}^n$ we define the **Banach-Mazur** distance

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between them as

$$d_{BM}(K,L) = \inf\Big\{\lambda > 0 \;\Big|\; K - a \subset T(L-b) \subset \lambda(K-a)\Big\}, \tag{2.3}$$

where the infimum is taken over all $a, b \in \mathbb{R}^n$ and all $T \in GL_n$.

This distance satisfies a multiplicative triangle inequality, i.e., for convex bodies K, L, and K' in $\mathbb{R}^n$, one has $d_{BM}(K,L) \leq d_{BM}(K,K')d_{BM}(K',L)$. It it also clear that the distance is invariant under affine transformations, i.e., $d_{BM}(K,L) = d_{BM}(K',L')$ for any affine images K' and L' of K and L respectively.

In most of this work we use another Banach-Mazur type distance suggested by Grünbaum [5] in 1963.

$$d(K,L) = \inf\Big\{|\lambda| \;\Big|\; \lambda \in \mathbb{R}, K - a \subset T(L-b) \subset \lambda(K-a)\Big\}. \tag{2.4}$$

This distance also satisfies a multiplicative triangle inequality and is affine invariant. Clearly $d(K,L) \leq d_{BM}(K,L)$.

In the next chapter we discuss how big the last distance can be and what the extremal case is.

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Chapter 3

Two convex bodies far apart

In this chapter we study the case where the distance defined in (2.4) between two convex bodies $K, L \in \mathbb{R}^n$ is maximal. In order to explore this situation we mainly use an extension of John's Theorem as well as some of its applications which we present next.

3.1 John's Theorem

Proved in 1948, John's theorem characterizes the unique ellipsoid of maximal volume inside a convex body K using the contact points between the boundary of K and its ellipsoid. This description is best expressed when after an affine transformation we take the ellipsoid of maximal volume to be equal to B_2^n. We next present John's Theorem (see e.g. [7], [1]).

Theorem 3.1. *Each convex body K contains a unique ellipsoid of maximal volume. This ellipsoid is B_2^n if and only if the following conditions are satisfied: $B_2^n \subset K$ and for some $m \in \mathbb{N}$ there are euclidian unit vectors $(u_i)_{i=1}^m$ on the boundary of K and positive numbers $(c_i)_{i=1}^m$ satisfying*

$$\sum_{i=1}^{m} c_i u_i = 0 \tag{3.1}$$

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and

$$x = \sum_{i=1}^{m} c_i \langle x, u_i \rangle u_i \quad \textit{for all } x, \tag{3.2}$$

or, equivalently in operator notation,

$$I_n = \sum_{i=1}^{n} c_i u_i \otimes u_i. \tag{3.3}$$

Note that, taking trace of (3.3) using (2.1) and the fact that $\langle u_i, u_i \rangle = 1$, we observe

$$\sum_{i=1}^{m} c_i = n.$$

For a symmetric convex body K Theorem 3.1 implies $K \subset \sqrt{n}B_2^n$, which gives us $\sqrt{n}$ as an upper bound for the distance between K and B_2^n.

John's Theorem has highly influenced the study of convex bodies; many of the results known nowadays are due to applications of this theorem. For the purpose of what is studied in this chapter we should mention one of the results that used John's Theorem as one of its ingredients. This result, found independently by Leichweiß [8] and years later by Palmon [11], states that an n-dimensional simplex S_n is the unique convex body (up to affine transformations) satisfying $d_{BM}(S_n, B_2^n) = n$.

John's Theorem was extended independently by Lewis (Theorem 1.3 in [9]) and Milman (Theorem 14.5 in [14]) to centrally symmetric convex bodies K and L. Later Giannopoulos, Perissinaki and Tsolomitis [4] proved a similar result for two smooth enough bodies. Some time later Bastero and Romance [2] presented another extension of John's Theorem for a convex body K and a compact connected set L, with $vol(L) > 0$. However in all these extension some information about the origin was missing.

In 2004 Gordon, Litvak, Meyer, and Pajor in [6] extended John's Theorem to the most general case. They proved the following theorem.

Theorem 3.2. *Let K and L be two convex bodies in $\mathbb{R}^n$ such that K is in a position of maximal volume in L, and $0 \in int(L)$. Then there exists $m \in \mathbb{N}$,*

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$m \leq n^2 + n$, $z \in int(K)$, u_i, and v_i for every $i \leq m$ such that

$$u_i \in \partial(K-z) \cap \partial(L-z),\ v_i \in \partial(L-z)^0 \cap \partial(K-z)^0 \quad \textit{with} \quad \langle u_i, v_i \rangle = 1,$$

and positive $a_1, a_2, ..., a_m \in \mathbb{R}$ such that

$$I_n = \sum_{i=1}^{m} a_i u_i \otimes v_i = \sum_{i=1}^{m} a_i v_i \otimes u_i$$

$$\left(i.e. \forall x \in \mathbb{R}^n \quad x = \sum_{i=1}^{m} a_i \langle v_i, x \rangle u_i \right)$$

and

$$\sum_{i=1}^{m} a_i u_i = \sum_{i=1}^{m} a_i v_i = 0. \tag{3.4}$$

Remark 1. Taking trace of I_n we obtain $\sum_{i=1}^{m} a_i = n$. Indeed, using (2.1) and $\langle u_i, v_i \rangle = 1$ we have

$$n = Tr(I_n) = Tr\left(\sum_{i=1}^{m} a_i u_i \otimes v_i\right) = \sum_{i=1}^{m} Tr(a_i u_i \otimes v_i) = \sum_{i=1}^{m} a_i Tr(u_i \otimes v_i) = \sum_{i=1}^{m} a_i.$$

Remark 2. As we see below, the new information on the center (equality (3.4)) is crucial for the applications.

3.2 Maximal Distance

In the study of numerical invariants on the set of convex bodies, a set of natural questions always emerges. We are usually interested in finding at which bodies the invariant attains its extremes. This section is devoted to the study of the maximum of one of these invariants: the distance defined in (2.4). We are basically interested in finding conditions that bodies at which this distance is maximal hold.

Using their extension of John's Theorem, Theorem 3.2 above, the authors of [6] were able to provide an upper bound for distance (2.4). They proved the

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following theorem.

Theorem 3.3. *Let K and L be two convex bodies in $\mathbb{R}^n$. Then $d(K, L) \leq n$. Moreover, if K is in a position of maximal volume in L then there exist z in $\mathbb{R}^n$ such that*

$$K - z \subset L - z \subset -n(K - z).$$

For the sake of completeness, we present its proof.

Proof. After an affine transformation, we may assume that K is in a position of maximal volume in L and that $0 \in int(L)$. We apply Theorem 3.2, and passing to $L - z$ and $K - z$, if needed, we assume that z given by this theorem is equal to 0. There exist $u_1, ..., u_m, v_1, ..., v_m$ such that

$$u_i \in \partial K \cap \partial L,\ v_i \in \partial L^0 \cap \partial K^0 \quad \text{with} \quad \langle u_i, v_i \rangle = 1,$$

and positive $a_1, a_2, ..., a_m \in \mathbb{R}$ such that

$$I_n = \sum_{i=1}^{m} a_i v_i \otimes u_i \qquad \Big(\text{i.e. } x = \sum_{i=1}^{m} a_i \langle v_i, x \rangle u_i\Big),$$

and

$$\sum_{i=1}^{m} a_i u_i = \sum_{i=1}^{m} a_i v_i = 0 \quad \text{with} \quad \sum_{i=1}^{m} a_i = n.$$

Then for every $x \in L$ one has

$$\begin{aligned} \|-x\|_K &= \|-\sum_{i=1}^{m} a_i \langle v_i, x \rangle u_i \|_K \\ &= \|-\sum_{i=1}^{m} a_i \langle v_i, x \rangle u_i + \left(\sum_{i=1}^{m} a_i u_i\right) \max_j \langle v_j, x \rangle \|_K \\ &= \|\sum_{i=1}^{m} \left(\max_j \langle v_j, x \rangle - \langle v_i, x \rangle\right) a_i u_i \|_K \end{aligned}$$

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$$
\begin{aligned}
&\leq \sum_{i=1}^{m} \left(\max_j \langle v_j, x \rangle - \langle v_i, x \rangle \right) a_i \|u_i\|_K && (3.5) \\
&= \sum_{i=1}^{m} \max_j \langle v_j, x \rangle a_i - \sum_{i=1}^{m} \langle v_i, x \rangle a_i \\
& \sum_{i=1} \max_j \langle v_j, x \rangle a_i - \sum_{i=1} \langle v_i, x \rangle a_i \\
&= \max_j \langle v_j, x \rangle \sum_{i=1} a_i - \langle \sum_{i=1} a_i v_i, x \rangle .
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\| - x \|_K \leq n \max_j \langle v_j, x \rangle &\leq n \max_j \|v_j\|_{L^0} \|x\|_L && (3.6) \\
&= n \|x\|_L \leq n. && (3.7)
\end{aligned}
$$

Thus $-x \in nK$ which implies $x \in -nK$. Since x was chosen arbitrarily, we obtain

$$L \subset -nK.$$

□

As the example involving the n-dimensional simplex and B_2^n shows, the upper bound n given by Theorem 3.3 can be obtained. The natural question that comes up is for which other pairs of bodies the distance is exactly n. The natural conjecture is that, in such a case, one of the bodies is a simplex. Let us note that in [6] (Theorem 5.8) it was proved that $d(S, B) = n$, where S is a simplex and B is an arbitrary symmetric convex body. We would also like to mention that the simplex is an extremal body for many other affine invariants. We refer to [13] for a survey in simplices as extremal bodies. In the rest of this section we explore the situation where the distance $d(K, L) = n$ for two arbitrary convex bodies $K, L \in \mathbb{R}^n$. Although the conjecture has not been proved, we find a few conditions that hold when the distance is maximal.

Roughly speaking we show that if K is in a position of maximal volume inside L and the distance between K and L is exactly n then there exist two hyperplanes P_A and P_B such that for all the contact points $u_1, .., u_m$ between the convex bodies K and L and $v_1, ..., v_m$ between K^0 and L^0 given by the

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Theorem 3.2 the following holds: for every $1 \le i \le m$ either u_i lies in P_A or v_i lies in P_B. Let us emphasize that in the case of the simplex we have exactly the same situation: all but one of the vertices of the simplex lie in one hyperplane. The existence of these two hyperplanes is a consequence of the following theorem.

Theorem 3.4. *Let $K \subset L$ and assume that*

$$I_n = \sum_{i=1}^{m} a_i u_i \otimes v_i = \sum_{i=1}^{m} a_i v_i \otimes u_i \tag{3.8}$$

and

$$\sum_{i=1}^{m} a_i u_i = \sum_{i=1}^{m} a_i v_i = 0, \tag{3.9}$$

for some $m \in \mathbb{N}$, $u_i \in \partial K \cap \partial L$, $v_i \in \partial L^0 \cap \partial K^0$, and positive $a_1, a_2, ..., a_m \in \mathbb{R}$. Assume that for every $0 < \varepsilon < n$ one has $L \not\subseteq -(n-\varepsilon)K$. Then there exist $x \in \partial L$ and $w \in \partial K^0$ such that for every $i \le m$ either $\langle u_i, w\rangle = 1$ or $\langle x, v_i\rangle = 1$. Moreover

$$\frac{-x}{n} \in Conv(u_i)_{i\in A} \qquad \text{and} \qquad \frac{-w}{n} \in Conv(v_i)_{i\in B},$$

where

$$A = \{i \,|\, \langle u_i, w\rangle = 1\} \qquad \text{and} \qquad B = \{i \,|\, \langle v_i, x\rangle = 1\}.$$

Remark 1. The planes P_A and P_B mentioned before are defined by x and w. That is,

$$P_A = \{u \,|\, \langle u_i, w\rangle = 1\} \text{ and } P_B = \{v \,|\, \langle v_i, x\rangle = 1\}.$$

Remark 2. By Theorem 3.2, the assumptions 3.8 and 3.9 in Theorem 3.4 are satisfied if K is of maximal volume in L (after some shift).

Proof. The condition for every $0 < \varepsilon < n$ one has $L \not\subseteq -(n-\varepsilon)K$ means that the inequalities (3.5), (3.6), and (3.7) are in fact equalities in the proof of Theorem 3.3 above.

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Thus, we observe

$$\|\sum_{i=1}^{m}(\max_j\langle v_j, x\rangle - \langle v_i, x\rangle)a_iu_i\|_K = \sum_{i=1}^{m}(\max_j\langle v_j, x\rangle - \langle v_i, x\rangle)a_i\|u_i\|_K, \quad (3.10)$$

and

$$n\max_j\langle v_j, x\rangle = n\max_j\|v_j\|_{L^0}\|x\|_L = n\|x\|_L = n. \quad (3.11)$$

From (3.11) we have

$$\|x\|_L = 1,$$

and

$$n\max_j\langle v_j, x\rangle = n,$$

that is,

$$\langle v_j, x\rangle = 1 \text{ for some } j.$$

Taking the inner product of the vector $w_1 = \sum_{i=1}^{m}(\max_j\langle v_j, x\rangle - \langle v_i, x\rangle)a_iu_i$ in (3.10) and the functional at which its norm is attained $w \in \partial K^0$ ($\|w\|_{K^0} = 1$), and also using that $\|u_i\|_K = 1$, we have

$$\sum_{i=1}^{m}\left(\max_j\langle v_j, x\rangle - \langle v_i, x\rangle\right)a_i\langle u_i, w\rangle = \sum_{i=1}^{m}\left(\max_j\langle v_j, x\rangle - \langle v_i, x\rangle\right)a_i.$$

Since $\langle u_i, w\rangle \leq 1$, we obtain that for every i either $\langle u_i, w\rangle = 1$ or $\langle v_i, x\rangle = \max_j\langle v_j, x\rangle = 1$.
Denote $B = \{i \in \{1, ..., m\} | \langle u_i, w\rangle = 1\}$ and $A = \{j \in \{1, ..., m\} | \langle x, v_j\rangle = 1\}$.
Now using (3.8) we express w as

$$\begin{aligned} w &= \sum_{i=1}^{m} a_i\langle u_i, w\rangle v_i = \sum_{B} a_i\langle u_i, w\rangle v_i + \sum_{B^C} a_i\langle u_i, w\rangle v_i \\ &= \sum_{B} a_iv_i + \sum_{B^C} a_i\langle u_i, w\rangle v_i. \end{aligned}$$

By (3.9)

$$w = -\sum_{B^C} a_iv_i + \sum_{B^C} a_i\langle u_i, w\rangle v_i.$$

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Thus

$$w = \sum_{B^C} a_i[\langle u_i, w\rangle - 1]v_i.$$

Now,

$$\begin{aligned}\sum_{B^C} a_i[\langle u_i, w\rangle - 1] &= \sum_{B^C} a_i\langle u_i, w\rangle - \sum_{B^C} a_i \\ &= \sum_{i=1}^{m} a_i\langle u_i, w\rangle - \sum_{B} a_i\langle u_i, w\rangle - \sum_{B^C} a_i.\end{aligned}$$

Since

$$\sum_{i=1}^{m} a_i\langle u_i, w\rangle = 0,$$

we have

$$\sum_{B^C} a_i[\langle u_i, w\rangle - 1] = -\sum_{B} a_i\langle u_i, w\rangle - \sum_{B^C} a_i = -\sum_{i=1}^{m} a_i = -n.$$

From above it is clear that

$$w = -n\sum_{i\in B^C} s_i v_i, \quad \text{where} \quad s_i \geq 0, \quad \text{and} \quad \sum_{i\in B^C} s_i = 1.$$

Using a similar reasoning we can also show that,

$$x = -n\sum_{i\in A^C} t_i u_i, \quad \text{where} \quad t_i \geq 0, \quad \text{and} \quad \sum_{i\in A^C} t_i = 1.$$

Since $A^C \subseteq B$ *and* $B^C \subseteq A$, completing (if necessary) with $t_i = 0$ for $i \in B\backslash A^C$ and $s_i = 0$ for $i \in A\backslash B^C$, we have that

$$w = -n\sum_{i\in A} s_i v_i \quad \text{and} \quad x = -n\sum_{i\in B} t_i u_i.$$

□

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3.3 Removing contact points

In this section we present a proposition that allows us to remove contact points from the John's decomposition of the identity given by Theorem 3.2 under special circumstances. Although some conditions on the contact points were already given in [6], the following proposition is almost as strong as the result in [6], but has a much simpler proof.

Proposition 3.5. *Let K, L be two convex bodies in $\mathbb{R}^n$. Let $m \in \mathbb{N}$. Assume $a_1, ..., a_m > 0, u_1, ..., u_m \in \partial K \cap \partial L, v_1, ..., v_m \in \partial K^0 \cap \partial L^0$ be such that*

$$I_n = \sum_{i=1}^{m} a_i u_i \otimes v_i = \sum_{i=1}^{m} a_i v_i \otimes u_i, \tag{3.12}$$

$$\sum_{i=1}^{m} a_i u_i = \sum_{i=1}^{m} a_i v_i = 0, \tag{3.13}$$

and $\langle u_i, v_i \rangle = 1$, for every $i \leq m$. If for some $j \leq m$ either $u_j \in Conv(u_i)_{\substack{1 \leq i \leq m \\ i \neq j}}$ or $v_j \in Conv(v_i)_{\substack{1 \leq i \leq m \\ i \neq j}}$, then for every $i \neq j$ there exist $b_i > 0$, $w_i \in \partial K \cap \partial L$, and $z_i \in \partial K^0 \cap \partial L^0$ such that $\langle w_i, z_i \rangle = 1$,

$$I_n = \sum_{\substack{i=1 \\ i \neq j}}^{m} b_i w_i \otimes z_i,$$

and

$$\sum_{\substack{i=1 \\ i \neq j}}^{m} b_i w_i = \sum_{\substack{i=1 \\ i \neq j}}^{m} b_i z_i = 0.$$

Proof. Without loss of generality, we assume that $u_m \in Conv(u_i)_{i \leq m-1}$. Then there exist $\lambda_1, ..., \lambda_{m-1}$, such that,

$$u_m = \sum_{i=1}^{m-1} \lambda_i u_i \quad \text{and} \quad \sum_{i=1}^{m-1} \lambda_i = 1, \quad \lambda_i \geq 0.$$

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By (3.12) we have then

$$\begin{aligned}
I_n = \sum_{i=1}^{m} a_i u_i \otimes v_i &= \sum_{i=1}^{m-1} a_i u_i \otimes v_i + a_m u_m \otimes v_m \\
&= \sum_{i=1}^{m-1} a_i u_i \otimes v_i + a_m \sum_{i=1}^{m-1} \lambda_i u_i \otimes v_m \\
&= \sum_{i=1}^{m-1} u_i \otimes a_i v_i + \sum_{i=1}^{m-1} u_i \otimes a_m \lambda_i v_m \\
&= \sum_{i=1}^{m-1} u_i \otimes (a_i v_i + a_m \lambda_i v_m).
\end{aligned}$$

Let

$$b_i = \langle u_i, a_i v_i + a_m \lambda_i v_m \rangle = a_i + a_m \lambda_i \langle u_i, v_m \rangle.$$

Note that $b_i > 0$. Indeed, since

$$1 = \langle u_m, v_m \rangle = \langle \sum_{i=1}^{m-1} \lambda_i u_i, v_m \rangle = \sum_{i=1}^{m-1} \lambda_i \langle u_i, v_m \rangle \leq \sum_{i=1}^{m-1} \lambda_i = 1,$$

we have $\langle u_i, v_m \rangle = 1$ for all $1 \leq i \leq m$. Thus, $b_i = a_i + a_m \lambda_i > 0$. Therefore

$$I_n = \sum_{i=1}^{m-1} u_i \otimes (a_i v_i + a_m \lambda_i v_m) = \sum_{i=1}^{m-1} b_i u_i \otimes \left(\frac{a_i v_i + a_m \lambda_i v_m}{b_i} \right).$$

Let

$$z_i = \frac{a_i v_i + a_m \lambda_i v_m}{b_i} \quad \text{and} \quad w_i = u_i.$$

This gives us

$$I_n = \sum_{i=1}^{m-1} b_i w_i \otimes z_i.$$

Since $\langle w_i, z_i \rangle = \langle u_i, z_i \rangle = b_i / b_i = 1$, all we need to show is that

$$\sum_{i=1}^{m-1} b_i w_i = \sum_{i=1}^{m-1} b_i z_i = 0,$$

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and that for every $1 \leq i \leq m-1$, we have

$$w_i \in \partial K \cap \partial L \quad \text{and} \quad z_i \in \partial L^0 \cap \partial K^0.$$

Since $w_i = u_i \in \partial K \cap \partial L$, we will only check that $z_i \in \partial L^0 \cap \partial K^0$. Note that

$$z_i = \frac{a_i v_i + a_m \lambda_i v_m}{a_i + a_m \lambda_i}.$$

This implies $z_i \in Conv(v_i)$ and, hence, $z_i \in L^0$.

Since $\langle u_i, z_i \rangle = 1$, we have $z_i \in \partial L^0 \cap \partial K^0$. Finally, we only need to show that

$$\sum_{i=1}^{m-1} b_i w_i = \sum_{i=1}^{m-1} b_i z_i = 0.$$

In order to do this, we observe that

$$\begin{aligned}
\sum_{i=1}^{m-1} b_i u_i &= \sum_{i=1}^{m-1} (a_i + a_m \lambda_i) u_i = \sum_{i=1}^{m-1} a_i u_i + \sum_{i=1}^{m-1} a_m \lambda_i u_i \\
&= -a_m u_m + a_m \sum_{i=1}^{m-1} \lambda_i u_i = -a_m u_m + a_m u_m = 0,
\end{aligned}$$

and

$$\begin{aligned}
\sum_{i=1}^{m-1} b_i z_i &= \sum_{i=1}^{m-1} (a_i v_i + a_m \lambda_i v_m) = \sum_{i=1}^{m-1} a_i v_i + \sum_{i=1}^{m-1} v_m a_m \lambda_i \\
&= -a_m v_m + a_m v_m \sum_{i=1}^{m-1} \lambda_i = -a_m v_m + a_m v_m = 0.
\end{aligned}$$

□

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