Characters of 2 x 2 Unitary Matrix Groups Over Quadratic Ring Extensions

by

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Abstract

In the Journal of Algebra 323(2010) R.Barrington Leigh et al. derive the number and degrees of the irreducible characters of G_l, the group of invertible 2x2 matrices over the ring $\mathbb{Z}/p^l\mathbb{Z}$ for p an odd prime. Here we generalize that work by finding the number and degrees of the irreducible characters of the groups of families of 2 x 2 unitary matrices over quadratic extensions of certain local rings. We will form the quadratic extension first by adjoining the square root of an invertible element of the ring, and then by adjoining the root of a nilpotent element.

The overarching argument is inductive: our unitary group will be denoted U_l, where l is a modulus of sorts. This argument requires that we know the results for $l = 1$, and in the case of the quadratic extension by the square root of a unit, the results are known from the author's own Masters's Thesis, but also from the work by V. Ennola. The results for $l = 1$ when the root of a nilpotent element is adjoined are developed as chapter 5 of the present work. The earlier work of Barrington Leigh et al. was based on Clifford theory, and we shall also follow this method, though many new technical difficulties arise in the unitary case, particularly when l is odd. We will depart from Clifford theory only when working out the nilpotent case for $l = 1$, since we will there be able to use a result from Serre concerning the characters of semi-direct product. We will also give a fuller explanation of certain aspects of the Barrington-Leigh work, in order that they might be adapted to the unitary groups.

To my wife Doris who will begin to see more of me now.

Acknowledgements

I would like to thank my supervisor Dr. Gerald Cliff for the help and guidance he has provided me for duration of this work. His patience sustained me during the more difficult parts of my investigations, and what I have been able to accomplish is due in no small way to his efforts.

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Chapter 1

Introduction

The irreducible characters of $GL_2(\mathbb{F}_q)$ are given in Fulton and Harris[FH] and those of the subgroup of unitary matrices are stated in the 1963 paper "On the Characters of the Finite Unitary Groups" by Veikko Ennola[E], in Annales Academiæ Scientiarum Fennicæ Mathematica. In their Journal of Algebra paper of 2010, Barrington Leigh et. al replace the finite field with $\mathbb{Z}/p^l\mathbb{Z}$ (for an odd prime p), and find the degrees, numbers, and values of the irreducible characters of the general linear group over that ring. The present work relaxes conditions on the ring, then takes a quadratic extensions by both invertible and non-invertible elements, so that one can consider the group of unitary 2×2 matrices over a conjugate bilinear form. We find the degrees and numbers of the irreducible characters of these unitary groups.

We will see that such unitary groups contain certain convenient abelian subgroups, and the plan will be to begin with irreducible characters of such a subgroup, and use Clifford theory to arrive at an irreducible characters of the unitary group.

The overall argument is inductive; we find the character degrees and numbers for U_l the unitary group, assuming that this information is known for U_{l-1}. When we adjoin the square root of an invertible element of the ring, the base case comes from Ennola [E], and a previous work of the present author [C]. For the case of adjoining a non-invertible element on the other hand, we work out the base case in the present work.

Chapter 2

Preliminaries

2.A Representations

Let G be a finite group and V be a finite dimensional $\mathbb{C}$ vector space; a *representation* of G is a homomorphism $\rho : G \rightarrow \mathrm{GL(V)}$. If we choose a basis for V, we can identify G with a group of invertible matrices. The dimension of V is called the *degree* of ρ. We can use representations, together with some of the machinery of linear algebra, to investigate the structure of G. When the context is clear, one often refers to V itself as the representation.

Given a representation ρ as above, a subspace W of V is a *subrepresentation* if, for all $g \in G, w \in W$ $\rho_g(w) \in W$; that is, W is a G invariant subspace. A representation that has no non-trivial subspaces is called *irreducible*. It is known that finite groups have finitely many irreducible representations, and that every representation of a finite group can be expressed as a direct sum of irreducible representations. Hence it suffices to know the irreducible representations of G. In fact the squares of the degrees of all irreducible representations of G sum to the order of G, and this is how we will demonstrate that we have found degrees and numbers of all irreducible characters of our group.

We can also view representations from the perspective of the group algebra which is denoted $\mathbb{C}[G]$: given G, take formal sums of elements in G as a $\mathbb{C}$ vector space, and define multiplication of basis vectors g, h to be consistent with the group multiplication; then extend by linearity to the multiplication of any two vectors. This is an associative algebra, and we can move freely from group to algebra representations (by linear extension), as well as from algebra to group representations (by restriction). When seen in the group algebra context, a representation makes V into a $\mathbb{C}[G]$ module.

Any representation ρ of G restricts to a representation of a subgroup H of G. It is possible to go in the other direction as well; given a representation of H, $r : H \to \mathrm{GL(V)}$ we can *induce* this to a representation on G. To see how this might be done, suppose first that we already have a representation on G, and that $W \subset V$ be a subspace of V invariant under the action of H. Given any $g \in G$, the subspapce gW will depend only on the coset gH that g lies in, since if $g' \in gH$, $g'W = (gh)W = g(hW) = gW$ where $h \in H$. If for some $\sigma \in G/H$ we write σW for this subspace, then if every $v \in V$ can be written uniquely as a sum of elements of such subspaces, we say that V has been *induced* by W, and we write $V = \mathrm{Ind}_H^G W$, or $\mathrm{Ind}\, W$. It can be shown (Fulton and Harris) that given a representation W of H, the induced representation V of G always exists and is unique. Unfortunately, a representation that is induced from an irreducible representation is not, in general, itself irreducible.

2.B Characters

With any representation V of G, we can associate a function $\chi : G \to \mathbb{C}$ defined by $\chi(g) = \text{trace}(\rho(g))$, which is called the *character* of the representation. Note that in general, only characters of degree 1 representations are homomorphisms. Characters have been used since the late 19th century to investigate representations of groups - the character will determine a representation up to isomorphism. A character is called irreducible if it comes from an irreducible representation.

We show first, that characters can be induced; in fact if $f : H \to \mathbb{C}$ is any class function (function constant on conjugacy classes of H), define:

$$\hat{f}(x) = \begin{cases} f(x) & x \in H \\ 0 & x \notin H \end{cases} \tag{2.1}$$

then we can define a linear map Ind_H^G from the space of class functions on H to the space of class functions on G, given by $\text{Ind}_H^G f(g) = \frac{1}{|H|} \sum_{x \in G} \hat{f}(x^{-1}gx)$. One can show that this map is adjoint to the restriction map sending class functions of G to class functions of H. This fact insures that induced characters are characters of G.

On the other hand, inducing an irreducible character of H does not usually produce an irreducible character of G. Clifford theory however, gives us a way of beginning with an irreducible character of a normal subgroup, and producing an irreducible character of G by using a combination of extension and induction.

2.C Clifford Theory

We briefly review some of the concepts of Clifford theory that provide the framework of the argument in this work. Let G be any finite group , $H \triangleleft G$, and ϕ an irreducible character of H. The elements of G act on the irreducible characters of H by conjugation: if $g \in G$, define ϕ^g by $\phi^g(h) = \phi(ghg^{-1})$. The subgroup T of G that acts trivially on ϕ is called the inertia group of ϕ in G, and is denoted $\mathrm{I}_U\phi$. A fundamental theorem in Clifford theory states that if ψ is an irreducible character of T such that $[\psi_H, \phi] \neq 0$ then ψ^G is irreducible. From [I], chapter six:

Theorem 2.C.1 (Clifford) Let N be normal in G, $\phi \in \mathrm{Irr}(N)$, and let T be the inertia group of ϕ in G. Let:

$$A = \{\psi \in \mathrm{Irr}(T)\ |[\psi_N, \phi] \neq 0\}, B = \{\chi \in \mathrm{Irr}(G)\ |[\chi_N, \phi] \neq 0\}$$

Then:

1. $\psi \in A \Rightarrow \psi^G \in \mathrm{Irr}(G)$
2. $\psi \to \psi^G$ is a bijection of A onto B.
3. if $\psi \in A$, $\psi^G = \chi$, then ψ is the unique irreducible component of χ_T in A.
4. if $\psi \in A$, $\psi^G = \chi$, then $[\psi_N, \phi] = [\chi_N, \phi]$

There can be many irreducible characters of the inertia group that lie over $\phi \in \mathrm{Irr}(N)$. The next theorem from Clifford helps us count these.

Theorem 2.C.2 (Gallagher) Let N be normal in G, $\phi \in \mathrm{Irr}(N)$, and let T be the inertia group of ϕ in G. If $\chi \in \mathrm{Irr}(T)$, with $\chi_N = \phi$, then for all

$\beta \in \mathrm{Irr}(T/N)$, the characters $\beta\chi$ are irreducible and distinct for distinct β, and are all of the irreducible constituents of ϕ^G.

This means that the number of irreducible characters of G derived from a character of the normal subgroup will be the index of that subgroup in the inertia group of the character, i.e. $[T : N]$ as in our work T/N will be abelian.

Some Extension Theorems

The method of the present work involves taking an irreducible character on a normal subgroup of the unitary group U_l, and finding an extension of this character to the inertia group. The method of this extension will depend on the parity of l: when we adjoin the root of an invertible element, it is easier for l even, and when we adjoin the root of a non-invertible element, it is easier for l odd. In any case, we will require a variety of techniques of extension of characters, and we will state and prove some of these techniques here.

We begin with a lemma from [S2]:

Lemma 2.C.1 If G is a finite abelian group, and N is a subgroup of G with irreducible character ϕ, then ϕ extends to G. That is, there is a character χ of G such that the restriction of χ to N is ϕ.

Proof. Let ϕ be a character (necessarily linear) on N. We will use induction on $[G : N]$: assume N is properly contained in G (else there is nothing to prove). Let $x \notin N$; there is a smallest positive integer k such that $x^k \in N$. Then $\phi(x^k) = c \in \mathbb{C}$, and since ϕ is a homomorphism, $\phi(x)^k = c$, thus setting $\phi(x)$ equal to a kth root of c[1] will extend ϕ to the subgroup generated by N and x. Calculation shows that this extended character is well defined, and

[1]This can be done since $\mathbb{C}^*$ is a divisible group.

since $[G :< N, x >] < [G : N]$ the Lemma is proved. (It can also be shown that the number of distinct such extensions is $[G : N]$). $\square$

We are now in a position to demonstrate a proposition that will be used many times in the arguments that follow:

Proposition 2.C.1 Let N be a normal subgroup of a finite group G, and ϕ a character on N of degree 1. If S is an abelian subgroup of G that is contained in the inertia subgroup of ϕ, then ϕ extends to NS. That is, there exists a linear character θ of NS such that θ restricted to N equals ϕ.

Proof. We restrict ϕ to the intersection $N \cap S$. This intersection is a normal subgroup of S (which is abelian), and we have seen that the character on $N \cap S$ extends to a character γ on S. Now define a character θ on NS by $\theta(ns) = \phi(n)\gamma(s)$. Note that θ is well defined, since if $n_1 s_1 = n_2 s_2$, then $n_2^{-1} n_1 = s_2 s_1^{-1} \in N \cap S$, and since ϕ, γ agree on $N \cap S$:

$$\phi(n_2^{-1})\phi(n_1) = \gamma(s_2)\gamma(s_1^{-1})$$

$$\phi(n_1)\gamma(s_1) = \phi(n_2)\gamma(s_2)$$

$$\theta(n_1 s_1) = \theta(n_2 s_2)$$

The following calculation shows that θ is a character that restricts to ϕ on N:

$$
\begin{aligned}
\theta(n_1 s_1 n_2 s_2) &= \theta(n_1 s_1 n_2 s_1^{-1} s_1 s_2) \\
&= \phi(n_1 s_1 n_2 s_1^{-1})\gamma(s_1 s_2) \\
&= \phi(n_1)\phi(s_1 n_2 s_1^{-1})\gamma(s_1)\gamma(s_2) \\
&= \phi(n_1)\phi^{s_1}(n_2)\gamma(s_1)\gamma(s_2) \\
&= \phi(n_1)\phi(n_2)\gamma(s_1)\gamma(s_2) \\
&= \theta(n_1 s_1)\theta(n_2 s_2)
\end{aligned}
$$

It follows from Lemma 2.C.1 and the above argument, that the number of extensions from N to NS is $\frac{|S|}{|N \cap S|} = \frac{|NS|}{|S|}$. □

The next result will be used only once, but it is indispensable:

Proposition 2.C.2 Let N be a normal subgroup of G and ϕ a G invariant irreducible character of N. If the degree of ϕ is relatively prime to $[G : N]$, then ϕ extends to G.

Proof. The proof of this fact is somewhat involved; it involves the concepts of *projective* representations as well as some group cohomology.

A projective representation of a group G is a map $\mathbb{X} : G \to \mathrm{GL}(n, \mathbb{C})$ such that for some scalar $\gamma(gh) \in \mathbb{C}$:

$$\mathbb{X}(g)\mathbb{X}(h) = \mathbb{X}(gh)\gamma(g, h)$$

The function $\gamma : G \times G \to \mathbb{C}^*$ is called the factor set of $\mathbb{X}$. Calculation shows that a necessary condition on factor sets is that for all $x, y, z \in G$ is that $\gamma(xy, z)\gamma(x, y) = \gamma(x, yz)\gamma(y, z)$. It can be shown that for any factor set

γ there is a projective representation of G having that factor set.

We can replace $\mathbb{C}$ above by any abelian group A, and consider the group C_1 of arbitrary maps from G to A (with point wise multiplication), as well as C_2, the group of maps from $G \times G$ to A. Following the terminology of cohomology, we have a boundary map $\delta : C_1 \to C_2$, so that for any $\mu \in C_1$, $\delta(\mu)(g,h) = \mu(g)\mu(h)\mu(gh)^{-1}$. It easy to see that $\delta(\mu)$ is a factor sets.

The factor sets are a subgroup of C_2; [2] this subgroup is called $Z^2(G,A)$, the 2 co-cycles. The image of C_1 under the boundary map is called $B^2(G,A)$, the 2 co-boundaries. Finally, the quotient $Z^2(G,A)/B^2(G,A) = H^2(G,A)$, the second co-homology group. With this framework in place, we can state the following:

Proposition 2.C.3 Let $N \lhd G$, with $\theta \in \mathrm{Irr}(N)$ invariant in G, and afforded by the representation $\mathbb{Y}$. Let $\mathbb{X}$ be a projective representation of G extending $\mathbb{Y}$, and satisfying the conditions above. If γ is the factor set of $\mathbb{X}$, we can define $\psi \in Z^2(G/N,\mathbb{C})$ by $\psi(gN,hN) = \gamma(g,h)$. Then ψ is well defined and the image $\overline{\psi} \in H^2(G/N,\mathbb{C})$ depends only on θ, and θ extends to G if and only if $\overline{\psi} = 1$.

Proposition 2.C.4 [I] Let $\mathbb{F}$ be an algebraically closed field, and G a finite group. Then $H^2(G,\mathbb{F})$ is finite and each of its elements has order dividing $|G|$.

Proof. Beginning with the statement of a projective representation:

$$\mathbb{X}(g)\mathbb{X}(h) = \mathbb{X}(gh)\gamma(g,h)$$

[2] actually the kernel of the boundary map from C_2 to C_3

Taking the determinant of both sides, and noting that the degree of $\mathbb{X}$ equals the degree of θ, we see that:

$$\gamma(g,h)^{\deg\theta} = \det(\mathbb{X}(g))\det(\mathbb{X}(h))\det(\mathbb{X}(gh)^{-1}) \in B^2(G,\mathbb{C}^*)$$

Thus $\gamma(g,h)^{\deg\theta}$ is congruent to 1 in $H^2(G,\mathbb{C}^*)$, so the order of the image of γ in $H^2(G,\mathbb{C}^*)$ divides the degree of θ. □

□

Chapter 3

The Ring

Let R be a finite commutative ring with $\pi \in R$ such that for every ideal I contained in R, $I = \pi^i R$ for some non-negative integer i, and such that for some minimal positive integer l, $\pi^l = 0$. Thus R is a local ring and πR is the unique maximal ideal. As R is finite, $R/\pi R$ is isomorphic to some finite field $\mathbb{F}_q$; we will always assume that q is a power of an odd prime. We will write R_l for R if we need to emphasize that l is the minimal power of π that equals zero; if there is no chance of confusion, we shall sometimes just write R for this ring. Let α be a non-square invertible element in R_l; we denote the quadratic extension ring $R_l[\sqrt{\alpha}\,]$ by $R_{l,\alpha}$. Our concern is to find the degrees of the irreducible characters of U_l, the group of unitary 2×2 matrices, with elements in $R_{l,\alpha}$, as well as the number of such characters. After this, we will turn to the case of a quadratic extension of R by the square root of a non-invertible element. We require a conjugate linear form on the module $R_{l,\alpha} \times R_{l,\alpha}$ and similarly in the case of extension by a non-invertible element. It would be convenient to have all such forms be equivalent in some sense, as this would give us freedom to choose any convenient matrix of the form. This equivalence will occur if the norm map $\mathcal{N} : R^*_{l,\alpha} \to R^*_l$ is surjective, where $\mathcal{N}(a + b\sqrt{\alpha}) = a^2 - \alpha b^2$

3.A Unique Expression of Ring Elements and Conditions of Invertible Elements

We fix a transversal $\mathcal{T}$ of the quotient $R_l/\pi R_l$ and include $0 \in R_l$ in T; this fixed transversal allows the following:

Lemma 3.A.1 If $a \in R_l$, a can be written as a unique sum:

$$a_{l-1}\pi^{l-1} + a_{l-2}\pi^{l-2} + \cdots + a_1\pi + a_0 \quad \text{for } \; a_i \in \mathcal{T} \quad 0 \leq i < l$$

.

Proof. We will write R for R_l; let $a \in R$. We have $a = \pi a' + a_0$ where $a_0 \in T$, and is thus unique. Similarly $a' = \pi a'' + a_1$ with $a_1 \in T$, hence unique, so that $a = \pi^2 a'' + \pi a_1 + a_0$. This can be continued until the form in the lemma is reached.

Definition 3.A.1 We shall refer to sums such as

$$a_{l-1}\pi^{l-1} + a_{l-2}\pi^{l-2} + \cdots + a_1\pi + a_0$$

from Lemma 3.A.1 as *quasi-polynomials*. They are not true polynomials because we can have, for example: $b\pi^i + c\pi^i = d\pi^{i+1}$.

□

Proposition 3.A.1 An element $a = a_{l-1}\pi^{l-1} + a_{l-2}\pi^{l-2} + \cdots + a_1\pi + a_0 \in R_l$ is invertible if and only if $a_0 \neq 0$, where 0 here is the additive identity in $R_l/\pi R_l$.

Proof. If $a_0 = 0$ then $a \in \pi R$ and is nilpotent, hence not invertible; if $a_0 \neq 0$ then $a \notin \pi R$ and if $(a) \neq R$ then we would have (a) contained in the (unique) maximal ideal generated by π, but this implies that $a \in \pi R$ which is a contradiction. Thus $(a) = R$ and a is invertible.

□

From the unique expression for each element $a \in R_l$, we have $|R_l| = q^l$ since there are q choices for each a_i. Since a is a unit if and only if $a_0 = 0$, then the number of units in R_l is $q^{l-1}(q-1)$.

At this point we consider only the quadratic extension $R_{l,\alpha}$; after dealing with its characters we will turn to $R_{l,\pi}$, the quadratic extension of R_l by $\sqrt{\pi}$.

For convenience in some of the arguments below, we define *pure roots* in $R_{l,\alpha}$:

Definition 3.A.2 An element $a + b\sqrt{\alpha} \in R_{l,\alpha}$ is a pure root if $a = 0$.

Chapter 4

Quadratic Extension by the Square Root of a Unit

The unitary groups that are the subject of this work are those matrices that preserve a conjugate linear form from, for example, $R_{l,\alpha} \times R_{l,\alpha}$ to $R_{l,\alpha}$. When we adjoin the square root of a unit of R, then all such forms are equivalent in some sense, and to show this we will need to show the surjectivity of the norm map $\mathcal{N} : R^*_{l,\alpha} \to R^*_l$ given by $\mathcal{N}(a + b\sqrt{\alpha}) = a^2 - b^2\alpha$. For this reason, we want to know the number of units in $R_{l,\alpha}$.

Proposition 4..1 $a + b\sqrt{\alpha} \in R_{l,\alpha}$ is invertible if and only if $d = a^2 - b^2\alpha \in R_l$ is invertible. By Proposition 3.A.1, d is invertible if and only if $d_0 \neq 0$.

Proof. If $a^2 - b^2\alpha$ is a unit in R_l then $(a - b\sqrt{\alpha})(a^2 - b^2\alpha)^{-1}$ is the inverse of $a + b\sqrt{\alpha}$. On the other hand, let $a^2 - b^2\alpha = \pi x$ for some $x \in R$, and suppose that $a + b\sqrt{\alpha}$ is invertible. Then there exists $c + d\sqrt{\alpha} \in R_{l,\alpha}$ such that

$$(c + d\sqrt{\alpha})(a + b\sqrt{\alpha}) = (ac + bd\alpha) + (ad + bc)\sqrt{\alpha} = 1$$

It follows that $ac + bd\alpha = 1$ and $bc + ad = 0$. From this system we get:

1. $a = c(a^2 - b^2\alpha) = \pi xc$
2. $b = -d(a^2 - b^2\alpha) = -\pi xd$

Then $a + b\sqrt{\alpha} = \pi x(c - d\sqrt{\alpha}) \in \pi R_{l,\alpha}$ is not invertible; a contradiction. □

Corollary 4..1 If $a, b \in R_l$, then $a + b\sqrt{\alpha}$ is a unit if and only if at least one of a or b is a unit in R_l.

Proof. Let $a = \sum_{i=0}^{l-1} a_i\pi^i$, $b = \sum_{i=0}^{l-1} b_i\pi^i$, and $a^2 - b^2\alpha = d = \sum_{i=0}^{l-1} d_i\pi^i$. It is clear that if both a, b are not invertible, they are both in πR, and $a + b\sqrt{\alpha}$ is not invertible. Next, suppose that one of a, b is a unit in R_l, but that $a + b\sqrt{\alpha}$ is not invertible, so that $a^2 - b^2\alpha \in \pi R_l$. We consider the natural projection map P from R_l to $R_l/\pi R_l \simeq \mathbb{F}_q$. Then $0 = P(a^2 - b^2\alpha) = (a^2)_0 - (b^2)_0\alpha_0$. But this is a contradiction, because one of $(a^2)_0, (b^2)_0$ is non-zero, and from the case of quadratic extensions over finite fields, we know that $(a^2)_0 - (b^2)_0\alpha_0 \neq 0$. Therefore $d = a^2 + b^2\alpha$ is a unit and $a + b\sqrt{\alpha}$ is a unit.

□

4.A The Kernel of the Norm Map

We denote by $\mathcal{L}$ the kernel of the norm map $\mathcal{N} : R_{l,\alpha}^* \to R_l^*$ where $\mathcal{N}(a + b\sqrt{\alpha}) = a^2 - b^2\alpha$. We will show the surjectivity of this map by counting the units in R_l and $R_{l,\alpha}$, as well as the size of the kernel.

Proposition 4.A.1 The size of the kernel of the norm map $\mathcal{N}$ is $q^{l-1}(q + 1)$.

Proof. We will give an algorithm for constructing norm 1 elements. Let $\mathcal{T}$ be our fixed set of coset representatives of $R_l/\pi R_l$ (which is isomorphic to $\mathbb{F}_q$).

The kernel of the norm map $n : \mathbb{F}_{q^2}^* \to \mathbb{F}_q^*$ has size $q+1$ ([Ca] p 9), so we can find that number of pairs (a_0, b_0) in $\mathcal{T} \times \mathcal{T}$ such that:

$$a_0^2 - \alpha b_0^2 = 1 + \pi r_1,\ r_1 \in R_l$$

We now construct elements $a, b \in R_l$ such that $a^2 - \alpha b^2 = 1$: choose any of the $q+1$ pairs a_0, b_0 such that $a_0^2 - b_0^2\alpha = 1 + \pi r_1$. Choose the other "coefficients" (b_i) of $b = b_{l-1}\pi^{l-1} + \cdots + b_1\pi + b_0$ arbitrarily; there are q^{l-1} ways to select these elements. Now solve successively for $a_1, a_2, \ldots, a_{l-1}$. For example, to find a_1, we require that $(a_1a_0 + a_0a_1) - \alpha(b_1b_0 + b_0b_1) + r_1 = 0$ where the zero is the additive identity of $R/\pi R$. Since only a_1 is unknown here, and a_0 is a unit, we can find a_1 that solves the above equations and replace it if necessary, with an element in the transversal. We can continue in this way to find all of the "coefficients" of a. As there were $q+1$ pairs (a_0, b_0) and q^{l-1} choices for b for each, the proposition holds.

$\square$

Since the number of elements of $R_{l,\alpha}^*$ is $q^{2l} - q^{2l-2} = q^{l-1}(q-1)q^{l-1}(q+1)$ then the size of the image of the norm map is:

$$\frac{q^{l-1}(q-1)q^{l-1}(q+1)}{q^{l-1}(q+1)} = q^{l-1}(q-1)$$

which is the number of elements in R_l^* so that we have proved:

Theorem 4.A.1 The norm map $\mathcal{N} : R_{l,\alpha}^* \to R_l^*$ is surjective.

For reference, we list the numbers of various types of elements of both R_l and $R_{l,\alpha}$.

Table 4.1: Enumerating Elements of R and $R_{l,\alpha}$

	R_l	$R_{l,\alpha}$
number of elements	q^l	q^{2l}
number of non-units	q^{l-1}	q^{2l-2}
number of units	$q^{l-1}(q-1)$	$q^{l-1}(q-1)q^{l-1}(q+1)$
number of norm 1 elements		$q^{l-1}(q+1)$

4.B The Form and the Group

Consider the additive group $\mathcal{M}$ given by $R_{l\alpha} \times R_{l,\alpha}$; this is an $R_{l,\alpha}$ module. By a hermitian form on $\mathcal{M}$, we mean a map from $\mathcal{M}$ to $R_{l,\alpha}$ so that for $u, v \in \mathcal{M},\ a \in R_{l,\alpha}$:

- $\mathcal{H}(u+v, w) = \mathcal{H}(u, w) + \mathcal{H}(v, w)$
- $\mathcal{H}(u, v+w) = \mathcal{H}(u, v) + \mathcal{H}(u, w)$
- $\mathcal{H}(au, v) = a\mathcal{H}(u, v) = \mathcal{H}(u, \overline{a}v)$
- $\mathcal{H}(v, u) = \overline{\mathcal{H}(u, v)}$

The bar above refers to conjugation in $R_{l,\alpha}$. Note that $v \in \mathcal{M}$ implies $\mathcal{H}(v, v) \in R_l$. A form is called non-degenerate if for all $v \neq 0 \in \mathcal{M}$, there exists $w \in \mathcal{M}$ such that $\mathcal{H}(v, w) \neq 0$, and a space having a non-degenerate Hermitian form is called a unitary space. As in the case of bilinear forms, if a form $\mathcal{H}$ on a module V is non-degenerate on a submodule W, then V is the direct sum of W and its orthogonal complement. It is known that all such forms are equivalent for a wide class of underlying rings (see [Cr] for example). We will give a demonstration of this for our case.

If a module V with a form $\mathcal{H}$ has a basis $(e_1, e_2, \ldots, e_n)$, then we can associate the matrix $\mathcal{B} = (\mathcal{H}(e_i, e_j))$ to the form, and for any $v, w \in V$:

$$\mathcal{H}(v, w) = v^T \mathcal{B} \overline{v}$$

If we change to a new basis $(f_1, f_2, \ldots, f_n)$ with change of basis matrix P, then the matrix of the form will change to $P^T \mathcal{B} \overline{P}$. Suppose that we have two modules $\mathcal{M}_1, \mathcal{M}_2$ (with bases) with corresponding forms $\mathcal{H}_1, \mathcal{H}_2$. We say that the forms are *equivalent* if there is an isomorphism $\tau : \mathcal{M}_1 \to \mathcal{M}_2$ such that for all $v, w \in \mathcal{M}_1$: $\mathcal{H}_1(v, w) = \mathcal{H}_2(\tau v, \tau w)$.

Proposition 4.B.1 For the module $\mathcal{M} = R_{l,\alpha} \times R_{l,\alpha}$, all non-degenerate Hermitian forms are equivalent.

Proof. We note that $\mathcal{M}$ does have a basis, for example $\{(1,0),(0,1)\}$. We claim there exist $v, w \in \mathcal{M}$ such that $\mathcal{H}(v, w) = 1$. To show this, it suffices to show that we can find v, w with $\mathcal{H}(v, w) \in R^*_{l,\alpha}$, since then an appropriate scaling of v or w will give the result. Suppose to the contrary, that for all $v, w \in \mathcal{M}$, $\mathcal{H}(v, w)$ is not a unit. Choose $x \in \mathcal{M}$ such that $\pi^{l-1}x \neq 0$. Then for all $y \in \mathcal{M}$, $\mathcal{H}(x, y) = \pi z$ for some z in $R_{l,\alpha}$, so $\mathcal{H}(\pi^{l-1}x, y) = 0$ contradicting the fact that the form is non-degenerate.

Next, we claim that for some $v \in \mathcal{M}$, $\mathcal{H}(v, v)$ is a unit (necessarily in R_l). Again, suppose not: let u, v be arbitrary in $\mathcal{M}$ and let a be any unit in $R_{l,\alpha}$. Thus $\pi | \mathcal{H}(au + v, au + v) = \mathcal{H}(au, au) + \mathcal{H}(au, v) + \mathcal{H}(v, au) + \mathcal{H}(v, v)$, and this implies that π is a factor of $\mathcal{H}(au, v) + \mathcal{H}(v, au)$, but we can rearrange this expression to get $a\mathcal{H}(u, v) + \overline{a\mathcal{H}(u, v)}$. If we now choose u, v such that $\mathcal{H}(u, v) = 1$, and let $a = 1$, we get $\pi | 2$ which is a contradiction since the characteristic of $R_{l,\alpha}$ is odd.

Now choose $v \in \mathcal{M}$ such that $\mathcal{H}(v, v) = c \in R^*_l$. Then the form $\mathcal{H}$ is non-degenerate on $W = < v >$, so that $\mathcal{M} = W \oplus W^\perp$. Note that the form must be non-degenerate on $W^\perp$: for if there were $x \in W^\perp$ such that for all $y \in W^\perp$, $\mathcal{H}(x, y) = 0$, then we could take the following element in $\mathcal{M}$: $z = 0 + x$ with $0 \in W$, and $x \in W^\perp$, and we would have $\mathcal{H}(z, y) = 0$ for all $y \in \mathcal{M}$; a contradiction. Thus, inductively, we see that there is a basis of $\mathcal{M}$ such that the matrix of the form is diagonal with units of R_l on the diagonal. Moreover, since the norm map is onto, $c^{-1} = d\overline{d}$ for some $d \in R^*_{l,\alpha}$. Thus replacing v by dv, and proceeding inductively, we see that there is a basis of $\mathcal{M}$ such that the matrix of the form is the identity matrix.

Finally, suppose that $\mathcal{M}_1$ and $\mathcal{M}_2$ are finite dimensional modules over $R_{l,\alpha}$ with corresponding non-degenerate forms $\mathcal{H}_1, \mathcal{H}_2$. Choose bases for each module so that the matrix for each form is the identity matrix. If P is a appropriately sized matrix with $P^T\overline{P} = I$, then identifying each module with its coordinate vectors, we see that P is an isomorphism from $\mathcal{M}_1$ to $\mathcal{M}_2$ such that for $u, v \in \mathcal{M}_1$, we have $\mathcal{H}_1(v, w) = \mathcal{H}_2(Pu, Pv)$, so that the forms are equivalent. $\square$

All of this justifies our use of any convenient hermitian matrix for the form. The matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ is the most convenient choice, as it allows us to use triangular matrices.

4.C The Unitary Group

Let $\mathcal{H} : R_{l,\alpha} \times R_{l,\alpha} \to R_{l,\alpha}$ be the form given by $\mathcal{H}(u, v) = u^T \mathcal{B} \overline{v}$, where $\mathcal{B} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Let $(u, v) \in R_{l,\alpha} \times R_{l,\alpha}$, and by U_l denote the 2×2 unitary matrices over $R_{l,\alpha}$:

$$U_l = \{g \in \mathcal{M}_{2\times 2}(R_{l,\alpha}) \,|\, \mathcal{H}(gu, gv) = \mathcal{H}(u, v)\}$$

Remark 4.C.1 Using $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ as the matrix of the form $\mathcal{H}$, the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is unitary if and only if it satisfies the conditions:

1. $a\overline{d} + c\overline{b} = 1$
2. $a\overline{b} + \overline{a}b = 0$
3. $a\overline{c} + \overline{a}c = 0$
4. $d\overline{b} + \overline{d}b = 0$
5. $d\overline{c} + \overline{d}c = 0$

We find the order of U_l by using the Borel subgroup: $\left\{\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)\right\} \cap U_l$. This subgroup has order $q^{3l-2}(q-1)(q+1)$: there are $q^{l-1}(q-1)q^{l-1}(q+1)$ choices for a since it is a unit, and d is determined by a. From remark 4.C.1, $a\overline{b} + \overline{a}b = 0$ and since a is a unit, we can divide both sides by $a\overline{a}$ to find that $\frac{b}{a}$ is a pure root, so that $b = a(r\sqrt{\alpha})$ for $r \in R_l$ with q^l choices for r. The coset representatives are of two forms: (these are adapted from [Ca])

1. $\left(\begin{smallmatrix} \pi t\sqrt{\alpha} & 1 \\ 1 & 0 \end{smallmatrix}\right)$, with q^{l-1} choices for t.
2. $\left(\begin{smallmatrix} 1 & 0 \\ t\sqrt{\alpha} & 1 \end{smallmatrix}\right)$, with q^l choices for t.

There are $q^{l-1} + q^l$ coset representatives so that $|U_l| = q^{4l-3}(q-1)(q+1)^2$.

4.D Surjectivity

Below we will define abelian subgroups of U_l (denoted by K_m or K_{m+1} depending on the parity of l). Using Clifford theory to find irreducible characters of U_l requires starting from the irreducible characters of a subgroup, and finding the stabilizer subgroups (under the conjugation action by U_l) of these subgroup characters. We will see that for every character of K_m (or K_{m+1}), the inertia group can always be written K_mS (or $K_{m+1}S$) for some abelian subgroup S that depends on the particular character. The proof of this fact will use the surjectivity of various projection maps. These include maps from $R_{l,\alpha} \to R_{i,\alpha}$, and maps from a subgroup of U_l to the corresponding subgroup of U_i. For example, if k is a positive integer strictly less than l, we can consider the quotient of R_l by the ideal generated by π^k and identify this quotient with R_k. Elements in the quotient can be though of as:

$$t_{k-1}\pi^{k-1} + t_{k-2}\pi^{k-2} + \cdots + t_1\pi + t_0 \quad \text{for } 0 \leq i < k \;\; t_i \in \mathcal{T}$$

.

where the same set of fixed coset representatives $\mathcal{T}$ can be used. We will often refer to the modulus π^l or π^k in such cases, by analogy with the case of $\mathbb{Z}/p^l\mathbb{Z}$. In turn we will refer to U_k and its subgroups as "modulo" k if the elements in it are in $R_{k,\alpha}$.

Lemma 4.D.1 The projection map $P : R_l^* \to R_k^*$ given by

$$P(\sum_{j=0}^{l-1} t_j\pi^j) = \sum_{j=0}^{k-1} t_j\pi^j$$

is a ring homomorphism and is surjective.

Proof. Let $x = \sum_{i=0}^{k-1} t_i\pi^i \in R_k^*$. Let $y = \sum_{j=0}^{l-1} t_j\pi^j \in R_l^*$ such that $t_j = t_i$ for $0 \leq i, j \leq k-1$. Then $P(y) = x$. □

The map P extends to a map from $R_{l,\alpha} \to R_{k,\alpha}$ by setting $P(a + b\sqrt{\alpha}) = P(a) + P(b)\sqrt{P(\alpha)}$ for $a, b \in R_l$. By the previous argument, this map is also surjective when restricted to units. Furthermore, we can extend P to matrix groups over $R_{l,\alpha}$ by applying it to each element of the matrix. If necessary we will write P_k to indicate a modulo k map. We now demonstrate the surjectivity of maps between various subgroups of U_l that are nevertheless all denoted by S. They are centralizers of certain elements of $\mathcal{M}_{2\times 2}(R_{l,\alpha})$. We could give each S subgroup an identifying index, but this is unnecessary because the context will always provide clarity.

Proposition 4.D.1 The natural projection map P_k from the group $S = \left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)$ in U_l to the analogous group in U_k is surjective.

Proof. The two S groups above are isomorphic, respectively, to $R^*_{l,\alpha}$ and $R^*_{k,\alpha}$, so the claims holds. □

We will denote by $\mathcal{L}$, the norm 1 elements in $R_{l,\alpha}$; that is $z \in \mathcal{L}$ if and only if $z\overline{z} = 1$. We might write this as $\mathcal{L}_l$ if the modulus needs to be made explicit. Then the set of norm 1 elements in $R_{k,\alpha}$ will be written $\mathcal{L}_k$.

Lemma 4.D.2 For $k \leq l$, the projection map $P : \mathcal{L}_l \to \mathcal{L}_k$ is surjective.

Proof. Let $a + b\sqrt{\alpha}$ be a norm 1 element in $R_{k,\alpha}$, so that $a^2 - b^2\alpha = 1$ modulo π^k. We assume that we have a and b chosen such that for $d = a^2 - b^2\alpha$ we have $d_0 = 1$, and $d_i = 0$ for $i = 1, 2, \ldots, k-1$. We can solve for pairs (a_j, b_j) for $j = k, k+1, \ldots, l-1$ as in Proposition 4.A.1. . □

Proposition 4.D.2 Let σ be a square unit in R_l. The natural projection map P from $S = \left(\begin{smallmatrix} a & c\sigma \\ c & a \end{smallmatrix}\right)$ in U_l to the analogous subgroup in U_k is surjective.

Proof. The two S groups above are isomorphic, respectively, to $\mathcal{L}_l \times \mathcal{L}_l$ and $\mathcal{L}_k \times \mathcal{L}_k$: from the conditions in remark 4.C.1 $a\overline{c} + \overline{a}c = 0$ so $\pm(a\overline{c}k + \overline{a}ck) = 0$, and $a\overline{a} + c\overline{c}k^2 = 1$. Combining these last two equations, we get

$$a\overline{a} + \pm(a\overline{c}k + \overline{a}ck) + c\overline{c}k^2 = 1$$
$$(a \pm ck)\overline{(a \pm ck)} = 1$$

so that $a \pm ck \in \mathcal{L}$, and in fact $\begin{pmatrix} a & c\sigma \\ c & a \end{pmatrix}$ could be written as ordered pairs $(a + ck, a - ck)$ with pointwise multiplication. □

Proposition 4.D.3 For any $\beta \in R_l$, the natural projection map P from $S = \begin{pmatrix} a & \pi\beta c \\ c & a \end{pmatrix}$ in U_l to the analogous subgroup in U_k is surjective.

Proof. We cannot describe this subgroup in terms of units or norm 1 elements as in the previous two cases, therefore we merely count S modulo π^l and π^k as well as the size of the kernel of the projection map; note that the argument does not depend on the parity of l. Taking S modulo π^l first, we see the following:

1. Since the matrix is invertible, a is a unit.
2. Since $a\overline{c} + \overline{a}c = 0$, we can divide both sides by $a\overline{a}$ to get $c = ar\sqrt{\alpha}$ for $r \in R_l$.
3. Since $a\overline{a} + \pi\beta c\overline{c} = 1$, and $c = ar\sqrt{\alpha}$, we can re-arrange to get

$$a\overline{a} = (1 - \pi\beta r^2\alpha)^{-1}$$

Therefore we can choose r freely from R_l, then choose a from the pre-image of $(1 - \pi\beta r^2\alpha)^{-1}$ in the norm map. This pre-image has the same size as the subgroup of norm 1 elements. Thus $|S|$ modulo π^l is $q^l q^{l-1}(q + 1)$, and $|S|$

modulo π^m is $q^m q^{m-1}(q+1)$.

The kernel of the projection map from S modulo π^l to S modulo π^m has the form:

$$\left(\begin{smallmatrix} 1+\pi^m a & \pi^{m+1}\beta c \\ \pi^m c & 1+\pi^m a \end{smallmatrix}\right),\ a, c \in R_{l,\alpha}$$

Since the product of the elements on the second diagonal is zero, then $1+\pi^m a$ must be a norm 1 element, therefore the number of choices for this element equals the size of the kernel of the projection map from norm 1 elements modulo π^l to the norm 1 elements modulo π^m. We have seen that this map is surjective, therefore the size of the kernel is:

$$\frac{q^{l-1}(q+1)}{q^{m-1}(q+1)} = q^{l-m}$$

Since $\pi^m c$ can be written $(1+\pi^m a)r\sqrt{\alpha}$, then π^m divides r therefore there are q^{l-m} choices for $\pi^m c$. We conclude that the kernel has size:

$$q^{l-m}q^{l-m}$$

Thus the index of the kernel in S modulo π^l equals the order of S modulo π^m, so the projection map is surjective.

□

Consider the map $P_i : U_l \to U_i$ given by sending each element of a matrix over U_l to its value modulo π; we would like to show that this map too is surjective, but we must first introduce the kernel of this map - the K_i subgroups. This is the subject of the next section.

4.E The K Subgroups: Characters and Inertia Groups

For a positive integer $i \leq l-1$, we define the K subgroups of U_l:

Definition 4.E.1

$$K_i = \{I + \pi^i B\} \cap U_l = \left\{ \begin{pmatrix} 1+\pi^i a & \pi^i b \\ \pi^i c & 1+\pi^i d \end{pmatrix} \right\} \cap U_l, \ a, b, c, d \in R_{l,\alpha}$$

It is clear that if $i \geq \frac{l}{2}$, K_i is abelian. We begin the Clifford method by defining irreducible (necessarily linear) characters of the largest abelian K group for U_l; this will be K_m when $l = 2m$, and K_{m+1} when $l = 2m+1$.

Proposition 4.E.1 For either parity of l, the order of K_i is $q^{4(l-i)}$.

Proof. For any unitary matrix $\begin{pmatrix} x & y \\ z & w \end{pmatrix}$ where x is a unit, remark 4.C.1 implies $x\overline{y} + \overline{x}y = 0$, and dividing both sides by $x\overline{x}$ gives $\frac{y}{x} + \overline{\left(\frac{y}{x}\right)} = 0$ so that $y = x(r\sqrt{\alpha}), r \in R_l$ and similarly $z = x(s\sqrt{\alpha})$, $s \in R_l$. For the K_i subgroups, this means that $\pi^i b = (1+\pi^i a)(r\sqrt{\alpha})$, and since $(1+\pi^i a)$ is a unit, π^i divides r, so that there are q^{l-i} choices for r, thus q^{l-i} choices for $\pi^i b$; likewise for $\pi^i c$. The element $1+\pi^i a$ is in the kernel of the natural projection map from $R^*_{l,\alpha}$ to $R^*_{i,\alpha}$. This is a surjective map, so the size of the kernel (and hence the number of choices for $1+\pi a$) is $q^{2(l-1)}$. Finally, $1+\pi^i d$ is determined by the unitary constraint $x\overline{w} + z\overline{y} = 1$, so the order of K_i is $q^{2(l-i)}q^{l-i}q^{l-i} = q^{4(l-i)}$.

□

If $i \geq l/2$ so that K_i is abelian, we can more precisely describe elements of the group.

Proposition 4.E.2 If $K_i = \left\{ \left(\begin{smallmatrix} 1+\pi^i a & \pi^i b \\ \pi^i c & 1+\pi^i d \end{smallmatrix} \right) \right\} \cap U_l$ with $i \geq l/2$ then

$$K_i = \left\{ \left(\begin{smallmatrix} 1+\pi^i a_1+\pi^i a_2\sqrt{\alpha} & \pi^i b\sqrt{\alpha} \\ \pi^i c\sqrt{\alpha} & 1-\pi^i a_1+\pi^i a_2\sqrt{\alpha} \end{smallmatrix} \right) a_1, a_2, b, c \in R_l \right\}$$

Proof. Since $I + \pi^i B$ is unitary, by remark 4.C.1:

1. since $1 + \pi^i a_1 + \pi^i a_2\sqrt{\alpha}$ is a unit, then $\pi^i b$ in the statement of the proposition must be a pure root with a factor of π^i, hence can be written $\pi^i b'\sqrt{\alpha}$, for $b' \in R_l$.

2. By remark 4.C.1, $(1 + \pi^i a)\overline{(1 + \pi^i d)} + (\pi^i c)\overline{(\pi^i b)} = 1 + \pi^i(a + \overline{d}) = 1$. Thus π^{l-i} divides $a + \overline{d}$, $\overline{d} = -a + \pi^{l-i}T$, so $\pi^i d = \pi^i(-\overline{a})$, and d can be assumed to be $-\overline{a}$.

□

We end this section with the following proposition:

Proposition 4.E.3 The modulo π^i map from U_l to U_i is surjective.

Proof. Since the kernel of the map is K_i, thus it suffices to show that $[U_l : K_i] = |U_i|$. The respective group orders are $q^{4l-3}(q-1)(q+1)^2$ and $q^{4i-3}(q-1)(q+1)^2$. The ratio of these is $\frac{q^{4l-3}(q-1)(q+1)^2}{q^{4i-3}(q-1)(q+1)^2} = q^{4(l-i)}$ which is $|K_i|$.

□

4.E.1 Characters on K_m and K_{m+1}

Following the method in [BL] for characters of the invertible matrices over $\mathbb{Z}/p^l\mathbb{Z}$, we will define a character on an abelian K group, starting with λ, a primitive character on the additive group of R_l. By *primitive* is meant that the kernel of λ contains no non-trivial ideal of R_l. There are two immediate consequences of this:

1. if, for some $x \in R_l$, and for all $r \in R_l$ $\lambda(xr) = 1$, then $x = 0$;

2. All of the characters of R_l^+ can be generated by λ over R_l by defining, for $r \in R_l$, $r\lambda(x) = \lambda(rx)$ - in this way the set $\{r\lambda, r \in R_l\}$ is IrrR_l^+.

It is not immediate that a primitive character exists on R_l, hence:

Proposition 4.E.4 There exists a character $\lambda : R_l^+ \to \mathbb{C}^\times$ such that the kernel of λ contains no ideal of R_l other than (0).

Proof. Any non-trivial ideal contained in the kernel of λ contains the minimal ideal. Thus λ may be considered to be the lift of a character of the quotient ring $R_l/\pi^{l-1}R_l$. But the number of characters of the quotient is strictly less than the number of characters of R_l^+, which must therefore have a character containing only the ideal (0).

□

The additive group of $R_{l,\alpha}$, is a direct sum of two copies of R_l^+, and any character γ on $R_{l,\alpha}$ can be expressed as $\gamma(a+b\sqrt{\alpha}) = \gamma_1(a)\gamma_2(b)$ where each γ_i is a character on R^+. There are many ways to extend λ to $R_{l,\alpha}$, but we want this extended character to be primitive, and the simplest choice is to use

$$\lambda(a + b\sqrt{\alpha}) = \lambda(a)\lambda(b) = \lambda(a + b)$$

Now we define $\phi_A \in \text{Irr}(K_m)$ by:

Definition 4.E.2 Let $A \in M_{2\times 2}(R_{l,\alpha})$; define $\phi_A \in \text{Irr}(K_m)$ or $\text{Irr}(K_{m+1})$ respectively, by:

$$\phi_A[I + \pi^m B] = \lambda[\text{tr}(\pi^m AB)]$$

$$\phi_A[I + \pi^{m+1}B] = \lambda[\mathrm{tr}(\pi^{m+1}AB)]$$

This sort of character is given in [BL] (for the general linear group) without much comment. It will be worthwhile here to demonstrate its reasonableness, that is, to show that it is a natural way to define characters on K_m and K_{m+1}. In what follows, we shall use K_m as our example but the argument does not depend on this. From Proposition 4.E.2, an element of K_m has the form

$$\begin{pmatrix} 1+\pi^m a_1+\pi^m a_2\sqrt{\alpha} & \pi^m b\sqrt{\alpha} \\ \pi^m c\sqrt{\alpha} & 1-\pi^m a_1+\pi^m a_2\sqrt{\alpha} \end{pmatrix} \; a_1, a_2, b, c \in R_l$$

and since $(I + \pi^m B)(I + \pi^m C) = I + \pi^m(B + C)$, K_m is isomorphic to the additive group whose elements are $M_{2\times 2}(R_l)$ (though it is only the modulo π^m value of each matrix entry that matters). The number of irreducible characters of K_m equals the order of the additive group of $M_{2\times 2}(R_l)$ modulo π^m. Moreover, it is clear that distinct A matrices over R_l modulo π^m give distinct ϕ_A characters on K_m, so while the A matrices of definition 4.E.2, can be over $R_{l,\alpha}$, we can account for all ϕ_A characters of K_m using only matrices over R_l. Any character on the additive group $\{\begin{pmatrix} \pi^m a_1 & \pi^m b \\ \pi^m c & \pi^m a_2 \end{pmatrix}\}$ $a_1, a_2, b, c \in R_l$ can be written as the product of characters on the elements $\pi^m a_1, \pi^m a_2, \pi^m b, \pi^m c \in R_l$. By using the following matrices:

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, A_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, A_4 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

For $g = \begin{pmatrix} \pi^m a_1 & \pi^m b \\ \pi^m c & \pi^m a_2 \end{pmatrix}$, $\lambda[\mathrm{tr}(A_i g)]$ is a character that applies λ to one of the entries of g. Thus in [BL], precisely these A matrices were used to form the ϕ_A characters of K_m. In the unitary case, an element of K_m has the form $I+\pi^m B = I+\pi^m \begin{pmatrix} a_1+a_2\sqrt{\alpha} & b\sqrt{\alpha} \\ c\sqrt{\alpha} & -a_1+a_2\sqrt{\alpha} \end{pmatrix}$, thus $\mathrm{tr}(AB)$ will pick out one of $\{\pi^m a_1, \pi^m a_2, \pi^m b, \pi^m c\}$ if A is one of the following:

$$A_1 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, A_2 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, A_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, A_4 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Hence the reasonableness of definition 4.E.2, since $\lambda[\mathrm{tr}(\pi^m A_i B)]$ is merely applying λ to an entry of B (multiplied by π^m).

The characters ϕ_A are permuted by U_l by conjugation; for $g \in U_l$,

$$\phi_A^g(I+\pi^m B) = \phi_A[g(I+\pi^m B)g^{-1}]$$

Conjugate characters on the K group lead to the same irreducible character $\chi \in \mathrm{Irr}U_l$, thus we are only concerned with non-conjugate characters on the K groups. The following important fact concerning conjugate ϕ_A characters comes from [BL] p 1292.

Proposition 4.E.5 The irreducible character ϕ_A on an abelian K group is conjugate to $\phi_{A'}$ if and only if A, A' are conjugate matrices:

$$\begin{aligned} (\phi_A)^g(I+\pi^m B) &= \phi_A(I+\pi^m gBg^{-1}) && (4.1) \\ &= \lambda(\mathrm{tr}(\pi^m AgBg^{-1})) && (4.2) \\ &= \lambda(\mathrm{tr}(\pi^m g^{-1}AgB)) && (4.3) \\ &= \phi_{A^{g^{-1}}}(I+\pi^m B) && (4.4) \end{aligned}$$

The same proof works for $l = 2m+1$ with ϕ_A on K_{m+1}.

4.F Selecting and Generalizing the A Matrices

We wish to use as few A matrices as possible to form the ϕ_A characters on K_m. One might think that the best candidates are the matrices that form the ϕ_A generators of the irreducible characters of the K group:

$$A_1 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, A_2 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, A_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, A_4 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

but this is not the case because, for example, A_3 and A_4 are conjugate. The following will be of some help. It assumes that $l = 2m$, but a similar argument would work for $l = 2m + 1$.

Proposition 4.F.1 For $l = 2m$, any character ϕ_A on K_m is conjugate (by U_l) to a character ϕ_B, where B is over R_l, and has one of the following forms:

1. $xI + \pi C$
2. $xI + \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, such that $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ is not a multiple of π.
3. $xI + \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$ where $\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$ is not a multiple of π, and is not diagonalizable.

Proof. We will count the characters for $B = xI + \pi c$, and then set them aside because they lead to characters of U_l that come from U_{l-1}. These ϕ_B characters are all distinct and number q^{4m-3}: to see this, we write

$$xI + \pi C = \begin{pmatrix} x+a & b \\ c & x+d \end{pmatrix}$$

Because of the π^m in the definition of ϕ_B, there are q^{m-1} choices for b, c and q^m choices for $x + a$ and finally, q^{m-1} choices for $x + d$, since $x + a$ has been selected. This gives q^{4m-3} choices in all.

For B matrices of the second type, we will see below in section 4.H.1, that the stabilizer T, of ϕ_B under the conjugation action of U_l has order $q^{3l-2}(q-1)(q+1)$, so the orbit size of such a character is $[U_l : T] = q^{l-1}(q+1)$. There are $q^{l-1}(q-1)$ of these B matrices, which are conjugate in pairs: $xI + \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$ is conjugate to $xI + \left(\begin{smallmatrix} b & 0 \\ 0 & a \end{smallmatrix}\right)$. Therefore the number of non-conjugate characters of this type is $\frac{q^{l-1}(q-1)}{2}$. Multiplying by the orbit size of each character gives $\frac{q^{2l-2}(q-1)(q+1)}{2}$ distinct characters on K_m.

Before counting the contribution from the third type, we examine more closely the matrix $\left(\begin{smallmatrix} 0 & a \\ b & 0 \end{smallmatrix}\right)$ when one of a, b is a unit, and the matrix is not diagonalizable. Conjugating by $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ if necessary, we assume b is a unit and and write $\left(\begin{smallmatrix} 0 & a \\ b & 0 \end{smallmatrix}\right) = b\left(\begin{smallmatrix} 0 & T \\ 1 & 0 \end{smallmatrix}\right)$ where $T = a(b)^{-1}$. For any $\left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right) \in U_m$ we have:

$$\left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right) b \left(\begin{smallmatrix} 0 & T \\ 1 & 0 \end{smallmatrix}\right) \left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right)^{-1} = b\left(\begin{smallmatrix} y & Tx \\ w & Tz \end{smallmatrix}\right)\left(\begin{smallmatrix} w & -y \\ -z & x \end{smallmatrix}\right)\frac{1}{\det}$$
$$= b\left(\begin{smallmatrix} .... & x^2T-y^2 \\ w^2-z^2T & .... \end{smallmatrix}\right)\frac{1}{\det}$$

Suppose that the result of conjugation is a diagonal matrix, so that

$$x^2T - y^2 = w^2 - z^2T = 0$$

Definition 4.F.1 In our 2×2 matrices we will denote by *neighbours*, any two elements horizontally or vertically adjacent. So that for example, in $\left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right)$ the pairs x, y and x, z are neighbours, but x and w are not.

We consider an exhaustive list of possibilites for $T \in R_l$: a square unit , a non-square unit, or a non-unit. In a unitary matrix over $R_{l,\alpha}$, the ratio of squares of neighbours (where defined) must always be a non-square in $R_{l,\alpha}$.

For example if x is a unit, by Proposition 4.E.1 $y = x(r\sqrt{\alpha})$, and $(y/x)^2 = r^2\alpha$ which is a non-square since α was chosen to be a non-square. Therefore if T above is a square the matrix cannot be diagonalized. If T is a non-square unit in R_m, it can be written $r^2\alpha$ for r a unit in R_m; then if s is a unit in R_m and $x\overline{x} = (-2rs\alpha),^{-1}$ we choose the unitary matrix:

$$\begin{pmatrix} x & (x)r\sqrt{\alpha} \\ (x)s\sqrt{\alpha} & x(-rs\alpha) \end{pmatrix}$$

which, by an unpleasant calculation, diagonalizes $\begin{pmatrix} 0 & T \\ 1 & 0 \end{pmatrix}$.

If T is a non unit, and the matrix is diagonalized, then $x^2T - y^2 = w^2 - z^2T = 0$ so both y and w would be non-units which is impossible in an invertible matrix. Therefore the matrix is not diagonalizable if and only if T is a non-unit or a square unit. Hence we use $\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$ where $a(b)^{-1}$ is a square unit in R_l or a non-unit in R_l.

For the case $xI + b\begin{pmatrix} 0 & \sigma \\ 1 & 0 \end{pmatrix}$ where σ is a square unit in R_l, the b is superfluous: since the norm map is onto the units of R_l, we can find $y \in R_{l,\alpha}$ with $y\overline{y} = b$. As a result, conjugating $xI + b\begin{pmatrix} 0 & \sigma \\ 1 & 0 \end{pmatrix}$ by $\begin{pmatrix} y & 0 \\ 0 & (\overline{y})^{-1} \end{pmatrix}$ gives $xI + \begin{pmatrix} 0 & \sigma' \\ 1 & 0 \end{pmatrix}$ where σ' is a square unit; thus merely varying σ over all squares, we get all characters of this type. They are all non-conjugate, and there are $q^m \frac{q^{m-1}(m-1)}{2}$ of them. The stabilizer of these elements are elements of U_m having the form $\begin{pmatrix} x & y\sigma \\ y & x \end{pmatrix}$. This subgroup is isomorphic to two copies of the norm 1 elements of $R_{m,\alpha}$ and thus has order $q^{m-1}(q+1)q^{m-1}(q+1)$, hence the orbit size is $\frac{q^{4m-3}(q-1)(q+1)^2}{q^{2m-2}(q+1)^2} = q^{2m-1}(q-1)$. As a result in this type we account for $\frac{q^{4m-2}(q-1)^2}{2}$ characters.

The final type is $xI + b\begin{pmatrix} 0 & \pi\beta \\ 1 & 0 \end{pmatrix}$ where, as before, the b is superfluous. We have qq^{m-1} non- conjugate matrices and the stabilizer is the subgroup of matrices of the form $\begin{pmatrix} x & \pi\beta y \\ y & x \end{pmatrix}$ having order $q^{2m-1}(q+1)$ resulting in an orbit size of $q^{2m-2}(q-1)(q+1)$, which gives us $q^{4m-3}(q-1)(q+1)$ characters. Let us list the number of characters from all cases:

1. $xI + \pi C$: q^{4m-3}
2. $xI + \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$: $\frac{q^{4m-2}(q-1)(q+1)}{2}$
3. $xI + b\begin{pmatrix} 0 & \sigma \\ 1 & 0 \end{pmatrix}$: $\frac{q^{4m-2}(q-1)(q-1)}{2}$
4. $xI + b\begin{pmatrix} 0 & \pi\beta \\ 1 & 0 \end{pmatrix}$: $q^{4m-3}(q-1)(q+1)$

Taking the sum of the second and third cases gives

$$\frac{q^{4m-2}(q-1)(q+1)}{2} + \frac{q^{4m-2}(q-1)(q-1)}{2} = \frac{q^{4m-2}(q-1)}{2}[q-1+q+1] = q^{4m-1}(q-1)$$

adding this to the fourth case gives

$$q^{4m-1}(q-1) + q^{4m-3}(q-1)(q+1) = q^{4m-3}(q-1)[q^2+q+1] = q^{4m-3}(q^3-1)$$

and adding this to the first case gives us q^{4m} which is the number of characters of K_m.

□

The matrices of the first type leads to characters of U_l that come from U_{l-1}; that is they are lifts of characters on U_l/K_{l-1} which is isomorphic to U_{l-1} and are thus supposed to be known by the inductive hypothesis. Their contribution to the sum of squares of degrees is found separately. Of the remaining three types, the scalar part does not affect the degrees or inertia groups of the characters that we will find, thus to simplify computations we will use the following A forms: (we well return later to the more general forms)

1. $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$
2. $\begin{pmatrix} 0 & \sigma \\ 1 & 0 \end{pmatrix}$ where σ is a square unit in R_l.
3. $\begin{pmatrix} 0 & \pi\beta \\ 1 & 0 \end{pmatrix}$ where $\beta \in R_{l-1}$.

4.G The Inertia Groups

Applying the method of Clifford theory to U_l requires that we find, for each character ϕ_A, the inertia group T; that is, all $g \in U_l$ such that

$$\phi_A[g(I + \pi^m B)g^{-1}] = \phi_A(I + \pi^m B)$$

The inertia group contains the abelian group K_m or K_{m+1} (depending on the parity of l), and the centralizer in U_l of the A matrix:

$$\begin{aligned} \phi_A[g(I + \pi^m B)g^{-1}] &= \phi_A[I + \pi^m gBg^{-1}] \\ &= \lambda[\mathrm{tr}(\pi^m A(gBg^{-1})] \\ &= \lambda[\mathrm{tr}(\pi^m (g^{-1}Ag)B] \end{aligned}$$

If we denote the centralizer of A by S, then $K_m S \leq T$ (for l even) or $K_{m+1}S \leq T$ (for l odd). Presently we will find an upper bound for T; to do so, it will be necessary to consider the parities of l separately, but first we need the following.

Lemma 4.G.1 If T is the inertia group of ϕ_A, and $g \in U_l$, then $\overline{g} \in T$. This does not depend on the parity of l.

Proof. For any of our three A matrices, let $g \in T$ so that (for ease of reading, we assume that π^m or π^{m+1} has been multiplied into the B matrix)

$$\lambda[\mathrm{tr}(AgBg^{-1})] = \lambda[\mathrm{tr}(AB)] \tag{4.5}$$

Note that for $A = \left(\begin{smallmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} a_1+a_2\sqrt{\alpha} & b\sqrt{\alpha} \\ c\sqrt{\alpha} & -a_1+a_2\sqrt{\alpha} \end{smallmatrix}\right)$ $\mathrm{tr}(A\overline{B}) = \mathrm{tr}(AB)$ and when $A = \left(\begin{smallmatrix} 0 & \sigma \\ 1 & 0 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 0 & \pi\beta \\ 1 & 0 \end{smallmatrix}\right)$ $\mathrm{tr}(A\overline{B}) = -\mathrm{tr}(AB)$.

Supposing that $g \in T$ where $A = \left(\begin{smallmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{smallmatrix}\right)$,

$$\begin{aligned}
\lambda[\mathrm{tr}(A(\overline{g}B\overline{g}^{-1})] &= \lambda[\mathrm{tr}(Ag\overline{B}g^{-1})] \\
&= \lambda[\mathrm{tr}(A\overline{B})] \\
&= \lambda[\mathrm{tr}(AB)]
\end{aligned}$$

Next, suppose $g \in T$ and $A = \left(\begin{smallmatrix} 0 & \sigma \\ 1 & 0 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 0 & \pi\beta \\ 1 & 0 \end{smallmatrix}\right)$

$$\begin{aligned}
\lambda[\mathrm{tr}(A(\overline{g}B\overline{g}^{-1})] &= \lambda[-\mathrm{tr}(Ag\overline{B}g^{-1})] \\
&= \lambda[-\mathrm{tr}(A\overline{B})] \\
&= \lambda[\mathrm{tr}(AB)]
\end{aligned}$$

therefore $\overline{g} \in T$ for all ϕ_A characters.

□

Finding the Inertia Groups

1. Let $l = 2m$, so that ϕ_A is defined on K_m. The centralizer of each the three A matrix types are precisely the S groups (one for each A matrix) mentioned in section 4.C. We demonstrate this below.

   (a)

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix}\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} x & y \\ z & w \end{pmatrix}$$
$$\begin{pmatrix} x & -y \\ z & -w \end{pmatrix} = \begin{pmatrix} x & y \\ -z & -w \end{pmatrix}$$

   implies that $y = z = 0$, so for $A = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$, the S group consists of the diagonal matrices, and so is isomorphic to $R^*_{l,\alpha}$.

   (b)

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix}\begin{pmatrix} 0 & \sigma \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x & y \\ z & w \end{pmatrix}$$
$$\begin{pmatrix} y & x\sigma \\ w & z\sigma \end{pmatrix} = \begin{pmatrix} z\sigma & w\sigma \\ x & y \end{pmatrix}$$

   implies that $y = z\sigma$ and $w = x$, so for $A = \begin{pmatrix} 0 & \sigma \\ 1 & 0 \end{pmatrix}$, the S group consists of all matrices of the form $\begin{pmatrix} x & z\sigma \\ z & x \end{pmatrix}$, and so is isomorphic to $\mathcal{L} \times \mathcal{L}$

   (c)

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix}\begin{pmatrix} 0 & \pi\beta \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \pi\beta \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x & y \\ z & w \end{pmatrix}$$
$$\begin{pmatrix} y & x\pi\beta \\ w & z\pi\beta \end{pmatrix} = \begin{pmatrix} z\pi\beta & w\pi\beta \\ x & y \end{pmatrix}$$

   implies that $w = x$ and $y = z\pi\beta$, so for $A = \begin{pmatrix} 0 & \pi\beta \\ 1 & 0 \end{pmatrix}$, the S group is all matrices of the form $\begin{pmatrix} x & \pi\beta z \\ z & x \end{pmatrix}$

We will show next, that when l is even, the upper bound for T is K_mS. Since $K_mS \leq T$ by construction, this will imply that $T = K_mS$.. In the following proof, we assume that A is a matrix over R_l so that $\overline{A} = A$.

Proposition 4.G.1 If $l = 2m$ and T is the inertia group of ϕ_A, then $T \leq K_mS$, where S is the centralizer in U_l of A.

Proof. Let $g \in T$ and $I + \pi^m B \in K_m$. Then

$$\lambda[\mathrm{tr}(\pi^m(AgBg^{-1}))] = \lambda[\mathrm{tr}(\pi^m(AB))]$$

and $\lambda[\mathrm{tr}(\pi^m(g^{-1}Ag - A)B)] = 1$; let X denote $\pi^m(g^{-1}Ag - A)$ so $\lambda[\mathrm{tr}(XB)] = 1$ where B is any matrix of the form

$$\begin{pmatrix} a_1 + a_2\sqrt{\alpha} & b\sqrt{\alpha} \\ c\sqrt{\alpha} & -a_1 + a_2\sqrt{\alpha} \end{pmatrix} \; a_1, a_2, b, c \in R_l$$

We can show that $X = 0$, and this will give us an upper bound for T.

Lemma 4.G.2 $X = \pi^m(g^{-1}Ag - A) = 0$

Proof. Since X has trace zero, we can write

$X = \begin{pmatrix} x_1 + x_2\sqrt{\alpha} & w \\ z & -x_1 - x_2\sqrt{\alpha} \end{pmatrix}$ where $x_1, x_2 \in R_l$, and $w, z \in R_{l,\alpha}$.

If, for any $r \in R_l$, we let $B = \begin{pmatrix} r/2 & 0 \\ 0 & -r/2 \end{pmatrix}$ then

$$1 = \lambda[\mathrm{tr}(XB)] = \lambda((x_1 + x_2)r)$$

Since since λ extended to $R_{l,\alpha}$ is primitive, $x_1 + x_2 = 0$. We can replace X with $\overline{X}$ because $\overline{g} \in T$, and $\overline{A} = A$. Now the preceding argument gives us $x_1 - x_2 = 0$, so that $x_1 = x_2 = 0$.

Next write $w = w_1 + w_2\sqrt{\alpha}$, and let $B = \begin{pmatrix} 0 & 0 \\ r\sqrt{\alpha} & 0 \end{pmatrix}$, so that $1 = \lambda(\mathrm{tr}XB)) = \lambda[((w_1 + w_2\sqrt{\alpha})r\sqrt{\alpha})] = \lambda[(w_2\alpha + w_1)r] = 1$ for all $r \in R_l$, implying that $w_2\alpha + w_1 = 0$. If we replace X by $\overline{X}$ we get $-w_2\alpha + w_1 = 0$ so that $w_1 = w_2 = 0$. A similar argument shows that $z = 0$.

□

Since $X = \pi^m(g^{-1}Ag - A) = 0$ then for any $g \in T$

$$\pi^m Ag = \pi^m gA \tag{4.6}$$

We assume that $\begin{pmatrix} x & y \\ z & w \end{pmatrix} \in T$, and consider the three A matrices separately in order to establish the upper bound for T:

(a) When $A = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$, from equation 4.6 we get:

$$\pi^m \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \pi^m \frac{1}{2}\begin{pmatrix} x & y \\ z & w \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

so

$$\pi^m \begin{pmatrix} x & y \\ -z & -w \end{pmatrix} = \pi^m \begin{pmatrix} x & -y \\ z & -w \end{pmatrix}$$

hence $\pi^m y = \pi^m z = 0$ which means y, z have a factor of π^m, so g can be written $\begin{pmatrix} x & \pi^m y \\ \pi^m z & w \end{pmatrix}$. Under the map $P_m|_T$: $U_l \to U_m$ that takes each entry of the matrix in U_l to its value modulo π^m,the image of the map is the subgroup $\begin{pmatrix} x & 0 \\ 0 & w \end{pmatrix}$ of U_m, and the kernel is K_m. We know this map is surjective because its domain contains the diagonal matrices in U_l, and we have seen that the diagonal matrices are isomorphic to the units (with the appropriate modulus), and the projection map from $R^*_{l,\alpha}$ to $R^*_{m,\alpha}$ is surjective. Thus every element in the domain can be written ks with $k \in K_m$ and $s \in S$, so that $T \leq K_m S$.

(b) When $A = \left(\begin{smallmatrix} 0 & \sigma \\ 1 & 0 \end{smallmatrix}\right)$, by equation 4.6 we get

$$\pi^m \left(\begin{smallmatrix} 0 & \sigma \\ 1 & 0 \end{smallmatrix}\right) \left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right) = \pi^m \left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right) \left(\begin{smallmatrix} 0 & \sigma \\ 1 & 0 \end{smallmatrix}\right)$$

so

$$\pi^m \left(\begin{smallmatrix} \sigma z & \sigma w \\ x & y \end{smallmatrix}\right) = \pi^m \left(\begin{smallmatrix} y & \sigma x \\ w & \sigma z \end{smallmatrix}\right)$$

and $w = x + \pi^m E$, $y = z\sigma + \pi^m D$ where E, D are elements in $R_{l,\alpha}$, so we can write g as $\left(\begin{smallmatrix} x & z\sigma+\pi^m D \\ z & x+pi^m T \end{smallmatrix}\right)$. Using the same map as before, and restricting to the matrices of the inertia group, we see that the image is the S group of matrices $\left(\begin{smallmatrix} x & z\sigma \\ z & x \end{smallmatrix}\right)$ modulo π^m. The kernel is again K_m, and the map is surjective because of the surjectivity of the projection map between the S groups of appropriate modulus. Thus again any g in the inertia group is in $K_m S$.

(c) Let $A = \left(\begin{smallmatrix} 0 & \pi\beta \\ 1 & 0 \end{smallmatrix}\right)$, so by the argument for the previous matrix $w = x + \pi^m E$ and $y = z\pi\beta + \pi^m D$, and we can write $g = \left(\begin{smallmatrix} x & z\pi\beta+\pi^m D \\ z & x+\pi^m E \end{smallmatrix}\right)$. The same argument applies here because of the surjectivity of the projection map between appropriate S groups. Thus $T \leq K_m S$.

□

2. If $l = 2m + 1$, K_m is not abelian so we define ϕ_A on the largest abelian

K group which is K_{m+1}. By construction $T \geq K_{m+1}S$, but in fact $T \geq K_m S$ since K_m centralizes K_{m+1}:

$$(I + \pi^m B)(I + \pi^{m+1}C) = I + \pi^m B + \pi^{m+1}C = (I + \pi^{m+1}C)(I + \pi^m B)$$

To get an upper bound for T, we start with ϕ_A and $g \in T$ so for any $(I + \pi^{m+1}B) \in K_{m+1}$

$$\lambda[\mathrm{tr}(\pi^{m+1}AgBg^{-1})] - \lambda[\mathrm{tr}(\pi^{m+1}AB)] = 1$$
$$\lambda[\mathrm{tr}(\pi^{m+1}(g^{-1}Ag - A)B)] = 1$$

If we let X denote $\pi^{m+1}(g^{-1}Ag - A)$ then

Lemma 4.G.3 Given any A matrix (over R_l), and $g \in T$,

$X = \pi^m(g^{-1}Ag - A) = 0$

Proof. Again the trace of X is zero, and

$X = \begin{pmatrix} x_1+x_2\sqrt{\alpha} & w \\ z & -x_1-x_2\sqrt{\alpha} \end{pmatrix}$ where $x_1, x_2 \in R_l$, and $w, z \in R_{l,\alpha}$.

The entire argument from Proposition 4.G.2 carries through here to give us following.

□

$$\pi^{m+1}Ag = \pi^{m+1}gA \tag{4.7}$$

Applying this condition to each A matrix shows that K_mS again contains the inertia group, thus for l odd and even, the inertia group is K_mS for the appropriate choice of S for each matrix.

4.H Finding the Character Degrees

For each ϕ_A character we will need to find the orders of subgroups having the form BC where B and C are themselves subgroups. We can always calculate this by

$$|BC|= \frac{|B||C|}{|B|\cap|C|}$$

but for K_mS (for either parity of l), we will sometimes use

$$|K_mS|= |K_m||S|_{\text{modulo } \pi^m}$$

which stems from the natural surjective projection map from U_l to U_m, as this is sometimes more convenient.

4.H.1 The Even Case

For each of the three A matrices

1. $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$
2. $\begin{pmatrix} 0 & \sigma \\ 1 & 0 \end{pmatrix}$ where σ is a square unit in R_l.
3. $\begin{pmatrix} 0 & \pi\beta \\ 1 & 0 \end{pmatrix}$ where $\beta \in R_{l-1}$.

we extend ϕ_A on K_m to ψ on K_mS by from Proposition 2.C.1. Then by Clifford theory, $\chi = \mathrm{Ind}_T^{U_l}\psi$ is an irreducible character of U_l of degree $[U_l : T]$. The schematic for each ϕ_A character is:

$$\underset{\phi_A}{K_m} \xrightarrow{\text{ext}} \underset{\psi}{K_mS} \xrightarrow{\text{ind}} \underset{\chi}{U_l}$$

1. $A = \left(\begin{smallmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{smallmatrix}\right)$

$$T = \left\{\begin{pmatrix} a & \pi^m b \\ \pi^m c & d \end{pmatrix}\right\} \cap U_l = K_mS,\ S = \left\{\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\right\} \cap U_l$$

   The group S is isomorphic to the units in $R_{l,\alpha}$. The order of the units modulo π^m is $q^{m-1}(q-1)q^{m-1}(q+1)$, so $|T| = |K_m||S|_{\text{modulo } \pi^m} = q^{4m}q^{m-1}(q-1)q^{m-1}(q+1) = q^{3l-2}(q-1)(q+1)$. Thus, $\chi = \mathrm{Ind}_T^{U_l}\psi$ is irreducible with degree $[U_l : T] = q^{l-1}(q+1)$

2. $A = \left(\begin{smallmatrix} 0 & \sigma \\ 1 & 0 \end{smallmatrix}\right)$, where $\sigma = k^2$, $k \in R_l^*$.

$$T = \left\{\begin{pmatrix} a & b\sigma+\pi^m c \\ b & a+\pi^m d \end{pmatrix}\right\} \cap U_l, = K_mS,\ \ S = \left\{\begin{pmatrix} a & b\sigma \\ b & a \end{pmatrix}\right\} \cap U_l$$

   From Proposition 4.D.2, S is isomorphic to two copies of $\mathcal{L}$, and $|S|$ modulo π^m is

$$q^{m-1}(q+1)q^{m-1}(q+1) = q^{l-2}(q+1)^2$$

   Since $|T| = |K_mS| = |K_m|(q^{l-2}(q+1)^2) = q^{3l-2}(q+1)^2$, $\chi = \mathrm{Ind}_T^{U_l}\psi$ has degree $[U_l : T] = q^{l-1}(q-1)$.

3. $A = \begin{pmatrix} 0 & \pi\beta \\ 1 & 0 \end{pmatrix}$, $\beta \in R_{l-1}$

$$T = \left\{ \left(\begin{smallmatrix} a & b\pi\beta+\pi^m c \\ b & a+\pi^m d \end{smallmatrix} \right) \right\} \cap U_l, = K_m S, \;\; S = \left\{ \left(\begin{smallmatrix} a & b\pi\beta, \\ b & a \end{smallmatrix} \right) \right\} \cap U_l$$

We can find $|S|$ by using the proof for Proposition 4.D.3, so the order of S modulo π^m is $q^{m-1}(q+1)q^m = q^{l-1}(q+1)$, and $|T| = q^{2l}q^{l-1}(q+1) = q^{3l-1}(q+1)$, and $\chi = \mathrm{Ind}_T^{U_l}\psi$ has degree $[U_l : T] = q^{l-2}(q-1)(q+1)$.

4.H.2 The Odd Case

When $l = 2m+1$, we define ϕ_A on K_{m+1}, the largest abelian K subgroup. In section 4.G we found the inertia group to be K_mS for S groups of the same form as in the even case. By Proposition 2.C.1, we can extend ϕ_A to $K_{m+1}S$ but not directly to the inertia group $T = K_mS$. Consequently, we interpose some intermediary subgroups of U_l and work our way in steps from K_{m+1} to K_mS. In anticipation of the calculation of the number of characters of U_l of each degree, we will mention the number of extensions as we move from K_{m+1} to K_mS. Calculations of the sizes of S groups follow the same methods used in the even case above.

1. $A = \left(\begin{smallmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{smallmatrix} \right)$

$$T = K_mS = \left\{ \left(\begin{smallmatrix} a & \pi^m b \\ \pi^m c & d \end{smallmatrix} \right) \right\} \cap U_l, \;\; S = \left\{ \left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix} \right) \right\} \cap U_l$$

$|T| = |K_m| \times |S|_{\text{modulo}\,\pi^m} = q^{3l-1}(q-1)(q+1)$.

We interpose the subgroup $N = \left\{ \left(\begin{smallmatrix} 1+\pi^m a & \pi^{m+1} b \\ \pi^m c & 1+\pi^m d \end{smallmatrix} \right) \right\} \cap U_l$ and extend ϕ_A to $\phi'_A \in \mathrm{Irr}(N)$. N is generated by K_{m+1} and the abelian subgroups of U_l

$$\mathcal{G}_1 = \left\{ \begin{pmatrix} 1+\pi^m a & 0 \\ 0 & \overline{(1+\pi^m a)}^{-1} \end{pmatrix} \right\} a \in R_{l,\alpha}, \quad \mathcal{G}_2 = \left\{ \begin{pmatrix} 1 & 0 \\ \pi^m c\sqrt{\alpha} & 1 \end{pmatrix} \right\} c \in R_l$$

We show that N is normal in $K_m S$ so we can apply Clifford theory to N. This requires finding the inertia group T_0 of ϕ'_A in $T = K_m S$. The extension to N will be done in several steps. The schematic is (omitting the steps from K_{m+1} to N for simplicity)

$$\underset{\phi_A}{K_{m+1}} \xrightarrow{\text{ext}} \underset{\phi'_A}{N} \xrightarrow{\text{ext}} \underset{\psi_0}{T_0} \xrightarrow{\text{ind}} \underset{\psi}{T} \xrightarrow{\text{ind}} \underset{\chi}{U_l}$$

Proposition 4.H.1 N is normal in $K_m S$

Proof. Write $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ for $\begin{pmatrix} 1+\pi^m a & \pi^{m+1} b \\ \pi^m c & 1+\pi^m d \end{pmatrix} \in N$ and let $\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \in S$ then

$$\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} X^{-1} & 0 \\ 0 & Y^{-1} \end{pmatrix} = \begin{pmatrix} a & (XY^{-1})b \\ (X^{-1}Y)c & d \end{pmatrix} \in N$$

Next write $\begin{pmatrix} x & y \\ z & w \end{pmatrix}$ for $\begin{pmatrix} 1+\pi^m x & \pi^m y \\ \pi^m z & 1+\pi^m w \end{pmatrix} \in K_m$. N is generated by $K_{m+1}, \mathcal{G}_1$, and $\mathcal{G}_2$, but K_m centralizes K_{m+1}, so we only need to check that when any elements of $\mathcal{G}_1$ and $\mathcal{G}_2$ are conjugated by K_m, the result is in N. Note that the subgroup generated by $\mathcal{G}_1$ and $\mathcal{G}_2$ is lower triangular, so for any $\begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \in N$,

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix}\begin{pmatrix} a & 0 \\ c & d \end{pmatrix}\begin{pmatrix} w & -y \\ -z & x \end{pmatrix}\frac{1}{xw-zy} = \begin{pmatrix} ax+cy & dy \\ .... & .... \end{pmatrix}\begin{pmatrix} ... & -y \\ ... & x \end{pmatrix}\frac{1}{xw-zy}$$
$$= \begin{pmatrix} .... & xy(d-a)-cyy \\ .... & .... \end{pmatrix}\frac{1}{xw-zy}$$

Both y and $(d-a)$ have a factor of π^m, and $cyy = 0$, therefore the result is in N.

Since conjugation by elements of both K_m and S produces elements of N, then N is normal in $T = K_mS$. Thus we can apply Clifford theory to ϕ'_A on N as a normal subgroup of K_mS; we find the stabilizer T_0 of this character, then induce to T. Next we show the details of extending ϕ_A to ϕ'_A on N.

□

The extension to N is accomplished by two applications of Proposition 2.C.1. Since $\mathcal{G}_1$ is diagonal, it stabilizes ϕ_A, which therefore extends to a character on the product $K_{m+1}\mathcal{G}_1$ by Proposition 2.C.1. For each non-conjugate ϕ_A character on K_{m+1} there will be $\frac{|K_{m+1}\mathcal{G}_1|}{|K_{m+1}|} = q^2$ characters on $K_{m+1}\mathcal{G}_1$. The subgroup $\mathcal{G}_2$ is abelian, and we will show that it stabilizes the character on $K_{m+1}\mathcal{G}_1$; this means that ϕ_A extends from K_{m+1} to ϕ'_A on N. In this second extension we assign the trivial character to $\mathcal{G}_2$; this choice was made to imitate part of the Barrington-Leigh paper. It is required in anticipation of the stablizer of ϕ'_A in K_mS. Hence the number of non-conjugate characters on N is greater by a factor of q^2 than the number on K_{m+1}. To show that $\mathcal{G}_2$ stabilizes the character on

$K_{m+1}\mathcal{G}_1$, it suffices to show that it stabilizes the character restricted to $\mathcal{G}_1$; this is because $\mathcal{G}_2$ is in K_m, and so centralizes K_{m+1}.

Write $\left(\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}\right)$ for $\left(\begin{smallmatrix} 1 & 0 \\ \pi^m c\sqrt{\alpha} & 1 \end{smallmatrix}\right) \in \mathcal{G}_2$, and write $\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)$ for $\left(\begin{smallmatrix} 1+\pi^m a & 0 \\ 0 & \overline{(1+\pi^m a)}^{-1} \end{smallmatrix}\right) \in \mathcal{G}_1$. Conjugating

$$\begin{aligned}\left(\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ -c & 1 \end{smallmatrix}\right) &= \left(\begin{smallmatrix} x & 0 \\ cx & y \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ -c & 1 \end{smallmatrix}\right) \\ &= \left(\begin{smallmatrix} x & 0 \\ c(x-y) & y \end{smallmatrix}\right)\end{aligned}$$

Since $x - y$ has a factor of π^m, we can write $c(x-y) = \pi^{2m}c'\sqrt{\alpha}$ so the result of conjugation can be written

$$\left(\begin{smallmatrix} 1+\pi^m a & 0 \\ \pi^{2m}c'\sqrt{\alpha} & \overline{(1+\pi^m a)}^{-1} \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ \pi^{2m}c'\sqrt{\alpha} & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1+\pi^m a & 0 \\ 0 & \overline{(1+\pi^m a)}^{-1} \end{smallmatrix}\right)$$

Since the first factor above is in K_{m+1} with a character value of 1, the elements in $\mathcal{G}_2$ stabilizes the character on $K_{m+1}\mathcal{G}_1$. Thus we can extend to $K_{m+1}\mathcal{G}_1\mathcal{G}_2 = N$.

The inertia group of ϕ'_A in $K_m S$ is $T_0 = NS$: since S is the diagonal subgroup, it centralizes $\mathcal{G}_1$. It also normalizes $\mathcal{G}_2$:

$$\begin{aligned}\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ \pi^m c\sqrt{\alpha} & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} x^{-1} & 0 \\ 0 & y^{-1} \end{smallmatrix}\right) &= \left(\begin{smallmatrix} x & 0 \\ y\pi^m c\sqrt{\alpha} & y \end{smallmatrix}\right)\left(\begin{smallmatrix} x^{-1} & 0 \\ 0 & y^{-1} \end{smallmatrix}\right) \\ &= \left(\begin{smallmatrix} 1 & 0 \\ (yx^{-1})\pi^m c\sqrt{\alpha} & 1 \end{smallmatrix}\right) \\ &= \left(\begin{smallmatrix} 1 & 0 \\ \pi^m c'\sqrt{\alpha} & 1 \end{smallmatrix}\right)\end{aligned}$$

but $\mathcal{G}_2$ was assigned the trivial character, thus S is in the inertia group of ϕ'_A. $K_m S$ is generated by NS and the abelian group $\mathcal{G}_3 = \left\{ \left(\begin{smallmatrix} 1 & \pi^m \sqrt{\alpha} \\ 0 & 1 \end{smallmatrix} \right) \right\}$, but $\mathcal{G}_3$ does not stabilize ϕ'_A:

$$\begin{aligned} \left(\begin{smallmatrix} 1 & \pi^m \sqrt{\alpha} \\ 0 & 1 \end{smallmatrix} \right) \left(\begin{smallmatrix} 1 & 0 \\ \pi^m c \sqrt{\alpha} & 1 \end{smallmatrix} \right) \left(\begin{smallmatrix} 1 & -\pi^m \sqrt{\alpha} \\ 0 & 1 \end{smallmatrix} \right) &= \left(\begin{smallmatrix} 1+\pi^{2m}\alpha & \pi^m \sqrt{\alpha} \\ \pi^m c \sqrt{\alpha} & 1 \end{smallmatrix} \right) \left(\begin{smallmatrix} 1 & -\pi^m \sqrt{\alpha} \\ 0 & 1 \end{smallmatrix} \right) \\ &= \left(\begin{smallmatrix} 1+\pi^{2m}\alpha & 0 \\ \pi^m c \sqrt{\alpha} & 1-\pi^{2m}\alpha \end{smallmatrix} \right) \\ &= \left(\begin{smallmatrix} 1 & 0 \\ \pi^m c \sqrt{\alpha} & 1 \end{smallmatrix} \right) \left(\begin{smallmatrix} 1+\pi^{2m}\alpha & 0 \\ 0 & 1-\pi^{2m}\alpha \end{smallmatrix} \right) \end{aligned}$$

Since the character value of the second factor is not identically 1, then $T_0 = NS$ is the inertia group of ϕ'_A in T. We can extend ϕ' from N to ψ_0 on $T_0 = NS$ by Proposition 2.C.1, there being $\frac{|NS|}{|N|} = q^{l-3}(q-1)(q+1)$ such extensions for each non-conjugate character on N. Now we induce from NS to $T = K_m S$; $\psi = \mathrm{Ind}_{T_0}^{T} \psi_0$ is irreducible of degree $[T : T_0] = q$, and $\chi = \mathrm{Ind}_T^{U_l} \psi$ is an irreducible character having degree $q[U_l : K_m S]$ or

$$q \frac{q^{4l-3}(q-1)(q+1)^2}{q^{3l-1}(q-1)(q+1)} = q^{l-1}(q+1)$$

2. $A = \left(\begin{smallmatrix} 0 & \sigma \\ 1 & 0 \end{smallmatrix} \right)$ with σ a square unit in R_l.

$$T = \left\{ \left(\begin{smallmatrix} a & b\sigma+\pi^m c \\ b & a+\pi^m d \end{smallmatrix} \right) \right\} \cap U_l = K_m S, \;\; S = \left\{ \left(\begin{smallmatrix} a & b\sigma \\ b & a \end{smallmatrix} \right) \right\} \cap U_l$$

The order of T is $q^{3l-1}(q+1)^2$. The schematic in this case is more complicated:

$$\underset{\phi_A}{K_{m+1}} \xrightarrow{\text{ext}} \underset{\phi'_A}{N_{m+1}} \xrightarrow{\text{ext}} \underset{\phi''_A}{H} \xrightarrow{\text{ind}} \underset{\psi_0}{N_m} \xrightarrow{\text{ext}} \underset{\psi}{K_m S} \xrightarrow{\text{ind}} \underset{\chi}{U_l}$$

where $N_i = K_i(K_1 \cap S)\left\{\left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)\right\} \cap U_l$ and H, as well as the various characters, will be defined below.

(a) We can extend ϕ_A to ϕ' on N_{m+1} by Proposition 2.C.1 because both the scalars and $K_1 \cap S$ are abelian and stabilize ϕ_A, and because the scalar matrices will stabilize any character that we assign to $K_1 \cap S$.

(b) We want to apply Clifford theory to H in N_m. To do this we will require

Proposition 4.H.2 N_{m+1} is normal in N_m, and every element of K_m stabilizes ϕ'_A.

Proof. Since $N_{m+1} = K_{m+1}(K_1 \cap S)\left\{\left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)\right\}$, it suffices to consider the K_m conjugation action on $K_1 \cap S$ (K_m centralizes K_{m+1}). In brief, the argument is that conjugation of $s \in K_1 \cap S$ by any element in K_m produces an element xs where $x \in K_{m+1}$ with $\phi_{A'}(x) = 1$.

Lemma 4.H.1 For $k \in K_m$ and $s \in K_1 \cap S$, $ksk^{-1} = xs$ for some $x \in K_{m+1}$.

Proof. Consider the natural projection map $P : U_l \to U_{m+1}$, that sends each entry of a matrix in U_l to its value modulo π^{m+1} and has kernel K_{m+1}. We claim that $f(ksk^{-1}) = f(s)$ which implies that $ksk^{-1} = xs$ for some $x \in K_{m+1}$: when the modulus is π^{m+1}, $f(k) = I + \pi^m A$ commutes with $f(s) = I + \pi B$, so that $f(ksk^{-1}) = f(s)$

and for some $x \in K_{m+1}$, $ksk^{-1} = xs$. An immediate consequence of this is that N_{m+1} is normal in N_m.

□

Corollary 4.H.1 For any $k \in K_m$, and $s \in K_1 \cap S$, there exists $x \in K_{m+1}$ such that $k^{-1}x = sk^{-1}s^{-1}$.

To show that K_m stabilizes ϕ'_A, it suffices to show that for x as in the corollary, $\phi'_A(x) = \phi_A(x) = 1$. For this we need:

Lemma 4.H.2 For any $k \in K_m$, $s \in K_1 \cap S$, $x \in K_{m+1}$, and $A = \left(\begin{smallmatrix} 0 & \sigma \\ 1 & 0 \end{smallmatrix}\right)$:

i. $\mathrm{tr}(A(kx)) = \mathrm{tr}(Ak) + \mathrm{tr}(Ax)$.

ii. $\mathrm{tr}(A(sks^{-1})) = \mathrm{tr}(Ak)$.

Proof. i. Let $k = I + \pi^m B$ and $x = I + \pi^{m+1} C$, so that

$$kx = I + \pi^m B + \pi^{m+1} C$$

and one sees that the elements on the second diagonal are additive.

ii. $\mathrm{tr}(A(sks^{-1})) = \mathrm{tr}(s^{-1}As)k)) = \mathrm{tr}(Ak)$ because S centralizes A.

□

Now we can show that for x as in 4.H.1, we have $\phi'_A(x) = \phi_A(x) = 1$. From Lemma 4.H.2 we have:

$$\begin{aligned}\mathrm{tr}(Ak^{-1}) + \mathrm{tr}(Ax) &= \mathrm{tr}(A(k^{-1}x)\\ &= \mathrm{tr}(A(sk^{-1}s^{-1}))\\ &= \mathrm{tr}(Ak^{-1})\end{aligned}$$

so that $\mathrm{tr}(Ax) = 0$, and $\phi'(x) = \phi_A(x) = \lambda[\mathrm{tr}(\pi^m Ax)] = \lambda(0) = 1$. Hence K_m stabilizes ϕ'_A.

□

(c) K_m is generated by K_{m+1} and these abelian subgroups of U_l (where $t \in R_l$):

i. $\mathcal{G}_0 = \left\{ \begin{pmatrix} 1+\pi^m t\sqrt{\alpha} & 0 \\ 0 & \frac{1}{1-\pi^m t\sqrt{\alpha}} \end{pmatrix} \right\}$

ii. $\mathcal{G}_1 = \left\{ \begin{pmatrix} 1 & 0 \\ \pi^m t\sqrt{\alpha} & 1 \end{pmatrix} \right\}$

iii. $\mathcal{G}_2 = \left\{ \begin{pmatrix} 1 & \pi^m t\sqrt{\alpha} \\ 0 & 1 \end{pmatrix} \right\}$

iv. $\mathcal{G}_3 = \left\{ \begin{pmatrix} 1+\pi^m t & 0 \\ 0 & (1+\pi^m t)^{-1} \end{pmatrix} \right\}$

We will now show how which of these subgroups (together with K_{m+1}) generate H, and then N_m.

(d) Of the subgroups above, only $\mathcal{G}_0$ is in N_{m+1}.

Proof. i. By calculation $\mathcal{G}_0 \in N_{m+1}$:

$$\begin{pmatrix} 1-\pi^{2m}b^2\frac{\alpha}{2} & 0 \\ 0 & 1+\pi^{2m}b^2\frac{\alpha}{2} \end{pmatrix}\begin{pmatrix} 1+\pi^{2m}b^2\frac{\alpha}{2}+\pi^m b\sqrt{\alpha} & 0 \\ 0 & 1+\pi^{2m}b^2\frac{\alpha}{2}+\pi^m b\sqrt{\alpha} \end{pmatrix} = \begin{pmatrix} 1+\pi^m b\sqrt{\alpha} & 0 \\ 0 & \frac{1}{1-\pi^m b\sqrt{\alpha}} \end{pmatrix}$$

ii. The subgroups $\mathcal{G}_1$ and $\mathcal{G}_2$ are either both in N_{m+1} or both not in N_{m+1}: suppose, for example, that $\mathcal{G}_2$ is in N_{m+1}. Then, since the element $\begin{pmatrix} 1+\pi^{2m}\frac{\alpha(\sigma)-1}{2} & -\pi^m\sqrt{\pi} \\ -\pi^m\sigma^{-1}\sqrt{\pi} & 1+\pi^{2m}\frac{\alpha(\sigma)-1}{2} \end{pmatrix}$ is in $K_1 \cap S$, we can take the following product:

$$\begin{pmatrix} 1+\pi^{2m}\frac{\alpha(\sigma)-1}{2} & -\pi^m\sqrt{\alpha} \\ -\pi^m\sigma^{-1}\sqrt{\alpha} & 1+\pi^{2m}\frac{\alpha(\sigma)-1}{2} \end{pmatrix}\begin{pmatrix} 1 & \pi^m\sqrt{\alpha} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1+\pi^{2m}\frac{\alpha(\sigma)-1}{2} & 0 \\ -\pi^m\sigma^{-1}\sqrt{\alpha} & 1-\pi^{2m}\frac{\alpha(\sigma)-1}{2} \end{pmatrix}$$

and rewrite it as

$$\begin{pmatrix} 1+\pi^{2m}\frac{\alpha(\sigma)-1}{2} & 0 \\ 0 & 1-\pi^{2m}\frac{\alpha(\sigma)-1}{2} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\pi^m\sigma^{-1}\sqrt{\alpha} & 1 \end{pmatrix}$$

Thus we get elements of $\mathcal{G}_1$.

iii. We claim that none of $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ are in N_{m+1}. The argument here requires the orders of N_{m+1} and N_m; we derive these, and then show that for $i = 1, 2, 3$, no $\mathcal{G}_i$ is in N_{m+1}.

The Orders of N_m and N_{m+1}

The Order of N_m:

A. $N_m = K_m(K_1 \cap S)$(scalar matrices)

B. K_m has order q^{2l+2}

C. $K_1 \cap S$: Since S can be considered $\mathcal{L} \times \mathcal{L}$, then $K_1 \cap S$ can be considered the kernel of the map from $(\mathcal{L} \times \mathcal{L})_{\text{modulo } \pi^l}$ to $(\mathcal{L} \times \mathcal{L})_{\text{modulo } \pi^1}$. This kernel has order

$$\frac{q^{l-1}(q+1)q^{l-1}(q+1)}{(q+1)(q+1)} = q^{2l-2}$$

D. $K_m \cap (K_1 \cap S) = K_m \cap S$ and the order of this last group (seen as a kernel of the projection map from $(\mathcal{L}\times\mathcal{L})_{\text{modulo } \pi^l}$ to $(\mathcal{L}\times\mathcal{L})_{\text{modulo } \pi^m}$ is

$$\frac{q^{l-1}(q+1)q^{l-1}(q+1)}{q^{m-1}(q+1)q^{m-1}(q+1)} = q^{l+1}$$

E. As a result: $|K_m(K_1 \cap S)|= \frac{q^{2l+2}q^{2l-2}}{q^{l+1}} = q^{3l-1}$

F. The scalar matrices can be identified with $\mathcal{L}$, and so have order $q^{l-1}(q+1)$

G. the intersection of the scalars and $K_m(K_1 \cap S)$ are of the form:

$$\begin{pmatrix} 1+\pi a & 0 \\ 0 & 1+\pi a \end{pmatrix}$$

where $1+\pi a \in \mathcal{L}$. The set $\{1+\pi a\}$ is the kernel of the (projection) map from $\mathcal{L}$ modulo l to $\mathcal{L}$ modulo 1. Hence the intersection has order q^{l-1}.

H. We conclude that the order of $N_m = K_m(K_1 \cap S)$(scalar matrices) is

$$\frac{q^{3l-1}q^{l-1}(q+1)}{q^{l-1}} = q^{3l-1}(q+1)$$

The order of N_{m+1} is calculated in the same way:

A. $|K_{m+1}|= q^{4m} = q^{2l-2}$; $|K_m|= q^{2l+2}$

B. $|K_1 \cap S|= q^{2l-2}$ since it can be considered the kernel of the natural projection map $\mathcal{L}_l \times \mathcal{L}_l \to \mathcal{L}_1 \times \mathcal{L}_1$.

C. $|[K_{m+1}(K_1 \cap S)]|= q^{l-1}$ consider as kernel of $\mathcal{L}_l \times \mathcal{L}_l \to \mathcal{L}_{m+1} \times \mathcal{L}_{m+1}$.

D. Scalars $q^{l-1}(q+1)$. Intersection of Scalars and $[K_{m+1}(K_1 \cap S)]$: scalar matrices with elements $1+\pi a$, norm 1 so consider as kernel $\mathcal{L}_l \to \mathcal{L}_1 = q^{l-1}$.

E. $|N_{m+1}| = \frac{q^{2l-2}q^{2l-2}}{q^{l-1}} = q^{3l-3} \times \frac{q^{l-1}(q+1)}{q^{l-1}} = q^{3l-3}(q+1)$

Note that $|N_m| = q^2|N_{m+1}|$. Having established the orders of these groups, we now suppose that $\mathcal{G}_1$ and $\mathcal{G}_2$ were in N_{m+1}. Then N_m would be generated by N_{m+1} and $\mathcal{G}_3$, and:

$$|N_m| = |N_{m+1}|\frac{|\mathcal{G}_3|}{|N_{m+1} \cap \mathcal{G}_3|} = |N_{m+1}|\frac{q^{m+1}}{q^m} = q|N_{m+1}|$$

which is a contradiction. If we had assumed that $\mathcal{G}_3$ were in N_{m+1} we would have arrived at a similar contradiction.

□

We now define H as the group generated by N_{m+1} and $\mathcal{G}_1$.

The order of $\mathcal{G}_1$ is q^{m+1}, and $\mathcal{G}_1 \cap N_{m+1}$ has order q^m, hence $|H| = \frac{|N_{m+1}| \times |\mathcal{G}_1|}{|N_{m+1} \cap \mathcal{G}_1|} = \frac{q^{3l-3}(q+1) \times q^{m+1}}{q^m} = q^{3l-2}(q+1)$. Moreover, $[N_m : H] = q$.

(e) Since every element of K_m stabilizes ϕ'_A then by Proposition 2.C.1 we can extend ϕ'_A to ϕ''_A on H.

We claim that H is normal in N_m: borrowing an idea from [BL], since N_m/N_{m+1} is abelian, any subgroup of N_m containing N_{m+1} is normal. Thus H is normal in N_m with index q. We can now apply Clifford theory to the group N_m with normal subgroup H and character ϕ''_A. We claim too, that the inertia group of $\phi_{A''}$ in N_m is H itself: N_m is generated by H and $\mathcal{G}_3$, and we will show that $\mathcal{G}_3$ does not stabilize ϕ''_A. Write $= \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$ for $= \begin{pmatrix} 1 & 0 \\ \pi^m\sqrt{\alpha} & 1 \end{pmatrix} \in \mathcal{G}_1 \leq H$, and $= \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix}$ for $= \begin{pmatrix} (1+\pi^m)^{-1} & 0 \\ 0 & (1+\pi^m) \end{pmatrix} \in \mathcal{G}_3$. Conjugating:

$$\begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a^2 c & 1 \end{pmatrix}$$

and

$$a^2 c = (1 + 2\pi^m + \pi^{2m})\pi^m \sqrt{\alpha} = \pi^m \sqrt{\alpha} + \pi^{2m} \sqrt{\alpha}$$

so that the product of conjugation can be written

$$\begin{pmatrix} 1 & 0 \\ \pi^m \sqrt{\alpha} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \pi^{2m} \sqrt{\alpha} & 1 \end{pmatrix}$$

where the second factor, being in K_{m+1} does not have a character value equal to 1. Consequently, the inertia group of ϕ''_A in N_m is H itself, and $\psi_0 = \mathrm{Ind}_H^{N_m} \phi''_A$ will be an irreducible character of degree $[N_m : H] = q$.

(f) We will extend the character ψ_0 from N_m to $T = K_m S$ by Proposition 2.C.2, which requires that ψ_0 be invariant in $K_m S$. The argument for this invariance is, in brief, that $\psi_0 \neq 0$ only on N_{m+1} and that $K_m S$ stabilizes the character on N_{m+1}. We include the schematic for this section for reference, followed by the proof of the invariance of ψ_0 in $K_m S$. Of particular importance is the fact that $[N_m : N_{m+1}] = q^2$.

$$\underset{\phi_A}{K_{m+1}} \xrightarrow{\text{ext}} \underset{\phi'_A}{N_{m+1}} \xrightarrow{\text{ext}} \underset{\phi''_A}{H} \xrightarrow{\text{ind}} \underset{\psi_0}{N_m} \xrightarrow{\text{ext}} \underset{\psi}{K_m S} \xrightarrow{\text{ind}} \underset{\chi}{U_l}$$

Proposition 4.H.3 The character ψ_0 is invariant in $K_m S$.

Proof. We know that $\frac{1}{|N_m|}\sum_{g\in N_m}|\psi_0(g)|^2 = 1$, but in fact we claim that $\frac{1}{|N_m|}\sum_{g\in N_{m+1}}|\psi_0(g)|^2 = 1$. For any $g \in N_{m+1}$, and $B = \{b_1, b_2, \ldots, b_q\}$ a fixed transversal of H in N_m, we have

$$\psi_0(g) = \sum_{b_i \in B} \phi''_A(b_i^{-1} g b_i)$$

But ϕ''_A on N_{m+1} is just ϕ'_A, and since each transversal element b_i is in K_m (which fixes ϕ'_A), then $\phi'_{\prime}(b_i^{-1} g b_i) = \phi'_A(g)$. Hence $\psi_0(g) = q\phi_A(g)$, and

$$\begin{aligned}
\frac{1}{|N_m|}\sum_{g\in N_{m+1}}|\psi_0(g)|^2 &= \frac{1}{|N_m|}\sum_{g\in N_{m+1}}|q\phi'_A(g)|^2 \\
&= \frac{1}{|N_m|}q^2\sum_{g\in N_{m+1}}|\phi'_A(g)|^2 \\
&= \frac{1}{|N_m|}q^2|N_{m+1}| \\
&= 1
\end{aligned}$$

Consequently, $\psi_0 = 0$ outside of N_{m+1}. We know already that K_m stabilizes ϕ'_A on N_{m+1}, but S also stabilizes ϕ'_A, since any $g \in N_{m+1}$ can be written as the product $g = hsa$ with $h \in K_{m+1}, s \in K_1 \cap S$, and a a scalar matrix, and S stabilizes the character on K_{m+1}, and commutes with both the scalar matrices as well as the elements of $K_1 \cap S$. Hence $K_m S$ stabilizes ϕ'_A on N_{m+1}, and so ψ_0 is invariant in $K_m S$.

□

Since $[K_mS : N_m] = q+1$ is prime to q, the degree of ψ_0 on N_m, and since ψ_0 is invariant in K_mS, then from Proposition 2.C.2, ψ_0 extends to an irreducible character ψ on K_mS of degree q. In turn $\chi = \mathrm{Ind}_{K_mS}^{U_l}\psi$ is an irreducible character of U_l with degree $q[U_l : K_mS] = (q)\frac{q^{4l-3}(q-1)(q+1)^2}{q^{3l-1}(q+1)^2} = (q)q^{l-2}(q-1) = q^{l-1}(q-1)$ as in the even case.

3. $A = \begin{pmatrix} 0 & \pi\beta \\ 1 & 0 \end{pmatrix}$, $\beta \in R_{l-1}$

$$T = \left\{ \begin{pmatrix} a & \pi\beta b+\pi^m c \\ c & a+\pi^m d \end{pmatrix} \right\} \cap U_l = K_mS$$

where $S = \left\{ \begin{pmatrix} a & \pi\beta b \\ b & a \end{pmatrix} \right\}$. and $|T| = q^{3l}(q+1)$. We follow the same schematic used with the first matrix, though in this case we do the extension to N in one step:

$$\underset{\phi_A}{K_{m+1}} \xrightarrow{\text{ext}} \underset{\phi'_A}{N} \xrightarrow{\text{ext}} \underset{\psi_0}{T_0} \xrightarrow{\text{ind}} \underset{\psi}{T} \xrightarrow{\text{ind}} \underset{\chi}{U_l}$$

where again

$$N = \left\{ \begin{pmatrix} 1+\pi^m a & \pi^{m+1} b \\ \pi^m c & 1+\pi^m d \end{pmatrix} \right\} \cap U_l, \; a,b,c,d \in R_{l,\alpha}$$

and, as we will show, $T_0 = NS$.

We can show that N is a normal subgroup in $T = K_mS$. We have seen in Proposition 4.H.1 that K_m normalizes N; to show that S does as well, write $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ for $\begin{pmatrix} 1+\pi^m a & \pi^{m+1} b \\ \pi^m c & 1+\pi^m d \end{pmatrix} \in N$ and conjugate by $\begin{pmatrix} x & y\pi\beta \\ y & x \end{pmatrix} \in S$ which we will write $\begin{pmatrix} x & y' \\ y & x \end{pmatrix}$

$$\begin{pmatrix} x & y' \\ y & x \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x & -y' \\ -y & w \end{pmatrix}\frac{1}{\det} = \begin{pmatrix} ax+cy' & bx+dy' \\ ay+cx & by+dx \end{pmatrix}\begin{pmatrix} x & -y; \\ -y & x \end{pmatrix}\frac{1}{\det}$$
$$= \begin{pmatrix} \cdots & bxx+dxy'-cy'y'-axy' \\ \cdots & \cdots \end{pmatrix}\frac{1}{\det}$$
$$= \begin{pmatrix} \cdots & bxx+xy'(d-a)-cy'y' \\ \cdots & \cdots \end{pmatrix}\frac{1}{\det}$$

Since each of bxx, $xy'(d-a)$, and $cy'y'$ has π^{m+1} as a factor then N is normal in $K_m S$.

We define a character ϕ'_A on N that is an extension of ϕ_A on K_{m+1}; for $n = \begin{pmatrix} 1+\pi^m a & \pi^{m+1}b \\ \pi^m c & 1+\pi^m d \end{pmatrix} \in N$, define $\phi'_A(n) = \lambda[\pi^{m+1}b + \pi^{m+1}\beta c]$. To show that this is a character on N, let n be as given, and take a second element of N: $r = \begin{pmatrix} 1+\pi^m e & \pi^{m+1}f \\ \pi^m g & 1+\pi^m h \end{pmatrix}$, so that $\phi'_A(nr) =$

$$\phi'_A\left[\begin{pmatrix} 1+\pi^m a & \pi^{m+1}b \\ \pi^m c & 1+\pi^m d \end{pmatrix}\begin{pmatrix} 1+\pi^m e & \pi^{m+1}f \\ \pi^m g & 1+\pi^m h \end{pmatrix}\right] = \phi'_A\begin{pmatrix} \cdots\cdots & \pi^{m+1}b+\pi^{m+1}f \\ \pi^m c+\pi^m g+\pi^{2m}(be+gd) & \cdots\cdots \end{pmatrix}$$
$$= \lambda[\pi\beta(\pi^m c + \pi^m g + \pi^{2m}(be+gd)) + \pi^{m+1}b + \pi^{m+1}f]$$

and the final line can be written

$$\lambda[\pi^{m+1}b + \pi^{m+1}\beta c] + \lambda[\pi^{m+1}f + \pi^{m+1}\beta g] = \phi'_A(n)\phi'_A(r)$$

It is clear that ϕ'_A restricts to ϕ_A on K_{m+1} and in fact we could write this character, applied to $n \in N$ as $\lambda[\mathrm{tr}(An)]$. We claim the inertia group of ϕ'_A in $K_m S$ is NS; this requires the following

(a) S stabilizes ϕ'_A

(b) Elements of the abelian group generated by matrices of the form $\left(\begin{smallmatrix} 1 & \pi^m b\sqrt{\alpha} \\ 0 & 1 \end{smallmatrix}\right)$, $b \in R_l^*$ do not stabilize ϕ'_A.

To prove the first point, for any $s \in S$ we have

$$\begin{aligned}
\phi'_A(sns^{-1}) &= \lambda[\mathrm{tr}(A(sns^{-1})] \\
&= \lambda[\mathrm{tr}(s^{-1}As)n] \\
&= \lambda[\mathrm{tr}(An)] \\
&= \phi'_A(n)
\end{aligned}$$

where we have used the fact that S centralizes A. For the second point, write $\left(\begin{smallmatrix} 1 & f \\ 0 & 1 \end{smallmatrix}\right)$ for some element $\left(\begin{smallmatrix} 1 & \pi^m f\sqrt{\alpha} \\ 0 & 1 \end{smallmatrix}\right)$ and consider the element $\left(\begin{smallmatrix} 1+\pi^m a & 0 \\ 0 & 1+\pi^m d \end{smallmatrix}\right)$ in N with a ϕ'_A value of 1. Conjugating gives

$$\begin{aligned}
\left(\begin{smallmatrix} 1 & f \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1+\pi^m a & 0 \\ 0 & 1+\pi^m d \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & -f \\ 0 & 1 \end{smallmatrix}\right) &= \left(\begin{smallmatrix} 1+\pi^m a & f(1+\pi^m d) \\ 0 & 1+\pi^m d \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & -f \\ 0 & 1 \end{smallmatrix}\right) \\
&= \left(\begin{smallmatrix} 1+\pi^m a & f(\pi^m d - \pi^m a) \\ 0 & 1+\pi^m d \end{smallmatrix}\right) \\
&= \left(\begin{smallmatrix} 1+\pi^m a & \pi^{2m} f(d-a)\sqrt{\alpha} \\ 0 & 1+\pi^m d \end{smallmatrix}\right)
\end{aligned}$$

Since the character value of the result is not 1, then elements of the form $\left(\begin{smallmatrix} 1 & \pi^m f\sqrt{\alpha} \\ 0 & 1 \end{smallmatrix}\right)$ are not in the inertia group (in $K_m S$) of ϕ'_A, and the inertia group is NS. By Proposition 2.C.1, ϕ'_A extends to to ψ_0 on $NS = T_0$. Then by Clifford theory ψ_0 induces to an irreducible character of ψ of $K_m S = T$, having degree $[K_m S : NS] = q$. Finally, $\chi = \mathrm{Ind}_T^{U_l}\psi$ will be an irreducible character of U_l whose degree is $q[U_l : T] = q^{l-2}(q-1)(q+1)$

as in the even case.

Below we summarize our results:

Table 4.2: Degrees From ϕ_A Characters

A Matrix	Character Degree
$\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$	$q^{l-1}(q+1)$
$\begin{pmatrix} 0 & \sigma \\ 1 & 0 \end{pmatrix}$	$q^{l-1}(q-1)$
$\begin{pmatrix} 0 & \pi\beta \\ 1 & 0 \end{pmatrix}$	$q^{l-2}(q-1)(q+1)$

4.I Extensions: Counting the Characters not Coming From U_{l-1}

In this section, we generalize each A to A', such that ϕ_A and ϕ'_A have the same inertia group, and lead to characters in $\mathrm{Irr}(U_l)$ of the same degree. One sees that if $A' = aI + bA$ where $a \in R_l$, $b \in R_l^*$, then ϕ_A and $\phi_{A'}$ will have the same inertia groups, and the characters of U_l arising from them will have the same degree, and will be equally numerous. From Proposition 4.E.5, we consider only non-conjugate A' matrices. Our assumption is that we know the character degrees and the number of them for the group U_{l-1}, the base case being given in [E]. Now we can state the following:

Theorem 4.I.1 The number of irreducible characters (and their degrees) of U_l not coming from U_{l-1} are as follows:

Proof. If u is a unit in $R_{l,\alpha}$, then from equations 4.6 and 4.7, it is clear that the inertia groups of ϕ_A and ϕ_{uA} are the same. It is also clear that ϕ_A and

Table 4.3: Character Numbers

degree of character	number of characters of this degree
$q^{l-1}(q+1)$	$\frac{1}{2}q^{2l-3}(q-1)^2(q+1)$
$q^{l-1}(q-1)$	$\frac{1}{2}q^{2l-3}(q-1)(q+1)^2$
$q^{l-2}(q-1)(q+1)$	$q^{2l-2}(q+1)$

ϕ_{I+A} will have the same inertia group. Therefore we generalize the A matrices follows:

1. $l = 2m$

   The schematic for each A matrix is

   $$\underset{\phi_A}{K_m} \xrightarrow{\text{ext}} \underset{\psi}{T} \xrightarrow{\text{ind}} \underset{\chi}{U_l}$$

   In order to count the number of non-conjugate characters on the K_m subgroups, we will need definition of ϕ_A:

   $$\phi_A[I + \pi^m B] = \lambda[\text{tr}(\pi^m AB)]$$

   (a) For $A = \left(\begin{smallmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{smallmatrix}\right)$, $A' = xI + bA$, where $x \in R_l$, $b \in R_l^*$. Because of the π^m in the definition of ϕ_A, there are q^m choices for x, and $q^{m-1}(q-1)$ choices for b. The only matrix of this type conjugate to $xI + bA$ is $xI - bA$. Therefore the number of non-conjugate A' is:

   $$\frac{1}{2}q^m q^{m-1}(q-1) = \frac{1}{2}q^{l-1}(q-1)$$

   .

   In order to count the number of characters of U_l that arise from these matrices, we multiply the number of non-conjugate matrices by the number of extensions from K_m to T, which is the index of K_m in T : $|S_m| = |R^{\times}_{m,\alpha}| = |\mathcal{L}_m| \cdot |R^{\times}_m| = q^{l-2}(q^2-1)$. As a result, the total number of characters of U_l arising from this case is

$$\frac{1}{2}q^{2l-3}(q-1)^2(q+1)$$

(b) For $A = \left(\begin{smallmatrix} 0 & \sigma \\ 1 & 0 \end{smallmatrix}\right)$, where σ is a square unit of R_l, let $A' = xI + bA$, where $x \in R_l$, and $b \in R_l^*$. We claim that b is superfluous and that we get all non-conjugate A' matrices by varying x and σ. To see this, note that for any unit $b \in R_l$, there is some $x \in R_{l,\alpha}$ such that $x\overline{x} = b$. Now

$$\begin{aligned}\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & b\sigma \\ b & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} x^{-1} & 0 \\ 0 & y^{-1} \end{smallmatrix}\right) &= \left(\begin{smallmatrix} 0 & \sigma \\ 1 & 0 \end{smallmatrix}\right) \\ &= \left(\begin{smallmatrix} 0 & x\overline{x}b\sigma \\ \frac{b}{x\overline{x}} & 0 \end{smallmatrix}\right) \\ &= \left(\begin{smallmatrix} 0 & b^2\sigma \\ 1 & 0 \end{smallmatrix}\right) \\ &= \left(\begin{smallmatrix} 0 & \sigma' \\ 1 & 0 \end{smallmatrix}\right)\end{aligned}$$

where σ' is a square. Therefore we count non-conjugate characters by varying only x and σ; there are q^m choices for the former, and $\frac{1}{2}q^{m-1}(q-1)$ choices for σ. Since no distinct A' matrices of this form are conjugate, the number of non-conjugate characters is

$$\frac{1}{2}q^m q^{m-1}(q-1) = \frac{1}{2}q^{l-1}(q-1)$$

The number of extensions of ϕ'_A, to T, is $|T|/|K_m| = |\mathcal{L}_m| \cdot |\mathcal{L}_m|$ or:

$$(q^{m-1}(q+1))(q^{m-1}(q+1)) = q^{l-2}(q+1)^2$$

Therefore the total number of characters of degree $q^{l-1}(q-1)$ is:

$$\frac{1}{2}q^{2l-3}(q-1)(q+1)^2$$

(c) For $A = \left(\begin{smallmatrix} 0 & \pi\beta \\ 1 & 0 \end{smallmatrix}\right)$ to $xI + bA$ with $x \in R_l$, $\beta \in R_l$,. We do not use b in front of A by the same argument used for the previous matrix; instead we vary β. There are q^m choices for x and q^{m-1} choices for β. No distinct matrices of this form are conjugate, thus we have $q^{m-1}q^m = q^{l-1}$ non conjugate characters. The number of extensions to T for each is $|T|/|K_m| = q^{l-1}(q+1)$, so the total number of characters of degree $q^{l-2}(q^2-1)$ is $q^{2l-2}(q+1)$.

The sum of the squares of the degrees of the characters we have found so far is:

$$q^{4l-6}(q-1)(q+1)(q^3-1)$$

The definition of ϕ_A on K_{m+1} is

$$\phi_A[I + \pi^{m+1}B] = \lambda[\mathrm{tr}(\pi^{m+1}AB)]$$

2. $l = 2m+1$

(a) $A = \left(\begin{smallmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{smallmatrix}\right)$.

The schematic is

$$\underset{\phi_A}{K_{m+1}} \xrightarrow{\text{ext}} \underset{\phi'_A}{N} \xrightarrow{\text{ext}} \underset{\psi_0}{T_0} \xrightarrow{\text{ind}} \underset{\psi}{T} \xrightarrow{\text{ind}} \underset{\chi}{U_l}$$

$A' = xI + bA$, with $x \in R_l, b \in R_l^*$. There are q^m choices for x and $q^{m-1}(q-1)$ choices for b. Since the matrices $xI \pm bA$ are conjugate, there are $\frac{1}{2}q^m q^{m-1}(q-1)$ or $\frac{1}{2}q^{l-2}(q-1)$ non -conjugate characters on K_{m+1}. We extended each of these to N, getting q^2 characters for each of the non-conjugate characters on K_{m+1}. Thus there are $\frac{1}{2}q^l(q-1)$ non-conjugate characters on N. Each of these extends, in turn, to $T_0 = NS$ and the number of such extensions is $|T_0|/|N| = q^{l-3}(q-1)(q+1)$. Thus we get

$$\frac{1}{2}q^{2l-3}(q-1)^2(q+1)$$

irreducible characters of U_l having degree $q^{l-1}(q+1)$.

(b) $A = \left(\begin{smallmatrix} 0 & \sigma \\ 1 & 0 \end{smallmatrix}\right)$.

The schematic

$$\underset{\phi_A}{K_{m+1}} \xrightarrow{\text{ext}} \underset{\phi'_A}{N_{m+1}} \xrightarrow{\text{ext}} \underset{\phi''_A}{H} \xrightarrow{\text{ind}} \underset{\psi_0}{N_m} \xrightarrow{\text{ext}} \underset{\psi}{K_m S} \xrightarrow{\text{ind}} \underset{\chi}{U_l}$$

$A' = xI + A$ with $x \in R_l$, and σ a square unit in R_l. There are q^m choices for x, and $\frac{1}{2}q^{m-1}(q-1)$ choices for σ, giving $\frac{1}{2}q^{l-2}(q-1)$ non-conjugate characters on K_{m+1}. The number of extensions to N_{m+1} is

$$[N_{m+1} : K_{m+1}] = \frac{q^{3l-3}(q+1)}{q^{2l-2}} = q^{l-1}(q+1)$$

for a total of $\frac{1}{2}q^{2l-3}(q-1)(q+1)$ non-conjugate characters on N_{m+1}. There are q extensions from N_{m+1} to H, but these can be ignored in the character count, because ψ_0 is induced from ϕ''_A on H, and we have shown that $\psi_0 = 0$ on $H - N_{m+1}$. Finally, the number of extensions from N_m to K_mS is $[K_mS : N_m] = (q+1)$, resulting in a total of $\frac{1}{2}q^{2l-3}(q-1)(q+1)^2$ distinct irreducible characters of U_l having degree $q^{l-1}(q-1)$.

(c) For $A = \left(\begin{smallmatrix} 0 & \pi\beta \\ 1 & 0 \end{smallmatrix}\right)$.

The schematic

$$\underset{\phi_A}{K_{m+1}} \xrightarrow{\text{ext}} \underset{\phi'_A}{N} \xrightarrow{\text{ext}} \underset{\psi_0}{T_0} \xrightarrow{\text{ind}} \underset{\psi}{T} \xrightarrow{\text{ind}} \underset{\chi}{U_l}$$

We extend to ϕ'_A in one step, and merely count the number of conjugate characters on $N = \left(\begin{smallmatrix} 1+\pi^m a & \pi^{m+1}b \\ \pi^m c & 1+\pi^m d \end{smallmatrix}\right)$ we can define for $n \in N$ as $\phi'_A(n) = \lambda[\mathrm{tr}A(n)]$. This means for $A' = xI + A$ that we have q^{m+1} choices for x and q^m choices for β. Hence there are $q^{m+1}q^m = q^l$ non-conjugate characters in N. The number of extensions for each is $|NS|/|N| = q^{l-2}(q+1)$. In all then, we get

$$q^{2l-2}(q+1)$$

irreducible characters of U_l of degree $q^{l-2}(q-1)(q+1)$.

The numbers of characters in U_l of each degree are the same as in the even case, so the sum of squares without the contribution from U_{l-1} will be

$$q^{4l-6}(q-1)(q+1)(q^3-1)$$

□

4.J Lifting Characters From U_{l-1}

From theorem 17.3 [JL], in any finite group G with normal subgroup H, there is a 1 to 1 correspondence between the irreducible characters of G/H and the irreducible characters of G having H in the kernel. The natural projection map $\phi : U_l \to U_{l-1})$ modulo π^{l-1} shows that $U_{l-1} \cong U_l/K_{l-1}$.

We now find the sum of the squares of those characters lifted from $U_{l-1} \cong U_l/K_{l-1}$; these are precisely the characters of U_l that have K_{l-1} in their kernel. If ϕ is such a character, and ψ is any irreducible character of U_l having degree 1, then $\psi\phi \in \mathrm{Irr}(U_l)$. Therefore we must find the number of distinct irreducible characters of the form $\psi\phi$. Note: in what follows we will identify the irreducible characters of U_l having K_{l-1} in the kernel with the irreducible characters of U_{l-1}.

Proposition 4.J.1 Let L_l and L_{l-1} be the linear characters of U_l and U_{l-1} respectively, and let $\mathcal{C} = \mathrm{Irr}(U_{l-1})$. The number of distinct irreducible characters of U_l of the form $l\psi$, where $l \in L_l$, $\psi \in \mathcal{C}$ is $[L_l : L_{l-1}]$.

Proof. Let l_1, l_2 be two elements of L_l that are in different cosets of the factor group L_l/L_{l-1}, and suppose that for some $\psi \in \mathcal{C}$ we have $l_1\psi = l_2\psi$. But then $\psi = l_1^{-1}l_2\psi$, which implies that $l_1^{-1}l_2 \in L_{l-1}$ which is a contradiction. Thus the number of distinct *cl* characters is not less than the index of L_{l-1} in L_l. On the other hand, if $l_1^{-1}l_2 \in L_{l-1}$ then $l_1^{-1}l_2\psi = \psi' \in \mathcal{C}$ and $l_2\psi = l_1\psi'$. Thus each $\psi \in \mathcal{C}$ produces $[L_l : L_{l-1}]$ irreducible characters of U_l of the form lc. Consequently, the contribution of the characters of U_{l-1} to the sum of squares of the degrees of the characters of U_l is $|U_{l-1}|[L_l : L_{l-1}]$. □

From this we can find the sum of squares of the characters inflated from U_{l-1}.

Proposition 4.J.2 The sum of the squares of the irreducible characters of U_l that are inflated from U_{l-1} is $q|U_{l-1}|$.

Proof. We claim that the A matrices that lead to linear characters of U_l are scalar matrices:

If $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $l = 2m$, the inertia group of ϕ_A is U_l. We can show that ϕ_A extends to U_l, since applying ϕ_A to $\left(\begin{smallmatrix} 1+\pi^m a_1+\pi^m a_2\sqrt{\alpha} & \pi^m b\sqrt{\alpha} \\ \pi^m c\sqrt{\alpha} & 1-\pi^m a_1+\pi^m a_2\sqrt{\alpha} \end{smallmatrix}\right) \in K_m$, gives $\lambda(\pi^m(2a_2))$. We can show that this is the restriction to K_m of a linear character on U_l: let λ^* be a character on the multiplicative subgroup of R_l, chosen so that $\lambda^*(1 + \pi^m(r)) = \lambda(\pi^m r)$. Now define the linear character χ on U_l thus: for all $g \in U_l$, $\chi(g) = \lambda^*(\det(g))$. Then χ restricted to K_m gives $\lambda^*(1 + \pi^m(2a_2)) = \lambda(\pi^m(2a_2))$. Hence ϕ_A extends to its stabilizer U_l and so leads to a linear character. The same argument applies when $l = 2m + 1$. To show that only scalar A matrices lead to linear characters of U_l, suppose that A is given, where ϕ_A leads, via Clifford theory, to a linear character of U_l. Then the inertia group of ϕ_A must be U_l itself. But we know from equation 4.6 that, modulo π^m the A matrix must be in the center of U_l, hence scalar.

The scalar A matrices that lead to linear characters of U_l having K_{l-1} in the kernel will be those scalars having π as a factor. Thus $A = \left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)$, $x \in R_m$ produces a linear character on U_l and there are q^m such A matrices, whereas $A = \left(\begin{smallmatrix} \pi x & 0 \\ 0 & \pi x \end{smallmatrix}\right)$, $x \in R_{m-1}$ produces a linear character on U_l with K_{l-1} in the kernel, and there are q^{m-1} such A matrices. From this we conclude that $[L_l : L_{l-1}] = q$. Thus the sum of the squares of the inflated characters is $q|U_{l-1}| = q^{4l-6}(q-1)(q+1)^2$

□

Adding this to the sum of squares previously determined gives:

$$q^{4l-6}(q-1)(q+1)(q^3-1)+q^{4l-6}(q-1)(q+1)^2=q^{4l-3}(q-1)(q+1)^2=|U_l|$$

It follows that we have found the degrees and numbers of all irreducible characters of U_l.

4.K Some Calculations of Conjugacy Classes

1. $g = \left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)$

   The number of such class representatives is the number of norm one elements, or $q^{l-1}(q+1)$. Since these elements are in the center of U_l, the centralizer for each representative will be U_l, so the number of elements accounted for is $q^{l-1}(q+1)$.

2. $g = \left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right),\ y \neq x$

   Note that such an element is conjugate to $\left(\begin{smallmatrix} y & 0 \\ 0 & x \end{smallmatrix}\right)$. To find the centralizer let:

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right) = \left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$$

   so that $b(x-y) = 0 = c(x-y)$, and since $y = (\overline{x})^{-1}$, we have $b(x\overline{x}-1) = 0 = c(x\overline{x}-1)$. By construction $x\overline{x}-1 \neq 0$; thus we consider the cases where $x\overline{x}-1$ is and is not a unit in R_l.

   (a) Let $x\overline{x}-1$ be a unit: There are $q^{l-1}(q-1)$ units in R_l, and by considering the kernel of the modulo π map $f : R_l \to R_1$, we see that q^{l-1} of them are congruent to 1 modulo π. Therefore there are $q^{l-1}(q-1) - q^{l-1} = q^{l-1}(q-2)$ units that are not congruent to 1 modulo π; thus if $x\overline{x}$ equals one of these units, then $x\overline{x}-1$ is not in πR_l, and is thus a unit. This gives us $\frac{1}{2}q^{2l-2}(q-2)(q+1)$ class representatives with $x\overline{x}-1$ a unit. Elements in the centralizer have the form: $\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)$ of which there are $q^{2l-2}(q-1)(q+1)$, so that each class has size $q^{2l-1}(q+1)$.

(b) Let π divide $x\overline{x}-1$, so that $x\overline{x}=1+\pi^i M$, where M is a unit. There are $q^{l-1-i}(q-1)$ choices for M, and $q^{l-1}(q+1)$ choices for x such that $x\overline{x}=1+\pi^i M$. This gives us $\frac{1}{2}q^{2l-2-i}(q-1)(q+1)$ class representatives. For a fixed i, the centralizer will be all matrices of the form:

$$\begin{pmatrix} a & \pi^{l-i}b \\ \pi^{l-i}c & d \end{pmatrix}$$

The size of the centralizer for fixed i is $q^{2l-2+2i}(q-1)(q+1)$ so that each class has size $q^{2l-1-2i}(q+1)$.

3. $g=\begin{pmatrix} x & y \\ y & x \end{pmatrix}$, $y\neq 0$

Again the centralizer depends on the highest power of π dividing y:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x & y \\ y & x \end{pmatrix}=\begin{pmatrix} x & y \\ y & x \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

implies $y(b-c)=0=y(a-d)$.

Since $x\overline{y}+\overline{x}y=0$, we can write $y=x\pi^i r\sqrt{\alpha}$ where r is a unit in R_l (note that $r=0$ has been counted above.) We now consider two cases:

(a) $i=0$: thus y is a unit. We will show that there are $q^{2l-1}(q+1)$ such matrices:

We consider the map $f: U_l \to U_1$ where U_l is the unitary matrices over $R_{1,\alpha}$, U_1 is the unitary matrices over $R_{1,\alpha}/\pi R_{1,\alpha}$, and under this map each element in a matrix in U_l is sent to its value modulo π. The kernel of this surjective map is $K_1=\begin{pmatrix} 1+\pi x & \pi y \\ \pi y & 1+\pi x \end{pmatrix}$, which has size q^{2l-2}. The matrices that we are counting (in which y is a unit) are the pre-images of the matrices in U_1 of the form $\left(\begin{smallmatrix} x & y \\ y & x \end{smallmatrix}\right)$ where $y \neq 0$. These number $(q+1)(q+1)-(q+1)=q(q+1)$, hence we are considering $q^{2l-2}q(q+1)=q^{2l-1}(q+1)$ matrices. We multiply by 1/2, because $\left(\begin{smallmatrix} x & y \\ y & x \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} x & -y \\ -y & x \end{smallmatrix}\right)$ are similar, getting $\frac{1}{2}q^{2l-1}(q+1)$ class representatives. The centralizer of each representative is the set of all matrices of the form $\left(\begin{smallmatrix} x & y \\ y & x \end{smallmatrix}\right)$, of which there are $q^{2l-2}(q+1)^2$ making the size of each class $q^{2l-1}(q-1)$.

(b) $i \neq 0$: we can write $y = x(\pi^i r)\sqrt{\alpha}$. There are $q^{l-i-1}(q-1)$ choices for $\pi^i r$, and since $x\overline{x}+y\overline{y}=1$ and $y=x\pi^i r\sqrt{\alpha}$, we can combine these equations to get $x\overline{x}=(1-\pi^{2i}r^2\alpha)^{-1}$. Thus x must lie in the pre-image of $(1-\pi^{2i}r^2\alpha)^{-1}$ in the norm map, which gives $q^{l-1}(q+1)$ choices for x. Furthermore, since $\frac{1}{2}q^{2l-2-i}(q-1)(q+1)$ class representatives because $\left(\begin{smallmatrix} x & y \\ y & x \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} x & -y \\ -y & x \end{smallmatrix}\right)$ are similar, there are $\frac{1}{2}q^{2l-2-i}(q-1)(q+1)$ class representatives. The centralizer, for fixed i, of a representative is the set of matrices with form $\left(\begin{smallmatrix} a & c+\pi^{l-i}W \\ c & a+\pi^{l-i}R \end{smallmatrix}\right)$. This set numbers $q^{2l-2-2i}(q+1)^2$ (Found by considering the map modulo π^{l-i} from U_l to U_{l-i}) so that the size of each class is $q^{2l-1-2i}(q-1)$.

4. $g = \left(\begin{smallmatrix} x & \pi^{i+1}\beta y \\ \pi^i y & x \end{smallmatrix}\right)$ where y is a unit in $R_{l,\alpha}$, and $\beta \in R_{l-i-1}$. As in previous cases, the centralizer depends on i. It is:

$$\left(\begin{smallmatrix} a & \pi\beta c+\pi^{l-i}N \\ c & a+\pi^{l-i}M \end{smallmatrix}\right)$$

where M, N are elements of $R_{l,\alpha}$. To see this, we can think of the above element in the form: $I + \pi^r B$ so that in calculating the size of the centralizer we need only be concerned with $\pi^r B$. From

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\pi^i\left(\begin{smallmatrix} 0 & \pi\beta y \\ y & 0 \end{smallmatrix}\right) = \pi^i\left(\begin{smallmatrix} 0 & \pi\beta y \\ y & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$$

we get:

$$\pi^i\left(\begin{smallmatrix} by & \pi\beta ay \\ dy & \pi\beta cy \end{smallmatrix}\right) = \pi^i\left(\begin{smallmatrix} \pi\beta cy & \pi\beta dy \\ ay & by \end{smallmatrix}\right)$$

Thus $\pi^i(ay - dy) = 0$, and $\pi^i(by - \pi\beta cy) = 0$; since y is a unit, we conclude that $d = a + \pi^{l-i}M$, and $b = \pi\beta c + \pi^{l-i}N$.

Under the projection map $f : U_l \to U_{l-i}$, (which takes the value modulo π^{l-i} of the matrix entries) restricted to the centralizer subgroup, the image of f is $\left(\begin{smallmatrix} a & \pi\beta c \\ c & a \end{smallmatrix}\right)$. The size of this image is $q^{2l-1-2i}(q+1)$, and the kernel is K_{l-i} with size q^{4i}. (These counts use the arguments from the section on surjectivity). Thus the size of the centralizer for fixed i is $q^{2l-1+2i}(q+1)$ and the size of each conjugacy class is $q^{2l-2-2i}(q-1)q+1)$.

For a class representative with i fixed we will show that all choices for y produce conjugate matrices. Recall that $y = \pi^i xr\sqrt{\alpha}$ where $r \in R_{l-i}$. Let i be fixed, and let y_1, y_2 be units in R_{l-i} with $y_2 = ky_1$ for k a unit in R_l. Choose $z \in R_{l,\alpha}$ such that $z\overline{z} = k$. Then conjugation of $\left(\begin{smallmatrix} x & \pi^{i+1}\beta y_1 \\ \pi^i y_1 & x \end{smallmatrix}\right)$ by $\left(\begin{smallmatrix} z & 0 \\ 0 & (\overline{z})^{-1} \end{smallmatrix}\right)$ gives $\left(\begin{smallmatrix} x & \pi^{i+1}\beta' ky_1 \\ \pi^i ky_1 & x \end{smallmatrix}\right) = \left(\begin{smallmatrix} x & \pi^{i+1}\beta' y_2 \\ \pi^i y_2 & x \end{smallmatrix}\right)$, where $\beta' = \beta(k^2)^{-1}$.

Therefore for a fixed i, we can only get non-conjugate matrices from our choices of β and x; there are q^{l-i-1} choices for β, and $q^{l-1}(q+1)$ choices for x which must satisfy $x\overline{x} + \pi^i y\overline{\pi^{i+1}\beta y} = 1$. This forces x to be in a particular coset of the norm 1 elements, hence the number of choices. In all, for a fixed i, there are $q^{2l-2-i}(q+1)$ conjugacy class representatives.

Below we summarize the conjugacy classes:

Table 4.4: Conjugacy Classes of U_l

Type	Number($i = 0$)	Number of classes ($i \neq 0$)	class size
1	-	$q^{l-1}(q+1)$	1
2	$q^{2l-1}(q+1)$	$q^{2l-2-i}(q+1)$	$q^{2l-2-2i}$
3	$\frac{1}{2}q^{2l-1}(q+1)$	$\frac{1}{2}q^{2l-2-i}(q-1)(q+1)$	$q^{2l-1-2i}(q-1)$
4	$\frac{1}{2}q^{2l-2}(q-2)(q+1)$	$\frac{1}{2}q^{2l-2-i}(q-1)(q+1)$	$q^{2l-1-2i}(q+1)$

Chapter 5

Adjoining $\sqrt{\pi}$ to $R/\pi R$

Let R_l be the ring defined in chapter 3; if the modulus is clearly l, we will write this simply as R. In chapter 4, we adjoined the square root of a unit of R, while in this chapter we adjoin the square root of π. Since the larger argument about the degrees of irreducible characters of U_l is inductive, we start with the base case; that is, we adjoin $\sqrt{\pi}$ to a finite field. By construction, $R/\pi R$ is isomorphic to some finite field $\mathbb{F}_q$, where q is a power of an odd prime p; we will write $\mathbb{F}$ for the quotient ring $R/\pi R$. We adjoin $\sqrt{\pi}$ to $\mathbb{F}$ to get a quadratic extension $\mathbb{F}_\pi = \mathbb{F}[\sqrt{\pi}] = \{a + b\sqrt{\pi}\}$, $a, b \in \mathbb{F}$.

$\mathbb{F}_\pi$ has $|\mathbb{F}|^2 = q^2$ elements and $q(q-1)$ units (i.e $a \neq 0$). Define conjugation in $\mathbb{F}_\pi$ by $\overline{(a + b\sqrt{\pi})} = a - b\sqrt{\pi}$, and let the norm map $N : \mathbb{F}_\pi^* \to \mathbb{F}^*$ be given by:

$$N(a + b\sqrt{\pi}) = (a + b\sqrt{\pi})\overline{(a + b\sqrt{\pi})} = a^2$$

Clearly $a + b\sqrt{\pi}$ has norm 1 if and only if $a = \pm 1$, therefore R_π contains $2q$ elements of norm 1. The image of N is the set of squares in $\mathbb{F}$, so the norm map is not surjective, and there might be distinct conjugate linear forms on the module $\mathbb{F}_\pi \times \mathbb{F}_\pi$. In what follows we will use the form whose associated matrix is $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Denote by U, the 2×2 unitary matrices over $\mathbb{F}_\pi$. The use of $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ as the matrix of the form means that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ over $\mathbb{F}_\pi$ will be unitary if and only if:

1. $a\overline{b} + \overline{a}b = a\overline{c} + \overline{a}c = 0$
2. $d\overline{b} + \overline{d}b = d\overline{c} + \overline{d}c = 0$
3. $a\overline{d} + c\overline{b} = 1$

These conditions make the unitary 2×2 matrices quite constrained and easy to count, for if $\left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right)$ is unitary, then $x\overline{y} + \overline{x}y = 0$ and since at least one of x, y is a unit, dividing both sides by, say, $x\overline{x}$ gives $\overline{(\frac{y}{x})} + \frac{y}{x} = 0$, so $\frac{y}{x} = r\sqrt{\pi},\ r \in \mathbb{F}$, and in all cases precisely one of x, y is a unit, while the other is a multiple of this unit and a pure root. We need consider only two cases:

1. If $x = x_1 + x_2\sqrt{\pi}$ is a unit, then $\frac{y}{x} = r\sqrt{\pi} =,\ r \in \mathbb{F}$ so

   $$y = (x)r\sqrt{\pi} = (x_1 + x_2\sqrt{\pi})r\sqrt{\pi} = (x_1)r\sqrt{\pi}$$

   so y is a pure root. Similarly z is a pure root. But then $w = \frac{1}{x}$ so we have:

   $$\left(\begin{smallmatrix} x_1+x_2\sqrt{\pi} & y\sqrt{\pi} \\ z\sqrt{\pi} & (x_1-x_2\sqrt{\pi})^{-1} \end{smallmatrix}\right)$$

   $x_1, x_2, y, z \in \mathbb{F}$ and $x_1 \neq 0$.

2. If x is not a unit, then $y = y_1 + y_2\sqrt{\pi}$ must be a unit, and by the argument above, x and w must be pure roots, and $z = \frac{1}{y}$, so the unitary matrix has the form:

$$\begin{pmatrix} x\sqrt{\pi} & y_1+y_2\sqrt{\pi} \\ (y_1-y_2\sqrt{\pi})^{-1} & w\sqrt{\pi} \end{pmatrix}$$

$x, y_1, y_2, w \in \mathbb{F}$ and $y_1 \neq 0$.

In each case we have $q^3(q-1)$ possible matrices, so the size of the unitary group U is $2q^3(q-1)$.

5.A Conjugacy Classes

Let H denote the subgroup of U with units on the main diagonal, and non units on the second diagonal; i.e. $H = \left\{\begin{pmatrix} x_1+x_2\sqrt{\pi} & y\sqrt{\pi} \\ z\sqrt{\pi} & (x_1-x_2\sqrt{\pi})^{-1} \end{pmatrix}\right\}$ $\cap\, U$. Since $[U : H] = 2$, H is normal and $U = H \cup \left(H\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$. To find the conjugacy classes of U, we begin with those classes that lie in H. In order to avoid the use of subscripts where possible, in what follows, x, y, a, b etc. will represent elements of R_π, but, for example, $y\sqrt{\pi}$ will represent a non- unit, with $y \in \mathbb{F}$.

5.A.1 Conjugacy Classes in H

1. $\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$

   Since x is norm 1, there are $2q$ such class representatives, all in the center of U, so size of each conjugacy class is 1, accounting for $2q$ elements of H.

2. $\left(\begin{smallmatrix} x & y\sqrt{\pi} \\ 0 & x \end{smallmatrix}\right)$, $y \in \mathbb{F}^*$. There are $(q-1)$ choices for $y \neq 0$ and $2q$ choices for the norm 1 element x. Conjugating of $\left(\begin{smallmatrix} x & y\sqrt{\pi} \\ 0 & x \end{smallmatrix}\right)$ by any diagonal element $\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$ gives $\left(\begin{smallmatrix} x & a\overline{a}y\sqrt{\pi} \\ 0 & x \end{smallmatrix}\right)$ where $a\overline{a}$ is a non zero square in $\mathbb{F}$. Thus the number of class representatives is:

$$2q(q-1)/\left(\frac{q-1}{2}\right) = 4q$$

The centralizer of these representatives is the set of matrices having the form $\left(\begin{smallmatrix} a & b\sqrt{\pi} \\ c\sqrt{\pi} & a \end{smallmatrix}\right)$ which has order $2q^3$:

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} x & y\sqrt{\pi} \\ 0 & x \end{smallmatrix}\right) = \left(\begin{smallmatrix} x & y\sqrt{\pi} \\ 0 & x \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$$
$$\left(\begin{smallmatrix} ax & ya_1\sqrt{\pi}+bx \\ cx & yc_1\sqrt{\pi}+dx \end{smallmatrix}\right) = \left(\begin{smallmatrix} ax+yc_1\sqrt{\pi} & bx+yd_1\sqrt{\pi} \\ cx & dx \end{smallmatrix}\right)$$

Thus $yc_1 = 0$ and since $y \neq 0$ then $c_1 = 0$ which means c is a pure roots. This implies that a, d are units and b is a pure root. In addition, $ya_1 = yd_1$ so that $a_1 = d_1$, and since b, c are pure roots, a, d are norm 1 so $a\overline{d} = 1$, we can write $a = \pm 1 a_2\sqrt{p}, d = \pm 1 d_2\sqrt{p}$ and $a\overline{d} = 1$ implies that $a_2 = d_2$. We have $2q$ choices for a, and q choices for each of b and c, hence the class size is $\frac{2q^3(q-1)}{2q^3} = q-1$, and we have accounted for $4q(q-1)$ elements of H.

3. $\left(\begin{smallmatrix} x & 0 \\ 0 & w \end{smallmatrix}\right)$, $x \neq w$

Since $w = \frac{1}{x}$ then if, say x were norm 1, $x\overline{x} = 1$ implies $w = \frac{1}{x} = x$. Thus x, w cannot be norm 1. There are $q(q-3)$ ways of choosing $x = x_1 + x_2\sqrt{\pi}$, since $x_1 \neq 0, \pm 1$, and since $\left(\begin{smallmatrix} x & 0 \\ 0 & w \end{smallmatrix}\right)$ is similar to $\left(\begin{smallmatrix} w & 0 \\ 0 & x \end{smallmatrix}\right)$, there are $\frac{q(q-3)}{2}$ such representatives. The centralizer is the subgroup of diagonal matrices, which has order $q(q-1)$, thus the class size is $2q^2$. This accounts for $\frac{q(q-3)}{2} \cdot 2q^2 = q^3(q-3)$ elements of H.

4. $\left(\begin{smallmatrix} x & y\sqrt{\pi} \\ z\sqrt{\pi} & x \end{smallmatrix}\right)$, $y, z \neq 0, z = e^2 y,\ e \in \mathbb{F}$

Note that distinguishing conjugacy classes according to whether the ratio of y, z is a square or non-square is due to the fact that the image of the norm map is the set of squares in $\mathbb{F}$. There are $2q(q-1)\frac{q-1}{2}$ class representatives when the ratio is a square. Conjugating by elements of H and its complement separately, we get (ignoring the scalar part of $\left(\begin{smallmatrix} x & y\sqrt{\pi} \\ z\sqrt{\pi} & x \end{smallmatrix}\right)$)

$$\begin{pmatrix} a & b\sqrt{\pi} \\ c\sqrt{\pi} & d \end{pmatrix}\begin{pmatrix} 0 & y\sqrt{\pi} \\ y\sqrt{\pi} & 0 \end{pmatrix}\begin{pmatrix} d & -b\sqrt{\pi} \\ -c\sqrt{\pi} & a \end{pmatrix}(ad)^{-1} = \begin{pmatrix} 0 & a_1^2 y\sqrt{\pi} \\ a_1^{-2} y\sqrt{\pi} & 0 \end{pmatrix}$$

$$\begin{pmatrix} a\sqrt{\pi} & b \\ c & d\sqrt{\pi} \end{pmatrix}\begin{pmatrix} 0 & y\sqrt{\pi} \\ y\sqrt{\pi} & 0 \end{pmatrix}\begin{pmatrix} d\sqrt{\pi} & -b \\ -c & a\sqrt{\pi} \end{pmatrix}(-bc)^{-1} = \begin{pmatrix} 0 & b_1^2 y\sqrt{\pi} \\ b_1^{-2} y\sqrt{\pi} & 0 \end{pmatrix}$$

Thus upper right entry of the conjugated matrix will be $e^2 y\sqrt{\pi}$ for every $e \in \mathbb{F}$, so these class representatives will be similar in sets of $\frac{q-1}{2}$. Therefore we have $2q(q-1)$ class representatives. The centralizer consists first, of all matrices having the $\left(\begin{smallmatrix} a & b\sqrt{\pi} \\ c\sqrt{\pi} & a \end{smallmatrix}\right)$, since $a_1^2 = 1 \Leftrightarrow a \in \mathcal{L}$. Also in the centralizer are matrices of the form $\left(\begin{smallmatrix} a\sqrt{\pi} & b \\ b & d\sqrt{\pi} \end{smallmatrix}\right)$, since we require $b_1^2 = 1$. As a result, the centralizer has order $4q^3$, the class size is $\frac{q-1}{2}$, and we have accounted for $2q(q-1)\frac{q-1}{2} = q(q-1)^2$ elements of H in this case.

5. $\left(\begin{smallmatrix} x & y\sqrt{\pi} \\ ey\sqrt{\pi} & x \end{smallmatrix}\right)$, $y \neq 0, e \notin \mathbb{F}^{*2}$

Again we can write $2q(q-1)$ such elements, and we conjugate as before:

$$\begin{pmatrix} a & b\sqrt{\pi} \\ c\sqrt{\pi} & d \end{pmatrix}\begin{pmatrix} 0 & y\sqrt{\pi} \\ ey\sqrt{\pi} & 0 \end{pmatrix}\begin{pmatrix} d & -b\sqrt{\pi} \\ -c\sqrt{\pi} & a \end{pmatrix}(ad)^{-1} = \begin{pmatrix} 0 & a_1^2 y\sqrt{\pi} \\ a_1^{-2} ey\sqrt{\pi} & 0 \end{pmatrix}$$

$$\begin{pmatrix} a\sqrt{\pi} & b \\ c & d\sqrt{\pi} \end{pmatrix}\begin{pmatrix} 0 & y\sqrt{\pi} \\ ey\sqrt{\pi} & 0 \end{pmatrix}\begin{pmatrix} d\sqrt{\pi} & -b \\ -c & a\sqrt{\pi} \end{pmatrix}(-bc)^{-1} = \begin{pmatrix} 0 & b_1^2 ey\sqrt{\pi} \\ b_1^{-2} y\sqrt{\pi} & 0 \end{pmatrix}$$

It is clear that the upper right elements $b_1^2 ey$ and $a_1^2 y$ will take on all $q-1$ values in $\mathbb{F}$, so that the conjugacy size is $q-1$. It is clear from the above that the centralizer consists only of the matrices of the form $\begin{pmatrix} a & b\sqrt{\pi} \\ c\sqrt{\pi} & a \end{pmatrix}$, and has size $2q^3$. This makes the class size $q-1$, and we account for $q(q-1)^2$ elements of H.

We have accounted for

$$2q + 4q(q-1) + q(q-1)^2 + q(q-1)^2 + q^3(q-3) = q^3(q-1)$$

elements, which is the order of H.

5.A.2 Conjugacy Classes Not in H

1. $\begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix}$

   Note that $y\overline{x} = 1$ implies that $yx\overline{x} = x$. There are $q(q-1)$ such matrices, similar to $\frac{q-1}{2}$ matrices of the form $\begin{pmatrix} 0 & a\overline{a}x \\ \frac{1}{a\overline{a}}y & 0 \end{pmatrix}$, and to another $\frac{q-1}{2}$ of the form $\begin{pmatrix} 0 & a\overline{a}y \\ \frac{1}{a\overline{a}}x & 0 \end{pmatrix}$. The second set is superfluous however, $\overline{x}xy = x$. It follows that there are $q(q-1)/((q-1)/2) = 2q$ class representatives.

   The centralizer has order $4q^2$; to show this, we consider separately, conjugation of $\begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix}$ by elements that are in H and by those that are not.

   (a) Elements in H:

$$\begin{pmatrix} a & b\sqrt{\pi} \\ c\sqrt{\pi} & d \end{pmatrix}\begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix} = \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix}\begin{pmatrix} a & b\sqrt{\pi} \\ c\sqrt{\pi} & d \end{pmatrix}$$

$$\begin{pmatrix} by_1\sqrt{\pi} & ax \\ dy & cx_1\sqrt{\pi} \end{pmatrix} = \begin{pmatrix} cx_1\sqrt{\pi} & dx \\ ay & by_1\sqrt{\pi} \end{pmatrix}$$

Thus $a = d$, and $b = c(x_1/y_1)$, and there are $2q^2$ such elements.

(b) Elements not in H:

$$\begin{pmatrix} a\sqrt{\pi} & b \\ c & d\sqrt{\pi} \end{pmatrix}\begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix} = \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix}\begin{pmatrix} a\sqrt{\pi} & b \\ c & d\sqrt{\pi} \end{pmatrix}$$

$$\begin{pmatrix} by & ax_1\sqrt{\pi} \\ dy_1\sqrt{\pi} & cx \end{pmatrix} = \begin{pmatrix} cx & dx_1\sqrt{\pi} \\ ay_1\sqrt{\pi} & by \end{pmatrix}$$

Hence $a = d$, and $by = cx \Rightarrow \frac{b}{\overline{x}} = \frac{x}{\overline{b}}$, so that $b\overline{b} = x\overline{x}$. Hence there are $2q$ choices for b (it is in the same pre image of the norm map as x). Therefore there are $2q^2$ elements of this form, and the centralizer has size $4q^2$.

The class size is $2q^3(q-1)/(4q^2) = \frac{q(q-1)}{2}$, and we have accounted for $2q\frac{q(q-1)}{2} = q^2(q-1)$ elements not in H.

2. $\begin{pmatrix} z\sqrt{\pi} & x \\ y & z\sqrt{\pi} \end{pmatrix}$, $z \neq 0$. Note that since this matrix is not trace zero, it's conjugacy class is distinct from that of the element above.

It is clear that the centralizer will be the same as that of $\begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix}$, so that the class size is $\frac{q(q-1)}{2}$. There are $q(q-1)$ choices for x and $q-1$ choices for z. Since $\begin{pmatrix} z\sqrt{\pi} & x \\ y & z\sqrt{\pi} \end{pmatrix}$ is similar to $\begin{pmatrix} z\sqrt{\pi} & a\overline{a}x \\ (a\overline{a})^{-1}y & z\sqrt{\pi} \end{pmatrix}$. This includes the matrix $\begin{pmatrix} z\sqrt{\pi} & y \\ x & z\sqrt{\pi} \end{pmatrix}$, so there are $2q(q-1)$ non-similar class representatives. This accounts for $q^2(q-1)^2$ elements, thus we have found all $q^2(q-1)+q^2(q-$

$1)^2 = q^3(q-1)$ elements in the complement of H.

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We summarize the conjugacy classes of U below: (Those in H are above the double line)

Table 5.1: Conjugacy Classes of U_1

Representative	Number of Representatives	Class Size	Elements
$\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$	$2q$	1	$2q$
$\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$	$\frac{q(q-3)}{2}$	$2q^2$	$q^3(q-3)$
$\begin{pmatrix} x & y\sqrt{\pi} \\ 0 & x \end{pmatrix}$	$4q$	$q-1$	$4q(q-1)$
$\begin{pmatrix} x & y\sqrt{\pi} \\ z\sqrt{\pi} & x \end{pmatrix}$, $y, z \neq 0, z = e^2y,\ e \in \mathbb{F}$	$2q(q-1)$	$\frac{q-1}{2}$	$q(q-1)^2$
$\begin{pmatrix} x & y\sqrt{\pi} \\ ey\sqrt{\pi} & x \end{pmatrix}$ $e \notin \mathbb{F}^{*2}$	$q(q-1)$	$q-1$	$q(q-1)^2$
$\begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix}$	$2q$	$\frac{q(q-1)}{2}$	$q^2(q-1)$
$\begin{pmatrix} z\sqrt{\pi} & x \\ y & z\sqrt{\pi} \end{pmatrix}$	$2q(q-1)$	$\frac{q(q-1)}{2}$	$q^2(q-1)^2$

5.B Generators

It is possible to list all generators of U

1. $g_1 = \begin{pmatrix} 1 & 0 \\ \sqrt{\pi} & 1 \end{pmatrix}$
2. $g_2 = \begin{pmatrix} 1 & \sqrt{\pi} \\ 0 & 1 \end{pmatrix}$
3. $g_3 = \begin{pmatrix} 1+\sqrt{\pi} & 0 \\ 0 & 1+\sqrt{\pi} \end{pmatrix}$
4. $g_4 = \left\{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right\},\ a \in \mathbb{F}_p^*$

5. $g_5 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$

Note: we can get by without $g_2 = g_5^{-1} g_1 g_5$, but it is useful for establishing a standard way of writing the elements of U.

5.C Characters Degrees of U

We find the degrees and numbers of the irreducible characters of U by exploiting the fact that U is a semi-direct product.

Theorem 5.C.1 The degrees of the irreducible characters of U, and the number of characters of each degree are:

1. $4q$ characters of degree 1.
2. $\frac{q(q-3)}{2}$ characters of degree 2.
3. $4q(q-1)$ characters of degree $\frac{q-1}{2}$.
4. $q(q+3)$ characters of degree $q-1$.

Proof. From [S2] (p62): Let A, H be two subgroups of a group G, with A normal and abelian, and $G = A \rtimes H$. We can express (and count) all of the irreducible characters of G in terms of those of A and certain subgroups of H. There is an H action on $\{\chi\}$, the set of (linear) characters of A; for all $h \in H$, $a \in A$, let $h(\chi)(a) = \chi(h^{-1}ah)$. Let $\{\chi_i\}$ be a set of orbit representatives of this action, and let H_i be the subgroup of H that stabilizes χ_i. Denote by G_i the group $A \rtimes H_i$. If ρ is an irreducible character of H_i, we may consider both χ_i and ρ to be characters of G_i:

1. For χ an irreducible character of A, and any $ah \in A \rtimes H_i$, define $\chi_i(ah) = \chi_i(a)$.

2. For ρ and irreducible character of H_i, and π the canonical projection from G_i to H_i, we see that $\rho \circ \pi$ is an irreducible character of G_i. For simplicity, we will also write ρ for this character of G_i.

We induce $\chi_i \otimes \rho$ to G, to get the character $\gamma_{i,\rho}$ and:

Proposition 5.C.1 $\gamma_{i,\rho}$ is an irreducible character of G_i; if $\gamma_{i,\rho}$ is isomorphic to $\gamma_{i',\rho'}$ then $i = i'$, and ρ is isomorphic to ρ', and finally, every irreducible character of G is isomorphic to some $\gamma_{i,\rho}$.

Proof. [S2] page 62

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To apply this method of finding characters to U, we note that $U = K_1 \rtimes \mathcal{D}$, where K_1 is the set of matrices in U with the form $\begin{pmatrix} 1+x\sqrt{\pi} & y\sqrt{\pi} \\ z\sqrt{\pi} & 1+x\sqrt{\pi} \end{pmatrix}$, $x, y, z \in \mathbb{F}$, and $\mathcal{D} = D \rtimes J$, where D is the group of diagonal matrices over $\mathbb{F}$ in U, that is, those of the form: $\begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix}$, $b \in \mathbb{F}^*$, while J is the group of order two generated by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

We first examine the irreducible characters K_1 and $\mathcal{D}$. Since $\mathcal{D}$ is itself a semi-direct product, it will require the method of Serre.

1. It is easy to see that K_1 is isomorphic to three copies of $\mathbb{F}^+$, so every character χ on K_1 can be written $\chi = \lambda_1(x)\lambda_2(y)\lambda_3(z)$, where each λ_i is a character of $\mathbb{F}^+$. In what follows, we will always keep to the same order for these characters, i.e. from left to right the lambdas operate on

x, y, z. We will denote by λ° the trivial character, and we will drop the subscript if λ is arbitrary in $\mathrm{Irr}(F^+)$.

2. We have $\mathcal{D} \cong D \rtimes J$; as D is isomorphic to $\mathbb{F}^*$, it has order $q-1$. For each character of D, we find its stabilizer under the J conjugation action, as well as the size of its orbit. Let α be a primitive generator of $\mathbb{F}^*$. For any character χ of D, the stabilizer will be J itself if and only if:

$$\chi\left(\begin{smallmatrix} b & 0 \\ 0 & b^{-1} \end{smallmatrix}\right) = \chi\left(\begin{smallmatrix} b^{-1} & 0 \\ 0 & b \end{smallmatrix}\right)$$

This implies that $\chi\left(\begin{smallmatrix} b & 0 \\ 0 & b^{-1} \end{smallmatrix}\right)$ must be $1, \text{or} -1$. Therefore the characters with stabilizer J are the trivial character and the character that sends $\left(\begin{smallmatrix} b & 0 \\ 0 & b^{-1} \end{smallmatrix}\right)$ to $(-1)^n$, where $b = \alpha^n$. We note that both of these characters have an orbit size of 1 under the J action, so that when each of these is tensored with the two characters of J, and the resulting character induced to $\mathcal{D} = D \rtimes J$, we get 4 characters of degree 1.

The remaining $q-3$ characters of D have only the trivial subgroup of J as a stabilizer, since each has the form:

$$\chi\left(\begin{smallmatrix} b & 0 \\ 0 & b^{-1} \end{smallmatrix}\right) = \lambda(b)$$

where λ is neither the trivial nor the alternating character on $\mathbb{F}^*$. These characters have an orbit size of two: χ is conjugate to χ^{-1}. Therefore these characters will result in $\frac{q-3}{2}$ characters of $\mathcal{D}$ having degree 2.

Now we find the number of degrees of the irreducible characters of U as well as the number of characters of each degree:

1. Let χ be the irreducible characters of K_1 given by $\lambda\lambda^\circ\lambda^\circ$ where the first λ is arbitrary; note that there are q such characters, and that under the $\mathcal{D}$ action, each is congruent only to itself. This results in $4q$ irreducible characters of degree 1, and $\frac{q(q-3)}{2}$ of degree 2. The degree 1 characters can easily be made explicit: write an arbitrary element of U as XJ^b, where $X \in H$, $J = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, and $b = 0$ or 1. We have the alternating character; $XJ^b \to (-1)^b$, and the determinant character. There are $2q$ determinant characters, since the determinant is a norm 1 element, and there are $2q$ such elements. Additionally, we have the tensor product of the determinant character and the alternating character, giving another $2q$ characters.

2. To get characters of degree $\frac{q-1}{2}$, let $a \in \mathbb{F}^*$ and $\chi = \lambda(a^2\lambda_3)\lambda_3$. Note that λ is arbitrary, but λ_3 is fixed, so that:

$$\chi\left(\begin{smallmatrix} 1+x\sqrt{\pi} & y\sqrt{\pi} \\ z\sqrt{\pi} & 1+x\sqrt{\pi} \end{smallmatrix}\right) = \lambda(x)\lambda_3(a^2y)\lambda_3(z)$$

   Each such character is stabilized by $\pm I$ and the two element subgroup generated by $\left(\begin{smallmatrix} 0 & a^{-1} \\ a & 0 \end{smallmatrix}\right)$, resulting in characters of U with degree $\frac{2(q-1)}{4} = \frac{q-1}{2}$. Each such character lies in an orbit of size $\frac{q-1}{2}$, since by conjugation, we can change λ_3 to $b^2\lambda_3$ for any non-zero square $b^2 \in \mathbb{F}$. This results in:

$$4q\frac{q-1}{2}(q-1)/\frac{q-1}{2} = 4q(q-1)$$

   characters of degree $\frac{q-1}{2}$.

3. For degree $q-1$, consider first $\chi = \lambda\lambda_2\lambda^\circ$; for $B = \left(\begin{smallmatrix} x & y\sqrt{\pi} \\ z\sqrt{\pi} & x \end{smallmatrix}\right)$, we have:

$$\chi\left(\begin{smallmatrix} 1+x\sqrt{\pi} & y\sqrt{\pi} \\ z\sqrt{\pi} & 1+x\sqrt{\pi} \end{smallmatrix}\right) = \lambda(x)\lambda_2(y)$$

The stabilizer of the $\mathcal{D}$ action is $\pm I$, having two linear characters, and resulting in characters of U with degree $\frac{2(q-1)}{2} = q-1$. Under the $\mathcal{D}$ action, $\lambda\lambda_2\lambda^\circ$ is congruent to $\lambda(a^2\lambda_2)\lambda^\circ$, where for $a \in \mathbb{F}$, $a^2\lambda(y) = \lambda(a^2y)$. Thus the orbits of these characters have size $\frac{q-1}{2}$; the number of distinct non-zero squares in $\mathbb{F}$. This gives us:

$$q(q-1)/\frac{q-1}{2}(2) = 4q$$

characters of degree $q-1$.

Next consider characters χ of K_1 of the form $\lambda(x)(k\lambda_3(y))\lambda_3(z)$, where k is a non-square in $\mathbb{F}$. The stabilizer of the $\mathcal{D}$ action is $\pm I$, giving characters on U of degree $q-1$, and these characters partition into equivalence classes of size $q-1$. Thus we get:

$$q\frac{q-1}{2}(q-1)/(q-1)(2) = q^2 - q$$

Thus in all we have $4q + q^2 - q = q(q+3)$ characters on U of degree $q-1$, and we have justified the numbers in table 5.A.2.

Finally, we note that:

1. The sum of squares of the degrees is $2q^3(q-1)$, the group order.

2. The number of characters is $\frac{11q^2+3q}{2}$, which equals the number of conjugacy classes.

Chapter 6

Adjoining $\sqrt{\pi}$ to R

In chapter 3, we defined a quadratic extension on R_l by adjoining the square root of a non-square unit of R_l. In the previous chapter we adjoined the root of a non-invertible element to the quotient $R/\pi R$. Here we form a quadratic extension of R_l itself by adjoining $\sqrt{\pi}$. We will consider the group of unitary matrices over the ring $R_{l,\pi} = \{a + b\sqrt{\pi}\ , a, b \in R_l\}$, and exploiting the arguments from chapter 3 regarding the expression of elements a, b as quasi polyomials over a fixed transversal $\mathcal{T}$ of πR_l, we find that the order of $R_{l,\pi}$ is q^{2l}, with $q^{2l-1}(q-1)$ units, and $2q^l$ norm 1 elements (we are using the same Hermitian form). From these counts it is clear that the norm map, which takes $a + b\sqrt{\pi}$ to $a^2 - \pi b^2$, surjects onto the square units of R_l; as a consequence we cannot claim that all conjugate linear forms are equivalent. In this work, we will use the same matrix for the form as was used in the $\sqrt{\alpha}$ case. Where there is no possibility of confusion, we will write R and R_π respectively, for R_l and $R_{l,\pi}$; where the modulus is not l we will be more precise. Denote by U_l the group of unitary 2×2 matrices over R_π, using the same matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ for the form.

6.A The Order of the Group

From remark 4.C.1, the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, $a, b, c, d \in R_{l,\pi}$ is unitary if and only if:

- $a\overline{c} + \overline{a}c = b\overline{d} + \overline{b}d = 0$
- $a\overline{b} + \overline{a}b = c\overline{d} + \overline{c}d = 0$
- $a\overline{d} + c\overline{b} = 1$

It is clear that at least one of a and c must be a unit (and similarly with b and d). If a is a unit, then from $a\overline{c} + \overline{a}c = 0$, we divide both sides by $a\overline{a}$ to get

$$\frac{a\overline{c}}{a\overline{a}} + \frac{\overline{a}c}{a\overline{a}} = 0 \tag{6.1}$$

$$\overline{\left(\frac{c}{a}\right)} + \frac{c}{a} = 0 \tag{6.2}$$

Thus $\frac{c}{a}$ is a pure root, and $c = a(r\sqrt{\pi})$, $r \in R$. Similarly, since $b\overline{d} + \overline{b}d = 0$ $b = d(s\sqrt{\pi})$, $s \in R$. Since we have non-invertible elements on the second diagonal, d must be a unit, so $a\overline{d} + c\overline{b} = 1$ and this can be written can be written:

$$a\overline{d} + (a(r\sqrt{\pi})\overline{(d(s\sqrt{\pi}))} = a\overline{d}(1 - \pi rs) = 1$$

Thus $a\overline{d} = (1 - \pi rs)^{-1}$; we choose any $r, s \in R$ and a to be any unit, then d is determined. Hence there are $q^{l-1}(q-1)q^{3l} = q^{4l-1}(q-1)$ such matrices. If we had assumed that c was a unit rather than a, we would have found the same number of elements, therefore the order of U_l is $2q^{4l-1}(q-1)$. It is a convenient feature of these unitary matrices that of any two vertically or horizontally adjacent elements, precisely one is a unit.

6.B The Abelian K Groups

In section 4.E we defined some abelian subgroups of the unitary group. In this section, to avoid fractional powers of π, we alter this definition somewhat and find the order of several important such subgroups.

We define for a positive integer $1 \leq i < 2l$

$$K_i = \{I + \sqrt{\pi}^i B\} \cap U_l$$

We index by the power of $\sqrt{\pi}$ instead π in order to avoid fractional powers of π later on. These subgroups are abelian for $i \geq l$, and we give the orders of the most important:

1. When $l = 2m$ we define characters on $K_l = \{I + \pi^m B\} \cap U_l$. If we consider a typical element: $\begin{pmatrix} 1+\pi^m M & \pi^m N \\ \pi^m Q & 1+\pi^m S \end{pmatrix}$, $M, N, Q, S \in R_\pi$ and note that the entries on the main diagonal are both units, then by section 6.A, $\pi^m N = (1 + \pi^m M) r\sqrt{\pi}$. This implies that r has a factor of π^m, therefore $\pi^m N$ can be written $\pi^m b\sqrt{\pi}$ with $b \in R$. Similarly $\pi^m Q$ can be written $\pi^m c\sqrt{\pi}$ for $c \in R$. In addition, $(1 + \pi^m M)\overline{(1 + \pi^m S)} = 1$, and this implies that $S = -\overline{M}$. Therefore we can write elements of K_l explicitly as

   $$I + \begin{pmatrix} \pi^m a_1 + \pi^m a_2\sqrt{\pi} & \pi^m b\sqrt{\pi} \\ \pi^m c\sqrt{\pi} & -\pi^m a_1 + \pi^m a_2\sqrt{\pi} \end{pmatrix},\ a_1, a_2, b, c \in R_l$$

   and the order of K_l is $q^{4m} = q^{2l}$.

   We will also need to consider the group $K_{l-1} = \{I + \pi^{m-1}\sqrt{\pi}B\} \cap U_l$. We count this subgroup, by writing a typical element explicitly:

$$I + \begin{pmatrix} \pi^m a_1 + \pi^{m-1} a_2 \sqrt{\pi} & \pi^{m-1} b \sqrt{\pi} \\ \pi^{m-1} c \sqrt{\pi} & \pi^m d_1 + \pi^{m-1} d_2 \sqrt{\pi} \end{pmatrix}, \; a_1, a_2, b, c \in R_l$$

thus the order of $|K_{l-1}|$ is $q^{4m+3} = q^{2l+3}$. It is useful to note that K_{l-1} is generated by K_l and the following subgroups:

(a) $\mathcal{G}_1 = \left\{ \begin{pmatrix} 1+\pi^{m-1} f\sqrt{\pi} & 0 \\ 0 & [1-\pi^{m-1} f\sqrt{\pi}]^{-1} \end{pmatrix} \right\}$

(b) $\mathcal{G}_2 = \left\{ \begin{pmatrix} 1 & 0 \\ \pi^{m-1} f\sqrt{\pi} & 1 \end{pmatrix} \right\}$

(c) $\mathcal{G}_3 = \left\{ \begin{pmatrix} 1 & \pi^{m-1} f\sqrt{\pi} \\ 0 & 1 \end{pmatrix} \right\}$

where $f \in R$.

2. When $l = 2m+1$ we define ϕ_A on $K_l = \{I + \pi^m \sqrt{\pi} B\}$. By the same analysis used in the even case, a typical element is

$$I + \begin{pmatrix} \pi^{m+1} a_1 + \pi^m a_2 \sqrt{\pi} & \pi^m b \sqrt{\pi} \\ \pi^m c \sqrt{\pi} & -\pi^{m+1} a_1 + \pi^m a_2 \sqrt{\pi} \end{pmatrix}, \; a_1, a_2, b, c \in R_l$$

so $|K_l|$ is $q^{4m+3} = q^{2l+1}$. We also use the group $K_{l-1} = \{I + \pi^m B\}$, a typical element of which is

$$I + \begin{pmatrix} \pi^m a_1 + \pi^m a_2 \sqrt{\pi} & \pi^m b \sqrt{\pi} \\ \pi^m c \sqrt{\pi} & \pi^m d_1 + \pi^m d_2 \sqrt{\pi} \end{pmatrix}, \; a_1, a_2, b, c, d_1, d_2 \in R_l$$

and $|K_{l-1}|$ is $q^{4m+4} = q^{2l+2}$. The generators of K_{l-1} are K_l and the subgroup

$$\mathcal{G}_1 = \left\{ \begin{pmatrix} 1+\pi^m f & 0 \\ 0 & (1+\pi^m f)^{-1} \end{pmatrix} \right\}$$

Since we always consider the different parities of l separately, there is no possibility of confusion by using the notation $\mathcal{G}_1$ again here.

Table 6.1: Some K Subgroup Orders

degree	order
K_l $(l = 2m)$	q^{2l}
K_{l-1} $(l = 2m)$	q^{2l+3}
K_l $(l = 2m + 1)$	q^{2l+1}
K_{l-1} $(l = 2m + 1)$	q^{2l+2}

When $l = 2m$, for $K_l = \{I + \pi^m B\}$ B must have the form

$$\begin{pmatrix} a_1+a_2\sqrt{\pi} & b\sqrt{\pi} \\ c\sqrt{\pi} & -a_1+a_2\sqrt{\pi} \end{pmatrix} a_1, a_2, r, s \in R_l$$

Whereas for $l = 2m + 1$, and $K_l = \{I + \pi^m\sqrt{\pi}B\}$ B has a different form:

$$\begin{pmatrix} a_1+a_2\sqrt{\pi} & b \\ c & a_1-a_2\sqrt{\pi} \end{pmatrix} a_1, a_2, r, s \in R_l$$

The nature of the B matrices will be important for getting an upper bound for the inertia groups.

Proposition 6.B.1 Let P_i be the map from unitary matrices over $R_{l,\pi}$ to the unitary matrices over $R_{i,\pi}, i \leq l$ that sends each entry of the domain matrix to its value modulo π^i. Then P_i is surjective.

Proof. The kernel of P_i is the subgroup K_{2i}, and this subgroup has order equal to the quotient of U_l and U_i. (Note the function P and the group U are indexed by π, but the K groups are indexed by $\sqrt{\pi}$.)

□

6.C Characters and Inertia Groups

Let λ be a primitive character on R_l^+; we extend λ to to a character to the additive group of $R_{l,\pi}$ by defining $\lambda(a+b\sqrt{\pi}) = \lambda(a+b) = \lambda(a)\lambda(b)$. For $l = 2m$ and $2m+1$, we define ϕ_A characters on K_l as follows: for any $A \in \mathrm{M}_{2\times 2}(R_{l,\pi})$ define ϕ_A on $K_l = \{I + \sqrt{\pi}^l B\} \cap U_l$ by:

$$\phi_A(I + \sqrt{\pi}^l B) = \lambda[\mathrm{tr}(\sqrt{\pi}^l AB)]$$

For such a character, whether l is odd or even, we have the following proposition which establishes an upper bound for T.

Proposition 6.C.1 Let ϕ_A be defined as above, and let $g \in T$ the inertia group of ϕ_A in U_l. Then

$$\sqrt{\pi}^{l+1} Ag = \sqrt{\pi}^{l+1} gA \tag{6.3}$$

Proof. As was the case in the discussion following 4.E.2 we can assume that $\overline{A} = A$ since any matrix $C \in \mathrm{M}_{2\times 2}(R_{l,\pi})$ will give a character ϕ_C that is equivalent to a character ϕ_A where $A \in \mathrm{M}_{2\times 2}(R_l)$ so that $\overline{A} = A$. In addition, the proof of Lemma 4.G.1 holds in the case of $R_{l,\pi}$, thus we know that $g \in T \iff \overline{g} \in T$.

1. Let $l = 2m$ If $g \in T$, by Proposition 4.G.1 $\overline{g}$ is also in T and we have

$$\lambda[\mathrm{tr}(\pi^m AB)] = \lambda[\mathrm{tr}(\pi^m (g^{-1}Ag)B)]$$

   or

$$\lambda[\mathrm{tr}[(\pi^m(g^{-1}Ag - A)B]] = 1$$

Let $X = \pi^m(g^{-1}Ag - A)$ so X has trace zero, and for some x_1, x_2, y_1, y_2 etc. in R_l, $X = \left(\begin{smallmatrix} x_1+x_2\sqrt{\pi} & y_1+y_2\sqrt{\pi} \\ z_1+z_2\sqrt{\pi} & -x_1-x_2\sqrt{\pi} \end{smallmatrix}\right)$, and we have:

$$\lambda[\mathrm{tr}(XB)] = 1 \tag{6.4}$$

for all $B = \left(\begin{smallmatrix} a_1+a_2\sqrt{\pi} & b\sqrt{\pi} \\ c\sqrt{\pi} & -a_1+a_2\sqrt{\pi} \end{smallmatrix}\right)$ $a_1, a_2, r, s \in R_l$.

Furthermore, since $g \in T$ implies $\overline{g} \in T$ and $\overline{A} = A$, then we can use $\overline{X}$ in place of X in equation 6.4:

$$\lambda[\mathrm{tr}\overline{X}B] = \lambda[\mathrm{tr}[(\pi^m(\overline{g}^{-1}A\overline{g} - A)B]] = 1,$$

and in the argument following, we can use X or $\overline{X}$ as required.

We find an upper bound for the inertia group by exploiting judicious choices for B, and by using both X and then $\overline{X}$ in equation 6.4.

Let $B = \left(\begin{smallmatrix} \frac{r}{2} & 0 \\ 0 & -\frac{r}{2} \end{smallmatrix}\right)$ for arbitrary $r \in R_l$ in equation 6.4 to get

$$\left(\begin{smallmatrix} x_1+x_2\sqrt{\pi} & y_1+y_2\sqrt{\pi} \\ z_1+z_2\sqrt{\pi} & -x_1-x_2\sqrt{\pi} \end{smallmatrix}\right)\left(\begin{smallmatrix} \frac{r}{2} & 0 \\ 0 & -\frac{r}{2} \end{smallmatrix}\right) = \left(\begin{smallmatrix} \frac{r}{2}(x_1+x_2\sqrt{\pi}) & y_1+y_2\sqrt{\pi} \\ z_1+z_2\sqrt{\pi} & -\frac{r}{2}(-x_1-x_2\sqrt{\pi}) \end{smallmatrix}\right)$$

so that

$$\lambda[\mathrm{tr}XB] = \lambda[r(x_1 + x_2\sqrt{\pi})] = \lambda[r(x_1 + x_2)] = 1$$

which implies that $x_1 + x_2 = 0$, since the extension of λ to $R_{l,\pi}$ is also primitive. Keeping B the same and replacing X with $\overline{X}$, we get

$$\lambda[r(x_1 - x_2\sqrt{\pi})] = \lambda[r(x_1 - x_2)] = 1$$

so that $x_1 - x_2 = 0$; combining both results implies that $x_1 = x_2 = 0$. Now letting $B = \left(\begin{smallmatrix} 0 & r\sqrt{\pi} \\ 0 & 0 \end{smallmatrix}\right)$ with X and then again with $\overline{X}$ in equation 6.4, we find that

$$\lambda[r(\pi y_2 + y_1\sqrt{\pi})] = \lambda[r(\pi y_2 + y_1)] = 1$$

and

$$\lambda[r(-\pi y_2 + y_1\sqrt{\pi})] = \lambda[r(-\pi y_2 + y_1)] = 1$$

Combining these, we have $y_1 = \pi y_2 = 0$. We get an analgous result for z_1, z_2 so that we can write:

$$X = \left(\begin{smallmatrix} 0 & \pi^{l-1} r\sqrt{\pi} \\ \pi^{l-1} s\sqrt{\pi} & 0 \end{smallmatrix}\right) \; r, s \in R_l$$

thus when l is even, we have $\sqrt{\pi}X = 0$, or

$$\sqrt{\pi}^{l+1} Ag = \sqrt{\pi}^{l+1} gA$$

as claimed.

2. Let l be odd so that for g in the stabilizer of ϕ_A

$$\lambda[\mathrm{tr}(\pi^m \sqrt{\pi}(g^{-1}Ag)B)] = \lambda[\mathrm{tr}(\pi^m \sqrt{\pi}AB)]$$

or

$$\lambda[\mathrm{tr}(\pi^m\sqrt{\pi}(g^{-1}Ag - A)B)] = 1$$

and $X = \pi^m\sqrt{\pi}(g^{-1}Ag - A)$ which is again trace zero. As in the even case we have $\lambda[\mathrm{tr}(XB)] = 1$ for all $B = \begin{pmatrix} a_1+a_2\sqrt{\pi} & b \\ c & a_1-a_2\sqrt{\pi} \end{pmatrix}$ $a_1, a_2, r, s \in R_l$. The form of B does not permit $B = \begin{pmatrix} r & 0 \\ 0 & -r \end{pmatrix}$ and using $B = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}$ gives no information about X. Thus we use $B\begin{pmatrix} r\sqrt{\pi} & 0 \\ 0 & -r\sqrt{\pi} \end{pmatrix}$ with first X, then $\overline{X}$ in equation 6.4 to get $x_1 = \pi x_2 = 0$. The same process using $B = \begin{pmatrix} 0 & r \\ 0 & 0 \end{pmatrix}$ with X then $\overline{X}$ shows that $y_1 = y_2 = 0$, and similarly for z_z, z_2. Therefore

$$X = \begin{pmatrix} \pi^{l-1}r\sqrt{\pi} & 0 \\ 0 & \pi^{l-1}s\sqrt{\pi} \end{pmatrix} \; r, s \in R_l$$

so that in the odd case we also have $\sqrt{\pi}X = 0$ and for either parity of l:

$$\sqrt{\pi}^{l+1}Ag = \sqrt{\pi}^{l+1}gA \tag{6.5}$$

□

6.C.1 The Character Degrees

The A matrices that we use will have one of two forms:

1. $A = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$
2. $A = \begin{pmatrix} 0 & f \\ 1 & 0 \end{pmatrix}$, where A cannot be diagonalized.

We claim that for all $f \in R_l$, the matrix $\left(\begin{smallmatrix} 0 & f \\ 1 & 0 \end{smallmatrix}\right)$ over $R_{l,\pi}$ cannot be diagonalized. Any defined ratio of neighbours (as mentioned in definition 4.F.1) will be a non-invertible element of $R_{l,\pi}$. This follows from section 6.A. Suppose that $\left(\begin{smallmatrix} 0 & f \\ 1 & 0 \end{smallmatrix}\right)$ were diagonalizable, so that for some unitary $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$:

$$\begin{aligned} \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & f \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^{-1} &= \left(\begin{smallmatrix} b & af \\ d & cf \end{smallmatrix}\right)\left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)\frac{1}{ad-bc} \\ &= \left(\begin{smallmatrix} .... & a^2f-b^2 \\ d^2-fc^2 & .... \end{smallmatrix}\right)\frac{1}{ad-bc} \end{aligned}$$

Since $a^2f - b^2 = d^2 - fc^2 = 0$, then f can be written as the square of a ratio of neighbours, and is therefore a non-unit. As a result b and d must be non-units, which is impossible since $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is invertible. Thus $\left(\begin{smallmatrix} 0 & f \\ 1 & 0 \end{smallmatrix}\right)$ cannot be diagonalized, and we have 3 possibilities for $f \in R_l$: a square unit, a non-square unit, and a non-unit.

We define ϕ_A characters on the abelian K subgroups, with the following A matrices:

1. $\left(\begin{smallmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{smallmatrix}\right)$
2. $\left(\begin{smallmatrix} 0 & \sigma \\ 1 & 0 \end{smallmatrix}\right)$, σ a non-square in R_l.
3. $\left(\begin{smallmatrix} 0 & \nu \\ 1 & 0 \end{smallmatrix}\right)$, ν a non-square in R_l.
4. $\left(\begin{smallmatrix} 0 & \pi\beta \\ 1 & 0 \end{smallmatrix}\right)$, β any element of R_{l-1}.

1. The Even Case

Let $l = 2m$ with ϕ_A defined on $K_l = \{I + \pi^m B\} \cap U_l$.

(a) Let $A = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$, For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in T$, equation 6.5 implies

$$\pi^m\sqrt{\pi}\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pi^m\sqrt{\pi}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$
$$\pi^m\sqrt{\pi}\begin{pmatrix} a & b \\ -c & -d \end{pmatrix} = \pi^m\sqrt{\pi}\begin{pmatrix} a & -b \\ c & -d \end{pmatrix}$$

Thus $\pi^m\sqrt{\pi}\, b = \pi^m\sqrt{\pi}\, c = 0$, and both b, c have a factor of $\pi^{m-1}\sqrt{\pi}$. Relabelling somewhat, g can be written

$$\begin{pmatrix} a & \pi^{m-1}b'\sqrt{\pi} \\ \pi^{m-1}c'\sqrt{\pi} & d \end{pmatrix},\ a, d \in R_\pi, b, c \in R$$

so that $T \leq K_{l-1}S$ where S is the subgroup of diagonal matrices. To show the reverse inclusion, recall that K_{l-1} is generated by K_l and the subgroups

i. $\mathcal{G}_1 = \left\{\begin{pmatrix} 1+\pi^{m-1}f\sqrt{\pi} & 0 \\ 0 & [1-\pi^{m-1}f\sqrt{\pi}]^{-1} \end{pmatrix}\right\}$

ii. $\mathcal{G}_2 = \left\{\begin{pmatrix} 1 & 0 \\ \pi^{m-1}f\sqrt{\pi} & 1 \end{pmatrix}\right\}$

iii. $\mathcal{G}_3 = \left\{\begin{pmatrix} 1 & \pi^{m-1}f\sqrt{\pi} \\ 0 & 1 \end{pmatrix}\right\}$

Conjugation by the diagonal elements of $\mathcal{G}_1$ stabilizes the main diagonal of elements in K_l, and therefore stabilizes ϕ_A. The subgroups $\mathcal{G}_2$ and $\mathcal{G}_3$ also stabilize ϕ_A: we give the argument for $\mathcal{G}_2$. A similar argument works for $\mathcal{G}_3$. Write $\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$ for $\begin{pmatrix} 1 & 0 \\ \pi^{m-1}f\sqrt{\pi} & 1 \end{pmatrix} \in \mathcal{G}_2$ and $\begin{pmatrix} x & y \\ z & w \end{pmatrix}$ for $\begin{pmatrix} \pi^m x & \pi^m y\sqrt{\pi} \\ \pi^m z\sqrt{\pi} & \pi^m(-\overline{x}) \end{pmatrix}$, $x \in R_{l,\pi},\ y, z \in R_l$ - this last matrix is, of course, the $\pi^m B$ in the element $I + \pi^m B \in K_l$.

Conjugating:

$$\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}\begin{pmatrix} x & y \\ z & w \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -c & 1 \end{pmatrix} = \begin{pmatrix} x-cy & .... \\ .... & w+cy \end{pmatrix}$$

But $cy = \pi^{m-1} f\sqrt{\pi}\pi^m y\sqrt{\pi} = 0$, so the main diagonal (and thus ϕ_A) is preserved.

Thus every $g \in K_{l-1}$ fixes the elements on the main diagonal of each element of K_l and so stabilizes ϕ_A. Hence $T = K_{l-1}S$. The order of T is

$$\frac{|K_{l-1}||S|}{|K_{l-1} \cap S|} = \frac{q^{2l+3}q^{2l-1}(q-1)}{q^{l+1}} = q^{3l+1}(q-1)$$

We cannot extend ϕ_A directly to T using 2.C.1, so we interpose subgroups between K_m and $K_{l-1}S$; the schematic is:

$$\underset{\phi_A}{K_l} \overset{\text{ext}}{\longrightarrow} \underset{\phi'_A}{N} \overset{\text{ext}}{\longrightarrow} \underset{\psi_0}{NS} \overset{\text{ind}}{\longrightarrow} \underset{\psi}{K_{l-1}S} \overset{\text{ind}}{\longrightarrow} \underset{\chi}{U_l}$$

where

$$N = \left\{ \begin{pmatrix} 1+\pi^{m-1}A & \pi^m B \\ \pi^{m-1}C & 1+\pi^{m-1}D \end{pmatrix} A, B, C, D \in R_\pi \right\} \cap U_l;$$

and (as we will show) NS is the inertia group of ϕ_A in $K_{l-1}S$. N is generated by K_l and the abelian subgroups

$$\mathcal{G}_1 = \left\{ \begin{pmatrix} 1+\pi^{m-1}f\sqrt{\pi} & 0 \\ 0 & [1-\pi^{m-1}f\sqrt{\pi}]^{-1} \end{pmatrix} \right\};\ \mathcal{G}_2 = \left\{ \begin{pmatrix} 1 & 0 \\ \pi^{m-1}f\sqrt{\pi} & 1 \end{pmatrix} \right\}$$

with $f \in R$. Both subgroups are contained in K_{l-1}, hence stabilize ϕ_A which therefore extends to a character on $K_l\mathcal{G}_1$ by Proposition

2.C.1. If we assign the trivial character to $\mathcal{G}_2$, then it will stabilize not only ϕ_A but also the character on $\mathcal{G}_1$:

Write $\begin{pmatrix} X & 0 \\ 0 & W \end{pmatrix}$ for $\begin{pmatrix} 1+\pi^{m-1}f\sqrt{\pi} & 0 \\ 0 & [1-\pi^{m-1}f\sqrt{\pi}]^{-1} \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$ for $\begin{pmatrix} 1 & 0 \\ \pi^{m-1}f\sqrt{\pi} & 1 \end{pmatrix}$.

Then:

$$\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}\begin{pmatrix} X & 0 \\ 0 & W \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -c & 1 \end{pmatrix} = \begin{pmatrix} X & 0 \\ cX & W \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -c & 1 \end{pmatrix}$$
$$= \begin{pmatrix} X & 0 \\ c(X-W) & W \end{pmatrix}$$

but $(X-W) = (X - \frac{1}{(\overline{X})^{-1}}) = \frac{X\overline{X}-1}{\overline{X}} = \frac{-\pi^{2m-1}f^2}{\overline{X}}$, and if we write $\frac{1}{\overline{X}} = x_1 + x_2\sqrt{\pi}$, then we have

$$c(X-W) = \pi^{m-1}c\sqrt{\pi}(-\pi^{2m-1}f^2(x_1+x_2\sqrt{\pi})) = \pi^{3m-2}c'\sqrt{\pi},\ c' \in R$$

We have two cases here: if $m \geq 2$ then $3m-2 \geq 2m$ and the result of conjugation can be written $\begin{pmatrix} X & 0 \\ 0 & W \end{pmatrix}$ so that $\mathcal{G}_2$ stabilizes the character on $\mathcal{G}_1$. If $l = 2$ so that $m = 1$, then we can write the conjugation product as

$$\begin{pmatrix} 1 & 0 \\ \pi c'\sqrt{\pi} & 1 \end{pmatrix}\begin{pmatrix} 1+\pi^{m-1}f\sqrt{\pi} & 0 \\ 0 & [1-\pi^{m-1}f\sqrt{\pi}]^{-1} \end{pmatrix}$$

since the first factor is in $\mathcal{G}_2$ which has been assigned the trivial character, we see that $\mathcal{G}_2$ stabilizes the character on $K_l\mathcal{G}_1$ therefore we get the extension to ϕ'_A on $(K_{l-1}\mathcal{G}_1)\mathcal{G}_2 = N$.

The number of extensions from K_l to $K_l\mathcal{G}_1$ is q, and there is only one extension from $K_l\mathcal{G}_1$ to N since we have assigned the trivial character to $\mathcal{G}_2$.

Next we can extend ϕ'_A to ψ_0 on NS. S will stabilize ϕ'_A since it stabilizes the character on $\mathcal{G}_1$ (both are diagonal), and because it normalizes $\mathcal{G}_2$, which has been assigned the trivial character: write $\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)$ for an element of S and $\left(\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}\right)$ for an element of $\mathcal{G}_2$, then

$$\begin{aligned}\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} x^{-1} & 0 \\ 0 & y^{-1} \end{smallmatrix}\right) &= \left(\begin{smallmatrix} x & 0 \\ cy & y \end{smallmatrix}\right)\left(\begin{smallmatrix} x^{-1} & 0 \\ 0 & y^{-1} \end{smallmatrix}\right) \\ &= \left(\begin{smallmatrix} 1 & 0 \\ x^{-1}yc & 1 \end{smallmatrix}\right) \\ &= \left(\begin{smallmatrix} 1 & 0 \\ \pi^{m-1}f'\sqrt{\pi} & 1 \end{smallmatrix}\right)\end{aligned}$$

Where the last equality follows because $c = \pi^{m-1}f\sqrt{\pi}$, and since $x = (\overline{y})^{-1}$ then $x^{-1}y = y\overline{y} \in R_l$, so $x^{-1}yc = \pi^{m-1}f'\sqrt{\pi},\ f' \in R_l$.

Thus, by Proposition 2.C.1, we extend to ψ_0 on NS. We show now that the inertia group of ψ_0 in $K_{l-1}S$ is NS itself. The group $K_{l-1}S$ is generated by NS and the subgroup $\mathcal{G}_3 = \left\{\left(\begin{smallmatrix} 1 & \pi^{m-1}f\sqrt{\pi} \\ 0 & 1 \end{smallmatrix}\right)\right\},\ f \in R$, but $\mathcal{G}_3$ does not stabilize the (trivial) character on $\mathcal{G}_2$, since if $\left(\begin{smallmatrix} 1 & \pi^{m-1}e\sqrt{\pi} \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & d \\ 0 & 1 \end{smallmatrix}\right)$ is an element of $\mathcal{G}_3$, and $\left(\begin{smallmatrix} 1 & 0 \\ \pi^{m-1}f\sqrt{\pi} & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}\right)$ is an element $\mathcal{G}_2$, then

$$\begin{aligned}\left(\begin{smallmatrix} 1 & d \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & -d \\ 0 & 1 \end{smallmatrix}\right) &= \left(\begin{smallmatrix} 1+cd & d \\ c & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & -d \\ 0 & 1 \end{smallmatrix}\right) \\ &= \left(\begin{smallmatrix} 1+cd & -fe^2 \\ c & 1 \end{smallmatrix}\right) \\ &= \left(\begin{smallmatrix} 1+\pi^{2m-1}ef & -\pi^{3m-2}fe^2\sqrt{\pi} \\ \pi^{m-1}f\sqrt{\pi} & 1-\pi^{2m-1}ef \end{smallmatrix}\right)\end{aligned}$$

If $m \geq 2$ so that $3m - 2 \geq 2m$, then the result can be written

$$\left(\begin{smallmatrix} 1 & 0 \\ \pi^{m-1} f\sqrt{\pi} & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1+\pi^{2m-1}ef & 0 \\ 0 & 1-\pi^{2m-1}ef \end{smallmatrix}\right)$$

Since the character value of the second factor is not necessarily 1, then $\mathcal{G}_3$ does not stabilize ϕ'_A. If $l = 2$ so that $m = 1$, we can write the conjugation product as

$$\left(\begin{smallmatrix} 1+\pi ef & -\pi ef^2\sqrt{\pi} \\ f\sqrt{\pi} & 1-\pi ef \end{smallmatrix}\right)$$

Under the natural projection map modulo π, this matrix maps to $\left(\begin{smallmatrix} 1 & 0 \\ f\sqrt{\pi} & 1 \end{smallmatrix}\right)$, hence the conjugations product is equal to $\left(\begin{smallmatrix} 1 & 0 \\ f\sqrt{\pi} & 1 \end{smallmatrix}\right)(Z)$ for some $Z \in K_l = \{I + \pi B\}$, and the main diagonal of Z must be the same as the main diagonal of $\left(\begin{smallmatrix} 1+\pi ef & -\pi ef^2\sqrt{\pi} \\ f\sqrt{\pi} & 1-\pi ef \end{smallmatrix}\right)$, hence $\phi_A(Z)$ is not identically 1, and $\mathcal{G}_3$ does not stabilize ϕ'_A.

Consequently, we can induce ψ_0 to an irreducible character ψ on $T = K_{l-1}S$ which will have degree

$$\frac{|NS||\mathcal{G}_3|}{|NS \cap \mathcal{G}_3|} = q$$

Finally, $\chi = \mathrm{Ind}_T^{U_l}\psi$ is an irreducible character of U_l having degree $q[U_l : T] = 2q^{l-1}$.

The next three A matrices are all of the form $A = \left(\begin{smallmatrix} 0 & t \\ 1 & 0 \end{smallmatrix}\right)$, and from equation 6.5, the inertia group for each ϕ_A is contained in the group

$K_{l-1}S$ for S the centralizer of A. Since $K_lS \leq T$, we need to check the generating subgroups of K_{l-1} that are not in K_l. These are: (f is in R)

i. $\mathcal{G}_1 = \left\{ \begin{pmatrix} 1+\pi^{m-1}f\sqrt{\pi} & 0 \\ 0 & (1-\pi^{m-1}f\sqrt{\pi})^{-1} \end{pmatrix} \right\}$

ii. $\mathcal{G}_2 = \left\{ \begin{pmatrix} 1 & 0 \\ \pi^{m-1}f\sqrt{\pi} & 1 \end{pmatrix} \right\}$

iii. $\mathcal{G}_3 = \left\{ \begin{pmatrix} 1 & \pi^{m-1}f\sqrt{\pi} \\ 0 & 1 \end{pmatrix} \right\}$

For any $B = \begin{pmatrix} 0 & t \\ 1 & 0 \end{pmatrix}$, we have

$$\phi_B[I+\pi^m \begin{pmatrix} x_1+x_2\sqrt{\pi} & y\sqrt{\pi} \\ z\sqrt{\pi} & -x_1+x_2\sqrt{\pi} \end{pmatrix}] = \lambda[\pi^m(tz\sqrt{\pi}+y\sqrt{\pi})] = \lambda[\pi^m(tz+y)]$$

We conjugate $I + \pi^m \begin{pmatrix} x_1+x_2\sqrt{\pi} & y\sqrt{\pi} \\ z\sqrt{\pi} & -x_1+x_2\sqrt{\pi} \end{pmatrix} \in K_l$ by elements of the above 3 subgroups, and we are only concerned with the second diagonal entries of the conjugation products:

i. The subgroup $\mathcal{G}_1$ is in the centralizer of K_l, and thus in T: to see this, take $\begin{pmatrix} F & 0 \\ 0 & \overline{F}^{-1} \end{pmatrix} = \begin{pmatrix} 1+\pi^{m-1}f\sqrt{\pi} & 0 \\ 0 & (1-\pi^{m-1}f\sqrt{\pi})^{-1} \end{pmatrix} \in \mathcal{G}_1$note that $F\overline{F} = 1 - \pi^{2m-1}f^2$, and $(F\overline{F})^{-1} = 1 + \pi^{2m-1}f^2$. Thus:

$$\begin{pmatrix} F & 0 \\ 0 & \overline{F}^{-1} \end{pmatrix} \pi^m \begin{pmatrix} x_1+x_2\sqrt{\pi} & y\sqrt{\pi} \\ z\sqrt{\pi} & -x_1+x_2\sqrt{\pi} \end{pmatrix} \begin{pmatrix} F & 0 \\ 0 & \overline{F}^{-1} \end{pmatrix}^{-1} = \pi^m \begin{pmatrix} (x_1+x_2\sqrt{\pi}) & F\overline{F}y\sqrt{\pi} \\ (F\overline{F})^{-1}z\sqrt{\pi} & (-x_1+x_2\sqrt{\pi}) \end{pmatrix}$$

The upper right element is:

$$F\overline{F}(\pi^m y\sqrt{\pi}) = (1 - \pi^{2m-1}f^2)(\pi^m y\sqrt{\pi}) = \pi^m y\sqrt{\pi}$$

By similar reasoning, the lower left element is $\pi^m z\sqrt{\pi}$.

ii. Checking to see whether or not $\mathcal{G}_2 \leq T$, we write $\left(\begin{smallmatrix} 1 & 0 \\ F & 1 \end{smallmatrix}\right)$ for $\left(\begin{smallmatrix} 1 & 0 \\ \pi^{m-1} f\sqrt{\pi} & 1 \end{smallmatrix}\right) \in \mathcal{G}_2$, and $\left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right)$ for $\pi^m \left(\begin{smallmatrix} x_1+x_2\sqrt{\pi} & y\sqrt{\pi} \\ z\sqrt{\pi} & -x_1+x_2\sqrt{\pi} \end{smallmatrix}\right)$, so

$$\begin{aligned} \left(\begin{smallmatrix} 1 & 0 \\ F & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ -F & 1 \end{smallmatrix}\right) &= \left(\begin{smallmatrix} x & y \\ Fx+z & Fy+w \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ -F & 1 \end{smallmatrix}\right) \\ &= \left(\begin{smallmatrix} x-Fy & y \\ Fx+z-F^2y-Fw & Fy+w \end{smallmatrix}\right) \end{aligned}$$

The upper right is preserved, but the lower left is

$$\begin{aligned} Fx + z - F^2y - Fw &= \pi^{m-1} f\sqrt{\pi}(x-w) + z \quad (F^2y \text{ is zero}) \\ &= \pi^{m-1} f\sqrt{\pi}(\pi^m(2x_1)) + \pi^m z\sqrt{\pi} \\ &= \pi^{2m-1}(2x_1 f)\sqrt{\pi} + \pi^m z\sqrt{\pi} \end{aligned}$$

so the lower left element is not fixed. Thus, for the A matrices $\left(\begin{smallmatrix} 0 & \sigma \\ 1 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & \nu \\ 1 & 0 \end{smallmatrix}\right)$, $\mathcal{G}_2$ is not in T. On the other hand, for $A = \left(\begin{smallmatrix} 0 & \pi\beta \\ 1 & 0 \end{smallmatrix}\right)$, the stabilizer of ϕ_A does, in fact, include $\mathcal{G}_2$ because when we calculate the character, the $\pi\beta$ entry will vanish the term $\pi^{2m-1}(2a_1 f)\sqrt{\pi}$.

iii. For $\mathcal{G}_3$, we write $\left(\begin{smallmatrix} 1 & F \\ 0 & 1 \end{smallmatrix}\right)$ for $\left(\begin{smallmatrix} 1 & \pi^{m-1} f\sqrt{\pi} \\ 0 & 1 \end{smallmatrix}\right) \in \mathcal{G}_2$, and $\left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right)$ for $\pi^m \left(\begin{smallmatrix} x_1+x_2\sqrt{\pi} & y\sqrt{\pi} \\ z\sqrt{\pi} & -x_1+x_2\sqrt{\pi} \end{smallmatrix}\right)$, so

$$\begin{aligned} \left(\begin{smallmatrix} 1 & F \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & -F \\ 0 & 1 \end{smallmatrix}\right) &= \left(\begin{smallmatrix} x+Fz & y+Fw \\ z & w \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & -F \\ 0 & 1 \end{smallmatrix}\right) \\ &= \left(\begin{smallmatrix} x-Fy & y+F(w-x) \\ z & w-Fz \end{smallmatrix}\right) \end{aligned}$$

In writing the upper right entry, we have used $FFz = 0$. Therefore the lower left entry is invariant under conjugation by $\mathcal{G}_3$, but the upper right entry is

$$\pi^m y\sqrt{\pi} + \pi^{m-1} f\sqrt{\pi}(\pi^m(-2x_1)) = \pi^m y\sqrt{\pi} - \pi^{2m-1} 2 f x_1 \sqrt{\pi}$$

Though the lower left entry is preserved, the upper right is not. Consequently, $\mathcal{G}_3$ is not in T for any of the three remaining A matrices. Note that the π in $\left(\begin{smallmatrix} 0 & \pi\beta \\ 1 & 0 \end{smallmatrix}\right)$ cannot save this subgroup.
Thus we must deal with $A = \left(\begin{smallmatrix} 0 & \pi\beta \\ 1 & 0 \end{smallmatrix}\right)$ separately, but for $A = \left(\begin{smallmatrix} 0 & \sigma \\ 1 & 0 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 0 & \nu \\ 1 & 0 \end{smallmatrix}\right)$, the inertia group will be $K_{l+}S$, where K_{l_+} is the subgroup generated by K_l and the subgroup $\mathcal{G}_1$. The extension schematic for these A matrices will be:

$$\underset{\phi_A}{K_l} \overset{\text{ext}}{\longrightarrow} \underset{\phi'_A}{K_{l+}} \overset{\text{ext}}{\longrightarrow} \underset{\phi''_A}{K_{l+}\, S} \overset{\text{ind}}{\longrightarrow} \underset{\chi}{U_l}$$

The first extension above is by Proposition 2.C.1, since $\mathcal{G}_1$ is abelian, and stabilizes ϕ_A. The second extension will be justified in the same way if we can show that S fixes the character on the $\mathcal{G}_1$ elements.

Proposition 6.C.2 For $A = \left(\begin{smallmatrix} 0 & t \\ 1 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & \sigma \\ 1 & 0 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 0 & \nu \\ 1 & 0 \end{smallmatrix}\right)$ and any $s \in S$ and $h = \left(\begin{smallmatrix} F & 0 \\ 0 & (\overline{F})^{-1} \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1+\pi^{m-1} f\sqrt{\pi} & 0 \\ 0 & (1-\pi^{m-1} f\sqrt{\pi})^{-1} \end{smallmatrix}\right) \in \mathcal{G}_1$, we have $shs^{-1} = hx$ where $x \in K_l$ such that $\phi'_A(x) = \phi_A(x) = 1$. Hence S stabilizes the character on $\mathcal{G}_1$.

Proof. Let $s = \left(\begin{smallmatrix} a & bt \\ b & a \end{smallmatrix}\right) \in S$ be an element of the centralizer of A. To show that $shs^{-1} = hx$, we consider the natural projection map $P : U_l \to U_m$, a surjective map with kernel K_l. We claim $P(hsh^{-1}) = P(s)$. Note that $F\overline{F} = 1 - \pi^{2m} f^2$ is congruent to 1 modulo π^m.

$$\begin{aligned} P\left[\begin{pmatrix} F & 0 \\ 0 & (\overline{F})^{-1} \end{pmatrix}\begin{pmatrix} a & bt \\ b & a \end{pmatrix}\begin{pmatrix} (F)^{-1} & 0 \\ 0 & (\overline{F}) \end{pmatrix}\right] &= P\left[\begin{pmatrix} Fa & Fbt \\ (\overline{F})^{-1}b & (\overline{F})^{-1}a \end{pmatrix}\begin{pmatrix} F^{-1} & 0 \\ 0 & (\overline{F}) \end{pmatrix}\right] \\ &= P\left[\begin{pmatrix} a & F\overline{F}bt \\ (F\overline{F})^{-1}b & a \end{pmatrix}\right] \\ &= P\left[\begin{pmatrix} a & bt \\ b & a \end{pmatrix}\right] \end{aligned}$$

Thus $hsh^{-1} = xs$, $x \in K_l$. To see that $\phi_A(x) = 1$, note that we can write $h^{-1}x = sh^{-1}s^{-1}$. Now we claim the following

- $\mathrm{tr}(A(hx)) = \mathrm{tr}(Ah) + \mathrm{tr}(Ax)$
- $\mathrm{tr}(Ashs^{-1}) = \mathrm{tr}(h)$

The second item above follows immediately because S centralizes A. For the first we note that for $h = \begin{pmatrix} F & 0 \\ 0 & (\overline{F})^{-1} \end{pmatrix} = \begin{pmatrix} 1+\pi^{m-1}f\sqrt{\pi} & 0 \\ 0 & (1-\pi^{m-1}f\sqrt{\pi})^{-1} \end{pmatrix}$, and $x = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \in K_l$, $hx = \begin{pmatrix} FX & FY \\ \overline{F}Z & \overline{F}W \end{pmatrix}$. But $FY = (1+\pi^{m-1}f\sqrt{\pi})(\pi^m y\sqrt{\pi}) = \pi^m y\sqrt{\pi}$, and similarly $\overline{F}Z = \pi^m z\sqrt{\pi}$. Since $\mathrm{tr}(Ah) = 0$, the result follows. Now we can show $\phi_A(x) = 1$.

$$\begin{aligned} \mathrm{tr}(Ah^{-1}) + \mathrm{tr}(Ax) &= \mathrm{tr}(A(h^{-1}x)) \\ &= \mathrm{tr}(Ash^{-1}s^{-1}) \\ &= \mathrm{tr}(s^{-1}Ash^{-1}) \\ &= \mathrm{tr}(Ah^{-1}) \end{aligned}$$

hence $\mathrm{tr}(Ax) = 0$ and $\phi_A(x) = 1$.

$\square$

Hence we get an extension to $T = K_{l_+}S$, followed by an induction to U_l. Now we can find character degrees for the cases $A = \left(\begin{smallmatrix} 0 & \sigma \\ 1 & 0 \end{smallmatrix}\right)$, and $A = \left(\begin{smallmatrix} 0 & \nu \\ 1 & 0 \end{smallmatrix}\right)$.

(b) Let $A = \left(\begin{smallmatrix} 0 & \sigma \\ 1 & 0 \end{smallmatrix}\right)$ with $\sigma \in R$ a square unit. The inertia group is $K_{l+}S$ for $S = \left(\begin{smallmatrix} a & c\sigma \\ c & a \end{smallmatrix}\right)$. From Proposition 4.D.2, S is isomorphic to two copies of $\mathcal{L}$, and thus has order $4q^{2m} = 4q^l$, then $|T| = 4q^{3l}$. We can extend ϕ_A to ψ on T, so $\chi = \mathrm{Ind}_T^{U_l}\psi$ is irreducible with degree $[U_l : T] = \frac{q^{l-1}(q-1)}{2}$.

(c) Let $A = \left(\begin{smallmatrix} 0 & \nu \\ 1 & 0 \end{smallmatrix}\right)$ with $\nu \in R$ a non-square. $T = K_{l+}S$ with $S = \left(\begin{smallmatrix} a & c\nu \\ c & a \end{smallmatrix}\right)$. The order of S modulo q^m is $2q^{2m} = 2q^l$; ϕ_A extends to ψ on K_lS which induces to χ on U_l of degree $[U_l : T] = q^{l-1}(q-1)$.

(d) For $A = \left(\begin{smallmatrix} 0 & \pi\beta \\ 1 & 0 \end{smallmatrix}\right)$ $S = \left(\begin{smallmatrix} a & c\pi\beta \\ c & a \end{smallmatrix}\right)$ the inertia group is generated by K_l, $\mathcal{G}_1$, $\mathcal{G}_2$, and $S = \left(\begin{smallmatrix} a & c\pi\beta \\ c & a \end{smallmatrix}\right)$. Once again we will let K_{l+} denote the group generated by K_l and $\mathcal{G}_1$. The extensions in this case will follow this schematic:

$$\underset{\phi_A}{K_l} \xrightarrow{\text{ext}} \underset{\phi'_A}{K_{l+}} \xrightarrow{\text{ext}} \underset{\phi''_A}{K_{l+}\,\mathcal{G}_2} \xrightarrow{\text{ext}} \underset{\psi}{K_{l+}\mathcal{G}_2S} \xrightarrow{\text{ind}} \underset{\chi}{U_l}$$

We have seen that both abelian subgroups $\mathcal{G}_1$ and $\mathcal{G}_2$ stabilize ϕ_A, so we can extend to ϕ'_A by Proposition 2.C.1. We can also do the second extension by Proposition 2.C.1, providing that we assign the trivial character to $\mathcal{G}_2$, because: $\mathcal{G}_2$ stabilizes ϕ_A, and we have seen in the work for the first A matrix that it will stabilize the character on $\mathcal{G}_1$ as well. To justify the last extension, we need to show that S will stabilize the characters on both $\mathcal{G}_1$ and $\mathcal{G}_2$. In the work for the first A matrix, we showed that the S group in that case stabilized the character on $\mathcal{G}_1$; the same proof for $S = \left(\begin{smallmatrix} a & c\pi\beta \\ c & a \end{smallmatrix}\right)$ works in the same way. We will use the same ideas to show that S stabilizes the (trivial) character on $\mathcal{G}_2$.

Proposition 6.C.3 Let $s \in S$ and let $h = \left(\begin{smallmatrix} 1 & 0 \\ F & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ \pi^{m-1} f\sqrt{\pi} & 1 \end{smallmatrix}\right) \in \mathcal{G}_2$. Then $shs^{-1} = hx, x \in K_l$ with $\phi_A(x) = 1$.

Proof. Let $s = \left(\begin{smallmatrix} a & \pi\beta c \\ c & a \end{smallmatrix}\right) \in S$ be an element of the centralizer of A. To show that $shs^{-1} = hx$, we consider the natural projection map $P : U_l \to U_m$, a surjective map with kernel K_l. We claim $P(hsh^{-1}) = P(s)$. Note that $F\pi = \pi^{m-1} f\sqrt{\pi}(\pi)$ is congruent to 0 modulo π^m.

$$\begin{aligned} P\left[\left(\begin{smallmatrix} 1 & 0 \\ F & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & \pi\beta c \\ c & a \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ -F & 1 \end{smallmatrix}\right)\right] &= P\left[\left(\begin{smallmatrix} a & \pi\beta c \\ aF+c & \pi\beta cF+a \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ -F & 1 \end{smallmatrix}\right)\right] \\ &= P\left[\left(\begin{smallmatrix} a-F\pi\beta c & \pi\beta c \\ c-\pi\beta cF & \pi\beta cF+a \end{smallmatrix}\right)\right] \\ &= P\left[\left(\begin{smallmatrix} a & \pi\beta c \\ c & a \end{smallmatrix}\right)\right] \end{aligned}$$

Thus $hsh^{-1} = xs$, $x \in K_l$. To see that $\phi_A(x) = 1$, note that we can write $h^{-1}x = sh^{-1}s^{-1}$, and:

- $\text{tr}(A(hx)) = \text{tr}(Ah) + \text{tr}(Ax)$
- $\text{tr}(Ashs^{-1}) = \text{tr}(h)$

The second item above follows immediately because S centralizes

A. For the first we note that for $h = \begin{pmatrix} 1 & 0 \\ F & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \pi^{m-1} f\sqrt{\pi} & 0 \end{pmatrix}$, and $x = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \in K_l$, $hx = \begin{pmatrix} \cdots\cdots & Y \\ FX+Z & \cdots\cdots \end{pmatrix}$. But

$$FX+Z = (\pi^{m-1} f\sqrt{\pi})(1+\pi^m X)+Z = \pi^{m-1} f\sqrt{\pi}+\pi^{2m-1} fX\sqrt{\pi}+Z$$

,

Note that the second diagonal would be additive here except for the term $\pi^{2m-1} fX\sqrt{\pi}$, but this zero is vanished by the A matrix, therefore the first point is proved. Now we have:

$$\begin{aligned} \operatorname{tr}(Ah^{-1}) + \operatorname{tr}(Ax) &= \operatorname{tr}(A(h^{-1}x)) \\ &= \operatorname{tr}(Ash^{-1}s^{-1}) \\ &= \operatorname{tr}(s^{-1}Ash^{-1}) \\ &= \operatorname{tr}(Ah^{-1}) \end{aligned}$$

hence $\operatorname{tr}(Ax) = 0$ and $\phi_A(x) = 1$.

□

Consequently, we get the extension to ψ, and then an induction to U_l. To find the order of $T = K_{l+}\mathcal{G}_2 S$, we note:

i. $|K_{l+}\mathcal{G}_2| = q^{2l+2}$

ii. $|S| = 2q^{2l}$

iii. $|K_{l+}\mathcal{G}_2 \cap S| = q^{l+2}$

Hence $|T|= 2q^{3l}$ and we can induce to an irreducible character χ of U_l of degree $[U_l : T] = q^{l-1}(q-1)$.

2. The Odd Case

Let $l = 2m+1$ and ϕ_A be defined on $K_l = \{I + \pi^m\sqrt{\pi}B\} \cap U_l$.

(a) Let $A = \left(\begin{smallmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{smallmatrix}\right)$. By construction, the inertia group T of ϕ_A contains K_lS. From equation 6.5, if $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in T$ then

$$\pi^{m+1}\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pi^{m+1}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$
$$\pi^{m+1}\begin{pmatrix} a & b \\ -c & -d \end{pmatrix} = \pi^{m+1}\begin{pmatrix} a & -b \\ c & -d \end{pmatrix}$$

so $\pi^{m+1}b = \pi^{m+1}c = 0$ and by relabelling somewhat, T is contained in the subgroup of the form $\left(\begin{smallmatrix} a & \pi^m b \\ \pi^m c & d \end{smallmatrix}\right) = K_{l-1}S$ where S is the subgroup of diagonal matrices. Thus

$$K_lS \leq T \leq K_{l-1}S$$

Here we find that the upper bound is achieved. K_{l-1} is generated K_l and the subgroup $\mathcal{G}_1 = \left\{\left(\begin{smallmatrix} 1+\pi^m f & 0 \\ 0 & (1+\pi^m f)^{-1} \end{smallmatrix}\right)\right\}$ $f \in R_l$, which is diagonal so it stabilizes ϕ_A and $T = K_{l-1}S$.

We can extend ϕ_A to $K_{l-1}S$ in two steps:

$$\underset{\phi_A}{K_l} \overset{\text{ext}}{\longrightarrow} \underset{\phi'_A}{K_{l-1}} \overset{\text{ext}}{\longrightarrow} \underset{\psi}{K_{l-1}S} \overset{\text{ind}}{\longrightarrow} \underset{\chi}{U_l}$$

By Proposition 2.C.1 ϕ_A extends to ϕ'_A on K_{l-1}, and we note that the number of such extensions is $[K_{l-1} : K_l] = q$. S centralizes $\mathcal{G}_1$ (both subgroups are diagonal), thus ϕ'_A extends to ψ on $K_{l-1}S$, and $\chi = \mathrm{Ind}_T^{U_l}\psi$ is an irreducible character of U_l of degree $\frac{|U_l|}{|T|} = \frac{2q^{4l-1}(q-1)}{q^{3l}(q-1)} = 2q^{l-1}$.

The remaining A matrices are all of the form $\left(\begin{smallmatrix} 0 & t \\ 1 & 0 \end{smallmatrix}\right)$ for some $t \in R_l$, and we claim that for all of them, the inertia group is K_lS. In each case, the inertia group T of ϕ_A is bounded thus:

$$K_lS \leq T \leq K_{l-1}S$$

Recall that K_{l-1} is generated by K_l and $\mathcal{G}_1$, but we will show that $\mathcal{G}_1$ does not stabilize ϕ_A, so that $T = K_lS$. To show this we conjugate, (ignoring the term I)
$I+\pi^m\sqrt{\pi}\left(\begin{smallmatrix} x_1+x_2\sqrt{\pi} & y \\ z & x_1-x_2\sqrt{\pi} \end{smallmatrix}\right)$, $\in K_l$ by $\left(\begin{smallmatrix} F & 0 \\ 0 & F^{-1} \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1+\pi^m f & 0 \\ 0 & (1+\pi^m f)^{-1} \end{smallmatrix}\right) \in \mathcal{G}_1$, getting:

$$\left(\begin{smallmatrix} F & 0 \\ 0 & F^{-1} \end{smallmatrix}\right)\pi^m\sqrt{\pi}\left(\begin{smallmatrix} x_1+x_2\sqrt{\pi} & y \\ z & x_1-x_2\sqrt{\pi} \end{smallmatrix}\right)\left(\begin{smallmatrix} F & 0 \\ 0 & F^{-1} \end{smallmatrix}\right)^{-1} = \pi^m\sqrt{\pi}\left(\begin{smallmatrix} x_1+x_2\sqrt{\pi} & yF^2 \\ z(F^2)^{-1} & x_1-x_2\sqrt{\pi} \end{smallmatrix}\right)$$

Note that $F^2 = 1 + \pi^m 2f + \pi^{2m} f^2$, so the upper right element in the conjugations becomes

$$\pi^m y\sqrt{\pi}(1 + \pi^m E) = \pi^m y\sqrt{\pi} + \pi^{2m} yE\sqrt{\pi}$$

thus ϕ_A is not stabilized. Consequently, for the next three A matrices, the inertia group of ϕ_A is $K_l S$ where S is the centralizer of A.

(b) Let $A = \left(\begin{smallmatrix} 0 & \sigma \\ 1 & 0 \end{smallmatrix}\right)$ where σ is a square unit in R and $T = K_l S$, with S the set of unitary matrices of the form $\left(\begin{smallmatrix} a & c\sigma \\ c & a \end{smallmatrix}\right)$. From Proposition 4.D.2, $S = \mathcal{L} \times \mathcal{L}$. We calculate $|T|= \frac{|K_l||S|}{|K_l \cap S|}$ From section 6.B $|K_l|= q^{2l+1}$, and from the form of S, we have $|S|= 2q^l \times 2q^l = 4q^{2l}$. An element in the intersection is

$$A = \left(\begin{smallmatrix} 1+\pi^m a\sqrt{\pi} & \pi^m c\sigma\sqrt{\pi} \\ \pi^m c\sqrt{\pi} & 1+\pi^m a\sqrt{\pi} \end{smallmatrix}\right), \ a, c \in R$$

Since there are q^{m+1} choices for both a and c, then $|K_l \cap S|= q^{l+1}$. As a result $|T|= \frac{q^{2l+1}4q^{2l}}{q^{l+1}} = 4q^{3l}$. Since S is abelian, ϕ_A extends to ψ, an irreducible degree 1 character of T, and $\chi = \mathrm{Ind}_T^{U_l}$ is irreducible of degree $[U_l : T] = \frac{2q^{4l-3}(q-1)}{4q^{3l}} = \frac{q^{l-1}(q-1)}{2}$.

(c) Let $A = \left(\begin{smallmatrix} 0 & \nu \\ 1 & 0 \end{smallmatrix}\right)$ with $\nu \in R_l$ a non-square unit, and $T = K_l S$, where S is the set of unitary matrices of the form $\left(\begin{smallmatrix} a & c\nu \\ c & a \end{smallmatrix}\right)$ To get the order of S note that, from section 6.A one of a, c must be a unit, but c cannot be a unit, for if it were, then we could write $a = r\sqrt{\pi}(c)$ so that:

$$a\bar{a} + c\bar{c}\nu = (r\sqrt{\pi}(c))\overline{(r\sqrt{\pi}(c))} + c\bar{c}\nu = c\bar{c}(\nu - \pi r^2) = 1$$

and this implies that $\nu - \pi r^2$ is a square in R_l; a contradiction. Therefore a must be a unit, and we write $c = r\sqrt{\pi}(a)$ to get:

$$a\bar{a}(1 - \pi r^2 \nu) = 1$$

Since we have q^l choices for r ($r \in R_l$), and $2q^l$ choices for a (it must come from the pre-image of $(1-\pi r^2\nu)^{-1}$ in the norm map, then $|S| = 2q^{2l}$. An element in the intersection of K_l and S looks like

$$\begin{pmatrix} 1+\pi^m a\sqrt{\pi} & \pi^m c\nu\sqrt{\pi} \\ \pi^m c\sqrt{\pi} & 1+\pi^m a\sqrt{\pi} \end{pmatrix}, \ a, c \in R$$

Since there are q^{m+1} choices for each of a, c then $|K_l \cap S| = q^{l+1}$, thus $|T| = \frac{q^{2l+1} 2q^{2l}}{q^{l+1}} = 2q^{3l}$. Again S is abelian and stabilizes ϕ_A, so we extend the ϕ_A to ψ, an irreducible character of T. Then $\chi = \mathrm{Ind}_T^{U_l}\psi$ is an irreducible character of U_l with degree $[U_l : T] = q^{l-1}(q-1)$.

(d) Let $A = \begin{pmatrix} 0 & \pi\beta \\ 1 & 0 \end{pmatrix}$ where $\beta \in R_{l-1}$. $T = K_l S$, where S is the set of unitary matrices of the form $\begin{pmatrix} a & \pi\beta c \\ c & a \end{pmatrix}$

The orders of S and $|K_l \cap S|$ are the same as for the previous matrix, hence $|T| = 2q^{3l}$, leading to an irreducible character of U_l having degree $q^{l-1}(q-1)$.

For reference, we give below the degrees of the characters found for the various A matrices.

6.C.2 Counting the Characters of the Unitary Group

Our inductive assumptions in this work is that both the character degrees of U_{l-1} and the number of these characters is known, the base case of $l = 1$ being given in the previous chapter. Since U_{l-1} is isomorphic to the factor group

Table 6.2: Character Degrees From ϕ_A Characters

A matrix	degree
$A = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$	$2q^{l-1}$
$A = \begin{pmatrix} 0 & \sigma \\ 1 & 0 \end{pmatrix}$	$\frac{q^{l-1}(q-1)}{2}$
$A = \begin{pmatrix} 0 & \nu \\ 1 & 0 \end{pmatrix}$	$q^{l-1}(q-1)$
$A = \begin{pmatrix} 0 & \pi\beta \\ 1 & 0 \end{pmatrix}$	$q^{l-1}(q-1)$

U_l/K_{l-1}, we can lift (so to speak), every character of U_{l-1} to U_l. Moreover, the product of each irreducible character from U_{l-1} with a linear character of U_l produces an irreducible character of U_l whose degree is known from the inductive hypothesis. Denote the linear characters of U_l as L_l, and the linear characters of U_{l-1} as L_{l-1}.

Proposition For any $\phi \in \mathrm{Irr}(U_{l-1})$, the number of distinct irreducible characters of U_l of the form $\psi\phi$ with $\psi \in L_l$ is $[L_l : L_{l-1}]$.

Proof. The proof used in Proposition 4.J.1 is valid for this case as well. □

Proposition $|L_l|/|L_{l-1}|= q$

Proof. We can use the proof from Proposition 4.J.2. □

Thus U_{l-1} will contribute $(q)2q^{4(l-1)-1}(q-1) = 2q^{4l-4}(q-1)$ to the sum of squares of character degrees which we will now calculate.

In order to show that we have found all character degrees of U_l with their respective numbers, we sum the squares of the degrees of all distinct irreducible characters of U_l to get the group order. The number of characters of a given degree will equal the number of non-conjugate ϕ_A characters on the abelian

K subgroup multiplied by the number of extensions to the inertia group. As a result we will demonstrate the following:

Theorem 6.C.1 The degrees of the irreducible characters of U_l found from the ϕ_A characters on K_l, and their respective numbers are:

Table 6.3: Degrees and Numbers

A matrix	degree	number
$A = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$	$2q^{l-1}$	$2q^{4l-4}(q-1)^2$
$A = \begin{pmatrix} 0 & \sigma \\ 1 & 0 \end{pmatrix}$	$\frac{q^{l-1}(q-1)}{2}$	$q^{4l-3}(q-1)^3$
$A = \begin{pmatrix} 0 & \nu \\ 1 & 0 \end{pmatrix}$	$q^{l-1}(q-1)$	$q^{4l-3}(q-1)^3$
$A = \begin{pmatrix} 0 & \pi\beta \\ 1 & 0 \end{pmatrix}$	$q^{l-1}(q-1)$	$4q^{4l-3}(q-1)^2$

Proof. 1. The Even Case

Let $l = 2m$.

We will need to count the number of non-conjugate ϕ_A characters; for all A matrices in the even case, we have

$$\phi_A[I + \pi^m B] = \lambda[\mathrm{tr}(\pi^m AB)]$$

thus when enumerating the parameters x, b below, we consider their value modulo π^m.

(a) Let $A = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$ with inertia group $K_{l-1}S$, the degree of the irreducible characters of U_l is $2q^{l-1}$, $A' = xI + bA$, with $x \in R_l, b \in R_l^*$. The schematic:

$$K_l \overset{\text{ext}}{\longrightarrow} N \overset{\text{ext}}{\longrightarrow} NS \overset{\text{ind}}{\longrightarrow} K_{l-1}S \overset{\text{ext}}{\longrightarrow} U_l$$
$$\phi_A \quad \phi'_A \quad \psi_0 \quad \psi \quad \chi$$

There are q^m choices for x, $q^{m-1}(q-1)$ choices for b and $xI \pm bA$ are conjugate. Hence the number of non-conjugate forms of A' is $\frac{q^{l-1}(q-1)}{2}$. The number of extensions from K_l to N is q, and the number of extensions from N to NS is $= q^{l-2}(q-1)$. Therefore there are $q^{l-1}(q-1)$ extensions in all. The number of characters of U_l of degree $2q^{l-1}$ is

$$\frac{q^{l-1}(q-1)}{2} \times q^{l-1}(q-1) = \frac{q^{2l-2}(q-1)^2}{2}$$

Multiplying by the degree squared gives $2p^{4l-4}(p-1)^2$.

The schematic for the next two A matrices is

$$K_l \overset{\text{ext}}{\longrightarrow} K_{l+} \overset{\text{ext}}{\longrightarrow} K_{l+}\ S \overset{\text{ind}}{\longrightarrow} U_l$$
$$\phi_A \quad \phi'_A \quad \phi''_A \quad \chi$$

(b) Let $A = \left(\begin{smallmatrix} 0 & \sigma \\ 1 & 0 \end{smallmatrix}\right)$ with inertia group $K_{l+}S$, and $A' = xI + bA$, $x \in R, b \in R_l^*$. The number of non-conjugate matrices is $q^{l-1}(q-1)$: to see this we consider two cases

i. Let $b = k^2 \in R_l^*$, so that for some $x \in R_\pi$, $x\overline{x} = b$. Then

$$\begin{pmatrix} x & 0 \\ 0 & (\overline{x})^{-1} \end{pmatrix}\begin{pmatrix} 0 & b\sigma \\ b & 0 \end{pmatrix}\begin{pmatrix} x^{-1} & \frac{\sigma}{\overline{x}} \\ 1 & \overline{x} \end{pmatrix} = \begin{pmatrix} 0 & x\overline{x}b\sigma \\ \frac{b}{x\overline{x}} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma' \\ 1 & 0 \end{pmatrix}$$

Thus for all squares bR_l, we have $bA \sim \left(\begin{smallmatrix} 0 & \sigma' \\ 1 & 0 \end{smallmatrix}\right)$, for some square $\sigma' \in R_l$.

ii. Let b be any non-square, and let n be some fixed non-square (both in R_l^*). We claim that $b\begin{pmatrix} 0 & \sigma \\ 1 & 0 \end{pmatrix} \sim n\begin{pmatrix} 0 & \sigma' \\ 1 & 0 \end{pmatrix}$. For some $x \in R_{l,\alpha}$, we have $x\overline{x}b = n$, and $\frac{b}{x\overline{x}}\sigma = n\sigma'$ for some σ', a square in R_l, so that

$$\begin{pmatrix} (\overline{x})^{-1} & 0 \\ 0 & x \end{pmatrix}\begin{pmatrix} 0 & \sigma \\ b & 0 \end{pmatrix}\begin{pmatrix} \overline{x} & 0 \\ 0 & x^{-1} \end{pmatrix} = \begin{pmatrix} 0 & b(x\overline{x})^{-1}\sigma' \\ bx\overline{x} & 0 \end{pmatrix} = \begin{pmatrix} 0 & n\sigma' \\ n & 0 \end{pmatrix}$$

There are q^m choices for x, and $\frac{q^{m-1}(q-1)}{2}$ choices for σ. We multiply by 2 to account for the b, so that the number of non-conjugate A' matrices is

$$q^m(2)\frac{q^{m-1}(q-1)}{2} = q^{l-1}(q-1)$$

The degree of the irreducible characters of U_l is $\frac{q^{l-1}(q-1)}{2}$. The number of extensions of each ϕ_A is $[K_{l+}S : K_l] = 4q^l$. Multiplying the numbers of non-conjugates, extensions and the degree squared gives:

$$q^{l-1}(q-1) \times 4q^l \times \frac{q^{2l-2}(q-1)^2}{4} = q^{4l-3}(q-1)^3$$

(c) Let $A = \begin{pmatrix} 0 & \nu \\ 1 & 0 \end{pmatrix}$ with inertia group $K_{l+}S$, the degree of the irreducible characters of U_l is $q^{l-1}(q-1)$. Again $A' = xI + bA$. In this case we can get all non-conjugate matrices by varying x and ν. If b is a square, then $\begin{pmatrix} 0 & b\nu \\ b & 0 \end{pmatrix}$ is conjugate to some $\begin{pmatrix} 0 & \nu' \\ 1 & 0 \end{pmatrix}$ by the same argument used for the previous A matrix. If b is a non-square then $b\nu$ is a square, so for some $x \in R_\pi$, $x\overline{x}b\nu = 1$ thus (let $y = \overline{x}^{-1}$)

$$\begin{pmatrix} 0 & y \\ x & 0 \end{pmatrix}\begin{pmatrix} 0 & b\nu \\ b & 0 \end{pmatrix}\begin{pmatrix} 0 & y^{-1} \\ x^{-1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & y\overline{y}\nu \\ x\overline{x}b\nu & 0 \end{pmatrix} = \begin{pmatrix} 0 & \nu' \\ 1 & 0 \end{pmatrix}$$

We have q^m choices for x and $\frac{q^{m-1}(q-1)}{2}$ choices for ν, giving $\frac{q^{l-1}(q-1)}{2}$ non-conjugate $\phi_{A'}$ characters. There are $\frac{|K_{l+1}S|}{|K_l|} = 2q^l$ extensions from K_l to $K_{l+}S$. Taking the product of these three values gives

$$\frac{q^{l-1}(q-1)}{2} \times 2q^l \times q^{2l-2}(q-1)^2 = q^{4l-3}(q-1)^3$$

(d) Let $A = \left(\begin{smallmatrix} 0 & \pi\beta \\ 1 & 0 \end{smallmatrix}\right)$ The schematic is:

$$\underset{\phi_A}{K_l} \overset{\text{ext}}{\longrightarrow} \underset{\phi'_A}{K_{l+}} \overset{\text{ext}}{\longrightarrow} \underset{\phi''_A}{K_{l+}\,\mathcal{G}_2} \overset{\text{ext}}{\longrightarrow} \underset{\psi}{K_{l+}\mathcal{G}_2 S} \overset{\text{ind}}{\longrightarrow} \underset{\chi}{U_l}$$

and $A' = xI + bA$. If b_1, b_2 are squares, then b_1A and b_2A are conjugate by the arguments given for previous A matrix. Similarly if b_1, b_2 are non squares, then b_1A and b_2A are conjugate. Hence there are $q^m(2)q^{m-1} = 2q^{l-1}$ non-conjugate characters. The inertia group is $K_{l+}\mathcal{G}_2S$, so the degree of the irreducible characters of U_l is $[U_l : K_{l+}\mathcal{G}_2S] = q^{l-1}(q-1)$. The number of extensions from K_l to K_{l+} is q, and from $K_{l+}\mathcal{G}_2$ to $K_{l+}\mathcal{G}_2S$, it is is $\frac{|S|}{|K_{l+}\mathcal{G}_2 \cap S|} = 2q^{l-1}$. Hence in all we have $2q^l$ extensions, so the product of characters, extensions, and degree squared is

$$2q^{l-1} \times 2q^l \times q^{2l-2}(q-1)^2 = 4q^{4l-3}(q-1)^2$$

We sum the four values above, together with the contribution from U_{l-1}:

$$2q^{4l-4}(q-1)^2 + q^{4l-3}(q-1)^3 + q^{4l-3}(q-1)^3 + 4q^{4l-3}(q-1)^2 + 2q^{4l-4}(q-1) = 2q^{4l-1}(q-1)$$

which is the order of U_l, hence for $l = 2m$ we have found the degrees and numbers of all irreducible characters of the unitary group.

2. The Odd Case

Let $A = 2m + 1$.

The K_l characters are

$$\phi_A[I + \pi^m \sqrt{\pi} B] = \lambda[\mathrm{tr}(\pi^m \sqrt{\pi} AB)]$$

and a typical element of K_l has the form

$$I + \begin{pmatrix} \pi^{m+1}e_1 + \pi^m e_2 \sqrt{\pi} & \pi^m f \sqrt{\pi} \\ \pi^m g \sqrt{\pi} & -\pi^{m+1}e_1 + \pi^m e_2 \sqrt{\pi} \end{pmatrix}, \; e_1, e_2, f, g \in R_l$$

and the precise form of the elements in the bracket will be relevant in counting the non-conjugate characters for each matrix.

(a) Let $A = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$ with inertia group $K_{l-1}S$. The degree of the irreducible characters of U_l is $2q^{l-1}$, $A' = xI + bA$, and the schematic:

$$\underset{\phi_A}{K_l} \xrightarrow{\text{ext}} \underset{\phi'_A}{K_{l-1}} \xrightarrow{\text{ext}} \underset{\psi}{K_{l-1}S} \xrightarrow{\text{ind}} \underset{\chi}{U_l}$$

To count the choices for x and b, we note

i. If $A = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$ then the applying ϕ_A to the element of K_l above gives $\lambda(\pi^m(2xa_2))$, hence x there are q^{m+1} choices for x.

ii. If $A = b\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{b} \end{pmatrix}$, applying ϕ_A gives $\lambda(\pi^{m+1}(ba_1)$. Since b must be a unit, there are $q^{m-1}(q-1)$ choices for b.

Since the matrices $xI \pm bA$ are conjugate, there are $q^{m+1}\frac{q^{m-1}(q-1)}{2} = \frac{q^{l-1}(q-1)}{2}$ non-conjugate characters on K_l. There are q extensions from K_l to K_{l-1}, and $q^{l-2}(q-1)$ from K_{l-1} to $K_{l-1}S$, giving $q^{l-1}(q-1)$ extensions in all. Hence the number of characters of U_l of degree $2q^{l-1}$ is

$$\frac{q^{l-1}(q-1)}{2} \times q^{l-1}(q-1) = \frac{q^{2l-2}(q-1)^2}{2}$$

Multiplying by the degree squared accounts for $2q^{4l-4}(q-1)^2$ elements of U_l.

(b) Let $A = \left(\begin{smallmatrix} 0 & \sigma \\ 1 & 0 \end{smallmatrix}\right)$ with inertia group K_lS. Applying ϕ_A as in the previous A matrix, we see that there are q^{m+1} choices for x, and $\frac{q^{m-1}(q-1)}{2}$ choices for σ. The argument for grouping squared values of b together, and non-square values of b together carries through here, so that the number of non-conjugate $A' = xI + bA$ is

$$q^{m+1}(2)\frac{q^{m-1}(q-1)}{2} = q^{l-1}(q-1)$$

and the number of extensions of each ϕ_A is $4q^{l-1} = [K_lS : K_l]$. The degree of the irreducible characters of U_l is $\frac{q^{l-1}}{2}$. Multiplying the numbers of non-conjugates, extensions and the degree squared gives:

$$q^{l-1}(q-1) \times 4q^{l-1} \times \frac{q^{2l-2}(q-1)^2}{4} = q^{4l-3}(q-1)^3$$

(c) Let $A = \left(\begin{smallmatrix} 0 & \nu \\ 1 & 0 \end{smallmatrix}\right)$ with inertia group K_lS, the degree of the irreducible characters of U_l is $q^{l-1}(q-1)$. There are q^{m+1} choices for x, and

$\frac{q^{m-1}(q-)}{2}$ choices for ν, and the argument from the even case that eliminates the effect of b carries through here, so that there are $\frac{q^{l-1}(q-1)}{2}$ non-conjugate $\phi_{A'}$ characters and $[K_lS : K_l] = 2q^{l-1}$ extensions from K_l to K_lS. Taking the product of these three values gives

$$\frac{q^{l-1}(q-1)}{2} \times 2q^{l-1} \times q^{2l-2}(q-1)^2 = q^{4l-3}(q-1)^3$$

(d) Let $A = \begin{pmatrix} 0 & \pi\beta \\ 1 & 0 \end{pmatrix}$ with inertia group K_lS, the degree of the irreducible characters of U_l is $q^{l-1}(q-1)$. The argument from the even case about grouping squared and non-square valued of b applies here. There are q^{m+1} choices for x, q^{m-1} choices for β, and a factor of 2 for the effect of b. Hence there are

$$q^{m+1}(2)q^{m-1} = 2q^l$$

non-conjugate characters. The number of extensions is $[K_lS : K_l]$ or $2q^{l-1}$. The product is

$$2q^l \times 2q^{l-1} \times q^{2l-2}(q-1)^2 = 4q^{4l-3}(q-1)^2$$

Summing the four values, and the contribution from U_{l-1}:

$$2q^{4l-4}(q-1)^2 + q^{4l-3}(q-1)^3 + q^{4l-3}(q-1)^3 + 4q^{4l-3}(q-1)^2 + 2q^{4l-4}(q-1) = 2q^{4l-1}(q-1)$$

which is $|U_l|$, hence for $l = 2m+1$ we have found the degrees and numbers of all irreducible characters of the unitary group.

□

Bibliography

[BL] R. Barrington Leigh, Gerald Cliff, Qianglong Wen, *Character Values for* $GL(2, \mathbb{Z}/p^l\mathbb{Z})$, Journal of Algebra, Vol. 323, 2010, pp 1288-1320.

[Ca] J. Campbell, *The Irreducible Characters of* 2×2 *Unitary Matrix Groups Over Finite Fields*, University of Alberta, 2014.

[Cr] J. Cruickshank, A. Herman, R. Quinlan, F. Szechtman, *Unitary Groups Over Local Rings*, Journal of Algebra and Its Applications, Vol. 13, 2014.

[E] V. Ennola, On the characters of the finite unitary groups, *Ann. Acad. Sci. Fenn. Ser. A I No.* **323** (1963) 35 pp.

[I] I. Martin Isaacs, *Character Theory of Finite Groups*, Dover Publications, New York, 1976.

[JL] G. James and M. Liebeck, *Representations and Characters of Groups*, Cambridge University Press, Cambridge, 2012.

[S1] Jean-Pierre Serre, *A Course in Arithmetic*, Springer, New York, 1973

[S2] Jean-Pierre Serre, *Linear Representations of Finite Groups*, Springer, New York, 1977