University of Alberta

Theory of Spectral Sequences of Exact Couples: Applications To Countably And Transfinitely Filtered Modules

by

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Dedication

To my mother, and two sisters,

those who do not need to know anything about spectral sequences.

Abstract

This thesis has two parts. In the first part we start from an arbitrary exact couple of R-modules and describe completely how the E^∞ terms of the associated spectral sequence relate to adjacent filtration stages of the universal (co-)augmenting objects of the exact couple. This advances earlier work, notably that of Boardman [3].

In the second part we use these insights to develop a framework which permits spectral sequence methods to gain information about suitably transfinitely filtered objects.

We offer several applications of this method:

1. We use Serre's idea of working relative to a class of modules while passing through the pages of the spectral sequence associated to an exact couple and we spell out conditions under which the filtration stages of countably or transfinitely filtered modules stay within such a class.

2. We extend Zeeman's comparison technique of spectral sequences to apply to a map between countably or transfinitely filtered modules.

3. Finally, we develop a general setting of reverse engineering information about finite pages in a spectral sequence from information about the universally filtered objects of the underlying exact couple.

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I am very grateful to Dr. Peschke for his endless insights in the subject and for his tireless motivation to get things accomplished. Although he had his own schedule to attend to, he always seemed to have time for my questions and concerns.

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Chapter 1

Introduction

"A spectral sequence is an algebraic gadget like an exact sequence, but more complicated."

J. F. Adams

Please be advised that, to reduce the complexity that Adams has mentioned, we will use colorful diagrams throughout this thesis, in particular, in the appendix Diagrams. If you print this thesis in black ink, you are strongly recommended to look at the diagrams in the PDF file to enjoy clarity provided by colors.

In appropriate settings spectral sequences help compute homology groups, cohomology groups, or homotopy groups of groups or topological spaces which are filtered by an ascending or a descending sequence of subgroups. Someone who wants to learn how to use spectral sequences faces a certain entrance hurdle of complexity that needs to be overcome.

We try to lower this hurdle here by first discussing components encountered in spectral sequences in isolation. Once these components are understood, it is easier to understand their mutually complementary roles. This enables us to extend the conventional theory of spectral sequences for objects with a finite, $\mathbb{N}$- or $\mathbb{Z}$-indexed filtration to a theory of spectral sequences for objects which have a filtration parametrized by an arbitrary limit ordinal.

• **Spectral Sequences and Exact Couples.** Spectral sequence methods are potentially relevant when one deals with a filtered module

$$\cdots \subseteq F_{p-1} \subseteq F_p \subseteq F_{p+1} \subseteq \cdots \subseteq \mathrm{H}, \tag{1.1}$$

where the F_p's are submodules of an R-module H; throughout the whole thesis the rings we choose are all commutative and unitary. Assume inductively, we have information about F_p and we want to extend this information to F_{p+1} and ultimately up the tower of inclusions to H. In the following short exact sequence

$$F_p \rightarrowtail F_{p+1} \twoheadrightarrow \frac{F_{p+1}}{F_p},$$

if $\dfrac{F_{p+1}}{F_p}$ is known then information about F_{p+1} is provided within the context of the *extension problem* in Homological Algebra, where possible candidates for F_{p+1} are studied. Thus we have two distinct tasks here:

1. the task to determine $\dfrac{F_{p+1}}{F_p}$, and
2. the task to infer properties of F_{p+1} from information about F_p and $\dfrac{F_{p+1}}{F_p}$.

One of the main purposes of a spectral sequence is to provide information about adjacent filtration quotients $\dfrac{F_p}{F_{p-1}}$.

A spectral sequence is similar to a "book". Each page in the book is an endomorphism d of a $(\mathbb{Z} \times \mathbb{Z})$-bigraded R-module which satisfies $d^2 = 0$. This turns each page in the book into a $\mathbb{Z}$-indexed family of chain complexes and flipping through pages is accomplished by taking homology. The pages in a spectral sequence determine a "limit page", which is again a bigraded R-module not equipped with an endomorphism d. The entries in this page are called the E^∞-terms of the spectral sequence.

In the second chapter, the definition and basic mechanics of a spectral sequence are discussed in isolation. However, the ultimate purpose of a spectral sequence is not yet visible from this isolated perspective.

A prominent approach to spectral sequences is based upon an interlocking system of long exact sequences, called an "exact couple", which was defined by Massey, in [22] and [23], and is outlined in the figure below; each subdiagram of the shape of same colored arrows forms a long exact sequence and the blue and red colored long exact sequences interlock at the green dots. However, the remaining pairs of neighboring long exact sequences interlock at the appropriate black dots.

$$
\begin{tikzcd}
 & \vdots \arrow[d, blue] & & \vdots \arrow[d] & & \vdots \arrow[d] & \\
\cdots \arrow[r, red] & {\color{green}\bullet} \arrow[r, blue] \arrow[d, red] & \diamond \arrow[r, blue] & \bullet \arrow[r] \arrow[d, blue] & \diamond \arrow[r] & \bullet \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \bullet \arrow[r, red] \arrow[d] & \diamond \arrow[r, red] & {\color{green}\bullet} \arrow[r, blue] \arrow[d, red] & \diamond \arrow[r, blue] & \bullet \arrow[r] \arrow[d, blue] & \cdots \\
\cdots \arrow[r] & \bullet \arrow[r] \arrow[d] & \diamond \arrow[r] & \bullet \arrow[r, red] \arrow[d] & \diamond \arrow[r, red] & {\color{green}\bullet} \arrow[r, blue] \arrow[d, red] & \cdots \\
 & \vdots & & \vdots & & \vdots &
\end{tikzcd}
$$

We will see three important objects obtained from an exact couple:

1. a $\mathbb{Z}$-graded R-module, called the universal augmentation L_*, which is the colimit of the vertical $\bullet$-columns,
2. a $\mathbb{Z}$-graded R-module, called the universal coaugmentation L^*, which is the limit of the vertical $\bullet$-columns, and
3. the first page of a spectral sequence, which are shown by $\diamond$ in the following figure.

$$
\begin{tikzcd}
 & L^{n+1} \arrow[d] & & L^{n} \arrow[d] & & L^{n-1} \arrow[d] & \\
 & \vdots \arrow[d, blue] & & \vdots \arrow[d] & & \vdots \arrow[d] & \\
\cdots \arrow[r, red] & {\color{green}\bullet} \arrow[r, blue] \arrow[d, red] & \diamond \arrow[r, blue] & \bullet \arrow[r] \arrow[d, blue] & \diamond \arrow[r] & \bullet \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \bullet \arrow[r, red] \arrow[d] & \diamond \arrow[r, red] & {\color{green}\bullet} \arrow[r, blue] \arrow[d, red] & \diamond \arrow[r, blue] & \bullet \arrow[r] \arrow[d, blue] & \cdots \\
\cdots \arrow[r] & \bullet \arrow[r] \arrow[d] & \diamond \arrow[r] & \bullet \arrow[r, red] \arrow[d] & \diamond \arrow[r, red] & {\color{green}\bullet} \arrow[r, blue] \arrow[d, red] & \cdots \\
 & \vdots \arrow[d] & & \vdots \arrow[d] & & \vdots \arrow[d] & \\
 & L_{n+1} & & L_{n} & & L_{n-1} &
\end{tikzcd}
$$

These L_* and L^* are the objects of interest and we use spectral sequences to compute them. For every $n \in \mathbb{Z}$, by universality of augmentation and coaugmentation, for each $\bullet$ there is a canonical morphism

$$\bullet \longrightarrow L_n \tag{1.2}$$

and

$$L^n \longrightarrow \bullet \tag{1.3}$$

and hence we can

- filter L_* by the images of the morphisms (1.2), and
- filter L^* by the kernels of the morphisms (1.3).

In [22] and [23], Massey introduces a recursive method to distill a spectral sequence from an exact couple. Hence, the limit page of the spectral sequence can be obtained from the corresponding exact couple. Therefore, we have

- the filtration of L_*,
- the filtration of L^*, and
- E^∞-page of the induced spectral sequence.

The question is

Question 1. *Is there any relationship between E^∞, L_* and L^*?*

To answer this question, Peschke [26] *matches* the filtrations of L_* and L^* with the induced spectral sequence; i.e., he connects the E^∞ to the quotient of adjacent filtration stages of L_* and L^*, which he denotes by $\epsilon_{-,-}$ and $\epsilon^{-,-}$, respectively. As a result, he uses the E^∞-terms of the spectral sequence to inductively carry some information through the filtration of L_* and L^* as we explained in the previous page. Boardman, [3], provides the relationship between $\epsilon_{-,-}$ and E^∞. Peschke [26], carried the discussion of exact couples, their associated spectral sequences and associated filtered objects a crucial step further. He shows that that there is a relationship between $\epsilon_{-,-}$, $\epsilon^{-,-}$ and E^∞ *simultaneously*. This phenomenon is explained in the second chapter as the *E^∞-Distribution Theorem*. Boardman and many others who work with spectral sequences are very interested in convergence issues. That is, they are interested in getting these three objects very close to each other. So the second question arises:

Question 2. *What are the necessary and sufficient conditions to have E^∞ isomorphic to $\epsilon_{-,-}$ or $\epsilon^{-,-}$? Is it possible to have both at the same time?*

The answer to this question is an immediate consequence of the E^∞-Distribution Theorem, which is not presented in Boardman's work.

The next question is:

Question 3. *What happens if one of $\epsilon_{-,-}$ or $\epsilon^{-,-}$ vanishes? How is the non-zero one related to E^∞?*

The E^∞-Distribution Theorem explains precisely what happens if one of these quotients vanishes and how the relationship between the non-zero one and E^∞ is improved. Boardman explains this situation by introducing a new type of convergence that he calls *conditional convergence*.

In fact, to answer the questions above, the E^∞-Distribution Theorem provides a diagram of where morphisms of the same color form an exact sequence

$$\begin{tikzcd} & & B_{-,-} & \\ \epsilon_{-,-} \arrow[r, tail] & E^\infty_{-,-} \arrow[r, two heads] & A_{-,-} \arrow[u, "f"] & \epsilon^{-,-}, \arrow[l, tail] \end{tikzcd}$$

where A and B are modules which will be defined later in the second chapter.

We will show that this theorem covers all known convergence types stated in [3]. Moreover, it also enables us to distill information from spectral sequences which fail to converge in any traditional sense of the word. At the end, we will compare Peschke's method which leads to the E^∞-Distribution Theorem with the approach that Boardman has taken.

• **Matching Convenient Exact Couples to a Tower of Modules.** The idea of the inductive argument in this chapter is originally stated in [26]. Climbing up the filtration stages in (1.1) by induction is enabled if we are able to *anchor* the induction at some ordinal; i.e., if F_{p_0} is known for some $p_0 \in \mathbb{Z}$. Such an anchor does not exist in general. But in many practical examples it does. See examples in sections 3.2.2 and 3.2.4. For simplicity, we assume $p_0 = 1$. Therefore, the filtration (1.1) looks like the following filtration indexed over ω, the least infinite ordinal:

$$F_1 \subseteq F_2 \subseteq \cdots \subseteq F_p \subseteq F_{p+1} \subseteq \cdots \subseteq F_\omega = \mathrm{H}\,. \tag{1.4}$$

In the following exact couple, the filtration of the universal augmentation is of the form (1.4)

$$
\begin{tikzcd}
 & 0 \arrow[d, equal] & & 0 \arrow[d, equal] & & 0 \arrow[d, equal] & \\
 & \vdots \arrow[d, equal] & & \vdots \arrow[d, equal] & & \vdots \arrow[d, equal] & \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[d, equal] & 0 \arrow[r] & 0 \arrow[r] \arrow[d, equal] & 0 \arrow[r] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] & 0 \arrow[r] \arrow[d, equal] & \diamond \arrow[r] & \bullet \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \bullet \arrow[r] \arrow[d] & \diamond \arrow[r] & 0 \arrow[r] \arrow[d, equal] & \diamond \arrow[r] & \bullet \arrow[r] \arrow[d] & \cdots \\
 & \vdots \arrow[d] & & \vdots \arrow[d, equal] & & \vdots \arrow[d] & \\
\cdots \arrow[r] & \bullet \arrow[r] \arrow[d] & \diamond \arrow[r] & 0 \arrow[r] \arrow[d] & \diamond \arrow[r] & \bullet \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \bullet \arrow[r] \arrow[d] & \diamond \arrow[r] & \bullet \arrow[r] \arrow[d] & \diamond \arrow[r] & \bullet \arrow[r] \arrow[d] & \cdots \\
 & \vdots \arrow[d] & & \vdots \arrow[d] & & \vdots \arrow[d] & \\
 & L_{n+1} & & L_n & & L_{n-1} &
\end{tikzcd}
$$

We will call it an *originally vanishing* exact couple. In the third chapter, for such exact couple and the filtration of its universal augmentation L_*, we start from the *known* step 0 and we assume we have information about F_p. To obtain information about F_{p+1} we look at the following short exact sequence

$$F_p \rightarrowtail F_{p+1} \twoheadrightarrow \frac{F_{p+1}}{F_p}.$$

We will see in the third chapter that there is a very close relationship between the quotients $\dfrac{F_{p+1}}{F_p}$ and the E^∞-terms of the spectral sequence induced by an originally vanishing exact couple. In fact, we have

$$\frac{F_{p+1}}{F_p} \cong E^\infty.$$

Now, if we have the desired information about the E^∞-terms, it can be inherited to the quotients $\dfrac{F_{p+1}}{F_p}$ and hence we have the information about F_{p+1}, up to extension.

This originally vanishing exact couple is a special case of an exact couple in which, instead of a zero range, we have isomorphisms in the blue range in the diagram above. We will call it an originally *stable* exact couple. The dual situations are called *eventually vanishing* and *eventually stable* exact couples.

Now, the following question arises

is this approach applicable to transfinitely filtered modules?

The answer is "Yes". Constructions associated to the homotopical (co-)localization lead exactly to settings of the kind we have just described: See Appendix B. In fact, the entire project was originally motivated by this observation. We will generalize to the case that H is transfinitely filtered over an arbitrary limit ordinal λ

$$F_1 \subseteq F_2 \subseteq \cdots \subseteq F_\omega \subseteq F_{\omega+1} \subseteq \cdots \subseteq F_\eta \subseteq F_{\eta+1} \subseteq \cdots \subseteq F_\lambda = \mathrm{H}\,. \tag{1.5}$$

We try to obtain information about F_λ through transfinite induction. For an arbitrary ordinal $\eta \leq \lambda$, when trying to take the inductive step from "information about F_β, for all $\beta < \eta$" to "information about F_η", we face one of the following two situations:

Situation 1: η is a non-limit ordinal and, hence, has a predecessor. In this case, we make the following elementary but crucial observation:

η has at most finitely many predecessors.

The foundation of the approach taken here is based on this technical observation. Therefore, there exist a limit ordinal η_0 and a positive integer r such that $\eta = \eta_0 + r$. Our method applies in cases that we are lucky to have an exact couple *matched* to the ω-length segment of the filtration

$$F_{\eta_0} \subseteq F_{\eta_0+1} \subseteq \cdots \subseteq F_{\eta_0+r-1} \subseteq F_{\eta_0+r} \subseteq \cdots \subseteq F_{\eta_0+\omega}. \tag{1.6}$$

That is, we assume there is an originally vanishing exact couple such that $F_{\eta_0+\omega}$ plays the role of its augmentation. This is, indeed, part of the strategy which motivated the entire approach. It is also the reason why ω-indexed filtrations play a key role in the development here. Now we can use the same method we used in the ω-indexed scenario to pass from F_{η_0+r-1} to $F_{\eta_0+r} = F_\eta$.

> Situation 2: η is a limit ordinal. The inductive step from filtration stages indexed by ordinals less than η to the filtration stage η is outside of the scope of the spectral sequence machine developed here. What we have here is a morphism $\rho_\eta : \operatorname{colim}_{\beta<\eta} \mathrm{H}_\beta \to \mathrm{H}_\eta$ and hence an inclusion $\operatorname{colim}_{\beta<\eta} F_\beta \rightarrowtail F_\eta$. We will make the passage from $\operatorname{colim}_{\beta<\eta} F_\beta$ to F_η possible by putting assumptions on ρ_η.

For an alternative spectral sequence based approach to countably transfinitely filtered objects see [18].

• **Classes of Modules Compatible with Spectral Sequences and Transfinite Induction.** In the fourth chapter, we extend the idea of Serre in [29] working modulo a class of *modules*. As an application of the inductive argument explained above, we will introduce a collection of closure properties which, when satisfied by any class of modules, ensures that the filtration stages of countably or transfinitely filtered modules stay within this class; i.e., these closure properties are "tailor-made" for the inductive argument above. For example, for a class $\mathcal{C}$ of modules with some closure properties

if the homology groups of all cofibers of a transfinite tower of cofibrations are in $\mathcal{C}$, then the homology groups of the colimit of the tower will be also in $\mathcal{C}$.

It turns out that this collection, even with fewer closure properties, is also "tailor-made" for the mechanics of *every* spectral sequence in the sense that

if the entries of some page of an arbitrary *spectral sequence are in this class then all entries of the successive pages will be in the class, even the entries of the limit page.*

• **Comparison Theorems Modulo a Class of Modules.** As the second application of the inductive argument above, for a morphism of two spectral sequences, in the fifth chapter we consider a variety of situations in which a choice of hypotheses about the effect of this morphism at the E^∞-page leads to conclusions about the effect of this morphism at the universal (co-)augmenting objects. For example, we will prove that if we have a morphism between two arbitrary exact couples such that

1. the morphisms between the E^∞-terms, and

2. the morphisms between the intersection of the filtrations of the universal augmentations

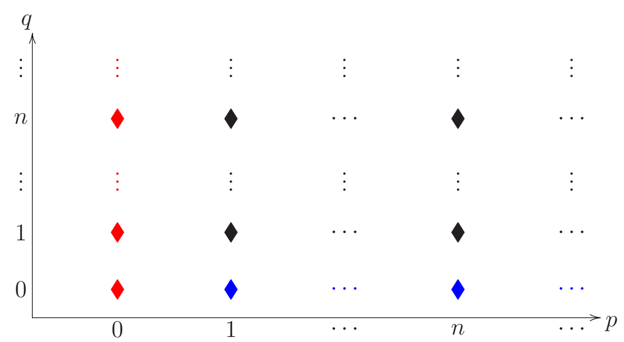
are $\mathcal{C}$-monomorphisms, then the morphism between the universal augmentations is a $\mathcal{C}$-monomorphism.

Then, given a morphism of two transfinitely filtered modules, we use the inductive argument above to carry some hypotheses about the effect of this morphism through the filtration stages to the effect of the morphism of the filtered modules.

All morphisms we work with in this chapter are morphisms with *tamed* kernels and/or cokernels; i.e., they are morphisms that their kernels and/or cokernels belong to a class $\mathcal{C}$ of modules with some closure properties. So a $\mathcal{C}$-monomorphism ($\mathcal{C}$-epimorphism) is a homomorphism whose kernel (cokernel) belongs to the class $\mathcal{C}$. A $\mathcal{C}$-isomorphism is a homomrphism that is both $\mathcal{C}$-monomorphism and $\mathcal{C}$-epimorphism. Note that if $\mathcal{C}$ is the trivial class consisting of only zero, then all these notions coincide with the standard notions of monomorphism, epimorphism and isomorphism. The idea of all proofs in this chapter are also originated from Peschke [26].

• **Reverse Engineering in Spectral Sequences.** Under some circumstances, it is possible to flip through the finite pages of a spectral sequence backward. That is, we can start with some information about the universal augmentation or coaugmentation of an exact couple and obtain information about some page of the induced spectral sequence. In chapter six, we see two scenarios that this reverse engineering phenomenon happens. The idea of the first scenario comes from the second scenario which was developed by Peschke [26]:

1. In the first scenario, we pick a class of modules with some closure properties and we assume in the r-th page of the following spectral sequence



- the red entries are in $\mathcal{C}$,
- if a blue entry is in $\mathcal{C}$ then all entries above it are also in $\mathcal{C}$, and
- the universal augmentation is in $\mathcal{C}$.

Then, we conclude that all entries of the r-th page also belong to $\mathcal{C}$.

2. In the second scenario, we pick two such spectral sequences and assume the following

   - the corresponding red entries are $\mathcal{C}$-isomorphic,
   - if a blue entry in one spectral sequence is $\mathcal{C}$-isomorphic to the corresponding blue entry of another, then all corresponding entries above them are also $\mathcal{C}$-isomorphic, and
   - the universal augmentations are $\mathcal{C}$-isomorphic.

   Then we conclude that all entries of the r-th page of the spectral sequences are $\mathcal{C}$-isomorphic.

   Note that Zeeman's comparison theorem is a special case: It assumes that the entries in the second page of the spectral sequence satisfy the following short exact sequence

$$E^2_{p,0} \otimes E^2_{0,q} \rightarrowtail E^2_{p,q} \twoheadrightarrow \mathrm{Tor}(E^2_{p-1,0}, E^2_{0,q}).$$

History. Spectral sequences were invented in 1946 by Jean Leray [21] to compute the (co-)homology of a graded chain complex. In 1947, Koszul [20] stated the definitions made by Leray in a more abstract term and in 1951, Serre [28] offered an important application of spectral sequences by providing the properties of the spectral (co-)homology sequence of a fibration. Using this spectral sequence, he could show that a simply-connected space has finitely generated homotopy groups iff it has finitely generated homology groups. As a result, the homotopy groups of a sphere are finitely generated; before this result, it was only known that these homotopy groups were countable. In 1952 and 1953, Massey, [22] and [23], introduced exact couples as a source of spectral sequences and the development of a general theory of spectral sequences. See [25] for a more detailed history of spectral sequences.

Later, more complicated spectral sequences of interest appeared and the old finiteness conditions which were imposed to guarantee convergence were replaced by some limit conditions; look at [8]. In 1999, Boardman published his paper [3], which had been in circulation for many years before publication, in which he describes the major players corresponded to an exact couple and the induced spectral sequence and, to some extent, clarifies the relationship between them. His paper has been a standard reference on convergence. Peschke [26] in his (unpublished) lecture notes improved upon the relationship between these major players and presented it in a single picture, which he called it E^∞-distribution diagram. This picture can describe all convergence types that exist in the literature and provide much more information.

Then series of successive spectral sequences arose and the need for a single structure capable of managing them was felt. Po Hu [18] introduced a generalization of spectral sequences called transfinite spectral sequences and also a good source of them, transfinite exact couples. As a result, under some conditions, we have a spectral sequence with transfinitely many pages and transfinitely long differentials. This is done for countable ordinals. Here, instead, we take a different approach and we consider a transfinitely filtered module such that the filtration stages are *suitably* related to the spectral sequence methods. We do not define any new type of spectral sequences or any differentials of new length. We also do not restrict the indexing ordinal to be countable.

Chapter 2

Spectral Sequences and Exact Couples

2.1 Introduction

In this chapter, we review the definition of a "spectral sequence" and develop its structural properties. We introduce the pages of a spectral sequence and, in particular, its infinity page, E^∞.

An outstanding source of spectral sequences is an interlocking system of long exact sequences, called an "exact couple", which is outlined in the figure below; each subdiagram of the shape of same colored arrows forms a long exact sequence of bigraded R-modules and each arrow has a bidegree **a**, **b** or **c**. Here we pick $\mathbf{a} = (-1, 1)$, $\mathbf{b} = (0, 0)$ and $\mathbf{c} = (0, -1)$ which are the bidegrees in a homology spectral sequence.

$$
\begin{tikzcd}
 & \vdots \arrow[d] & & \vdots \arrow[d] & & \vdots \arrow[d] & \\
\cdots \arrow[r] & D_{(p,q)} \arrow[r] \arrow[d, "\mathbf{a}"'] & E_{(p,q)} \arrow[r] & D_{(p,q-1)} \arrow[r] \arrow[d] & E_{(p,q-1)} \arrow[r] & D_{(p,q-2)} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & D_{(p-1,q+1)} \arrow[r, "\mathbf{b}"] \arrow[d] & E_{(p-1,q+1)} \arrow[r, "\mathbf{c}"] & D_{(p-1,q)} \arrow[r] \arrow[d] & E_{(p-1,q)} \arrow[r] & D_{(p-1,q-1)} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & D_{(p-2,q+2)} \arrow[r] \arrow[d] & E_{(p-2,q+2)} \arrow[r] & D_{(p-2,q+1)} \arrow[r] \arrow[d] & E_{(p-2,q+1)} \arrow[r] & D_{(p-2,q)} \arrow[r] \arrow[d] & \cdots \\
 & \vdots & & \vdots & & \vdots &
\end{tikzcd}
$$

Every exact couple delivers three important objects:

1. a $\mathbb{Z}$-graded R-module L_*, where $L_n = \operatorname{colim}_r D_{(p-r,q+r)}$, called the universal augmentation,

2. a $\mathbb{Z}$-graded R-module L^*, where $L^n = \lim_r D_{(p_r,q+r)}$, called the universal coaugmentation, and

3. the first page of a spectral sequence, which are shown by $E_{p,q}$.

$$
\begin{tikzcd}
 & L^n \arrow[d] & & L^{n-1} \arrow[d] & & L^{n-2} \arrow[d] & \\
 & \vdots \arrow[d] & & \vdots \arrow[d] & & \vdots \arrow[d] & \\
\cdots \arrow[r] & D_{(p,q)} \arrow[r] \arrow[d, "\mathbf{a}"'] & E_{(p,q)} \arrow[r] & D_{(p,q-1)} \arrow[r] \arrow[d] & E_{(p,q-1)} \arrow[r] & D_{(p,q-2)} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & D_{(p-1,q+1)} \arrow[r, "\mathbf{b}"] \arrow[d] & E_{(p-1,q+1)} \arrow[r, "\mathbf{c}"] & D_{(p-1,q)} \arrow[r] \arrow[d] & E_{(p-1,q)} \arrow[r] & D_{(p-1,q-1)} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & D_{(p-2,q+2)} \arrow[r] \arrow[d] & E_{(p-2,q+2)} \arrow[r] & D_{(p-2,q+1)} \arrow[r] \arrow[d] & E_{(p-2,q+1)} \arrow[r] & D_{(p-2,q)} \arrow[r] \arrow[d] & \cdots \\
 & \vdots \arrow[d] & & \vdots \arrow[d] & & \vdots \arrow[d] & \\
 & L_n & & L_{n-1} & & L_{n-2} &
\end{tikzcd}
$$

For every $n \in \mathbb{Z}$, by universality of augmentation and coaugmentation, for each $D_{(p,q)}$ there is a canonical morphism

$$D_{(p,q)} \longrightarrow L_n \tag{2.1}$$

and

$$L^n \longrightarrow D_{(p,q)} \tag{2.2}$$

and hence we can

- filter L_* by the images of the morphisms (2.1), and

- filter L^* by the kernels of the morphisms (2.2).

Then we show how to distill a spectral sequence from an exact couple and, in particular, how to distill the E^∞-page of the induced spectral sequence. Much of this review is necessarily standard material covered in

[3], [22] and [23]. However, in the end we carry the discussion of exact couples, their associated spectral sequences and associated filtered objects a significant step further. In the E^∞-Distribution Theorem, Peschke [26], in his (unpublished) lecture notes, shows that E^∞ is related to the filtrations of L_* and L^* *simultaneously* and we describe this relationship completely for all spectral sequences, and without any assumptions. In most applications, one is interested in either the universally augmenting object L_* and its image filtration or the universally coaugmenting object L^* and its kernel filtration, not both at the same time. In this situation one hopes to gain information about the quotient of adjacent filtration stages from E^∞, and this information is strongest if these quotients and E^∞-terms are isomorphic. The question whether or not this happens is the "convergence question" for a spectral sequence.

In [3], Boardman presents the relationship between E^∞-terms of a spectral sequence and the quotients of the adjacent filtration stages of L_*, which we denote by $\epsilon_{p,q}$. Peschke [26] provides a relationship between E^∞-terms of a spectral sequence and the quotients of adjacent filtration stages of L^*, which we denote by $\epsilon^{p,q-1}$. In the E^∞-Distribution Theorem both of these relationships are offered. In this theorem, we see that we have the following diagram, where morphisms of the same color form an exact sequence and i and j are homomorphisms defined by the corresponding exact couple

$$\begin{tikzcd}
 & & \lim^1 \ker(i^r_{(p+r,q-r-1)}) & \\
\epsilon_{(p,q)} \arrow[r, tail] & E^\infty_{(p,q)} \arrow[r, two heads] & \dfrac{Z^\infty_{(p,q)}}{\operatorname{im}(j_{(p,q)})} \arrow[u, "f"] & \epsilon^{(p,q-1)} \arrow[l, tail]
\end{tikzcd}$$

This provides a complete answer to the first question we mentioned in the first chapter. To answer the second question, we will provide necessary and sufficient conditions for

1. $\epsilon_{(p,q)} \cong E^\infty_{(p,q)}$ (convergence to L_*), and

2. $E^\infty_{(p,q)} \cong \epsilon^{(p,q-1)}$ (convergence to L^*),

as follows:

1. $\epsilon_{(p,q)} \cong E^\infty_{(p,q)}$ if and only if $\dfrac{Z^\infty_{(p,q)}}{\operatorname{im}(j_{(p,q)})}$ vanishes.

2. $E^{\infty}_{(p,q)} \cong \epsilon^{(p,q-1)}$ if and only if $\epsilon_{(p,q)}$ and the map

$$f : \frac{Z^{\infty}_{(p,q)}}{\mathrm{im}(j_{(p,q)})} \longrightarrow \lim{}^1 \ker(i^r_{(p+r,q-r-1)})$$

vanish.

In particular, if only the map f vanishes then we obtain the following short exact sequence

$$\epsilon_{(p,q)} \rightarrowtail E^{\infty}_{(p,q)} \twoheadrightarrow \epsilon^{(p,q-1)}.$$

This is the closest possible non-trivial relationship between these three objects which is not in the scope of what Boardman has provided. To complete the answer to the second question, we will show that E^{∞} can be isomorphic to only one of $\epsilon_{(p,q)}$ or $\epsilon^{(p,q)}$, because by the E^{∞}-Distribution Theorem we can see easily that

- if $E^{\infty}_{(p,q)} \cong \epsilon_{(p,q)}$ then $\epsilon^{(p,q-1)} = 0$, and
- if $E^{\infty}_{(p,q)} \cong \epsilon^{(p,q-1)}$ then $\epsilon_{(p,q)} = 0$.

So E^{∞} is *uniquely* distributed over either $\epsilon_{(p,q)}$ or $\epsilon^{(p,q-1)}$.

Then to provide the answer to the third question we will see what happens if $\epsilon_{(p,q)}$ or $\epsilon^{(p,q-1)}$ vanishes and, in particular, the fact that vanishing of one of these quotients is the motivation behind the definition of *conditional convergence* in [3].

We will also show that this result implies all known convergence results. For example, in Boardman's terminology:

1. Vanishing of $\dfrac{Z^{\infty}_{(p,q)}}{\mathrm{im}(j_{(p,q)})}$ means weak convergence to L_*.
2. Vanishing of f and $\epsilon_{(p,q-1)}$ means weak (and strong) convergence to L^*.

We will also extend this theorem to the case that an exact couple has an arbitrary augmentation and coaugmentation and explain the relationships between these augmentation and coaugmentation and the E^{∞}-terms of the induced spectral sequence.

We will finally compare our method with Boardman's and state the limitations which result from the absence of the E^{∞}-Distribution Theorem and the gains which result from its availability.

All results and proofs in this chapter are part of what Peschke [26] has developed in [26].

2.2 Spectral Sequences

2.2.1 Definition of a Spectral Sequence

Definition 2.2.1. *A* $(\mathbb{Z} \times \mathbb{Z})-$bigraded *$R$-module is given by a family of R-modules* $(A_{(p,q)}|(p,q) \in \mathbb{Z} \times \mathbb{Z})$.

Definition 2.2.2. *A* morphism $f : A \to B$ *of* $(\mathbb{Z}\times\mathbb{Z})$*-graded R-modules of bidegree* $(\mu, \nu) = \mathbf{u}$ *is given by a family of R-module maps* $(f_{(p,q)} : A_{(p,q)} \to B_{(p+\mu,q+\nu)} \mid (p,q) \in \mathbb{Z} \times \mathbb{Z}) = (f_{(p,q)} : A_{(p,q)} \to B_{(p,q)+\mathbf{u}} \mid (p,q) \in \mathbb{Z} \times \mathbb{Z})$.

Definition 2.2.3. *A* spectral sequence *is given by*

1. *a family* $\{E^r_{(p,q)} \mid 0 \leq r_0 \leq r, (p,q) \in \mathbb{Z} \times \mathbb{Z}\}$ *of bigraded R-modules, where r_0 is a fixed integer, and*
2. *a sequence of endomorphisms* $d^r : E^r_{(p,q)} \to E^r_{(p,q)+\mathbf{u}_r}$ *of bidegree* $\mathbf{u}_r = (\mu_r, \nu_r) \in \mathbb{Z}\times\mathbb{Z}$*; i.e.,* $d^r_{(p,q)} : E^r_{(p,q)} \to E^r_{(p,q)+\mathbf{u}_r}$*, such that the following hold*
   (a) $d^r_{(p,q)} \circ d^r_{(p,q)-\mathbf{u}_r} = 0,$ *and*
   (b) $E^{r+1}_{(p,q)} = \dfrac{\ker(d^r_{(p,q)})}{\mathrm{im}(d^r_{(p,q)-\mathbf{u}_r})}.$

We show each $E^r_{(p,q)}$ by a point with coordinates (p,q) in the plane.

$$
\begin{tikzcd}
E^r_{(p,q)+\mathbf{u}_r} & & & & & \\
\vdots & & \vdots & \vdots & \vdots & \\
& E^r_{(p-1,q)} & E^r_{(p,q)} \arrow[uull, "d^r_{(p,q)}"'] & E^r_{(p+1,q)} & \vdots & \vdots \\
& \vdots & E^r_{(p,q-1)} & & \vdots & \vdots \\
& \vdots & \vdots & \vdots & E^r_{(p,q)-\mathbf{u}_r} \arrow[uull, "d^r_{(p,q)-\mathbf{u}_r}"'] &
\end{tikzcd}
$$

Terminology 2.2.4. For a fixed r, the bigraded objects $E^r_{(p,q)}$ form the *r-th page* of the spectral sequence and the bigraded morphism d^r is called the *differential* of the r-th page of the spectral sequence.

Definition 2.2.5. *A spectral sequence is called*

1. first quadrant *if $E^1_{(p,q)} = 0$ for $p < 0$ or $q < 0$,*
2. second quadrant *if $E^1_{(p,q)} = 0$ for $p > 0$ or $q < 0$,*
3. third quadrant *if $E^1_{(p,q)} = 0$ for $p > 0$ or $q > 0$,*
4. fourth quadrant *if $E^1_{(p,q)} = 0$ for $p < 0$ or $q > 0$.*

2.2.2 Limit Page of a Spectral Sequence

We define a sequence of modules in each position (p, q) of the form

$$0 \subseteq B^{r_0-1}_{(p,q)} \subseteq \cdots \subseteq B^r_{(p,q)} \subseteq \cdots \subseteq \bigcup_r B^r_{(p,q)} \subseteq \bigcap_r Z^r_{(p,q)} \subseteq \cdots \subseteq Z^r_{(p,q)} \subseteq \cdots \subseteq Z^{r_0-1}_{(p,q)} \subseteq E^{r_0}_{(p,q)}$$

and then define $E^\infty_{(p,q)} = \dfrac{Z^\infty_{(p,q)}}{B^\infty_{(p,q)}}$, where $Z^\infty_{(p,q)} = \bigcap_r Z^r_{(p,q)}$ and $B^\infty_{(p,q)} = \bigcup_r B^r_{(p,q)}$. We proceed by induction on r:

- Define $Z^{r_0-1}_{(p,q)} = E^{r_0}_{(p,q)}$ and $B^{r_0-1}_{(p,q)} = 0$.

- Suppose inductively for $r_0 \leq r$ we have defined
$$0 = B^{r_0-1}_{(p,q)} \subseteq \cdots \subseteq B^r_{(p,q)} \subseteq Z^r_{(p,q)} \subseteq \cdots \subseteq Z^{r_0-1}_{(p,q)} = E^{r_0}_{(p,q)}$$
satisfying $E^{k+1}_{(p,q)} = \dfrac{Z^k_{(p,q)}}{B^k_{(p,q)}}$, for $k \leq r$.

- Since $E^{r+1}_{(p,q)}$ is a quotient of $Z^r_{(p,q)}$ there is an epimorphism $\tau^r_{(p,q)} : Z^r_{(p,q)} \twoheadrightarrow E^{r+1}_{(p,q)}$

$$\begin{tikzcd}
 & Z^r_{(p,q)} \arrow[d, "\tau^r_{(p,q)}"] & \\
E^{r+1}_{(p,q)-\mathbf{u}_{r+1}} \arrow[r, "d^{r+1}_{(p,q)-\mathbf{u}_{r+1}}"] & E^{r+1}_{(p,q)} \arrow[r, "d^{r+1}_{(p,q)}"] & E^{r+1}_{(p,q)+\mathbf{u}_{r+1}}.
\end{tikzcd}$$

Now define
$$B^{r+1}_{(p,q)} := (\tau^r_{(p,q)})^{-1}(\mathrm{im}(d^{r+1}_{(p,q)-\mathbf{u}_{r+1}}))$$
and
$$Z^{r+1}_{(p,q)} := (\tau^r_{(p,q)})^{-1}(\ker(d^{r+1}_{(p,q)})).$$
By passing from r-th page to the $(r+1)$-th page we collapse $B^r_{(p,q)}$ to zero and hence
$$B^r_{(p,q)} = (\tau^r_{(p,q)})^{-1}(0) \subseteq B^{r+1}_{(p,q)} \subseteq Z^{r+1}_{(p,q)} \subseteq Z^r_{(p,q)}.$$
Therefore,
$$\frac{Z^{r+1}_{(p,q)}}{B^{r+1}_{(p,q)}} = \frac{(\tau^r_{(p,q)})^{-1}(\ker(d^{r+1}_{(p,q)}))}{(\tau^r_{(p,q)})^{-1}(\mathrm{im}(d^{r+1}_{(p,q)-\mathbf{u}_{r+1}}))}.$$
Now, consider $\tau^r_{(p,q)}| : Z^{r+1}_{(p,q)} \twoheadrightarrow \dfrac{Z^{r+1}_{(p,q)}}{B^r_{(p,q)}}$. The homomorphism
$$\overline{\tau^r_{(p,q)}|} : \frac{Z^{r+1}_{(p,q)}}{B^{r+1}_{(p,q)}} \longrightarrow \frac{\tau^r_{(p,q)}|((\tau^r_{(p,q)})^{-1}(\ker(d^{r+1}_{(p,q)})))}{\tau^r_{(p,q)}|((\tau^r_{(p,q)})^{-1}(\mathrm{im}(d^{r+1}_{(p,q)-\mathbf{u}_{r+1}})))} = \frac{\ker(d^{r+1}_{(p,q)})}{\mathrm{im}(d^{r+1}_{(p,q)-\mathbf{u}_{r+1}})}$$
is a surjection. If $\overline{\tau^r_{(p,q)}|}(z + B^{r+1}_{(p,q)}) = 0$, then $\tau^r_{(p,q)}(z) \in \mathrm{im}(d^{r+1}_{(p,q)-\mathbf{u}_{r+1}})$ and hence $z \in (\tau^r_{(p,q)})^{-1}(\mathrm{im}(d^{r+1}_{(p,q)-\mathbf{u}_{r+1}})) = B^{r+1}_{(p,q)}$; i.e., $\overline{\tau^r_{(p,q)}|}$ is an

injection and hence

$$\frac{(\tau^r_{(p,q)})^{-1}(\ker(d^{r+1}_{(p,q)}))}{(\tau^r_{(p,q)})^{-1}(\mathrm{im}(d^{r+1}_{(p,q)-\mathbf{u}_{r+1}}))} \cong \frac{\ker(d^{r+1}_{(p,q)})}{\mathrm{im}(d^{r+1}_{(p,q)-\mathbf{u}_{r+1}})}.$$

Therefore,

$$E^{r+2}_{(p,q)} = \frac{Z^{r+1}_{(p,q)}}{B^{r+1}_{(p,q)}}.$$

Definition 2.2.6. *A spectral sequence* collapses *on page r if $d^{r+k} = 0$ for all $k \geq 0$.*

Proposition 2.2.7. *If (E^r, d^r) collapses on page r, then*

$$B^r = B^{r+1} = \cdots = B^{r+k} = B^\infty \quad \text{and} \quad Z^r = Z^{r+1} = \cdots = Z^{r+k} = Z^\infty$$

and hence

$$E^r = E^{r+1} = E^{r+2} = \cdots = E^{r+k} = E^\infty \qquad \forall k \geq 0.$$

Lemma 2.2.8. *The E^∞-term of a spectral sequence can be represented as*

1. *a colimit via*

$$E^\infty_{(p,q)} \cong colim \left\{ \frac{Z^\infty_{(p,q)}}{B^{r_0}_{(p,q)}} \twoheadrightarrow \frac{Z^\infty_{(p,q)}}{B^{r_0+1}_{(p,q)}} \twoheadrightarrow \cdots \twoheadrightarrow \frac{Z^\infty_{(p,q)}}{B^r_{(p,q)}} \twoheadrightarrow \cdots \right\}$$

2. *a limit via*

$$E^\infty_{(p,q)} \cong lim \left\{ \frac{Z^{r_0}_{(p,q)}}{B^\infty_{(p,q)}} \hookleftarrow \frac{Z^{r_0+1}_{(p,q)}}{B^\infty_{(p,q)}} \hookleftarrow \cdots \hookleftarrow \frac{Z^r_{(p,q)}}{B^\infty_{(p,q)}} \hookleftarrow \cdots \right\}.$$

Proof. 1. We have the following diagram of short exact sequences

$$\begin{tikzcd}
B^{r_0-1}_{(p,q)} \arrow[r, tail] \arrow[d, tail] & Z^{\infty}_{(p,q)} \arrow[r, two heads] \arrow[d, equal] & \dfrac{Z^{\infty}_{(p,q)}}{B^{r_0-1}_{(p,q)}} \arrow[d, two heads] \\
\vdots \arrow[d, tail] & \vdots \arrow[d, equal] & \vdots \arrow[d, two heads] \\
B^{r-1}_{(p,q)} \arrow[r, tail] \arrow[d, tail] & Z^{\infty}_{(p,q)} \arrow[r, two heads] \arrow[d, equal] & \dfrac{Z^{\infty}_{(p,q)}}{B^{r-1}_{(p,q)}} \arrow[d, two heads] \\
B^{r}_{(p,q)} \arrow[r, tail] \arrow[d, tail] & Z^{\infty}_{(p,q)} \arrow[r, two heads] \arrow[d, equal] & \dfrac{Z^{\infty}_{(p,q)}}{B^{r}_{(p,q)}} \arrow[d, two heads] \\
\vdots & \vdots & \vdots .
\end{tikzcd}$$

It is a well-known fact that the colimit functor on morphisms of diagrams over directed towers is exact; see Appendix A. Therefore, we obtain the following short exact sequence

$$B^{\infty}_{(p,q)} \rightarrowtail Z^{\infty}_{(p,q)} \twoheadrightarrow \operatorname{colim}_r \left\{ \frac{Z^{\infty}_{(p,q)}}{B^{r}_{(p,q)}} \right\}$$

and hence $E^{\infty}_{(p,q)} \cong \operatorname{colim}_r \left\{ \dfrac{Z^{\infty}_{(p,q)}}{B^{r}_{(p,q)}} \right\}$.

2. We have the following diagram of short exact sequences

$$\begin{tikzcd}
\vdots \arrow[d, equal] & \vdots \arrow[d, tail] & \vdots \arrow[d, tail] \\
B^{\infty}_{(p,q)} \arrow[r, tail] \arrow[d, equal] & Z^{r}_{(p,q)} \arrow[r, two heads] \arrow[d, tail] & \dfrac{Z^{r}_{(p,q)}}{B^{\infty}_{(p,q)}} \arrow[d, tail] \\
B^{\infty}_{(p,q)} \arrow[r, tail] \arrow[d, equal] & Z^{r-1}_{(p,q)} \arrow[r, two heads] \arrow[d, tail] & \dfrac{Z^{r-1}_{(p,q)}}{B^{\infty}_{(p,q)}} \arrow[d, tail] \\
\vdots \arrow[d, equal] & \vdots \arrow[d, tail] & \vdots \arrow[d, tail] \\
B^{\infty}_{(p,q)} \arrow[r, tail] & Z^{r_0-1}_{(p,q)} \arrow[r, two heads] & \dfrac{Z^{r_0-1}_{(p,q)}}{B^{\infty}_{(p,q)}} .
\end{tikzcd}$$

It is a well-known fact that the limit functor on morphisms of diagrams over directed towers is left-exact; see Appendix A. Therefore, we obtain the following exact sequence

$$B^{\infty}_{(p,q)} \rightarrowtail Z^{\infty}_{(p,q)} \longrightarrow \lim_r \left\{ \frac{Z^r_{(p,q)}}{B^{\infty}_{(p,q)}} \right\} \longrightarrow \lim^1_r B^{\infty}_{(p,q)} \longrightarrow \lim^1_r Z^{\infty}_{(p,q)} \twoheadrightarrow \lim^1_r \left\{ \frac{Z^r_{(p,q)}}{B^{\infty}_{(p,q)}} \right\},$$

where $\lim^1$ is the first derived functor of the limit functor; see Appendix A. Since $\lim^1$ of a constant diagram is zero we have $\lim^1_r B^{\infty}_{(p,q)} = 0$. Therefore, we obtain the following short exact sequence

$$B^{\infty}_{(p,q)} \rightarrowtail Z^{\infty}_{(p,q)} \twoheadrightarrow \lim_r \left\{ \frac{Z^r_{(p,q)}}{B^{\infty}_{(p,q)}} \right\}$$

and hence

$$E^{\infty}_{(p,q)} \cong \lim_r \left\{ \frac{Z^r_{(p,q)}}{B^{\infty}_{(p,q)}} \right\}.$$

□

2.2.3 Morphisms of Spectral Sequences

Definition 2.2.9. *A morphism of bidegree* $\mathbf{m} \in \mathbb{Z} \oplus \mathbb{Z}$ *of one spectral sequence* $(E^r(1), d^r(1))$ *to another one* $(E^r(2), d^r(2))$ *is a family of morphisms* $(f^r : E^r(1) \to E^r(2) \mid r \geq r_0)$ *of bidegree* $\mathbf{m}$ *of bigraded* R*-modules satisfying:*

1. *each* f^r *commutes with the differentials in the* r*-th page; i.e.,* $f^r \circ d^r = d^r \circ f^r$, *and*
2. f^r *induces* f^{r+1} *in homology; i.e.,* $f^{r+1} = H(f^r)$.

Lemma 2.2.10. *A morphism of spectral sequences* $(f^r : E^r(1) \to E^r(2) \mid r \geq r_0)$ *of bidegree* $\mathbf{m}$ *induces a morphism*

$$f^{\infty} : E^{\infty}(1) \to E^{\infty}(2)$$

which is also of bidegree $\mathbf{m}$.

Proof. For each (p, q) we build inductively the commutative diagram of solid arrows

$$
\begin{tikzcd}
B^r_{(p,q)}(1) \arrow[r, tail] \arrow[d] & B^\infty_{(p,q)}(1) \arrow[r, tail] \arrow[d, dotted] & Z^\infty_{(p,q)}(1) \arrow[r, tail] \arrow[d, dotted] & Z^r_{(p,q)}(1) \arrow[d] \\
B^r_{(p,q)}(2) \arrow[r, tail] & B^\infty_{(p,q)}(2) \arrow[r, tail] & Z^\infty_{(p,q)}(2) \arrow[r, tail] & Z^r_{(p,q)}(2)
\end{tikzcd}
\tag{2.3}
$$

where all vertical arrows are restrictions of $f^{r_0}_{(p,q)}$.

- For $r = r_0 - 1$ the vertical map on the left is zero and the one on the right is f_{r_0}.
- Suppose the diagram is established for $r - 1 \geq r_0 - 1$. Look at the diagram below

$$
\begin{tikzcd}[column sep=small]
 & & & & & & \mathrm{im}(d^r_{p-\mu_r,q-\nu_r}(1)) \arrow[dd, "f^r|"] \\
 & B^{r-1}_{(p,q)}(1) \arrow[rr, tail] \arrow[dd] & & Z^{r-1}_{(p,q)}(1) \arrow[rr, two heads] \arrow[dd] & & E^r_{(p,q)}(1) \arrow[ur, two heads] \arrow[dd, "f^r"] & \\
B^{r-1}_{(p,q)}(1) \arrow[ur, equals] \arrow[rr, tail] \arrow[dd] & & Z^r_{(p,q)}(1) \arrow[ur] \arrow[rr, two heads] \arrow[dd, dotted, "f^{r_0}|"] & & \ker(d^r_{(p,q)}(1)) \arrow[ur] \arrow[dd, "f^r|"] & & \mathrm{im}(d^r_{p-\mu_r,q-\nu_r}(2)) \\
 & B^{r-1}_{(p,q)}(2) \arrow[rr, tail] & & Z^{r-1}_{(p,q)}(2) \arrow[rr, two heads] & & E^r_{(p,q)}(2) \arrow[ur, two heads] & \\
B^{r-1}_{(p,q)}(2) \arrow[ur, equals] \arrow[rr, tail] & & Z^r_{(p,q)}(2) \arrow[ur] \arrow[rr, two heads] & & \ker(d^r_{(p,q)}(2)) \arrow[ur] & &
\end{tikzcd}
$$

We show that $f^{r_0}_{(p,q)}(Z^r_{(p,q)}(1)) \subseteq Z^r_{(p,q)}(2)$: Take

$$f^{r_0}_{(p,q)}(x) \in f^{r_0}_{(p,q)}(Z^r_{(p,q)}(1)).$$

Then

$$x \in Z^r_{(p,q)}(1) = (\tau^r_{(p,q)}(1))^{-1}(\ker(d^{r+1}_{(p,q)}(1)));$$

i.e., $d^{r+1}_{(p,q)}(1)(\tau^r_{(p,q)}(1)(x)) = 0$. Now

$$
\begin{aligned}
d^{r+1}_{(p,q)}(2)(\tau^r_{(p,q)}(2)(f^{r_0}_{(p,q)}(x))) &= d^r_{(p,q)}(2)(f^{r_0}_{(p,q)}(\tau^r_{(p,q)}(1)(x))) \\
&= f^{r_0}_{(p,q)}(d^{r+1}_{(p,q)}(1)(\tau^r_{(p,q)}(1)(x))) \\
&= 0.
\end{aligned}
$$

Therefore, $f^{r_0}_{(p,q)}(x) \in (\tau^r_{(p,q)}(2))^{-1}(\ker(d^{r+1}_{(p,q)}(2))) = Z^r_{(p,q)}(2)$.

To see that $f^{r_0}_{(p,q)}(B^r_{(p,q)}(1)) \subseteq B^r_{(p,q)}(2)$ consider the diagram below

$$\begin{tikzcd}
& Z^{r-1}_{(p,q)}(1) \arrow[rr, two heads] \arrow[dd, "f^{r_0}|"] & & E^r_{(p,q)}(1) \arrow[dd, "f^r"] \\
B^r_{(p,q)}(1) \arrow[ur, tail] \arrow[rr, two heads, crossing over] \arrow[dd, dotted, "f^{r_0}|"] & & \operatorname{im}(d^r_{p-\mu_r,q-\nu_r}(1)) \arrow[ur, tail] & \\
& Z^{r-1}_{(p,q)}(2) \arrow[rr, two heads] & & E^r_{(p,q)}(2) \\
B^r_{(p,q)}(2) \arrow[ur, tail] \arrow[rr, two heads] & & \operatorname{im}(d^r_{p-\mu_r,q-\nu_r}(2)) \arrow[ur, tail] \arrow[from=uu, crossing over, "f^r|"] &
\end{tikzcd}$$

We know that $f^r| : \operatorname{im}(d^r_{p-\mu_r,q-\nu_r}(1)) \to \operatorname{im}(d^r_{p-\mu_r,q-\nu_r}(2))$. Thus $f^{r_0}(B^r(1)) \subseteq B^r_{(p,q)}(2)$. The dotted arrows $f^r| : B^\infty_{(p,q)}(1) \to B^\infty_{(p,q)}(2)$ and $f^r| : Z^\infty_{(p,q)} \to Z^\infty_{(p,q)}(2)$ in diagram (2.3) follow from purely set theoretic considerations. Thus f^∞ is given by the universal property of cokernel:

$$\begin{tikzcd}
B^\infty_{(p,q)}(1) \arrow[r, tail] \arrow[d, "f^{r_0}|"'] & Z^\infty_{(p,q)}(1) \arrow[r, two heads] \arrow[d, "f^{r_0}|"'] & E^\infty_{(p,q)}(1) \arrow[d, dotted, "f^\infty"] \\
B^\infty_{(p,q)}(2) \arrow[r, tail] & Z^\infty_{(p,q)}(2) \arrow[r, two heads] & E^\infty_{(p,q)}(2).
\end{tikzcd}$$

□

Corollary 2.2.11. *Given a morphism* $(f^r : E^r(1) \to E^r(2) \mid r \geq r_0)$ *of spectral sequences, suppose that* f^r *is an isomorphism of bigraded* R*-modules for some* $r \geq 1$*. Then*

$$f^\infty : E^\infty(1) \to E^\infty(2)$$

is also an isomorphism.

Proof. Using the diagram below we show inductively that

- f^{r+k+1} is an isomorphism of bigraded objects, for each $k \geq 0$,
- f induces isomorphisms.

$$
\begin{tikzcd}[column sep=small]
 & & & & & & \mathrm{im}(d^{r+k}_{p-\mu_r,q-\nu_r}(1)) \arrow[dd, "f^{r+k}|"] \\
 & \frac{B^{r+k-1}_{(p,q)}(1)}{B^{r-1}_{(p,q)}(1)} \arrow[rr, tail] \arrow[dd, "\cong"'] & & \frac{Z^{r+k-1}_{(p,q)}(1)}{B^{r-1}_{(p,q)}(1)} \arrow[rr, two heads] \arrow[dd, "\cong"'] & & E^{r+k}_{(p,q)}(1) \arrow[ur] \arrow[dd, "f^{r+k}", "\cong"'] & \\
\frac{B^{r+k-1}_{(p,q)}(1)}{B^{r-1}_{(p,q)}(1)} \arrow[ur, equal] \arrow[rr, tail] \arrow[dd] & & \frac{Z^{r+k}_{(p,q)}(1)}{B^{r-1}_{(p,q)}(1)} \arrow[ur, tail] \arrow[rr, two heads] \arrow[dd, dotted, "f^{r_0}|", "\overset{3}{\cong}"'] & & \ker(d^{r+k}_{(p,q)}(1)) \arrow[ur, tail] \arrow[dd, "f^r|", "\overset{2}{\cong}"'] & & \mathrm{im}(d^{r+k}_{p-\mu_r,q-\nu_r}(2)) \\
 & \frac{B^{r+k-1}_{(p,q)}(2)}{B^{r-1}_{(p,q)}(2)} \arrow[rr, tail] & & \frac{Z^{r-1}_{(p,q)}(2)}{B^{r-1}_{(p,q)}(2)} \arrow[rr, two heads] & & E^{r+k}_{(p,q)}(2) \arrow[ur] & \\
\frac{B^{r+k-1}_{(p,q)}(2)}{B^{r-1}_{(p,q)}(2)} \arrow[ur, equal] \arrow[rr, tail] & & \frac{Z^{r+k}_{(p,q)}(2)}{B^{r-1}_{(p,q)}(2)} \arrow[ur, tail] \arrow[rr, two heads] & & \ker(d^{r+k}_{(p,q)}(2)) \arrow[ur, tail] & &
\end{tikzcd}
$$

Indeed,

$$f^{r+k}| : \mathrm{im}(d^{r+k}_{p-\mu_r,q-\nu_r}(1)) \to \mathrm{im}(d^{r+k}_{p-\mu_r,q-\nu_r}(2))$$

is one-to-one, being the restriction of an isomorphism. It is onto because the diagram commutes. The isomorphisms $\overset{2}{\cong}$ and, subsequently $\overset{3}{\cong}$, come from the 5-lemma. In particular, f^{r+k+1} is an isomorphism. It remains to verify the isomorphism

$$\frac{B^{r+k}_{(p,q)}(1)}{B^{r-1}_{(p,q)}(1)} \to \frac{B^{r+k}_{(p,q)}(2)}{B^{r-1}_{(p,q)}(2)}.$$

We have the commutative diagram

$$
\begin{tikzcd}
\frac{B^{r+k-1}_{(p,q)}(1)}{B^{r-1}_{(p,q)}(1)} \arrow[r, tail] \arrow[d] & \frac{B^{r+k}_{(p,q)}(1)}{B^{r-1}_{(p,q)}(1)} \arrow[r, two heads] \arrow[d] & \mathrm{im}(d^{r+k}_{p-\mu_r,q-\nu_r}(1)) \arrow[d, "\cong"] \\
\frac{B^{r+k-1}_{(p,q)}(2)}{B^{r-1}_{(p,q)}(2)} \arrow[r, tail] & \frac{B^{r+k}_{(p,q)}(2)}{B^{r-1}_{(p,q)}(2)} \arrow[r, two heads] & \mathrm{im}(d^{r+k}_{p-\mu_r,q-\nu_r}(2)).
\end{tikzcd}
$$

The arrow on the right is an isomorphism as we have just seen. The arrow on the left is an isomorphism by induction hypothesis. The arrow in the middle is an isomorphism by the 5-lemma. Therefore, f induces isomorphisms

$$\frac{Z^{r+k}_{(p,q)}(1)}{B^{r-1}_{(p,q)}(1)} \xrightarrow{\cong} \frac{Z^{r+k}_{(p,q)}(2)}{B^{r-1}_{(p,q)}(2)}$$

and

$$\frac{B^{r+k}_{(p,q)}(1)}{B^{r-1}_{(p,q)}(1)} \xrightarrow{\cong} \frac{B^{r+k}_{(p,q)}(2)}{B^{r-1}_{(p,q)}(2)}.$$

This yields isomorphisms

$$\begin{aligned}\frac{Z^{r+k}_{(p,q)}(1)}{B^{\infty}_{(p,q)}(1)} \cong \frac{Z^{r+k}_{(p,q)}(1)/B^{r-1}_{(p,q)}(1)}{B^{\infty}_{(p,q)}(1)/B^{r-1}_{(p,q)}(1)} &\xrightarrow{\cong} \frac{Z^{r+k}_{(p,q)}(2)/B^{r}_{(p,q)}(2)}{B^{\infty}_{(p,q)}(2)/B^{r}_{(p,q)}(2)} \\ &\cong \frac{Z^{r+k}_{(p,q)}(2)/B^{r-1}_{(p,q)}(2)}{B^{\infty}_{(p,q)}(2)/B^{r-1}_{(p,q)}(2)} \cong \frac{Z^{r+k}_{(p,q)}(2)}{B^{\infty}_{(p,q)}(2)}.\end{aligned}$$

□

Lemma 2.2.12. *Suppose a morphism $f : E(1) \to E(2)$ of spectral sequences induces the diagram below*

$$\begin{tikzcd} E^r_{(p,q)-\mathbf{b}_r}(1) \arrow[r, "d^r_{(p,q)-\mathbf{u}_r}(1)"] \arrow[d, "f^r_{(p,q)-\mathbf{u}_r}"'] & E^r_{(p,q)}(1) \arrow[r, "d^r_{(p,q)}(1)"] \arrow[d, "f^r_{(p,q)}"', "\cong"] & E^r_{(p,q)+\mathbf{u}_r}(1) \arrow[d, "f^r_{(p,q)+\mathbf{u}_r}"] \\ E^r_{(p,q)-\mathbf{u}_r}(2) \arrow[r, "d^r_{(p,q)-\mathbf{u}_r}(2)"'] & E^r_{(p,q)}(2) \arrow[r, "d^r_{(p,q)}(2)"'] & E^r_{(p,q)+\mathbf{u}_r}(2). \end{tikzcd}$$

Then $f^{r+1}_{(p,q)} : E^{r+1}_{(p,q)}(1) \to E^{r+1}_{(p,q)}(2)$ is an isomorphism.

Proof. We have the two commutative diagrams

$$\begin{tikzcd} \ker(d^r_{(p,q)-\mathbf{u}_r}(1)) \arrow[r, tail] \arrow[dd] & E^r_{(p,q)-\mathbf{u}_r}(1) \arrow[rr, "d^r_{(p,q)-\mathbf{u}_r}(1)"] \arrow[dd, "f^r_{(p,q)-\mathbf{u}_r}"'] \arrow[dr] & & E^r_{(p,q)}(1) \arrow[dd, "f^r_{(p,q)}"', "\cong"] \\ & & \operatorname{im}(d^r_{(p,q)-\mathbf{u}_r}(1)) \arrow[ur, hook] \arrow[dd, dashed, "f^r_{(p,q)}|"'] & \\ \ker(d^r_{(p,q)-\mathbf{u}_r}(1)) \arrow[r, tail] & E^r_{(p,q)-\mathbf{u}_r}(2) \arrow[rr, "d^r_{(p,q)-\mathbf{u}_r}(2)"'] \arrow[dr] & & E^r_{(p,q)}(2) \\ & & \operatorname{im}(d^r_{(p,q)-\mathbf{u}_r}(2)) \arrow[ur, hook] & \end{tikzcd} \tag{2.4}$$

and

$$\begin{tikzcd}
\ker(d^r_{(p,q)}(1)) \arrow[r, tail] \arrow[d, dashed, "f^r_{(p,q)}|"'] & E^r_{(p,q)}(1) \arrow[r, two heads, "d^r_{(p,q)}(1)"] \arrow[d, "f^r_{(p,q)}"', "\cong"] & \operatorname{im}(d^r_{(p,q)}(1)) \arrow[d] \\
\ker(d^r_{(p,q)}(2)) \arrow[r, tail] & E^r_{(p,q)}(2) \arrow[r, two heads, "d^r_{(p,q)}(2)"'] & \operatorname{im}(d^r_{(p,q)}(2)).
\end{tikzcd} \tag{2.5}$$

In (2.4), the vertical dashed arrow is a monomorphism because $f^r_{(p,q)}$ is an isomorphism and an epimorphism because $f^r_{(p,q)-\mathbf{b}_r}$ is an epimorphism. Therefore,

$$f^r_{(p,q)}| : \operatorname{im}(d^r_{(p,q)-\mathbf{u}_r}(1)) \to \operatorname{im}(d^r_{(p,q)-\mathbf{u}_r}(2))$$

is an isomorphism. In (2.5), the vertical dashed arrow is a monomorphism by isomorphicity of $f^r_{(p,q)}$ and an epimorphism by five-lemma.Therefore,

$$f^r_{(p,q)}| : \ker(d^r_{(p,q)}(1)) \to \ker(d^r_{(p,q)}(2))$$

is an isomorphism. Thus we obtain the commutative diagram below

$$\begin{tikzcd}
\operatorname{im}(d^r_{(p,q)-\mathbf{u}_r}(1)) \arrow[r, tail] \arrow[d, "f^r_{(p,q)}|"', "\cong"] & \ker(d^r_{(p,q)}(1)) \arrow[r, two heads] \arrow[d, "f^r_{(p,q)}|"', "\cong"] & E^{r+1}_{(p,q)}(1) \arrow[d, "f^{r+1}_{(p,q)}"] \\
\operatorname{im}(d^r_{(p,q)-\mathbf{u}_r}(2)) \arrow[r, tail] & \ker(d^r_{(p,q)}(2)) \arrow[r, two heads] & E^{r+1}_{(p,q)}(2).
\end{tikzcd}$$

By five-lemma, we see that $f^{r+1}_{(p,q)}$ is an isomorphism as well.

□

2.3 Exact Couples: Source of Spectral Sequences

Many times a spectral sequence arises from a family of long exact sequences of R-modules which are intertwined in a particular way. The sequences together with their intertwinement form a new type of structured object, called an *exact couple*, which was introduced by Massey in [22] and [23]. However, in the previous section, spectral sequences were considered in their standalone nature. From now on, we will specialize to spectral sequences that arise from an exact couple. We will see that from an exact couples we obtain the following:

1. *Two $\mathbb{Z}$-graded R-modules L_* and L^*:*
   (a) L_{p+q} is filtered by
   $$\cdots \subseteq F_{(p,q)-\mathbf{a}} \subseteq F_{(p,q)} \subseteq F_{(p,q)+\mathbf{a}} \subseteq \cdots \subset L_{p+q}.$$
   (b) L^{p+q} is filtered by
   $$\cdots \subseteq F^{(p,q)-\mathbf{a}} \subseteq F^{(p,q)} \subseteq F^{(p,q)+\mathbf{a}} \subseteq \cdots \subseteq L^{p+q}.$$
   Here $\mathbf{a}$ is a bidegree which is determined by the exact couple.
2. *A spectral sequence* (E^r, d^r): The E^∞-terms of this induced spectral sequence are related to the quotients of adjacent stages of the filtrations L_* and L^* as is shown in the following diagram:
   $$\frac{F_{(p,q)}}{F_{(p,q)-\mathbf{a}}} \rightarrowtail E^\infty_{(p,q)+\mathbf{b}} \twoheadrightarrow A_{(p,q)+\mathbf{b}} \leftarrowtail \frac{F^{(p,q)+\mathbf{a}+\mathbf{b}+\mathbf{c}}}{F^{(p,q)+\mathbf{b}+\mathbf{c}}}$$
   where $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are certain bidegrees that are determined by the spectral sequence and $A_{(p,q)+\mathbf{b}}$ is an R-module obtained from the exact couple.

In the E^∞-Distribution Theorem 2.3.13 on page 35 we will see how these information combine and produce a diagram that looks like a bird; the left and right wings carry the filtration stages of L_* and L^*, respectively, and the body shows how the quotients of adjacent filtration stages of L_* and L^* are related to the E^∞-terms of the corresponding spectral sequence. Later in this chapter, we will compare this theorem with what Boardman has offered in [3].

2.3.1 Definition of an Exact Couple

Definition 2.3.1. *An* exact couple *consists of bigraded R-modules D^1 and E^1 with the following R-modules homomorphisms*

$$i^1_{(p,q)} : D^1_{(p,q)} \to D^1_{(p,q)+\mathbf{a}}, \quad \textit{of bidegree } \mathbf{a} \in \mathbb{Z} \oplus \mathbb{Z},$$
$$j^1_{(p,q)} : D^1_{(p,q)} \to E^1_{(p,q)+\mathbf{b}}, \quad \textit{of bidegree } \mathbf{b} \in \mathbb{Z} \oplus \mathbb{Z},$$
$$k^1_{(p,q)} : E^1_{(p,q)} \to D^1_{(p,q)+\mathbf{c}}, \quad \textit{of bidegree } \mathbf{c} \in \mathbb{Z} \oplus \mathbb{Z},$$

such that $j^1 \circ i^1, k^1 \circ j^1$ and $i^1 \circ k^1$ are zero homomorphisms. We require $(\mathbf{b}+\mathbf{c}) \cdot \hat{\mathbf{a}} \neq 0$.

It turns out that in most examples $(\mathbf{b}+\mathbf{c})\cdot\hat{\mathbf{a}} = \pm 1$. An exact couple may be displayed as in the following diagram: here the sequence of blue colored terms and arrows is long exact

$$
\begin{tikzcd}[column sep=small]
 & \vdots \arrow[d] & & \vdots \arrow[d] & \\
E^1_{(p,q)-(r-1)\mathbf{a}-\mathbf{c}} \arrow[r] & D^1_{(p,q)-(r-1)\mathbf{a}} \arrow[r] \arrow[d] & E^1_{(p,q)-(r-1)\mathbf{a}+\mathbf{b}} \arrow[r] & D^1_{(p,q)-(r-1)\mathbf{a}+\mathbf{b}+\mathbf{c}} \arrow[r] \arrow[d] & E^1_{(p,q)-(r-1)\mathbf{a}+2\mathbf{b}+\mathbf{c}} \\
 & \vdots \arrow[d] & & \vdots \arrow[d] & \\
E^1_{(p,q)-\mathbf{a}-\mathbf{c}} \arrow[r] & D^1_{(p,q)-\mathbf{a}} \arrow[r] \arrow[d, "i^1_{(p,q)-\mathbf{a}}"'] & E^1_{(p,q)-\mathbf{a}+\mathbf{b}} \arrow[r] & D^1_{(p,q)-\mathbf{a}+\mathbf{b}+\mathbf{c}} \arrow[r] \arrow[d] & E^1_{(p,q)-\mathbf{a}+2\mathbf{b}+\mathbf{c}} \\
E^1_{(p,q)-\mathbf{c}} \arrow[r] & D^1_{(p,q)} \arrow[r, "j^1_{(p,q)}"] \arrow[d] & E^1_{(p,q)+\mathbf{b}} \arrow[r, "k^1_{(p,q)+\mathbf{b}}"] & D^1_{(p,q)+\mathbf{b}+\mathbf{c}} \arrow[r] \arrow[d] & E^1_{(p,q)+2\mathbf{b}+\mathbf{c}} \\
E^1_{(p,q)+\mathbf{a}-\mathbf{c}} \arrow[r] & D^1_{(p,q)+\mathbf{a}} \arrow[r] \arrow[d] & E^1_{(p,q)+\mathbf{a}+\mathbf{b}} \arrow[r] & D^1_{(p,q)+\mathbf{a}+\mathbf{b}+\mathbf{c}} \arrow[r] \arrow[d] & E^1_{(p,q)+\mathbf{a}+2\mathbf{b}+\mathbf{c}} \\
 & \vdots \arrow[d] & & \vdots \arrow[d] & \\
E^1_{(p,q)+(r-1)\mathbf{a}-\mathbf{c}} \arrow[r] & D^1_{(p,q)+(r-1)\mathbf{a}} \arrow[r] \arrow[d] & E^1_{(p,q)+(r-1)\mathbf{a}+\mathbf{b}} \arrow[r] & D^1_{(p,q)+(r-1)\mathbf{a}+\mathbf{b}+\mathbf{c}} \arrow[r] \arrow[d] & E^1_{(p,q)+(r-1)\mathbf{a}+2\mathbf{b}+\mathbf{c}} \\
 & \vdots & & \vdots &
\end{tikzcd}
$$

2.3.2 Distilling a Spectral Sequence from an Exact Couple

The following theorem is proved by Massey in [23] for a spectral sequence of bidegrees $\mathbf{a} = (-1, 1)$, $\mathbf{b} = (0, 0)$ and $\mathbf{c} = (1, 0)$. Here we state and prove it for the general bidegrees. It shows the process of distilling a spectral sequence from an exact couple.

Theorem 2.3.2. *[26] Associated to an exact couple there is a spectral sequence with*

$$E^{r+1}_{(p,q)+\mathbf{b}} = \frac{(k^1_{(p,q)+\mathbf{b}})^{-1}\operatorname{im}(i^r_{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}})}{j^1_{(p,q)}(\ker(i^r_{(p,q)}))}.$$

If we show $(k^1_{(p,q)+\mathbf{b}})^{-1}\operatorname{im}(i^r_{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}})$ by $\bar{Z}^r_{(p,q)+\mathbf{b}}$, then the differential on the r-th page is determined by the commutative diagram

$$
\begin{tikzcd}[column sep=huge]
\bar{Z}^{r-1}_{(p,q)+\mathbf{b}} \arrow[rrr, "\Delta^r_{(p,q)+\mathbf{b}} := j^1_{(p,q)+\mathbf{b}+\mathbf{c}-(r-1)\mathbf{a}} \circ (i^{r-1}_{(p,q)+\mathbf{b}+\mathbf{c}})^{-1} \circ k^1_{(p,q)+\mathbf{b}}"] \arrow[d] & & & \bar{Z}^{r-1}_{(p,q)+2\mathbf{b}+\mathbf{c}-(r-1)\mathbf{a}} \arrow[d] \\
E^r_{(p,q)+\mathbf{b}} \arrow[rrr, "d^r_{(p,q)+\mathbf{b}}"'] & & & E^r_{(p,q)+2\mathbf{b}+\mathbf{c}-(r-1)\mathbf{a}}
\end{tikzcd}
$$

$$
\textit{where}\;\; i^r_{(p,q)} := (D^1_{(p,q)} \xrightarrow{i^1_{(p,q)}} D^1_{(p,q)+\mathbf{a}} \xrightarrow{i^1_{(p,q)+\mathbf{a}}} \cdots \xrightarrow{i^1_{(p,q)+(r-1)\mathbf{a}}} D^1_{(p,q)+r\mathbf{a}}) \,.
$$

Proof. Let us show $j^1_{(p,q)}(\ker(i^r_{(p,q)}))$ by $\bar{B}^r_{(p,q)+\mathbf{b}}$. We proceed by induction on r:

- Put

$$
\bar{Z}^0_{(p,q)+\mathbf{b}} := E^1_{(p,q)+\mathbf{b}} \quad \text{and} \quad \bar{B}^0_{(p,q)+\mathbf{b}} := 0.
$$

Then $d^1 = \Delta^1$ is a differential on E^1 because $k^1 \circ j^1 = 0$. Further, $\bar{Z}^1$ and $\bar{B}^1$ are as claimed.

- Suppose $r \geq 1$ and for $0 \leq s \leq r$, E^s, d^s and Δ^s satisfy the claim.

- We show that $E^{r+1} = \dfrac{\bar{Z}^r_{(p,q)+\mathbf{b}}}{\bar{B}^r_{(p,q)+\mathbf{b}}}$:

$$
\begin{aligned}
\frac{\bar{Z}^r_{(p,q)+\mathbf{b}}}{\bar{B}^r_{(p,q)+\mathbf{b}}} &= \frac{(k^1_{(p,q)+\mathbf{b}})^{-1}\operatorname{im}(i^r_{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}})}{j^1_{(p,q)}\ker(i^r_{(p,q)})} \\
&= \frac{\bar{Z}^r_{(p,q)+\mathbf{b}} \cap (k^1_{(p,q)+\mathbf{b}})^{-1}\operatorname{im}(i^r_{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}})}{j^1_{(p,q)} \circ (i^{r-1}_{(p,q)})^{-1}\ker(i^1_{(p,q)+(r-1)\mathbf{a}})} \\
&= \frac{\bar{Z}^{r-1}_{(p,q)+\mathbf{b}} \cap (k^1_{(p,q)+\mathbf{b}})^{-1} i^{r-1}_{(p,q)+\mathbf{b}+\mathbf{c}+(r-1)\mathbf{a}}(\ker(i^{r-1}_{(p,q)+\mathbf{b}+\mathbf{c}-(r-1)\mathbf{a}}) + \operatorname{im}(i^1_{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}}))}{j^1_{(p,q)} \circ (i^{r-1}_{(p,q)})^{-1}(\ker(i^1_{(p,q)+(r-1)\mathbf{a}}) \cap \operatorname{im}(i^{r-1}_{(p,q)}))} \\
&= \frac{\bar{Z}^{r-1}_{(p,q)+\mathbf{b}} \cap (k^1_{(p,q)+\mathbf{b}})^{-1} i^{r-1}_{(p,q)+\mathbf{b}+\mathbf{c}+(r-1)\mathbf{a}}(\ker(i^{r-1}_{(p,q)+\mathbf{b}+\mathbf{c}-(r-1)\mathbf{a}}) + \ker(j^1_{(p,q)+\mathbf{b}+\mathbf{c}-(r-1)\mathbf{a}}))}{j^1_{(p,q)} \circ (i^{r-1}_{(p,q)})^{-1}(\operatorname{im}(k^1_{(p,q)+(r-1)\mathbf{a}-\mathbf{c}}))} \\
&= \frac{\bar{Z}^{r-1}_{(p,q)+\mathbf{b}} \cap (j^1_{(p,q)+\mathbf{b}+\mathbf{c}-(r-1)\mathbf{a}} \circ (i^{r-1}_{(p,q)+\mathbf{b}+\mathbf{c}-(r-1)\mathbf{a}})^{-1} \circ k^1_{(p,q)+\mathbf{b}})^{-1} j^1_{(p,q)+\mathbf{b}+\mathbf{c}-(r-1)\mathbf{a}}(\ker(i^{r-1}_{(p,q)+\mathbf{b}+\mathbf{c}-(r-1)\mathbf{a}}))}{j^1_{(p,q)} \circ (i^{r-1}_{(p,q)})^{-1} \circ k^1_{(p,q)+(r-1)\mathbf{a}-\mathbf{c}}(k^1_{(p,q)+(r-1)\mathbf{a}-\mathbf{c}})^{-1}\operatorname{im}(i^{r-1}_{(p,q)})} \\
&= \frac{\bar{Z}^{r-1}_{(p,q)+\mathbf{b}} \cap (j^1_{(p,q)+\mathbf{b}+\mathbf{c}-(r-1)\mathbf{a}} \circ (i^{r-1}_{(p,q)+\mathbf{b}+\mathbf{c}-(r-1)\mathbf{a}})^{-1} \circ k^1_{(p,q)+\mathbf{b}})^{-1}(\bar{B}^{r-1}_{(p,q)+\mathbf{c}+2\mathbf{b}-(r-1)\mathbf{a}})}{j^1_{(p,q)} \circ (i^{r-1}_{(p,q)})^{-1} \circ k^1_{(p,q)+(r-1)\mathbf{a}-\mathbf{c}}(\bar{Z}^{r-1}_{(p,q)+(r-1)\mathbf{a}-\mathbf{c}})} \\
&= \frac{(\Delta^r_{(p,q)+\mathbf{b}})^{-1}(\bar{B}^{r-1}_{(p,q)+2\mathbf{b}+\mathbf{c}-(r-1)\mathbf{a}})/\bar{B}^{r-1}_{(p,q)+\mathbf{b}}}{\Delta^r_{(p,q)+(r-1)\mathbf{a}-\mathbf{c}}(\bar{Z}^{r-1}_{(p,q)+(r-1)\mathbf{a}-\mathbf{c}})/\bar{B}^{r-1}_{(p,q)+\mathbf{b}}} \\
&= \frac{\ker(d^r_{(p,q)+\mathbf{b}})}{\operatorname{im}(d^r_{(p,q)+(r-1)\mathbf{a}-\mathbf{c}})} \\
&= E^{r+1}_{(p,q)+\mathbf{b}}
\end{aligned}
$$

- About d^{r+1}: We first show that d^{r+1} is a homomorphism. To this end, we need to show that $\Delta^{r+1}_{(p,q)+\mathbf{b}}(\bar{B}^r_{(p,q)+\mathbf{b}}) \subseteq \bar{B}^r_{(p,q)+2\mathbf{b}+\mathbf{c}-r\mathbf{a}}$:

$$\begin{aligned}\Delta^{r+1}_{(p,q)+\mathbf{b}}(\bar{B}^r_{(p,q)+\mathbf{b}}) &= j^1_{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}}((i^r_{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}})^{-1}k^1_{(p,q)+\mathbf{b}}(\bar{B}^r_{(p,q)+\mathbf{b}}))\\ &= j^1_{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}}((i^r_{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}})^{-1}(k^1_{(p,q)+\mathbf{b}}(j^1_{(p,q)}(\ker(i^r_{(p,q)})))))\\ &= j^1_{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}}((i^r_{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}})^{-1}(0))\\ &= j^1_{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}}(\ker(i^r_{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}}))\\ &= \bar{B}^r_{(p,q)+2\mathbf{b}+\mathbf{c}-r\mathbf{a}}.\end{aligned}$$

So d^{r+1} is a bigraded homomorphism and it is related to Δ^{r+1} as required.

It remains to check that d^{r+1} is a differential. Thus we need to show that

$$\Delta^{r+1}_{(p,q)+\mathbf{b}} \circ \Delta^{r+1}_{(p,q)-\mathbf{c}+r\mathbf{a}}(\bar{Z}^r_{(p,q)-\mathbf{c}+r\mathbf{a}}) \subseteq \bar{B}^r_{(p,q)+2\mathbf{b}-\mathbf{c}-r\mathbf{a}}.$$

Indeed,

$$\begin{aligned}\Delta^{r+1}_{(p,q)+\mathbf{b}} \circ \Delta^{r+1}_{(p,q)-\mathbf{c}+r\mathbf{a}}(\bar{Z}^r_{(p,q)-\mathbf{c}+r\mathbf{a}}) &\subseteq j^1_{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}}(i^r_{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}})^{-1}k^1_{(p,q)+\mathbf{b}}j^1_{(p,q)}(D^1_{(p,q)})\\ &= j^1_{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}}(i^r_{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}})^{-1}(0)\\ &= \bar{B}^r_{(p,q)+2\mathbf{b}-\mathbf{c}-r\mathbf{a}}.\end{aligned}$$

$\square$

2.3.3 Morphism of Exact Couples

Definition 2.3.3. *A morphism of bidegree* $\mathbf{m} \in \mathbb{Z} \oplus \mathbb{Z}$ *from one exact couple* $\{D^1_{(p,q)}(1), E^1_{(p,q)}(1)\}_{(p,q)\in\mathbb{Z}\times\mathbb{Z}}$ *to another* $\{D^1_{(p,q)}(2), E^1_{(p,q)}(2)\}_{(p,q)\in\mathbb{Z}\times\mathbb{Z}}$ *is a family of morphisms*

$$\left\{(f_{(p,q)}, g_{(p,q)}) \mid f_{(p,q)} : D^1_{(p,q)}(1) \to D^1_{(p,q)+\mathbf{m}}(2),\ g_{(p,q)} : E^1_{(p,q)}(1) \to E^1_{(p,q)+\mathbf{m}}(2)\right\}_{(p,q)\in\mathbb{Z}\times\mathbb{Z}}$$

satisfying:

1. $f_{(p,q)+\mathbf{a}} \circ i^1_{(p,q)}(1) = i^1_{(p,q)+\mathbf{m}}(2) \circ f_{(p,q)}$,
2. $g_{(p,q)+\mathbf{b}} \circ j^1_{(p,q)}(1) = j^1_{(p,q)+\mathbf{m}}(2) \circ f_{(p,q)}$,
3. $f_{(p,q)+\mathbf{c}} \circ k^1_{(p,q)}(1) = k^1_{(p,q)+\mathbf{m}}(2) \circ g_{(p,q)}$.

Proposition 2.3.4. *A morphism between exact couples induces a morphism between the induced spectral sequences.*

Proof. Note that, by Theorem 2.3.2 on page 28 and the proof of Lemma 2.2.10 on page 21, for every $r \geq r_0$, we obtain a morphism $g_{(p,q)}| : Z^r_{(p,q)}(1) \to Z^r_{(p,q)+\mathbf{m}}(2)$ and hence a morphism $g^r_{(p,q)} : E^r_{(p,q)}(1) \to E^r_{(p,q)+\mathbf{m}}(2)$. Therefore, the following relations hold:

1. $f_{(p,q)+\mathbf{a}}| \circ i^1_{(p,q)}(1) = i^1_{(p,q)+\mathbf{m}}(2) \circ f_{(p,q)}|$,
2. $g_{(p,q)+\mathbf{b}}| \circ j^1_{(p,q)}(1) = j^1_{(p,q)+\mathbf{m}}(2) \circ f_{(p,q)}|$,

3. $f_{(p,q)+\mathbf{c}}| \circ k^1_{(p,q)}(1) = k^1_{(p,q)+\mathbf{m}}(2) \circ g_{(p,q)}|$.

We need to check the following:

- $g^r \circ d^r(1) = d^r(2) \circ g^r$: By Theorem 2.3.2 on page 28, we are done if we prove $g_{(p,q)-(r-1)\mathbf{a}+\mathbf{b}+\mathbf{c}}| \circ \Delta^r_{(p,q)}(1) = \Delta^r_{(p,q)+\mathbf{m}}(2) \circ g_{(p,q)}|$:

$$
\begin{aligned}
g_{(p,q)-(r-1)\mathbf{a}+\mathbf{b}+\mathbf{c}}| \circ \Delta^r_{(p,q)}(1) &= g_{(p,q)-(r-1)\mathbf{a}+\mathbf{b}+\mathbf{c}}| \circ j^1_{(p,q)-(r-1)\mathbf{a}+\mathbf{c}}(1) \circ (i^{r-1}_{(p,q)+\mathbf{c}}(1))^{-1} \circ k^1_{(p,q)}(1) \\
&= j^1_{(p,q)-(r-1)\mathbf{a}+\mathbf{c}+\mathbf{m}}(2) \circ f_{(p,q)-(r-1)\mathbf{a}+\mathbf{c}}| \circ (i^{r-1}_{(p,q)+\mathbf{c}}(1))^{-1} \circ k^1_{(p,q)}(1) \\
&= j^1_{(p,q)-(r-1)\mathbf{a}+\mathbf{c}+\mathbf{m}}(2) \circ (i^{r-1}_{(p,q)+\mathbf{c}+\mathbf{m}}(2))^{-1} \circ f_{(p,q)+\mathbf{c}}| \circ k^1_{(p,q)}(1) \\
&= j^1_{(p,q)-(r-1)\mathbf{a}+\mathbf{c}+\mathbf{m}}(2) \circ (i^{r-1}_{(p,q)+\mathbf{c}+\mathbf{m}}(2))^{-1} \circ k^1_{(p,q)+\mathbf{m}}(2) \circ g_{(p,q)}| \\
&= \Delta^r_{(p,q)+\mathbf{m}}(2) \circ g_{(p,q)}|.
\end{aligned}
$$

- $g^{r+1} = \mathrm{H}(g^r)$: From the homomorphism $g^r_{(p,q)} : E^r_{(p,q)}(1) \to E^r_{(p,q)+\mathbf{m}}(2)$ we obtain a homomorphism $\mathrm{H}_*(g^r_{(p,q)}) : \mathrm{H}_*(E^r_{(p,q)}(1)) \to \mathrm{H}_*(E^r_{(p,q)+\mathbf{m}}(2))$. From the proof of the Theorem 2.3.2 on page 28, we know that $Z^r_{(p,q)}(t) \cong \ker(d^r_{(p,q)}(t))$ and $B^r_{(p,q)}(t) \cong \mathrm{im}(d^r_{(p,q)+(r-1)\mathbf{a}-\mathbf{c}}(t))$, for $t = 1, 2$. Therefore, the homomorphism $g^{r+1}_{(p,q)} : E^{r+1}_{(p,q)}(1) \to E^{r+1}_{(p,q)+\mathbf{m}}(2)$ is the same as $\mathrm{H}_*(g^r_{(p,q)})$.

□

2.3.4 (Co-)Augmented Exact Couples

Now that we know how to distill a spectral sequence from an exact couple, we focus on the two objects of interest L_* and L^* that we have talked about in the introduction to this thesis.

Note 2.3.5. For $\mathbf{a} = (a_1, a_2)$, if we define $\hat{\mathbf{a}} := (-a_2, a_1)$ and look at $\mathbf{a}$ and $\hat{\mathbf{a}}$ as *vectors* then their *dot product* is $\mathbf{a} \cdot \hat{\mathbf{a}} = -a_1a_2 + a_2a_1 = 0$. Note that going up/down in a D^1-column in an exact couple is by subtracting/adding $\mathbf{a}$ to the bidegree of the D^1-terms. The way $\hat{\mathbf{a}}$ is defined guarantees that $D^1_{(u,v)}$ and $D^1_{(p,q)}$ are in the same D^1-column if and only if $(u,v)\cdot\hat{\mathbf{a}} = (p,q)\cdot\hat{\mathbf{a}}$. So if we pick a D^1-column, then for every $D^1_{(p,q)}$ in it, $(p,q) \cdot \hat{\mathbf{a}}$ is a fixed integer, say n. We call this column *the n-th D^1-column* of the exact couple. The D^1-column on the right of the n-th D^1-column is reached by adding $\mathbf{b} + \mathbf{c}$ to every (p,q). Therefore, the D^1-column to the right of the n-th D^1-column is the $(n+\sigma)$-th column where $\sigma = (\mathbf{b}+\mathbf{c}) \cdot \hat{\mathbf{a}}$. This is the reason in Definition 2.3.1 on page 27 we required $\sigma \neq 0$.

Definition 2.3.6. *An* augmentation *of an exact couple is a morphism* $D^1_{*,*} \to X_*$ *of the form*

$$\begin{tikzcd} \cdots \arrow[r] & D^1_{(p,q)-\mathbf{a}} \arrow[r, "i^1_{(p,q)-\mathbf{a}}"] & D^1_{(p,q)} \arrow[r] \arrow[rrr, bend right, "Q_{(p,q)}"'] & D^1_{(p,q)+\mathbf{a}} \arrow[r] & \cdots \arrow[r] & X_n \end{tikzcd}$$

where this is the n-th D^1*-column of the exact couple and* X_n *is an* R*-module and* $Q_{(p,q)-\mathbf{a}} = Q_{(p,q)} \circ i^1_{(p,q)-\mathbf{a}}$.

Definition 2.3.7. *A* coaugmentation *of an exact couple is a morphism* $X^* \to D^1_{*,*}$ *of the form*

$$\begin{tikzcd} X^n \arrow[r] \arrow[rrr, bend right, "Q^{(p,q)}"'] & \cdots \arrow[r] & D^1_{(p,q)-\mathbf{a}} \arrow[r] & D^1_{(p,q)} \arrow[r, "i^1_{(p,q)}"] & D^1_{(p,q)+\mathbf{a}} \arrow[r] & \cdots \end{tikzcd}$$

where this is the n-th D^1*-column of the exact couple and* X^n *is an* R*-module and* $Q^{(p,q)+\mathbf{a}} = i^1_{(p,q)} \circ Q^{(p,q)}$.

Remark 2.3.8. An exact couple always has a *universal* augmentation given by

$$\begin{tikzcd} \cdots \arrow[r] & D^1_{(p,q)-\mathbf{a}} \arrow[r, "i^1_{(p,q)-\mathbf{a}}"] & D^1_{(p,q)} \arrow[r] \arrow[rrr, bend right, "I_{(p,q)}"'] & D^1_{(p,q)+\mathbf{a}} \arrow[r] & \cdots \arrow[r] & L_n, \end{tikzcd}$$

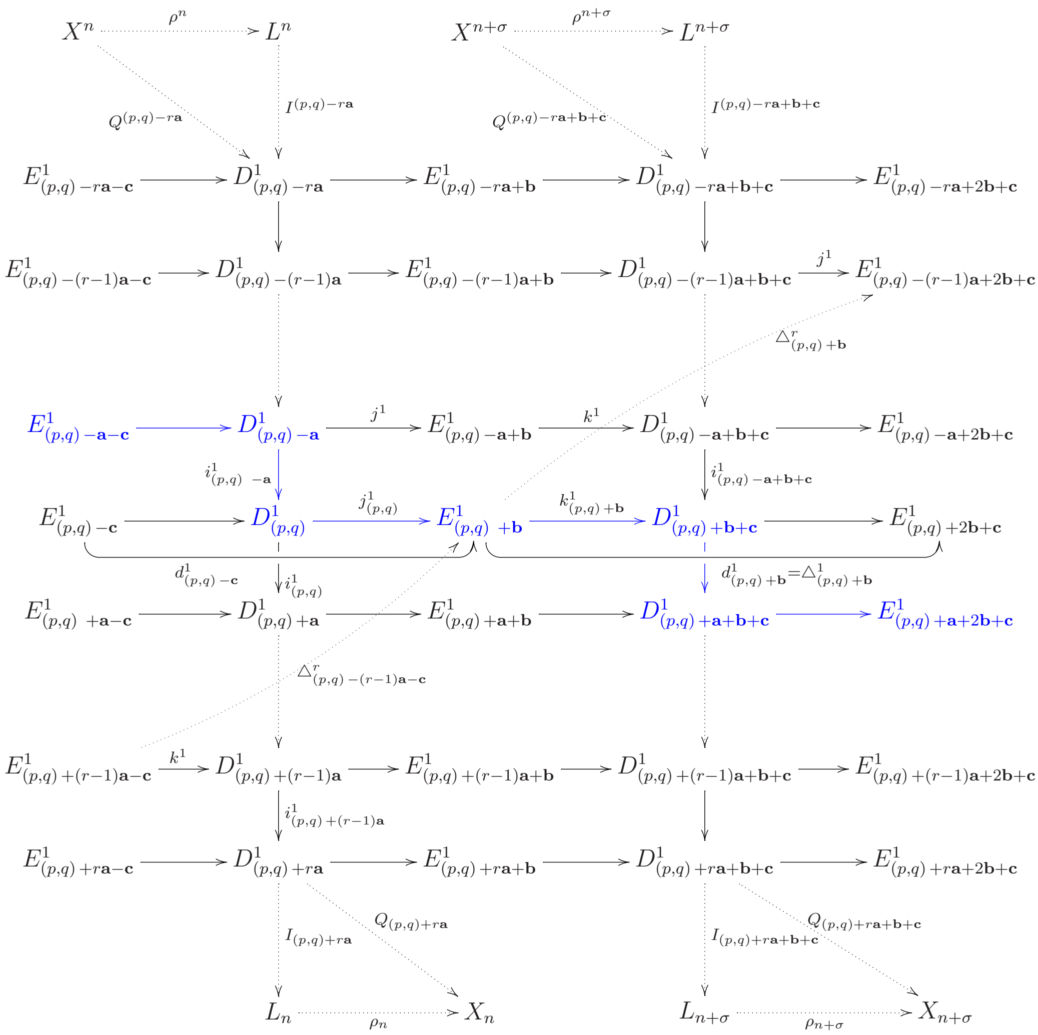
where $n = (p,q) \cdot \hat{\mathbf{a}}$ and $L_n = \operatorname{colim}_{(u,v)\cdot\hat{\mathbf{a}}=n} D^1_{u,v}$.

An exact couple always also has a *universal* coaugmentation given by

$$\begin{tikzcd} L^n \arrow[r] \arrow[rrr, bend right, "I^{(p,q)}"'] & \cdots \arrow[r] & D^1_{(p,q)-\mathbf{a}} \arrow[r] & D^1_{(p,q)} \arrow[r, "i^1_{(p,q)}"] & D^1_{(p,q)+\mathbf{a}} \arrow[r] & \cdots \end{tikzcd}$$

where $n = (p,q) \cdot \hat{\mathbf{a}}$ and $L^n := \lim_{(u,v)\cdot\hat{\mathbf{a}}=n} D^1_{u,v}$.

Therefore, if we take X_* as an arbitrary augmentation and X^* as an arbitrary coaugmentation we have the following diagram.



We can filter these augmentations and coaugmentations as follows.

Definition 2.3.9. *The* image filtration *of L_n is the sequence of modules*

$$0 \subseteq \cdots \subseteq \phi_{(p,q)-\mathbf{a}} \subseteq \phi_{(p,q)} \subseteq \phi_{(p,q)+\mathbf{a}} \subseteq \cdots \subseteq L_n$$

determined by

$$\phi_{(p,q)} = im(I_{(p,q)} : D^1_{(p,q)} \to L_n),$$

where $(p,q) \cdot \hat{\mathbf{a}} = n$.

Each pair of adjacent terms in this image filtration gives a short exact sequence

$$\phi_{(p,q)-\mathbf{a}} \rightarrowtail \phi_{(p,q)} \twoheadrightarrow \frac{\phi_{(p,q)}}{\phi_{(p,q)-\mathbf{a}}}.$$

Definition 2.3.10. *The* kernel filtration *of L^n is the sequence of modules*

$$0 \subseteq \cdots \subseteq \phi^{(p,q)-\mathbf{a}} \subseteq \phi^{(p,q)} \subseteq \phi^{(p,q)+\mathbf{a}} \subseteq \cdots \subseteq L^n$$

determined by

$$\phi^{(p,q)} = ker(I^{(p,q)} : L^n \to D^1_{(p,q)}),$$

wher $(p,q) \cdot \hat{\mathbf{a}} = n$. so that

Each pair of adjacent terms in this kernel filtration gives a short exact sequence

$$\phi^{(p,q)-\mathbf{a}} \rightarrowtail \phi^{(p,q)} \twoheadrightarrow \frac{\phi^{(p,q)}}{\phi^{(p,q)-\mathbf{a}}}.$$

Definition 2.3.11. *The* image filtration *of* $\mathrm{im}(\rho_n)$ *is the sequence of modules*

$$0 \subseteq \cdots \subseteq F_{(p,q)-\mathbf{a}} \subseteq F_{(p,q)} \subseteq F_{(p,q)+\mathbf{a}} \subseteq \cdots \subseteq im(\rho_n)$$

determined by

$$F_{(p,q)} = im(Q_{(p,q)} : D^1_{(p,q)} \to X_n),$$

where $(p,q) \cdot \hat{\mathbf{a}} = n$.

Each pair of adjacent terms in this image filtration gives a short exact sequence

$$F_{(p,q)-\mathbf{a}} \rightarrowtail F_{(p,q)} \twoheadrightarrow \frac{F_{(p,q)}}{F_{(p,q)-\mathbf{a}}}.$$

Definition 2.3.12. *The* kernel filtration *of X^n is the sequence of modules*

$$0 \subseteq \cdots \subseteq F^{(p,q)-\mathbf{a}} \subseteq F^{(p,q)} \subseteq F^{(p,q)+\mathbf{a}} \subseteq \cdots \subseteq X^n$$

determined by

$$F^{(p,q)} = ker(Q^{(p,q)} : X^n \to D^1_{(p,q)}),$$

where $(p,q) \cdot \hat{\mathbf{a}} = n$.

Each pair of adjacent terms in this kernel filtration gives a short exact sequence

$$F^{(p,q)-\mathbf{a}} \rightarrowtail F^{(p,q)} \twoheadrightarrow \frac{F^{(p,q)}}{F^{(p,q)-\mathbf{a}}}.$$

2.3.5 E^∞-Distribution Theorem

In Lemma 5.6 in [3], Boardman connects the quotient of the adjacent image filtration stages of the universal augmentation L_* to the E^∞-term.

Peschke [26], in the E^∞-Distribution Theorem, states the *simultaneous* relation between the quotient of adjacent image filtration stages of the universal augmentation, E^∞-term and the quotient of adjacent kernel filtration stages of the universal coaugmentation. He does so for an *arbitrary* spectral sequence, without any assumptions. This theorem plays a pivotal role in the theory of exact couple based spectral sequences and the entire thesis.

We state the E^∞-Distribution Theorem here and postpone the proof until the last section of this chapter. As we promised, it provides the answer to the following question from chapter 1 on page 4.

Theorem 2.3.13. *(E^∞-**Distribution Theorem** [26]) For an exact couple the following hold:*

1. *The differential on E^r has bidegree $\mathbf{b}+\mathbf{c}-(r-1)\mathbf{a}$.*

2. *In the image filtration of L_n, a pair of adjacent filtration terms fits into the 4-term exact sequence*

$$\begin{tikzcd}
\phi_{(p,q)-\mathbf{a}} \arrow[r, tail] & \phi_{(p,q)} \arrow[rr] \arrow[dr, two heads] & & E^\infty_{(p,q)+\mathbf{b}} \arrow[r, two heads] & \dfrac{Z^\infty_{(p,q)+\mathbf{b}}}{\mathrm{im}(j^1_{(p,q)})} = \dfrac{Z^\infty_{(p,q)+\mathbf{b}}}{\ker(k^1_{(p,q)+\mathbf{b}})}, \\
& & \epsilon_{(p,q)} \arrow[ur, tail] & &
\end{tikzcd}$$

where $\epsilon_{(p,q)} = Coker(\phi_{(p,q)-\mathbf{a}} \rightarrowtail \phi_{(p,q)})$.

3. *In the kernel filtration of $L^{n+\sigma}$, a pair of adjacent filtration terms fits into the 6-term exact sequence*

$$\begin{tikzcd}
\phi^{(p,q)+\mathbf{b}+\mathbf{c}} \arrow[r, tail] & \phi^{(p,q)+\mathbf{a}+\mathbf{b}+\mathbf{c}} \arrow[rr] \arrow[dr, two heads] & & \dfrac{Z^\infty_{(p,q)+\mathbf{b}}}{\mathrm{im}(j^1_{(p,q)})} \arrow[r] & \lim^1_r \ker(i^r_{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}}) \arrow[d] \\
& & \epsilon^{(p,q)+\mathbf{b}+\mathbf{c}} \arrow[ur, tail] & & \lim^1_r \ker(i^{r+1}_{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}}) \arrow[d] \\
& & & & \lim^1_r Z^r_{(p,q)+\mathbf{b}}
\end{tikzcd}$$

where $\epsilon^{(p,q)} = Coker(\phi^{(p,q)} \rightarrowtail \phi^{(p,q)+\mathbf{a}})$.

4. *For (u,v)'s where $(u,v)\cdot\hat{\mathbf{a}} = n$, we have $\mathrm{colim}_{(u,v)}\,\phi_{u,v} = L_n$ and $\lim_{(u,v)}\phi_{u,v}$ fit into the following 6-term exact sequence*

$$\begin{tikzcd}
\ker(L^n \to L_n) \arrow[r, tail] & L^n \arrow[r] & \lim_{(u,v)} \phi_{u,v} \arrow[r] & \lim^1_{(u,v)} \ker(I_{u,v}) \arrow[d] \\
& & & \lim^1_{(u,v)} D^1_{u,v} \arrow[d, two heads] \\
& & & \lim^1_{(u,v)} \phi_{u,v}
\end{tikzcd}$$

where $I_{u,v} : D^1_{u,v} \to L_n$.

5. *For* (u, v)*'s where* $(u, v) \cdot \hat{\mathbf{a}} = n$*, we have* $\lim_{(u,v)} \phi^{u,v} = 0 = \lim^1_{(u,v)} \phi^{u,v}$ *and* $\operatorname{colim}_{(u,v)} \phi^{u,v} = \ker(L^n \to L_n)$.

All of the assertions above are natural with respect to morphisms of exact couples.

We can summarize the Theorem 2.3.13 in the following diagram where sequences of the same color are exact: See [26].

$$\begin{tikzcd}
& & & & & 0 \arrow[d, equal] \\
\bigcap_r \phi_{(p,q)-r\mathbf{a}} \arrow[dd, hook, dotted] & & & \lim^1_r Z^r_{(p,q)+\mathbf{b}} & & \bigcap_r \phi^{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}} \arrow[dd, hook, dotted] \\
& & & \lim^1_r \ker(i^{r+1}_{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}}) \arrow[u, two heads, green] & & \\
\phi_{(p,q)-\mathbf{a}} \arrow[d, tail, blue] & & & \lim^1_r \ker(i^{r}_{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}}) \arrow[u, green] & & \phi^{(p,q)+\mathbf{b}+\mathbf{c}} \arrow[d, tail, blue] \\
\phi_{(p,q)} \arrow[r, two heads, blue] \arrow[d, tail] & \epsilon_{(p,q)} \arrow[r, tail, red] & E^\infty_{(p,q)+\mathbf{b}} \arrow[r, two heads, red] & \dfrac{Z^\infty_{(p,q)+\mathbf{b}}}{\operatorname{im}(j^1_{(p,q)})} \arrow[u, green] & \epsilon^{(p,q)+\mathbf{b}+\mathbf{c}} \arrow[l, tail, green] & \phi^{(p,q)+\mathbf{b}+\mathbf{c}+\mathbf{a}} \arrow[l, two heads, blue] \arrow[d, tail] \\
\phi_{(p,q)+\mathbf{a}} \arrow[r, two heads] \arrow[d, hook, dotted] & \epsilon_{(p,q)+\mathbf{a}} \arrow[r, tail] & E^\infty_{(p,q)+\mathbf{b}+\mathbf{a}} \arrow[r, two heads] & \dfrac{Z^\infty_{(p,q)+\mathbf{b}+\mathbf{a}}}{\operatorname{im}(j^1_{(p,q)+\mathbf{a}})} & \epsilon^{(p,q)+\mathbf{b}+\mathbf{c}+\mathbf{a}} \arrow[l, tail] & \phi^{(p,q)+\mathbf{b}+\mathbf{c}+2\mathbf{a}} \arrow[l, two heads] \arrow[d, hook, dotted] \\
\bigcup_r \phi_{(p,q)+r\mathbf{a}} \arrow[d, equal] & & & & & \bigcup_r \phi^{(p,q)+\mathbf{b}+\mathbf{c}+r\mathbf{a}} \arrow[d, equal] \\
L_n & & & & & \ker(L^{n+\sigma} \to L_{n+\sigma})
\end{tikzcd}$$

E^∞-Distribution Diagram

In analogy with the anatomy of a butterfly or a bird we refer

- to the image filtration stages of L_n and their quotients as the "left wing",
- to the kernel filtration stages of L^n and their quotients as the "right wing", and
- to the E^∞-terms, the $\dfrac{Z^\infty_{-,-}}{\mathrm{im}(j^1_{-,-})}$'s and the $\lim^1$-terms as the "body"

of the E^∞-Distribution Diagram.

Remark 2.3.14. On the one hand, the monomorphism

$$\epsilon_{(p,q)} \rightarrowtail E^\infty_{(p,q)+\mathbf{b}}$$

shows that $E^\infty_{(p,q)+\mathbf{b}}$ measures the contribution of $D^1_{(p,q)}$ to the image filtration of L_n. On the other hand, the epimorphism

$$E^\infty_{(p,q)+\mathbf{b}} \twoheadrightarrow \frac{Z^\infty_{(p,q)+\mathbf{b}}}{\mathrm{im}(j^1_{(p,q)})},$$

in presence of the monomorphism $\dfrac{Z^\infty_{(p,q)+\mathbf{b}}}{\mathrm{im}(j^1_{(p,q)})} \leftarrowtail \epsilon^{(p,q)+\mathbf{b}+\mathbf{c}}$, puts an upper limit on the contribution of $D^1_{(p,q)+\mathbf{b}+\mathbf{c}+\mathbf{a}}$ to the kernel filtration of $L^{n+\sigma}$, where $\sigma = (\mathbf{b}+\mathbf{c})\cdot\hat{\mathbf{a}}$.

For example, assume we have an exact couple whose D^1-columns are originally stable; i.e., for sufficiently large r we have $D^1_{(p,q)-r\mathbf{a}} \cong D^1_{(p,q)-(r+s)\mathbf{a}}$, for every $s > 1$. Then $\lim^1_r \ker(i^r_{-,-})$ vanishes and hence we have the following short exact sequence

$$\epsilon_{(p,q)} \rightarrowtail E^\infty_{(p,q)+\mathbf{b}} \twoheadrightarrow \epsilon^{(p,q)+\mathbf{b}+\mathbf{c}}.$$

The following example shows how little it takes for an exact couple to make the E^∞-term distribute itself *non-trivially* over $\epsilon_{-,-}$ and $\epsilon^{-,-}$.

Example 2.3.15. Assume we have a short exact sequence

$$A \rightarrowtail B \twoheadrightarrow C$$

and the following exact couple

$$\begin{tikzcd}
 & \vdots \arrow[d] & & \vdots \arrow[d] & & \vdots \arrow[d, "\cong"] & & \vdots \arrow[d] & \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] & C \arrow[r] \arrow[d, "\cong"] & 0 \arrow[r] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] & A \arrow[r, tail] \arrow[d, "\cong"'] & B \arrow[r, two heads] & C \arrow[r] \arrow[d] & 0 \arrow[r] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] & A \arrow[r] \arrow[d, "\cong"'] & 0 \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] & 0 \arrow[r] \arrow[d] & \cdots \\
 & \vdots & & \vdots & & \vdots & & \vdots &
\end{tikzcd}$$

Let us denote the position of B, the only nonzero E^1-term, by $(p, q+\mathbf{b})$. So we have the following

1. For $r \geq 0$, we have $\phi_{(p,q)+r\mathbf{a}} = A$ and zero otherwise. Hence $\epsilon_{(p,q)} = A$ and for every nonzero $s \in \mathbb{Z}$, $\epsilon_{(p,q)+s\mathbf{a}}$ vanishes.

2. For $r \geq 0$, we have $\phi^{(p,q)-r\mathbf{a}+\mathbf{b}+\mathbf{c}} = C$ and zero otherwise. Hence $\epsilon^{(p,q)+\mathbf{b}+\mathbf{c}} = C$ and for every nonzero $s \in \mathbb{Z}$, $\epsilon^{(p,q)+s\mathbf{a}+\mathbf{b}+\mathbf{c}}$ vanishes.

3. The spectral sequence collapses on the first page, so $E^{\infty}_{(p,q)+\mathbf{b}} = B$.

4. Since $\lim^1_r \ker(i^r_{(p,q)-r\mathbf{a}+\mathbf{b}+\mathbf{c}})$ vanishes, we have the following short exact sequence
$$\epsilon_{(p,q)} \rightarrowtail E^{\infty}_{(p,q)+\mathbf{b}} \twoheadrightarrow \epsilon^{(p,q)+\mathbf{b}+\mathbf{c}},$$
that is,
$$A \rightarrowtail B \twoheadrightarrow C.$$

Even in a spectral sequence with only one nonzero E-term, E^{∞} can be distributed non-trivially over both quotients of adjacent filtration stages.

We can generalize Theorem 2.3.13 to the case of an arbitrary (co-)augmentation. The proof will be similar.

Proposition 2.3.16. *[26] For an augmented and coaugmented exact couple the following hold.*

1. *There is a filtration of* X_n *given by*

$$F_{(p,q)} = im(Q_{(p,q)} : D^1_{(p,q)} \to X_n).$$

Adjacent filtration terms are related by the following diagram of exact sequences

$$\begin{tikzcd}
F_{(p,q)-\mathbf{a}} \arrow[r, tail] & F_{(p,q)} \arrow[rr, two heads] & & \delta_{(p,q)} & \\
\phi_{(p,q)-\mathbf{a}} \arrow[u, two heads, "\rho_n|"] \arrow[r, tail] & \phi_{(p,q)} \arrow[u, two heads, "\rho_n|"] \arrow[rr, two heads] \arrow[dr] & & \epsilon_{(p,q)} \arrow[u, two heads] \arrow[dl] & \\
& & E^\infty_{(p,q)+\mathbf{b}} \arrow[rr, two heads] & & \dfrac{Z^\infty_{(p,q)+\mathbf{b}}}{\mathrm{im}(j^1_{(p,q)})} \\
\ker(\rho_n|) \arrow[uu, tail] \arrow[r, tail, "u_{(p,q)}"'] & \ker(\rho_n|) \arrow[uu, tail] \arrow[rr, two heads] & & \mathrm{coker}(u_{(p,q)}) \arrow[uu, tail] &
\end{tikzcd}$$

where $\delta_{(p,q)} = \mathrm{coker}(F_{(p,q)-\mathbf{a}} \rightarrowtail F_{(p,q)})$.

2. *There is a filtration of* $X^{n+\sigma}$ *given by*

$$F^{(p,q)+\mathbf{b}+\mathbf{c}} = \ker(Q^{(p,q)+\mathbf{b}+\mathbf{c}} : X^{n+\sigma} \to D^1_{(p,q)+\mathbf{b}+\mathbf{c}}).$$

Adjacent filtration terms are related by the following diagram of exact sequences

$$\begin{tikzcd}
\ker(\rho^{n+\sigma}|) \arrow[r, equal] \arrow[d, tail] & \ker(\rho^{n+\sigma}|) \arrow[d, tail] & & & \\
F^{(p,q)+\mathbf{b}+\mathbf{c}} \arrow[r, tail] \arrow[d, "\rho^{n+\sigma}|"'] & F^{(p,q)+\mathbf{a}+\mathbf{b}+\mathbf{c}} \arrow[rr, two heads] \arrow[d, "\rho^{n+\sigma}|"'] & & \delta^{(p,q)+\mathbf{b}+\mathbf{c}} \arrow[d, tail] & \\
\phi^{(p,q)+\mathbf{b}+\mathbf{c}} \arrow[r, tail] \arrow[dd] & \phi^{(p,q)+\mathbf{a}+\mathbf{b}+\mathbf{c}} \arrow[rr, two heads] \arrow[dd] \arrow[dr] & & \epsilon^{(p,q)+\mathbf{b}+\mathbf{c}} \arrow[dd] \arrow[dl] & \\
& & \dfrac{Z^\infty_{(p,q)+\mathbf{b}}}{\ker(k^1_{(p,q)+\mathbf{b}})} \arrow[rr, tail] & & \lim^1_r \ker i^r_{(p,q)-r\mathbf{a}+\mathbf{b}+\mathbf{c}} \arrow[d] \\
\mathrm{coker}(\rho^{n+\sigma}|) \arrow[r, tail, "\nu_{(p,q)+\mathbf{b}+\mathbf{c}}"'] & \mathrm{coker}(\rho^{n+\sigma}|) \arrow[rr, two heads] & & \mathrm{coker}(\nu_{(p,q)+\mathbf{b}+\mathbf{c}}) & \lim^1_r \ker i^{r+1}_{(p,q)-r\mathbf{a}+\mathbf{b}+\mathbf{c}} \arrow[d] \\
& & & & \lim^1_r Z^r_{(p,q)+\mathbf{b}}
\end{tikzcd}$$

3. $\bigcup_r F_{(p,q)+r\mathbf{a}} = \mathrm{im}(\rho_n : L_n \to X_n)$ *and* $\bigcap_r F_{(p,q)+r\mathbf{a}}$ *fit into the exact* lim^n*-sequences*

$$\begin{tikzcd}
\lim_r \ker(Q_{(p,q)+r\mathbf{a}}) \arrow[r, tail] & L^n \arrow[r] & \bigcap_r F_{(p,q)+r\mathbf{a}} \arrow[r] & \lim^1_r \ker(Q_{(p,q)+r\mathbf{a}}) \arrow[r] & \lim^1_r D^1_{(p,q)+r\mathbf{a}} \arrow[d, two heads] \\
 & & & & \lim^1_r F_{(p,q)+r\mathbf{a}}.
\end{tikzcd}$$

4. We have

$$\bigcap_r F^{(p,q)+r\mathbf{a}+\mathbf{b}+\mathbf{c}} = \ker(\rho^{n+\sigma} : X^{n+\sigma} \to L^{n+\sigma})$$

and

$$\bigcup_r F^{(p,q)+r\mathbf{a}+\mathbf{b}+\mathbf{c}} = \ker(X^{n+\sigma} \to L_{n+\sigma}).$$

Moreover, all of the exact sequences in the assertions above are natural with respect to morphisms of (co-)augmented exact couples.

We can summarize this theorem and the E^∞-Distribution Theorem in the diagram on the first fold-out diagram in Appendix D, where $\rho^{n+\sigma}_{n+\sigma} : X^{n+\sigma} \to L_{n+\sigma}$ and $I^{n+\sigma}_{n+\sigma} : L^{n+\sigma} \to L_{n+\sigma}$.

2.4 Outcomes of the E^∞-Distribution Theorem

The E^∞-Distribution Theorem shows that there are three major players corresponding to every spectral sequence: $\epsilon_{-,-}$, $\epsilon^{-,-}$ and E^∞, and it shows the relationship between them. In this section, we provide necessary and sufficient conditions to obtain $\epsilon_{-,-} \cong E^\infty$ or $E^\infty \cong \epsilon^{-,-}$, and we see that it is impossible to have both at the same time. Then we will see when we obtain the closest possible relationship between these three; i.e., we will see a necessary and sufficient condition for availability of the following short exact sequence

$$\epsilon_{-,-} \rightarrowtail E^\infty_{-,-} \twoheadrightarrow \epsilon^{-,-}.$$

We will also see what happens if $\epsilon_{-,-} = 0$ or $\epsilon^{-,-} = 0$. Therefore, the Questions 2 and 3 stated on page 5 in the first chapter are answered.

Remember the following question from chapter 1 on page 5:

Question 2. *What are the necessary and sufficient conditions to have E^∞ isomorphic to $\epsilon_{-,-}$ or $\epsilon^{-,-}$? Is it possible to have both at the same time?*

Here is the answer to the first part which is a result of the E^∞-Distribution Theorem:

Proposition 2.4.1. *$\epsilon_{(p,q)} \cong E^\infty_{(p,q)+\mathbf{b}}$ if and only if $\dfrac{Z^\infty_{(p,q)+\mathbf{b}}}{\operatorname{im}(j^1_{(p,q)})} = 0$; i.e., if and only if $Z^\infty_{(p,q)+\mathbf{b}} = \operatorname{im}(j^1_{(p,q)})$.*

Definition 2.4.2. *Given an exact couple, we say that the induced spectral sequence is* convergent *to its universal augmentation if $\epsilon_{-,-} \cong E^\infty$.*

When the spectral sequence converges to L_*, the E^∞-Distribution Diagram turns into the following diagram:

$$\begin{tikzcd}
\bigcap_r \phi_{(p,q)-r\mathbf{a}} \arrow[d, hook, dotted] & & \\
\phi_{(p,q)-\mathbf{a}} \arrow[d, hook] & & \\
\phi_{(p,q)} \arrow[r, two heads] \arrow[d, hook] & \epsilon_{(p,q)} \arrow[r, "\cong"] & E^\infty_{(p,q)+\mathbf{b}} \\
\phi_{(p,q)+\mathbf{a}} \arrow[r, two heads] \arrow[d, hook, dotted] & \epsilon_{(p,q)+\mathbf{a}} \arrow[r, "\cong"] & E^\infty_{(p,q)+\mathbf{b}+\mathbf{a}} \\
\bigcup_r \phi_{(p,q)+r\mathbf{a}} \arrow[d, equal] & & \\
L_n & &
\end{tikzcd}$$

◀ Convergence to Augmentation $\epsilon_{-,-} \cong E^\infty$ ▶

Remark 2.4.3. This is what Boardman calls *weak convergence* to L_*. If also $\lim_r \phi_{(p,q)-r\mathbf{a}} = \bigcap_r \phi_{(p,q)-r\mathbf{a}} = 0$ he calls it *convergence* to L_* and if $\lim_r \phi_{(p,q)-r\mathbf{a}} = \lim^1_r \phi_{(p,q)-r\mathbf{a}} = 0$ he calls it *strong convergence* to L_*. See [3].

We also have the following special case:

Corollary 2.4.4. *If $L^{n+\sigma} = 0$, then any of the following conditions implies that in the induced spectral sequence we have $\epsilon_{(p,q)} \cong E^\infty_{(p,q)+\mathbf{b}}$:*

1. $Z^\infty_{(p,q)+\mathbf{b}} = \operatorname{im}(j^1_{(p,q)})$,

2. $\operatorname{im}(D^r_{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}} \to D^1_{(p,q)+\mathbf{b}+\mathbf{c}}) \cap \operatorname{im}(k^1_{(p,q)+\mathbf{b}}) = 0$ *for some* $r > 0$ *(in the literature it is called the Mittag-Leffler condition, see [24]),*

3. $D^1_{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}} = 0$ *for some* $r > 0$ *sufficiently large,*

4. $\lim^1_r \ker(i^r_{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}}) = 0.$

Proposition 2.4.5. $E^\infty_{(p,q)+\mathbf{b}} \cong \epsilon^{(p,q)+\mathbf{b}+\mathbf{c}}$ *if and only if* $\epsilon_{(p,q)}$ *and the map* $\dfrac{Z^\infty_{(p,q)+\mathbf{b}}}{\operatorname{im}(j^1_{(p,q)})} \to$ *lim*$^1_r \ker(i^r_{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}})$ *vanish.*

Definition 2.4.6. *Given an exact couple, we say that the induced spectral sequence is* convergent *to its universal coaugmentation if* $E^\infty \cong \epsilon^{-,-}$.

When the spectral sequence converges toL^*, the E^∞-Distribution Diagram turns into the following diagram:

$$
\begin{tikzcd}
 & & & 0 \arrow[d, equal] \\
 & \lim^1_r Z^r_{(p,q)+\mathbf{b}} & & \bigcap_r \phi^{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}} \arrow[dd, hook, dotted] \\
 & \lim^1_r \ker(i^{r+1}_{(p,q)-r\mathbf{a}}) \arrow[u, two heads] & & \\
 & \lim^1_r \ker(i^r_{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}}) \arrow[u, tail] & & \phi^{(p,q)+\mathbf{b}+\mathbf{c}} \arrow[d, tail] \\
E^\infty_{(p,q)+\mathbf{b}} \arrow[r, "\cong"] & \dfrac{Z^\infty_{(p,q)+\mathbf{b}}}{\operatorname{im}(j^1_{(p,q)})} \arrow[u, "0"] & \epsilon^{(p,q)+\mathbf{b}+\mathbf{c}} \arrow[l, "\cong"'] & \phi^{(p,q)+\mathbf{b}+\mathbf{c}+\mathbf{a}} \arrow[l, two heads] \arrow[d, tail] \\
E^\infty_{(p,q)+\mathbf{b}+\mathbf{a}} \arrow[r, "\cong"] & \dfrac{Z^\infty_{(p,q)+\mathbf{b}+\mathbf{a}}}{\operatorname{im}(j^1_{(p,q)+\mathbf{a}})} & \epsilon^{(p,q)+\mathbf{b}+\mathbf{c}+\mathbf{a}} \arrow[l, "\cong"'] & \phi^{(p,q)+\mathbf{b}+\mathbf{c}+2\mathbf{a}} \arrow[l, two heads] \arrow[d, hook, dotted] \\
 & & & \bigcup_r \phi^{(p,q)+\mathbf{b}+\mathbf{c}+r\mathbf{a}} \arrow[d, equal] \\
 & & & L^{n+\sigma}
\end{tikzcd}
$$

◀ Convergence to Coaugmentation $E^\infty \cong \epsilon^{-,-}$ ▶

Remark 2.4.7. This is what Boardman calls *(weak) convergence* to L^*. See [3].

We also have the following special case:

Corollary 2.4.8. *Suppose $L_n = 0$. Then any of the following conditions implies that $\epsilon^{(p,q)+\mathbf{b}+\mathbf{c}} \cong E^\infty_{(p,q)+\mathbf{b}}$:*

1. $\lim^1_r \ker(i^r_{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}}) = 0$,
2. *all $D^1_{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}} \to D^1_{(p,q)+\mathbf{b}+\mathbf{c}}$ are onto.*

The answer to the second part of Question 2 is "No", because:

- if $\epsilon_{(p,q)} \cong E^\infty_{(p,q)+\mathbf{b}}$ then $\dfrac{Z^\infty_{(p,q)+\mathbf{b}}}{\mathrm{im}(j^1_{(p,q)})} = 0$ and hence $\epsilon^{(p,q)+\mathbf{b}+\mathbf{c}}$ vanishes, and
- if $E^\infty_{(p,q)+\mathbf{b}} \cong \epsilon^{(p,q)+\mathbf{b}+\mathbf{c}}$ then $E^\infty_{(p,q)+\mathbf{b}} \cong \dfrac{Z^\infty_{(p,q)+\mathbf{b}}}{\mathrm{im}(j^1_{(p,q)})} \cong \epsilon^{(p,q)+\mathbf{b}+\mathbf{c}}$ and hence $\epsilon_{(p,q)}$ vanishes.

Remark 2.4.9. The map

$$\frac{Z^\infty_{(p,q)+\mathbf{b}}}{\mathrm{im}(j^1_{(p,q)})} \longrightarrow \lim\nolimits^1_r \ker(i^r_{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}})$$

vanishes if and only if

$$\frac{Z^\infty_{(p,q)+\mathbf{b}}}{\mathrm{im}(j^1_{(p,q)})} \cong \epsilon^{(p,q)+\mathbf{b}+\mathbf{c}}$$

and, equivalently, if and only if we have the following short exact sequence

$$\epsilon_{(p,q)} \rightarrowtail E^\infty_{(p,q)+\mathbf{b}} \twoheadrightarrow \epsilon^{(p,q)+\mathbf{b}+\mathbf{c}}.$$

This is the closest possible relationship between the E^∞-terms of a spectral sequence and the quotients of the adjacent filtration stages of the universal augmentation and coaugmentation of the corresponding exact couple. Therefore, E^∞ distributes itself uniquely over either $\epsilon_{(p,q)}$ or $\epsilon^{(p,q)+\mathbf{b}}$.

Definition 2.4.10. *Given an exact couple, the induced spectral sequence is called* purely distributed *over the universal augmentation and coaugmentation if we have the following short exact sequence for every $(p,q) \in \mathbb{Z}\times\mathbb{Z}$*

$$\epsilon_{(p,q)} \rightarrowtail E^\infty_{(p,q)+\mathbf{b}} \twoheadrightarrow \epsilon^{(p,q)+\mathbf{b}+\mathbf{c}}.$$

Now we consider the third question we stated in the first chapter on page 5:

Question 3. *What happens if $\epsilon_{-,-}$ or $\epsilon^{-,-}$ vanishes?*

Here is the answer:

Proposition 2.4.11. *If $\epsilon^{(p,q)+\mathbf{b}+\mathbf{c}} = 0$ then E^∞-Distribution Diagram has no right wing:*

$$
\begin{tikzcd}
\bigcap_r \phi_{(p,q)-r\mathbf{a}} \arrow[dd, dotted, tail] & & & \lim^1_r Z^r_{(p,q)+\mathbf{b}} \\
& & & \lim^1_r \ker(i^{r+1}_{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}}) \arrow[u] \\
\phi_{(p,q)-\mathbf{a}} \arrow[d, tail] & & & \lim^1_r \ker(i^{r}_{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}}) \arrow[u] \\
\phi_{(p,q)} \arrow[r, two heads] \arrow[d, tail] & \epsilon_{(p,q)} \arrow[r, tail] & E^\infty_{(p,q)+\mathbf{b}} \arrow[r, two heads] & \dfrac{Z^\infty_{(p,q)+\mathbf{b}}}{\mathrm{im}(j^1_{(p,q)})} \arrow[u] \\
\phi_{(p,q)+\mathbf{a}} \arrow[r, two heads] \arrow[d, dotted, tail] & \epsilon_{(p,q)+\mathbf{a}} \arrow[r, tail] & E^\infty_{(p,q)+\mathbf{b}+\mathbf{a}} \arrow[r, two heads] & \dfrac{Z^\infty_{(p,q)+\mathbf{b}+\mathbf{a}}}{\mathrm{im}(j^1_{(p,q)+\mathbf{a}})} \\
L_n \arrow[d, equal] & & & \\
\bigcup_r \phi_{(p,q)+r\mathbf{a}} & & &
\end{tikzcd}
$$

Proposition 2.4.12. *If $\epsilon_{(p,q)} = 0$, then the E^∞-Distribution Diagram has no left wing:*

$$
\begin{tikzcd}
& \lim^1_r Z^r_{(p,q)+\mathbf{b}} & & \bigcap_r \phi^{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}} = 0 \arrow[ddd, dotted, tail] \\
& \lim^1_r \ker(i^{r+1}_{(p,q)-r\mathbf{a}}) \arrow[u] & & \\
& \lim^1_r \ker(i^{r}_{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}}) \arrow[u] & & \\
& & & \phi^{(p,q)+\mathbf{b}+\mathbf{c}} \arrow[d, tail] \\
E^\infty_{(p,q)+\mathbf{b}} \arrow[r, "\cong"] & \dfrac{Z^\infty_{(p,q)+\mathbf{b}}}{\mathrm{im}(j^1_{(p,q)})} \arrow[uu] & \epsilon^{(p,q)+\mathbf{b}+\mathbf{c}} \arrow[l, tail] & \phi^{(p,q)+\mathbf{b}+\mathbf{c}+\mathbf{a}} \arrow[l, two heads] \arrow[d, tail] \\
E^\infty_{(p,q)+\mathbf{b}+\mathbf{a}} \arrow[r, "\cong"] & \dfrac{Z^\infty_{(p,q)+\mathbf{b}+\mathbf{a}}}{\mathrm{im}(j^1_{(p,q)+\mathbf{a}})} & \epsilon^{(p,q)+\mathbf{b}+\mathbf{c}+\mathbf{a}} \arrow[l, tail] & \phi^{(p,q)+\mathbf{b}+\mathbf{c}+2\mathbf{a}} \arrow[l, two heads] \arrow[d, dotted, tail] \\
& & & \bigcup_r \phi^{(p,q)+\mathbf{b}+\mathbf{c}+r\mathbf{a}} \arrow[d, equal] \\
& & & L^{n+\sigma}
\end{tikzcd}
$$

Definition 2.4.13. *Given an exact couple, the induced spectral sequence is called*

1. augmentation concentrated *if* $\epsilon^{-,-} = 0$, *and*
2. coaugmentation concentrated *if* $\epsilon_{-,-} = 0$.

The following corollary is an immediate outcome of the E^∞-Distribution Theorem and Remark 2.4.9 on page 43.

Corollary 2.4.14. *Given an exact couple, if the induced spectral sequence is convergent to the augmentation or coaugmentation, then it is augmentation or coaugmentation concentrated, respectively. That is,*

1. *if* $\epsilon_{-,-} \cong E^\infty_{-,-}$ *then* $\epsilon^{-,-} = 0$, *and*
2. *if* $E^\infty_{-,-} \cong \epsilon^{-,-}$ *then* $\epsilon_{-,-} = 0$.

The converse holds if and only if the map $\dfrac{Z^\infty_{-,-}}{\mathrm{im}(j_{-,-})} \longrightarrow lim^1_r \ker(i^r_{-,-})$ *is zero (e.g.,* $lim^1_r \ker(i^r_{-,-}) = 0$*).*

2.5 E^∞-Distribution Theorem vs Boardman Approach

In this section, we compare Boardman's approach in [3] and our approach that led to the E^∞-Distribution Theorem. We will see that Boardman's method enabled him to realize that there are three major players $\epsilon_{-,-}$, $\epsilon^{-,-}$ and E^∞ related to any spectral sequence. He successfully describes the relationship between two of them; i.e., he shows that there is a monomorphism from $\epsilon_{-,-}$ to E^∞. However, supported by the E^∞-Distribution Theorem, here we can also provide the relationship between $\epsilon^{-,-}$ and E^∞. Then we will see that his emphasis on providing necessary and sufficient conditions for what he calls *strong convergence* keeps one away from the fundamental fact that in convergence of any type what matters most is the isomorphism between $\epsilon_{-,-}$ or $\epsilon^{-,-}$ and E^∞, not requiring the limit and/or $\lim^1$ of the filtrations to vanish.

We will see that he is also aware that if one of $\epsilon_{-,-}$ or $\epsilon^{-,-}$ vanishes then there is a good chance to relate the non-zero quotient to E^∞ and it gives him the motivation to define the notion of *conditional convergence*.

We consider the following issues:

1. Relationship between $\epsilon_{-,-}$, $\epsilon^{-,-}$ and E^∞.
2. When do we have $\epsilon_{-,-} \cong E^\infty$ or $\epsilon^{-,-} \cong E^\infty$?
3. What happens if $\epsilon_{-,-} = 0$ or $\epsilon^{-,-} = 0$?
4. Boardman's tools (RE^∞ and W) vs our tool ($\lim^1 \ker(i)$).

2.5.1 Relationship between $\epsilon_{-,-}$, $\epsilon^{-,-}$ and E^∞

Boardman offers Lemma 5.6 that provides a monomorphism $\epsilon_{-,-} \rightarrowtail E^\infty$. Here, using the E^∞-Distribution Theorem, we can also describe the relationship between $\epsilon^{-,-}$ and E^∞. The E^∞-Distribution Theorem offers the following relationship between the monomorphism $\epsilon_{-,-} \rightarrowtail E^\infty$, E^∞ and $\epsilon^{-,-}$:

$$\begin{tikzcd}
 & & \lim^1_r \ker(i^r_{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}}) & \\
\epsilon_{(p,q)} \arrow[r, tail] & E^\infty_{(p,q)+\mathbf{b}} \arrow[r, two heads] & \dfrac{Z^\infty_{(p,q)+\mathbf{b}}}{\operatorname{im}(j^1_{(p,q)})} \arrow[u] & \epsilon^{(p,q)+\mathbf{b}+\mathbf{c}} \arrow[l, tail]
\end{tikzcd}$$

$\checkmark$ *The first advantage of the E^∞-Distribution Theorem is providing a relationship between $\epsilon^{-,-}$ and E^∞.*

2.5.2 When do we have $\epsilon_{-,-} \cong E^\infty$ or $\epsilon^{-,-} \cong E^\infty$?

Boardman is mostly interested in conditions that guarantee convergence of different types. He defines four types of convergence

1. weak convergence,
2. convergence,
3. strong convergence, and
4. conditional convergence.

See Definition 5.2 and 5.10 in [3]. In the first three types, the main role is played by the isomorphisms $E^\infty \cong \epsilon_{-,-}$ or $E^\infty \cong \epsilon^{-,-}$. The other conditions (e.g., vanishing of limit and/or $\lim^1$ of the filtrations) in the definition of convergence or strong convergence are only for extra comfort; look at Definition 2.1, Proposition 2.2, the explanation after Theorem 2.6 and Definition 5.2 in [3]. For example, as we have seen in the E^∞-Distribution Theorem, the limit and $\lim^1$ of the kernel filtration of the universal coaugmentation always vanish. So, with Boardman's terminology, being weakly convergent, convergent or strongly convergent to the universal coaugmentation are all the same concepts. We will talk about the fourth type of convergence in the next part.

To provide necessary and sufficient conditions to guarantee $\epsilon_{-,-} \cong E^\infty$ or $\epsilon^{-,-} \cong E^\infty$, he introduces an object RE^∞, which in our notation is $\lim^1_r Z^r_{(p,q)+\mathbf{b}}$; see section 5 in [3]. However, he often obtains more than just an isomorphism. Look at Theorems 6.1 and 7.4 in [3].

But, what are the necessary and sufficient conditions to have only $\epsilon_{-,-} \cong E^\infty$ or $\epsilon^{-,-} \cong E^\infty$?

The answer to **Question 2** in the previous section, Corollary 2.4.4 on page 41 and Corollary 2.4.8 on page 43 provide the necessary and sufficient conditions. Therefore,

$\surd$ *The second advantage of the E^∞-Distribution Theorem is providing necessary and sufficient conditions for $\epsilon_{-,-} \cong E^\infty$ or $\epsilon^{-,-} \cong E^\infty$.*

2.5.3 What happens if $\epsilon_{-,-} = 0$ or $\epsilon^{-,-} = 0$?

By defining the notion of conditional convergence, Boardman uses the fact that if L_* or L^* vanishes then the quotients $\epsilon_{-,-}$ or $\epsilon^{-,-}$ vanish, respectively, and then he can relate the E^∞-terms to the non-zero quotients; look at the proof of Theorems 7.2, 7.3 and 7.4 and also the statements of Theorems 8.10 and 8.13 in [3]. Vanishing of $\lim^1$ of the D^1-columns in the definition of conditionally convergence is only because of his policy to always mention both limit and $\lim^1$ for every sequence; look at page 63 in [3].

To describe how to relate the E^∞-terms isomorphically to the non-zero quotients $\epsilon_{-,-}$ or $\epsilon^{-,-}$, he introduces another object W, that he calls an "obstruction group". Look at Lemma 8.5 in [3]. But, again, he obtains "sufficient" but "not necessary" conditions for existence of such isomorphism. Look at Theorems 8.2, 8.10 and 8.13 in [3].

Again the same question arises:

What are the necessary and sufficient conditions to have only $\epsilon_{-,-} \cong E^\infty$ when $\epsilon^{-,-} = 0$ or $\epsilon^{-,-} \cong E^\infty$ when $\epsilon_{-,-} = 0$?

Proposition 2.4.11 on page 44, Proposition 2.4.12 on page 44 and Remark 2.4.9 on page 43 provide the necessary and sufficient conditions.

2.5.4 RE_∞ and W vs $\lim^1 \ker(i)$

Boardman introduces two objects RE_∞ and W and uses them as some sort of "obstructions" against the pleasant convergence. The object that we introduce and work with is $\lim^1_r \ker(i^r_{-,-})$; look at the E^∞-Distribution Theorem 2.3.13, part 3. Vanishing of $\lim^1_r\{\ker(i^r_{-,-})\}$ provides a sufficient condition for the closest possible relationship between the quotient of adjacent kernel filtration stages of $L^{n+\sigma}$, quotient of adjacent image filtration stages of L_n and the E^∞-terms; i.e., we have the following short exact sequence

$$\epsilon_{(p,q)} \rightarrowtail E^\infty_{(p,q)+\mathbf{b}} \twoheadrightarrow \epsilon^{(p,q)+\mathbf{b}+\mathbf{c}}$$

and hence the spectral sequence is purely distributed over the universal augmentation and coaugmentation; recall the Definition 2.4.10 on page 43. Whereas, vanishing of RE_∞ or W, or even both of them, does not imply any immediate relationship between $\epsilon_{-,-}$, $\epsilon^{-,-}$ and E^∞. See Lemma 5.9, Theorem 8.10, Lemma 8.11 and Theorem 8.13 in [3].

We should mention that RE_∞ and $\lim^1 \ker(i)$ are very similar:

1. Remember that RE_∞ is $\lim^1_r Z^r_{-,-}$. By the E^∞-Distribution Theorem, part 3, they both appear in a 6-term exact sequence.

2. If in an exact couple, for sufficiently large r we have $D^1_{(p,q)-r\mathbf{a}} \cong D^1_{(p,q)-(r+s)\mathbf{a}}$, for every $s > 1$, then $\lim^1 \ker(i)$ vanishes. Compare this with Section 6 in [3] where RE_∞ also vanishes for these types of exact couples. For an exact couple that the D^1-columns turn into isomorphisms or vanish as we go down, neither $\lim^1 \ker(i)$ nor RE_∞ necessarily vanish. Compare this with Section 7 in [3]. At this point, we do not know if there are examples where one vanishes and the other does not.

Therefore, the close connection between $\lim^1 \ker(i)$ and the quotients $\epsilon_{-,-}$ and $\epsilon^{-,-}$ turns $\lim^1 \ker(i)$ into a more efficient tool than RE_∞ and W.

$\surd$ *The third advantage of the E^∞-Distribution Theorem is providing an immediate relationship between $\epsilon_{-,-}$, $\epsilon^{-,-}$ and E^∞ when $\lim^1 \ker(i)$ vanishes.*

It also shows the following:

Corollary 2.5.1. *E^∞ can be isomorphic to only one of $\epsilon_{-,-}$ or $\epsilon^{-,-}$.*

2.6 Proof of the E^∞-Distribution Theorem

Before providing the proof of the E^∞-Distribution Theorem from [26], we state a few preliminary results first.

Definition 2.6.1. *For a fixed $n \in \mathbb{Z}$ and any $r \in \mathbb{N}$, define*

$$g^r_{(p,q)} = i^r_{(p,q)-r\mathbf{a}}|_{\mathrm{im}(Q^{(p,q)-r\mathbf{a}})}.$$

$$\begin{tikzcd}
X^n \arrow[rr, dotted, "\rho^n"] \arrow[drr, dotted, "Q^{(p,q)-r\mathbf{a}}"] \arrow[dddrr, dotted, "Q^{(p,q)}"'] & & L^n \arrow[d, dotted, "I^{(p,q)-r\mathbf{a}}"] & \\
& \cdots \arrow[r] & D^1_{(p,q)-r\mathbf{a}} \arrow[r] \arrow[d] \arrow[dd, bend left, "i^r_{(p,q)-r\mathbf{a}}"] & \cdots \\
& & \vdots \arrow[d] & \\
& \cdots \arrow[r] & D^1_{(p,q)} \arrow[r] \arrow[d] & \cdots \\
& & \vdots &
\end{tikzcd}$$

Note that

$$i^r_{(p,q)-r\mathbf{a}} : D^1_{(p,q)-r\mathbf{a}} \to D^1_{(p,q)}$$

and hence $g^r_{(p,q)}$ is defined from $\mathrm{im}(Q^{(p,q)-r\mathbf{a}})$ to $\mathrm{im}(Q^{(p,q)})$, because $i^r_{(p,q)-r\mathbf{a}} \circ Q^{(p,q)-r\mathbf{a}} = Q^{(p,q)}$. Also, $g^r_{(p,q)}$ is surjective, because for $Q^{(p,q)}(x) \in \mathrm{im}(Q^{(p,q)})$, we have

$$g^r_{(p,q)}(Q^{(p,q)-r\mathbf{a}}(x)) = Q^{(p,q)}(x).$$

For any $y \in \ker(g^1_{(p,q)})$, we have $y \in \ker(i^1_{(p,q)-\mathbf{a}}) = \mathrm{im}(k^1_{(p,q)-\mathbf{a}-\mathbf{c}})$, because $\ker(g^1_{(p,q)}) \subseteq \ker(i^1_{(p,q)-\mathbf{a}})$.

$$\begin{tikzcd}
 & & \vdots \arrow[d] & & \\
\cdots \arrow[r] & E^1_{(p,q)-2\mathbf{a}-\mathbf{c}} \arrow[r] & D^1_{(p,q)-2\mathbf{a}} \arrow[r] \arrow[d, "i^1_{(p,q)-2\mathbf{a}}"'] & E^1_{(p,q)\,-2\mathbf{a}+\mathbf{b}} \arrow[r] & \cdots \\
\cdots \arrow[r] & E^1_{(p,q)-\mathbf{a}-\mathbf{c}} \arrow[r, "k^1_{(p,q)-\mathbf{a}-\mathbf{c}}"] & D^1_{(p,q)-\mathbf{a}} \arrow[r, "j^1_{(p,q)-\mathbf{a}}"] \arrow[d, "i^1_{(p,q)-\mathbf{a}}"'] & E^1_{(p,q)-\mathbf{a}+\mathbf{b}} \arrow[r] & \cdots \\
\cdots \arrow[r] & E^1_{(p,q)-\mathbf{c}} \arrow[r] & D^1_{(p,q)} \arrow[r] \arrow[d] & E^1_{(p,q)+\mathbf{b}} \arrow[r] & \cdots \\
 & & \vdots & &
\end{tikzcd}$$

Therefore, for some $\hat{y} \in E^1_{(p,q)-\mathbf{a}-\mathbf{c}}$ we have

$$k^1_{(p,q)-\mathbf{a}-\mathbf{c}}(\hat{y}) = y.$$

We are trying to show that $\ker(g^1_{(p,q)})$ is isomorphic to the quotient of adjacent kernel filtration stages of X^n and also it is mapped monomorphically into $\dfrac{Z^{\infty}_{(p,q)-\mathbf{a}-\mathbf{c}}}{\ker(k^1_{(p,q)-\mathbf{a}-\mathbf{c}})}$.

Lemma 2.6.2. *For every $r \in \mathbb{N} \setminus \{0\}$, we have*

$$y \in im(g^r_{(p,q)-\mathbf{a}}).$$

Proof. The statement holds for $r = 1$, since $g^1_{(p,q)-\mathbf{a}}$ is surjective. Now assume that $y \in \operatorname{im}(g^{r-1}_{(p,q)-\mathbf{a}})$. So for some $y^{r-1} \in \operatorname{im}(Q^{(p,q)-r\mathbf{a}})$ we have $g^{r-1}_{(p,q)-\mathbf{a}}(y^{r-1}) = y$. Since $g^1_{(p,q)-r\mathbf{a}}$ is surjective, there is an element $y^r \in \operatorname{im}(Q^{(p,q)-(r+1)\mathbf{a}})$ such that $g^1_{(p,q)-r\mathbf{a}}(y^r) = y^{r-1}$. Therefore, by commutativity we have

$$g^r_{(p,q)-\mathbf{a}}(y^r) = g^{r-1}_{(p,q)-\mathbf{a}} \circ g^1_{(p,q)-r\mathbf{a}}(y^r) = g^{r-1}_{(p,q)-\mathbf{a}}(y^{r-1}) = y;$$

i.e., $y \in \operatorname{im}(g^r_{(p,q)-\mathbf{a}})$. $\square$

Corollary 2.6.3. *For every $y \in \ker(g^1_{(p,q)})$, we have*

$$\hat{y} \in (k^1_{(p,q)-\mathbf{a}-\mathbf{c}})^{-1}(\lim_r \operatorname{im}(g^r_{(p,q)-\mathbf{a}})) \subseteq Z^\infty_{(p,q)-\mathbf{a}-\mathbf{c}}.$$

Lemma 2.6.4. *The function $\varphi : \ker(g^1_{(p,q)}) \longrightarrow \dfrac{Z^\infty_{(p,q)-\mathbf{a}-\mathbf{c}}}{\ker(k^1_{(p,q)-\mathbf{a}-\mathbf{c}})}$ defined by*

$$\varphi(y) = [\hat{y}] = \hat{y} + \ker(k^1_{(p,q)-\mathbf{a}-\mathbf{c}})$$

is a monomorphism of R-modules.

Proof.

- *φ is well-defined:* If $y = z \in \ker(g^1_{(p,q)})$, then $\varphi(y) = [\hat{y}]$, $\varphi(z) = [\hat{z}]$ and

$$k^1_{(p,q)-\mathbf{a}-\mathbf{c}}(\hat{y}) = y = z = k^1_{(p,q)-\mathbf{a}-\mathbf{c}}(\hat{z}).$$

  So $\hat{y} - \hat{z} \in \ker(k^1_{(p,q)-\mathbf{a}-\mathbf{c}})$; i.e., $\varphi(y) = \varphi(z)$.

- *φ is a homomorphism:* Let $y, z \in \ker(g^1_{(p,q)})$ and $t \in R$. We should show that

$$\varphi(y) + t\varphi(z) = \varphi(y + tz).$$

  We have $\varphi(y) = [\hat{y}]$, $\varphi(z) = [\hat{z}]$ and $\varphi(y + tz) = [y \hat{+} tz]$. So we should show that

$$[\hat{y}] + t[\hat{z}] = [y \hat{+} tz]$$

  or equivalently,

$$\hat{y} + t\hat{z} - \widehat{y + tz} \in \ker(k^1_{(p,q)-\mathbf{a}-\mathbf{c}}).$$

  We have

$$\begin{aligned} k^1_{(p,q)-\mathbf{a}-\mathbf{c}}(\hat{y} + t\hat{z} - \widehat{y + tz}) &= k^1_{(p,q)-\mathbf{a}-\mathbf{c}}(\hat{y}) + k^1_{(p,q)-\mathbf{a}-\mathbf{c}}(t\hat{z}) - k^1_{(p,q)-\mathbf{a}-\mathbf{c}}(\widehat{y + tz}) \\ &= k^1_{(p,q)-\mathbf{a}-\mathbf{c}}(\hat{y}) + tk^1_{(p,q)-\mathbf{a}-\mathbf{c}}(\hat{z}) - k^1_{(p,q)-\mathbf{a}-\mathbf{c}}(\widehat{y + tz}) \\ &= y + tz - (y + tz) \\ &= 0. \end{aligned}$$

  So $\hat{y} + t\hat{z} - \widehat{y + tz} \in \ker(k^1_{(p,q)-\mathbf{a}-\mathbf{c}})$.

- *φ is a monomorphism:* Take $y \in \ker \varphi$. So for some

$$\hat{y} \in (k^1_{(p,q)-\mathbf{a}-\mathbf{c}})^{-1}(\lim_r \operatorname{im}(g^r_{(p,q)-\mathbf{a}}))$$

  we have $0 = \varphi(y) = [\hat{y}]$. Therefore, $\hat{y} \in \ker(k^1_{(p,q)-\mathbf{a}-\mathbf{c}})$. So there is $w \in D^{1,\eta_0}_{(p,q)-\mathbf{a}-\mathbf{b}-\mathbf{c}}$ such that $j^1_{(p,q)-\mathbf{a}-\mathbf{b}-\mathbf{c}}(w) = \hat{y}$. But by exactness we have

$$0 = (k^1_{(p,q)-\mathbf{a}-\mathbf{c}} \circ j^1_{(p,q)-\mathbf{a}-\mathbf{b}-\mathbf{c}})(w) = k^1_{(p,q)-\mathbf{a}-\mathbf{c}}(\hat{y}) = y.$$

Therefore, φ is a monomorphism.

$\square$

Proposition 2.6.5. *There is a short exact sequence* $F^{(p,q)-\mathbf{a}} \rightarrowtail F^{(p,q)} \twoheadrightarrow \ker(g^1_{(p,q)})$.

Proof. Define

$$\psi : F^{(p,q)} \to \ker(g^1_{(p,q)})$$

by $\psi = Q^{(p,q)-\mathbf{a}}|_{F^{(p,q)}}$. Let $x \in F^{(p,q)}$. Then $Q^{(p,q)-\mathbf{a}}(x) \in \operatorname{im}(Q^{(p,q)-\mathbf{a}})$. By commutativity, we have

$$g^1_{(p,q)}(Q^{(p,q)-\mathbf{a}}(x)) = Q^{(p,q)}(x) = 0;$$

i.e., $Q^{(p,q)-\mathbf{a}}(x) \in \ker(g^1_{(p,q)})$. Also, $x \in \ker(\psi)$ iff $Q^{(p,q)-\mathbf{a}}(x) = 0$; i.e., $\ker(\psi) = F^{(p,q)-\mathbf{a}}$.

Take $y \in \ker(g^1_{(p,q)})$. Then there is some $x \in X^n$ such that $Q^{(p,q)-\mathbf{a}}(x) = y$. By commutativity, we have

$$Q^{(p,q)}(x) = g^1_{(p,q)}(Q^{(p,q)-\mathbf{a}}(x)) = g^1_{(p,q)}(y) = 0;$$

i.e., $x \in F^{(p,q)}$. Since $\psi(x) = Q^{(p,q)-\mathbf{a}}(x) = y$, we then obtain the desired epimorphism. $\square$

Corollary 2.6.6. *For any* $(p,q) \in \mathbb{Z} \times \mathbb{Z}$,

$$\frac{F^{(p,q)}}{F^{(p,q)-\mathbf{a}}} \cong \ker(g^1_{(p,q)}) \rightarrowtail \frac{Z^\infty_{(p,q)-\mathbf{a}-\mathbf{c}}}{\ker(k^1_{(p,q)-\mathbf{a}-\mathbf{c}})}.$$

Remark 2.6.7. Note that those elements of the D^1-terms of a D^1-column that are in the images of all $Q^{(p,q)-r\mathbf{a}}$'s are the only elements that are involved in building $\lim_r D^1_{(p,q)-r\mathbf{a}}$: The inclusion maps $\operatorname{im}(Q^{(p,q)-r\mathbf{a}}) \subseteq D^1_{(p,q)-r\mathbf{a}}$ induce a monomorphism

$$\lim_r \operatorname{im}(Q^{(p,q)-r\mathbf{a}}) \rightarrowtail \lim_r D^1_{(p,q)-r\mathbf{a}}.$$

This is actually an isomorphism, since for any thread $(x_n) \in \lim_r D^1_{(p,q)-r\mathbf{a}}$, we have $Q^{(p,q)-r\mathbf{a}}(x_n) = x_r$ and hence $x_r \in \operatorname{im}(Q^{(p,q)-r\mathbf{a}})$. Since for any $s >$

0 we have $i^s_{(p,q)-r\mathbf{a}}(x_r) = x_{r+s}$, we get $(x_n) \in \lim_r \operatorname{im}(Q^{(p,q)-r\mathbf{a}})$. Therefore,

$$\lim_r \operatorname{im}(Q^{(p,q)-r\mathbf{a}}) \cong \lim_r D^1_{(p,q)-r\mathbf{a}}.$$

Lemma 2.6.8. *We have the following triangle of isomorphisms*

$$\begin{tikzcd}
\dfrac{\operatorname{im}(i^r_{(p,q)})}{\operatorname{im}(i^{r+1}_{(p,q)-\mathbf{a}})} \arrow[rr, "\cong"] & & \dfrac{\ker(k^1_{(p,q)+\mathbf{b}})}{B^r_{(p,q)+\mathbf{b}}} \\
& \dfrac{D^1_{(p,q)}}{\ker(i^r_{(p,q)}) + \operatorname{im}(i^1_{(p,q)-\mathbf{a}})} \arrow[ul, "\cong"] \arrow[ur, "\cong"'] &
\end{tikzcd}$$

Proof. We have

$$\begin{aligned}
\frac{\ker(k^1_{(p,q)+\mathbf{b}})}{B^r_{(p,q)+\mathbf{b}}} &= \frac{\operatorname{im}(j^1_{(p,q)})}{j^1_{(p,q)} \circ (i^{r-1}_{(p,q)})^{-1}(\operatorname{im}(k^1_{(p,q)-\mathbf{c}+(r-1)\mathbf{a}}))} \\
&= \frac{\operatorname{im}(j^1_{(p,q)})}{j^1_{(p,q)} \circ (i^{r-1}_{(p,q)})^{-1}(\ker(i^1_{(p,q)+(r-1)\mathbf{a}}))} \\
&= \frac{\operatorname{im}(j^1_{(p,q)})}{j^1_{(p,q)}(\ker(i^r_{(p,q)}))} \\
&= \frac{\operatorname{im}(j^1_{(p,q)})}{\operatorname{im}(j^1_{(p,q)}|_{\ker(i^r_{(p,q)})})} \\
&\cong \frac{\dfrac{D^1_{(p,q)}}{\ker(j^1_{(p,q)})}}{\dfrac{\ker(i^r_{(p,q)}) + \ker(j^1_{(p,q)})}{\ker(j^1_{(p,q)})}} \\
&= \frac{\dfrac{D^1_{(p,q)}}{\operatorname{im}(i^1_{(p,q)-\mathbf{a}})}}{\dfrac{\ker(i^r_{(p,q)}) + \operatorname{im}(i^1_{(p,q)-\mathbf{a}})}{\operatorname{im}(i^1_{(p,q)-\mathbf{a}})}} \\
&\cong \frac{D^1_{(p,q)}}{\ker(i^r_{(p,q)}) + \operatorname{im}(i^1_{(p,q)-\mathbf{a}})}.
\end{aligned}$$

Also

$$
\begin{aligned}
\frac{\operatorname{im}(i^r_{(p,q)})}{\operatorname{im}(i^{r+1}_{(p,q)-\mathbf{a}})} &= \frac{\operatorname{im}(i^r_{(p,q)})}{\operatorname{im}(i^r_{(p,q)} \circ i^1_{(p,q)-\mathbf{a}})} \\
&\cong \frac{\dfrac{D^1_{(p,q)}}{\ker(i^r_{(p,q)})}}{\dfrac{\ker(i^r_{(p,q)}) + \operatorname{im}(i^1_{(p,q)-\mathbf{a}})}{\ker(i^r_{(p,q)})}} \\
&\cong \frac{D^1_{(p,q)}}{\ker(i^r_{(p,q)}) + \operatorname{im}(i^1_{(p,q)-\mathbf{a}})}.
\end{aligned}
$$

□

Lemma 2.6.9. *If $f : A \to B$ and $g : B \to C$ are two $R-$module homomorphisms, then*
$$\frac{\ker(g \circ f)}{\ker(f)} \cong \operatorname{im}(f) \cap \ker(g).$$

Proof. The vertical arrow on the right in the following diagram is an isomorphism

$$
\begin{tikzcd}
\ker(f) \arrow[r, tail] \arrow[d, equal] & \ker(g \circ f) \arrow[r, two heads] \arrow[d, equal] & \dfrac{\ker(g \circ f)}{\ker(f)} \arrow[d, dashed] \\
\ker(f) \arrow[r, tail] & f^{-1}(\ker(g)) \arrow[r, two heads] & \operatorname{im}(f) \cap \ker(g).
\end{tikzcd}
$$

□

Lemma 2.6.10. *We have the following triangle of isomorphisms*

$$
\begin{tikzcd}
\dfrac{\ker(i^{r+1}_{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}})}{\ker(i^{r}_{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}})} \arrow[rr, "\cong"] \arrow[dr, "\cong"] & & \dfrac{Z^r_{(p,q)+\mathbf{b}}}{\ker(k^1_{(p,q)+\mathbf{b}})} \arrow[dl, "\cong"'] \\
& \ker(i^1_{(p,q)+\mathbf{b}+\mathbf{c}}) \cap \operatorname{im}(i^r_{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}}) &
\end{tikzcd}
$$

Proof. By lemma 2.6.9 we have

$$\begin{aligned}
\frac{\ker(i^{r+1}_{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}})}{\ker(i^{r}_{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}})} &\cong \ker(i^1_{(p,q)+\mathbf{b}+\mathbf{c}}) \cap \operatorname{im}(i^r_{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}}) \\
&= \operatorname{im}(k^1_{(p,q)+\mathbf{b}}) \cap \operatorname{im}(i^r_{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}}) \\
&\cong \frac{(k^1_{(p,q)+\mathbf{b}})^{-1}(\operatorname{im}(i^r_{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}}))}{\ker(k^1_{(p,q)+\mathbf{b}})} \\
&= \frac{(k^1_{(p,q)+\mathbf{b}})^{-1}(i^{r-1}_{(p,q)+\mathbf{b}+\mathbf{c}-(r-1)\mathbf{a}}(\operatorname{im}(i^1_{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}})))}{\ker(k^1_{(p,q)+\mathbf{b}})} \\
&= \frac{(k^1_{(p,q)+\mathbf{b}})^{-1}(i^{r-1}_{(p,q)+\mathbf{b}+\mathbf{c}-(r-1)\mathbf{a}}(\ker(j^1_{(p,q)+\mathbf{b}+\mathbf{c}-(r-1)\mathbf{a}})))}{\ker(k^1_{(p,q)+\mathbf{b}})} \\
&= \frac{Z^r_{(p,q)+\mathbf{b}}}{\ker(k^1_{(p,q)+\mathbf{b}})}.
\end{aligned}$$

□

We are finally ready to prove the E^∞-Distribution Theorem 2.3.13.

Proof. (*Theorem 2.3.13*)

1. Remember that d^r is defined in Theorem 2.3.2 on page 28 to be a morphism
$$d^r_{(p,q)} : E^r_{(p,q)} \to E^r_{(p,q)+\mathbf{b}+\mathbf{c}-(r-1)\mathbf{a}}.$$
Therefore, it is of bidegree $\mathbf{b}+\mathbf{c}-(r-1)\mathbf{a}$.

2. We have the following short exact sequence of diagrams of towers
$$\{\operatorname{im}(i^{r+1}_{(p,q)-\mathbf{a}})\}_{r\in\mathbb{Z}} \rightarrowtail \{\operatorname{im}(i^{r}_{(p,q)})\}_{r\in\mathbb{Z}} \twoheadrightarrow \left\{\frac{\operatorname{im}(i^r_{(p,q)})}{\operatorname{im}(i^{r+1}_{(p,q)-\mathbf{a}})}\right\}_{r\in\mathbb{Z}}.$$
Using lemma 2.6.8 we can write it as
$$\{\operatorname{im}(i^{r+1}_{(p,q)-\mathbf{a}})\}_{r\in\mathbb{Z}} \rightarrowtail \{\operatorname{im}(i^{r}_{(p,q)})\}_{r\in\mathbb{Z}} \twoheadrightarrow \left\{\frac{\ker(k^1_{(p,q)+\mathbf{b}})}{B^r_{(p,q)+\mathbf{b}}}\right\}_{r\in\mathbb{Z}}.$$
This diagram of short exact sequences is directed. Therefore, the colimit functor preserves exactness and so we obtain
$$\operatorname{colim}_r \operatorname{im}(i^{r+1}_{(p,q)-\mathbf{a}}) \rightarrowtail \operatorname{colim}_r \operatorname{im}(i^{r}_{(p,q)}) \twoheadrightarrow \operatorname{colim}_r \frac{\ker(k^1_{(p,q)+\mathbf{b}})}{B^r_{(p,q)+\mathbf{b}}}$$

because the colimit functor is exact. Note that

$$\operatorname{colim}_r \operatorname{im}(i^r_{(p,q)} : D^1_{(p,q)} \to D^1_{(p,q)+r\mathbf{a}}) = \operatorname{im}(Q_{(p,q)} : D^1_{(p,q)} \to L_n) = \phi_{(p,q)}$$

and

$$\operatorname{colim}_r \frac{\ker(k^1_{(p,q)+\mathbf{b}})}{B^r_{(p,q)+\mathbf{b}}} = \frac{\ker(k^1_{(p,q)+\mathbf{b}})}{\operatorname{colim}_r B^r_{(p,q)+\mathbf{b}}} = \frac{\ker(k^1_{(p,q)+\mathbf{b}})}{B^\infty_{(p,q)+\mathbf{b}}}.$$

So we obtain the short exact sequence

$$\phi_{(p,q)-\mathbf{a}} \rightarrowtail \phi_{(p,q)} \twoheadrightarrow \frac{\ker(k^1_{(p,q)+\mathbf{b}})}{B^\infty_{(p,q)+\mathbf{b}}} \tag{2.6}$$

and hence $\epsilon_{(p,q)} \cong \dfrac{\ker(k^1_{(p,q)+\mathbf{b}})}{B^\infty_{(p,q)+\mathbf{b}}}$. Since $\ker(k^1_{(p,q)+\mathbf{b}}) = \operatorname{im}(j^1_{(p,q)}) \subseteq Z^\infty_{(p,q)+\mathbf{b}}$, we have the following short exact sequence

$$\frac{\operatorname{im}(j^1_{(p,q)})}{B^\infty_{(p,q)+\mathbf{b}}} \rightarrowtail \frac{Z^\infty_{(p,q)+\mathbf{b}}}{B^\infty_{(p,q)+\mathbf{b}}} \twoheadrightarrow \frac{Z^\infty_{(p,q)+\mathbf{b}}}{\ker(k^1_{(p,q)+\mathbf{b}})}. \tag{2.7}$$

But

$$E^\infty_{(p,q)+\mathbf{b}} = \frac{Z^\infty_{(p,q)+\mathbf{b}}}{B^\infty_{(p,q)+\mathbf{b}}} \tag{2.8}$$

So we can combine (2.6), (2.7) and (2.8) in the following exact sequence

$$\begin{tikzcd}
\phi_{(p,q)-\mathbf{a}} \arrow[r, tail] & \phi_{(p,q)} \arrow[rr] \arrow[dr, two heads] & & E^\infty_{(p,q)+\mathbf{b}} \arrow[r, two heads] & \dfrac{Z^\infty_{(p,q)+\mathbf{b}}}{\ker(k^1_{(p,q)+\mathbf{b}})} \\
& & \epsilon_{(p,q)} \arrow[ur, tail] & &
\end{tikzcd}$$

3. By lemma 2.6.10, we have the following short exact sequence of diagrams of inverse towers
$$\{\ker(i^r_{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}})\}_{r\in\mathbb{Z}} \rightarrowtail \{\ker(i^{r+1}_{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}})\}_{r\in\mathbb{Z}} \twoheadrightarrow \left\{\frac{Z^r_{(p,q)+\mathbf{b}}}{\ker(k^1_{(p,q)+\mathbf{b}})}\right\}_{r\in\mathbb{Z}}.$$
If we take inverse limit, by Theorem A.0.7 in Appendix A, we obtain the following exact sequence

$$\lim_r \ker(i^r_{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}}) \rightarrowtail \lim_r \ker(i^{r+1}_{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}}) \longrightarrow \lim_r \frac{Z^r_{(p,q)+\mathbf{b}}}{\ker(k^1_{(p,q)+\mathbf{b}})} \longrightarrow$$

$$\lim^1_r \ker(i^r_{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}}) \longrightarrow \lim^1_r \ker(i^{r+1}_{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}}) \twoheadrightarrow \lim^1_r \frac{Z^r_{(p,q)+\mathbf{b}}}{\ker(k^1_{(p,q)+\mathbf{b}})}$$

where $\lim^1$ is the first derived functor of the inverse limit functor. Note that for $i^r_{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}} : D^1_{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}} \to D^1_{(p,q)+\mathbf{b}+\mathbf{c}}$ we have

$$\begin{aligned}\lim_r \ker(i^r_{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}}) &= \ker(Q^{(p,q)+\mathbf{b}+\mathbf{c}} : L^{n+\sigma} \to D^1_{(p,q)+\mathbf{b}+\mathbf{c}}) \\ &= \phi^{(p,q)+\mathbf{b}+\mathbf{c}}\end{aligned}$$

and

$$\lim_r \frac{Z^r_{(p,q)+\mathbf{b}}}{\ker(k^1_{(p,q)+\mathbf{b}})} = \frac{Z^\infty_{(p,q)+\mathbf{b}}}{\ker(k^1_{(p,q)+\mathbf{b}})}.$$

Therefore, the initial segment of the lim-sequence turns into

$$\phi^{(p,q)+\mathbf{b}+\mathbf{c}} \rightarrowtail \phi^{(p,q)+\mathbf{b}+\mathbf{c}+\mathbf{a}} \to \frac{Z^\infty_{(p,q)+\mathbf{b}}}{\ker(k^1_{(p,q)+\mathbf{b}})} \to \lim^1_r \ker(i^r_{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}}).$$

Therefore, part 3 follows.

4. If we take the inverse limit of the short exact sequence of towers

$$\{\ker(I_{u,v})\}_{(u,v)\cdot\hat{\mathbf{a}}=n} \rightarrowtail \{D^1_{u,v}\}_{(u,v)\cdot\hat{\mathbf{a}}=n} \twoheadrightarrow \{\phi_{u,v}\}_{(u,v)\cdot\hat{\mathbf{a}}=n}$$

we obtain the mentioned exact sequence.

5. If we take inverse limits of the short exact sequence of towers

$$\{\phi^{u,v}\}_{(u,v)\cdot\hat{\mathbf{a}}=n} \rightarrowtail L^n \twoheadrightarrow \{\mathrm{im}(Q^{u,v})\}_{(u,v)\cdot\hat{\mathbf{a}}=n}$$

we obtain the exact sequence

$$\lim_{(u,v)\cdot\hat{\mathbf{a}}=n} \phi^{u,v} \rightarrowtail L^n \to \lim_{(u,v)\cdot\hat{\mathbf{a}}=n} \mathrm{im}(Q^{u,v}) \twoheadrightarrow \lim^1_{(u,v)\cdot\hat{\mathbf{a}}=n}\phi^{u,v}.$$

Now by Remark 2.6.7 we have $L^n \cong \lim_{(u,v)\cdot\hat{\mathbf{a}}=n} \mathrm{im}(Q^{u,v})$ and hence

$$\lim_{(u,v)\cdot\hat{\mathbf{a}}=n} \phi^{u,v} = \lim^1_{(u,v)\cdot\hat{\mathbf{a}}=n}\phi^{u,v} = 0.$$

That $\mathrm{colim}_{(u,v)\cdot\hat{\mathbf{a}}=n} F_{u,v} = L_n$ is immediate from the construction of L_n. Now if we apply the colimit functor to the following short exact sequence of directed diagrams

$$\{\phi^{u,v}\}_{(u,v)\cdot\hat{\mathbf{a}}=n} \rightarrowtail L^n \twoheadrightarrow \{\mathrm{im}(Q^{u,v})\}_{(u,v)\cdot\hat{\mathbf{a}}=n}$$

we have the short exact sequence

$$\mathrm{colim}_{(u,v)\cdot\hat{\mathbf{a}}=n}\, \phi^{u,v} \rightarrowtail L^n \twoheadrightarrow \mathrm{colim}_{(u,v)\cdot\hat{\mathbf{a}}=n}\, \mathrm{im}(Q^{u,v})$$

that implies

$$\mathrm{colim}_{(u,v)\cdot\hat{\mathbf{a}}=n}\, \phi^{u,v} = \bigcup_{(u,v)\cdot\hat{\mathbf{a}}=n} \phi^{u,v} = \ker(L^n \to L_n).$$

$\square$

Chapter 3

Matching Convenient Exact Couples to Towers of Modules

3.1 Introduction

We now turn to applications of the machinery of exact couples and spectral sequences. So consider an R-module H, filtered by an ascending or descending sequence of submodules:

$$\cdots \subseteq F_{p-1} \subseteq F_p \subseteq F_{p+1} \subseteq \cdots \subseteq \mathrm{H}, \tag{3.1}$$

where $p \in \mathbb{Z}$. Here is an outline of the procedure, taken from [26], by which we hope to use spectral sequence methods to gain information about adjacent filtration steps and, eventually, about H itself:

1. we assume, inductively, some information about F_p,

2. we assume an exact couple is *matched* to this filtration; i.e., we assume (3.1) is the filtration of an augmentation or coaugmentation of an exact couple,

3. we use the relationship between the E^∞-terms of the spectral sequence and $\dfrac{F_{p+1}}{F_p}$, explained in the E^∞-Distribution Theorem, to obtain information about $\dfrac{F_{p+1}}{F_p}$,

4. and then using the following short exact sequence

$$F_p \rightarrowtail F_{p+1} \twoheadrightarrow \frac{F_{p+1}}{F_p}$$

   we carry the information to F_{p+1}, up to extension.

This inductive argument for climbing up the filtration stages of (3.1) is enabled when the filtration has a starting step which is known; i.e., when the filtration is of the form

$$\cdots = F_{p_0} = F_{p_0} \subseteq F_{p_0+1} \subseteq F_{p_0+2} \subseteq \cdots \subseteq F_{p-1} \subseteq F_p \subseteq F_{p+1} \subseteq \cdots \subseteq \mathrm{H} \tag{3.2}$$

and we have information about F_{p_0}. In this case, the filtration is indexed over ω, the least infinite ordinal. Let us focus on the filtration of type

$$\cdots = 0 = F_0 \subset F_1 \subseteq \cdots \subseteq F_{p-1} \subseteq F_p \subseteq F_{p+1} \subseteq \cdots \subseteq \mathrm{H} \tag{3.3}$$

which is a special case of (3.2). In the second section of this chapter, we introduce an exact couple with the property that the filtration of its universal augmentation is of the form (3.3). This exact couple looks like the following diagram. Here, we are assuming that $\mathrm{H} = L_*$.

$$\begin{tikzcd}
 & 0 \arrow[d, equal] & & 0 \arrow[d, equal] & & 0 \arrow[d, equal] & \\
 & \vdots \arrow[d, equal] & & \vdots \arrow[d, equal] & & \vdots \arrow[d, equal] & \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[d, equal] & 0 \arrow[r] & 0 \arrow[r] \arrow[d, equal] & 0 \arrow[r] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] & 0 \arrow[r] \arrow[d, equal] & E_{-,-} \arrow[r] & D_{-,-} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & D_{-,-} \arrow[r] \arrow[d] & E_{-,-} \arrow[r] & 0 \arrow[r] \arrow[d, equal] & E_{-,-} \arrow[r] & D_{-,-} \arrow[r] \arrow[d] & \cdots \\
 & \vdots \arrow[d] & & \vdots \arrow[d, equal] & & \vdots \arrow[d] & \\
\cdots \arrow[r] & D_{-,-} \arrow[r] \arrow[d] & E_{-,-} \arrow[r] & 0 \arrow[r] \arrow[d] & E_{-,-} \arrow[r] & D_{-,-} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & D_{-,-} \arrow[r] \arrow[d] & E_{-,-} \arrow[r] & D_{-,-} \arrow[r] \arrow[d] & E_{-,-} \arrow[r] & D_{-,-} \arrow[r] \arrow[d] & \cdots \\
 & \vdots \arrow[d] & & \vdots \arrow[d] & & \vdots \arrow[d] & \\
 & L_{n+1} & & L_n & & L_{n-1} &
\end{tikzcd}$$

Many familiar spectral sequences can be induced from this type of exact couples; e.g., Serre spectral sequence, Grothendieck spectral sequence.

We will call it an *originally vanishing* exact couple and will see that we have a close connection between the E^∞-terms of the induced spectral sequence and the quotient of adjacent filtration stages of (3.3); i.e., we have an isomorphism $\dfrac{F_{p+1}}{F_p} \xrightarrow{\cong} E^{\infty}_{\_,\_}$. Even if H is an arbitrary augmentation of an originally vanishing exact couple, we can still argue similarly and obtain information about $\operatorname{colim}_p F_p$. In this case, we have an epimorphism $\dfrac{F_{p+1}}{F_p} \twoheadleftarrow E^{\infty}_{\_,\_}$. We will make the passage from $\operatorname{colim}_p F_p$ to H possible by providing sufficient conditions that are available by arguments not based on spectral sequences or exact couples.

An originally vanishing exact couple is a special case of an exact couple called *originally stable* in which the D^1-columns turn into isomorphisms as we go up. Look at the following diagram.

$$\begin{tikzcd}
 & L^{n+1} \arrow[d, equal] & & L^{n} \arrow[d, equal] & & L^{n-1} \arrow[d, equal] & \\
 & \vdots \arrow[d, equal] & & \vdots \arrow[d, equal] & & \vdots \arrow[d, equal] & \\
\cdots \arrow[r] & D_{_,_} \arrow[r] \arrow[d, equal] & 0 \arrow[r] & D_{_,_} \arrow[r] \arrow[d, equal] & E_{_,_} \arrow[r] & D_{_,_} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & D_{_,_} \arrow[r] \arrow[d] & 0 \arrow[r] & D_{_,_} \arrow[r] \arrow[d, equal] & E_{_,_} \arrow[r] & D_{_,_} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & D_{_,_} \arrow[r] \arrow[d] & E_{_,_} \arrow[r] & D_{_,_} \arrow[r] \arrow[d, equal] & E_{_,_} \arrow[r] & D_{_,_} \arrow[r] \arrow[d] & \cdots \\
 & \vdots \arrow[d] & & \vdots \arrow[d, equal] & & \vdots \arrow[d] & \\
\cdots \arrow[r] & D_{_,_} \arrow[r] \arrow[d] & E_{_,_} \arrow[r] & D_{_,_} \arrow[r] \arrow[d] & E_{_,_} \arrow[r] & D_{_,_} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & D_{_,_} \arrow[r] \arrow[d] & E_{_,_} \arrow[r] & D_{_,_} \arrow[r] \arrow[d] & E_{_,_} \arrow[r] & D_{_,_} \arrow[r] \arrow[d] & \cdots \\
 & \vdots \arrow[d] & & \vdots \arrow[d] & & \vdots \arrow[d] & \\
 & L_{n+1} & & L_{n} & & L_{n-1} &
\end{tikzcd}$$

Such an exact couple allows us to draw conclusions stated above about a filtration of the form (3.2). Here, again, H could be an arbitrary augmentation or coaugmentation.

Then, we introduce an exact couple with the property that the filtration of its (co-)augmentation has the following *descending* form

$$\mathrm{H} = F_0 \supseteq F_1 \supseteq \cdots \supseteq F_{p-1} \supseteq F_p \supseteq F_{p+1} \supseteq \cdots . \tag{3.4}$$

We refer to these exact couples as *convenient* exact couples.

In the third section of this chapter, we generalize this method in [26] to the case where we have bigraded R-modules with *ascending* filtrations indexed over an arbitrary limit ordinal λ. We consider a transfinite tower of bigraded R-modules indexed over a limit ordinal λ

$$\mathrm{H}_1 \to \mathrm{H}_2 \to \cdots \to \mathrm{H}_p \to \mathrm{H}_{p+1} \to \cdots \to \mathrm{H}_\eta \to \mathrm{H}_{\eta+1} \to \cdots \to \mathrm{H}_\lambda = \mathrm{H}, \tag{3.5}$$

the image filtration of H_λ

$$0 \subset F_1 \subseteq \cdots \subseteq F_p \subseteq F_{p+1} \subseteq \cdots \subseteq F_\eta \subseteq F_{\eta+1} \subseteq \cdots \subseteq F_\lambda = \mathrm{H}_\lambda \tag{3.6}$$

and the kernel filtration of H_1

$$0 \subset F^1 \subseteq \cdots \subseteq F^p \subseteq F^{p+1} \subseteq \cdots \subseteq F^\eta \subseteq F^{\eta+1} \subseteq \cdots \subseteq F^\lambda \subseteq \mathrm{H}_1 . \tag{3.7}$$

Let us focus on the filtration (3.6). We use transfinite induction to obtain information about F_λ. Given a property $\mathcal{P}$, when trying to take the inductive step from "ordinals less than η satisfy property $\mathcal{P}$" to "ordinal η satisfies property $\mathcal{P}$", we face one of the following two situations:

Situation 1: η is a non-limit ordinal and, hence, has at most finitely many predecessors. Therefore, there exist a limit ordinal η_0 and a positive integer r such that $\eta = \eta_0 + r$. We assume a convenient exact couple is *matched* to the ω-length segment of the filtration

$$F_{\eta_0} \subseteq F_{\eta_0+1} \subseteq \cdots \subseteq F_{\eta_0+r-1} \subseteq F_{\eta_0+r} \subseteq \cdots \subseteq F_{\eta_0+\omega}. \tag{3.8}$$

That is, we assume there is an originally vanishing or stable exact couple such that H_λ plays the role of its augmentation. Now we can use the same method we used in the ω-indexed scenario to pass from F_{η_0+r-1} to $F_{\eta_0+r} = F_\eta$.

Situation 2: η is a limit ordinal. Here we have a morphism ρ_η : $\operatorname{colim}_{\beta<\eta} \mathrm{H}_\beta \to \mathrm{H}_\eta$ and hence an inclusion $\operatorname{colim}_{\beta<\eta} F_\beta \rightarrowtail F_\eta$. We will make the passage from $\operatorname{colim}_{\beta<\eta} F_\beta$ to F_η possible by putting assumptions on $\mathrm{coker}(\rho_\eta)$.

Look at Remark 3.3.4 on page 81 for more explanation and the proof of Proposition 5.8.1 on page 141 or Corollary 5.8.3 on page 143 for an example of such argument.

At the end, we explain the same inductive argument for a *descending* filtration indexed over an arbitrary limit ordinal.

We need to be more specific about the type of “information” we would like to carry through the filtration stages. Here is one example: Let $\mathcal{C}$ be a class of R-modules with the following properties:

1. $\mathcal{C}$ is closed under isomorphism; i.e., if $A \cong B$ then $A \in \mathcal{C}$ if and only if $B \in \mathcal{C}$,
2. $\mathcal{C}$ is closed under subobject; i.e., if $A \rightarrowtail B$ and $B \in \mathcal{C}$, then $A \in \mathcal{C}$,
3. $\mathcal{C}$ is closed under quotient; i.e., if $B \twoheadrightarrow C$ and $B \in \mathcal{C}$, then $C \in \mathcal{C}$,
4. $\mathcal{C}$ is closed under extension; i.e., if in a short exact sequence $A \rightarrowtail B \twoheadrightarrow C$ we have $A \in \mathcal{C}$ and $C \in \mathcal{C}$, then $B \in \mathcal{C}$,
5. $\mathcal{C}$ is closed under colimit of λ-length directed towers; i.e., if $\{A_\eta\}_{\eta<\lambda}$ is a directed tower of R-modules in which $A_\eta \in \mathcal{C}$ for every $\eta < \lambda$, then $\mathrm{colim}_{\eta<\lambda} A_\eta \in \mathcal{C}$.

We are interested in “staying in $\mathcal{C}$” as we climb up the filtration (3.6). This is a generalization of the idea of Serre, [29], which will be explained in Chapter 4.

To see another example, assume we have a morphism of filtered objects, where the filtrations are over a potentially transfinite limit ordinal λ. In other words, assume we have two filtered objects $\mathrm{H}(1)$ and $\mathrm{H}(2)$ with potentially transfinite filtrations

$$F_0(1) \subseteq \cdots \subseteq F_p(1) \subseteq \cdots \subseteq F_\eta(1) \subseteq F_{\eta+1}(1) \subseteq \cdots \subseteq F_\lambda(1) = \mathrm{H}(1)$$

and

$$F_0(2) \subseteq \cdots \subseteq F_p(2) \subseteq \cdots \subseteq F_\eta(2) \subseteq F_{\eta+1}(2) \subseteq \cdots \subseteq F_\lambda(2) = \mathrm{H}(2)$$

for an arbitrary limit ordinal λ, such that for every $\eta \leq \lambda$, there is a morphism from $F_\eta(1)$ to $F_\eta(2)$ and for every $\eta < \lambda$, the morphism from $F_\eta(1)$ to $F_\eta(2)$ has certain properties; e.g., it is an (epi-, mono-)isomorphism, its (co-)kernel has a certain property, etc. We are interested in carrying this type of information also to the morphism from H(1) to H(2). This leads to the "comparison theorems" that will be covered in Chapter 5.

3.2 Convenient Exact Couples

In this section, we investigate the properties of special exact couples with the property that their (universal) augmentation and coaugmentation are induction-friendly; i.e., there is a starting point for the *ascending* or *descending* filtrations of their (universal) augmentation and coaugmentation.

3.2.1 Originally Stable or Vanishing Exact Couples

An originally stable exact couple is an exact couple in which every D^1-column consists of isomorphisms from the universal coaugmentation down until some D^1-term. An originally vanishing exact couple is an originally stable exact couple with 0 universal coaugmentation.

Definition 3.2.1.

- *An* originally stable exact couple *is an exact couple $(D^1_{*,*}, E^1_{*,*})$ in which for every $n \in \mathbb{Z}$ there exists some $(u,v) \in \mathbb{Z} \times \mathbb{Z}$, where $(u,v) \cdot \hat{\mathbf{a}} = n$, such that*

$$\forall\ t > 0 \quad i^t_{u,v-t\mathbf{a}} : D^1_{u,v-t\mathbf{a}} \longrightarrow D^1_{u,v}$$

is an isomorphism. In this situation, we say that $D^1_{u,v-t\mathbf{a}}$ is in the originally stable range *of the n-th D^1-column.*

- *An* originally vanishing exact couple *is an exact couple $(D^1_{*,*}, E^1_{*,*})$ in which for every $n \in \mathbb{Z}$ there exists some $(u,v) \in \mathbb{Z} \times \mathbb{Z}$, where $(u,v) \cdot \hat{\mathbf{a}} = n$, such that*

$$\forall\ t \geq 0 \quad D^1_{u,v-t\mathbf{a}} = 0.$$

There is a list of examples at the end of this section.

Remark 3.2.2. For an originally stable exact couple the following hold:

1. In the stable scenario depicted below, the E^1-term vanishes:

$$\begin{tikzcd}
D^1_{u,v+(t-1)\mathbf{a}} \arrow[d, "\cong"'] & & \vdots \arrow[d, "\cong"'] \\
D^1_{u,v+t\mathbf{a}} \arrow[r] \arrow[d] & E^1_{u,v+\mathbf{b}+t\mathbf{a}} \arrow[r] & D^1_{u,v+\mathbf{b}+\mathbf{c}+t\mathbf{a}} \arrow[d, "\cong"'] \\
\vdots & & D^1_{u,v+\mathbf{b}+\mathbf{c}+(t+1)\mathbf{a}}
\end{tikzcd}$$

2. Assume the n-th D^1-column stabilizes from $D^1_{(p,q)}$ up. Therefore, L^n, the universal coaugmentation in degree n, is isomorphic to $D^1_{(p,q)}$ and hence the kernel filtration of $\ker(L^n \to L_n)$ is of the form

$$0 \subseteq \phi^{(p,q)+\mathbf{a}} \subseteq \cdots \subseteq \phi^{(p,q)+r\mathbf{a}} \subseteq \phi^{(p,q)+(r+1)\mathbf{a}} \subseteq \cdots \subseteq \ker(L^n \to L_n).$$

The universal augmentation L_n has image filtration

$$\cdots = \phi_{(p,q)} \subseteq \phi_{(p,q)+\mathbf{a}} \subseteq \cdots \subseteq \phi_{(p,q)+r\mathbf{a}} \subseteq \phi_{(p,q)+(r+1)\mathbf{a}} \subseteq \cdots \subseteq L_n.$$

Since every D^1-column stabilizes, we have

$$\lim^1_r \ker(i^r_{-,-}) = 0.$$

Therefore, in an originally stable exact couple, the three major players $\epsilon_{-,-}$, $\epsilon^{-,-}$ and E^∞ form a short exact sequence

$$\epsilon_{-,-} \rightarrowtail E^\infty \twoheadrightarrow \epsilon^{-,-};$$

i.e., the E^∞-Distribution Diagram turns into

$$
\begin{tikzcd}
 & & & & 0 \arrow[d, tail] \\
\bigcap_r \phi_{(p,q)-r\mathbf{a}} \arrow[d, equal] & & & & \vdots \arrow[d, tail] \\
\vdots \arrow[d, equal] & & & & \phi^{(p,q)+\mathbf{b}+\mathbf{c}} \arrow[d, tail] \\
\phi_{(p,q)} \arrow[r, two heads] \arrow[d, tail] & \epsilon_{(p,q)} = 0 \arrow[r, tail, "0"] & E^\infty_{(p,q)+\mathbf{b}} \arrow[r, two heads, "\cong"] & \epsilon^{(p,q)+\mathbf{b}+\mathbf{c}} & \phi^{(p,q)+\mathbf{b}+\mathbf{c}+\mathbf{a}} \arrow[l, two heads] \arrow[d, tail] \\
\phi_{(p,q)+\mathbf{a}} \arrow[r, two heads] \arrow[d, tail] & \epsilon_{(p,q)+\mathbf{a}} \arrow[r, tail] & E^\infty_{(p,q)+\mathbf{b}+\mathbf{a}} \arrow[r, two heads] & \epsilon^{(p,q)+\mathbf{b}+\mathbf{c}+\mathbf{a}} & \phi^{(p,q)+\mathbf{b}+\mathbf{c}+2\mathbf{a}} \arrow[l, two heads] \arrow[d, tail] \\
\vdots \arrow[d, tail] & & & & \vdots \arrow[d, tail] \\
\phi_{(p,q)+(r-1)\mathbf{a}} \arrow[r, two heads] \arrow[d, tail] & \epsilon_{(p,q)+(r-1)\mathbf{a}} \arrow[r, tail] & E^\infty_{(p,q)+\mathbf{b}+(r-1)\mathbf{a}} \arrow[r, two heads] & \epsilon^{(p,q)+\mathbf{b}+\mathbf{c}+(r-1)\mathbf{a}} & \phi^{(p,q)+\mathbf{b}+\mathbf{c}+r\mathbf{a}} \arrow[l, two heads] \arrow[d, tail] \\
\phi_{(p,q)+r\mathbf{a}} \arrow[r, two heads] \arrow[d, tail] & \epsilon_{(p,q)+r\mathbf{a}} \arrow[r, tail] & E^\infty_{(p,q)+\mathbf{b}+r\mathbf{a}} \arrow[r, two heads] & \epsilon^{(p,q)+\mathbf{b}+\mathbf{c}+r\mathbf{a}} & \phi^{(p,q)+\mathbf{b}+\mathbf{c}+(r+1)\mathbf{a}} \arrow[l, two heads] \arrow[d, tail] \\
\vdots \arrow[d, tail] & & & & \vdots \arrow[d, tail] \\
\bigcup_r \phi_{(p,q)+r\mathbf{a}} \arrow[d, equal] & & & & \bigcup_r \phi^{(p,q)+\mathbf{b}+\mathbf{c}+r\mathbf{a}} \arrow[d, equal] \\
L_n & & & & \ker(L^{n+\sigma} \to L_{n+\sigma})
\end{tikzcd}
$$

E^∞-Distribution Diagram of an Originally Stable Exact Couple

Note that, in general, E^∞ may be non purely distributed over kernel and image filtration stages of the universal (co-)augmentations of the exact couple: See Remark 2.4.9 on page 43 and Definition 2.4.10 on page 43.

3. For every originally vanishing exact couple, we always have $\epsilon^{-,-} = 0$ and hence $\epsilon_{-,-} \cong E^\infty$. Therefore, every originally vanishing exact couple converges to its universal augmentation. If the n-th D^1-column vanishes from $D^1_{(p,q)-\mathbf{a}}$ up, the E^∞-Distribution Diagram looks like the following diagram:

$$\begin{tikzcd}
\bigcap_r \phi_{(p,q)-r\mathbf{a}} = 0 \arrow[d] & & \\
\phi_{(p,q)} \arrow[r, equal] \arrow[d] & \epsilon_{(p,q)} \arrow[r, "\cong"] & E^{\infty}_{(p,q)+\mathbf{b}} \\
\phi_{(p,q)+\mathbf{a}} \arrow[r, two heads] \arrow[d] & \epsilon_{(p,q)+\mathbf{a}} \arrow[r, "\cong"] & E^{\infty}_{(p,q)+\mathbf{b}+\mathbf{a}} \\
\vdots \arrow[d] & & \\
\phi_{(p,q)+(r-1)\mathbf{a}} \arrow[r, two heads] \arrow[d] & \epsilon_{(p,q)+(r-1)\mathbf{a}} \arrow[r, "\cong"] & E^{\infty}_{(p,q)+\mathbf{b}+(r-1)\mathbf{a}} \\
\phi_{(p,q)+r\mathbf{a}} \arrow[r, two heads] \arrow[d] & \epsilon_{(p,q)+r\mathbf{a}} \arrow[r, "\cong"] & E^{\infty}_{(p,q)+\mathbf{b}+r\mathbf{a}} \\
\vdots \arrow[d] & & \\
\bigcup_r \phi_{(p,q)+r\mathbf{a}} \arrow[d, equal] & & \\
L_n & &
\end{tikzcd}$$

E^{∞}-Distribution Diagram of an Originally Vanishing Exact Couple

3.2.2 Examples

Example 3.2.3. (**Tower of Cofibers**) Assume, for $r \geq 1$, X_r and F_r are path connected topological spaces and consider the following tower of cofibers

$$\begin{tikzcd}
X_1 \arrow[r] & X_2 \arrow[r] & X_3 \arrow[r] & \cdots \arrow[r] & X_r \arrow[r] & X_{r+1} \arrow[r] & \cdots \\
F_1 \arrow[u] & F_2 \arrow[u] & F_3 \arrow[u] & & F_r \arrow[u] & F_{r+1} \arrow[u] &
\end{tikzcd}$$

where every $F_r \to X_r$ is a map with homotopy cofiber X_{r+1} and $X := \mathrm{hocolim}_r\, X_r$. We can extend it to get

$$\begin{tikzcd}
\cdots \arrow[r, equal] & X_1 \arrow[r, equal] & X_1 \arrow[r, equal] & X_1 \arrow[r] & X_2 \arrow[r] & X_3 \arrow[r] & \cdots \arrow[r] & X_r \arrow[r] & X_{r+1} \arrow[r] & \cdots \\
& \{pt\} \arrow[u] & \{pt\} \arrow[u] & F_1 \arrow[u] & F_2 \arrow[u] & F_3 \arrow[u] & & F_r \arrow[u] & F_{r+1} \arrow[u] &
\end{tikzcd}$$

For any choice of a (generalized) additive homology theory h, where $\mathrm{h}_q(\{pt\}) = 0$ for $q < 0$, each cofibration $F_r \to X_r \to X_{r+1}$ induces a long exact sequence of homology groups

$$\cdots \to \mathrm{h}_k(F_r) \to \mathrm{h}_k(X_r) \to \mathrm{h}_k(X_{r+1}) \to \mathrm{h}_{k-1}(F_r) \to \cdots .$$

If we put

$$D^1_{(p,q)} = \mathrm{h}_{p+q}(X_q) \quad \text{and} \quad E^1_{(p,q)} = \mathrm{h}_{p+q}(F_q)$$

then we obtain an originally stable exact couple with homomorphisms of bidegrees $\mathbf{a} = (-1, 1)$, $\mathbf{b} = (0, -1)$ and $\mathbf{c} = (0, 0)$ and also $\sigma = (\mathbf{b}+\mathbf{c})\cdot\hat{\mathbf{a}} = (0, -1)\cdot(-1, -1) = 1$. The differential on page r is of bidegree

$$\mathbf{b} + \mathbf{c} - (r-1)\mathbf{a} = (0, -1) - (r-1)(-1, 1) = (r-1, -r).$$

Since additive homology theories commute with directed colimits, therefore we have $\mathrm{colim}_r\, \mathrm{h}_*(X_r) \cong \mathrm{h}_*(X)$: See [31].

$$\begin{tikzcd}
 & & \vdots \arrow[d, "\cong"'] & & \vdots \arrow[d, "\cong"'] & & \\
\cdots \arrow[r] & 0 \arrow[r] & \mathrm{h}_{k+1}(X_1) \arrow[r] \arrow[d, "\cong"'] & 0 \arrow[r] & \mathrm{h}_k(X_1) \arrow[r] \arrow[d, "\cong"'] & 0 \arrow[r] & \cdots \\
\cdots \arrow[r] & \mathrm{h}_{k+1}(F_1) \arrow[r] & \mathrm{h}_{k+1}(X_1) \arrow[r] \arrow[d, "i"] & 0 \arrow[r] & \mathrm{h}_k(X_1) \arrow[r] \arrow[d, "\cong"'] & 0 \arrow[r] & \cdots \\
\cdots \arrow[r] & \mathrm{h}_{k+1}(F_2) \arrow[r] & \mathrm{h}_{k+1}(X_2) \arrow[r, "j"] \arrow[d] & \mathrm{h}_k(F_1) \arrow[r, "k"] & \mathrm{h}_k(X_1) \arrow[r] \arrow[d] & 0 \arrow[r] & \cdots \\
\cdots \arrow[r] & \mathrm{h}_{k+1}(F_3) \arrow[r] & \mathrm{h}_{k+1}(X_3) \arrow[r] \arrow[d] & \mathrm{h}_k(F_2) \arrow[r] & \mathrm{h}_k(X_2) \arrow[r] \arrow[d] & \mathrm{h}_{k-1}(F_1) \arrow[r] & \cdots \\
 & & \vdots \arrow[d] & & \vdots \arrow[d] & & \\
 & & \mathrm{h}_{k+1}(X) & & \mathrm{h}_k(X) & &
\end{tikzcd}$$

Exact Couple of a Tower of Cofibers

Example 3.2.4. (**Tower of Cofibrations**) Assume, for $r \geq 0$, X_r and F_r are path connected topological spaces and consider the following tower of cofibrations

$$\begin{tikzcd}
X_0 \arrow[r] & X_1 \arrow[r] \arrow[d] & X_2 \arrow[r] \arrow[d] & \cdots \arrow[r] & X_r \arrow[r] \arrow[d] & X_{r+1} \arrow[r] \arrow[d] & \cdots \\
 & F_1 & F_2 & & F_r & F_{r+1} &
\end{tikzcd}$$

where F_{r+1} is the cofiber of the cofibration $X_r \to X_{r+1}$. Since the tower consists of cofibrations we have $X := \operatorname{hocolim}_r X_r$. We can extend this tower to get

$$\begin{tikzcd} \cdots \arrow[r] & \{pt\} \arrow[r] \arrow[d] & \{pt\} \arrow[r] \arrow[d] & X_0 \arrow[r] \arrow[d, equal] & X_1 \arrow[r] \arrow[d] & X_2 \arrow[r] \arrow[d] & \cdots \arrow[r] & X_r \arrow[r] \arrow[d] & X_{r+1} \arrow[r] \arrow[d] & \cdots . \\ & \{pt\} & \{pt\} & F_0 & F_1 & F_2 & & F_r & F_{r+1} & \end{tikzcd}$$

For any choice of a (generalized) homology theory h, where $\mathrm{h}_q(\{pt\}) = 0$ for $q < 0$, each cofibration $X_r \to X_{r+1} \to F_{r+1}$ induces a long exact sequence of homology groups

$$\cdots \to \mathrm{h}_k(X_r) \to \mathrm{h}_k(X_{r+1}) \to \mathrm{h}_k(F_{r+1}) \to \mathrm{h}_{k-1}(X_r) \to \cdots .$$

If we put

$$D^1_{(p,q)} = \mathrm{h}_{p+q}(X_q) \quad , \quad E^1_{(p,q)} = \mathrm{h}_{p+q}(F_q)$$

then we obtain an originally vanishing exact couple with homomorphisms of bidegrees $\mathbf{a} = (-1, 1)$, $\mathbf{b} = (0, 0)$ and $\mathbf{c} = (0, -1)$ and also $\sigma = (\mathbf{b} + \mathbf{c}) \cdot \hat{\mathbf{a}} = (0, -1) \cdot (-1, -1) = 1$. The differential on page r is of bidegree

$$\mathbf{b} + \mathbf{c} - (r - 1)\mathbf{a} = (r - 1, -r).$$

By part 3 of Remark 3.2.2 on page 64, this spectral sequence is convergent to $\mathrm{h}_*(X)$.

$$
\begin{tikzcd}
 & & \vdots \arrow[d] & & \vdots \arrow[d] & & \vdots \arrow[d] & \\
\cdots \arrow[r, "\cong"] & \mathrm{h}_{k+1}(F_0) \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathrm{h}_{k+1}(F_1) \arrow[r] & \mathrm{h}_k(X_0) \arrow[r, "\cong"] \arrow[d, "i"] & \mathrm{h}_k(F_0) \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathrm{h}_{k+1}(F_2) \arrow[r] & \mathrm{h}_k(X_1) \arrow[r, "j"] \arrow[d] & \mathrm{h}_k(F_1) \arrow[r, "k"] & \mathrm{h}_{k-1}(X_0) \arrow[r, "\cong"] \arrow[d] & \mathrm{h}_{k-1}(F_0) \arrow[r] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathrm{h}_{k+1}(F_3) \arrow[r] & \mathrm{h}_k(X_2) \arrow[r] \arrow[d] & \mathrm{h}_k(F_2) \arrow[r] & \mathrm{h}_{k-1}(X_1) \arrow[r] \arrow[d] & \mathrm{h}_{k-1}(F_1) \arrow[r] & \mathrm{h}_{k-2}(X_0) \arrow[r, "\cong"] \arrow[d] & \cdots \\
 & & \vdots \arrow[d] & & \vdots \arrow[d] & & \vdots \arrow[d] & \\
\cdots \arrow[r] & \mathrm{h}_{k+1}(F_q) \arrow[r] & \mathrm{h}_k(X_{q-1}) \arrow[r] \arrow[d] & \mathrm{h}_k(F_{q-1}) \arrow[r] & \mathrm{h}_{k-1}(X_{q-2}) \arrow[r] \arrow[d] & \mathrm{h}_{k-1}(F_{q-2}) \arrow[r] & \mathrm{h}_{k-2}(X_{q-3}) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathrm{h}_{k+1}(F_{q+1}) \arrow[r] & \mathrm{h}_k(X_q) \arrow[r] \arrow[d] & \mathrm{h}_k(F_q) \arrow[r] & \mathrm{h}_{k-1}(X_{q-1}) \arrow[r] \arrow[d] & \mathrm{h}_{k-1}(F_{q-1}) \arrow[r] & \mathrm{h}_{k-2}(X_{q-2}) \arrow[r] \arrow[d] & \cdots \\
 & & \vdots \arrow[d] & & \vdots \arrow[d] & & \vdots \arrow[d] & \\
 & & \mathrm{h}_k(X) & & \mathrm{h}_{k-1}(X) & & \mathrm{h}_{k-2}(X) &
\end{tikzcd}
$$

Exact Couple of a Tower of Cofibrations

This example can be generalized by taking F_{r+1} as the homotopy cofiber of an arbitrary map $X_r \to X_{r+1}$.

Example 3.2.5. (**Leray-Serre Spectral Sequence**) Look at [31] for the terminology: Let h_* be an additive homology theory satisfying the weak homotopy equivalence axiom and $\pi : E \to B$ be a homotopy fibration, with fiber F, such that the action of $\pi_1(B, b_0)$ on $\mathrm{h}_*(F)$ is trivial for all $b_0 \in B$ and B is a 0-connected CW-complex with skeleta B^p. If we define

$$E^1_{(p,q)} = \mathrm{h}_{p+q}(E^p, E^{p-1}) \quad \text{and} \quad D^1_{(p,q)} = \mathrm{h}_{p+q}(E^p),$$

where $E^p = \pi^{-1}(B^p)$, then we obtain an originally vanishing exact couple, shown below, with bidegrees $\mathbf{a} = (1, -1)$, $\mathbf{b} = (0, 0)$ and $\mathbf{c} = (-1, 0)$. Also, the differential in the r-th page of the induced spectral sequence is of bidegree

$$\mathbf{b} + \mathbf{c} - (r-1)\mathbf{a} = (-1, 0) - (r-1)(1, -1) = (-r, r-1).$$

We also have $E^2_{(p,q)} = \mathrm{H}_p(B; \mathrm{h}_q(F))$: See [31], p. 350. By part 3 of Remark 3.2.2 on page 64, this spectral sequence is convergent to $\mathrm{h}_*(E)$.

$$
\begin{tikzcd}[column sep=small]
 & & \vdots \arrow[d] & & \vdots \arrow[d] & & \vdots \arrow[d] & \\
\cdots \arrow[r] & \mathrm{h}_{k+1}(E^1,E^0) \arrow[r] & \mathrm{h}_k(E^0) \arrow[r, "\cong"] \arrow[d] & \mathrm{h}_k(E^0,\emptyset) \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathrm{h}_{k+1}(E^2,E^1) \arrow[r] & \mathrm{h}_k(E^1) \arrow[r] \arrow[d, "i"] & \mathrm{h}_k(E^1,E^0) \arrow[r] & \mathrm{h}_{k-1}(E^0) \arrow[r, "\cong"] \arrow[d] & \mathrm{h}_{k-1}(E^0,\emptyset) \arrow[r] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathrm{h}_{k+1}(E^3,E^2) \arrow[r] & \mathrm{h}_k(E^2) \arrow[r, "j"] \arrow[d] & \mathrm{h}_k(E^2,E^1) \arrow[r, "k"] & \mathrm{h}_{k-1}(E^1) \arrow[r] \arrow[d] & \mathrm{h}_{k-1}(E^1,E^0) \arrow[r] & \mathrm{h}_{k-2}(E^0) \arrow[r, "\cong"] \arrow[d] & \cdots \\
 & & \vdots \arrow[d] & & \vdots \arrow[d] & & \vdots \arrow[d] & \\
\cdots \arrow[r] & \mathrm{h}_{k+1}(E^{q+1},E^q) \arrow[r] & \mathrm{h}_k(E^q) \arrow[r] \arrow[d] & \mathrm{h}_k(E^q,E^{q-1}) \arrow[r] & \mathrm{h}_{k-1}(E^{q-1}) \arrow[r] \arrow[d] & \mathrm{h}_{k-1}(E^{q-1},E^{q-2}) \arrow[r] & \mathrm{h}_{k-2}(E^{q-2}) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathrm{h}_{k+1}(E^{q+2},E^{q+1}) \arrow[r] & \mathrm{h}_k(E^{q+1}) \arrow[r] \arrow[d] & \mathrm{h}_k(E^{q+1},E^q) \arrow[r] & \mathrm{h}_{k-1}(E^q) \arrow[r] \arrow[d] & \mathrm{h}_{k-1}(E^q,E^{q-1}) \arrow[r] & \mathrm{h}_{k-2}(E^{q-1}) \arrow[r] \arrow[d] & \cdots \\
 & & \vdots \arrow[d] & & \vdots \arrow[d] & & \vdots \arrow[d] & \\
 & & \mathrm{h}_k(E) & & \mathrm{h}_{k-1}(E) & & \mathrm{h}_{k-2}(E) &
\end{tikzcd}
$$

Exact Couple Inducing Leray-Serre Spectral Sequence

Example 3.2.6. (**Atiyah-Hirzebruch-Whitehead Spectral Sequence**) If, in the previous example, we take $\pi = id : X \to X$, where X is a 0-connected CW-complex, and h_* is an additive homology theory, then we obtain the Atiyah-Hirzebruch-Whitehead spectral sequence induced by the originally vanishing exact couple in which

$$E^1_{(p,q)} = \mathrm{h}_{p+q}(X^p, X^{p-1}) \quad \text{and} \quad D^1_{(p,q)} = \mathrm{h}_{p+q}(X^p),$$

where X^p is the p-skeleton of X. We also have $E^2_{(p,q)} = \mathrm{H}_p(X; \mathrm{h}_q(pt))$. Again, by part 3 of Remark 3.2.2 on page 64, this spectral sequence is convergent to $\mathrm{h}_*(X)$.

$$
\begin{tikzcd}
 & & \vdots \arrow[d] & & \vdots \arrow[d] & & \vdots \arrow[d] & \\
\cdots \arrow[r, "\cong"] & h_{k+1}(X^0,\emptyset) \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & h_{k+1}(X^1,X^0) \arrow[r] & h_k(X^0) \arrow[r, "\cong"] \arrow[d, "i"] & h_k(X^0,\emptyset) \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & h_{k+1}(X^2,X^1) \arrow[r] & h_k(X^1) \arrow[r, "j"] \arrow[d] & h_k(X^1,X^0) \arrow[r, "k"] & h_{k-1}(X^0) \arrow[r, "\cong"] \arrow[d] & h_{k-1}(X^0,\emptyset) \arrow[r] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & h_{k+1}(X^3,X^2) \arrow[r] & h_k(X^2) \arrow[r] \arrow[d] & h_k(X^2,X^1) \arrow[r] & h_{k-1}(X^1) \arrow[r] \arrow[d] & h_{k-1}(X^1,X^0) \arrow[r] & h_{k-2}(X^0) \arrow[r, "\cong"] \arrow[d] & \cdots \\
 & & \vdots \arrow[d] & & \vdots \arrow[d] & & \vdots \arrow[d] & \\
 & & h_k(X) & & h_{k-1}(X) & & h_{k-2}(X) &
\end{tikzcd}
$$

Exact Couple Inducing Atiyah-Hirzebruch-Whitehead Spectral Sequence

3.2.3 Eventually Stable or Vanishing Exact Couples

An eventually stable exact couple is an exact couple in which every D^1-column consists of isomorphisms from some D^1-term down to the universal augmentation. An eventually vanishing exact couple is an eventually stable exact couple that stabilizes at 0.

Definition 3.2.7. *1. An* eventually stable exact couple $(D^1_{*,*}, E^1_{*,*})$ *is an exact couple in which for every $n \in \mathbb{Z}$ there exists some $(u,v) \in \mathbb{Z} \times \mathbb{Z}$, where $(u,v) \cdot \hat{\mathbf{a}} = n$, such that*

$$\forall\ t > 0 \ \ i^t_{u,v} : D^1_{u,v} \longrightarrow D^1_{u,v+t\mathbf{a}}$$

is an isomorphism. We say that $D^1_{u,v+t\mathbf{a}}$ is in the eventually stable range *of the n-th D^1-column.*

2. An eventually vanishing exact couple *is an exact couple $(D^1_{*,*}, E^1_{*,*})$ in which for every $n \in \mathbb{Z}$ there exists some $(u,v) \in \mathbb{Z} \times \mathbb{Z}$, where $(u,v) \cdot \hat{\mathbf{a}} = n$, such that*

$$\forall\ t \geq 0 \ \ D^1_{u,v+t\mathbf{a}} = 0.$$

There is a list of examples at the end of this section.

Remark 3.2.8. For an eventually stable exact couple the following hold:

1. In the stable scenario depicted below, the E^1-term vanishes:

$$\begin{tikzcd}
D^1_{u,v+(t-1)\mathbf{a}} \arrow[d, "\cong"'] & & \vdots \arrow[d] \\
D^1_{u,v+t\mathbf{a}} \arrow[r] \arrow[d, "\cong"'] & E^1_{u,v+\mathbf{b}+t\mathbf{a}} \arrow[r] & D^1_{u,v+\mathbf{b}+\mathbf{c}+t\mathbf{a}} \arrow[d, "\cong"] \\
D^1_{u,v+(t+1)\mathbf{a}} & & D^1_{u,v+\mathbf{b}+\mathbf{c}+(t+1)\mathbf{a}}
\end{tikzcd}$$

2. Assume the n-th D^1-column stabilizes from $D^1_{(p,q)}$ down. Then L_n, the universal augmentation in degree n, is isomorphic to $D^1_{(p,q)}$ and hence the image filtration of L_n is of the form

$$\cap_r \phi_{(p,q)-r\mathbf{a}} \subseteq \cdots \subseteq \phi_{(p,q)-(r+1)\mathbf{a}} \subseteq \phi_{(p,q)-r\mathbf{a}} \subseteq \cdots \subseteq \phi_{(p,q)-2\mathbf{a}} \subseteq \phi_{(p,q)-\mathbf{a}} \subseteq \phi_{(p,q)} = L_n.$$

   The kernel filtration of $\ker(L^n \to L_n)$ is of the form

$$0 \subseteq \cdots \subseteq \phi^{(p,q)-(r+1)\mathbf{a}} \subseteq \phi^{(p,q)-r\mathbf{a}} \subseteq \cdots \subseteq \phi^{(p,q)-2\mathbf{a}} \subseteq \phi^{(p,q)-\mathbf{a}} \subseteq \phi^{(p,q)} = \ker(L^n \to L_n).$$

   As we have mentioned at the introduction to this chapter, we are interested in carrying some information through the filtration stages and to enable this induction we need a *known* starting point. Here, for both objects L_* and L^*, the first steps of the filtrations are the filtered objects themselves! So we cannot think of them as the first known steps. However, if we "turn the table" we can start from the known step 1, as it is explained in the diagram on page 196, where we assume that the $(n+\sigma)$-th D^1-column, the column on the right of the n-th D^1-column, stabilizes at $D^1_{(p,q)+s\mathbf{a}+\mathbf{b}+\mathbf{c}}$; i.e., it stabilizes s steps lower or higher than the n-th D^1-column depending on s being positive or negative, respectively.

3. For every eventually vanishing exact couple, we have $\epsilon_{-,-} = 0$ and hence it is coaugmentation concentrated: See Definition 2.4.13 on page 45. It converges to its universal coaugmentation if and only if $f : E^\infty_{-,-} \to \lim^1_t \ker(i^t_{-,-})$ vanishes. The E^∞-Distribution Diagram turns into the diagram on page 197.

3.2.4 Examples

Example 3.2.9. (**Tower of Fibers**) Assume, for $r \geq 0$, X_r and F_r are path connected topological spaces with Abelian fundamental groups and consider the following tower of fibers

$$\begin{tikzcd}
X_1 \arrow[d] & X_2 \arrow[l] \arrow[d] & X_3 \arrow[l] \arrow[d] & \cdots \arrow[l] & X_n \arrow[l] \arrow[d] & X_{n+1} \arrow[l] \arrow[d] & \cdots \arrow[l] \\
F_1 & F_2 & F_3 & \cdots & F_n & F_{n+1} &
\end{tikzcd}$$

where each $X_n \to F_n$ is a homotopy fibration with fiber X_{n+1}. We can extend it to get a long tower of fibers

$$\begin{tikzcd}
\cdots \arrow[r, equal] & X_{-1} \arrow[r, equal] \arrow[d] & X_0 \arrow[r, equal] \arrow[d] & X_1 \arrow[d] & X_2 \arrow[l] \arrow[d] & X_3 \arrow[l] \arrow[d] & \cdots \arrow[l] & X_n \arrow[l] \arrow[d] & X_{n+1} \arrow[l] \arrow[d] & \cdots \arrow[l] \\
\cdots & \{pt\} & \{pt\} & F_1 & F_2 & F_3 & \cdots & F_n & F_{n+1} &
\end{tikzcd}$$

Every fibration in the tower yields the following long exact sequence of homotopy groups

$$\cdots \to \pi_{n+1}(F_r) \to \pi_n(X_{r+1}) \to \pi_n(X_r) \to \pi_n(F_r) \to \cdots .$$

These exact sequences give rise to an eventually stable exact couple and hence a homotopy spectral sequence where

$$D^1_{p,q} = \pi_{p+q}(X_q) \quad \text{and} \quad E^1_{(p,q)} = \pi_{p+q}(F_q)$$

with bidegrees $\mathbf{a} = (1, -1)$, $\mathbf{b} = (0, 0)$, $\mathbf{c} = (-2, 1)$, and $\sigma = (\mathbf{b} + \mathbf{c}) \cdot \hat{\mathbf{a}} = (-2, 1) \cdot (1, 1) = -1$.

$$\begin{tikzcd}
 & \vdots \arrow[d] & & \vdots \arrow[d] & & \vdots \arrow[d] & \\
\cdots \arrow[r] & \pi_{p+q}(X_q) \arrow[r] \arrow[d, "f^q_*"] & \pi_{p+q}(F_q) \arrow[r] & \pi_{p+q-1}(X_{q+1}) \arrow[r] \arrow[d] & \pi_{p+q-1}(F_{q+1}) \arrow[r] & \pi_{p+q-2}(X_{q+2}) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \pi_{p+q}(X_{q-1}) \arrow[r, "g^q_*"] \arrow[d] & \pi_{p+q}(F_{q-1}) \arrow[r, "\partial"] & \pi_{p+q-1}(X_q) \arrow[r] \arrow[d] & \pi_{p+q-1}(F_q) \arrow[r] & \pi_{p+q-2}(X_{q+1}) \arrow[r] \arrow[d] & \cdots \\
 & \vdots \arrow[d] & \vdots & \vdots \arrow[d] & \vdots & \vdots \arrow[d] & \\
\cdots \arrow[r] & \pi_{p+q}(X_1) \arrow[r] \arrow[d, "\cong"] & \pi_{p+q}(F_1) \arrow[r] & \pi_{p+q-1}(X_2) \arrow[r] \arrow[d] & \pi_{p+q-1}(F_2) \arrow[r] & \pi_{p+q-2}(X_3) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \pi_{p+q}(X_0) \arrow[r] \arrow[d, "\cong"] & 0 \arrow[r] & \pi_{p+q-1}(X_1) \arrow[r] \arrow[d, "\cong"] & \pi_{p+q-1}(F_1) \arrow[r] & \pi_{p+q-2}(X_2) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \pi_{p+q}(X_{-1}) \arrow[r] \arrow[d, "\cong"] & 0 \arrow[r] & \pi_{p+q-1}(X_0) \arrow[r] \arrow[d, "\cong"] & 0 \arrow[r] & \pi_{p+q-2}(X_1) \arrow[r] \arrow[d, "\cong"] & \cdots \\
 & \vdots & & \vdots & & \vdots &
\end{tikzcd}$$

Also, the differential in the r-th page is of bidegree

$$\mathbf{b} + \mathbf{c} - (r-1)\mathbf{a} = (-2, 1) - (r-1)(1, -1) = (-r-1, r).$$

Note that the induced spectral sequence is not necessarily convergent.

Example 3.2.10. (**Tower of Homotopy Fibrations**) Let

$$\begin{tikzcd}
\{pt\} = X_0 & X_1 \arrow[l] & X_2 \arrow[l] & \cdots \arrow[l] & X_n \arrow[l] & X_{n+1} \arrow[l] & \cdots \arrow[l] \\
& F_1 \arrow[u, equal] & F_2 \arrow[u] & \cdots & F_n \arrow[u] & F_{n+1} \arrow[u] &
\end{tikzcd}$$

be a tower of homotopy fibrations of path-connected spaces with Abelian fundamental groups, where every $X_{n+1} \to X_n$ is a fibration with fiber F_{n+1}. In the literature, there are a few spectral sequences corresponded to this tower. We mention three of them here.

1. (**Adams-like Spectral Sequence of a Tower of Fibrations**) If we let
$$D^1_{(p,q)} = \pi_p(X_q) \ \text{ and } \ E^1_{(p,q)} = \pi_p(F_q)$$
we will obtain an eventually vanishing exact couple, shown in the following diagram, and the induced spectral sequence with bidegrees $\mathbf{a} = (0, -1)$, $\mathbf{b} = (-1, 1)$ and $\mathbf{c} = (0, 0)$.

$$\begin{tikzcd}
& \lim_q \pi_p(X_q) \arrow[d] & & \lim_q \pi_{p-1}(X_q) \arrow[d] & & \lim_q \pi_{p-2}(X_q) \arrow[d] & \\
& \vdots \arrow[d] & & \vdots \arrow[d] & & \vdots \arrow[d] & \\
\cdots \arrow[r] & \pi_p(X_q) \arrow[r] \arrow[d] & \pi_{p-1}(F_{q+1}) \arrow[r] & \pi_{p-1}(X_{q+1}) \arrow[r] \arrow[d] & \pi_{p-2}(F_{q+2}) \arrow[r] & \pi_{p-2}(X_{q+2}) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \pi_p(X_{q-1}) \arrow[r] \arrow[d] & \pi_{p-1}(F_q) \arrow[r] & \pi_{p-1}(X_q) \arrow[r] \arrow[d] & \pi_{p-2}(F_{q+1}) \arrow[r] & \pi_{p-2}(X_{q+1}) \arrow[r] \arrow[d] & \cdots \\
& \vdots \arrow[d] & & \vdots \arrow[d] & & \vdots \arrow[d] & \\
\cdots \arrow[r, "\cong"] & \pi_p(X_1) \arrow[r] \arrow[d] & \pi_{p-1}(F_2) \arrow[r] & \pi_{p-1}(X_2) \arrow[r] \arrow[d] & \pi_{p-2}(F_3) \arrow[r] & \pi_{p-2}(X_3) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & \pi_{p-1}(F_1) \arrow[r, "\cong"] & \pi_{p-1}(X_1) \arrow[r] \arrow[d] & \pi_{p-2}(F_2) \arrow[r] & \pi_{p-2}(X_2) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] & 0 \arrow[r] \arrow[d] & \pi_{p-2}(F_1) \arrow[r, "\cong"] & \pi_{p-2}(X_1) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] & 0 \arrow[r] \arrow[d] & \cdots \\
& \vdots & & \vdots & & \vdots &
\end{tikzcd}$$

Also, the differential in the r-th page is of bidegree
$$\mathbf{b} + \mathbf{c} - (r-1)\mathbf{a} = (-1, 1) - (r-1)(0, -1) = (-1, r).$$

2. (**Bousfield-Friedlander Spectral Sequence.**) If we let

$$D^1_{p,q} = \pi_{p+q}(X_q) \quad \text{and} \quad E^1_{(p,q)} = \pi_{p+q}(F_q)$$

we will obtain an eventually vanishing exact couple, shown in the following diagram, and the induced spectral sequence defined in [5] with bidegrees $\mathbf{a} = (1,-1)$, $\mathbf{b} = (-2,1)$ and $\mathbf{c} = (0,0)$. Also, the differential in the r-th page is of bidegree

$$\mathbf{b} + \mathbf{c} - (r-1)\mathbf{a} = (-2,1) - (r-1)(1,-1) = (-r-1, r).$$

$$\begin{tikzcd}
 & \lim_q \pi_n(X_q) \arrow[d] & & \lim_q \pi_{n-1}(X_q) \arrow[d] & & \lim_q \pi_{n-2}(X_q) \arrow[d] & \\
 & \vdots \arrow[d] & & \vdots \arrow[d] & & \vdots \arrow[d] & \\
\cdots \arrow[r] & \pi_n(X_q) \arrow[r] \arrow[d] & \pi_{n-1}(F_{q+1}) \arrow[r] & \pi_{n-1}(X_{q+1}) \arrow[r] \arrow[d] & \pi_{n-2}(F_{q+2}) \arrow[r] & \pi_{n-2}(X_{q+2}) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \pi_n(X_{q-1}) \arrow[r] \arrow[d] & \pi_{n-1}(F_q) \arrow[r] & \pi_{n-1}(X_q) \arrow[r] \arrow[d] & \pi_{n-2}(F_{q+1}) \arrow[r] & \pi_{n-2}(X_{q+1}) \arrow[r] \arrow[d] & \cdots \\
 & \vdots \arrow[d] & & \vdots \arrow[d] & & \vdots \arrow[d] & \\
\cdots \arrow[r, "\cong"] & \pi_n(X_1) \arrow[r] \arrow[d] & \pi_{n-1}(F_2) \arrow[r] & \pi_{n-1}(X_2) \arrow[r] \arrow[d] & \pi_{n-2}(F_3) \arrow[r] & \pi_{n-2}(X_3) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & \pi_{n-1}(F_1) \arrow[r, "\cong"] & \pi_{n-1}(X_1) \arrow[r] \arrow[d] & \pi_{n-2}(F_2) \arrow[r] & \pi_{n-2}(X_2) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] & 0 \arrow[r] \arrow[d] & \pi_{n-2}(F_1) \arrow[r, "\cong"] & \pi_{n-2}(X_1) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] & 0 \arrow[r] \arrow[d] & \cdots \\
 & \vdots & & \vdots & & \vdots &
\end{tikzcd}$$

3. (**Bousfield-Kan Spectral Sequence**) If we let

$$D_1^{p,q} = \pi_{q-p}(X_p) \quad \text{and} \quad E_1^{(p,q)} = \pi_{q-p}(F_p)$$

we will obtain an eventually vanishing exact couple, shown in the following diagram, and the induced spectral sequence defined in [6] with bidegrees $\mathbf{a} = (-1,-1)$, $\mathbf{b} = (1,0)$ and $\mathbf{c} = (0,0)$. Also, the differential in the r-th page is of bidegree

$$\mathbf{b} + \mathbf{c} - (r-1)\mathbf{a} = (1,0) - (r-1)(-1,-1) = (r, r-1).$$

$$
\begin{tikzcd}
 & \lim_p \pi_n(X_p) \arrow[d] & & \lim_p \pi_{n-1}(X_p) \arrow[d] & & \lim_p \pi_{n-2}(X_p) \arrow[d] & \\
 & \vdots \arrow[d] & & \vdots \arrow[d] & & \vdots \arrow[d] & \\
\cdots \arrow[r] & \pi_n(X_p) \arrow[r] \arrow[d] & \pi_{n-1}(F_{p+1}) \arrow[r] & \pi_{n-1}(X_{p+1}) \arrow[r] \arrow[d] & \pi_{n-2}(F_{p+2}) \arrow[r] & \pi_{n-2}(X_{p+2}) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \pi_n(X_{p-1}) \arrow[r] \arrow[d] & \pi_{n-1}(F_p) \arrow[r] & \pi_{n-1}(X_p) \arrow[r] \arrow[d] & \pi_{n-2}(F_{p+1}) \arrow[r] & \pi_{n-2}(X_{p+1}) \arrow[r] \arrow[d] & \cdots \\
 & \vdots \arrow[d] & & \vdots \arrow[d] & & \vdots \arrow[d] & \\
\cdots \arrow[r, "\cong"] & \pi_n(X_1) \arrow[r] \arrow[d] & \pi_{n-1}(F_2) \arrow[r] & \pi_{n-1}(X_2) \arrow[r] \arrow[d] & \pi_{n-2}(F_3) \arrow[r] & \pi_{n-2}(X_3) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & \pi_{n-1}(F_1) \arrow[r, "\cong"] & \pi_{n-1}(X_1) \arrow[r] \arrow[d] & \pi_{n-2}(F_2) \arrow[r] & \pi_{n-2}(X_2) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] & 0 \arrow[r] \arrow[d] & \pi_{n-2}(F_1) \arrow[r, "\cong"] & \pi_{n-2}(X_1) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] & 0 \arrow[r] \arrow[d] & \cdots \\
 & \vdots & & \vdots & & \vdots &
\end{tikzcd}
$$

See [6] or [12] for more properties of the last two spectral sequences.

It is well-known ([9]) that there is a short exact sequence

$$\lim\nolimits_n^1 \pi_{k+1}(X_n) \rightarrowtail \pi_k(X) \xrightarrow{\rho^k} \lim\nolimits_n \pi_k(X_n)$$

where X is the homotopy inverse limit of the tower. So, in all three examples corresponding to the tower of fibrations, $\lim_n \pi_*(X_n)$ is the universal coaugmentation and $\pi_*(X)$ is a coaugmentation of the exact couples. By part 3 of Remark 3.2.8 on page 72, they are all coaugmentation concentrated.

Example 3.2.11. (**Grothendieck Spectral Sequence**) Given two additive functors $\mathcal{A} \xrightarrow{G} \mathcal{B} \xrightarrow{F} \mathcal{D}$, where $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{D}$ are Abelian categories with enough injectives or projectives, the Grothendieck spectral sequence expresses the derived functors of GF in terms of the derived fuctors of G and of F. For the definitions of Abelian category, additive functor, enough injective-projective and derived functors look at [27].

Here we assume that $\mathcal{A}$ and $\mathcal{B}$ are categories of S-modules and R-modules, for two commutative unitary rings S and R and let $F : \mathcal{B} \to \mathcal{A}b$

be an additive functor of either variance. We denote the left and right derived functors of F by LF and RF, respectively.

- An object B in $\mathcal{B}$ is called *right F-acyclic* if $(R^pF)B = \{0\}$, for every $p \geq 1$.
- An object C in $\mathcal{B}$ is called *left F-acyclic* if $(L_pF)C = \{0\}$, for every $p \geq 1$.

For example, if $F = \hom_R(A, -)$, then every injective R-module E is right F-acyclic because $\mathrm{Ext}^p_R(A, E) = \{0\}$, for every $p \geq 1$. Also, if $F = A \otimes_R -$, then every projective R-module P is left F-acyclic, because $\mathrm{Tor}^R_p(A, P) = \{0\}$, for every $p \geq 1$. Every flat R-module is also left F-acyclic.

There are four Grothendieck spectral sequences, depending on the variances of the functors involved. Here are two types of functor compositions that have corresponding spectral sequences induced by originally vanishing and eventually stable exact couples:

1. Let $\mathcal{A} \xrightarrow{G} \mathcal{B} \xrightarrow{F} \mathcal{A}b$ be covariant additive functors. Assume that F is right exact and that GP is left F-acyclic for every projective P in $\mathcal{A}$. Then, for every object A in $\mathcal{A}$, there is a first quadrant (homology) spectral sequence with
$$E^2_{(p,q)} = (L_pF)(L_qG)A \Longrightarrow L_n(FG)A.$$
This spectral sequence is functorial in A.

2. Let $\mathcal{A} \xrightarrow{G} \mathcal{B} \xrightarrow{F} \mathcal{A}b$ be additive contravariant functors. Assume that F is left exact and that GP is right F-acyclic for every projective P in $\mathcal{A}$. Then, for every object A in $\mathcal{A}$, there is a first quadrant (homology) spectral sequence with
$$E^2_{(p,q)} = (R^pF)(R^qG)A \Longrightarrow L_n(FG)A.$$
This spectral sequence is functorial in A.

The following are two types of functor compositions that the corresponding spectral sequences are induced by originally stable and eventually vanishing exact couples:

3. Let $\mathcal{A} \xrightarrow{G} \mathcal{B} \xrightarrow{F} \mathcal{A}b$ be covariant additive functors. Assume that F is left exact and that GE is right F-acyclic for every injective object

E in $\mathcal{A}$. Then, for every object A in $\mathcal{A}$, there is a third quadrant (cohomology) spectral sequence with

$$E_2^{(p,q)} = (R^p F)(R^q G)A \Longrightarrow R^n(FG)A.$$

This spectral sequence is functorial in A.

4. Let $\mathcal{A} \xrightarrow{G} \mathcal{B} \xrightarrow{F} \mathcal{A}b$ be additive functors. Assume that F is contravariant left exact, G is covariant, and GP is right F-acyclic for every projective P in $\mathcal{A}$. Then, for every object A in $\mathcal{A}$, there is a third quadrant (cohomology) spectral sequence with

$$E_2^{(p,q)} = (R^p F)(L_q G)A \Longrightarrow R^n(FG)A.$$

This spectral sequence is functorial in A.

3.3 Matching Originally Stable or Vanishing Exact Couples to a Tower

In this section, we present the structural background required for carrying information through the ascending filtration stages of the first and last term of a (potentially transfinite) *directed* tower of bigraded modules. To avoid duplication, we do not distinguish the special case that the tower of modules is indexed over ω, the least infinite ordinal, and work with towers that are indexed over an arbitrary limit ordinal.

3.3.1 Matching Originally Stable Exact Couples to a Tower

Assume we have a directed tower of bigraded modules

$$\mathrm{H}_1 \to \mathrm{H}_2 \to \cdots \to \mathrm{H}_p \to \mathrm{H}_{p+1} \to \cdots \to \mathrm{H}_\eta \to \mathrm{H}_{\eta+1} \to \cdots \to \mathrm{H}_\lambda = \mathrm{H} \tag{3.9}$$

where λ is a limit ordinal. For every $\eta \leq \lambda$, we define $F_\eta = \mathrm{im}(Q_\eta : \mathrm{H}_\eta \to \mathrm{H}_\lambda)$ to obtain the ascending transfinite image filtration of H_λ

$$F_1 \subseteq \cdots \subseteq F_p \subseteq F_{p+1} \subseteq \cdots \subseteq F_\eta \subseteq F_{\eta+1} \subseteq \cdots \subseteq F_\lambda = \mathrm{H}_\lambda \tag{3.10}$$

and define $F^\eta = \ker(Q^\eta : \mathrm{H}_1 \to \mathrm{H}_\eta)$ to obtain the ascending transfinite kernel filtration of H_1

$$F^1 \subseteq \cdots \subseteq F^p \subseteq \cdots \subseteq F^\eta \subseteq F^{\eta+1} \subseteq \cdots \subseteq F^\lambda = \ker(\mathrm{H}_1 \to \mathrm{H}_\lambda) \subseteq \mathrm{H}_1\,. \tag{3.11}$$

Definition 3.3.1. *We* matchoriginally stable exact couples *to the tower (3.9) if for every limit ordinal $\eta_0 < \lambda$, there is an originally stable exact couple $\mathcal{EC}(\eta_0)$ (with nonzero stable range) and homomorphisms α and β, as depicted in the diagram on page 198, such that*

- $F_{\eta_0+r}(n) = \mathrm{im}(D^1_{u,v+r\mathbf{a}} \xrightarrow{\alpha_{u,v+r\mathbf{a}}} \mathrm{H}_{\eta_0+r}(n) \xrightarrow{Q_{\eta_0+r}} \mathrm{H}_\lambda(n))$ *, and*
- $F^{\eta_0+r}(n) = \ker(\mathrm{H}_1(n) \xrightarrow{Q^{\eta_0+r}} \mathrm{H}_{\eta_0+r}(n) \xrightarrow{\beta^{u,v+r\mathbf{a}}} D^1_{u,v+r\mathbf{a}})$ *,*

where $\mathrm{H}_*(n)$, $F^*(n)$ and $F_*(n)$ are the n-th degree of H_*, F^* and F_*, respectively, and $(u,v)\cdot\hat{\mathbf{a}} = n$. The homomorphisms α and β are called the matching homomorphisms.

Remark 3.3.2. By Definition 3.3.1 on page 80, we look at H_λ (in fact, $F_{\eta_0+\omega}$) as a *far* augmentation of $\mathcal{EC}(\eta_0)$ and we consider the following ω-length filtration segment of H_λ

$$F_{\eta_0} \subseteq F_{\eta_0+1} \subseteq \cdots \subseteq F_{\eta_0+r} \subseteq F_{\eta_0+r+1} \subseteq \cdots \subseteq F_{\eta_0+\omega}.$$

Similarly, we look at H_1 (in fact, $F^{\eta_0+\omega}$) as a *far* coaugmentation of $\mathcal{EC}(\eta_0)$ and we consider the following ω-length filtration segment of H_1

$$F^{\eta_0} \subseteq F^{\eta_0+1} \subseteq \cdots \subseteq F^{\eta_0+r} \subseteq F^{\eta_0+r+1} \subseteq \cdots \subseteq F^{\eta_0+\omega}.$$

Note that $\mathcal{EC}(\eta_0)$ has its own universal augmentation and coaugmentation. By Remark 3.2.2 on page 64, for every $r \geq 0$, we have the short exact sequence

$$\epsilon_{u,v+r\mathbf{a}-\mathbf{b}-\mathbf{c}} \rightarrowtail E^\infty_{u,v+r\mathbf{a}-\mathbf{c}} \twoheadrightarrow \epsilon^{u,v+r\mathbf{a}}.$$

Therefore, by the E^∞-Distribution Theorem we have

$$\begin{tikzcd}
\frac{F_{\eta_0+r}(n)}{F_{\eta_0+r-1}(n)} & & \frac{F^{\eta_0+r+1}(n)}{F^{\eta_0+r}(n)} \arrow[d, tail, two heads] \\
\epsilon_{u,v+r\mathbf{a}-\mathbf{b}-\mathbf{c}} \arrow[u, two heads] \arrow[r, tail] & E^\infty_{u,v+r\mathbf{a}-\mathbf{c}} \arrow[r, two heads] & \epsilon^{u,v+r\mathbf{a}}.
\end{tikzcd} \tag{3.12}$$

For every limit ordinal $\eta_0 < \lambda$, the universal property of colimit provides a homomorphism $\rho_{\eta_0} : \operatorname{colim}_{\beta<\eta_0} \mathrm{H}_\beta \to \mathrm{H}_{\eta_0}$.

Definition 3.3.3. *The homomorphism ρ_{η_0} is called the* clutching *homomorphism and H_{η_0} is called the* clutching term*.*

These filtrations and the information provided by the originally stable exact couples can be combined in the diagram on page 199, where arrows of the same color represent exactness. We postpone the proof of the ingredients of the mentioned diagram to the end of this section.

Remark 3.3.4. Remember that, as we explained in the introduction to this chapter, we are ultimately interested in carrying some information through the filtration stages of H_1 or H_λ and we use transfinite induction as our tool. For example, if we want to carry some information through the filtration stages of H_λ, we assume for an ordinal $\eta \leq \lambda$ we have information about F_β's, where $\beta < \eta$, and we consider the following two situations:

Situation 1: η is a non-limit ordinal and hence it has finitely many predecessors. So for a limit ordinal η_0 and a positive integer r we have $\eta = \eta_0 + r$ and hence $F_\eta = F_{\eta_0+r}$. In the following short exact sequence

$$F_{\eta_0+r-1} \rightarrowtail F_{\eta_0+r} \twoheadrightarrow \frac{F_{\eta_0+r}}{F_{\eta_0+r-1}},$$

by induction assumption, we have information about F_{η_0+r-1} and using the diagram (3.12) we try to obtain information about $\dfrac{F_{\eta_0+r}}{F_{\eta_0+r-1}}$ from the information we can assume about the E^∞-terms of the intermediate exact couple $\mathcal{EC}(\eta_0)$. Look at the diagram on page 199. Therefore, information about $F_{\eta_0+r} = F_\eta$ is provided, up to extension.

Situation 2: η is a limit ordinal. Our method works only for those types of information about F_β's that can be carried to $\operatorname{colim}_{\beta<\eta} F_\beta$. Therefore, in the following short exact sequence

$$\operatorname{colim}_{\beta<\eta} F_\beta \rightarrowtail F_\eta \twoheadrightarrow \frac{F_\eta}{\operatorname{colim}_{\beta<\eta} F_\beta}$$

we have information about $\operatorname{colim}_{\beta<\eta} F_\beta$. Then, we obtain information about $\dfrac{F_\eta}{\operatorname{colim}_{\beta<\eta} F_\beta}$ by putting assumptions on $\operatorname{coker}(\rho_\eta)$. Look at the diagram on page 199. Therefore, the information about F_η is provided, up to extension.

Look at the proof of Proposition 4.3.2 on page 100, Proposition 4.5.2 on page 109 and Section 5.8 for examples of such argument.

Example 3.3.5. (**Transfinite Tower of Cofibers**) Let

$$\begin{tikzcd} X_1 \arrow[r] & X_2 \arrow[r] & X_3 \arrow[r] & \cdots \arrow[r] & X_\eta \arrow[r] & X_{\eta+1} \arrow[r] & X_{\eta+2} \arrow[r] & \cdots \\ F_1 \arrow[u] & F_2 \arrow[u] & F_3 \arrow[u] & & F_\eta \arrow[u] & F_{\eta+1} \arrow[u] & F_{\eta+2} \arrow[u] & \end{tikzcd}$$

be a tower of cofibers indexed over a limit ordinal λ where each $F_\eta \to X_\eta$ is a cofibration with cofiber $X_{\eta+1}$ and all spaces involved are path-connected. We assume that for every limit ordinal $\eta_0 \leq \lambda$ we have $X_{\eta_0} = \mathrm{hocolim}_{\beta<\eta_0} X_\beta$. For every limit ordinal $\eta_0 < \lambda$, the following ω-length segment of the tower of cofibers

$$\begin{tikzcd} X_{\eta_0} \arrow[r] & X_{\eta_0+1} \arrow[r] & X_{\eta_0+2} \arrow[r] & \cdots \arrow[r] & X_{\eta_0+r} \arrow[r] & X_{\eta_0+r+1} \arrow[r] & \cdots \\ F_{\eta_0} \arrow[u] & F_{\eta_0+1} \arrow[u] & F_{\eta_0+2} \arrow[u] & \cdots & F_{\eta_0+r} \arrow[u] & F_{\eta_0+r+1} \arrow[u] & \end{tikzcd}$$

defines an originally stable exact couple and hence induces a homology spectral sequence as stated in Example 3.2.3 on page 67. Therefore, we obtain transfinite towers of homology groups of cofibers that are shown in blue in the diagram on page 200. As a result, the matching homomorphisms α and β in Definition 3.3.1 on page 80 are isomorphisms. Note that, for every integer n we have the following

$$\rho_{\eta_0}(n) : \mathrm{colim}_{\beta<\eta_0} \mathrm{h}_n(X_\beta) \xrightarrow{\cong} \mathrm{h}_n(\mathrm{colim}_{\beta<\eta_0} X_\beta) = \mathrm{h}_n(X_{\eta_0}).$$

Therefore, the clutching homomorphisms are also isomorphisms.

3.3.2 Matching Originally Vanishing Exact Couples to a Tower

In this section, we match originally vanishing exact couples to the tower (3.9) on page 79, with a minor modification of what we have provided in the last section.

Definition 3.3.6. *We* match originally vanishing exact couples *to the tower (3.9) if for every limit ordinal* $\eta_0 < \lambda$*, there is an originally vanishing exact couple* $\mathcal{EC}(\eta_0)$ *and a homomorphism* α*, as depicted in the diagram on page 201, such that*

$$F_{\eta_0+r}(n) = \mathrm{im}(D^1_{u,v+r\mathbf{a}} \xrightarrow{\alpha_{u,v+r\mathbf{a}}} \mathrm{H}_{\eta_0+r}(n) \xrightarrow{Q_{\eta_0+r}} \mathrm{H}_\lambda(n)) ,$$

where $\mathrm{H}_*(n)$ *and* $F_*(n)$ *are the* n*-th degree of* H_* *and* F_**, respectively, and* $(u, v)\cdot\hat{\mathbf{a}} = n$*. The homomorphism* α *is called the* matching homomorphism*.*

Remark 3.3.7. By Definition 3.3.6, we look at H_λ (in fact $F_{\eta_0+\omega}$) as a *far* augmentation of $\mathcal{EC}(\eta_0)$ and we consider the following ω-length filtration segment of H_λ

$$F_{\eta_0} \subseteq F_{\eta_0+1} \subseteq \cdots \subseteq F_{\eta_0+r} \subseteq F_{\eta_0+r+1} \subseteq \cdots \subseteq F_{\eta_0+\omega}.$$

Unlike the case we matched originally stable exact couples to the tower (3.9), H_1 does not play the role of a non-trivial coaugmentation of $\mathcal{EC}(\eta_0)$, because any homomorphism from a coaugmentation to the D^1-terms of $\mathcal{EC}(\eta_0)$ must factor through the universal coaugmentation of $\mathcal{EC}(\eta_0)$, which is zero.

Remember that by part 3 of Remark 3.2.2 on page 64, the spectral sequence induced by the intermediate originally vanishing exact couple $\mathcal{EC}(\eta_0)$ is convergent to its universal augmentation. Therefore, by the E^∞-Distribution Theorem we have the following diagram

$$\begin{tikzcd} \dfrac{F_{\eta_0+r}(n)}{F_{\eta_0+r-1}(n)} & \\ \epsilon_{u,v+r\mathbf{a}} \arrow[u, two heads] \arrow[r, "\cong"] & E^\infty_{u,v+\mathbf{b}+r\mathbf{a}}. \end{tikzcd} \tag{3.13}$$

These filtrations and the information provided by the originally vanishing exact couples can be combined in the diagram on page 202, where arrows of the same color represent exactness.

We can modify the argument given in Remark 3.3.4 on page 81 when we match originally vanishing exact couples to a tower. Look at the proof of Proposition 5.8.1 on page 141 or Corolarry 5.8.3 on page 143 for examples of such argument.

Example 3.3.8. (**Transfinite Tower of Cofibrations**) Let

$$\begin{tikzcd} X_1 \arrow[r] & X_2 \arrow[r] \arrow[d] & X_3 \arrow[r] \arrow[d] & \cdots \arrow[r] & X_\eta \arrow[r] \arrow[d] & X_{\eta+1} \arrow[r] \arrow[d] & X_{\eta+2} \arrow[r] \arrow[d] & \cdots \\ & F_2 & F_3 & & * & F_{\eta+1} & F_{\eta+2} & \end{tikzcd}$$

be a tower of cofibrations indexed over a limit ordinal λ where each $X_\eta \to X_{\eta+1}$ is a cofibration with cofiber $F_{\eta+1}$ and all spaces involved are path-conneced. We assume that for every limit ordinal $\eta_0 \leq \lambda$, we have

$$\mathrm{hocolim}_{\beta<\eta_0} X_\beta = X_{\eta_0}.$$

For every limit ordinal $\eta_0 < \lambda$, the following ω-length segment of the tower of cofibrations

$$\begin{tikzcd} X_{\eta_0} \arrow[r] & X_{\eta_0+1} \arrow[r] \arrow[d] & \cdots \arrow[r] & X_{\eta_0+r} \arrow[r] \arrow[d] & X_{\eta_0+r+1} \arrow[r] \arrow[d] & \cdots \\ & F_{\eta_0+1} & & F_{\eta_0+r} & F_{\eta_0+r+1} & \end{tikzcd}$$

defines an originally vanishing exact couple and hence induces a homology spectral sequence as stated in Example 3.2.4 on page 68. Therefore, we obtain transfinite towers of homology groups of cofibrations that are shown in blue in the diagram on page 203. As a result, the matching homomorphism α in Definition 3.3.6 on page 83 is isomorphism. Note that, for every integer n we have the following

$$\rho_{\eta_0}(n) : \mathrm{colim}_{\beta<\eta_0} \mathrm{h}_n(X_\beta) \xrightarrow{\cong} \mathrm{h}_n(\mathrm{hocolim}_{\beta<\eta_0} X_\beta) = \mathrm{h}_n(X_{\eta_0}).$$

Therefore, the clutching homomorphisms are also isomorphisms. Look at Appendix B for an important example of a transfinite tower of cofibrations.

3.3.3 Proof of the Ingredients of the Diagram on Page 199

Let $\eta_0 \leq \lambda$ be a limit ordinal

$$\underline{F}_{\eta_0} = \mathrm{colim}_{\beta<\eta_0} F_\beta$$

and

$$\underline{F}^{\eta_0} = \mathrm{colim}_{\beta<\eta_0} F^\beta.$$

$\ker(\rho_{\eta_0}(n)) \leftarrowtail \dfrac{F^{\eta_0}(n)}{\underline{F}^{\eta_0}(n)}$:

Note that for the homomorphism $\underline{Q}^{\eta_0}(n) : \mathrm{H}_1(n) \to \operatorname{colim}_{\beta<\eta_0} \mathrm{H}_\beta(n)$ we have $\underline{F}^{\eta_0}(n) = \ker(\underline{Q}^{\eta_0}(n))$.

Lemma 3.3.9. $\dfrac{F^{\eta_0}(n)}{\underline{F}^{\eta_0}(n)} \cong \operatorname{im}(\underline{Q}^{\eta_0}(n)) \cap \ker(\rho_{\eta_0}(n))$. *In particular, we have* $\dfrac{F^{\eta_0}(n)}{\underline{F}^{\eta_0}(n)} \rightarrowtail \ker(\rho_{\eta_0}(n))$.

Proof. $\dfrac{F^{\eta_0}(n)}{\underline{F}^{\eta_0}(n)} \cong \dfrac{\ker(\rho_{\eta_0}(n) \circ \underline{Q}^{\eta_0}(n))}{\ker(\underline{Q}^{\eta_0}(n))} \cong \operatorname{im}(\underline{Q}^{\eta_0}(n)) \cap \ker(\rho_{\eta_0}(n))$. □

Corollary 3.3.10. $F^{\eta_0}(n) \cong \underline{F}^{\eta_0}(n)$ *if and only if* $\operatorname{im}(\underline{Q}^{\eta_0}(n)) \cap \ker(\rho_{\eta_0}(n)) = \emptyset$. *In particular, if the clutching homomorphisms are monomorphisms, then* $F^{\eta_0} \cong \underline{F}^{\eta_0}$.

$\dfrac{F_{\eta_0}(n)}{\underline{F}_{\eta_0}(n)} \twoheadleftarrow \operatorname{coker}(\rho_{\eta_0}(n))$:

Lemma 3.3.11. $\dfrac{F_{\eta_0}(n)}{\underline{F}_{\eta_0}(n)} \cong \dfrac{\mathrm{H}_{\eta_0}(n)}{\ker(Q_{\eta_0}(n)) + \operatorname{im}(\rho_{\eta_0}(n))}$. *In particular,*

$$\operatorname{coker}(\rho_{\eta_0}(n)) \twoheadrightarrow \frac{F_{\eta_0}(n)}{\underline{F}_{\eta_0}(n)}.$$

Proof. We have

$$\begin{aligned}\frac{F_{\eta_0}(n)}{\underline{F}_{\eta_0}(n)} &\cong \frac{\operatorname{im}(Q_{\eta_0}(n))}{\operatorname{im}(Q_{\eta_0}(n) \circ \rho_{\eta_0}(n))} \\ &\cong \frac{\dfrac{\mathrm{H}_{\eta_0}(n)}{\ker(Q_{\eta_0}(n))}}{\dfrac{\ker(Q_{\eta_0}(n)) + \operatorname{im}(\rho_{\eta_0}(n))}{\ker(Q_{\eta_0}(n))}} \\ &\cong \frac{\mathrm{H}_{\eta_0}(n)}{\ker(Q_{\eta_0}(n)) + \operatorname{im}(\rho_{\eta_0}(n))}.\end{aligned}$$

Note that

$$\operatorname{coker}(\rho_{\eta_0}(n)) = \frac{\mathrm{H}_{\eta_0}(n)}{\operatorname{im}(\rho_{\eta_0}(n))} \twoheadrightarrow \frac{\mathrm{H}_{\eta_0}(n)}{\ker(Q_{\eta_0}(n)) + \operatorname{im}(\rho_{\eta_0}(n))} \cong \frac{F_{\eta_0}(n)}{\underline{F}_{\eta_0}(n)}.$$

□

Corollary 3.3.12. *$\underline{F}_{\eta_0}(n) = F_{\eta_0}(n)$ if and only if $\mathrm{H}_{\eta_0}(n) = \ker(Q_{\eta_0}(n)) + \mathrm{im}(\rho_{\eta_0}(n))$. In particular, if the clutching structures are epimorphisms, then $F_{\eta_0} = \mathrm{colim}_{\beta<\eta_0} F_\beta$.*

3.4 Matching Eventually Stable or Vanishing Exact Couples to a Tower

In this section, we consider a transfinite *inverse* tower of bigraded modules and the *descending* filtration of its first and last term. Again, we match *convenient* exact couples to this tower. Since the limit functor on inverse towers is not exact, the situation here is much more complicated than the one in Section (3.3.1) and Section (3.3.2). In particular, dualizing the development in those two sections does not handle the situation here. So we explain everything in details.

3.4.1 Matching Eventually Stable Exact Couples to a Tower

Assume we have an inverse tower of bigraded modules

$$\mathrm{H} = \mathrm{H}^\lambda \to \cdots \to \mathrm{H}^{\eta+1} \to \mathrm{H}^\eta \to \cdots \to \mathrm{H}^{p+1} \to \mathrm{H}^p \to \cdots \to \mathrm{H}^1 \tag{3.14}$$

where λ is a limit ordinal. For every $\eta \leq \lambda$, we define $F_\eta = \mathrm{im}(Q_\eta : \mathrm{H}^\eta \to \mathrm{H}^1)$ to obtain the descending transfinite image filtration of H^1

$$\mathrm{H}^1 = F_1 \supseteq F_2 \supseteq \cdots \supseteq F_p \supseteq \cdots \supseteq F_\eta \supseteq F_{\eta+1} \supseteq \cdots \supseteq F_\lambda \tag{3.15}$$

and we define $F^\eta = \ker(Q^\eta : \mathrm{H}^\lambda \to \mathrm{H}^\eta)$ to obtain the descending transfinite kernel filtration of H^λ

$$\mathrm{H}^\lambda \supseteq \ker(\mathrm{H}^\lambda \to \mathrm{H}^1) = F^1 \supseteq \cdots \supseteq F^p \supseteq \cdots \supseteq F^\eta \supseteq F^{\eta+1} \supseteq \cdots \supseteq F^\lambda. \tag{3.16}$$

Definition 3.4.1. *We* match eventually stable exact couples *to the tower (3.14) if for every limit ordinal $\eta_0 < \lambda$, there is an eventually stable exact couple $\mathcal{EC}(\eta_0)$ (with nonzero stable range) and homomorphisms α and β, as depicted in the diagram on page 204, such that*

- $F_{\eta_0+r}(n) = \mathrm{im}(D^1_{u,v+r\mathbf{a}} \xrightarrow{\alpha_{u,v+r\mathbf{a}}} \mathrm{H}^{\eta_0+r}(n) \xrightarrow{Q_{\eta_0+r}} \mathrm{H}^1(n))$, *and*

- $F^{\eta_0+r}(n) = \ker(\mathrm{H}^\lambda(n) \xrightarrow{Q^{\eta_0+r}} \mathrm{H}^{\eta_0+r}(n) \xrightarrow{\beta^{u,v+r\mathbf{a}}} D^1_{u,v+r\mathbf{a}})$,

where $\mathrm{H}^*(n)$, $F^*(n)$ *and* $F_*(n)$ *are the* n*-th degree of* H^*, F^* *and* F_*, *respectively, and* $(u, v) \cdot \hat{\mathbf{a}} = n$*. The homomorphisms* α *and* β *are called the* matching homomorphisms*.*

Remark 3.4.2. By Definition 3.4.1, we look at H^1 (in fact F_{η_0}) as a *far* augmentation of $\mathcal{EC}(\eta_0)$ and we consider the following ω-length filtration segment of H^1

$$F_{\eta_0} \supseteq F_{\eta_0+1} \supseteq \cdots \supseteq F_{\eta_0+r} \supseteq F_{\eta_0+r+1} \supseteq \cdots \supseteq F_{\eta_0+\omega}.$$

Similarly, we look at H^λ (in fact F^{η_0}) as a *far* coaugmentation of $\mathcal{EC}(\eta_0)$ and we consider the following ω-length filtration segment of H^λ

$$F^{\eta_0} \supseteq F^{\eta_0+1} \supseteq \cdots \supseteq F^{\eta_0+r} \supseteq F^{\eta_0+r+1} \supseteq \cdots \supseteq F^{\eta_0+\omega}.$$

Note that $\mathcal{EC}(\eta_0)$ has its own universal augmentation and coaugmentation. By part 2 of Remark 3.2.8 on page 72, for every $r \geq 0$, we have the following diagram, where, as usual, arrows of the same color form exact sequences

$$\begin{tikzcd}
\dfrac{F_{\eta_0+r}(n)}{F_{\eta_0+r+1}(n)} & & \lim^1_t \ker(i^t_{u,v-r\mathbf{a}+\mathbf{b}+\mathbf{c}}) & \dfrac{F^{\eta_0+r-1}(n+\sigma)}{F^{\eta_0+r}(n+\sigma)} \arrow[d, tail] \\
\epsilon_{u,v-r\mathbf{a}} \arrow[u, two heads] \arrow[r, tail, red] & E^\infty_{u,v+\mathbf{b}-r\mathbf{a}} \arrow[r, two heads, red] & \dfrac{Z^\infty_{u,v+\mathbf{b}-r\mathbf{a}}}{\mathrm{im}(j^1_{u,v-r\mathbf{a}})} \arrow[u, blue] & \epsilon^{u,v-r\mathbf{a}+\mathbf{b}+\mathbf{c}} . \arrow[l, tail, blue]
\end{tikzcd} \tag{3.17}$$

For every limit ordinal $\eta_0 < \lambda$, the universal property of colimit provides a homomorphism $\rho^{\eta_0} : \mathrm{H}^{\eta_0} \to \lim_{\beta<\eta_0} \mathrm{H}^\beta$.

Definition 3.4.3. *The homomorphism* ρ^{η_0} *is called the* clutching *homomorphism and* H^{η_0} *is called the* clutching term*.*

These filtrations and the information provided by the eventually stable exact couples can be combined in the diagram on page 205, where arrows of the same color represent exactness.

Remark 3.4.4. We can modify the discussion in Remark 3.4.4 on page 88 for eventually stable exact couples matched to inverse towers. However, we need a "twist" here: Assume, for example, we are interested in carrying some information through the filtration stages of H^1. For an ordinal $\eta \leq \lambda$, instead of assuming some information about F_β's, for every $\beta < \eta$, we assume information about $\dfrac{\mathrm{H}^1}{F_\beta}$'s. Look at the diagram on page 205. Again, we consider two situations:

- Situation 1: η is a non-limit ordinal and hence it has finitely many predecessors. So for a limit ordinal η_0 and a positive integer r we have $\eta = \eta_0 + r$ and hence $\dfrac{\mathrm{H}^1}{F_\eta} = \dfrac{\mathrm{H}^1}{F_{\eta_0+r}}$. In the following short exact sequence

$$\frac{F_{\eta_0+r-1}}{F_{\eta_0+r}} \rightarrowtail \frac{\mathrm{H}^1}{F_{\eta_0+r}} \twoheadrightarrow \frac{\mathrm{H}^1}{F_{\eta_0+r-1}}$$

  by induction assumption, we have information about $\dfrac{\mathrm{H}^1}{F_{\eta_0+r-1}}$ and using the diagram (3.17) we try to obtain information about $\dfrac{F_{\eta_0+r-1}}{F_{\eta_0+r}}$ from the information we can assume about the E^∞-terms of the intermediate exact couple $\mathcal{EC}(\eta_0)$. Therefore, information about $\dfrac{\mathrm{H}^1}{F_{\eta_0+r}}$ is provided, up to extension.

- Situation 2: η is a limit ordinal. Our method works only for those types of information about $\dfrac{\mathrm{H}^1}{F_\beta}$'s that can be carried to $\lim_{\beta<\eta} \dfrac{\mathrm{H}^1}{F_\beta}$. Therefore, in the following short exact sequence from the diagram on page 205

$$\frac{\lim_{\beta<\eta} F_\beta}{F_\eta} \rightarrowtail \frac{\mathrm{H}^1}{F_\eta} \longrightarrow \lim_{\beta<\eta} \frac{\mathrm{H}^1}{F_\beta} \twoheadrightarrow \lim^1_{\beta<\eta} F_\beta$$

  we have information about $\lim_{\beta<\eta} \dfrac{\mathrm{H}^1}{F_\beta}$. Then, we obtain information about $\dfrac{\lim_{\beta<\eta} F_\beta}{F_\eta}$ by putting assumptions on $\operatorname{coker}(\rho^\eta)$. Look at the diagram on page 205. Then, we try to obtain information about $\dfrac{\mathrm{H}^1}{F_\eta}$, up to extension. Sometimes we have to put some assumptions on

$\lim^1_{\beta<\eta} F_\beta$ too.

Look at the proof of Proposition 4.6.3 on page 112, Proposition 4.4.3 on page 104 and Section 5.7 for examples of such argument.

Example 3.4.5. (**Transfinite Tower of Fibers**) Let

$$\begin{tikzcd} X_1 \arrow[d] & X_2 \arrow[l] \arrow[d] & \cdots \arrow[l] & X_\eta \arrow[l] \arrow[d] & X_{\eta+1} \arrow[l] \arrow[d] & \cdots \arrow[l] \\ F_1 & F_2 & \cdots & F_\eta & F_{\eta+1} & \end{tikzcd}$$

be a tower of fibers indexed over a limit ordinal λ where each $X_\eta \to F_\eta$ is a homotopy fibration with fiber $X_{\eta+1}$ and all spaces involved are path-connected with Abelian fundamental groups. We assume that for every limit ordinal $\eta_0 \leq \lambda$, $\mathrm{holim}_{\beta<\eta_0} X_\beta = X_{\eta_0}$. For every limit ordinal $\eta_0 < \lambda$, the following ω-length stage of the tower of fibers

$$\begin{tikzcd} X_{\eta_0} \arrow[d] & X_{\eta_0+1} \arrow[l] \arrow[d] & X_{\eta_0+2} \arrow[l] \arrow[d] & \cdots \arrow[l] & X_{\eta_0+r} \arrow[l] \arrow[d] & X_{\eta_0+r+1} \arrow[l] \arrow[d] & \cdots \arrow[l] \\ F_{\eta_0} & F_{\eta_0+1} & F_{\eta_0+2} & \cdots & F_{\eta_0+r} & F_{\eta_0+r+1} & \end{tikzcd}$$

defines an eventually stable exact couple and hence induces a homotopy spectral sequence as stated in Example 3.2.9 on page 73. Therefore, we obtain transfinite towers of homotopy groups of cofibers that are shown in blue in the diagram on page 206. As a result, the matching homomorphisms α and β in Definition 3.4.1 on page 86 are isomorphisms. Note that, for every integer n, $\rho^{\eta_0}(n) : \pi_n(X_{\eta_0}) \to \lim\limits_{\beta<\eta_0} \pi_n(X_\beta)$ is the clutching homomorphism.

3.4.2 Proof of the Ingredients of the Diagram on Page 205

Let $\eta_0 \leq \lambda$ be a limit ordinal.

$$\frac{\lim_{\beta<\eta_0} F_\beta(n)}{F_{\eta_0}(n)} \twoheadleftarrow \mathrm{coker}(\rho^{\eta_0}(n)) :$$

Lemma 3.4.6. $\dfrac{\lim_{\beta<\eta_0} F_\beta(n)}{F_{\eta_0}(n)} \cong \dfrac{\lim_{\beta<\eta_0} \mathrm{H}^\beta(n)}{\ker(\bar{Q}_{\eta_0}(n)) + \operatorname{im}(\rho^{\eta_0}(n))}$, *where*

$$\bar{Q}_{\eta_0}(n) : \lim_{\beta<\eta_0} \mathrm{H}^\beta(n) \longrightarrow \mathrm{H}^1(n).$$

In particular, $\operatorname{coker}(\rho^{\eta_0}(n)) \twoheadrightarrow \dfrac{\lim_{\beta<\eta_0} F_\beta(n)}{F_{\eta_0}(n)}$.

Proof. We have

$$\begin{aligned}
\frac{\lim_{\beta<\eta_0} F_\beta(n)}{F_{\eta_0}(n)} &\cong \frac{\operatorname{im}(\bar{Q}_{\eta_0}(n))}{\operatorname{im}(\bar{Q}_{\eta_0}(n) \circ \rho^{\eta_0}(n))} \\
&\cong \frac{\dfrac{\lim_{\beta<\eta_0} \mathrm{H}^\beta(n)}{\ker(\bar{Q}_{\eta_0}(n))}}{\dfrac{\ker(\bar{Q}_{\eta_0}(n)) + \operatorname{im}(\rho^{\eta_0}(n))}{\ker(\bar{Q}_{\eta_0}(n))}} \\
&\cong \frac{\lim_{\beta<\eta_0} \mathrm{H}^\beta(n)}{\ker(\bar{Q}_{\eta_0}(n)) + \operatorname{im}(\rho^{\eta_0}(n))}.
\end{aligned}$$

Note that

$$\operatorname{coker}(\rho^{\eta_0}(n)) = \frac{\lim_{\beta<\eta_0} \mathrm{H}^\beta(n)}{\operatorname{im}(\rho^{\eta_0}(n))} \twoheadrightarrow \frac{\lim_{\beta<\eta_0} \mathrm{H}^\beta(n)}{\ker(\bar{Q}_{\eta_0}(n)) + \operatorname{im}(\rho^{\eta_0}(n))} \cong \frac{\lim_{\beta<\eta_0} F_\beta(n)}{F_{\eta_0}(n)}.$$

□

Corollary 3.4.7. $\lim_{\beta<\eta_0} F_\beta(n) = F_{\eta_0}(n)$ *if and only if* $\lim_{\beta<\eta_0} \mathrm{H}^\beta(n) = \ker(\bar{Q}_{\eta_0}(n)) + \operatorname{im}(\rho^{\eta_0}(n))$. *In particular, if the clutching homomorphisms are epimorphisms, then* $F_{\eta_0} = \lim_{\beta<\eta_0} F_\beta$.

$$\frac{\lim_{\beta<\eta_0} F_\beta}{F_{\eta_0}} \rightarrowtail \frac{\mathrm{H}^1}{F_{\eta_0}} \longrightarrow \lim_{\beta<\eta_0} \frac{\mathrm{H}^1}{F_\beta} \twoheadrightarrow \lim{}^1_{\beta<\eta_0} F_\beta \ :$$

Lemma 3.4.8. $\dfrac{\lim_{\beta<\eta_0} F_\beta(n)}{F_{\eta_0}(n)} \cong \lim_{\beta<\eta_0} \dfrac{F_\beta(n)}{F_{\eta_0}(n)}$.

Proof. If we take the inverse limit of the short exact sequence of towers

$$F_{\eta_0}(n) \rightarrowtail \{F_\beta(n)\}_{\beta<\eta_0} \twoheadrightarrow \left\{\frac{F_\beta(n)}{F_{\eta_0}(n)}\right\}_{\beta<\eta_0}$$

we obtain

$$F_{\eta_0}(n) \rightarrowtail \lim_{\beta<\eta_0} F_\beta(n) \twoheadrightarrow \lim_{\beta<\eta_0} \frac{F_\beta(n)}{F_{\eta_0}(n)},$$

because $\lim^1_{\beta<\eta_0} F_{\eta_0}(n) = 0$. $\square$

Lemma 3.4.9. *We have the exact sequence*

$$\frac{\lim_{\beta<\eta_0} F_\beta(n)}{F_{\eta_0}(n)} \rightarrowtail \frac{\mathrm{H}^1(n)}{F_{\eta_0}(n)} \longrightarrow \lim_{\beta<\eta_0} \frac{\mathrm{H}^1(n)}{F_\beta(n)} \twoheadrightarrow \lim^1_{\beta<\eta_0} F_\beta(n)\ .$$

Proof. If we take inverse limits of the short exact sequence of towers

$$\left\{\frac{F_\beta(n)}{F_{\eta_0}(n)}\right\}_{\beta<\eta_0} \rightarrowtail \frac{\mathrm{H}^1(n)}{F_{\eta_0}(n)} \twoheadrightarrow \left\{\frac{\mathrm{H}^1(n)}{F_\beta(n)}\right\}_{\beta<\eta_0},$$

we are done by the previous lemma. Note that

$$\lim_{\beta<\eta_0}^{1} F_\beta(n) \cong \lim_{\beta<\eta_0}^{1} \frac{F_\beta(n)}{F_{\eta_0}(n)}.$$

$\square$

$\dfrac{\lim_{\beta<\eta_0} F^\beta}{F^{\eta_0}} \rightarrowtail \ker(\rho^{\eta_0})$:

Lemma 3.4.10. $\dfrac{\lim_{\beta<\eta_0} F^\beta(n)}{F^{\eta_0}(n)} \cong \mathrm{im}(Q^{\eta_0}(n)) \cap \ker(\rho^{\eta_0}(n))$. *In particular, we have a monomorphism* $\dfrac{\lim_{\beta<\eta_0} F^\beta(n)}{F^{\eta_0}(n)} \rightarrowtail \ker(\rho^{\eta_0}(n))$.

Proof. In the following diagram with short exact rows the vertical arrow on the right is an isomorphism

$$\begin{tikzcd}
\ker(Q^{\eta_0}(n)) \arrow[r, tail] \arrow[d, equal] & \ker(\rho^{\eta_0}(n) \circ Q^{\eta_0}(n)) \arrow[r, two heads] \arrow[d, equal] & \dfrac{\ker(\rho^{\eta_0}(n) \circ Q^{\eta_0}(n))}{\ker(Q^{\eta_0}(n))} \arrow[d, "\cong"] \\
\ker(Q^{\eta_0}(n)) \arrow[r, tail] & (Q^{\eta_0}(n))^{-1}(\ker(\rho^{\eta_0}(n))) \arrow[r, two heads] & \mathrm{im}(Q^{\eta_0}(n)) \cap \ker(\rho^{\eta_0}(n))
\end{tikzcd}$$

and we have $\ker(Q^{\eta_0}(n)) = F^{\eta_0}(n)$ and $\ker(\rho^{\eta_0}(n) \circ Q^{\eta_0}(n)) = \lim_{\beta<\eta_0} F^\beta(n)$. Therefore,

$$\frac{\lim_{\beta<\eta_0} F^\beta(n)}{F^{\eta_0}(n)} = \frac{\ker(\rho^{\eta_0}(n) \circ Q^{\eta_0}(n))}{\ker(Q^{\eta_0}(n))} \cong \mathrm{im}(Q^{\eta_0}(n)) \cap \ker(\rho^{\eta_0}(n)).$$

$\square$

Corollary 3.4.11. $\lim_{\beta<\eta_0} F^\beta(n) = F^{\eta_0}(n)$ *if and only if* $\mathrm{im}(Q^{\eta_0}(n)) \cap \ker(\rho^{\eta_0}(n)) = 0$. *In particular, if the clutching structures are monomorphisms (i.e.,* $\ker(\rho^{\eta_0}(n)) = 0$*), then* $\lim_{\beta<\eta_0} F^\beta(n) = F^{\eta_0}(n)$.

$$\frac{\lim_{\beta<\eta_0} F^\beta(n)}{F^{\eta_0}(n)} \rightarrowtail \frac{\ker(\mathrm{H}^\lambda(n) \to \mathrm{H}^1(n))}{F^{\eta_0}(n)} \longrightarrow \lim_{\beta<\eta_0} \frac{\ker(\mathrm{H}^\lambda(n) \to \mathrm{H}^1(n))}{F^\beta(n)} \twoheadrightarrow \lim^1_{\beta<\eta_0} F^\beta(n) \;:$$

Lemma 3.4.12. $\dfrac{\lim_{\beta<\eta_0} F^\beta(n)}{F^{\eta_0}(n)} \cong \lim_{\beta<\eta_0} \dfrac{F^\beta(n)}{F^{\eta_0}(n)}$.

Proof. Take inverse limits of the short exact sequence

$$F^{\eta_0}(n) \rightarrowtail \left\{F^\beta(n)\right\}_{\beta<\eta_0} \twoheadrightarrow \left\{\frac{F^\beta(n)}{F^{\eta_0}(n)}\right\}_{\beta<\eta_0}.$$

□

Lemma 3.4.13. *We have the exact sequence*

$$\frac{\lim_{\beta<\eta_0} F^\beta(n)}{F^{\eta_0}(n)} \rightarrowtail \frac{\ker(\mathrm{H}^\lambda(n) \to \mathrm{H}^1(n))}{F^{\eta_0}(n)} \longrightarrow \lim_{\beta<\eta_0} \frac{\ker(\mathrm{H}^\lambda(n) \to \mathrm{H}^1(n))}{F^\beta(n)} \twoheadrightarrow \lim^1_{\beta<\eta_0} F^\beta(n).$$

Proof. If we take inverse limits of the short exact sequence of towers

$$\left\{\frac{F^\beta(n)}{F^{\eta_0}(n)}\right\}_{\beta<\eta_0} \rightarrowtail \frac{\ker(\mathrm{H}^\lambda(n) \to \mathrm{H}^1(n))}{F^{\eta_0}(n)} \twoheadrightarrow \left\{\frac{\ker(\mathrm{H}^\lambda(n) \to \mathrm{H}^1(n))}{F^\beta(n)}\right\}_{\beta<\eta_0}$$

we are done by the previous lemma. Note that

$$\lim_{\beta<\eta_0}{}^1 F^\beta(n) \cong \lim_{\beta<\eta_0}{}^1 \left\{\frac{F^\beta(n)}{F^{\eta_0}(n)}\right\}.$$

□

3.4.3 Matching Eventually Vanishing Exact Couples to a Tower

In this section, we match eventually vanishing exact couples to the tower (3.14), with a minor modification of what we have provided in the last section.

Definition 3.4.14. *We* match eventually vanishing exact couples *to the tower (3.14) if for every limit ordinal* $\eta_0 < \lambda$*, there is an eventually vanishing exact couple* $\mathcal{EC}(\eta_0)$ *and homomorphism* β*, as depicted in the diagram on page 207, such that*

- $F^{\eta_0+r}(n) = \ker(\mathrm{H}^\lambda(n) \xrightarrow{Q^{\eta_0+r}} \mathrm{H}^{\eta_0+r}(n) \xrightarrow{\beta^{u,v+r\mathbf{a}}} D^1_{u,v+r\mathbf{a}})$,

where $\mathrm{H}^*(n)$ *and* $F^*(n)$ *are the* n*-th degree of* H^* *and* F^**, respectively, and* $(u,v)\cdot\hat{\mathbf{a}} = n$*. The homomorphism* β *is called the* matching homomorphism*.*

Remark 3.4.15. By Definition 3.4.14, we look at H^λ (in fact F^{η_0}) as a *far* coaugmentation of $\mathcal{EC}(\eta_0)$ and we consider the ω-length filtration segment of H^λ

$$F^{\eta_0} \supseteq F^{\eta_0+1} \supseteq \cdots \supseteq F^{\eta_0+r} \supseteq F^{\eta_0+r+1} \supseteq \cdots \supseteq F^{\eta_0+\omega}.$$

Unlike the case we matched eventually stable exact couples to the tower (3.14), here H^1 does not play the role of a non-trivial augmentation of $\mathcal{EC}(\eta_0)$, because any homomorphism from the D^1-terms of $\mathcal{EC}(\eta_0)$ to an augmentation must factor through the universal augmentation of $\mathcal{EC}(\eta_0)$, which is zero.

We know that an eventually vanishing exact couple is coaugmentation concentrated: See Remark 3.2.8 on page 72. Therefore, by the E^∞-Distribution Theorem, we have

$$E^\infty_{u,v-\mathbf{c}-r\mathbf{a}} \leftarrowtail \epsilon^{u,v-r\mathbf{a}} \leftarrowtail \frac{F^{\eta_0+(r-1)}(n)}{F^{\eta_0+r}(n)}$$

These filtrations and the information provided by the eventually vanishing exact couples can be combined in the diagram on page 208, where arrows of the same color represent exactness.

Look at the proof of Proposition 5.7.1 on page 134 and Proposition 5.7.3 on page 137 for examples of the argument given in Remark 3.4.4 on page 88, modified to the case we match eventually vanishing exact couples to a tower.

Example 3.4.16. (**Transfinite Tower of Fibrations**) Let

$$\begin{tikzcd}
X_1 & X_2 \arrow[l] & \cdots \arrow[l] & X_\eta \arrow[l] & X_{\eta+1} \arrow[l] & X_{\eta+2} \arrow[l] & \cdots \arrow[l] \\
F_1 \arrow[u, equal] & F_2 \arrow[u] & \cdots & \{pt\} \arrow[u] & F_{\eta+1} \arrow[u] & F_{\eta+2} \arrow[u] &
\end{tikzcd}$$

be a tower of homotopy fibrations indexed over a limit ordinal λ where each $X_{\eta+1} \to X_\eta$ is a fibration with fiber $F_{\eta+1}$ and all spaces involved are path-connected with Abelian fundamental groups. Let $X := \mathrm{holim}_{\eta<\lambda} X_\eta$ be the homotopy inverse limit of X_η's. We assume that for every limit ordinal $\eta_0 \leq \lambda$, $\mathrm{holim}_{\beta<\eta_0} X_\beta = X_{\eta_0}$. For every limit ordinal $\eta_0 < \lambda$, the following ω-length stage of the tower of fibrations

$$\begin{tikzcd}
X_{\eta_0} & X_{\eta_0+1} \arrow[l] & X_{\eta_0+2} \arrow[l] & \cdots \arrow[l] & X_{\eta_0+r} \arrow[l] & X_{\eta_0+r+1} \arrow[l] & \cdots \arrow[l] \\
& F_{\eta_0+1} \arrow[u] & F_{\eta_0+2} \arrow[u] & \cdots & F_{\eta_0+r} \arrow[u] & F_{\eta_0+r+1} \arrow[u] &
\end{tikzcd}$$

defines an eventually vanishing exact couple and hence induces a homotopy spectral sequence as stated in Example 3.2.10 on page 75. Therefore, we have transfinite towers of homotopy groups of fibrations that are shown in blue in the diagram on page 209, where we have picked Bousfield-Friedlander spectral sequence. As a result, the matching homomorphism β in Definition 3.4.14 on page 92 is isomorphism. Note that, for every integer n, $\rho^{\eta_0}(n) : \pi_n(X_{\eta_0}) \to \lim_{\beta<\eta_0} \pi_n(X_\beta)$ is the clutching homomorphism.

It is well-known ([9]) that if λ is an arbitrary limit ordinal, then for every $n \in \mathbb{N}$ the following sequence that is exact on the first and second positions

$$\lim^1_{\eta<\lambda} \pi_{n+1}(X_\eta) \rightarrowtail \pi_n(X) \xrightarrow{\rho^\lambda(n)} \lim_{\eta<\lambda} \pi_n(X_\eta)$$

exists if for all limit ordinals $\eta_0 < \lambda$ and integers $k \geq 2$, $\lim^1_{\beta<\eta_0} \pi_{n+k}(X_\beta) = 0$. If, in addition, we have $\lim^1_{\beta<\eta_0} \pi_{n+1}(X_\beta) = 0$, then $\rho^\lambda(n)$ is an epimorphism and hence we obtain a short exact sequence

$$\lim^1_{\eta<\lambda} \pi_{n+1}(X_\eta) \rightarrowtail \pi_n(X) \xrightarrow{\rho^\lambda(n)} \mathrel{\mkern-14mu}\twoheadrightarrow \lim_{\eta<\lambda} \pi_n(X_\eta) \; .$$

In this case, the clutching homomorphisms are epimorphisms.

Chapter 4

Class of Modules Compatible with Spectral Sequences and Transfinite Induction

4.1 Introduction

The contribution of Serre to homotopy theory is very well-known. One of the wonderful notions that he has used is the notion of a class of Abelian groups as a generalization of 0: A *class* of Abelian groups is a non-empty collection of Abelian groups that is closed under isomorphism, sub-object, quotient, extension, Tor, tensor and homology with integer coefficients: See [29]. By the Universal Coefficient Theorem, for a fibration $E \to B$, with fiber F, the second page of the Serre spectral sequence is of the form

$$E^2_{(p,q)} = \mathrm{H}_p(B; \mathrm{H}_q(F)) \cong \mathrm{H}_p(B) \otimes \mathrm{H}_q(F) \oplus \mathrm{Tor}(\mathrm{H}_{p-1}(B) \otimes \mathrm{H}_q(F)).$$

See Example 4.3.6 on page 101. This class is "tailor-made" for the Serre spectral sequence in the sense that while we flip through the pages of the Serre spectral sequence, we stay in a given class. He used this class beautifully to show that if the integral homology groups of any two of E, B and F belong to a given class of Abelian groups, so do the integral homology groups of the third. As an immediate application, he could show that the homotopy groups of a sphere are finitely generated : See [29].

Hu [19] defines a class of Abelian groups to be a non-empty collection of Abelian groups that is closed under isomorphism, sub-object, quotient and extension and shows that if an E-term of the Serre spectral sequence belongs to this class then the E-terms in the successive pages also belong to this class, even the E^∞-term: See [19], p. 301. That is, he offers a more general class that is "tailor-made" for the Serre spectral sequence, in the sense mentioned above.

In this chapter, we start from a different platform and we consider matching convenient exact couples to a (potentially transfinite) tower of bigraded R-modules. For example, assume we have the following ascending filtration of H

$$F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n \subseteq F_{n+1} \subseteq \cdots \subseteq F_\eta \subseteq F_{\eta+1} \subseteq \cdots \subseteq F_\lambda = \mathrm{H}\,. \tag{4.1}$$

Here, we are interested in "staying in a class $\mathcal{C}$ of *modules*" as we climb up the filtration (4.1) by transfinite induction. The idea comes from the discussions with Peschke and the ideas behind the inductive arguments in [26]. We define $\mathcal{C}$ to be a non-empty collection of modules that is closed under isomorphism, sub-object, quotient, extension and colimit of directed towers and we show that if convenient exact couples are *matched* to the filtration (4.1), as we have explained in Section 3.3 and Section 3.4, then this class is "tailor-made" for the mechanics of transfinite induction; i.e., if one filtration stage of (4.1) is in $\mathcal{C}$, with some extra assumptions, so are all other filtration stages, in particular, H itself. When trying to take the inductive step from "$F_\beta \in \mathcal{C}$, for every $\beta < \eta$" to "$F_\eta \in \mathcal{C}$", we face one of the following two situations:

Situation 1: η is a non-limit ordinal and hence it has at most finitely many predecessors. So there exists a limit ordinal η_0 and a positive integer r such that $\eta = \eta_0 + r$. Since we have assumed that we have matched a convenient exact couple to the ω-length filtration segment

$$F_{\eta_0} \subseteq F_{\eta_0+1} \subseteq \cdots \subseteq F_{\eta_0+r-1} \subseteq F_{\eta_0+r} \subseteq \cdots \subseteq F_{\eta_0+\omega},$$

we can use the relationship between the E^∞-terms of the corresponding spectral sequence and $\dfrac{F_\eta}{F_\beta}$ to take the inductive step from F_β to F_η.

Situation 2: η is a limit ordinal. The inductive step from $\operatorname{colim}_{\beta<\eta} F_\beta$ to F_η can be taken by putting assumptions on the clutching homomorphisms.

It turns out that even if we drop the last two conditions from the list of the closure properties of a class, that is, if we have a class $\mathcal{C}$ of modules closed under isomorphism, sub-object and quotient, then $\mathcal{C}$ is "tailor-made" for the mechanics of *every* spectral sequence; i.e.,

- *if in an arbitrary spectral sequence, $E^r_{(p,q)}$ is in $\mathcal{C}$ then, for every $r \le s \le \infty$, $E^s_{(p,q)}$ also belongs to $\mathcal{C}$.*

This is a generalization of the idea of Serre [29] that works for *every* spectral sequence. We will explain this case in the first section of this chapter.

In the second and third sections, we consider the task of "staying in a class of modules" as we climb up the ω-length filtration of the universal augmentation and coaugmentation of our convenient exact couples. For example, we show that

- *if the E^∞-terms of an originally (eventually) vanishing exact couple belong to a class of modules with some closure properties, then the universal augmentation (coaugmentation) also belong to that class.*

In the third and fourth sections, we generalize this task of "staying in a class of modules" to the case that we have a module which is transfinitely filtered by a tower of modules where we can match convenient exact couples to it. Using the inductive argument explained above, we will show, for example,

- *in a transfinite tower of cofibrations, if the homology groups of the cofibers are in $\mathcal{C}$, then so are the homology groups of the homotopy colimit of the tower.*

4.2 Compatibility with Spectral Sequences

In [29], Serre considered the spectral sequence associated to a fibration, named after him as Serre spectral sequence, and showed that if $E^r_{(p,q)}$ is in a class of Abelian groups - closed under sub-object, quotient, extension, Tor, tensor and homology with integer coefficients - then $E^{r+1}_{(p,q)}$ also belongs to this class; see Example 3.2.5 on page 70 and Example 4.3.6 on page 101. In other words, he showed that this class of Abelian groups is compatible with the mechanics of the spectral sequence associated to a fibration.

We flip through the pages of a spectral sequence by taking homology; i.e., we find the kernel of the d^r, the differential on page r, that leaves the position (p,q) and then we find its quotient modulo the image of the

d^r that enters the position (p,q). In fact, every $E^{r+1}_{(p,q)}$ is a "sub-quotient" of $E^r_{(p,q)}$. This is our motivation for the next definition. Throughout this chapter we assume that R is a commutative unitary ring.

Definition 4.2.1. *A* class *of R-modules is a collection $\mathcal{C}$ of R-modules containing the trivial module that is closed under:*

1. *isomorphisms; i.e., if $A \cong B$ then $A \in \mathcal{C}$ if and only if $B \in \mathcal{C}$,*
2. *sub-objects; i.e., if $A \rightarrowtail B$ and $B \in \mathcal{C}$, then $A \in \mathcal{C}$,*
3. *quotients; i.e., if $B \twoheadrightarrow C$ and $B \in \mathcal{C}$, then $C \in \mathcal{C}$.*

The following proposition shows that $\mathcal{C}$ can pass through pages of *any* spectral sequence, even to the limit page. Serre [29] and Hu [19] proved this proposition only for the Serre spectral sequence and with more hypotheses on $\mathcal{C}$.

Proposition 4.2.2. *Let $(E^r, d^r \mid r \geq r_0)$ be a spectral sequence. If for every $(p,q) \in \mathbb{Z} \times \mathbb{Z}$ there is some $r \in \mathbb{N}$ such that $E^r_{(p,q)} \in \mathcal{C}$, then for every $r \leq s \leq \infty$ we have $E^s_{(p,q)} \in \mathcal{C}$.*

Proof. Take $r < s < \infty$. Since we have $E^r_{(p,q)} \in \mathcal{C}$, then, using the notation in section 2.2.2 in Chapter 1, the inclusion

$$\frac{Z^{s-1}_{(p,q)}}{B^{r-1}_{(p,q)}} \rightarrowtail \frac{Z^{r-1}_{(p,q)}}{B^{r-1}_{(p,q)}} = E^r_{(p,q)}$$

shows that $\dfrac{Z^{s-1}_{(p,q)}}{B^{r-1}_{(p,q)}} \in \mathcal{C}$. From the following epimorphism

$$\frac{Z^{s-1}_{(p,q)}}{B^{r-1}_{(p,q)}} \twoheadrightarrow \frac{Z^{s-1}_{(p,q)}}{B^{s-1}_{(p,q)}} = E^s_{(p,q)}$$

we obtain $E^s_{(p,q)} \in \mathcal{C}$.

Let $s = \infty$. Since we have $E^r_{(p,q)} \in \mathcal{C}$, then, using the notation in section 2.2.2 in Chapter 1, the inclusion

$$\frac{Z^{\infty}_{(p,q)}}{B^{r-1}_{(p,q)}} \rightarrowtail \frac{Z^{r-1}_{(p,q)}}{B^{r-1}_{(p,q)}} = E^r_{(p,q)}$$

shows that $\dfrac{Z^{\infty}_{(p,q)}}{B^{r-1}_{(p,q)}} \in \mathcal{C}$. From the following epimorphism

$$\frac{Z^{\infty}_{(p,q)}}{B^{r-1}_{(p,q)}} \twoheadrightarrow \frac{Z^{\infty}_{(p,q)}}{B^{\infty}_{(p,q)}} = E^{\infty}_{(p,q)}$$

we obtain $E^{\infty}_{(p,q)} \in \mathcal{C}$. □

In the applications of class theory, the objects in a class $\mathcal{C}$ are usually to be neglected in a certain sense, [19]. The following definition is classical for class of Abelian groups that is closed under isomorphism, subobject, quotient and extension: See [19] or [30]. Here we state it for a class of R-modules.

Definition 4.2.3. *Let $\mathcal{C}$ be a given class of R-modules and $f : A \to B$ be a homomorphism. Then f is said to be*

1. *a* $\mathcal{C}$-monomorphism *if* $\ker(f) \in \mathcal{C}$*,*
2. *a* $\mathcal{C}$-epimorphism *if* $\operatorname{coker}(f) \in \mathcal{C}$ *and*
3. *a* $\mathcal{C}$-isomorphism *if it is both $\mathcal{C}$-monomorphism and $\mathcal{C}$-epimorphism.*

If in this definition $\mathcal{C}$ is $\{0\}$, then these notions coincide with the corresponding classical notions.

Here is a rather trivial example:

Example 4.2.4. Let $f : A \to B$ be a module homomorphism and $\mathcal{C}$ be a class of modules.

1. If $A \in \mathcal{C}$, then f is a $\mathcal{C}$-monomorphism.
2. If $B \in \mathcal{C}$, then f is a $\mathcal{C}$-epimorphism.
3. If $A, B \in \mathcal{C}$, then f is a $\mathcal{C}$-isomorphism.

4.3 Compatibility with Originally Stable or Vanishing Exact Couples

In this section, we consider the spectral sequence induced by an originally stable or vanishing exact couple and we show that with a few more assumptions on a class $\mathcal{C}$ of modules, not only do we stay in $\mathcal{C}$ when we flip through the pages of the spectral sequence, but we can also show that the universal augmentation and coaugmentation of the exact couple are also in $\mathcal{C}$.

Definition 4.3.1. 1. *An* ω-cocomplete *class of* R*-modules is a class* $\mathcal{C}$ *of* R*-modules that is closed under the colimit of* ω*-length directed towers of* R*-modules; i.e., if* $\{A_n\}_{n\in\mathbb{N}}$ *is a directed tower of* R*-modules such that for every* $n \in \mathbb{N}$ *we have* $A_n \in \mathcal{C}$*, then* $\operatorname{colim}_n A_n \in \mathcal{C}$*.*

2. *A class* $\mathcal{C}$ *of* R*-modules is closed under* extensions *when in a short exact sequence* $A \rightarrowtail B \twoheadrightarrow C$*, if* $A, C \in \mathcal{C}$ *then* $B \in \mathcal{C}$*.*

Look at the E^∞-Distribution Theorem 2.3.13 on page 35 for notations.

Proposition 4.3.2. *Let* $\mathcal{C}$ *be an* ω*-cocomplete class of* R*-modules that is closed under extensions and assume the* E^∞*-terms of the spectral sequence induced by an originally stable exact couple are in* $\mathcal{C}$*. Then, for every* $n \in \mathbb{Z}$*,*

1. *if* $\cap_s \phi_{(p,q)+s\mathbf{a}} \in \mathcal{C}$ *then* $L_n \in \mathcal{C}$*, where* $(p,q)\cdot\hat{\mathbf{a}} = n$*, and*
2. $\ker(L^n \to L_n) \in \mathcal{C}$*.*

Proof. Let $D^1_{u,v}$ be the lowest stable term of the n-th D^1-column.

1. We show that for every $s \in \mathbb{N}$, we have $\phi_{u,v+s\mathbf{a}} \in \mathcal{C}$:

- For $s = 0$ we know that $\cap_s \phi_{u,v+s\mathbf{a}} = \phi_{u,v}$. Therefore, $\phi_{u,v} \in \mathcal{C}$.
- Assume $\phi_{u,v+(s-1)\mathbf{a}} \in \mathcal{C}$. By the following diagram from Remark 3.2.2 on page 65

$$\begin{tikzcd} \phi_{u,v+(s-1)\mathbf{a}} \arrow[r, tail] & \phi_{u,v+s\mathbf{a}} \arrow[r, two heads] & \dfrac{\phi_{u,v+s\mathbf{a}}}{\phi_{u,v+(s-1)\mathbf{a}}} \arrow[d, tail] \\ & & E^\infty_{u,v+\mathbf{b}+s\mathbf{a}} \end{tikzcd}$$

  $E^\infty_{u,v+\mathbf{b}+s\mathbf{a}} \in \mathcal{C}$ implies $\dfrac{\phi_{u,v+s\mathbf{a}}}{\phi_{u,v+(s-1)\mathbf{a}}} \in \mathcal{C}$ and by induction assumption we obtain $\phi_{u,v+s\mathbf{a}} \in \mathcal{C}$.

Since $L_n = \operatorname{colim}_s \phi_{u,v+s\mathbf{a}} \in \mathcal{C}$, we are done.

2. We show that for every $s \geq 1$, we have $\phi^{u,v+s\mathbf{a}} \in \mathcal{C}$:

- For $s = 1$ we know that $\cap_s \phi^{u,v+s\mathbf{a}} = \phi^{u,v} = 0$. Therefore, by the following diagram from part 2 of Remark 3.2.2 on page 65

$$\begin{tikzcd} & & E^\infty_{u,v+\mathbf{b}} \arrow[d, two heads] \\ 0 = \phi^{u,v} \arrow[r, tail] & \phi^{u,v+\mathbf{a}} \arrow[r, two heads] & \dfrac{\phi^{u,v+\mathbf{a}}}{\phi^{u,v}} \end{tikzcd}$$

$E^\infty_{u,v+\mathbf{b}} \in \mathcal{C}$ implies $\dfrac{\phi^{u,v+\mathbf{a}}}{\phi^{u,v}} = \phi^{u,v+\mathbf{a}} \in \mathcal{C}$.

- Assume $\phi^{u,v+(s-1)\mathbf{a}} \in \mathcal{C}$. By the following diagram from part 2 of Remark 3.2.2 on page 65

$$\begin{tikzcd} & & E^\infty_{u,v+\mathbf{b}+s\mathbf{a}} \arrow[d, two heads] \\ \phi^{u,v+(s-1)\mathbf{a}} \arrow[r, tail] & \phi^{u,v+s\mathbf{a}} \arrow[r, two heads] & \dfrac{\phi^{u,v+s\mathbf{a}}}{\phi^{u,v+(s-1)\mathbf{a}}} \end{tikzcd}$$

$E^\infty_{u,v+\mathbf{b}+s\mathbf{a}} \in \mathcal{C}$ implies $\dfrac{\phi^{u,v+s\mathbf{a}}}{\phi^{u,v+(s-1)\mathbf{a}}} \in \mathcal{C}$ and by induction assumption we obtain $\phi^{u,v+s\mathbf{a}} \in \mathcal{C}$.

Since $\ker(L^n \to L_n) = \operatorname{colim}_s \phi^{u,v+s\mathbf{a}} \in \mathcal{C}$, we are done.

$\square$

Corollary 4.3.3. *Let $\mathcal{C}$ be an ω-cocomplete class of R-modules closed under extensions and the E^∞-terms of the spectral sequence induced by an originally vanishing exact couple be in $\mathcal{C}$. Then $L_* \in \mathcal{C}$.*

Corollary 4.3.4. *Let $\mathcal{C}$ be a class of R-modules closed under extensions and the E^∞-terms of the spectral sequence induced by an originally vanishing and eventually stable exact couple be in $\mathcal{C}$. Then $L_* \in \mathcal{C}$.*

Example 4.3.5. Let $\mathcal{C}$ be an ω-cocomplete class of Abelian groups closed under extensions. In a tower of cofibrations, if the homology groups of the cofibers are in $\mathcal{C}$, then the homology groups of the colimit of the tower are also in $\mathcal{C}$: See Example 3.2.4 on page 68.

Example 4.3.6. In the setting of Example 3.2.5 on page 70, for a fibration $E \to B$ with fiber F, where B is a finite CW-complex, the Leray-Serre spectral sequence is induced by an originally vanishing and eventually stable exact couple. For such a fibration, Serre picked a class of Abelian groups $\mathcal{C}_{\mathcal{S}}$ that, in addition to closure under isomorphism, sub-object, quotient and extension, is also closed under tensor product, Tor and integral homology; i.e., for every $A, B \in \mathcal{C}_{\mathcal{S}}$, $A \otimes B$, $\mathrm{Tor}(A, B)$ and $\mathrm{H}_*(A; \mathbb{Z})$ are in $\mathcal{C}_{\mathcal{S}}$. We call this class of Abelian groups a *Serre class*. The Universal Coefficient Theorem ensures that if for every $(p, q) \in \mathbb{N} \times \mathbb{N}$, we have $\mathrm{H}_p(B), \mathrm{H}_q(F) \in \mathcal{C}_{\mathcal{S}}$, then $E^2_{(p,q)} \in \mathcal{C}_{\mathcal{S}}$, because

$$E^2_{(p,q)} = \mathrm{H}_p(B; \mathrm{H}_q(F)) \cong \mathrm{H}_p(B) \otimes \mathrm{H}_q(F) \oplus \mathrm{Tor}(\mathrm{H}_{p-1}(B), \mathrm{H}_q(F)).$$

In [29], Serre showed that if, for every $(p, q) \in \mathbb{N} \times \mathbb{N}$, we have $\mathrm{H}_p(B)$ and $\mathrm{H}_q(F) \in \mathcal{C}_{\mathcal{S}}$, then $\mathrm{H}_*(E)$ is also in $\mathcal{C}_{\mathcal{S}}$.

Here, if we take $\mathcal{C}$ to be only a class of Abelian groups that is closed under isomorphism, sub-object, quotient and extension and not necessarily closed under tensor, Tor and integral homology, and also $E^2_{(p,q)} = \mathrm{H}_p(B; \mathrm{H}_q(F)) \in \mathcal{C}$ for every $p, q \in \mathbb{N}$, then by Corollary 4.3.4, $\mathrm{H}_*(E) \in \mathcal{C}$. Closure under tensor, Tor and integral homology are dictated only because of the special description of the second page of the Leray-Serre spectral sequence.

Example 4.3.7. In the setting of the Grothendieck spectral sequence in Example 3.2.11 on page 77, let $\mathcal{C}$ be a class of Abelian groups closed under extensions and $\mathcal{A} \xrightarrow{G} \mathcal{B} \xrightarrow{F} \mathcal{A}b$ be covariant additive functors. Assume that F is right exact and that GP is left F-acyclic for every projective object P in $\mathcal{A}$. By Corollary 4.3.4 on page 101, if for every $p, q \in \mathbb{Z}$ and $A \in \mathcal{A}$ we have $(L_pF)(L_qG)A \in \mathcal{C}$, then for every $n \in \mathbb{N}$, $L_n(FG)A \in \mathcal{C}$.

As a special case, if $F = A \otimes_R -$, then every projective R-module P is left F-acyclic because $\mathrm{Tor}^R_p(A, P) = \{0\}$, for every $p \geq 1$. So we have the following

- (**Change of Basis**) Let $f : R \to S$ be a ring homomorphism. Then there is a spectral sequence

  $$E_2^{(p,q)} = \mathrm{Tor}^S_p(\mathrm{Tor}^R_q(A, S), B) \Rightarrow \mathrm{Tor}^R_{p+q}(A, B)$$

  for every S-module A and R-module B.

By Corollary 4.3.4, if for every $(p, q) \in \mathbb{Z} \times \mathbb{Z}$ and every S-module A and R-module B, we have $\mathrm{Tor}^S_p(\mathrm{Tor}^R_q(A, S), B) \in \mathcal{C}$, then $\mathrm{Tor}^R_{p+q}(A, B) \in \mathcal{C}$.

Corollary 4.3.8. *Let $\mathcal{C}$ be a class of R-modules closed under extensions and $\{D^1_{*,*}, E^1_{*,*}\}$ be an originally vanishing and eventually stable exact cou-*

ple. If for every $(p,q) \in \mathbb{Z} \times \mathbb{Z}$ there is some $r \leq \infty$ such that $E^r_{(p,q)} \in \mathcal{C}$, then

1. *the dashed arrow*

$$\begin{tikzcd} L_n \arrow[r, dashed] \arrow[d, two heads] & E^t_{u,v+\mathbf{b}} \\ \dfrac{L_n}{\mathrm{im}(i^1_{u,v-\mathbf{a}})} \arrow[r, "\cong"] & E^\infty_{u,v+\mathbf{b}} \arrow[u, tail] \end{tikzcd}$$

   is a $\mathcal{C}$-monomorphism for any $2 \leq t$, where $D^1_{u,v}$ is a stable term and $(u,v) \cdot \hat{\mathbf{a}} = n$.

2. *$L_n \to E^s_{u,v+\mathbf{b}}$ is a $\mathcal{C}$-isomorphism for large enough s, where $D^1_{u,v}$ is a stable term and $(u,v) \cdot \hat{\mathbf{a}} = n$.*

Proof. First remember from Remark 3.2.2 on page 64 that every eventually vanishing exact couple converges to its universal augmentation. By Corollary 4.3.4 on page 101, we know that $L_n \in \mathcal{C}$. Since $L_n \cong D^1_{u,v}$ we also have a morphism

$$L_n \twoheadrightarrow \frac{L_n}{\mathrm{im}(i^1_{u,v-\mathbf{a}})} = \epsilon_{u,v} \cong E^\infty_{u,v+\mathbf{b}} \rightarrowtail E^t_{u,v+\mathbf{b}}$$

for every $2 \leq t$; the inclusion is given by the fact that the exact couple is downward stable.

1. It follows from Example 4.2.4 on page 99.

2. Let r be the integer that $E^r_{u,v+\mathbf{b}} \in \mathcal{C}$. Then, by Proposition 4.2.2 on page 98, for every s that $r \leq s \leq \infty$, we have $E^s_{u,v+\mathbf{b}} \in \mathcal{C}$. By part 1 and Example 4.2.4 on page 99 we are done.

$\square$

Example 4.3.9. Let $\mathcal{C}$ be a class of Abelian groups closed under extensions.

1. In the setting of the Leray-Serre spectral sequence in Example 3.2.5 on page 70, for a fibration $E \to B$ with fiber F and finite CW-complex B, if for every (p,q) we have $\mathrm{H}_p(B; \mathrm{H}_q(F)) \in \mathcal{C}$, then $\mathrm{H}_{p+q}(E) \to \mathrm{H}_{p+q}(B)$ is a $\mathcal{C}$-isomorphism. If $\mathcal{C}$ is a Serre class $\mathcal{C}_{\mathcal{S}}$

then we can restate the last sentence as: if for every (p, q), $\mathrm{H}_p(B)$ and $\mathrm{H}_q(F)$ are in $\mathcal{C}_\mathcal{S}$, then $\mathrm{H}_{p+q}(E) \to \mathrm{H}_{p+q}(B)$ is a $\mathcal{C}_\mathcal{S}$-isomorphism: See Example 4.3.6 on page 101.

2. In the setting of the Atiyah-Hirzebruch-Whitehead spectral sequence in Example 3.2.6 on page 71, for a finite CW-complex X, if for every (p, q) we have $\mathrm{H}_p(X; h_q(pt)) \in \mathcal{C}$, then $h_{p+q}(X) \to \mathrm{H}_{p+q}(X; h_0(pt))$ is a $\mathcal{C}$-isomorphism. Note that the finiteness of X guarantees that the corresponding exact couple is eventually stable. If $\mathcal{C}$ is a Serre class $\mathcal{C}_\mathcal{S}$ then we can restate the last sentence as: if for every (p, q), $\mathrm{H}_p(X)$ and $h_q(pt)$ are in $\mathcal{C}_\mathcal{S}$, then $h_{p+q}(X) \to \mathrm{H}_{p+q}(X; h_0(pt))$ is a $\mathcal{C}_\mathcal{S}$-isomorphism: See Example 4.3.6 on page 101.

4.4 Compatibility with Eventually Stable or Vanishing Exact Couples

In this section, we consider the spectral sequence induced by an eventually stable or vanishing exact couple and we prove the analogous results we had in the previous section for these exact couples too.

Definition 4.4.1. *An* ω-complete *class of R-modules is a class of R-modules $\mathcal{C}$ that is also closed under the limit of ω-length inverse towers of R-modules; i.e., if $\{A_n\}_{n\in\mathbb{N}}$ is an inverse tower of R-modules such that for every $n \in \mathbb{N}$ we have $A_n \in \mathcal{C}$, then* $\lim_n A_n \in \mathcal{C}$.

Remark 4.4.2. The cardinality of the product of ω-indexed towers of objects is bounded above by $\aleph_1$: See Theorem 3.8 in [17]. So an ω-complete class of modules closed under extension is not bigger than the class containing all limits of towers indexed by a category whose objects are of cardinality $\aleph_1$.

Look at the E^∞-Distribution Theorem 2.3.13 on page 35 for notations.

Proposition 4.4.3. *Let $\mathcal{C}$ be an ω-complete class of R-modules that is closed under extensions and assume the E^∞-terms of the spectral sequence induced by an eventually stable exact couple are in $\mathcal{C}$. Then, for every $n \in \mathbb{Z}$*

1. *if* $\lim_s \phi_{(p,q)+s\mathbf{a}} \in \mathcal{C}$*, where $(p, q) \cdot \hat{\mathbf{a}} = n$, then $L_n \in \mathcal{C}$.*
2. $\ker(L^n \to L_n) \in \mathcal{C}$*.*

Proof. Let $D^1_{(u,v)}$ be the highest stable term of the n-th D^1-column.

1. We show that for every $s \geq 1$, we have $\dfrac{L_n}{\phi_{(u,v)-s\mathbf{a}}} \in \mathcal{C}$:

- Since $L_n = \phi_{(u,v)}$, for $s = 1$ using the following diagram

$$\begin{tikzcd} \dfrac{\phi_{(u,v)}}{\phi_{(u,v)-\mathbf{a}}} \arrow[r, tail, "\cong"] \arrow[d, tail] & \dfrac{L_n}{\phi_{(u,v)-\mathbf{a}}} \arrow[r, two heads] & 0 \\ E^{\infty}_{(u,v)+\mathbf{b}} & & \end{tikzcd}$$

  $E^{\infty}_{(u,v)+\mathbf{b}} \in \mathcal{C}$ implies $\dfrac{L_n}{\phi_{(u,v)-\mathbf{a}}} \in \mathcal{C}$.

- Assume $\dfrac{L_n}{\phi_{(u,v)-(s-1)\mathbf{a}}} \in \mathcal{C}$. By the following part of the diagram on page 196

$$\begin{tikzcd} \dfrac{\phi_{(u,v)-(s-1)\mathbf{a}}}{\phi_{(u,v)-s\mathbf{a}}} \arrow[r, tail] \arrow[d, tail] & \dfrac{L_n}{\phi_{(u,v)-s\mathbf{a}}} \arrow[r, two heads] & \dfrac{L_n}{\phi_{(u,v)-(s-1)\mathbf{a}}} \\ E^{\infty}_{(u,v)+\mathbf{b}-s\mathbf{a}} & & \end{tikzcd}$$

  $E^{\infty}_{(u,v)+\mathbf{b}-s\mathbf{a}} \in \mathcal{C}$ implies $\dfrac{\phi_{(u,v)-(s-1)\mathbf{a}}}{\phi_{(u,v)-s\mathbf{a}}} \in \mathcal{C}$ and by induction assumption we obtain $\dfrac{L_n}{\phi_{(u,v)-s\mathbf{a}}} \in \mathcal{C}$.

Therefore, we have $\lim_s \dfrac{L_n}{\phi_{(u,v)-s\mathbf{a}}} \in \mathcal{C}$.

If we take the inverse limit of the short exact sequence of towers

$$\{\phi_{(u,v)-s\mathbf{a}}\}_{s\geq 1} \rightarrowtail L_n \twoheadrightarrow \left\{\frac{L_n}{\phi_{(u,v)-s\mathbf{a}}}\right\}_{s\geq 1}$$

we obtain the following exact sequence

$$\begin{tikzcd}
\lim_s \phi_{(u,v)-s\mathbf{a}} \arrow[r, tail] & L_n \arrow[rr] \arrow[dr] & & \lim_s \dfrac{L_n}{\phi_{(u,v)-s\mathbf{a}}} \arrow[r, two heads] & \lim^1_s \phi_{(u,v)-s\mathbf{a}}. \\
& & \dfrac{L_n}{\lim_s \phi_{(u,v)-s\mathbf{a}}} \arrow[ur, tail] & &
\end{tikzcd}$$

Since $\lim_s \dfrac{L_n}{\phi_{(u,v)-s\mathbf{a}}} \in \mathcal{C}$, we obtain $\dfrac{L_n}{\lim_s \phi_{(u,v)-s\mathbf{a}}} \in \mathcal{C}$. Using the assumption of this part, we have $\lim_s \phi_{(u,v)-s\mathbf{a}} \in \mathcal{C}$. Therefore, we obtain $L_n \in \mathcal{C}$.

2. We show that for every $s \geq 1$, we have $\dfrac{\ker(L^n \to L_n)}{\phi^{u,v-s\mathbf{a}}} \in \mathcal{C}$:

- Let $s = 1$. We know that $\dfrac{\ker(L^n \to L_n)}{\phi^{u,v-\mathbf{a}}} = \dfrac{\phi^{u,v}}{\phi^{u,v-\mathbf{a}}}$. By the E^∞-Distribution Theorem, we have

  $$\begin{tikzcd}
  E^\infty_{(u,v)-\mathbf{a}-\mathbf{c}} \arrow[r, two heads] & \dfrac{Z^\infty_{(u,v)-\mathbf{a}-\mathbf{c}}}{\mathrm{im}(j^1_{(u,v)-\mathbf{a}-\mathbf{b}-\mathbf{c}})} & \dfrac{\ker(L^n \to L_n)}{\phi^{u,v-\mathbf{a}}}. \arrow[l, tail]
  \end{tikzcd}$$

  Since $E^\infty_{(u,v)-\mathbf{a}-\mathbf{c}} \in \mathcal{C}$, we obtain $\dfrac{Z^\infty_{(u,v)-\mathbf{a}-\mathbf{c}}}{\mathrm{im}(j^1_{(u,v)-\mathbf{a}-\mathbf{b}-\mathbf{c}})} \in \mathcal{C}$ and hence $\dfrac{\ker(L^n \to L_n)}{\phi^{u,v-\mathbf{a}}} \in \mathcal{C}$.

- Assume $\dfrac{\ker(L^n \to L_n)}{\phi^{u,v-s\mathbf{a}}} \in \mathcal{C}$. By the following part of the diagram on page 196

  $$\begin{tikzcd}
  E^\infty_{(u,v)-(s+1)\mathbf{a}-\mathbf{c}} \arrow[r, two heads] & \dfrac{Z^\infty_{(u,v)-(s+1)\mathbf{a}-\mathbf{c}}}{\mathrm{im}(j^1_{(u,v)-(s+1)\mathbf{a}-\mathbf{b}-\mathbf{c}})} & \dfrac{\phi^{u,v-s\mathbf{a}}}{\phi^{u,v-(s+1)\mathbf{a}}} \arrow[l, tail] \arrow[r, tail] & \dfrac{\ker(L^n \to L_n)}{\phi^{u,v-(s+1)\mathbf{a}}} \arrow[d, two heads] \\
  & & & \dfrac{\ker(L^n \to L_n)}{\phi^{u,v-s\mathbf{a}}}
  \end{tikzcd}$$

  $E^\infty_{(u,v)-(s+1)\mathbf{a}-\mathbf{c}} \in \mathcal{C}$ implies $\dfrac{Z^\infty_{(u,v)-(s+1)\mathbf{a}-\mathbf{c}}}{\mathrm{im}(j^1_{(u,v)-(s+1)\mathbf{a}-\mathbf{b}-\mathbf{c}})} \in \mathcal{C}$ and hence we have $\dfrac{\phi^{u,v-s\mathbf{a}}}{\phi^{u,v-(s+1)\mathbf{a}}} \in \mathcal{C}$. By induction hypothesis, $\dfrac{\ker(L^n \to L_n)}{\phi^{u,v-(s+1)\mathbf{a}}} \in \mathcal{C}$.

Since $\ker(L^n \to L_n) \cong \lim_s \dfrac{\ker(L^n \to L_n)}{\phi^{u,v-s\mathbf{a}}}$, we obtain $\ker(L^n \to L_n) \in \mathcal{C}$.

$\square$

Corollary 4.4.4. *Let $\mathcal{C}$ be an ω-complete class of R-modules closed under extensions and the E^∞-terms of the spectral sequence induced by an eventually vanishing exact couple be in $\mathcal{C}$. Then $L^* \in \mathcal{C}$.*

Corollary 4.4.5. *Let $\mathcal{C}$ be a class of R-modules closed under extensions and the E^∞-terms of the spectral sequence induced by an eventually vanishing and originally stable exact couple be in $\mathcal{C}$. Then $L^* \in \mathcal{C}$.*

Example 4.4.6. Let $\mathcal{C}$ be an ω-complete class of Abelian groups closed under extensions. In a tower of fibrations as in Example 3.2.10 on page 75, if for every $k \in \mathbb{N}$ the k-th homotopy groups of the fibers are in $\mathcal{C}$ then $\lim_n \pi_k(X_n) \in \mathcal{C}$. By the following short exact sequence from [9]

$$\lim{}^1_n \pi_{k+1}(X_n) \rightarrowtail \pi_k(\lim_n X_n) \twoheadrightarrow \lim_n \pi_k(X_n)$$

if $\lim^1_n \pi_{k+1}(X_n) < in < CC$, then $\pi_k(\lim_n X_n)$ is also in $\mathcal{C}$.

Example 4.4.7. In the setting of the Grothendieck spectral sequence in Example 3.2.11 on page 77, let $\mathcal{C}$ be a class of Abelian groups closed under extensions and $\mathcal{A} \xrightarrow{G} \mathcal{B} \xrightarrow{F} \mathcal{A}b$ be covariant additive functors. Assume that F is left exact and that GE is right F-acyclic for every injective object E in $\mathcal{A}$. By Corollary 4.4.5, if for every $p, q \in \mathbb{Z}$ and $A \in \mathcal{A}$ we have $(R^pF)(R^qG)A \in \mathcal{C}$, then for every $n \in \mathbb{N}$, $R^n(FG)A \in \mathcal{C}$.

As a special case, if $F = \hom_R(A, -)$, then every injective R-module E is right F-acyclic because $\operatorname{Ext}^p_R(A, E) = \{0\}$, for every $p \geq 1$. So we have the following

- (**Change of Basis**) Let $f : R \to S$ be a ring homomorphism. Then there is a spectral sequence

$$E_2^{(p,q)} = \operatorname{Ext}^p_S(A, \operatorname{Ext}^q_R(S, B)) \Rightarrow \operatorname{Ext}^{p+q}_R(A, B)$$

for every S-module A and R-module B.

By Corollary 4.4.5 on page 107, if for every $(p, q) \in \mathbb{Z} \times \mathbb{Z}$ and every S-module A and R-module B, we have $\operatorname{Ext}^p_S(A, \operatorname{Ext}^q_R(S, B)) \in \mathcal{C}$, then $\operatorname{Ext}^{p+q}_R(A, B) \in \mathcal{C}$.

Corollary 4.4.8. *Let $\mathcal{C}$ be a class of R-modules closed under extensions and also $(D^1_{*,*}, E^1_{*,*})$ be an eventually vanishing and originally stable exact couple. If for every $(p, q) \in \mathbb{Z} \times \mathbb{Z}$ there is some $r \leq \infty$ such that $E^r_{(p,q)} \in \mathcal{C}$, then*

1. $E^t_{(u,v)-\mathbf{c}} \to L^n$ *is a* $\mathcal{C}$*-epimorphism for every* $2 \le t$*, where* $D^1_{(u,v)}$ *is the stable term and* $(u,v)\cdot\hat{\mathbf{a}} = n$*.*

2. $E^s_{(u,v)-\mathbf{c}} \to L^n$ *is a* $\mathcal{C}$*-isomorphism for large enough* s*, where* $D^1_{(u,v)}$ *is a stable term and* $(u,v)\cdot\hat{\mathbf{a}} = n$*.*

Proof. First remember from Remark 3.2.8 on page 72 and Remark 3.2.2 on page 64 that every eventually vanishing and originally stable exact couple converges to its universal coaugmentation. By Corollary 4.4.5 on page 107, we know that $L^n \in \mathcal{C}$. Since $L^n \cong D^1_{(u,v)}$ we also have a morphism

$$E^t_{(u,v)-\mathbf{c}} \twoheadrightarrow E^\infty_{(u,v)-\mathbf{c}} \cong \epsilon^{u,v} = \ker(i^1_{(u,v)}) \rightarrowtail D^1_{(u,v)}$$

for every $2 \le t$. The rest of the proof is similar to the proof of Corollary 4.3.8 on page 102. $\square$

Example 4.4.9. Let $\mathcal{C}$ be a class of Abelian groups and $f : R \to S$ be a ring homomorphism. If for every $(p,q) \in \mathbb{Z}\times\mathbb{Z}$ and every S-module A and R-module B, we have $\mathrm{Ext}^p_S(A, \mathrm{Ext}^q_R(S,B)) \in \mathcal{C}$, then, by Example 4.4.7 on page 107, $\hom_S(A, \mathrm{Ext}^q_R(S,B)) \to \mathrm{Ext}^{p+q}_R(A,B)$ is a $\mathcal{C}$-isomorphism.

4.5 Compatibility with Matching Originally Stable or Vanishing ECs

In this section, we generalize the task of "staying in a class of R-modules" to the case that we have an *ascending* filtration of the first and last term of a tower of bigraded R-modules, indexed over an arbitrary limit ordinal, where we can match originally stable or vanishing exact couples to this tower. We need the following definition.

Definition 4.5.1. *An* ordinal-cocomplete *class of* R*-modules is a class* $\mathcal{C}$ *of* R*-modules that is also closed under the colimit of directed towers of* R*-modules; i.e., for any limit ordinal* λ*, if* $\{A_\eta\}_{\eta<\lambda}$ *is a directed tower of* R*-modules such that for every* $\eta < \lambda$ *we have* $A_\eta \in \mathcal{C}$*, then* $\operatorname{colim}_{\eta<\lambda} A_\eta \in \mathcal{C}$*.*

Assume we can match originally stable exact couples to the directed tower of bigraded modules

$$\mathrm{H}_1 \to \mathrm{H}_2 \to \cdots \to \mathrm{H}_p \to \mathrm{H}_{p+1} \to \cdots \to \mathrm{H}_\eta \to \mathrm{H}_{\eta+1} \to \cdots \to \mathrm{H}_\lambda = \mathrm{H}, \tag{4.2}$$

where λ is a limit ordinal: See Definition 3.3.1 on page 80. Also, look at Remark 3.3.2 on page 80.

Proposition 4.5.2. *Let $\mathcal{C}$ be an ordinal-cocomplete class of R-modules closed under extensions. If*

1. *the E^∞-terms of all intermediate spectral sequences, induced by the originally stable exact couples, are in $\mathcal{C}$, and*
2. *for every limit ordinal $\eta_0 \leq \lambda$, $\operatorname{coker}(\rho_{\eta_0}) \in \mathcal{C}$,*

then $\mathrm{H}_\lambda \in \mathcal{C}$.

Proof. Fix $n \in \mathbb{N}$. We show that for every $\eta \leq \lambda$ we have $F_\eta(n) \in \mathcal{C}$:

For an arbitrary $\eta \leq \lambda$, assume $F_\beta(n) \in \mathcal{C}$, for every $\beta < \eta$, and consider the following two situations:

- η is a non-limit ordinal: Therefore, η has an immediate predecessor; i.e., for some $\beta < \eta$ we have $\eta = \beta + 1$. Let $D^1_{(u,v)}$ be the lowest stable D^1-term of the n-th D^1-column of the intermediate originally stable exact couple corresponding to the limit ordinal η_0. Using the diagram on page 199, we can see that $E^\infty_{(u,v)+r\mathbf{a}+\mathbf{b}} \in \mathcal{C}$ implies $\epsilon_{(u,v)+r\mathbf{a}} \in \mathcal{C}$ and hence $\dfrac{F_\eta(n)}{F_\beta(n)} \in \mathcal{C}$. By the following short exact sequence

$$F_\beta(n) \rightarrowtail F_\eta(n) \twoheadrightarrow \frac{F_\eta(n)}{F_\beta(n)}$$

  and the induction assumption $F_\beta(n) \in \mathcal{C}$, we conclude $F_\eta(n) \in \mathcal{C}$.

- η is a limit ordinal: By induction assumption $\operatorname{colim}_{\beta<\eta} F_\beta(n) \in \mathcal{C}$. From the epimorphism $\operatorname{coker}(\rho_{\eta_0}(n)) \twoheadrightarrow \dfrac{F_\eta(n)}{\operatorname{colim}_{\beta<\eta} F_\beta(n)}$ on the diagram on page 199 we obtain $\dfrac{F_\eta(n)}{\operatorname{colim}_{\beta<\eta} F_\beta(n)} \in \mathcal{C}$. From the short exact sequence

$$\operatorname{colim}_{\beta<\eta} F_\beta(n) \rightarrowtail F_\eta(n) \twoheadrightarrow \frac{F_\eta(n)}{\operatorname{colim}_{\beta<\eta} F_\beta(n)},$$

  and $\operatorname{colim}_{\beta<\eta} F_\beta(n) \in \mathcal{C}$ we see that $F_\eta(n) \in \mathcal{C}$.

Therefore, for every $\eta \leq \lambda$ we have $F_\eta(n) \in \mathcal{C}$. In particular, $\mathrm{H}_\lambda(n) = F_\lambda(n) \in \mathcal{C}$. $\square$

Look at Definition 3.3.6 on page 83.

Corollary 4.5.3. *Assume we can match originally vanishing exact couples to the direct tower (4.2) and let $\mathcal{C}$ be an ordinal-cocomplete class of R-modules closed under extensions. If*

1. *the E^∞-terms of all intermediate spectral sequences, induced by the originally vanishing exact couples, are in $\mathcal{C}$, and*
2. *for every limit ordinal $\eta_0 \leq \lambda$, $\operatorname{coker}(\rho_{\eta_0}) \in \mathcal{C}$,*

then $\mathrm{H}_\lambda \in \mathcal{C}$.

Look at Definition 3.3.6 on page 83 and Remark 3.3.7 on page 83.

Proposition 4.5.4. *Assume we can match originally stable exact couples to the directed tower (4.2) and let $\mathcal{C}$ be an ordinal-cocomplete class of R-modules closed under extensions. If*

1. *the E^∞-terms of all intermediate spectral sequences, induced by the originally stable exact couples, are in $\mathcal{C}$, and*
2. *for every limit ordinal $\eta_0 \leq \lambda$, $\ker(\rho_{\eta_0}) \in \mathcal{C}$,*

then $\ker(\mathrm{H}_1 \to \mathrm{H}_\lambda) \in \mathcal{C}$.

Proof. Fix $n \in \mathbb{N}$. We show that for every $\eta \leq \lambda$ we have $F^\eta(n) \in \mathcal{C}$:

For an arbitrary $\eta \leq \lambda$, assume $F^\beta(n) \in \mathcal{C}$, for every $\beta < \eta$, and consider the following two situations:

- η is a non-limit ordinal: Therefore, η has an immediate predecessor; i.e., for some $\beta < \eta$ we have $\eta = \beta + 1$. Let $D^1_{(u,v)}$ be the lowest stable D^1-term of the n-th D^1-column of the intermediate originally stable exact couple corresponding to the limit ordinal η_0. Using the diagram on page 199, we can see that $E^\infty_{(u,v)+r\mathbf{a}-\mathbf{c}} \in \mathcal{C}$ implies $\epsilon^{u,v+r\mathbf{a}} \in \mathcal{C}$ and hence $\dfrac{F^\eta(n)}{F^\beta(n)} \in \mathcal{C}$. By the following short exact sequence

$$F^\beta(n) \rightarrowtail F^\eta(n) \twoheadrightarrow \frac{F^\eta(n)}{F^\beta(n)}$$

and the induction assumption $F^\beta(n) \in \mathcal{C}$, we conclude $F^\eta(n) \in \mathcal{C}$.

- η is a limit ordinal: By induction assumption $\operatorname{colim}_{\beta<\eta} F^{\beta}(n) \in \mathcal{C}$. From the monomorphism $\dfrac{F^{\eta}(n)}{\operatorname{colim}_{\beta<\eta} F^{\beta}(n)} \rightarrowtail \ker(\rho_{\eta_0}(n))$ on the diagram on page 199 we obtain $\dfrac{F^{\eta}(n)}{\operatorname{colim}_{\beta<\eta} F^{\beta}(n)} \in \mathcal{C}$. From the short exact sequence

$$\operatorname{colim}_{\beta<\eta} F^{\beta}(n) \rightarrowtail F^{\eta}(n) \twoheadrightarrow \frac{F^{\eta}(n)}{\operatorname{colim}_{\beta<\eta} F^{\beta}(n)},$$

and $\operatorname{colim}_{\beta<\eta} F^{\beta}(n) \in \mathcal{C}$ we see that $F^{\eta}(n) \in \mathcal{C}$.

Therefore, for every $\eta \leq \lambda$ we have $F^{\eta}(n) \in \mathcal{C}$. In particular, $\ker(\mathrm{H}_1(n) \to \mathrm{H}_{\lambda}(n)) = F^{\lambda}(n) \in \mathcal{C}$. □

Corollary 4.5.5. *Assume we can match originally vanishing exact couples to the directed tower (4.2) and let $\mathcal{C}$ be an ordinal-cocomplete class of R-modules closed under extensions. If*

1. *the E^{∞}-terms of all intermediate spectral sequences, induced by the originally vanishing exact couples, are in $\mathcal{C}$, and*
2. *for every limit ordinal $\eta_0 \leq \lambda$, $\ker(\rho_{\eta_0}) \in \mathcal{C}$,*

then $\ker(\mathrm{H}_1 \to \mathrm{H}_{\lambda}) \in \mathcal{C}$.

Example 4.5.6. Look at the Example 3.3.8 on page 84 and let $\mathcal{C}$ be an ordinal-cocomplete class of Abelian groups closed under extensions. In a transfinite tower of cofibrations indexed over a limit ordinal λ, if the homology groups of the cofibers are in $\mathcal{C}$, then the homology groups of the colimit of the tower are also in $\mathcal{C}$; note that here the clutching homomorphisms are isomorphisms. In particular, for a connected space X and a given cofibration $f : A \to B$ of CW-complexes, we obtain a transfinite tower of cofibrations, indexed over a limit ordinal λ, that yields the homotopy localization of X, $\mathbf{L}_f X$. If the homology groups of the cofibers of this tower are in a class $\mathcal{C}$, then the homology groups of $\mathbf{L}_f X$ are in $\mathcal{C}$: See Appendix B.

4.6 Compatibility with Matching Eventually Stable or Vanishing ECs

In this section, we generalize the task of "staying in a class of R-modules" to the case that we have a *descending* filtration of a tower of bigraded

R-modules, indexed over an arbitrary limit ordinal, where we can match eventually stable or vanishing exact couples to it. We need the following definition.

Definition 4.6.1. *An* ordinal-complete *class of R-modules is a class of R-modules $\mathcal{C}$ that is also closed under the limit of inverse towers of R-modules; i.e., for any limit ordinal λ, if $\{A_\eta\}_{\eta<\lambda}$ is an inverse tower of R-modules such that for every $\eta < \lambda$ we have $A_\eta \in \mathcal{C}$, then $\lim_{\eta<\lambda} A_\eta \in \mathcal{C}$.*

Remark 4.6.2. The cardinality of the product of λ-indexed towers of objects is bounded above by $\aleph_{k+1}$, where cardinality of λ is $\aleph_k$: See Theorem 3.8 in [17]. So an ordinal-complete class closed under extension is not bigger than the class containing all limits of towers indexed by a category whose objects are of cardinality $\aleph_{k+1}$, when we pick an ordinal of cardinality $\aleph_k$.

Assume we can match eventually stable exact couples to the inverse tower of bigraded modules

$$\mathrm{H} = \mathrm{H}^\lambda \to \cdots \to \mathrm{H}^{\eta+1} \to \mathrm{H}^\eta \to \cdots \to \mathrm{H}^{p+1} \to \mathrm{H}^p \to \cdots \to \mathrm{H}^1, \quad (4.3)$$

where λ is a limit ordinal: See Definition 3.4.1 on page 86. Also, look at Remark 3.4.2 on page 87.

Proposition 4.6.3. *Consider an ordinal-complete class $\mathcal{C}$ of R-modules closed under extensions. If*

1. *the E^∞-terms of all intermediate spectral sequences, induced by the eventually stable exact couple, are in $\mathcal{C}$,*
2. *for every limit ordinal $\eta_0 \leq \lambda$, $\mathrm{coker}(\rho^{\eta_0}) \in \mathcal{C}$, and*
3. *$\lim_{\eta<\lambda} F_\eta \in \mathcal{C}$,*

then $\mathrm{H}^1 \in \mathcal{C}$.

Proof. Fix $n \in \mathbb{N}$. We show that for every $\eta \leq \lambda$ we have $\dfrac{\mathrm{H}^1(n)}{F_\eta(n)} \in \mathcal{C}$:

For an arbitrary $\eta \leq \lambda$, assume $\dfrac{\mathrm{H}^1(n)}{F_\beta(n)} \in \mathcal{C}$, for every $\beta < \eta$, and consider the following two situations:

- η is a non-limit ordinal: Therefore, η has an immediate predecessor; i.e., for some $\beta < \eta$ we have $\eta = \beta+1$. Let $D^1_{(u,v)}$ be the highest stable D^1-term of the n-th D^1-column of the intermediate eventually stable exact couple corresponding to the limit ordinal η_0. Using the diagram on page 205, we can see that $E^{\infty}_{(u,v)-r\mathbf{a}+\mathbf{b}} \in \mathcal{C}$ implies $\epsilon_{(u,v)-r\mathbf{a}} \in \mathcal{C}$ and hence $\dfrac{F_\beta(n)}{F_\eta(n)} \in \mathcal{C}$. By the following short exact sequence

$$\frac{F_\beta(n)}{F_\eta(n)} \rightarrowtail \frac{\mathrm{H}^1(n)}{F_\eta(n)} \twoheadrightarrow \frac{\mathrm{H}^1(n)}{F_\beta(n)}$$

  and the induction assumption $\dfrac{\mathrm{H}^1(n)}{F_\beta(n)} \in \mathcal{C}$, we conclude $\dfrac{\mathrm{H}^1(n)}{F_\eta(n)} \in \mathcal{C}$.

- η is a limit ordinal: Look at the following part of the diagram on page 205. The epimorphism $\operatorname{coker}(\rho^{\eta_0}(n)) \twoheadrightarrow \dfrac{\lim_{\beta<\eta} F_\beta(n)}{F_\eta(n)}$ shows that $\dfrac{\lim_{\beta<\eta} F_\beta(n)}{F_\eta(n)} \in \mathcal{C}$. The following part of the diagram on page 205

$$\frac{\lim_{\beta<\eta} F_\beta(n)}{F_\eta(n)} \rightarrowtail \frac{\mathrm{H}^1(n)}{F_\eta(n)} \longrightarrow \lim_{\beta<\eta} \frac{\mathrm{H}^1(n)}{F_\beta(n)}$$

  and $\lim_{\beta<\eta} \dfrac{\mathrm{H}^1(n)}{F_\beta(n)} \in \mathcal{C}$ show that $\dfrac{\mathrm{H}^1(n)}{F_\eta(n)} \in \mathcal{C}$.

Therefore, for every $\eta \leq \lambda$ we have $\dfrac{\mathrm{H}^1(n)}{F_\eta(n)} \in \mathcal{C}$, in particular, $\dfrac{\mathrm{H}^1(n)}{F_\lambda(n)} \in \mathcal{C}$. Since $F_\lambda(n) \subseteq \lim_{\eta<\lambda} F_\eta(n)$, we have $F_\lambda(n) \in \mathcal{C}$. From the following short exact sequence

$$F_\lambda(n) \rightarrowtail \mathrm{H}^1(n) \twoheadrightarrow \frac{\mathrm{H}^1(n)}{F_\lambda(n)}$$

we conclude that $\mathrm{H}^1(n) \in \mathcal{C}$.

□

Look at Definition 3.4.14 on page 92.

Corollary 4.6.4. *Assume we can match eventually vanishing exact couples to the inverse tower (4.3) and let $\mathcal{C}$ be an ordinal-complete class of R-modules closed under extensions. If*

1. *the E^∞-terms of all intermediate spectral sequences, induced by the eventually vanishing exact couple, are in $\mathcal{C}$,*

2. *for every limit ordinal $\eta_0 \leq \lambda$, $\operatorname{coker}(\rho^{\eta_0}) \in \mathcal{C}$, and*

3. $\lim_{\eta<\lambda} F_\eta \in \mathcal{C}$,

then $\mathrm{H}^1 \in \mathcal{C}$.

Look at Definition 3.4.1 on page 86 and Remark 3.4.2 on page 87.

Proposition 4.6.5. *Assume we can match eventually stable exact couples to the inverse tower (4.3) and let $\mathcal{C}$ be an ordinal-complete class of R-modules closed under extensions. If*

1. *the E^∞-terms of all intermediate spectral sequences, induced by the eventually stable exact couples, are in $\mathcal{C}$, and*

2. *for every limit ordinal $\eta_0 \leq \lambda$, $\ker(\rho^{\eta_0}) \in \mathcal{C}$, and*

3. $\lim_{\eta<\lambda} F^\eta \in \mathcal{C}$,

then $\ker(\mathrm{H}^\lambda \to \mathrm{H}^1) \in \mathcal{C}$.

Proof. Fix $n \in \mathbb{N}$. We show that for every $\eta \leq \lambda$ we have

$$\frac{\ker(\mathrm{H}^\lambda(n) \to \mathrm{H}^1(n))}{F^\eta(n)} \in \mathcal{C}:$$

For an arbitrary $\eta \leq \lambda$, assume $\dfrac{\ker(\mathrm{H}^\lambda(n) \to \mathrm{H}^1(n))}{F^\beta(n)} \in \mathcal{C}$, for every $\beta < \eta$, and consider the following two situations:

- η is a non-limit ordinal: Therefore, η has an immediate predecessor; i.e., for some $\beta < \eta$ we have $\eta = \beta + 1$. Let $D^1_{(u,v)}$ be the lowest nonzero D^1-term of the n-th D^1-column of the intermediate eventually stable exact couple corresponding to the limit ordinal η_0. Using the following part of the diagram on page 205

$$E^\infty_{(u,v)-r\mathbf{a}-\mathbf{c}} \twoheadrightarrow \frac{Z^\infty_{(u,v)-r\mathbf{a}-\mathbf{c}}}{\operatorname{im}(j^1_{(u,v)-r\mathbf{a}-\mathbf{b}-\mathbf{c}})} \leftarrowtail \epsilon^{u,v-r\mathbf{a}} \leftarrowtail \frac{F^\beta(n)}{F^\eta(n)}$$

from $E^{\infty}_{(u,v)-r\mathbf{a}-\mathbf{c}} \in \mathcal{C}$ we conclude that $\dfrac{Z^{\infty}_{(u,v)-r\mathbf{a}-\mathbf{c}}}{\mathrm{im}(j^{1}_{(u,v)-r\mathbf{a}-\mathbf{b}-\mathbf{c}})} \in \mathcal{C}$ and hence $\epsilon^{u,v-r\mathbf{a}} \in \mathcal{C}$. Therefore, $\dfrac{F^{\beta}(n)}{F^{\eta}(n)} \in \mathcal{C}$. By the following short exact sequence

$$\frac{F^{\beta}(n)}{F^{\eta}(n)} \rightarrowtail \frac{\ker(\mathrm{H}^{\lambda}(n) \to \mathrm{H}^{1}(n))}{F^{\eta}(n)} \twoheadrightarrow \frac{\ker(\mathrm{H}^{\lambda}(n) \to \mathrm{H}^{1}(n))}{F^{\beta}(n)}$$

and the induction assumption $\dfrac{\ker(\mathrm{H}^{\lambda}(n) \to \mathrm{H}^{1}(n))}{F^{\beta}(n)} \in \mathcal{C}$, we conclude that $\dfrac{\ker(\mathrm{H}^{\lambda}(n) \to \mathrm{H}^{1}(n))}{F^{\eta}(n)} \in \mathcal{C}$.

- η is a limit ordinal: Look at the diagram on page 205. By induction assumption, we have $\lim_{\beta<\eta} \dfrac{\ker(\mathrm{H}^{\lambda}(n) \to \mathrm{H}^{1}(n))}{F^{\beta}(n)} \in \mathcal{C}$. The monomorphism $\dfrac{\lim_{\beta<\eta} F^{\beta}(n)}{F^{\eta}(n)} \rightarrowtail \ker(\rho^{\eta_0}(n))$ shows that we must have $\dfrac{\lim_{\beta<\eta} F^{\beta}(n)}{F^{\eta}(n)} \in \mathcal{C}$. The following part of the diagram on page 205

$$\frac{\lim_{\beta<\eta} F^{\beta}(n)}{F^{\eta}(n)} \rightarrowtail \frac{\ker(\mathrm{H}^{\lambda}(n) \to \mathrm{H}^{1}(n))}{F^{\eta}(n)} \longrightarrow \lim_{\beta<\eta} \frac{\ker(\mathrm{H}^{\lambda}(n) \to \mathrm{H}^{1}(n))}{F^{\beta}(n)}$$

shows that $\dfrac{\ker(\mathrm{H}^{\lambda}(n) \to \mathrm{H}^{1}(n))}{F^{\eta}(n)} \in \mathcal{C}$.

Therefore, for every $\eta \leq \lambda$ we have $\dfrac{\ker(\mathrm{H}^{\lambda}(n) \to \mathrm{H}^{1}(n))}{F^{\eta}(n)} \in \mathcal{C}$ and, in particular, $\dfrac{\ker(\mathrm{H}^{\lambda}(n) \to \mathrm{H}^{1}(n))}{F^{\lambda}(n)} \in \mathcal{C}$. Since $\lim_{\eta<\lambda} F^{\eta}(n) \in \mathcal{C}$ and $F^{\lambda}(n) \subseteq \lim_{\eta<\lambda} F^{\eta}$, we have $F^{\lambda}(n) \in \mathcal{C}$. From the following short exact sequence

$$F^{\lambda}(n) \rightarrowtail \ker(\mathrm{H}^{\lambda}(n) \to \mathrm{H}^{1}(n)) \twoheadrightarrow \frac{\ker(\mathrm{H}^{\lambda}(n) \to \mathrm{H}^{1}(n))}{F^{\lambda}(n)}$$

we obtain $\ker(\mathrm{H}^{\lambda}(n) \to \mathrm{H}^{1}(n)) \in \mathcal{C}$.

□

Look at Definition 3.4.14 on page 92.

Proposition 4.6.6. *Assume we can match eventually vanishing exact couples to the inverse tower (4.3) and let $\mathcal{C}$ be an ordinal-complete class of R-modules closed under extensions. If*

1. *the E^∞-terms of all intermediate spectral sequences, induced by the eventually vanishing exact couples, are in $\mathcal{C}$, and*
2. *for every limit ordinal $\eta_0 \leq \lambda$, $\ker(\rho^{\eta_0}) \in \mathcal{C}$, and*
3. *$\lim_{\eta<\lambda} F^\eta \in \mathcal{C}$,*

then $\ker(\mathrm{H}^\lambda \to \mathrm{H}^1) \in \mathcal{C}$.

Example 4.6.7. Look at Example 3.4.16 on page 93 and let $\mathcal{C}$ be an ordinal-complete class of Abelian groups closed under extensions. In a transfinite tower of fibrations, if for every $n \in \mathbb{N}$, every integer $k \geq 1$ and every limit ordinal $\eta_0 < \lambda$, we have

- $\lim^1_{\beta<\eta_0} \pi_{n+k}(X_\beta) = 0$, and
- the n-th homotopy groups of the fibers are in $\mathcal{C}$,

then Proposition 4.6.6 on page 116 shows that $\lim_{\eta<\lambda} \pi_n(X_\eta)$ is in $\mathcal{C}$. If $\lim^1_{\eta<\lambda} \pi_{n+1}(X_\eta)$ is in $\mathcal{C}$, by the following short exact sequence from [9]

$$\lim^1_{\eta<\lambda} \pi_{n+1}(X_\eta) \rightarrowtail \pi_n(X) \twoheadrightarrow \lim_{\eta<\lambda} \pi_n(X_\eta),$$

where X is the homotopy inverse limit of the tower of fibrations, we see that $\pi_n(X)$ is in $\mathcal{C}$.

Chapter 5

Comparison Theorems Modulo a Class of Modules

5.1 Introduction

In the third chapter, we explained that when we deal with a module filtered over a limit ordinal, in lucky cases, we can use transfinite induction and spectral sequence methods to carry some information through the filtration stages. In the previous chapter, we carried the property of "being in a class of modules" through the filtration stages of a filtered module as an example.

In this chapter, we give another example in which we assume we have two filtered objects H(1) and H(2) with (potentially) transfinite filtrations

$$F_0(1) \subseteq \cdots \subseteq F_p(1) \subseteq \cdots \subseteq F_\eta(1) \subseteq F_{\eta+1}(1) \subseteq \cdots \subseteq F_\lambda(1) = \mathrm{H}(1)$$

and

$$F_0(2) \subseteq \cdots \subseteq F_p(2) \subseteq \cdots \subseteq F_\eta(2) \subseteq F_{\eta+1}(2) \subseteq \cdots \subseteq F_\lambda(2) = \mathrm{H}(2)$$

such that for every $\eta \leq \lambda$, there is a morphism from $F_\eta(1)$ to $F_\eta(2)$ and for every $\eta < \lambda$, the morphism from $F_\eta(1)$ to $F_\eta(2)$ has certain properties. We are interested in carrying this property also to the morphism from H(1) to H(2). To this end, we use the convenient intermediate exact couples *matched* to these filtrations. The idea of all of the proofs are coming from the proofs in [26].

We state all our results modulo a class of modules closed under extensions. We will use the following generalized 5-Lemma whose proof can be found in Appendix C. It is also stated for a class of Abelian groups in [19] and [30].

Lemma 5.1.1. (5-Lemma mod $\mathcal{C}$) *Let $\mathcal{C}$ be a class of modules closed under extensions and consider a commutative diagram with exact rows*

$$\begin{tikzcd} A_1 \arrow[r, "f_1"] \arrow[d, "h_1"'] & A_2 \arrow[r, "f_2"] \arrow[d, "h_2"'] & A_3 \arrow[r, "f_3"] \arrow[d, "h_3"'] & A_4 \arrow[r, "f_4"] \arrow[d, "h_4"'] & A_5 \arrow[d, "h_5"'] \\ B_1 \arrow[r, "g_1"'] & B_2 \arrow[r, "g_2"'] & B_3 \arrow[r, "g_3"'] & B_4 \arrow[r, "g_4"'] & B_5. \end{tikzcd}$$

1. *If h_2 and h_4 are $\mathcal{C}$-monomorphisms and h_1 is a $\mathcal{C}$-epimorphism, then h_3 is a $\mathcal{C}$-monomorphism.*

2. *If h_2 and h_4 are $\mathcal{C}$-epimorphisms and h_5 is a $\mathcal{C}$-monomorphism, then h_3 is $\mathcal{C}$-epimorphism.*

3. *If h_1, h_2, h_4 and h_5 are $\mathcal{C}$-isomorphisms, then so is h_3.*

Proof. See Appendix C. □

5.2 Comparing Arbitrary Spectral Sequences

Let $\mathcal{C}$ be an ω-complete class of modules. Look at the E^∞-Distribution Theorem 2.3.13 on page 35 for the notations.

Proposition 5.2.1. *Let $\varphi : \mathcal{EC}(1) \to \mathcal{EC}(2)$ be a morphism of exact couples such that*

1. $\varphi^\infty : E^\infty_{-,-}(1) \to E^\infty_{-,-}(2)$*, and*

2. $\bar{\varphi}| : \lim_r \phi_r(1) \to \lim_r \phi_r(2)$

are $\mathcal{C}$-monomorphisms. Then the induced homomorphism $\bar{\varphi} : L_*(1) \to L_*(2)$ is a $\mathcal{C}$-monomorphism.

Proof. Fix $n \in \mathbb{Z}$ and $\phi_{(u,v)}$, where $(u,v) \cdot \hat{\mathbf{a}} = n$. The first condition and the E^∞-Distribution Theorem imply that for every $r \in \mathbb{N}$, the dashed arrow in the following diagram is a $\mathcal{C}$-monomorphism

$$\begin{tikzcd}
\ker(\varphi^{\infty}_{(u,v)-\mathbf{c}-r\mathbf{a}}) \arrow[r, tail] & E^{\infty}_{(u,v)-\mathbf{c}-r\mathbf{a}}(1) \arrow[r, "\varphi^{\infty}_{(u,v)-\mathbf{c}-r\mathbf{a}}"] & E^{\infty}_{(u,v)-\mathbf{c}-r\mathbf{a}}(2) \\
\ker(\varphi^{\infty}_{(u,v)-\mathbf{c}-r\mathbf{a}}) \arrow[r, tail] \arrow[u, tail] & \dfrac{\phi_{(u,v)-r\mathbf{a}}(1)}{\phi_{(u,v)-(r+1)\mathbf{a}}(1)} \arrow[r, dashed, "\varphi^{\infty}_{(u,v)-\mathbf{c}-r\mathbf{a}}|"] \arrow[u, tail] & \dfrac{\phi_{(u,v)-r\mathbf{a}}(2)}{\phi_{(u,v)-(r+1)\mathbf{a}}(2)}. \arrow[u, tail]
\end{tikzcd} \tag{5.1}$$

Claim 1. *For every $r \geq 1$, $f^r : \dfrac{\phi_{(u,v)}(1)}{\phi_{(u,v)-r\mathbf{a}}(1)} \to \dfrac{\phi_{(u,v)}(2)}{\phi_{(u,v)-r\mathbf{a}}(2)}$ is a $\mathcal{C}$-monomorphism.*

Proof. We proceed by induction on r. For $r = 1$ the claim holds by the following diagram and hypothesis 1

$$\begin{tikzcd}
\ker(\varphi^{\infty}_{(u,v)-\mathbf{c}}) \arrow[r, tail] & E^{\infty}_{(u,v)-\mathbf{c}}(1) \arrow[r, "\varphi^{\infty}_{(u,v)-\mathbf{c}}"] & E^{\infty}_{(u,v)-\mathbf{c}}(2) \\
\ker(\varphi^{\infty}_{(u,v)-\mathbf{c}}) \arrow[r, tail] \arrow[u, tail] & \dfrac{\phi_{(u,v)}(1)}{\phi_{(u,v)-\mathbf{a}}(1)} \arrow[r, "f^1=\varphi^{\infty}_{(u,v)-\mathbf{c}}|"] \arrow[u, tail] & \dfrac{\phi_{(u,v)}(2)}{\phi_{(u,v)-\mathbf{a}}(2)}. \arrow[u, tail]
\end{tikzcd}$$

Assume the claim holds for r. Using diagram (5.1), in the following diagram

$$\begin{tikzcd}
\dfrac{\phi_{(u,v)-r\mathbf{a}}(1)}{\phi_{(u,v)-(r+1)\mathbf{a}}(1)} \arrow[r, tail] \arrow[d, "\varphi^{\infty}_{(u,v)-\mathbf{c}-r\mathbf{a}}|"', "(5.1)"] & \dfrac{\phi_{(u,v)}(1)}{\phi_{(u,v)-(r+1)\mathbf{a}}(1)} \arrow[r, two heads] \arrow[d, "f^{r+1}"'] & \dfrac{\phi_{(u,v)}(1)}{\phi_{(u,v)-r\mathbf{a}}(1)} \arrow[d, "f^r"', "\text{Ind. Hyp.}"] \\
\dfrac{\phi_{(u,v)-r\mathbf{a}}(2)}{\phi_{(u,v)-(r+1)\mathbf{a}}(2)} \arrow[r, tail] & \dfrac{\phi_{(u,v)}(2)}{\phi_{(u,v)-(r+1)\mathbf{a}}(2)} \arrow[r, two heads] & \dfrac{\phi_{(u,v)}(2)}{\phi_{(u,v)-r\mathbf{a}}(2)}
\end{tikzcd}$$

induction hypothesis and 5-Lemma mod $\mathcal{C}$ imply that f^{r+1} is also a $\mathcal{C}$-monomorphism. $\diamond$

Claim 2. *For every $s \geq 1$, $f_s : \dfrac{\phi_{(u,v)+s\mathbf{a}}(1)}{\phi_{(u,v)}(1)} \to \dfrac{\phi_{(u,v)+s\mathbf{a}}(2)}{\phi_{(u,v)}(2)}$ is a $\mathcal{C}$-monomorphism.*

Proof. The proof is exactly the same as the proof of Claim 1. $\diamond$

If we take the direct limit of the following tower of $\mathcal{C}$-monomorphisms

$$\begin{tikzcd}
\dfrac{\phi_{(u,v)+\mathbf{a}}(1)}{\phi_{(u,v)}(1)} \arrow[r, "f_1"] \arrow[d] & \dfrac{\phi_{(u,v)+\mathbf{a}}(2)}{\phi_{(u,v)}(2)} \arrow[d] \\
\vdots \arrow[d] & \vdots \arrow[d] \\
\dfrac{\phi_{(u,v)+s\mathbf{a}}(1)}{\phi_{(u,v)}(1)} \arrow[r, "f_s"] \arrow[d] & \dfrac{\phi_{(u,v)+s\mathbf{a}}(2)}{\phi_{(u,v)}(2)} \arrow[d] \\
\dfrac{\phi_{(u,v)+(s+1)\mathbf{a}}(1)}{\phi_{(u,v)}(1)} \arrow[r, "f_{s+1}"] \arrow[d] & \dfrac{\phi_{(u,v)+(s+1)\mathbf{a}}(2)}{\phi_{(u,v)}(2)} \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$

we obtain

$$\frac{L_n(1)}{\phi_{(u,v)}(1)} \longrightarrow \frac{L_n(2)}{\phi_{(u,v)}(2)}$$

with kernel $\operatorname{colim}_s \ker(f_s) \in \mathcal{C}$ and hence it is a $\mathcal{C}$-monomorphism. If we take the inverse limit of the following tower of $\mathcal{C}$-monomorphisms

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
\dfrac{L_n(1)}{\phi_{(u,v)-r\mathbf{a}}(1)} \arrow[r, "f^r"] \arrow[d, two heads] & \dfrac{L_n(2)}{\phi_{(u,v)-r\mathbf{a}}(2)} \arrow[d, two heads] \\
\dfrac{L_n(1)}{\phi_{(u,v)-(r-1)\mathbf{a}}(1)} \arrow[r, "f^{r-1}"] \arrow[d, two heads] & \dfrac{L_n(2)}{\phi_{(u,v)-(r-1)\mathbf{a}}(2)} \arrow[d, two heads] \\
\vdots \arrow[d, two heads] & \vdots \arrow[d, two heads] \\
\dfrac{L_n(1)}{\phi_{(u,v)}(1)} \arrow[r, "f^1"] & \dfrac{L_n(2)}{\phi_{(u,v)}(2)}
\end{tikzcd}$$

we obtain

$$\lim_r \frac{L_n(1)}{\phi_{(u,v)-r\mathbf{a}}(1)} \xrightarrow{f} \lim_r \frac{L_n(2)}{\phi_{(u,v)-r\mathbf{a}}(2)}$$

with $\ker(f) = \lim_r \ker(f^r) \in \mathcal{C}$ and hence a $\mathcal{C}$-monomorphism. By 5-Lemma mod $\mathcal{C}$ in Appendix C, in the following diagram

$$\begin{tikzcd}
\lim_r \phi_{(u,v)-r\mathbf{a}}(1) \arrow[r, tail] \arrow[d, "\bar{\varphi}_n"', "\mathcal{C}\text{-mono}"] & L_n(1) \arrow[r] \arrow[d, "\bar{\varphi}_n"'] & \lim_r \dfrac{L_n(1)}{\phi_{(u,v)-r\mathbf{a}}(1)} \arrow[r, two heads] \arrow[d, "\mathcal{C}\text{-mono}"', "f"] & \lim^1_r \phi_{(u,v)-r\mathbf{a}}(1) \arrow[d] \\
\lim_r \phi_{(u,v)-r\mathbf{a}}(2) \arrow[r, tail] & L_n(2) \arrow[r] & \lim_r \dfrac{L_n(2)}{\phi_{(u,v)-r\mathbf{a}}(2)} \arrow[r, two heads] & \lim^1_r \phi_{(u,v)-r\mathbf{a}}(2)
\end{tikzcd}$$

$\bar{\varphi}_n$ is a $\mathcal{C}$-monomorphism. $\square$

Here, we used the immediate relationship between the quotient of adjacent filtration stages of the universal augmentation and the E^∞-term. Since the quotient of adjacent filtration stages of the universal coaugmentation is not closely related to the E^∞-term, we cannot say anything in general about a comparison theorem for universal coaugmentations. But, for the rest of this chapter we focus on special cases where these two are closely related.

5.3 Comparing Convergent Spectral Sequences

We consider two cases of convergence to the universal augmentation and the universal coaugmentation. Look at Definition 2.4.2 on page 41 and definition 2.4.6 on page 42.

5.3.1 Comparing Spectral Sequences Convergent to the Universal Augmentation

Let $\mathcal{C}$ be an ω-complete class of modules closed under extensions. Look at the E^∞-Distribution Theorem 2.3.13 on page 35 for the notations.

Proposition 5.3.1. *Let $\varphi : \mathcal{EC}(1) \to \mathcal{EC}(2)$ be a morphism of exact couples such that the induced spectral sequences are convergent to their universal augmentations and*

1. $\varphi^\infty_{-,-} : E^\infty_{-,-}(1) \to E^\infty_{-,-}(2)$ *are $\mathcal{C}$-epimorphisms, and*

2. $\lim_r \phi_r(1) \to \lim_r \phi_r(2)$ *are* $\mathcal{C}$*-epimorphisms, and*

3. $\lim_r^1 \phi_r(1) \to \lim_r^1 \phi_r(2)$ *is a* $\mathcal{C}$*-monomorphism.*

Then the induced homomorphism $\bar{\varphi} : L_*(1) \to L_*(2)$ *is a* $\mathcal{C}$*-epimorphism.*

Proof. Fix $n \in \mathbb{Z}$ and $\phi_{(u,v)}$, where $(u,v) \cdot \hat{\mathbf{a}} = n$. The first condition and 5-Lemma mod $\mathcal{C}$ imply that for every $r \in \mathbb{Z}$, the dashed arrow in the following diagram is a $\mathcal{C}$-epimorphism

$$\begin{tikzcd}
E^\infty_{(u,v)+r\mathbf{a}+\mathbf{b}}(1) \arrow[r, "\varphi^\infty_{(u,v)+r\mathbf{a}+\mathbf{b}}"] & E^\infty_{(u,v)+r\mathbf{a}+\mathbf{b}}(2) \arrow[r, two heads] & \operatorname{coker}(\varphi^\infty_{(u,v)+r\mathbf{a}+\mathbf{b}}) \\
\dfrac{\phi_{(u,v)+r\mathbf{a}}(1)}{\phi_{(u,v)+(r-1)\mathbf{a}}(1)} \arrow[u, "\cong"] \arrow[r, dashed, "\varphi^\infty_{(u,v)+r\mathbf{a}+\mathbf{b}}|"] & \dfrac{\phi_{(u,v)+r\mathbf{a}}(2)}{\phi_{(u,v)+(r-1)\mathbf{a}}(2)} \arrow[u, "\cong"] \arrow[r, two heads] & \operatorname{coker}(\varphi^\infty_{(u,v)+r\mathbf{a}+\mathbf{b}}|) \arrow[u, "\cong"]
\end{tikzcd} . \tag{5.2}$$

Claim 1. *For every* $r \geq 1$, $f^r : \dfrac{\phi_{(u,v)}(1)}{\phi_{(u,v)-r\mathbf{a}}(1)} \to \dfrac{\phi_{(u,v)}(2)}{\phi_{(u,v)-r\mathbf{a}}(2)}$ *is a* $\mathcal{C}$*-epimorphism.*

Proof. We proceed by induction on r. For $r = 1$ the claim holds by the following diagram and hypothesis 1

$$\begin{tikzcd}
E^\infty_{(u,v)+\mathbf{b}}(1) \arrow[r, "\varphi^\infty_{(u,v)+\mathbf{b}}"] & E^\infty_{(u,v)+\mathbf{b}}(2) \arrow[r, two heads] & \operatorname{coker}(\varphi^\infty_{(u,v)+\mathbf{b}}) \\
\dfrac{\phi_{(u,v)}(1)}{\phi_{(u,v)-\mathbf{a}}(1)} \arrow[u, "\cong"] \arrow[r, "f^1=\varphi^\infty_{(u,v)+\mathbf{b}}|"] & \dfrac{\phi_{(u,v)}(2)}{\phi_{(u,v)-\mathbf{a}}(2)} \arrow[u, "\cong"] \arrow[r, two heads] & \operatorname{coker}(\varphi^\infty_{(u,v)+\mathbf{b}}) \arrow[u, "\cong"]
\end{tikzcd}$$

Assume the claim holds for r. Using diagram (5.2), in the following diagram

$$\begin{tikzcd}
\dfrac{\phi_{(u,v)-r\mathbf{a}}(1)}{\phi_{(u,v)-(r+1)\mathbf{a}}(1)} \arrow[r, tail] \arrow[d, "\varphi^\infty_{(u,v)-\mathbf{c}-r\mathbf{a}}|"', "(5.2)"] & \dfrac{\phi_{(u,v)}(1)}{\phi_{(u,v)-(r+1)\mathbf{a}}(1)} \arrow[r, two heads] \arrow[d, "f^{r+1}"'] & \dfrac{\phi_{(u,v)}(1)}{\phi_{(u,v)-r\mathbf{a}}(1)} \arrow[d, "f^r"', "\text{Ind. Hyp.}"] \\
\dfrac{\phi_{(u,v)-r\mathbf{a}}(2)}{\phi_{(u,v)-(r+1)\mathbf{a}}(2)} \arrow[r, tail] & \dfrac{\phi_{(u,v)}(2)}{\phi_{(u,v)-(r+1)\mathbf{a}}(2)} \arrow[r, two heads] & \dfrac{\phi_{(u,v)}(2)}{\phi_{(u,v)-r\mathbf{a}}(2)}
\end{tikzcd}$$

induction assumption and 5-Lemma mod $\mathcal{C}$ imply that f^{r+1} is also a $\mathcal{C}$-epimorphism. $\diamond$

Claim 2. *For every $s \geq 1$, $f_s : \dfrac{\phi_{(u,v)+s\mathbf{a}}(1)}{\phi_{(u,v)}(1)} \to \dfrac{\phi_{(u,v)+s\mathbf{a}}(2)}{\phi_{(u,v)}(2)}$ is a $\mathcal{C}$-epimorphism.*

Proof. The proof is exactly the same as the proof of Claim 1. $\diamond$

If we take the direct limit of $\mathcal{C}$-epimorphisms

$$\left\{ \frac{\phi_{(u,v)+s\mathbf{a}}(1)}{\phi_{(u,v)}(1)} \xrightarrow{f_s} \frac{\phi_{(u,v)+s\mathbf{a}}(2)}{\phi_{(u,v)}(2)} \right\}_{s\in\mathbb{N}}$$

over s, we obtain

$$\frac{L_n(1)}{\phi_{(u,v)}(1)} \longrightarrow \frac{L_n(2)}{\phi_{(u,v)}(2)}$$

with cokernel $\operatorname{colim}_s \operatorname{coker}(f_s) \in \mathcal{C}$ and hence it is $\mathcal{C}$-epimorphism. If we take the inverse limit of the $\mathcal{C}$-epimorphisms

$$\left\{ \frac{L_n(1)}{\phi_{(u,v)-r\mathbf{a}}(1)} \xrightarrow{f^r} \frac{L_n(2)}{\phi_{(u,v)-r\mathbf{a}}(2)} \right\}_{r\in\mathbb{Z}}$$

over r, we obtain

$$\lim_r \frac{L_n(1)}{\phi_{(u,v)-r\mathbf{a}}(1)} \xrightarrow{f} \lim_r \frac{L_n(2)}{\phi_{(u,v)-r\mathbf{a}}(2)}$$

with $\operatorname{coker}(f)$ a *sub-object* of $\lim_r \operatorname{coker}(f^r) \in \mathcal{C}$ and hence $\operatorname{coker}(f) \in \mathcal{C}$; i.e., f is a $\mathcal{C}$-epimorphism. By 5-Lemma mod $\mathcal{C}$, in the following diagram

$$\begin{tikzcd}
\lim_r \phi_{(u,v)-r\mathbf{a}}(1) \arrow[r, tail] \arrow[d, "\bar{\varphi}_n|"', "\mathcal{C}\text{-epi}"] & L_n(1) \arrow[r] \arrow[d, "\bar{\varphi}_n"'] & \lim_r \dfrac{L_n(1)}{\phi_{(u,v)-r\mathbf{a}}(1)} \arrow[r, two heads] \arrow[d, "\mathcal{C}\text{-epi}"', "f"] & \lim^1_r \phi_{(u,v)-r\mathbf{a}}(1) \arrow[d, "\mathcal{C}\text{-mono}"] \\
\lim_r \phi_{(u,v)-r\mathbf{a}}(2) \arrow[r, tail] & L_n(2) \arrow[r] & \lim_r \dfrac{L_n(2)}{\phi_{(u,v)-r\mathbf{a}}(2)} \arrow[r, two heads] & \lim^1_r \phi_{(u,v)-r\mathbf{a}}(2)
\end{tikzcd}$$

$\bar{\varphi}_n$ is a $\mathcal{C}$-epimorphism. $\square$

Corollary 5.3.2. *Let $\varphi : \mathcal{C}(1) \to \mathcal{C}(2)$ be a morphism of exact couples such that the induced spectral sequences are convergent to their universal augmentations and*

1. $E^{\infty}_{-,-}(1) \to E^{\infty}_{-,-}(2)$,
2. $\lim_r \phi_r(1) \to \lim_r \phi_r(2)$, *and*
3. $\lim^1_r \phi_r(1) \to \lim^1_r \phi_r(2)$

are $\mathcal{C}$*-isomorphisms. Then their universal augmentations are* $\mathcal{C}$*-isomorphic.*

5.3.2 Comparing Spectral Sequences Convergent to the Universal Coaugmentation

Let $\mathcal{C}$ be an ω-complete class of modules closed under extensions.

Proposition 5.3.3. *Let* $\varphi : \mathcal{C}(1) \to \mathcal{C}(2)$ *be a morphism of exact couples such that the induced spectral sequences are convergent to their universal coaugmentations and* $E^{\infty}_{-,-}(1) \to E^{\infty}_{-,-}(2)$ *is* $\mathcal{C}$*-(mono-,epi-)isomorphism. Then the induced homomorphism* $\bar{\varphi} : L^*(1) \to L^*(2)$ *is a* $\mathcal{C}$*-(mono-, epi-)isomorphism.*

Proof. The proof is similar to the proof of Proposition 5.3.1 on page 121. □

5.4 Comparing Concentrated Spectral Sequences

We consider two cases of the universal augmentation and the universal coaugmentation concentration. Look at Definition 2.4.13 on page 45.

5.4.1 Comparing Universal Augmentation Concentrated Spectral Sequences

Let $\mathcal{C}$ be an ω-complete class of modules closed under extensions. Look at the E^{∞}-Distribution Theorem 2.3.13 on page 35 for the notations.

Proposition 5.4.1. *Let* $\varphi : \mathcal{EC}(1) \to \mathcal{EC}(2)$ *be a morphism of exact couples such that the induced spectral sequences are universal augmentation concentrated and, for every* $(p, q) \in \mathbb{Z} \times \mathbb{Z}$*, the following hold:*

1. $\varphi^{\infty}_{(p,q)} : E^{\infty}_{(p,q)}(1) \to E^{\infty}_{(p,q)}(2)$ *and*

2. *$\bar{\varphi}| : \lim_r \phi_r(1) \to \lim_r \phi_r(2)$ are $\mathcal{C}$-epimorphisms,*

and

3. *$\varphi'_{(p,q)} : \lim^1_r\{\ker(i^r_{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}}(1))\} \to \lim^1_r\{\ker(i^r_{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}}(2))\}$ and*

4. *$\bar{\varphi}| : \lim^1_r \phi_r(1) \to \lim^1_r \phi_r(2)$ are $\mathcal{C}$-monomorphisms.*

Then the induced homomorphism $\bar{\varphi} : L_*(1) \to L_*(2)$ is a $\mathcal{C}$-epimorphism.

Proof. Fix $n \in \mathbb{Z}$ and $\phi_{(p,q)}$, where $(p,q) \cdot \hat{\mathbf{a}} = n$. By the E^∞-Distribution Theorem on page 35, in the following diagram, $\varphi'_{(p,q)}|$ is a $\mathcal{C}$-monomorphism

$$\begin{tikzcd}
\ker(\varphi'_{(p,q)}) \arrow[r, tail] & \lim^1_r\{\ker(i^r_{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}}(1))\} \arrow[r, "\varphi'_{(p,q)}"] & \lim^1_r\{\ker(i^r_{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}}(2))\} \\
\ker(\varphi'_{(p,q)}|) \arrow[r, tail] \arrow[u, tail] & \dfrac{Z^\infty_{(p,q)+\mathbf{b}}(1)}{\operatorname{im}(j^1_{(p,q)}(1))} \arrow[r, "\varphi'_{(p,q)}|"] \arrow[u, tail] & \dfrac{Z^\infty_{(p,q)+\mathbf{b}}(2)}{\operatorname{im}(j^1_{(p,q)}(2))} \arrow[u, tail]
\end{tikzcd}$$

Conditions 1 and 2 and 5-Lemma mod $\mathcal{C}$ imply that, for every $s \in \mathbb{Z}$, the dashed arrow in the following diagram of exact sequences is a $\mathcal{C}$-epimorphism

$$\begin{tikzcd}
\dfrac{\phi_{(u,v)+s\mathbf{a}}(1)}{\phi_{(u,v)+(s-1)\mathbf{a}}(1)} \arrow[r, tail] \arrow[d, dashed] & E^\infty_{(u,v)+s\mathbf{a}+\mathbf{b}}(1) \arrow[r] \arrow[d, "\mathcal{C}-\mathrm{epi}"] & \dfrac{Z^\infty_{(p,q)+\mathbf{b}}(1)}{\operatorname{im}(j^1_{(p,q)}(1))} \arrow[d, "\mathcal{C}-\mathrm{mono}"] \\
\dfrac{\phi_{(u,v)+s\mathbf{a}}(2)}{\phi_{(u,v)+(s-1)\mathbf{a}}(1)} \arrow[r, tail] & E^\infty_{(u,v)+s\mathbf{a}+\mathbf{b}}(2) \arrow[r] & \dfrac{Z^\infty_{(p,q)+\mathbf{b}}(2)}{\operatorname{im}(j^1_{(p,q)}(2))}.
\end{tikzcd}$$

The rest of the proof is similar to the proof of Proposition 5.2.1 on page 118. □

Corollary 5.4.2. *Let $\varphi : \mathcal{EC}(1) \to \mathcal{EC}(2)$ be a morphism of exact couples such that the induced spectral sequences are universal augmentation concentrated and, for every $(p,q) \in \mathbb{Z} \times \mathbb{Z}$, the following homomorphisms are $\mathcal{C}$-isomorphisms:*

1. *$E^\infty_{(p,q)}(1) \to E^\infty_{(p,q)}(2)$,*

2. *$\lim_r \phi_r(1) \to \lim_r \phi_r(2)$,*

3. $\lim^1_r\{\ker(i^r_{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}}(1))\} \to \lim^1_r\{\ker(i^r_{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}}(2))\}$, *and*

4. $\lim^1_r\phi_r(1) \to \lim^1_r\phi_r(2)$.

Then their universal augmentations are $\mathcal{C}$-isomorphic.

5.4.2 Comparing Universal Coaugmentation Concentrated Spectral Sequences

Let $\mathcal{C}$ be an ω-complete and ω-cocomplete class of modules closed under extensions. Look at the E^∞-Distribution Theorem 2.3.13 on page 35 for the notations.

Proposition 5.4.3. *Let $\varphi : \mathcal{EC}(1) \to \mathcal{EC}(2)$ be a morphism of exact couples such that the induced spectral sequences are universal coaugmentation concentrated and $E^\infty_{-,-}(1) \to E^\infty_{-,-}(2)$ is a $\mathcal{C}$-monomorphism. Then the induced homomorphism $\bar\varphi : L^*(1) \to L^*(2)$ is a $\mathcal{C}$-monomorphism.*

Proof. For $i = 1, 2$, let $\{\phi^r(i)\}_{r\in\mathbb{Z}}$ be the kernel filtration of $L^*(i)$. In the proof of Proposition 5.2.1 on page 118, if we replace $\phi_{(p,q)}$ by $\phi^{(p,q)}$ and the last diagram by the following diagram

$$\begin{array}{ccc} L^n(1) & \xrightarrow{\cong} & \lim_r \dfrac{L^n(1)}{\phi^{u,v+r\mathbf{a}}(1)} \\ \Big\downarrow{\scriptstyle \bar\varphi_n} & & \Big\downarrow{\scriptstyle f} \\ L^n(2) & \xrightarrow{\cong} & \lim_r \dfrac{L_n(2)}{\phi^{u,v+r\mathbf{a}}(2)} \end{array}$$

the proof is similar. □

Proposition 5.4.4. *Let $\varphi : \mathcal{EC}(1) \to \mathcal{EC}(2)$ be a morphism of exact couples such that the induced spectral sequences are universal coaugmentation concentrated and, for every $(p,q) \in \mathbb{Z}\times\mathbb{Z}$, the following hold:*

1. $\varphi^\infty_{(p,q)} : E^\infty_{(p,q)}(1) \to E^\infty_{(p,q)}(2)$ *is a $\mathcal{C}$-epimorphism, and*

2. *the morphism*

$$\varphi'_{(p,q)} : \lim^1_r\{\ker(i^r_{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}}(1))\} \to \lim^1_r\{\ker(i^r_{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}}(2))\}$$

is a $\mathcal{C}$-monomorphism.

Then the induced homomorphism $\bar{\varphi} : L^*(1) \to L^*(2)$ is a $\mathcal{C}$-epimorphism.

Proof. Fix $n \in \mathbb{Z}$ and $\phi_{(p,q)}$, where $(p,q) \cdot \hat{\mathbf{a}} = n$. By the diagram on page 45, conditions 1 and 2 and 5-Lemma mod $\mathcal{C}$, for every $s \in \mathbb{Z}$, the dashed arrow in the following diagram of exact sequences is a $\mathcal{C}$-epimorphism

$$\begin{tikzcd}
\frac{\phi^{u,v+(s+1)\mathbf{a}+\mathbf{b}+\mathbf{c}}(1)}{\phi^{u,v+s\mathbf{a}+\mathbf{b}+\mathbf{c}}(1)} \arrow[r, tail] \arrow[d, dashed] & E^{\infty}_{(u,v)+s\mathbf{a}+\mathbf{b}}(1) \arrow[r] \arrow[d, "\mathcal{C}-\mathrm{epi}"] & \lim^1_r\{\ker(i^r_{(u,v)+\mathbf{b}+\mathbf{c}-r\mathbf{a}}(1))\} \arrow[d, "\mathcal{C}-\mathrm{mono}"] \\
\frac{\phi^{u,v+(s+1)\mathbf{a}+\mathbf{b}+\mathbf{c}}(2)}{\phi^{u,v+s\mathbf{a}+\mathbf{b}+\mathbf{c}}(1)} \arrow[r, tail] & E^{\infty}_{(u,v)+s\mathbf{a}+\mathbf{b}}(2) \arrow[r] & \lim^1_r\{\ker(i^r_{(u,v)+\mathbf{b}+\mathbf{c}-r\mathbf{a}}(2))\}.
\end{tikzcd} \tag{5.3}$$

Claim 1. *For every $r \geq 1$, $f^r : \dfrac{\phi^{u,v+r\mathbf{a}+\mathbf{b}+\mathbf{c}}(1)}{\phi^{u,v+\mathbf{b}+\mathbf{c}}(1)} \to \dfrac{\phi^{u,v+r\mathbf{a}+\mathbf{b}+\mathbf{c}}(2)}{\phi^{u,v+\mathbf{b}+\mathbf{c}}(2)}$ is a $\mathcal{C}$-epimorphism.*

Proof. We proceed by induction on r. For $r = 1$ the claim holds by the following diagram and hypothesis 1

$$\begin{tikzcd}
\ker(\varphi^{\infty}_{(u,v)+\mathbf{b}}) \arrow[r, tail] & E^{\infty}_{(u,v)+\mathbf{b}}(1) \arrow[r, "\varphi^{\infty}_{(u,v)+\mathbf{b}}"] & E^{\infty}_{(u,v)+\mathbf{b}}(2) \\
\ker(\varphi^{\infty}_{(u,v)+\mathbf{b}}) \arrow[r, tail] \arrow[u, tail] & \frac{\phi^{u,v+\mathbf{a}+\mathbf{b}+\mathbf{c}}(1)}{\phi^{u,v+\mathbf{b}+\mathbf{c}}(1)} \arrow[r, "f^1=\varphi^{\infty}_{(u,v)+\mathbf{b}}|"] \arrow[u, tail] & \frac{\phi^{u,v+\mathbf{a}+\mathbf{b}+\mathbf{c}}(2)}{\phi^{u,v+\mathbf{b}+\mathbf{c}}(2)}. \arrow[u, tail]
\end{tikzcd}$$

Assume the claim holds for r. Using diagram (5.3), in the following diagram

$$\begin{tikzcd}
\frac{\phi^{u,v+r\mathbf{a}+\mathbf{b}+\mathbf{c}}(1)}{\phi^{u,v+\mathbf{b}+\mathbf{c}}(1)} \arrow[r, tail] \arrow[d, "\varphi^{\infty}_{(u,v)+r\mathbf{a}+\mathbf{b}}|"', "\text{Ind. Hyp.}"] & \frac{\phi^{u,v+(r+1)\mathbf{a}+\mathbf{b}+\mathbf{c}}(1)}{\phi^{u,v+\mathbf{b}+\mathbf{c}}(1)} \arrow[r, two heads] \arrow[d, "f^{r+1}"'] & \frac{\phi^{u,v+(r+1)\mathbf{a}+\mathbf{b}+\mathbf{c}}(1)}{\phi^{u,v+r\mathbf{a}+\mathbf{b}+\mathbf{c}}(1)} \arrow[d, "f^r"', "(5.3)"] \\
\frac{\phi^{u,v+r\mathbf{a}+\mathbf{b}+\mathbf{c}}(2)}{\phi^{u,v+\mathbf{b}+\mathbf{c}}(2)} \arrow[r, tail] & \frac{\phi^{u,v+(r+1)\mathbf{a}+\mathbf{b}+\mathbf{c}}(2)}{\phi^{u,v+\mathbf{b}+\mathbf{c}}(2)} \arrow[r, two heads] & \frac{\phi^{u,v+(r+1)\mathbf{a}+\mathbf{b}+\mathbf{c}}(2)}{\phi_{(u,v)+r\mathbf{a}+\mathbf{b}+\mathbf{c}}(2)}
\end{tikzcd}$$

induction hypothesis and 5-Lemma mod $\mathcal{C}$ imply that f^{r+1} is also a $\mathcal{C}$-epimorphism. $\diamond$

Claim 2. *For every $s \geq 1$, $f_s : \dfrac{\phi^{u,v+\mathbf{b}+\mathbf{c}}(1)}{\phi^{u,v-s\mathbf{a}+\mathbf{b}+\mathbf{c}}(1)} \to \dfrac{\phi^{u,v+\mathbf{b}+\mathbf{c}}(2)}{\phi^{u,v-s\mathbf{a}+\mathbf{b}+\mathbf{c}}(2)}$ is a $\mathcal{C}$-epimorphism.*

Proof. The proof is exactly the same as the proof of Claim 1. $\diamond$

If we take the direct limit of the following tower of $\mathcal{C}$-epimorphisms

$$\begin{tikzcd}
\dfrac{\phi^{u,v+\mathbf{a}+\mathbf{b}+\mathbf{c}}(1)}{\phi^{u,v+\mathbf{b}+\mathbf{c}}(1)} \arrow[r, "f_1"] \arrow[d, tail] & \dfrac{\phi^{u,v+\mathbf{a}+\mathbf{b}+\mathbf{c}}(2)}{\phi^{u,v+\mathbf{b}+\mathbf{c}}(2)} \arrow[d, tail] \\
\vdots \arrow[d, tail] & \vdots \arrow[d, tail] \\
\dfrac{\phi^{u,v+s\mathbf{a}+\mathbf{b}+\mathbf{c}}(1)}{\phi^{u,v+\mathbf{b}+\mathbf{c}}(1)} \arrow[r, "f_s"] \arrow[d, tail] & \dfrac{\phi^{u,v+s\mathbf{a}+\mathbf{b}+\mathbf{c}}(2)}{\phi^{u,v+\mathbf{b}+\mathbf{c}}(2)} \arrow[d, tail] \\
\dfrac{\phi^{u,v+(s+1)\mathbf{a}+\mathbf{b}+\mathbf{c}}(1)}{\phi^{u,v+\mathbf{b}+\mathbf{c}}(1)} \arrow[r, "f_{s+1}"] \arrow[d, tail] & \dfrac{\phi^{u,v+(s+1)\mathbf{a}+\mathbf{b}+\mathbf{c}}(2)}{\phi^{u,v+\mathbf{b}+\mathbf{c}}(2)} \arrow[d, tail] \\
\vdots & \vdots
\end{tikzcd}$$

we obtain

$$\frac{L^{n+\sigma}(1)}{\phi^{u,v+\mathbf{b}+\mathbf{c}}(1)} \longrightarrow \frac{L^{n+\sigma}(2)}{\phi^{u,v+\mathbf{b}+\mathbf{c}}(2)}$$

with cokernel $\operatorname{colim}_s \operatorname{coker}(f_s) \in \mathcal{C}$ and hence it is a $\mathcal{C}$-epimorphism. If we take the inverse limit of the following tower of $\mathcal{C}$-epimorphisms

$$\begin{tikzcd}
\vdots \arrow[d, two heads] & \vdots \arrow[d, two heads] \\
\dfrac{L^{n+\sigma}(1)}{\phi^{u,v-(r+1)\mathbf{a}+\mathbf{b}+\mathbf{c}}(1)} \arrow[r, "f^{r+1}"] \arrow[d, two heads] & \dfrac{L^{n+\sigma}(2)}{\phi^{u,v-(r+1)\mathbf{a}+\mathbf{b}+\mathbf{c}}(2)} \arrow[d, two heads] \\
\dfrac{L^{n+\sigma}(1)}{\phi^{u,v-r\mathbf{a}+\mathbf{b}+\mathbf{c}}(1)} \arrow[r, "f^{r}"] \arrow[d, two heads] & \dfrac{L^{n+\sigma}(2)}{\phi^{u,v-r\mathbf{a}+\mathbf{b}+\mathbf{c}}(2)} \arrow[d, two heads] \\
\vdots \arrow[d, two heads] & \vdots \arrow[d, two heads] \\
\dfrac{L^{n+\sigma}(1)}{\phi^{u,v+\mathbf{b}+\mathbf{c}}(1)} \arrow[r, "f^{1}"] & \dfrac{L^{n+\sigma}(2)}{\phi^{u,v+\mathbf{b}+\mathbf{c}}(2)}
\end{tikzcd}$$

we obtain

$$\lim_r \frac{L^{n+\sigma}(1)}{\phi^{u,v-r\mathbf{a}+\mathbf{b}+\mathbf{c}}(1)} \xrightarrow{f} \lim_r \frac{L^{n+\sigma}(2)}{\phi^{u,v-r\mathbf{a}+\mathbf{b}+\mathbf{c}}(2)}$$

with $\operatorname{coker}(f) = \lim_r \operatorname{coker}(f^r) \in \mathcal{C}$ and hence a $\mathcal{C}$-epimorphism. By the following diagram

$$\begin{tikzcd}
L^{n+\sigma}(1) \arrow[r, "\cong"] \arrow[d, "\bar{\varphi}_n"'] & \lim_r \dfrac{L^{n+\sigma}(1)}{\phi^{u,v-r\mathbf{a}+\mathbf{b}+\mathbf{c}}(1)} \arrow[d, "\mathcal{C}\text{-epi}"', "f"] \\
L^{n+\sigma}(2) \arrow[r, "\cong"] & \lim_r \dfrac{L^{n+\sigma}(2)}{\phi^{u,v-r\mathbf{a}+\mathbf{b}+\mathbf{c}}(2)}
\end{tikzcd}$$

$\bar{\varphi}_n$ is a $\mathcal{C}$-epimorphism. □

Corollary 5.4.5. *Let $\varphi : \mathcal{EC}(1) \to \mathcal{EC}(2)$ be a morphism of exact couples such that the induced spectral sequences are universal coaugmentation concentrated and, for every $(p,q) \in \mathbb{Z} \times \mathbb{Z}$, the following homomorphisms are $\mathcal{C}$-isomorphisms:*

1. $\varphi^\infty : E^\infty_{(p,q)}(1) \to E^\infty_{(p,q)}(2)$, *and*
2. $\lim^1_r\{\ker(i^r_{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}}(1))\} \to \lim^1_r\{\ker(i^r_{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}}(2))\}$.

Then their universal coaugmentations are $\mathcal{C}$-isomorphic

5.5 Comparing Originally Stable or Vanishing Exact Couples

5.5.1 Comparing Originally Stable Exact Couples

Let $\mathcal{C}$ be an ω-cocomplete class of modules closed under extensions.

Proposition 5.5.1. *Let $\varphi : \mathcal{EC}(1) \to \mathcal{EC}(2)$ be a morphism of originally stable exact couples and $E^\infty_{-,-}(1) \to E^\infty_{-,-}(2)$ is a $\mathcal{C}$-epimorphism. Then the induced homomorphism* $\ker(L^{n+\sigma}(1) \to L_{n+\sigma}(1)) \to \ker(L^{n+\sigma}(2) \to L_{n+\sigma}(2))$ *is a $\mathcal{C}$-epimorphism.*

Proof. Fix $n \in \mathbb{Z}$ and let $D^1_{(u,v)+\mathbf{b}+\mathbf{c}}$ is in the originally stable range of the $n+\sigma$-th D^1-column, where $(u, v+\mathbf{b}+\mathbf{c}) \cdot \hat{\mathbf{a}} = n+\sigma$. For simplicity, we take $D^1_{(u,v)+\mathbf{b}+\mathbf{c}}$ to be the lowest D^1-term of the mentioned column. The first condition and 5-Lemma mod $\mathcal{C}$ imply that for every $r \in \mathbb{Z}$, the dashed arrow in the following diagram is a $\mathcal{C}$-epimorphism

$$\begin{tikzcd}
E^{\infty}_{(u,v)+r\mathbf{a}+\mathbf{b}}(1) \arrow[r, "\varphi^{\infty}_{(u,v)+r\mathbf{a}+\mathbf{b}}"] \arrow[d] & E^{\infty}_{(u,v)+r\mathbf{a}+\mathbf{b}}(2) \arrow[r, two heads] \arrow[d] & \operatorname{coker}(\varphi^{\infty}_{(u,v)+r\mathbf{a}+\mathbf{b}}) \arrow[d] \\
\frac{\phi^{u,v+(r+1)\mathbf{a}+\mathbf{b}+\mathbf{c}}(1)}{\phi^{u,v+r\mathbf{a}+\mathbf{b}+\mathbf{c}}(1)} \arrow[r, dashed, "\varphi^{\infty}_{(u,v)+r\mathbf{a}+\mathbf{b}}|"] & \frac{\phi^{u,v+(r+1)\mathbf{a}+\mathbf{b}+\mathbf{c}}(2)}{\phi^{u,v+r\mathbf{a}+\mathbf{b}+\mathbf{c}}(2)} \arrow[r, two heads] & \operatorname{coker}(\varphi^{\infty}_{(u,v)+r\mathbf{a}+\mathbf{b}}|) .
\end{tikzcd} \tag{5.4}$$

Claim 1. *For every $s \geq 1$, $f^s : \phi^{u,v+s\mathbf{a}+\mathbf{b}+\mathbf{c}}(1) \to \phi^{u,v+s\mathbf{a}+\mathbf{b}+\mathbf{c}}(2)$ is a $\mathcal{C}$-epimorphism.*

Proof. We proceed by induction on s. For $s = 1$ the claim holds by the following diagram and hypothesis 1

$$\begin{tikzcd}
E^{\infty}_{(u,v)+\mathbf{b}}(1) \arrow[r, "\varphi^{\infty}_{(u,v)+\mathbf{b}}"] \arrow[d] & E^{\infty}_{(u,v)+\mathbf{b}}(2) \arrow[r, two heads] \arrow[d] & \operatorname{coker}(\varphi^{\infty}_{(u,v)+\mathbf{b}}) \arrow[d] \\
\frac{\phi^{u,v+\mathbf{a}+\mathbf{b}+\mathbf{c}}(1)}{\phi^{u,v+\mathbf{b}+\mathbf{c}}(1)} \arrow[r, "f^1=\varphi^{\infty}_{(u,v)+\mathbf{a}+\mathbf{b}}|"] & \frac{\phi^{u,v+\mathbf{a}+\mathbf{b}+\mathbf{c}}(2)}{\phi^{u,v+\mathbf{b}+\mathbf{c}}(2)} \arrow[r, two heads] & \operatorname{coker}(\varphi^{\infty}_{(u,v)+\mathbf{b}}).
\end{tikzcd}$$

Note that by part 2 of Remark 3.2.2 on page 64, $\phi^{u,v+\mathbf{b}+\mathbf{c}}(i) = 0$, for $i = 1, 2$. Assume the claim holds for s. Using diagram (5.4), in the following diagram

$$\begin{tikzcd}
\phi^{u,v+s\mathbf{a}+\mathbf{b}+\mathbf{c}}(1) \arrow[r, tail] \arrow[d, "f^s"', "\text{Ind. Hyp.}"] & \phi^{u,v+(s+1)\mathbf{a}+\mathbf{b}+\mathbf{c}}(1) \arrow[r, two heads] \arrow[d, "f^{s+1}"'] & \frac{\phi^{u,v+(s+1)\mathbf{a}+\mathbf{b}+\mathbf{c}}(1)}{\phi^{u,v+s\mathbf{a}+\mathbf{b}+\mathbf{c}}(1)} \arrow[d, "(5.4)"] \\
\phi^{u,v+s\mathbf{a}+\mathbf{b}+\mathbf{c}}(2) \arrow[r, tail] & \phi^{u,v+(s+1)\mathbf{a}+\mathbf{b}+\mathbf{c}}(2) \arrow[r, two heads] & \frac{\phi^{u,v+(s+1)\mathbf{a}+\mathbf{b}+\mathbf{c}}(2)}{\phi^{u,v+s\mathbf{a}+\mathbf{b}+\mathbf{c}}(2)}
\end{tikzcd}$$

induction assumption and 5-Lemma mod $\mathcal{C}$ imply that f^{s+1} is also a $\mathcal{C}$-epimorphism. $\diamond$

If we take the direct limit of $\mathcal{C}$-epimorphisms

$$\left\{ \phi^{u,v+s\mathbf{a}+\mathbf{b}+\mathbf{c}}(1) \xrightarrow{f^s} \phi^{u,v+s\mathbf{a}+\mathbf{b}+\mathbf{c}}(2) \right\}_{s\in\mathbb{N}}$$

over s, we obtain the following $\mathcal{C}$-epimorphism

$$\operatorname{colim}_s \phi^{u,v+s\mathbf{a}+\mathbf{b}+\mathbf{c}}(1) \longrightarrow \operatorname{colim}_s \phi^{u,v+s\mathbf{a}+\mathbf{b}+\mathbf{c}}(2).$$

We are done by the following diagram:

$$\begin{tikzcd}
\ker(L^{n+\sigma}(1) \to L_{n+\sigma}(1)) \arrow[r, "\cong"] \arrow[d, "\bar{\varphi}_n"'] & \operatorname{colim}_s \phi^{u,v+s\mathbf{a}+\mathbf{b}+\mathbf{c}}(1) \arrow[d, "\mathcal{C}\text{-epi}"] \\
\ker(L^{n+\sigma}(2) \to L_{n+\sigma}(2)) \arrow[r, "\cong"] & \operatorname{colim}_s \phi^{u,v+s\mathbf{a}+\mathbf{b}+\mathbf{c}}(2).
\end{tikzcd}$$

□

5.5.2 Comparing Originally Vanishing Exact Couples

Let $\mathcal{C}$ be an ω-complete class of modules closed under extensions. From Remark 3.2.2 on page 64, we know that every originally vanishing exact couple is convergent to it universal augmentation. Therefore, the results in section 5.3.1 on "Comparing Spectral Sequences Convergent to the Universal Augmentations" can be stated about the spectral sequences induced by originally vanishing exact couples as the following corollary shows.

Corollary 5.5.2. *Let $\varphi : \mathcal{EC}(1) \to \mathcal{EC}(2)$ be a morphism of originally vanishing exact couples and, for every $(p,q) \in \mathbb{Z} \times \mathbb{Z}$, $E^\infty_{(p,q)}(1) \to E^\infty_{(p,q)}(2)$ is a $\mathcal{C}$-(mono-,epi-)isomorphism. Then $L_*(1) \to L_*(2)$ is $\mathcal{C}$-(mono-,epi-)isomorphism.*

Compare this with Theorem 5.3. in [3] where Boardman says "*No new comparison theorem is needed for such spectral sequences, because Theorem 5.3 is entirely satisfactory*:

Theorem.5.3. [3] *Suppose given a morphism f of spectral sequences, with components $f_r : (D^1_{*,*}(1), E^1_{*,*}(1)) \to (D^1_{*,*}(2), E^1_{*,*}(2))$, where [the induced spectral sequence of] $(D^1_{*,*}(1), E^1_{*,*}(1))$ converges strongly to $G(1)$ and $(D^1_{*,*}(2), E^1_{*,*}(2))$ converges to $G(2)$ (not necessarily strongly), together with a compatible morphism $f : G(1) \to G(2)$ of filtered target groups. If $f^m : E^m_{*,*}(1) \to E^m_{*,*}(2)$ is an isomorphism for some $m \leq \infty$, then $f : G(1) \to G(2)$ is an isomorphism of filtered groups."*

Example 5.5.3. In the setting of Example 3.2.4 on page 68, if we have two towers of cofibrations and the corresponding exact couples are $\mathcal{EC}(1)$ and $\mathcal{EC}(2)$ and if for every $(p,q) \in \mathbb{Z} \times \mathbb{Z}$, $\varphi^\infty_{(p,q)} : E^\infty_{(p,q)}(1) \to E^\infty_{(p,q)}(2)$

is an $\mathcal{C}$-(mono-, epi-)isomorphism, then, for every $n \in \mathbb{Z}$, the induced homomorphism

$$\bar{\varphi}_n : \mathrm{h}_n(X(1)) \to \mathrm{h}_n(X(2))$$

is an $\mathcal{C}$-(mono-,epi-)isomorphism.

5.6 Comparing Eventually Stable or Vanishing Exact Couples

5.6.1 Comparing Eventually Stable Exact Couples

Let $\mathcal{C}$ be an ω-complete class of modules closed under extensions.

Proposition 5.6.1. *Let $\varphi : \mathcal{EC}(1) \to \mathcal{EC}(2)$ be a morphism of eventually stable exact couples and, for every $(p,q) \in \mathbb{Z} \times \mathbb{Z}$, the following hold:*

1. $\varphi^{\infty}_{(p,q)} : E^{\infty}_{(p,q)}(1) \to E^{\infty}_{(p,q)}(2)$ *and*
2. $\bar{\varphi}| : \lim_r \phi_r(1) \to \lim_r \phi_r(2)$ *are $\mathcal{C}$-epimorphisms, and*
3. $\varphi'_{(p,q)} : \lim^1_r\{\ker(i^r_{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}}(1))\} \to \lim^1_r\{\ker(i^r_{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}}(2))\}$ *and*
4. $\bar{\varphi}| : \lim^1_r \phi_r(1) \to \lim^1_r \phi_r(2)$ *are $\mathcal{C}$-monomorphisms.*

Then the induced homomorphism $\bar{\varphi} : L_*(1) \to L_*(2)$ is a $\mathcal{C}$-epimorphism.

Proof. The proof is similar to the proof of Proposition 5.4.1 on page 124. □

Proposition 5.6.2. *Let $\varphi : \mathcal{EC}(1) \to \mathcal{EC}(2)$ be a morphism of eventually stable exact couples and, for every $(p,q) \in \mathbb{Z} \times \mathbb{Z}$,*

1. $\varphi^{\infty}_{(p,q)} : E^{\infty}_{(p,q)}(1) \to E^{\infty}_{(p,q)}(2)$,
2. $\bar{\varphi}| : \lim_r \phi_r(1) \to \lim_r \phi_r(2)$,
3. $\varphi'_{(p,q)} : \lim^1_r\{\ker(i^r_{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}}(1))\} \to \lim^1_r\{\ker(i^r_{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}}(2))\}$, *and*
4. $\bar{\varphi}| : \lim^1_r \phi_r(1) \to \lim^1_r \phi_r(2)$

are $\mathcal{C}$-isomorphisms. Then their universal augmentations are $\mathcal{C}$-isomorphic.

Again, since the E^∞-terms and the quotients of adjacent *kernel* filtrations of the universal coaugmentation are not closely related we can not provide any comparison between universal coaugmentations of two eventually stable exact couples in general. Look at the diagram on page 196.

5.6.2 Comparing Eventually Vanishing Exact Couples

From Remark 3.2.8 on page 72, we know that the spectral sequence induced by an eventually vanishing exact couple is coaugmentation concentrated. Therefore, the results in section 5.4.2 on "Comparing Coaugmentation Concentrated Spectral Sequences" can be stated about the spectral sequences induced by eventually vanishing exact couples.

Example 5.6.3. Look at Example 3.2.10 on page 75. Let $\mathcal{C}$ be an ω-complete class of Abelian groups closed under extensions and assume we have two towers of fibrations $\{X_n(1), F_n(1)\}_{n\in\mathbb{N}}$ and $\{X_n(2), F_n(2)\}_{n\in\mathbb{N}}$, with $X(1)$ and $X(2)$ the (homotopy) inverse limit of them, a morphism between these two towers and the induced morphism between the corresponding eventually vanishing exact couples. Assume, for every $(p,q) \in \mathbb{Z}\times\mathbb{Z}$, the following homomorphism are $\mathcal{C}$-isomorphisms

1. $E^\infty_{(p,q)}(1) \to E^\infty_{(p,q)}(2)$,
2. $\lim^1_n \pi_{k+1}(X_n(1)) \to \lim^1_n \pi_{k+1}(X_n(2))$, and
3. $\lim^1_r \ker(i^r_{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}}(1)) \to \lim^1_r \ker(i^r_{(p,q)+\mathbf{b}+\mathbf{c}-r\mathbf{a}}(2))$.

Then the homotopy groups of $X(1)$ and $X(2)$ are $\mathcal{C}$-isomorphic.

Compare it with the "Mapping Lemma" in [6], p.261, or [12], p.325:

- *Suppose $f : \{X_n(1)\} \to \{X_n(2)\}$ is a map of towers of pointed fibrations and suppose there is a (finite) $N \geq 1$ such that*

$$f_* : E_N^{(p,q)}(1) \to E_N^{(p,q)}(2)$$

is an isomorphism for all (p,q). If $E_r^{p,p}(1) = 0$ for all p, then the map $\lim_n X_n(1) \to \lim_n X_n(2)$ is a weak equivalence of connected spaces.

5.7 Comparing Towers with Matched Eventually Vanishing Exact Couples

Let λ be a limit ordinal and $\mathcal{C}$ be an ordinal-complete class of modules closed under extensions. Also, assume we have inverse towers of bigraded modules

$$\mathrm{H} = \mathrm{H}^{\lambda} \to \cdots \to \mathrm{H}^{\eta+1} \to \mathrm{H}^{\eta} \to \cdots \to \mathrm{H}^{p} \to \cdots \to \mathrm{H}^{1} \tag{5.5}$$

and

$$\bar{\mathrm{H}} = \bar{\mathrm{H}}^{\lambda} \to \cdots \to \bar{\mathrm{H}}^{\eta+1} \to \bar{\mathrm{H}}^{\eta} \to \cdots \to \bar{\mathrm{H}}^{p} \to \cdots \to \bar{\mathrm{H}}^{1}. \tag{5.6}$$

Look at Definition 3.4.14 on page 92, Remark 3.4.15 on page 93 and the diagram on page 208.

Lemma 5.7.1. *Let φ be a morphism from the tower (5.5) to the tower (5.6) where we can match eventually vanishing exact couples to them such that*

1. *the E^{∞}-terms of all intermediate spectral sequences, induced by the originally vanishing exact couples, are $\mathcal{C}$-monomorphic, and*
2. $\hat{\varphi}^{\eta_0} : \ker(\rho^{\eta_0}(1)) \to \ker(\bar{\rho}^{\eta_0}(2))$, *for every limit ordinal $\eta_0 \leq \lambda$, and*
3. $\lim_{\eta<\lambda} F^{\eta}(1) \to \lim_{\eta<\lambda} \bar{F}^{\lambda}(2)$

are $\mathcal{C}$-monomorphisms. Then $\ker(\mathrm{H}^{\lambda} \to \mathrm{H}^{1}) \to \ker(\bar{\mathrm{H}}^{\lambda} \to \bar{\mathrm{H}}^{1})$ is a $\mathcal{C}$-monomorphism.

Proof. Fix $n \in \mathbb{Z}$ and let $\eta = \eta_0 + r$ for a limit ordinal η_0 and a non-negative integer r. Then, the first condition and Remark 3.4.15 on page 93 imply that the dashed arrow in the following diagram is a $\mathcal{C}$-monomorphism

$$\begin{tikzcd}
\ker(\varphi^{\infty}_{(u,v)-\mathbf{c}-r\mathbf{a}}) \arrow[r, tail] & E^{\infty}_{(u,v)-\mathbf{c}-r\mathbf{a}} \arrow[r, "\varphi^{\infty}_{(u,v)-\mathbf{c}-r\mathbf{a}}"] & \bar{E}^{\infty}_{(u,v)-\mathbf{c}-r\mathbf{a}} \\
\ker(\varphi^{\infty}_{(u,v)-\mathbf{c}-r\mathbf{a}}|) \arrow[u] \arrow[r, tail] & \dfrac{F^{\eta_0+r-1}(n)}{F^{\eta_0+r}(n)} \arrow[u] \arrow[r, dashed, "\varphi^{\infty,\eta_0}_{(u,v)-\mathbf{c}-r\mathbf{a}}|"] & \dfrac{\bar{F}^{\eta_0+r-1}(n)}{\bar{F}^{\eta_0+r}(n)} \arrow[u]
\end{tikzcd} \tag{5.7}$$

where $D^{1}_{(u,v)}$ is the lowest nonzero D^{1}-term of the n-th D^{1}-column of the eventually vanishing exact couple corresponding to the limit ordinal η_0.

Claim. *For every* $\eta \le \lambda$, $\dfrac{\ker(\mathrm{H}^\lambda(n) \to \mathrm{H}^1(n))}{F^\eta(n)} \to \dfrac{\ker(\bar{\mathrm{H}}^\lambda(n) \to \bar{\mathrm{H}}^1(n))}{\bar{F}^\eta(n)}$ *is a* $\mathcal{C}$*-monomorphism.*

Proof. We proceed by induction. Let the claim hold for all

$$\frac{\ker(\mathrm{H}^\lambda(n) \to \mathrm{H}^1(n))}{F^\beta(n)} \to \frac{\ker(\bar{\mathrm{H}}^\lambda(n) \to \bar{\mathrm{H}}^1(n))}{\bar{F}^\beta(n)},$$

where $\beta < \eta$, and consider the following two situations:

- η is a non-limit ordinal: Therefore, η has an immediate predecessor; i.e., there is an ordinal β such that $\eta = \beta + 1$. By the 5-Lemma mod $\mathcal{C}$ and the diagram on page 208 the dashed arrow in the following diagram is a $\mathcal{C}$-monomorphism

$$\begin{tikzcd}
\dfrac{F^\beta(n)}{F^\eta(n)} \arrow[r, tail] \arrow[d, "(5.7)"', "\mathcal{C}-\text{mono}"] & \dfrac{\ker(\mathrm{H}^\lambda(n) \to \mathrm{H}^1(n))}{F^\eta(n)} \arrow[r, two heads] \arrow[d, dashed] & \dfrac{\ker(\mathrm{H}^\lambda(n) \to \mathrm{H}^1(n))}{F^\beta(n)} \arrow[d, "\mathcal{C}-\text{mono}"', "\text{Ind. Hypo.}"] \\
\dfrac{\bar{F}^\beta(n)}{\bar{F}^\eta(n)} \arrow[r, tail] & \dfrac{\ker(\bar{\mathrm{H}}^\lambda(n) \to \bar{\mathrm{H}}^1(n))}{\bar{F}^\eta(n)} \arrow[r, two heads] & \dfrac{\ker(\bar{\mathrm{H}}^\lambda(n) \to \bar{\mathrm{H}}^1(n))}{\bar{F}^\beta(n)}.
\end{tikzcd}$$

- η is a limit ordinal: By induction hypothesis, for every $\beta < \eta$, there exists a $\mathcal{C}$-monomorphism

$$\frac{\ker(\mathrm{H}^\lambda(n) \to \mathrm{H}^1(n))}{F^\beta(n)} \xrightarrow{f^\beta} \frac{\ker(\bar{\mathrm{H}}^\lambda(n) \to \bar{\mathrm{H}}^1(n))}{\bar{F}^\beta(n)}.$$

If we take the inverse limit of these $\mathcal{C}$-monomorphisms, we obtain a $\mathcal{C}$-monomorphism

$$\lim_{\beta<\eta} \frac{\ker(\mathrm{H}^\lambda(n) \to \mathrm{H}^1(n))}{F^\beta(n)} \xrightarrow{\bar{f}^\eta} \lim_{\beta<\eta} \frac{\ker(\bar{\mathrm{H}}^\lambda(n) \to \bar{\mathrm{H}}^1(n))}{\bar{F}^\beta(n)}.$$

By condition 2 and the following piece of the diagram on page 208

$$\begin{tikzcd}
\ker(\rho^\eta(n)) \arrow[r, "\mathcal{C}\text{-mono}"] & \ker(\bar{\rho}^\eta(n)) \\
\dfrac{\lim_{\beta<\eta} F^\beta(n)}{F^\eta(n)} \arrow[u, tail] \arrow[r, "\hat{f}^\eta"] & \dfrac{\lim_{\beta<\eta} \bar{F}^\beta(n)}{\bar{F}^\eta(n)} \arrow[u, tail]
\end{tikzcd}$$

we see that $\bar{f}^\eta$ is a $\mathcal{C}$-monomorphism. By the 5-Lemma mod $\mathcal{C}$, the dashed arrow in the following diagram is a $\mathcal{C}$-monomorphism

$$\begin{tikzcd}
\frac{\lim_{\beta<\eta} F^{\beta}(n)}{F^{\eta}(n)} \arrow[r, tail] \arrow[d, "\hat{f}^{\eta}"'] & \frac{\ker(\mathrm{H}^{\lambda}(n)\to\mathrm{H}^{1}(n))}{F^{\eta}(n)} \arrow[r] \arrow[d, dashed] & \lim_{\beta<\eta}\frac{\ker(\mathrm{H}^{\lambda}(n)\to\mathrm{H}^{1}(n))}{F^{\beta}(n)} \arrow[r, two heads] \arrow[d, "\bar{f}^{\eta}"] & \lim^{1}_{\beta<\eta}F^{\beta}(n) \arrow[d] \\
\frac{\lim_{\beta<\eta} \bar{F}^{\beta}(n)}{\bar{F}^{\eta}(n)} \arrow[r, tail] & \frac{\ker(\bar{\mathrm{H}}^{\lambda}(n)\to\bar{\mathrm{H}}^{1}(n))}{\bar{F}^{\eta}(n)} \arrow[r] & \lim_{\beta<\eta}\frac{\ker(\bar{\mathrm{H}}^{\lambda}(n)\to\bar{\mathrm{H}}^{1}(n))}{\bar{F}^{\beta}(n)} \arrow[r, two heads] & \lim^{1}_{\beta<\eta}\bar{F}^{\beta}(n).
\end{tikzcd}$$

$\diamond$

Therefore, for every $\eta \leq \lambda$, $\frac{\ker(\mathrm{H}^{\lambda}(n)\to\mathrm{H}^{1}(n))}{F^{\eta}(n)} \to \frac{\ker(\bar{\mathrm{H}}^{\lambda}(n)\to\bar{\mathrm{H}}^{1}(n))}{\bar{F}^{\eta}(n)}$ is a $\mathcal{C}$-monomorphism. In particular, $\frac{\ker(\mathrm{H}^{\lambda}(n)\to\mathrm{H}^{1}(n))}{F^{\lambda}(n)} \to \frac{\ker(\bar{\mathrm{H}}^{\lambda}(n)\to\bar{\mathrm{H}}^{1}(n))}{\bar{F}^{\lambda}(n)}$ is a $\mathcal{C}$-monomorphism.

Using condition 3 and the following diagram

$$\begin{tikzcd}
F^{\lambda}(n) \arrow[r, tail] \arrow[d] & \lim_{\eta<\lambda} F^{\eta}(n) \arrow[d, "\mathcal{C}-\text{mono}"] \\
\bar{F}^{\lambda}(n) \arrow[r, tail] & \lim_{\eta<\lambda} \bar{F}^{\eta}(n)
\end{tikzcd}$$

$F^{\lambda}(n) \to \bar{F}^{\lambda}(n)$ is a $\mathcal{C}$-monomorphism. By the 5-Lemma mod $\mathcal{C}$, the dashed arrow in the following diagram is a $\mathcal{C}$-monomorphism

$$\begin{tikzcd}
F^{\lambda}(n) \arrow[r, tail] \arrow[d, "\mathcal{C}-\text{mono}"'] & \ker(\mathrm{H}^{\lambda}(n)\to\mathrm{H}^{1}(n)) \arrow[r, two heads] \arrow[d, dashed] & \frac{\ker(\mathrm{H}^{\lambda}(n)\to\mathrm{H}^{1}(n))}{F^{\lambda}(n)} \arrow[d, "\mathcal{C}-\text{mono}"'] \\
\bar{F}^{\lambda}(n) \arrow[r, tail] & \ker(\bar{\mathrm{H}}^{\lambda}(n)\to\bar{\mathrm{H}}^{1}(n)) \arrow[r, two heads] & \frac{\ker(\bar{\mathrm{H}}^{\lambda}(n)\to\bar{\mathrm{H}}^{1}(n))}{\bar{F}^{\lambda}(n)}.
\end{tikzcd}$$

$\square$

We will use the following lemma in the proof of the next proposition. Look at Remark 3.4.15 on page 93 for notations.

Lemma 5.7.2. *Assume we can match eventually vanishing exact couples to the towers in (5.5) and, for every limit ordinal $\eta_0 < \lambda$, the composite*

$$\mathrm{H}^{\lambda} \xrightarrow{Q^{\eta_0+\omega}} \mathrm{H}^{\eta_0+\omega} \xrightarrow{\rho^{\eta_0+\omega}} \lim_{\beta<\eta_0} \mathrm{H}^{\beta}$$

is an epimorphism. If $D^1_{(u,v)}$ is the lowest nonzero D^1-term of the n-th D^1-column of the eventually vanishing exact couple corresponding to η_0,

then for every $r \in \mathbb{N}$
$$\frac{F^{\eta_0+r-1}(n)}{F^{\eta_0+r}(n)} \cong \epsilon^{u,v-r\mathbf{a}}.$$
and hence $\dfrac{F^{\eta_0+r-1}(n)}{F^{\eta_0+r}(n)} \rightarrowtail E^{\infty}_{(u,v)-r\mathbf{a}-\mathbf{c}}.$

Proof. Note that the homomorphism $\rho^{\eta_0+\omega}(n) \circ Q^{\eta_0+\omega}(n)$ is defined from $\mathrm{H}^{\lambda}(n)$ to $\lim_{\beta<\eta_0} \mathrm{H}^{\beta}(n)$. By E^{∞}-Comparison Theorem we have the following diagram with exact rows and columns

$$\begin{tikzcd}
\operatorname{coker}(h^{u,v-r\mathbf{a}}) & \epsilon^{u,v-r\mathbf{a}} \arrow[l, two heads] & \frac{F^{\eta_0+r-1}(n)}{F^{\eta_0+r}(n)} \arrow[l, tail] \\
\operatorname{coker}(\rho^{\eta_0+\omega}(n) \circ Q^{\eta_0+\omega}(n)|) \arrow[u, two heads] & \phi^{u,v-(r-1)\mathbf{a}} \arrow[l, two heads] \arrow[u, two heads] & F^{\eta_0+r-1}(n) \arrow[l, "\rho^{\eta_0+\omega}(n)\circ Q^{\eta_0+\omega}(n)|"'] \arrow[u, two heads] \\
\operatorname{coker}(\rho^{\eta_0+\omega}(n) \circ Q^{\eta_0+\omega}(n)|) \arrow[u, tail, "h^{u,v-r\mathbf{a}}"] & \phi^{u,v-r\mathbf{a}} \arrow[l, two heads] \arrow[u, tail] & F^{\eta_0+r}(n) \arrow[l, "\rho^{\eta_0+\omega}(n)\circ Q^{\eta_0+\omega}(n)|"'] \arrow[u, tail]
\end{tikzcd}$$

If $\rho^{\eta_0+\omega}(n) \circ Q^{\eta_0+\omega}(n)$ is an epimorphism, then $\operatorname{coker}(h^{u,v-r\mathbf{a}}) = 0$ and we are done.

□

Proposition 5.7.3. *Let* φ *be a morphism from the tower (5.5) to the tower (5.6) where we can match eventually vanishing exact couples to them such that for every limit ordinal*

1. *the* E^{∞}*-terms of all intermediate spectral sequences, induced by the originally vanishing exact couples, are* $\mathcal{C}$*-isomorphic, and*
2. *for every limit ordinal* $\eta_0 \leq \lambda$, $\lim^1_{\beta<\eta_0} F^{\beta} \to \lim^1_{\beta<\eta_0} \bar{F}^{\beta}$ *is a* $\mathcal{C}$*-isomorphism and* ρ^{η_0} *and* $\bar{\rho}^{\eta_0}$ *are isomorphisms,*
3. *for every limit ordinal* $\eta_0 < \lambda$, $Q^{\eta_0+\omega}$ *and* $\bar{Q}^{\eta_0+\omega}$ *are epimorphisms, and*
4. $\lim_{\eta<\lambda} F^{\eta} \to \lim_{\eta<\lambda} \bar{F}^{\eta}$ *is a* $\mathcal{C}$*-isomorphism.*

Then $\ker(\mathrm{H}^{\lambda} \to \mathrm{H}^{1})$ *and* $\ker(\bar{\mathrm{H}}^{\lambda} \to \bar{\mathrm{H}}^{1})$ *are* $\mathcal{C}$*-isomorphic.*

Proof. Fix $n \in \mathbb{Z}$ and assume that $\eta = \eta_0 + r < \lambda$, for a limit ordinal $\eta_0 < \lambda$ and a non-negative integer r. Then, the first two conditions, Remark 3.4.15 on page 93 and 5-Lemma mod $\mathcal{C}$ imply that the dashed arrow in the following diagram is a $\mathcal{C}$-isomorphism

$$\begin{tikzcd} \dfrac{F^{\eta_0+r-1}(n)}{F^{\eta_0+r}(n)} \arrow[r, tail] \arrow[d, dashed] & E^{\infty}_{(u,v)-\mathbf{c}-r\mathbf{a}} \arrow[r] \arrow[d, "\mathcal{C}-\text{iso}"] & \lim^1_r \ker(i^{r,\eta_0}_{(u,v)-r\mathbf{a}}) \arrow[d, "\mathcal{C}-\text{iso}"] \\ \dfrac{\bar{F}^{\eta_0+r-1}(n)}{\bar{F}^{\eta_0+r}(n)} \arrow[r, tail] & \bar{E}^{\infty}_{(u,v)-\mathbf{c}-r\mathbf{a}} \arrow[r] & \lim^1_r \ker(\bar{i}^{r,\eta_0}_{(u,v)-r\mathbf{a}}) \end{tikzcd} \tag{5.8}$$

where $D^1_{(u,v)}$ is the lowest nonzero D^1-term of the n-th D^1-column of the eventually vanishing exact couple corresponding to the limit ordinal η_0.

Claim. *For every* $\eta \leq \lambda$, $\dfrac{\ker(\mathrm{H}^{\lambda}(n) \to \mathrm{H}^1(n))}{F^{\eta}(n)} \to \dfrac{\ker(\bar{\mathrm{H}}^{\lambda}(n) \to \bar{\mathrm{H}}^1(n))}{\bar{F}^{\eta}(n)}$ *is a* $\mathcal{C}$*-isomorphism.*

Proof. We proceed by induction. Let the claim hold for all

$$\frac{\ker(\mathrm{H}^{\lambda}(n) \to \mathrm{H}^1(n))}{F^{\beta}(n)} \to \frac{\ker(\bar{\mathrm{H}}^{\lambda}(n) \to \bar{\mathrm{H}}^1(n))}{\bar{F}^{\beta}(n)},$$

where $\beta < \eta$, and consider the following situations:

- η is a non-limit ordinal: Therefore, η has an immediate predecessor; i.e., there is an ordinal β such that $\eta = \beta + 1$. By the 5-Lemma mod $\mathcal{C}$ and the diagram on page 208 the dashed arrow in the following diagram is a $\mathcal{C}$-isomorphism

$$\begin{tikzcd} \dfrac{F^{\beta}(n)}{F^{\eta}(n)} \arrow[r, tail] \arrow[d, "(5.8)"'] & \dfrac{\ker(\mathrm{H}^{\lambda}(n) \to \mathrm{H}^1(n))}{F^{\eta}(n)} \arrow[r, two heads] \arrow[d, dashed] & \dfrac{\ker(\mathrm{H}^{\lambda}(n) \to \mathrm{H}^1(n))}{F^{\beta}(n)} \arrow[d, "\text{Ind. Hypo.}"] \\ \dfrac{\bar{F}^{\beta}(n)}{\bar{F}^{\eta}(n)} \arrow[r, tail] & \dfrac{\ker(\bar{\mathrm{H}}^{\lambda}(n) \to \bar{\mathrm{H}}^1(n))}{\bar{F}^{\eta}(n)} \arrow[r, two heads] & \dfrac{\ker(\bar{\mathrm{H}}^{\lambda}(n) \to \bar{\mathrm{H}}^1(n))}{\bar{F}^{\beta}(n)} \end{tikzcd}.$$

- η is a limit ordinal: In this case, by induction hypothesis there is a $\mathcal{C}$-isomorphism

$$\frac{\ker(\mathrm{H}^{\lambda}(n) \to \mathrm{H}^1(n))}{F^{\beta}(n)} \xrightarrow{f^{\beta}} \frac{\ker(\bar{\mathrm{H}}^{\lambda}(n) \to \bar{\mathrm{H}}^1(n))}{\bar{F}^{\beta}(n)}$$

for every $\beta < \eta$. If we take the inverse limit of these $\mathcal{C}$-isomorphisms, we get a $\mathcal{C}$-isomorphism

$$\lim_{\beta<\eta} \frac{\ker(\mathrm{H}^\lambda(n) \to \mathrm{H}^1(n))}{F^\beta(n)} \xrightarrow{\bar{f}^\eta} \lim_{\beta<\eta} \frac{\ker(\bar{\mathrm{H}}^\lambda(n) \to \bar{\mathrm{H}}^1(n))}{\bar{F}^\beta(n)}.$$

By condition 2 and the following piece of the diagram on page 208

$$\frac{\lim_{\beta<\eta} F^\beta(n)}{F^\eta(n)} \cong \frac{\lim_{\beta<\eta} \bar{F}^\beta(n)}{\bar{F}^\eta(n)} = 0.$$

By the 5-Lemma mod $\mathcal{C}$, the dashed arrow in the following part of the diagram on page 208 is a $\mathcal{C}$-isomorphism

$$\begin{tikzcd}
\frac{\ker(\mathrm{H}^\lambda(n) \to \mathrm{H}^1(n))}{F^\eta(n)} \arrow[r, tail] \arrow[d, dashed] & \lim_{\beta<\eta} \frac{\ker(\mathrm{H}^\lambda(n) \to \mathrm{H}^1(n))}{F^\beta(n)} \arrow[r, two heads] \arrow[d, "\mathcal{C}-\mathrm{iso}"', "\bar{f}^\eta"] & \lim^1_{\beta<\eta} F^\beta(n) \arrow[d, "\mathcal{C}-\mathrm{iso}"'] \\
\frac{\ker(\bar{\mathrm{H}}^\lambda(n) \to \bar{\mathrm{H}}^1(n))}{\bar{F}^\eta(n)} \arrow[r, tail] & \lim_{\beta<\eta} \frac{\ker(\bar{\mathrm{H}}^\lambda(n) \to \bar{\mathrm{H}}^1(n))}{\bar{F}^\beta(n)} \arrow[r, two heads] & \lim^1_{\beta<\eta} \bar{F}^\beta(n).
\end{tikzcd}$$

◇

Therefore, for every $\eta \le \lambda$, $\frac{\ker(\mathrm{H}^\lambda(n) \to \mathrm{H}^1(n))}{F^\eta(n)} \to \frac{\ker(\bar{\mathrm{H}}^\lambda(n) \to \bar{\mathrm{H}}^1(n))}{\bar{F}^\eta(n)}$ is a $\mathcal{C}$-monomorphism. In particular, $\frac{\ker(\mathrm{H}^\lambda(n) \to \mathrm{H}^1(n))}{F^\lambda(n)} \to \frac{\ker(\bar{\mathrm{H}}^\lambda(n) \to \bar{\mathrm{H}}^1(n))}{\bar{F}^\lambda(n)}$ is a $\mathcal{C}$-monomorphism.

Using condition 4 and the following diagram

$$\begin{tikzcd}
F^\lambda(n) \arrow[r, "\cong"] \arrow[d] & \lim_{\eta<\lambda} F^\eta(n) \arrow[d, "\mathcal{C}-\mathrm{iso}"] \\
\bar{F}^\lambda(n) \arrow[r, "\cong"] & \lim_{\eta<\lambda} \bar{F}^\eta(n)
\end{tikzcd}$$

$F^\lambda(n) \to \bar{F}^\lambda(n)$ is a $\mathcal{C}$-isomorphism. By the 5-Lemma mod $\mathcal{C}$, the dashed arrow in the following diagram is a $\mathcal{C}$-isomorphism

$$\begin{tikzcd}
F^\lambda(n) \arrow[r, tail] \arrow[d, "\mathcal{C}-\mathrm{iso}"'] & \ker(\mathrm{H}^\lambda(n) \to \mathrm{H}^1(n)) \arrow[r, two heads] \arrow[d, dashed] & \frac{\ker(\mathrm{H}^\lambda(n) \to \mathrm{H}^1(n))}{F^\lambda(n)} \arrow[d, "\mathcal{C}-\mathrm{iso}"'] \\
\bar{F}^\lambda(n) \arrow[r, tail] & \ker(\bar{\mathrm{H}}^\lambda(n) \to \bar{\mathrm{H}}^1(n)) \arrow[r, two heads] & \frac{\ker(\bar{\mathrm{H}}^\lambda(n) \to \bar{\mathrm{H}}^1(n))}{\bar{F}^\lambda(n)}.
\end{tikzcd}$$

□

Example 5.7.4. Look at Example 3.4.16 on page 93 and assume we have two transfinite towers of fibrations $\{X_\eta(i), F_\eta(i)\}_{\eta<\lambda}$, for $i = 1, 2$ and a morphism from one tower to another and $\mathcal{C}$ be an ordinal-complete class of Abelian groups closed under extensions. Then we have the following:

1. If

   (a) in every intermediate eventually vanishing exact couple the morphism $E^\infty_{-,-}(1) \to E^\infty_{-,-}(2)$ is a $\mathcal{C}$-monomorphism, and

   (b) for every $n \in \mathbb{N}$ and every limit ordinal $\eta_0 \le \lambda$,

   $$\lim^1_{\beta<\eta_0} \pi_n(X_\beta(1)) = \lim^1_{\beta<\eta_0} \pi_n(X_\beta(2)) = 0,$$

   then $\pi_*(X(1)) \to \pi_*(X(2))$ is a $\mathcal{C}$-monomorphism.

2. Assume in every intermediate eventually vanishing exact couple

   (a) $E^\infty_{-,-}(1) \to E^\infty_{-,-}(2)$ is a $\mathcal{C}$-isomorphism, and

   (b) $\lim^1_r \ker(i^r_{(u,v)-r\mathbf{a}}(1)) = \lim^1_r \ker(i^r_{(u,v)-r\mathbf{a}}(2)) = 0$,

   and for every $n \in \mathbb{N}$, $i = 1, 2$ and every limit ordinal $\eta_0 \le \lambda$

   (c) $\lim^1_{\beta<\eta_0} F^\beta(1) \to \lim^1_{\beta<\eta_0} F^\beta(2)$ is a $\mathcal{C}$-isomorphism,

   (d) $\pi_n(X(i)) \twoheadrightarrow \pi_n(X_{\eta_0+\omega}(i))$ is an epimorphism, and

   (a) $\lim^1_{\beta<\eta_0} \pi_n(X_\beta(i)) = 0$.

   Then $\pi_*(X(1))$ and $\pi_*(X(2))$ are $\mathcal{C}$-isomorphic.

5.8 Comparing Towers with Matched Originally Vanishing Exact Couples

Let λ be a limit ordinal and $\mathcal{C}$ be an ordinal-complete class of modules closed under extensions. Also, assume we have directed towers of bigraded modules

$$\mathrm{H}_1 \to \cdots \to \mathrm{H}_p \to \cdots \to \mathrm{H}_\eta \to \mathrm{H}_{\eta+1} \to \cdots \to \mathrm{H}_\lambda = \mathrm{H} \tag{5.9}$$

and

$$\bar{\mathrm{H}}_1 \to \cdots \to \bar{\mathrm{H}}_p \to \cdots \to \bar{\mathrm{H}}_\eta \to \bar{\mathrm{H}}_{\eta+1} \to \cdots \to \bar{\mathrm{H}}_\lambda = \bar{\mathrm{H}}. \tag{5.10}$$

Look at Definition 3.3.6 on page 83, Remark 3.3.7 on page 83 and the diagram on page 202.

Proposition 5.8.1. *Let φ be a morphism from the tower (5.9) to the tower (5.10) where we can match originally vanishing exact couples to these towers such that*

1. *the E^{∞}-terms of all intermediate spectral sequences, induced by the originally vanishing exact couples, are $\mathcal{C}$-epimorphic, and*
2. *$\hat{\varphi}^{\eta_0} : \mathrm{coker}(\rho_{\eta_0}(1)) \to \mathrm{coker}(\bar{\rho}_{\eta_0}(2))$, for every limit ordinal $\eta_0 \leq \lambda$, is a $\mathcal{C}$-epimorphism.*

Then $\mathrm{H}(1) \to \bar{\mathrm{H}}(2)$ *is a $\mathcal{C}$-epimorphism.*

Proof. Fix $n \in \mathbb{Z}$ and assume $\eta = \eta_0 + r$ for a limit ordinal η_0 and a non-negative integer r. Then, the first condition and Remark 3.3.7 on page 83 imply that the dashed arrow in the following diagram is a $\mathcal{C}$-epimorphism

$$\begin{tikzcd}
E^{\infty}_{(u,v)+r\mathbf{a}+\mathbf{b}} \arrow[r, "\varphi^{\infty}_{(u,v)+r\mathbf{a}+\mathbf{b}}"] \arrow[d] & \bar{E}^{\infty}_{(u,v)+r\mathbf{a}+\mathbf{b}} \arrow[r, two heads] \arrow[d] & \mathrm{coker}(\varphi^{\infty}_{(u,v)+r\mathbf{a}+\mathbf{b}}) \arrow[d] \\
\dfrac{F_{\eta_0+r}(n)}{F_{\eta_0+r-1}(n)} \arrow[r, dashed, "\varphi^{\infty,\eta_0}_{(u,v)+r\mathbf{a}+\mathbf{b}}|"] & \dfrac{\bar{F}_{\eta_0+r}(n)}{\bar{F}_{\eta_0+r-1}(n)} \arrow[r, two heads] & \mathrm{coker}(\varphi^{\infty}_{(u,v)+r\mathbf{a}+\mathbf{b}}|)
\end{tikzcd} \tag{5.11}$$

where $D^1_{(u,v)}$ is the highest nonzero D^1-term of the n-th D^1-column of the originally vanishing exact couple corresponding to the limit ordinal η_0.

Claim. *For every $\eta \leq \lambda$, $F_{\eta}(n) \to \bar{F}_{\eta}(n)$ is a $\mathcal{C}$-epimorphism.*

Proof. We proceed by induction. Assume the claim holds for all $F_{\beta}(n) \to \bar{F}_{\beta}(n)$, where $\beta < \eta$, and consider the following two situations:

- η is a non-limit ordinal: Therefore, η has an immediate predecessor; i.e., there is an ordinal β such that $\eta = \beta + 1$. By Five-Lemma mod $\mathcal{C}$ and the diagram on page 202 the dashed arrow in the following diagram is a $\mathcal{C}$-epimorphism

$$\begin{tikzcd}
F_{\beta}(n) \arrow[r, tail] \arrow[d, "\mathcal{C}-\mathrm{epi}", "\text{Ind. Hypo.}"'] & F_{\eta}(n) \arrow[r, two heads] \arrow[d, dashed] & \dfrac{F_{\eta}(n)}{F_{\beta}(n)} \arrow[d, "(5.11)", "\mathcal{C}-\mathrm{epi}"'] \\
\bar{F}_{\beta}(n) \arrow[r, tail] & \bar{F}_{\eta}(n) \arrow[r, two heads] & \dfrac{\bar{F}_{\eta}(n)}{\bar{F}_{\beta}(n)}.
\end{tikzcd}$$

- η is a limit ordinal: In this case, by induction hypothesis, for every $\beta < \eta$, there is a $\mathcal{C}$-epimorphism $F_\beta(n) \xrightarrow{f_\beta} \bar{F}_\beta(n)$. If we take the direct limit of these $\mathcal{C}$-epimorphisms, we obtain a $\mathcal{C}$-epimorphism

$$\mathrm{colim}_{\beta<\eta} F_\beta(n) \xrightarrow{\bar{f}_\eta} \mathrm{colim}_{\beta<\eta} \bar{F}_\beta(n).$$

By condition 2 and the following piece of the diagram on page 202

$$\begin{tikzcd} \mathrm{coker}(\rho_\eta(n)) \arrow[r, "\mathcal{C}\text{-epi}"] \arrow[d] & \mathrm{coker}(\bar{\rho}_\eta(n)) \arrow[d] \\ \dfrac{F_\eta(n)}{\mathrm{colim}_{\beta<\eta} F_\beta(n)} \arrow[r, "\hat{f}_\eta"] & \dfrac{\bar{F}_\eta(n)}{\mathrm{colim}_{\beta<\eta} \bar{F}_\beta(n)} \end{tikzcd}$$

we see that $\bar{f}_\eta$ is a $\mathcal{C}$-epimorphism. By Five-Lemma mod $\mathcal{C}$, the dashed arrow in the following diagram is a $\mathcal{C}$-epimorphism

$$\begin{tikzcd} \mathrm{colim}_{\beta<\eta} F_\beta(n) \arrow[r, tail] \arrow[d, "\bar{f}_\eta"', "\mathcal{C}\text{-epi}"] & F_\eta(n) \arrow[r] \arrow[d, dashed] & \dfrac{F_\eta(n)}{\mathrm{colim}_{\beta<\eta} F_\beta(n)} \arrow[d, "\mathcal{C}\text{-epi}"', "\hat{f}_\eta"] \\ \mathrm{colim}_{\beta<\eta} \bar{F}_\beta(n) \arrow[r, tail] & \bar{F}_\eta(n) \arrow[r] & \dfrac{\bar{F}_\eta(n)}{\mathrm{colim}_{\beta<\eta} \bar{F}_\beta(n)}. \end{tikzcd}$$

$\diamond$

Therefore, for every $\eta \le \lambda$, $F_\eta(n) \to \bar{F}_\eta(n)$ is a $\mathcal{C}$-epimorphism, in particular, $\mathrm{H}_\lambda(n) \to \bar{\mathrm{H}}_\lambda(n)$ is a $\mathcal{C}$-epimorphism.

$\square$

We will use the following lemma in the proof of the next proposition. Look at Remark 3.3.7 on page 83 for notations.

Lemma 5.8.2. *Assume we can match originally vanishing exact couples to the tower in (5.9) and, for every limit ordinal $\eta_0 < \lambda$, the composite*

$$\mathrm{colim}_{\beta<\eta_0+\omega} \mathrm{H}_\beta \xrightarrow{\rho_{\eta_0+\omega}} \mathrm{H}_{\eta_0+\omega} \xrightarrow{Q_{\eta_0+\omega}} \mathrm{H}_\lambda$$

is a monomorphism. If $D^1_{(u,v)}$ is the highest nonzero D^1-term of the n-th D^1-column of the originally vanishing exact couple corresponding to η_0, *then for every* $r \in \mathbb{N}$

$$\frac{F_{\eta_0+r}(n)}{F_{\eta_0+r-1}(n)} \cong E^{\infty}_{(u,v)+\mathbf{b}-r\mathbf{a}}.$$

Proof. The proof is similar to the proof of Lemma 5.7.2 on page 136 and a result of part 1 of Proposition 2.3.16 on page 38 applied to the diagram (3.13) on page 83. □

Corollary 5.8.3. *Let* φ *be a morphism from the tower (5.9) to the tower (5.10) where we can match originally vanishing exact couples to them such that*

1. *the* E^∞*-terms of all intermediate spectral sequences, induced by the originally vanishing exact couples, are* $\mathcal{C}$*-isomorphic, and*
2. *for every limit ordinal* $\eta_0 < \lambda$*,* $Q_{\eta_0+\omega}$ *and* $\bar{Q}_{\eta_0+\omega}$ *are monomorphisms, and*
3. *for every limit ordinal* $\eta_0 \leq \lambda$*,* ρ_{η_0} *and* $\bar{\rho}_{\eta_0}$ *are isomorphisms.*

Then H *and* $\bar{\mathrm{H}}$ *are* $\mathcal{C}$*-isomorphic.*

Proof. Fix $n \in \mathbb{Z}$ and assume $\eta = \eta_0 + r$ for a limit ordinal η and a non-negative integer r. Then, the first condition, Lemma 5.8.2 on page 142 and Remark 3.3.7 on page 83 imply that the dashed arrow in the following diagram is a $\mathcal{C}$-isomorphism

$$\begin{array}{ccc} E^{\infty}_{(u,v)+r\mathbf{a}+\mathbf{b}} & \xrightarrow{\varphi^{\infty}_{(u,v)+r\mathbf{a}+\mathbf{b}}} & \bar{E}^{\infty}_{(u,v)+r\mathbf{a}+\mathbf{b}} \\ \cong \uparrow & & \cong \uparrow \\ \dfrac{F_{\eta_0+r}(n)}{F_{\eta_0+r-1}(n)} & \xrightarrow{\varphi^{\infty,\eta_0}_{(u,v)+r\mathbf{a}+\mathbf{b}}|} & \dfrac{\bar{F}_{\eta_0+r}(n)}{\bar{F}_{\eta_0+r-1}(n)} \end{array} \tag{5.12}$$

where $D^1_{(u,v)}$ is the highest nonzero D^1-term of the n-th D^1-column of the originally vanishing exact couple corresponding to the limit ordinal η_0.

Claim. *For every* $\eta \leq \lambda$*,* $F_\eta(n) \to \bar{F}_\eta(n)$ *is a* $\mathcal{C}$*-isomorphism.*

Proof. We proceed by induction. Let the claim hold for all $F_\beta(n) \to \bar{F}_\beta(n)$, where $\beta < \eta$, and consider the following two situations:

- η is a non-limit ordinal: Therefore, η has an immediate predecessor; i.e., there is an ordinal β such that $\eta = \beta + 1$. By the 5-Lemma mod $\mathcal{C}$ and the diagram on page 202 the arrow in the middle in the following diagram is a $\mathcal{C}$-epimorphism

$$\begin{tikzcd}
F_\beta(n) \arrow[r, tail] \arrow[d, "\text{Ind. Hypo.}"', "\mathcal{C}-\text{iso}"] & F_\eta(n) \arrow[r, two heads] \arrow[d] & \frac{F_\eta(n)}{F_\beta(n)} \arrow[d, "\mathcal{C}-\text{iso}"', "(5.12)"] \\
\bar{F}_\beta(n) \arrow[r, tail] & \bar{F}_\eta(n) \arrow[r, two heads] & \frac{\bar{F}_\eta(n)}{\bar{F}_\beta(n)}.
\end{tikzcd}$$

- η is a limit ordinal: In this case, by induction hypothesis there is a $\mathcal{C}$-isomorphism $F_\beta(n) \xrightarrow{f_\beta} \bar{F}_\beta(n)$, for every $\beta < \eta$. If we take the direct limit of these $\mathcal{C}$-isomorphisms, we obtain a $\mathcal{C}$-isomorphism

$$\operatorname{colim}_{\beta<\eta} F_\beta(n) \xrightarrow{\bar{f}_\eta} \operatorname{colim}_{\beta<\eta} \bar{F}_\beta(n).$$

By condition 2 and the diagram on page 202 we have

$$\frac{F_\eta(n)}{\operatorname{colim}_{\beta<\eta} F_\beta(n)} \cong \frac{\bar{F}_\eta(n)}{\operatorname{colim}_{\beta<\eta} \bar{F}_\beta(n)} = 0.$$

Therefore,

$$F_\eta(n) = \operatorname{colim}_{\beta<\eta} F_\beta(n) \xrightarrow{\bar{f}_\eta} \operatorname{colim}_{\beta<\eta} \bar{F}_\beta(n) = \bar{F}_\eta(n).$$

is a $\mathcal{C}$-isomorphism. $\diamond$

Therefore, for every $\eta \leq \lambda$, $F_\eta(n) \to \bar{F}_\eta(n)$ is a $\mathcal{C}$-isomorphism, in particular, $\mathrm{H}_\lambda \to \bar{\mathrm{H}}_\lambda$ is a $\mathcal{C}$-isomorphism. $\square$

Example 5.8.4. Look at Example 3.3.8 on page 84 and assume we have two transfinite towers of cofibrations $\{X_\eta(i), F_\eta(i)\}_{\eta<\lambda}$, for $i = 1, 2$. Let φ be a morphism from one tower to another and $\mathcal{C}$ be an ordinal-cocomplete class of Abelian groups closed under extensions. Then we have the following:

- If in every intermediate originally vanishing exact couple, $E^\infty_{-,-}(1) \to E^\infty_{-,-}(2)$ is a $\mathcal{C}$-epimorphism, then the induced morphism $\mathrm{H}_*(X(1)) \to \mathrm{H}_*(X(2))$ is a $\mathcal{C}$-epimorphism.

- If

– in every intermediate originally vanishing exact couple, $E^{\infty}_{-,-}(1)$ and $E^{\infty}_{-,-}(2)$ are $\mathcal{C}$-isomorphic, and
– for every limit ordinal $\eta_0 < \lambda$ and $i = 1, 2$, the morphism $Q_{\eta_0+\omega}(i) : \mathrm{H}_*(X_{\eta_0+\omega}(i)) \to \mathrm{H}_*(X(i))$ is a monomorphism,

then $\mathrm{H}_*(X(1))$ and $\mathrm{H}_*(X(2))$ are $\mathcal{C}$-isomorphic.

Chapter 6

Reverse Engineering

6.1 Introduction

In the last two chapters, we explained examples of carrying a property from some page of a spectral sequence to the limit page and, ultimately, to the universal augmentation or coaugmentation of the corresponding exact couple. In this chapter, we go backward; i.e., we carry a property from the universal augmentation or coaugmentation of an exact couple to the limit page and, ultimately, to one of the pages of the induced spectral sequence. We do so for some spectral sequences that collapse at some page.

In this chapter, we see two scenarios where this reverse engineering technique can be applied. In both scenarios, we deal with spectral sequences induced by "super convenient" exact couples; i.e., exact couples that are either originally vanishing and eventually stable or eventually vanishing and originally stable. Let us consider, for example, a first quadrant spectral sequence induced by an originally vanishing and eventually stable exact couple. From the third chapter, we know that this spectral sequence is convergent to its universal augmentation. We then pick a class $\mathcal{C}$ of modules that is closed under extensions and we look at the r-th page of the spectral sequence

$$\begin{array}{c|cccccc}
q & & & & & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
n & E^r_{0,n} & E^r_{1,n} & \cdots & E^r_{n,n} & \cdots & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
1 & E^r_{0,1} & E^r_{1,1} & \cdots & E^r_{n,1} & \cdots & \\
0 & E^r_{0,0} & E^r_{1,0} & \cdots & E^r_{n,0} & \cdots & \\
\hline
 & 0 & 1 & \cdots & n & \cdots & p
\end{array}$$

with the following properties:

- the red entries are in $\mathcal{C}$,
- if a blue entry is in $\mathcal{C}$ then all entries above it are also in $\mathcal{C}$, and
- the universal augmentation is in $\mathcal{C}$.

In the first scenario, we conclude that all entries of this page also belong to $\mathcal{C}$. In fact, the idea of the first scenario comes from the second scenario which is developed by Peschke [26].

In the second scenario, we compare two such spectral sequences backward; i.e., we pick two such spectral sequences and assume the following

- the corresponding red entries are $\mathcal{C}$-isomorphic,
- if a blue entry in one spectral sequence is $\mathcal{C}$-isomorphic to the corresponding blue entry of another, then all corresponding entries above them are also $\mathcal{C}$-isomorphic, and
- the universal augmentations are $\mathcal{C}$-isomorphic.

Then we conclude that all entries of these pages of the spectral sequences are $\mathcal{C}$-isomorphic. This scenario is developed in Peschke [26]. Here we just state it modulo $\mathcal{C}$. We will see that Zeeman's comparison theorem is a special case of these comparison theorems. At the end, we will see a few examples that are not compatible with the structure of Zeeman's comparison theorem.

It should be mentioned that the results of this chapter are stated and proved for spectral sequences of bidegrees $(-r, r-1)$ and $(r, -r+1)$, which cover many examples in the literature. However, there is a brief section which supplies the interested reader with a list of the tools we have used in the proofs, revised for arbitrary bidegrees, and he/she can use them to adjust the proofs for the spectral sequences induced by the mentioned "super convenient" exact couples with arbitrary bidegrees.

6.2 Reverse Engineering Using a Class of Modules

In this section, we cover the first scenario; i.e., we choose a class $\mathcal{C}$ of modules closed under extensions and start from the universal augmentation or coaugmentation of a first quadrant spectral sequence. Our goal is to show that, under some assumptions, if the universal augmentation or coaugmentation is in $\mathcal{C}$ then all entries of some page of the spectral sequence and hence subsequent pages belong to $\mathcal{C}$.

6.2.1 Reverse Engineering for Originally Vanishing Eventually Stable Exact Couples

Let $\{D^1_{*,*}, E^1_{*,*}\}$ be an originally vanishing and eventually stable exact couple that induces a first quadrant spectral sequence with differentials of bidegree $(-r, r-1)$ on page r and $\mathcal{C}$ be a class of modules that is closed under extensions. From Remark 3.2.2 on page 64 and the diagram on page 66 we know that this spectral sequence is convergent to its universal augmentation.

Proposition 6.2.1. *Let $\mathcal{C}$ be a class of modules closed under extensions and assume, for some $r_0 \geq 1$,*

1. *$L_n \in \mathcal{C}$ for every $n \in \mathbb{N}$,*
2. *$E^{r_0}_{0,q} \in \mathcal{C}$ for every $q \geq 0$, and*
3. *whenever $E^{r_0}_{p,0} \in \mathcal{C}$, then $E^{r_0}_{(p,q)} \in \mathcal{C}$ for every $q \geq 0$.*

Then, for every $p, q \geq 0$ and every $r \geq r_0$, we have $E^r_{(p,q)} \in \mathcal{C}$.

Proof. Look at the E^∞-Distribution Diagram on page 36. For every $(p,q) \in \mathbb{Z} \times \mathbb{Z}$, where $(p,q) \cdot \hat{\mathbf{a}} = n$, we have $\phi_{(p,q)} \in \mathcal{C}$, because $\phi_{(p,q)} \subseteq L_n$. Therefore, $\dfrac{\phi_{(p,q)}}{\phi_{p-1,q+1}} \in \mathcal{C}$ and since $\dfrac{\phi_{(p,q)}}{\phi_{p-1,q+1}} \cong E^\infty_{(p,q)}$ we have $E^\infty_{(p,q)} \in \mathcal{C}$.

In view of property 3, we only need to show that for every $p \geq 0$ we have $E^{r_0}_{p,0} \in \mathcal{C}$. We induct on p.

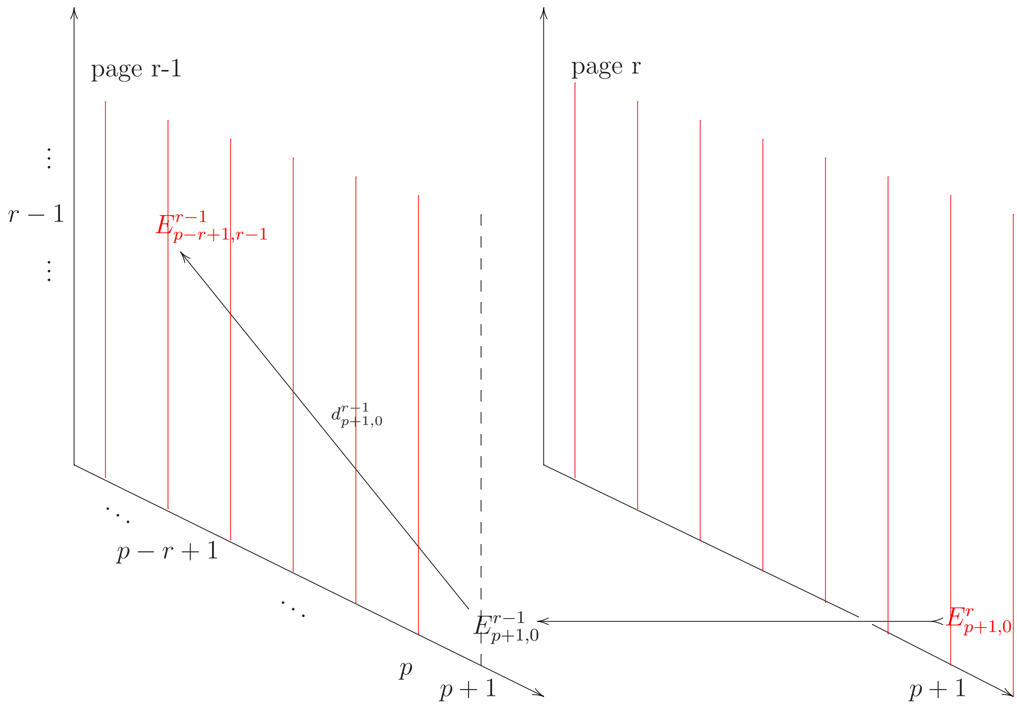
- $\underline{p=0}$: It holds by property 2.
- $\underline{p}$: Assume inductively that $E^{r_0}_{k,0} \in \mathcal{C}$ for every $0 \leq k \leq p$. Therefore, by property 3 and Proposition 4.2.2 on page 98, for every $r \geq r_0$ and every $q \geq 0$ we have $E^r_{k,q} \in \mathcal{C}$.
- $\underline{p+1}$: We show that $E^{r_0}_{p+1,0} \in \mathcal{C}$: Note that we have the following tower of monomorphisms

$$E^\infty_{p+1,0} = E^{p+2}_{p+1,0} \rightarrowtail \cdots \rightarrowtail E^{r_0+1}_{p+1,0} \rightarrowtail E^{r_0}_{p+1,0}$$

and we know that $E^{\infty}_{p+1,0} \in \mathcal{C}$. Assume inductively that $E^{r}_{p+1,0} \in \mathcal{C}$ and look at the following figure, where the entries in $\mathcal{C}$ are in red. In the following exact sequence

$$E^{r}_{p+1,0} = \ker(d^{r-1}_{p+1,0}) \rightarrowtail E^{r-1}_{p+1,0} \xrightarrow{d^{r-1}_{p+1,0}} \ker(d^{r-1}_{p-r+1,r-1})$$

$\ker(d^{r-1}_{p-r+1,r-1})$ is in $\mathcal{C}$ because it is included in $E^{r-1}_{p-r+1,r-1}$.



Since $p-r+1 < p$ by induction assumption we have $E^{r-1}_{p-r+1,r-1} \in \mathcal{C}$. This implies that $E^{r-1}_{p+1,0} \in \mathcal{C}$. Therefore, inductively $E^{r_0}_{p+1,0} \in \mathcal{C}$.

By property 3 and Proposition 4.2.2 on page 98, for every $r \geq r_0$ and every $q \geq 0$ we have $E^{r}_{p+1,q} \in \mathcal{C}$. This completes the proof. □

Example 6.2.2. Let $\mathcal{C}$ be a class of Abelian groups closed under extensions.

1. In the setting of the Leray-Serre spectral sequence in Example 3.2.5 on page 70, let $E \to B$ be a fibration with fiber F and $\mathrm{h} = \mathrm{H}$, an ordinary homology theory. If
   (a) $\mathrm{H}_n(E) \in \mathcal{C}$ for every $n \in \mathbb{N}$, and

   (b) $\mathrm{H}_q(F) \in \mathcal{C}$ for every $q \in \mathbb{N}$, and
   (c) whenever $\mathrm{H}_p(B; \mathrm{H}_0(F)) \in \mathcal{C}$, then $\mathrm{H}_p(B; \mathrm{H}_q(F)) \in \mathcal{C}$ for every $q \in \mathbb{N}$,

   then $\mathrm{H}_p(B; \mathrm{H}_q(F)) \in \mathcal{C}$, for every $p, q \in \mathbb{N}$.

2. In the setting of the Atiyah-Hirzebruch-Whitehead spectral sequence in Example 3.2.6 on page 71, if X is a finite CW-complex and

   (a) $\mathrm{h}_n(X) \in \mathcal{C}$ for every $n \in \mathbb{N}$, and
   (b) $\mathrm{h}_q(pt) \in \mathcal{C}$ for every $q \geq 0$, and
   (c) whenever $\mathrm{H}_p(X; \mathrm{h}_0(pt)) \in \mathcal{C}$, then $\mathrm{H}_p(X; \mathrm{h}_q(pt)) \in \mathcal{C}$, for every $q \geq 0$,

   then $\mathrm{H}_p(X; \mathrm{h}_q(pt)) \in \mathcal{C}$ for every $p, q \geq 0$.

3. Let $f : R \to S$ be a ring homomorphism and look at Example 3.2.11 on page 77. If $F = A \otimes_R -$, then every projective R-module P is left F-acyclic, because $\mathrm{Tor}_p^R(A, P) = \{0\}$, for every $p \geq 1$. Assume for every S-module A and R-module B

   (a) $\mathrm{Tor}_n^R(A, B) \in \mathcal{C}$, for every $n \in \mathbb{N}$, and
   (b) $\mathrm{Tor}_q^R(A, S) \otimes_S B \in \mathcal{C}$, for every $q \geq 0$, and
   (c) whenever $\mathrm{Tor}_p^S(A \otimes_R S, B) \in \mathcal{C}$, then $\mathrm{Tor}_p^S(\mathrm{Tor}_q^R(A, S), B) \in \mathcal{C}$ for every $q \geq 0$.

   Then $\mathrm{Tor}_p^S(\mathrm{Tor}_q^R(A, S), B) \in \mathcal{C}$ for every $p, q \geq 0$.

Corollary 6.2.3. *Let $\mathcal{C}$ be a class of modules closed under extensions and tensor product. If, for some $r_0 \geq 0$,*

1. *$L_n \in \mathcal{C}$, and*
2. *$E^{r_0}_{0,q} \in \mathcal{C}$ for every $q \geq 0$, and*
3. *$E^{r_0}_{(p,q)} \cong E^{r_0}_{p,0} \otimes E^{r_0}_{0,q}$,*

then $E^r_{(p,q)} \in \mathcal{C}$, for every $p, q \geq 0$ and every $r \geq r_0$.

Example 6.2.4. Let $\mathcal{C}$ be a class of Abelian groups closed under extensions and tensor product.

1. In the setting of the Leray-Serre spectral sequence in Example 3.2.5 on page 70, let $E \to B$ be a fibration with fiber F where $\mathrm{H}_*(B)$ is torsion-free and $\mathrm{h} = \mathrm{H}$, an ordinary homology theory. If for every $n \geq 0$ we have $\mathrm{H}_n(E), \mathrm{H}_n(F) \in \mathcal{C}$, then for every $p, q \geq 0$ we have $\mathrm{H}_p(B; \mathrm{H}_q(F)) \in \mathcal{C}$. This is because by the Universal Coefficient Theorem, condition $c)$ in the first part of Example 6.2.2 on page 149 holds.

2. In the setting of the Atiyah-Hirzebruch-Whitehead spectral sequence in Example 3.2.6 on page 71, for a finite CW-complex X and a homology theory h_*, if $\mathrm{H}_*(X)$ is torsion-free and $\mathrm{h}_n(X), \mathrm{h}_n(pt) \in \mathcal{C}$ for every $n \geq 0$, then $\mathrm{H}_p(X; \mathrm{h}_q(pt)) \in \mathcal{C}$ for every $p, q \geq 0$.

In the following proposition we consider the "dual" of Proposition 6.2.1 on page 148.

Proposition 6.2.5. *Let $\mathcal{C}$ be a class of modules closed under extensions and assume, for some $r_0 \geq 1$,*

1. *$L_n \in \mathcal{C}$ for every $n \in \mathbb{N}$, and*
2. *$E^{r_0}_{p,0} \in \mathcal{C}$ for $p \geq 0$, and*
3. *whenever $E^{r_0}_{0,q} \in \mathcal{C}$, then $E^{r_0}_{(p,q)} \in \mathcal{C}$ for $p \geq 0$.*

Then $E^{r_0}_{(p,q)} \in \mathcal{C}$ for every $p, q \geq 0$ and every $r \geq r_0$.

Proof. Note that for every $(p,q) \in \mathbb{Z} \times \mathbb{Z}$, where $(p,q) \cdot \hat{\mathbf{a}} = n$, we have $\phi_{(p,q)} \in \mathcal{C}$, because $\phi_{(p,q)} \subseteq L_n$. Therefore, $\dfrac{\phi_{(p,q)}}{\phi_{p-1,q+1}} \in \mathcal{C}$ and since $\dfrac{\phi_{(p,q)}}{\phi_{p-1,q+1}} \cong E^{\infty}_{(p,q)}$ we have $E^{\infty}_{(p,q)} \in \mathcal{C}$.

In view of property 3, we only need to show that for every $q \geq 0$ we have $E^{r_0}_{0,q} \in \mathcal{C}$. We induct on q:

- $\underline{q = 0}$: It holds by property 2.
- $\underline{q}$: Assume inductively that $E^{r_0}_{0,k} \in \mathcal{C}$ for every $0 \leq k \leq q$. Therefore, by property 3 and Proposition 4.2.2 on page 98, for every $r \geq r_0$ and every $p \geq 0$ we have $E^{r}_{p,k} \in \mathcal{C}$.

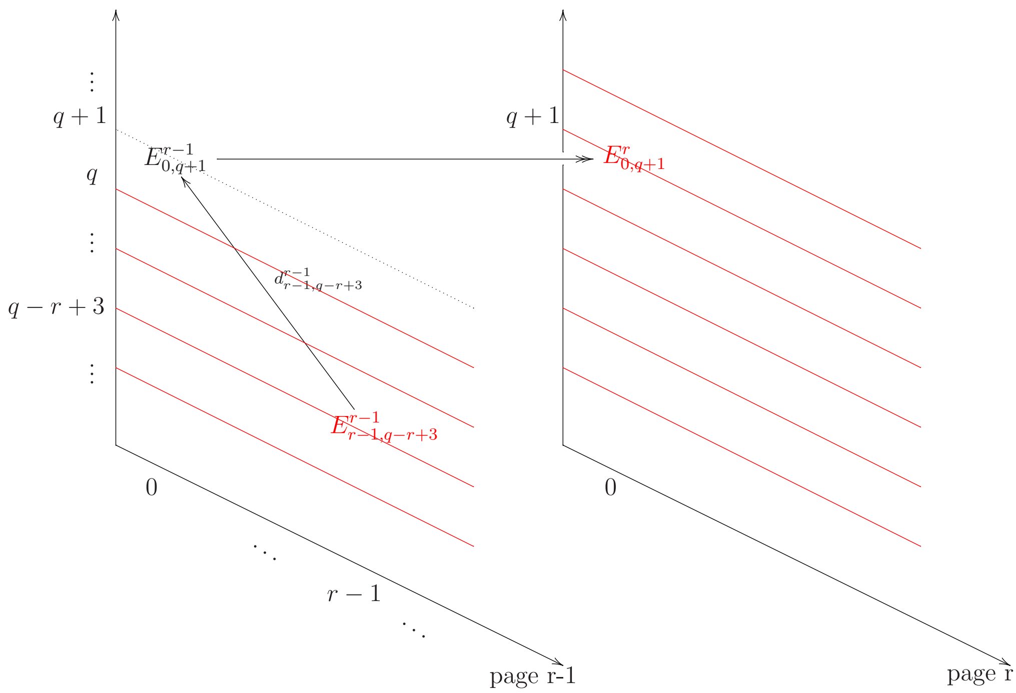
- $\underline{q+1}$: We show that $E^{r_0}_{0,q+1} \in \mathcal{C}$: Note that we have the following tower of epimorphisms

$$E^{r_0}_{0,q+1} \twoheadrightarrow E^{r_0+1}_{0,q+1} \twoheadrightarrow \cdots \twoheadrightarrow E^{q+2}_{0,q+1} \twoheadrightarrow E^{q+3}_{0,q+1} = E^{\infty}_{0,q+1}$$

and we know that $E^{\infty}_{0,q+1} \in \mathcal{C}$. Assume inductively that $E^{r}_{0,q+1} \in \mathcal{C}$. Look at the following exact sequence

$$E^{r-1}_{r-1,q-r+3} \xrightarrow{d^{r-1}_{r-1,q-r+3}} E^{r-1}_{0,q+1} \longrightarrow\!\!\!\!\!\rightarrow E^{r}_{0,q+1}$$

and the following picture, where entries in $\mathcal{C}$ are in red.



Since $q-r+3 \le q$ by induction assumption we have $E^{r-1}_{r-1,q-r+3} \in \mathcal{C}$. This implies that $E^{r-1}_{0,q+1} \in \mathcal{C}$.

Therefore, $E^{r_0}_{0,q+1} \in \mathcal{C}$ and hence, by property 3 and Proposition 4.2.2 on page 98, for every $r \geq r_0$ and every $p \geq 0$ we have $E^{r}_{(p,q)+1} \in \mathcal{C}$. This completes the proof of the proposition. □

Example 6.2.6. 1. In the setting of the Leray-Serre spectral sequence in Example 3.2.5 on page 70, let $E \to B$ be a fibration with fiber F and $\mathrm{h} = \mathrm{H}$, an ordinary homology theory. If

(a) $\mathrm{H}_n(E) \in \mathcal{C}$ for every $n \in \mathbb{N}$, and
(b) $\mathrm{H}_p(B; \mathrm{H}_0(F)) \in \mathcal{C}$ for every $p \in \mathbb{N}$, and
(c) whenever $\mathrm{H}_q(F) \in \mathcal{C}$, then $\mathrm{H}_p(B; \mathrm{H}_q(F)) \in \mathcal{C}$ for every $p \in \mathbb{N}$,

then $\mathrm{H}_p(B; \mathrm{H}_q(F)) \in \mathcal{C}$, for every $p, q \in \mathbb{N}$.

2. Let $f : R \to S$ be a ring homomorphism and look at Example 3.2.11 on page 77. Assume for every S-module A and R-module B

(a) $\mathrm{Tor}_n^R(A, B) \in \mathcal{C}$, for every $n \in \mathbb{N}$, and
(b) $\mathrm{Tor}_p^S(A \otimes_R S, B) \in \mathcal{C}$, for every $p \geq 0$, and
(c) whenever $\mathrm{Tor}_q^R(A, S) \otimes_S B \in \mathcal{C}$, then $\mathrm{Tor}_p^S(\mathrm{Tor}_q^R(A, S), B) \in \mathcal{C}$ for every $p \geq 0$.

Then $\mathrm{Tor}_p^S(\mathrm{Tor}_q^R(A, S), B) \in \mathcal{C}$ for every $p, q \geq 0$.

Corollary 6.2.7. *Let $\mathcal{C}$ be a class of modules that is closed under extensions and tensor product and for some $r_0 \geq 1$*

1. *$L_n \in \mathcal{C}$ for every $n \in \mathbb{N}$, and*
2. *$E_{p,0}^{r_0} \in \mathcal{C}$ for $p \geq 0$, and*
3. *$E_{(p,q)}^{r_0} \cong E_{p,0}^{r_0} \otimes E_{0,q}^{r_0}$.*

Then $E_{(p,q)}^{r_0} \in \mathcal{C}$ for every $p, q \geq 0$ and every $r \geq r_0$.

Example 6.2.8. Let $\mathcal{C}$ be a class of Abelian groups closed under extensions and tensor product. In the setting of the Leray-Serre spectral sequence in Example 3.2.5 on page 70, let $E \to B$ be a fibration with fiber F where $\mathrm{H}_*(B)$ is torsion-free and $\mathrm{h} = \mathrm{H}$, an ordinary homology theory. If $\mathrm{H}_n(E), \mathrm{H}_n(B; \mathrm{H}_0(F)) \in \mathcal{C}$ for every $n \geq 0$, then $\mathrm{H}_p(B; \mathrm{H}_q(F)) \in \mathcal{C}$, for every $p, q \geq 0$.

6.2.2 Reverse Engineering for Eventually Vanishing Originally Stable Exact Couples

Let $\{D^1_{*,*}, E^1_{*,*}\}$ be an eventually vanishing and originally stable exact couples that induces a first quadrant spectral sequence with differentials of bidegree $(r, -r+1)$ on page r and $\mathcal{C}$ be a class of modules closed under extensions. The proof of the following propositions is the "mirror image" of the proof of the propositions in the previous section.

Proposition 6.2.9. *Assume, for some $r_0 \geq 1$,*

1. *$L^n \in \mathcal{C}$ for every $n \in \mathbb{N}$, and*
2. *$E^{r_0}_{p,0} \in \mathcal{C}$ for $p \geq 0$, and*
3. *whenever $E^{r_0}_{0,q} \in \mathcal{C}$, then $E^{r_0}_{(p,q)} \in \mathcal{C}$ for $p \geq 0$.*

Then $E^{r_0}_{(p,q)} \in \mathcal{C}$ for every $p, q \geq 0$ and every $r \geq r_0$.

Proposition 6.2.10. *Assume, for some $r_0 \geq 1$,*

1. *$L^n \in \mathcal{C}$ for every $n \in \mathbb{N}$, and*
2. *$E^{r_0}_{0,q} \in \mathcal{C}$ for every $q \geq 0$, and*
3. *whenever $E^{r_0}_{p,0} \in \mathcal{C}$, then $E^{r_0}_{(p,q)} \in \mathcal{C}$ for every $q \geq 0$.*

Then $E^{r_0}_{(p,q)} \in \mathcal{C}$ for every $p, q \geq 0$ and every $r \geq r_0$.

Example 6.2.11. Let $f : R \to S$ be a ring homomorphism and look at Example 3.2.11 on page 77. Assume $\mathcal{C}$ is a class of modules closed under extensions and look at the Example 4.4.7 on page 107.

- Assume for every S-module A and R-module B

  1. $\mathrm{Ext}^n_R(A, B) \in \mathcal{C}$ for every $n \in \mathbb{N}$, and
  2. $\mathrm{Ext}^p_S(A, \hom_R(S, B)) \in \mathcal{C}$ for $p \geq 0$, and
  3. whenever $\hom_S(A, \mathrm{Ext}^q_R(S, B)) \in \mathcal{C}$, then $\mathrm{Ext}^p_S(A, \mathrm{Ext}^q_R(S, B)) \in \mathcal{C}$ for $p \geq 0$.

  Then $\mathrm{Ext}^p_S(A, \mathrm{Ext}^q_R(S, B)) \in \mathcal{C}$ for every $p, q \geq 0$.

- Assume for every S-module A and R-module B

  1. $\mathrm{Ext}^n_R(A, B) \in \mathcal{C}$ for every $n \in \mathbb{N}$, and
  2. $\hom_S(A, \mathrm{Ext}^q_R(S, B)) \in \mathcal{C}$ for $q \geq 0$, and
  3. whenever $\mathrm{Ext}^p_S(A, \hom_R(S, B)) \in \mathcal{C}$, then $\mathrm{Ext}^p_S(A, \mathrm{Ext}^q_R(S, B)) \in \mathcal{C}$ for $q \geq 0$.

  Then $\mathrm{Ext}^p_S(A, \mathrm{Ext}^q_R(S, B)) \in \mathcal{C}$ for every $p, q \geq 0$.

6.3 Reverse Engineering for Comparison Theorems

Here we explain the second scenario that we talked about in the introduction to this chapter, presented in [26], and we furnish reverse engineering results via comparison of spectral sequences. More precisely, suppose $(f, \varphi) : (\mathcal{C}(1), L_*(1)) \to (\mathcal{C}(2), L_*(2))$ is a morphism of universally augmented exact couples. Assume $\varphi : L_*(1) \to L_*(2)$ is an isomorphism.

Under which circumstances is $f : E^2_{*,*}(1) \to E^2_{*,*}(2)$ *an isomorphism?*

We can state a more general question: Let $\mathcal{C}$ be a class of modules closed under extensions.

Under which circumstances is $f : E^2_{*,*}(1) \to E^2_{*,*}(2)$ *a* $\mathcal{C}$*-isomorphism?*

6.3.1 Reverse Comparison Theorems for Originally Vanishing Eventually Stable Exact Couples

For $i = 1, 2$, let $\{D^1_{*,*}(i), E^1_{*,*}(i)\}$ be an originally vanishing and eventually stable exact couples that induces a first quadrant spectral sequence with differentials of bidegree $(-r, r-1)$ on page r and $\mathcal{C}$ be a class of modules closed under extensions.

Proposition 6.3.1. *[26] Let* $(f, \varphi) : (\mathfrak{C}(1), L_*(1)) \to (\mathfrak{C}(2), L_*(2))$ *be a morphism of universally augmented first quadrant exact couples with differentials* $d^r(i) : E^r_{(p,q)}(i) \to E^r_{p-r,q+r-1}(i)$ *satisfying the following hypotheses:*

1. $\varphi : L_*(1) \to L_*(2)$ *is a* $\mathcal{C}$*-isomorphism;*
2. $f_{0,q} : E^2_{0,q}(1) \to E^2_{0,q}(2)$ *is a* $\mathcal{C}$*-isomorphism for* $q \geq 0$*;*
3. *Whenever* $f_{p,0} : E^2_{p,0}(1) \to E^2_{p,0}(2)$ *is a* $\mathcal{C}$*-isomorphism, then* $f_{(p,q)} : E^2_{(p,q)}(1) \to E^2_{(p,q)}(2)$ *is also a* $\mathcal{C}$*-isomorphism for* $q \geq 0$*.*

Then

$$f_{(p,q)} : E^2_{(p,q)}(1) \to E^2_{(p,q)}(2)$$

is a $\mathcal{C}$*-isomorphism for all* $p, q \geq 0$*.*

Proof. We induct on p.

- $p = 0$: *Claim. The homomorphism*

$$f^r_{0,0} : E^r_{0,0}(1) \to E^r_{0,0}(2) \tag{6.1}$$

is a $\mathcal{C}$-isomorphism for all $r \geq 0$ and

$$f^r_{k,1-k} : E^r_{k,1-k}(1) \to E^r_{k,1-k}(2) \tag{6.2}$$

is a $\mathcal{C}$-epimorphism for all $r \geq 2$ and $0 \leq k \leq 1$.

Proof (Claim). (6.1) follows from 2 and the fact that $E^2_{0,0} = E^r_{0,0}$ for all $r \geq 2$. To see (6.2), remember that

$$E^3_{0,1} = E^r_{0,1}, \quad r \geq 3$$

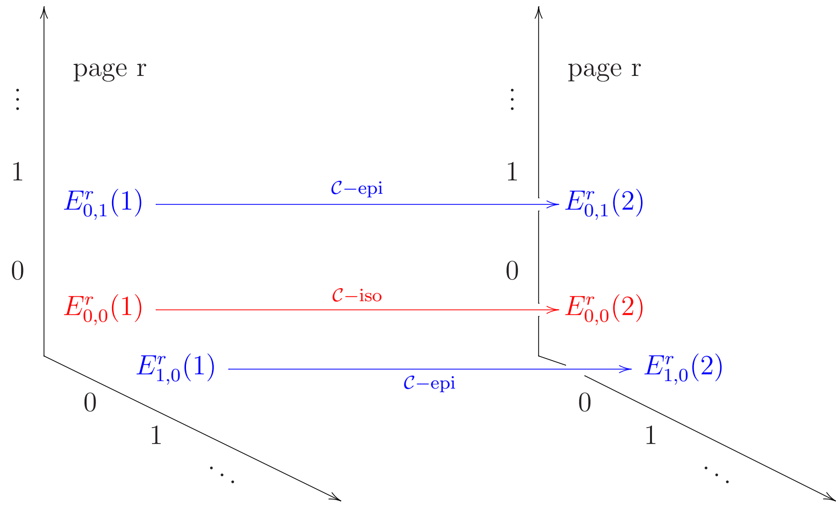
and

$$E^2_{1,0} = E^r_{1,0}, \quad r \geq 2$$

and consider the commutative diagram with exact rows below

$$\begin{tikzcd}
\ker(f_{0,1}) \arrow[rr] \arrow[d, tail] & & \ker(\varphi) \arrow[r] \arrow[d, tail] & \ker(f_{1,0}) \arrow[d, tail] \\
E^2_{0,1}(1) \arrow[rr] \arrow[dr, two heads] \arrow[dd, "f_{0,1}"] & & L_1(1) \arrow[r, two heads] \arrow[dd, "\varphi"] & E^2_{1,0}(1) \arrow[dd, "f_{1,0}"] \\
& E^3_{0,1}(1) \arrow[ur, tail] \arrow[dd] & & \\
E^2_{0,1}(2) \arrow[rr] \arrow[dr, two heads] \arrow[dd] & & L_1(2) \arrow[r, two heads] \arrow[dd] & E^2_{1,0}(2) \arrow[dd] \\
& E^3_{0,1}(2) \arrow[ur, tail] & & \\
\operatorname{coker}(f_{0,1}) \arrow[rr] & & \operatorname{coker}(\varphi) \arrow[r, two heads] & \operatorname{coker}(f_{1,0})
\end{tikzcd}$$

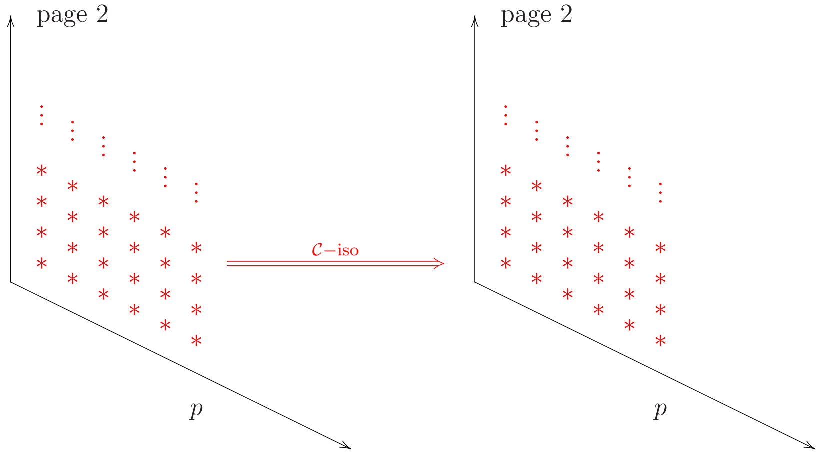
Since $\operatorname{coker}(\varphi) \in \mathcal{C}$ then $\operatorname{coker}(f_{1,0}) \in \mathcal{C}$. Note that since $f_{0,1}$ is a $\mathcal{C}$-isomorphism we already know that $\operatorname{coker}(f_{0,1}) \in \mathcal{C}$. We can summarize this part in the following picture.

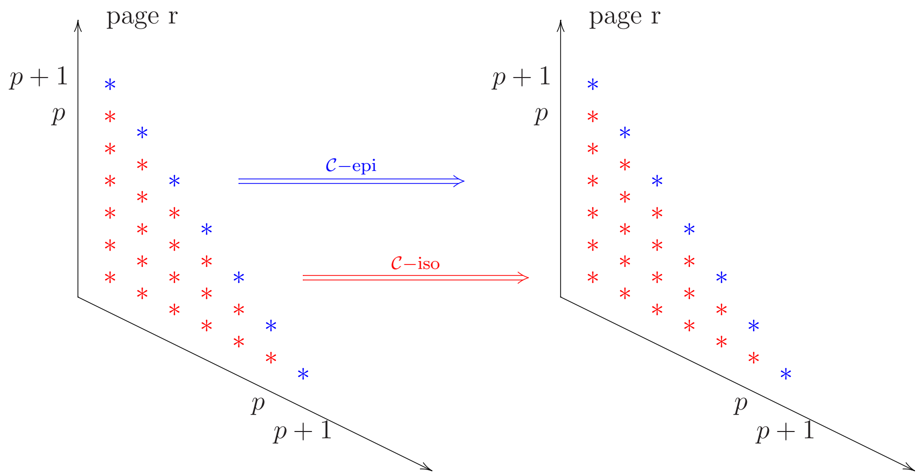


- p: Assume inductively that

1. $f^2_{k,q} : E^2_{k,q}(1) \to E^2_{k,q}(2)$ is a $\mathcal{C}$-isomorphism for $0 \leq k \leq p$ and $q \geq 0$.
2. $f^r_{k,l} : E^r_{k,l}(1) \to E^r_{k,l}(2)$ is a $\mathcal{C}$-isomorphism for $0 \leq k + l \leq p$ and $r \geq 2$.
3. $f^r_{k,p+1-k} : E^r_{k,p+1-k}(1) \twoheadrightarrow E^r_{k,p+1-k}(2)$ is a $\mathcal{C}$-epimorphism for $0 \leq k \leq p + 1$ and $r \geq 2$.

We can summarize these three conditions in the following picture, where red entries are $\mathcal{C}$ isomorphic and the morphisms between the blue entries are $\mathcal{C}$-epimorphisms.





• $p+1$: We need to show that 1, 2 and 3 hold with p replaced by $p+1$. The argument is locally trivial and globally complicated, involving nested and simultaneous induction arguments. We proceed through the following steps:

<u>*Step 1*</u> $f^{\infty}_{k,p+1-k} : E^{\infty}_{k,p+1-k}(1) \to E^{\infty}_{k,p+1-k}(2)$ is a $\mathcal{C}$-isomorphism for $0 \le k \le p+1$.

<u>*Step 2*</u> $f^{r}_{p+1,0}(1) : E^{r}_{p+1,0}(1) \to E^{r}_{p+1,0}(2)$ is a $\mathcal{C}$-isomorphism for $r \ge 2$. This implies that $f^{2}_{p+1,q} : E^{2}_{p+1,q}(1) \to E^{2}_{p+1,q}(2)$ is a $\mathcal{C}$-isomorphism for $q \ge 0$, using 3.

<u>*Step 3*</u> $f^{r}_{p+1-l,l} : E^{r}_{p+1-l,l}(1) \to E^{r}_{p+1-l,l}(2)$ is a $\mathcal{C}$-isomorphism for $0 \le l \le p+1$ and $r \ge 2$.

<u>*Step 4*</u> $f^{r}_{p+1-\lambda,\lambda+1} : E^{r}_{p+1-\lambda,\lambda+1}(1) \to E^{r}_{p+1-\lambda,\lambda+1}(2)$ is a $\mathcal{C}$-epimorphism for $0 \le \lambda \le p+1$ and $r \ge 2$.

<u>*Step 5*</u> $f^{r}_{p+2,0} : E^{r}_{p+2,0}(1) \to E^{r}_{p+2,0}(2)$ is a $\mathcal{C}$-epimorphism for $r \ge 2$.

<u>*Step 1* Proof</u>: We have the commutative filtration diagram for $\mathrm{H}_{p+1}(1)$ and $\mathrm{H}_{p+1}(2)$

$$\begin{tikzcd}
\ker(f^{\infty}_{0,p+1}) \arrow[r, equal] \arrow[d, tail] & \ker(\bar{\varphi}_{0,p+1}) \arrow[r, tail] \arrow[d, tail] & \cdots \arrow[r, tail] & \ker(\bar{\varphi}_{p,1}) \arrow[r, tail] \arrow[d, tail] & \ker(\varphi) \arrow[d, tail] \\
E^{\infty}_{0,p+1}(1) \arrow[r, equal] \arrow[d, "f^{\infty}_{0,p+1}"'] & F_{0,p+1}(1) \arrow[r, tail] \arrow[d, "\bar{\varphi}_{0,p+1}"'] & \cdots \arrow[r, tail] & F_{p,1}(1) \arrow[r, tail] \arrow[d, "\bar{\varphi}_{p,1}"'] & L_{p+1}(1) \arrow[d, "\varphi"] \\
E^{\infty}_{0,p+1}(2) \arrow[r, equal] \arrow[d, two heads] & F_{0,p+1}(2) \arrow[r, tail] \arrow[d, two heads] & \cdots \arrow[r, tail] & F_{p,1}(2) \arrow[r, tail] \arrow[d, two heads] & L_{p+1}(2) \arrow[d, two heads] \\
\mathrm{coker}(f^{\infty}_{0,p+1}) \arrow[r, equal] & \mathrm{coker}(\bar{\varphi}_{0,p+1}) \arrow[r] & \cdots \arrow[r] & \mathrm{coker}(\bar{\varphi}_{p,1}) \arrow[r] & \mathrm{coker}(\varphi).
\end{tikzcd}$$

Since $\ker(\varphi) \in \mathcal{C}$ then $\ker(f^{\infty}_{0,p+1}) \in \mathcal{C}$ and $\ker(\bar{\varphi}_{k,p+1-k}) \in \mathcal{C}$, for every $0 \leq k \leq p+1$; i.e., every $\bar{\varphi}_{k,p+1-k}$ is a $\mathcal{C}$-monomorphism.

Part 3 of the induction assumption shows that $\operatorname{coker}(f^{\infty}_{k+1,p-k}) \in \mathcal{C}$, for every $0 \leq k \leq p+1$. Therefore, $\operatorname{coker}(\bar{\varphi}_{0,p+1}) \in \mathcal{C}$. Assume inductively that $\operatorname{coker}(\bar{\varphi}_{k,p+1-k}) \in \mathcal{C}$. The diagram of short exact sequences below

$$\begin{tikzcd}
\ker(\bar{\varphi}_{k,p+1-k}) \arrow[r, tail] \arrow[d, tail] & \ker(\bar{\varphi}_{k+1,p-k}) \arrow[r] \arrow[d, tail] & \ker(f^{\infty}_{k+1,p-k}) \arrow[d, tail] \\
F_{k,p+1-k}(1) \arrow[r, tail] \arrow[d, "\bar{\varphi}_{k,p+1-k}"'] & F_{k+1,p-k}(1) \arrow[r, two heads] \arrow[d, "\bar{\varphi}_{k+1,p-k}"'] & E^{\infty}_{k+1,p-k}(1) \arrow[d, "f^{\infty}_{k+1,p-k}"'] \\
F_{k,p+1-k}(2) \arrow[r, tail] \arrow[d, two heads] & F_{k+1,p-k}(2) \arrow[r, two heads] \arrow[d, two heads] & E^{\infty}_{k+1,p-k}(2) \arrow[d, two heads] \\
\operatorname{coker}(\bar{\varphi}_{k,p+1-k}) \arrow[r] & \operatorname{coker}(\bar{\varphi}_{k+1,p-k}) \arrow[r, two heads] & \operatorname{coker}(f^{\infty}_{k+1,p-k})
\end{tikzcd}$$

has the following consequences:

- $\operatorname{coker}(\bar{\varphi}_{k+1,p-k}) \in \mathcal{C}$. This shows inductively that every $\bar{\varphi}_{k,p+1-k}$ is a $\mathcal{C}$-epimorphism and hence a $\mathcal{C}$-isomorphism.
- Using the Snake Lemma we see that $\ker(f^{\infty}_{k,p+1-k}) \in \mathcal{C}$, for every $0 \leq k \leq p+1$.

Therefore, $f^{\infty}_{k,p+1-k}$ is a $\mathcal{C}$-isomorphism, for every $0 \leq k \leq p+1$.

Step 2 Proof: We have the following tower of inclusions for $i = 1, 2$

$$E^{\infty}_{p+1,0}(i) = E^{r+2}_{p+1,0}(i) \rightarrowtail E^{r+1}_{p+1,0}(i) \rightarrowtail \cdots \rightarrowtail E^{r+2-s}_{p+1,0}(i) \rightarrowtail \cdots \rightarrowtail E^{2}_{p+1,0}.$$

From step 1 we know that

$$E^{r+2}_{p+1,0}(1) = E^{\infty}_{p+1,0}(1) \to E^{\infty}_{p+1,0}(2) = E^{r+2}_{p+1,0}(2)$$

is a $\mathcal{C}$-isomorphism. Let assume inductively that

$$E^{r+2-s}_{p+1,0}(1) \to E^{r+2-s}_{p+1,0}(2)$$

is a $\mathcal{C}$-isomorphism. We have the commutative diagram with exact rows

$$\begin{tikzcd}
E^{r+2-s}_{p+1,0}(1) \arrow[r, tail] \arrow[d, "f^{r+2-s}_{p+1,0}"'] & E^{r+2-(s+1)}_{p+1,0}(1) \arrow[r] \arrow[d, dashed] & \ker(d^{r+2-(s+1)}_{p-r+s,r-s}(1)) \arrow[r, two heads] \arrow[d] & E^{r+2-s}_{p-r+s,r-s}(1) \arrow[d, "f^{r+2-s}_{p-r+s,r-s}"'] \\
E^{r+2-s}_{p+1,0}(2) \arrow[r, tail] & E^{r+2-(s+1)}_{p+1,0}(2) \arrow[r] & \ker(d^{r+2-(s+1)}_{p-r+s,r-s}(2)) \arrow[r, two heads] & E^{r+2-s}_{p-r+s,r-s}(2)
\end{tikzcd}$$

The three solid vertical arrows are $\mathcal{C}$-isomorphisms by induction hypothesis. Note that $\ker(d^{r+2-(s+1)}_{p-r+s,r-s}(i))$ is in a position where $(p-r+s)+(r-s)=p$ and hence it satisfies the induction hypothesis. By $\mathcal{C}$-5-Lemma the dashed arrow is also a $\mathcal{C}$-isomorphism.

<u>*Step 3* Proof:</u> The claim holds for $l=0$ by step 2. Suppose inductively that

$$E^r_{p+1-\lambda,\lambda}(1) \to E^r_{p+1-\lambda,\lambda}(2)$$

are $\mathcal{C}$-isomorphisms for $0 \leq \lambda \leq l$ and $r \geq 2$. To prove the claim for $l+1$ we distinguish two cases.

Case 1. <u>$2 \leq r \leq l+2$</u>: We induct on r. By induction assumption 1 we know that

$$E^2_{p-l,l+1}(1) \to E^2_{p-l,l+1}(2)$$

is a $\mathcal{C}$-isomorphism. Suppose that

$$E^\rho_{p-l,l+1}(1) \to E^\rho_{p-l,l+1}(2)$$

is a $\mathcal{C}$-isomorphism for $2 \leq \rho \leq r < l+2$. To see that

$$E^{r+1}_{p-l,l+1}(1) \to E^{r+1}_{p-l,l+1}(2)$$

is a $\mathcal{C}$-isomorphism as well, we use the commutative diagram with exact rows below

$$\begin{tikzcd}
\ker(f^r_{p-l+r,l+1-(r-1)}) \arrow[r] \arrow[d] & \ker(\bar{f}^r_{p-l,l+1}) \arrow[r] \arrow[d] & \ker(f^{r+1}_{p-l,l+1}) \arrow[d] \\
E^r_{p-l+r,l+1-(r-1)}(1) \arrow[r, "d^r_{p-l+r,l+1-(r-1)}"] \arrow[d, "f^r_{p-l+r,l+1-(r-1)}"'] & \ker(d^r_{p-l,l+1}(1)) \arrow[r, two heads] \arrow[d, "\mathcal{C}-iso"', "\bar{f}^r_{p-l,l+1}"] & E^{r+1}_{p-l,l+1}(1) \arrow[d, "f^{r+1}_{p-l,l+1}"] \\
E^r_{p-l+r,l+1-(r-1)}(2) \arrow[r, "d^r_{p-l+r,l+1-(r-1)}"] \arrow[d] & \ker(d^r_{p-l,l+1}(2)) \arrow[r, two heads] \arrow[d] & E^{r+1}_{p-l,l+1}(2) \arrow[d] \\
\operatorname{coker}(f^r_{p-l+r,l+1-(r-1)}) \arrow[r] & \operatorname{coker}(\bar{f}^r_{p-l,l+1}) \arrow[r, two heads] & \operatorname{coker}(f^{r+1}_{p-l,l+1})
\end{tikzcd}$$

The vertical arrow in the middle is a $\mathcal{C}$-isomorphism because it is on the

r-th page and it satisfies the induction hypothesis. So the vertical arrow on the right is a $\mathcal{C}$-epimorphism. If we can prove that $f^r_{p-l+r,l+1-(r-1)}$ is a $\mathcal{C}$-epimorphism then by $\mathcal{C}$-5-Lemma we get $\ker(f^{r+1}_{p-l,l+1}) \in \mathcal{C}$.

Therefore, the vertical arrow on the right is a $\mathcal{C}$-isomorphism once we have shown the step 4. We restate the step 4 here.

Step 4: If $E^r_{p+1-\lambda,\lambda}(1) \to E^r_{p+1-\lambda,\lambda}(2)$ is a $\mathcal{C}$-isomorphism for $0 \leq \lambda \leq l$ and $r \geq 2$, then

$$E^r_{p+1-\lambda,\lambda+1}(1) \to E^r_{p+1-\lambda,\lambda+1}(2)$$

is a $\mathcal{C}$-epimorphism for $0 \leq \lambda \leq p+1$ and

$$2 \leq r \leq \max\{2, l-\lambda\} + 1.$$

Step 4 Proof: For $r = 2$ it follows from step 2. Now suppose step 4 holds for $2 \leq r \leq l - \lambda + 1$. To establish step 4 consider the commutative diagram with exact rows below

$$\begin{tikzcd}
\ker(d^r_{p+1-\lambda,\lambda+1}(1)) \arrow[r, tail] \arrow[d] & E^r_{p+1-\lambda,\lambda+1}(1) \arrow[r, "d^r_{p+1-\lambda,\lambda+1}(1)"] \arrow[d, "\mathcal{C}-epi"'] & \ker(d^r_{p+1-\lambda-r,\lambda+r}(1)) \arrow[r, two heads] \arrow[d, "\mathcal{C}-iso"] & E^{r+1}_{p+1-\lambda-r,\lambda+r}(1) \arrow[d, "\mathcal{C}-iso"] \\
\ker(d^r_{p+1-\lambda,\lambda+1}(2)) \arrow[r, tail] & E^r_{p+1-\lambda,\lambda+1}(2) \arrow[r, "d^r_{p+1-\lambda,\lambda+1}(2)"] & \ker(d^r_{p+1-\lambda-r,\lambda+r}(2)) \arrow[r, two heads] & E^{r+1}_{p+1-\lambda-r,\lambda+r}(2)
\end{tikzcd}$$

The two vertical arrows on the right are $\mathcal{C}$-isomorphisms by induction hypothesis 2 and *Step 2*. The second vertical arrow from the left is $\mathcal{C}$-epimorphism by induction hypothesis. Consequently the vertical arrow on the left is $\mathcal{C}$-epimorphism, implying the induction step. Thus *Step 4* holds and *Case 1* of *Step 3* follows.

Case 2. $r \geq l+3$: From step 1 we have the isomorphism

$$E^r_{p+1-l,l}(1) = E^\infty_{p+1-l,l}(1) \to E^\infty_{p+1-l,l}(2) = E^r_{p+1-l,l}(2)$$

for $r \geq l+3$ sufficiently large. But so long as $s \geq l+3$ we have the commutative diagram with exact row below

$$\begin{tikzcd}
E^{s+1}_{p+1-l,l}(1) \arrow[r, tail] \arrow[d, "\mathcal{C}-epi"'] & E^s_{p+1-l,l}(1) \arrow[r, "d^s_{p+1-l,l}(1)"] \arrow[d, dashed] & \ker(d^s_{p+1-l-s,l+(s-1)}(1)) \arrow[r, two heads] \arrow[d, "\mathcal{C}-iso"] & E^{s+1}_{p+1-l-s,l+(s-1)}(1) \arrow[d, "\mathcal{C}-iso"] \\
E^{s+1}_{p+1-l,l}(2) \arrow[r, tail] & E^s_{p+1-l,l}(2) \arrow[r, "d^s_{p+1-l,l}(2)"] & \ker(d^s_{p+1-l-s,l+(s-1)}(2)) \arrow[r, two heads] & E^{s+1}_{p+1-l-s,l+(s-1)}(2)
\end{tikzcd}$$

Again the two vertical arrow on the right are $\mathcal{C}$-isomorphims by induction hypothesis 2. So the dashed arrow is a $\mathcal{C}$-epimorphis which proves *Case 2*.

Thus the proof of step 3 is complete and the argument of step 4 shows

$$f^r_{k,p+1-k} : E^r_{k,p+1-k}(1) \twoheadrightarrow E^r_{k,p+1-k}(2)$$

is a $\mathcal{C}$-epimorphism for $1 \leq k \leq p+1$.

<u>*Step 5* Proof:</u> We begin by observing that since φ is a $\mathcal{C}$-epimorphism, then $f^\infty_{p+2,0}$ is a $\mathcal{C}$-epimorphism in the commutative diagram below

$$\begin{tikzcd}
\ker(\varphi) \arrow[r] \arrow[d, tail] & \ker(f^\infty_{p+2,0}) \arrow[d, tail] & \\
L_{p+2}(1) \arrow[r, two heads] \arrow[d, "\varphi"'] & E^\infty_{p+2,0}(1) \arrow[r, equal] \arrow[d, "f^\infty_{p+2,0}"] & E^{p+3-\mu}_{p+2,0}(1) \\
L_{p+2}(2) \arrow[r, two heads] \arrow[d, two heads] & E^\infty_{p+2,0}(2) \arrow[r, equal] \arrow[d, two heads] & E^{p+3-\mu}_{p+2,0}(2) \\
\operatorname{coker}(\varphi) \arrow[r, two heads] & \operatorname{coker}(f^\infty_{p+2,0}) &
\end{tikzcd}$$

where $\mu \geq 0$. Assume inductively that

$$E^{p+3-\lambda}_{p+2,0}(1) \twoheadrightarrow E^{p+3-\lambda}_{p+2,0}(2)$$

is a $\mathcal{C}$-epimorphism for $0 \leq \lambda \leq l < p+1$ and consider the commutative diagram with exact row below

$$\begin{tikzcd}
E^{p+3-l}_{p+2,0}(1) \arrow[r, tail] \arrow[d, "f^{p+3-l}_{p+2,0}"'] & E^{p+3-(l+1)}_{p+2,0}(1) \arrow[r, "d^{p+3-(l+1)}(1)"] \arrow[d, dashed] & \ker(d^{p+3-(l+1)}_{p+2,0}(1)) \arrow[r, two heads] \arrow[d] & E^{p+3-l}_{l,p+1-l}(1) \arrow[d, "{}^{p+3-l}_{l,p+1-l}"] \\
E^{p+3-l}_{p+2,0}(2) \arrow[r, tail] & E^{p+3-(l+1)}_{p+2,0}(2) \arrow[r, "d^{p+3-(l+1)}(2)"] & \ker(d^{p+3-(l+1)}_{p+2,0}(2)) \arrow[r, two heads] & E^{p+3-l}_{l,p+1-l}(2)
\end{tikzcd}$$

The two vertical arrow on the right are $\mathcal{C}$-isomorphisms by step 3 and induction hypothesis 2. The vertical arrow on the left is $\mathcal{C}$-epimorphism by hypothesis of the current induction. Therefore, the dashed vertical arrow is a $\mathcal{C}$-epimorphism. This completes the proof of the step 5 and the proof of the proposition. $\square$

Corollary 6.3.2. *Zeeman's comparison theorem follows from Proposition 6.3.1.*

Proof. Here $\mathcal{C} = \{0\}$. Zeeman assumed that the entries in the second page of the spectral sequence satisfy the following short exact sequence

$$E^2_{p,0} \otimes E^2_{0,q} \rightarrowtail E^2_{(p,q)} \twoheadrightarrow \mathrm{Tor}(E^2_{p-1,0}, E^2_{0,q}).$$

See [33]. $\square$

Example 6.3.3. Let $\mathcal{C}$ be a class of Abelian groups closed under extensions. In the setting of the Leray-Serre spectral sequence in Example 3.2.5 on page 70, let $E(i) \to B(i)$ be a fibration with fiber $F(i)$, for $i = 1, 2$, with $\mathrm{h} = \mathrm{H}$, an ordinary homology theory. Also assume that there is a map from the first fibration to the second one that induces a morphism of the induced exact couples. If

1. $\mathrm{H}_n(E(1)) \to \mathrm{H}_n(E(2))$ is a $\mathcal{C}$-isomorphism for every $n \in \mathbb{N}$, and
2. $\mathrm{H}_q(F(1)) \to \mathrm{H}_q(F(2))$ is a $\mathcal{C}$-isomorphism for every $q \in \mathbb{N}$, and
3. whenever
$$\mathrm{H}_p(B(1); \mathrm{H}_0(F(1))) \to \mathrm{H}_p(B(2); \mathrm{H}_0(F(2)))$$
is a $\mathcal{C}$-isomorphism then $\mathrm{H}_p(B(1); \mathrm{H}_q(F(1))) \to \mathrm{H}_p(B(2); \mathrm{H}_q(F(2)))$ is a $\mathcal{C}$-isomorphism, for every $q \geq 0$,

then
$$\mathrm{H}_p(B(1); \mathrm{H}_q(F(1))) \to \mathrm{H}_p(B(2); \mathrm{H}_q(F(2)))$$
is a $\mathcal{C}$-isomorphism for all $p, q \geq 0$.

Proposition 6.3.4. *[26] Let $(f, \varphi) : (\mathfrak{C}(1), L_*(1)) \to (\mathfrak{C}(2), L_*(2))$ be a morphism of universally augmented first quadrant exact couples with differentials $d^r(i) : E^r_{(p,q)}(i) \to E^r_{p-r,p+r-1}(i)$ satisfying the following hypotheses:*

1. *$\varphi : L_*(1) \to L_*(2)$ is a $\mathcal{C}$-isomorphism;*
2. *$f^2_{p,0} : E^2_{p,0}(1) \to E^2_{p,0}(2)$ is a $\mathcal{C}$-isomorphism for $p \geq 0$;*
3. *Whenever $f^2_{0,q} : E^2_{0,q}(1) \to E^2_{0,q}(2)$ is a $\mathcal{C}$-isomorphism, then*
$$f^2_{(p,q)} : E^2_{(p,q)}(1) \to E^2_{(p,q)}(2)$$
is also a $\mathcal{C}$-isomorphism for $p \geq 0$.

Then

$$f^2_{(p,q)} : E^2_{(p,q)}(1) \to E^2_{(p,q)}(2)$$

is a $\mathcal{C}$-isomorphism for all $p, q \geq 0$.

Proof. We apply induction on q:

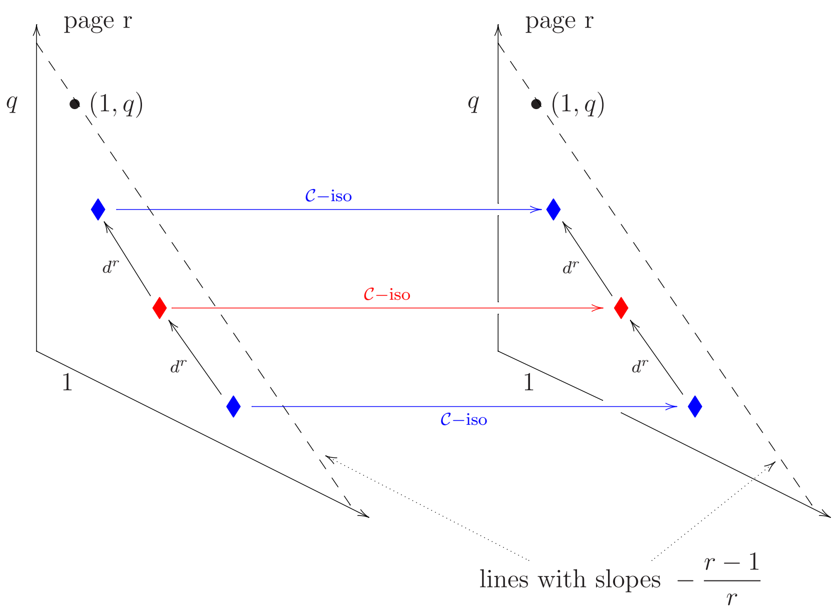
<u>*q=0*</u>: By 2 we know that $f^2_{p,0}$ is a $\mathcal{C}$-isomorphism for $p \geq 0$.

<u>*q*</u>: Assume inductively that $f^2_{p,\nu}$ is a $\mathcal{C}$-isomorphism for $p \geq 0$ and $0 \leq \nu \leq q$.

<u>*q+1*</u>: To complete the induction, we need to show that $f^2_{(p,q)+1}$ is a $\mathcal{C}$-isomorphism for all $p \geq 0$. In view of 3, it suffices to show that $f^2_{0,q+1}$ is a $\mathcal{C}$-isomorphism. Our argument involves several steps.

Claim. *On the r-th page, for all (μ, ν) that are under the line passing through $(1, q)$ with slope $-\dfrac{r-1}{r}$, if $f^r_{\mu,\nu}$ is a $\mathcal{C}$-isomorphism then $f^r_{\mu-r,\nu+r-1}$ and $f^r_{\mu+r,\nu-r+1}$ are also $\mathcal{C}$-isomorphisms.*

That is, in the following picture, if the red entries are $\mathcal{C}$-isomorphic then the blue ones are also $\mathcal{C}$-isomorphic.



Proof (Claim). Note that such (μ, ν) satisfies

$$0 \leq \nu \leq q \quad \text{and} \quad \nu \leq q - (\mu - 1)\frac{r-1}{r}.$$

We apply induction on r:

- The claim holds for $r = 2$.
- Now suppose inductively that it holds for $r \geq 2$. Then $f^r_{\mu,\nu}$ is a $\mathcal{C}$-isomorphism whenever

$$0 \leq \nu \leq q \text{ and } \nu \leq q - (\mu - 1)\frac{r-1}{r}.$$

- If (μ_0, ν_0) satisfies $\nu_0 \leq q$ and $\nu_0 \leq q - (\mu_0 - 1)\dfrac{r}{r+1}$, then using $\dfrac{r}{r+1} > \dfrac{r-1}{r}$ we have

$$\nu_0 \leq q - (\mu_0 - 1)\frac{r-1}{r}.$$

Thus $f^{r+1}_{\mu_0,\nu_0}$ is a $\mathcal{C}$-isomorphism. But also

$$(\mu, \nu) := (\mu_0 - (r+1), \nu_0 + r)$$

satisfies

$$\begin{aligned} \nu_0 + r &\leq q - (\mu_0 - 1)\frac{r}{r+1} + r \\ &= q - (\mu_0 - (r+1) - 1)\frac{r}{r+1} - (r+1)\frac{r}{r+1} + r \\ &= q - (\mu_0 - (r+1) - 1)\frac{r}{r+1}. \end{aligned}$$

So $f^{r+1}_{\mu_0-(r+1),\nu_0+r}$ is a $\mathcal{C}$-isomorphism. Similarly,

$$(\mu, \nu) := (\mu_0 + (r+1), \nu_0 - r)$$

satisfies

$$\begin{aligned} \nu_0 - r &\leq q - (\mu_0 - 1)\frac{r}{r+1} - r \\ &= q - (\mu_0 + (r+1) - 1)\frac{r}{r+1} + (r+1)\frac{r}{r+1} - r \\ &= q - (\mu_0 + (r+1) - 1)\frac{r}{r+1} \end{aligned}$$

implying that $f^{r+1}_{\mu_0+r+1,\nu_0-r}$ is a $\mathcal{C}$-isomorphism as well. This completes the induction and hence the proof of the claim. $\diamond$

<u>*Step 1*</u>

1. $f^r_{\mu,\nu}$ *is a* $\mathcal{C}$*-isomorphism for all* $r \geq 2$ *whenever* $\mu + \nu \leq q + 1$ *and*

$\nu \leq q$.

2. *$f^r_{q+2-l,l}$ is a $\mathcal{C}$-isomorphism for $0 \leq l \leq q$ and $2 \leq r \leq q+2-l$.*

<u>*Step 1 Proof*</u>:

1. If $\mu + \nu \leq q+1$, then

$$\nu \leq q+1-\mu = q-(\mu-1) < q-(\mu-1)\frac{r-1}{r}$$

for all $r \geq 2$. So this case holds by applying the Claim repeatedly.

2. If $(\mu, \nu) = (q+2-l, l)$ and $2 \leq r \leq q+2-l$, then

$$\begin{aligned}\nu \ &= q+2-\mu \\ &q-(\mu-1)+1 \\ &= q-(\mu-1)\frac{r-2}{r-1} - (\mu-1)\frac{2}{r-1} + 1.\end{aligned}$$

Thus the Claim is applicable in the $(r-1)$-th page so long as

$$-(\mu-1)\frac{2}{r-1} + 1 \leq 0;$$

i.e., $r-1 \leq 2(\mu-1)$. But we know that

$$1 \leq r-1 \leq q+1-l \quad \text{and} \quad \mu-1 = q+1-l > 0.$$

We know that

$$r-1 \leq q+1-l < 2(q+1-l) = 2(\mu-1),$$

implying that $f^r_{\mu,\nu}$ is a $\mathcal{C}$-isomorphism.

<u>*Step 2*</u>. *$f^\infty_{0,q+1}$ is a $\mathcal{C}$-isomorphism.*

<u>*Step 2 Proof*</u>: φ induces the morphism of filtrations for $L_{q+1}(1)$ and $L_{q+1}(2)$ displayed below

$$\begin{tikzcd}
F_{1,q+1}(1) \arrow[r, tail] \arrow[d] & F_{1,q}(1) \arrow[r, tail] \arrow[d] & \cdots \arrow[r, tail] & F_{k,q+1-k}(1) \arrow[r, tail] \arrow[d] & \cdots \arrow[r, tail] & F_{q+1,0}(1) \arrow[r, equal] \arrow[d] & L_{q+1}(1) \arrow[d, "\varphi_{q+1}", "\mathcal{C}-iso"'] \\
F_{1,q+1}(2) \arrow[r, tail] & F_{1,q}(2) \arrow[r, tail] & \cdots \arrow[r, tail] & F_{k,q+1-k}(2) \arrow[r, tail] & \cdots \arrow[r, tail] & F_{q+1,0}(2) \arrow[r, equal] & L_{q+1}(2).
\end{tikzcd}$$

From *Step 1* we know that $f^\infty_{q+1-l,l}$ is a $\mathcal{C}$-isomorphism whenever $0 \leq l \leq q$. Knowing that φ_{q+1} is a $\mathcal{C}$-isomorphism, let $0 \leq l \leq q+1$ and suppose inductively that

$$\varphi| : F_{q+1-\lambda,\lambda}(1) \to F_{q+1-\lambda,\lambda}(2)$$

is a $\mathcal{C}$-isomorphism for $0 \le \lambda \le l$. The diagram of short exact sequences

$$\begin{tikzcd}
F_{q-l,l+1}(1) \arrow[r, tail] \arrow[d] & F_{q+1-l,l}(1) \arrow[r, two heads] \arrow[d, "\mathcal{C}-iso"] & E^{\infty}_{q+1-l,l}(1) \arrow[d, "\mathcal{C}-iso"] \\
F_{q-l,l+1}(2) \arrow[r, tail] & F_{q+1-l,l}(2) \arrow[r, two heads] & E^{\infty}_{q+1-l,l}(2)
\end{tikzcd}$$

shows that $\varphi| : F_{q-l,l+1}(1) \to F_{q-l,l+1}(2)$ is a $\mathcal{C}$-isomorphism as well. Setting $l = q$ gives the $\mathcal{C}$-isomorphism

$$\begin{tikzcd}
F_{0,q+1}(1) \arrow[r, "\varphi_{q+1}|", "\mathcal{C}-iso"'] \arrow[d, "\cong"'] & F_{0,q+1}(2) \arrow[d, "\cong"] \\
E^{\infty}_{0,q+1}(1) \arrow[r, "f^{\infty}_{0,q+1}"'] & E^{\infty}_{0,q+1}(2)
\end{tikzcd}$$

which proves *Step 2*.

Step 3. $f^2_{0,q+1}$ is a $\mathcal{C}$-isomorphism.

Step 3 Proof. From *Step 2* we know that $f^{q+3}_{0,q+1} = f^{\infty}_{0,q+1}$ is a $\mathcal{C}$-isomorphism. We will inductively work our way down to $f^2_{0,q+1}$. Part of the induction is to show that $f^r_{q+2-l,l}$ is a $\mathcal{C}$-isomorphism for $0 \le l \le q$ and $r > q + 2 - l$.

Part 1. $F^{\infty}_{q+2,0}$ and $\varphi_{q+2}| : F_{q+1,1}(1) \to F_{q+1,1}(2)$ are $\mathcal{C}$-isomorphisms.

Part 1 Proof. The morphism of short exact sequences below

$$\begin{tikzcd}
F_{q+1,1}(1) \arrow[r, tail] \arrow[d, "\varphi_{q+2}|"'] & \mathrm{H}_{q+2}(1) \arrow[r, two heads] \arrow[d, "\mathcal{C}-iso"'] & E^{\infty}_{q+2,0}(1) \arrow[d, "f^{\infty}_{q+2,0}"] \\
F_{q+1,1}(2) \arrow[r, tail] & \mathrm{H}_{q+2}(2) \arrow[r, two heads] & E^{\infty}_{q+2,0}(2)
\end{tikzcd}$$

shows that $f^{\infty}_{q+2,0}$ is a $\mathcal{C}$-epimorphism. The commutative diagram with exact rows below

$$\begin{tikzcd}
E^{\infty}_{q+2,0}(1) \arrow[r, equal] \arrow[d, "f^{\infty}_{q+2,0}"'] & E^{q+3}_{q+2,0}(1) \arrow[r, tail] \arrow[d, "f^{q+3}_{q+2,0}"'] & E^{q+2}_{q+2,0}(1) \arrow[r, "d^{q+2}(1)"] \arrow[d, "\mathcal{C}-iso"] & E^{q+2}_{0,q+1}(1) \arrow[r, two heads] \arrow[d, "f^{q+2}_{0,q+1}"] & E^{\infty}_{0,q+1}(1) \arrow[d, "\mathcal{C}-iso"] \\
E^{\infty}_{q+2,0}(2) \arrow[r, equal] & E^{q+3}_{q+2,0}(2) \arrow[r, tail] & E^{q+2}_{q+2,0}(2) \arrow[r, "d^{q+2}(2)"'] & E^{q+2}_{0,q+1}(2) \arrow[r, two heads] & E^{\infty}_{0,q+1}(2)
\end{tikzcd}$$

shows that $f^{\infty}_{q+2,0}$ is also a $\mathcal{C}$-monomorphism. Thus $f^{q+2}_{0,q+1}$ is a $\mathcal{C}$-isomorphism. Moreover,

$$\varphi_{q+2}| : F_{q+1,1}(1) \to F_{q+1,2}(2)$$

is a $\mathcal{C}$-isomorphism.

Part 2 Suppose inductively that for $0 \leq l-1 < q$ we have

1. $f^{q+2-\lambda}_{0,q+1}$ is a $\mathcal{C}$-isomorphism, for $0 \leq \lambda \leq l-1$,
2. $\varphi_{q+2}| : F_{q+2-\lambda}(1) \to F_{q+2-\lambda}(2)$ is a $\mathcal{C}$-isomorphism for $0 \leq \lambda \leq l$,
3. (a) $f_{q+2-\lambda+tr,\lambda-t(r-1)}$ is a $\mathcal{C}$-isomorphism for all $r \geq 2$, $0 \leq \lambda < l$ and $t \geq 0$,

   (b) $f^{r}_{q+2-\lambda+tr,\lambda-t(r-1)}$ is a $\mathcal{C}$-isomorphism for all $r \geq 2$ and $t \geq 1$.

We complete the induction through the following items:

(A) For $\rho \geq q+3-l$, $f^{\rho}_{q+2-l,l}$ is a $\mathcal{C}$-epimorphism.

Proof. $f^{\infty}_{q+2-l,l}$ is $\mathcal{C}$-epimorphism because of the morphism of short exact sequences

$$\begin{tikzcd}
F_{q+1-l,l+1}(1) \arrow[r, tail] \arrow[d] & F_{q+1-l,l}(1) \arrow[r, two heads] \arrow[d, "\mathcal{C}-iso"] & E^{\infty}_{q+2-l,l}(1) \arrow[d, "f^{\infty}_{q+2-l,l}"] \\
F_{q+1-l,l+1}(2) \arrow[r, tail] & F_{q+1-l,l}(2) \arrow[r, two heads] & E^{\infty}_{q+2-l,l}(2)
\end{tikzcd} \quad . \tag{6.3}$$

Next, for $s \geq 0$ we have the commutative diagram with exact rows

$$\begin{tikzcd}
\ker(d^{q+3-l+s}_{-,-}(1)) \arrow[r, tail] \arrow[d] & E^{q+3-l+s}_{-,-}(1) \arrow[r, "d^{q+3-l+s}_{-,-}(1)"] \arrow[d, "\mathcal{C}-iso"'] & E^{q+3-l+s}_{q+2-l,l}(1) \arrow[r, two heads] \arrow[d, "f^{q+3-l+s}_{q+2-l,l}"] & E^{q+3-l+s+1}_{q+2-l,l}(1) \arrow[d, "f^{q+3-l+s+1}_{q+2-l,l}"] \\
\ker(d^{q+3-l+s}_{-,-}(2)) \arrow[r, tail] & E^{q+3-l+s}_{-,-}(2) \arrow[r, "d^{q+3-l+s}_{-,-}(2)"'] & E^{q+3-l+s}_{q+2-l,l}(2) \arrow[r, two heads] & E^{q+3-l+s+1}_{q+2-l,l}(2)
\end{tikzcd} \tag{6.4}$$

From induction hypothesis $3.(b)$ we know that the second vertical arrow from the left is a $\mathcal{C}$-isomorphism. Knowing that $f^{\infty}_{q+2-l,l}$ is a $\mathcal{C}$-epimorphism, assume inductively that $f^{q+3-l+s+1}_{q+2-l,l}$ is $\mathcal{C}$-epimorphism. It follows that the morphism $f^{q+3-l+s}_{q+2-l,l}$ is $\mathcal{C}$-epimorphism for $s \geq 0$. $\diamond$

(B) For $\rho \geq q+3-l$, $f^{\rho}_{q+2-l,l}$ is a $\mathcal{C}$-isomorphism and $f^{q+2-l}_{0,q+1}$ is a $\mathcal{C}$-isomorphism.

Proof. Consider the commutative diagram with exact rows below

$$\begin{tikzcd}
\ker(d^{q+2-l}_{q+2-l,l}(1)) \arrow[r, tail] \arrow[d] & E^{q+2-l}_{q+2-l,l}(1) \arrow[r, "d^{q+2-l}_{q+2-l,l}(1)"] \arrow[d, "\mathcal{C}-iso"'] & E^{q+2-l}_{0,q+1}(1) \arrow[r, two heads] \arrow[d, "f^{q+2-l}_{0,q+1}"] & E^{q+3-l}_{0,q+1}(1) \arrow[d, "\mathcal{C}-iso"] \\
\ker(d^{q+2-l}_{q+2-l,l}(2)) \arrow[r, tail] & E^{q+2-l}_{q+2-l,l}(2) \arrow[r, "d^{q+2-l}_{q+2-l,l}(2)"'] & E^{q+2-l}_{0,q+1}(2) \arrow[r, two heads] & E^{q+3-l}_{0,q+1}(2)
\end{tikzcd} \tag{6.5}$$

$$\begin{tikzcd}
\ker(d^{q+2-l}_{-,-}(1)) \arrow[r, tail] \arrow[d] & E^{q+2-l}_{-,-}(1) \arrow[r, "d^{q+2-l}_{-,-}(1)"] \arrow[d, "\mathcal{C}-iso"'] & \ker(d^{q+2-l}_{q+2-l,l}(1)) \arrow[r, two heads] \arrow[d] & E^{q+3-l}_{q+2-l,l}(1) \arrow[d, "f^{q+3-l}_{q+2-l,l}"] \\
\ker(d^{q+2-l}_{q+2-l}(2)) \arrow[r, tail] & E^{q+2-l}_{q+2-l,l}(2) \arrow[r, "d^{q+2-l}_{-,-}(2)"'] & \ker(d^{q+2-l}_{0,q+1}(2)) \arrow[r, two heads] & E^{q+3-l}_{0,q+1}(2)
\end{tikzcd} \tag{6.6}$$

From (6.5) we see that the second vertical arrow from the right in (6.6) is a $\mathcal{C}$-monomorphism. Chasing (6.6) shows that it is also $\mathcal{C}$-epimorphism. Consequently, $f^{q+3-l}_{q+2-l,l}$ is a $\mathcal{C}$-isomorphism. Feed this information into (6.5) to see that $f^{q+2-l}_{0,q+1}$ is a $\mathcal{C}$-isomorphism. Feed this information into (6.4) to see that $f^{\rho}_{q+2-l,l}$ is a $\mathcal{C}$-isomorphism whenever $\rho \geq q+3-l$. $\diamond$

(C) $\varphi| : F_{q+1-l,l+1}(1) \to F_{q+1-l,l+1}(2)$ *is a* $\mathcal{C}$*-isomorphism.*

Proof. Wwe know that $f^{\rho}_{q+2-l,l}$ is a $\mathcal{C}$-isomorphism whenever $\rho \geq q+3-l$. Thus $f^{\infty}_{q+2-l,l}$ is a $\mathcal{C}$-isomorphism. Using (6.3) we see that $F_{q+1-l,l+1}(1) \to F_{q+1-l,l+1}(2)$ is a $\mathcal{C}$-isomorphism. $\diamond$

In part (B), we took the induction step for 1, *Part 2*. In part (C), we took the induction step for 2, *Part 2*. It remains to take the induction step for 3, *Part 2*:

- (a): It only remains to show that $f^r_{q+2-l+tr,l-t(r-1)}$ is a $\mathcal{C}$-isomorphism for all $t \geq 0$. Combining *Step 1* and part (A), we see that $f^r_{q+2-l,l}$ is a $\mathcal{C}$-isomorphism for all $r \geq 2$. This settles the case $t = 0$. The case $t \geq 1$ is hypothesis (b).

- (b): We need to show that $f^r_{q+2-(l+1)+tr,l+1-t(r-1)}$ is a $\mathcal{C}$-isomorphism for all $r \geq 2$ and $t \geq 1$. The case $r = 2$ follows because $f^2_{\mu,\nu}$ is a $\mathcal{C}$-isomorphism for all $\nu \geq q$. If $r \geq 3$, we have

$$\begin{aligned}(q+2-(l+1)+tr, l+1-t(r-1)) &= (q+2-[(l+1)+t]+t(r-1), [l+1+t]-t(r-2)) \\ &=: (\mu,\nu).\end{aligned}$$

Now $t \geq 1$. So $[(l+1)-t] \leq q$. Thus induction hypotheses (a) and (b), together with the completed induction step (a) show that the Claim is applicable to $f^{r-1}_{\mu,\nu}$. Thus $f^r_{\mu,\nu}$ is a $\mathcal{C}$-isomorphism as required. $\diamond$

$\square$

Example 6.3.5. Let $\mathcal{C}$ be a class of Abelian groups closed under extensions.

1. In the setting of the Leray-Serre spectral sequence in Example 3.2.5 on page 70, let $E(i) \to B(i)$ be a fibration with fiber $F(i)$, for $i = 1, 2$, and $\mathrm{h} = \mathrm{H}$, an ordinary homology theory. Also assume that there is a map from the first fibration to the second one that induces a morphism of the induced exact couples. If

   (a) $\mathrm{H}_n(E(1)) \to \mathrm{H}_n(E(2))$ is a $\mathcal{C}$-isomorphism for every $n \in \mathbb{N}$, and

   (b) $\mathrm{H}_p(B(1); \mathrm{H}_0(F(1))) \to \mathrm{H}_p(X(2); \mathrm{H}_0(F(2)))$ is a $\mathcal{C}$-isomorphism for every $p \in \mathbb{N}$, and

   (c) whenever $\mathrm{H}_q(F(1)) \to \mathrm{H}_q(F(2))$ is a $\mathcal{C}$-isomorphism then
   $$\mathrm{H}_p(B(1); \mathrm{H}_q(F(1))) \to \mathrm{H}_p(B(2); \mathrm{H}_q(F(2)))$$
   is a $\mathcal{C}$-isomorphism, for every $p \geq 0$,

   then
   $$\mathrm{H}_p(B(1); \mathrm{H}_q(F(1))) \to \mathrm{H}_p(B(2); \mathrm{H}_q(F(2)))$$
   is a $\mathcal{C}$-isomorphism for all $p, q \geq 0$.

2. In the setting of the Atiyah-Hirzebruch-Whitehead spectral sequence in Example 3.2.6 on page 71, if X is a finite CW-complex and

   (a) $\mathrm{h}_n(X(1)) \to \mathrm{h}_n(X(2))$ is a $\mathcal{C}$-isomorphism for every $n \in \mathbb{N}$, and

   (b) $\mathrm{H}_p(X(1); \mathrm{h}_0(1)(pt)) \to \mathrm{H}_p(X(2); \mathrm{h}_0(2)(pt))$ is a $\mathcal{C}$-isomorphism for every $p \in \mathbb{N}$, and

   (c) whenever $\mathrm{h}_q(1)(pt) \to \mathrm{h}_q(2)(pt)$ is a $\mathcal{C}$-isomorphism, then
   $$\mathrm{H}_p(X(1); \mathrm{h}_q(1)(pt)) \to \mathrm{H}_p(X(2); \mathrm{h}_q(2)(pt))$$
   is a $\mathcal{C}$-isomorphism, for every $q \geq 0$,

   then
   $$\mathrm{H}_p(X(1); \mathrm{h}_q(1)(pt)) \to \mathrm{H}_p(X(2); \mathrm{h}_q(2)(pt))$$
   is a $\mathcal{C}$-isomorphism for every $p, q \geq 0$.

6.3.2 Reverse Comparison Theorems for Eventually Vanishing Originally Stable Exact Couples

The proof of the following propositions is the "mirror image" of the proof of the propositions in the previous section.

Proposition 6.3.6. *[26] Let $(f, \varphi) : (\mathfrak{C}(1), L^*(1)) \to (\mathfrak{C}(2), L^*(2))$ be a morphism of universally coaugmented first quadrant exact couples with differentials $d^r(i) : E^r_{(p,q)}(i) \to E^r_{p+r,q-r+1}(i)$ satisfying the following hypotheses:*

1. *$\varphi : L^*(1) \to L^*(2)$ is a $\mathcal{C}$-isomorphism;*
2. *$f_{0,q} : E^2_{0,q}(1) \to E^2_{0,q}(2)$ is a $\mathcal{C}$-isomorphism for $q \geq 0$;*
3. *Whenever $f_{p,0} : E^2_{p,0}(1) \to E^2_{p,0}(2)$ is a $\mathcal{C}$-isomorphism, then $f_{(p,q)} : E^2_{(p,q)}(1) \to E^2_{(p,q)}(2)$ is also a $\mathcal{C}$-isomorphism for $q \geq 0$.*

Then

$$f_{(p,q)} : E^2_{(p,q)}(1) \to E^2_{(p,q)}(2)$$

is a $\mathcal{C}$-isomorphism for all $p, q \geq 0$.

Proposition 6.3.7. *[26] Let $(f, \varphi) : (\mathfrak{C}(1), L^*(1)) \to (\mathfrak{C}(2), L^*(2))$ be a morphism of universally coaugmented first quadrant exact couples with differentials $d^r(i) : E^r_{(p,q)}(i) \to E^r_{p+r,p-r+1}(i)$ satisfying the following hypotheses:*

1. *$\varphi : H(1) \to H(2)$ is a $\mathcal{C}$-isomorphism;*
2. *$f^2_{p,0} : E^2_{p,0}(1) \to E^2_{p,0}(2)$ is a $\mathcal{C}$-isomorphism for $p \geq 0$;*
3. *Whenever $f^2_{0,q} : E^2_{0,q}(1) \to E^2_{0,q}(2)$ is a $\mathcal{C}$-isomorphism, then $f^2_{(p,q)} : E^2_{(p,q)}(1) \to E^2_{(p,q)}(2)$ is also a $\mathcal{C}$-isomorphism for $p \geq 0$.*

Then

$$f^2_{(p,q)} : E^2_{(p,q)}(1) \to E^2_{(p,q)}(2)$$

is a $\mathcal{C}$-isomorphism for all $p, q \geq 0$.

6.4 Generalization to Arbitrary Bidegrees

We can generalize the previous sections to exact couples with arbitrary bidegrees **a**, **b** and **c**. To this end, we state the general form of the tools we have used so far.

• Let $\{D^1_{*,*}, E^1_{*,*}\}$ be an originally vanishing and eventually stable exact couple.

1. If $D^1_{(u,v)}$ is the highest nonzero term of the n-th D^1-column then there is a tower of epimorphisms

$$E^2_{(u,v)-\mathbf{c}} \twoheadrightarrow E^3_{(u,v)-\mathbf{c}} \twoheadrightarrow \cdots \twoheadrightarrow E^s_{(u,v)-\mathbf{c}} = E^\infty_{(u,v)-\mathbf{c}},$$

   for some $s \in \mathbb{N}$. Also, for every $3 \leq r \leq s$, we have the following exact sequence

$$\ker(d^{r-1}_{(u,v)-\mathbf{b}-2\mathbf{c}+(r-2)\mathbf{a}}) \rightarrowtail E^{r-1}_{(u,v)-\mathbf{b}-2\mathbf{c}+(r-2)\mathbf{a}} \xrightarrow{d^{r-1}_{(u,v)-\mathbf{b}-2\mathbf{c}+(r-2)\mathbf{a}}} E^{r-1}_{(u,v)-\mathbf{c}} \twoheadrightarrow E^r_{(u,v)-\mathbf{c}}.$$

2. If $D^1_{(p,q)}$ is the highest stable term of the n-th D^1-column, there is a tower of monomorphisms

$$E^\infty_{(p,q)+\mathbf{b}} = E^t_{(p,q)+\mathbf{b}} \rightarrowtail \cdots \rightarrowtail E^3_{(p,q)+\mathbf{b}} \rightarrowtail E^2_{(p,q)+\mathbf{b}}$$

   for some $t \in \mathbb{N}$. Also, for every $3 \leq r \leq t$, we have the following exact sequence

$$E^r_{(p,q)+\mathbf{b}} \rightarrowtail E^{r-1}_{(p,q)+\mathbf{b}} \xrightarrow{d^{r-1}_{(p,q)+\mathbf{b}}} \ker(d^{r-1}_{(p,q)+2\mathbf{b}+\mathbf{c}-(r-2)\mathbf{a}}) \twoheadrightarrow E^r_{(p,q)+2\mathbf{b}+\mathbf{c}-(r-2)\mathbf{a}}.$$

3. We also have $\dfrac{\phi_{(p,q)}}{\phi_{(p,q)-\mathbf{a}}} \cong E^\infty_{(p,q)+\mathbf{b}}$.

4. To apply the induction, we need to find the highest stable term of the n-th D^1-column, say $D^1_{(p,q)}$. We start from an E-term in position $p, q + \mathbf{b}$ that is equal to the E^∞-term in that position and apply induction on k to carry the information through a tower of inclusions to E-terms in positions $p, q - k\mathbf{a} + \mathbf{b}$.

To see how the results of last sections look, here is Proposition 6.2.1 on page 148 for general bidegrees:

Proposition 6.4.1. *Let $\{D^1_{*,*}, E^1_{*,*}\}$ be an originally vanishing and eventually stable exact couple. Assume, for some $r_0 \geq 1$,*

1. $L_n \in \mathcal{C}$ for every $n \in \mathbb{N}$, and

2. $E^{r_0}_{(p,q)} \in \mathcal{C}$, where $E^1_{(p,q)}$ runs through all nonzero leftmost E-terms, and

3. *if* $D^1_{(u,v)+s\mathbf{a}}$ *is the highest stable term of the* n*-th* D*-column, then whenever* $E^{r_0}_{(u,v)+s\mathbf{a}+\mathbf{b}} \in \mathcal{C}$*, we have* $E^{r_0}_{u+p,v+q+s\mathbf{a}+\mathbf{b}-\mathbf{c}} \in \mathcal{C}$*, for every* p, q*.*

Then $E^{r_0}_{*,*} \in \mathcal{C}$ *for every* $r \geq r_0$*.*

• Let $\{D^1_{*,*}, E^1_{*,*}\}$ be an eventually vanishing and originally stable exact couple.

1. Let $D^1_{(u,v)}$ be the lowest nonzero term of the n-th D^1-column. Then there is a tower of monomorphisms

$$E^{\infty}_{(u,v)+\mathbf{b}} = E^{s}_{(u,v)+\mathbf{b}} \rightarrowtail \cdots \rightarrowtail E^{3}_{(u,v)+\mathbf{b}} \rightarrowtail E^{2}_{(u,v)+\mathbf{b}}$$

for some $s \in \mathbb{N}$.

2. Let $D^1_{(p,q)}$ be the lowest stable term of the n-th D^1-column, there is a tower of monomorphisms

$$E^{2}_{(p,q)-\mathbf{c}} \twoheadrightarrow E^{3}_{(p,q)-\mathbf{c}} \twoheadrightarrow \cdots \twoheadrightarrow E^{t}_{(p,q)-\mathbf{c}} = E^{\infty}_{(p,q)-\mathbf{c}}$$

for some $t \in \mathbb{N}$.

3. Since the exact couple is upward stable we have $\lim^1_r\{\ker(i^r_{(p,q)-r\mathbf{a}})\} = 0$ and hence the exact sequence

$$\frac{\phi^{(p,q)}}{\phi^{(p,q)-\mathbf{a}}} \rightarrowtail E^{\infty}_{(p,q)-\mathbf{a}-\mathbf{c}} \to \lim^1_r\{\ker(i^r_{(p,q)-r\mathbf{a}})\}$$

implies that $\dfrac{\phi^{(p,q)}}{\phi^{(p,q)-\mathbf{a}}} \cong E^{\infty}_{(p,q)-\mathbf{a}-\mathbf{c}}$.

4. To apply the induction, we need to find the lowest stable term of the n-th D^1-column, say $D^1_{(u,v)}$, start from an E-term in position $p, q-\mathbf{c}$ and apply induction on k to carry the information through a tower of inclusions to E-terms in positions $p, q+k\mathbf{a}-\mathbf{c}$.

6.5 Zeeman-Unfriendly Examples

Zeeman's comparison theorem is applicable to those spectral sequences where the entries on the second page have a specific form; i.e., $\mathrm{H}_*(-;\mathrm{H}_*(-))$. But, for example, if we consider the spectral sequence of the total complex of a double complex, then Zeeman's comparison theorem is not applicable anymore.

In the following three sections, we offer examples of spectral sequences where their second pages are *Zeeman-unfriendly*, that is, they are not of the form $\mathrm{H}_p(-;\mathrm{H}_q(-))$, and we can apply the results of this chapter.

6.5.1 Total Complex

Look at [27] for terminologies. Let A be a bicomplex. If $A_{(p,q)} = 0$ for $p < 0$ or $q < 0$ (first quadrant bicomplex), then there is a spectral sequence

$$E^2_{(p,q)} \cong \mathrm{H}_p(\mathrm{H}_q(A_{p,-}, d^v_{p.-}), d^h_*) = \mathrm{H}^h_p(\mathrm{H}^v_q A) \Rightarrow \mathrm{H}_n \operatorname{Tot}(A).$$

For a class $\mathcal{C}$ of modules closed under extensions we have the following

- If
  1. $\mathrm{H}_n \operatorname{Tot}(A) \in \mathcal{C}$ for every $n \in \mathbb{N}$, and
  2. $\mathrm{H}^h_0(\mathrm{H}^v_q A) \in \mathcal{C}$, for every $q \geq 0$, and
  3. whenever $\mathrm{H}^h_p(\mathrm{H}^v_0 A) \in \mathcal{C}$, then $\mathrm{H}^h_p(\mathrm{H}^v_q A) \in \mathcal{C}$, for every $q \geq 0$,

  then $\mathrm{H}^h_p(\mathrm{H}^v_q A) \in \mathcal{C}$, for every $p, q \geq 0$.
- If $f : A \to B$ is a morphism of first quadrant bicomplexes and
  1. $\mathrm{H}_n \operatorname{Tot}(A) \to \mathrm{H}_n \operatorname{Tot}(B)$ is a $\mathcal{C}$-isomorphism for every $n \in \mathbb{N}$, and
  2. $\mathrm{H}^h_0(\mathrm{H}^v_q A) \to \mathrm{H}^h_0(\mathrm{H}^v_q B)$ is a $\mathcal{C}$-isomorphism, for every $q \geq 0$, and
  3. whenever $\mathrm{H}^h_p(\mathrm{H}^v_0 A) \to \mathrm{H}^h_p(\mathrm{H}^v_0 B)$ is a $\mathcal{C}$-isomorphism, then
  $$\mathrm{H}^h_p(\mathrm{H}^v_q A) \to \mathrm{H}^h_p(\mathrm{H}^v_q B)$$
  is a $\mathcal{C}$-isomorphism, for every $q \geq 0$,

  then $\mathrm{H}^h_p(\mathrm{H}^v_q A) \to \mathrm{H}^h_p(\mathrm{H}^v_q B)$ is a $\mathcal{C}$-isomorphism, for every $p, q \geq 0$.

Similarly, we can state the rest of the results of this chapter for the spectral sequence of the total complex of A.

6.5.2 Grothendieck Spectral Sequence

All four Grothendieck spectral sequences are Zeeman-unfriendly. We just mention one of them here.

Let $\mathcal{A} \xrightarrow{G} \mathcal{B} \xrightarrow{F} \mathcal{D}$ be covariant additive functors, where $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{D}$ are Abelian categories with enough projectives. Assume that F is right exact and that GP is left F-acyclic for every projective P in $\mathcal{A}$. Then, by part 1 in Example 3.2.11 on page 77, for every homomorphism $f : A \to B$ in $\mathcal{A}$, there are first quadrant (homology) spectral sequences with

$$(L_pF)(L_qG)A \Longrightarrow L_n(FG)A$$

and

$$(L_pF)(L_qG)B \Longrightarrow L_n(FG)B.$$

Now we have the following:

- Let $\mathcal{C}$ be a class of modules closed under extensions and
  1. $L_n(FG)A \to L_n(FG)B$ is a $\mathcal{C}$-isomorphism, for every $n \geq 0$, and
  2. $(L_0F)(L_qG)A \to (L_0F)(L_qG)B$ is a $\mathcal{C}$-isomorphism for every $q \geq 0$, and
  3. whenever $(L_pF)(L_0G)A \to (L_pF)(L_0G)B$ is a $\mathcal{C}$-isomorphism, then
  $$(L_pF)(L_qG)A \to (L_pF)(L_qG)B$$
  is a $\mathcal{C}$-isomoprhism, for every $q \geq 0$.

  Then $(L_pF)(L_qG)A \to (L_pF)(L_qG)B$ is a $\mathcal{C}$-isomorphism, for every $p, q \geq 0$.

6.5.3 Bousfield-Friedlander

For the terminologies look at [12] and [5].

Theorem [Bousfield-Friedlander]. *Let X be a bisimplicial set satisfying the π_*-Kan condition, and let $* \in X_{0,0}$ be a base vertex (whose degeneracies are taken as the basepoints of the sets $X_{m,n}$.) Then there is a first quadrant spectral sequence*

$$E^2_{(p,q)} = \pi^h_p(\pi^v_q(X)) \Rightarrow \pi_{p+q}(diagX).$$

The term $E^r_{(p,q)}$ is a set for $p+q=0$, a group for $p+q=1$ and an Abelian group for $p+q=2$.

If we are lucky and for $p+q=0$ and $p+q=1$, $E^r_{(p,q)}$ is an Abelian group then we have the following

- Let $\mathcal{C}$ be a class of modules closed under extensions and X be a bisimplicial set satisfying the π_*-Kan condition, and

  1. $\pi_n(diagX) \in \mathcal{C}$ for every $n \in \mathbb{N}$, and
  2. $\pi^h_0(\pi^v_q X) \in \mathcal{C}$ for $q \geq 0$, and
  3. whenever $\pi^h_p(\pi^v_0 X) \in \mathcal{C}$, then $\pi^h_p(\pi^v_q X) \in \mathcal{C}$ for $q \geq 0$.

  Then $\pi^h_p(\pi^v_q X) \in \mathcal{C}$ for every $p, q \geq 0$ and every $r \geq r_0$.

- Let $\mathcal{C}$ be a class of modules closed under extensions and $f : X \to Y$ be a morphism of bisimplicial sets where both sets satisfy the π_*-Kan condition, and

  1. $\pi_n(diagX) \to \pi_n(diagY)$ is a $\mathcal{C}$-isomorphism for every $n \in \mathbb{N}$, and
  2. $\pi^h_0(\pi^v_q X) \to \pi^h_0(\pi^v_q Y)$ be a $\mathcal{C}$-isomorphism for $q \geq 0$, and
  3. whenever $\pi^h_p(\pi^v_0 X) \to \pi^h_p(\pi^v_0 Y)$ is a $\mathcal{C}$-isomorphism, then the morphism $\pi^h_p(\pi^v_q X) \to \pi^h_p(\pi^v_q Y)$ is a $\mathcal{C}$-isomorphism for $q \geq 0$.

  Then $\pi^h_p(\pi^v_q X) \to \pi^h_p(\pi^v_q Y)$ is a $\mathcal{C}$-isomorphism for every $p, q \geq 0$ and every $r \geq r_0$.

Chapter 7

Future Projects

The ideas and the questions presented in this chapter are the outcome of hours of discussions with Dr. Peschke.

7.1 An Abelian Subcategory of Exact Couples

As Massey has mentioned in [22], p. 369, exact couples and the morphisms between them form a category, which we show it here by $\mathcal{EC}$.

Proposition 7.1.1. *$\mathcal{EC}$ is an additive category.*

Proof. See [15], p. 260. $\square$

Unfortunately, kernel and cokernel of morphisms in $\mathcal{EC}$ need not exist and hence $\mathcal{EC}$ does not form an Abelian category, as the following example shows:

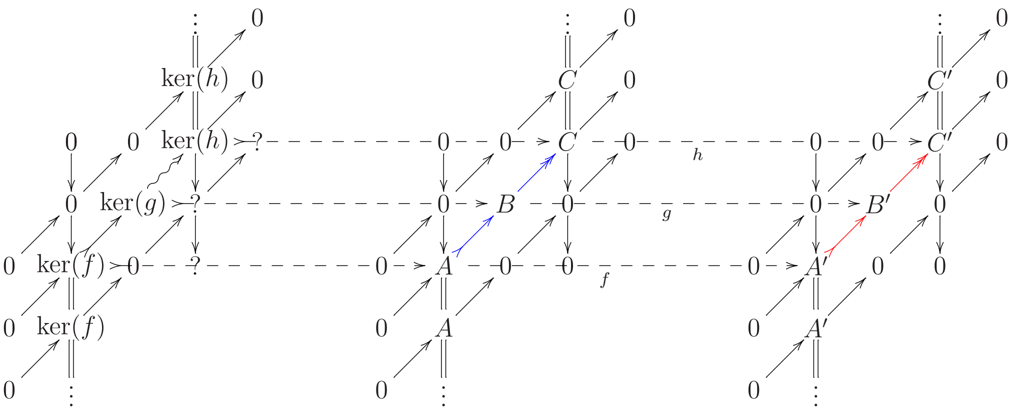
Example 7.1.2. Assume we have a morphism between two short exact sequences of modules $F : SES(1) \to SES(2)$, where, as usual, arrows of the same color represent exactness

$$\begin{tikzcd} SES(1): \arrow[d, dashed, "F"'] & A \arrow[r, tail] \arrow[d, dashed, "f"'] & B \arrow[r, two heads] \arrow[d, dashed, "g"'] & C \arrow[d, dashed, "h"'] \\ SES(2): & A' \arrow[r, tail] & B' \arrow[r, two heads] & C'. \end{tikzcd}$$

By the Snake Lemma, the kernel of this morphism is not necessarily a short exact sequence; the wavy arrow is not necessarily an epimorphism

$$
\begin{tikzcd}
\ker(F) \arrow[d, tail, dashed] & \ker(f) \arrow[r, tail] \arrow[d, tail, dashed] & \ker(g) \arrow[r, squiggly] \arrow[d, tail, dashed] & \ker(h) \arrow[d, tail, dashed] \\
SES(1): \arrow[d, dashed, "F"'] & A \arrow[r, tail, blue] \arrow[d, dashed, "f"'] & B \arrow[r, two heads, blue] \arrow[d, dashed, "g"'] & C \arrow[d, dashed, "h"'] \\
SES(2): & A' \arrow[r, tail, red] & B' \arrow[r, two heads, red] & C'.
\end{tikzcd}
$$

Therefore, if we look at the two short exact sequences above as parts of two exact couples, as shown below, then not every arbitrary nontrivial morphism between these two exact couples has kernel.



Not an Exact Couple Exact Couple Exact Couple

That is, not every morphism in $\mathcal{EC}$ has kernel.

However, if we can find a "large enough" subcategory of $\mathcal{EC}$ that is Abelian, then we can define the notion of "exactness" for the morphisms of exact couples in that subcategory and, as a result, we can talk about "the short and hence long exact sequence of exact couples" and "the short and hence long exact sequence of their induced spectral sequences". In particular, the comparison theorems we stated in the fifth chapter become special cases of more general theorems.

First we try to define the notion of an *exact morphism* of exact couples:

Definition 7.1.3. *A morphism of exact couples is called* exact *if its image is an exact couple.*

Lemma 7.1.4. *A morphism of exact couples is exact if and only if its kernel is exact.*

Proof. Consider the following segment of a ladder of long exact sequences of modules

$$\begin{tikzcd}
\cdots \arrow[r] & A \arrow[r, "i"] \arrow[d, "f"] & B \arrow[r, "j"] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & \cdots \\
\cdots \arrow[r] & A' \arrow[r, "i'"] & B' \arrow[r, "j'"] & C' \arrow[r] & \cdots
\end{tikzcd}$$

and consider the following diagram where the middle row is exact

$$\begin{tikzcd}
\cdots \arrow[r] & \ker(f) \arrow[r] \arrow[d, tail] & \ker(g) \arrow[r] \arrow[d, tail] & \ker(h) \arrow[r] \arrow[d, tail] & \cdots \\
\cdots \arrow[r] & A \arrow[r, "i"] \arrow[d, two heads, "f"] & B \arrow[r, "j"] \arrow[d, two heads, "g"] & C \arrow[r] \arrow[d, two heads, "h"] & \cdots \\
\cdots \arrow[r] & \mathrm{im}(f) \arrow[r, "i'"] & \mathrm{im}(g) \arrow[r, "j'"] & \mathrm{im}(h) \arrow[r] & \cdots\ .
\end{tikzcd} \tag{7.1}$$

Note that by exactness of the middle row, we know that the composite of the dashed arrows in the middle of the following diagram is zero. By applying the Snake Lemma on the dashed morphisms between the following two vertical short exact sequences and some diagram chasing

$$\begin{tikzcd}
\cdots \arrow[r] & \ker(f) \arrow[rr, "i|"] \arrow[dd, tail] \arrow[dr, two heads] & & \ker(g) \arrow[rr, "j|"] \arrow[dd, tail] \arrow[dr, dashed, two heads] & & \ker(h) \arrow[r] \arrow[dd, tail] & \cdots \\
& & \mathrm{im}(i|) \arrow[ur, dashed, tail] \arrow[dd, tail] & & \mathrm{im}(j|) \arrow[ur, tail] \arrow[dd, tail] & & \\
\cdots \arrow[r] & A \arrow[rr, "i"] \arrow[dd, two heads, "f"] \arrow[dr, two heads] & & B \arrow[rr, "j"] \arrow[dd, two heads, "g"] \arrow[dr, dashed, two heads] & & C \arrow[r] \arrow[dd, two heads, "h"] & \cdots \\
& & \mathrm{im}(i) \arrow[ur, dashed, tail] \arrow[dd, two heads] & & \mathrm{im}(j) \arrow[ur, tail] \arrow[dd, two heads] & & \\
\cdots \arrow[r] & \mathrm{im}(f) \arrow[rr, "i'"] \arrow[dr, two heads] & & \mathrm{im}(g) \arrow[rr, "j'"] \arrow[dr, dashed, two heads] & & \mathrm{im}(h) \arrow[r] & \cdots \\
& & \mathrm{im}(i') \arrow[ur, dashed, tail] & & \mathrm{im}(j') \arrow[ur, tail] & &
\end{tikzcd}$$

we can show that in (7.1), the first row is exact if and only if the third row is exact.

Using this observation, we see that the kernel of a morphism between two exact couples forms an exact couple if and only if its image forms an exact couple. $\square$

Corollary 7.1.5. *A morphism of exact couples is exact if and only if its cokernel is exact.*

Proof. The image of a morphism from the first exact couple to the second one is the kernel of the morphism from the second exact couple onto the cokernel of the original morphism. So we are done by the previous lemma. □

Now, for every two exact couples $EC(1)$ and $EC(2)$ in $\mathcal{EC}$, we can define the subset of $\hom_{\mathcal{EC}}(EC(1), EC(2))$ consisting of exact morphisms from $EC(1)$ to $EC(2)$, and show it by $\hom_{\mathrm{EX}}(EC(1), EC(2))$. Since the zero morphism is exact, the set $\hom_{\mathrm{EX}}(EC(1), EC(2))$ is nonempty. The question is

Do exact couples with morphism sets $\hom_{EX}(-,-)$ *form an Abelian subcategory of* $\mathcal{EC}$*?*

The following questions also arise, where some of them can be answered regardless of the answer to the question above,

- *What is the effect of an exact morphism on the* E^∞*-distribution diagrams?*
- *How can we define the short exact sequence of exact couples and the induced spectral sequences?*
- *How we can compare two short exact sequences of exact couples and the induced spectral sequences?*

7.2 Multiplicative Pairing of Spectral Sequences

As a generalization of the already existing notion of multiplication on a spectral sequence, e.g., cup product in cohomology spectral sequences, we can "pair" two spectral sequences into another spectral sequence.

Definition 7.2.1. *A* multiplicative pairing *of two spectral sequences* $(E^r(1), d^r(1))$ *and* $(E^r(2), d^r(2))$ *in* (E^r, d^r) *is a family of morphisms*

$$\mu^r_{(p,q);(u,v)} : E^r_{(p,q)}(1) \otimes E^r_{(u,v)}(2) \to E^r_{(p+u,q+v)}$$

for $r \geq k > 0$, such that for $x \otimes y \in E^r_{(p,q)}(1) \otimes E^r_{(u,v)}(2)$, we have

$$d^r(\mu^r(x \otimes y)) = \mu^r(d^r(1)(x \otimes y)) + (-1)^{p+q}\mu^r(x \otimes d^r(2)(y)).$$

Now, the following questions arise:

- *What is the interaction between the differentials and pages of these three spectral sequences?*
- *What is the relationship between the E^∞-distribution diagrams of these spectral sequences?*
- *What is the interaction with the notion of exact morphism in the previous section?*

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Index

Appendix A

Colimit and Limit

Colimit. Let $\mathcal{C}$ be a category, $\mathcal{J}$ a small category, $X \in Obj(\mathcal{C})$ and $\varphi : \mathcal{J} \to \mathcal{C}$ a functor. The family of morphisms $u_j : \varphi(j) \to X$, $j \in Obj(\mathcal{J})$, are called *cone over* X or *cone from* φ *to* X, if the triangles

$$\begin{tikzcd}
\varphi(j) \arrow[rr, "\varphi(u)"] \arrow[dr, "u_j"'] & & \varphi(j') \arrow[dl, "u_{j'}"] \\
& X &
\end{tikzcd}$$

commute for all $u : j \to j'$ in $\mathcal{J}$. Let

$$\text{cone}(\varphi, X) := \{\text{all cones from } \varphi \text{ to } X\},$$

then $\text{cone}(\varphi, X)$ is a proper set as $\mathcal{J}$ is small.

Definition A.0.2. *A* colimit *for* φ *is given by*

- *an object* $L = \operatorname{colim} \varphi$ *in* $\mathcal{C}$ *and*
- *a cone* λ *in* $\text{cone}(\varphi, L)$

with the following property: for every cone u *from* φ *to an arbitrary element*

X in $\mathcal{C}$ where the following diagram commutes

$$\begin{tikzcd}
\varphi(j) \arrow[dd, "\varphi(u)"'] \arrow[dr, "\lambda_j"] \arrow[drr, "u_j"] & & \\
& L & X \\
\varphi(j') \arrow[ur, "\lambda_{j'}"] \arrow[urr, "u_{j'}"'] & &
\end{tikzcd}$$

there exists a unique function $g : L \to X$ such that $u_j = g \circ \lambda_j$ and $u_{j'} = g \circ \lambda_{j'}$ for every $j, j' \in Obj(\mathcal{J})$.

The proof of the following theorems can be found in standard literatures on homological algebra, for example see [27].

Theorem A.0.3. *Let $\mathcal{J}$ be an arbitrary small category and φ be a functor from $\mathcal{J}$ to the category of R-modules, for an arbitrary ring R. Then* $\operatorname{colim} \varphi$ *exists.*

Theorem A.0.4. *If the set of objects of a small category $\mathcal{J}$ is directly ordered, then* $\operatorname{colim}_{\mathcal{J}}$ *is an exact functor; i.e., if α, β and γ are three functors from $\mathcal{J}$ to a category $\mathcal{C}$ such that for every $j \in \mathcal{J}$ we have a short exact sequence*

$$\alpha(j) \rightarrowtail \beta(j) \twoheadrightarrow \gamma(j)$$

then we get a short exact sequence

$$\operatorname{colim}_{j\in\mathcal{J}} \alpha \rightarrowtail \operatorname{colim}_{j\in\mathcal{J}} \beta \twoheadrightarrow \operatorname{colim}_{j\in\mathcal{J}} \gamma.$$

Limit. Let $\mathcal{C}$ be a category, $\mathcal{J}$ a small category, $X \in Obj(\mathcal{C})$ and $\varphi : \mathcal{J} \to \mathcal{C}$ a functor. The family of morphisms $v_j : X \to \varphi(j)$, $j \in Obj(\mathcal{J})$ are called *cone from X to φ* if the triangles

$$\begin{tikzcd}
\varphi(j) \arrow[rr, "\varphi(v)"] & & \varphi(j') \\
& X \arrow[ul, "v_j"] \arrow[ur, "v_{j'}"'] &
\end{tikzcd}$$

commute for all $v : j \to j'$ in $\mathcal{J}$. Let

$$\operatorname{cone}(X, \varphi) := \{ \text{ cones from } X \text{ to } \varphi \},$$

then $\operatorname{cone}(X, \varphi)$ is a set as $\mathcal{J}$ is small. $\operatorname{cone}(-, \varphi)$ is a contravariant functor.

Definition A.0.5. *A* limit *for φ is given by*

- *an object $Z = \lim \varphi$ in $\mathcal{C}$ and*
- *a cone ρ in* $\mathrm{cone}(Z, \varphi)$

with the following property: for every cone u from an arbitrary element X in $\mathcal{C}$ to φ where the following diagram commutes

$$\begin{tikzcd}
 & & \varphi(j) \arrow[dd, "\varphi(v)"] \\
X \arrow[urr, "v_j"] \arrow[drr, "v_{j'}"'] & Z \arrow[ur, "\rho_j"] \arrow[dr, "\rho_{j'}"'] & \\
 & & \varphi(j')
\end{tikzcd}$$

there exists a unique function $g : X \to Z$ such that $v_j = \rho_j \circ g$ and $v_{j'} = \rho_{j'} \circ g$, for every $j, j' \in \mathcal{J}$.

The proof of the following theorems can be found in standard literatures on homological algebra, for example see [27].

Theorem A.0.6. *Let $\mathcal{J}$ be an arbitrary small category and φ a functor from $\mathcal{J}$ to the category of R-modules, for an arbitrary ring R. Then* $\lim \varphi$ *exists.*

Theorem A.0.7. *If the set of objects of a small category $\mathcal{J}$ is inversely ordered,* $\lim_{\mathcal{J}}$ *is a left-exact functor; i.e., if α, β and γ are three functors from a small category $\mathcal{J}$ to a category $\mathcal{C}$ such that for every $j \in \mathcal{J}$ we have a short exact sequence*

$$\alpha(j) \rightarrowtail \beta(j) \twoheadrightarrow \gamma(j)$$

then we get an exact sequence

$$\lim_{j\in\mathcal{J}} \alpha \rightarrowtail \lim_{j\in\mathcal{J}} \beta \to \lim_{j\in\mathcal{J}} \gamma \to \lim_{j\in\mathcal{J}}{}^{1} \alpha \to \lim_{j\in\mathcal{J}}{}^{1} \beta \twoheadrightarrow \lim_{j\in\mathcal{J}}{}^{1} \gamma.$$

Note that $\lim^1$ is defined as follows: Let $\bar{\alpha} : \prod_{j\in\mathcal{J}} \alpha(j) \to \prod_{j\in\mathcal{J}} \alpha(j)$ be defined by $\bar{\alpha}(x_j) = (x_j - \alpha(u)(x_{j+1}))$, for $u : j+1 \to j$. Then $\lim\limits_{j\in\mathcal{J}} \alpha(j) = \ker(\bar{\alpha})$ and $\lim\limits_{j\in\mathcal{J}}{}^{1} \alpha(j) = \mathrm{coker}(\bar{\alpha})$.

Appendix B

Homotopy Localization

We borrow the following from [11] and [16]: Let X be a CW-complex and $f : A \to B$ be a cofibration between two CW-complexes. We say that X is *f-local* if f induces a weak homotopy equivalence

$$f^X : B^X \to A^X$$

where B^X and A^X are given compact-open-K-topology. Let λ be a limit ordinal whose cardinality is greater than that of $[0,1] \times (A \sqcup B)$. Define $\mathbf{L}^0 X := X$ and assume that $\mathbf{L}^\eta X$ is given, where η is an ordinal less than the limit ordinal λ. Define $\mathbf{L}^{\eta+1} X := \mathbf{L}^1 \mathbf{L}^\eta X$, where $\mathbf{L}^1 X$ is the homotopy pushout in the following diagram

$$\begin{tikzcd}
A \times X^B \arrow[r, "f\times 1"] \arrow[d, "1\times f^*"'] & B \times X^B \arrow[d, dashed] \arrow[dr, bend left, "1\times f^*"] & \\
A \times X^A \arrow[r, dashed] \arrow[rr, bend right, "f\times 1"'] & A \times X^A \coprod_{A\times X^B} B \times X^B \arrow[r, dashed, "f\coprod f^*"] \arrow[d, "ev"'] & B \times X^A \arrow[d, squiggly] \\
 & X \arrow[r, squiggly] & \mathbf{L}^1 X
\end{tikzcd}$$

and ev is the evaluation map. If η is a limit ordinal, define $\mathbf{L}^\eta X := \operatorname{colim}_{\beta<\eta} \mathbf{L}^\beta X$. We define the *homotopy localization* of X to be the homotopy colimit of the following transfinite tower

$$X = \mathbf{L}^0 X \to \mathbf{L}^1 X \to \cdots \to \mathbf{L}^\eta X \to \mathbf{L}^{\eta+1} X \to \cdots$$

and we show it by $\mathbf{L}_f X$. We have the following facts

1. $\mathbf{L}_f$ is a homotopy functor and for every CW-complex X, $\mathbf{L}_f X$ is f-local.

2. For every ordinal $\eta < \lambda$, $\mathbf{L}^\eta X \to \mathbf{L}^{\eta+1} X$ is a cofibration. Therefore, we can define $\mathbf{L}_f X$ as the *colimit* of the tower of cofibrations above.

3. If the map $\varphi : X \to Y$ is a weak homotopy equivalence, then so is $\mathbf{L}_f(\varphi)$.

4. If X is f-local, then $X \to \mathbf{L}^1 X$ is a weak homotopy equivalence and thus so is $X \to \mathbf{L}^\eta X$, for every $\eta < \lambda$. For such X we have a weak homotopy equivalence $X \to \mathbf{L}_f X$.

Appendix C

5-Lemma Modulo a Class of Modules

The following is an exercise in [19] where $\mathcal{C}$ is a class of Abelian groups: See [19] for Hu's definition of a class of Abelina groups. Here, we prove it when $\mathcal{C}$ is a class of modules closed under extensions.

Lemma C.0.8. *(**5-Lemma mod** $\mathcal{C}$) Consider a commutative diagram with exact rows*

$$\begin{tikzcd} A_1 \arrow[r, "f_1"] \arrow[d, "\mathrm{H}_1"'] & A_2 \arrow[r, "f_2"] \arrow[d, "\mathrm{H}_2"'] & A_3 \arrow[r, "f_3"] \arrow[d, "\mathrm{H}_3"'] & A_4 \arrow[r, "f_4"] \arrow[d, "\mathrm{H}_4"'] & A_5 \arrow[d, "\mathrm{H}_5"'] \\ B_1 \arrow[r, "g_1"'] & B_2 \arrow[r, "g_2"'] & B_3 \arrow[r, "g_3"'] & B_4 \arrow[r, "g_4"'] & B_5. \end{tikzcd}$$

1. *If* H_2 *and* H_4 *are* $\mathcal{C}$*-monomorphisms and* H_1 *is a* $\mathcal{C}$*-epimorphism, then* H_3 *is a* $\mathcal{C}$*-monomorphism.*

2. *If* H_2 *and* H_4 *are* $\mathcal{C}$*-epimorphisms and* H_5 *is a* $\mathcal{C}$*-monomorphism, then* H_3 *is* $\mathcal{C}$*-epimorphism.*

3. *If* H_1*,* H_2*,* H_4 *and* H_5 *are* $\mathcal{C}$*-isomorphisms, then so is* H_3*.*

Proof. 1. Since $f_3(\ker(\mathrm{H}_3)) \subset \ker \mathrm{H}_4$ and $\ker(\mathrm{H}_4) \in \mathcal{C}$, then $f_3(\ker(\mathrm{H}_3)) \in \mathcal{C}$. Then if we show $f_3|_{\ker(\mathrm{H}_3)}$ by i_3, we have the following short exact sequence

$$\ker(i_3) \rightarrowtail \ker(\mathrm{H}_3) \xrightarrow{i_3} \mathrm{im}(i_3)$$

and $\mathrm{im}(i_3) \in \mathcal{C}$. If we show that $\ker(i_3) \in \mathcal{C}$, then $\ker(\mathrm{H}_3) \in \mathcal{C}$ and H_3 is a $\mathcal{C}$-monomorphism:

Since $\ker(i_3) \subset \ker(f_3)$ and $\ker(f_3) = \mathrm{im}(f_2)$, then for some $\bar{A}_2 \subset A_2$ we have a surjection $\bar{A}_2 \twoheadrightarrow \ker(i_3)$. Since

$$g_2(\mathrm{H}_2(\bar{A}_2)) = \mathrm{H}_3(f_2(\bar{A}_2)) = \mathrm{H}_3(\ker(i_3)) = \{0\}.$$

So $\mathrm{H}_2(\bar{A}_2) \subset \ker(g_2)$. Since $\ker(g_2) = \mathrm{im}(g_1)$, then for some $\bar{B}_1 \subset B_1$ we have a surjection $\bar{B}_1 \twoheadrightarrow \mathrm{H}_2(\bar{A}_2)$. There is an injection

$$\frac{\bar{B}_1 + \mathrm{im}(\mathrm{H}_1)}{\mathrm{im}(\mathrm{H}_1)} \rightarrowtail \frac{B_1}{\mathrm{im}(\mathrm{H}_1)}$$

and since $\dfrac{B_1}{\mathrm{im}(\mathrm{H}_1)} = \mathrm{coker}(\mathrm{H}_1) \in \mathcal{C}$ we have $\dfrac{\bar{B}_1 + \mathrm{im}(\mathrm{H}_1)}{\mathrm{im}(\mathrm{H}_1)} \in \mathcal{C}$ and since

$$\frac{\bar{B}_1 + \mathrm{im}(\mathrm{H}_1)}{\mathrm{im}(\mathrm{H}_1)} \cong \frac{\bar{B}_1}{\bar{B}_1 \cap \mathrm{im}(\mathrm{H}_1)}$$

then $\dfrac{\bar{B}_1}{\bar{B}_1 \cap \mathrm{im}(\mathrm{H}_1)} \in \mathcal{C}$. We have the following cases:

(a) $\bar{B}_1 \cap \mathrm{im}(\mathrm{H}_1) = 0$: In this case, $\bar{B}_1 \in \mathcal{C}$ and hence $\mathrm{H}_2(\bar{A}_2) \in \mathcal{C}$. Since $\ker(\bar{A}_2 \to \mathrm{H}_2(\bar{A}_2)) \subset \ker(\mathrm{H}_2)$ and $\ker(\mathrm{H}_2) \in \mathcal{C}$, then $\ker(\bar{A}_2 \to \mathrm{H}_2(\bar{A}_2))$ and hence $\bar{A}_2$ are in $\mathcal{C}$. Therefore, by the surjection $\bar{A}_2 \twoheadrightarrow \ker(i_3)$ we have $\ker(i_3) \in \mathcal{C}$.

(b) $\bar{B}_1 \cap \mathrm{im}(\mathrm{H}_1) \neq 0$: In this case, for some $\bar{A}_1 \subset A_1$ we have a surjection $\bar{A}_1 \twoheadrightarrow \bar{B}_1 \cap \mathrm{im}(\mathrm{H}_1)$. We also know that $f_1(\bar{A}_1) \subseteq \ker(f_2)$. So we have an epimorphism $\bar{A}_2 + f_1(\bar{A}_1) \twoheadrightarrow \ker(i_3)$ and hence an epimorphism

$$\frac{\bar{A}_2 + f_1(\bar{A}_1)}{f_1(\bar{A}_1)} \twoheadrightarrow \ker(i_3). \tag{C.1}$$

We have

$$\mathrm{H}_2(f_1(\bar{A}_1)) = g_1(\mathrm{H}_1(\bar{A}_1)) = g_1(\bar{B}_1 \cap \mathrm{im}(\mathrm{H}_1)) \subseteq \mathrm{H}_2(\bar{A}_2).$$

Therefore, $\mathrm{H}_2(\bar{A}_2 + f_1(\bar{A}_1)) = \mathrm{H}_2(\bar{A}_2)$. There is an epimorphism

$$\frac{\bar{B}_1}{\bar{B}_1 + \mathrm{im}(\mathrm{H}_1)} \twoheadrightarrow \frac{g(\bar{B}_1)}{g(\bar{B}_1 + \mathrm{im}(\mathrm{H}_1))}$$

and since $\dfrac{\bar{B}_1}{\bar{B}_1 + \mathrm{im}(\mathrm{H}_1)} \in \mathcal{C}$, we have $\dfrac{g(\bar{B}_1)}{g(\bar{B}_1 + \mathrm{im}(\mathrm{H}_1))} \in \mathcal{C}$. Since we have

$$\frac{g(\bar{B}_1)}{g(\bar{B}_1 + \mathrm{im}(\mathrm{H}_1))} = \frac{\mathrm{H}_2(\bar{A}_2)}{\mathrm{H}_2(f_1(\bar{A}_1))}$$

we also have $\dfrac{\mathrm{H}_2(\bar{A}_2)}{\mathrm{H}_2(f_1(\bar{A}_1))} \in \mathcal{C}$. In the following diagram

$$\begin{tikzcd}
f_1(\bar{A}_1) \cap \ker(\mathrm{H}_2) \arrow[r, tail] \arrow[d, tail] & f_1(\bar{A}_1) \arrow[r, two heads] \arrow[d, tail] & \mathrm{H}_2(f_1(\bar{A}_1)) \arrow[d, tail] \\
(\bar{A}_2 + f_1(\bar{A}_1)) \cap \ker(\mathrm{H}_2) \arrow[r, tail] \arrow[d, two heads] & \bar{A}_2 + f_1(\bar{A}_1) \arrow[r, two heads] \arrow[d, two heads] & \mathrm{H}_2(\bar{A}_2) \arrow[d, two heads] \\
\dfrac{(\bar{A}_2 + f_1(\bar{A}_1)) \cap \ker(\mathrm{H}_2)}{f_1(\bar{A}_1) \cap \ker(\mathrm{H}_2)} \arrow[r, tail] & \dfrac{\bar{A}_2 + f_1(\bar{A}_1)}{f_1(\bar{A}_1)} \arrow[r, two heads] & \dfrac{\mathrm{H}_2(\bar{A}_2)}{\mathrm{H}_2(f_1(\bar{A}_1))}
\end{tikzcd}$$

since $\ker(\mathrm{H}_2) \in \mathcal{C}$, then $\dfrac{(\bar{A}_2 + f_1(\bar{A}_1)) \cap \ker(\mathrm{H}_2)}{f_1(\bar{A}_1) \cap \ker(\mathrm{H}_2)} \in \mathcal{C}$ and hence the last row of the diagram shows that $\dfrac{\bar{A}_2 + f_1(\bar{A}_1)}{f_1(\bar{A}_1)} \in \mathcal{C}$. Therefore, by epimorphism (C.1) we have $\ker(i_3) \in \mathcal{C}$ and we are done.

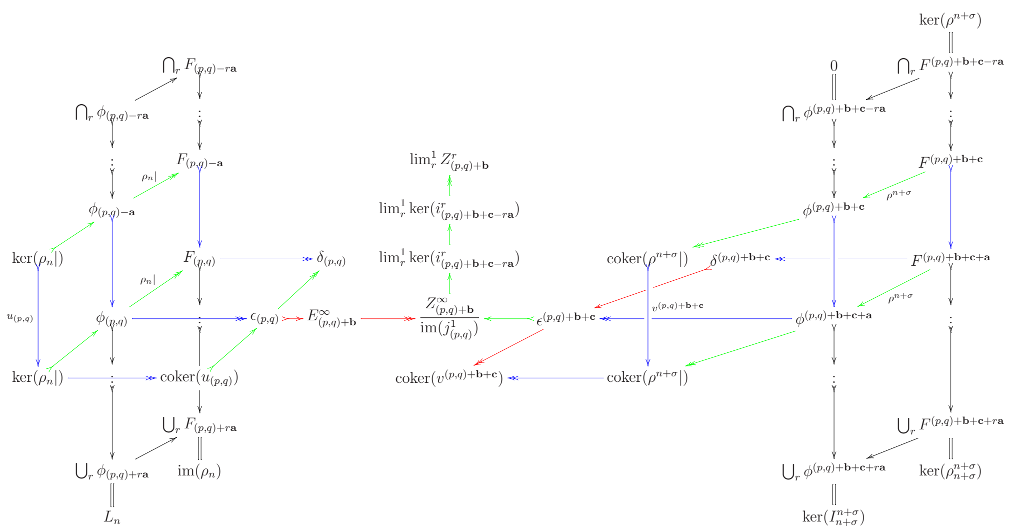
2. It can be proved similarly.

3. It follows from parts 1 and 2.

$\square$

Appendix D

Diagrams



Generalized E^∞-Distribution Diagram

$$
\begin{tikzcd}[column sep=small]
L_n \arrow[d, two heads] & \lim_r \phi_{(p,q)-r\mathbf{a}} \arrow[l, tail] & & & & \ker(L^{n+\sigma} \to L_{n+\sigma}) \arrow[d, "\cong"] \\
\lim_r \dfrac{L_n}{\phi_{(p,q)-r\mathbf{a}}} \arrow[r, two heads] \arrow[d, two heads] & \lim^1_r \phi_{(p,q)-r\mathbf{a}} & & \lim^1_t Z^t_{(p,q)-r\mathbf{a}+\mathbf{b}} & & \lim_r \dfrac{\ker(L^{n+\sigma} \to L_{n+\sigma})}{\phi^{(p,q)-r\mathbf{a}+\mathbf{b}+\mathbf{c}}} \arrow[d, two heads] \\
\vdots \arrow[d, two heads] & & & \lim^1_t \ker(i^{t+1}_{(p,q)+\mathbf{b}+\mathbf{c}-(r+t)\mathbf{a}}) \arrow[u] & & \vdots \arrow[d, two heads] \\
\dfrac{L_n}{\phi_{(p,q)-(r+2)\mathbf{a}}} \arrow[d, two heads] & & & \lim^1_t \ker(i^{t}_{(p,q)+\mathbf{b}+\mathbf{c}-(r+t)\mathbf{a}}) \arrow[u] & & \dfrac{\ker(L^{n+\sigma} \to L_{n+\sigma})}{\phi^{(p,q)-(r+1)\mathbf{a}+\mathbf{b}+\mathbf{c}}} \arrow[d, two heads] \\
\dfrac{L_n}{\phi_{(p,q)-(r+1)\mathbf{a}}} \arrow[d, two heads] & \dfrac{\phi_{(p,q)-r\mathbf{a}}}{\phi_{(p,q)-(r+1)\mathbf{a}}} \arrow[l, tail] \arrow[r, tail] & E^\infty_{(p,q)-r\mathbf{a}+\mathbf{b}} \arrow[r, two heads] & \dfrac{Z^\infty_{(p,q)-r\mathbf{a}+\mathbf{b}}}{\mathrm{im}(j^1_{(p,q)-r\mathbf{a}})} \arrow[u] & \dfrac{\phi^{(p,q)-(r-1)\mathbf{a}+\mathbf{b}+\mathbf{c}}}{\phi^{(p,q)-r\mathbf{a}+\mathbf{b}+\mathbf{c}}} \arrow[l, tail] \arrow[r, tail] & \dfrac{\ker(L^{n+\sigma} \to L_{n+\sigma})}{\phi^{(p,q)-r\mathbf{a}+\mathbf{b}+\mathbf{c}}} \arrow[d, two heads] \\
\dfrac{L_n}{\phi_{(p,q)-r\mathbf{a}}} \arrow[d, two heads] & \dfrac{\phi_{(p,q)-(r-1)\mathbf{a}}}{\phi_{(p,q)-r\mathbf{a}}} \arrow[l, tail] \arrow[r, tail] & E^\infty_{(p,q)-(r-1)\mathbf{a}+\mathbf{b}} \arrow[r, two heads] & \dfrac{Z^\infty_{(p,q)-(r-1)\mathbf{a}+\mathbf{b}}}{\mathrm{im}(j^1_{(p,q)-(r-1)\mathbf{a}})} & \dfrac{\phi^{(p,q)-(r-2)\mathbf{a}+\mathbf{b}+\mathbf{c}}}{\phi^{(p,q)-(r-1)\mathbf{a}+\mathbf{b}+\mathbf{c}}} \arrow[l, tail] \arrow[r, tail] & \dfrac{\ker(L^{n+\sigma} \to L_{n+\sigma})}{\phi^{(p,q)-(r-1)\mathbf{a}+\mathbf{b}+\mathbf{c}}} \arrow[d, two heads] \\
\vdots \arrow[d, two heads] & & & & & \vdots \arrow[d, two heads] \\
\dfrac{L_n}{\phi_{(p,q)-\mathbf{a}}} \arrow[d, two heads] & & & & & \dfrac{\ker(L^{n+\sigma} \to L_{n+\sigma})}{\phi^{(p,q)+(s-1)\mathbf{a}+\mathbf{b}+\mathbf{c}}} \arrow[d, two heads] \\
\dfrac{L_n}{\phi_{(p,q)}} \arrow[d, two heads] & & & & & \dfrac{\ker(L^{n+\sigma} \to L_{n+\sigma})}{\phi^{(p,q)+s\mathbf{a}+\mathbf{b}+\mathbf{c}}} \arrow[d, two heads] \\
0 & & & & & 0
\end{tikzcd}
$$

E^∞-Distribution Diagram of an Eventually Stable Exact Couple

$$
\begin{tikzcd}
 & & L^n \arrow[d, "\cong"] \\
\lim_t^1 Z^t_{(p,q)-r\mathbf{a}-\mathbf{c}} & & \lim_r \dfrac{L^n}{\phi^{(p,q)-r\mathbf{a}}} \arrow[d, two heads] \\
\lim_t^1 \ker(i^{t+1}_{(p,q)-(r+t)\mathbf{a}}) \arrow[u, two heads] & & \vdots \arrow[d, two heads] \\
\lim_t^1 \ker(i^{t}_{(p,q)-(r+t)\mathbf{a}}) \arrow[u] & & \dfrac{L^n}{\phi^{(p,q)-(r+1)\mathbf{a}}} \arrow[d, two heads] \\
E^\infty_{(p,q)-r\mathbf{a}-\mathbf{c}} \arrow[u] & \dfrac{\phi^{(p,q)-(r-1)\mathbf{a}}}{\phi^{(p,q)-r\mathbf{a}}} \arrow[l, tail] \arrow[r, tail] & \dfrac{L^n}{\phi^{(p,q)-r\mathbf{a}}} \arrow[d, two heads] \\
E^\infty_{(p,q)-(r-1)\mathbf{a}-\mathbf{c}} & \dfrac{\phi^{(p,q)-(r-2)\mathbf{a}}}{\phi^{(p,q)-(r-1)\mathbf{a}}} \arrow[l, tail] \arrow[r, tail] & \dfrac{L^n}{\phi^{(p,q)-(r-1)\mathbf{a}}} \arrow[d, two heads] \\
 & & \vdots \arrow[d, two heads] \\
 & & \dfrac{L^n}{\phi^{(p,q)-\mathbf{a}}} \arrow[d, two heads] \\
 & & \dfrac{L^n}{\phi^{(p,q)}} \arrow[d, two heads] \\
 & & 0
\end{tikzcd}
$$

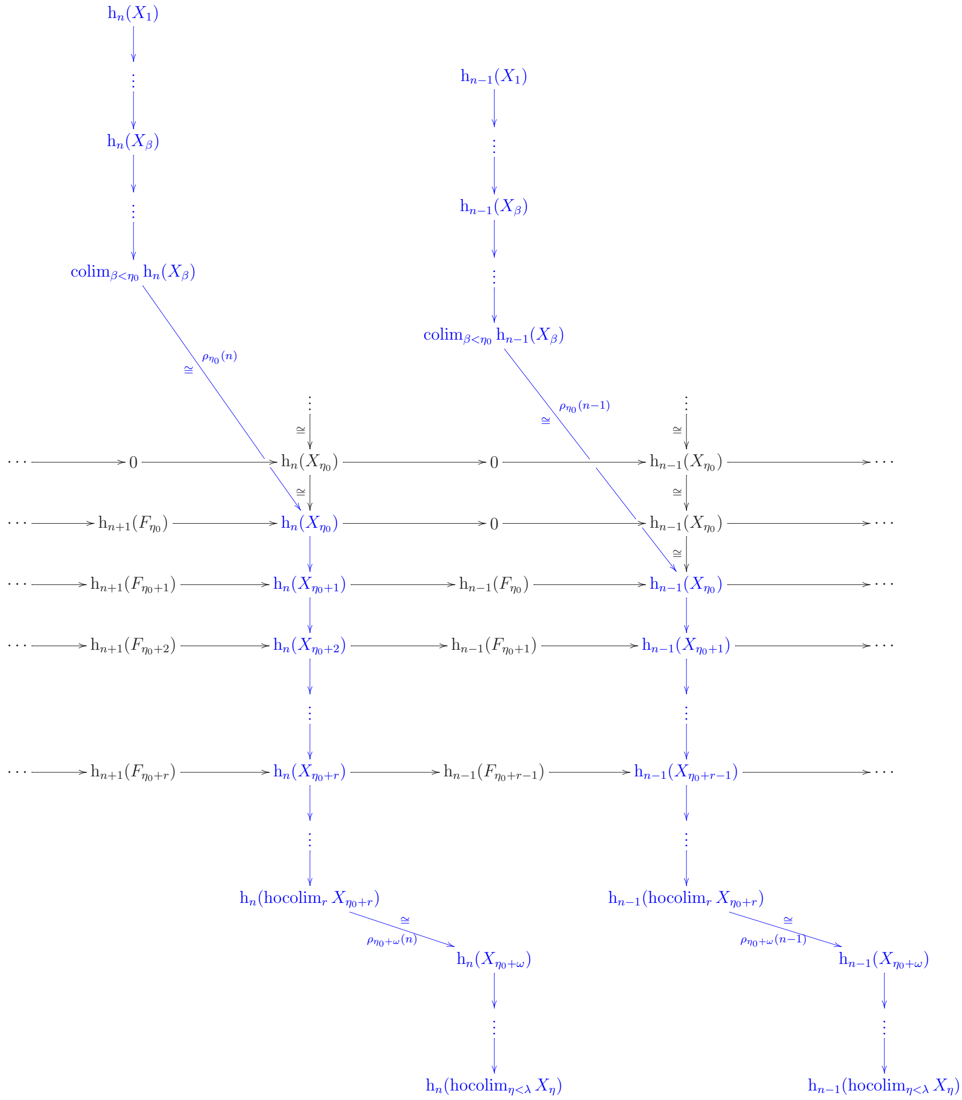
E^∞-Distribution Diagram of an Eventually Vanishing Exact Couple

$$
\begin{tikzcd}
\mathrm{H}_1(n) \arrow[d] & & & & \\
\vdots \arrow[d] & & & & \\
\mathrm{H}_\beta(n) \arrow[d] & & & & \\
\vdots \arrow[d] & & & & \\
\operatorname{colim}_{\beta<\eta_0} \mathrm{H}_\beta(n) \arrow[ddr, "\rho_{\eta_0}(n)"'] & & & & \\
 & & \ddots \arrow[dr] & \vdots \arrow[d, equal] & \\
 & \mathrm{H}_{\eta_0}(n) \arrow[d] \arrow[rr, bend left, "\beta^{u,v}"] & \ddots \arrow[dr] & D^1_{(u,v)} \arrow[d] \arrow[dr] \arrow[ll, bend left, "\alpha_{(u,v)}"] & \\
 & \mathrm{H}_{\eta_0+1}(n) \arrow[d] \arrow[rr, bend left, "\beta^{u,v+\mathbf{a}}"] & & D^1_{(u,v)+\mathbf{a}} \arrow[d] \arrow[dr] \arrow[ll, bend left, "\alpha_{(u,v)+\mathbf{a}}"] & \ddots \\
 & \vdots \arrow[d] & \ddots \arrow[dr] & \vdots \arrow[d] & \ddots \\
 & \mathrm{H}_{\eta_0+r}(n) \arrow[d] \arrow[rr, bend left, "\beta^{u,v+r\mathbf{a}}"] & & D^1_{(u,v)+r\mathbf{a}} \arrow[d] \arrow[dr] \arrow[ll, bend left, "\alpha_{(u,v)+r\mathbf{a}}"] & \\
 & \vdots \arrow[d] & & \vdots \arrow[d] & \ddots \\
 & \operatorname{colim}_r \mathrm{H}_{\eta_0+r}(n) \arrow[dr, "\rho_{\eta_0+\omega}(n)"'] \arrow[rr, bend left, "\beta^{\eta_0+\omega}"] & & L_n \arrow[ll, bend left, "\alpha_{\eta_0+\omega}"] & \\
 & & \mathrm{H}_{\eta_0+\omega}(n) \arrow[d] & & \\
 & & \vdots \arrow[d] & & \\
 & & \mathrm{H}_\lambda(n) & &
\end{tikzcd}
$$

Matching Originally Stable Exact Couples to a Directed Tower

$$
\begin{tikzcd}[column sep=small]
F_0(n-\sigma) \arrow[d, tail] & & & & & & F^0(n) \arrow[d, tail] \\
\vdots \arrow[d, tail] & & & & & & \vdots \arrow[d, tail] \\
F_\beta(n-\sigma) \arrow[d, tail] & & & & & & F^\beta(n) \arrow[d, tail] \\
\vdots \arrow[d, tail] & & & & & & \vdots \arrow[d, tail] \\
\operatorname{colim}_{\beta<\eta_0} F_\beta(n-\sigma) \arrow[d, tail] & & & & & & \operatorname{colim}_{\beta<\eta_0} F^\beta(n) \arrow[d, tail] \\
F_{\eta_0}(n-\sigma) \arrow[r, two heads] \arrow[d, tail] & \dfrac{F_{\eta_0}(n-\sigma)}{\operatorname{colim}_{\beta<\eta_0} F_\beta(n-\sigma)} & \operatorname{coker}(\rho_{\eta_0}(n-\sigma)) \arrow[l, two heads] & & \ker(\rho_{\eta_0}(n)) & \dfrac{F^{\eta_0}(n)}{\operatorname{colim}_{\beta<\eta_0} F^\beta(n)} \arrow[l, tail] & F^{\eta_0}(n) \arrow[l, two heads] \arrow[d, tail] \\
\vdots \arrow[d, tail] & & & & & & \vdots \arrow[d, tail] \\
F_{\eta_0+r-1}(n-\sigma) \arrow[d, tail] & & & & & & F^{\eta_0+r}(n) \arrow[d, tail] \\
F_{\eta_0+r}(n-\sigma) \arrow[r, two heads] \arrow[d, tail] & \dfrac{F_{\eta_0+r}(n-\sigma)}{F_{\eta_0+r-1}(n-\sigma)} & \epsilon_{(u,v)+r\mathbf{a}-\mathbf{b}-\mathbf{c}} \arrow[l, two heads] \arrow[r, tail] & E^\infty_{(u,v)+r\mathbf{a}-\mathbf{c}} \arrow[r, tail] & \epsilon^{u,v+r\mathbf{a}} & \dfrac{F^{\eta_0+r+1}(n)}{F^{\eta_0+r}(n)} \arrow[l, tail] & F^{\eta_0+r+1}(n) \arrow[l, two heads] \arrow[d, tail] \\
\vdots \arrow[d, tail] & & & & & & \vdots \arrow[d, tail] \\
\operatorname{colim}_{\eta<\lambda} F_\eta(n-\sigma) \arrow[d, tail] & & & & & & \operatorname{colim}_{\eta<\lambda} F^\eta(n) \arrow[d, tail] \\
F_\lambda(n-\sigma) \arrow[r, two heads] \arrow[d, equal] & \dfrac{F_\lambda(n-\sigma)}{\operatorname{colim}_{\eta<\lambda} F_\eta(n-\sigma)} & \operatorname{coker}(\rho_\lambda(n-\sigma)) \arrow[l, two heads] & & \ker(\rho_\lambda(n)) & \dfrac{F^\lambda(n)}{\operatorname{colim}_{\eta<\lambda} F^\eta(n)} \arrow[l, tail] & F^\lambda(n) \arrow[l, two heads] \arrow[d, equal] \\
\mathrm{H}_\lambda(n-\sigma) & & & & & & \ker(\mathrm{H}_1(n) \to \mathrm{H}_\lambda(n))
\end{tikzcd}
$$

E^∞-Distribution Diagram Corresponding to Matching Originally Stable Exact Couples to a Directed Tower



Transfinite Tower of Cofibers

$$
\begin{tikzcd}
\mathrm{H}_1(n) \arrow[d] & & & & & \\
\vdots \arrow[d] & & & & & \\
\mathrm{H}_\beta(n) \arrow[d] & & & & & \\
\vdots \arrow[d] & & & & & \\
\operatorname{colim}_{\beta<\eta_0} \mathrm{H}_\beta(n) \arrow[ddr, "\rho_{\eta_0}(n)"'] & & & \ddots \arrow[dr] & \vdots \arrow[d, equal] & \\
& & & \ddots \arrow[dr] & 0 \arrow[d] \arrow[dr] & \\
& \mathrm{H}_{\eta_0}(n) \arrow[d] & & \ddots \arrow[dr] & D^1_{(u,v)} \arrow[d] \arrow[dr] \arrow[lll, bend left, "\alpha_{(u,v)}"'] & \ddots \\
& \mathrm{H}_{\eta_0+1}(n) \arrow[d] & & & D^1_{(u,v)+\mathbf{a}} \arrow[d] \arrow[dr] \arrow[lll, bend left, "\alpha_{(u,v)+\mathbf{a}}"'] & \ddots \\
& \vdots \arrow[d] & & \ddots \arrow[dr] & \vdots \arrow[d] & \ddots \\
& \mathrm{H}_{\eta_0+r}(n) \arrow[d] & & & D^1_{(u,v)+r\mathbf{a}} \arrow[d] \arrow[dr] \arrow[lll, bend left, "\alpha_{(u,v)+r\mathbf{a}}"'] & \\
& \vdots \arrow[d] & & & \vdots \arrow[d] & \ddots \\
& \operatorname{colim}_r \mathrm{H}_{\eta_0+r}(n) \arrow[dr, "\rho_{\eta_0+\omega}(n)"'] & & & L_n \arrow[lll, bend left, "\alpha_{\eta_0+\omega}"'] & \\
& & \mathrm{H}_{\eta_0+\omega}(n) \arrow[d] & & & \\
& & \vdots \arrow[d] & & & \\
& & \mathrm{H}_\lambda(n) & & &
\end{tikzcd}
$$

Matching Originally Vanishing Exact Couples to a Directed Tower

$$\begin{tikzcd}
F_0(n) \arrow[d, tail] & & \\
\vdots \arrow[d, tail] & & \\
F_\beta(n) \arrow[d, tail] & & \\
\vdots \arrow[d, tail] & & \\
\operatorname{colim}_{\beta<\eta_0} F_\beta(n) \arrow[d, tail] & & \\
F_{\eta_0}(n) \arrow[r, two heads] \arrow[d, tail] & \dfrac{F_{\eta_0}(n)}{\underline{F}_{\eta_0}(n)} & \operatorname{coker}(\rho_{\eta_0}(n)) \arrow[l, two heads] \\
\vdots \arrow[d, tail] & & \\
F_{\eta_0+r-1}(n) \arrow[d, tail] & & \\
F_{\eta_0+r}(n) \arrow[r, two heads] \arrow[d, tail] & \dfrac{F_{\eta_0+r}(n)}{F_{\eta_0+r-1}(n)} & E^\infty_{(u,v)+r\mathbf{a}+\mathbf{b}} \arrow[l, two heads] \\
F_{\eta_0+r+1}(n) \arrow[r, two heads] \arrow[d, tail] & \dfrac{F_{\eta_0+r+1}(n)}{F_{\eta_0+r}(n)} & E^\infty_{(u,v)+(r+1)\mathbf{a}+\mathbf{b}} \arrow[l, two heads] \\
\vdots \arrow[d, tail] & & \\
\operatorname{colim}_{\eta<\lambda} F_\eta(n) \arrow[d, tail] & & \\
F_\lambda(n) \arrow[r, two heads] \arrow[d, equal] & \dfrac{F_\lambda(n)}{\operatorname{colim}_{\eta<\lambda} F_\eta(n)} & \operatorname{coker}(\rho_\lambda(n)) \arrow[l, two heads] \\
\mathrm{H}_\lambda(n) & &
\end{tikzcd}$$

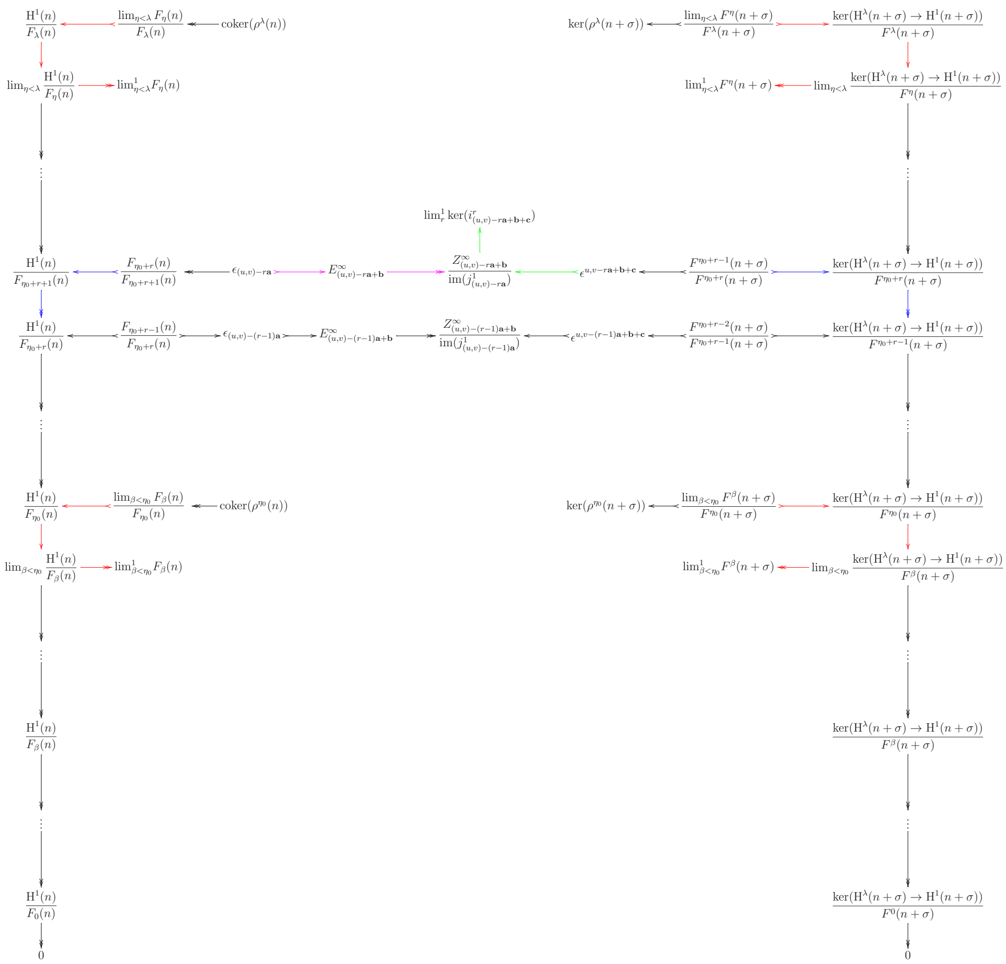
E^∞-Distribution Diagram Corresponding to Matching Originally Vanishing Exact Couples to a Directed Tower

$$
\begin{tikzcd}
 & \mathrm{h}_n(X_1) \arrow[d] & & & & \\
 & \vdots \arrow[d] & & \mathrm{h}_{n-1}(X_1) \arrow[d] & & \\
 & \mathrm{h}_n(X_\beta) \arrow[d] & & \vdots \arrow[d] & & \\
 & \vdots \arrow[d] & & \mathrm{h}_{n-1}(X_\beta) \arrow[d] & & \\
 & \operatorname{colim}_{\beta<\eta_0} \mathrm{h}_n(X_\beta) \arrow[ddddr, "\rho_{\eta_0}(n)", "\cong"'] & & \vdots \arrow[d] & & \\
 & & & \operatorname{colim}_{\beta<\eta_0} \mathrm{h}_{n-1}(X_\beta) \arrow[ddddr, "\rho_{\eta_0}(n-1)", "\cong"'] & & \\
 & & \vdots \arrow[d] & & \vdots \arrow[d] & \\
\cdots \arrow[r, "\cong"] & \mathrm{h}_{n+1}(F_{\eta_0}) \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathrm{h}_{n+1}(F_{\eta_0+1}) \arrow[r] & \mathrm{h}_n(X_{\eta_0}) \arrow[r, "\cong"] \arrow[d] & \mathrm{h}_n(F_{\eta_0}) \arrow[r] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathrm{h}_{n+1}(F_{\eta_0+2}) \arrow[r] & \mathrm{h}_n(X_{\eta_0+1}) \arrow[r] \arrow[d] & \mathrm{h}_n(F_{\eta_0+1}) \arrow[r] & \mathrm{h}_{n-1}(X_{\eta_0}) \arrow[r, "\cong"] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathrm{h}_{n+1}(F_{\eta_0+3}) \arrow[r] & \mathrm{h}_n(X_{\eta_0+2}) \arrow[r] \arrow[d] & \mathrm{h}_n(F_{\eta_0+2}) \arrow[r] & \mathrm{h}_{n-1}(X_{\eta_0+1}) \arrow[r] \arrow[d] & \cdots \\
 & & \vdots \arrow[d] & & \vdots \arrow[d] & \\
\cdots \arrow[r] & \mathrm{h}_{n+1}(F_{\eta_0+r+1}) \arrow[r] & \mathrm{h}_n(X_{\eta_0+r}) \arrow[r] \arrow[d] & \mathrm{h}_n(F_{\eta_0+r}) \arrow[r] & \mathrm{h}_{n-1}(X_{\eta_0+r-1}) \arrow[r] \arrow[d] & \cdots \\
 & & \vdots \arrow[d] & & \vdots \arrow[d] & \\
 & & \mathrm{h}_n(\operatorname{hocolim}_r X_{\eta_0+r}) \arrow[dr, "\cong", "\rho_{\eta_0}(n)"'] & & \mathrm{h}_{n-1}(\operatorname{hocolim}_r X_{\eta_0+r}) \arrow[dr, "\cong", "\rho_{\eta_0}(n-1)"'] & \\
 & & & \mathrm{h}_n(X_{\eta_0+\omega}) \arrow[d] & & \mathrm{h}_{n-1}(X_{\eta_0+\omega}) \arrow[d] \\
 & & & \vdots \arrow[d] & & \vdots \arrow[d] \\
 & & & \mathrm{h}_n(X_\theta) \arrow[d] & & \mathrm{h}_{n-1}(X_\theta) \arrow[d] \\
 & & & \vdots \arrow[d] & & \vdots \arrow[d] \\
 & & & \mathrm{h}_n(\operatorname{hocolim}_{\eta<\lambda} X_\eta) & & \mathrm{h}_{n-1}(\operatorname{hocolim}_{\eta<\lambda} X_\eta)
\end{tikzcd}
$$

Transfinite Tower of Cofibrations

$$
\begin{tikzcd}
\mathrm{H}^{\lambda}(n) \arrow[d] & & & & & \\
\vdots \arrow[d] & & & & & \\
\mathrm{H}^{\eta_0+\omega}(n) \arrow[dr, "\rho^{\eta_0+\omega}(n)"'] & & & & & \\
& \lim_r \mathrm{H}^{\eta_0+r}(n) \arrow[rrr, bend left, "\beta^{\eta_0+\omega}"] \arrow[d] & & & L^n \arrow[lll, bend left, "\alpha_{\eta_0+\omega}"] \arrow[d] & \\
& \vdots \arrow[d] & & \ddots \arrow[dr] & \vdots \arrow[d] & \\
& \mathrm{H}^{\eta_0+r}(n) \arrow[rrr, bend left, "\beta^{u,v-r\mathbf{a}}"] \arrow[d] & & & D^1_{(u,v)-r\mathbf{a}} \arrow[lll, bend left, "\alpha_{(u,v)-r\mathbf{a}}"] \arrow[d] \arrow[dr] & \\
& \vdots \arrow[d] & & \ddots \arrow[dr] & \vdots \arrow[d] & \ddots \\
& \mathrm{H}^{\eta_0+1}(n) \arrow[rrr, bend left, "\beta^{u,v-\mathbf{a}}"] \arrow[d] & & \ddots \arrow[dr] & D^1_{(u,v)-\mathbf{a}} \arrow[lll, bend left, "\alpha_{(u,v)-\mathbf{a}}"] \arrow[d] \arrow[dr] & \\
& \mathrm{H}^{\eta_0}(n) \arrow[rrr, bend left, "\beta^{u,v}"] \arrow[ddr, "\rho^{\eta_0}(n)"'] & & & D^1_{(u,v)} \arrow[lll, bend left, "\alpha_{(u,v)}"] \arrow[d, equal] \arrow[dr] & \ddots \\
& & & & \vdots & \ddots \\
& & \lim_{\beta<\eta_0} \mathrm{H}^{\beta}(n) \arrow[d] & & & \\
& & \vdots \arrow[d] & & & \\
& & \mathrm{H}^{\beta}(n) \arrow[d] & & & \\
& & \vdots \arrow[d] & & & \\
& & \mathrm{H}^{1}(n) & & &
\end{tikzcd}
$$

Matching Eventually Stable Exact Couples to an Inverse Tower



E^{∞}-Distribution Diagram Corresponding to Matching Eventually Stable Exact Couples to an Inverse Tower

$$
\begin{tikzcd}[column sep=small]
 & \lim_{\eta<\lambda} \pi_n(X_\eta) \arrow[d] & & \lim_{\eta<\lambda} \pi_{n-1}(X_\eta) \arrow[d] & & \\
 & \vdots \arrow[d] & & \vdots \arrow[d] & & \\
 & \lim_r \pi_n(X_{\eta_0+r}) \arrow[d] & & \lim_r \pi_{n-1}(X_{\eta_0+r}) \arrow[d] & & \\
 & \vdots \arrow[d] & & \vdots \arrow[d] & & \\
\cdots \arrow[r] & \pi_n(X_{\eta_0+r}) \arrow[r] \arrow[d] & \pi_n(F_{\eta_0+r}) \arrow[r] & \pi_{n-1}(X_{\eta_0+r+1}) \arrow[r] \arrow[d] & \pi_{n-1}(F_{\eta_0+r+1}) \arrow[r] & \cdots \\
\cdots \arrow[r] & \pi_n(X_{\eta_0+r-1}) \arrow[r] \arrow[d] & \pi_n(F_{\eta_0+r-1}) \arrow[r] & \pi_{n-1}(X_{\eta_0+r}) \arrow[r] \arrow[d] & \pi_{n-1}(F_{\eta_0+r}) \arrow[r] & \cdots \\
 & \vdots \arrow[d] & & \vdots \arrow[d] & & \\
\cdots \arrow[r] & \pi_n(X_{\eta_0}) \arrow[r] \arrow[d, "\cong"'] \arrow[dddr] & \pi_n(F_{\eta_0}) \arrow[r] & \pi_{n-1}(X_{\eta_0+1}) \arrow[r] \arrow[d] & \pi_{n-1}(F_{\eta_0+1}) \arrow[r] & \cdots \\
\cdots \arrow[r] & \pi_n(X_{\eta_0}) \arrow[r] \arrow[d, "\cong"'] & 0 \arrow[r] & \pi_{n-1}(X_{\eta_0}) \arrow[r] \arrow[d, "\cong"'] \arrow[dddr, "\rho^{\eta_0}(n-1)"'] & \pi_{n-1}(F_{\eta_0}) \arrow[r] & \cdots \\
\cdots \arrow[r] & \pi_n(X_{\eta_0}) \arrow[r] \arrow[d, "\cong"'] & 0 \arrow[r] & \pi_{n-1}(X_{\eta_0}) \arrow[r] \arrow[d, "\cong"'] & 0 \arrow[r] & \cdots \\
 & \vdots & \lim_{\beta<\eta_0} \pi_n(X_\beta) \arrow[d] & \vdots & & \\
 & & \vdots \arrow[d] & & \lim_{\beta<\eta_0} \pi_{n-1}(X_\beta) \arrow[d] & \\
 & & \pi_n(X_\beta) \arrow[d] & & \vdots \arrow[d] & \\
 & & \vdots \arrow[d] & & \pi_{n-1}(X_\beta) \arrow[d] & \\
 & & \pi_n(X_1) & & \vdots \arrow[d] & \\
 & & & & \pi_{n-1}(X_1) &
\end{tikzcd}
$$

Transfinite Tower of Fibers

$$
\begin{tikzcd}
\mathrm{H}^{\lambda}(n) \arrow[d] &&&&& \\
\vdots \arrow[d] &&&&& \\
\mathrm{H}^{\eta_0+\omega}(n) \arrow[dr, "\rho^{\eta_0+\omega}(n)"'] &&&&& \\
& \lim_r \mathrm{H}^{\eta_0+r}(n) \arrow[rrr, bend left, "\beta^{\eta_0+\omega}"] \arrow[d] &&& L^n \arrow[d] & \\
& \vdots \arrow[d] && \ddots \arrow[dr] & \vdots \arrow[d] & \\
& \mathrm{H}^{\eta_0+r}(n) \arrow[rrr, bend left, "\beta^{u,v-r\mathbf{a}}"] \arrow[d] &&& D^1_{(u,v)-r\mathbf{a}} \arrow[d] \arrow[dr] & \\
& \vdots \arrow[d] && \ddots \arrow[dr] & \vdots \arrow[d] & \ddots \\
& \mathrm{H}^{\eta_0+1}(n) \arrow[rrr, bend left, "\beta^{u,v-\mathbf{a}}"] \arrow[dd] &&& D^1_{(u,v)-\mathbf{a}} \arrow[dd] \arrow[dr] & \\
&&& \ddots \arrow[dr] && \ddots \\
& \mathrm{H}^{\eta_0}(n) \arrow[rrr, bend left, "\beta^{u,v}"] \arrow[dddr, "\rho^{\eta_0}(n)"'] &&& D^1_{(u,v)} \arrow[dd] \arrow[dr] & \\
&&& \ddots \arrow[dr] && \ddots \\
&&&& 0 \arrow[d, equal] \arrow[dr] & \\
&& \lim_{\beta<\eta_0} \mathrm{H}^{\beta}(n) \arrow[d] && \vdots & \ddots \\
&& \vdots \arrow[d] &&& \\
&& \mathrm{H}^{\beta}(n) \arrow[d] &&& \\
&& \vdots \arrow[d] &&& \\
&& \mathrm{H}^{1}(n) &&&
\end{tikzcd}
$$

Matching Eventually Vanishing Exact Couples to an Inverse Tower

$$
\begin{tikzcd}
 & \ker(\rho^{\lambda}(n)) & \frac{\lim_{\eta<\lambda} F^{\eta}(n)}{F^{\lambda}(n)} \arrow[l, tail] \arrow[r, tail] & \frac{\ker(\mathrm{H}^{\lambda}(n) \to \mathrm{H}^{1}(n))}{F^{\lambda}(n)} \arrow[d] \\
 & & \lim^{1}_{\eta<\lambda} F^{\eta}(n) & \lim_{\eta<\lambda} \frac{\ker(\mathrm{H}^{\lambda}(n) \to \mathrm{H}^{1}(n))}{F^{\eta}(n)} \arrow[l, two heads] \arrow[d] \\
 & & & \vdots \arrow[d] \\
\lim^{1}_{r} \ker(i^{r}_{(u,v)-r\mathbf{a}}) & & & \\
E^{\infty}_{(u,v)-r\mathbf{a}-\mathbf{b}} \arrow[u] & \epsilon^{u,v-r\mathbf{a}} \arrow[l, tail] & \frac{F^{\eta_0+r-1}(n)}{F^{\eta_0+r}(n)} \arrow[l, tail] \arrow[r, tail] & \frac{\ker(\mathrm{H}^{\lambda}(n) \to \mathrm{H}^{1}(n))}{F^{\eta_0+r}(n)} \arrow[d] \\
E^{\infty}_{(u,v)-(r-1)\mathbf{a}-\mathbf{b}} & \epsilon^{u,v-(r-1)\mathbf{a}} \arrow[l, tail] & \frac{F^{\eta_0+r-2}(n)}{F^{\eta_0+r-1}(n)} \arrow[l, tail] \arrow[r, tail] & \frac{\ker(\mathrm{H}^{\lambda}(n) \to \mathrm{H}^{1}(n))}{F^{\eta_0+r-1}(n)} \arrow[d] \\
 & & & \vdots \arrow[d] \\
 & \ker(\rho^{\eta_0}(n)) & \frac{\lim_{\beta<\eta_0} F^{\beta}(n)}{F^{\eta_0}(n)} \arrow[l, tail] \arrow[r, tail] & \frac{\ker(\mathrm{H}^{\lambda}(n) \to \mathrm{H}^{1}(n))}{F^{\eta_0}(n)} \arrow[d] \\
 & & \lim^{1}_{\beta<\eta_0} F^{\beta}(n) & \lim_{\beta<\eta_0} \frac{\ker(\mathrm{H}^{\lambda}(n) \to \mathrm{H}^{1}(n))}{F^{\beta}(n)} \arrow[l, two heads] \arrow[d] \\
 & & & \vdots \arrow[d] \\
 & & & \frac{\ker(\mathrm{H}^{\lambda}(n) \to \mathrm{H}^{1}(n))}{F^{\beta}(n)} \arrow[d] \\
 & & & \vdots \arrow[d] \\
 & & & \frac{\ker(\mathrm{H}^{\lambda}(n) \to \mathrm{H}^{1}(n))}{F^{0}(n)} \arrow[d] \\
 & & & 0
\end{tikzcd}
$$

E^{∞}-Distribution Diagram Corresponding to Matching Eventually Vanishing Exact Couples to an Inverse Tower

$$
\begin{tikzcd}
& \lim_{\eta<\lambda} \pi_n(X_\eta) \arrow[d] & & \lim_{\eta<\lambda} \pi_{n-1}(X_\eta) \arrow[d] & & \\
& \vdots \arrow[d] & & \vdots \arrow[d] & & \\
& \lim_r \pi_n(X_{\eta_0+r}) \arrow[d] & & \lim_r \pi_{n-1}(X_{\eta_0+r}) \arrow[d] & & \\
& \vdots \arrow[d] & & \vdots \arrow[d] & & \\
\cdots \arrow[r] & \pi_n(X_{\eta_0+r}) \arrow[r] \arrow[d] & \pi_{n-1}(F_{\eta_0+r+1}) \arrow[r] & \pi_{n-1}(X_{\eta_0+r+1}) \arrow[r] \arrow[d] & \pi_{n-2}(F_{\eta_0+r+2}) \arrow[r] & \cdots \\
\cdots \arrow[r] & \pi_n(X_{\eta_0+r-1}) \arrow[r] \arrow[d] & \pi_{n-1}(F_{\eta_0+r}) \arrow[r] & \pi_{n-1}(X_{\eta_0+r}) \arrow[r] \arrow[d] & \pi_{n-2}(F_{\eta_0+r+1}) \arrow[r] & \cdots \\
& \vdots \arrow[d] & & \vdots \arrow[d] & & \\
\cdots \arrow[r, "\cong"] & \pi_n(X_{\eta_0}) \arrow[r] \arrow[d] \arrow[dddr] & \pi_{n-1}(F_{\eta_0+1}) \arrow[r] & \pi_{n-1}(X_{\eta_0+1}) \arrow[r] \arrow[d] & \pi_{n-2}(F_{\eta_0+2}) \arrow[r] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & \pi_{n-1}(F_{\eta_0}) \arrow[r, "\cong"] & \pi_{n-1}(X_{\eta_0}) \arrow[r] \arrow[d] \arrow[dddr, "\rho^{\eta_0}(n-1)"'] & \pi_{n-2}(F_{\eta_0+1}) \arrow[r] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] & 0 \arrow[r] \arrow[d] & \pi_{n-2}(F_{\eta_0}) \arrow[r, "\cong"] & \cdots \\
& \vdots & \lim_{\beta<\eta_0} \pi_n(X_\beta) \arrow[d] & \vdots & & \\
& & \vdots \arrow[d] & & \lim_{\beta<\eta_0} \pi_{n-1}(X_\beta) \arrow[d] & \\
& & \pi_n(X_\beta) \arrow[d] & & \vdots \arrow[d] & \\
& & \vdots \arrow[d] & & \pi_{n-1}(X_\beta) \arrow[d] & \\
& & \pi_n(X_1) & & \vdots \arrow[d] & \\
& & & & \pi_{n-1}(X_1) &
\end{tikzcd}
$$

Transfinite Tower of Fibrations